ON LEVEL CURVES AND CONFORMAL EQUIVALENCE OF MEROMORPHIC FUNCTIONS

By

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To my little baby Abigail, that I may get a job and buy you food

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In this dissertation we study the level curves of the class of meromorphic functions (f, G) which are conformally equivalent to a finite Blaschke product on the disk, and, in particular, conformal equivalence of these functions in light of their respective level curve structures. We begin with a general study, and then apply our findings to the study of the equivalence classes of such functions modulo conformal equivalence. We develop the notion of a level curve structure, and show that conformal equivalence of the meromorphic functions referred to above may be determined entirely in terms of level curve structure.

CHAPTER 1
HISTORY AND OVERVIEW

The study of the level sets of a meromorphic function f (sets of the form $\{z : |f(z)| = \epsilon\}$) or tracts of the function (sets of the form $\{z : |f(z)| < \epsilon\}$) has tended to cluster around two major classes of questions. We will briefly mention several results from each class to give a taste of the existing research on this subject.

The first class of questions has to do with properties of a level curve (or tract) of a single function such as arc length of the level curve, area of the tract, and convexity or star-shapeness of the tract. In 1981, W. K. Hayman and J. M. Wu [1] showed that if $f : \mathbb{D} \to \mathbb{C}$ is an injective analytic function, then any level set of f has length less than $2 \cdot 10^{35}$. On the other hand, in 1980, P. W. Jones [2] constructed an analytic bounded function $f : \mathbb{D} \to \mathbb{C}$ each of whose level sets is either empty or of infinite length. Questions about convexity and star-shapeness have mostly had to do with polynomial tracts, and several counter-examples have been given by G. Pirinian and others [3–5].

The other class of questions has to do with meromorphic functions which share a level curve or a tract. A number of people have given results showing the relationship between two meromorphic functions which share a level curve, starting with G. Valiron [6], whose work implied the following result.

Theorem 1.1. *Let f_1 and f_2 be entire functions such that $|f_1| \equiv 1 \equiv |f_2|$ on a simple closed curve Γ. Then there is an entire function h with $|h| \equiv 1$ on Γ, and $f_1 \equiv b_1 \circ h$ and $f_2 \equiv b_2 \circ h$ for some finite Blaschke products b_1 and b_2.*

A nice summation of this result and following results of the same flavor may be found in a 1986 paper by K. Stephenson and C. Sundberg [7] in which they give the following stronger result for inner functions.

Theorem 1.2. *Let ϕ_1 and ϕ_2 be inner functions such that the sets $\{z : |\phi_1(z)| = c\}$ and $\{z : |\phi_2(z)| = c\}$ have a sub-arc in common for some $c \in (0, 1)$. Then $\phi_1 = \lambda\phi_2$ for some unimodular constant λ.*

Finally, in 1986, K. Stephenson [8] proved the following general result which implies much of the earlier work. Here if $\mathfrak{W}$ is a simply connected Riemann surface, and Γ is an arc in $\mathfrak{W}$, then we let $\mathfrak{F}_\Gamma$ denote the collection of all non-constant meromorphic functions on $\mathfrak{W}$ having modulus one on Γ.

Theorem 1.3 (Level Curve Structure Theorem)**.** *Let $\mathfrak{W}$ be a simply connected Riemann surface, Γ an arc on $\mathfrak{W}$, and suppose $\mathfrak{F}_\Gamma$ is nonempty. Then there exists a unique simply connected surface $\mathfrak{S}$ (one of $\mathbb{C}$, cl$(\mathbb{C})$, or $\mathbb{D}$) and an analytic function $\phi : \mathfrak{W} \to \mathfrak{S}$ such that $\mathfrak{F}_\Gamma \equiv \mathfrak{F}_\mathbb{R} \circ \phi \equiv \{\mathfrak{F} \circ \phi : \phi \in \mathfrak{F}_\mathbb{R}\}$.*

There is a somewhat different class of questions which has arisen and which concerns smooth shapes in general, but in which the level curves of meromorphic (and in particular rational and polynomial) functions play a large part. These have to do with a "fingerprint" which a smooth curve imposes on the unit circle $\mathbb{T}$, which was introduced by A. A. Kirillov in 1987 [9, 10]. Let Γ be a smooth simple closed curve in $\mathbb{C}$, with bounded face Ω_- and unbounded face Ω_+. Let ϕ_-, ϕ_+ denote Riemann maps from $\mathbb{D}, \mathbb{D}_+$ to Ω_-, Ω_+ respectively (here $\mathbb{D}_+$ is defined as $\mathbb{C} \setminus$ cl$(\mathbb{D})$). With certain normalizations on the Reimann maps, we define the fingerprint k of Γ by $k := {\phi_+}^{-1} \circ \phi_-$. Since Γ is smooth it is easy to show that k is a diffeomorphism from $\mathbb{T}$ to $\mathbb{T}$. Moreover if $\hat{\Gamma}$ equals the image of Γ under an affine transformation $f(z) = az + b$, with corresponding fingerprint $\hat{k}$, then $k = \hat{k} \circ \phi$ for some automorphism $\phi : \mathbb{D} \to \mathbb{D}$. Therefore we may define a function $\mathcal{F}$ which maps a smooth simple closed curves (modulo composition with affine transformation) to the corresponding diffeomorphism of $\mathbb{T}$ which is its fingerprint (modulo pre-composition with an automorphism of $\mathbb{D}$). (Note: This and more background may be found in [11].) Kirillov proved the following theorem [9, 10].

Theorem 1.4. *$\mathcal{F}$ is a bijection.*

If we restrict our attention to smooth curves which arise as level curves of polynomials, a similar result may be obtained. One first shows that if Γ is a proper polynomial lemniscate (ie $\Gamma = \{z : |p(z)| = 1\}$ for some n-degree polynomial p such that

$\{z : |p(z)| = 1\}$ is smooth and connected) then the corresponding fingerprint $k = B^{\frac{1}{n}}$ for some n-degree Blaschke product. Then if we let $\mathcal{F}_p$ denote the function $\mathcal{F}$ viewed as having as its domain the equivalence classes of simple smooth closed curves which arise as proper polynomial lemniscates, and having codomain the equivalence classes of diffeomorphisms of $\mathbb{T}$ consisting of n^{th} roots of n-degree Blaschke products ($n \in \mathbb{N}$), then one may prove the following theorem.

Theorem 1.5. *$\mathcal{F}_p$ is a bijection.*

This result was stated in [11] and follows rather directly from Theorem 1.4, though the authors proved it using other means[1] . In Chapter 5 and Chapter 6, we will prove a somewhat stronger result which has Theorem 1.5 as a corollary, though in the following equivalent form.

Theorem 1.6. *For every finite Blaschke product B with degree n, there is some n degree polynomial p such that the set $G := \{z : |p(z)| < 1\}$ is connected, and some conformal map $\phi : \mathbb{D} \to G$ such that $B = p \cdot \phi$ on $\mathbb{D}$.*

Our main goal in this paper, however, is to explore the way in which the configuration of level curves of a meromorphic function characterize that function modulo conformal equivalence. In Chapter 2 we build up several preliminary properties of the level curves of a meromorphic function. Our setting will be a meromorphic function f with a bounded finitely connected domain G such that f is meromorphic across the boundary of G, $|f|$ is constant on any component of ∂G, and $f' \neq 0$ on ∂G. In this setting, we make the following definition.

Definition 1. *If $w \in \mathbb{C}$, and f is meromorphic at w, then we let Λ_w denote the component of $\{z \in dom(f) : |f(z)| = |f(w)|\}$ which contains w.*

One of the main result from this chapter is Theorem 2.1, which states the following.

Theorem 1.7. *Let each of L_1 and L_2 be a level curve of f contained in G or a bounded component of G^c. Let F_1 denote the unbounded face of L_1 and F_2 the unbounded face* *of L_2. If $L_1 \subset F_2$, and $L_2 \subset F_1$, then there is some $w \in G$ which lies in $F_1 \cap F_2$, such that $f'(w) = 0$ and L_1 and L_2 are contained in different bounded faces of Λ_w.*

This existence theorem for critical points of f is proved by means of the separation result Proposition 2.4, which states broadly speaking that if L is a level curve of f in G, and K is a compact set which does not intersect L, then K is separated from L by another level curve of f in G. The other main result (Theorem 2.1) from Chapter 2 states in brief the following.

Theorem 1.8. *For any meromorphic function (f, G), if the finitely many critical level curves of f in G (that is the level curves of f in G which contain critical points of f) are removed, then in each of the remaining regions f is conformally equivalent to a pure power of z.*

More specifically, if D is one of the remaining regions then there is an annulus A and a conformal map $\phi : D \to A$ and some $n \in \mathbb{Z} \setminus \{0\}$ such that $f \equiv \phi^n$ on D. This may be viewed as the natural extension of the fact that if w is a zero or pole of any meromorphic function g, then there is a neighborhood of w on which g is conformally equivalent to a pure power of z. The proof of this theorem relies heavily on Theorem 2.1.

In Chapter 3, we make the further assumption on our functions (f, G) that G is simply connected, and we rigorously construct a set PC which represents all possible configurations of critical level curves of (f, G). We then define a function Π which maps (f, G) to the corresponding configuration in PC. Chapter 4 contains the main result (Theorem 4.1) of the paper, namely that Π respects conformal equivalence.

Theorem 1.9. *If (f_1, G_1) and (f_2, G_2) are two functions as described above, then $(f_1, G_1) \sim (f_2, G_2)$ if and only if $\Pi(f_1, G_1) = \Pi(f_2, G_2)$.*

Here $(f_1, G_1) \sim (f_2, G_2)$ means that there is some conformal map $\phi : G_1 \to G_2$ such that $f_1 = f_2 \circ \phi$ on G_1. This result implies that if we view Π as having for its domain the set of equivalence classes of meromorphic functions modulo conformal equivalence, then first Π is well defined, and second Π is injective.

In Chapter 5 and Chapter 6, we show that in a limited sense, Π is surjective. That is, we define a subset $PC_a \subset PC$ of configurations which naturally correspond to analytic functions. Then if we view Π as having for its domain the equivalence classes of analytic (f, G) modulo conformal equivalence, and having codomain PC_a, then Π is surjective. In Chapter 5, we show that the image of Π contains each configuration in PC_a all of whose critical values are non-zero and have different moduli. In Chapter 6, we extend this to all of PC_a by approximating a general member of PC_a by a generic one.

Finally, in the first appendix, we catalogue several facts about level curves and applications to level curves which were not of direct use in proving the main results of the paper, but have some independent interest, including an apparently new proof, using level curves, of the following Gauss-Lucas theorem.

Theorem 1.10. *The critical points of a polynomial p are contained in the convex hull of the zeros of p.*

And in the second appendix we prove several lemmata used throughout the paper.

CHAPTER 2
PRELIMINARIES

2.1 Level Curves as Planar Graphs

Definition 2. *By planar graph we mean a finite graph embedded in a plane or sphere, whose edges do not cross. For the purposes of this paper we will include simple closed paths with no specified vertices as planar graphs.*

Definition 3. *Let $G \subset \mathbb{C}$ be a fixed open set, and let $f : G \to \mathbb{C}$ be a non-constant meromorphic function. Let $\epsilon > 0$ be given, and define $E_{f,\epsilon} := \{z \in G : |f(z)| = \epsilon\}$. By an ϵ level curve of f, we mean a component of $E_{f,\epsilon}$. Let Λ be ϵ some level curve of f which is bounded, and such that cl$(\Lambda) \subset G$. Let $\epsilon > 0$ denote the value $|f|$ takes on Λ. In general, by a level curve of f we mean a component of $E_{f,\eta}$ for some $\eta > 0$. For any $z \in G$, let Λ_z denote the level curve of f which contains z.*

In the existing literature on the subject of level curves of a meromorphic function, it is generally assumed without proof that Λ is a planar graph whose vertices are the critical points of f in Λ, and whose edges are paths in Λ with endpoints at the critical points of f. However because part of the subject of this dissertation is precisely the geometry of the level curves of a meromorphic function, we make that fact explicit in Proposition 2.1, and give an indication as to the nature of the proof. Because much of the proof amounts to an exercise in analytic continuation and compactness, we leave out most of the details.

Proposition 2.1. *Λ is a planar graph, whose vertices are the critical points of f. If Λ does not contain any critical point of f, then Λ is a simple closed path.*

Proof. Since Λ is bounded and f is non-constant, there are only finitely many critical points of f in Λ. Choose some $z_0 \in \Lambda$ which is not a critical point of f. We wish to show that either z_0 lies in a path in Λ whose endpoints are critical points of f, or that Λ contains no critical points of f, in which case Λ consists of a simple closed path.

Since $f'(z_0) \neq 0$, there is some open simply connected set $D \subset G$ which contains z_0 such that an inverse g of f may be defined on $f(D)$. $f(z_0)$ lies in the circle of radius ϵ around 0. Our strategy is to continue g analytically around that circle as far as possible, and consider the path that this traces out back in the domain of f.

Case 2.0.1. *g may be continued indefinitely around the ϵ-circle in at least one direction.*

Assume that g may be continued indefinitely around the ϵ-circle in the positive direction. By the compactness of Λ, there are only finitely many points in Λ at which f takes the value $f(z_0)$. Therefore if g is continued analytically along the ϵ-circle in the positive direction enough times, the path traced out in the domain intersects itself. Let σ denote this path. Because f is invertible at each point in σ we may conclude that σ is simple (as f would not be injective at a crossing point of such a path) and does not contain any critical point of f. Furthermore, we can use the invertibility of f at each point in σ, and the connectedness of Λ, to show that the path σ is all of Λ. Thus Λ is a simple closed path which does not contain any critical points of f. If g may be continued indefinitely around the ϵ-circle in the negative direction we obtain the same result.

Case 2.0.2. *g may not be continued indefinitely around the ϵ-circle in either direction.*

Again let σ denote the path obtained by continuing g around the ϵ-circle as far as may be in the positive direction. Clearly σ is contained in Λ and does not contain any critical points of f (because f is invertible at each point in σ). One can use the compactness of Λ to show that σ approaches a unique point in Λ, and one can show that this point is a critical point of f (otherwise one could continue g further along the ϵ-circle). g may not be continued indefinitely around the ϵ circle in the opposite direction because this would imply that Λ does not contain a critical point of f (as seen in Case 2.0.1 above). Therefore if we now allow σ to denote the path obtained by continuing g along the ϵ-circle in the opposite direction, we obtain another path from z_0 to a critical point of f. We conclude that z_0 lies in a path in Λ whose end points are critical points of f. Furthermore this path is simple because f is invertible at each point in this path other than the end points.

Finally we may again use the compactness of Λ to show that Λ contains only finitely many edges and vertices, which gives us the desired result. $\square$

Due to the nice behavior of rational functions near ∞, one can easily extend the result of the above proposition to all level curves of a rational function. Since the proof merely requires one to pre-compose the rational function with a Möbius function, we omit it here.

Corollary 1. *If $g : cl(\mathbb{C}) \to cl(\mathbb{C})$ is a rational function, every level curve of g is a planar graph.*

We will now uncover some restrictions on which planar graphs we may see as level curves of our function f. First in Proposition 2.2 we see that Λ must have bounded faces, and that each edge of Λ is adjacent to at least one of its bounded faces. In Proposition 2.3, we see that each vertex of Λ must have evenly many edges of Λ incident to it.

Proposition 2.2. *Λ has one or more bounded faces and a single unbounded face, and each edge of Λ is adjacent to two distinct faces of Λ.*

Proof. Since Λ is bounded, there is a single unbounded face of Λ, so to show that Λ has bounded faces, it suffices to show the second result of the proposition, that each edge of Λ is adjacent to two distinct faces of Λ. Define $E := \{z \in E_{f,\epsilon} : z \notin \Lambda\}$. Then $cl(E)$ is a closed set contained in $cl(G)$, and $cl(E)$ does not intersect Λ.

Choose some $z_0 \in \Lambda$ which is not a zero of f'. That is, z_0 is a point in one of the edges of Λ. Choose some $\delta > 0$ small enough so that $B_\delta(z_0) \subset G \setminus E$. (So for all $z \in B_\delta(z_0)$, if $|f(z)| = \epsilon$, then $z \in \Lambda$.) By the Open Mapping Theorem, there are points $x, y \in B_\delta(z_0)$ such that $|f(x)| < \epsilon$ and $|f(y)| > \epsilon$. Suppose by way of contradiction that x and y are in the same face of Λ. Lemma 1 (with $\Lambda = X$ and $E \cup G^c = Y$) gives that x and y are in the same component of $(\Lambda \cup E \cup G^c)^c$ which is equal to $G \setminus E_{f,\epsilon}$. Since open connected sets in $\mathbb{C}$ are path connected, there is a path $\gamma : [0, 1] \to G \setminus E_{f,\epsilon}$ from x to y. But f is continuous, and $|f(x)| < \epsilon < |f(y)|$, so there is some $s \in (0, 1)$ such that $|f(\gamma(s))| = \epsilon$, and thus $\gamma(s) \in E_{f,\epsilon}$, and we have our contradiction. So we conclude that x and y are in different faces of Λ, and thus Λ has bounded faces.

Furthermore, since δ was arbitrarily small, and the points x and y we found in $B_\delta(z_0)$ were shown to be in different components of Λ^c, we have that z_0 is in the boundary of more than one component of Λ^c. Since the edge of Λ containing z_0 is locally conformally equivalent to the ϵ-circle, we have that the edge of Λ containing z_0 is in the boundary of at most two faces of Λ, and we are done. □

One more restriction may be made on which planar graphs we may see as the bounded level curves of f. As the method of proof is largely the same as that used in the proof of Proposition 2.1, we will leave out most of the details. First a definition.

Definition 4. *If $z \in G$ is a zero of f', then let mult(z) denote the multiplicity of z as a zero of f'.*

Proposition 2.3. *If z is a vertex of Λ, then the number of edges of Λ which are incident to z is exactly $2(\text{mult}(z) + 1)$.*

Proof. As already noted, if $z \in \Lambda$ is not a zero of f', then z is not a vertex of Λ. Suppose $z \in \Lambda$ is a zero of f'. Then f is precisely $(\text{mult}(z) + 1)$-to-1 in a neighborhood of z. One can show (using the same tricks found in the proof of Proposition 2.1 of analytic continuation of branches of the inverse of f) that there are precisely $2(\text{mult}(z)+1)$ edges of Λ incident to z. □

In Figure 2-1 at the end of this chapter, we have several examples of planar graphs that satisfy the restrictions noted in Proposition 2.2 and in Proposition 2.3, and in Figure 2-2 we have several examples of planar graphs which do not satisfy these restrictions.

Here are several examples of these graphs which arise as level curves of analytic functions.

Example: Let $f(z) = z^n$ for some $n \in \mathbb{Z} \setminus \{0\}$, and $\epsilon \in (0, \infty)$. Then $E_{f,\epsilon}$ is the circle $\{z \in \mathbb{C} : |z| = \epsilon^{\frac{1}{n}}\}$.

Example: Let $f(z) = z^n - 1$ for some $n \in \{2, 3, ...\}$. If $\epsilon \in (0, 1)$, then $E_{f,\epsilon}$ has n components, each a simple closed path which contains a single zero of f in its bounded face. If $\epsilon \in (1, \infty)$, then $E_{f,\epsilon}$ consists of a single simple closed path which contains all n zeros of f in its bounded face. Finally, $E_{f,1}$ consists of a single component with a single vertex (at 0), n edges, and n bounded faces, each of which contains a single zero of f. In Figure 2-3 we see the example $f(z) = z^5 - 1$. The level sets shown are the sets $E_{f,.5}$, $E_{f,1}$, and $E_{f,1.5}$.

2.2 Assorted Properties of Level Curves

2.2.1 Setting

We now impose the following additional restrictions on the set G.

- G is bounded.
- There is some open set G' such that cl$(G) \subset G'$ and f is meromorphic on G'.
- For each component K of ∂G, there is some $r \in (0, \infty)$ such that $|f| \equiv r$ on K. (That is, each component of ∂G is contained in some level curve of f in G'.)
  Note: If one of the components of G^c is a single point $\{z\}$ for some $z \in \mathbb{C}$ (and thus z is a removable discontinuity of f restricted to G), then we replace G with $G \cup \{z\}$.
- Any level curve of f in G' that contains part of the boundary of G does not extend into G.
- G is connected and G^c has finitely many components.

2.2.2 Properties

In the previous section we examined one level curve of f in isolation from the others. In this section we catalogue some facts about the level curves of f globally in G. We will begin by introducing some vocabulary.

Definition 5. *A level curve of f which is not a zero or pole of f, and contains a critical point of f, we will call a critical level curve of f. A level curve of f which is not a critical level curve we call a non-critical level curve of f.*

In the next result we show that a closed set contained in the complement of a level curve of f may be separated from that level curve by a different non-critical level curve of f. This begins to show that the level sets of f vary continuously.

Definition 6. *For $D \subset \mathbb{C}$, let $sc(D)$ denote the union of D with each bounded component of D^c. (The "sc" stands for "simply connected.")*

Proposition 2.4. *Let L denote some level curve of f in G, or some component of ∂G, and let $r_0 \in [0, \infty]$ be the value that $|f|$ takes on L. Let $K \subset sc(G)$ denote some closed set contained in a single component of L^c. If $z \in G$ is sufficiently close to L and in the same component of L^c as K, the following holds.*

- $|f(z)| \neq r_0$.
- *Λ_z is a non-critical level curve of f in G.*
- *K and L are in different faces of Λ_z.*
- *If K is in the unbounded face of L, then K is in the unbounded face of Λ_z. Otherwise K is in the bounded face of Λ_z.*

Proof. Let D denote the intersection of the face of L which contains K with $sc(G)$.

Case 2.0.3. *K is contained in a bounded face of L.*

Then in this case, D equals the bounded face of L which contains K. Let $K_1 \subset D$ be the set of all points $z \in D$ such that z satisfies at least one of the following.

- $z \in K$.
- $z \in G$ and $f(z) = 0$ or $f(z) = \infty$.
- $z \in G$ and $|f(z)| = r_0$.
- $z \in G$ and $f'(z) = 0$.
- $z \in G^c$.

K_1 is closed (as it is a finite union of closed sets) and bounded, and thus compact. Therefore we may find some compact connected set $\widetilde{K_1} \subset D$ which contains K_1 (a simple point set topology argument, relying on the fact that open connected sets in $\mathbb{C}$ are path connected.). We can then find a compact connected set $K_2 \subset D$ such that $\widetilde{K_1}$ is contained in the interior of K_2 (for example a finite union of closed balls contained in D). Since $\widetilde{K_1}$ does not intersect ∂K_2, $|f|$ does not take the value r_0 on ∂K_2. Thus since ∂K_2 is compact, if we set $\iota := \inf_{z \in \partial K_2}(||f(z)| - r_0|)$, then $\iota > 0$. We will now use this ι to determine how close the "sufficiently close" from the statement of the proposition is.

Since L and K_2 are compact, some $\delta > 0$ may be found small enough so that if $z \in \mathbb{C}$ is within δ of L, then $z \in G \setminus K_2$, and $||f(z)| - r_0| \in (0, \iota)$.

Claim 2.0.1. *If $z \in D$ is less than δ away from L, then Λ_z is a non-critical level curve of f contained in G, such that L is in the unbounded face of Λ_z, and K_2 is in the bounded face of Λ_z, and $|f(z)| \neq r_0$.*

Let $z \in D$ be less than δ away from L. By the definition of δ, $||f(z)| - r_0| \in (0, \iota)$ (and thus $|f(z)| \neq r_0$), so by definition of ι, Λ_z does not intersect ∂K_2. Since Λ_z does intersect ${K_2}^c$ (namely at z), and Λ_z is connected, we may conclude that Λ_z does not intersect K_2. And K_2 is connected, so K_2 is entirely contained in one of the faces of Λ_z. Since K_2 contains all critical points of f in D, Λ_z is a non-critical level curve of f, and therefore has only one bounded face. Let F denote the bounded face of Λ_z. By the Maximum Modulus Theorem F must contain either a zero or pole of f, or a point in G^c. But each zero and pole of f and point in G^c which is in D is contained in K_2, so we may conclude that K_2 is contained in F. And since Λ_z is contained in D, and D is a bounded face of L, L is contained in the unbounded face of Λ_z.

Case 2.0.4. *K is contained in the unbounded face of L.*

We use a similar argument as with the previous case, but we want to construct K_2 in a way that will guarantee that when we have found our δ, if $z \in D$ is within δ of L, then

K_2 will be contained in the unbounded face of Λ_z. Therefore we define K_1 identically as before except in addition we include ∞ in K_1. Constructing K_2 as before, we obtain a closed connected set contained in D which contains K_1 in its interior. Let ι and δ be defined as above. The same argument as above with minor changes allows us to conclude if $z \in D$ is within δ of L, then Λ_z is a non-critical level curve of f in G with $|f| \neq r_0$ on Λ_z, and L is in the bounded face of Λ_z, and K_2 (and therefore K) is in the unbounded face of Λ_z. □

The next theorem states that if any two level curves of f in G are exterior to each other, then there is a critical level curve of f in G which contains the two level curves in different bounded faces. We will conclude from this that if we remove all critical level curves from G, each component of the remaining set will be conformally equivalent to a disk or an annulus. If, in addition, we remove the zeros and poles of f from G, then each component of the remaining set will be conformally equivalent to an annulus.

Definition 7. *If Λ_1 and Λ_2 are level curves of f in G then we say $\Lambda_1 \prec \Lambda_2$ if Λ_1 lies in one of the bounded faces of Λ_2. Let D be an open sub-set of G. If Λ_1 is a critical level curve of f contained in D, then we say that Λ_1 is $\prec$-maximal with respect to D if there is no other critical level curve Λ_2 of f contained in D such that $\Lambda_1 \prec \Lambda_2$.*

Theorem 2.1. *Let each of L_1 and L_2 be a level curve of f contained in G or a component of ∂G which is adjacent to a bounded component of G^c. Let F_1 denote the unbounded face of L_1 and F_2 the unbounded face of L_2. If $L_1 \subset F_2$, and $L_2 \subset F_1$, then there is some $w \in G$ which lies in $F_1 \cap F_2$, such that $f'(w) = 0$ and L_1 and L_2 are contained in different bounded faces of Λ_w.*

Proof. Define $\epsilon_1 := |f(L_1)|$, and $\epsilon_2 := |f(L_2)|$.

Case 2.1.1. *Both L_1 and L_2 are level curves of f in G.*

$G \cap F_1 \cap F_2$ is open, and by Lemma 1, it is connected. From this it is easy to show that we can find a path $\gamma : [0, 1] \to G$ such that $\gamma(0) \in L_1$ and $\gamma(1) \in L_2$, and for all

$r \in (0,1)$, $\gamma(r) \in G \cap F_1 \cap F_2$. Define $A \subset (0,1)$ be the set such that $r \in A$ if and only if $\Lambda_{\gamma(r)}$ contains L_1 in one of its bounded faces and L_2 in its unbounded face. Clearly if L is any level curve of f in G such that L_1 and L_2 are in different faces of L, then L intersects the path γ. Thus Proposition 2.4 guarantees that A is non-empty. So if we define $r_1 := \sup(r \in (0,1) : r \in A)$, we have $r_1 \in (0,1]$.

Claim 2.1.1. *$r_1 < 1$.*

Proposition 2.4 also implies that we may find some $s \in (0,1)$ such that $\Lambda_{\gamma(s)}$ contains L_2 in one of its bounded faces and L_1 in its unbounded face. Let D denote the face of $\Lambda_{\gamma(s)}$ which contains L_2. Since $\Lambda_{\gamma(s)}$ and L_2 are compact, the distance between $\Lambda_{\gamma(s)}$ and L_2 is greater than zero, so, because γ is continuous, there is some $\eta > 0$ such that if $r \in (1-\eta, 1]$, $\gamma(r)$ is contained in D. Therefore for all $r \in (1-\eta, 1)$, $\Lambda_{\gamma(r)}$ is contained in D, so the bounded faces of $\Lambda_{\gamma(r)}$ are contained in D. Thus for all $r \in (1-\eta, 1)$, r is not in A. From this we may conclude that $r_1 < 1$.

We will now show that $\Lambda_{\gamma(r_1)}$ contains a critical point of f, and that L_1 and L_2 are contained in different bounded faces of $\Lambda_{\gamma(r_1)}$.

Claim 2.1.2. *$L_1 \prec \Lambda_{\gamma(r_1)}$.*

Since $\gamma(r_1) \in F_1 \cap F_2$, $\Lambda_{\gamma(r_1)} \subset F_1 \cap F_2$. Suppose by way of contradiction that L_1 is contained in the unbounded face of $\Lambda_{\gamma(r_1)}$. Then by Proposition 2.4, there is some non-critical level curve Λ_1 of f contained in G such that $\Lambda_{\gamma(r_1)}$ is contained in the bounded face of Λ_1 and L_1 is contained in the unbounded face of Λ_1. Then one may easily use the continuity of γ to show there is some $s \in (0, r_1)$ such that for each $r \in (s, r_1]$, $\Lambda_{\gamma(r)}$ does not contain L_1 in any of its bounded faces (since $\gamma(r)$ is in the bounded face of Λ_1). Therefore for each $r \in (s, r_1]$, $r \notin A$, which is a contradiction of the definition of r_1. Thus we conclude that $L_1 \prec \Lambda_{\gamma(r_1)}$.

Claim 2.1.3. *$L_2 \prec \Lambda_{\gamma(r_1)}$.*

Suppose by way of contradiction that L_2 is in the unbounded face of $\Lambda_{\gamma(r_1)}$. Then Proposition 2.4 gives that there is some non-critical level curve Λ_2 of f in G such that $\Lambda_{\gamma(r_1)}$ in the bounded face of Λ_2, and L_2 in the unbounded face of Λ_2. Thus $\gamma(r_1)$ is in the bounded face of Λ_2, and $\gamma(1)$ is in the unbounded face of Λ_2, so by the continuity of γ, there is some $r \in (r_1, 1)$ such that $\gamma(r) \in \Lambda_2$ (and thus $\Lambda_{\gamma(r)} = \Lambda_2$). But L_1 is in one of the bounded faces of $\Lambda_{\gamma(r_1)}$, so L_1 is in the bounded face of Λ_2 as well, and thus $r \in A$. However this is a contradiction of the definition of r_1. We conclude that L_2 is contained in one of the bounded faces of $\Lambda_{\gamma(r_1)}$. That is, $L_2 \prec \Lambda_{\gamma(r_1)}$.

Thus L_1 and L_2 are each contained in bounded faces of $\Lambda_{\gamma(r_1)}$.

Claim 2.1.4. *L_1 and L_2 are contained in different bounded faces of $\Lambda_{\gamma(r_1)}$.*

We use an almost identical argument to that found in Claim 2.1.2. Suppose by way of contradiction that L_1 and L_2 are contained in the same bounded face of $\Lambda_{\gamma(r_1)}$. Let D denote the face of $\Lambda_{\gamma(r_1)}$ containing L_1 and L_2. Again by Proposition 2.4, there is some non-critical level curve Λ_3 of f contained in D such that L_1 and L_2 are in the bounded face of Λ_3, and $\Lambda_{\gamma(r_1)}$ is in the unbounded face of Λ_3. Therefore $\gamma(0)$ is in the bounded face of Λ_3 and $\gamma(r_1)$ is in the unbounded face of Λ_3. Thus by the continuity of γ there is some $s \in (0, r_1)$ such that for each $r \in (s, r_1]$, $\gamma(r)$ is in the unbounded face of Λ_3. Fix for the moment some $r \in (s, r_1]$. $\Lambda_{\gamma(r)}$ is contained in the unbounded face of Λ_3, and L_1 and L_2 are both contained in the bounded face of Λ_3. Therefore, since $\Lambda_{\gamma(r)}$ does not intersect Λ_3, L_1 is in the bounded face of $\Lambda_{\gamma(r)}$ if and only if L_2 is as well. Thus $r \notin A$ for each $r \in (s, r_1]$, which contradicts the definition of r_1.

We conclude that L_1 and L_2 are contained in different bounded faces $\Lambda_{\gamma(r_1)}$, which implies that $\Lambda_{\gamma(r_1)}$ contains a zero of f' by Proposition 2.1, and we are done.

Case 2.1.2. *At least one of L_1 and L_2 are components of ∂G.*

Suppose that L_1 is a component of ∂G. L_2 is compact, so by Proposition 2.4, we may find a non-critical level curve ${L_1}'$ of f in G such that L_1 is contained in the bounded face of ${L_1}'$ and L_2 is contained in the unbounded face of ${L_1}'$. If L_2 is a component of ∂G, the same use of Proposition 2.4 gives us a level curve ${L_2}'$ of f in G such that L_2 is contained in the bounded face of ${L_2}'$ and ${L_1}'$ (and thus L_1) is contained in the unbounded face of ${L_1}'$. Then we can use the preceding case with ${L_1}'$ and ${L_2}'$, which then gives us the desired result for L_1 and L_2. □

Let us now extend the $\prec$-ordering to the bounded components of ∂G as follows.

Definition 8. *If $K \subset G$ is a component of ∂G, and L is a level curve of f in G such that K is contained in one of the bounded faces of L, then we write $K \prec L$.*

Next we have two corollaries to the previous theorem guaranteeing the existence of a unique $\prec$-maximal critical level curve of f in certain regions of G, but first a definition.

Definition 9. *Define*

$$\mathcal{B} := \{w \in G : f'(w) = 0 \text{ or } f(w) = 0 \text{ or } f(w) = \infty\},$$

and define

$$\mathcal{C} := \left(\bigcup_{w\in\mathcal{B}} \Lambda_w\right) \cup \{w \in \partial G : w \text{ is in } \partial D \text{ for some bounded component } D \text{ of } G^c\}.$$

Corollary 2. *There is a unique $\prec$-maximal component of $\mathcal{C}$.*

Proof. By the Maximum Modulus Theorem, $\mathcal{C}$ is non-empty. Since $\mathcal{C}$ has finitely many components, it must have at least one $\prec$-maximal component. Suppose by way of contradiction that there are two distinct components L_1 and L_2 of $\mathcal{C}$ which are both $\prec$-maximal. Then each is in the unbounded component of the other, so Theorem 2.1 guarantees that there is some other critical level curve $\Lambda' \subset \mathcal{C}$ such that $L_i \prec \Lambda'$ for $i = 1, 2$. But this is a contradiction of the $\prec$-maximality of L_1 and L_2. □

More generally we have the following corollary.

Corollary 3. *If D is a bounded face of Λ then there is a unique component of $\mathcal{C}$ in D which is $\prec$-maximal with respect to D.*

Proof. Just replace G with D, and then the desired result is just the result from Corollary 2 in the new setting. □

In light of our following final result, the name "critical level curve" has new meaning. The following theorem says that on any component of $G \setminus \mathcal{C}$, f is very simple, being conformally equivalent to the function $z \mapsto z^n$ for some $n \in \{1, 2, ...\}$. First a bit of notation.

Definition 10. *For any path γ, let $-\gamma$ denote the same curve γ equipped with the opposite orientation.*

Theorem 2.2. *Let D be a component of $G \setminus \mathcal{C}$. Then the following hold.*

1. *D is conformally equivalent to some annulus A.*
2. *Let E_1 denote the inner boundary of D, and let E_2 denote the outer boundary of D. Then there is some $\epsilon_1, \epsilon_2 \in [0, \infty]$ such that $\epsilon_1 \neq \epsilon_2$, and $|f| \equiv \epsilon_1$ on E_1, and $|f| \equiv \epsilon_2$ on E_2.*
3. *Let $i_1, i_2 \in \{1, 2\}$ be chosen so that $\epsilon_{i_1} < \epsilon_{i_2}$. Then there is some $N \in \{1, 2, ...\}$ such that $A = ann(0; {\epsilon_{i_1}}^{\frac{1}{N}}, {\epsilon_{i_2}}^{\frac{1}{N}})$, and some conformal mapping $\phi : D \to ann(0; \epsilon_1^{\frac{1}{N}}, \epsilon_2^{\frac{1}{N}})$ such that $f \equiv \phi^M$ on D, where $M = \pm N$.*
4. *The conformal map ϕ described in Item 3 extends continuously to E_2 and to all points in E_1 which are not zeros of f'. If $z \in E_1$ is a zero of f', and $\gamma : [0, 1] \to G$ is a path such that $\gamma([0, 1)) \subset D$, and $\gamma(1) = z$, then $\lim_{r \to 1} \gamma(r)$ exists.*

Proof. Let D be some component of $G \setminus \mathcal{C}$. Since D does not contain a zero or pole of f, the Maximum Modulus Theorem implies that D^c must have at least one bounded component. Suppose by way of contradiction that D^c has two distinct bounded components. Replacing G with D in the statement of Theorem 2.1, we may conclude that D contains a zero of f', which is a contradiction because all zeros of f' in G are contained in $\mathcal{C}$. We conclude that D^c has exactly one bounded component. Thus D is conformally equivalent to an annulus (see for example [12]). Let E_1 denote the interior boundary of D and E_2 denote the exterior boundary of D. Each component of the boundary of D is contained in a level curve of f or a component of ∂G. Therefore we may define $\epsilon_1 \in [0, \infty]$ to be the value of $|f|$ on E_1 and $\epsilon_2 \in [0, \infty]$ to be the value of $|f|$ on E_2. By the Maximum Modulus Theorem since D does not contain a zero or pole of f and $D \subset G$, we conclude that $\epsilon_1 \neq \epsilon_2$. Assume throughout that $\epsilon_1 < \epsilon_2$, otherwise make the appropriate minor changes.

Claim 2.2.1. *There is some* $N \in \{\pm 1, \pm 2, \ldots\}$ *such that for any* $z \in D$*, the change in* $\arg(f)$ *as* Λ_z *is traversed in the positive direction is exactly* $2\pi N$*.*

Let $z_1, z_2 \in D$ be given such that $\Lambda_{z_1} \neq \Lambda_{z_1}$. Since D contains no critical points of f, Λ_{z_1} and Λ_{z_2} are non-critical level curves of f, and Theorem 2.1 implies that either $\Lambda_{z_1} \prec \Lambda_{z_2}$ or $\Lambda_{z_2} \prec \Lambda_{z_1}$. Rename z_1 and z_2 if necessary so that $\Lambda_{z_1} \prec \Lambda_{z_2}$. By the Maximum Modulus Theorem, the bounded face of Λ_{z_1} contains some point in $\mathcal{C}$. However D contains no points of $\mathcal{C}$, so E_1 is a contained in the bounded face of Λ_{z_1}. Let Figure 2-4 (at the end of this chapter) represent this choice of D, z_1, and z_2. Let σ denote the path in Figure 2-5.

Since σ may be retracted to a point in D, and D contains no zero or pole of f, the total change in $\arg(f)$ along σ is zero. Therefore the sum of the changes in $\arg(f)$ along Λ_{z_2} and $-\Lambda_{z_1}$ equals zero. Therefore the change in $\arg(f)$ along Λ_{z_1} is the same as the change in $\arg(f)$ along Λ_{z_2}. We conclude that the change in $\arg(f)$ along Λ_z is independent of the choice of $z \in D$. Let β denote this common number. It is well known (see for example [13]) that if γ is any closed path in $\mathbb{C}$ and g is a function analytic on γ with no zeros on γ, the change in $\arg(g)$ along γ is well defined and there is some $n \in \mathbb{Z}$ such that the change in $\arg(g)$ along γ is $2\pi n$. Thus we may choose $N \in \mathbb{Z}$ such that $\beta = 2\pi N$. By the same argument as above, the changes in $\arg(f)$ along E_1 and E_2 are both $2\pi N$ as well.

Furthermore since D does not contain any zero of f', f is injective at each point in D, so if L is some level curve of f in D, $\arg(f)$ is either strictly increasing or strictly decreasing as L is traversed in the positive direction. Therefore $N \neq 0$, so $N \in \{\pm 1, \pm 2, ...\}$.

Let us now adopt the convention that unless otherwise specified, any time we are referring to a level curve of f in D as a path, we are traversing that level curve in the direction in which $\arg(f)$ is increasing. By the claim above, this orientation is either the positive orientation for all level curves of f in D or the negative orientation for all level curves of f in D. Thus with this assumed orientation we may say that the change in $\arg(f)$ along any level curve L in D is $2\pi N$ for some $N \in \{1, 2, ...\}$ which is independent of L.

Claim 2.2.2. *Let γ be a closed path in D. Then the change in $\arg(f)$ along γ is an integer multiple of $2\pi N$.*

Let k denote the number of times γ winds around E_1. Then since D contains no zeros or poles of f, the Argument Principle implies that the change in $\arg(f)$ along γ is k times the change in $\arg(f)$ along E_1. Thus the change in $\arg(f)$ along γ is $k2\pi N$.

We now wish to define the conformal mapping described in the statement of the theorem. Fix some z_0 in D at which f takes a positive real value. (To see that such a point z_0 exists, observe that for any $z \in D$, the change in $\arg(f)$ along Λ_z is $2\pi N$, so there are N different points in Λ_z at which f takes positive real values.) We wish to define a map $\phi : D \to \mathbb{C}$ which we will show is a conformal map with $\phi(D) = \text{ann}(0; {\epsilon_1}^{\frac{1}{N}}, {\epsilon_2}^{\frac{1}{N}})$. For $w \in D$, let $\gamma : [0, 1] \to D$ be a path such that $\gamma(0) = z_0$ and $\gamma(1) = w$. Define α to be the change in $\arg(f)$ along γ. Then define $\phi(w) := |f(w)|^{\frac{1}{N}} e^{i\frac{\alpha}{N}}$. This may be shown to be the value at w of the analytic continuation of $f^{\frac{1}{N}}$ along the path γ, and thus (if well defined) is analytic in a neighborhood of w. Therefore if ϕ defined as such is well defined, then ϕ is analytic on D. Let $w \in D$ be given, and let γ_1 and γ_2 be paths in D from z_0 to w. Let $\phi_1(w)$ denote the value of ϕ at w obtained using $\gamma = \gamma_1$ and let $\phi_2(w)$ denote the value of ϕ at w obtained by using $\gamma = \gamma_2$. Let α_1 denote the change in $\arg(f)$ along γ_1 and let α_2 denote the change in $\arg(f)$ along γ_2. Let γ_3 denote the path in D obtained by traversing γ_1 and then traversing $-\gamma_2$. γ_3 is a closed path in D, so by the claim above, the change in $\arg(f)$ along γ_3 (which is the change in $\arg(f)$ along γ_1 minus the change in $\arg(f)$ along γ_2) is an integer multiple of $2\pi N$. Thus $\alpha_1 = \alpha_2 + k2\pi N$ for some $k \in \mathbb{Z}$. Thus

$$\phi_1(w) = |f(w)|^{\frac{1}{N}} e^{i\frac{\alpha_1}{N}} = |f(w)|^{\frac{1}{N}} e^{i\frac{\alpha_2 + k2\pi N}{N}} = |f(w)|^{\frac{1}{N}} e^{i\frac{\alpha_2}{N}} e^{\frac{k2\pi N}{N}} = \phi_2(w).$$

Therefore whether we define $\phi(w)$ using γ_1 or using γ_2 we obtain the same value. (Note that we are essentially just showing that we may take an N^{th} root of f on D.)

Claim 2.2.3. $\phi(D) = \mathit{ann}(0; {\epsilon_1}^{\frac{1}{N}}, {\epsilon_2}^{\frac{1}{N}})$.

Note that by the Maximum Modulus Theorem, for each $z \in D$, since $D \subset G$ and D does not contain any zero or pole of f, $|f(z)| \in (\epsilon_1, \epsilon_2)$, and thus $|\phi(z)| \in ({\epsilon_1}^{\frac{1}{N}}, {\epsilon_2}^{\frac{1}{N}})$. Therefore $\phi(D) \subset \text{ann}(0; {\epsilon_1}^{\frac{1}{N}}, {\epsilon_2}^{\frac{1}{N}})$.

Let $\xi \in \text{ann}(0; {\epsilon_1}^{\frac{1}{N}}, {\epsilon_2}^{\frac{1}{N}})$ be given. Choose some $x \in D$ such that $|f(x)|^{\frac{1}{N}} = |\xi|$. Such an x exists because $|f| \equiv \epsilon_1$ on E_1 and $|f| \equiv \epsilon_2$ on E_2, and $|f|$ is continuous. Let $\rho \in [0, 2\pi)$ be such that $\arg(\xi) = \arg(\phi(x)) + \rho$. (And thus $\phi(x)e^{i\rho} = \xi$.) Since the change in $\arg(f)$ along Λ_x is $2\pi N$ with $N \neq 0$, there is some point x' in Λ_x such that if Λ_x is traversed from x to x', then the change in $\arg(f)$ along this portion of Λ_x is exactly ρN. One can easily then show that $\phi(x') = \xi$.

Therefore we conclude that $\phi(D) = \text{ann}(0; {\epsilon_1}^{\frac{1}{N}}, {\epsilon_2}^{\frac{1}{N}})$.

Claim 2.2.4. *ϕ is injective.*

Let $w_1, w_2 \in D$ be given such that $\phi(w_1) = \phi(w_2)$. From the definition of ϕ, $|f(w_1)| = |f(w_2)|$. The Maximum Modulus Theorem and Theorem 2.1 together imply that there is only one level curve of f in D on which $|f|$ takes the value $|f(w_1)|$, and therefore w_1 and w_2 are in the same level curve of f. Let L denote the level curve of f in D which contains both w_1 and w_2. Let γ_1 be a path in D from z_0 to w_1. Let γ_2 be a path obtained by traversing L in the direction of increase of $\arg(f)$ from w_1 to w_2. (If $w_1 = w_2$

let γ_2 be constant.) Let γ_3 denote the path from z_0 to w_2 obtained by first traversing γ_1 and then traversing γ_2. Let α_i denote the change in $\arg(f)$ along γ_i for $i \in \{1, 2\}$. Since $\arg(f)$ is strictly increasing as γ_2 is traversed, and the total change in $\arg(f)$ as all of L is traversed is $2\pi N$, we conclude that $\alpha_2 \in [0, 2\pi N)$. Since $\phi(w_1) = \phi(w_2)$, we have the following equation, calculating $\phi(w_1)$ using γ_1, and calculating $\phi(w_2)$ using γ_3.

$$|f(w_1)|^{\frac{1}{N}} e^{i\frac{\alpha_1}{N}} = \phi(w_1) = \phi(w_2) = |f(w_2)|^{\frac{1}{N}} e^{i\frac{\alpha_1+\alpha_2}{N}} = |f(w_1)|^{\frac{1}{N}} e^{i\frac{\alpha_1}{N}} e^{i\frac{\alpha_2}{N}}.$$

Dividing both sides by $|f(w_1)|^{\frac{1}{N}} e^{i\frac{\alpha_1}{N}}$, we obtain $1 = e^{i\frac{\alpha_2}{N}}$. Thus α_2 is an integer multiple of $2\pi N$. Since $\alpha_2 \in [0, 2\pi N)$, we conclude that $\alpha_2 = 0$, and thus $w_1 = w_2$.

It remains to show that ϕ extends continuously all points in the boundary of D except possibly the zeros of f' in E_1. To see this we just observe that the definition of ϕ may be defined in the same way as above for all points in ∂D. This extension is well defined (by the same argument used above) except at the zeros of f' in E_1. If z is one of the zeros of f' in E_1, then for any path $\gamma : [0, 1] \to G$ such that $\gamma([0, 1)) \subset D$, and $\gamma(1) = z$, it may be shown using the continuity of f that $\lim_{r \to 1^-} \phi(\gamma(r))$ exists by the compactness of cl(ann$(0; \epsilon_{i_1}{}^{\frac{1}{N}}, \epsilon_{i_2}{}^{\frac{1}{N}})$). $\square$

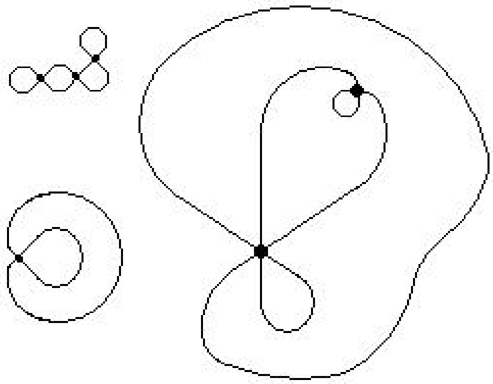

Figure 2-1. Admissible Graphs

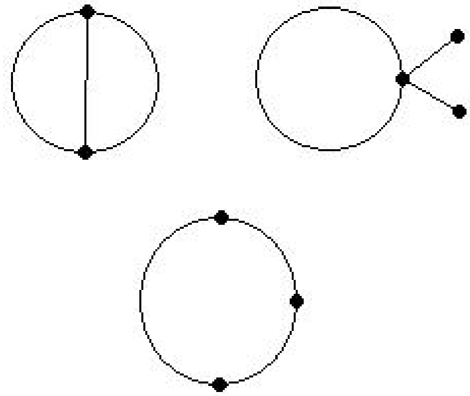

Figure 2-2. Non-Admissible Graphs

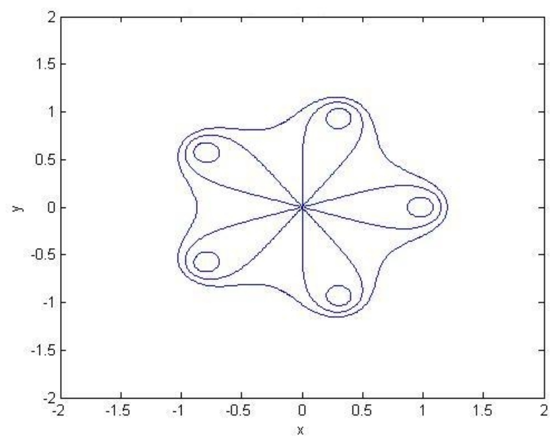

Figure 2-3. $f(z) = z^5 - 1$

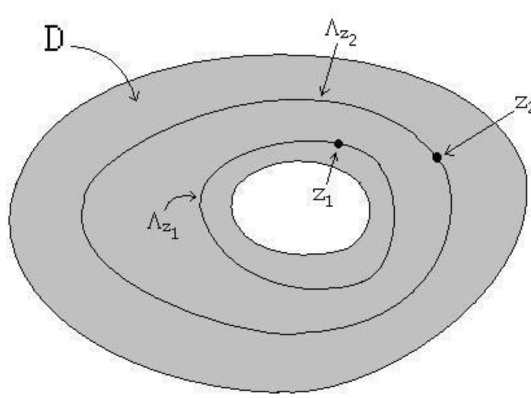

Figure 2-4. D, z_1, and z_2

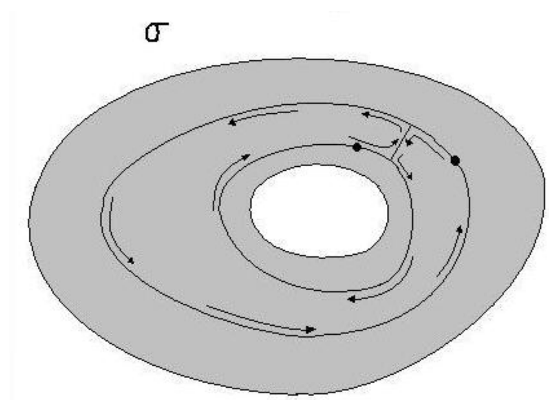

Figure 2-5. Definition of σ

CHAPTER 3
THE POSSIBLE LEVEL CURVE CONFIGURATIONS OF A MEROMORPHIC FUNCTION

We will now consider only meromorphic functions whose domains are simply connected.

Definition 11.

- *Let G be an open bounded simply connected set in $\mathbb{C}$, and let $f : G \to \mathbb{C}$ be meromorphic, and such that f can be extended to an meromorphic function on an open set containing the closure of G. Call such a pair (f, G) a function element.*
- *Say (f, G) is a special type function element if $|f| \equiv 1$ and $f' \neq 0$ on ∂G.*
- *If (f_1, G_1) and (f_2, G_2) are function elements, and there is some conformal map $\phi : G_1 \to G_2$ such that $f_1 \equiv f_2 \circ \phi$, then we say that (f_1, G_1) and (f_2, G_2) are conformally equivalent, and we write $(f_1, G_1) \sim (f_2, G_2)$.*

It is easy to see that $\sim$ is an equivalence relation on the collection of all function elements, and we make the following definition.

Definition 12. *Let H' denote the set of all special type function elements, and define $H := H'/\sim$. Let $H'_a \subset H'$ denote the set of all special type function elements (f, G) such that f is analytic on G, and define $H_a := H'_a/\sim$.*

In Chapter 4, we will show that two special type function elements are in the same member of H if and only if they have the same level curve structure. We will see that to fully describe the configuration of level curves of a special type function element (f, G), it suffices to consider only the configuration of the critical level curves of f. In order to rigorously define the configuration of critical level curves of (f, G), in the next section we will define a mathematical object PC (for "Possible Level Curve Configurations") which will parameterize the different possible level curve configurations of a special type function element.

3.1 Construction of PC

We begin by defining a set $\check{P}$ which will represent the different possible graphs one may obtain as a level curve of a special type function element. Members of $\check{P}$ are certain connected finite graphs (that is, graphs with finitely many vertices with finitely many edges), and may be viewed as sub-sets of $\mathbb{C}$, but are defined modulo orientation preserving homeomorphism (which will be defined shortly). We will now describe which finite graphs are contained in $\check{P}$.

We first include finite graphs consisting of a single vertex and no edges. Of course all single points are the same modulo homeomorphism, so there is only one member of $\check{P}$ which consists of a single point. Beyond this, a connected finite graph ξ embedded in $\mathbb{C}$ is contained in $\check{P}$ if and only if it has the following properties.

- Each edge of ξ is incident to at least one bounded face of ξ.
- For each vertex v of ξ, the number of edges of ξ incident to v is even and greater than 2 (where we count an edge twice if both endpoints of the edge are at v).

When modding out our graphs embedded in $\mathbb{C}$ by orientation preserving homeomorphisms, we mean that two graphs which meet the above restrictions are considered the same if there is an orientation preserving homeomorphism of $\mathbb{C}$ to itself which maps the one graph to the other. (In the setting of a special type level curve (f, G), the single points will be used to represent zeros or poles of f, and the graphs will be used to represent the critical level curves of f.)

Note: Throughout when we refer to the single point element of $\check{P}$, or constructions from that element, we will just refer to it as w, though $\{w\}$ may be technically more accurate.

Given that the members of $\check{P}$ will be used to help represent the critical level curves of a special type function element (f, G), we now form another set P by associating some auxiliary data to the members of $\check{P}$. To each member of $\check{P}$, we will associate auxiliary data to represent the following.

- The modulus of f on the level curve being represented.
- The points in the level curve being represented at which f takes non-negative real values. (These points we will call "distinguished points" of the graph.) If the member of $\check{P}$ in question is the single point member, it will be used to represent a

zero or pole of f, and is thus automatically distinguished. (Note: we will view ∞ as a non-negative real value.)

To the members of $\check{P}$ which are not single points, we will additionally associate auxiliary data to represent the following.

- The number of zeros minus the number of poles in each bounded face of the level curve being represented. (This will of course be equal to the number of distinguished points in the boundary of that face.)
- The argument of f at each vertex (critical point of f) of the level curve being represented.

We begin this process with the single point members of $\check{P}$. Let w denote the single point member of $\check{P}$. From w, we will construct a member $\langle w \rangle_P$ of P. We do this by associating the following pieces of data to w.

- We define $H(\langle w \rangle_P)$ to be a value in $\{0, \infty\}$ (depending on whether $\langle w \rangle_P$ will represent a zero of f or a pole of f).
- We write $Z(\langle w \rangle_P) = n$ for some non-zero $n \in \mathbb{Z}$. This represents the multiplicity of the point being represented as a zero or pole of f (positive if a zero, negative if a pole).
- We say that w is distinguished with multiplicity $|n|$ to represent that f is non-negative real on w, and the ramification of f at w is $|n|$.

The resulting object we denote $\langle w \rangle_P$.

If λ is a member of $\check{P}$ which is not the single point, then we construct a member $\langle \lambda \rangle_P$ of P from λ by associating the following pieces of data to λ.

- We define $H(\langle \lambda \rangle_P) = \epsilon$ for some value $\epsilon \in (0, \infty)$ to denote the value of $|f|$ on λ.
- If D is a bounded face of λ, we associate an integer $z(D) \in \mathbb{Z} \setminus \{0\}$. (This represents the number of zeros of f in D minus the number of poles of f in D.) If $D_1, D_2, \ldots, D_k$ denote the bounded face of λ, we define $Z(\langle \lambda \rangle_P) = \sum_{i=1}^{k} z(D_i)$. This assignment must be done in such a way that $Z(\langle \lambda \rangle_P) \neq 0$ and if D_1 and D_2 are bounded faces of λ which share a common edge, then $z(D_1)$ and $z(D_2)$ are not both positive or both negative. (This is the case for level curves of f in view of the Open Mapping Theorem.)

- For each bounded face D of λ, we distinguish $z(D)$ points (but finitely many) in ∂D (to represent the points in λ at which f takes non-negative real values).
- If $x \in \lambda$ is a vertex of λ, we designate a value $a(x) \in [0, 2\pi)$. (This will represent the argument of f at x.) We require that this assignment follows the following rules.
  - For a vertex x of λ, $a(x) = 0$ if and only if x is a distinguished point.
  - If D is a face of λ, and $z(D) > 0$, and x_1, x_2 are vertices of λ in ∂D such that $a(x_1) \geq a(x_2)$, then there is some distinguished point $z \in \partial D$ such that x_1, z, x_2 is written in increasing order as they appear in ∂D. (This reflects the fact that if λ is a level curve of f, and D contains more zeros of f than poles of f, then the argument of f is increasing as ∂D is traversed with positive orientation.)
  - If D is a face of λ, and $z(D) < 0$, and x_1, x_2 are vertices of λ in ∂D such that $a(x_1) \geq a(x_2)$, then there is some distinguished point $z \in \partial D$ such that x_2, z, x_1 is written in increasing order as they appear in ∂D. (This reflects the fact that if λ is a level curve of f, and D contains more poles of f than zeros of f, then the argument of f is decreasing as ∂D is traversed with positive orientation.)

The resulting object with the above auxiliary data we denote $\langle\lambda\rangle_P$, and we define P to be the set of all such $\langle\lambda\rangle_P$ and $\langle w\rangle_P$. We also define $P_a \subset P$ by $\langle w\rangle_P \in P_a$ if and only if $Z(\langle w\rangle_P) > 0$, and $\langle\lambda\rangle_P \in P_a$ if and only if $z(D) > 0$ for each bounded face D of λ.

Throughout this paper, $\langle w\rangle_P$ will be used to refer to single point members of P, $\langle\lambda\rangle_P$ will be used for graph members of P, and $\langle\xi\rangle_P$ will be used when we do not wish to distinguish between the two types of members of P.

We will now construct PC. Each member of PC will be a collection of members of P arranged in different ways according to certain rules, with certain auxiliary data which will be described. As noted before, this will be used to represent the different ways in which the critical level curves of a special type function element may lie with respect to each other. There are two steps to this. First, determine which graphs lie in which bounded faces of which other graphs, and second, determine the orientations of each graph with respect to the others. We begin by describing the different ways in which the graphs may lie with respect to each other recursively.

Note: Each member of $\langle\xi\rangle_P$ of P will give rise to possibly multiple different members of PC, but when it will not cause confusion, we will use $\langle\xi\rangle_{PC}$ to denote a member of PC

which arises from $\langle\xi\rangle_P$. All members of PC will arise from some member of P, so if we use $\langle\xi\rangle_{PC}$ to denote some member of PC, then by $\langle\xi\rangle_P$ we mean the member of P which gave rise to $\langle\xi\rangle_{PC}$.

A level 0 member of PC will be a single point member of P viewed as a member of PC, with no additional data. For $n > 0$, level n members of PC are constructed by taking $\langle\lambda\rangle_P$ a graph member of P, and assigning to each bounded face D of λ a level k member $\langle\xi_D\rangle_{PC}$ of PC for some $k < n$. This assignment must follow the following restrictions.

- For each bounded face D of λ, $Z(\langle\xi_D\rangle_P) = z(D)$.
- If $z(D) > 0$, then $H(\langle\xi_D\rangle_P) < H(\langle\lambda\rangle_P)$.
- If $z(D) < 0$, then $H(\langle\xi_D\rangle_P) > H(\langle\lambda\rangle_P)$.

Furthermore, let D be any bounded face of λ. Then $\langle\lambda\rangle_{PC}$ also comes equipped with a surjective map g_D from the distinguished points in ∂D to the distinguished points in ξ_D. This map we call the gradient map, (since in the study of a special type function element (f, G), the gradient maps will be determined by the gradient lines of f) and we require that g_D preserve the orientation of the distinguished points. That is, if ξ_D is a graph embedded in $\mathbb{C}$, and we read the distinguished points in ξ_D off as they appear around ξ_D when ξ_D is traversed one full time in positive order from the outside, let $x_1, \ldots, x_{z(D)}$ be their enumeration as they appear in this way (if some vertex of ξ_D is distinguished, then it will appear more than one time in this list). Then there must be some point $y_1 \in \partial D$ which is distinguished as a point in λ, and when the distinguished points in ∂D are listed by their appearence in positive order starting with y_1, namely $y_1, \ldots, y_{z(D)}$, then $g_D(y_i) = x_i$ for each $i \in \{1, \ldots, z(D)\}$. If ξ_D is just a single point with associated number 0, then g_D is just the map that takes every distinguished point in ∂D to that single point. (In the context of level curves of meromorphic functions, $g_D(w) = z$ means that w and z are connected by a gradient line of f. This keeps track of the "orientation" of a critical level curve of f with respect to the other critical level curves of f.)

Note: If a member of PC has been formed at level n for some $n \geq 0$, we do not form it again at any later level, so we may say in a well defined way that a member has a given level.

We call this assignment of members of PC to the bounded faces of λ, along with the associated gradient maps, $\langle \lambda \rangle_{PC}$. The set of all such $\langle \lambda \rangle_{PC}$ and $\langle w \rangle_{PC}$ we denote PC, and we call this the set of possible level curve configurations. We define $PC_a \subset PC$ to be the collection of members of PC which is constructed entirely using members of P_a. That is, $\langle \lambda \rangle_{PC} \in PC_a$ if and only if $\langle \lambda \rangle_P \in P_a$, and each member of PC which is assigned to a bounded face of λ is in PC_a.

We adopt the same convention of w, λ or ξ for members of PC as we did for members of P, namely that level 0 members of PC we denote by $\langle w \rangle_{PC}$. Level $n > 0$ members of PC we denote by $\langle \lambda \rangle_{PC}$, and if we do not wish to specify the level of a member of PC we will denote it by $\langle \xi \rangle_{PC}$.

3.2 Construction of Π

We now make explicit the way in which PC parameterizes the possible level curve configurations of a special type function element. We do this by defining a function $\Pi : H' \to PC$. In Chapter 4, we will show that Π takes the same value on conformally equivalent members of H', and therefore we may view Π as acting on H. It is fairly easy to show that Π acting on H is injective, and that $\Pi(H_a) \subset PC_a$. One of the major goals of this paper will be to show that $\Pi : H_a \to PC_a$ is a bijection. We now define the action of Π on H'. Let (f, G) be some member of H'.

Recall that $\mathcal{B} = \{w \in G : f'(w) = 0 \text{ or } f(w) = 0 \text{ or } f(w) = \infty\}$. Proposition 2.1 shows that for each $w \in \mathcal{B}$, Λ_w is either a single point (if w is a zero of f), or is a planar graph of the type used to form members of $\check{P}$.

We begin by picking members of P to represent Λ_w for each $w \in \mathcal{B}$. If w is either a zero or a pole of f, then $\Lambda_w = \{w\}$, and we define $Z(\langle w \rangle_P) := k$ where k is the multiplicity of w as a zero of f, ($k < 0$ if w is a pole of f,) and define $H(\langle w \rangle_P) := 0$ if w

is a zero of f and $H(\langle w \rangle_P) = \infty$ if w is a pole of f. The resulting object $\langle w \rangle_P$ is now a member of P.

If Λ_w is not a single point, then let D be a bounded face of Λ_w. We define $z(D) := k$, where k is the number of zeros of f which are contained in D minus the number of poles of f in D. We distinguish the points in ∂D at which f takes positive real values (there will be exactly $|k|$ of them). And to each vertex $w' \in \Lambda_w$, we associate the number $a(w') \in [0, 2\pi)$ where $a(w')$ is the choice of the argument of $f(w')$ which lies in $[0, 2\pi)$. Clearly $a(w') = 0$ if and only if w' is distinguished. It is also the case that if $k > 0$, and z, z' are vertices of Λ_w with $z \neq z'$ and $0 < a(z) \leq a(z')$, then there must be some $w' \in \partial D \subset \Lambda_w$ which is distinguished, and such that z, w', z' is written as they appear in increasing order around ∂D. Similarly, if $k < 0$, and z, z' are vertices of Λ_w with $z \neq z'$ and $0 < a(z) \leq a(z')$, then there must be some $w' \in \partial D \subset \Lambda_w$ which is distinguished, and such that z', w', z is written as they appear in increasing order around ∂D. This can be shown by the interested reader using the the uniform continuity of f on the closure of D, and the Argument Principle. And we define $H(\langle \Lambda_w \rangle_P)$ to be the value that $|f|$ takes on Λ_w. Thus we obtain $\langle \Lambda_w \rangle_P \in P$.

Now we wish to stitch together the members of P obtained from $\mathcal{B}$ as above in such a way as to obtain a member of PC (which we will call $\Pi(f, G)$)). We will then identify a critical level curve of (f, G) with the graph that it gives rise to in $\Pi(f, G)$. We will also identify the critical points of f with the corresponding vertices, and the points in the critical level curves of (f, G) at which f takes non-negative real values with the corresponding distinguished points in $\Pi(f, G)$.

From each of the members w of $\mathcal{B}$ which is a zero or pole of f, form $\langle w \rangle_{PC}$ the level 0 member of PC formed from $\langle w \rangle_P$. If $w \in \mathcal{B}$ is a zero of f' which is not a zero of f, then we form $\langle \Lambda_w \rangle_{PC}$ as follows.

Let D be a bounded face of Λ_w. Corollary 3 implies that one of the two following cases hold.

Case 3.0.1. *There is a single distinct zero or pole of f contained in D, and there is no zero of f' contained in D which is not a zero of f.*

In this case, let w' denote this single distinct zero or pole of f. $Z(\langle w' \rangle_P) = z(D)$, since w' is the only zero or pole of f in D, and if $k > 0$, then $H(\langle w' \rangle_P) = 0 < H(\langle \Lambda_w \rangle_P)$, so we may associate $\langle w' \rangle_{PC}$ to D. Similarly if $k < 0$, then $H(\langle w' \rangle_{PC}) = \infty > H(\langle \Lambda_w \rangle_P)$, so we may associate $\langle w' \rangle_{PC}$ to D. Finally we define g_D to map all the distinguished points in ∂D to w'.

Case 3.0.2. *There is some critical point w' of f in G which is not a zero of f, and such that each member of $\mathcal{B}$ which is in D is either in $\Lambda_{w'}$ or in one of the bounded faces of $\Lambda_{w'}$.*

Proceed recursively. Assume that $\langle \Lambda_{w'} \rangle_{PC}$ has been already been formed. Since each zero and pole of f is in some bounded face of $\Lambda_{w'}$, $Z(\langle \Lambda_{w'} \rangle_{PC}) = z(D)$. Furthermore, if $k > 0$, then the value of $|f|$ on $\Lambda_{w'}$ is strictly less than the value of $|f|$ on Λ_w, and therefore $H(\langle \Lambda_{w'} \rangle_P) < H(\langle \Lambda_w \rangle_P)$, so we may associate $\langle \Lambda_{w'} \rangle_{PC}$ to D. On the other hand, if $k < 0$, then $H(\langle \Lambda_{w'} \rangle_P) > H(\langle \Lambda_w \rangle_P)$, so we may associate $\langle \Lambda_{w'} \rangle_{PC}$ to D. Now we wish to define g_D.

Let $z \in \partial D$ be distinguished (thus $f(z) > 0$). The fact that no gradient lines may intersect in the region in D which is exterior to $\Lambda_{w'}$ (since there are no zeros of f' in that region) gives us that there is only a single distinguished point in $\Lambda_{w'}$ which is connected to z by a portion of a gradient line of f which lies entirely in D but exterior to $\Lambda_{w'}$. Call that distinguished point $z' \in \Lambda_{w'}$. Then we define $g_D(z) := z'$. Since the gradient lines of f do not cross in the region of D exterior to $\Lambda_{w'}$, the map g_D so defined respects the order of the distinguished points as they appear in ∂D.

Do this assignment process, and definition of the gradient map, for each bounded face of Λ_w. The resulting object we call $\langle \Lambda_w \rangle_{PC}$. Since $\mathcal{B}$ has finitely many members, this process terminates. Corollary 3 implies that there is some point $w \in \mathcal{B}$ such that each $w' \in \mathcal{B}$ is either in Λ_w or in one of the bounded faces of Λ_w. Furthermore, if w, w' are any two such points, it is easy to see that $\langle \Lambda_w \rangle_{PC}$ and $\langle \Lambda_{w'} \rangle_{PC}$ as defined above are equal. Therefore we may define $\Pi(f, G) := \langle \Lambda_w \rangle_{PC}$.

Thus we classify the ways in which the critical level curves of a function f may be configured in its domain G by $\Pi(f, G)$.

CHAPTER 4
Π RESPECTS CONFORMAL EQUIVALENCE

Our goal in this chapter is to show that conformal equivalence of special type function elements may be determined entirely by their respective level curve structures. That is, we have the following theorem.

Theorem 4.1. *If* (f_1, G_1) *and* (f_2, G_2) *are two special type function elements, then* $(f_1, G_1) \sim (f_2, G_2)$ *if and only if* $\Pi(f_1, G_1) = \Pi(f_2, G_2)$.

Proof. The forward implication (that if $(f_1, G_1) \sim (f_2, G_2)$, then $\Pi(f_1, G_1) = \Pi(f_2, G_2)$) follows fairly straight forwardly from the definition of $\sim$ and the definition of Π. Let $\phi : G_1 \to G_2$ be a conformal map intertwining f_1 and f_2 (ie. $f_1 = f_2 \circ \phi$). Then if λ is a level curve of f_1 in G_1, then $\phi(\lambda)$ is a level curve of f_2 in G_2. If w is a zero or pole or critical point of f_1 in G_1, then $\phi(w)$ is a zero or pole or critical point respectively of f_2 in G_2 with the same multiplicity, and ϕ carries gradient lines of f_1 to gradient lines of f_2. It follows immediately that the construction of $\Pi(f_1, G_1)$ proceeds in exactly the same manner as the construction of $\Pi(f_2, G_2)$, so we proceed to the more difficult backward implication.

Assume that $\Pi(f_1, G_1) = \Pi(f_2, G_2)$. Let $\langle\lambda\rangle_{PC}$ denote this member of PC.

For $i \in \{1, 2\}$, define $\mathcal{B}_i := \{w \in G_i : f_1'(w) = 0$ or $f_i(w) = 0$ or $f_i(w) = \infty\}$. Define $\mathcal{C}_i \subset G_i$ by $\mathcal{C}_i := \bigcup_{w \in \mathcal{B}_i} \Lambda_w$. Let $\langle\xi\rangle_P$ be some member of P used in the construction of $\langle\lambda\rangle_{PC}$. For $i \in \{1, 2\}$, let ξ_i denote the level curve of f_i which gives rise to ξ. Since ξ_1 and ξ_2 are the same when viewed as members of P, there is an orientation preserving homeomorphism $\phi : \xi_1 \to \xi_2$. Furthermore, if E_1 is some edge in ξ_1, and E_2 is the corresponding edge in ξ_2, then E_1 and E_2 contain the same number of distinguished points. Therefore by reparameterizing ϕ, we may assume that ϕ maps the distinguished points of ξ_1 to the distinguished points of ξ_2. Since we may form this orientation preserving homeomorphism for each $\langle\xi\rangle_P$ used to construct $\langle\lambda\rangle_{PC}$, we may stitch these homeomorphisms together to obtain an orientation preserving homeomorphism $\phi : \mathcal{C}_1 \to \mathcal{C}_2$.

Furthermore, since Π preserves information about how distinguished points are connected by gradient lines, we may assume that ϕ respects gradient lines. That is, if D_1 is a bounded face of ξ_1, and D_2 is the corresponding bounded face of ξ_2 (that is, $D_2 = \phi(D_1)$), and w_1 is a distinguished point in ∂D_1, then ϕ may be chosen so that $\phi(g_{D_1}(w_1)) = g_{D_2}(\phi(w_1))$.

We now show that we can assume that $f_1 = f_2 \circ \phi$ on $\mathcal{C}_1$. Let ξ_1 be some component of $\mathcal{C}_1$. If ξ_1 is just a single point, then $\phi(\xi_1)$ is a single point, and because ξ_1 and $\phi(\xi_1)$ give rise to the same member of P, ξ_1 and $\phi(\xi_1)$ are either both zeros or both poles of f_1 and f_2 respectively. Therefore $f_1 = f_2 \circ \phi$ on ξ_1.

Now assume that ξ_1 is a critical level curve of f_1 such that $H(\langle \xi_1 \rangle_P) \in (0, \infty)$ (where $\langle \xi_1 \rangle_P$ refers to the member of P used to construct $\Pi(f_1, G_1)$ which arises from ξ_1). Then let D be a bounded face of ξ_1. Let C denote the collection of distinguished points and vertices of ξ_1 which are contained in ∂D. That is, C is the collection of points in ∂D at which f_1 takes positive real values, or $f_1{}' = 0$. Then since ϕ is a homeomorphism which preserves the information recorded by Π, (ie what points f_i takes positive real values at, and what argument f_i takes at the vertices of the level curve in question, and what the modulus of f_i is on that level curve for $i \in \{1, 2\}$), we have that $f_1 = f_2 \circ \phi$ at each point in C. Let T be a segment of ∂D whose end points are in C and such that no other point in T is in C. So T does not contain any vertex of S, except possibly at the endpoints. Let w, w' be the endpoints of T chosen so that w, T, w' is written in increasing order. If w' is distinguished, view its argument as being 2π rather than 0 for the moment. Assume that $z(D) > 0$. Then $\arg(f_1(\cdot))$ is a continuous function on T that is increasing as T is traversed in the positive direction, and the same may be said of f_2 on $\phi(T)$, and it is a well known fact that under these circumstances, there is a homeomorphism $\hat{\phi}$ mapping T to $\phi(T)$ such that $\arg(f_1(\cdot)) = \arg(f_2(\hat{\phi}(\cdot)))$. Replace ϕ by $\hat{\phi}$ on T. In this way it may be seen that $f_1 = f_2 \circ \phi$ on ξ_1. If $z(D) < 0$, the same result holds by an analogous arguments. So we may assume that $f_1 = f_2 \circ \phi$ on all of $\mathcal{C}_1$.

We now wish to extend ϕ to $G_1 \setminus \mathcal{C}_1$ in such a way that $\phi|_{G_1\setminus\mathcal{C}_1}$ is analytic, and $f_1 = f_2 \circ \phi$ on $G_1 \setminus \mathcal{C}_1$. We then show that this extended ϕ is analytic on all of G_1.

Let F be a component of $G_1 \setminus \mathcal{C}_1$. By Theorem 2.2, F is homeomorphic to an annulus. Let K denote the bounded component of F^c, and let n denote the number of zeros of f_1 contained in K minus the number of poles of f_1 contained in K. Let L_i and L_e denote the interior and exterior boundaries of F. Let $H_i, H_e > 0$ be the numbers such that $|f| \equiv H_i$ on L_i, and $|f| \equiv H_e$ on L_e. Assume that $n > 0$, and choose some distinguished point w in L_e. Then $H_i < H_e$, and by Theorem 2.2, we may find a conformal map $\phi_1 : F \to ann(0, (H_i)^{\frac{1}{n}}, (H_e)^{\frac{1}{n}})$ that satisfies the following.

- $(\phi_1)^n = f_1$.
- ϕ_1 extends continuously to all points in L_i and L_e with the exception of critical points of f_1 in L_i.
- If z is a critical point of f_1 in L_i, and $\gamma : [0,1] \to G_1$ is a path with $\gamma([0,1)) \subset F$ and $\gamma(1) = z$, then $\lim_{r\to 1} \phi_1(\gamma(r))$ exists.
- $\phi_1(w) > 0$.

Let $\phi(F)$ denote the component of $G_2 \setminus \mathcal{C}_2$ which is bounded by $\phi(L_i)$ and $\phi(L_e)$. Then by the same reasoning as above, there is a conformal map $\phi_2 : \phi(F) \to ann(0, (H_i)^{\frac{1}{n}}, (H_e)^{\frac{1}{n}})$ that satisfies the following.

- $(\phi_2)^n = f_2$.
- ϕ_2 extends continuously to all points in $\phi(L_i)$ and $\phi(L_e)$ with the exception of the critical points of f_2 in $\phi(L_i)$.
- If z is a critical point of f_2 in $\phi(L_i)$, and $\gamma : [0,1] \to G_2$ is a path with $\gamma([0,1)) \subset \phi(F)$ and $\gamma(1) = z$, then $\lim_{r\to 1} \phi_2(\gamma(r))$ exists.
- $\phi_2(\phi(w)) > 0$. (Note: We can do this because $\phi(w)$ is a distinguished point in $\phi(L_e)$.)

We extend ϕ to F by $\phi = {\phi_2}^{-1} \circ \phi_1$. This is clearly a conformal map from F to $\phi(F)$. Note that $f_2 \circ {\phi_2}^{-1}(z) = z^n$ for each point z in $ann(0, (H_i)^{\frac{1}{n}}, (H_e)^{\frac{1}{n}})$. Therefore, for each $z \in F$, $f_2(\phi(z)) = f_2({\phi_2}^{-1}(\phi_1(z))) = (f_2 \circ {\phi_2}^{-1})(\phi_1(z)) = \phi_1(z)^n = f_1(z)$.

We now have that ϕ is a map from G_1 to G_2 which is bijective (since ϕ is a homeomorphism on $\mathcal{C}_1$ and conformal on each component of $G_1 \setminus \mathcal{C}_1$), and that $f_1 = f_2 \circ \phi$ on all of G_1. We now wish to show that ϕ is analytic on all of G_1. Let $z \in G_1 \setminus \mathcal{B}_1$ be given. If $z \in G_1 \setminus \mathcal{C}_1$, then we already know that ϕ is analytic at z. Suppose that $z \in \mathcal{C}_1$. Since $z \notin \mathcal{B}_1$, z is not a critical point of f, so as noted above, ϕ is continuous in a neighborhood of z. Then as an application of the Schwartz Reflection Principle, we may conclude that ϕ is analytic at z. Thus ϕ is analytic at all points in G_1 except possibly the finitely many points in $\mathcal{B}_1$.

Let z be one of the points in $\mathcal{B}_1$. It suffices to show that ϕ is continuous at z. If $\lim_{w \to z} \phi(w)$ exists, then this limit equals $\phi(z)$ because $\phi|_{\Lambda_z}$ is a homeomorphism. Let us assume that this limit does not exist. Then there is a some $\iota > 0$ and some sequence of points $\{z_k\}_{k=1}^{\infty} \subset G_1$ such that $\lim_{k \to \infty} z_k = z$, and $|\phi(z_k) - \phi(z_{k+1})| > \iota$ for each k. Then since since there are only finitely many faces of Λ_z which are incident to z, by dropping to a sub-sequence, we can assume that all z_k are in the same face component of $G_1 \setminus \mathcal{C}_1$, or that all z_k are in Λ_z. If all the z_k are in Λ_z, then we have a contradiction because ϕ is a homeomorphism on Λ_z. If all z_k are in some single component D of $G_1 \setminus \mathcal{C}_1$, then it is easy to show that we may find some path $\gamma : [0, 1] \to G_1$ such that $\gamma([0, 1)) \subset D$, and $\gamma(1) = z$, and $\gamma(1 - \frac{1}{k}) = z_k$ for each k. But by definition of ϕ_1 and ϕ_2, we may conclude that $\lim_{r \to 1^-} \phi(\gamma(r))$ exists, so we have a contradiction and we are done.

Thus we conclude that ϕ is analytic on G_1, and as already noted ϕ is a bijection which intertwines f_1 and f_2, so we conclude that $(f_1, G_1) \sim (f_2, G_2)$. □

The fact observed at the beginning of this section that $(f_1, G_1) \sim (f_2, G_2)$ implies that $\Pi(f_1, G_1) = \Pi(f_2, G_2)$ gives us that we may view Π as acting on the equivalence classes of special type function elements. That is, defining $\Pi : H \to PC$ by $\Pi([(f, G)]) := \Pi(f, G)$ is well defined. The backwards implication of Theorem 4.1 (that $\Pi(f_1, G_1) = \Pi(f_2, G_2)$ implies that $(f_1, G_1) \sim (f_2, G_2)$) gives us that $\Pi : H \to PC$ is injective.

CHAPTER 5
Π IS SURJECTIVE: THE GENERIC CASE

It is fairly easy to show from the definition of Π that $\Pi(H_a) \subset PC_a$. To show that $\Pi(H_a) = PC_a$, we begin by considering subset of H_a which contains special type function elements (f, G) where f is a polynomial.

Definition 13. *For $G \subset \mathbb{C}$ an open simply connected set, and $f : G \to \mathbb{C}$ analytic on G, and $\epsilon > 0$, define $G_{f,\epsilon} := \{z \in G : |f(z)| < \epsilon\}$. Because of the definition of a special type function element, we define $G_f := G_{f,1}$.*

Definition 14. *Let H'_p be the set of all special type function elements $(p|_{G_p}, G_p)$ where $p \in \mathbb{C}[z]$. We write (p, G_p) for $(p|_{G_p}, G_p)$. We also define $H_p := H'_p/\sim$.*

Since $H_p \subset H_a$, if we can show that $\Pi(H_p) = PC_a$, then we are done. That is, we wish to show that for any $\langle\lambda\rangle_{PC} \in PC$, there is a polynomial $p \in \mathbb{C}$ such that $\Pi(p, G_p) = \langle\lambda\rangle_{PC}$. Our method, broadly speaking, will be to partition H_p by critical values. That is, a partition set will be the collection of members of H_p which have a given list of critical values. We then define a notion of critical values for members of PC_a, and partition PC_a by these critical values. We then show that for any finite list of critical values, $\{v_1, \ldots, v_{n-1}\}$, Π maps the partition set of H_p corresponding to this list of critical values bijectively to the partition set of PC_a corresponding to this list of critical values. Having shown this, we can conclude that Π maps H_p bijectively to PC_a.

We begin by building up some notation for dealing with the critical values we will be working with.

Definition 15. *For n a positive integer, define $V_n \subset \mathbb{C}^n$ by $V_n = \{v = (v^{(1)}, \ldots, v^{(n)}) \in \mathbb{C}^n : 0 \le |v^{(1)}| \le \cdots \le |v^{(n)}| < 1\}$. Then define $V := \bigcup_{n=1}^{\infty} V_n$.*

Definition 16. *For $x, y \in \mathbb{C}^n$, we use the metric $|x - y| := \max(|x^{(i)} - y^{(i)}| : 1 \le i \le n)$.*

We now define the partition of H_p which we will use.

Definition 17. *For $n \geq 2$ an integer, and $v \in V_{n-1}$, let $H'_{p,v}$ denote the subset of members $(f, G) \in H'_p$ such that the critical values of f are exactly $v^{(1)}, v^{(2)}, \dots, v^{(n-1)}$. Further, define $H_{p,v} := H'_{p,v}/\sim$.*

We now will work out a notion of critical values for a member of PC_a. In essence, the critical values of a member $\langle\xi\rangle_{PC}$ are the critical values of any member of $\Pi^{-1}(\langle\xi\rangle_{PC})$. Since we do not yet know that this set is non-empty, we will have to define the critical values of $\langle\xi\rangle_{PC}$ directly from $\langle\xi\rangle_{PC}$. We begin with some definitions having to do with graphs.

Definition 18. *For $\lambda \in \check{P}$, and w a vertex in λ, let $m(w)$ denote the number of edges of λ incident to w where we count the edge twice if both of its endpoints are at w.*

Note: As mentioned earlier, if λ is a member of $\check{P}$, and w is a vertex of λ, then $m(w)$ is even, and greater than or equal to 4.

Definition 19. *For λ a member of $\check{P}$, and w a vertex of λ, then we let $m(w)$ denote the number of edges of λ incident to z. Furthermore, we say that w is a vertex of λ with multiplicity $\frac{m(w)}{2} - 1$. Note that if $w \in \lambda$ is not a vertex of λ, then this definition still makes sense: if we count the edge of λ which contains w as meeting w two times (ie from either side), then we would say the multiplicity of w as a vertex of λ is $\frac{2}{2} - 1 = 0$.*

Note that if (f, G) is a special type function element, and $w \in G$ is a zero of f' with multiplicity k, then f is $(k+1)$-to-1 in a neighborhood of w. Therefore there are $2(k+1)$ edges of Λ_w which are incident to w. Thus the multiplicity of w as a vertex of Λ_w is $\frac{2(k+1)}{2} - 1 = k$.

Definition 20. *Let $\langle\lambda\rangle_{PC} \in PC_a$ be given. If $\langle w\rangle_{PC}$ is one of the level 0 members of PC used to form $\langle\lambda\rangle_{PC}$, then we say that 0 is a critical value of $\langle\lambda\rangle_{PC}$ with multiplicity $Z(\langle w\rangle_P) - 1$. Suppose that $\langle\lambda'\rangle_P$ is a member of P used to build $\langle\lambda\rangle_{PC}$. Then if w is a vertex of λ', we say that $H(\langle\lambda'\rangle_P)e^{ia(w)}$ is a critical value of $\langle\lambda\rangle_{PC}$ of multiplicity equal to the multiplicity of w as a vertex of λ'.*

With this notion of critical values of a member of PC_a built up, we may now partition PC_a as follows.

Definition 21. *For $v = (v^{(1)}, \dots, v^{(n-1)}) \in V_{n-1}$, define $PC_{a,v}$ to be the collection of members of PC_a whose critical values listed according to multiplicity are $v^{(1)}, \dots, v^{(n-1)}$, and let $|PC_{a,v}|$ denote the number of elements of $PC_{a,v}$. (In the context of polynomials, n is the number of zeros of the poynomial, and thus there would be $n-1$ critical values of the polynomial.)*

From the definition of critical values of a member of PC, it should be clear that $\Pi(H_{p,v}) \subset PC_{a,v}$. Then to show that Π is surjective, we show that $\Pi(H_{p.v}) = PC_{a,v}$ for each $v \in V$. In this chapter we show that this equality holds for any v in a dense subset U of V about to be defined, and then extend this to all of V in Chapter 6.

Definition 22. *For n a positive integer, define $U_n \subset V_n$ to be the collection of $u = (u^{(1)}, \dots, u^{(n)}) \in V_n$ such that $0 < |u^{(1)}| < \cdots < |u^{(n)}| < 1$. Then define $U := \bigcup_{n=1}^{\infty} U_n$.*

Fix some positive integer $n \geq 2$ and $v_0 = ({v_0}^{(1)}, \dots, {v_0}^{(n-1)})$ in U_{n-1}. Since Π is injective, it suffices to show that $|H_{p,v_0}| = |PC_{a,v_0}|$.

In a paper by Beardon, Carne, and Ng, [14] it was shown that if $n = 2$, and $v \in V_{n-1}$, then $|H_{p,v}| = 1$. And if $n \in \mathbb{N}$ with $n \geq 3$, and $v \in V_{n-1}$, $H_{a,v}$ has exactly n^{n-3} elements according to multiplicity, where multiplicity arises through a use of Bezout's theorem. As an easy corollary to what was shown in this paper, one may prove that if $n = 2$, and $v \in U_{n-1}$, then $|H_{p,v}| = 1$, and if $n \geq 3$, then $H_{p,v}$ contains exactly n^{n-3} distinct members.

Note: Since $0 < |{v_0}^{(1)}| < \cdots < |{v_0}^{(n-1)}|$, if $\langle \lambda \rangle_{PC} \in PC_{a,v_0}$, and $\langle \lambda' \rangle_P$ is a member of P used in the construction of $\langle \lambda \rangle_{PC}$, then λ' contains only a single vertex. (Since if there were two vertices in λ', each would give rise to a critical value, and they would have the same modulus.) If w is the vertex of λ', then $H(\langle \lambda' \rangle_P)e^{ia(w)}$ is a critical value of $\langle \lambda \rangle_{PC}$ with multiplicity 1, so by definition of multiplicity, the number of edges of λ' which meet at w is $2 * (1+1) = 4$. There is only one member of $\check{P}$ which has a single vertex at which exactly 4 edges meet, namely the "figure eight" graph. (Recall that an edge is counted twice if both ends meet at the vertex.). Thus each graph used while constructing $\langle\lambda\rangle_{PC}$ is this figure eight graph.

We will use an induction argument to count the number of members of PC_{a,v_0}. In order to do this, it will be helpful to have an ordering on the members of P used to construct a given member of PC_a, which we define now.

Definition 23. *Fix some $\langle\lambda\rangle_{PC} \in PC_a$, and let $\langle\xi_1\rangle_{PC}, \dots, \langle\xi_n\rangle_{PC}$ with $n \geq 2$ be the members of PC which are used in constructing $\langle\lambda\rangle_{PC}$. Then we say $\xi_i \prec \lambda$ with respect to $\langle\lambda\rangle_{PC}$ for each $i \in \{1, 2, \dots, n\}$. Furthermore, if some $\langle\xi_i\rangle_{PC}$ has been associated to some bounded face of some ξ_j while constructing $\langle\lambda\rangle_{PC}$ for some i, j, then we say $\xi_i \prec \xi_j$ with respect to $\langle\lambda\rangle_{PC}$ (this "with respect to $\langle\lambda\rangle_{PC}$" will usually be suppressed when the member $\langle\lambda\rangle_{PC}$ in question is clear). We extend this to be a transitive relation. That is, if $\xi_{i_1} \prec \xi_{i_2} \prec \cdots \prec \xi_{i_k}$ for some $2 \leq k \leq n$, then we say $\xi_{i_1} \prec \xi_{i_k}$.*

If $\langle\lambda\rangle_{PC}$ is a member of PC_{a,v_0}, since all critical values of $\langle\lambda\rangle_{PC}$ are non-zero, and thus come from a vertices of a graph used in constructing $\langle\lambda\rangle_{PC}$, and each planar graph used in the construction of $\langle\lambda\rangle_{PC}$ contains a single vertex, and each of these vertices gives rise to a critical value of $\langle\lambda\rangle_{PC}$, there must be $n-1$ distinct planar graphs used to construct $\langle\lambda\rangle_{PC}$. Therefore we make the following definition in order to be able to refer to the vertex which gives rise to a given critical value of $\langle\lambda\rangle_{PC}$.

Definition 24. *For $\langle\lambda\rangle_{PC} \in PC_{a,v_0}$, and $i \in \{1, \dots, N-1\}$, let z_i denote the point or vertex from which the critical value ${v_0}^{(i)}$ arose. Furthermore, let λ_i denote the planar graph which contains the vertex z_i. (Note that since $v_0 \in U_{n-1}$, this is well defined.)*

We now wish to show that PC_{a,v_0} has exactly 1 member if $n = 2$, and n^{n-3} distinct members if $n \geq 3$. We will have to handle the different possible values of n separately up to $n = 6$. For $n \geq 6$ we will be able to make a general argument.

Case 5.0.1. *$n = 2$.*

Let $\langle\lambda\rangle_{PC}$ be a member of PC_{a,v_0}. Since $\langle\lambda\rangle_{PC}$ has a single critical value, $\langle\lambda\rangle_{PC}$ is constructed from a single figure eight graph, namely λ. Let D be either bounded face of λ, and let $\langle w\rangle_{PC}$ denote the level 0 member of PC associated with D. Since 0 is not a critical value of $\langle\lambda\rangle_{PC}$, $Z(\langle w\rangle_{PC}) = 1$, so there is only one distinguished point in ∂D, and thus there is only one possible choice of gradient map g_D, namely the one that takes the single distinguished point in ∂D to w. Further if z is the single vertex of λ, then $a(z) = \arg({v_0}^{(1)})$ and $H(\langle\lambda\rangle_P) = |{v_0}^{(1)}|$ since ${v_0}^{(1)}$ is the only critical value of $\langle\lambda\rangle_{PC}$. So all the data pertaining to $\langle\lambda\rangle_{PC}$ is determined entirely by v_0. Thus PC_{a,v_0} contains only a single element.

For the future cases we will need the following definition.

Definition 25. *Let $\langle\lambda\rangle_{PC}$ be a member of PC, and let D denote one of the bounded faces of λ. For some $\langle\xi\rangle_P$ used in constucting $\langle\lambda\rangle_{PC}$, if $\langle\xi\rangle_{PC}$ were associated to D, then we say $\xi \prec D$. We extend this as follows. If $\langle\xi_1\rangle_P, \dots, \langle\xi_k\rangle_P$ were used in the construction of $\langle\lambda\rangle_{PC}$, and $\xi_1 \prec \cdots \prec \xi_k \prec D$, then we say $\xi_1 \prec D$.*

Note: Let $\langle\lambda\rangle_{PC}$ be any member of PC, let $\langle\lambda'\rangle_P$ be any member of P used in the construction of $\langle\lambda\rangle_{PC}$, and let D be any face of λ'. An easy induction argument gives that the number of single point elements $\langle w\rangle_P$ of P such that $w \prec D$ is exactly $z(D)$ (where these single point members of P are counted according to multiplicity), and that the number of critical values of $\langle\lambda\rangle_{PC}$ which come from members of PC associated to D is exactly $z(D) - 1$.

Definition 26. *Let $\langle\lambda\rangle_{PC}$ be any member of PC_{a,v_0}. Since v_0 was taken from U_{n-1}, λ only contains one vertex and has only two bounded faces. Let D_1 denote the bounded face of λ from which fewer critical values come. Let D_2 denote the other one. That is, the naming is done so that $z(D_1) \leq z(D_2)$. (If both bounded faces of λ give rise to the same number of critical values, then this naming is arbitrary.)*

Note: For any member $\langle\lambda\rangle_{PC}$ of PC_{a,v_0}, $z(D_1) + z(D_2) = n$. There are $n - 1$ total critical values of $\langle\lambda\rangle_{PC}$, and one of those critical values comes from the vertex of λ, so $n - 2$ of them come from the two regions D_1 and D_2. This together with the fact that

$z(D_1) \leq z(D_2)$ immediately gives that the possible values for $z(D_1) - 1$ to take are exactly $\{k \in \mathbb{Z} : 0 \leq k \leq \frac{n-2}{2}\}$.

Case 5.0.2. *$n = 3$.*

Since $\lambda_1 \prec \lambda_2$, and $\langle \lambda_1 \rangle_P$ and $\langle \lambda_2 \rangle_P$ are the only planar graph members of P used to construct $\langle \lambda_2 \rangle_{PC}$, we have that $\langle \lambda_1 \rangle_{PC}$ has been associated to D_2. Let $\langle w \rangle_{PC}$ denote the level zero member of PC associated to D_1. Then as in Case 5.0.1, there is only one possible choice of g_{D_1}, namely the one that maps the single distinguished point of ∂D_1 to w. We claim that there is only a single choice of g_{D_2} as well. Since λ_1 has exactly two bounded faces, and each of these faces has only a single level zero member of PC associated to it, we have that $Z(\langle \lambda_1 \rangle_P) = 2$, and thus $z(D_2) = 2$. And thus there are exactly two distinguished points in ∂D_2. The two distinguished points in λ_1 are either both at the vertex, or neither at the vertex, and in either case there is a orientation preserving homeomorphism of λ_1 which exchanges the distinguished points. Thus modulo orientation preserving homeomorphism of λ_1 (and recall that PC is formed modulo these homeomorphisms) there is only one possible choice of g_{D_2}. And it is easy to see as in Case 5.0.1 that all the rest of the data (ie the values $H(\cdot)$ and $a(\cdot)$ take) for $\langle \lambda_2 \rangle_{PC}$ is fully determined by v_0 (up to orientation preserving homeomorphism), and thus PC_{a,v_0} has exactly one member.

Note: Our general way of counting the number of elements in PC_{a,v_0} when $n \geq 3$ will be to partition PC_{v_0} by the value $z(D_1) - 1$ takes. For a given value of $z(D_1) - 1$, we find out how many ways the $z(D_1) - 1$ different critical values may be chosen from the $n - 2$ critical values available to come from D_1 and D_2 (which is of course $\binom{n-2}{z(D_1)-1}$). For that choice of critical values coming from D_1, we count the number of members of PC which may be associated to D_1 and the number which may be associated to D_2 (a natural induction step). We then count the number of choices of g_{D_1} and of g_{D_2} ($z(D_1)$ and $z(D_2)$ respectively, except in the case where $z(D_1) - 1 = 1$ or $z(D_2) - 1 = 1$ as we will see).

We then multiply these numbers to find the number of members of PC_{v_0} with the given value of $z(D_1) - 1$.

Case 5.0.3. *$n = 4$.*

As noted above, the only possible values of $z(D_1) - 1$ in this case are 0 and 1.

If $z(D_1) - 1 = 0$, then there is a single level 0 member of PC which could be associated to D_1. Let this member be called $\langle w \rangle_{PC}$. Then $Z(\langle w \rangle_P) = 1$, so $z(D) = 1$. Thus there is a single choice of g_{D_1}, namely the map that takes the single distinguished point in ∂D_1 to w. On the other hand, since $z(D_1) - 1 = 0$, $z(D_2) - 1 = 2$, and from Case 5.0.2 we know that then there is only a single member of PC whose critical values are ${v_0}^{(1)}, {v_0}^{(2)}$. However since $z(D_2) - 1 = 2$, ∂D_2 has 3 distinguished points, so there are three different choices of g_{D_2}. Thus there are $\binom{2}{0} * 1 * 1 * 1 * 3 = 3$ members of PC_{v_0} for which $z(D_1) - 1 = 0$.

Suppose $z(D_1) - 1 = 1$, and thus $z(D_2) - 1 = 1$. Hence there are $\binom{2}{1} = 2$ possible choices of the critical value which comes from D_1. By the work done for Case 5.0.1, there is a single possible member of PC which may be associated to D_1 and a single member of PC which may be associated to D_2. Also by the work done in Case 5.0.1, there is a single possible choice of g_{D_1} and a single possible choice of g_{D_2}. And hence we count $\binom{2}{1} 1 * 1 * 1 * 1 = 2$ members of PC_{a,v_0} for which $z(D_1) - 1 = 1$. But in this case, one critical value comes from D_1 and one comes from D_2, so we may switch the roles of D_1 and D_2 without breaking the restriction $z(D_1) \leq z(D_2)$. That is, we are overcounting by a factor of 2. So finally we get that there is $2 * \frac{1}{2} = 1$ member of PC_{v_0} for which $z(D_1) - 1 = 1$.

So $|PC_{a,v_0}| = 3 + 1 = 4 = 4^{4-3}$.

Case 5.0.4. *$n = 5$.*

Here again the only possible values of $z(D_1) - 1$ are 0 and 1. For the sake of brevity we will leave out much of the explanation that can easily be translated from the previous case.

If $z(D_1) - 1 = 0$, there is only one member of PC which may be associated to D_1, and one choice of g_{D_1}. From Case 5.0.3, above (prefiguring an inductive argument here), there are 4 members of PC that may be associated to D_2, and for each choice of the member of PC associated to D_2, there are 4 possible choices of g_{D_2}. Hence there are $\binom{3}{0} * 1 * 1 * 4 * 4 = 16$ members for which $z(D_1) - 1 = 0$.

If $z(D_1) - 1 = 1$, there are $\binom{3}{1}$ different choices of the critical value that comes from D_1. For that choice, there is a single member of PC_a which may be associated to D_1, and a single choice of g_{D_1}. For that choice of the critical value in D_1 (and thus the two critical values which come from D_2), there is a single member of PC_a which may be associated to D_2, and 3 choices of g_{D_2}. Hence PC_{a,v_0} has $\binom{3}{1} * 1 * 1 * 1 * 3 = 9$ members for which $z(D_1) - 1 = 1$.

So $|PC_{a,v_0}| = 16 + 9 = 25 = 5^{5-3}$.

Case 5.0.5. *$n \geq 6$.*

In the following calculations, the first number will be the number of ways of choosing the critical values which come from D_1. The second number will be the number of members of PC which may be associated to D_1 (induction step) for the given choice of critical values coming from D_1. The third number will be the number of possible choices of g_{D_1}. The fourth number will be the number of members of PC which may be associated to D_2 (induction step) The fifth number will be the number of possible choices of g_{D_2}.

Assume first that n is odd. Then $z(D_1) - 1$ can take any value in the set

$$\{0, 1, \dots, \frac{(n-2)-1}{2} = \frac{n-3}{2}\}.$$

The number of members of PC_{a,v_0} for which $z(D_1) - 1 = 0$ is

$$\binom{n-2}{0} * 1 * 1 * (n-2+1)^{(n-2+1)-3} * (n-2+1) = (n-1)^{n-3}.$$

The number of members of PC_{a,v_0} for which $z(D_1) - 1 = 1$ is

$$\binom{n-2}{1} * 1 * 1 * (n-2-1+1)^{(n-2-1+1)-3} * (n-2-1+1),$$

which is equal to

$$\binom{n-2}{1}(n-2)^{n-4}.$$

If $2 \leq i \leq \frac{n-3}{2}$, then the number of members of PC_{a,v_0} for which $z(D_1) - 1 = i$ is

$$\binom{n-2}{i} * (i+1)^{i+1-3} * (i+1) * (n-2-i+1)^{(n-2-i+1)-3} * (n-2-i+1).$$

Simplifying this, we conclude that the number of members of PC_{a,v_0} for which $z(D_1) - 1 = i$ is

$$\binom{n-2}{i}(i+1)^{i-1}(n-i-1)^{n-i-3}.$$

Hence we get that

$$|PC_{a,v_0}| = \binom{n-2}{0}(n-1)^{n-3} + \binom{n-2}{1}(n-2)^{n-4} +$$

$$\sum_{i=2}^{\frac{n-3}{2}} \binom{n-2}{i}(i+1)^{i-1}(n-i-1)^{n-i-3}.$$

However,

$$\binom{n-2}{0}(n-1)^{n-3} = \binom{n-2}{0}(0+1)^{0-1}(n-0-1)^{n-0-3},$$

and

$$\binom{n-2}{1}(n-2)^{n-4} = \binom{n-2}{1}(1+1)^{1-1}(n-1-1)^{n-1-3},$$

so we may include these terms in the sum. That is,

$$|PC_{a,v_0}| = \sum_{i=0}^{\frac{n-3}{2}} \binom{n-2}{i} (i+1)^{i-1}(n-i-1)^{n-i-3}.$$

By performing the substitution $m = n-2$, we obtain

$$|PC_{a,v_0}| = \sum_{i=0}^{\frac{m-1}{2}} \binom{m}{i} (i+1)^{i-1}(m-i+1)^{m-i-1},$$

where the length of v_0 is now $m+1$. A brief examination of this sum should then convince the reader that due to the symmetric nature of the above sum, we have

$$|PC_{a,v_0}| = \sum_{i=0}^{m-1} \frac{1}{2}\binom{m}{i} (i+1)^{i-1}(m-i+1)^{m-i-1}.$$

From there we may invoke a result in [15] which then gives that $|PC_{a,v_0}| = (m+2)^{m-1} = n^{n-3}$. Thus we have the desired result when n is odd.

Assume now that n is even. Our calculations here are identical to the case where n is odd except in calculating the last term of the sum. Since n is even, $z(D_1)-1$ can take any value in $\{0, 1 \ldots, \frac{n-2}{2}\}$. In counting the number of members of PC_{a,v_0} with a given value of $z(D_1)-1$ we get the same results as when n is odd if $0 \leq i = z(D_1)-1 < \frac{m-2}{2}$, namely that the number of members of PC_{a,v_0} for which $z(D_1)-1 = i$ is $\binom{n-2}{i}(i+1)^{i-1}(n-i-1)^{n-i-3}$. However if $z(D_1)-1 = \frac{n-2}{2}$, as described in Case 5.0.3 the roles of D_1 and D_2 may be reversed, so we are over counting by a factor of 2. Thus we need to include a factor of $\frac{1}{2}$ when we count the number of ways in which $\frac{n-2}{2}$ of the $n-2$ critical values may be chosen as the ones arising from vertices in D_1. Hence when n is even, the number of members of PC_{a,v_0} for which $z(D_1)-1 = \frac{n-2}{2}$ is exactly

$$\frac{1}{2}\binom{n-2}{\frac{n-2}{2}} \left(\frac{n-2}{2}+1\right)^{\frac{n-2}{2}-1} \left(n - \frac{n-2}{2} - 1\right)^{n-\frac{n-2}{2}-3},$$

and therefore $|PC_{a,v_0}|$ equals

$$\left(\sum_{i=0}^{\frac{n-2}{2}-1}\binom{n-2}{i}(i+1)^{i-1}(n-i-1)^{n-i-3}\right)+$$

$$\left(\frac{1}{2}\binom{n-2}{\frac{n-2}{2}}\left(\frac{n-2}{2}+1\right)^{\frac{n-2}{2}-1}\left(n-\frac{n-2}{2}-1\right)^{n-\frac{n-2}{2}-3}\right)$$

Again using the substitution $m=n-2$, and taking advantage of the symmetry in the sum, we find that

$$|PC_{a,v_0}|=\sum_{i=0}^{m}\frac{1}{2}\binom{m}{i}(i+1)^{i-1}(m-1+1)^{m-i-1}.$$

Again by [15] we conclude that $|PC_{a,v_0}|=(m+2)^{m-1}=n^{n-3}$.

Thus under the assumption that $v_0\in U_{n-1}$, we conclude that PC_{a,v_0} has precisely n^{n-3} members.

We now have that $|PC_{a,v_0}|=n^{n-3}=|H_{p,v_0}|\leq|H_{a,v_0}|$, and $\Pi : H_{a,v_0}\to PC_{a,v_0}$ is injective, so we conclude that $\Pi : H_{a,v_0}\to PC_{a,v_0}$ is also surjective.

So we have the desired result for each $v\in U_{n-1}$.

An example is in order here. Unfortunately it is very difficult either to determine the critical level curve configuration of a function element, or to find a polynomial with a given critical leve curve configuration. Therefore our example is quite simple.

Example: Consider the function

$$f(z)=\frac{1}{.6}\left(z^2+\frac{9}{25}\right)e^z.$$

Then the shaded region G in Figure 5-1 is one of the components of the set $\{w:|f(w)|<1\}$, and the boundary of G is one of the level curves of f on which $|f|\equiv 1$. The critical point of f in G is at $z=-.2$.

Consider also the polynomial

$$p(z)=\frac{1}{.6}\left(z^2+f(-.2)\right).$$

The shaded region D in Figure 5-2 is the set $\{w : |p(w)| < 1\}$, and the boundary of D is one of the level curve of p on which $|p| \equiv 1$. The critical point of p is at $z = 0$. It is easy to see that the critical value which arises from the critical point of f in G is equal to the critical value of p. Since there is only one member of PC which has a given critical value, it follows that $\Pi(f, G) = \Pi(p, D)$. Therefore by Theorem 4.1, there is some conformal function $\phi : G \to D$ such that $f \equiv p \circ \phi$ on G.

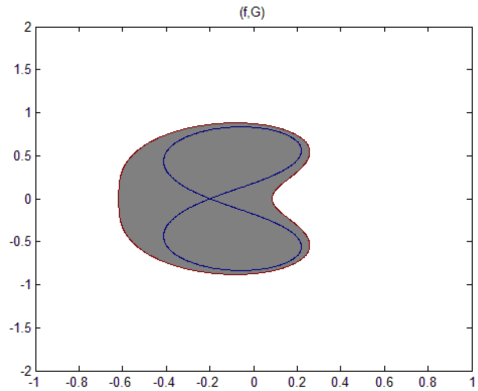

Figure 5-1. Tract of f

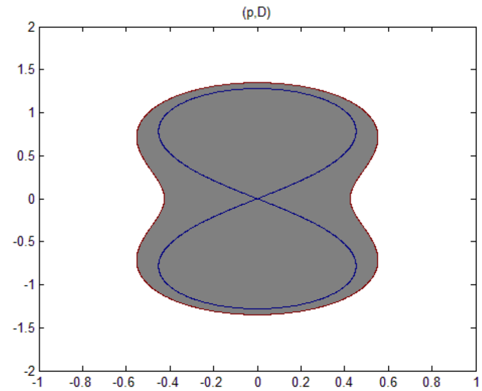

Figure 5-2. Tract of p

CHAPTER 6
Π IS SURJECTIVE: THE GENERAL CASE

We now wish to show that $\Pi : H_{a,v_0} \to PC_{a,v_0}$ is surjective for choices of v_0 in $V \setminus U$. This will also be an induction argument, and we need several definitions.

Definition 27. *For $v \in V_{n-1}$ say v is typical if $v \in U_{n-1}$, in which case we say v has atypicallity degree 0 (so $0 < |v^{(1)}| < \cdots < |v^{(n-1)}|$). Say v has atypicallity degree 1 if $0 = |v^{(1)}| < |v^{(2)}| < |v^{(3)}| < \cdots < |v^{(n-1)}|$. Say v has atypicallity degree k for $2 \leq k \leq n-1$ if $0 \leq |v^{(1)}| \leq \cdots \leq |v^{(k-1)}| = |v^{(k)}| < |v^{(k+1)}| < \cdots < |v^{(n-1)}|$.*

Thus the result of Chapter 5 may be restated as for any $n > 0$, and any $v = (v^{(1)}, \dots, v^{(n-1)}) \in V_{n-1}$ with atypicallity degree 0, and any $\langle\lambda\rangle_{PC} \in PC_{a,v}$, there is some $u \in \Theta^{-1}(v)$ such that $\Pi(p_u, G_{p_u}) = \langle\lambda\rangle_{PC}$.

Definition 28. *For $u = (u^{(1)}, \dots, u^{(n-1)}) \in \mathbb{C}^{n-1}$, define a polynomial p_u by $p_u(w) := \int_0^w \prod_{i=1}^{n-1}(z - u^{(i)})dz$. Then define $\Theta : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ by $\Theta(u) = (p_u(u^{(1)}), \dots, p_u(u^{(n-1)}))$.*

Of course for any $u \in \mathbb{C}^{n-1}$, the critical points of p_u are exactly $u^{(1)}, \dots, u^{(n-1)}$, so Θ may be thought of as taking a prescribed list of critical points to a list of critical values via a normalized polynomial (p_u is the polynomial with critical points $u^{(1)}, \dots, u^{(n-1)}$ normalized so that $p_u(0) = 0$ and p_u' is monic). This function Θ was studied in [14] in which it was shown that for any $n > 0$, and any $v = (v^{(1)}, \dots, v^{(n-1)}) \in V$, and any $(p, G_p) \in H_p$ whose critical values are $v^{(1)}, \dots, v^{(n-1)}$, there is some $u \in \Theta^{-1}(v)$ such that $(p, G_p) \sim (p_u, G_{p_u})$.

Definition 29. *For any $i \geq 0$, let $\mathcal{J}(i)$ denote the statement "For any $\langle\lambda\rangle_{PC} \in PC_a$ whose vector of critical values $v \in V_{n-1}$ has atypicallity degree less than or equal to i, there is a $u \in \Theta^{-1}(v)$ such that $(p_u, G_{p_u}) \in H_p$ and $\Pi(p_u, G_{p_u}) = \langle\lambda\rangle_{PC}$.*

We wish to show $\mathcal{J}(i)$ holds for each $i \geq 0$. We have already shown in Chapter 5 that $\mathcal{J}(0)$ holds. Fix some $M \geq 1$ and assume inductively that $\mathcal{J}(i)$ holds for all $i \in \{0, \dots, M-1\}$. We now wish to show that $\mathcal{J}(M)$ holds. Fix some $N - 1 \geq M$ and some $v_1 = ({v_1}^{(1)}, \dots, {v_1}^{(N-1)}) \in V_{N-1}$ with atypicallity degree M. Assume that

$v_1 \neq (0, \dots, 0)$, since if this were the case then it is easy to show the desired result holds. Now fix some member $\langle \Lambda \rangle_{PC}$ of PC_{a,v_1}. Our plan is to choose another member $\langle \widehat{\Lambda} \rangle_{PC}$ of PC_a which is in some sense to be determined very close to $\langle \Lambda \rangle_{PC}$, but whose list of critical values $\widehat{v_1}$ has atypicallity degree strictly less than M. By the induction assumption, there is some $\widehat{u_1} \in \Theta^{-1}(\widehat{v_1})$ such that $\Pi(p_{\widehat{u_1}}, G_{p_{\widehat{u_1}}}) = \langle \widehat{\Lambda} \rangle_{PC}$. If we choose $\langle \widehat{\Lambda} \rangle_{PC}$ so that $\widehat{v_1}$ is sufficiently close to v_1, this will ensure that the members of $\Theta^{-1}(\widehat{v_1})$ are close to the members of $\Theta^{-1}(v_1)$. If we let u_1 denote a member of $\Theta^{-1}(v_1)$ which $\widehat{u_1}$ is close to, then we wil show that $\Pi(p_{u_1}, G_{p_{u_1}}) = \langle \Lambda \rangle_{PC}$. First a couple of definitions.

Definition 30. *For non-zero $x^{(1)}, x^{(2)} \in \mathbb{C}$, define*

$$d_{\arg}(x^{(1)}, x^{(2)}) := \begin{cases} |\arg(x^{(1)}) - \arg(x^{(2)})|, & \text{if } x^{(1)} \neq 0 \neq x^{(2)} \text{ and } \arg(x^{(1)}) \neq \arg(x^{(2)}) \\ 2\pi, & \text{if } x^{(1)} = 0 \text{ or } x^{(2)} = 0 \text{ or } \arg(x^{(1)}) = \arg(x^{(2)}) \end{cases},$$

where the choice of $\arg(x^{(1)})$ and $\arg(x^{(2)})$ in the definition above is made so as to minimize $d_{\arg}(x^{(1)}, x^{(2)})$. For $x = (x^{(1)}, \dots, x^{(m)}) \in \mathbb{C}^m$, define

$$d_{\arg}(x) := \min(d_{\arg}(x^{(i)}, x^{(j)}) : 1 \leq i, j \leq m).$$

Definition 31. *For $a, b \in \mathbb{R}$ with $a < b$, and for $I : a = i_0 < i_1 < \cdots < i_n = b$ a partition on $[a, b]$, define $|I| := \min(i_k - i_{k-1} : 1 \leq k \leq n)$. Let $\gamma : [a, b] \to \mathbb{C}$ be a path and let f be a function analytic and non-zero on the image of γ. Define $\Delta_{\arg}(f, \gamma, I) := \sum_{k=1}^{n} \arg(f(\gamma(i_k))) - \arg(f(\gamma(i_{k-1})))$, where the choice of the arguments in each summand is made so as to minimize the magnitude of the summand. Since f is non-zero on γ, it is not hard to show that the limit as $|I| \to 0$ of $\Delta_{\arg}(f, \gamma, I)$ exists (and is finite). Let $\Delta_{\arg}(f, \gamma)$ denote this limit. We call $\Delta_{\arg}(f, \gamma)$ the winding number of f along γ. Define $|\Delta_{\arg}|(f, \gamma, I) := \sum_{k=1}^{n} |\arg(f(\gamma_k)) - \arg(f(\gamma(i_{k-1})))|$, where again the choice of arguments is made so as to minimize the summands. Again it is not hard to show that* *the limit as $|I| \to 0$ of $|\Delta_{\arg}|(f, \gamma, I)$ exists (although possibly infinite). We let $|\Delta_{\arg}|(f, \gamma)$ denote this limit, and we call $|\Delta_{\arg}|(f, \gamma)$ the total variation of $\arg(f)$ along γ.*

Definition 32. *For $x = (x^{(1)}, \dots, x^{(m)}) \in \mathbb{C}^m$, define*

$$\text{minmod}(x) := \begin{cases} 0, & \text{if } x^{(1)} = \cdots = x^{(m)} = 0 \\ \min(|x^{(i)}| : 1 \le i \le m, x^{(i)} \ne 0), & \text{otherwise} \end{cases}.$$

Definition 33. *For any $m \ge 2$ and $r = (r^{(1)}, \dots, r^{(m)}) \in \mathbb{C}^m$, define*

$$\text{mindiff}(r^{(1)}, \dots, r^{(m)}) := \begin{cases} 0, & \text{if } r^{(1)} = \cdots = r^{(m)} \\ \min(|r^{(i)} - r^{(j)}| : r^{(i)} \ne r^{(j)}, 1 \le i < j \le m), & \text{otherwise} \end{cases}.$$

We may write this as mindiff(r) as well.

We will now determine how close $\widehat{v_1}$ must be to v_1.

We begin by choosing a $\delta_1 > 0$ small enough that the following hold.

1. Let $u = (u^{(1)}, \dots, u^{(N-1)})$ be some point in $\Theta^{-1}(v_1)$. Fix some $i \in \{1, \dots, N-1\}$ such that ${v_1}^{(i)} \ne 0$, and let $\widehat{u} \in B_{\delta_1}(u)$ be given. Let L be a line segment contained in $B_{4\delta_1}(u^{(i)})$. Then $|\Delta_{\arg}|(p_{\widehat{u}}, L) < \frac{d_{\arg}(1, v_1)}{4}$. This follows from the compactness of cl$(B_{\delta_1}(u))$ and cl$(B_{4\delta_1}(u^{(i)}))$.

2. For any $v = (v^{(1)}, \dots, v^{(N-1)}) \in V_{N-1}$ such that $|v - v_1| < 1$ and $d_{\arg}(v) \ge \frac{d_{\arg}(1, {v_1}^{(1)}, \dots, {v_1}^{(N-1)})}{2}$, if $u \in \Theta^{-1}(v)$, and λ is a critical level curve of (p_u, G_{p_u}) with $|p_u| \equiv r$ on λ for some $r \ge \text{minmod}(v_1)$, then in each edge E of λ there is some point z which is greater than δ_1 away from each critical point of p_u, and greater than δ_1 away from each edge of a critical level curve of p_u other than E. We may do this by Lemma 4.

3. For $u \in \Theta^{-1}(v_1)$, let D denote either all of G_{p_u}, or a bounded face of one of the critical level curves λ of p_u such that D contains a critical point of p_u whose corresponding critical value is non-zero. Let λ_D be the critical level curve of p_u in D which is maximal with respect to D. Let m denote the number of distinct edges in λ_D. Let $E^{(1)}, \dots, E^{(m)}$ be some enumeration of the edges of λ_D. For each $i \in \{1, \dots, m\}$, choose some point $z^{(i)}$ in $E^{(i)}$ such that $\arg(p_u(z^{(i)}))$ is greater that $\frac{d_{\arg(1, v_1)}}{4}$ away from each of $\{\arg({v_1}^{(1)}), \dots, \arg({v_1}^{(N-1)})\}$. Let $y^{(i)}$ be the point in ∂D which is connected to $z^{(i)}$ by a section of a gradient line of p_u, and let $\sigma^{(i)}$ denote

this section of gradient line which connects $z^{(i)}$ and $y^{(i)}$. Since $\sigma^{(i)}$ is a portion of a gradient line of p_u, $\arg(p_u(y^{(i)})) = \arg(p_u(z^{(i)}))$, so $y^{(i)}$ is not a critical point of p_u. Since there are only finitely many such u, λ, and D, we may construct such a collection of paths for each such choice of u, λ, D, and choose δ_1 so that for each such u, λ, and D, if $i \in \{1, \ldots, m\}$ (m depends on the choice of u, λ, and D) and $t \in [0, 1]$, there is no $j \in \{1, \ldots, m\} \setminus \{i\}$ and $s \in [0, 1]$ such that $\sigma^{(j)}(s)$ is within $2\delta_1$ of $\sigma^{(i)}(t)$, and no critical point of p_u is within $2\delta_1$ of $\sigma^{(i)}(t)$, and there is no edge of any critical level curve of p_u other than $E^{(i)}$ within $2\delta_1$ of $\sigma^{(i)}(t)$.

4. For each $u \in \Theta^{-1}(v_1)$, no critical level curve of p_u is within $2\delta_1$ of ∂G_{p_u}, and no critical level curve of p_u is within $2\delta_1$ of any zero of p_u, and no critical level curve of p_u is within $3\delta_1$ of any other critical level curve of p_u.

5. For each $u \in \Theta^{-1}(v_1)$, and each $k \in \{1, \ldots, N-1\}$, there is no point $B_{2\delta^{(1)}}(u^{(k)}) \setminus \{u^{(k)}\}$ at which p_u takes the value ${v_1}^{(k)}$.

6. For each $u \in \Theta^{-1}(v_1)$, if $|\widehat{u} - u| < \delta_1$, then for each $k \in \{1, \ldots, N-1\}$ such that ${v_1}^{(k)} \neq 0$, if $|z - u^{(k)}| < 2\delta_1$, then $|p_{\widehat{u}}(z)| > \frac{\text{minmod}(v_1)}{2}$.

7. $\delta_1 < \frac{\text{minmod}(v_1) d_{\arg}(1, v_1)}{4\pi^2}$.

8. $\delta_1 < \frac{\text{mindiff}(v_1)}{4}$.

9. Let $u \in \Theta^{-1}(v_1)$ be given. If $x_1, x_2 \in G_{p_u}$ are both in critical level curves of p_u, and $\arg(p_u(x_1)) = \arg(p_u(x_2)) = 0$, then either $x_1 = x_2$ or $|x_1 - x_2| > 2\delta_1$.

We now choose $\delta_2 > 0$ small enough so that each of the following holds.

1. $\delta_2 < \delta_1$.

2. For each $u \in \Theta^{-1}(v_1)$, for each $k \in \{1, \ldots, N-1\}$, for each $z \in B_{3\delta_2}(u^{(k)})$, we have $|p_u(u^{(k)}) - p_u(z)| < \delta_1$.

3. By Lemma 10, we may choose $\delta_2 > 0$ and $\rho_1 > 0$ so that for any $u \in \Theta^{-1}(v_1)$, let $\widehat{u}$ be any point in $B_{\rho_1}(u)$ and let $\widehat{x_1}, \widehat{x_2} \in G_{p_{\widehat{u}}}$ be given such that $\arg(p_u(x_1)) = \arg(p_u(x_2)) = 0$, and such that there is a path $\widehat{\sigma} : [0, 1] \to G_{p_u}$ such that $\widehat{\sigma}(0) = \widehat{x_1}$ and $\widehat{\sigma}(1) = \widehat{x_2}$ and $\arg(p_{\widehat{u}}(\widehat{\sigma}(r))) = 0$ for all $r \in [0, 1]$. Then if $x_1, x_2 \in G_{p_{\widehat{u}}}$ are such that $\arg(p_u(x_1)) = \arg(p_u(x_1)) = 0$ and $|\widehat{x_1} - x_1| < \delta_2$ and $|\widehat{x_2} - x_2| < \delta_2$, then there is a path $\sigma : [0, 1] \to G_{p_u}$ such that $\sigma(0) = x_1$, $\sigma(1) = x_2$, and for all $r \in [0, 1]$, $\arg(p_u(\sigma(r))) = 0$ and $|\widehat{\sigma}(r) - \sigma(r)| < \delta_1$. Moreover, if $|p_{\widehat{u}}|$ is strictly increasing or strictly decreasing along $\widehat{\sigma}$, then σ may be chosen so that $|p_u|$ is increasing or decreasing along σ respectively.

In Item 3 above we chose a $\rho_1 > 0$. We now require that $\rho_1 > 0$ be chosen so that the following holds.

1. $\rho_1 < \delta_2$.

2. We will use this second item to refer to the restriction on ρ_1 described in Item 3 for the choice of δ_2 above.

3. Let $u = (u^{(1)}, \dots, u^{(N-1)}) \in \Theta^{-1}(v_1)$ be chosen. For $\widehat{u} \in B_{\rho_1}(u)$ define $\widehat{v} = (\widehat{v^{(1)}}, \dots, \widehat{v^{(n-1)}}) := \Theta(\widehat{u})$. Suppose that $\arg(\widehat{v^{(k)}}) = \arg(v^{(k)})$ for each $k \in \{1, \dots, N-1\}$. For some $k \in \{1, \dots, N-1\}$ with $|v^{(k)}| \neq 0$, let $\widehat{\lambda}$ denote the level curve of $p_{\widehat{u}}$ which contains $\widehat{u^{(k)}}$. Then the following holds. Let $\widehat{E}$ denote some edge of $\widehat{\lambda}$ which is incident to $\widehat{u^{(k)}}$, and let $\widehat{\gamma}$ denote a parameterization of $\widehat{E}$ beginning at $\widehat{u^{(k)}}$ parameterized with respect to $\arg(p_{\widehat{u}})$. That is, if Δ is the total change in argument of $\arg(p_{\widehat{u}})$ along $\widehat{E}$, and $\alpha \in [0, 2\pi)$ is the argument of $\widehat{v^{(k)}}$, then $\widehat{\gamma} : [\alpha, \alpha + \Delta] \to \widehat{\lambda}$ and satisfies $\widehat{\gamma}(\alpha) = \widehat{u^{(k)}}$ and $\arg(p_{\widehat{u}}(\widehat{\gamma}(t))) = t$ for all $t \in [\alpha, \alpha + \Delta]$. Then if we let λ denote the critical level curve of p_u containing $u^{(k)}$, there is a path $\gamma : [\alpha, \alpha + \Delta] \to \lambda$ such that $\gamma(\alpha) = u^{(k)}$, and for each $r \in [\alpha, \alpha + \Delta]$, $\arg(p_u(\gamma(r))) = r$ and $|\gamma(r) - \widehat{\gamma}(r)| < \delta_2$. This may be done by Lemma 9.

4. For each $u \in \Theta^{-1}(v_1)$, for each $i \in \{1, \dots, N-1\}$, $B_{\rho_1}(u^{(i)}) \subset G_{p_u}$.

5. For each $u \in \Theta^{-1}(v_1)$, if $|\widehat{u} - u| < \rho_1$, and $z \in G_{p_{\widehat{u}}}$, and $|z - z'| < \rho_1$, then $|p_{\widehat{u}}(z') - p_u(z)| < \delta_2$.

6. $\rho_1 < \text{mindiff}(0, u^{(1)}, \dots, u^{(N-1)})$ for each $u = (u^{(1)}, \dots, u^{(N-1)}) \in \Theta^{-1}(v_1)$.

Finally, choose some $\nu_1 > 0$ small so that the following holds.

1. $\nu_1 < \rho_1$.

2. If $\widehat{v_1} \in V_{N-1}$ satisfies $|v_1 - \widehat{v_1}| < \nu_1$, and $\widehat{u} \in \Theta^{-1}(\widehat{v_1})$, then there is some $u \in \Theta^{-1}(v_1)$ such that $|u - \widehat{u}| < \frac{\rho_1}{4}$. This may be done by Lemma 5.

3. $\nu_1 < \text{mindiff}(0, |{v_1}^{(1)}|, \dots, |{v_1}^{(N-1)}|)$.

4. $\nu_1 < 1$.

This ν_1 just found will be how close $\widehat{v_1}$ must be to v_1 to make the argument described above and obtain the desired result. We now proceed to construct a critical level curve configuration $\langle\widehat{\Lambda}\rangle_{PC} \in PC$ with critical values $\widehat{v_1} \in V_{N-1}$ satisfying $|v_1 - \widehat{v_1}| < \nu_1$, and such that $\widehat{v_1}$ has atypicallity degree strictly less than M.

We will first introduce some notation. Recall that all single point members of P are identical except for the value that $Z(\cdot)$ takes. Therefore we make the following definition.

Definition 34. *For each non-zero integer k, let $\langle w_k \rangle_P$ denote the single point member of P such that $Z(\langle w_k \rangle_P) = k$.*

Definition 35. *For $\langle \xi \rangle_{PC} \in PC$, and $\epsilon > 0$ we define $E_{\langle \xi \rangle_{PC}, \epsilon}$ to be the collection of members $\langle \psi \rangle_P \in P$ used to construct $\langle \xi \rangle_{PC}$ such that $H(\langle \psi \rangle_P) = \epsilon$.*

We will construct $\langle \widehat{\Lambda} \rangle_{PC} \in PC$ differently depending on which of the following three cases into which $\langle \Lambda \rangle_{PC} \in PC$ falls.

- $|{v_1}^{(M)}| = 0$.
- $|{v_1}^{(M)}| > 0$ and for each $\langle \lambda \rangle_P \in E_{\langle \Lambda \rangle_{PC}, |{v_1}^{(M)}|}$, $\langle \lambda \rangle_P$ only contains a single vertex (counting multiplicity).
- $|{v_1}^{(M)}| > 0$ and there is some member of $E_{\langle \Lambda \rangle_{PC}, |{v_1}^{(M)}|}$ which contains more than one vertex (counted with multiplicity).

Case 6.0.6. $|{v_1}^{(M)}| = 0$.

Since 0 is a critical value of $\langle \Lambda \rangle_{PC}$, there is some level 0 member $\langle w_k \rangle_{PC} \in PC$ used in the construction of $\langle \Lambda \rangle_{PC}$ such that $k \geq 2$. That is, in the construction of $\langle \Lambda \rangle_{PC}$, $\langle w_k \rangle_{PC}$ was associated to a face of some member of P. Let $\langle \psi \rangle_P$ denote this member of P, and let D denote the face of ψ to which $\langle w_k \rangle_{PC}$ was associated. Then g_D mapped each distinguished point in ∂D to w_k. We will define $\langle \widehat{\lambda} \rangle_{PC}$, another member of PC, to replace $\langle w_k \rangle_{PC}$ as we construct $\langle \widehat{\Lambda} \rangle_{PC}$, and in every other respect we construct $\langle \widehat{\Lambda} \rangle_{PC}$ in the same manner as $\langle \Lambda \rangle_{PC}$.

Let $\widehat{\lambda}$ denote the ”figure eight” planar graph. Let x denote the vertex of $\widehat{\lambda}$. Define $H(\langle \widehat{\lambda} \rangle_P) := \frac{v_1}{2}$, and $a(x) := 0$. Let $D^{(1)}$ denote one of the bounded faces of $\widehat{\lambda}$, and $D^{(2)}$ the other. Distinguish x and distinguish $k-2$ distinct points other than x in the boundry of $D^{(1)}$.

With this auxiliary data we have formed a member of P, namely $\langle \widehat{\lambda} \rangle_P$. To $D^{(1)}$ we associate $\langle w_{k-1} \rangle_{PC}$, and define $g_{D^{(1)}}$ by mapping each distinguished point in $\partial D^{(1)}$ to w_{k-1}, and associate $\langle w_1 \rangle_{PC}$ to $D^{(2)}$, and define $g_{D^{(2)}}$ to map the single distinguished point in $\partial D^{(2)}$ (namely x) to w_1. The resulting object is a member of PC, namely $\langle \widehat{\lambda} \rangle_{PC}$.

We wish to construct $\langle\widehat{\Lambda}\rangle_{PC}$ in exactly the same manner as $\langle\Lambda\rangle_{PC}$, except by replacing $\langle w_k\rangle_{PC}$ with $\langle\widehat{\lambda}\rangle_{PC}$. We may do this because by construction, $Z(\langle\widehat{\lambda}\rangle_P) = k = Z(\langle w_k\rangle_P)$. The only thing remaining to do in the construction of $\langle\widehat{\Lambda}\rangle_{PC}$ is specify g_D. Let $w^{(1)}$ be any fixed distinguished point in ∂D. Then define $g_D(w^{(1)}) := x$, and if w is the i^{th} distinguished point in ∂D (for some $i \in \{1, \dots, k-1\}$) in the positive direction after $w^{(1)}$, define $g_D(w)$ to be the i^{th} distinguished point in $\partial D^{(1)}$ in the positive direction after x (where the $(k-1)^{\text{st}}$ distinguished point in $\partial D^{(1)}$ after x is interpreted as being x itself). Then proceeding with the construction in every other way the same as with $\langle\Lambda\rangle_{PC}$, we obtain a member of PC, namely $\langle\widehat{\Lambda}\rangle_{PC}$.

Note that the critcal values of $\langle\widehat{\Lambda}\rangle_{PC}$ will be exactly $\widehat{v_1} := (0, \dots, 0, \frac{\nu_1}{2}, {v_1}^{(M+1)}, \dots, {v_1}^{(N-1)})$ (with $M-1$ copies of 0), while $v_1 = (0, \dots, 0, {v_1}^{(M+1)}, \dots, {v_1}^{(N-1)})$, so $|v_1 - \widehat{v_1}| = \frac{\nu_1}{2} < \nu_1$. Note also that since $\frac{\nu_1}{2} < \text{minmod}(v_1) < |{v_1}^{(M+1)}|$, $\widehat{v_1}$ has atypicallity degree $M-1 < M$.

Case 6.0.7. $|{v_1}^{(M)}| > 0$ *and for each* $\langle\lambda\rangle_P \in E_{\langle\Lambda\rangle_{PC}, |{v_1}^{(M)}|}$, $\langle\lambda\rangle_P$ *only contains a single vertex (counting multiplicity).*

Let $\langle\lambda\rangle_P$ be some fixed member of $E_{\langle\Lambda\rangle_{PC}, |{v_1}^{(M)}|}$. We construct $\langle\widehat{\Lambda}\rangle_{PC}$ identically to the construction of $\langle\Lambda\rangle_{PC}$, except we replace $\langle\lambda\rangle_{PC}$ in the construction with $\langle\widehat{\lambda}\rangle_{PC}$, where $\langle\widehat{\lambda}\rangle_{PC}$ is identical to $\langle\lambda\rangle_{PC}$ except that $H(\langle\widehat{\lambda}\rangle_P) := (1 + \frac{\nu_1}{2})H(\langle\lambda\rangle_P) = (1 + \frac{\nu_1}{2})|{v_1}^{(M)}|$.

Note that with this construction, the critical values of $\langle\widehat{\Lambda}\rangle_{PC}$ are exactly $\widehat{v_1} := ({v_1}^{(1)}, \dots, {v_1}^{(M-1)}, (1 + \frac{\nu_1}{2}){v_1}^{(M)}, {v_1}^{(M+1)}, \dots, {v_1}^{(N-1)})$, so $|v_1 - \widehat{v_1}| = |\frac{\nu_1}{2}{v_1}^{(M)}| \le \frac{\nu_1}{2} < \nu_1$. For each $k \in \{1, \dots, N-1\}$, let $\widehat{{v_1}^{(k)}}$ denote the k^{th} entry of $\widehat{v_1}$. Then

$$|\widehat{{v_1}^{(M-1)}}| = |{v_1}^{(M-1)}| = |{v_1}^{M}| < |(1 + \frac{\nu_1}{2}){v_1}^{(M)}| = |\widehat{{v_1}^{(M)}}|.$$

And $\nu_1 < \text{mindiff}(0, |{v_1}^{(1)}|, \dots, |{v_1}^{(N-1)}|)$, so we have

$$|\widehat{{v_1}^{(M+1)}}| - |\widehat{{v_1}^{(M)}}| = |{v_1}^{(M+1)}| - (1 + \frac{\nu_1}{2})|{v_1}^{(M)}| = |{v_1}^{(M+1)}| - |{v_1}^{(M)}| - \frac{\nu_1}{2}|{v_1}^{(M)}|.$$

And $|{v_1}^{(M+1)}| - |{v_1}^{(M)}| \ge \nu_1$, so

$$|\widehat{v_1{}^{(M+1)}}| - |\widehat{v_1{}^{(M)}}| \geq \nu_1 - \frac{\nu_1}{2}|v_1{}^{(M)}| > \frac{\nu_1}{2} > 0.$$

Thus we conclude that $\widehat{v_1}$ has atypicallity degree less than M.

Case 6.0.8. *$|v_1{}^{(M)}| > 0$ and some member of $E_{\langle\Lambda\rangle_{PC},|v_1{}^{(M)}|}$ contains more than one vertex (counting multiplicity).*

Let $\langle\lambda\rangle_{PC}$ denote one of the members of PC used in constructing $\langle\Lambda\rangle_{PC}$ such that $\langle\lambda\rangle_P \in E_{\langle\Lambda\rangle_{PC},|v_1{}^{(M)}|}$, and such that λ contains more than one vertex. (Possibly $\langle\lambda\rangle_{PC} = \langle\Lambda\rangle_{PC}$.) We now construct $\langle\widehat{\lambda}\rangle_{PC}$ which will take the place of $\langle\lambda\rangle_{PC}$ as we construct $\langle\widehat{\Lambda}\rangle_{PC}$. First a definition.

Definition 36. *Let $\langle\lambda\rangle_{PC} \in PC_a$ be given with the assumption that λ has more than two edges. By Lemma 2, we may find some bounded face F of λ such that the boundary of F consists of a single edge E of λ.*

- *We define $\lambda \setminus E$ to be the member of $\check{P}$ which arises from the graph λ when the edge E is removed.*
- *We define $\langle\lambda \setminus E\rangle_P$ to be the member of P which arises from $\lambda \setminus E$, and inherits all of its auxiliary data from $\langle\lambda\rangle_P$. Note that if x is the vertex of λ which E has as its endpoints, if the multiplicity of x as a vertex of λ equals 1, then x is no longer a vertex of $\lambda \setminus E$, and thus $a(x)$ no longer has any meaning for $\langle\lambda \setminus E\rangle_P$.*
- *We define $\langle\lambda \setminus E\rangle_{PC}$ to be the member of P which arises from $\lambda \setminus E$, and inherits all of its auxiliary data from $\langle\lambda\rangle_{PC}$. For example, if D is a bounded face of λ other than F, and $\langle\xi\rangle_{PC}$ is the member of PC associated to D in $\langle\lambda\rangle_{PC}$, then we associate $\langle\xi\rangle_{PC}$ to D in the construction of $\langle\lambda \setminus E\rangle_{PC}$, and we carry over g_D to $\langle\lambda \setminus E\rangle_{PC}$ as well.*

By Lemma 2, there is some face of λ which has only one edge of λ in its boundary. Let $F^{(1)}, \ldots, F^{(h)}$ be the bounded faces of λ, ordered so that $F^{(h)}$ has only one edge of λ in its boundry, and let E denote the edge that forms the boundry of $F^{(h)}$. Let z denote the vertex of λ which is incident to $F^{(h)}$. Let m denote the multiplicity of z as a vertex of λ. As noted above, if $m = 1$, then z is not a vertex of $\lambda \setminus E$ (or one might say that z is a vertex of $\lambda \setminus E$ with multiplicity 0), while if $m > 1$ then z is a vertex of $\lambda \setminus E$ with multiplicity $m-1$. Note that $Z(\langle \lambda \setminus E \rangle_P) = \sum_{k=1}^{h-1} z(F^{(k)})$.

Let $\widehat{\lambda}$ denote the "figure eight" graph, and let $D^{(1)}, D^{(2)}$ denote its two faces. Let x denote the vertex in $\widehat{\lambda}$. From $\widehat{\lambda}$ we will form a member of P, and eventually a member of PC which will replace $\langle \lambda \rangle_{PC}$ in the construction of $\langle \Lambda \rangle_{PC}$. Define $H(\langle \widehat{\lambda} \rangle_P) := (1 + \frac{\nu_1}{2})H(\langle \lambda \rangle_P)$. Define $z(D^{(1)}) := Z(\langle \lambda \setminus E \rangle_P)$, and $z(D^{(2)}) := z(F^{(h)})$ (where $F^{(h)}$ is viewed as a face of λ). Distinguish $z(D^{(i)})$ points in $\partial D^{(i)}$ for $i = 1, 2$, distinguishing x if and only if z is distinguished as a vertex in $\langle \lambda \rangle_P$. Define $a(x) := a(z)$ where $a(z)$ comes from $\langle \lambda \rangle_P$. With this data, we have a member of P, namely $\langle \widehat{\lambda} \rangle_P$.

Let $\langle \xi \rangle_{PC}$ be the member of PC which was associated to $F^{(h)}$ in the construction of $\langle \lambda \rangle_{PC}$. Then we associate $\langle \lambda \setminus E \rangle_{PC}$ to $D^{(1)}$, and $\langle \xi \rangle_{PC}$ to $D^{(2)}$. We now wish to define $g_{D^{(1)}}$ and $g_{D^{(2)}}$.

In order to define $g_{D^{(1)}}$, we define an enumeration of the distinguished points in $\lambda \setminus E$. Let E' denote the edge in λ which is directly after E as λ is traversed with a positive orientation. Define $y^{(1)} := z$ if z is a distinguished point in $\lambda \setminus E$. Otherwise define $y^{(1)}$ to be the first distinguished point after z in $\lambda \setminus E$ as $\lambda \setminus E$ is traversed with a positive orientation beginning with E'. Continue traversing $\lambda \setminus E$ with a positive orientation, and let $y^{(2)}, \ldots, y^{(z(D^{(1)}))}$ be the distinguished points after $y^{(1)}$ of $\lambda \setminus E$ as they appear as $\lambda \setminus E$ is traversed one full time beginning with $E^{(1)}$. Note that a distinguished point will appear in this list exactly n times if it is a distinguished point of $\lambda \setminus E$ with multiplicity n. Now let $x^{(1)}, \ldots, x^{(z(D^{(1)}))}$ be an enumeration of the distinguished points in $\partial D^{(1)}$ as they appear in increasing order beginning with x if x is a distinguished point, and beginning with the first distinguished point after x otherwise. Finally we define $g_{D^{(1)}}(x^{(i)}) := y^{(i)}$ for each i.

We will now define $g_{D^{(2)}}$. Now let $y^{(1)}, \ldots, y^{(z(D^{(2)}))}$ be the distinguished points in $\partial F^{(h)}$ listed in increasing order, with $y^{(1)} = z$ if z is distinguished in $\langle \lambda \rangle_P$, and $y^{(1)}$ the first distinguished point after z in $\partial F^{(h)}$ otherwise. Let $x^{(1)}, \ldots, x^{(z(D^{(2)}))}$ be the distinguished points in $\partial D^{(2)}$ listed in increasing order, with $x^{(1)} = x$ if x is distinguished, and $x^{(1)}$

the first distinguished point in the positive direction from x in $\partial D^{(2)}$ otherwise. Then for $1 \leq i \leq z(D^{(2)})$, we define $g_{D^{(2)}}(x^{(i)}) := g_{F^{(h)}}(y^{(i)})$. With this data we now have a member of PC, namely $\langle\widehat{\lambda}\rangle_{PC}$.

We now wish to construct $\langle\widehat{\Lambda}\rangle_{PC}$ in every respect the same as $\langle\Lambda\rangle_{PC}$, except that $\langle\lambda\rangle_{PC}$ will be replaced in this construction with $\langle\widehat{\lambda}\rangle_{PC}$. If $\langle\lambda\rangle_{PC} = \langle\Lambda\rangle_{PC}$, then we are done, and we define $\langle\widehat{\Lambda}\rangle_{PC} := \langle\widehat{\lambda}\rangle_{PC}$. If $\langle\lambda\rangle_{PC} \neq \langle\Lambda\rangle_{PC}$, then $\langle\lambda\rangle_{PC}$ was associated to some face D of $\langle\psi\rangle_P$ a member of P during the construction of $\langle\Lambda\rangle_{PC}$. Then $\langle\widehat{\Lambda}\rangle_{PC}$ may inherit all of its data from $\langle\Lambda\rangle_{PC}$ (other than $\langle\lambda\rangle_{PC}$, which we have exchanged for $\langle\widehat{\lambda}\rangle_{PC}$) except the gradient function g_D. Let g_D denote the gradient map for D in $\langle\Lambda\rangle_{PC}$ (which maps the distinguished points in ∂D to the distinguished points in λ), and let $\widehat{g_D}$ denote the gradient map for D in $\langle\widehat{\Lambda}\rangle_{PC}$ (which will map the distinguished points in ∂D to the distinguished points in $\widehat{\lambda}$). To construct $\widehat{g_D}$, we have two possible cases, first that z is a distinguished point in λ, and, in fact, the only distinct distinguished point in λ, and second that there are distinguished points in λ which are distinct from z.

Subcase 6.0.8.1. *z is a distinguished point in λ, and z is the only distinct distinguished point in λ.*

Let $x^{(1)}, \dots, x^{(z(D))}$ be an enumeration of the distinguished points in ∂D listed in increasing order. Let $y^{(1)}, \dots, y^{(z(D))}$ be an enumeration of the distinguished points in $\widehat{\lambda}$ listed in the order in which they appear as $\widehat{\lambda}$ is traversed, beginning and ending with x. Then we define $\widehat{g_D}(x^{(i)}) := y^{(i)}$ for each i.

Subcase 6.0.8.2. *There are distinguished points in λ which are distinct from z.*

We define an enumeration of the distinguished points of λ which will be used to define $\widehat{g_D}$. Let $\widehat{y^{(1)}}, \dots, \widehat{y^{(z(D^{(1)}))}}$ be an enumeration of the distinguished points of $\widehat{\lambda}$ which are in $\partial D^{(1)}$, beginning at x if x is a distinguished point of $\widehat{\lambda}$, and beginning at the first distinguished point past x otherwise. For each $i \in \{1, \dots, z(D^{(1)})\}$, $g_{D^{(1)}}(\widehat{y^{(i)}})$ is a distinguished point in $\lambda \setminus E$. Define $y^{(i)}$ to be the distinguished point in λ which corresponds to the distinguished point $g_{D^{(1)}}(\widehat{y^{(i)}})$ in $\lambda \setminus E$. For $i \in \{1, \dots, z(D^{(2)})\}$, let $y^{(z(D^{(1)})+i)}$ denote the i^{th} distinguished point in $\partial F^{(h)}$ in increasing order, beginning with z if z is distinguished, and beginning with the first distinguished point past z otherwise. Let $y^{\widehat{(z(D^{(1)})+i)}}$ denote the i^{th} distiguished point in $\partial D^{(2)}$, beginning with x if x is a distinguished point of $\widehat{\lambda}$, and with the first distinguished point past x in increasing order around $\partial D^{(2)}$ otherwise. Then $\{y^{(1)}, \dots, y^{(z(D^{(1)})+z(D^{(2)}))}\}$ is an enumeration of the distinguished points in λ in increasing order, and $\{\widehat{y^{(1)}}, \dots, y^{\widehat{(z(D^{(1)})+z(D^{(2)}))}}\}$ is an enumeration of the distinguished points in $\widehat{\lambda}$ in order as they appear around $\widehat{\lambda}$. Let $z^{(1)}, \dots, z^{(z(D^{(1)})+z(D^{(2)}))}$ be any enumeration of the distinguished points in ∂D such that $g_D(z^{(i)}) = y^{(i)}$ for each $i \in \{1, \dots, z(D^{(1)})+z(D^{(2)})\}$. Now for $i \in \{1, \dots, z(D^{(1)})+z(D^{(2)})\}$, we define $\widehat{g_D}(z^{(i)}) := \widehat{y^{(i)}}$. With this definition, we have all the data needed for a member of PC, namely $\langle\widehat{\Lambda}\rangle_{PC}$.

Notice, then, that by the construction of $\langle\widehat{\Lambda}\rangle_{PC}$, the critical values of $\langle\widehat{\Lambda}\rangle_{PC}$ are exactly

$$\widehat{v_1} := ({v_1}^{(1)}, \dots, {v_1}^{(M-1)}, (1 + \frac{\nu_1}{2}){v_1}^{(M)}, {v_1}^{(M+1)}, \dots, {v_1}^{(N-1)}) \in V_{N-1}.$$

Therefore by the same argument as in Case 6.0.7, $\widehat{v_1}$ has atypicallity degree less than M, and

$$|v_1 - \widehat{v_1}| = \frac{\nu_1}{2}|{v_1}^{(M)}| < \frac{\nu_1}{2} < \nu_1.$$

So now in any case we have a member of PC, $\langle\widehat{\Lambda}\rangle_{PC}$, with critical values $\widehat{v_1}$ such that $|v_1 - \widehat{v_1}| < \nu_1$ and the atypicallity degree of $\widehat{v_1}$ is strictly less than M. Note also that by construction, for each $k \in \{1, \dots, N-1\}$, $\arg(\widehat{{v_1}^{(k)}}) = \arg({v_1}^{(k)})$ (where we adopt the convention that $\arg(0) = 0$).

By the induction assumption there is some $\widehat{u_1} = (\widehat{{u_1}^{(1)}}, \dots, \widehat{{u_1}^{(N-1)}}) \in \Theta^{-1}(\widehat{v_1})$ such that $\Pi(p_{\widehat{u_1}}, G_{p_{\widehat{u_1}}}) = \langle\widehat{\Lambda}\rangle_{PC}$. By Item 2 in the choice of ν_1, there is a $u_1 = ({u_1}^{(1)}, \dots, {u_1}^{(N-1)}) \in \mathbb{C}^{N-1}$ such that $\Theta(u_1) = v_1$ and $|u_1 - \widehat{u_1}| < \rho_1$. Define $\langle\widetilde{\Lambda}\rangle_{PC} := \Pi(p_{u_1}, G_{p_{u_1}})$. Our goal is to show that $\langle\Lambda\rangle_{PC} = \langle\widetilde{\Lambda}\rangle_{PC}$. First two definitions.

Definition 37. *Let $\langle\lambda\rangle_{PC}$ be some member of PC, and let D denote some face of λ. Then we let $\langle\lambda_D\rangle_{PC}$ denote the member of PC which was associated to D in the construction of $\langle\lambda\rangle_{PC}$.*

Definition 38. *Let $\langle\lambda\rangle_P$ be a graph member of P, and let $n \geq 2$ denote the number of edges of λ. We say an enumeration $E^{(1)}, \dots, E^{(n)}$ of these edges is in order with respect to λ (or just "in order" when λ is obvious) if the order in which the edges appear when λ is traversed one full time with positive orientation beginning with $E^{(1)}$ is exactly $E^{(1)}, \dots, E^{(n)}$. Let D be a bounded face of λ, and let $k \geq 1$ denote the number of edges of λ which are in ∂D. We say that an enumeration $E^{(1)}, \dots, E^{(k)}$ of these edges is in order with respect to D if $E^{(1)}, \dots, E^{(k)}$ is the order in which these edges appear as ∂D is traversed one full time with positive orientation beginning with $E^{(1)}$.*

We will show that $\langle\Lambda\rangle_{PC} = \langle\widetilde{\Lambda}\rangle_{PC}$ recursively, working "outside in", by doing the following steps.

1. Show that $\langle\Lambda\rangle_P = \langle\widetilde{\Lambda}\rangle_P$, and establish a correspondence between the vertices of $\langle\Lambda\rangle_P$ and the vertices of $\langle\widetilde{\Lambda}\rangle_P$ which respects the data contained in a member of P. (That is, if $k \geq 1$ is the number of vertices in Λ and $\widetilde{\Lambda}$, find enumerations $u^{(1)}, \dots, u^{(k)}$ and $\widetilde{u^{(1)}}, \dots, \widetilde{u^{(k)}}$ of the vertices of Λ and $\widetilde{\Lambda}$ respectively such that the following holds. For each $i \in \{1, \dots, k\}$, $a(u^{(i)}) = a(\widetilde{u^{(i)}})$. For each $i, j \in \{1, \dots, k\}$, $\{u^{(i)}u^{(j)}\}$ is an edge in Λ if and only if $\{\widetilde{u^{(i)}}\widetilde{u^{(j)}}\}$ is an edge in $\widetilde{\Lambda}$ and, moreover, if $\{u^{(i)}u^{(j)}\}$ is an edge in Λ, then $\{u^{(i)}u^{(j)}\}$ and $\{\widetilde{u^{(i)}}\widetilde{u^{(j)}}\}$ contain the same number of distinguished points. Finally, if $n \geq 2$ is the number of edges in Λ, and $\{u^{(i_1)}u^{(i_2)}\}, \dots, \{u^{(i_n)}u^{(i_{n+1})}\}$ is the list of edges of Λ as they appear in order around Λ, then $\{\widetilde{u^{(i_1)}}\widetilde{u^{(i_2)}}\}, \dots, \{\widetilde{u^{(i_n)}}\widetilde{u^{(i_{n+1})}}\}$ is the order in which the edges of $\widetilde{\Lambda}$ appear around $\widetilde{\Lambda}$. Note that this will immediately provide a well defined correspondence between the bounded faces D of Λ and the bounded faces $\widetilde{D}$ of $\widetilde{\Lambda}$ and between the distinguished points x of $\langle\Lambda\rangle_P$ and the distinguished points $\widetilde{x}$ of $\langle\widetilde{\Lambda}\rangle_P$.)

2. Let D be one of the bounded faces of Λ, and let $\langle\lambda_D\rangle_{PC}$ and $\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_{PC}$ denote the members of PC assigned to D and $\widetilde{D}$ during the construction of $\langle\Lambda\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$ respectively. Show that $\langle\lambda_D\rangle_P = \langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$, and establish a correspondence between the vertices of λ_D and the vertices of $\widetilde{\lambda_{\widetilde{D}}}$ as described in Step 1. Then show that the correspondence between $\langle\lambda_D\rangle_P$ and $\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$ respects the gradient maps g_D and $g_{\widetilde{D}}$. That is, if x is one of the distinguished points of Λ in ∂D, and $\widetilde{x}$ is

the corresponding distinguished point of $\widetilde{\Lambda}$ in $\partial\widetilde{D}$, then show that $g_{\widetilde{D}}(\widetilde{x})$ is the distinguished point in $\widetilde{\lambda_{\widetilde{D}}}$ which corresponds to the distinguished point $g_D(x)$ in λ_D.

3. Iterate Step 2 for each face D of Λ, then again with each face of each λ_D, etc.

Since $\langle\Lambda\rangle_{PC}, \langle\widetilde{\Lambda}\rangle_{PC}$ are constructed with finitely many steps, this process will terminate after finitely many steps. When this process terminates, we will have shown that $\langle\Lambda\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$ have all the same data, and are therefore equal. Notice that the base case (Step 1 and Step 2 as written) is just a simpler case of the recursive step (Step 1 and Step 2 with any $\langle\lambda\rangle_{PC}$ used in the construction of $\langle\Lambda\rangle_{PC}$ inserted in the place of $\langle\Lambda\rangle_{PC}$). Therefore we just do the recursive step.

Now for any $\langle\lambda\rangle_{PC}$ used to construct $\langle\Lambda\rangle_{PC}$, we will see that in the process of establishing the correspondence of the edges and vertices described above between $\langle\lambda\rangle_{PC}$ and the corresponding $\langle\widetilde{\lambda}\rangle_{PC}$ used to construct $\langle\widetilde{\Lambda}\rangle_{PC}$ we will do the following. Let D be a face of λ. Let $\widetilde{D}$ be the corresponding face of $\widetilde{\lambda}$. We will select a face $\widehat{D}$ of some graph member $\langle\widehat{\xi}\rangle_{PC}$ used to construct $\langle\widehat{\Lambda}\rangle_{PC}$, such that $\widehat{D}$ corresponds naturally to D in the construction of $\langle\widehat{\Lambda}\rangle_{PC}$. We will view $\widetilde{\lambda}$ and $\widehat{\xi}$ as embedded in $\mathbb{C}$ as critical level curves of p_{u_1} and $p_{\widehat{u_1}}$ respectively, and find paths σ and $\widehat{\sigma}$ which parameterize $\widetilde{D}$ and $\widehat{D}$ respectively such that $|\sigma(r) - \widehat{\sigma}(r)| < \delta_1$ for each r. This then implies that for any $i \in \{1, \dots, N-1\}$, if ${u_1}^{(i)} \in \widetilde{D}$, then $\widehat{{u_1}^{(i)}} \in \widehat{D}$. To see this implication, observe that, by Item 5 in the choice of δ_1, if ${u_1}^{(i)} \in \widetilde{D}$, then $B_{3\delta_1}({u_1}^{(i)}) \subset \widetilde{D}$. But $|{u_1}^{(i)} - \widehat{{u_1}^{(i)}}| < \rho_1 < \delta_1$, so $B_{2\delta_1}(\widehat{{u_1}^{(i)}}) \subset \widetilde{D}$. Since $|\sigma(r) - \widehat{\sigma}(r)| < \delta_1$ for all r, the winding number of $\widehat{\sigma}$ around $\widehat{{u_1}^{(i)}}$ is the same as the winding number of σ around ${u_1}^{(i)}$. Thus we conclude that $\widehat{{u_1}^{(i)}} \in \widehat{D}$. Now let us return to our induction argument.

Suppose that Step 2 has just been completed for some $\langle\lambda\rangle_P$ used to construct $\langle\Lambda\rangle_{PC}$, with corresponding $\langle\widetilde{\lambda}\rangle_P$ used to construct $\langle\widetilde{\Lambda}\rangle_{PC}$. Let D be one of the bounded faces of λ, and let $\widetilde{D}$ be the corresponding bounded face of $\widetilde{\Lambda}$. We will now describe the selection of the face $\widehat{D}$ referred to above. Let $\langle\widehat{\lambda}\rangle_{PC}$ denote the member of PC which replaced $\langle\lambda\rangle_{PC}$ in the construction of $\langle\widehat{\Lambda}\rangle_{PC}$.

Case 6.0.9. $\langle\widehat{\lambda}\rangle_P = \langle\lambda\rangle_P$.

Then let $\widehat{D}$ denote the face D of λ viewed as a face of $\widehat{\lambda}$.

Case 6.0.10. $\langle\widehat{\lambda}\rangle_P \neq \langle\lambda\rangle_P$.

Then either $\langle\widehat{\lambda}\rangle_P$ was formed by merely changing the value of $H(\cdot)$ (that is, $\langle\widehat{\lambda}\rangle_P = \langle\lambda\rangle_P$ except that $H(\langle\widehat{\lambda}\rangle_P) = (1 + \frac{\nu_1}{2})H(\langle\lambda\rangle_P))$ or the graph λ itself was changed to form $\langle\widehat{\lambda}\rangle_{PC}$. If $\langle\widehat{\lambda}\rangle_P$ was formed by merely changing the value of $H(\cdot)$, then we may choose $\widehat{D}$ to be the face D of λ viewed as a face of $\widehat{\lambda}$ as in the previous case.

Suppose that λ itself was changed while forming $\langle\widehat{\lambda}\rangle_{PC}$. Recall that in this case, $\langle\widehat{\lambda}\rangle_{PC}$ was formed by selecting a bounded face F of λ such that ∂F consists of only a single edge $E^{(1)}$ of λ. We then set $\widehat{\lambda}$ to be the figure eight graph, and assign $\langle\lambda_F\rangle_{PC}$ to one face of $\widehat{\lambda}$, and $\langle\lambda \setminus E^{(1)}\rangle_{PC}$ to the other face of $\widehat{\lambda}$. We define auxiliary data (values of $a(\cdot)$, values of $H(\cdot)$, distinguished points, gradient maps) as described earlier in this chapter, and the resulting member of PC we call $\langle\widehat{\lambda}\rangle_{PC}$. (From now on we will refer to $\langle\lambda \setminus E^{(1)}\rangle_{PC}$ as $\langle\widehat{\lambda \setminus E^{(1)}}\rangle_{PC}$ to remind us that we are viewing $\langle\widehat{\lambda \setminus E^{(1)}}\rangle_{PC}$ as a member of PC used to construct $\langle\widehat{\Lambda}\rangle_{PC}$.) Let us call this method of forming $\langle\widehat{\lambda}\rangle_{PC}$ just described the "scattering method", as it "scatters" one of the vertices of λ.

With the above description, if $D = F$ we define $\widehat{D}$ to be the face of $\widehat{\lambda}$ to which $\langle\lambda_F\rangle_{PC}$ was assigned. If $D \neq F$, we define $\widehat{D}$ to be the face of $\widehat{\lambda \setminus E^{(1)}}$ which corresponds to D in the construction of $\langle\widehat{\lambda \setminus E^{(1)}}\rangle_{PC}$.

We now let $\langle\lambda_D\rangle_{PC}$, $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$, and $\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_{PC}$ be the members of PC which are assigned to D, $\widehat{D}$, and $\widetilde{D}$ in the construction of $\langle\Lambda\rangle_{PC}$, $\langle\widehat{\Lambda}\rangle_{PC}$, and $\langle\widetilde{\Lambda}\rangle_{PC}$ respectively.

Before we dive into the next argument, let us step back for a moment and review our strategy. $\langle\Lambda\rangle_{PC}$ was given to us as a member of PC with the prescribed critical values v_1, and our goal is to find a polynomial p such that $\Pi(p, G_p) = \langle\Lambda\rangle_{PC}$. We constructed $\langle\widehat{\Lambda}\rangle_{PC}$ in such a way as to be in some sense very similar to $\langle\Lambda\rangle_{PC}$, have critical values $\widehat{v_1}$ very close to v_1, and so that, by the induction assumption, there is some $\widehat{u_1} \in \mathbb{C}^{N-1}$ such that $\Theta(\widehat{u_1}) = \widehat{v_1}$ and $\Pi(p_{\widehat{u_1}}, G_{p_{\widehat{u_1}}}) = \langle\widehat{\Lambda}\rangle_{PC}$. We then used Lemma 5 to find a point $u_1 \in \mathbb{C}^{N-1}$

close to $\widehat{u_1}$ and such that $\Theta(u_1) = v_1$. We define $\langle\widetilde{\Lambda}\rangle_{PC} := \Pi(p_{u_1}, G_{p_{u_1}})$. We view $\langle\widehat{\Lambda}\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$ as being embedded in $\mathbb{C}$ as the critical level curves of $p_{\widehat{u_1}}$ and p_{u_1} respectively, and we wish to use the fact that $\widehat{u_1}$ and u_1 are so close, along with the fact that $\langle\Lambda\rangle_{PC}$ and $\langle\widehat{\Lambda}\rangle_{PC}$ are close (in some sense), to show that $\langle\Lambda\rangle_{PC} = \langle\widetilde{\Lambda}\rangle_{PC} = \Pi(p_{u_1}, G_{p_{u_1}})$, which is our desired result.

Right now we wish to show that $\langle\lambda_D\rangle_P = \langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$.

Case 6.0.11. *$\langle\widehat{\lambda_{\widehat{D}}}\rangle_P \neq \langle\lambda_D\rangle_P$, and $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$ was formed using the scattering method.*

Let $L \geq 2$ denote the number of edges in λ_D. We will again let F denote the bounded face of λ_D which was removed during the formation of $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$, and let $\widehat{F}$ denote the face of $\widehat{\lambda_{\widehat{D}}}$ to which we assigned $\langle\lambda_F\rangle_{PC}$. Let $\widehat{G}$ denote the other face of $\widehat{\lambda_{\widehat{D}}}$, namely the one to which $\langle\widehat{\lambda_D \setminus E^{(1)}}\rangle_{PC}$ was assigned. Recall that $E^{(1)}$ is the edge of λ_D which forms the boundary of F and let $E^{(2)}, \ldots, E^{(L)}$ be the enumeration of the other edges so that the order in which the edges of λ_D appear in order around λ_D beginning with $E^{(1)}$ is exactly $E^{(1)}, \ldots, E^{(L)}$. Let $K \geq 1$ denote the number of distinct vertices of λ_D, and let $x^{(1)}, \ldots, x^{(K)}$ be any enumeration of these vertices such that $E^{(1)}$ has both its endpoints at $x^{(1)}$. Now for each $i \in \{1, \ldots, L\}$, we wish to choose an edge (or a portion of an edge) of a graph used to construct $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$ which will correspond to the edge $E^{(i)}$ in λ_D.

Subcase 6.0.11.1. *$i = 1$.*

Recall that $E^{(1)}$ forms the boundary of the face F of λ_D. In this case we define $\widehat{E^{(i)}}$ to be the edge of $\widehat{\lambda_{\widehat{D}}}$ which forms the boundary of $\widehat{F}$.

Subcase 6.0.11.2. *$i \neq 1$, and $x^{(1)}$ is not an end point of $E^{(i)}$.*

In this case we define $\widehat{E^{(i)}}$ to be the edge $E^{(i)}$ viewed as an edge of $\widehat{\lambda_D \setminus E^{(1)}}$.

Subcase 6.0.11.3. *$i \neq 1$, and $x^{(1)}$ is an end point of $E^{(i)}$.*

The difficulty in this case arises from the fact that if $x^{(1)}$ is incident to only two bounded faces of λ_D, then $x^{(1)}$ is no longer a vertex of $\widehat{\lambda_D \setminus E^{(1)}}$.

Subcase 6.0.11.4. *There are more than two bounded faces of λ_D which are incident to $x^{(1)}$.*

Then $x^{(1)}$ is still a vertex of of $\widehat{\lambda_D \setminus E^{(1)}}$, so we may define $\widehat{E^{(i)}}$ to be the edge $E^{(i)}$ viewed as an edge in the graph $\widehat{\lambda_D \setminus E^{(1)}}$.

Subcase 6.0.11.5. *There are only two bounded faces of λ_D which are incident to $x^{(1)}$.*

First a definition.

Definition 39. *Let $\langle \xi \rangle_P$ be a graph member of P, and let E be an edge of ξ. Let x denote the initial point of E and let y denote the final point of E. We wish to define a quantity which we will call the change in argument along E (with respect to $\langle \xi \rangle_P$). Define $r_1 := a(x)$ and define $r_2 := a(y)$. If $a(y) = 0$, then we instead define $r_2 := 2\pi$. Then we define the change in argument along E with respect to $\langle \xi \rangle_P$ to be $r_2 - r_1 + 2\pi n$, where n denotes the number of distinguished points in E which are not end points of E. Note that if $\langle \xi \rangle_P$ arises as a critical level curve of some analytic function f, then the change in argument along E with respect to $\langle \xi \rangle_P$ is the same as the change in $\arg(f)$ along E.*

Let $j \in \{1, \dots, L\}$ be the index of the other end point of $E^{(i)}$. Recall that in this sub-case $x^{(1)}$ is one end point of $E^{(i)}$. Recall that F is one of the bounded faces of λ_D which is incident to $x^{(1)}$. Let G denote the other. If ∂G is formed by a single edge of λ_D, then λ_D is the figure eight graph. However since $\widehat{\lambda_{\widehat{D}}}$ was formed using the scattering method, λ_D is not the figure eight graph, so ∂G contains more than one edge of λ_D. Since $E^{(i)}$ does not form the boundary of F, we conclude that $E^{(i)}$ is contained in ∂G. Therefore $E^{(i)}$ does not have both ends at $x^{(1)}$, and therefore $j \neq 1$. Therefore also we may view $x^{(j)}$ as a vertex in $\lambda_D \setminus E^{(1)}$.

Let $\Delta > 0$ denote the change in argument along $E^{(i)}$ where $E^{(i)}$ is traversed from $x^{(j)}$ to $x^{(1)}$. Let $\widehat{E}$ denote the edge of $\widehat{\lambda_D \setminus E^{(1)}}$ which contains the point $x^{(1)}$ (which is no longer a vertex of $\widehat{\lambda_D \setminus E^{(1)}}$). Then let z denote the point in $\widehat{E}$ such that the change in $\arg(p_{\widehat{u}_1})$ along the portion of $\widehat{E}$ beginning at $x^{(j)}$ and ending at z is exactly Δ. Then we define $\widehat{E^{(i)}}$ to be this portion of $\widehat{E}$.

Note that in each case by the construction of $\langle \widehat{\Lambda} \rangle_{PC}$, the change in argument along $\widehat{E^{(i)}}$ is the same as the change in argument along $E^{(i)}$.

We now wish for each $i \in \{1, \ldots, K\}$ to choose a vertex $\widehat{x^{(i)}}$ of one of the graphs used to construct $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$ in such a way that $x^{(i)}$ gives rise to $\widehat{x^{(i)}}$ during the construction of $\langle\widehat{\Lambda}\rangle_{PC}$. Let $\widehat{x^{(1)}}$ denote the vertex in $\widehat{\lambda_{\widehat{D}}}$ and, for each $i \in \{2, \ldots, K\}$, let $\widehat{x^{(i)}}$ denote the vertex $x^{(i)}$ viewed as a vertex of $\lambda_{\widehat{D \setminus E^{(1)}}}$. We now have a name for the vertex in $\widehat{\lambda_{\widehat{D}}}$ and for each vertex in $\lambda_{\widehat{D \setminus E^{(1)}}}$ viewed as a graph used to form $\langle\widehat{\Lambda}\rangle_{PC}$, unless $x^{(1)}$ is still a vertex of $\lambda_{\widehat{D \setminus E^{(1)}}}$. In this case, let $\widehat{x^{(0)}}$ denote the vertex $x^{(1)}$ in $\lambda_{\widehat{D \setminus E^{(1)}}}$. Now view $\langle\widehat{\Lambda}\rangle_{PC}$ as embedded in $\mathbb{C}$ as the critical level curves of $p_{\widehat{u_1}}$. For each $i \in \{1, \ldots, K\}$, fix some choice of $t_i \in \{1, \ldots, N-1\}$ such that $\widehat{{u_1}^{(t_i)}}$ is the critical point of $p_{\widehat{u_1}}$ which gives rise to $\widehat{x^{(i)}}$ in the construction of $\Pi(p_{\widehat{u_1}}, G_{p_{\widehat{u_1}}})$. Then we define $\widetilde{x^{(i)}} := {u_1}^{(t_i)}$ (note that this implies that $t_1 = M$ by the construction of $\langle\widehat{\Lambda}\rangle_{PC}$). If $\widehat{x^{(0)}}$ is defined, then let $t_0 \in \{1, \ldots, N-1\}$ be some choice of index so that $\widehat{{u_1}^{(t_0)}}$ is the critical point of $p_{\widehat{u_1}}$ which gives rise to $\widehat{x^{(0)}}$ in the construction of $\Pi(p_{\widehat{u_1}}, G_{p_{\widehat{u_1}}})$. Then we define $\widetilde{x^{(0)}} := {u_1}^{(t_0)}$.

Define $\epsilon := H(\langle\lambda_D\rangle_P)$. We now wish to show that for each $k \in \{1, \ldots, K\}$ (and for $k = 0$ if $\widehat{x^{(0)}}$ is defined), $\widetilde{x^{(k)}} \in \widetilde{\lambda_{\widetilde{D}}}$. If $k \in \{2, \ldots, K\}$ (or $k = 0$ if $\widehat{x^{(0)}}$ is defined), then $t_k \neq M$, so

$$|p_{u_1}(\widetilde{x^{(k)}})| = |p_{u_1}({u_1}^{(t_k)})| = |{v_1}^{(t_k)}| = |\widehat{{v_1}^{(t_k)}}| = H(\langle\lambda_{\widehat{D \setminus E^{(1)}}}\rangle_P) = H(\langle\lambda_D\rangle_P) = \epsilon.$$

For $k = 1$, $t_k = M$, so

$$|p_{u_1}(\widetilde{x^{(k)}})| = |p_{u_1}({u_1}^{(M)})| = |{v_1}^{(M)}| = |\frac{1}{1+\frac{\nu_1}{2}}\widehat{{v_1}^{(t_k)}}| = \frac{1}{1+\frac{\nu_1}{2}}H(\langle\widehat{\lambda_{\widehat{D}}}\rangle_P) = H(\langle\lambda_D\rangle_P) = \epsilon.$$

Now suppose by way of contradiction that there is some critical point z of p_{u_1} in $\widetilde{D}$ such that $|p_{u_1}(z)| > \epsilon$. Then $|p_{u_1}(z)| \geq \epsilon + \text{mindiff}(v_1)$ by definition of mindiff. Choose some $t \in \{1, \ldots, N-1\}$, such that ${u_1}^{(t)} = z$. As mentioned above, this implies that $\widehat{{u_1}^{(t)}} \in \widehat{D}$ and, since $t_k = M$, $t \neq M$, so $|\widehat{{v_1}^{(t)}}| = |{v_1}^{(t)}| \geq \epsilon + \text{mindiff}(v_1) > H(\langle\widehat{\lambda_{\widehat{D}}}\rangle_P)$ by Item 3 in the choice of ν_1. Thus $\widehat{{u_1}^{(t)}}$ is not in one of the bounded faces of $\widehat{\lambda_{\widehat{D}}}$, which is a contradiction by the definition of $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$. Therefore the value that $|p_{u_1}|$ takes at each point in $\{\widetilde{x^{(1)}}, \ldots, \widetilde{x^{(K)}}\}$ (and $\widetilde{x^{(0)}}$ if it is defined) is ϵ, and there is no critical point of p_{u_1} in $\widetilde{D}$ at which $|p_{u_1}|$ takes a value larger than ϵ, so we conclude by Theorem 2.1 that each point $\widetilde{x^{(i)}}$ is in the critical level curve in $\widetilde{D}$ on which $|p_{u_1}|$ takes its largest value, namely $\widetilde{\lambda_{\widetilde{D}}}$.

Suppose now that $\widehat{x^{(0)}}$ is defined. We have already seen that $|p_{u_1}(\widetilde{x^{(1)}})| = |p_{u_1}(\widetilde{x^{(0)}})|$. Now for each $k \in \{1, \ldots, N-1\}$, $\arg(\widehat{{v_1}^{(k)}}) = \arg({v_1}^{(k)})$. Therefore by the construction of $\langle\widehat{\Lambda}\rangle_{PC}$, we have

$$\arg(p_{u_1}(\widetilde{x^{(1)}})) = \arg({v_1}^{(t_1)}) = \arg(\widehat{{v_1}^{(t_1)}}) = a(\widehat{x^{(1)}}) = a(\widehat{x^{(0)}}) = \arg(\widehat{{v_1}^{(t_0)}}) =$$

$$\arg({v_1}^{(t_0)}) = \arg(p_{u_1}(\widetilde{x^{(0)}}))$$

Therefore $p_{u_1}(\widetilde{x^{(1)}}) = p_{u_1}(\widetilde{x^{(0)}})$. We now wish to show that $\widetilde{x^{(1)}} = \widetilde{x^{(0)}}$. Define L_1 to be the straight line path from $\widetilde{x^{(1)}}$ to $\widehat{x^{(1)}}$. Let L denote the portion of the gradient line of $p_{\widehat{u_1}}$ which connects $\widehat{x^{(1)}}$ with $\widehat{x^{(0)}}$. Let L_0 denote the straight line path from $\widehat{x^{(0)}}$ to $\widetilde{x^{(0)}}$. By Item 1 in the choice of ρ_1, Item 5 in the choice of δ_1, and Item 3 in the choice of ν_1, it can be shown that this path from $\widetilde{x^{(1)}}$ to $\widetilde{x^{(0)}}$ does not intersect any critical level curve of p_{u_1} other than $\widetilde{\lambda_{\widetilde{D}}}$. Therefore we can project this path along gradient lines to a path $\sigma : [0,1] \to \widetilde{\lambda_{\widetilde{D}}}$ from $\widetilde{x^{(1)}}$ to $\widetilde{x^{(0)}}$. Then it can easily be shown that either $\widetilde{x^{(1)}} = \widetilde{x^{(0)}}$, or there is some $r \in (0,1)$ such that $\sigma(r)$ is a critical point of p_{u_1} or $\arg(p_{u_1}(\sigma(r))) = \arg(p_{u_1}(x^{(1)}) + \pi$. However by Item 8 in the choice of δ_1, Item 1 in the choice of δ_2, Item 5 in the choice of ρ_1, and Item 3 in the choice of ν_1, no such r exists, so we conclude that $\widetilde{x^{(1)}} = \widetilde{x^{(0)}}$.

Now choose some $k_0 \in \{1, \ldots, L\}$. We will now find an edge $\widetilde{E^{(k_0)}}$ of $\widetilde{\lambda_{\widetilde{D}}}$ which corresponds to the edge $E^{(k_0)}$ in λ_D.

Case 6.0.12. *$x^{(1)}$ is not an end point of $E^{(k_0)}$.*

Let $i, j \in \{1, \dots, L\}$ be the indices such that $x^{(i)}$ is the initial point of $E^{(k_0)}$ and $x^{(j)}$ is the final point of $E^{(k_0)}$. Recall that as we are viewing $\langle\widehat{\Lambda}\rangle_{PC}$ as embedded in $\mathbb{C}$ as the critical level curves of $p_{\widehat{u_1}}$, we have $\widehat{x^{(i)}} = \widehat{u_1{}^{(t_i)}}$ and $\widehat{x^{(j)}} = \widehat{u_1{}^{(t_j)}}$. And $a(x^{(i)}) = a(\widehat{x^{(i)}})$ and $a(x^{(j)}) = a(\widehat{x^{(j)}})$ by the construction of $\langle\widehat{\Lambda}\rangle_{PC}$. Thus

$$a(x^{(i)}) = a(\widehat{x^{(i)}}) = \arg(p_{\widehat{u_1{}^{(1)}}}(\widehat{u_1{}^{(t_i)}})),$$

and similarly

$$a(x^{(j)}) = a(\widehat{x^{(j)}}) = \arg(p_{\widehat{u_1}}(\widehat{u_1{}^{(t_j)}})).$$

Assume that $a(x^{(i)}) = 0$ (otherwise make the appropriate minor changes throughout the argument). Let $\Delta > 0$ denote the change in $\arg(p_{\widehat{u_1}})$ along $\widehat{E^{(k_0)}}$, and let $\widehat{\gamma}$ be a parameterization of $\widehat{E^{(k_0)}}$ according to $\arg(p_{\widehat{u_1}})$. That is, $\widehat{\gamma} : [0, \Delta] \to \widehat{E^{(k_0)}}$, and for each $r \in [0, \Delta]$, $\arg(p_{\widehat{u_1}}(\widehat{\gamma}(r))) = r$. By Item 3 in the choice of ρ_1, we may find a path $\gamma : [0, \Delta] \to \widetilde{\lambda_{\widetilde{D}}}$ such that $\gamma(0) = u_1{}^{(t_i)}$, and for each $r \in [0, \Delta]$, $\arg(p_{u_1}(\gamma(r))) = r$, and $|\gamma(r) - \widehat{\gamma}(r)| < \delta_2$.

Now $\widehat{\gamma}(\Delta) = \widehat{u_1{}^{(t_j)}}$, so $\Delta = \arg(\widehat{v_1{}^{(t_j)}}) \bmod 2\pi$. Therefore $p_{u_1}(\gamma(\Delta)) = |v_1{}^{(t_j)}|e^{i \arg(v_1{}^{(t_j)})} = v_1{}^{(t_j)}$. Moreover,

$$|\gamma(\Delta) - u_1{}^{(t_j)}| \leq |\gamma(\Delta) - \widehat{\gamma}(\Delta)| + |\widehat{\gamma}(\Delta) - \widehat{u_1{}^{(t_j)}}| + |\widehat{u_1{}^{(t_j)}} - u_1{}^{(t_j)}|.$$

However $\widehat{u_1{}^{(t_j)}} = \widehat{\gamma}(\Delta)$, so we have

$$|\gamma(\Delta) - u_1{}^{(t_j)}| \leq |\gamma(\Delta) - \widehat{\gamma}(\Delta)| + |\widehat{u_1{}^{(t_j)}} - u_1{}^{(t_j)}| < \delta_2 + \rho_1 < 2\delta_2.$$

By Item 6 in the choice of δ_1 and Item 1 in the choice of δ_2, there is no point in $B_{2\delta_2}(u_1{}^{(t_j)}) \setminus \{u_1{}^{(t_j)}\}$ at which p_{u_1} takes the value $v_1{}^{(t_j)}$, we conclude that $\gamma(\Delta) = u_1{}^{(t_j)}$. Therefore we have that γ is a path from $u_1{}^{(t_i)}$ to $u_1{}^{(t_j)}$ through $\widetilde{\lambda_{\widetilde{D}}}$. We now wish to show that the image of γ consists of a single edge of $\widetilde{\lambda}$. That is, we wish to show that for any $r \in (0, \Delta)$, $\gamma(r)$ is not a critical point of p_{u_1}.

Suppose by way of contradiction that there is some $r^{(0)} \in (0, \Delta)$ such that $\gamma(r^{(0)})$ is a critical point of p_{u_1}. Therefore we have $r_0, 0, \Delta \in \{\arg(v_1), \ldots, \arg(v_{N-1})\}$, and thus both r_0 and $\Delta - r_0$ are greater than $d_{\arg}(1, {v_1}^{(1)}, \ldots, {v_1}^{(N-1)})$. Since by Item 8 in the choice of δ_1 and Item 1 in the choice of δ_2, $\delta_2 < \frac{\operatorname{minmod}(v_1) d_{\arg}(1, {v_1}^{(1)}, \ldots, {v_1}^{(N-1)})}{2\pi}$, we conclude that both $r^{(0)}$ and $\Delta - r^{(0)}$ are greater than $\frac{\pi \delta_2}{\operatorname{minmod}(v_1)}$.

Choose some $l_0 \in \{1, \ldots, N-1\}$ so that $\gamma(r^{(0)}) = {u_1}^{(l_0)}$. We now wish to show that ${u_1}^{(l_0)} \in \widetilde{\lambda_{\widetilde{D}}}$. By construction of $\langle \widehat{\Lambda} \rangle_{PC}$, there is only one critical level curve of $p_{\widehat{u_1}}$ in $\widehat{D}$ on which $|p_{\widehat{u_1}}| = \epsilon$, namely $\widehat{\lambda_D \setminus E^{(1)}}$. If $l_0 \neq M$, then $\widehat{{v_1}^{(l_0)}} = {v_1}^{(l_0)}$, so $|\widehat{{v_1}^{(l_0)}}| = |{v_1}^{(l_0)}| = \epsilon$. Thus we conclude that either $l_0 = M$ or $\widehat{{u_1}^{(l_0)}} \in \widehat{\lambda_D \setminus E^{(1)}}$. Therefore there is some $k \in \{1, \ldots, K\}$ such that $\widehat{{u_1}^{(l_0)}} = \widehat{x^{(k)}}$, and therefore, as shown earlier, ${u_1}^{(l_0)} \in \widetilde{\lambda_{\widetilde{D}}}$.

Let L denote the straight line segment from $\widehat{{u_1}^{(l_0)}}$ to $\widehat{\gamma}(r^{(0)})$. Since $\gamma(r^{(0)}) = {u_1}^{(l_0)}$, we have

$$|\widehat{{u_1}^{(l_0)}} - \widehat{\gamma}(r_0)| \leq |\widehat{{u_1}^{(l_0)}} - \gamma(r_0)| + |\gamma(r_0) - \widehat{\gamma}(r_0)| = |\widehat{{u_1}^{(l_0)}} - {u_1}^{(l_0)}| + |\gamma(r_0) - \widehat{\gamma}(r_0)|.$$

So

$$|\widehat{{u_1}^{(l_0)}} - \widehat{\gamma}(r_0)| < \rho_1 + \delta_2 < 2\delta_2.$$

Therefore $L \subset B_{2\delta_2}(\widehat{{u_1}^{(l_0)}})$. Now for all $z \in L$,

$$|p_{\widehat{u_1}}(z) - \widehat{{v_1}^{(l_0)}}| \leq |p_{\widehat{u_1}}(z) - p_{u_1}(z)| + |p_{u_1}(z) - {v_1}^{(l_0)}| + |{v_1}^{(l_0)} - \widehat{{v_1}^{(l_0)}}|,$$

and

$$|z - {u_1}^{(l_0)}| \leq |z - \widehat{{u_1}^{(l_0)}}| + |\widehat{{u_1}^{(l_0)}} - {u_1}^{(l_0)}| < 2\delta_2 + \rho_1 < 3\delta_2.$$

So by Item 2 in the choice of δ_2 and Item 4 in the choice of ρ_1, we have

$$|p_{\widehat{u_1}}(z) - \widehat{v_1{}^{(l_0)}}| < \delta_1 + \delta_1 + \frac{\nu_1}{2} < 3\delta_1.$$

Therefore the path $p_{\widehat{u_1}}(L)$ is contained in $B_{3\delta_1}(\widehat{v_1{}^{(l_0)}})$, and $|\widehat{v_1{}^{(l_0)}}| \geq \text{minmod}(v_1)$, so by Item 1 in the choice of δ_1, we conclude that the net change in argument of $p_{\widehat{u_1}}$ along L is less than $\frac{d_{\arg}(1,v_1)}{4}$.

As noted above, $r^{(0)}$ and $\Delta - r^{(0)}$ are both greater than $d_{\arg}(1, v_1)$. Therefore, we can choose some $s_1 \in (0, r_0)$, and some $s_2 \in (r_0, \Delta)$, each of which is greater than $\frac{d_{\arg}(1,v_1)}{2}$ away from each member of $\{\arg(v_1{}^{(1)}), \ldots, \arg(v_1{}^{(N-1)})\}$. Therefore $\widehat{\gamma}(s_1), \widehat{\gamma}(s_2) \notin L$. We now consider the set $\widehat{\lambda_D \setminus E^{(1)}} \cup L$ as a graph whose vertices are the vertices of $\widehat{\lambda_D \setminus E^{(1)}}$ along with any intersections of $\widehat{\lambda_D \setminus E^{(1)}}$ and L. (Note that the smoothness of the edges of $\widehat{\lambda_D \setminus E^{(1)}}$ and the smoothness of L imply that $\widehat{\lambda_D \setminus E^{(1)}}$ and L intersect at only finitely many places.) Let K_1 and K_2 denote the edges of $\widehat{\lambda_D \setminus E^{(1)}} \cup L$ which contain $\widehat{\gamma}(s_1)$ and $\widehat{\gamma}(s_2)$ respectively. Since L intersects $\widehat{E^{(k_0)}}$ at $\widehat{\gamma}(r_0)$, we conclude that $\widehat{\gamma}(s_1)$ and $\widehat{\gamma}(s_2)$ are in different edges of $\widehat{\lambda_D \setminus E^{(1)}} \cup L$, so $K_1 \neq K_2$.

Let G_0 denote the bounded face of $\widehat{\lambda_D \setminus E^{(1)}}$ such that $\widehat{E^{(k_0)}}$ is contained in ∂G_0.

Subcase 6.0.12.1. *Both K_1 and K_2 are adjacent to the unbounded face of $\widehat{\lambda_D \setminus E^{(1)}} \cup L$.*

Let D_1 and D_2 denote the bounded faces of $\widehat{\lambda_D \setminus E^{(1)}} \cup L$ which are adjacent to K_1 and K_2 respectively. Then in this case, L must intersect some portion of $\partial G_0 \setminus \widehat{\gamma}(s_1, s_2)$, and $D_1 \neq D_2$. By choice of s_j, the change in argument of $p_{\widehat{u_1}}$ along K_j is greater than $\frac{d_{\arg}(1,v_1)}{2}$ for $j = 1, 2$. Let $\Gamma_{\widehat{\lambda_D \setminus E^{(1)}}}$ denote the portion of ∂D_1 which is contained in $\widehat{\lambda_D \setminus E^{(1)}}$. Let Γ_L denote the portion of ∂D_1 which is contained in L. Since the argument of $p_{\widehat{u_1}}$ is strictly increasing on $\widehat{\lambda_D \setminus E^{(1)}}$, and $K_1 \subset \widehat{\lambda_D \setminus E^{(1)}}$, the net change in the argument of $p_{\widehat{u_1}}$ along $\Gamma_{\widehat{\lambda_D \setminus E^{(1)}}}$ is greater than $\frac{d_{\arg}(1,v_1)}{2}$. And since the total variation of $\arg(p_{\widehat{u_1}})$ on L is less than $\frac{d_{\arg}(1,v_1)}{2}$, the magnitude of the net change in $\arg(p_{\widehat{u_1}})$ along Γ_L is less than $\frac{d_{\arg}(1,v_1)}{2}$. Therefore we conclude that the net change in $\arg(p_{\widehat{u_1}})$ along ∂D_1 is greater than zero. We conclude that D_1 contains a zero of $p_{\widehat{u_1}}$. The exactly similar argument implies that D_2 contains a zero of $p_{\widehat{u_1}}$. Let z_1 denote one of the zeros of $p_{\widehat{u_1}}$ in D_1, and let z_2 denote one of the zeros of $p_{\widehat{u_1}}$ in D_2. Then Theorem 2.1 implies that there is a critical level curve $\widehat{\xi}$ of $p_{\widehat{u_1}}$ in G_0, such that each of z_1 and z_2 is in a bounded face of $\widehat{\xi}$. Then $\widehat{\xi}$ intersects D_1 and ${D_1}^c$ (because $z_2 \notin \text{sc}(D_1)$), and thus $\widehat{\xi}$ intersects D_1. $\widehat{\xi}$ also intersects $D_2 \subset {D_1}^c$, and is connected, thus $\widehat{\xi}$ intersects ∂D_1. However $\widehat{\xi}$ does not intersect $\lambda_{\widehat{D \setminus E^{(1)}}}$, so therefore we conclude that $\widehat{\xi}$ intersects L. We have already shown that for all $z \in L$, $|p_{\widehat{u_1}}(z) - \widehat{{v_1}^{(l_0)}}| < 3\delta_1$. Therefore the value that $|p_{\widehat{u_1}}|$ takes on $\widehat{\xi}$ is contained in $(\epsilon - 3\delta_1, \epsilon + 3\delta_1)$. By the Maximum Modulus Principle, since $\widehat{\xi} \subset G_0$, the value that $|p_{\widehat{u_1}}|$ takes on $\widehat{\xi}$ is contained in $(\epsilon - 3\delta_1, \epsilon)$. However by the construction of $\langle\widehat{\Lambda}\rangle_{PC}$ and Item 9 in the choice of δ_1, no critical values of $p_{\widehat{u_1}}$ have moduli in $(\epsilon - 3\delta_1, \epsilon)$, which supplies us with our contradiction.

Subcase 6.0.12.2. *One of K_1 or K_2 is not adjacent to the unbounded face of $\lambda_{\widehat{D \setminus E^{(1)}}} \cup L$.*

Assume that K_1 is not adjacent to the unbounded face of $\lambda_{\widehat{D \setminus E^{(1)}}} \cup L$. Since K_1 is adjacent to the unbounded face of $\lambda_{\widehat{D \setminus E^{(1)}}}$, one of the faces of $\lambda_{\widehat{D \setminus E^{(1)}}} \cup L$ which is adjacent to K_1 is contained in the unbounded face of $\lambda_{\widehat{D \setminus E^{(1)}}}$. Let D_1 denote this face. Let $\Gamma_{\lambda_{\widehat{D \setminus E^{(1)}}}}$ denote the portion of ∂D_1 which is contained in $\lambda_{\widehat{D \setminus E^{(1)}}}$. Let Γ_L denote the portion of ∂D_1 which is contained in L. Then since D_1 is contained on the unbounded portion of $\widehat{\Lambda}$, $\arg(p_{\widehat{u_1}})$ is strictly decreasing as $\Gamma_{\lambda_{\widehat{D \setminus E^{(1)}}}}$ is traversed with positive orientation, and the change in $\arg(\widehat{p_{u_1}})$ as K_1 is traversed as a portion of ∂D_1 (thus with the opposite orientation as before) is less than $-\frac{d_{\arg}(1,v_1)}{2}$. Therefore the net change in $\arg(p_{\widehat{u_1}})$ as $\Gamma_{\lambda_{\widehat{D \setminus E^{(1)}}}}$ is traversed is less than $-\frac{d_{\arg}(1,v_1)}{2}$. And since the total variation of $\arg(p_{\widehat{u_1}})$ along L is less than $\frac{d_{\arg}(1,v_1)}{2}$, the net change in $\arg(p_{\widehat{u_1}})$ along Γ_L is less than $\frac{d_{\arg}(1,v_1)}{2}$ in magnitude. Therefore the net change in $\arg(p_{\widehat{u_1}})$ along ∂D_1 is strictly less than 0. Therefore there is a pole of $p_{\widehat{u_1}}$ in D_1, which is obviously a contradiction as $p_{\widehat{u_1}}$ is a polynomial.

Therefore by the previous two subcases, we conclude that there is no $r \in (0, \Delta)$ such that $\gamma(r)$ is a critical point of p_{u_1}. Therefore $\gamma(0, \Delta)$ is an edge of $\widetilde{\lambda_{\widetilde{D}}}$. Let $\widetilde{E^{(k_0)}}$ denote this edge.

Case 6.0.13. *$x^{(1)}$ is an end point of $E^{(k_0)}$.*

In this case, there are three subcases to consider, two of which are essentially identical to the previous case, and which we list first.

Subcase 6.0.13.1. *$k_0 = 1$ (so $E^{(k_0)} = \partial F$).*

Subcase 6.0.13.2. *$k_0 \neq 1$, but $x^{(1)}$ is still a vertex of $\lambda_D \setminus E^{(1)}$.*

For the previous two cases, both end points of $\widehat{E^{(k_0)}}$ are critical points of $p_{\widehat{u}_1}$, and $\widehat{E^{(k_0)}}$ is completely contained in a single critical level curve of $p_{\widehat{u}_1}$. ($\widehat{\lambda_{\widehat{D}}}$ for Subcase 6.0.13.1 and $\widehat{\lambda_D \setminus E^{(1)}}$ in Subcase 6.0.13.2.) Therefore the method of Case 6.0.12 may be applied with minor changes, and the conclusion is that if $i, j \in \{0, \dots, K\}$ are the indices such that $x^{(i)}$ is the initial vertex of $E^{(k_0)}$ and $x^{(j)}$ is the final point of $E^{(k_0)}$, then there is a path $\widehat{\gamma}$ which parameterizes $\widehat{E^{(k_0)}}$ with respect to $\arg(p_{\widehat{u}_1})$ and an edge $\widetilde{E^{(k_0)}}$ of $\widetilde{\lambda_{\widetilde{D}}}$ from $\widetilde{x^{(i)}}$ to $\widetilde{x^{(j)}}$ which we call $\widetilde{E^{(k_0)}}$ with a parameterization $\widetilde{\gamma}$ which parameterizes $\widetilde{E^{(k_0)}}$ with respect to $\arg(p_{u_1})$ such that for each r, $|\widehat{\gamma}(r) - \widetilde{\gamma(r)}| < \delta_2$.

Subcase 6.0.13.3. *$k_0 \neq 1$ and $x^{(1)}$ is not a vertex of $\lambda_D \setminus E^{(1)}$.*

Since $k_0 \neq 1$, $E^{(k_0)}$ does not form ∂F. As noted earlier, the fact that $\langle \widehat{\lambda_{\widehat{D}}} \rangle_{PC}$ was formed using the scattering method implies that both end points of $E^{(k_0)}$ are not at $x^{(1)}$. Let $i \in \{2, \dots, K\}$ be the index so that $x^{(i)}$ is the other end point of $E^{(k_0)}$. Assume during the following argument that $x^{(i)}$ is the initial point of $E^{(k_0)}$ (otherwise make the appropriate minor changes, such as reversing orientations of paths, etc.). Let $\Delta \in \mathbb{R} \setminus \{0\}$ denote change in argument along $E^{(k_0)}$ from $x^{(i)}$ to $x^{(1)}$. Assume that $a(x^{(i)}) = 0$ (otherwise make the appropriate minor changes). Let E denote the edge of $\lambda_D \setminus E^{(1)}$ which contains $x^{(1)}$ (which is no longer a vertex of $\lambda_D \setminus E^{(1)}$). Let $\widehat{E}$ denote the edge of $\widehat{\lambda_D \setminus E^{(1)}}$ which corresponds to E. Recall that $\widehat{E^{(k_0)}}$ is the portion of $\widehat{E}$ formed in the following way. Let $\Delta^{(1)}$ denote the change in argument of $p_{\widehat{u}_1}$ along $\widehat{E}$ beginning at

$\widehat{x^{(i)}}$. Let $\widehat{\gamma^{(1)}}$ be a parameterization of $\widehat{E}$ according to $\arg(p_{\widehat{u_1}})$ and beginning at $\widehat{x^{(i)}}$. So $\widehat{\gamma^{(1)}} : [0, \Delta^{(1)}]$, and $\arg(p_{\widehat{u_1}}(\widehat{\gamma}(r))) = r$ for each $r \in [0, \Delta^{(1)}]$. Then we define $\widehat{E^{(k_0)}}$ to be $\widehat{\gamma^{(1)}}([0, \Delta])$. Let $\widehat{\gamma}$ denote the path $\widehat{\gamma^{(1)}}$ restricted to $[0, \Delta]$.

Again by Item 3 in the choice of ρ_1, there is a path $\gamma : [0, \Delta] \to \widetilde{\lambda_{\widetilde{D}}}$ such that $\gamma(0) = {u_1}^{(t_i)}$, for each $r \in [0, \Delta]$ $\arg(p_{u_1}(\gamma(r))) = r$ and $|\gamma(r) - \widehat{\gamma}(r)| \leq \delta_1$. The argument for this subcase is very similar to the argument for Case 6.0.12. The major difference is in the method by which we show that $\gamma(\Delta) = \widetilde{{u_1}^{(t_1)}}$. Our method here is similar to the way in which we showed that $\widetilde{x^{(0)}} = \widetilde{x^{(1)}}$ (when $\widetilde{x^{(0)}}$ is defined).

Since the image of γ is contained in $\widetilde{\lambda_{\widetilde{D}}}$, we conclude that $|p_{u_1}(\gamma(\Delta))| = |p_{u_1}(\widetilde{x^{(1)}})|$. By definition of γ, $\arg(p_{u_1}(\gamma(\Delta))) = \Delta$. And by definition of Δ, $\arg(p_{u_1}(x^{(1)})) = a(\widetilde{x^{(1)}}) = \Delta$ (since we are assuming that $\arg(p_{u_1}(\widetilde{x^{(i)}})) = 0$). Therefore $p_{u_1}(\gamma(\Delta)) = p_{u_1}(x^{(1)})$. Let L_1 denote the straight line path from $\widetilde{x^{(1)}}$ to $\widehat{x^{(1)}}$. Let L denote the portion of the gradient line of $p_{\widehat{u_1}}$ which connects $\widehat{x^{(1)}}$ to $\widehat{\gamma}(\Delta)$. Let L_2 denote the straight line path from $\widehat{\gamma}(\Delta)$ to $\gamma(\Delta)$. Let L_1, L, L_2 denote the path obtained by concatinating the three paths L_1, L, and L_2. By Item 5 in the choice of δ_1, Item 1 in the choice of δ_2, Item 1 in the choice of ρ_1, and Item 3 in the choice of ν_1, it can be shown (by considering the value of $|p_{u_1}|$ on L_1, L, L_2) that the path L_1, L, L_2 does not intersect any critical level curve of p_{u_1} other than $\widetilde{\lambda_{\widetilde{D}}}$. Therefore we can project this path along gradient lines to a path $\sigma : [0, 1] \to \widetilde{\lambda_{\widetilde{D}}}$ from $\widetilde{x^{(1)}}$ to $\gamma(\Delta)$. Then it can easily be shown that either $\widetilde{x^{(1)}} = \gamma(\Delta)$, or there is some $r \in (0, 1)$ such that $\sigma(r)$ is a critical point of p_{u_1} or $\arg(p_{u_1}(\sigma(r))) = \arg(p_{u_1}(x^{(1)})) + \pi$. However by Item 8 in the choice of δ_1, Item 1 in the choice of δ_2, Item 5 in the choice of ρ_1, and Item 3, no such r can exist, so we conclude that ${u_1}^{(t_1)} = \widetilde{x^{(1)}} = \gamma(\Delta)$. The rest of the argument for this subcase is essentially the same as for Case 6.0.11.2, so we omit it.

The conclusion which we draw is as before. Namely, there is a path $\widehat{\gamma}$ which parameterizes $\widehat{E^{(k_0)}}$ according to $\arg(p_{\widehat{u_1}})$ and an edge $\widetilde{E^{(k_0)}}$ of $\widetilde{\lambda_{\widetilde{D}}}$ from $\widetilde{x^{(i)}}$ to $\widetilde{x^{(1)}}$ with a parameterization γ which parameterizes $\widetilde{E^{(k_0)}}$ according to $\arg(p_{u_1})$ such that for each r,

$|\widehat{\gamma}(r)-\gamma(r)|<\delta_2$. Now for each $k \in \{1, \ldots, L\}$, let $\widetilde{\gamma^{(k)}}$ be the path which parameterizes $\widetilde{E^{(k)}}$. Let $\widehat{\gamma^{(k)}}$ be the path which parameterizes $\widehat{E^{(k)}}$.

We now wish to show that the order in which the edges $\widetilde{E^{(1)}}, \ldots, \widetilde{E^{(L)}}$ appear around $\widetilde{\lambda_{\widetilde{D}}}$ is the same as the order of their corresponding edges around λ_D.

Now it does not quite make sense to speak of the order in which the edges $\widehat{E^{(1)}}, \ldots, \widehat{E^{(L)}}$ appear, because these edges are not all contained in a single level curve of $p_{\widehat{u}_1}$. However there is some way to make sense of the order of appearance of $\widehat{E^{(1)}}, \ldots, \widehat{E^{(L)}}$. Begin with $\widehat{E^{(1)}}$. Now for each $k \in \{2, \ldots, L\}$, $\widehat{E^{(k)}}$ is contained in $\widehat{\lambda_D \setminus E^{(1)}}$. Select one point $z^{(k)}$ in $\widehat{E^{(k)}}$ which is not an end point of $\widehat{E^{(k)}}$.

Define for the moment $y^{(k)}$ to be the point in $\partial \widehat{G}$ which connects to $z^{(k)}$ by a gradient line of $p_{\widehat{u}_1}$. Then by definition of the edges $\widehat{E^{(2)}}, \ldots, \widehat{E^{(L)}}$, the order in which the points $y^{(2)}, \ldots, y^{(L)}$ appear around $\partial \widehat{G}$ is exactly $y^{(2)}, \ldots, y^{(L)}$. Therefore, by the construction of $\langle\widehat{\Lambda}\rangle_{PC}$, if we begin at $\widehat{x^{(1)}}$, and we travel down the gradient line containing $x^{(1)}$ into $\widehat{G}$ until we reach $\widehat{\lambda_D \setminus E^{(1)}}$, and begin traversing $\widehat{\lambda_D \setminus E^{(1)}}$, the order in which we traverse the edges of $\widehat{\lambda_D \setminus E^{(1)}}$ is exactly $\widehat{E^{(2)}}, \ldots \widehat{E^{(L)}}$.. This is the sense in which we will say that the edges $\widehat{E^{(1)}}, \ldots, \widehat{E^{(L)}}$ appear in the order $\widehat{E^{(1)}}, \ldots, \widehat{E^{(L)}}$.

We will make further use of the process just described of "parameterizing" the order in which the edges of a member of $\check{P}$ appear by points contained in the boundary of a region which contains this member of $\check{P}$. Let us first describe this process more precisely. Let ξ denote a member of $\check{P}$, and let n denote the number of edges $e^{(1)}, \ldots, e^{(n)}$ of ξ. Choose some simple closed path such that the bounded face G of the path contains ξ in its bounded face, and choose n distinct points in ∂G. For each edge e in ξ, draw a path from a point in e which is not an end point of e to one of the chosen points in ∂G. Do this in such a way that the paths are contained in the portion of G exterior to ξ except at the endpoints, so that each edge connects to a different point in ∂G, and so that they do not intersect. For each $i \in \{1, \ldots, n\}$, let $y^{(i)}$ denote the point in

∂G which is connected to $e^{(i)}$. Then the orientation of the edges $e^{(1)}, \dots, e^{(n)}$ in ξ is the same as the orientation of the points $y^{(1)}, \dots, y^{(n)}$ in ∂G.

Recall that by Item 4 in the choice of δ_1, there is a choice of points $z^{(1)}, \dots, z^{(L)}$ such that for each $k \in \{1, \dots, L\}$, the following holds.

- $z^{(k)}$ is in $\widetilde{E^{(k)}}$ but is not an end point of $\widetilde{E^{(k)}}$.
- $\arg(p_{u_1}(z^{(k)}))$ is more than $\frac{d_{\arg}(1, v_1)}{4}$ away from each of $\{\arg({v_1}^{(1)}), \dots, \arg({v_1}^{(N-1)})\}$.
- $z^{(k)}$ is more than $2\delta_1$ away from each critical point of p_{u_1}.

Now for each $k \in \{1, \dots, L\}$, let $\sigma^{(k)} : [0, 1] \to \mathbb{C}$ be a parameterization of the portion of the gradient line of p_{u_1} which connects $z^{(k)}$ to a point in $\partial\widetilde{D}$. Let $y^{(k)}$ denote this point in $\partial\widetilde{D}$. Then recall that δ_1 is chosen so that for any $j, k \in \{1, \dots, L\}$ with $j \neq k$, and for any $s, t \in [0, 1]$, $|\sigma^{(j)}(s) - \sigma^{(k)}(t)| > 2\delta_1$, and there is no edge of a critical level curve of p_{u_1} other than $E^{(j)}$ within $2\delta_1$ of $\sigma^{(j)}(s)$. Let $\sigma^{(k)}$ parameterize this gradient line so that $\sigma^{(k)} : [0, 1] \to \mathbb{C}$ with $\sigma^{(k)}(0) = y^{(k)}$ and $\sigma^{(k)}(1) = z^{(k)}$.

Define $i_1 := 1$ and choose distinct indices $i_2, \dots, i_L \in \{2, \dots, L\}$ so that the order in which the edges of $\widetilde{\lambda_{\widetilde{D}}}$ appear around $\widetilde{\lambda_{\widetilde{D}}}$ is $\widetilde{E^{(i_1)}}, \dots, \widetilde{E^{(i_L)}}$. Now by Item 5 in the choice of δ_1 and Item 1 in the choice of δ_2, $\widehat{\lambda_{\widehat{D}}} \subset \widetilde{D}$. We are now going to alter each $\sigma^{(k)}$ so that it is a path from $y^{(k)}$ to a point in $\widehat{E^{(k)}}$. By Item 1 in the choice of δ_2, there is no point in the path $\sigma^{(k)}$ which intersects $\widehat{E^{(l)}}$ for any $l \neq k$. If there is any $s \in [0, 1]$ such that $\sigma^{(k)}(s) \in \widehat{E^{(k)}}$, then let s_0 denote the smallest such s. Then define $\widehat{\sigma^{(k)}}$ to be the restriction of $\sigma^{(k)}$ to $[0, s_0]$. If there is no such s, let $\Delta^{(k)}$ denote the change in $\arg(p_{u_1})$ along $\widetilde{E^{(k)}}$ and let $i \in \{1, \dots, K\}$ denote the index such that $\widetilde{x^{(i)}}$ is the initial point of $\widetilde{E^{(k)}}$. Let $r_1 \in [a(\widetilde{x^{(i)}}), a(\widetilde{x^{(i)}}) + \Delta^{(k)}]$ be chosen so that $\widetilde{\gamma^{(k)}}(r_1) = z^{(k)}$. Let $L^{(k)}$ denote the straight line path from $\widetilde{\gamma^{(k)}}(r_1)$ to $\widehat{\gamma^{(k)}}(r_1)$. Let $\sigma^{(k)}L^{(k)}$ denote the path obtained by first traversing $\sigma^{(k)}$, and then traversing $L^{(k)}$ from $z^{(k)}$ to $\widehat{\gamma^{(k)}}(r_1)$. Then $\sigma^{(k)}L^{(k)}$ is a path from $y^{(k)}$ to $\widehat{\gamma^{(k)}}(r_1) \in \widehat{E^{(k)}}$. Let s_0 be the smallest number in the domain of the path $\sigma^{(k)}L^{(k)}$ such that $\sigma^{(k)}L^{(k)}(s_0) \in \widehat{E^{(k)}}$. Then define $\widehat{\sigma^{(k)}}$ to be this path $\sigma^{(k)}L^{(k)}$ restricted to $[0, s_0]$. Now since $L^{(k)} \subset B_{\delta_1}(z^{(k)})$, by Item 4 in the choice of δ_1, $\widehat{\sigma^{(k)}}$ does not intersect $\sigma^{(l)}$

for any $l \neq k$, and it can be shown that $L^{(k)}$ does not intersect the gradient line which connects $x^{(1)}$ to $\widehat{\lambda_D \setminus E^{(1)}}$. Define $\widehat{z^{(k)}} := \widehat{\sigma^{(k)}}(s_0)$.

By definition of the $\sigma^{(k)}$, the order in which the points $y^{(1)}, \ldots, y^{(L)}$ appear around $\partial\widetilde{D}$ beginning with $y^{(1)}$ is $y^{(1)} = y^{(i_1)}, y^{(i_2)}, \ldots, y^{(i_L)}$. We wish to show that for each $k \in \{2, \ldots, L\}$, $i_k = k$. Recall that if one begins at $x^{(k)}$, traverses the gradient line down to $\widehat{\lambda_D \setminus E^{(1)}}$, and begins traversing $\widehat{\lambda_D \setminus E^{(1)}}$ with positive orientation, the first edge one traverses is $\widehat{E^{(2)}}$. Therefore the first point of the set $\{\widehat{z^{(2)}}, \ldots, \widehat{z^{(L)}}\}$ which one encounters while traversing $\widehat{\lambda_D \setminus E^{(1)}}$ is $\widehat{z^{(2)}}$. Now consider the path one obtains by traversing $\widehat{\sigma^{(1)}}$ from $y^{(1)}$ to $\widehat{z^{(1)}}$, traversing $\widehat{E^{(1)}}$ with positive orientation from $\widehat{z^{(1)}}$ to $\widehat{x^{(1)}}$, traversing the gradient line of $p_{\widehat{u_1}}$ from $\widehat{x^{(1)}}$ to the point where it intersects $\widehat{\lambda_D \setminus E^{(1)}}$ (let us call that point z for the moment), traversing $\widehat{E^{(2)}}$ from z to $\widehat{z^{(2)}}$, and finally traversing $\widehat{\sigma^{(2)}}$ from $\widehat{z^{(2)}}$ to $y^{(2)}$. Let σ denote this path. Since no $\widehat{z^{(l)}}$ is in this path for $l \notin \{1, 2\}$, and no $\sigma^{(l)}$ for $l \notin \{1, 2\}$ can intersect this path, we conclude that if one traverses $\partial\widetilde{D}$ from $y^{(1)}$ to $y^{(2)}$ with positive orientation, then one does not encounter any $y^{(l)}$ for $l \notin \{1, 2\}$. Therefore since i_2 is the index such that $y^{(i_2)}$ is the next point in $\partial\widetilde{D}$ after $y^{(1)}$ (with respect to a positive orientation), therefore we conclude that $i_2 = 2$. The same argument gives us that for each $k \in \{3, \ldots, L\}$, $i_k = k$. Therefore the order in which the edges $\widetilde{E^{(1)}}, \ldots, \widetilde{E^{(L)}}$ appear around $\widetilde{\lambda_{\widetilde{D}}}$ is exactly $\widetilde{E^{(1)}}, \ldots, \widetilde{E^{(L)}}$. Therefore the edges $\widetilde{E^{(k)}}$ appear in the same order around $\widetilde{\lambda_{\widetilde{D}}}$ as the corresponding edges $E^{(k)}$ appear around λ_D.

Let G denote for the moment either the face $\widehat{F}$ of $\widehat{\lambda_{\widehat{D}}}$ or one of the bounded faces of $\widehat{\lambda_D \setminus E^{(1)}}$. Let $n \geq 1$ be the number of edges of $\widehat{\lambda}$ which are contained in ∂G. Let $i_1, \ldots, i_n \in \{1, \ldots, L\}$ be the indices such that $\widehat{E^{(i_1)}}, \ldots, \widehat{E^{(i_n)}}$ are the edges which form ∂G listed in order of their appearance around ∂G. Then $\widetilde{E^{(i_1)}}, \ldots, \widetilde{E^{(i_n)}}$ forms a simple closed path and, by the Maximum Modulus Principle, no edge of $\widetilde{\lambda_{\widetilde{D}}}$ intersects the bounded face of this path. Let $\widetilde{G}$ denote the face of $\widetilde{\lambda_{\widetilde{D}}}$ which has this path as its boundary. By definition of the paths $\widetilde{\gamma^{(i)}}$ defined earlier, the change of $\arg(p_{u_1})$ along $\partial\widetilde{G}$ is the same as the change in $\arg(p_{\widehat{u_1}})$ along ∂G. Therefore the number of zeros of

p_{u_1} in $\widetilde{G}$ is the same as the number of zeros $p_{\widehat{u_1}}$ in G. Let $m \geq 2$ denote the number of bounded faces of λ_D. Then $\widehat{\lambda_D \setminus E^{(1)}}$ has $m-1$ bounded faces. Define $\widehat{D^{(1)}} := \widehat{F}$ and let $\widehat{D^{(2)}}, \ldots, \widehat{D^{(m)}}$ be an enumeration of the bounded faces of $\widehat{\lambda_D \setminus E^{(1)}}$. Then we have $\sum_{k=1}^{m} z(\widehat{D^{(k)}}) = \sum_{k=1}^{m} z(\widetilde{D^{(k)}})$ where $z(\widehat{D^{(k)}})$ denotes the number of zeros of $p_{\widehat{u_1}}$ in $\widehat{D^{(k)}}$ and $z(\widetilde{D^{(k)}})$ denotes the number of zeros of p_{u_1} in $\widetilde{D^{(k)}}$. Therefore the number of zeros of p_{u_1} contained in the bounded faces of $\widetilde{\lambda_{\widetilde{D}}}$ is greater than or equal to the number of zeros of $p_{\widehat{u_1}}$ in the bounded faces of $\widehat{\lambda_{\widehat{D}}}$. However by definition of $\widehat{\lambda_{\widehat{D}}}$ and the map Π, each zero of $p_{\widehat{u_1}}$ which is contained in $\widehat{D}$ is contained in one of $\widehat{D^{(1)}}, \ldots, \widehat{D^{(m)}}$. Moreover, by the same argument as above it may easily be shown that there are the same number of zeros of $p_{\widehat{u_1}}$ in $\widehat{D}$ as the number of zeros of p_{u_1} in $\widetilde{D}$. These two facts together imply that for each bounded face $\widetilde{G}$ of $\widetilde{\lambda_{\widetilde{D}}}$, $\widetilde{G} = \widetilde{D^{(k)}}$ for some $k \in \{1, \ldots, m\}$. Therefore $\widetilde{\lambda_{\widetilde{D}}}$ contains no edges other than $\widetilde{E^{(1)}}, \ldots, \widetilde{E^{(L)}}$.

We have already seen that for each $k \in \{1, \ldots, K\}$,

$$a(x^{(k)}) = a(\widehat{x^{(k)}}) = \arg(p_{\widehat{u_1}}(\widehat{x^{(k)}})) = \arg(p_{u_1}(\widetilde{x^{(k)}})) = a(\widetilde{x^{(k)}})$$

and, by the definition of the paths $\widetilde{\gamma^{(k)}}$ and $\widehat{\gamma^{(k)}}$, each $\widetilde{E^{(k)}}$ contains the same number of distinguished points as $\widehat{E^{(k)}}$, which contains the same number of distinguished points as $E^{(k)}$ by construction of $\langle\widehat{\Lambda}\rangle_{PC}$. We have also already seen that $H(\langle\lambda_D\rangle_P)$ equals $|{v_1}^{(t_k)}|$ for each $k \in \{1, \ldots, K\}$ and this value is in turn equal to $H(\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P)$. Therefore $\langle\lambda_D\rangle_P$ and $\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$ share all their auxiliary data, and we conclude that $\langle\lambda_D\rangle_P = \langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$.

Case 6.0.14. $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$ *was not formed using the scattering method.*

In this case the graph $\widehat{\lambda_{\widehat{D}}}$ is actually equal to λ_D as members of $\check{P}$, which removes many of the difficulties encountered in Case 6.0.11. Therefore the argument needed to show that $\langle\lambda_D\rangle_P = \langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$ and establish the correspondence between the vertices, edges, and distinguished points of $\langle\lambda_D\rangle_P$ and $\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$ is a much simplified version of that found in Case 6.0.11, so we omit it here.

Note that Case 6.0.11 and Case 6.0.14 assume that $\langle\lambda_D\rangle_P$ is a graph member of P. If $\langle\lambda_D\rangle_P$ is a single point member of P it is easy to show that $\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_P$ must be the same single point member of P by simply considering the different values that $|{v_1}^{(k)}|$ can take, and using the fact that by the construction of $\langle\widehat{\Lambda}\rangle_{PC}$, $z(D) = z(\widehat{D})$, and, as described above, $z(\widehat{D}) = z(\widetilde{D})$.

We now wish to show that the correspondence we have established between the graphs, vertices, and distinguished points of $\langle\Lambda\rangle_{PC}$ and the graphs, vertices, and distinguished points of $\langle\widetilde{\Lambda}\rangle_{PC}$ respects the gradient maps of $\langle\Lambda\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$.

As before, let $\langle\lambda\rangle_{PC}$ be a member of PC used to construct $\langle\Lambda\rangle_{PC}$. Let D be a bounded face of λ, and let $\langle\widehat{\lambda}\rangle_{PC}$, $\widehat{D}$, and $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$ and $\langle\widetilde{\lambda}\rangle_{PC}$, $\widetilde{D}$, and $\langle\widetilde{\lambda_{\widetilde{D}}}\rangle_{PC}$ be the objects for $\langle\widehat{\Lambda}\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$ which correspond to $\langle\lambda\rangle_{PC}$, D, and $\langle\lambda_D\rangle_{PC}$ respectively. Choose some edge E_1 of ∂D and let $\widehat{E_1}$ and $\widetilde{E_1}$ be the corresponding edges in $\partial\widehat{D}$ and $\partial\widetilde{D}$. Let y_1 and y_2 be the initial and final points of E_1, and let $\widehat{y_1}, \widehat{y_2}, \widetilde{y_1}$, and $\widetilde{y_2}$ be the corresponding points for $\langle\widehat{\Lambda}\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$. Let Δ_1 denote the change in argument along E_1. Assume that $a(y_1) = 0$, (otherwise make the appropriate minor changes). Then let $\widehat{\gamma^{(1)}} : [0, \Delta_1] \to \mathbb{C}$ be the path which parameterizes $\widehat{E_1}$ according to $\arg(p_{\widehat{u_1}})$. Let $\widetilde{\gamma^{(1)}} : [0, \Delta_1] \to \mathbb{C}$ be the path which parameterizes $\widetilde{E_1}$ according to $\arg(p_{\widetilde{u_1}})$. Let y be a distinguished point in E_1. Let $\widehat{y}$ and $\widetilde{y}$ be the corresponding points in $\widehat{E_1}$ and $\widetilde{E_1}$. Then by choice of $\widehat{\gamma^{(1)}}$ and $\widetilde{\gamma^{(1)}}$, $|\widehat{y} - \widetilde{y}| < \delta_2$. Define z to be the distinguished point in λ_D such that $g_D(y) = z$. Let $\widehat{z}$ and $\widetilde{z}$ be the distinguished points corresponding to z for $\langle\widehat{\Lambda}\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$. Then since $g_D(y) = z$, the goal is to show that $g_{\widetilde{D}}(\widetilde{y}) = \widetilde{z}$. Let E_2 denote one of the edges of λ_D which contains z (if z is a vertex of λ_D then it will be contained in more than one edge of λ_D). Let z_1 and z_2 be the initial and final points of E_2. Let Δ_2 denote the change in argument along E_2. Let $\widehat{\gamma^{(2)}}, \widetilde{\gamma^{(2)}} : [a(z_1), a(z_1) + \Delta_2] \to \mathbb{C}$ be the paths which parameterize $\widehat{E_2}$ and $\widetilde{E_2}$ with respect to $\arg(p_{\widehat{u_1}})$ and $\arg(p_{u_1})$ respectively. Then by choice of $\widehat{\gamma^{(2)}}$ and $\widetilde{\gamma^{(2)}}$, $|\widehat{z} - \widetilde{z}| < \delta_2$. We will show the desired result in the case where $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$ was formed using the scattering method. As before, the other cases are just simpler versions of this case.

Case 6.0.15. *$\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$ was formed using the scattering method.*

Recall that $\widehat{F}$ denotes the face of $\widehat{\lambda_{\widehat{D}}}$ to which $\langle\lambda_F\rangle_{PC}$ was assigned, and $\widehat{G}$ denotes the other face of $\widehat{\lambda_{\widehat{D}}}$. We now will define a path $\widehat{\sigma}$ from $\widehat{y}$ to $\widehat{z}$.

Subcase 6.0.15.1. *$z \in \partial F$.*

In this case $\widehat{z}$ is a distinguished point in $\partial\widehat{F}$. By definition of $\widehat{y}$ and $\widehat{z}$, $g_{\widehat{D}}(\widehat{y}) = \widehat{z}$. Therefore there is a portion of a gradient line $\widehat{\sigma} : [0, 1] \to \mathbb{C}$ of $p_{\widehat{u_1}}$ which connects $\widehat{y}$ and $\widehat{z}$ and such that $\widehat{\sigma}((0, 1))$ is contained in the portion of $\widehat{D}$ which is exterior to $\widehat{\lambda_{\widehat{D}}}$.

Subcase 6.0.15.2. *$z \notin \partial F$.*

In this case by the definition of the correspondence already established, $\widehat{z}$ is a point in an edge of $\widehat{\lambda_D \setminus E^{(1)}}$. Recall that $\langle\widehat{\lambda_D \setminus E^{(1)}}\rangle_{PC}$ has been assigned to $\widehat{G}$ during the construction of $\langle\widehat{\lambda_{\widehat{D}}}\rangle_{PC}$, and by this construction, $g_{\widehat{D}}(\widehat{y})$ is a point in $\partial\widehat{G}$, and one can show that $g_{\widehat{G}}(g_{\widehat{D}}(\widehat{y})) = \widehat{z}$. Therefore there is a portion of a gradient line $\widehat{\sigma_1} : [0, 1] \to \mathbb{C}$ of $p_{\widehat{u_1}}$ which connects $\widehat{y}$ to $g_{\widehat{D}}(\widehat{y})$, and another portion of a gradient line $\widehat{\sigma_2} : [0, 1] \to \mathbb{C}$ of $p_{\widehat{u_1}}$ which connects $g_{\widehat{D}}(\widehat{y})$ to $\widehat{z}$. Let $\widehat{\sigma}$ denote the concatenation of these two paths.

Therefore we have the desired path $\widehat{\sigma}$. By Item 3 in the choice of δ_2 and Item 2 in the choice of ρ_1, we conclude that there is a path $\sigma : [0, 1] \to \mathbb{C}$ such that $\sigma(0) = \widetilde{y}$ and $\sigma(1) = \widetilde{z}$ and, for all $r \in [0, 1]$, $\arg(p_u(\sigma(r))) = 0$ and $|\sigma(r) - \widehat{\sigma}(r)| < \delta_1$. Moreover, since $|p_{\widehat{u_1}}|$ is strictly decreasing on $\widehat{\sigma}$, we may assume that $|p_{u_1}|$ is strictly decreasing on σ. Therefore for each $r \in (0, 1)$, $|p_{u_1}(\sigma(r))| \in (|p_{u_1}(\widetilde{z})|, |p_{u_1}(\widetilde{y})|)$. Therefore for each $r \in (0, 1)$, $\sigma(r)$ is in the portion of $\widetilde{D}$ which is in the unbounded face of $\widetilde{\lambda_{\widetilde{D}}}$. Therefore by definition of $g_{\widetilde{D}}$, we conclude that $g_{\widetilde{D}}(\widetilde{y}) = \widetilde{z}$.

Therefore the correspondence established above between the graphs, vertices, and distinguished points of $\langle\Lambda\rangle_{PC}$ and those of $\langle\widetilde{\Lambda}\rangle_{PC}$ respects the gradient maps of $\langle\Lambda\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$. Finally we conclude that $\langle\Lambda\rangle_{PC}$ and $\langle\widetilde{\Lambda}\rangle_{PC}$ share all auxiliary data, and are thus equal.

By inspecting this proof we immediately have the following corollary.

Corollary 4. *For any* $(f, G) \in H_a$ *there is a polynomial* (p, G_p) *such that* $(p, G_p) \sim (f, G)$*.*

APPENDIX A
SEVERAL RESULTS

Theorem A.1. *Let p be a polynomial. The critical points of p are contained in the convex hull of the zeros of p.*

Proof. Let $w_1, w_2, \ldots, w_n \in \mathbb{C}$ be the zeros of p repeated according to multiplicity. Assume by way of contradiction that there is some critical point z_0 of p which is not in the convex hull of $\{w_i : 1 \leq i \leq n\}$. By pre-composing with a linear map, we may assume that all zeros of p are contained in the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and that $z_0 \in (1, \infty)$. Let G denote one of the bounded faces of Λ_{z_0} which is incident to z_0, and let $\gamma : [0, 1] \to \mathbb{C}$ be a parameterization of the boundary of G. Thus, γ is a simple closed path with $\gamma(0) = \gamma(1) = z_0$. The Maximum Modulus Theorem implies that G contains a zero of p, and therefore ∂G intersects the line $L := \{z \in \mathbb{C} : Re(z) = 1\}$. Define $r_1, r_2 \in (0, 1)$ by $r_1 = \min(r \in [0, 1] : \gamma(r) \in L)$ and $r_2 = \max(r \in [0, 1] : \gamma(r) \in L)$.

Then $\gamma(0, r_1)$ and $\gamma(r_2, 1)$ are paths from z_0 to L which do not intersect except at z_0 (and possibly in L). Therefore since there are at least two bounded faces of Λ_{z_0} which are incident to z_0, there are at least four paths in Λ_{z_0} from z_0 to L which do not intersect except at z_0 and in L. It is not hard to see then that there is some $s \in (-1, 1) \setminus \{0\}$ such that two of the paths intersect the set $\{z \in \mathbb{C} : Re(z) > 1, Im(z) = s\}$.

Let z_1, z_2 be distinct points in $\{z \in \mathbb{C} : Re(z) > 1, Im(z) = s\}$ which are contained in Λ_{z_0}, with $Re(z_1) < Re(z_2)$, as in Figure A-1. Then for each zero w_i of p, $|z_1 - w_i| < |z_2 - w_i|$. Therefore

$$|p(z_1)| = \prod_{i=1}^{n} |z_1 - w_i| < \prod_{i=1}^{n} |z_2 - w_i| = |p(z_2)|,$$

which is a contradiction because z_1 and z_2 are in the same level curve of p. Thus we conclude that there is no critical point z_0 of p outside of the convex hull of the zeros of p.

□

Definition 40. *For (f, G) a function element, and $K \subset G$, then we define $\Lambda_K := \bigcup_{z \in K} \Lambda_z$.*

Corollary 5. *Let (f, G) be a special type function element. Let $K \subset G$ be compact. Then Λ_K is compact.*

Proof. Since G is bounded, Λ_K is bounded, and thus it suffices to show that Λ_K is closed in $\mathbb{C}$. Let $z_0 \in {\Lambda_K}^c$ be given. We will show that there is some $\delta > 0$ such that $B_\delta(z_0) \subset {\Lambda_K}^c$.

Case A.1.1. $z_0 \in cl(G)$.

Since $z_0 \notin \Lambda_K$, $\Lambda_{z_0} \cap K = \emptyset$. Thus by Proposition 2.4, there is some non-critical level curve L of f in G such that Λ_{z_0} is contained in one face of L and K is contained in the other face of Λ_{z_0}. Let D_1 denote the face of L which contains z_0 and let D_2 denote the face of L which contains K. For each $z \in K$, $z \in D_2$ and Λ_z is connected and does not intersect L, so Λ_z is contained in D_2. Therefore $\Lambda_K \subset D_2$. If we choose $\delta > 0$ small enough that $B_\delta(z_0) \subset D_1$, then $B_\delta(z_0) \subset {\Lambda_K}^c$.

Case A.1.2. $z_0 \notin cl(G)$.

Since $z_0 \notin cl(G)$, there is some $\delta > 0$ such that $B_\delta(z_0) \subset G^c$, and thus $B_\delta(z_0) \subset {\Lambda_K}^c$. We conclude that ${\Lambda_K}^c$ is open in $\mathbb{C}$, and thus Λ_K is closed in $\mathbb{C}$. □

Definition 41. *If (f, G) is a special type function element, and Λ is a level curve of f in G, and D is a face of Λ, then we say that f is increasing into D if there is some $\iota > 0$ such that $|f| > \epsilon$ on $\{z \in D : d(\{z\}, \Lambda) < \iota\}$. We say that f is decreasing into D if there is some $\iota > 0$ such that $|f| < \epsilon$ on $\{z \in D : d(\{z\}, \Lambda) < \iota\}$.*

Corollary 6. *Let (f, G) be a special type function element, and let Λ be a level curve of f in D. Let D be a face of Λ. Then either f is increasing into D or f is decreasing into D.*

Proof. We begin by assuming that D is one of the bounded faces of Λ.

Define A to be the union of the set $\{z \in D \cap G : |f(z)| = \epsilon\}$ with any components of G^c contained in D. If A defined as such is empty, then let A be just some single point in D. A is closed, so by Proposition 2.4, there is some non-critical level curve of f contained in $D \cap G$ which separates Λ from A. Call this level curve L. Let $\eta > 0$ be such that $|f| \equiv \eta$ on L. Since L separates A from Λ, L does not intersect A, so $\eta \neq \epsilon$.

Case A.1.3. *$\eta > \epsilon$.*

Let D_1 denote the face of L which contains Λ, and let D_2 denote the face of L which contains A. Define $D' := D_1 \cap D$, the portion of D which is "between" L and Λ. $D \setminus D_1 = cl(D_2)$, so $D \setminus D_1$ is compact and simply connected. Moreover, we may use Lemma 1 to show that $D \setminus cl(D_2) = D \cap D_1$ is connected. Therefore since $|f| \neq \epsilon$ on $D \cap D_1$, and $|f|$ is continuous, either $|f| > \epsilon$ on $D \cap D_1$, or $|f| < \epsilon$ on $D \cap D_1$. But there are points in $\partial(D \cap D_1)$ at which $|f|$ takes values greater than ϵ, namely any point in L, so there must be points in $D \cap D_1$ at which $|f|$ takes values greater than ϵ by the continuity of $|f|$. Thus $|f| > \epsilon$ on $D \cap D_1$. Define $\iota := d(L, \Lambda)$. If $w \in \{z \in D : d(\{z\}, \Lambda) < \iota\}$, then $w \in D \cap D_1$, so $|f(w)| > \epsilon$. Thus f is increasing into D.

Case A.1.4. *$\eta < \epsilon$.*

The same argument as above works, with the conclusion that f is decreasing into D.

If D is the unbounded face of Λ, the same argument works with the appropriate minor changes. □

Definition 42. *For $X, Y \subset \mathbb{C}$, define $d_1(X, Y) := \sup_{x \in X}(d(\{x\}, Y))$, and define $d_2(X, Y) := \sup_{y \in Y}(d(X, \{y\}))$. Then we let $\check{d}$ denote the Hausdorff metric $\check{d}(X, Y) := \max(d_1(X, Y), d_2(X, Y))$. If either X or Y are empty, we define $\check{d}(X, Y) := \infty$.*

Proposition A.1. *Let $\delta > 0$ be given. Then there is some $\eta \in (0, \epsilon)$ such that for each $\zeta \in (\epsilon - \eta, \epsilon + \eta)$, there is some collection $L_1, \ldots, L_N$ of level curves of f contained in G such that $|f| \equiv \zeta$ on each L_i, and $\check{d}\left(\left(\bigcup_{i=1}^{N} L_i\right), \Lambda\right) < \delta$.*

Proof. Let $\delta > 0$ be given. Let $E_1, E_2, \dots, E_N$ be an enumeration of the edges of Λ, and for each $i \in \{1, 2, \dots, N\}$, let z_i be some fixed interior point of E_i (that is, a point in E_i which is not an endpoint of E_i). Our first goal is to find an $\iota > 0$ small enough so that each of the following hold:

1. For each $i \in \{1, 2, \dots, N\}$, the only faces of Λ that intersect $B_\iota(z_i)$ are the two faces adjacent to E_i.

2. If D is one of the faces of Λ, and $z \in D$ is less than ι away from Λ, then $\check{d}(\Lambda_z, \partial D) < \frac{\delta}{2}$.

3. If D is one of the faces of Λ, and we define $A := \{z \in D : d(\{z\}, \Lambda) < \iota\}$, then either $|f| < \epsilon$ on A or $|f| > \epsilon$ on A.

Suppose an ι may be found which satisfies each of these three items. Our next goal is to find some $\eta \in (0, \epsilon)$ small enough so that if $\zeta \in (\epsilon - \eta, \epsilon + \eta)$, then for each $i \in \{1, 2, \dots, N\}$, there is some point in $B_\iota(z_i)$ at which $|f|$ takes the value ζ. Suppose such an $\eta > 0$ may be found. We now show that the statement of the proposition holds for this δ and η.

Let $\zeta \in (\epsilon - \eta, \epsilon + \eta)$ be given. Of course if $\zeta = \epsilon$, then by putting $N = 1$ and $L_1 = \Lambda$, the statement of the proposition obviously holds, so let us assume that $\zeta \neq \epsilon$. For each $i \in \{1, 2, \dots, N\}$, let w_i be a fixed point in $B_\iota(z_i)$ at which $|f|$ takes the value ζ, which may be found by Item 3 above. Define $\mathcal{L} := \bigcup_{i=1}^{N} \Lambda_{w_i}$.

Claim A.1.1. *$\check{d}(\mathcal{L}, \Lambda) < \delta$.*

Let $x \in \mathcal{L}$ be given. Let $i \in \{1, 2, \dots, N\}$ be such that $x \in \Lambda_{w_i}$. Since $d(\{w_i\}, \Lambda) < \iota$, we have that $\check{d}(\Lambda_{w_i}, \Lambda) < \frac{\delta}{2}$ by Item 2, and thus $d(\{x\}, \Lambda) < \frac{\delta}{2}$. Therefore $d_1(\mathcal{L}, \Lambda) < \delta$.

Let $x \in \Lambda$ be given. Let $i \in \{1, 2, \dots, N\}$ be such that $x \in E_i$. Let D denote the face of Λ which contains w_i. $d(\{z_i\}, \{w_i\}) < \iota$, so by Item 1 above, $E_i \subset \partial D$. And $d(\{w_i\}, \Lambda) < \iota$, so by Item 2 above, $\check{d}(\Lambda_{w_i}, \partial D) < \frac{\delta}{2}$. Since $x \in E_i \subset \partial D$, we conclude that $d(\Lambda_{w_i}, \{x\}) < \frac{\delta}{2}$. Since $x \in \Lambda$ was chosen arbitrarily, we conclude that $d_2(\mathcal{L}, \Lambda) < \delta$.

Finally we conclude that $\check{d}(\mathcal{L}, \Lambda) < \delta$, and we are done, subject to finding the specified ι and η.

Of course if we show that a positive constant may be chosen to satisfy each of the three above items individually, then the minimum of the three constants will have the properties desired in a choice of ι.

We first show that ι may be chosen to satisfy Item 1. Let $i \in \{1, 2, \dots, N\}$ be given. Define E_i' to be the points in E_i which are not endpoints of E_i. Then $\Lambda \setminus E_i'$ is closed, so $r_i := d(\{z_i\}, \Lambda \setminus E_i') > 0$. Now if D is a face of Λ such that $z_i \notin \partial D$ (and thus $\partial D \subset \Lambda \setminus E_i'$), and $w \in D$, then any path from w to z_i intersects ∂D, in particular the straight line path. Therefore

$$d(\{z_i\}, \{w\}) \geq d(\{z_i\}, \partial D) \geq d(\{z_i\}, \Lambda \setminus E_i') = r_i,$$

and therefore we conclude that D does not intersect $B_{r_i}(z_i)$. If we choose $\iota > 0$ smaller than each r_i, this ι satisfies Item 1.

We now show that ι may be chosen to satisfy Item 2. Let D be one of the faces of Λ. By the compactness of ∂D, there is a finite sequence of points $x_1, x_2, \dots, x_k \in D$ such that $\check{d}\left(\bigcup_{i=1}^{k}\{x_i\}, \partial D\right) < \frac{\delta}{4}$. Define

$$K := \{x_i\}_{i=1}^{k} \cup \left\{z \in D : d(\{z\}, \partial D) \geq \frac{\delta}{4}\right\}.$$

By Proposition 2.4, there is some $\iota_D > 0$ such that if $z \in D$ is less than ι_D away from Λ, then Λ_z contains Λ in one face and K in the other. Fix some $w \in D$ less than ι_D away from Λ.

Claim A.1.2. $\check{d}(\Lambda_w, \partial D) < \frac{\delta}{2}$.

Let $w_0 \in \Lambda_w$ be given. Since Λ_w is contained in $D \setminus K$, we have

$$\Lambda_w \subset \left\{z \in D : d(\{z\}, \partial D) < \frac{\delta}{4}\right\}.$$

Thus $d(\{w_0\}, \partial D) < \frac{\delta}{4}$, and thus $d_1(\Lambda_w, \partial D) < \frac{\delta}{2}$.

Let $z_0 \in \partial D$ be given. Then for some $i \in \{1, 2, \dots, k\}$, $d(\{x_i\}, \{z_0\}) < \frac{\delta}{4}$. But $x_i \in K$, so x_i and z_0 are in different faces of Λ_w. Since x_i and z_0 are in different faces of Λ_w, the straight line path from z_0 to x_i intersects Λ_w, and thus $d(\Lambda_w, \{z_0\}) \leq d(\{x_i\}, \{z_0\}) < \frac{\delta}{4}$. Thus $d_2(\Lambda_w, \partial D) < \frac{\delta}{2}$.

Therefore we conclude that $\check{d}(\Lambda_w, \partial D) < \frac{\delta}{2}$. Define $\iota := \min(\iota_D : D \text{ is a face of } \Lambda)$. Then this ι satisfies Item 2.

Finally the fact that ι may be chosen to satisfy Item 3 follows directly from Corollary 6.

The fact that an $\eta > 0$ may be found with the desired property follows directly from the Open Mapping Theorem. Since the desired $\iota > 0$ and $\eta > 0$ may be found, we are done. □

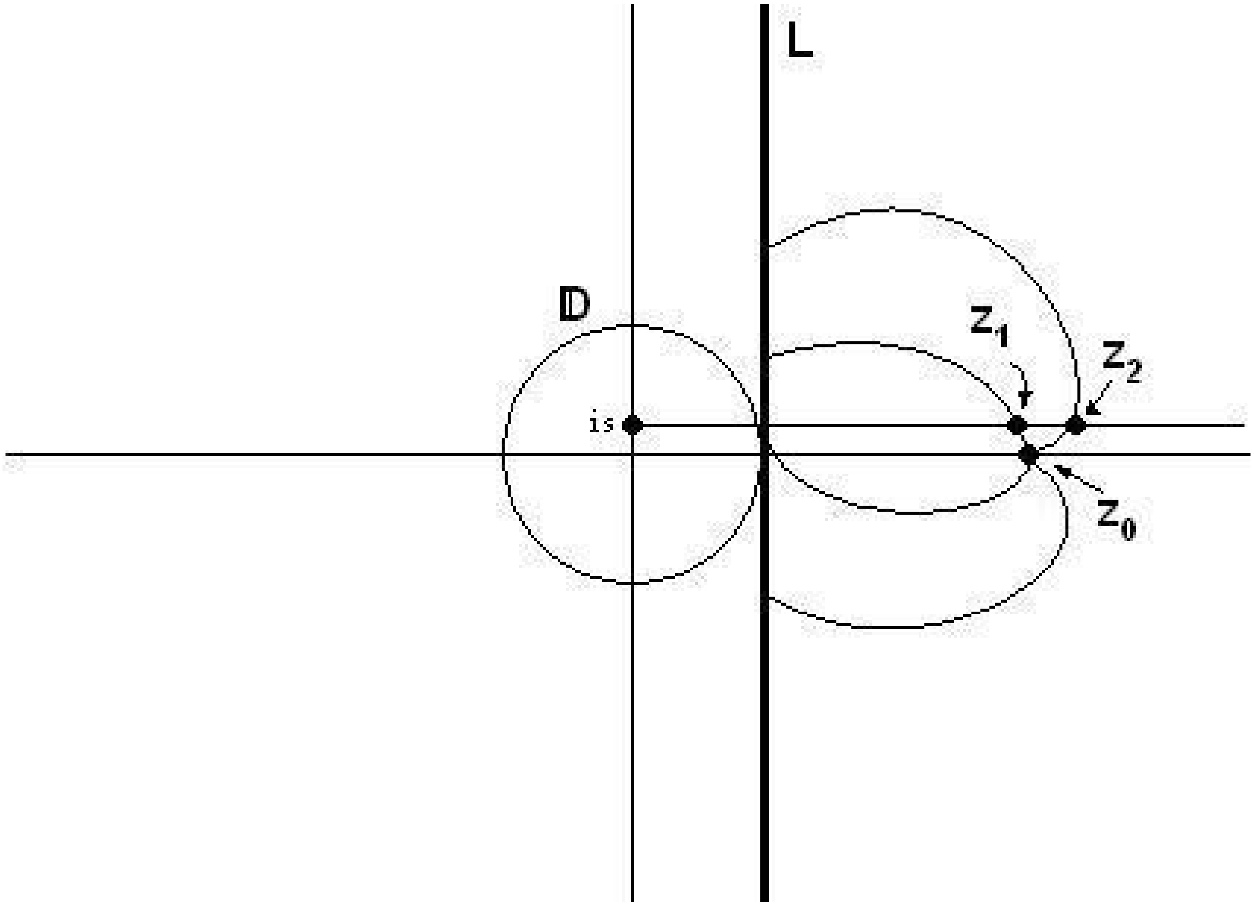

Figure A-1. Gauss' Theorem

APPENDIX B
SEVERAL LEMMATA

Lemma 1. *For any disjoint closed sets $X, Y \subset cl(\mathbb{C})$, and $x, y \in cl(\mathbb{C}) \setminus (X \cup Y)$, if x and y are in the same component of X^c and the same component of Y^c, then x and y are in the same component of $(X \cup Y)^c$.*

Proof. Suppose by way of contradiction that x and y are in different components of $(X \cup Y)^c$. Assume without loss of generality that $y = \infty$. Let A_1 denote the component of $(X \cup Y)^c$ which contains x, and let B_1 denote the component of $(X \cup Y)^c$ which contains y. Let Z denote the union of all bounded components of ${A_1}^c$. Define $A_2 := A_1 \cup Z$. Since B_1 is open and contains ∞, and A_2 does not intersect B_1, we may conclude that A_2 is bounded. Therefore A_2 has only a single unbounded component, so A_2 is simply connected. And the boundary of a simply connected set is connected, so $\partial A_2 \subset X \cup Y$ is connected. Because X and Y are disjoint and compact, this implies that ∂A_2 is either contained in X or contained in Y. However this is a contradiction because x and y are in the same component of X^c and in the same component of Y^c. □

It may easily be seen that this lemma implies that if A is an open connected set, and $X \subset A$ is compact, such that X^c has a single component, then $A \setminus X$ is connected. This fact will be used several times in this paper, and we will cite the above lemma when it is needed.

Lemma 2. *Let λ be a finite connected graph embedded in the plane. If λ has the property that each edge of λ is in the boundary both of a bounded and the unbounded face of λ, some bounded face of λ has a single edge of λ as its boundary.*

Proof. We begin by constructing a graph $\mathcal{T}$ from λ. $\mathcal{T}$ will have two kinds ov vertices. We place a V-type vertex for $\mathcal{T}$ at each vertex of λ, and we place one F-type vertex for $\mathcal{T}$ in each bounded face of λ. Let u be an F-type vertex of $\mathcal{T}$. Let D denote the bounded face of λ which contains u. Then we draw an edge from u to each V-type vertex of $\mathcal{T}$ which arises from a vertex of λ which is contained in ∂D. We draw these edges in such a way that they are contained in D (except at the end points) and do not intersect (except at u). Having done this for each F type vertex in $\mathcal{T}$, the resulting connected graph is $\mathcal{T}$.

We now wish to show that $\mathcal{T}$ is a tree. Suppose by way of contradiction that $\mathcal{T}$ contains some cycle $C : u_1 E_1 u_2 \cdots u_n E_n u_1$. Consider this cycle as a path in $\mathbb{C}$. Reduce this cycle if necessary so that it forms a simple closed path. Let D_1 denote the face of λ which contains u_1. Since C is a simple path, the only edges in C which have u_1 as an end point are E_1 and E_n. Therefore C bisects D_1. Let E be one of the edges of λ which is in ∂D_1 and which is contained in the bounded face of the path C. Then since $\mathcal{T}$ is contained in the closure of the bounded faces of λ, E is not adjacent to the unbounded face of λ, which is a contradiction of the definition of λ.

Now that we have shown that $\mathcal{T}$ is a tree, and $\mathcal{T}$ is certainly finite, let u denote one of the leaves of $\mathcal{T}$. Since each vertex of λ is incident to more than one bounded face of λ, u must be an F-type vertex of $\mathcal{T}$. Let D now denote this face of λ. Since u is a leaf of $\mathcal{T}$, ∂D only contains one vertex of λ, and thus ∂D consists of a single edge of λ. □

Lemma 3. *Given any special type function element (f, G) and $\eta > 0$, and any compact set $G' \subset G$ which does not contain any critical points of f, there exists $\tau > 0$ such that if g is analytic on G, and $|f(z) - g(z)| < \tau$ for all $z \in G$, then the following hold:*

1If $z^{(0)} \in G'$, and $w^{(1)} \in B_\tau(f(z^{(0)}))$, then there is a point $z^{(1)} \in B_\eta(z^{(0)})$ such that $g(z^{(1)}) = w^{(1)}$. (In particular we may put $w^{(1)} = f(z^{(0)})$.)

2If $z^{(0)} \in G'$, and $w^{(1)} \in B_\tau(g(z^{(0)}))$, then there is a point $z^{(1)} \in B_\eta(z^{(0)})$ such that $f(z^{(1)}) = w^{(1)}$. (In particular we may put $w^{(1)} = g(z^{(0)})$.)

Proof. For $z \in G'$, since f is analytic, there is a $\kappa \in (0, \eta)$ such that f is injective on $B_\kappa(z)$. Let $h^{(1)}(z)$ denote the supremum over G' of all such κ. Since G' does not contain any critical points of f, $h^{(1)}$ is continuous on G'. Therefore if we define $\kappa^{(1)} := \dfrac{\inf(h^{(1)}(z) : z \in G')}{2}$, we conclude that $\kappa^{(1)} \in (0, \eta)$. Now define $h^{(2)}(z) := \inf(|f(z') - f(z)| : |z' - z| = \frac{\kappa^{(1)}}{2})$. By definition of $\kappa^{(1)}$, $h^{(2)}$ is non-zero on G', and $h^{(2)}$ is continuous, so if we define $\kappa^{(2)} := \inf(h^{(2)}(z) : z \in G')$, we may conclude that $\kappa^{(2)} > 0$. Now define $\tau := \dfrac{\kappa^{(2)}}{100}$.

Let g be analytic on G, with $|f-g| < \tau$ on G, and let $z^{(0)} \in G'$ and $w^{(1)} \in B_\tau(f(z^{(0)}))$ be given. Consider the function $h^{(3)}(z) := f(z^{(0)}) - g(z)$. For any $z \in \partial B_{\frac{\kappa^{(1)}}{2}}(z^{(0)})$,

$$|h^{(3)}(z)| = |f(z^{(0)}) - g(z)| = |f(z^{(0)}) - f(z) + f(z) - g(z)| \geq |f(z^{(0)}) - f(z)| - |f(z) - g(z)|.$$

Thus we have

$$|h^{(3)}(z)| \geq \kappa^{(2)} - \frac{\kappa^{(2)}}{100} = \frac{99\kappa^{(2)}}{100}.$$

However $|h^{(3)}(z^{(0)})| = |f(z^{(0)}) - g(z^{(0)})| \leq \tau = \frac{\kappa^{(2)}}{100}$. Then by the Maximum Modulus Principle, $h^{(3)}$ has a zero in $B_{\frac{\kappa^{(1)}}{2}}(z^{(0)})$, which proves that Item 1 holds for this τ.

Let us now define $h^{(4)}(z) := g(z^{(0)}) - f(z)$. For any $z \in \partial B_{\frac{\kappa^{(2)}}{2}}(z^{(0)})$,

$$|h^{(4)}(z)| = |g(z^{(0)}) - f(z)| = |g(z^{(0)}) - f(z^{(0)}) + f(z^{(0)}) - f(z)|.$$

By using the reverse triangle inequality, we obtain

$$|h^{(4)}(z)| \geq |f(z^{(0)}) - f(z)| - |g(z^{(0)}) - f(z^{(0)})| \geq \frac{99\kappa^{(2)}}{100}.$$

However $|h^{(4)}(z^{(0)})| = |g(z^{(0)}) - f(z^{(0)})| \leq \tau = \frac{\kappa^{(2)}}{100}$. Thus by the Maximum Modulus Principle, $h^{(4)}$ has a zero in $B_{\frac{\kappa^{(1)}}{2}}(z^{(0)})$, which proves that Item 2 holds for the chosen τ.

□

Lemma 4. *Let $K \subset \mathbb{C}^{n-1}$ be a compact set, and let $\iota^{(1)} \in [0, \pi)$ and $r^{(1)} > 0$ be given. There is a $\delta^{(1)} > 0$ such that the following holds. Let $v \in K$ be chosen, such that $d_{\arg}(v) > \iota^{(1)}$. Let $u \in \Theta^{-1}(v)$ be chosen. Then if $r \geq r^{(1)}$, and λ is any component of $E_{p_u,r}$, and E is any edge in λ, then there is some point z in E which is greater than $\delta^{(1)}$ away from each critical point of p_u.*

Proof. By definition of p_u, there are polynomials $P^{(1)}, \dots, P^{(n-1)}$ in $n-1$ variables such that for $u \in \mathbb{C}^{n-1}$, $p_u(z) = \frac{1}{n}z^n + \sum_{k=1}^{n-1} P^{(k)}(u)z^k$. Therefore for $z \in \mathbb{C}$,

$$|p_u(z)| \geq \frac{1}{n}|z|^n - \left(\sum_{i=k}^{n-1} |P^{(k)}(u)||z|^k \right). \tag{B–1}$$

Since Θ is proper (by [14]), $\Theta^{-1}(K)$ is compact. By inspecting Equation B–1, we conclude that there is some constant $S > 0$ such that if $u \in \Theta^{-1}(K)$, and $|z| > S$, then $|p_u(z)| \geq 2$. Therefore $G_{p_u} \subset B_S(0)$ for each $u \in \Theta^{-1}(K)$. Increase S further if necessary so that $\Theta^{-1}(K) \subset B_S(0)$. Finally we set

$$T := \sup(|p_u{}'(z)| : u \in B_S(0) \text{ and } z \in B_S(0)).$$

By a similar argument as above, this T is finite by the compactness of the sets involved. We now define

$$\delta^{(1)} := \frac{r^{(1)} \sin(\frac{\iota^{(1)}}{2})}{T}.$$

Now choose any $v \in K$ such that $d_{\arg}(v) > \iota^{(1)}$, and any $u \in \Theta^{-1}(v)$. Let $r \in [r^{(1)}, 1)$ be chosen, and let λ be some component $E_{p_u,r}$, and let E be any edge of λ. Since the endpoints of E are critical points of p_u, the change in argument of p_u along E is greater than or equal to $\iota^{(1)}$. Therefore there is some point $z^{(1)}$ in E such that $d_{\arg}(p_u(z), v^{(k)}) > \frac{\iota^{(1)}}{2}$ for each $k \in \{1, \dots, n-1\}$. Fix some $i \in \{1, \dots, n-1\}$. The angle between $p_u(z^{(1)})$ and $p_u(u^{(i)})$ is greater than or equal to $\frac{\iota^{(1)}}{2}$, and $|p_u(z^{(1)})| = r \geq r^{(1)}$. Therefore by geometry, $|p_u(z^{(1)}) - p_u(u^{(i)})| \geq r^{(1)} \sin(\frac{\iota^{(1)}}{2})$. Let L denote the straight line path from $z^{(1)}$ to $u^{(i)}$. Then

$$\frac{|p_u(z^{(1)}) - p_u(u^{(i)})|}{|z^{(1)} - u^{(i)}|} \leq \max(|p_u{}'(z)| : z \in L) \leq T.$$

Therefore

$$|z^{(1)} - u^{(i)}| \geq \frac{|p_u(z^{(1)}) - p_u(u^{(i)})|}{T} \geq \frac{r^{(1)} \sin(\frac{\iota^{(1)}}{2})}{T} = \delta^{(1)}.$$

Since this holds for each $i \in \{1, \dots, n-1\}$, we are done. ☐

Lemma 5. *Let $v \in \mathbb{C}^{n-1}$ and $\rho > 0$ be given. Then there exists a $\nu > 0$ such that if $\widehat{v} \in \mathbb{C}^{n-1}$ and $|v - \widehat{v}| < \nu$, and $\widehat{u} \in \Theta^{-1}(\widehat{v})$, then there is a $u \in \Theta^{-1}(v)$ such that $|u - \widehat{u}| < \rho$.*

Proof. It was shown in [14] that Θ is continuous, open, and proper, and that $\Theta^{-1}(v)$ is finite. Since Θ is open and $\Theta^{-1}(v)$ is finite, there is some $\nu > 0$ small enough that $B_\nu(v) \subset \bigcap_{u \in \Theta^{-1}(v)} \Theta(B_\rho(u))$. Since Θ is proper, $\Theta^{-1}(cl(B_\nu(v)))$ is compact. Suppose by way of contradiction that there is a sequence of $\{v_k\}_{k=0}^\infty \subset B_\nu(v)$ such that $v_k \to v$, and for each $k \geq 0$ there is a $u_k \in \Theta^{-1}(v_k)$ such that $|u_k - u| > \rho$ for each $u \in \Theta^{-1}(v)$. Since $\Theta^{-1}(cl(B_\nu(v)))$ is compact, there is some sub-sequence $\{u_{k_l}\}_{l=0}^\infty$ which converges to some point u. Since Θ is continuous, $\Theta(u) = v$, so $u \in \Theta^{-1}(v)$, which gives us our contradiction. ☐

Definition 43. *If $\gamma : [\alpha, \beta] \to \mathbb{C}$ is a path, and f is a function which is analytic and non-zero on the image of γ, then we say that γ is parameterized according to $\arg(f)$ if for each $r \in [\alpha, \beta]$, $\arg(f(\gamma(r))) = r$.*

Lemma 6. *Let $v \in V_{n-1}$, and $\delta^{(1)} > 0$ be given. There exists some $\delta^{(2)} \in (0, \delta^{(1)})$ such that if $u \in \Theta^{-1}(v)$, and λ is a critical level curve of (p_u, G_{p_u}) (with $|f| \equiv \epsilon > 0$ on λ), and $x \in \lambda$, then if $y \in B_{\delta^{(2)}}(x)$ satisfies $|f(y)| = \epsilon$, then there is a path σ from y to x which is contained in $\lambda \cap B_{\delta^{(1)}}(x)$. Moreover, we may choose σ so that $\arg(p_u)$ is strictly increasing or strictly decreasing along σ, and parameterized according to $\arg(p_u)$.*

Proof. Since $\Theta^{-1}(v)$ is finite ([14]), we need only show the result for some fixed $u \in \Theta^{-1}(v)$. Let $u \in \Theta^{-1}(v)$, and let λ be one of the critical level curves of (p_u, G_{p_u}), (with $|f| \equiv \epsilon > 0$ on λ). Let $x \in \lambda$ be given. Let $k \in \mathbb{N}$ denote the multiplicity of x as a zero of $p_u{}'$ (possibly $k = 0$). Then there is some neighborhood $D \subset B_{\delta^{(1)}}(x)$ of x and $S > 0$

and conformal map $\phi : D \to B_S(p_u(0))$ such that $p_u(z) = \phi(z)^{k+1} + p_u(x)$ for all $z \in D$. Define $f(w) := w^{k+1} + p_u(x)$. The level curves of f are well understood. Let L denote the level curve of f which contains 0. Then if $w \in L$, there is a path in L from w to 0 which is contained in $B_{|w|}(0)$, along which $\arg(f)$ is either strictly increasing or strictly decreasing. Choose some $r > 0$ such that $B_r(x) \subset D$. Let $y \in B_r(x)$ be any point such that $|p_u(y)| = \epsilon$. Let $\sigma^{(1)}$ denote the path in $B_{|\phi(y)|}(0)$ from $\phi(y)$ to 0 along which $\arg(f)$ is strictly increasing or strictly decreasing. Then if we define $\sigma := \phi^{-1} \circ \sigma^{(1)}$, $\sigma \subset \phi^{-1}(B_{|\phi(y)|}(0)) \subset D \subset B_{\delta^{(1)}}(x)$, and for each $t \in [0, 1]$, $p_u(\sigma(t)) = f(\sigma^{(1)}(t))$, so $\arg(p_u)$ is either strictly increasing or strictly decreasing along σ.

Now for $x \in \lambda$, let $h(x)$ denote the supremum over all $r > 0$ such that for $y \in B_r(x)$ with $|p_u(y)| = \epsilon$, a path σ with the desired properties may be found. We have just shown that $h(x) > 0$ for all $x \in \lambda$, and it is easy to see that h is continuous, so if we define $h(\lambda) := \inf(h(x) : x \in \lambda)$, the compactness of λ implies that $h(\lambda) > 0$. Then choosing $\delta^{(2)} > 0$ to be less than $h(\lambda)$ for each critical level curve λ of p_u on which $|p_u| \neq 0$, it is clear that $\delta^{(2)}$ has the desired property.

□

Lemma 7. *Let $v \in V_{n-1}$, and $\delta^{(1)} > 0$ be given. There exists some $\delta^{(2)} \in (0, \delta^{(1)})$ such that if $u \in \Theta^{-1}(v)$, and λ is a critical level curve of (p_u, G_{p_u}) (with $|p_u| \equiv \epsilon > 0$ on λ), and $x \in \lambda$, then if $y \in B_{\delta^{(2)}}(x)$ satisfies $\arg(p_u(y)) = \arg(p_u(x))$, then there is a path σ from y to x which is contained in $B_{\delta^{(1)}}(x)$ and such that $\arg(p_u(\sigma(r))) = \arg(p_u(x))$ for all r. Moreover we may choose σ so that $|p_u|$ is strictly increasing or strictly decreasing along σ, and parameterized according to $|p_u|$.*

Proof. Essentially the same argument for Lemma 6 works here. □

Lemma 8. *Let $v \in V_{n-1}$, and $\tau > 0$ and be given. Then there exists a $\rho > 0$ such that if $u \in \Theta^{-1}(v)$, and $\widehat{u} \in \Theta^{-1}(V_{n-1})$ such that $|u - \widehat{u}| < \rho$, then the following holds. $G_{p_{\widehat{u}},1} \subset G_{p_u,2}$, and $|p_u(z) - p_{\widehat{u}}(z')| < \tau$ for all $z, z' \in G_{p_u,2}$ satisfying $|z - z'| < \rho$.*

Proof. This follows from the fact that the coefficients of p_u are polynomials in the components of u. Therefore if $u^{(n)} \to u$ in $\Theta^{-1}(V_{n-1})$, then $p_{u^{(n)}} \to p_u$ uniformly on any compact set. □

Definition 44. *For $u \in \mathbb{C}^{n-1}$, if $\gamma : [0,1] \to \mathbb{C}$ is a path, and $0 < a < b < 1$, then for $0 < \epsilon^{(1)} < \epsilon^{(2)}$, we say that γ takes an $(\epsilon^{(1)}, \epsilon^{(2)})$ trip on $[a,b]$ if the following hold.*

- *There is some $\iota > 0$ such that for all $r \in (a-\iota, a) \cup (b, b+\iota)$, $\gamma(r)$ is less than $\epsilon^{(1)}$ away from any critical point of p_u.*
- *For each $r \in (a,b)$, $\gamma(r)$ is greater than or equal to $\epsilon^{(1)}$ away from every critical point of p_u.*
- *There is some $r \in (a,b)$ such that $\gamma(r)$ is greater than $\epsilon^{(2)}$ away from every critical point of p_u.*

Definition 45. *Let $u \in \mathbb{C}^{n-1}$ be given. Let γ be a path in $E_{p_u,\epsilon}$ for some $\epsilon > 0$, such that $\gamma(0)$ and $\gamma(1)$ are critical points of p_u. For $0 \le a < b \le 1$, we say γ has a p_u-edge on $[a,b]$ if $\gamma(a)$ and $\gamma(b)$ are critical points of p_u, and for all $r \in (a,b)$, $\gamma(r)$ is not a critical point of p_u.*

Lemma 9. *Fix some $v = (v^{(1)}, \dots, v^{(n-1)}) \in V_{n-1}$ not the zero vector, and $\delta^{(1)} > 0$. Then there exists a constant $\rho > 0$ such that the following hold. Let $u \in \Theta^{-1}(v)$ be chosen, and fix some $\widehat{u} \in B_\rho(u)$ such that if we define $\widehat{v} = (\widehat{v^{(1)}}, \dots, \widehat{v^{(n-1)}}) := \Theta(\widehat{u})$, then $\arg(\widehat{v^{(k)}}) = \arg(v^{(k)})$ for each $k \in \{1, \dots, n-1\}$. For some $k \in \{1, \dots, n-1\}$ with $|v^{(k)}| \neq 0$, let $\widehat{\lambda}$ denote the level curve of $p_{\widehat{u}}$ which contains $\widehat{u^{(k)}}$. Let $\widehat{E}$ denote some edge of $\widehat{\lambda}$ which is incident to $\widehat{u^{(k)}}$, and let $\widehat{\gamma}$ denote a parameterization of $\widehat{E}$ such that $\widehat{\gamma} : [\alpha, \beta] \to \widehat{\lambda}$ (for some $\alpha, \beta \in \mathbb{R}$) satisfies $\widehat{\gamma}(\alpha) = \widehat{u^{(k)}}$ and $\arg(p_{\widehat{u}}(\widehat{\gamma}(t))) = t$ for all $t \in [\alpha, \beta]$. (Note: if $\arg(p_{\widehat{u}})$ is increasing as this portion of $\widehat{E}$ is traversed, then $\alpha < \beta$, otherwise $\alpha > \beta$.) Then if we let λ denote the critical level curve of p_u containing $u^{(k)}$, there is a path $\gamma : [\alpha, \beta] \to \lambda$ such that $\gamma(\alpha) = u^{(k)}$, and for each $t \in [\alpha, \beta]$, $\arg(p_u(\gamma(t))) = t$ and $|\gamma(t) - \widehat{\gamma}(t)| < \delta^{(1)}$.*

Proof. We assume that $\arg(p_{\widehat{u}})$ is increasing as $\widehat{\gamma}$ is traversed. Otherwise make the appropriate changes. We will show that the result of the lemma holds for any fixed $u \in \Theta^{-1}(v)$, which will suffice because $\Theta^{-1}(v)$ is finite by [14]. Reduce $\delta^{(1)} > 0$ if necessary so that for each $k \in \{1, \dots, n-1\}$ with $|v^{(k)}| \neq 0$, if $|z - u^{(k)}| < \delta^{(1)}$, then $|p_u(z) - v^{(k)}| < \frac{\text{mindiff}(0,|v^{(1)}|,\dots,|v^{(n-1)}|)}{4}$. Of course $\frac{\text{mindiff}(0,|v^{(1)}|,\dots,|v^{(n-1)}|)}{4} \leq \frac{|v^{(k)}|}{4}$, so by geometry, $|\arg(p_u(z)) - \arg(v^{(k)})| < \frac{\pi}{4}$.

By Lemma 6, we may choose $\delta^{(2)} \in (0, \frac{\delta^{(1)}}{4})$ such that the following holds. If $y \in B_{\delta^{(2)}}(u^{(k)})$ for some $k \in \{1, \dots, n-1\}$ such that $|p_u(y)| = |v^{(k)}|$, then there is a path σ from y to $u^{(k)}$ contained in $B_{\frac{\delta^{(1)}}{2}}(u^{(k)}) \cap E_{p_u,|v^{(k)}|}$ such that $\arg(p_u)$ is strictly increasing or strictly decreasing along σ.

Since p_u is an open mapping, we may choose some $M > 0$ small enough so that for each $k \in \{1, \dots, n-1\}$, $B_{2M}(v^{(k)}) \subset p_u(B_{\delta^{(2)}}(u^{(k)}))$. By Lemma 8, we may choose a $\rho^{(1)} > 0$ so that $\rho^{(1)} < \frac{\delta^{(2)}}{2}$, and if $\widehat{u} \in B_{\rho^{(1)}}(u)$, then $|p_u(z) - p_{\widehat{u}}(\widehat{z})| < M$ for all $z, \widehat{z} \in G_{p_u}$ such that $|z - \widehat{z}| < \rho^{(1)}$.

Let K denote the set of all points x in G_{p_u} such that the following hold.

- $x \in E_{p_u,|v^{(k)}|}$ for some $k \in \{1, \dots, n-1\}$.
- $|x - u^{(k)}| \geq \frac{\delta^{(2)}}{2}$ for each $k \in \{1, \dots, n-1\}$.

Then by the compactness of K, we may choose an $\eta > 0$ such that for each $x \in K$ the following holds. Let $l \in \{1, \dots, n-1\}$ be chosen so that $|p_u(x)| = |v^{(l)}|$.

- $\eta < \min(d(\{z\}, \partial G_{p_u}) : z \in K)$.
- p_u is injective on $B_\eta(x)$.
- $\eta < \rho^{(1)}$.
- $|x - u^{(k)}| > \eta$ for each $k \in \{1, \dots, n-1\}$.

Define $G' := \{x \in G_{p_u} : d(x, \partial G_{p_u}) \geq \eta, d(x, u^{(k)}) \geq \eta \text{ for each } k\}$. By Lemma 3, we may choose $\tau > 0$ so that $\tau < \min(M, \frac{\text{minmod}(v)}{4})$, $\tau < \text{mindiff}(0, |v^{(1)}|, \dots, |v^{(n-1)}|)$, and if f is analytic on G' with $|f - p_u| < \tau$ of G', then for all x in G', the following hold.

- For any $w \in B_\tau(p_u(x))$, there is a $y \in B_\eta(x)$ with $f(y) = w$.
- For any $w \in B_\tau(f(x))$, there is a $y \in B_\eta(x)$ with $p_u(y) = w$.

By Lemma 8 and the continuity of Θ, we may choose $\rho \in (0, \delta^{(1)})$ so that if $\widehat{u} \in B_\rho(u)$, then $|p_u(z) - p_{\widehat{u}}(\widehat{z})| < \tau$ for all $z, \widehat{z} \in G_{p_u}$ such that $|z - \widehat{z}| < \rho$, and for $\widehat{v} = (\widehat{v^{(1)}}, \dots, \widehat{v^{(n-1)}}) := \Theta(\widehat{u})$, $|\widehat{v} - v| < \tau$. We now show that the statement of the lemma holds for the chosen ρ.

Let $\widehat{u} \in B_\rho(u)$ be chosen. Fix some $k \in \{1, \dots, n-1\}$ such that $|v^{(k)}| \neq 0$. Then $|v^{(k)}| \geq \text{minmod}(v)$. Note that since $\tau > \frac{\text{minmod}(v)}{4}$, and $|v - \widehat{v}| < \frac{\tau}{4}$, we have $|\widehat{v^{(k)}}| > \frac{\text{minmod}(v)}{2}$. Let $\widehat{\lambda}$ denote the level curve of $p_{\widehat{u}}$ which contains $\widehat{u^{(k)}}$. Let $\widehat{E}$ denote some edge of $\widehat{\lambda}$ which is incident to $\widehat{\lambda}$. Let α denote some choice of the argument of $p_{\widehat{u}}(\widehat{u^{(k)}})$, and let $\widehat{\gamma} : [\alpha, \beta] \to \widehat{\lambda}$ (for some $\beta > \alpha$ because $\arg(p_{\widehat{u}})$ is increasing as $\widehat{\gamma}$ is traversed) be a path which parameterizes $\widehat{E}$ according to the argument of $p_{\widehat{u}}$ (that is, $\arg(p_{\widehat{u}}(\widehat{\gamma}(t))) = t$ for all $t \in [\alpha, \beta]$).

Note that by the definition of a $(\rho^{(1)}, \delta^{(1)})$ trip over an interval, if $\widehat{\gamma}$ takes $(\rho^{(1)}, \delta^{(1)})$ trips over two intervals $I^{(1)}, I^{(2)} \subset [0, 1]$, then either $I^{(1)} = I^{(2)}$, or $I^{(1)}$ and $I^{(2)}$ are disjoint. Therefore since $\widehat{\gamma}$ is a rectifiable path, $\widehat{\gamma}$ takes at most finitely many distinct $(\rho^{(1)}, \delta^{(1)})$ trips.

Case B.0.5. $\widehat{\gamma}$ *takes a* $[\rho^{(1)}, \delta^{(1)}]$ *trip on some sub-interval of* $[\alpha, \beta]$.

Let $[r^{(1)}, s^{(1)}], \dots, [r^{(N)}, s^{(N)}] \subset [\alpha, \beta]$ be the disjoint subintervals of $[\alpha, \beta]$ over which γ takes $(\rho^{(1)}, \delta^{(1)})$ trips, ordered so that $s^{(k)} < r^{(k+1)}$ for each $k \in \{1, \dots, N-1\}$. We begin by defining γ on $\bigcup_{i=1}^{N} [r^{(i)}, s^{(i)}]$. Fix for the moment some $j \in \{1, \dots, N\}$.

For all $t \in [r^{(j)}, s^{(j)}]$, define $w(t) := |v^{(k)}| e^{it}$. Then by choice of τ

$$|w(t) - p_{\widehat{u}}(\widehat{\gamma}(t))| = \left| |v^{(k)}| e^{it} - |\widehat{v^{(k)}}| e^{it} \right| = \left| |v^{(k)}| - |\widehat{v^{(k)}}| \right| < \tau$$

Thus there is some $y \in B_\eta(\widehat{\gamma}(r))$ such that $p_u(y) = w(r)$. Moreover, since p_u is injective in $B_\eta(\widehat{\gamma}(r))$, this choice of y is unique. Define $\gamma(r) = y$.

Since p_u is injective on $B_\eta(\widehat{\gamma}(r))$ for each $r \in [r^{(j)}, s^{(j)}]$, and p_u is an open mapping, it is easy to show that γ is a continuous function, and thus a path from $\gamma(r^{(j)})$ to $\gamma(s^{(j)})$. Further, if $r \in [r^{(j)}, s^{(j)}]$, $|p_u(\gamma(r))| = |w(r)| = |v^{(k)}|$. Therefore we conclude that $\gamma|_{[r^{(j)}, s^{(j)}]}$ is a path in $E_{p_u, |v^{(k)}|}$, and by construction, for each $r \in [r^{(j)}, s^{(j)}]$, $|\widehat{\gamma}(r) - \gamma(r)| < \eta$ and $\arg(p_u(\gamma(r))) = r$. Having done this for each $j \in \{1, \ldots, N\}$, we now wish to define γ on $(s^{(j)}, r^{(j+1)})$ for each $j \in \{1, \ldots, N-1\}$.

Again fix for the moment some new $j \in \{0, \ldots, N\}$.

Since there is no sub-interval of $(s^{(j)}, r^{(j+1)})$ on which $\widehat{\gamma}$ takes a $(\rho^{(1)}, \delta^{(1)})$ trip, $\widehat{\gamma}(r)$ is within $\frac{\delta^{(1)}}{4}$ of some critical point of p_u for each $r \in (s^{(j)}, r^{(j+1)})$. However $\delta^{(1)} < \frac{\text{mindiff}(u)}{2}$, thus there is some unique $l \in \{1, \ldots, n-1\}$ such that for each $r \in (s^{(j)}, r^{(j+1)})$, $|\widehat{\gamma}(r) - u^{(l)}| \leq \delta^{(1)}$. Since $\widehat{\gamma}$ takes a $(\rho^{(1)}, \delta^{(1)})$ trip over $[r^{(j)}, s^{(j)}]$, $|\widehat{\gamma}(s^{(j)}) - u^{(l)}| = \rho^{(1)}$. Therefore

$$|\gamma(s^{(j)}) - u^{(l)}| \leq |\gamma(s^{(j)}) - \widehat{\gamma}(s^{(j)})| + |\widehat{\gamma}(s^{(j)}) - u^{(l)}| < \eta + \rho^{(1)} < \delta^{(2)}.$$

In addition to this, $|p_u(\gamma(s^{(j)}))| = |v^{(k)}|$, but by choice of $\delta^{(1)}$,

$$||v^{(k)}| - |v^{(l)}|| = ||p_u(\gamma(s^{(j)}))| - |v^{(l)}|| < |p_u(\gamma(s^{(j)})) - v^{(l)}| < \delta^{(2)},$$

and

$$\delta^{(2)} < \text{mindiff}(0, |v^{(1)}|, \ldots, |v^{(n-1)}|).$$

Therefore we conclude that $|p_u(\gamma(s^{(j)}))| = |v^{(l)}| = |v^{(k)}|$. Then by choice of $\delta^{(2)}$, there is some path $\sigma^{(1)}$ from $\gamma(s^{(j)})$ to $u^{(l)}$ contained in $B_{\frac{\delta^{(1)}}{2}}(u^{(l)}) \cap E_{p_u, |v^{(l)}|}$ such that $\arg(p_u)$ is strictly monotonic on $\sigma^{(1)}$, and $\sigma^{(1)}$ is parameterized according to $\arg(p_u)$. Since $\arg(p_{\widehat{u}})$ is increasing along $\widehat{\gamma}$, $\arg(p_u)$ is increasing along the portions of γ which have already been defined. Let D denote an open region containing $\gamma(s^{(j)})$ on which p_u is injective. Choose some $t^{(0)} \in (r^{(j)}, s^{(j)})$ such that $\gamma(t^{(0)}, s^{(j)}) \subset D$. If $\arg(p_u)$ is decreasing on $\sigma^{(1)}$, then since p_u is injective on D, for each $r \in (s^{(j)}, t^{(0)})$, $\sigma^{(1)}(r) = \gamma(r)$. Furthermore, since p_u is injective in a neighborhood of each point of $\gamma([r^{(j)}, s^{(j)}])$, $\sigma^{(1)}$ must continue to trace back along the entire length of $\gamma([r^{(j)}, s^{(j)}])$. This is because both $\sigma^{(1)}$ and γ are parameterized according to $\arg(p_u)$, so any branching off of $\sigma^{(1)}$ from γ would have to be a critical point of p_u. However $\sigma^{(1)}$ may not trace back along $\gamma([r^{(j)}, s^{(j)}])$ because the image of $\sigma^{(1)}$ is contained in $B_{\delta^{(1)}}(u^{(l)})$. Therefore we conclude that $\arg(p_u)$ is increasing on $\sigma^{(1)}$.

By very similar reasoning we may obtain a path $\sigma^{(2)}$ from $u^{(l)}$ to $\gamma(r^{(j+1)})$ contained in $B_{\delta^{(1)}}(u^{(l)}) \cap E_{p_u,|v^{(l)}|}$ parameterized according to $\arg(p_u)$, and along which $\arg(p_u)$ is increasing.

Let $s^{(j)}$ be the choice of $\arg(p_u(\gamma(s^{(j)})))$ which is the starting point for the domain of $\sigma^{(1)}$, and choose some $t^{(1)} > 0$ so that the domain of $\sigma^{(1)}$ is $[s^{(j)}, s^{(j)} + t^{(1)}]$. Now let $s^{(j)} + t^{(1)}$ be the choice of $\arg(v^{(l)})$ which is the starting point for the domain of $\sigma^{(2)}$, and choose $t^{(2)} > 0$ so that the domain of $\sigma^{(2)}$ is $[s^{(j)} + t^{(1)}, s^{(j)} + t^{(1)} + t^{(2)}]$. Let $\sigma : [s^{(j)}, s^{(j)} + t^{(1)} + t^{(2)}] \to B_{\delta^{(1)}}(u^{(l)})$ denote the concatenation of $\sigma^{(1)}$ and $\sigma^{(2)}$. Then $s^{(j)} + t^{(1)} + t^{(2)} = r^{(j+1)} \pmod{2\pi}$, so $t^{(1)} + t^{(2)} = r^{(j+1)} - s^{(j)} \pmod{2\pi}$. However by choice of $\delta^{(1)}$, the total change in argument of p_u along σ must be less than π. And since $\widehat{\gamma}(s^{(j)}, r^{(j+1)}) \subset B_{\delta^{(1)}}(u^{(l)})$, the total change in argument of p_u along $\widehat{\gamma}|_{(s^{(j)}, r^{(j+1)})}$ is less than π, and thus the total change in argument of $p_{\widehat{u}}$ along $\widehat{\gamma}|_{(s^{(j)}, r^{(j+1)})}$ (which of course equals $r^{(j+1)} - s^{(j)}$) is less than 2π in magnitude (since $|p_u - p_{\widehat{u}}| < \tau$ on the image of $\widehat{\gamma}$). Thus we have that $t^{(1)} + t^{(2)} = r^{(j+1)} - s^{(j)} \pmod{2\pi}$ and both sides are contained in $[0, 2\pi)$, so $t^{(1)} + t^{(2)} = r^{(j+1)} - s^{(j)}$. Therefore we may define $\gamma(r) := \sigma(r)$ for each $r \in (s^{(j)}, r^{(j+1)})$. With this definition, we have that for each $r \in (s^{(j)}, r^{(j+1)})$, $\arg(p_u(\gamma(r))) = r$, and

$$|\gamma(r) - \widehat{\gamma(r)}| \le |\gamma(r) - u^{(l)}| + |u^{(l)} - \widehat{u^{(l)}}| + |\widehat{u^{(l)}} - \widehat{\gamma(r)}| < \delta^{(1)}.$$

We extend γ in this manner to $(s^{(j)}, r^{(j+1)})$ for each $j \in \{1, \dots, N-1\}$. Moreover, we may extend γ in using the exactly similar construction to $[\alpha, r^{(1)})$ and $(s^{(N)}, \beta]$, and this extended γ has all of the desired properties.

Case B.0.6. *There is no sub-interval of $[\alpha, \beta]$ along which $\widehat{\gamma}$ takes a $(\rho^{(1)}, \delta^{(1)})$ trip.*

Then either $|\widehat{\gamma}(r) - u^{(k)}| \leq \delta^{(1)}$ for all $r \in [\alpha, \beta]$, or there is some $r^{(0)} \in (\alpha, \beta)$ such that for all $r \in [\alpha, r^{(0)}]$, $|\widehat{\gamma}(r) - u^{(k)}| \leq \delta^{(1)}$, and for all $r \in (r^{(0)}, \beta]$, $\widehat{\gamma}$ is greater than $\rho^{(1)}$ from any critical point of p_u.

Subcase B.0.6.1. *$|\widehat{\gamma}(r) - u^{(k)}| \leq \delta^{(1)}$ for all $r \in [\alpha, \beta]$.*

In this case, we construct γ using the same method as in the second part of Case B.0.5.

Subcase B.0.6.2. *There is some $r^{(0)} \in (\alpha, \beta)$ such that for all $r \in [\alpha, r^{(0)}]$, $|\widehat{\gamma}(r) - u^{(k)}| \leq \delta^{(1)}$, and for all $r \in (r^{(0)}, \beta]$, $\widehat{\gamma}$ is greater than $\rho^{(1)}$ from any critical point of p_u.*

In this case, we construct γ on $[\alpha, r^{(0)})$ using the same method as in the second part of Case B.0.5, and we construct γ on $[r^{(0)}, \beta]$ using the same method as in the first part of Case B.0.5. $\square$

Lemma 10. *Fix some $v = (v^{(1)}, \dots, v^{(n-1)}) \in V_{n-1}$ not the zero vector, and $\delta^{(1)} > 0$. Then there exists constants $\rho, \delta^{(2)} > 0$ such that the following hold. Let $u \in \Theta^{-1}(v)$ be chosen, and fix some $\widehat{u} \in B_\rho(u)$. Let $\widehat{x_1}, \widehat{x_2} \in G_{p_{\widehat{u}}}$ be given such that $\arg(p_u(x_1)) = \arg(p_u(x_2)) = 0$, and such that there is a path $\widehat{\sigma} : [0,1] \to G_{p_u}$ such that $\widehat{\sigma(0)} = \widehat{x_1}$ and $\widehat{\sigma(1)} = \widehat{x_2}$ and $\arg(p_{\widehat{u}}(\widehat{\sigma}(r))) = 0$ for all $r \in [0,1]$. Then if $x_1, x_2 \in G_{p_{\widehat{u}}}$ are such that $\arg(p_u(x_1)) = \arg(p_u(x_1)) = 0$ and $|\widehat{x_1} - x_1| < \delta^{(2)}$ and $|\widehat{x^{(2)}} - x^{(2)}| < \delta^{(2)}$, then there is a path $\sigma : [0,1] \to G_{p_u}$ such that $\sigma(0) = x_1$, $\sigma(1) = x_2$, and for all $r \in [0,1]$, $\arg(p_u(\sigma(r))) = 0$ and $|\widehat{\sigma}(r) - \sigma(r)| < \delta^{(1)}$. Moreover, if $|p_{\widehat{u}}|$ is strictly increasing or strictly decreasing on $\widehat{\sigma}$, then we may assume that $|p_u|$ is strictly increasing or strictly deacreasing on σ respectively.*

Proof. The exact same method of proof used for Lemma 9 works here except that instead of invoking Lemma 6 we would invoke the gradient line version Lemma 7. □

Lemma 11. *Fix some* $v = (v^{(1)}, \dots, v^{(n-1)}) \in V_{n-1}$ *and some* $u \in \Theta^{-1}(v)$*. Let* $\delta > 0$ *be given, and choose some point* $x \in G_{p_u}$*. Then there are constants* $\rho, \nu > 0$ *small enough so that for any* $\widehat{u} \in B_\rho(u)$*, if* $\widehat{y} \in B_\nu(p_u(x))$ *then there is some* $\widehat{x} \in B_\delta(x)$ *such that* $p_{\widehat{u}}(\widehat{x}) = \widehat{y}$*.*

Proof. Note that the statement of the lemma is similar to the statement of Lemma 3, but more general in that the point x which is chosen may be a critical point of p_u. The proof is similar to a portion of the proof of Lemma 3, but we will reproduce it here.

Reduce δ if necessary so that $B_\delta(x) \subset G_{p_u}$ and there is no point w such that $|w - x| = \delta$ and $p_u(w) = p_u(x)$. Then define $\eta > 0$ to be the minimum that $|p_u(w) - p_u(x)|$ takes on the set $\{w \in \mathbb{C} : |w - x| = \delta\}$. Now choose $\rho > 0$ so that if $\widehat{u} \in B_\rho(u)$, then for all $w \in G_{p_u}$, $|p_u(w) - p_{\widehat{u}}(w)| < \frac{\eta}{4}$. Define $h(z) := p_u(x) - p_{\widehat{u}}(z)$. On the set $\{w \in \mathbb{C} : |w - x| = \delta\}$ by using the reverse triangle inequality we have

$$|h(z)| = |p_u(x) - p_{\widehat{u}}(z)| = |p_u(x) - p_u(z) + p_u(z) - p_{\widehat{u}}(z)| \geq |p_u(x) - p_u(z)| + |p_u(z) - p_{\widehat{u}}(z)|,$$

and thus

$$|h(z)| \geq \eta - \frac{\eta}{2} = \frac{\eta}{2}.$$

However $|h(x)| = |p_u(x) - p_{\widehat{u}}(z)| \leq \frac{\eta}{4}$, so by the Maximum Modulus Theorem, we conclude that h contains a zero in the set $\{w : |w - x| < \delta\}$, which is the desired result. □

Lemma 12. *Fix some compact set* $K \subset \mathbb{C}^{n-1}$ *and some* $\tau > 0$ *and* $R^{(0)} \in (0, 1)$*. There exists some* $\delta > 0$ *so that the following holds. Let* $u \in K$*, and let* $z \in \mathbb{C}$ *be such that*

$|p_u(z)| := R \in [R^{(0)}, 1]$*. For all* $w \in B_\delta(z)$*,* $|p_u(w)| \in (R-\tau, R+\tau)$*. Fix some* $w \in B_\delta(z)$*, and let* L *denote the straight line path from* z *to* w*, then* $|\Delta_{\arg}|(p_u, L) < \tau$*.*

Proof. Reduce τ if necessary so that $\tau \in (0, \pi)$. Since K is compact, there is some $S > 0$ such that for each $u \in K$, $G_{p_u} \subset B_{\frac{S}{2}}(0)$. Again since K is compact, and $cl(B_{\frac{S}{2}}(0))$ is compact, there is some $M > 0$ such that for all $u \in K$, for all $z \in B_S(0)$, $|p_u{}'(z)| < M$. Choose $\delta > 0$ so that the following hold.

- $\delta < \frac{S}{2}$.
- $\delta < \frac{R^{(0)}\tau}{2\pi M}$.

Fix some $u \in K$, and let $z \in \mathbb{C}$ be chosen so that $R := |p_u(z)| \in [R^{(0)}, 1]$. Then $B_\delta(z) \subset B_S(0)$, so if w is any point in $B_\delta(z)$, then $|p_u(w) - p_u(z)| \le |w-z|M < \delta M < \tau$. Therefore $|p_u(w)| \in (R-\tau, R+\tau)$.

Now fix some $w \in B_\delta(z)$, and let L denote the straight line path from z to w. Let $P : 0 = x^{(0)} < x^{(1)} < \cdots < x^{(N)} = 1$ be any fixed partition of $[0,1]$. Then for each $i \in \{0, \dots, N\}$, $|L(x^{(i)}) - z| < \delta$, so $|f(L(x^{(i)})) - f(z)| \le M\delta < \frac{R^{(0)}}{2}$. Therefore $|p_u(L(x^{(i)}))| \ge \frac{R^{(0)}}{2}$. Therefore, by geometry, for each $i \in \{1, \dots, N\}$,

$$|\arg(f(L(x^{(i)}))) - \arg(f(L(x^{(i-1)})))| < \frac{2\pi}{R^{(0)}}|f(L(x^{(i)})) - f(L(x^{(i-1)}))|.$$

Then since L is contained in $B_S(0)$, we have

$$|\arg(f(L(x^{(i)}))) - \arg(f(L(x^{(i-1)})))| \le \frac{2\pi M}{R^{(0)}}|L(x^{(i)}) - L(x^{(i-1)})|.$$

Since L is the straight line path from z to w, $\sum_{i=1}^{N} |L(x^{(i)}) - L(x^{(i-1)})| = |z-w| < \delta$. Therefore

$$\sum_{i=1}^{N} |\arg(f(L(x^{(i)}))) - \arg(f(L(x^{(i-1)})))| < \frac{2\pi M\delta}{R^{(0)}} < \tau,$$

which gives us the desired result. □

Lemma 13. *Let* (f, G) *be a special type function element, and let* λ *be a level curve of* f *in* G*. Let* D *be some face of* λ*, and let* $x, y \in \partial D$ *be given such that the line segment* (x, y) *is contained entirely in* D*. Define* $R^{(0)} := \min(|f(z)| : z \in [x, y])$*. Define* $R^{(1)} := \max(|f'(z)| : z \in [x, y])$*. Let* σ^+ *denote the path in* λ *obtained by traversing* ∂D *from* y *to* x *with a positive orientation, and let* σ^- *denote the path through* λ *obtained by traversing* ∂D *from* y *to* x *with a negative orientation. Define* $\delta^+ := \Delta_{\arg}(f, \sigma^+)$*, and* $\delta^- := \Delta_{\arg}(f, \sigma^-)$*. Then either there are zeros of* f *in* D *on both sides of* $[x, y]$*, or* $|x - y| \geq \dfrac{2R^{(0)} \min(|\delta^+|, |\delta^-|)}{\pi R^{(1)}}$.

Proof. Assume that all the zeros of f in D are on one side of (x, y) or the other. Let σ denote the path obtained by concatinating $[x, y]$ with σ^+. Then σ is a simple closed path. Let $\widehat{D}$ denote the bounded face of σ. Assume that $\widehat{D}$ does not contain any zeros of f. (Otherwise, we make the exactly similar argument with σ^+ replaced by σ^-.) Since $\widehat{D}$ does not contain any zero of f, $\Delta_{\arg}(f, \sigma) = 0$, and thus $\Delta_{\arg}(f, [x, y]) = -\Delta_{\arg}(f, \sigma^+)$, and thus $|\Delta_{\arg}(f, [x, y])| = |\delta^+|$.

Let γ denote the standard parameterization of $[x, y]$. Let $0 = t^{(0)} < \cdots < t^{(N)} = 1$ be a partition of $[0, 1]$ such that $\Delta_{\arg}(f, \gamma|_{[t^{(i-1)}, t^{(i)}]}) < \frac{\pi}{2}$ for each $i \in \{1, \ldots, N\}$, and define $\delta_i^+ := \Delta_{\arg}(f, \gamma|_{[t^{(i-1)}, t^{(i)}]})$. For each $i \in \{1, \ldots, N\}$, define $z^{(i)} := \gamma(t^{(i)})$. Since $[x, y]$ is a straight line, $|x - y| = \sum_{k=1}^{N} |z^{(i)} - z^{(i-1)}|$. Now since the $|\delta_i^+| < \frac{\pi}{2}$, we may use elementary trigonometry to show that

$$|f(x^{(i)}) - f(x^{(i-1)})| < 2R^{(0)} \sin(\frac{|\delta_i^+|}{2}).$$

Moreover, for $\alpha \in [0, \frac{\pi}{2}]$, $\sin(\alpha) \geq \frac{2\alpha}{\pi}$. Therefore we have

$$R^{(1)} \geq \left| \frac{f(z^{(i)}) - f(z^{(i-1)})}{z^{(i)} - z^{(i-1)}} \right| \geq \frac{2R^{(0)} |\delta_i^+|}{\pi |z^{(i)} - z^{(i-1)}|}.$$

Solving for $|z^{(i)} - z^{(i-1)}|$, we obtain

$$|z^{(i)} - z^{(i-1)}| \geq \frac{2R^{(0)}|\delta_i^+|}{\pi R^{(1)}}.$$

And if we sum over all i, we obtain

$$|x - y| \geq \frac{2R^{(0)}}{\pi R^{(1)}} \sum_{i=1}^{N} |\delta_i^+| \geq \frac{2R^{(0)}|\delta^+|}{\pi R^{(1)}} \geq \frac{2R^{(0)} \min(, |\delta^-|)}{\pi R^{(1)}},$$

which is the desired result. □

Lemma 14. *Let* (f, G) *be a special type function element, and let* λ *be a level curve of* f *in* G*. Let* $x, y \in \lambda$ *be given such that the line segment* (x, y) *is contained entirely in the unbounded face of* λ*. Let* $\widehat{D}$ *denote the bounded face of* $\lambda \cup [x, y]$ *which is not a bounded face of* λ*. Let* $\sigma^{(0)}$ *be a parameterization of the portion of* $\partial\widehat{D}$ *which is in* λ *from* y *to* x*, and define* $\delta := \Delta_{\arg}(f, \sigma^{(0)})$*. Define* $R^{(0)} := \min(|f(z)| : z \in [x, y])$*. Define* $R^{(1)} := \max(|f'(z)| : z \in [x, y])$*. Then either* $\widehat{D}$ *contains zeros of* f*, or* $|x - y| \geq \frac{2R^{(0)}|\delta|}{\pi R^{(1)}}$*.*

Proof. By the same reasoning found in Lemma 13. □

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BIOGRAPHICAL SKETCH

Trevor Richards was born in Lansing, Michigan in 1983. Trevor moved to Florida in 2002 to pursue under-graduate studies at the University of Florida, and graduated with a BS in mathematics with a minor in philosophy in 2006. Trevor began graduate study in the University of Florida Mathematics Department in 2007.