CLASSES OF DENSELY DEFINED MULTIPLICATION AND TOEPLITZ OPERATORS WITH APPLICATIONS TO EXTENSIONS OF RKHS'S

By

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I dedicate this to mother, Katherine Vann and my brother, Spencer Rosenfeld.

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While bounded multiplication has been extensively researched, unbounded multiplication has received little attention until more recently. We develop a framework for densely defined multiplication over reproducing kernel Hilbert spaces, and we find an application toward extending reproducing kernels. We also extend a result of Allen Shields, who showed that the multipliers for the Sobolev space are precisely the elements of that space. We show that this holds even if multipliers are merely densely defined.

In connection with multiplication operators, we explore densely defined Toeplitz operators. Here we find simpler proofs of theorems from Sarason and Suarez. We also produce a partial answer to a problem posed by Donald Sarason by constructing an algorithm for recovering the symbol for a class of densely defined Toeplitz operators.

CHAPTER 1
INTRODUCTION

We have two goals to reach in this Dissertation. The first is the discussion of reproducing kernel presentations and extensions of reproducing kernel Hilbert spaces. The second goal is to explore densely defined Toeplitz operators over the Hardy space. The link between both of these ideas is the concept of densely defined multiplication operators and their interplay with the reproducing kernels of a reproducing kernel Hilbert space.

The topic of bounded multiplication on function spaces is a subject that has received extensive study. The landmark result of Pick in the space of bounded analytic functions is approaching its centennial anniversary, and the study of bounded multiplication is still older than this. Around the same time Pick and Nevanlinna both proved what is now called the Nevanlinna-Pick Interpolation theorem [1]:

Theorem 1.1. *Given two collections of points in the unit disc* $\{z_1, z_2, ..., z_k\}$ *and* $\{w_1, w_2, ... w_k\}$ *there exists a bounded analytic function for which* $f(z_i) = w_i$ *if and only if the matrix* $((1 - \bar{w}_i w_j)(1 - \bar{z}_i z_j)^{-1})_{i,j=1}^{k}$ *is positive definite.*

Work on this interpolation result has brought much attention to bounded multipliers. In particular the space of bounded analytic functions in the disc, H^∞, is precisely the collection of bounded multiplication operators on the Hardy Space. This work has found applications not only in pure mathematics, but also in signal processing where multipliers appear as transfer functions for linear systems. There has also been a great deal of development of the aptly named branch of control theory: H^∞-control. Here the Nevanlinna-Pick property is used explicitly.

Unbounded multiplication has received much less attention. Until recently there were only a handful of exceptions. One is in the Fock space, where the multiplication operator M_z is densely defined and whose adjoint is the derivative. In the Bergman space over a region Ω, M_z is well recognized to be unbounded when the domain Ω itself is unbounded. In [2], Kouchekian has investigated the question of when this operator is densely defined. Densely defined multiplication has also been explored in de Branges Rovnyak spaces [3]. Recently Aleman, Martin, and Ross worked on symmetric and self adjoint densely defined operators (which included multiplication operators) [4].

The first explicit characterization of a collection of densely defined multiplication operators was given by Sarason in 2008 when he characterized the analytic Toeplitz operators over the Hardy space. He found that these were precisely the operators with symbols in the Smirnov class, N^+.

$$N^+ := \{f : \mathbb{D} \to \mathbb{C} \mid f(z) = b(z)/a(z) \text{ where } b, a \in H^2 \text{ and } a \text{ is outer. } \}$$

Here we begin a systematic study of densely defined multiplication operators over reproducing kernel Hilbert spaces. These multipliers, while not necessarily continuous, are all closed operators given their natural domain.

In the second chapter we visit several preliminary results about reproducing kernel Hilbert spaces (RKHS), the Hardy space, the Fock space, and unbounded operators. Chapter 3 introduces reproducing kernel presentations and an application of densely defined multipliers toward extending a RKHS. Chapter 4 explores these extensions and discusses densely defined multipliers over the Hardy, Fock and the Polylogarithmic Hardy space.

In chapter 5, we turn our attention to Sobolev spaces. We improve upon a result of Allen Shields found in Halmos' *A Hilbert Space Problems Book* [5]. There Halmos asked if there was a space of functions whose bounded multipliers were exactly the functions in the space. For instance, in the Hardy space only a proper subspace of functions are its multipliers.

Shields showed that the Sobolev Space is such a space of functions. It turns out that in this same space, every densely defined multiplication operator is bounded (and hence in the space). This sharpening of Shields' result is the content of chapter 5. In addition we show that there are unbounded multipliers for some subspaces arising from choices of boundary conditions. This same Sobolev space was shown to have the Nevanlinna-Pick property as well. This was proved by Agler in 1990 [6].

Finally in Chapter 6, we return to Sarason's Unbounded Toeplitz operators. At the end of his expository article in 2008, Sarason proposed a collection of algebraic conditions to classify Toeplitz operators. He left with a question: Are these algebraic properties enough to recover a symbol (broadly interpreted) of a Toeplitz operator? We introduce the Sarason sub-symbol and use it to find simpler proofs for some known results by Suarez and Sarason. We also show that it can recover the symbols for a large class of densely defined Toeplitz operators.

CHAPTER 2
PRELIMINARIES

2.1 Reproducing Kernel Hilbert Spaces and Common Examples

Let X be a set and let $\mathcal{H}$ be a Hilbert space of functions $f : X \to \mathbb{C}$. We say $\mathcal{H}$ is a *reproducing kernel Hilbert space* (RKHS) if for each $x \in X$ the linear functional $E_x : \mathcal{H} \to \mathbb{C}$ given by $E_x f = f(x)$ is bounded. Recall that for any bounded linear functional $L : \mathcal{H} \to \mathbb{C}$ can be represented as $Lf = \langle f, y \rangle$ for a unique $y \in \mathcal{H}$. This means for each $x \in X$ there is a unique k_x for which $\langle f, k_x \rangle = E_x f = f(x)$.

We call k_x the reproducing kernel for x, and we call $K(x, y) = \langle k_x, k_y \rangle$ the reproducing kernel function of $\mathcal{H}$. The kernel functions have a nice interplay with what are called function theoretic operators. We are principally concerned with multiplication operators, which are defined as follows.

Definition 1. *Let $\phi : X \to \mathbb{C}$ for which $\phi f \in \mathcal{H}$ for every $f \in \mathcal{H}$. We say that the operator $M_\phi : \mathcal{H} \to \mathcal{H}$ given by $M_\phi f = \phi f$ is a (bounded) multiplication operator.*

When M_ϕ can be applied to all of the functions in $\mathcal{H}$, the closed graph theorem tells us that M_ϕ is a bounded operator. Bounded multiplication is an extensively studied area of operator theory that goes at least as far back as Pick in 1916 [1]. A comprehensive reference for this study can be found in Agler and McCarthy's book published in 2002 [7]. This also includes much of what we will discuss about reproducing kernels. However, the standard reference on reproducing kernels was published by Aronszajn in 1950 [8].

We will be discussing several RKHSs in this dissertation, but we will focus on three in particular. The first is the Hardy space H^2, which can be viewed as those functions f in

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{C} \mid \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta < \infty \right\}$$

for which

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} d\theta = \langle f, z^n \rangle = 0$$

for all $n < 0$.

Another convenient way to define the Hardy space is as those functions f which are analytic in the disc (writing $f(z) = \sum_{n=0}^{\infty} a_n z^n$) and for which the quantity

$$(2\pi)^{-1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} r^{2n} |a_n|^2$$

stays bounded as $r \to 1^-$. Fatou's theorem tells us that for such f, the auxiliary function $\tilde{f}(e^{i\theta}) := \lim_{r \to 1^-} f(re^{i\theta})$ is defined almost everywhere and $\tilde{f} \in L^2(\mathbb{T})$. Moreover $\left\langle \tilde{f}, z^n \right\rangle = a_n$ for $n \geq 0$.

This enables us to express the inner product in two forms. Take $f = \sum_{n=0}^{\infty} a_n z^n$ and $g = \sum_{n=0}^{\infty} b_n z^n$ we write their inner product as:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

Note that we know $\{z^n\}_{n=0}^{\infty}$ is an orthonormal basis for H^2. This follows from the fact that it is a closed subspace of $L^2(\mathbb{T})$. The properties of H^2 that we will be exploring in these preliminaries can be found in Douglas's book *Banach Algebra Techniques in Operator Theory* [9], Hoffman's *Banach Spaces of Analytic Functions* [10], and also Duren's *Theory of H^p spaces* [11]. We can also find them in Rudin's *Real and Complex Analysis* [12]. The Hardy space was first introduced by Reisz in 1923 [13].

Another space of analytic functions that is important to our discussions is the Fock space, denoted by F^2. This is also known as the Bargmann-Segal space. The Fock space is a Hilbert space of entire functions $f(z)$ for which the following inequality holds:

$$\|f\|_{F^2}^2 = (2\pi)^{-1} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} dA(z) < \infty.$$

Here the quantity $dA(z)$ indicates that we are taking the standard area integral over the complex plane. This is contrasted with the Hardy space, where we were integrating with respect to arc length on the circle. As with any Hilbert space, the inner product for the

Fock space can be determined by its norm and is given by:

$$\langle f, g \rangle_{F^2} = (2\pi)^{-1} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} dA(z).$$

It is easy to see that the monomials are in the Fock space and are mutually orthogonal (this follows after a computation in polar coordinates). The norm of z^n can be computed recursively by the use of integration by parts and yields $\|z^n\|_{F^2} = \sqrt{n!}$. The set $\{z^n/\sqrt{n!}\}$ is an orthonormal basis for the Fock space. Moreover the Taylor polynomials of f converge to f in the F^2 norm. We can use this to determine if an entire function is in F^2:

Lemma 1. *Let f be an entire function. Then $f \in F^2$ iff $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and its Taylor coefficients satisfy $\sum_{n=0}^{\infty} n!|a_n|^2 < \infty$.*

These facts can be found in Kehe Zhu's book *Analysis on the Fock Spaces* [14], and in J. Tung's dissertation *Fock Spaces* [15]. A lot of the initial work on this space was done by Bargmann in 1961 [16].

Finally the third space we will be considering is the Sobolev space $W^{1,2}[0,1]$. This will be the focus of Chapter 5. The Sobolev space is the collection of functions $f : [0,1] \to \mathbb{C}$ that are absolutely continuous and whose derivatives are in $L^2[0,1]$. Absolute continuity guarantees the existence of the derivative almost everywhere (with respect to Lebesgue measure), and it also is the type of continuity that guarantees $f(x) = \int_0^x f'(t)dt + f(0)$. The inner product is given by

$$\langle f, g \rangle_{W^{1,2}[0,1]} = \int_0^1 f(t)\overline{g(t)} + f'(t)\overline{g'(t)} dt.$$

2.2 Reproducing Kernels and Kernel Functions

Having laid out the definitions of the spaces in the previous section, we now set out to determine the kernel functions of each space. There is a useful theorem in this direction for determining $K(x, y)$ for a general RKHS $\mathcal{H}$.

Theorem 2.1 (Bergman)**.** *Let $K(z,w)$ be the reproducing kernel for a RKHS, $\mathcal{H}$, over a set X. If $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$ then we can write $K(z,w) = \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z)$.*

Proof. For each $w \in X$ we have $K(z,w) = k_w(z) \in H$. We can then write

$$k_w(z) = \sum_{n=0}^{\infty} \langle k_w, e_n \rangle e_n(z) = \sum_{n=0}^{\infty} \overline{\langle e_n, k_w \rangle} e_n(z) = \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z).$$

□

This expedites the process of finding the kernel function once a space is known to have bounded point evaluations. Notice that for the Hardy space the monomials form an orthonormal basis. Take $e_n(z) = z^n$ and we see that for $w \in \mathbb{D}$,

$$K_{H^2}(z,w) = \sum_{n=0}^{\infty} \overline{e_n(w)} e_n(z) = \sum_{n=0}^{\infty} \bar{w}^n z^n = \frac{1}{1 - \bar{w}z}.$$

We can prove that the Hardy space has bounded point evaluations retroactively. Let $w \in \mathbb{D}$ and set $k_w = (1 - \bar{w}z)^{-1} \in H^2$. If $f \in H^2$, then

$$|E_w f| = |f(w)| = \left| \sum_{n=0}^{\infty} a_n w^n \right| = \left| \sum_{n=0}^{\infty} a_n \overline{\overline{w}^n} \right| = |\langle f, k_w \rangle| \leq \|f\| \|k_w\|.$$

Thus by Cauchy-Schwarz we see for arbitrary $w \in \mathbb{D}$ we have that E_w is a bounded functional. Similarly we can produce kernels for the Fock space by the same calculations finding that

$$K_{F^2}(z,w) = \sum_{n=0}^{\infty} \frac{\bar{w}^n z^n}{n!} = e^{\bar{w}z} = k_w(z).$$

It can be verified that $k_w(z) \in F^2$ for all $w \in \mathbb{C}$ by Lemma 1. In the Fock space, the reproducing kernels have a physical significance: The reproducing kernels of the Fock space are the coherent states to the quantum harmonic oscillator.

For the Sobolev space the reproducing kernel for $s \in [0,1]$ is the function

$$K_{W^{1,2}[0,1]}(t,s) = k_s(t) = \begin{cases} a_s e^t + b_s e^{-t} & : \quad t \leq s \\ c_s e^t + d_s e^{-t} & : \quad t \geq s \end{cases}$$

where

$$a_s = b_s = \frac{e^s + e^2 e^{-s}}{2(e^2-1)},\ c_s = \frac{e^s + e^{-s}}{e^2-1}, \text{and } d_s = e^2 c_s.$$

This can be found in [6] among other places. The reproducing kernel is sometimes called the Green's function in this context.

Now that we have established some examples of RKHSs, let's take a look at a more abstract definition. Recall that a $n \times n$ matrix A is positive if for every vector $\alpha = (\alpha_1, ...\alpha_n)^T \in \mathbb{C}^n$ we have

$$\langle A\alpha, \alpha \rangle = \sum_{i,j=1}^{n} \bar{\alpha}_j \alpha_i A_{i,j} \geq 0.$$

Let us fix $x_1, x_2, ..., x_n \in \mathbb{C}$ and consider the matrix $A = (K(x_i, x_j))_{i,j=1}^n$. The following computation tells us that this matrix is positive:

$$\begin{aligned} \langle A\alpha, \alpha \rangle &= \sum_{i,j=1}^{n} \bar{\alpha}_j \alpha_i K(x_i, x_j) = \sum_{i,j=1}^{n} \bar{\alpha}_j \alpha_i \left\langle k_{x_i}, k_{x_j} \right\rangle \\ &= \left\langle \sum_{i=1}^{n} \alpha_i k_{x_i}, \sum_{j=1}^{n} \alpha_j k_{x_j} \right\rangle = \left\| \sum_{i=1}^{n} \alpha_i k_{x_i} \right\|^2 \geq 0. \end{aligned}$$

This leads us to the definition of a *kernel function*.

Definition 2. *Let X be a set and $K : X \times X \to \mathbb{C}$ be a function of two variables. Then K is called a* kernel function *if for every n and every choice of distinct points $\{x_1, ..., x_n\} \in \mathbb{C}^n$ the matrix $(K(x_i, x_j))_{i,j=1}^n \geq 0$.*

The **Moore-Aronszajn Theorem** tells us that every kernel function is a reproducing kernel for some RKHS. The proof of this involves a GNS style construction to make the Hilbert Space, and can be found in Paulsen's notes [17]. In every instance we use a kernel function the space will be explicitly defined.

2.3 Bounded Multiplication Operators

We will be exploring the multiplication operators over these spaces. The following lemma helps us see an important difference between the multipliers of these spaces.

Lemma 2. *If $\mathcal{H}$ is a reproducing kernel Hilbert space over a set X and ϕ is the symbol of a bounded multiplication operator M_ϕ, then $M_\phi^* k_x = \overline{\phi(x)} k_x$ for all $x \in X$. Moreover, ϕ is a bounded function.*

Proof. Let $f \in \mathcal{H}$ and $x \in X$. Now let us consider

$$\left\langle f, M_\phi^* k_x \right\rangle = \langle \phi \cdot f, k_x \rangle = \phi(x) f(x) = \phi(x) \langle f, k_x \rangle = \left\langle f, \overline{\phi(x)} k_x \right\rangle.$$

This holds for all $f \in \mathcal{H}$, so $M_\phi^* k_x = \overline{\phi(x)} k_x$ as desired. To see that ϕ is bounded, recall that the spectral radius of a bounded operator V is

$$r(V) = \sup\{|\lambda| \in \mathbb{C} \mid V - \lambda I \text{ is not invertible } \}.$$

In particular $r(V)$ is at least as large as any eigenvalue. Also the norm of an operator is in general larger than the spectral radius. Thus for all $x \in X$ we have:

$$|\phi(x)| \leq r(M_\phi) \leq \|M_\phi\|.$$

□

We also need the following lemma:

Lemma 3. *If D is a dense subspace of a RKHS $\mathcal{H}$ over a set X, then for all $x \in X$ there is a function $f \in D$ for which $f(x) \neq 0$.*

Proof. Suppose this were not true, then there is an $x \in X$ such that for all $f \in D$ we have $\langle f, k_x \rangle = f(x) = 0$. Thus $D \subset \{k_x\}^\perp$ and is not dense. □

From this we can conclude that if M_ϕ is a bounded multiplication operator for a RKHS $\mathcal{H}$ of analytic functions on a set X, then ϕ is analytic on X as well. Indeed, note that for each $x \in X$ there is a function $f \in \mathcal{H}$ such that $f(x) \neq 0$. Since $M_\phi f = \phi f \in \mathcal{H}$, the function $h := \phi f$ is analytic on X. Finally $\phi(x) = h(x)/f(x)$ is a ratio of analytic function where the denominator is nonzero. Thus ϕ is analytic on X.

Now let us look at the Hardy space. If ϕ is the symbol for a bounded multiplicaiton operator $M_\phi : H^2 \to H^2$, then $\phi = M_\phi \cdot 1 \in H^2$. Moreover, ϕ is a bounded analytic function on the disc.

On the other hand, if ϕ is a bounded analytic function on the disc, then there is an $M > 0$ so that $|\phi(z)| < M$ for all $z \in \overline{\mathbb{D}}$. If $f \in H^2$, then

$$\|M_\phi f(z)\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})\phi(e^{i\theta})|^2 dz \leq M^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2 dz = M^2\|f\|^2.$$

Thus we have proved:

Theorem 2.2. *$\phi \in H^\infty$ iff M_ϕ is a bounded multiplication operator on H^2.*

For the Fock space, we find something very different. There are enough multipliers over the Hardy space that the collection of symbols is dense in H^2. However, if ψ is the symbol for a bounded multiplication operator over the Fock space, then ψ is a bounded entire function. By Liouville's theorem, ψ is constant. Thus we see the Fock space has only trivial bounded multipliers.

The Sobolev space provides a different kind of example for bounded multiplication operators. In Halmos' *A Hilbert Space Problems Book* [5], he asks if there is a Hilbert space of functions for which the collection symbols of the bounded multiplication operators is the space itself.

Allen Shields sent a note to Halmos that proved the Sobolev space $W^{1,2}[0, 1]$ is such a space. The content of Chapter 3 is a sharpening of this result to include densely defined multiplication operators, which we will define shortly. Lemma 3 is essential to the proof of this fact.

2.4 Densely Defined Operators

A bounded linear operator over a normed vector space N is a linear operator L for which there exists a $C > 0$ such that for all $y \in N$ we have $\|Ly\| \leq C\|y\|$. In particular,

L is Lipshitz continuous with respect to the norm metric on N. A stronger connection is given by the classic result:

Theorem 2.3. *[18] Let N be a vector space and $L : N \to N$ a linear operator over that space. The following are equivalent:*

1. *L is bounded.*
2. *L is continuous.*
3. *L is continuous at 0.*

In this sense when we say an operator is unbounded it is synonymous with saying it is discontinuous. The unbounded operators we will be discussing here are all densely defined.

Definition 3. *Let $\mathcal{H}$ be a Hilbert space and let $D(T)$ be a dense subspace of $\mathcal{H}$. We say an operator whose domain is $D(T)$, $T : D(T) \to \mathcal{H}$, is* densely defined*.*

When we define a densely defined operator, we must also define the domain it is over. If T and T' are operators on $\mathcal{H}$ with domains $D(T) \subset D(T')$ and $Tf = T'f$ for all $f \in D(T)$, we say that T' extends T. We denote this with $T \subset T'$.

Example 1. *Let P be the set of all polynomials, and let $\frac{d}{dz}^{\circ} : P \to H^2$ be the operation of differentiation applied to the polynomials. Since P is dense in H^2, we see that $\frac{d}{dz}^{\circ}$ is densely defined. However, it is clear that the natural domain of the differentiation operator should be larger than just the space of polynomials.*

Indeed, consider the space of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in H^2 for which $\sum n^2 |a_n|^2 < \infty$. The derivative clearly takes this collection of functions into H^2. We will call this collection by $D(\frac{d}{dz})$. The operator $\frac{d}{dz} : D(\frac{d}{dz}) \to H^2$ extends $\frac{d}{dz}^{\circ}$ and so we write $\frac{d}{dz}^{\circ} \subset \frac{d}{dz}$.

While a densely defined operator may not be continous, we may still hold out hope for some sort of limit properties.

Definition 4. *Let $T : D(T) \to \mathcal{H}$ be a densely defined operator on a Hilbert space $\mathcal{H}$. We say that T is a* closed *operator if whenever $\{f_n\} \subset D(T)$ with $f_n \to f \in \mathcal{H}$ and $Tf_n \to h \in \mathcal{H}$ we have $f \in D(T)$ and $Tf = h$.*

Equivalently, we define an operator as closed if its graph $G(T) = \{(f, Tf) \mid f \in D(T)\}$ is a closed subset of $\mathcal{H} \times \mathcal{H}$ with respect to the norm topology. The closed graph theorem from Functional Analysis tells us that if T is everywhere defined and closed, then T is bounded. It can be shown that $\frac{d}{dz}$ is closed, whereas $\frac{d}{dz}^{\circ}$ is not.

The definition of adjoints for densely defined operators is a little bit more delicate. We define the domain of the adjoint as those functions $h \in \mathcal{H}$ for which the functional $L : D(T) \to \mathbb{C}$ given by $L(f) = \langle Tf, h\rangle$ is continuous. We write this domain as $D(T^*)$. The Riesz Representation Theorem tells us there is a unique vector, call it T^*h for which $L(f) = \langle f, T^*h\rangle$.

In many cases it might turn out that $D(T^*)$ contains only the zero vector. We would like to have a condition that would guarantee that $D(T^*)$ is not only nontrivial but is also dense in $\mathcal{H}$. The following is this condition. Note by *closable* we mean that T has a closed extension.

Theorem 2.4. *Let $T : D(T) \to \mathcal{H}$ be a densely defined operator on $\mathcal{H}$. If T is closable, then T^* is closed an densely defined. Moreover, T^{**} is the smallest closed extension of T.*

A more extensive treatment of densely defined operators can be found in Pedersen's *Analysis NOW* [19] and Conway's *A Course in Functional Analysis* [20].

2.5 Inner-Outer Factorization of H^2 Functions

A well known result from Complex Analysis is the Weierstrass factorization theorem. This theorem tells us that any entire function f can be factored into two other entire functions one of which is non-vanishing, and the other is a particular kind of product containing all of the zeros of f. Specifically:

Theorem 2.5 (Weierstrass Factorization Theorem)**.** *If f is an entire function with a root of order k at zero and $\{\omega_n\}_{n=1}^{\infty}$ is the sequence of non-zero roots of f, then*

$$f(z) = \left(z^k \prod_{n=1}^{\infty} E_n(z/\omega_n) \right) e^{h(z)}$$

where $h(z)$ is an entire function and

$$E_n(z) = (1-z) \exp\left\{ z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n} \right\}.$$

There is an analogous theorem for the Hardy space with one additional advantage. The exponential piece of the Weierstrass factorization can affect the average growth of f in an obvious way. It is less obvious that the average density of the zeros also contribute to the growth of our entire function f. In the Hardy space, the inner-outer factorization allows us to separate the zeros from the growth of a function in the disc.

Let $\alpha \in \mathbb{D}$. We call a function of the form

$$B_\alpha(z) = \frac{|\alpha|}{\alpha} \frac{\alpha - z}{1 - \bar{\alpha} z}$$

a Blaschke factor. A Blaschke factor has a zero at α and a simple pole at $1/\bar{\alpha}$ outside the disc. Notice that for all θ:

$$|B_\alpha(e^{i\theta})| = \left| \frac{\alpha - e^{i\theta}}{1 - \bar{\alpha} e^{i\theta}} \right| = |e^{-i\theta}| \left| \frac{\alpha - e^{i\theta}}{e^{-i\theta} - \bar{\alpha}} \right| = \frac{|\alpha - e^{i\theta}|}{|\overline{e^{i\theta} - \alpha}|} = 1.$$

If $f \in H^2$ and $\{\alpha_n\}_{n=1}^{\infty}$ is the zero sequence of f, then we call $B(z) = \prod B_{\alpha_n}(z)$ the Blaschke factor of f.

Proposition 2.1. *If B is the Blaschke factor of f, then $B(z)$ has the following properties:*

1. $\lim_{r \to 1^-} |B(re^{i\theta})| = 1$ *for almost every θ and $|B(z)| < 1$ for all $z \in \mathbb{D}$.*
2. $B \in H^2$ *and* $\|B\| = 1$.
3. $f/B \in H^2$, *f/B is nonvanishing, and* $\|f/B\| = \|f\|$.

We say that a function satisfying (1) in the above proposition is an inner function. For a function $f \in H^2$, we call

$$F(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|d\theta\right\}$$

the outer factor of f. We will also call $F(z)$ an outer function.

Proposition 2.2. *$F(z)$ does not vanish inside the disc. Moreover, $F(z) \in H^2$ is an outer function iff the set $\{F(z) \cdot p(z) : p(z)$ is a polynomial. $\}$ is dense in H^2.*

Finally if we wish to factor f, we need one other type of factor called the singular inner factor of f. This is an inner function without zeros that is positive at the origin.

Theorem 2.6 (Beurling's Inner-Outer Factorization [10])**.** *Every function $f \in H^2$ can be expressed as $f = BSF$ where B is the Blasche factor of f, F is the outer factor, and S is a singular inner factor. We call BS the inner factor of f.*

CHAPTER 3
REPRODUCING KERNEL PRESENTATIONS

Let f be an analytic function of the disc, and suppose that we have a sequence of points $\{\lambda_n\}$ that converge at the origin. A theorem from complex analysis tells us that if $f(\lambda_n) = 0$ for all n, then f is identically zero. This also tells us that if there are two functions f, g analytic in the disc and $f(\lambda_n) = g(\lambda_n)$ for all n, then $f(z) = g(z)$ for all $z \in \mathbb{D}$. We call such a set a uniqueness set for analytic functions in the disc. In particular what this tells us is that f is completely determined by a small subset of its domain. Given that we know f is an analytic function of the disc, we could start with the restriction of f to $\{\lambda_n\}$ and recover f by extension.

In this chapter we set out to define a notion of extension of a RKHS and work out a reasonable extension theory. One of the key features of a RKHS $\mathcal{H}$ over a set X is that the reproducing kernels have dense span inside of $\mathcal{H}$. For instance suppose that $\langle f, k_x \rangle = 0$ for all $x \in X$. This means that $f(x) = 0$ for each x by the reproducing property and hence $f \equiv 0$. Sometimes the set of kernels is referred to as an *over-complete* basis. A justification of this terminology can be seen by considering the Hardy space H^2 taking a subset $\{k_{\lambda_n}\}$ ($\{\lambda_n\}$ as above) of it's kernels. By the argument in the above paragraphs, we see that this collection also has a dense span in H^2.

The main question of this section is given a small subset of the kernel functions, is there a way of recovering the whole set of kernels? If we are given a RKHS $\mathcal{H}$ over a set X, can we determine a more natural domain Y that contains X for the functions in $\mathcal{H}$? In this chapter, we will answer this question affirmatively.

3.1 Definitions

Definition 5. *Let $\mathcal{H}$ be a Hilbert Space and $X \subset \mathcal{H}$ such that $\operatorname{span} X$ is dense in $\mathcal{H}$. The* reprodcing kernel presentation *(RKP) is defined to be the pair $(\mathcal{H}, X)$. It is viewed as a reproducing kernel Hilbert space where evaluation is defined as: $f(a) = \langle f, a \rangle$ for all $f \in \mathcal{H}$ and $a \in X$.*

Given any reproducing kernel Hilbert space $\mathcal{H}$ whose functions take values in $\mathbb{C}$ and its collection of kernels X, then the RKP defined by $(\mathcal{H}, X)$ is the same reproducing kernel Hilbert space (where we say $f(k_w) \equiv f(w)$). In this way we are abstracting the notion of a RKHS.

Definition 6. *Let* $(\mathcal{H}, X)$ *be a RKP and suppose* $g : X \to \mathbb{C}$*. If the set*

$$D(M_g) = \{f \in \mathcal{H} \mid \exists h \in \mathcal{H} \textit{ st. } h(x) = g(x)f(x) \forall x \in X\}$$

is dense in $\mathcal{H}$*, then we say that* $M_g : D(M_g) \to \mathcal{H}$*,* $M_g f = gf$*, is a* densely defined multiplier*. The collection of all multipliers for* $(\mathcal{H}, X)$ *is denoted by* $\mathcal{M}_X$*.*

When there is no confusion, we may simply say such a function g is a multiplier. Also for simplicity we will often say $g \in \mathcal{M}_X$ for $M_g \in \mathcal{M}_X$. For a densely defined operator the first question to ask is if it is closed. It turns out that this follows immediately from the reproducing property.

Proposition 3.1. *Let* $(\mathcal{H}, X)$ *be an RKP and let* M_g *be a multiplication operator on* $\mathcal{H}$ *with domain* $D(M_g)$ *given above. The operator* M_g *with this domain is closed.*

Proof. Let $\{f_n\}$ be a sequence of functions in $D(M_g)$ that converge in norm to a function $f \in \mathcal{H}$, and suppose that $M_g f_n \to h \in \mathcal{H}$. We wish to show that $f \in D(M_g)$ and $gf = h$. We start with $h(x)$:

$$h(x) = \langle h, k_x \rangle = \lim_{n \to \infty} \langle g f_n, k_x \rangle =$$

$$\lim_{n \to \infty} g(x) f_n(x) = \lim_{n \to \infty} g(x) \langle f_n, k_x \rangle = g(x) \langle f, k_x \rangle = g(x) f(x).$$

The limits above are justified since weak convergence (convergence inside the inner product) is controlled by norm convergence. Hence we have shown that $h(x) = g(x)f(x)$ for all $x \in X$, which means $f \in D(M_g)$ and $M_g f = h$. □

Once we know that an operator is closed and densely defined, we also know that M_g^* is closed and densely defined. Like their bounded counterparts, the reproducing kernels of a RKHS are eigenvectors of M_g^*.

Proposition 3.2. *Let $(\mathcal{H}, X)$ be an RKP, and let $g \in \mathcal{M}_X$. Every $x \in X$ is an eigenvector for M_g^* with eigenvalue $\overline{g(x)}$.*

Proof. In order to place the kernel $x \in X$ inside of $D(M_g^*)$ we must show that the linear operator $L : D(T) \to \mathbb{C}$ given by $L(f) = \langle M_g f, x\rangle$ is continuous. Note that $\langle M_g f, x\rangle = g(x)f(x)$ for all f. Since $f_n \to f$ in norm implies that $f_n \to f$ pointwise: $L(f_n) = g(x)f_n(x) \to g(x)f(x)$. Thus L is continuous.

Moreover,

$$\left\langle f, M_g^* x\right\rangle = \langle gf, x\rangle = g(x)f(x) = g(x)\langle f, x\rangle = \left\langle f, \overline{g(x)}x\right\rangle$$

for all $f \in D(M_g)$. Since $D(M_g)$ is dense in $\mathcal{H}$, this means $M_g^* x = \overline{g(x)}x$. □

We now see that there is a connection between densely defined multiplication operators and reproducing kernels. We define an extension of a RKP as follows:

Definition 7. *Let $(\mathcal{H}, X)$ be a RKP and suppose $X \subset Y \subset \mathcal{H}$, then the RKP $(\mathcal{H}, Y)$ is called an* extension *of $(\mathcal{H}, X)$.*

Given an RKP it is easy to find an extension. If $(\mathcal{H}, X)$ is an RKP then for any Y for which $X \subset Y \subset \mathcal{H}$ the pair $(\mathcal{H}, Y)$ immediately satisfies the definition of a RKP. For this reason alone, extensions as defined above are not interesting. However, there is a more compelling reason to look for another definition of extension.

Example 2. *Consider the trivlal RKP $(\mathcal{H}, \mathcal{H})$, and suppose that $g \in \mathcal{M}_{\mathcal{H}}$. M_g is a closed operator on $\mathcal{H}$ and has a closed densely defined adjoint for which every vector $f \in \mathcal{H}$ is an eigenvector. Immediately it follows that g must be a constant function.*

We can quickly justify this by taking two linearly independent vectors in $f, h \in \mathcal{H}$. This means $M_g^* f = \overline{g(f)}f$ and $M_g^* h = \overline{g(h)}h$, but also $M_g^*(f + h) = \overline{g(f + h)}(f + h)$. Expanding the left hand side we find:

$$\overline{g(f)}f + \overline{g(h)}h = \overline{g(f + h)}(f + h)$$

moving everything to the left,

$$(\overline{g(f)} - \overline{g(f+h)})f + (\overline{g(h)} - \overline{g(f+h)})h = 0.$$

Since f and h are linearly independent, $g(f) = g(f+h) = g(h)$. The vectors f and h were arbitrary linearly independent vectors, thus g is constant.

From this example we see that if we are trying to extend $(\mathcal{H}, X)$ to $(\mathcal{H}, \mathcal{H})$ then $\mathcal{M}_{\mathcal{H}}$ can be much smaller than $\mathcal{M}_X$. If we wish to have an interesting connection between an RKP and its extension, we should start by examining the densely defined multipliers. Before we define this stronger extension, we need a lemma.

Lemma 4. *Suppose $(\mathcal{H}, Y)$ is an RKP extending $(\mathcal{H}, X)$. If $g_1, g_2 \in \mathcal{M}_Y$ and $g_1|_X = g_2|_X$, then $g_1 = g_2$.*

Proof. Let g_1 and g_2 be as in the hypothesis. Let $g = g_1|_X = g_2|_X$. Then g is a densely defined multiplication operator for the RKP $(\mathcal{H}, X)$. Let $D(M_g)$ be the domain of M_g as a multiplier in $\mathcal{M}_X$, and take $D(M_{g_1})$ to be the domain of M_{g_1} as a multiplier in $\mathcal{M}_Y$. Note that $D(M_{g_1}) \subset D(M_g)$.

Take a point $y \in Y$ and let $f \in D(M_{g_1})$ be a function for which $f(y) = \langle f, y \rangle \neq 0$. If we fix $h = M_{g_1} f = M_g f$, this means $\langle h, y \rangle = g_1(y) \langle f, y \rangle$ and

$$g_1(y) = \frac{\langle h, y \rangle}{\langle f, y \rangle}$$

. However, the values of $\langle h, y \rangle$ and $\langle f, y \rangle$ do not depend on how g was extended. □

Lemma 4 tells us that X is a set of uniqueness for the multipliers in $\mathcal{M}_Y$. Compare this to the example at the beginning of the chapter where we saw that a sequence of points accumulating at the origin was a set of uniqueness for functions analytic in the disc.

Definition 8. *Let $(\mathcal{H}, Y)$ be an extension of $(\mathcal{H}, X)$. If every multiplier in $\mathcal{M}_X$ can be extended to be a multiplier in $\mathcal{M}_Y$, then we say $(\mathcal{H}, Y)$ is a* respectful extension *of $(\mathcal{H}, X)$.*

From Example 2 we see that not every extension is a respectful extension, and there should be some upper bound to how far we may extend a given presentation. In the next section we will explore these extensions, and find the maximal respectful extension of a presentation. The following two lemmas help us simplify our search for an extension:

Lemma 5. *Let* $(\mathcal{H}, X)$ *be an RKP and* $x_1, x_2 \in X$*. If* $x_1 + x_2 \in X$ *and* $g \in \mathcal{M}_X$*, then* $g(x_1) = g(x_2) = g(x_1 + x_2)$*.*

Proof. Let $f \in D(M_g)$ and set $h = gf \in \mathcal{H}$

First we have:

$$h(x_1 + x_2) = g(x_1 + x_2)f(x_1 + x_2) = g(x_1 + x_2)\langle f, x_1 + x_2\rangle = g(x_1 + x_2)(f(x_1) + f(x_2)).$$

We also can see that

$$h(x_1 + x_2) = \langle h, x_1 + x_2\rangle = h(x_1) + h(x_2) = g(x_1)f(x_1) + g(x_2)f(x_2).$$

Rearranging terms, we find the following relation:

$$(g(x_1 + x_2) - g(x_1))f(x_1) = (g(x_2) - g(x_1 + x_2))f(x_2).$$

Since f is an arbitrary element of $D(M_g)$ and $D(M_g)$ is a dense subspace of $\mathcal{H}$, this tells us that $g(x_1 + x_2) - g(x_1) = g(x_1 + x_2) - g(x_2) = 0$ and

$$g(x_1) = g(x_1 + x_2) = g(x_2).$$

□

Lemma 6. *If* $(\mathcal{H}, X)$ *is a RKP and* $\alpha : X \to \mathbb{C}$ *is a function such that* $\alpha(x) \neq 0$ *for all* $x \in X$*, then* $\mathcal{M}_X = \mathcal{M}_{X_\alpha}$ *where* $X_\alpha = \{\alpha(x)x \mid x \in X\}$*.*

Proof. Let $g \in \mathcal{M}_{X_\alpha}$, then identify $g(x) \equiv g(\alpha(x)x)$. For each element $f \in D(M_g)$ and for $gf = h \in \mathcal{H}$ the following holds for every $x_0 \in X$:

$$\langle h, \alpha(x_0)x_0 \rangle = g(x_0) \langle f, \alpha(x_0)x_0 \rangle$$

but then multiplying both sides by $\overline{\alpha(x_0)}^{-1}$:

$$\langle h, x_0 \rangle = g(x_0) \langle f, x_0 \rangle$$

Thus we see $g \in \mathcal{M}_X$, with the same dense domain. The other inclusion is shown identically. □

Lemma 6 allows us to only consider $x \in X$ for which $\|x\| = 1$. This also means for element in $\mathrm{span}(x) \cap X$ we only need to consider a single representative, since each of these elements are equivalent under the densely defined multipliers. We end this section with another straightforward example. This is complimentary to Example 2.

Example 3. *Let $X = \{e_n\}_{n\in\mathbb{N}}$ where $\{e_n\}$ is an orthonormal basis for a separable Hilbert space $\mathcal{H}$. Consider the RKP (H, X) and its set of multipliers $\mathcal{M}_X$. In this case we can view each multiplier g as a sequence indexed by the natural numbers.*

In this case, every sequence $\{g(e_n)\} \equiv g$ in the complex plane will be a multiplier. We can see this by showing that the subset $\mathcal{H}_0 := \{f \in \mathcal{H} \mid f = \sum_{n=0}^{N} \alpha_n e_n\}$ is in the domain of g. Since this subset is dense in $\mathcal{H}$, the multiplier will be densely defined. This follows trivially, since $h = \sum_{n=0}^{N} g(e_n)\alpha_n e_n \in \mathcal{H}$ is a function for which $h(e_n) = g(e_n)f(e_n)$ where $f = \sum_{n=0}^{N} \alpha_n e_n$.

3.2 RKP Extensions

Now that we have some concept of an extension of a RKP our next goal is to discover how we might find a respectful extension. The following is a minimal type of extension that takes advantage of continuity. In a sense, it tells us that if we are not already working with a weakly closed set of kernels, then our collection is too small.

Theorem 3.1 (The Trivial Extension)**.** *Let $(\mathcal{H}, K)$ be a RKP, and suppose $g \in M_K$. In this case, g is weakly continuous on $K \setminus \{0_\mathcal{H}\}$. Moreover g extends to be a multiplier on $\overline{K}^{wk} \setminus \{0_\mathcal{H}\}$.*

Proof. Let $\{k_n\} \subset K \subset \mathcal{H}$ and suppose $k_n \to^{wk} k \in K$. Therefore we have $\langle f, k_n \rangle \to \langle f, k \rangle$ for all $f \in \mathcal{H}$. Thus f is weakly continuous on K when we define $f(z) := \langle f, z \rangle$.

Let $g \in \mathcal{M}_K$, and let $z_0 \in K \setminus \{0_\mathcal{H}\}$. There exists some $f \in D(M_g)$ such that $f(z_0) \neq 0$, since $D(M_g)$ is a dense subset of $\mathcal{H}$. Let $h \in \mathcal{H}$ where $h(z) = g(z)f(z)$. From the previous paragraph, h and f are weakly continuous at z_0, so $h(z)/f(z) = g(z)$ is weakly continuous at z_0. Since z_0 was an arbitrary element of K, g is weakly continuous on K.

Let $z_0 \in \overline{K}^{wk} \setminus \{0_\mathcal{H}\}$ and suppose $\{k_n\}$ is a sequence in K converging to z_0 weakly. Let $f \in D(M_g)$ such that $f(z_0) \neq 0$ and let $h(z) = g(z)f(z) \in H$ for all $z \in K$. Since $f(k_n) \to f(z_0) \neq 0$ and $h(k_n) \to h(z_0)$ this means $h(k_n)/f(k_n)$ converges in $\mathbb{C}$. In addition the values of $h(k_n)/f(k_n) = g(k_n)$ are independent of the choice of f. Thus we can define $g(z_0)$ uniquely as this limit, and g extends to be a multiplier on $\overline{K}^{wk} \setminus \{0_\mathcal{H}\}$. □

Theorem 3.1 tells us that we can extend our multipliers from K to its weak closure. It also shows us that every multiplier must be continuous on K at the very least. This notion has a lot of value when we set out to classify collections of densely defined multipliers.

For the general case, we can only extract the continuity of our multipliers. However, if we are dealing with a space of analytic functions, then the same proof yields that multipliers must also be analytic. In particular if $g \in \mathcal{M}_\mathbb{C}$ for $(F^2, \mathbb{C})$, then g must be entire. So we can conclude that, loosely speaking, if $g \in \mathcal{M}_X$ for some RKP $(\mathcal{H}, X)$, then g is as smooth as the members of $\mathcal{H}$.

We will see that this method of extension is weaker than the one we will find for the Hardy Space in Example 4. There we will take advantage of the analyticity of the functions in H^2 to extend the multipliers. This brings up the question: how far can we expect to extend the space in the general case? We know there is some limitation, since not all extensions are respectful. Jury and McCullough found that there is a maximal extension that can be found through the densely defined multiplication operators themselves.

Theorem 3.2 (JM-Extension Theorem)**.** *Let $(\mathcal{H}, X)$ be an RKP with multipliers $\mathcal{M}_X$. Then there is a unique largest set Y with $X \subset Y \subset \mathcal{H}$ such that $(\mathcal{H}, Y)$ is a respectful extension of $(\mathcal{H}, X)$; this Y is equal to the set of all common eigenvectors for the operators M_g^*, $g \in \mathcal{M}_X$.*

Proof. Suppose that $g \in \mathcal{M}_X$. If g can be extended to be in $\mathcal{M}_Y$, then each $y \in Y$ is an eigenvector for M_g^* by Proposition 3.2.

On the other hand suppose Y is the collection of all vectors y for which $y \in D(M_g^*)$ for all $g \in \mathcal{M}_X$ and y is an eigenvector for each $g \in \mathcal{M}_X$. For each $g \in \mathcal{M}_X$ we have $M_g^* y = \lambda y$. Take $g(y) = \bar{\lambda}$. We must show that each g extended in such a fashion is in $\mathcal{M}_Y$.

Let D denote the domain of M_g for the RKP $(\mathcal{H}, Y)$. We will show that $D = D(M_g)$, which means that $g \in \mathcal{M}_Y$. It follows immediately by taking restrictions to X that $D \subset D_X(M_g)$ (the domain as a multiplier in $\mathcal{M}_X$). For the other inclusion we will use the fact that M_g with the domain $D_X(M_g)$ is a closed densely defined operator, hence $M_g = M_g^{**}$.

Thus $f \in D(M_g^{**})$, and it follows from the definition of the domain of an adjoint that the following linear functional is bounded on $\operatorname{span} Y$:

$$L_f : \sum c_j y_j \to \left\langle f, M_g^* \sum c_j y_j \right\rangle = \sum c_j g(y_j) \langle f, y_j \rangle .$$

Therefore there is a unique $h \in \mathcal{H}$ for which $L_f y = \langle h, y \rangle$ for all y. In particular, $h(y) := \langle h, y \rangle = \left\langle f, M_g^* y \right\rangle = g(y) \langle f, y \rangle = g(y) f(y)$. Thus $f \in D$ and $D_X(M_g) \subset D$. Hence $D = D_X(M_g)$ and $g \in \mathcal{M}_Y$.

Finally notice that since $X \subset Y$, $h(x) = g(x)f(x)$ for all x. Which means $M_g f = h$ whether g is taken to be in $\mathcal{M}_X$ or $\mathcal{M}_Y$. The same h that worked for X works for Y. $\square$

This theorem provides a target for our extension theory. We wish not only to find a respectful extension of a given RKP, but we want to find a maximal one. Using this theorem can be overwhelming. Often the space of multipliers is uncountable, so barring special circumstances, we would not expect to be able to find out if a given RKP is indeed maximal. In spite of this, later we will show that $(H^2, \mathbb{D})$ is a maximal RKP.

CHAPTER 4
CLASSES OF DENSELY DEFINED MULTIPLICATION OPERATORS

4.1 Hardy Space

Recall that a function $\phi \in N^+$ is the ratio of two H^2 functions, where the denominator can be chosen to be outer. The choice of functions can be made much more specific and leads to a canonical representation for ϕ.

Theorem 4.1. *[21] If $\phi \in N^+$, then there exists a unique pair of functions $b, a \in H^\infty$ for which $|b|^2 + |a|^2 = 1$, a is outer and $\phi = b/a$.*

In Sarason's paper *Unbounded Toeplitz Operators* [21] he characterized the densely defined multiplication operators of H^2 as follows:

Theorem 4.2 (Sarason)**.** *The function $\phi : \mathbb{D} \to \mathbb{C}$ is the symbol for a densely defined multiplication operator on H^2 iff ϕ is in the Smirnov class N^+. Moreover, if $\phi = b/a$ is the canonical representation of ϕ, then $D(M_\phi) = aH^2$.*

Proof. Let $D(M_\phi)$ be the domain of M_ϕ. Since M_ϕ is densely defined, this means that $D(M_\phi)$ is nontrivial. Let $f \in D(M_\phi)$ and suppose $f \not\equiv 0$. Then $\phi = \phi f/f$ is the ratio of two H^2 functions and is in the Nevanlinna class N.

Now suppose that $\phi = \psi/\chi$ where ψ and χ have relatively prime inner factors. Take $f \in D(M_\phi)$, and set $g = \phi f$. Then $\chi g = \psi f$.

Now let ψ_0 and χ_1 be the outer factor for ψ and the inner factor for χ respectively. Since ψ and χ have relatively prime inner factors this means that $\psi_0 f \in \chi_1 H^2$, and hence $f \in \chi_1 H^2$. Thus we conclude that $D(M_\phi) \subset \chi_1 H^2$. Since $D(M_\phi)$ is dense, and $\chi_1 H^2$ is dense only if χ_1 is constant. Thus we have χ_1 is constant. We conclude that χ is an outer function.

For the other direction, suppose $\phi \in N^+$. Let $\phi = b/a$ be its canonical representation where a is outer, then aH^2 is dense in H^2. Moreover, $aH^2 \subset D(M_\phi)$ trivially, and ϕ is the symbol for a densely defined multiplication operator.

For the other containment, take $f \in D(M_\phi)$. Then

$$|\phi f|^2 = \frac{|b|^2|f|^2}{|a|^2} = \left|\frac{f}{a}\right|^2 - |f|^2.$$

Since ϕf and f are in L^2, so is f/a. Moreover, f/a is analytic in the disc, since a is outer. Thus $f/a \in H^2$. From this we conclude that $D(M_\phi) \subset aH^2$. This completes the proof. □

We now will use Sarason's theorem as a launching point for extensions of RKPs involving the Hardy space.

Example 4. *Let H^2 be the usual hardy space of the disc and let $K = \{k_{w_n}(z)\}$ for some uniqueness set $\{w_n\} \in \mathbb{D}$. A uniqueness set is a collection of points in the disc for which $\sum(1 - |w_n|) = \infty$. These are points who do not approach the boundary quickly. For example, if a sequence converges in the interior of the disc, then it is a uniqueness set for H^2.*

For the RKP (H^2, K), it can be shown that $\mathcal{M}_K$ consists of functions in the Smirnov class that have been restricted to $\{w_n\}$. This follows from the proof of Theorem 4.2. Formally elements of $\mathcal{M}_K$ are functions defined on K, not necessarily on $\mathbb{D}$. However, since N^+ is a quotient of H^2 functions, it has the same uniqueness sets as H^2. Therefore every multiplier in $\mathcal{M}_K$ extends to be a multiplier in $\mathcal{M}_\mathbb{D}$ and this extension is unique. Thus $(H^2, \mathbb{D})$ is an extension that respects (H^2, K).

The notions used in this example stem from H^2 as a space of analytic functions. Sets of uniqueness exist for every space of analytic functions, in the case of H^2 this is any set $\{w_n\} \subset \mathbb{D}$ where $\sum(1 - |w_n|) = \infty$. We also used the factorization theory of H^2 functions implicitly through in invocation of Theorem 4.2.

Both of these ideas should carry over to a large extent for any RKHS of analytic functions. However, for spaces that do not involve analytic functions, other techniques must come forward, as we will see when we characterize the densely defined multiplication operators for the Sobolev space. Now we show that H^2 is maximal in regards to respectful extensions.

Example 5. *Let* (H^2, Y) *be a respectful extension of* $(H^2, \mathbb{D})$*. This means* $\mathcal{M}_X = \mathcal{M}_Y$ *and* $M_z \in \mathcal{M}_Y$*. By Lemma 3.2, every vector in* $Y \subset H^2$ *is an eigenvector of* M_z^**. If we consider the eigenvector equation for* M_z^* *we find:*

$$M_z^* f(z) = \frac{f(z) - f(0)}{z} = \bar{\lambda} f(z).$$

This yields: $f(z) = f(0)(1 - \bar{\lambda} z)^{-1}$*, which is simply a scaled copy of one of the original reproducing kernels. Thus* $(H^2, Y) = (H^2, \mathbb{D})$ *and it is maximal.*

This method of extension is strictly stronger than the Trivial Extension in Theorem 3.1. For example if $\{w_n\}$ was a sequence of distinct points converging in the disc, the Trivial Extension would only add the limit point.

4.2 Fock Space

As was shown in the preliminaries, the symbols for bounded multipliers over the Fock space were the constant functions. This means bounded multiplication on the Fock space is not an interesting concept. However, it is easy to find some examples of densely defined multiplication operators over the Fock space. For instance a well studied operator is M_z. We can see this operator is densely defined, since the polynomials are a dense subspace of F^2.

More interestingly the densely defined operator M_z^* is d/dz. It follows that the eigenvectors for M_z^* are precisely $k_w = e^{\bar{w}z}$, and the Fock space RKP $(F^2, \mathbb{C})$ is maximal. This also demonstrates the need of densely defined multipliers in the extensions of RKPs, since if we only collected the common eigenvectors of the bounded multiplication operators we would extend to (F^2, F^2).

The triviality of the bounded multipliers followed directly from the fact that any multiplier must be entire (bounded or densely defined). It turns out the problems with the

Fock space are deeper than this. The norm on the Fock space induces a growth limit for the entire functions in this space.

In a recent preprint [22], Kehe Zhu explored what are called *maximal zero sequences*. We say a sequence is a zero sequence for F^2 if there is a function in F^2 that vanishes at exactly those points with appropriate multiplicity. For a zero sequence Z we define $I_Z := \{f \in F^2 \mid f(w) = 0 \text{ for all } w \in Z\}$. The following is Corollary 8 in [22].

Theorem 4.3 (Zhu)**.** *Let Z be a zero sequence for F^2 and $k \in \mathbb{N}$. The following are equivalent:*

1. $\dim(I_Z) = k$
2. *For any $\{a_1, ..., a_k\}$ the sequence $Z \cup \{a_1, ..., a_k\}$ is a uniqueness set for F^2, but the sequence $Z \cup \{a_1, ..., a_{k-1}\}$ is not.*
3. *For some $\{a_1, ..., a_{k-1}\}$ the sequence $Z \cup \{a_1, ..., a_{k-1}\}$ is not a uniqueness set for F^2, but $Z \cup \{b_1, ..., b_k\}$ is a uniqueness set for some $\{b_1, ..., b_k\}$.*

Kehe Zhu showed the existence of these *maximal zero sequences* via the Weierstrass sigma function. The following are standard in the theory of Fock spaces and can be found in Zhu [14, 22] among others.

Definition 9. *For $w, w_0, w_1 \in \mathbb{C}$ we call the subset of the complex plane $\Lambda(w, w_0, w_1) = \{w + nw_0 + mw_1 \mid n, m \in \mathbb{Z}\}$ a lattice of points centered at w.*

Definition 10. *Fix the lattice $\Lambda(0, \sqrt{\pi}, i\sqrt{\pi}) = \Lambda = \{w_{mn} = \sqrt{\pi}(m + in) \mid (m, n) \in \mathbb{Z}^2\}$. We define the Weierstrass sigma function by*

$$\sigma(z) = z \prod_{(m,n)\neq(0,0)} \left[\left(1 - \frac{z}{w_{mn}}\right) \exp\left(\frac{z}{w_{mn}} + \frac{z^2}{2w_{mn}^2}\right)\right].$$

Lemma 7. *$|\sigma(z)|e^{-|z|^2}$ is doubly periodic with periods $\sqrt{\pi}$ and $i\sqrt{\pi}$.*

Proposition 4.1. *[22] The Weierstrass sigma function is not in F^2. More generally if $f \in F^2$ and f vanishes on Λ then $f \equiv 0$.*

Note that the double periodicity of $|\sigma(z)|e^{-|z|^2}$ implies that $\sigma(z) \notin F^2$. However, $\sigma(z)/z(z - w_{1,1})$ is in F^2 since the integrand in $\|\sigma(z)/z(z - w_{1,1})\|_{F^2}$ will be $O(1/z^2)$ for large z. Thus $\Lambda' := \Lambda \setminus \{0, w_{1,1}\}$ is a zero set for F^2, but by Proposition 4.1 Λ is not a zero set of F^2. Hence, Λ' is a finite dimensional zero set.

This yields the counter intuitive result that multiplication by a polynomial can affect the growth rate of a Fock space function enough so that the product is not in the Fock space. We can draw a corollary to Theorem 4.3 that can apply to densely defined multiplication operators.

Corollary 1. *If Z is a maximal zero sequence for F^2 and $g \in I_Z$, then g is not a densely defined multiplication operator.*

Proof. If $D(M_g) = \{f \in F^2 \mid gf \in F^2\}$ were dense, then $D(M_g)$ is infinite dimensional. Moreover $gD(M_g) \subset I_Z$. Thus we have a contradiction. □

This demonstrates that unlike the Hardy space, not every Fock space function is a symbol for a densely defined multiplier. We can draw another conclusion. Note that we are treating Z as a sequence and not a set of points. Thus if Z is a zero sequence for the Fock space with $\dim(I_Z) = \infty$, then $\dim(I_{Z'}) = \infty$ for $Z' = Z \cup \{\underbrace{0, 0, \ldots, 0}_{n \text{ times}}\}$. This leads us to the following conjecture:

Conjecture 1. *If Z is a zero sequence for F^2 for which $\dim(I_Z) = \infty$, then I_Z contains the symbol for some densely defined multiplication operator.*

We now take a small diversion to discuss some algebraic properties of multipliers. We will find that the collection of densely defined multipliers over the Fock space is not an algebra. From our experience with bounded multipliers, we are trained to view collections of multipliers as algebras.

This intuition would not betray us if we restricted our view to the multipliers over the Hardy space or even the Sobolev space. In fact we will see for the Sobolev space all of our multipliers are bounded. We set out to show that the multipliers over the Fock space lack these algebraic properties. We begin with an optimistic statement, that products of densely defined multipliers with bounded multipliers always yield new multipliers.

Proposition 4.2. *Let $(\mathcal{H}, X)$ be an RKP. If $g \in \mathcal{M}_X$, $M_\phi \in B(\mathcal{H})$, then $M_\phi M_g$ and $M_g M_\phi$ are densely defined operators with domain $D(M_g)$.*

Proof. M_ϕ is everywhere defined, so $M_\phi M_g$ is densely defined. Now suppose that $f \in D(M_g)$. For such a function $gf \in \mathcal{H}$ and since $M_\phi : \mathcal{H} \to \mathcal{H}$, $\phi \cdot (gf) \in \mathcal{H}$. Notice that $\phi \cdot (gf) = g \cdot (\phi f) \in \mathcal{H}$, so by definition $\phi f \in D(M_g)$. We now see that ϕ leaves $D(M_g)$ invariant, and $M_g M_\phi : D(M_g) \to \mathcal{H}$ is a densely defined operator. □

For spaces like the Fock space, this sheds little light. The Fock space has only constant bounded multipliers, by Liouville's theorem. This tells us for any densely defined multiplier we can multiply by a constant and have a new densely defined multiplier. This is not an interesting result.

In this case we see Proposition 4.2 does not help us. It is in fact much worse for the Fock space. We use Lemma 7 and Proposition 4.1 to prove the following.

Theorem 4.4. *The densely defined multiplication operators in the Fock space are not an algebra.*

Proof. We prove this by constructing a counterexample. The function $\sigma(z)$ is entire and has zeros exactly at the points in Λ. Moreover the function $|\sigma(z)|e^{-|z|^2}$ is doubly periodic with periods $\sqrt{\pi}$ and $i\sqrt{\pi}$.

Consider the function $\sigma(z/2)$. This function has zeros at the points $w_{(2m)(2n)}$ in Λ. From the statement in the previous paragraph, $|\sigma(z/2)|e^{-|z|^2/4}$ is also doubly periodic. The following calculation demonstrates that $M_{\sigma(z/2)}$ is densely defined multiplier for the Fock space by establishing that the polynomials are in its domain:

$$\begin{aligned} \int_{\mathbb{C}} |z^k \sigma(z/2)|^2 e^{-|z|^2} dA(z) &= \int_{\mathbb{C}} |z|^{2k} |\sigma(z/2) e^{-|z|^2/4}|^2 e^{-|z|^2/2} dA(z) \\ &\leq C \int_{\mathbb{C}} \frac{|z|^{2k}}{e^{|z|^2/2}} dA(z) < \infty. \end{aligned}$$

The inequality follows from double periodicity and continuity. The space of polynomials are dense in the Fock space, so $D_{\sigma(z/2)}$ is a dense subspace of F^2.

Similar calculations show $M_{\sigma((z-1)/2)}$, $M_{\sigma((z-i)/2)}$, and $M_{\sigma((z-(i+1))/2)}$ are densely defined multipliers. However $\sigma(z/2)\sigma((z-1)/2)\sigma((z-i)/2)\sigma((z-(i+1))/2)f \in F^2$ for $f \in F^2$ only if $f \equiv 0$. This is because there are no nonzero Fock space functions with the zero set Λ. □

4.3 The Dirichlet-Hardy Space and the Polylogarithm

A Dirichlet series is a series of the form $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ where $a_n \in \mathbb{C}$ for all n. These series are central to the study of Analytic Number Theory, and in Operator Theory have been studied as a starting point for RKHS's. In [23] Hedenmalm, Lindqvist and Seip showed that if the coefficients a_n are square summable, then $f(s)$ converges in the half plane $\sigma > 1/2$. (We adopt the standard convention that $\sigma = Re(s)$.) We say that such functions are in the Dirichlet-Hardy space which we will denote by H^2_d. The reproducing kernel for H^2_d is given by:

$$K(s,t) = \sum_{n=1}^{\infty} \frac{1}{n^{s+\bar{t}}} = \zeta(s+\bar{t}).$$

In [23] they also showed that the bounded multiplication operators over H^2_d are precisely those functions that are representable as a Dirichlet series in the right half plane $\sigma > 0$ and are bounded in that half plane. McCarthy also did some work in this direction in [24], but generalized to weighted Dirichlet-Hardy spaces. In [24] McCarthy also found a weight that would yield a space with the Nevanlinna-Pick property.

Now we introduce a new RKHS. We begin constructing it by taking the tensor product of H^2 with H^2_d. This is a RKHS with orthonormal basis $z^n m^{-s}$ for all $n \geq 0$ and $m \geq 1$. The tensor product space has reproducing kernel

$$K(z,w,s,t) = \frac{\zeta(s+\bar{t})}{1-z\bar{w}}.$$

We are concerned with the closed subspace made from those basis vectors for which $n = m$. In particular we define the following:

Definition 11. *The* Polylogarithmic Hardy Space *is the Hilbert space of functions in two variables given by*

$$PL^2 = \left\{ f(z,s) = \sum_{k=1}^{\infty} a_k \frac{z^k}{k^s} : \sum_{k=1}^{\infty} |a_k|^2 < \infty \right\}$$

The space PL^2 is a RKHS, since it is a closed subspace of a RKHS. Moreover, each $f(z,s)$ inherits convergence on $|z| < 1$ and $\sigma > 1/2$. The reproducing kernel for this space is

$$K(z,w,s,t) = \sum_{k=1}^{\infty} \frac{z^k \bar{w}^k}{k^{s+\bar{t}}} = L_{s+\bar{t}}(z\bar{w}).$$

Here $L_s(z)$ denotes is the polylogarithm.

The space PL^2 has some additional properties. First note that if $f(z,s) = \sum_{k=1}^{\infty} a_k z^k k^{-s} \in PL^2$, then $f(0,s) = 0$. Also since $a_k \to 0$ and $|z| < 1$, the Dirichlet series converges absolutely for all s in $\mathbb{C}$. In other words for fixed z_0, $f(z_0, s)$ has an abscissa of absolute convergence of $-\infty$ and $f(z_0,s)$ is entire. This buys us the following proposition:

Proposition 4.3. *If* $\phi(z,s)$ *is the symbol for a bounded multiplication operator on* PL^2 *then* $\phi(z,s)$ *must be constant in both variables.*

Proof. Let $\phi(z,s)$ be a the symbol for a bounded multiplication operator on PL^2. First fix $z_0 \in \mathbb{D} \setminus \{0\}$ and $s_0 \in \mathbb{C}$. Take the function $f(z,s) = \sum_{k=1}^{\infty} a_k z^k k^{-s} \in PL^2$ for which $f(z_0, s_0) \neq 0$.

Since M_ϕ is a multiplier on PL^2, take $h(z,s) = \sum_{k_1}^{\infty} b_k z^k k^{-s} = M_\phi f$. We may write $\phi(z_0,s) = h(z_0,s)/f(z_0,s)$ and $\phi(z_0,s)$ is analytic at s_0. Since s_0 was arbitrary, $\phi(z_0,s)$ must be entire with respect to s. The function $\phi(z_0,s)$ is bounded, since it is the symbol for a bounded multiplier. By Liouville's theorem, $\phi(z_0,s)$ is constant.

Therefore $\phi(z,s) = \phi(z) \in H^\infty$. Now

$$\phi(z) f(z,s) = \sum_{k=1}^{\infty} a_k(\phi(z) z^k) k^{-s} = \sum_{k=1}^{\infty} b_k z^k k^{-s}.$$

By the uniqueness of Dirichlet series we have $\phi(z)a_k = b_k$ for all k. Since $f \not\equiv 0$ we have that $\phi(z)$ is constant. Hence the proposition. $\square$

This proposition is subsumed under Theorem 4.6. There we will establish that there are no non-constant densely defined multiplication operators over this space. Since bounded multipliers are all densely defined, it follows that any bounded multiplier is constant in both variables. The proof of Theorem 4.6 is more algebraic in flavor than some of the previous results. We will also need some facts about Dirichlet series.

To begin with we will be taking advantage of the Dirichlet convolution for the product of Dirichlet spaces. Recall that if $f(s) = \sum_{k=1}^{\infty} a_k k^{-s}$ and $g(s) = \sum_{k=1}^{\infty} b_k k^{-s}$ are two Dirichlet series that converge in some common half plane, then in that half plane we have $(f \cdot g)(s) = \sum_{k=1}^{\infty} c_k k^{-s}$ where $c_k = \sum_{nm=k} a_n b_m$.

We will also be concerned with the Dirichlet series for the multiplicative inverse of $f(s)$. It is a standard exercise to show that if $f(s) = \sum_{k=1}^{\infty} a_k k^{-s}$ and $a_1 \neq 0$, then we can define a formal inverse series $g(s) = \sum_{k=1}^{\infty} b_k k^{-s}$ recursively. However, there is no guarantee that this series will converge for any s. To find a convegent half plane we utilize the following theorem due to Hewitt and Williamson [25]:

Theorem 4.5 (Hewitt and Williamson)**.** *Suppose that $\sum_{k=1}^{\infty} |a_k| < \infty$. If the Dirichlet series $f(s) = \sum_{k=1}^{\infty} a_k k^{-s}$ is bounded away from zero in absolute value on $\sigma \geq 0$ then $(f(s))^{-1} = \sum_{k=1}^{\infty} b_k k^{-s}$ for $k \geq 0$ where $\sum_{k=0}^{\infty} |b_k| < \infty$ and this series converges in $\sigma \geq 0$.*

This is a specialization of their result, but it is what we need in order to proceed. Recall that if $f(s) = \sum_{k=1}^{\infty} a_k k^{-s}$ converges for some s and is not identically zero, then there is some half plane where $f(s)$ is nonvanishing. Moreover $f(s) \to a_1$ as $\sigma = Re(s) \to \infty$. This means when $a_1 \neq 0$ we can graduate the result from the theorem to this half plane and not just $\sigma \geq 0$. Other facts about Dirichlet series can be found in Apostol's book *Introduction to Analytic Number Theory* [26]. As a warm up to Theorem 4.6 we first establish the following proposition:

Proposition 4.4. *No function in PL^2 is the symbol for a densely defined operator over PL^2.*

Proof. Let $f(z,s) = \sum_{k=1}^{\infty} a_k z^k k^{-s}$ and $\phi(z,s) = \sum_{k=1}^{\infty} b_k z^k k^{-s}$ be in PL^2. Suppose that their product $h = \phi f$ is in PL^2, which means $h = \sum_{k=1}^{\infty} c_k z^k k^{-s}$ where $c_k = \sum_{nm=k} a_n b_m z^{m+n}$. This follows from Dirichlet convolution.

For each c_k the largest term is $(a_k b_1 + b_k a_1) z^{k+1}$. This coefficient must be zero since c_k is a scalar. Thus we find that $b_k/b_1 = -a_k/a_1$ for each k.

However, this means that the functions in the domain of M_ϕ must have the same ratios of its coefficients as ϕ. In particular, if $g(z,s) = \sum_{k=1}^{\infty} d_k z^k k^{-s}$ we must have $d_2/d_1 = -b_2/b_1$. From this we see that

$$D(M_\phi) \subset \left\{ f \in PL^2 : f(z,s) = a_1 \left(z - \frac{b_n}{b_1} z^n n^{-s} \right) + \sum_{k>1:k\neq n}^{\infty} a_k z^k k^{-s} \right\}.$$

This implies that $(b_n/b_1)z + z^n n^{-s} \in D(M_\phi)^\perp$. Finally we conclude the domain is not dense in PL^2, since otherwise $D(M_\phi)^\perp = \{0\}$. □

A convenience of the last proposition came from knowing the form of ϕ beforehand. In order to proceed to the full theorem we must show that if $\phi(z,s)$ is a multiplier over PL^2, then $\phi(z,s)$ is of the form $\sum_{k=1}^{\infty} \phi_k(z) z^k k^{-s}$ where $\phi_k(z)$ is some function of $z \in \mathbb{D}$. To this end we begin with the following lemmata:

Lemma 8. *Let $z_0 \in \mathbb{D}$, $\gamma > 0$ and $f(z,s) \in PL^2$. For every $\epsilon > 0$ there is a $\delta > 0$ so that for all $z \in B_\delta(z_0)$ and all s for which $\sigma = Re(s) \geq 3/2 + \gamma$ we have $|f(z,s) - f(z_0,s)| < \epsilon$.*

Proof. Fix $z_0 \in \mathbb{D}$ and $\epsilon > 0$. And consider

$$\begin{aligned}
|f(z,s) - f(z_0,s)| &= \left| \sum_{k=1}^{\infty} a_k \frac{z^k - z_0^k}{k^s} \right| \\
&= |z - z_0| \left| \sum_{k=1} a_k \frac{z^{k-1} z_0^0 + z^{k-2} z_0^1 + \cdots + z^0 z_0^{k-1}}{k^s} \right| \\
&\leq |z - z_0| \sum_{k=1}^{\infty} |a_k| \frac{1}{k^{\sigma-1}} \\
&\leq |z - z_0| \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k^{2\sigma-2}} \right)^{1/2} \\
&= |z - z_0| \cdot \|f\| \cdot \zeta(2\sigma - 2)^{1/2}.
\end{aligned}$$

The zeta function is defined for $\sigma > 3/2$, and $\zeta(2\sigma - 2)$ is a decreasing function of σ. For $\sigma = Re(s) \geq 3/2 + \gamma$ we have:

$$|f(z,s) - f(z_0,s)| < |z - z_0| \zeta(1 + 2\gamma)^{1/2}$$

Since, $\zeta(1 + 2\gamma) \neq 0$ we may choose $0 < \delta < \epsilon \zeta(1 + 2\gamma)^{-1/2}$, and we have proved the theorem. □

Lemma 9. *Let K be a compact subset of the punctured disc $\mathbb{D} \setminus \{0\}$. If $f(z,s) = \sum_{k=1}^{\infty} a_k z^k k^{-s} \in PL^2$ with $a_1 \neq 0$, then there is a real number σ_K and $\alpha_k : K \to \mathbb{C}$ for which $(f(z,s))^{-1} = \sum_{k=1}^{\infty} \alpha_k(z) k^{-s}$ for all $z \in K$ and $\sigma = Re(s) > \sigma_K$.*

Proof. Fix a real number $0 < \delta < |a_1| \min_K |z|$. For any fixed $z_0 \in K$, there is a half plane $\sigma \geq \sigma_{z_0}$ for which $|f(z_0,s)| > \delta > 0$.

By virtue of Lemma 8, we can choose σ_z so that it varies continuously with respect to z (possibly requiring $\sigma_z \geq 3/2 + \gamma$ with $\gamma > 0$ fixed). In particular σ_z takes a maximum value on K, we call this σ_K. In every half plane $\sigma > \sigma_K$ the function $|f(z,s)|$ is bounded away from zero. For every $z \in \mathbb{D}$, $\sum_{k=1}^{\infty} |a_k z^k| < \infty$. We now apply Theorem 4.5 to obtain the representation for $(f(z,s))^{-1}$. □

Theorem 4.6. *The only densely defined multipliers over PL^2 are the constant functions.*

Proof. Suppose that $\phi(z,s)$ is a densely defined multiplier over PL^2. Let K be a compact subset of the punctured disk. Let $f(z,s) = \sum a_k z^k k^{-s} \in D(M_\phi)$ and set $h(z,s) = (\phi \cdot f)(z,s) = \sum b_k z^k k^{-s}$. Further, assume that $a_1 \neq 0$. Such a function must be in $D(M_\phi)$ since it is dense in PL^2. Let σ_K be the half plane described in Lemma 9. Then $(f(z,s))^{-1} = \sum_{k=1}^{\infty} \alpha_k(z) k^{-s}$ in this half plane and $\phi(z,s) = h(z,s)/f(z,s)$.

In particular $\phi(z,s) = \sum_{k=1}^{\infty} \phi_k(z) k^{-s}$ for all $z \in K$, where $\phi_k(z)$ is some function on K. We will show that ϕ must be a constant function by taking advantage of the incompatibility of Dirichlet convolution and the convolution of coefficients of power functions. The proof proceeds by induction on the number of prime factors of k. Here we will show that each coefficient $\phi_k(z)$ with $k > 1$ is zero. Along the way we will also establish $\phi_{2k}(z) = c_{2k} z^{2k-1}$, but c_{2k} will turn out to be zero in the next step.

For all k not prime, in proof below it is essential to notice that if $k = p_1 \cdot p_2 \cdots p_n$ is the prime factorization for k, then $k + 1 > 2\frac{k}{p_i} + p_i - 1$. This will isolate a term in the polynomials, and the result will follow. We start with $k = 1$, using Dirichlet convolution we find the coefficient $b_1 z^1 = \phi_1(z) a_1 z^1$. This means $\phi_1(z) = b_1/a_1 := c_1$. Now if $k = 2$ we find that $b_2 z^2 = \phi_2(z) a_1 z^1 + \phi_1(z) a_2 z^2$. Since $\phi_1(z)$ is a constant function, this means that $\phi_2(z) = c_2 z^1$ for some c_2.

If we take $k = 4$ we find that

$$b_4 z^4 = \phi_4(z) a_1 z^1 + \phi_2(z) a_2 z^2 + \phi_1(z) a_4 z^4 = \phi_4(z) a_1 z^1 + c_2 a_2 z^2 + c_1 a_4 z^4.$$

Thus

$$\phi_4(z) = \frac{b_4 - c_1 a_4}{a_1} z^3 - \frac{c_2 a_2}{a_1} z.$$

However, since M_ϕ is densely defined, there is another function $g = \sum d_k z^k k^{-s} \in D(M_\phi)$ for which $d_1 \neq 0$ and $d_2/d_1 \neq a_2/a_1$. Using the same algorithm we would find that the z coefficient is $c_2 d_2 d_1^{-1}$. Since $\phi_4(z)$ is fixed, we must have $c_2 d_2 d_1^{-1} = c_2 a_2 a_1^{-1}$.

Thus we see that since M_ϕ is densely defined $c_2 = 0$ and $\phi_4(z) = c_4 z^3$. In the same manner we can show that $\phi_p(z) = 0$ for every prime p and $\phi_{2p}(z) = c_{2p} z^{2p-1}$. Suppose for each $m < n$ and $k = p_1 \cdot p_2 \cdots p_m$ (p_i prime not necessarily distinct) we have $\phi_k(z) = 0$ and $\phi_{2k}(z) = c_{2k} z^{2k-1}$.

Take $k' = p_1 \cdot p_2 \cdots p_n$. Then by our induction assumption:

$$b_{k'} z^{k'} = \phi_{k'}(z) a_1 z^1 + c_1 a_{k'} z^{k'}.$$

This yields $\phi_{k'}(z) = c_{k'} z^{k'-1}$.

Now consider the $2k'$ coefficient. Again using our induction assumption the only terms that remain in the convolution are those that have n or more prime factors and of course $\phi_1(z)$. Thus we find the following:

$$\begin{aligned} b_{2k'} z^{2k'} &= \phi_{2k'}(z) a_1 z^1 + \phi_{k'}(z) a_2 z^2 + \phi_{2k'/p_1}(z) a_{p_1} z^{p_1} + \cdots \\ &+ \phi_{2k'/p_n}(z) a_{p_n} z^{p_n} + \phi_1(z) a_{2k'} z^{2k'} \\ &= \phi_{2k'}(z) a_1 z^1 + c_{k'} a_2 z^{k'+1} + c_{2k'/p_1} a_{p_1} z^{(2k'/p_1)+p_1-1} + \cdots \\ &+ c_{2k'/p_n} a_{p_n} z^{(2k'/p_n)+p_n-1} + c_1 a_{2k'} z^{2k'}. \end{aligned}$$

Now we solve for $\phi_{2k'}(z)$ and we have:

$$\phi_{2k'}(z) = \frac{b_{2k'} - c_1 a_{2k'}}{a_1} z^{2k'-1} - \frac{c_{k'} a_2}{a_1} z^{k'+1} - \text{ other terms}.$$

Since none of the other terms of our polynomial has a $z^{k'+1}$ (by our earlier comment), we find that the $k'+1$ coefficient is $c_{k'} a_2 a_1^{-1}$ which depends on f. We conclude $c_{k'} = 0$ for all k' with n prime factors. Moreover $\phi_{2k'} = c_{2k'} z^{2k'-1}$.

Therefore by strong induction we conclude that $\phi_k(z) = 0$ for all $k \neq 1$, and $\phi(z, s)$ is constant for all z in the compact set K. This constant does not depend on the choice of K, so $\phi(z, s) = b_1/a_1$ for all s and $z \in \mathbb{D} \setminus \{0\}$. Finally we define $\phi(0, s) = b_1/a_1$ and we have the theorem. $\square$

Corollary 2. *No multiplication operator (with nonconstant symbol) on the tensor product of H^2 with H_d^2 has PL^2 as an invariant subspace.*

Let us take a moment to summarize these results. For the Fock space we did not have any nontrivial bounded multipliers because of Liouville's theorem, but we found that there is a large collection of nontrivial densely defined multipliers. The Hardy space has the Nevanlinna class as its collection of densely defined operators, and thus every element of the Hardy space and many more functions are symbols of densely defined multiplication operators. In the case of the Polylogarthmic Hardy space, even when we allow for a multiplier to be densely defined, we still have only trivial multipliers. Though this fact follows from the algebraic properties of Dirichlet series and not from the analytic properties of the space.

CHAPTER 5
UNBOUNDED MULTIPLICATION ON THE SOBOLEV SPACE

Here we will investigate unbounded multiplication over the Sobolev Space. That is the space $W^{1,2}[0,1]$ of absolutely continuous functions on $[0,1]$ whose almost everywhere defined derivative is contained in $L^2[0,1]$. The Sobolev space has a Nevanlinna-Pick kernel, as found by Agler in [6]. This makes it a good next step after the classification of the Hardy space.

It was shown by Shields [5] that the collection of bounded multipliers over the Sobolev space is the Sobolev space itself. We sharpen this by showing that by relaxing bounded to densely defined, the collection of multipliers remains the Sobolev space. In particular, no densely defined multiplier over the Sobolev space is an unbounded multiplier.

5.1 Densely Defined Multipliers for the Sobolev Space

First we will classify the densely defined multiplication operators for the subspace

$$W = \{f \in W^{1,2}[0,1] \mid f(0) = f(1) = 0\}.$$

The investigation of this space will expose the complications that can arise in the general case, but ultimately turn out not to be present.

Example 6. *As we will see in the following theorem, the densely defined multipliers of W are those functions that are well behaved everywhere but the endpoints of $[0,1]$. Take for instance the topologist's sine curve $g(x) = \sin(1/x)$.*

On any interval bounded away from zero, $\sin(1/x)$ is well behaved. To determine that $D(M_g)$ is dense, it is enough to recognize that the set of functions that vanish in a neighborhood of zero are in $D(M_g)$ and this collection of functions is dense in W.

Example 7. *Two other examples of exotic functions that are symbols for densely defined multiplication operators are $g(x) = 1/x$ and $\exp(1/x)$.*

Theorem 5.1. *A function* $g : (0,1) \to \mathbb{C}$ *is the symbol for a densely defined multiplier on* W *iff* $g \in W^{1,2}[a,b]$ *for all* $[a,b] \subset (0,1)$.

Proof. First suppose that g is a densely defined multiplier on W. For each $x_0 \in (0,1)$ there is a function $f \in D(M_g)$ such that $f(x_0) \neq 0$, this follows from the density of the domain. Let $h = M_g f$, so that $g(x) = h(x)/f(x)$. The functions h and f are differentiable almost everywhere in a neighborhood of x_0, so then is g. Since x_0 is arbitrary, g is differentiable almost everywhere on $(0,1)$.

Fix $[a,b] \subset (0,1)$. By way of compactness, there exists a finite collection $\{f_1, ..., f_k\} \subset D(M_g)$ together with subsets

$$[a, t_1),\ (s_2, t_2),\ (s_3, t_3),\ \ldots,\ (s_{k-1}, t_{k-1}),\ (s_k, b]$$

so that the subsets cover $[a,b]$ and f_i does not vanish on $[s_i, t_i]$, here we take $s_1 = a$ and $t_k = b$.

Since f_i does not vanish on $[s_i, t_i]$, g is absolutely continuous on each $[s_i, t_i]$ and hence on $[a,b]$. We wish to show that $g \in W^{1,2}[a,b]$ so we set out to show g and g' are in $L^2[a,b]$. Set $h_i = gf_i$ and by the product rule we find $h_i' = g' f_i + f_i' g$ almost everywhere.

Since g is continuous on $[s_i, t_i]$, $g \in L^2[s_i, t_i]$. The function f' is also in $L^2[s_i, t_i]$ which implies $gf' \in L^2[s_i, t_i]$, because g is bounded. Therefore $h - gf' = g'f \in L^2[s_i, t_i]$. By construction, f_i does not vanish on $[s_i, t_i]$, so

$$\inf_{[s_i, t_i]} |f_i(x)|^2 \int_0^1 |g'|^2 dx \leq \int_0^1 |g' f_i|^2 dx < \infty.$$

Thus $g' \in L^2[s_i, t_i]$ and $g \in W^{1,2}[a,b]$.

For the other direction, suppose that $g \in W^{1,2}[a,b]$ for all $[a,b] \subset (0,1)$. Let $f \in W$ such that f has compact support in $(0,1)$. Let $[a,b]$ be a compact subset of $(0,1)$ containing the support of f. Outside of $[a,b]$, f is identically zero and so $f' \equiv 0$ as well.

The function gf is in $L^2[a,b]$ since it is continuous. Also the function $gf' \in L^2[a,b]$ since g is continuous and $f' \in L^2[a,b]$, and $g'f \in L^2[a,b]$ for the opposite reason. Thus

$h := gf \in W^{1,2}[a,b]$, and since it vanishes outside the interval, $h \in W$. Therefore $f \in D(M_g)$, and compactly supported functions are dense in W. Thus g is a densely defined multiplication operator. $\square$

It was essential to the proof above to find for each point x_0 ($\neq 0$ or 1) a function $f \in D(M_g)$ that did not vanish at that point. We can then conclude that for some closed neighborhood $[a,b]$ of x_0 the symbol g is in $W^{1,2}[a,b]$. In the case where we consider the whole Sobolev space, we can find $f, \tilde{f} \in D(M_g)$ so that they do not vanish at 1 and 0 repsectively. This produces the following theorem:

Theorem 5.2. *For the Sobolev space, $W^{1,2}[0,1]$, the collection of symbols of densely defined multipliers is $W^{1,2}[0,1]$. In particular, all the densely defined multipliers are bounded.*

Thus far this is the only nontrivial collection of densely defined multipliers for which every one is bounded. The same methods can be used to show the following corollaries:

Corollary 3. *Let $\{x_1, ..., x_n\} \subset [0,1]$ and $V \subset W^{1,2}[0,1]$ such that*

$$V = \{f \in W^{1,2}[0,1] \mid f(x_i) = 0 \textit{ for each } i\}.$$

A function $g : [0,1] \setminus \{x_1, ..., x_n\} \to \mathbb{C}$ is the symbol of a densely defined multiplication operator iff $g \in W^{1,2}(E)$ for all compact subsets of $[0,1] \setminus \{x_1, ..., x_n\}$.

Corollary 4. *Given the Sobolev space $W^{1,2}(\mathbb{R})$, a function g is a multiplier for $(W^{1,2}(\mathbb{R}), \mathbb{R})$ iff $g \in W^{1,2}(E)$ for all E a compact subset of $\mathbb{R}$.*

5.2 Local to Global Non-Vanishing Denominator

Ideally given any densely defined multiplication operator over a Hilbert function space H, we would like to express its symbol as a ratio of two functions from H such that the denominator is non-vanishing. For the Hardy space this was achieved through an application of the inner-outer factorization, but there is no such factorization theorem for functions in the Sobolev space. Thus we need to try something a little different.

We saw in Theorem 5.1 that for any point $x \in (0,1)$ we can find a function in the domain that does not vanish in a neighborhood of that point. In other words, we used a local non-vanishing property. Now that we have an explicit description of the densely defined multipliers of W, we can sharpen this to finding a global nonvanishing denominator.

Looking at our three "exotic" functions we can rewrite them as quotients of functions in W as follows:

$$\sin(1/x) = \frac{x^2(1-x)\sin(1/x)}{x^2(1-x)}$$

$$1/x = \frac{x(1-x)}{x^2(1-x)}$$

$$\exp(1/x) = \frac{x(1-x)}{x(1-x)\exp(-1/x)}$$

The proof of the following theorem feels like an application of l'Hopital's rule applied by hand:

Theorem 5.3. *If g is a densely defined multiplier for W, then there exists $f \in D(M_g)$ such that $f(x) \neq 0$ on $(0,1)$.*

Proof. First we suppose that $g \notin W^{1,2}[0,1]$. In our construction we will only consider the left half of the interval. The other half can be handled identically. We now know by Theorem 5.1 that $g \in W^{1,2}[a, \frac{1}{2}]$ for each $a > 0$, but $g \notin W^{1,2}[0,1]$. This means $g \notin L^2[0, \frac{1}{2}]$ and/or $g' \notin L^2[0, \frac{1}{2}]$.

Suppose $\int_0^{1/2} |g|^2 + |g'|^2 dx = \infty$. Since both g and g' are in $L^2[a, \frac{1}{2}]$ for all $a > 0$: $\int_a^{.5} |g|^2 + |g'|^2 dx < \infty$. Using this information, we will construct a sequence:

$$a_n = \int_{\frac{1}{2^{n+1}}}^{\frac{1}{2^n}} |g|^2 + |g'|^2 dx.$$

By our construction, $\sum a_n$ is a divergent series. Define $b_n = \min\{(a_n)^{-1}, (a_{n-1})^{-1}, 1\}$. Notice that $a_n b_{n+1}$, $a_n b_n$ and $b_n \leq 1$ for all n.

Now we can begin constructing our non-vanishing function f. Define $L_n(x)$ by:

$$L_n(x) = \begin{cases} \frac{4(b_n)-(b_{n+1})}{2^{n+1}}(x-2^{-n}) + (2^{-2n}(b_n)) & : \quad x \in (2^{-(n+1)}, 2^{-n}] \\ 0 & : \quad \text{otherwise} \end{cases}$$

Now set $f = \sum_{n=1}^{\infty} L_n(x)$. In other words f interpolates the points $\left\{(\frac{1}{2^n}, \frac{b_n}{2^{2n}})\right\}_{n=1}^{\infty}$ linearly. The function f is continuous on $[0, \frac{1}{2}]$ and differentiable almost everywhere. Further $f, f' \in L^2[0, \frac{1}{2}]$. Thus $f \in W^{1,2}[0, \frac{1}{2}]$ and $f(0) = 0$.

The function gf is continuous on $(0, 1/2)$ and differentiable almost everywhere. We wish to show that both gf and $(gf)' = g'f + f'g$ are in $L^2[0, \frac{1}{2}]$.

$$\int_0^{0.5} |gf|^2 dx = \sum_{n=1}^{\infty} \int_{1/2^{n+1}}^{1/2^n} |gL_n|^2 dx \leq \sum_{n=1}^{\infty} a_n \max\left\{ \left(\frac{b_n}{2^{2n}}\right)^2, \left(\frac{b_{n+1}}{2^{2(n+1)}}\right)^2 \right\} < \infty$$

$$\int_0^{0.5} |g'f|^2 dx = \sum_{n=1}^{\infty} \int_{1/2^{n+1}}^{1/2^n} |g'L_n|^2 dx \leq \sum_{n=1}^{\infty} a_n \max\left\{ \left(\frac{b_n}{2^{2n}}\right)^2, \left(\frac{b_{n+1}}{2^{2(n+1)}}\right)^2 \right\} < \infty$$

$$\int_0^{0.5} |gf'|^2 dx = \sum_{n=1}^{\infty} \int_{1/2^{n+1}}^{1/2^n} |gL_n'|^2 dx \leq \sum_{n=1}^{\infty} a_n \left(\frac{4(b_n)-(b_{n+1})}{2^{n+1}}\right)^2 < \infty$$

Here we see each integral is dominated by a geometric series, and so $gf, (gf)' \in L^2[0, 1/2]$. Thus far we haven't yet established that f is in D_g. Indeed gf may not vanish at 0. To rectify this notice that gf and $(gf)'$ is in $L^2[0, 1/2]$ and gf is absolutely continuous on $[a, 1/2]$ for any $0 < a < 1/2$. For the moment call $h = gf$. Then the following two statements are true:

$$h(x) = \int_{\frac{1}{2}}^{x} h'(t)dt + h(1/2)$$

$$\lim_{x\to 0} |x^2 h(x)| = \lim_{x\to 0} \left| x^2 \int_{\frac{1}{2}}^{x} h'(t)dt + x^2 h(1/2) \right| \leq \lim_{x\to 0} |x|^2 \int_{\frac{1}{2}}^{x} |h'(t)|dt + |x^2||h(1/2)|$$

The last limit is zero, since $h' \in L^2[0, 1/2] \subset L^1[0, 1/2]$. So now we see $x^2 h \in W^{1,2}[0, 1/2]$ and $h(0) = 0$. Thus if we construct f as above for both the left half of $[0, 1]$ and the right half, we can make a function in $D(M_g)$ by $x^2(1-x)^2 f$.

One final note to this proof. It might happen that $g \in W^{1,2}[a,b]$ for all $[a,b] \subset (0,1)$ and $g, g' \in L^2[0,1]$. In this case the above argument yields $x^2(1-x)^2 g \in W$, $x^2(1-x)^2 \in D(M_g)$ and is non-vanishing on $(0,1)$. $\square$

5.3 Alternate Boundary Conditions

We can also consider other subspaces of $W^{1,2}[0,1]$ where we do not have zero boundary conditions. These ideas can be extended to show that certain subspaces satisfying other boundary conditions lead to every densely defined multiplier being bounded.

5.3.1 Sturm-Liouville Boundary Conditions

Consider the Sturm-Liouville boundary conditions:

$$\begin{cases} py(0) + qy'(0) &= 0 \\ sy(1) + ty'(1) &= 0 \end{cases}$$

Let $W(p,q,s,t)$ be the subspace of $W^{1,2}[0,1]$ of functions satisfying the above conditions. Under this definition, the space W we have been studying corresponds to $W(1,0,1,0)$. We can explore all of these spaces with only a few cases. For example if $a \neq 0$ and $c \neq 0$ then $W(p,q,s,t) = W(1, q/p, 1, t/s)$. Therefore if we wish to examine all of these spaces we must check $W(1,q,1,t)$, $W(1,q,0,1)$, $W(0,1,1,t)$, and $W(0,1,0,1)$. Moreover, because the boundary conditions are separated, it is sufficient to determine what happens to the multipliers at 0 for $W(1,q,\cdot,\cdot)$ and $W(0,1,\cdot,\cdot)$ by symmetry.

In the following theorem we will find that some of the subspaces with Sturm-Liouville boundary conditions have only bounded multipliers. This gives us more nontrivial collections of densely defined multipliers where none are unbounded. For the other subspaces, we find that they have unbounded densely defined multipliers in the style of W.

Theorem 5.4. *We have the following connections between these subspaces and their densely defined multiplication operators.*

Table 5-1. Multipliers associated with Sturm-Liouville boundary conditions

Subspace	Densely Defined Multipliers
$W(1,0,1,0)$	$\cap_{0<a<b<1} W^{1,2}[a,b]$
$W(1,q,1,t)$ $q,t \neq 0$	$W(0,1,0,1)$
$W(1,q,1,0)$ $q \neq 0$	$\cap_{b<1} W^{1,2}[0,b]$ and $g'(0) = 0$.
$W(1,q,0,1)$ $q \neq 0$	$W(0,1,0,1)$
$W(1,0,0,1)$	$\cap_{0<a} W^{1,2}[a,1]$ and $g'(1) = 0$.
$W(1,0,1,t)$ $t \neq 0$	$\cap_{0<a} W^{1,2}[a,1]$ and $g'(1) = 0$.

All other subspaces with Sturm-Liouville boundary conditions are equivalent to one of the above spaces.

Proof. **The space $W(0,1,\cdot,\cdot)$:** Now suppose that g is the symbol for a densely defined multiplication operator over the space $W(0,1,\cdot,\cdot)$. Since for each point $x_0 \in [0,1]$ this space has a function that does not vanish at x_0, $g \in W^{1,2}[0,1]$. Let $f \in D(M_g)$ and set $h = gf$. Since

$$0 = h'(0) = f'(0)g(0) + g'(0)f(0) = g'(0)f(0)$$

for all $f \in D(M_g)$, $g'(0) = 0$. We see in this case g satisfies the same boundary conditions at zero.

The space $W(1,q,\cdot,\cdot)$: First take $q \neq 0$. Now suppose g is the symbol of a densely defined multiplication operator over this space. Again $g \in W^{1,2}[0,1]$ for the same reasons as above. Now for every $f \in D(M_g)$ we have (setting $h = gf$):

$$h'(0) = g'(0)f(0) + f'(0)g(0) = f(0)(g'(0) - b^{-1}g(0)).$$

Also we find from the boundary conditions: $h'(0) = -q^{-1}h(0) = -q^{-1}g(0)f(0)$. Thus $-q^{-1}g(0) = g'(0) - b^{-1}g(0)$, and $g'(0) = 0$.

In the second case if we let $q = 0$ then we are in the same scenario as in Theorem 5.1. Hence $g \in W^{1,2}[a,1]$ for all $a > 0$ (disregarding the right boundary condition.) All the others can be compared through symmetry. □

5.3.2 Mixed Boundary Conditions

In addition to the separated Sturm-Liouville boundary conditions, we can examine the mixed boundary conditions:

$$\begin{cases} py(0) + qy(1) &= 0 \\ sy'(0) + ty'(1) &= 0 \end{cases}$$

We will designate the subspaces of $W^{1,2}[0,1]$ corresponding to these conditions as $W'(p,q,s,t)$.

Theorem 5.5. *We have the following connections between these subspaces and their densely defined multiplication operators.*

Table 5-2. Multipliers associated with mixed boundary conditions

Subspace	Densely Defined Multipliers
$W'(1,0,1,0)$	$\cap_{0<a<1} W^{1,2}[a,1]$.
$W'(1,0,0,1)$	$\cap_{0<a<1} W^{1,2}[a,1]$ and $g'(1)=0$.
$W'(1,0,1,t)$ $t \neq 0$	$\cap_{0<a<1} W^{1,2}[a,1]$ and $g'(1)=0$.
$W'(1,0,0,0)$	$\cap_{0<a<1} W^{1,2}[a,1]$.
$W'(1,q,0,0)$ $q \neq 0$	$W'(1,-1,0,0)$.
$W'(1,q,1,t)$ $q,t \neq 0$	$W'(1,-1,q,-t)$.
$W'(0,0,1,t)$ $t \neq 0$	$W'(1,-1,1,-1)$.

All other subspaces with mixed boundary conditions are equivalent to one of the above spaces.

Proof. **The space $W'(1,0,1,0)$:** Suppose that g is the symbol of a densely defined multiplication operator corresponding to $W'(1,0,1,0)$. This is the subspace of functions f for which $f(0) = 0$ and $f'(0) = 0$. Then by the same arguments as in Theorem 5.1 we find that g is in $W^{1,2}[a,1]$ for all $a > 0$. Here we can see that the functions $f \in W'(1,0,1,0)$ for which $f(x) = 0$ in a neighborhood of zero are in the domain of these multipliers.

The space $W'(1,0,0,1)$: Again we find the multipliers for this space are those functions in $W^{1,2}[a,1]$ for all $a > 0$. In addition, if g is the symbol for a densely defined multiplier on this space and $f \in D(M_g)$ with $h = gf$, then

$$0 = h'(1) = g'(1)f(1) + g(1)f'(1) = g'(1)f(1).$$

This is true for all $f \in D(M_g)$ which means $g'(1) = 0$.

The space $W'(1,0,1,t)$, $t \neq 0$: Just as above the multipliers for this space must be in $W^{1,2}[a,1]$ for all $a > 0$. If g is the symbol for a densely defined multiplier on this space and $f \in D(M_g)$ so that $h = fg$, then we have two subcases. If g has a limit at zero (and redefining g to be continuous if necessary) then

$$-tg(0)f'(1) = g(0)f'(0) = h'(0) = -th'(1) = g(1)f'(1) + g'(1)f(1)$$

and so

$$f'(1)(tg(0) + g(1)) = -g'(1)f(1).$$

If $g'(1) \neq 0$ then

$$f'(1)\left(\frac{tg(0) + g(1)}{g'(1)}\right) = f(1).$$

The fraction in this equation is a fixed quantity for a given g, and the equality must hold for every $f \in D(M_g)$. This implies that M_g is not densely defined. If $g'(1) = 0$ then $-tg(0)f'(1) = g(1)f'(1)$ and $g(1) + tg(0) = 0$. Which gives us boundary conditions on g.

On the other hand, using the common dense domain of functions f that vanish in a neighborhood of zero we can find unbounded multipliers. Let f be such a function in $W'(1,0,1,t)$ and let $h = gf$. In this case $h'(0) = f'(0) = 0$. This means that $h'(1) = f'(1) = 0$ as well. Using this we can find the boundary condition at 1 on g:

$$0 = h'(1) = f'(1)g(1) + g'(1)f(1) = g'(1)f(1).$$

This holds for all functions in the domain of g which implies $g'(1) = 0$.

The space $W'(1,0,0,0)$: This is the space of Sobolev space functions where $f(0) = 0$. This can be handled by Corollary 3.

The space $W'(1, q, 0, 0)$: Now let us consider the densely defined multiplication operators over $W'(1, q, 0, 0)$, where $q \neq 0$. If $f \in D(M_g)$ for a densely defined multiplier M_g and letting $h = gf$, then

$$-qg(1)f(1) = -qh(1) = h(0) = g(0)f(0) = -qg(0)f(1),$$

and $g(0) = g(1)$. Also note that since a function in $W'(1, q, 0, 0)$ may have nonzero boundary values, $g \in W^{1,2}[0, 1]$. Thus $g \in W'(1, -1, 0, 0)$.

The space $W'(1, q, 1, t)$: Next let us consider the space $W'(1, q, 1, t)$ for $q, t \neq 0$. Again we find that $g(0) = g(1)$ for every densely defined multiplication operator. Suppose also that $f \in D(M_g)$ and $h = fg$, then

$$g'(0)f(0) + f'(0)g(0) = h'(0) = -th'(1) = -t(g'(1)f(1) + f'(1)g(1)), \text{ and so}$$

$$-qg'(0)f(1) - tf'(1)g(0)) = -tg'(1)f(1) - tf'(1)g(0).$$

This yields $qg'(0) = tg'(1)$. Thus $g \in W'(1, -1, q, -t)$.

The space $W'(0, 0, 1, t)$: Finally examining the space $W'(0, 0, 1, t)$ for $t \neq 0$. Let g, f and h be as before. This yields

$$g'(0)f(0) + f'(0)g(0) = h'(0) = -th'(1) = -t(g'(1)f(1) + f'(1)g(1)), \text{ and so}$$

$$g'(0)f(0) - tf'(1)g(0) = g'(1)f(0) - tf'(1)g(1).$$

Rearranging terms we find $f(0)(g'(0) - g'(1)) = -tf'(1)(g(0) - g(1))$. This holds for all $f \in D(M_g) = W'(0, 0, 1, t)$, which means $g'(0) = g'(1)$ and $g(0) = g(1)$. Thus $g \in W'(1, -1, 1, -1)$. □

5.4 Remarks

We leave with one last note concerning densely defined multipliers on the Sobolev space. We know from Lemma 2 that if a multiplier is bounded, then it's symbol is bounded by the norm of the operator. The question arrises, if g is known to be a densely defined multiplication operator over W and $\sup_{x\in(0,1)} |g(x)| < \infty$ is M_g a bounded multiplier?

The answer is: not necessarily. We can produce a counterexample by examining

$$g(x) = \sqrt{1/4 - (x - 1/2)^2}$$

which is bounded on $[0, 1]$ by 1/2. By the work above, we know that M_g is a densely defined multiplier, since $g \in W^{1,2}[a, b]$ for every $[a, b] \subset (0, 1)$. However, since g' is not bounded on $[0, 1]$, $M_g 1 = g \cdot 1 \notin W$. Therefore even though g is a bounded function, the multiplier M_g is not bounded.

CHAPTER 6
UNBOUNDED TOEPLITZ OPERATORS

6.1 Sarason's Problem

Toeplitz operators come up naturally in many areas of mathematics. They also appears in physics, signal processing and many other contexts. In the finite dimensional case by Toeplitz operator we mean an operator whose $n \times n$ matrix is constant down the diagonals. Extending on this idea, we say that a bounded operator T on H^2 is Toeplitz if the matrix representation for T (with respect to the standard orthonormal basis) is constant down the diagonals. Moreover we have the following equivalent definitions:

Theorem 6.1. *Let T be a bounded operator on H^2. The following are equivalent:*

1. *T is a Toeplitz operator.*
2. *For some $\phi \in L^\infty$ we have $T = T_\phi := P_{H^2} M_\phi$*
3. *$S^* T S = T$*

Most of the work involving Toeplitz operators is concerned with those operators who have bounded symbol ϕ. In a 1960 paper [27] Rosenblum demonstrated the absolute continuity of semi-bounded Toeplitz operators. In 1997 Suarez [28] characterized the closed densely defined operators that commute with S^*, and Sarason [21] investigated the analytic unbounded Toeplitz operators. Helson [29] also investigated the properties of some symmetric Toeplitz operators, which we will use as examples shortly.

For the unbounded Toeplitz operators, the properties in Theorem 6.1 are no longer equivalent. Not to mention these properties need some rewording. We start by looking for Toeplitz operators in the form of (2) in the above theorem.

Let's start with one plausible definition of an unbounded Toeplitz operator.

Definition 12. *We say that T is a Toeplitz operator of multiplication type if there is a ϕ that is the symbol for a densely defined multiplication operator for which $T = T_\phi := P_{H^2} M_\phi$ with domain $D(M_\phi)$.*

Note that if $\phi \in H^\infty$ then $P_{H^2} M_\phi = M_\phi$. We call such an operator an analytic Toeplitz operator, and we extend this terminology to T_ϕ with $\phi \in N^+$. We call T a co-analytic Toeplitz operator if $T = M_\phi^*$ for some $\phi \in N^+$. The next example will demonstrate that M_ϕ^* with domain $D(M_\phi^*)$ is not the same as the operator $M_{\overline{\phi}}$ with domain $D(M_{\overline{\phi}})$.

Example 8. *If we examine the function $\phi_0 = i(1+z)(1-z)^{-1}$ as Helson did, we see that $\phi_0 \in N^+$. This is because $1+z \in H^2$ and $1-z$ is an outer function in H^2. In addition*

$$\overline{\phi_0(e^{i\theta})} = \overline{i\frac{1+e^{i\theta}}{1-e^{i\theta}}} = -i\frac{1+e^{-i\theta}}{1-e^{-i\theta}} = -i\frac{e^{i\theta}+1}{e^{i\theta}-1} = i\frac{1+e^{i\theta}}{1-e^{i\theta}} = \phi_0(e^{i\theta}).$$

Now suppose that f and g are in $D(M_{\phi_0})$. We can show directly that T_{ϕ_0} is symmetric:

$$\langle T_{\phi_0} f, g \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_0(e^{i\theta}) f(e^{i\theta}) \overline{g(e^{i\theta})} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta}) \overline{\phi_0(e^{i\theta}) g(e^{i\theta})} d\theta = \langle f, T_{\phi_0} g \rangle .$$

If T_{ϕ_0} were not only symmetric but also self adjoint, then we would have $T_{\phi_0}^* = T_{\phi_0}$ and $D(T_{\phi_0}^*) = D(T_{\phi_0})$. One way of determining whether a closed symmetric operator is self adjoint is to show its defect indices are 0. Where the defect indices are given by the quantities:

$$\Delta_+ = \dim\{\ker(T_{\phi_0}^* + iI)\} \qquad\qquad \Delta_- = \dim\{\ker(T_{\phi_0}^* - iI)\}.$$

However, the defect indices Δ_+ and Δ_- are both nonzero. The index $\Delta_+ = 0$, but the index $\Delta_- = 1$. This means that $T_{\phi_0}^* \neq T_{\phi_0} = T_{\overline{\phi_0}}$. The operator $T_{\phi_0}^*$ contains the domain of T_{ϕ_0}, which tells us that there are $f \in D(T_{\phi_0}^*)$ that are not in $D(M_{\phi_0})$.

A more exhaustive treatment of symmetric non-selfadjoint operators can be found in Aleman, Martin and Ross's 2013 paper [4]. Suarez [28] showed that the domain for T_ϕ^* is the de Branges-Rovnyak space $\mathcal{H}(b)$, where b is the H^∞ function in the numerator of the standard form $b/a = \phi \in N^+$. Note that the de Branges-Rovnyak space contains aH^2, the domain for M_ϕ. This indicates that we need a notion more general than *multiplication type* Toeplitz operators if we wish to include all of the natural cases. Let's give a name to each of these notions of Toeplitzness.

Definition 13.

1. *If an operator T contains the polynomials in its domain, and its matrix representation with respect to the standard orthonormal basis is constant down diagonals we call T a* matrix-type *Toeplitz operator.*

2. *If an operator is the adjoint of a multiplication type Toeplitz operator with analytic symbol we say that T is a coanalytic Toeplitz operator.*

In his expository article on Unbounded Toeplitz Operators [21], Donald Sarason investigated several classes of unbounded Toeplitz Operators. In each case, he found that these Toeplitz Operators shared some of the algebraic properties of their bounded counter parts. At the end of the paper he posed the following as a problem for characterization:

Problem 1 (The Sarason Problem)**.** *Characterize the closed densely defined operators T on H^2 with the properties:*

1. *$D(T)$ is S-invariant,*

2. *$S^*TS = T|_{D(T)}$,*

3. *If f is in $D(T)$ and $f(0) = 0$ then $S^*f \in D(T)$.*

Is every closed densely defined operator on H^2 that satisfies the above conditions determined in some sense by a symbol?

If we take b/a to be the canonical representation for $\phi \in N^+$, then we see that the domain of $T_\phi = M_\phi$ is aH^2. This is shift invariant. The domain of T_ϕ^* is $\mathcal{H}(b)$, which is also shift invariant. Both of these spaces also share property two. Since co-analytic and analytic Toeplitz operators commute with S^* and S respectively, property three also follows. Thus these *Sarason conditions* encompass both analytic and co-analytic Toeplitz operators. We will call an operator that satisfies Sarason's conditions a *Sarason-Toeplitz operator*.

Recall that for a bounded Toeplitz operator $S^*TS = T$ completely characterizes these operators. By this we mean, if an operator satisfies this algebraic condition, then the matrix representation of such an operator is constant down the diagonals (with respect to the standard basis). It can also be represented by $T = T_\phi = P_{H^2} M_\phi$ for some $\phi \in L^\infty$.

However in the unbounded case, having such a matrix representation does not guarantee that T will agree with PM_ϕ even on a dense domain for any ϕ. If multiplication by this matrix on polynomials is closable, the closed operator will have property three. In [28] the author characterized all such matrices as those whose coefficients are from an analytic function in N^+. We will provide a different proof in this chapter. First let's establish the following lemma:

Lemma 10. *If T satisfies the Sarason conditions, then so does T^*.*

Proof. Since T satisfies the Sarason conditions, it is closed and densely defined. Thus it's adjoint is densely defined and closed as well. Now let us establish the shift invariance of the domain. Take $g \in D(T^*)$. We want to show that the operator $L : D(T) \to \mathbb{C}$ given by $L(f) = \langle Tf, zg \rangle$ is continuous. We know that $\tilde{L}(f) := \langle Tf, g \rangle$ is continuous since, $g \in D(T^*)$.

First suppose that $f \in SD(T)$. That is $f = Sh$ for some $h \in D(T)$. In this case we can see that $L(f) = \langle Tf, zg \rangle = \langle S^* TSh, g \rangle = \langle Th, g \rangle = \tilde{L}(h)$ is continuous, since it coincides with $\tilde{L}$. The space $SD(T)$ is co-dimension 1 as a subspace of $D(T)$. This means there is a vector f_0 in $D(T)$ such that for each $f \in D(T)$ there exists $\alpha \in \mathbb{C}$ and $h \in SD(T)$ so that:

$$f(z) = \alpha f_0(z) + h(z).$$

We know that L is continuous on the span of f_0 since the restriction to this space is one dimensional. L is also continuous on $SD(T)$ as we have established. Therefore L is continuous on $D(T)$ and $zg(z) \in D(T^*)$.

Now we need to show that if $g \in D(T^*)$ and $g(0) = 0$ then $S^* g \in D(T)$. Take such a $g \in D(T^*)$. We now set out to show the operator $F : D(T) \to \mathbb{C}$ given by $F(f) = \langle Tf, S^* g \rangle$ is continuous. What we have to work with is that the operator $\tilde{F}(f) = \langle Tf, g \rangle$ is continuous, since $g \in D(T^*)$.

This time take $f \in D(T)$. This yields:

$$F(f) = \langle Tf, S^*g \rangle = \langle S^*TSf, S^*g \rangle = \langle TSf, SS^*g \rangle = \langle TSf, g \rangle = \tilde{F}(Sf).$$

The last equality between the inner products follows since $SS^*g = g$ for every function satisfying $g(0) = 0$. $\tilde{F}$ is continuous by construction and S is a bounded operator. Thus we see that F is continuous on $D(T)$. Finally the second condition follows by taking adjoints. □

As for recovering a symbol for such a Toeplitz operator, a good first attempt would be to use the Berezin transform of T. The Berezin transform is a function obtained from an operator whose properties reflect that of the operator it came from. The Berezin transform was first used in the context of the Fock space. More information can be found in [14].

Definition 14. *Let T be an operator with $k_\lambda \in D(T)$ for each $\lambda \in \mathbb{D}$. We define the Berezin transform of T to be*

$$\tilde{T}(\lambda) = (1 - |\lambda|^2) \langle Tk_\lambda, k_\lambda \rangle .$$

In many cases this will reproduce the symbol of a Toeplitz operator. For instance if $\phi \in H^\infty$ we have the following:

$$\tilde{T}_\phi(\lambda) = (1 - |\lambda|^2) \langle T_\phi k_\lambda, k_\lambda \rangle = (1 - |\lambda|^2) \left\langle k_\lambda, \overline{\phi(\lambda)} k_\lambda \right\rangle = \phi(\lambda) \frac{1 - |\lambda|^2}{1 - |\lambda|^2} = \phi(\lambda).$$

The Berezin transform will reproduce the symbol in many other cases. However, for a general densely defined Toeplitz operator we cannot apply the Berezin transform as defined above. The problem is that we know k_λ will not be in the domain of T for general unbounded Toeplitz Operators. In fact they are not in the domain of a multiplication type Toeplitz operators with analytic symbol.

In this case there is still hope. In the equations above, we saw that we used the fact that k_λ was an eigenvalue for M_ϕ^* which is still true for densely defined multiplication operators. In this case we can see that for a densely defined analytic Toeplitz operator we have a dual Berezin transform:

$$\tilde{T}_\phi(z)^\dagger := (1 - |\lambda|^2) \left\langle k_\lambda, T_\phi^* k_\lambda \right\rangle = \phi(\lambda)$$

which would reproduce the symbol of an analytic Toeplitz operator. In the next section we will use another method of recovering the symbol of a Toeplitz operator that uses only the algebraic properties stated in the Sarason Problem.

6.2 Sarason Sub-Symbol

Let us begin with the bounded case. This is the traditional Toeplitz operator with L^∞ symbol: T_ϕ. We set out to recover ϕ by playing with its domain. First we begin with an example.

Example 9. *Let $\phi = \sum_{n=-\infty}^{\infty} \gamma_n z^n \in L^\infty$ and consider the bounded Toeplitz operator T_ϕ. The matrix representation of this operator is given by*

$$\begin{pmatrix} \gamma_0 & \gamma_{-1} & \gamma_{-2} & \\ \gamma_1 & \gamma_0 & \gamma_{-1} & \cdots \\ \gamma_2 & \gamma_1 & \gamma_0 & \\ & \vdots & & \ddots \end{pmatrix}$$

Given only this matrix, if we wished to find the nth Fourier coefficient of ϕ for $n \geq 0$ then we would apply this matrix to $(1, 0, 0, 0, \ldots)^T \equiv 1 \in H^2$ and take the inner product with $(0, 0, \ldots 0, 0, 1, 0, 0, \ldots)^T \equiv z^n \in H^2$. This intuition is easily verified:

$$\langle T_\phi 1, z^n \rangle_{H^2} = \langle \phi, z^n \rangle_{L^2} = \gamma_n.$$

Similarly if we wished to find the $-n$th coefficient for ϕ we need to compute

$$\langle T_\phi z^n, 1 \rangle_{H^2} = \left\langle \phi, z^{-n} \right\rangle_{L^2} = \gamma_{-n}.$$

Thus for a bounded Toeplitz operator we can reproduce the symbol of this operator by writing

$$\phi(z) = \sum_{n=1}^{\infty} \langle T_\phi z^n, 1 \rangle \bar{z}^n + \sum_{n=0}^{\infty} \langle T_\phi 1, z^n \rangle z^n.$$

Using this method we see that we require z^n to be in the domain of the operator for all $n \geq 0$. This is not generally true in the unbounded case, but with a shift invariant domain as in Sarason's Problem we can produce some symbol. We define the *Sarason Sub-Symbol* as follows:

Definition 15 (Sarason Sub-Symbol)**.** *Let $V : D(V) \to H^2$ be an operator on H^2 for which $zD(T) \subset D(T)$. The* Sarason Sub-Symbol *depending on $f \in D(T)$ is defined as $R_f = h/f$ where h is given formally by:*

$$h \sim \sum_{k=1}^{\infty} \left\langle Vfz^k, 1 \right\rangle \bar{z}^k + \sum_{k=0}^{\infty} \left\langle Vf, z^k \right\rangle z^k$$

It should be emphasized that this symbol is defined only formally in terms of Fourier series on the circle. In most cases the negative frequency components of h are not square summable, and the hypotheses we will choose for the succeeding theorems allow us to find h in L^2. Of course when h is in L^2, the sub-symbol is well defined. Every example of Toeplitz operators explored so far admits a dense collection of functions f for which h in the sub-symbol is in L^2. We call this collection $D_2(T)$ for general operators T with shift invariant domain.

It has not yet been proven that for an operator with shift invariant domain has nontrivial $D_2(T)$. We hope that the closability of an operator is enough to guarantee this. In the bounded case, it is straightforward to find elements in $D_2(T)$, and we use this to prove the following theorem. This adds yet another equivalent characterization of Toeplitz operators to the list in Theorem 6.1.

Theorem 6.2. *Let V be a bounded operator on H^2. In this case, the Sarason Sub-Symbol is unique iff V is a Toeplitz operator.*

Proof. For the forward implication, suppose that V is not a Toeplitz operator. This yields two cases. In the first case we assume there is a pair $n, m \in \mathbb{N}$ with $n < m$ and $\langle Vz^n, z^m \rangle \neq \langle V1, z^{m-n} \rangle$.

For this case we consider two Sarason Sub-symbols:

$$\phi_1 = \sum_{k=1}^{\infty} \left\langle Vz^k, 1 \right\rangle \bar{z}^k + \sum_{k=0}^{\infty} \left\langle V1, z^k \right\rangle z^k = \sum_{k=-\infty}^{\infty} a_k z^k, \text{ and}$$

$$\phi_{z^n} = z^{-n} \left(\sum_{k=1}^{\infty} \left\langle Vz^{k+n}, 1 \right\rangle \bar{z}^k + \sum_{k=0}^{\infty} \left\langle Vz^n, z^k \right\rangle z^k \right) = z^{-n} \sum_{k=-\infty}^{\infty} b_k z^k.$$

Now we wish to show $\phi_1 - \phi_{z^n} \neq 0$ which amounts to showing the following is nonzero:

$$z^n \sum_{k=-\infty}^{\infty} a_k z^k - \sum_{k=-\infty}^{\infty} b_k z^k = \sum_{k=-\infty}^{\infty} a_k z^{k+n} - \sum_{k=-\infty}^{\infty} b_k z^k = \sum_{k=-\infty}^{\infty} (a_{k-n} - b_k) z^k.$$

Of interest here is $a_{m-n} = \langle V1, z^{m-n} \rangle$ and $b_m = \langle Vz^n, z^m \rangle$. By our construction, these are not equal and the coefficient on z^m is nonzero. Hence the sub-symbol is not unique.

For the second case we examine when there is a pair $n, m \in \mathbb{N}$ such that $n \geq m$ and $\langle Vz^n, z^m \rangle \neq \langle Vz^{n-m}, 1 \rangle$. In this case we proceed as above but use the sub-symbols ϕ_{z^n} and $\phi_{z^{n-m}}$.

Now suppose that V is a bounded Toeplitz operator with symbol ϕ, that is $V = T_\phi$. We will show that $R_f = \phi$ for all $f \in H^2$. Let $R_f = h/f$ this yields:

$$h = \sum_{k=1}^{\infty} \left\langle T_\phi f z^k, 1 \right\rangle_{H^2} \bar{z}^k + \sum_{k=0}^{\infty} \left\langle T_\phi f, z^k \right\rangle_{H^2} z^k = \sum_{k=1}^{\infty} \left\langle \phi f z^k, 1 \right\rangle_{L^2} \bar{z}^k + \sum_{k=0}^{\infty} \left\langle \phi f, z^k \right\rangle_{L^2} z^k$$

$$= \sum_{k=1}^{\infty} \left\langle \phi f, \bar{z}^k \right\rangle_{L^2} \bar{z}^k + \sum_{k=0}^{\infty} \left\langle \phi f, z^k \right\rangle_{L^2} z^k = \phi f.$$

Thus we have shown that if V is Toeplitz then the Sub-symbol is unique. Hence the theorem. □

By the above argument, we can use this Sarason Sub-Symbol to recover the symbol of a bounded Toeplitz operator. However, for the bounded case we can always recover the Toeplitz symbol by simply using the subsymbol ϕ_1. Which would make the Sarason Sub-symbol redundant.

This is not the case for unbounded Toeplitz Operators, since we cannot guarantee that 1 is in the domain of the operator.

Theorem 6.3. *Given an Sarason-Toeplitz operator T , there exists a symbol $\phi \in N^+$ for which $T_\phi = T$ iff $\langle Tfz^m, 1\rangle = 0$ for all $m > 0$ and all $f \in D(T)$. Moreover this symbol is unique.*

Remark 1. *The sufficient condition here is equivalent to T being S-analytic. That is $TS = ST$.*

Proof. The forward direction follows since $T_\phi = M_\phi$ for $\phi \in N^+$, and $M_\phi f$ is an analytic L^2 function for $f \in D(T)$.

For the other direction let $f_1 = \sum a_n z^n$ and $f_2 = \sum b_n z^n$ be nonzero functions in the domain of T, and define two corresponding functions h_1 and h_2 as the numerators of the Sarason sub-symbols corresponding to f_1 and f_2 respectively:

$$h_{1,2} = \sum_{n=1}^{\infty} \langle Tf_{1,2}z^n, 1\rangle \bar{z}^n + \sum_{n=0}^{\infty} \langle Tf_{1,2}1, z^n\rangle z^n.$$

However by our hypothesis:

$$h_{1,2} = \sum_{n=0}^{\infty} \langle Tf_{1,2}1, z^n\rangle z^n.$$

The functions h_1 and h_2 are in H^2, since they are simply the images of f_1 and f_2 respectively.

Our goal is to show that $h_1/f_1 = h_2/f_2$ and thus the symbol we find for T is independent of the choice of $f \in D(T)$. Equivalently we will demonstrate that $h_1 f_2 - h_2 f_1 = 0$ by way of its Fourier series. The Fourier series for $h_1 f_2$ and $h_2 f_1$ can be computed by using the convolution product:

$$\begin{aligned} h_1 f_2 &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\langle Tf_1 1, z^{n-k}\right\rangle b_k\right) z^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\langle Tf_1 z^k, z^n\right\rangle b_k\right) z^n \\ h_2 f_1 &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\langle Tf_2 z^k, z^n\right\rangle a_k\right) z^n \end{aligned}$$

We now have the Fourier series for $h_1 f_2 - h_2 f_1$ (for brevity call this function H):

$$H = h_1 f_2 - h_2 f_1 = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\langle Tf_1 z^k, z^n\right\rangle b_k - \sum_{k=0}^{n}\left\langle Tf_2 z^k, z^n\right\rangle a_k\right) z^n.$$

We now set out to show that each of the coefficients are zero. Let us examine the nth coefficient:

$$\begin{aligned} \hat{H}(n) &= \sum_{k=0}^{n}\left\langle Tf_1 z^k, z^n\right\rangle b_k - \sum_{k=0}^{n}\left\langle Tf_2 z^k, z^n\right\rangle a_k \\ &= \sum_{k=0}^{n}\left(\left\langle Tf_1 z^k, z^n\right\rangle b_k - \left\langle Tf_2 z^k, z^n\right\rangle a_k\right) \\ &= \left\langle T\left(f_1 \sum_{k=0}^{n} b_k z^k - f_2 \sum_{k=0}^{n} a_k z^k\right), z^n\right\rangle \\ &= \left\langle T\left(f_1\left[\sum_{k=0}^{n} b_k z^k - f_2\right] - f_2\left[\sum_{k=0}^{n} a_k z^k - f_1\right]\right), z^n\right\rangle \end{aligned}$$

The H^2 function inside of T is in fact in the domain of T by our assumptions. Moreover it is a function with a zero at zero of order greater than n. Write it as $z^{n+1}F_n$ and by the hypothesis we arrive at:

$$\hat{H}(n) = \left\langle Tz^{n+1}F_N, z^n\right\rangle = \langle TzF_N, 1\rangle = 0.$$

Thus $h_1/f_1 = h_2/f_2$ for all choices of (nonzero) f_1 and f_2. Call $h_1/f_1 = \phi$. ϕ is analytic, since for any $f \in D(T)$, $Tf \in H^2$ and for any $z \in \mathbb{D}$ there is some $f \in D(T)$ such that $f(z) \neq 0$, thus $\phi = (Tf)/f$ is analytic at every point $z \in \mathbb{D}$. Further note that for any $f \in D(T)$, $\phi = (Tf)/f$ and

$$P_{H_2} M_\phi f = P_{H_2} Tf = Tf.$$

Thus T_ϕ is a densely defined Toeplitz operator with an analytic symbol, which means $\phi \in N^+$. $\square$

The strategy used above can be summarized as follows. From the Sarason conditions we believe that T is in some sense a (densely defined) Toeplitz operator. As such, the matrix representation of T should be constant down the diagonals, and we can find the Fourier coefficients of its symbol by simply reading the entries in the first row and column.

However, since T does not necessarily contain the monomials in its domain (it does in the co-analytic case, but not the analytic case) we first take an element from the domain of T and instead examine the operator $T \circ M_f$. The latter operator contains the polynomials and satisfies Sarason's conditions. With the new operator, we make an unbounded Toeplitz operator that agrees with T on $\{f(z) \cdot p(z) \mid p(z) \text{ is a polynomial }\}$. In the case that f is outer, T extends the densely defined Toeplitz operator we have created.

Lemma 11. *Let ϕ be a function on the unit circle that can be written as the ratio of an L^2 function and an H^2 outer function. Let*

$$D(M_\phi) = \{f \in H^2 \mid \phi f \in L^2\}.$$

The operator $M_\phi : D(M_\phi) \to L^2$ is a closed densely defined operator.

Proof. Write $\phi = h/g$ where $h \in L^2$ and $g \in H^2$ with g outer. Since $h \cdot p \in L^2$ for every polynomial $p(z)$, we see that $g \cdot p \in D(M_\phi)$, so the domain of M_ϕ is dense in H^2.

Now suppose that $\{f_n\} \subset D(M_\phi)$ and $f_n \to f \in H^2$. Suppose further that $M_\phi f_n \to F \in L^2$. We wish to show that $f \in D(M_\phi)$ and $\phi f = F$. Since $f_n \to f$ in L^2 norm, there must be a subsequence $f_{n_j} \to f$ almost everywhere. Since g is an outer function, $g(e^{i\theta}) \neq 0$ for almost every θ. Thus $\phi f_{n_j} \to \phi f$ almost everywhere.

The subsequence $\phi f_{n_j} \to F$ in norm and has itself a subsequence converging to F almost everywhere. Call this subsequence $\phi f_{n_{j_k}}$. However this subsequence also converges to ϕf. Thus $\phi f = F$ almost everywhere, and the conclusion follows. □

Theorem 6.4. *Let T satisfy Sarason's conditions. Suppose further that there is an outer function $f \in D(T)$ such that $\sum_{n=1}^{\infty} \langle Tfz^n, 1\rangle \bar{z}^n \in L^2$. In this case, T extends an closed densely defined Toeplitz operator with symbol ϕ that is a ratio of an L^2 function and f.*

Remark 2. *These conditions on the domain of T are not unreasonable. For a bounded Toeplitz operator this holds trivially, and for the unbounded Toeplitz operators examined by Sarason the conditions also hold. In each of these cases, the symbol produced is exactly the one we start with.*

Proof. Let f be as in the statement above. Define

$$h = \sum_{n=1}^{\infty} \langle Tfz^n, 1\rangle \bar{z}^n + \sum_{n=0}^{\infty} \langle Tf, z^n\rangle z^n.$$

For brevity we will write $h = \sum_{n=-\infty}^{\infty} b_n z^n$ and by the Toeplitzness of T we have $b_{n-m} = \langle Tz^m, z^n\rangle$. Finally declare $\phi = R_f = h/f$, and consider the unbounded Toeplitz operator $T_\phi = P_{H^2} M_\phi$ with the domain F.

By the Sarason conditions, $f(z)p(z) \in D(T)$ for every polynomial $p(z)$. This means that $D(T)$ contains the dense subset of H^2: $F = \{f(z) \cdot p(z) \mid p(z)$ is a polynomial $\}$. The following calculations show that T agrees with T_ϕ on this domain.

Let $p(z) = a_k z^k + \cdots + a_1 z + a_0$.

$$\begin{aligned} h(z) \cdot p(z) &= \sum_{n=-\infty}^{\infty} \left(\sum_{n=0}^{k} b_{n-m} a_m \right) z^n \\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{n=0}^{k} \langle T f a_m z^m, z^n \rangle \right) z^n \\ &= \sum_{n=-\infty}^{\infty} \langle T f p(z), z^n \rangle \, z^n \\ &= w(z) + T(fp(z)). \end{aligned}$$

Here $w(z) \in \overline{H_0^2}$ by our hypothesis. In particular this means $T_\phi(fp(z)) = P_{H^2}(hp(z)) = T(fp(z))$. Hence T agrees with T_ϕ on a dense domain, and T extends $T_\phi|_F$.

Finally note that by virtue of the previous lemma, $T_\phi|_F$ is closable. Further $T_\phi|_F \subset T$ which means $T_\phi|_F^{**} \subset T^{**} = T$. □

The above argument doesn't depend on T having the Toeplitz properties as much as TM_f having these properties. We could adjust the hypothesis accordingly, and we can also arrive at the following corollary.

Corollary 5. *Suppose V is a bounded operator on H^2. If f is an outer function such that VM_f is a Toeplitz operator and $V^*1 \in D(M_f^*)$, then $V = T_{R_f}$.*

Proof. Given the hypothesis, we see that $\sum_{n=1}^{\infty} |\langle Vfz^n, 1\rangle|^2 = \sum_{n=1}^{\infty} |\langle z^n, (VM_f)^*1\rangle|^2 < \infty$ since $(VM_f)^*1 \in H^2$. Thus by the previous proposition V agrees with T_{R_f} on a dense domain. Hence $V = T_{R_f}$ by continuity. □

This theorem gives an answer to whether T can be related to a symbol. However, we had to assume that there was not only an outer function $f \in D(T)$ but also one for which $\sum_{n=1}^{\infty} \langle Tfz^n, 1\rangle \bar{z}^n$ was in L^2. Without the above conditions, we can still find that T extends a pointwise limit of unbounded Toeplitz operators.

Theorem 6.5. *Suppose T satisfies Sarason's conditions, let $f \in D(T)$ and consider the set $F = \{f(z) \cdot p(z) : p(z)$ polynomial$\}$. There exists a sequence of densely defined multiplication type Toeplitz operators that converge strongly to T on F. Moreover these Toeplitz operators are densely defined with common domain $\{f_o(z) \cdot p(z) :$ $p(z)$ polynomial $\}$ where f_o is the H^2 outer factor of f.*

Proof. Let $f \in D(T)$ and define $h_N = \sum_{n=1}^{N} \langle Tfz^n, 1\rangle \bar{z}^n + \sum_{n=0}^{\infty} \langle Tf, z^n\rangle z^n = \sum_{n=-N}^{\infty} b_n z^n$. Also define $\phi_N = h_N/f$. Let $p(z) = a_k z^k + \cdots + a_1 z + a_0$, and consider the product $h_N p(z)$:

$$\begin{aligned} h_N(z)p(z) &= \sum_{n=-N}^{\infty} \left(\sum_{m=0}^{\min(k,n+N)} b_{n-m} a_m \right) z^n \\ &= \sum_{n=-N}^{k-N-1} \left\langle Tf\left(a_0 + \cdots + a_{n+N} z^{n+N}\right), z^n \right\rangle z^n + \sum_{n=k-N}^{\infty} \langle Tfp(z), z^n\rangle z^n. \end{aligned}$$

Write $c_n = \left\langle Tf\left(a_0 + \cdots + a_{n+N} z^{n+N}\right), z^n \right\rangle$. Thus $M_{\phi_N} fp(z) = h_N p(z) \in L^2$ and

$$T_{\phi_N}(fp(z)) = P_{H^2}(h_N p(z)) =$$

$$\left(\left(c_0 + c_1 z + \cdots + c_{k-N-1} z^{k-N-1}\right) + \sum_{n=\max(k-N,0)}^{\infty} \langle Tfp(z), z^n\rangle z^n \right) \to T(fp(z))$$

where the limit happens as $N \to \infty$. The limit is possible simply because the sequence is constant for large N.

For the common domain, let $f = f_i f_o$ be the inner-outer factorization of f in H^2. Let $\tilde{h}_N = h_N/f_i$. The function $\tilde{h}_N \in L^2(\mathbb{T})$ since f_i has modulus one on the circle. Thus $\phi_N f_o p(z) = \tilde{h}_N p(z) \in L^2$ and $T_{\phi_N} f_o p(z) = P_{H^2}(\tilde{h}_N p(z))$. □

6.3 Extending Co-analytic Toeplitz Operators

Following Sarason, we begin a study of Co-analytic Toeplitz operators by using a matrix representation of the operator. Though it is not strictly true, we often think of this matrix as being the conjugate transpose of the matrix of a densely defined multiplication operator whose coefficients (with respect to the standard orthonormal basis for H^2)

is composed of its Fourier coefficients. As we have seen before, the symbol for this multiplication must be analytic in the disc, and this puts a constraint on the coefficients.

Here we will view the matrix as an object abstracted from a multiplication operator. Instead this will be an upper triangular matrix that is constant down the diagonals:

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \\ 0 & \gamma_0 & \gamma_1 & \cdots \\ 0 & 0 & \gamma_0 & \\ & \vdots & & \ddots \end{pmatrix}.$$

If these were the coefficients for an L^∞ function $\bar{\phi}$, then this would be the matrix representation for the adjoint of a bounded multiplication operator M_ϕ. However, if this is not such a sequence then the operator can only be densely defined. Using the analogy of matrix multiplication, we can see that multiplying this matrix to a vector corresponding to a polynomial yields another polynomial. Hence this operator is densely defined with the polynomials in its domain.

We may extend this operator by the operator T defined as:

$$Tf = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \gamma_n \hat{f}(n+m) \right) z^m$$

where we say the domain of T is the collection of functions in H^2 for which Tf is in H^2. This operator extends the operator we defined on the polynomials above, and for a large number of cases we can describe the domain simply.

In Sarason's paper, he was concerned only with functions that were the adjoints for densely defined multiplication operators. These place conditions on the sequence $\{\gamma_n\}$. First, these must be the Taylor coefficients for a function that is analytic in the disc. Second, the domain of this operator must contain the kernel functions (if it is closed). The following can be found in [21]:

Theorem 6.6. *If $f \in H(\bar{\mathbb{D}})$, then each series $\sum_{n=0}^{\infty} \gamma_n \hat{f}(m+n)$ converges absolutely for $m = 0, 1, 2, \ldots$, and the function represented by the power series $Tf = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \gamma_n \hat{f}(m+n) \right) z^m$ is in $H(\bar{\mathbb{D}})$.*

Note that if a function is in $H(\bar{\mathbb{D}})$, the set of functions analytic in a neighborhood of the closed unit disc, then it is a bounded analytic function and hence in H^2. This set contains not only the polynomials but also all of the H^2 kernel functions. If we no longer require the coefficients to be that of an analytic function of the disc, the domain of the operator T becomes much easier to describe. First let's examine the case with $\gamma_n = n!$. This is clearly not covered by Theorem 6.6, since the power series $\sum_{n=0}^{\infty} n! z^n$ has a radius of convergence of 0.

Theorem 6.7. *Let T be the extension of multiplication by an upper triangular matrix given by $Tf = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} n! \hat{f}(n+m) \right) z^m$. We define the domain of T to be*

$$D(T) = \{ f \in H^2 \mid Tf \in H^2 \}.$$

Every function $f \in D(T)$ is an entire function and can be written as $f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ where $\sum_{n=0}^{\infty} a_n$ converges.

Lemma 12. *The sequence $\{ c_m = \sum_{n=1}^{\infty} (n+1)^{-m} \}_{m=2}^{\infty}$ is an l^2 sequence.*

Proof. By virtue of the integral test we can bound c_m by

$$\int_0^{\infty} \frac{1}{(1+x)^m} dx = \lim_{b \to \infty} \int_0^b \frac{1}{(1+x)^m} dx = \lim_{b \to \infty} \frac{1}{m-1} \left(1 - \frac{1}{(1+b)^{m-1}} \right) = \frac{1}{m-1}.$$

Thus we bound c_m by a harmonic sequence, which by a gift from Euler is in l^2. □

Proof of Theorem 6.7. First we suppose that $f \in D(T)$. By definition the zeroth coefficient of Tf is given by $\sum_{n=0}^{\infty} n! \hat{f}(n)$, which must be a convergent series. Declaring $a_n = n! \hat{f}(n)$ we see that $\sum a_n$ is convergent. Moreover, since $f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ the function f must be entire by the root test.

For the other direction, suppose that $f(n) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ where $\sum a_n$ converges. Call $d_0 = \sum_{n=0}^{\infty} n!\hat{f}(n) = \sum a_n$. Note also that since $\sum a_n$ converges, so does

$$d_1 = \sum_{n=0}^{\infty} n!\hat{f}(n+1) = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n+1}, \text{ and}$$

$$d_m = \sum_{n=0}^{\infty} n!\hat{f}(n+m) = \sum_{n=0}^{\infty} \frac{a_{n+m}}{(n+1)(n+2)\cdots(n+m)}.$$

This follows since $\frac{1}{(n+1)(n+2)\cdots(n+m)}$ is decreasing to zero monotonically. This enables us to define Tf formally as $\sum_{m=0}^{\infty} d_m z^m$. Finally we can show that Tf is in H^2, by the following observation:

$$d_m = \frac{a_m}{m!} + \sum_{n=1}^{\infty} \frac{a_{n+m}}{(n+1)(n+2)\cdots(n+m)} := s_m + t_m.$$

The sequence $\{s_m\}$ is in l^2 since these are the Fourier coefficients of f. For the t_m sequence we use the comparison:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+m)} \leq \sum_{n=1}^{\infty} \frac{1}{(n+1)^m} = c_m.$$

Thus t_m is an l^2 sequence by comparison with Lemma 12. This completes the theorem.

□

Provided the γ_n's are growing fast enough, we can apply the same proof to find the domain of this operator. We arrive at the following:

Theorem 6.8. *Let γ_n be a sequence of positive real numbers such that $\gamma_{n+1} \geq (n+1)\gamma_n$ for all n, and let T be the operator given by $Tf = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \gamma_n \hat{f}(n+m) \right) z^m$. We define the domain of T to be*

$$D(T) = \{f \in H^2 \mid Tf \in H^2\}.$$

Every function $f \in D(T)$ is an entire function and can be written as $f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{\gamma_n}$ where $\sum_{n=0}^{\infty} a_n$ converges.

Using the same techniques we can find a slightly weaker result in the case that the sequence $\{\gamma_n\}$ is complex with the growth condition $|\gamma_{n+1}| > (n+1)|\gamma_n|$.

Theorem 6.9. *Let $\{\gamma_n\}$ be a sequence of complex numbers as described above and define the operator $Tf = \sum_{m=0}^{\infty}(\sum_{n=0}^{\infty}\gamma_n \hat{f}(n+m))z^m$ with the domain $D(T) = \{f \in H^2 \mid Tf \in H^2\}$. In this case if $f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{\gamma_n}$ and $\sum |a_n| < \infty$ then $f \in D(T)$.*

Proof. Let f be as stated in the hypothesis. Define $d_m = \sum_{n=0}^{\infty}\gamma_n \hat{f}(n+m)$. This makes $d_0 = \sum a_n$, and is well defined since the series is absolutely convergent. We see that d_m is well defined by comparison with $\sum a_n$,

$$|d_m| = \left|\sum_{n=0}^{\infty}\gamma_n \frac{a_{n+m}}{\gamma_{n+m}}\right| \le \sum_{n=0}^{\infty}\frac{|a_{n+m}|}{(n+1)(n+2)\cdots(n+m)} \le \sum_{n=0}^{\infty}|a_{n+m}|.$$

To verify that $\{d_m\}$ is an l^2 sequence, we split it as in Theorem 6.7. □

Now that we have the entire domain for the operator in Theorem 6.7, we can arrive at an alarming conclusion. The domain is not shift invariant. To see this take $f \in D(T)$ and consider

$$zf(x) = z\sum_{n=0}^{\infty}\frac{a_n}{n!}z^n = \sum_{n=0}^{\infty}\frac{(n+1)a_n}{(n+1)!}z^{n+1}.$$

If zf was in $D(T)$, the series $\sum_{n=0}^{\infty}(n+1)a_n$ must be convergent. However simply setting $a_n = (-1)^{n+1}(n+1)^{-1}$, we see that $f \in D(T)$ but zf is not.

We have yet to address the closure of these co-analytic Toeplitz operators. If the γ_n grow too fast, then it is clear that closability should be much harder. We conclude by proving that if multiplication by a Toeplitz matrix is closable, then the symbol the matrix coefficients come from must be in N^+. This was first recognized by Suarez, but the proof here is different.

Theorem 6.10. *Let T be the operator corresponding to the upper triangular matrix with the polynomials as its domain. If T is closable, then the coeficients in the matrix are the Taylor coefficients of a Smirnov class function ϕ. In particular, they are the coefficients of a function analytic in the disc.*

Proof. If T was closable, then T^* is closed, densely defined, and satisfies Sarason's Conditions by the proof of Lemma 10. Since T commutes with S^*, this means T^* commutes with S. By Theorem 6.3 we see that T^* is a densely defined multiplier on the Hardy space with symbol $\phi \in N^+$. This completes the proof. □

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BIOGRAPHICAL SKETCH

Joel Rosenfeld was born in Gainesville, Florida. Born to parents Katherine Vann and Perry Rosenfeld with one sibling, Spencer Rosenfeld. He has spent the past seven years studying mathematics and the last five in graduate school. Joel has enjoyed his time at the University of Florida immensely.

Before he became a mathematician, he worked as a graphic designer for several years at an educational technology company. He also has worked as a programmer in several contract jobs. Joel is passionate about education, mathematics, and the history of mathematics. He received his Ph.D. degree from the University of Florida in the spring of 2013.