MODEL INDEPENDENT PARTICLE MASS MEASUREMENTS
IN MISSING ENERGY EVENTS AT HADRON COLLIDERS

By

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I dedicate this to my grand mother SunAe Seo and my wife Kelly Chung.
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C-5  A study of the sharpness of the $M_{T2}$ ridge for the case when missing particles are the same.
This dissertation describes several new kinematic methods to measure the masses of new particles in events with missing transverse energy at hadron colliders. Each method relies on the measurement of some feature (a peak or an endpoint) in the distribution of a suitable kinematic variable. The first method makes use of the "Gator" variable $\sqrt{s_{\text{min}}}$, whose peak provides a global and fully inclusive measure of the production scale of the new particles. In the early stage of the LHC, this variable can be used both as an estimator and a discriminator for new physics over the standard model backgrounds.

The next method studies the invariant mass distributions of the visible decay products from a cascade decay chain and the shapes and endpoints of those distributions. Given a sufficient number of endpoint measurements, one could in principle attempt to invert and solve for the mass spectrum. However, the non-linear character of the relevant coupled quadratic equations often leads to multiple solutions. In addition, there is a combinatorial ambiguity related to the ordering of the decay products from the cascade decay chain. We propose a new set of invariant mass variables which are less sensitive to these problems. We demonstrate how the new particle mass spectrum can be extracted from the measurement of their kinematic endpoints.

The remaining methods described in the dissertation are based on "transverse" invariant mass variables like the "Cambridge" transverse mass $M_{T2}$, the "Sheffield"
contrasverse mass $M_{CT}$ and their corresponding one-dimensional projections $M_{T2\perp}$, $M_{T2\parallel}$, $M_{CT\perp}$, and $M_{CT\parallel}$ with respect to the upstream transverse momentum $\vec{U}_T$.

The main advantage of all those methods is that they can be applied to very short (single-stage) decay topologies, as well as to a subsystem of the observed event. The methods can also be generalized to the case of non-identical missing particles, as demonstrated in Chapter 7. A complete set of analytical results for the calculation of the relevant variables in each event, as well as the dependence of their endpoints on the underlying mass spectrum is given for each case. In some circumstances, the whole shape of the differential distribution can be theoretically predicted as well. The methods are illustrated with examples from supersymmetry and from top quark production in the standard model.
CHAPTER 1
INTRODUCTION

The Large Hadron Collider (LHC) at CERN has begun its long awaited exploration of the TeV scale and reached integrated luminosity of $\mathcal{L} \approx 40\,pb^{-1}$ at 7 TeV. Starting with the hunt for standard model “Higgs” particle, we expect to see Beyond the Standard Model "(BSM)" phenomena at the LHC which may hold the key to our understanding of some very basic questions about our universe: What is the dark matter? What are the fundamental symmetries of Nature? Are there any hidden dimensions of space? A potential discovery of a missing energy signal at the LHC may relate to all three of these questions. Perhaps the most compelling phenomenological evidence for BSM particles and interactions at the TeV scale is provided by the dark matter problem [1]. It is a tantalizing coincidence that a neutral, weakly interacting massive particle (WIMP) in the TeV range can explain all of the observed dark matter in the Universe. A typical WIMP does not interact in the detector and can only manifest itself as missing energy. The WIMP idea therefore greatly motivates the study of missing energy signatures at the Tevatron and the LHC [2].

The long lifetime of the dark matter WIMPs is typically ensured by some new exact symmetry, e.g. $R$-parity in supersymmetry [3], KK parity in models with extra dimensions [4], $T$-parity in Little Higgs models [5, 6] etc. The particles of the Standard Model (SM) are not charged under this new symmetry, but the new particles are, and the lightest among them is the dark matter WIMP. This setup guarantees that the WIMP cannot decay, and more importantly, that WIMPs are always pair-produced at colliders. The cross-sections for direct production of WIMPs (tagged with a jet or a photon from initial state radiation) at hadron colliders are typically too small to allow observation above the SM backgrounds [7]. Therefore one typically concentrates on the pair production of the other, heavier particles (e.g. superpartners, KK-partners, or $T$-partners), which also carry nontrivial new quantum numbers just like the WIMPs. Once produced, those
heavier partners will cascade decay down, emitting SM particles which are in principle observable in the detector. However, each such cascade also inevitably ends up with an invisible WIMP, whose energy and momentum are unknown. Since the heavy partners are being pair-produced, there are two such cascades in each event, and therefore, two unknown WIMP momenta. In addition, at hadron colliders the total parton level energy and momentum in the center of mass frame are also unknown, and thus the exact reconstruction of the decay chains on an event by event basis is a very challenging task\(^1\). In this dissertation, we present how we can resolve these difficulties more systematically. Our approach is described schematically in the Figure 1-1.

**Systematic Approaches for Mass Determinations.** The very first question that we may want to ask would be what is the scale of the new physics. To answer this question, it would be best to avoid any assumptions about the event-topology. We studied variables which mimic true production energy \(\sqrt{s}\) of new particles. The reference \[50\] provides \(\sqrt{s}_{\text{min}}\) that is a minimization of \(\sqrt{s}\) with only one condition from missing energy constraint. It turned out that the peak of \(\sqrt{s}_{\text{min}}\) provides a global and fully inclusive measure of the production scale of the new particles. In the early stage of the LHC, this variable can be used both as an estimator and a discriminator for new physics over the standard model backgrounds. More detailed studies on \(\sqrt{s}_{\text{min}}\) will be provided in Chapter 2.

Since we have not specified on the event-topology, \(\sqrt{s}_{\text{min}}\) itself is not precise enough to determine the full mass spectrum of the new particles. In addition to \(\sqrt{s}_{\text{min}}\) one can also study invariant mass of various visible particles. As we will point out, it is possible to reconstruct the intermediate particles’ masses when the cascade decay chain is long enough. The problem of using invariant-mass is that we need

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\(^1\) See “Dark matter and collider phenomenology of universal extra dimensions” (Phys. Rept. 453, 29 (2007)) \[8\] for a recent review.
to specify which visible particle comes from which intermediate particle, namely we suffer from combinatorial problems, thus if $N_{\text{cascade}}$ (the length of decaying chain) is large, it becomes more difficult to choose the right set of visible particles to form the invariant mass.\(^2\) On top of this problem, when we invert the invariant mass endpoints to solve for the mass spectra, there will be multiple solutions coming from the non-linear characteristics of coupled quadratic equations [9]. Those difficulties initiate our projects on invariant mass methods, and detailed studies will be presented in Chapter 3.

While invariant mass methods rely on a single cascade decay mode of a new particle, usually new physics will come with pair produced particles due to the new exact symmetry. Thus if we use this extra condition about produced particles, we can reconstruct new particles’ mass even for short decaying modes like as $N_{\text{cascade}} = 2$ which is not possible with the invariant mass endpoints method. We developed a “subsystem” concept which we can apply to specific decay modes. We applied the “subsystem” to a Cambridge transverse variable called $M_{T^2}$ [10] in Chapter 4.

While a subsystem $M_{T^2}$ in principle works for $N_{\text{cascade}} \geq 1$, when $N_{\text{cascade}} = 1$ the resolution of the mass determination is not so good. This is because the mass determination depends on the hardness of the upstream momentum, which is not big enough when upstream objects come from the initial state radiation. We proposed orthogonal “decompositions” of known kinematic variables such as $M_{CT}$ and $M_{T^2}$ onto that special transverse direction [11, 12]. We realized that the “doubly transverse” quantities like $M_{CT\perp}$ and $M_{T^2\perp}$ are particularly useful, since their kinematic endpoints are independent of $\vec{U}_T$ and make it possible to use the whole structure of phase space so that we can determine the mass spectra of related particles in a very short decay chain. We provide these methods in Chapter 5.

\(^2\) $\sqrt{s_{\text{min}}}$ does not suffer from this kind of combinatorial problem.
Given our utter ignorance about the structure of the dark matter sector, we set out to develop the necessary formalism for carrying out missing energy studies at hadron colliders in a very general and model-independent way, without relying on any assumptions about the nature of the missing particles. In particular, we did not assume that the two missing particles in each event are the same. We generalized the $M_{T2}$ idea to asymmetric events with different missing particles in Chapter 6.
Figure 1-1. A schematic description of the mass-determination methods described in this dissertation.
CHAPTER 2
GENERAL ANALYSIS WITHOUT ANY ASSUMPTIONS

2.1 The Need for a Universal, Global and Inclusive Mass Variable

Most methods are model-dependent in the sense that each method crucially relies on the assumption of a very specific event topology. One common flaw of all methods on the market is that they usually do not allow any SM neutrinos to enter the targeted event topology, and the missing energy is typically assumed to arise only as a result of the production of (two) new dark matter particles. Furthermore, each method has its own limitations. For example, the traditional invariant mass endpoint methods \cite{9, 13–22} require the identification of a sufficiently long cascade decay chain, with at least three successive two-body decays \cite{23}. The polynomial methods \cite{24–31} also require such long decay chains and furthermore, the events must be symmetric, i.e. must have two identical decay chains per event, or else the decay chain must be even longer \cite{23}. The recently popular $M_{T2}$ methods \cite{10, 12, 32–39} do not require long decay chains \cite{23}, but typically assume that the parent particles are the same and decay to two identical invisible particles\footnote{See “Dark Matter Particle Spectroscopy at the LHC: Generalizing MT2 to Asymmetric Event Topologies (JHEP 1004, 086 (2010)) \cite{41} for a more general approach which avoids this assumption.}. The limitations of the $M_{CT}$ methods \cite{11, 42, 43} are rather similar. The kinematic cusp method \cite{44} is limited to the so called “antler” event topology, which contains two symmetric one-step decay chains originating from a single $s$-channel resonance. In light of all these various assumptions, it is certainly desirable to have a universal method which can be applied to any event topology.

The $\sqrt{s_{min}}$ variable is defined in terms of the total energy $E$ and 3-momentum $\vec{p}$ observed in the event, and thus does not make any reference to the actual event topology. It is completely general, universal and fully inclusive, and to the fullest extent makes use of the available experimental information.
Figure 2-1. The generic event topology used to define the $\sqrt{s}_{\text{min}}$ variable in “$\sqrt{s}_{\text{min}}$: A Global inclusive variable for determining the mass scale of new physics in events with missing energy at hadron colliders, (JHEP 0903, 085 (2009))”. Black (red) lines correspond to SM (BSM) particles. The solid lines denote SM particles $X_i, i = 1, 2, ..., n_{\text{vis}}$, which are visible in the detector, e.g. jets, electrons, muons and photons. The SM particles may originate either from initial state radiation (ISR), or from the hard scattering and subsequent cascade decays (indicated with the green-shaded ellipse). The dashed lines denote neutral stable particles $\chi_i, i = 1, 2, ..., n_{\text{inv}}$, which are invisible in the detector. In general, the set of invisible particles consists of some number $n_\chi$ of BSM particles (indicated with the red dashed lines), as well as some number $n_\nu = n_{\text{inv}} - n_\chi$ of SM neutrinos (denoted with the black dashed lines). The identities and the masses $m_i$ of the BSM invisible particles $\chi_i, (i = 1, 2, ..., n_\chi)$ do not necessarily have to be all the same, i.e. we allow for the simultaneous production of several different species of dark matter particles. The global event variables describing the visible particles are: the total energy $E$, the transverse components $P_x$ and $P_y$ and the longitudinal component $P_z$ of the total visible momentum $\vec{P}$. The only experimentally available information regarding the invisible particles is the missing transverse momentum $\vec{P}_T$. 
2.1.1 Definition of $\sqrt{s}_{\text{min}}$

Consider the most generic missing energy event topology shown in Figure 2-1. In defining $\sqrt{s}_{\text{min}}$, one imagines a completely general setup – each event contains some number $n_{\text{vis}}$ of Standard Model (SM) particles $X_i, i = 1, 2, \ldots, n_{\text{vis}}$, which are visible in the detector, i.e. their energies and momenta are in principle measured. Examples of such visible SM particles are the basic reconstructed objects, e.g. jets, photons, electrons and muons. The visible particles $X_i$ are denoted in Figure 2-1 with solid black lines and may originate either from ISR, or from the hard scattering and subsequent cascade decays (indicated with the green-shaded ellipse). In turn, the missing transverse momentum $\vec{P}_T$ arises from a certain number $n_{\text{inv}}$ of stable neutral particles $\chi_i, i = 1, 2, \ldots, n_{\text{inv}}$, which are invisible in the detector. In general, the set of invisible particles consists of some number $n_\chi$ of BSM particles (indicated with the red dashed lines), as well as some number $n_\nu = n_{\text{inv}} - n_\chi$ of SM neutrinos (denoted with the black dashed lines).

As already mentioned earlier, the $\vec{P}_T$ measurement alone does not reveal the number $n_{\text{inv}}$ of missing particles, nor how many of them are neutrinos and how many are BSM (dark matter) particles. This general setup also allows the identities and the masses $m_i$ of the BSM invisible particles $\chi_i, (i = 1, 2, \ldots, n_\chi)$ in principle to be different, as in models with several different species of dark matter particles [45–49]. Of course, the neutrino masses can be safely taken to be zero

$$m_i = 0, \text{ for } i = n_\chi + 1, n_\chi + 2, \ldots, n_{\text{inv}}. \quad (2-1)$$

Given this very general setup, if we try to minimize the parton-level Mandelstam invariant mass variable $\sqrt{s}$ which is consistent with the observed visible 4-momentum vector $P^\mu \equiv (E, \vec{P})$, we will get the following minimum of $\sqrt{s}$ as

$$\sqrt{s}_{\text{min}}(\mathcal{M}) \equiv \sqrt{E^2 - \vec{P}_Z^2} + \sqrt{\mathcal{M}^2 + \vec{P}_T^2}, \quad (2-2)$$
where the mass parameter $\mathcal{M}$ is nothing but the total mass of all invisible particles in the event:

$$\mathcal{M} \equiv \sum_{i=1}^{n_{\text{inv}}} m_i = \sum_{i=1}^{n_{\chi}} m_i,$$

and the second equality follows from the assumption of vanishing neutrino masses (2–1). The result (2–2) can be equivalently rewritten in a more symmetric form

$$\sqrt{s_{\text{min}}}(\mathcal{M}) = \sqrt{\mathcal{M}^2 + P_T^2} + \sqrt{\mathcal{M}^2 + \overline{P}_T^2}$$

in terms of the total visible invariant mass $M$ defined as

$$M^2 \equiv E^2 - P_x^2 - P_y^2 - P_z^2 \equiv E^2 - P_T^2 - \overline{P}_T^2.$$

Notice that in spite of the complete arbitrariness of the invisible particle sector at this point, the definition of $\sqrt{s_{\text{min}}}$ depends on a single unknown parameter $\mathcal{M}$ - the sum of all the masses of the invisible particles in the event. For future reference, one should keep in mind that transverse momentum conservation at this point implies that

$$\overline{P}_T + \overline{P}_T = 0.$$

The main result from “$\sqrt{s_{\text{min}}}$ : A Global inclusive variable for determining the mass scale of new physics in events with missing energy at hadron colliders, (JHEP 0903, 085 (2009))[50] was that in the absence of ISR and MPI, the peak in the $\sqrt{s_{\text{min}}}$ distribution nicely correlates with the mass threshold of the newly produced particles. This observation provides one generic relation between the total mass of the produced particles and the total mass $\mathcal{M}$ of the invisible particles.

### 2.1.2 $\sqrt{s_{\text{min}}}$ and the Underlying Event Problem

At the same time, it was also recognized that effects from the underlying event (UE), most notably ISR and MPI, severely jeopardize this measurement. The problem is that in the presence of the UE, the $\sqrt{s_{\text{min}}}$ variable would be measuring the total energy of the full system shown in Figure 2-1, while for studying any new physics we are mostly
interested in the energy of the hard scattering, as represented by the green-shaded ellipse in Figure 2-1. The inclusion of the UE causes a drastic shift of the peak of the $\sqrt{s_{\text{min}}}$ distribution to higher values, often by as much as a few TeV [50–52]. As a result, it appeared that unless effects from the underlying event could somehow be compensated for, the proposed measurement of the $\sqrt{s_{\text{min}}}$ peak would be of no practical value.

The main purpose of this chapter is to propose two fresh new approaches to dealing with the underlying event problem which has plagued the $\sqrt{s_{\text{min}}}$ variable and prevented its more widespread use in hadron collider physics applications. We propose two new variants of the $\sqrt{s_{\text{min}}}$ variable, which we label $\sqrt{s_{\text{min}}^{(\text{reco})}}$ and $\sqrt{s_{\text{min}}^{(\text{sub})}}$ and define in Sections 2.2 and 2.3, correspondingly. We illustrate the properties of these two variables with several examples in Sections 2.4-2.6. These examples will show that both $\sqrt{s_{\text{min}}^{(\text{reco})}}$ and $\sqrt{s_{\text{min}}^{(\text{sub})}}$ are unharmed by the effects from the underlying event, thus resurrecting the original idea of $\sqrt{s_{\text{min}}}$ proposed in Reference [50] to use the peak in the $\sqrt{s_{\text{min}}}$ distribution as a first, quick, model-independent estimate of the new physics mass scale. In Section 2.7 we compare the performance of $\sqrt{s_{\text{min}}}$ against some other inclusive variables which are commonly used in hadron collider physics for the purpose of estimating the new physics mass scale.

2.2 Definition of the RECO level Variable $\sqrt{s_{\text{min}}^{(\text{reco})}}$

In the first approach, we shall not modify the original definition of $\sqrt{s_{\text{min}}}$ and will continue to use the Equation (2–2) (or its equivalent Equation (2–4)), preserving the desired universal, global and inclusive character of the $\sqrt{s_{\text{min}}}$ variable. Then we shall concentrate on the question, how should one calculate the observable quantities $E$, $\vec{P}$ and $\vec{P}_T$ entering the defining Equations (2–2) and (2–4).

The previous $\sqrt{s_{\text{min}}}$ studies [50–52] used calorimeter-based measurements of the total visible energy $E$ and momentum $\vec{P}$ as follows. The total visible energy in the
calorimeter \(E_{(\text{cal})}\) is simply a scalar sum over all calorimeter deposits

\[
E_{(\text{cal})} \equiv \sum_{\alpha} E_{\alpha}, \quad (2-7)
\]

where the index \(\alpha\) labels the calorimeter towers, and \(E_{\alpha}\) is the energy deposit in the \(\alpha\) tower. As usual, since muons do not deposit significantly in the calorimeters, the measured \(E_{\alpha}\) should first be corrected for the energy of any muons which might be present in the event and happen to pass through the corresponding tower \(\alpha\). The three components of the total visible momentum \(\vec{P}\) were also measured from the calorimeters as

\[
P_{x(\text{cal})} = \sum_{\alpha} E_{\alpha} \sin \theta_{\alpha} \cos \varphi_{\alpha}, \quad (2-8)
\]

\[
P_{y(\text{cal})} = \sum_{\alpha} E_{\alpha} \sin \theta_{\alpha} \sin \varphi_{\alpha}, \quad (2-9)
\]

\[
P_{z(\text{cal})} = \sum_{\alpha} E_{\alpha} \cos \theta_{\alpha}, \quad (2-10)
\]

where \(\theta_{\alpha}\) and \(\varphi_{\alpha}\) are correspondingly the polar and azimuthal angular coordinates of the \(\alpha\) calorimeter tower. The missing transverse momentum can similarly be measured from the calorimeter as (Equation (2–6))

\[
\vec{P}_{T(\text{cal})} \equiv -\vec{P}_{T(\text{cal})}. \quad (2-11)
\]

Using these calorimeter-based measurements (2–7-2–11), one can make the identification

\[
E \equiv E_{(\text{cal})}, \quad (2-12)
\]

\[
\vec{P} \equiv \vec{P}_{(\text{cal})}, \quad (2-13)
\]

\[
\vec{P}_{T} \equiv \vec{P}_{T(\text{cal})} \quad (2-14)
\]
in the definition (2–2) and construct the corresponding “calorimeter-based” \( \sqrt{s_{\text{min}}} \) variable as

\[
\sqrt{s_{\text{min}}^{(\text{cal})}}(\mathcal{M}) \equiv \sqrt{E_{\text{cal}}^2 - P_{z,\text{cal}}^2 + \sqrt{M_{\text{cal}}^2 + P_{T,\text{cal}}^2}}. \tag{2–15}
\]

This was precisely the quantity which was studied in [50–52] and shown to exhibit extreme sensitivity to the physics of the underlying event.

Here we propose to evaluate the visible quantities \( E \) and \( \vec{P} \) at the RECO level, i.e. in terms of the reconstructed objects, namely jets, muons, electrons and photons. To be precise, let there be \( N_{\text{obj}} \) reconstructed objects in the event, with energies \( E_i \) and 3-momenta \( \vec{P}_i \), \( i = 1, 2, \ldots, N_{\text{obj}} \), correspondingly. Then in place of Equations (2–12–2–14), let us instead identify

\[
E \equiv E_{\text{(reco)}}(\mathcal{M}) \equiv \sum_{i=1}^{N_{\text{obj}}} E_i, \tag{2–16}
\]

\[
\vec{P} \equiv \vec{P}_{\text{(reco)}}(\mathcal{M}) \equiv \sum_{i=1}^{N_{\text{obj}}} \vec{P}_i, \tag{2–17}
\]

\[
\vec{P}_T \equiv \vec{P}_{T,\text{(reco)}} = -\vec{P}_{T,\text{(reco)}}, \tag{2–18}
\]

and correspondingly define a “RECO-level” \( \sqrt{s_{\text{min}}} \) variable as

\[
\sqrt{s_{\text{min}}^{(\text{reco})}}(\mathcal{M}) \equiv \sqrt{E_{\text{reco}}^2 - P_{z,\text{reco}}^2 + \sqrt{M_{\text{reco}}^2 + P_{T,\text{reco}}^2}}, \tag{2–19}
\]

which can also be rewritten in analogy to Equation (2–4) as

\[
\sqrt{s_{\text{min}}^{(\text{reco})}}(\mathcal{M}) \equiv \sqrt{M_{\text{reco}}^2 + P_{T,\text{reco}}^2} + \sqrt{M_{\text{reco}}^2 + P_{T,\text{reco}}^2}, \tag{2–20}
\]

where \( P_{T,\text{reco}} \) and \( P_{T,\text{reco}} \) are related as in Equation (2–18) and the RECO-level total visible mass \( M_{\text{reco}} \) is defined by

\[
M_{\text{reco}}^2 \equiv E_{\text{reco}}^2 - \vec{P}_{\text{reco}}^2. \tag{2–21}
\]
What are the benefits from the new RECO-level $\sqrt{s}_{\text{min}}$ defined as in Equations (2–19,2–20) in comparison to the old calorimeter-based $\sqrt{s}_{\text{min}}$ definition in an Equation (2–15)? In order to understand the basic idea, it is worth comparing the calorimeter-based missing transverse momentum $P_T$ (which in the literature is commonly referred to as “missing transverse energy” $E_T$) and the analogous RECO-level variable $\vec{H}_T$, the “missing $H_T$”. The $\vec{H}_T$ vector is defined as the negative of the vector sum of the transverse momenta of all reconstructed objects in the event:

$$\vec{H}_T \equiv - \sum_{i=1}^{N_{\text{obj}}} \vec{P}_{Ti}.$$  (2–22)

Then it is clear that in terms of our notation here, $\vec{H}_T$ is nothing but $P_{T(\text{reco})}$.

It is known that $\vec{H}_T$ performs better than $E_T$ [53]. First, $\vec{H}_T$ is less affected by a number of adverse instrumental factors such as: electronic noise, faulty calorimeter cells, pile-up, etc. These effects tend to populate the calorimeter uniformly with unclustered energy, which will later fail the basic quality cuts during object reconstruction. In contrast, the true missing momentum is dominated by clustered energy, which will be successfully captured during reconstruction. Another advantage of $\vec{H}_T$ is that one can easily apply the known jet energy corrections to account for the nonlinear detector response. For both of these reasons, CMS is now using $\vec{H}_T$ at both the trigger level and offline [53].

Now realize that $\sqrt{s}_{\text{cal}}(\text{cal})$ is analogous to the calorimeter-based $E_T$, while our new variable $\sqrt{s}_{\text{min}}^{(\text{reco})}$ is analogous to the RECO-level $\vec{H}_T$. Thus we may already expect that $\sqrt{s}_{\text{min}}^{(\text{reco})}$ will inherit the advantages of $\vec{H}_T$ and will be better suited for determining the new physics mass scale than the calorimeter-based quantity $\sqrt{s}_{\text{min}}^{(\text{cal})}$. This expectation is confirmed in the explicit examples studied below in Sections 2.4 and 2.5. Apart from the already mentioned instrumental issues, the most important advantage of $\sqrt{s}_{\text{min}}^{(\text{reco})}$ from the physics point of view is that it is much less sensitive to the effects from the underlying event, which had doomed its calorimeter-based $\sqrt{s}_{\text{min}}^{(\text{cal})}$ cousin.
Strictly speaking, the idea of $\sqrt{s}_{\text{min}}^{(\text{reco})}$ does not solve the underlying event problem completely and as a matter of principle. Every now and then the underlying event will still produce a well-defined jet, which will have to be included in the calculation of $\sqrt{s}_{\text{min}}^{(\text{reco})}$. Because of this effect, we cannot any more guarantee that $\sqrt{s}_{\text{min}}^{(\text{reco})}$ provides a lower bound on the true value $\sqrt{s}_{\text{true}}$ of the center-of-mass energy of the hard scattering — the additional jets formed out of ISR, pile-up, and so on, will sometimes cause $\sqrt{s}_{\text{min}}^{(\text{reco})}$ to exceed $\sqrt{s}_{\text{true}}$. Nevertheless we find that this effect modifies only the shape of the $\sqrt{s}_{\text{min}}^{(\text{reco})}$ distribution, but leaves the location of its peak largely intact. To the extent that one is mostly interested in the peak location, $\sqrt{s}_{\text{min}}^{(\text{reco})}$ should already be good enough for all practical purposes.

2.3 Definition of the Subsystem Variable $\sqrt{s}_{\text{min}}^{(\text{sub})}$

In this section we propose an alternative modification of the original $\sqrt{s}_{\text{min}}$ variable, which solves the underlying event problem completely and as a matter of principle. The downside of this approach is that it is not as general and universal as the one discussed in the previous section, and can be applied only in cases where one can unambiguously identify a subsystem of the original event topology which is untouched by the underlying event. The basic idea is schematically illustrated in Figure 2-2, which is nothing but a slight rearrangement of Figure 2-1 exhibiting a well defined subsystem (delineated by the black rectangle). The original $n_{\text{vis}}$ visible particle $X_i$ from Figure 2-1 have now been divided into two groups as follows:

1. There are $n_{\text{sub}}$ visible particles $X_1, \ldots, X_{n_{\text{sub}}}$ originating from within the subsystem. Their total energy and total momentum are denoted by $E_{(\text{sub})}$ and $\vec{P}_{(\text{sub})}$. The subsystem particles are chosen so that to guarantee that they could not have come from the underlying event.

2. The remaining $n_{\text{vis}} - n_{\text{sub}}$ visible particles $X_{n_{\text{sub}}+1}, \ldots, X_{n_{\text{vis}}}$ are created upstream (outside the subsystem) and have total energy $E_{(\text{up})}$ and total momentum $\vec{P}_{(\text{up})}$. The upstream particles may originate from the underlying event or from decays of heavier particles upstream – this distinction is inconsequential at this point.
Figure 2-2. A rearrangement of Figure 2-1 into an event topology exhibiting a well defined subsystem (delineated by the black rectangle) with total invariant mass \( \sqrt{s^{(sub)}} \). There are \( n_{\text{sub}} \) visible particles \( X_i \), \( i = 1, 2, \ldots, n_{\text{sub}} \), originating from within the subsystem, while the remaining \( n_{\text{vis}} - n_{\text{sub}} \) visible particles \( X_{n_{\text{sub}} + 1}, \ldots, X_{n_{\text{vis}}} \) are created upstream, outside the subsystem. The subsystem results from the production and decays of a certain number of parent particles \( P_j \), \( j = 1, 2, \ldots, n_p \), (some of) which may decay semi-invisibly. All invisible particles \( \chi_1, \ldots, \chi_{n_{\text{inv}}} \) are then assumed to originate from within the subsystem.

We also assume that all invisible particles \( \chi_1, \ldots, \chi_{n_{\text{inv}}} \) originate from within the subsystem, i.e. that no invisible particles are created upstream. In effect, all we have done in Figure 2-2 is to partition the original measured values of the total visible energy \( E \) and 3-momentum \( \vec{p} \) from Figure 2-1 into two separate components as

\[
E = E_{(up)} + E_{(sub)} , \tag{2–23}
\]

\[
\vec{p} = \vec{p}_{(up)} + \vec{p}_{(sub)} , \tag{2–24}
\]
Notice that now the missing transverse momentum is defined as

\[ \vec{p}_T \equiv -\vec{p}_{T(up)} - \vec{p}_{T(sub)}, \] (2–25)

while the total visible invariant mass \( M_{(\text{sub})} \) of the subsystem is given by

\[ M_{(\text{sub})}^2 = E_{(\text{sub})}^2 - \vec{p}_{(\text{sub})}^2. \] (2–26)

There would be ambiguities in categorizing a given visible particle \( X_i \) as a subsystem or an upstream particle. Since our goal is to identify a subsystem which is shielded from the effects of the underlying event, the safest way to do the partition of the visible particles is to require that all QCD jets belong to the upstream particles, while the subsystem particles consist of objects which are unlikely to come from the underlying event, such as isolated electrons, photons and muons (and possibly identified \( \tau \)-jets and, to a lesser extent, tagged \( b \)-jets).

With those preliminaries, we are now ready to ask the usual \( \sqrt{s}_{\text{min}} \) question: Given the measured values of \( E_{(up)}, E_{(sub)}, \vec{p}_{(up)} \) and \( \vec{p}_{(sub)} \), what is the minimum value \( \sqrt{s}_{\text{min}}^{(sub)} \) of the subsystem Mandelstam invariant mass variable \( \sqrt{s}_{\text{sub}} \), which is consistent with those measurements? Proceeding as in [50], once again we find a very simple universal answer, which, with the help of Equations (2–25) and (2–26), can be equivalently written in several different ways as follows:

\[ \sqrt{s}_{\text{min}}^{(sub)} (M) = \left\{ \left( \sqrt{E_{(sub)}^2 - \vec{p}_{z(sub)}^2 + \sqrt{M^2 + \vec{p}_{T}^2}} \right)^2 - \vec{p}_{T(up)}^2 \right\}^{\frac{1}{2}} \] (2–27)

\[ = \left\{ \left( \sqrt{M_{(sub)}^2 + \vec{p}_{T(sub)}^2} + \sqrt{M^2 + \vec{p}_{T}^2} \right)^2 - \vec{p}_{T(up)}^2 \right\}^{\frac{1}{2}} \] (2–28)

\[ = \left\{ \left( \sqrt{M_{(sub)}^2 + \vec{p}_{T(sub)}^2} + \sqrt{M^2 + \vec{p}_{T}^2} \right)^2 - (\vec{p}_{T(sub)} + \vec{p}_{T})^2 \right\}^{\frac{1}{2}} \] (2–29)

\[ = \big| |\vec{p}_{T(sub)} + \vec{p}_{T}| \big|, \] (2–30)
where in the last line we have introduced the Lorentz 1+2 vectors

\[ p_T^{(sub)} \equiv \left( \sqrt{M_{(sub)}^2 + p_T^{2 (sub)}} , \tilde{P}_T^{(sub)} \right), \quad (2–31) \]

\[ p_T' \equiv \left( \sqrt{M^2 + p_T'^2} , \tilde{P}_T' \right). \quad (2–32) \]

As usual, the length of a 1+2 vector is computed as \( ||p|| = \sqrt{p \cdot p} = \sqrt{p_0^2 - p_1^2 - p_2^2} \).

Before we proceed to the examples of the next few sections, as a sanity check of the obtained result it is useful to consider some limiting cases. First, by taking the upstream visible particles to be an empty set, i.e. \( \tilde{P}_{T(\text{up})} \to 0 \), we recover the usual expression for \( \sqrt{s_{\text{min}}} \) given in Equations (2–2,2–4). Next, consider a case with no invisible particles, i.e. \( \mathcal{M} = 0 \) and correspondingly, \( \tilde{P}_T = 0 \). In that case we obtain that \( \sqrt{s_{\text{min}}^{(sub)}} = M_{(sub)} \), which is of course the correct result. Finally, suppose that there are no visible subsystem particles, i.e. \( E_{(sub)} = \tilde{P}_{(sub)} = M_{(sub)} = 0 \). In that case we obtain \( \sqrt{s_{\text{min}}^{(sub)}} = \mathcal{M} \), which is also the correct answer.

As we shall see, the subsystem concept of Figure 2-2 will be most useful when the subsystem results from the production and decays of a certain number \( n_p \) of parent particles \( P_j \) with masses \( M_{P_j}, j = 1, 2, \ldots, n_p \), correspondingly. Then the total combined mass of all parent particles is given by

\[ M_p \equiv \sum_{j=1}^{n_p} M_{P_j}. \quad (2–33) \]

By the conjecture of Reference [50], the location of the peak of the \( \sqrt{s_{\text{min}}^{(sub)}} ( \mathcal{M} ) \) distribution will provide an approximate measurement of \( M_p \) as a function of the unknown parameter \( \mathcal{M} \). By construction, the obtained relationship \( M_p ( \mathcal{M} ) \) will then be completely insensitive to the effects from the underlying event.

At this point it may seem that by excluding all QCD jets from the subsystem, we have significantly narrowed down the number of potential applications of the \( \sqrt{s_{\text{min}}^{(sub)}} \) variable. Furthermore, we have apparently reintroduced a certain amount of
model-dependence which the original $\sqrt{s_{\text{min}}}$ approach was trying so hard to avoid. Those are in principle valid objections, which can be overcome by using the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ variable introduced in the previous section. Nevertheless, we feel that the $\sqrt{s_{\text{min}}^{(\text{sub})}}$ variable can prove to be useful in its own right, and in a wide variety of contexts. To see this, note that the typical hadron collider signatures of the most popular new physics models (supersymmetry, extra dimensions, Little Higgs, etc.) are precisely of the form exhibited in Figure 2-2. One typically considers production of colored particles (squarks, gluinos, KK-quarks, etc.) whose cross-sections dominate. In turn, these colored particles shed their color charge by emitting jets and decaying to lighter, uncolored particles in an electroweak sector. The decays of the latter often involve electromagnetic objects, which could be targeted for selection in the subsystem. The $\sqrt{s_{\text{min}}^{(\text{sub})}}$ variable would then be the perfect tool for studying the mass scales in the electroweak sector (in the context of supersymmetry, for example, the electroweak sector is composed of the charginos, neutralinos and sleptons).

Before we move on to some specific examples illustrating these ideas, one last comment is in order. One may wonder whether the $\sqrt{s_{\text{min}}^{(\text{sub})}}$ variable should be computed at the RECO-level or from the calorimeter. Since the subsystem will usually be defined in terms of reconstructed objects, the more logical option is to calculate $\sqrt{s_{\text{min}}^{(\text{sub})}}$ at the RECO-level and label it as $\sqrt{s_{\text{min}}^{(\text{sub, reco})}}$. However, to streamline our notation, in what follows we shall always omit the “reco” part of the superscript and will always implicitly assume that $\sqrt{s_{\text{min}}^{(\text{sub})}}$ is computed at RECO-level.

2.4 SM example: Dilepton Events from $t\bar{t}$ production

In this and the next two sections we illustrate the properties of the new variables $\sqrt{s_{\text{min}}^{(\text{reco})}}$ and $\sqrt{s_{\text{min}}^{(\text{sub})}}$ with some specific examples. In this section we discuss an example taken from the Standard Model, which is guaranteed to be available for early studies at the LHC. We consider dilepton events from $t\bar{t}$ pair production, where both $W$‘s decay leptonically. In this event topology, there are two missing particles (two neutrinos).
Therefore, these events very closely resemble the typical SUSY-like events, in which there are two missing dark matter particles. In the next two sections, we shall also consider some SUSY examples. In all cases, we perform detailed event simulation, including the effects from the underlying event and detector resolution.

2.4.1 Event Simulation Details

Events are generated with PYTHIA [54] (using its default model of the underlying event) at an LHC of 14 TeV, and then reconstructed with the PGS detector simulation package [55]. We have made certain modifications in the publicly available version of PGS to better match it to the CMS detector. For example, we take the hadronic calorimeter resolution to be [56]

\[ \frac{\sigma}{E} = 120\% - \frac{E}{\sqrt{E}}, \]  

(2–34)

while the electromagnetic calorimeter resolution is [56]

\[ \left( \frac{\sigma}{E} \right)^2 = \left( \frac{S}{\sqrt{E}} \right)^2 + \left( \frac{N}{E} \right)^2 + C^2, \]  

(2–35)

where the energy \( E \) is measured in GeV, \( S = 3.63\% \) is the stochastic term, \( N = 0.124 \) is the noise and \( C = 0.26\% \) is the constant term. Muons are reconstructed within \( |\eta| < 2.4 \), and we use the muon global reconstruction efficiency quoted in [56]. We use default \( p_T \) cuts on the reconstructed objects as follows: 3 GeV for muons, 10 GeV for electrons and photons, and 15 GeV for jets. For the \( t\bar{t} \) example presented in this section, we use the approximate next-to-next-to-leading order \( t\bar{t} \) cross-section of \( \sigma_{t\bar{t}} = 894 \pm 4^{+73+12}_{-46-12} \) pb at a top mass of \( m_t = 175 \) GeV [57]. For the SUSY examples in the next two sections we use leading order cross-sections.

2.4.2 \( \sqrt{s_{\text{min}}^{(\text{reco})}} \) Variable

We first consider SUSY-like missing energy events arising from \( t\bar{t} \) production, where each \( W \)-boson is forced to decay leptonically (to an electron or a muon).
Figure 2-3. Distributions of various $\sqrt{s_{\text{min}}}$ quantities discussed in the text, for the dilepton $t\bar{t}$ sample at the LHC with 14 TeV CM energy and 0.5 fb$^{-1}$ of data. The dotted (yellow-shaded) histogram gives the true $\sqrt{s}$ distribution of the $t\bar{t}$ pair. The blue histogram is the distribution of the calorimeter-based $\sqrt{s_{\text{min}}^{(\text{cal})}}$ variable in the ideal case when all effects from the underlying event are turned off. The red histogram shows the corresponding result for $\sqrt{s_{\text{min}}^{(\text{cal})}}$ in the presence of the underlying event. The black histogram is the distribution of the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ variable introduced in Section 2.2. All $\sqrt{s_{\text{min}}}$ distributions are shown for $\sqrt{\hat{s}} = 0$.

We do not impose any trigger or offline requirements, and simply plot directly the output from PGS$^2$. We show various $\sqrt{s}$ quantities of interest in Figure 2-3, setting $\hat{M} = 0$, since in this case the missing particles are neutrinos and are massless. The dotted (yellow-shaded) histogram represents the true $\sqrt{s}$ distribution of the $t\bar{t}$ pair. It quickly rises at the $t\bar{t}$ mass threshold

$$M_p \equiv 2m_t = 350 \text{ GeV}$$  \hspace{1cm} (2–36)

$^2$ Therefore, our plots in this subsection are normalized to a total number of events equal to $\sigma_{t\bar{t}} \times BR(W \rightarrow e, \mu)^2$.\hspace{1cm}
and then eventually falls off at large $\sqrt{s}$ due to the parton density function suppression. Because the top quarks are typically produced with some boost, the $\sqrt{s_{\text{true}}}$ distribution in Figure 2-3 peaks a little bit above threshold:

$$(\sqrt{s_{\text{true}}})_{\text{peak}} > M_p.$$  \hfill (2–37)

It is clear that if one could directly measure the $\sqrt{s_{\text{true}}}$ distribution, or at least its onset, the $t\bar{t}$ mass scale will be easily revealed. Unfortunately, the escaping neutrinos make such a measurement impossible, unless one is willing to make additional model-dependent assumptions\textsuperscript{3}.

Figure 2-3 also shows two versions of the calorimeter-based $\sqrt{s}^{(\text{cal})}_{\text{min}}$ variable: the blue (red) histogram is obtained by switching off (on) the underlying event (ISR and MPI). These curves reveal two very interesting phenomena. First, without the UE, the peak of the $\sqrt{s}^{(\text{cal})}_{\text{min}}$ distribution (blue histogram) is very close to the parent mass threshold [50]:

$$\text{no UE} \implies (\sqrt{s}^{(\text{cal})}_{\text{min}})_{\text{peak}} \approx M_p.$$  \hfill (2–38)

The main observation of Partha et al. [50] was that this correlation offers an alternative, fully inclusive and model-independent, method of estimating the mass scale $M_p$ of the parent particles, even when some of their decay products are invisible and not seen in the detector.

\textsuperscript{3} For example, one can use the known values of the neutrino, $W$ and top masses to solve for the neutrino kinematics (up to discrete ambiguities). However, this method assumes that the full mass spectrum is already known, and furthermore, uses the knowledge of the top decay topology to perfectly solve the combinatorics problem discussed in the Introduction. As an example, consider a case where the lepton is produced first and the $b$-quark second, i.e. when the top first decays to a lepton and a leptoquark, which in turn decays to a neutrino and a $b$-quark. The kinematic method would then be using the wrong on-shell conditions. The advantage of the $\sqrt{s}_{\text{min}}$ approach is that it is fully inclusive and does not make any reference to the actual decay topology.
Unfortunately, the “no UE” limit of Equation (2–38) is unphysical, and the corresponding $\sqrt{s}_{\text{min}}^{(\text{cal})}$ distribution (blue histogram in Figure 2-3) is unobservable. What is worse, when one tries to measure the $\sqrt{s}_{\text{min}}^{(\text{cal})}$ distribution in the presence of the UE (red histogram in Figure 2-3), the resulting peak is very far from the physical threshold:

$$\text{with UE} \implies \left( \sqrt{s}_{\text{min}}^{(\text{cal})} \right)_{\text{peak}} \gg M_p.$$  \hfill (2–39)

In the $t\bar{t}$ example of Figure 2-3, the shift is on the order of 1 TeV! It appears therefore that in practice the $\sqrt{s}_{\text{min}}^{(\text{cal})}$ peak would be uncorrelated with any physical mass scale, and instead would be completely determined by the (uninteresting) physics of the underlying event. Once the nice model-independent correlation of Equation (2–38) is destroyed by the UE, it becomes of only academic value [8, 50–52, 58].

However, Figure 2-3 also suggests the solution to this difficult problem. If we look at the distribution of the $\sqrt{s}_{\text{min}}^{(\text{reco})}$ variable (black solid histogram), we see that its peak has returned to the desired value:

$$\left( \sqrt{s}_{\text{min}}^{(\text{reco})} \right)_{\text{peak}} \approx M_p,$$  \hfill (2–40)

In order to measure physical mass thresholds, one simply needs to investigate the distribution of the inclusive $\sqrt{s}_{\text{min}}^{(\text{reco})}$ variable, which is calculated at RECO-level. Each peak in that distribution signals the opening of a new channel, and from Equation (2–40) the location of the peak provides an immediate estimate of the total mass of all particles involved in the production.

Our first main result is therefore nicely summarized in Figure 2-3, which shows a total of 4 distributions, 3 of which are either unphysical (the blue histogram of $\sqrt{s}_{\text{min}}^{(\text{cal})}$ in the absence of the UE), unobservable (the yellow-shaded histogram of $\sqrt{s}_{\text{true}}$), or useless (the red histogram of $\sqrt{s}_{\text{min}}^{(\text{cal})}$ in the presence of the UE). The only distribution in Figure 2-3 which is physical, observable and useful at the same time, is the distribution of $\sqrt{s}_{\text{min}}^{(\text{reco})}$ (solid black histogram).
Figure 2-4. PGS calorimeter map of the energy deposits, as a function of pseudorapidity $\eta$ and azimuthal angle $\phi$, for a dilepton $t\bar{t}$ event with only two reconstructed jets. At the parton level, this particular event has two $b$-quarks and two electrons. The location of a $b$-quark (electron, muon) is marked with the letter “q” (“e”, “$\mu$”). A grey circle delineates (the cone of) a reconstructed jet, while a green dotted circle denotes a reconstructed lepton. In the upper two plots the calorimeter is filled at the parton level directly from PYTHIA, while the lower two plots contain results after PGS simulation. The left plots show absolute energy deposits $E_\alpha$, while in the right plots the energy in each tower is shown projected on the transverse plane as $E_\alpha \cos \theta_\alpha$. 
Figure 2-5. PGS calorimeter map of the energy deposits for a dilepton $t\bar{t}$ event with more than two reconstructed jets
Table 2-1. Selected $\sqrt{s}$ quantities (in GeV) for the events shown in Figures 2-4, 2-5 and 2-12. The second column shows the true invariant mass $\sqrt{s}_{\text{true}}$ of the parent system: top quark pair in case of Figures 2-4 and 2-5, or gluino pair in case of Figure 2-12. The third column shows the value of the $\sqrt{s}_{\text{min}}^{(\text{cal})}$ variable (2-15) calculated at the parton level, without any PGS detector simulation, but with the full detector acceptance cut of $|\eta| < 4.1$. The fourth column lists the value of $\sqrt{s}_{\text{min}}^{(\text{cal})}$ obtained after PGS detector simulation, while the last column shows the value of the $\sqrt{s}_{\text{min}}^{(\text{reco})}$ variable defined in Equation (2–19).

<table>
<thead>
<tr>
<th>Event type</th>
<th>PYTHIA parton level</th>
<th>after PGS simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sqrt{s}_{\text{true}}$</td>
<td>$\sqrt{s}_{\text{min}}^{(\text{cal})}$</td>
</tr>
<tr>
<td>$t\bar{t}$ event in Figure 2-4</td>
<td>427</td>
<td>1110</td>
</tr>
<tr>
<td>$t\bar{t}$ event in Figure 2-5</td>
<td>638</td>
<td>2596</td>
</tr>
<tr>
<td>SUSY event in Figure 2-12</td>
<td>1954</td>
<td>3539</td>
</tr>
</tbody>
</table>

Before concluding this subsection, we explain the reason for the improved performance of the $\sqrt{s}_{\text{min}}^{(\text{reco})}$ variable in comparison to the $\sqrt{s}_{\text{min}}^{(\text{cal})}$ version. As already anticipated in Section 2.2, the basic idea is that energy deposits which are due to hard particles originating from the hard scattering, tend to be clustered, while the energy deposits due to the UE tend to be more uniformly spread throughout the detector. In order to see this pictorially, in Figures 2-4 and 2-5 we show a series of calorimeter maps of the combined ECAL+HCAL energy deposits as a function of the pseudorapidity $\eta$ and azimuthal angle $\phi$. Since the calorimeter in PGS is segmented in cells of $(\Delta\eta, \Delta\phi) = (0.1, 0.1)$, each calorimeter tower is represented by a square pixel, which is color-coded according to the amount of energy present in the tower. We have chosen the color scheme so that larger deposits correspond to darker colors. Each calorimeter map in Figures 2-4 and 2-5 has four panels. In the upper two panels the calorimeter is filled at the parton level directly from PYTHIA. This corresponds to a perfect detector, where we ignore any smearing effects due to the finite energy resolution. The lower two plots in Figures 2-4 and 2-5 show the corresponding results after PGS simulation. Thus by comparing the plots in the upper row to those in the bottom row, one can see the effect of the detector resolution. While the finite detector resolution does play some role,
we find that it is of no particular importance for understanding the reason behind the big swings in the $\sqrt{s_{\text{min}}}$ peaks observed in Figure 2-3.

Let us instead concentrate on comparing the plots in the left column versus those in the right column. The left plots show the absolute energy deposit $E_\alpha$ in the $\alpha$ calorimeter tower, while in the right plots this energy is shown projected on the transverse plane as $E_\alpha \cos \theta_\alpha$. The difference between the left and the right plots is quite striking. The plots on the left exhibit lots of energy, which is deposited mostly in the forward calorimeter cells (at large $|\eta|$) [50]. The plots on the right, on the other hand, show only a few clusters of energy, concentrated mostly in the central part of the detector. Those energy clusters give rise to the objects (jets, electrons and photons) which are reconstructed from the calorimeter. Furthermore, each energy cluster can be easily identified with a parton-level particle in the top decay chain. In order to exhibit this correlation, in Figures 2-4 and 2-5 we use the following notation for the parton-level particles: a $b$-quark (electron, muon) is marked with the letter “q” (“e”, “$\mu$”). A grey circle delineates (the cone of) a reconstructed jet, while a green dotted circle marks a reconstructed lepton (electron or muon). The lepton isolation requirement implies that green circles should be void of large energy deposits off-center, and indeed we observe this to be the case.

In particular, Figure 2-4 shows a bare-bone dilepton $t\bar{t}$ event with just two reconstructed jets and two reconstructed leptons (which happen to be both electrons). As seen in the Figure 2-4, the two jets can be easily traced back to the two $b$-quarks at the parton level, and there are no additional reconstructed jets due to the UE activity. Because the event is so clean and simple, one might expect to obtain a reasonable value for $\sqrt{s_{\text{min}}}$, i.e. close to the $t\bar{t}$ threshold. However, this is not the case, if we use the calorimeter-based measurement $\sqrt{s_{\text{min}}^{\text{cal}}}$ As seen in Table 2-1, the measured value of $\sqrt{s_{\text{min}}^{\text{cal}}}$ is very far off — on the order of 1 TeV, even in the case of a perfect detector.
The reason for this discrepancy is now easy to understand from Figure 2-4. Recall that \( \sqrt{s}^{(\text{cal})}_{\text{min}} \) is defined in terms of the total energy \( E^{(\text{cal})}_{\text{min}} \) in the calorimeter, which in turn is dominated by the large deposits in the forward region, which came from the underlying event. More importantly, those contributions are more or less equally spread over the forward and backward region of the detector, leading to cancellations in the calculation of the corresponding longitudinal \( P_{z}^{(\text{cal})} \) momentum component. As a result, the first term in Equation (2–15) becomes completely dominated by the UE contributions [51].

Let us now see how the calculation of \( \sqrt{s}^{(\text{reco})}_{\text{min}} \) is affected by the UE. Since object reconstruction is done with the help of minimum transverse cuts (for clustering and object id), the relevant calorimeter plots are the maps on the right side in Figure 2-4. We see that the large forward energy deposits which were causing the large shift in \( \sqrt{s}^{(\text{cal})}_{\text{min}} \) are not incorporated into any reconstructed objects, and thus do not contribute to the \( \sqrt{s}^{(\text{reco})}_{\text{min}} \) calculation at all. In effect, the RECO-level prescription for calculating \( \sqrt{s}_{\text{min}} \) is leaving out precisely the unwanted contributions from the UE, while keeping the relevant contributions from the hard scattering. As seen from Table 2-1, the calculated value of \( \sqrt{s}_{\text{min}} \) for that event is 363 GeV, which is indeed very close to the \( t\bar{t} \) threshold. It is also smaller than the true \( \sqrt{s} \) value of 427 GeV in that event, which is to be expected, since by design \( \sqrt{s}_{\text{min}} \leq \sqrt{s} \), and this event does not have any extra ISR jets to spoil this relation.

It is instructive to consider another, more complex \( t\bar{t} \) dilepton event, such as the one shown in Figure 2-5. The corresponding calculated values for \( \sqrt{s}_{\text{min}}^{(\text{cal})} \) and \( \sqrt{s}_{\text{min}}^{(\text{reco})} \) are shown in the second row of Table 2-1. As seen in Figure 2-5, this event has additional jets and a lot more UE activity. As a result, the calculated value of \( \sqrt{s}_{\text{min}}^{(\text{cal})} \) is shifted by almost 2 TeV from the nominal \( \sqrt{s}_{\text{true}} \) value. Nevertheless, the RECO-level prescription nicely compensates for this effect, and the calculated \( \sqrt{s}_{\text{min}}^{(\text{reco})} \) value is only 736 GeV, which is within 100 GeV of the nominal \( \sqrt{s}_{\text{true}} = 638 \) GeV. Notice that in this example
we end up with a situation where $\sqrt{s_{\text{min}}^{(\text{reco})}} > \sqrt{s_{\text{true}}}$. Figure 2-3 indicates that this happens quite often — the tail of the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ distribution is more populated than the (yellow-shaded) $\sqrt{s_{\text{true}}}$ distribution. This should be no cause for concern. First of all, we are only interested in the peak of the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ distribution, and we do not need to make any comparisons between $\sqrt{s_{\text{min}}^{(\text{reco})}}$ and $\sqrt{s_{\text{true}}}$. Second, any such comparison would be meaningless, since the value of $\sqrt{s_{\text{true}}}$ is a priori unknown, and unobservable.

### 2.4.3 $\sqrt{s_{\text{min}}^{(\text{sub})}}$ Variable

Before concluding this section, we shall use the $t\bar{t}$ example to also illustrate the idea of the subsystem $\sqrt{s_{\text{min}}^{(\text{sub})}}$ variable developed in Section 2.3. Dilepton $t\bar{t}$ events are a perfect testing ground for this idea, since the $WW$ subsystem decays leptonically, without any jet activity. We therefore define the subsystem as the two hard isolated leptons resulting from the decays of the $W$-bosons. Correspondingly, we require two reconstructed leptons (electrons or muons) at the PGS level\(^4\), and plot the distribution of the leptonic subsystem $\sqrt{s_{\text{min}}^{(\text{sub})}}$ variable in Figure 2-6.

As before, the dotted (yellow-shaded) histogram represents the true $\sqrt{s}$ distribution of the $W^+W^-$ pair. As expected, it quickly rises at the $WW$ threshold (denoted by the vertical arrow), then falls off at large $\sqrt{s}$. Since the $\sqrt{s_{\text{true}}^{(WW)}}$ distribution is unobservable, the best we can do is to study the corresponding $\sqrt{s_{\text{min}}^{(\text{sub})}}$ distribution shown with the solid black histogram. In this subsystem example, all UE activity is lumped together with the upstream $b$-jets from the top quarks decays, and thus has no bearing on the properties of the leptonic $\sqrt{s_{\text{min}}^{(\text{sub})}}$. In particular, we find that the value of $\sqrt{s_{\text{min}}^{(\text{sub})}}$ is always smaller than the true $\sqrt{s_{\text{true}}^{(WW)}}$. More importantly, Figure 2-6 demonstrates that the peak in the $\sqrt{s_{\text{min}}^{(\text{sub})}}$ distribution is found precisely at the mass threshold of the particles (in this case the two $W$ bosons) which initiated the subsystem.

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\(^4\) The selection efficiency for the two leptons is on the order of 60\%, which explains the different normalization of the distributions in Figures 2-3 and 2-6.
Figure 2-6. [Distribution of various $\sqrt{s_{\text{min}}}$ for the dilepton subsystem in dilepton $t\bar{t}$ events with two reconstructed leptons in PGS. The dotted (yellow-shaded) histogram gives the true $\sqrt{s}$ distribution of the $W^+W^-$ pair in those events. The black histogram shows the distribution of the (leptonic) subsystem variable $\sqrt{s_{\text{min}}^{(\text{sub})}}$ defined in Section 2.3. In this case, the subsystem is defined by the two isolated leptons, while all jets are treated as upstream particles. The vertical arrow marks the $W^+W^-$ mass threshold.

In analogy to (2–40) we can also write

$\left(\sqrt{s_{\text{min}}^{(\text{sub})}}\right)_{\text{peak}} \approx M_{p}^{(\text{sub})}, \quad (2–41)$

where $M_{p}^{(\text{sub})}$ is the combined mass of all the parents initiating the subsystem. Figure 2-6 shows that in the $t\bar{t}$ example just considered, this relation holds to a very high degree of accuracy.

This example should not leave the reader with the impression that hadronic jets are never allowed to be part of the subsystem. On the contrary — the subsystem may very well include reconstructed jets as well. The $t\bar{t}$ case considered here in fact provides a perfect example to illustrate the idea.
Figure 2-7. Unit-normalized distribution of jet multiplicity in dilepton $t\bar{t}$ events.

Figure 2-8. Distributions of various $\sqrt{s_{min}}$ for the dilepton $t\bar{t}$ sample, in addition to the two leptons, the subsystem now also includes: exactly two $b$-tagged jets (black histogram); the two highest $p_T$ jets (blue histogram); or all jets (red histogram). The dotted (yellow-shaded) histogram gives the true $\sqrt{s}$ distribution of the $t\bar{t}$ pair.
Let us reconsider the $t\bar{t}$ dilepton sample, and redefine the subsystem so that we now target the two top quarks as the parents initiating the subsystem. Correspondingly, in addition to the two leptons, let us allow the subsystem to include two jets, presumably coming from the two top quark decays. Unfortunately, in doing so, we must face a variant of the partitioning\(^5\) combinatorial problem discussed in the introduction: as seen in Figure 2-7, the typical jet multiplicity in the events is relatively high, and we must therefore specify the exact procedure how to select the two jets which would enter the subsystem.

We shall consider three different approaches.

- **B-tagging.** We can use the fact that the jets from top quark decay are $b$-jets, while the jets from ISR are typically light flavor jets. Therefore, by requiring exactly two $b$-tags, and including only the two $b$-tagged jets as part of the subsystem, we can significantly increase the probability of selecting the correct jets. Of course, ISR will sometimes also contribute $b$-tagged jets from gluon splitting, but that happens rather rarely and the corresponding contribution can be suppressed by a further invariant mass cut on the two $b$-jets. The resulting $\sqrt{s}_{\text{min}}^{(\text{sub})}$ distribution for the subsystem of 2 leptons and 2 $b$-tagged jets is shown in Figure 2-8 with the black histogram. We see that, as expected, the distribution peaks at the $t\bar{t}$ threshold and this time provides a measurement of the top quark mass:

$$\left(\sqrt{s}_{\text{min}}^{(\text{sub})}\right)_{\text{peak}} \approx M_{p}^{(\text{sub})} = 2m_t = 350 \text{ GeV} \ .$$

(2–42)

The disadvantage of this method is the loss in statistics: compare the normalization of the black histogram in Figure 2-8 after applying the two $b$-tags, to the dotted (yellow-shaded) distribution of the true $t\bar{t}$ distribution in the selected inclusive dilepton sample (without $b$-tags).

- **Selection by jet $p_T$.** Here one can use the fact that the jets from top decays are on average harder than the jets from ISR. Correspondingly, by choosing the two highest $p_T$ jets (regardless of $b$-tagging), one also increases the probability to select the correct jet pair. The corresponding distribution is shown in Figure 2-8 with the blue histogram, and is also seen to peak at the $t\bar{t}$ threshold. An important

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\(^5\) By construction, the $\sqrt{s}_{\text{min}}$ and $\sqrt{s}_{\text{min}}^{(\text{sub})}$ variables never have to face the ordering combinatorial problem.
advantage of this method is that one does not have to pay the price of reduced
statistics due to the two additional $b$-tags.

- No selection. The most conservative approach would be to apply no selection
criteria on the jets, and include all reconstructed jets in the subsystem. Then the
subsystem $\sqrt{s_{\text{min}}}^{(\text{sub})}$ variable essentially reverts back to the RECO-level inclusive
variable $\sqrt{s_{\text{min}}}^{(\text{reco})}$ already discussed in the previous subsection. Not surprisingly, we
find the peak of its distribution (red histogram in Figure 2-8) near the $t\bar{t}$ threshold
as well.

All three of these examples show that jets can also be usefully incorporated into the
subsystem. The only question is whether one can find a reliable way of preferentially
selecting jets which are more likely to originate from within the intended subsystem,
as opposed to from the outside. As we see in Figure 2-8, in the $t\bar{t}$ case this is quite
possible, although in general it may be difficult in other settings, like the SUSY examples
discussed in the next section.

2.5 An Exclusive SUSY Example: Multijet Events From Gluino Production

Since $\sqrt{s_{\text{min}}}$ is a fully inclusive variable, arguably its biggest advantage is that it
can be applied to purely jetty events with large jet multiplicities, where no other method
on the market would seem to work. In order to simulate such a challenging case, we
consider gluino pair production in supersymmetry, with each gluino forced to undergo
a cascade decay chain involving only QCD jets and nothing else. In this section, two
different possibilities for the gluino decays were considered:

- In one scenario, the gluino $\tilde{g}$ is forced to undergo a two-stage cascade decay to
  the LSP. In the first stage, the gluino decays to the second-lightest neutralino $\tilde{\chi}^0_2$ and
two quark jets: $\tilde{g} \rightarrow q\bar{q}\tilde{\chi}^0_2$. In turn, $\tilde{\chi}^0_2$ itself is then forced to decay via a 3-body
decay to 2 quark jets and the LSP: $\tilde{\chi}^0_2 \rightarrow q\bar{q}\tilde{\chi}^0_1$. The resulting gluino signature is 4
jets plus missing energy:

$$\tilde{g} \rightarrow jjj\tilde{\chi}^0_1 .$$

Therefore, gluino pair production will nominally result in 8 jet events. Of course,
as shown in Figure 2-9, the actual number of reconstructed jets in such events
is even higher, due to the effects of ISR, FSR and/or string fragmentation. As in
Figure 2-9, each such event has on average $\sim 10$ jets, presenting a formidable
combinatorics problem. We suspect that all\(^6\) mass reconstruction methods on the market are doomed if they were to face such a scenario. It is therefore of particular interest to see how well the \(\sqrt{s}_{\text{min}}\) method (which is advertised as universally applicable) would fare under such dire circumstances.

- In the second scenario, the gluino decays directly to the LSP via a three-body decay

\[
\tilde{g} \rightarrow jj\tilde{\chi}_1^0, \tag{2–44}
\]

so that gluino pair-production events would nominally have 4 jets and missing energy.

For concreteness, in each scenario we fix the mass spectrum as was done in [50]: we use the approximate gaugino unification relations to relate the gaugino and neutralino masses as

\[
m_{\tilde{g}} = 3m_{\tilde{\chi}_2^0} = 6m_{\tilde{\chi}_1^0}. \tag{2–45}
\]

We can then vary one of these masses, and choose the other two in accord with these relations. Since we assume three-body decays in Equations (2–44) and (2–43), we do not need to specify the SUSY scalar mass parameters, which can be taken to be very large. In addition, as implied by Equation (2–45), we imagine that the lightest two neutralinos are gaugino-like, so that we do not have to specify the higgsino mass parameter either, and it can be taken to be very large as well.

After these preliminaries, our results for these two scenarios are shown in Figures 2-10 and 2-11, correspondingly. In Figure 2-10 (Figure 2-11) we consider the 8-jet signature arising from (2–43) (the 4-jet signature arising from (2–44)). Panels (a) correspond to a light mass spectrum \(m_{\tilde{g}} = 600\) GeV, \(m_{\tilde{\chi}_2^0} = 200\) GeV and \(m_{\tilde{\chi}_1^0} = 100\) GeV; while panels (b) correspond to a heavy mass spectrum \(m_{\tilde{g}} = 2400\) GeV, \(m_{\tilde{\chi}_2^0} = 800\) GeV and \(m_{\tilde{\chi}_1^0} = 400\) GeV. Each plot shows the same four distributions

\(^6\) With the possible exception of the \(M_{\text{Tgen}}\) method of reference, C. Lester and A. Barr, “MTGEN : Mass scale measurements in pair-production at colliders, (JHEP 0712, 102 (2007)). [33], see Section 2.7 below.
as in Figure 2-3. The $\sqrt{s}_{\text{min}}$ distributions are all plotted for the correct value of the missing mass parameter, namely $\mathcal{M} = 2m_{\chi_1^0}$.

Overall, the results seen in Figures 2-10 and 2-11 are not too different from what we already witnessed in Figure 2-3 for the $t\bar{t}$ example. The (unobservable) distribution $\sqrt{s}_{\text{true}}$ shown with the dotted yellow-shaded histogram has a sharp turn-on at the physical mass threshold $M_p = 2m_{\tilde{g}}$. If the effects of the UE are ignored, the position of this threshold is given rather well by the peak of the $\sqrt{s}_{\text{min}}^{(\text{cal})}$ distribution (blue histogram). Unfortunately, the UE shifts the peak in $\sqrt{s}_{\text{min}}^{(\text{cal})}$ by 1-2 TeV (red histogram). Fortunately, the distribution of the RECO-level variable $\sqrt{s}_{\text{min}}^{(\text{reco})}$ (black histogram) is stable against UE contamination, and its peak is still in the right place (near $M_p$).

Having already seen a similar behavior in the $t\bar{t}$ example of the previous section, these results may not seem very impressive, until one realizes just how complicated those events are. For illustration, Figure 2-12 shows the previously discussed calorimeter maps for one particular “8 jet” event. This event happens to have 11 reconstructed jets, which is consistent with the typical jet multiplicity seen in Figure 2-9. The values of the $\sqrt{s}$ quantities of interest for this event are listed in Table 2-1. We see that the RECO prescription for calculating $\sqrt{s}_{\text{min}}$ is able to compensate for a shift in $\sqrt{s}$ of more than 1.5 TeV! A casual look at Figure 2-12 should be enough to convince the reader just how daunting the task of mass reconstruction in such events is. In this sense, the ease with which the $\sqrt{s}_{\text{min}}$ method reveals the gluino mass scale in Figures 2-10 and 2-11 is quite impressive.
Figure 2-9. Unit-normalized distribution of jet multiplicity in gluino pair production events, with each gluino decaying to four jets and a $\tilde{\chi}^0_1$ LSP as in (2–43).
Figure 2-10. Distribution of various $\sqrt{s}_{\text{min}}$ with a SUSY example of gluino pair production, with each gluino decaying to four jets and a $\tilde{\chi}_0^1$ LSP as indicated in (2–43). The mass spectrum is chosen as: (a) $m_{\tilde{g}} = 600$ GeV, $m_{\tilde{\chi}_2^0} = 200$ GeV and $m_{\tilde{\chi}_1^0} = 100$ GeV; or (b) $m_{\tilde{g}} = 2400$ GeV, $m_{\tilde{\chi}_2^0} = 800$ GeV and $m_{\tilde{\chi}_1^0} = 400$ GeV. All three $\sqrt{s}_{\text{min}}$ distributions are plotted for the correct value of the missing mass parameter, in this case $M = 2m_{\tilde{\chi}_1^0}$.

Figure 2-11. Distribution of various $\sqrt{s}_{\text{min}}$ with a SUSY example of gluino pair production, with each gluino decaying to two jets and a $\tilde{\chi}_0^1$ LSP as in (2–44).
Figure 2-12. PGS calorimeter map of the energy deposit for a SUSY event of gluino pair production, with each gluino forced to decay to 4 jets and the LSP as in (2–43). The SUSY mass spectrum is as in Figures 2-10(a) and 2-11(a): $m_\tilde{g} = 600$ GeV, $m_{\tilde{\chi}_0^2} = 200$ GeV and $m_{\tilde{\chi}_1^0} = 100$ GeV. As in Figures 2-4 and 2-5, the circles denote jets reconstructed in PGS, and here “q” marks the location of a quark from a gluino decay chain. Therefore, a circle without a “q” inside corresponds to a jet resulting from ISR or FSR, while a letter “q” without an accompanying circle represents a quark in the gluino decay chain which was not subsequently reconstructed as a jet.
Figure 2-13. Distribution of the $\sqrt{s_{\text{min}}}^{\text{(cal)}}$ (dotted red) and $\sqrt{s_{\text{min}}}^{\text{(reco)}}$ (solid black) variables in inclusive SUSY production for the GMSB GM1a benchmark study point with parameters $\Lambda = 80$ TeV, $M_{\text{mes}} = 160$ TeV, $N_{\text{mes}} = 1$, $\tan \beta = 15$ and $\mu > 0$. The dotted yellow-shaded histogram is the true $\sqrt{s}$ distribution of the parent pair of SUSY particles produced at the top of each decay chain (the identity of the parent particles varies from event to event). The $\sqrt{s_{\text{min}}}^{\text{d}}$ distributions are shown for $\mathcal{M} = 0$ and are normalized to $1 \text{ fb}^{-1}$ of data. The vertical arrows mark the mass thresholds for a few dominant SUSY pair-production processes.

Table 2-2. Masses (in GeV) of the SUSY particles at the GM1b study point. Here $\tilde{u}$ and $\tilde{d}$ ($\tilde{\ell}$ and $\tilde{\nu}_\ell$) stand for either of the first two generations squarks (sleptons).

<table>
<thead>
<tr>
<th>Particle</th>
<th>Mass (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{u}_L$</td>
<td>908</td>
</tr>
<tr>
<td>$\tilde{d}_L$</td>
<td>911</td>
</tr>
<tr>
<td>$\tilde{u}_R$</td>
<td>872</td>
</tr>
<tr>
<td>$\tilde{d}_R$</td>
<td>870</td>
</tr>
<tr>
<td>$\tilde{\ell}_L$</td>
<td>289</td>
</tr>
<tr>
<td>$\tilde{\nu}_\ell$</td>
<td>278</td>
</tr>
<tr>
<td>$\tilde{\ell}_R$</td>
<td>145</td>
</tr>
<tr>
<td>$\tilde{\tau}_L$</td>
<td>371</td>
</tr>
<tr>
<td>$\tilde{\tau}_R$</td>
<td>371</td>
</tr>
<tr>
<td>$\tilde{\tau}_L$</td>
<td>348</td>
</tr>
<tr>
<td>$\tilde{\nu}_\tau$</td>
<td>690</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Particle</th>
<th>Mass (GeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{u}_1$</td>
<td>806</td>
</tr>
<tr>
<td>$\tilde{d}_1$</td>
<td>863</td>
</tr>
<tr>
<td>$\tilde{u}_2$</td>
<td>895</td>
</tr>
<tr>
<td>$\tilde{d}_2$</td>
<td>878</td>
</tr>
<tr>
<td>$\tilde{\ell}_1$</td>
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<tr>
<td>$\tilde{\nu}_\ell$</td>
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<tr>
<td>$\tilde{\tau}_1$</td>
<td>138</td>
</tr>
<tr>
<td>$\tilde{\tau}_2$</td>
<td>206</td>
</tr>
<tr>
<td>$\tilde{\nu}_\tau$</td>
<td>206</td>
</tr>
<tr>
<td>$\tilde{\chi}_1^0$</td>
<td>106</td>
</tr>
<tr>
<td>$\tilde{\chi}_1^0$</td>
<td>0</td>
</tr>
</tbody>
</table>

2.6 An Inclusive SUSY Example: GMSB Study Point GM1b

In the Introduction we already mentioned that $\sqrt{s_{\text{min}}}$ is a fully inclusive variable. Here we would like to point out that there are two different aspects of the inclusivity property of $\sqrt{s_{\text{min}}}$:

- **Object-wise inclusivity:** $\sqrt{s_{\text{min}}}^{\text{(cal)}}$ is inclusive with regards to the type of reconstructed objects. The definition of $\sqrt{s_{\text{min}}}^{\text{(reco)}}$ does not distinguish between the different types...
of reconstructed objects (and $\sqrt{s_{\text{min}}^{(\text{cal})}}$ makes no reference to any reconstructed objects at all). This makes $\sqrt{s_{\text{min}}}$ a very convenient variable to use in those cases where the newly produced particles have many possible decay modes, and restricting oneself to a single exclusive signature would cause loss in statistics. For illustration, consider the gluino pair production example from the previous section. Even though we are always producing the same type of parent particles (two gluinos), in general they can have several different decay modes, leading to a very diverse sample of events with varying number of jets and leptons. Nevertheless, the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ distribution, plotted over this whole signal sample, will still be able to pinpoint the gluino mass scale, as explained in Section 2.5.

- Event-wise inclusivity: $\sqrt{s_{\text{min}}}$ is inclusive also with regards to the type of events, i.e. the type of new particle production. For simplicity, in our previous examples we have been considering only one production mechanism at a time, but this is not really necessary — $\sqrt{s_{\text{min}}}$ can also be applied in the case of several simultaneous production mechanisms.

In order to illustrate the last point, in this section we shall consider the simultaneous production of the full spectrum of SUSY particles at a particular benchmark point. We chose the GM1b CMS study point [59], which is nothing but a minimal gauge-mediated SUSY-breaking (GMSB) scenario on the SPS8 Snowmass slope [81]. The input parameters are $\Lambda=80$ TeV, $M_{\text{mes}}=160$ TeV, $N_{\text{mes}}=1$, $\tan \beta = 15$ and $\mu > 0$. The physical mass spectrum is given in Table 2-2. Point GM1b is characterized by a neutralino NLSP, which promptly decays (predominantly) to a photon and a gravitino. Therefore, a typical event has two hard photons and missing energy, which provide good handles for suppressing the SM backgrounds.

We now consider inclusive production of all SUSY subprocesses and plot the $\sqrt{s_{\text{min}}}$ distributions of interest in Figure 2-13. As usual, the dotted yellow-shaded histogram is the true $\sqrt{s}$ distribution of the parent pair of SUSY particles produced at the top of each decay chain. Since we do not fix the production subprocess, the identity of the parent particles varies from event to event. Naturally, the most common parent particles are the ones with the highest production cross-sections. For point GM1b, at a 14 TeV LHC, strong SUSY production dominates, and is 87% of the total cross-section. A few of the dominant subprocesses and their cross-sections are listed in Table 2-3.
Table 2-3. Cross-sections (in pb) and parent mass thresholds (in GeV) for the dominant production processes at the GM1b study point. The listed squark cross-sections are summed over the light squark flavors and conjugate states. The total SUSY cross-section at point GM1b is 9.4 pb.

<table>
<thead>
<tr>
<th>Process</th>
<th>$\tilde{\chi}_1^\pm \tilde{\chi}_2^0$</th>
<th>$\tilde{\chi}_1^f \tilde{\chi}_1^f$</th>
<th>$\tilde{g} \tilde{g}$</th>
<th>$\tilde{g} \tilde{q}_R$</th>
<th>$\tilde{g} \tilde{q}_L$</th>
<th>$\tilde{q}_R \tilde{q}_R$</th>
<th>$\tilde{q}_L \tilde{q}_R$</th>
<th>$\tilde{q}_L \tilde{q}_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$ (pb)</td>
<td>0.83</td>
<td>0.43</td>
<td>2.03</td>
<td>2.17</td>
<td>1.90</td>
<td>0.36</td>
<td>0.50</td>
<td>0.28</td>
</tr>
<tr>
<td>$M_p$ (GeV)</td>
<td>412</td>
<td>412</td>
<td>1380</td>
<td>$\sim 1560$</td>
<td>$\sim 1600$</td>
<td>$\sim 1740$</td>
<td>$\sim 1780$</td>
<td>$\sim 1820$</td>
</tr>
</tbody>
</table>

The true $\sqrt{s}$ distribution in Figure 2-13 exhibits an interesting double-peak structure, which is easy to understand as follows. As we have seen in the exclusive examples from Sections 2.4 and 2.5, at hadron colliders the particles tend to be produced with $\sqrt{s}$ close to their mass threshold. As seen in Table 2-2, the particle spectrum of the GM1b point can be broadly divided (according to mass) into two groups of superpartners: electroweak sector (the lightest chargino $\tilde{\chi}_1^\pm$, second-to-lightest neutralino $\tilde{\chi}_2^0$ and sleptons) with a mass scale on the order of 200 GeV and a strong sector (squarks and gluino) with masses of order 700 – 900 GeV. The first peak in the true $\sqrt{s}$ distribution (near $\sqrt{s} \sim 500$ GeV) arises from the pair production of two particles from the electroweak sector, while the second, broader peak in the range of $\sqrt{s} \sim 1500 – 2300$ GeV is due to the pair production of two colored superpartners$^7$. Each one of those peaks is made up of several contributions from different individual subprocesses, but because their mass thresholds$^8$ are so close, they cannot be individually resolved, and appear as a single bump.

If one could somehow directly observe the true $\sqrt{s}$ SUSY distribution (the dotted yellow-shaded histogram in Figure 2-13), this would lead to some very interesting conclusions. First, from the presence of two separate peaks one would know immediately

$^7$ The attentive reader may also notice two barely visible bumps (near 950 GeV and 1150 GeV) reflecting the associated production of one colored and one uncolored particle: $\tilde{g} \tilde{\chi}_1^\pm$, $\tilde{g} \tilde{\chi}_2^0$ and $\tilde{q} \tilde{\chi}_1^\pm$, $\tilde{q} \tilde{\chi}_2^0$, correspondingly.

$^8$ A few individual mass thresholds are indicated by vertical arrows in Figure 2-13.
that there are two widely separated scales in the problem. Second, the normalization of each peak would indicate the relative size of the total inclusive cross-sections (in this example, of the particles in the electroweak sector versus those in the strong sector). Finally, the broadness of each peak is indicative of the total number of contributing subprocesses, as well as the typical mass splittings of the particles within each sector. It may appear surprising that one is able to draw so many conclusions from a single distribution of an inclusive variable, but this just comes to show the importance of $\sqrt{s}$ as one of the fundamental collider physics variables. Unfortunately, because of the missing energy due to the escaping invisible particles, the true $\sqrt{s}$ distribution cannot be observed, and the best one can do to approximate it is to look at the distributions of our inclusive $\sqrt{s_{\text{min}}}$ variables discussed in Section 2.2: the calorimeter-based $\sqrt{s_{\text{min}}^{(\text{cal})}}$ variable (dotted red histogram in Figure 2-13) and the RECO-level $\sqrt{s_{\text{min}}^{(\text{reco})}}$ variable (solid black histogram in Figure 2-13).

First let us concentrate on the calorimeter-based version $\sqrt{s_{\text{min}}^{(\text{cal})}}$ (dotted red histogram). We can immediately see the detrimental effects of the UE: first, the electroweak production peak has been almost completely smeared out, while the strong production peak has been shifted upwards by more than a TeV! This behavior is not too surprising, since the same effect was already encountered in our previous examples in Sections 2.4 and 2.5. Fortunately, we now also know the solution to this problem: one needs to consider the RECO-level variable $\sqrt{s_{\text{min}}^{(\text{reco})}}$ instead, which tracks the true $\sqrt{s}$ distribution much better. We can see evidence of this in Figure 2-13 as well. In particular, $\sqrt{s_{\text{min}}^{(\text{reco})}}$ does show two separate peaks (indicating that SUSY production takes place at two different mass scales), the peaks are in their proper locations (relative to the missing mass scale $M$), and have the correct relative width, hinting at the size of the mass splittings in each sector. We thus conclude that all of the interesting physics conclusions that one would be able to reach from looking at the true $\sqrt{s}$ distributions, can still be made based on the inclusive distribution of our RECO-level $\sqrt{s_{\text{min}}^{(\text{reco})}}$ variable.
Figure 2-14. Distributions of various $\sqrt{s_{\text{min}}}$ for the GMSB SUSY example considered in Figure 2-13. Here the subsystem is defined in terms of the two hard photons resulting from the two $\tilde{\chi}^0_1 \rightarrow \tilde{G} + \gamma$ decays. The vertical arrow marks the onset for inclusive $\tilde{\chi}^0_1 \tilde{\chi}^0_1$ production.

Before concluding this section, we shall take the opportunity to use the GM1b example to also illustrate the $\sqrt{s}^{(\text{sub})}$ variable proposed in Section 2.3. As already mentioned, the GM1b study point corresponds to a GMSB scenario with a promptly decaying Bino-like $\tilde{\chi}_1^0$ NLSP. Most events therefore contain two hard photons from the two $\tilde{\chi}_1^0$ decays to gravitinos. Then it is quite natural to define the exclusive subsystem in Figure 2-2 in terms of these two photons. The corresponding $\sqrt{s}^{(\text{sub})}$ distribution is shown in Figure 2-14 with the black solid histogram. For completeness, we also show the true $\sqrt{s}$ distribution of the $\tilde{\chi}_1^0$ pair (dotted yellow-shaded histogram). The vertical arrow marks the location of the $\tilde{\chi}_1^0 \tilde{\chi}_1^0$ mass threshold. We notice that the peak of the $\sqrt{s}^{(\text{sub})}$ distribution nicely reveals the location of the neutralino mass threshold, and from there the neutralino mass itself. We see that the method of $\sqrt{s}^{(\text{sub})}$ provides a very simple way of measuring the NLSP mass in such GMSB scenarios (for an alternative approach based on $M_{T2}$, see [61]).
2.7 Comparison to Other Inclusive Collider Variables

Having discussed the newly proposed variables $\sqrt{s}_{\text{min}}^{(\text{reco})}$ and $\sqrt{s}_{\text{min}}^{(\text{sub})}$ in various settings in Sections 2.4-2.6, we shall now compare them to some other global inclusive variables which have been discussed in the literature in relation to determining a mass scale of the new physics. For simplicity here we shall concentrate only on the most model-independent variables, which do not suffer from the topological and combinatorial ambiguities mentioned in the Introduction.

At the moment, there are only a handful of such variables. Depending on the treatment of the unknown masses of the invisible particles, they can be classified into one of the following two categories:

- Variables which do not depend on an unknown invisible mass parameter. The most popular members of this class are the “missing $H_T$” variable

$$H_T \equiv \sqrt{\mathcal{H}_T} \equiv \left| -\sum_{i=1}^{N_{\text{obj}}} \vec{P}_{Ti} \right|, \quad (2-46)$$

which is simply the magnitude of the $\vec{H}_T$ vector from Equation (2–22), and the scalar $H_T$ variable

$$H_T \equiv \sqrt{\mathcal{H}_T} + \sum_{i=1}^{N_{\text{obj}}} P_{Ti}. \quad (2-47)$$

Here we follow the notation from Section 2.2, where $\vec{P}_{Ti}$ is the measured transverse momentum of the $i$-th reconstructed object in the event ($i = 1, 2, \ldots, N_{\text{obj}}$). The main advantage of $H_T$ and $H_T$ is their simplicity: both are very general, and are defined purely in terms of observed quantities, without any unknown mass parameters. The downside of $H_T$ and $H_T$ is that they cannot be directly correlated with any physical mass scale in a model-independent way\(^9\).

\(^9\) Some early studies of $H_T$-like variables found interesting linear correlations between the peak in the $H_T$ distribution and a suitably defined SUSY mass scale in the context of specific SUSY models, e.g. minimal supergravity (MSUGRA) [13, 63, 64], minimal GMSB [63], or mixed moduli-mediation [65]. However, any such correlations do not survive further scrutiny in more generic SUSY scenarios, see e.g. [66].
Variables which exhibit dependence on one or more invisible mass parameters. As two representatives from this class we shall consider $M_{Tgen}$ from Reference [33] and $\sqrt{s_{\text{min}}^{(\text{reco})}}$ from Section 2.2 here. We shall not repeat the technical definition of $M_{Tgen}$, and instead refer the uninitiated reader to the original paper [33]. Suffice it to say that the method of $M_{Tgen}$ starts out by assuming exactly two decay chains in each event. The arising combinatorial problem is then solved by brute force — by considering all possible partitions of the event into two sides, computing $M_{T2}$ for each such partition, and taking the minimum value of $M_{T2}$ found in the process. Both $M_{Tgen}$ and $\sqrt{s_{\text{min}}^{(\text{reco})}}$ introduce a priori unknown parameters related to the mass scale of the missing particles produced in the event. In the case of $\sqrt{s_{\text{min}}^{(\text{reco})}}$, this is simply the single parameter $\mathcal{M}$, measuring the total invisible mass (in the sense of a scalar sum as defined in Equation (2–3)). The $M_{Tgen}$ variable, on the other hand, must in principle introduce two separate missing mass parameters $\mathcal{M}_1$ and $\mathcal{M}_2$ (one for each side of the event). However, the existing applications of $M_{Tgen}$ in the literature have typically made the assumption that $\mathcal{M}_1 = \mathcal{M}_2$, although this is not really necessary and one could just as easily work in terms of two separate inputs $\mathcal{M}_1$ and $\mathcal{M}_2$ [40, 41]. The inconvenience of having to deal with unknown mass parameters in the case of $M_{Tgen}$ and $\sqrt{s_{\text{min}}^{(\text{reco})}}$ is greatly compensated by the luxury of being able to relate certain features of their distributions to a fundamental physical mass scale in a robust, model-independent way. In particular, the upper endpoint $M_{Tgen}^{(\text{max})}$ of the $M_{Tgen}$ distribution gives the larger of the two parent masses $\max\{M_{P_1}, M_{P_2}\}$ [62]. Therefore, if the two parent masses are the same, i.e. $M_{P_1} = M_{P_2}$, then the parent mass threshold $M_p = M_{P_1} + M_{P_2}$ is simply given by

$$M_p = 2M_{Tgen}^{(\text{max})}. \quad (2–48)$$

On the other hand, as we have already seen in Sections 2.4-2.6, the peak of the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ is similarly correlated with the parent mass threshold, see Equation (2–40).

In principle, all four\(^\text{10}\) of these variables are inclusive both object-wise and event-wise. It is therefore of interest to compare them with respect to:

1. The degree of correlation with the new physics mass scale $M_p$.
2. Stability of this correlation against the detrimental effects of the UE.

Figures 2-15, 2-16 and 2-17 allow for such comparisons.

\(^{10}\text{We caution the reader that } H_T \text{ is often defined in a more narrow sense than Equation (2–47). For example, sometimes the } \sum_TH_T \text{ term is omitted, sometimes the sum in Equation (2–47) is limited to the reconstructed jets only; or to the four highest } p_T \text{ jets only; or to all jets, but starting from the second-highest } p_T \text{ one.}\)
Figure 2-15. Comparison various $\sqrt{s}_{\text{min}}$ with other transverse variables for $t\bar{t}$ production. In addition to the true $\sqrt{s}$ (yellow shaded) and $\sqrt{s}^{\text{(reco)}}_{\text{min}}$ (black) distribution, we also plot the distributions of $2M_{T_{\text{gen}}}$ (red dots), $2M_{T_{Tgen}}$ (magenta dots), $H_T$ (green dots) and $\pT$ (blue dots), all calculated at the RECO-level. All results include the full simulation of the underlying event. For plotting convenience, the $\pT$ distribution is shown scaled down by a factor of 2. The vertical dotted line marks the $t\bar{t}$ mass threshold $M_p = 2m_t = 350$ GeV.

In Figure 2-15 we first revisit the case of the dilepton $t\bar{t}$ sample discussed in Section 2.4. In addition to the true $\sqrt{s}$ (yellow shaded) and $\sqrt{s}^{\text{(reco)}}_{\text{min}}$ (black) distribution already appearing in Figure 2-3, we now also plot the distributions of $2M_{T_{\text{gen}}}$ (red dots), $H_T$ (green dots) and $\pT$ (blue dots), all calculated at the RECO-level. For completeness, in Figure 2-15 we also show a variant of $M_{T_{\text{gen}}}$, called $M_{T_{Tgen}}$ (magenta dots), where all visible particle momenta are first projected on the transverse plane, before computing $M_{T_{\text{gen}}}$ in the usual way [33]. All results include the full simulation of the underlying event. For plotting convenience, the $\pT$ distribution is shown scaled down by a factor of 2.

Based on the results from Figure 2-15, we can now address the question, which inclusive distribution shows the best correlation with the parent mass scale (in this case the parent mass scale is the $t\bar{t}$ mass threshold $M_p = 2m_t = 350$ GeV marked
by the vertical dotted line in Figure 2-15). Let us begin with the two variables, $H_T$ and $H_T$, which do not depend on any unknown mass parameters. Figure 2-15 reveals that the $H_T$ distribution peaks very far from threshold, and therefore does not reveal much information about the new physics mass scale. Consequently, any attempt at extracting new physics parameters out of the missing energy distribution alone, must make some additional model-dependent assumptions [2]. On the other hand, the $H_T$ distribution appears to correlate better with $M_p$, since its peak is relatively close to the $t\bar{t}$ threshold. However, this relationship is purely empirical, and it is difficult to know what is the associated systematic error.

Moving on to the variables which carry a dependence on a missing mass parameter, $\sqrt{s_{\text{min}}^{(\text{reco})}}$, $2M_{T_{\text{gen}}}$ and $2M_{TT_{\text{gen}}}$, we see that all three are affected to some extent by the presence of the UE. In particular, the distributions of $2M_{T_{\text{gen}}}$ and $2M_{TT_{\text{gen}}}$ are now smeared and extend significantly beyond their expected endpoint (2–48). Not surprisingly, the UE has a larger impact on $2M_{T_{\text{gen}}}$ than on $2M_{TT_{\text{gen}}}$. In either case, there is no obvious endpoint. Nevertheless, one could in principle try to extract an endpoint through a straight-line fit, for example, but it is clear that the obtained value will be wrong by a certain amount (depending on the chosen region for fitting and on the associated backgrounds). All these difficulties with $2M_{T_{\text{gen}}}$ and $2M_{TT_{\text{gen}}}$ are simply a reflection of the challenge of measuring a mass scale from an endpoint as in (2–48), instead of from a peak as in (2–40). By comparison, the determination of the new physics mass scale from the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ distribution is much more robust. As shown in Figure 2-15, the $\sqrt{s_{\text{min}}^{(\text{reco})}}$ peak is barely affected by the UE, and is still found precisely in the right location.

All of the above discussion can be directly applied to the SUSY examples considered in Section 2.5 as well. As an illustration, Figures 2-16 and 2-17 revisit two of the gluino examples from Section 2.5. We consider gluino pair-production with a light SUSY spectrum ($m_{\tilde{\chi}_1^0} = 100$ GeV, $m_{\tilde{\chi}_2^0} = 200$ GeV and $m_{\tilde{g}} = 600$ GeV). Then in Figure 2-16 each gluino decays to 4 jets as in Equation (2–43), while in Figure 2-17
each gluino decays to 2 jets as in Equation (2–44). (Thus Figure 2-16 is the analogue of Figure 2-10(a), while Figure 2-17 is the analogue of Figure 2-11(a).)

The conclusions from Figures 2-16 and 2-17 are very similar. These results confirm that $H_T$ is not very helpful in determining the gluino mass scale $M_p = 2m_g = 1200$ GeV (indicated by the vertical dotted line). The $H_T$ distribution, on the other hand, has a nice well-defined peak, but the location of the $H_T$ peak always underestimates the gluino mass scale (by about 250 GeV in each case). Figures 2-16 and 2-17 also confirm the effect already seen in Figure 2-15: that the underlying event causes the $2M_{Tgen}$ and $2M_{TTgen}$ distributions to extend well beyond their upper kinematic endpoint, thus violating (2–48) and making the corresponding extraction of $M_p$ rather problematic. In fact, just by looking at Figures 2-16 and 2-17, one might be tempted to deduce that, if anything, it is the peak in $2M_{Tgen}$ that perhaps might indicate the value of the new physics mass scale and not the $2M_{Tgen}$ endpoint. Finally, the $\sqrt{s}_\text{min}^{(reco)}$ distribution also feels to some extent the effects from the UE, but always has its peak in the near vicinity of $M_p$. Therefore, among the five inclusive variables under consideration here, $\sqrt{s}_\text{min}^{(reco)}$ appears to provide the best estimate of the new physics mass scale. The correlation of Equation (2–40) is seen to hold very well in Figure 2-17 and reasonably well in Figure 2-16.
Figure 2-16. Comparison of various $\sqrt{s_{\text{min}}}$ with other transverse variables for the gluino pair production example from Section 2.5, with each gluino decaying to 4 jets as in (2–43). We use the light SUSY mass spectrum from Figure 2-10(a). The vertical dotted line now shows the $\tilde{g}\tilde{g}$ mass threshold $M_p = 2m_{\tilde{g}} = 1200 \text{ GeV}$.

Figure 2-17. Comparison of various $\sqrt{s_{\text{min}}}$ with other transverse variables for gluino pair production with each gluino decaying to 2 jets as in (2–44). Compare to Figure 2-11(a).
CHAPTER 3
INVARIANT MASS ENDPOINTS METHOD

In this chapter we concentrate on the classic method of kinematical endpoints [13].

We set out to redesign this standard algorithm for performing these studies, by pursuing two main objectives:

- Improving on the experimental precision of the SUSY mass determination. For example, we required that our analysis be based exclusively on upper invariant mass endpoints, which are expected to be measured with a greater precision than the corresponding lower endpoints (a.k.a. thresholds). Consequently, we did not make use of the “threshold” measurement \( m_{\ell \ell (\theta > \pi/2)} \), which has been an integral part of most SUSY studies since Reference [16]. In the same vein, we also demanded that we should not rely on any features observed in a two- or a three-dimensional invariant mass distribution — such measurements are expected to be less precise than the (upper) endpoints extracted from simple one-dimensional histograms.

- Avoiding any parameter space region ambiguities. It is well known that some of the invariant mass endpoints used in the conventional analyses are piecewise-defined functions. This feature may sometimes lead to multiple solutions for the SUSY mass spectrum in the “LHC inverse problem” [9, 17, 67–69]. In order to safeguard against this possibility, we conservatively demanded from the outset that none of our endpoint measurements be given by piecewise defined functions. This rather strict requirement rules out three of the standard endpoint measurements \( m_{\ell \ell (\mu)} \), \( m_{\ell \ell (\tau)} \), and \( m_{\ell \ell (\chi)} \).

In order to meet these objectives, in Section 3.2 we proposed a set of new invariant mass variables whose upper kinematic endpoints can be alternatively used for SUSY mass reconstruction studies. Then in Section 3.3 we outlined a simple analysis which was based on the particular set of four invariant mass variables (3–41), all of which satisfy our requirements. In Section 3.3.1 we provided simple analytical formulas for the SUSY mass spectrum in terms of the four measured endpoints in Equation (3–41). Our solutions revealed a surprise: in spite of the two-fold ambiguity as in Equations (3–22, 3–23) in the interpretation of two of our endpoints \( M_{J \ell (\mu)} \) and \( m_{J \ell (\tau)} \), the answer for three \( (m_D, m_C \text{ and } m_A) \) out of the four SUSY masses is unique!
Figure 3-1. The typical cascade decay chain under consideration in this chapter. Here $D$, $C$, $B$ and $A$ are new BSM particles, while the corresponding SM decay products are: a QCD jet $j$, a “near” lepton $\ell_n^\pm$ and a “far” lepton $\ell_f^\mp$. This chain is quite common in SUSY, with the identification $D = \tilde{q}$, $C = \tilde{\chi}_0^2$, $B = \tilde{\ell}$ and $A = \tilde{\chi}_1^0$, where $\tilde{q}$ is a squark, $\tilde{\ell}$ is a slepton, and $\tilde{\chi}_0^2$ ($\tilde{\chi}_1^0$) is the first (second) lightest neutralino. In what follows we shall quote our results in terms of the $D$ mass $m_D$ and the three dimensionless squared mass ratios $R_{CD}$, $R_{BC}$ and $R_{AB}$ defined in Equation (3–6).

The fourth mass ($m_B$) is also known, up to the two-fold ambiguity as in Equation (3–55), which can be easily resolved by a variety of methods discussed and illustrated in Sections 3.3.2 and 3.4.2. In Section 3.4 we applied our technique to two specific examples — the LM1 and LM6 CMS study points. Following the previous SUSY studies, for illustration of our results we shall use the generic decay chain $D \rightarrow jC \rightarrow j\ell_n^\pm B \rightarrow j\ell_n^\pm \ell_f^\mp A$ shown in Figure 3-1. Here $D$, $C$, $B$ and $A$ are new BSM particles with masses $m_D$, $m_C$, $m_B$ and $m_A$. Their corresponding SM decay products are: a QCD jet $j$, a “near” lepton $\ell_n^\pm$ and a “far” lepton $\ell_f^\mp$. This decay chain is quite common in SUSY, with the identification $D = \tilde{q}$, $C = \tilde{\chi}_0^2$, $B = \tilde{\ell}$ and $A = \tilde{\chi}_1^0$, where $\tilde{q}$ is a squark, $\tilde{\ell}$ is a slepton, and $\tilde{\chi}_0^2$ ($\tilde{\chi}_1^0$) is the first (second) lightest neutralino. However, our analysis is not limited to SUSY only, since the chain in Figure 3-1 also appears in other BSM scenarios, e.g. Universal Extra Dimensions [71]. For concreteness, we shall assume that all three decays exhibited in Figure 3-1 are two-body, i.e. we shall consider the mass hierarchy

$$m_D > m_C > m_B > m_A > 0.$$  

(3–1)
This presents the most challenging case, in which one has to determine all four masses $m_D$, $m_C$, $m_B$ and $m_A$.

The idea of the kinematic endpoint method is very simple. Given the SM decay products $j$, $\ell_n$ and $\ell_f$ exhibited in Figure 3-1, form the invariant mass\footnote{We shall see below that the formulas simplify considerably if we use invariant masses squared instead. This distinction is not central to our analysis.} of every possible combination, $m_{\ell\ell}$, $m_{j\ell_n}$, $m_{j\ell_f}$, and $m_{j\ell\ell}$, plot the resulting distributions and measure the corresponding upper kinematic endpoints [13, 16, 17]

\[
(m_{\ell\ell}^{\text{max}})^2 = m_D^2 R_{CD} (1 - R_{BC})(1 - R_{AB}); \tag{3–2}
\]

\[
(m_{j\ell_n}^{\text{max}})^2 = m_D^2 (1 - R_{CD})(1 - R_{BC}); \tag{3–3}
\]

\[
(m_{j\ell_f}^{\text{max}})^2 = m_D^2 (1 - R_{CD})(1 - R_{AB}); \tag{3–4}
\]

\[
(m_{j\ell\ell}^{\text{max}})^2 = \begin{cases} 
m_D^2(1 - R_{CD})(1 - R_{AC}), & \text{for } R_{CD} < R_{AC}, \text{ case (1, -)}, \\
m_D^2(1 - R_{BC})(1 - R_{AB} R_{CD}), & \text{for } R_{BC} < R_{AB} R_{CD}, \text{ case (2, -)}, \\
m_D^2(1 - R_{AB})(1 - R_{BD}), & \text{for } R_{AB} < R_{BD}, \text{ case (3, -)}, \\
m_D^2 \left(1 - \sqrt{R_{AD}}\right)^2, & \text{otherwise, case (4, -)}. \end{cases} \tag{3–5}
\]

Here and below we write all results in terms of an overall mass scale (given by the mass $m_D$ of the heaviest BSM particle $D$) and three dimensionless squared mass ratios

\[
R_{ij} \equiv \frac{m_i^2}{m_j^2}, \quad i, j \in \{A, B, C, D\}. \tag{3–6}
\]
Note that there are only three independent ratios in Equation (3–6). We shall take those to be $R_{AB}$, $R_{BC}$, and $R_{CD}$ (see Figure 3-1), and their definition domain will be the interval $(0, 1)$.\(^2\)

### 3.1 Three Generic Problems in Invariant-mass Endpoint Methods

In spite of their transparent theoretical meaning, the set of four endpoints in Equations (3–2 through 3–5) by themselves have (justifiably) never been used as the sole basis for a SUSY mass determination analysis. This is due to three generic problems, which are all very well known, and are separately reviewed in the next three subsections 3.1.1, 3.1.2 and 3.1.3. Our new approach to resolving these three problems, and the outline of the rest of the paper are presented in Section 3.1.4.

#### 3.1.1 Near-far Lepton Ambiguity

The first problem is that one cannot differentiate between the “near” and “far” leptons $\ell_n$ and $\ell_f$ on an event-by-event basis. Since all decays in Figure 3-1 are prompt, both leptons point back to the primary interaction vertex and there is no way to tell which came first and which came second. Consequently, one cannot separately construct the individual $m_{j\ell_n}$ and $m_{j\ell_f}$ invariant mass distributions, whose upper endpoints would be given by Equations (3–3) and (3–4). This problem has motivated most of the previous invariant mass studies in the literature, beginning with [16], to introduce an alternative definition of the two $j\ell$ distributions, simply by ordering the two $m_{j\ell}$ entries in each event by invariant mass as follows

$$m_{j\ell(lo)} \equiv \min \{m_{j\ell_n}, m_{j\ell_f}\}, \quad (3–7)$$

$$m_{j\ell(hi)} \equiv \max \{m_{j\ell_n}, m_{j\ell_f}\}. \quad (3–8)$$

---

\(^2\) As seen in Equation (3–5), at times we shall also utilize one or more of the other three ratios, $R_{AC}$, $R_{AD}$ and $R_{BD}$, whenever this will lead to a simplification of the formulas. Of course, the latter three ratios are related to our preferred set \{\(R_{AB}, R_{BC}, R_{CD}\)\} due to the transitivity property $R_{ij}R_{jk} = R_{ik}$.

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Both of the newly defined quantities $m_{j\ell(lo)}$ and $m_{j\ell(hi)}$ also exhibit upper kinematic endpoints ($m_{j\ell(lo)}^{\text{max}}$ and $m_{j\ell(hi)}^{\text{max}}$, correspondingly). Since the individual $m_{j\ell(lo)}$ and $m_{j\ell(hi)}$ distributions are observable, their endpoints are experimentally measurable and can be related to the underlying SUSY mass spectrum as follows [16, 17]

$$
(m_{j\ell(lo)}^{\text{max}})^2 = \begin{cases} 
(m_{j\ell_n}^{\text{max}})^2, & \text{for } (2 - R_{AB})^{-1} < R_{BC} < 1, \text{ case } (-, 1), \\
(m_{j\ell(eq)}^{\text{max}})^2, & \text{for } R_{AB} < R_{BC} < (2 - R_{AB})^{-1}, \text{ case } (-, 2), \\
(m_{j\ell(eq)}^{\text{max}})^2, & \text{for } 0 < R_{BC} < R_{AB}, \text{ case } (-, 3); 
\end{cases}
$$

(3–9)

$$
(m_{j\ell(hi)}^{\text{max}})^2 = \begin{cases} 
(m_{j\ell_f}^{\text{max}})^2, & \text{for } (2 - R_{AB})^{-1} < R_{BC} < 1, \text{ case } (-, 1), \\
(m_{j\ell}^{\text{max}})^2, & \text{for } R_{AB} < R_{BC} < (2 - R_{AB})^{-1}, \text{ case } (-, 2), \\
(m_{j\ell_n}^{\text{max}})^2, & \text{for } 0 < R_{BC} < R_{AB}, \text{ case } (-, 3); 
\end{cases}
$$

(3–10)

where

$$
(m_{j\ell(eq)}^{\text{max}})^2 = m_D^2 (1 - R_{CD}) (1 - R_{AB}) (2 - R_{AB})^{-1}
$$

(3–11)

and $m_{j\ell_n}^{\text{max}}$ and $m_{j\ell_f}^{\text{max}}$ were already defined in Equations (3–3) and (3–4), correspondingly.

With this approach, the original set of 4 endpoints in Equations (3–2-3–5) is replaced by

$$
m_{j\ell}, m_{j\ell_n}^{\text{max}}, m_{j\ell(hi)}, m_{j\ell(eq)}^{\text{max}}.
$$

(3–12)

In contrast to this conventional approach in the literature, we shall adopt a very different attitude towards resolving the problem of the near-far lepton ambiguity. We will do the simplest possible thing, namely, we shall do nothing. We shall never ask the question “which lepton was $\ell_n$ and which one was $\ell_f$?”. We shall also not use the ordering in Equations (3–7,3–8). Instead, we shall simply take the two $m_{j\ell}$ entries in each event, and always treat them in a symmetric fashion. For example, any observable invariant mass distribution that we will build out of the two measured quantities $m_{j\ell_n}$ and
$m_{j\ell}$ should be invariant under the symmetry

$$m_{j\ell_n} \leftrightarrow m_{j\ell} \quad .$$

(3–13)

The advantages of our approach may not be immediately obvious at this point, but will become clear in the process of our mass determination analysis in Section 3.3 below.

3.1.2 Insufficient Number of Measurements.

The second problem associated with the original set of four measurements in Equations (3–2-3–5), as well as the alternative set in Equation (3–12), is that the measured endpoints may not all be independent from each other. Indeed, there are certain regions of parameter space where one finds the following correlation [17]

$$\left( m_{j\ell}^{\text{max}} \right)^2 = \left( m_{j\ell(hi)}^{\text{max}} \right)^2 + \left( m_{\ell\ell}^{\text{max}} \right)^2 .$$

(3–14)

In this case, the four measurements in Equation (3–12) are clearly insufficient to pin down all four independent input parameters $m_D, m_C, m_B$ and $m_A$. Therefore, one has to measure an additional independent endpoint. To this end, it has been suggested to consider the constrained distribution $m_{j\ell\ell(\theta > \pi/2)}$, which exhibits a useful lower kinematic endpoint $m_{j\ell\ell(\theta > \pi/2)}^{\text{min}}$ [16]

$$\left( m_{j\ell\ell(\theta > \pi/2)}^{\text{min}} \right)^2 = \frac{1}{4} m_D^2 \left\{ (1 - R_{AB})(1 - R_{BC})(1 + R_{CD}) \right. \right.$$

$$+ \left. 2 (1 - R_{AC})(1 - R_{CD}) - (1 - R_{CD}) \sqrt{(1 + R_{AB})^2(1 + R_{BC})^2 - 16 R_{AC}} \right\} .$$

(3–15)

The distribution $m_{j\ell\ell(\theta > \pi/2)}$ is nothing but the usual $m_{j\ell}$ distribution over a subset of the original events, subject to the additional dilepton mass constraint

$$\frac{m_{\ell\ell}^{\text{max}}}{\sqrt{2}} < m_{\ell\ell} < m_{\ell\ell}^{\text{max}} .$$

(3–16)
In the rest frame of particle $B$, this cut implies the following restriction on the opening angle $\theta$ between the two leptons [70]

$$\theta > \frac{\pi}{2},$$

thus justifying the notation for $m_{j\ell\ell}(\theta > \frac{\pi}{2})$.

The advantage of the “threshold” endpoint measurement (3–15) is that it is always independent of the other four measurements in (3–12). As a result, it would appear that the enlarged set of five kinematic endpoint measurements

$$m_{\ell\ell}^{\text{max}}, m_{j\ell\ell}^{\text{max}}, m_{j\ell}(\nu), m_{j\ell}(\nu), m_{j\ell\ell}(\theta > \frac{\pi}{2})$$

(3–18)

should be in principle sufficient to determine all four unknown masses (see, however Section 3.1.3 below).

Unfortunately, the “threshold” (3–15) also suffers from certain disadvantages, which are mostly of experimental nature. It is generally expected that the experimental precision on the determination of the lower kinematic endpoint (3–15) will be rather inferior compared to the precision on the other four upper kinematic endpoints (3–12) [17]. There are several generic reasons for such a pessimistic attitude. First, the region in the $m_{j\ell\ell}(\theta > \frac{\pi}{2})$ distribution near its lower endpoint (3–15) is rather sparsely populated, resulting in a shallow edge and sizable statistical errors. To make matters worse, the $m_{j\ell\ell}(\theta > \frac{\pi}{2})$ distribution near its lower edge is a convex function [72], which makes it even more difficult to tell where the signal ends and the tails from various sources begin [17]. Finally, the low mass region of almost any invariant mass distribution in SUSY is generally associated with larger SM (as well as SUSY combinatorial) backgrounds compared to its high mass counterpart.

Overall we find all these disadvantages sufficiently convincing so that we will drop the measurement in Equation (3–15) altogether and will never use it in the course of our analysis in Section 3.3 below. We will be justified in doing so, since the linear
dependence problem (3–14), which has plagued previous studies and was the prime motivation for introducing the \( m_{j\ell\ell}^{\min} > \frac{\pi}{2} \) measurement in the first place, will have no effect on our analysis. In fact, we will not be using the endpoint measurement \( m_{j\ell\ell}^{\max} \) (for the reasons given in the previous subsection 3.1.1) and we will not be using the endpoint measurement \( m_{j\ell\ell}^{\max} \) (for the reasons given in the following subsection 3.1.3). Once these two problematic measurements are removed from consideration, the linear dependence problem (3–14) does not arise, and the “threshold” measurement (3–15) is not central to the analysis any more.

3.1.3 Parameter Space Region Ambiguity

The third problem with the conventional set of measurements (3–18) is immediately obvious from the defining Equations (3–5), (3–9) and (3–10) for the kinematic endpoints \( m_{j\ell\ell}^{\max} \), \( m_{j\ell(lo)}^{\max} \), and \( m_{j\ell(hi)}^{\max} \), correspondingly. One can see that the relevant expressions are piecewise-defined functions, i.e. they depend on the values of the independent variables \( m_A, m_B, m_C \) and \( m_D \). For example, there are four different cases for \( m_{j\ell\ell}^{\max} \), and three different cases for the pair of \( (m_{j\ell(lo)}^{\max}, m_{j\ell(hi)}^{\max}) \). Altogether, these give rise to 9 different cases\(^3\) which must be separately considered [9, 17]. Of course, this represents a problem, since the masses are a priori unknown, and it is not clear which case is the relevant one. Barring any model-dependent assumptions, one is forced to consider all possibilities, obtain a solution for the spectrum, and only at the very end, test whether the solution falls within the parameter space applicable for the case at hand. This procedure may often result in several alternative solutions [9, 17, 67–69, 73, 74]. In fact, we recently proved that there exists a sizable parameter space region in which even the full set of measurements (3–18) would always yield two alternative solutions, even under ideal experimental conditions[9]. The problem is further exacerbated by the inevitable

\(^3\) The remaining 3 cases are always unphysical [17].
experimental errors on the measurements (3–18), which would allow for an even larger number of “fake” or “duplicate” solutions [9, 68, 69].

Having identified the root of the duplication problem as the piecewise definition of the mathematical formulas in (3–5,3–9,3–10), our solution to the problem will be again very simple and conservative. We will simply avoid using any kinematic endpoints which are given in terms of piecewise-defined expressions. This requirement automatically eliminates from consideration the three conventional endpoints $m_{j\ell\ell}^{\text{max}}$, $m_{j\ell(\omega)}^{\text{max}}$, and $m_{j\ell(hi)}^{\text{max}}$. Since we already gave up on $m_{j\ell\ell(\theta > \frac{\pi}{2})}^{\text{min}}$ in the previous subsection, this leaves $m_{j\ell\ell}^{\text{max}}$ as the only measurement out of the conventional set (3–18) that we shall use in our analysis. This is perhaps the most drastic difference between our approach and all previous studies in the literature.

### 3.1.4 Posing The Problem

In the previous three subsections we discussed each of the three generic theoretical\(^4\) problems with the previous applications of the kinematic endpoint method for mass determination. We are now ready to explicitly formulate our main goal in this paper. We aim to design a method for measuring the masses of the particles in the decay chain of Figure 3-1, which is based on kinematic endpoint information, and satisfies the following requirements:

---

\(^4\) In addition, there are problems which are of experimental nature, e.g. identifying the correct jet and the correct lepton pair resulting from the decay chain in Figure 3-1. There exists a set of standard experimental techniques which are aimed at overcoming these problems, e.g. the opposite flavor subtraction for the two leptons and the mixed event subtraction for the jet [75]. Wrong $\ell\ell$ and $j\ell$ pairings can also be identified and a posteriori removed whenever an invariant mass entry for $m_{\ell\ell}$, $m_{j\ell}$ or $m_{j\ell\ell}$ exceeds the corresponding kinematic endpoint $m_{j\ell\ell}^{\text{max}}$, $m_{j\ell(\omega)}^{\text{max}}$ or $m_{j\ell(hi)}^{\text{max}}$. In what follows we shall assume that those preliminary steps have already been done and the samples we are dealing with have already been appropriately subtracted to remove the combinatorial background.
• It does not make use of any kinematic endpoints whose interpretation is ambiguous, i.e. whose expressions in terms of the physical masses are piecewise-defined functions.

• It does not make use of any lower kinematic endpoints such as the “threshold" \( m_{j\ell\ell}(\theta > \frac{\pi}{2}) \), due to the experimental challenges with such measurements.

• It relies solely on 1-dimensional distributions, unlike the methods recently advertised in \([9, 21, 73, 74, 76]\), which utilize 2-dimensional correlation plots. While the latter do provide a wealth of valuable information, they also typically require more data in order to obtain good enough statistics for drawing any robust conclusions from them. In contrast, the one-dimensional distributions should be available rather early on, and with sufficient statistics for endpoint measurements.

As already alluded to in the previous subsections, the first two requirements already eliminate four out of the five conventional inputs (3–18). Obviously, we will need to find a way to replace those with an alternative set of kinematic endpoint measurements which nevertheless satisfy the above requirements. In Section 3.2 we introduce and investigate a new set of invariant mass variables whose upper endpoints can be useful for our analysis. Then in Section 3.3 we outline our basic method, which makes use of some of these new variables. We illustrate our discussion in Section 3.4 with two numerical examples: the LM1 and LM6 CMS study points. In Appendix A we supply the analytic expressions for the shapes of the 1-dimensional invariant mass distributions used in our main analysis in Section 3.3.1. Those results can be useful in improving the precision on the extraction of the kinematical endpoints.

### 3.2 New Variables

In this section we propose a new set of invariant mass (squared) variables. As already explained in the Introduction, our variables should be composed of \( m_{j\ell\ell,n}^2 \) and \( m_{j\ell\ell,f}^2 \) in a symmetric way, in accordance with (3–13). Consequently, any plotting manipulations or mathematical operations involving \( m_{j\ell\ell,n}^2 \) and \( m_{j\ell\ell,f}^2 \) should obey the symmetry implied by Equation (3–13).
3.2.1 The Union $m_{j\ell n}^2 \sqcup m_{j\ell f}^2$

We begin with the simplest case, where we postpone applying any mathematical operations to $m_{j\ell n}^2$ and $m_{j\ell f}^2$, and instead simply plot them. The requirement of Equation (3–13) implies that the only possibility is to place both of them together on the same plot, in essence forming the union

\[ m_{jl(u)}^2 \equiv m_{j\ell n}^2 \sqcup m_{j\ell f}^2 \]  

(3–19)

of the individual $m_{j\ell n}^2$ and $m_{j\ell f}^2$ distributions. Since each individual distribution is smooth and has a kinematic endpoint, the same two kinematic endpoints should be visible on the combined distribution $m_{jl(u)}^2$ as well\(^5\). We shall denote the larger of the two endpoints with

\[ (M_{jl(u)}^{\text{max}})^2 \equiv \max \left\{ (m_{j\ell n}^{\text{max}})^2, (m_{j\ell f}^{\text{max}})^2 \right\} \]  

(3–20)

and the smaller of the two endpoints with

\[ (m_{jl(u)}^{\text{max}})^2 \equiv \min \left\{ (m_{j\ell n}^{\text{max}})^2, (m_{j\ell f}^{\text{max}})^2 \right\} . \]  

(3–21)

The newly introduced quantities $M_{jl(u)}^{\text{max}}$ and $m_{jl(u)}^{\text{max}}$ are nothing but the usual kinematic endpoints $m_{j\ell n}^{\text{max}}$ and $m_{j\ell f}^{\text{max}}$, given by (3–3) and (3–4), correspondingly. Of course, at this point we do not know which is which, and we have an apparent two-fold ambiguity: we can have either

\[ M_{jl(u)}^{\text{max}} = m_{j\ell n}^{\text{max}}, \quad m_{jl(u)}^{\text{max}} = m_{j\ell f}^{\text{max}}, \quad \text{if } R_{AB} \geq R_{BC}, \]  

(3–22)

or

\[ M_{jl(u)}^{\text{max}} = m_{j\ell f}^{\text{max}}, \quad m_{jl(u)}^{\text{max}} = m_{j\ell n}^{\text{max}}, \quad \text{if } R_{AB} \leq R_{BC}. \]  

(3–23)

Notice that both (3–20) and (3–21) are officially upper kinematic endpoints, and thus satisfy our basic requirements.

\(^5\) For specific numerical examples, refer to Section 3.4.
The benefits of our alternative treatment (3–19) in response to the near-far lepton ambiguity problem of Section 3.1.1, are now starting to emerge. With the conventional ordering (3–7,3–8) one has to deal with a three-fold ambiguity in the interpretation of the endpoints $m_{j\ell(lo)}^\text{max}$ and $m_{j\ell(hi)}^\text{max}$, as seen in Equations (3–9,3–10). Instead, the simple union (3–19) leads only to the two-fold ambiguity of Equations (3–22,3–23). More importantly, the analysis of Section 3.3.1 below will reveal that in spite of the remaining two-fold ambiguity in Equations (3–22,3–23), one can nevertheless uniquely determine all three of the masses $m_D$, $m_C$ and $m_A$. We consider this to be one of the important results of this paper.

3.2.2 The Product $m_{j\ell_n} \times m_{j\ell_f}$

In the remainder of this section, we shall construct new invariant mass squared variables out of the two entries $m_{j\ell_n}^2$ and $m_{j\ell_f}^2$, simply by applying various mathematical operations on them in a symmetric fashion. We begin with the product

$$m_{j\ell(p)}^2 \equiv m_{j\ell_n} m_{j\ell_f}$$

whose endpoint is given by

$$\left( m_{j\ell(p)}^\text{max} \right)^2 \equiv \begin{cases} \frac{1}{2} m_D^2 (1 - R_{CD}) \sqrt{1 - R_{AB}}, & \text{for } R_{BC} \leq 0.5, \\ m_D^2 (1 - R_{CD}) \sqrt{R_{BC}(1 - R_{BC})(1 - R_{AB})}, & \text{for } R_{BC} \geq 0.5. \end{cases}$$

(3–25)

Unfortunately, this endpoint also turns out to be piecewise-defined, thus failing one of our basic requirements from the Introduction. Therefore we shall not use this endpoint in the course of our analysis.

3.2.3 The Sums $m_{j\ell_n}^{2\alpha} + m_{j\ell_f}^{2\alpha}$

Another possibility is to consider various sums, for example $m_{j\ell_n}^2 + m_{j\ell_f}^2$ or $(m_{j\ell_n} + m_{j\ell_f})^2$, as originally proposed in [76]. Here we generalize the discussion in [76] and introduce a whole set of new variables, $m_{j\ell(a)}^2(\alpha)$, labelled by the continuous parameter...
\( \alpha \), which are defined as
\[
m_{j\ell(s)}^2(\alpha) \equiv \left( m_{j\ell_n}^{2\alpha} + m_{j\ell_f}^{2\alpha} \right)^\frac{1}{\alpha}.
\] (3–26)

Since \( \alpha \) is a continuous parameter, in principle there are infinitely many \( m_{j\ell(s)} \) variables!

Notice that the conventional variables \( m_{j\ell(lo)}^2 \) and \( m_{j\ell(hi)}^2 \) from (3–7) and (3–8) are also included in our set, and are simply given by
\[
m_{j\ell(lo)}^2 \equiv m_{j\ell(s)}^2(-\infty),
\] (3–27)
\[
m_{j\ell(hi)}^2 \equiv m_{j\ell(s)}^2(\infty).
\] (3–28)

We see that our new set (3–26) is a very broad generalization of the conventional definitions (3–7) and (3–8), which just correspond to the two extreme cases \( \alpha = \pm \infty \).

Of course, the user is free to choose \( \alpha \) at will, and any finite value of \( \alpha \) will lead to a new variable \( m_{j\ell(s)}^2(\alpha) \).

In order to make the new variables \( m_{j\ell(s)}^2(\alpha) \) useful for mass spectrum studies, we need to provide the formulas for their kinematic endpoints \( (m_{j\ell(s)}^\text{max}(\alpha))^2 \). These formulas are easy to derive, using the results from [9], and we present them in the next two subsections, where it is convenient to consider separately the following two cases: \( \alpha \geq 1 \) (in Section 3.2.3.1) and \( \alpha < 1 \), but \( \alpha \neq 0 \) (in Section 3.2.3.2).

### 3.2.3.1 Kinematic Endpoints of \( m_{j\ell(s)}^2(\alpha) \) with \( \alpha \geq 1 \)

When one chooses a value of \( \alpha \geq 1 \), the \( m_{j\ell(s)}^2(\alpha) \) endpoint is given by the following expression
\[
(m_{j\ell(s)}^\text{max}(\alpha \geq 1))^2 \equiv \begin{cases} 
(m_{j\ell_f}^\text{max})^2, & R_{AB} \leq 1 - (1 - R_{BC})(1 - R_{BC}^\alpha)^{-\frac{1}{\alpha}}, \\
(m_{j\ell}^\text{max}(\alpha))^2, & R_{AB} \geq 1 - (1 - R_{BC})(1 - R_{BC}^\alpha)^{-\frac{1}{\alpha}},
\end{cases}
\] (3–29)

where \( m_{j\ell_f}^\text{max} \) was already defined in Equation (3–4), and \( m_{j\ell}^\text{max}(\alpha) \) is a newly defined, \( \alpha \)-dependent quantity
\[
(m_{j\ell}^\text{max}(\alpha))^2 \equiv m_D^2(1 - R_{CD})\left[R_{BC}^\alpha(1 - R_{AB}) + (1 - R_{BC})^\alpha \right]^{\frac{1}{\alpha}}.
\] (3–30)
As a cross-check, one can verify that in the limit $\alpha \to \infty$ the expression in Equation (3–29) reduces to Equation (3–10), in agreement with Equation (3–28). In that case, the upper line in Equation (3–29) corresponds to options $(-, 1)$ and $(-, 2)$ in Equation (3–10), where $m_{j\ell(h)}^{\text{max}} = m_{j\ell}^{\text{max}}$, while the lower line in Equation (3–29) corresponds to option $(-, 3)$ in Equation (3–10), where $m_{j\ell(h)}^{\text{max}} = m_{j\ell(n)}^{\text{max}}$. Unfortunately, just like the product endpoint of Equation (3–25), the endpoint of Equation (3–29) is in general piecewise-defined, and does not meet our criteria.

However, there is one important exception, namely the case of $\alpha = 1$, in which we do get a singly defined function. According to the general definition in Equation (3–26), $m_{j\ell(s)}^2(\alpha = 1)$ is simply the sum of the two $m_{j\ell}^2$ entries in each event:

$$m_{j\ell(s)}^2(\alpha = 1) \equiv m_{j\ell(n)}^2 + m_{j\ell(r)}^2. \quad (3–31)$$

Using the identity

$$m_{j\ell\ell}^2 = m_{j\ell(n)}^2 + m_{j\ell(r)}^2 + m_{\ell\ell}^2, \quad (3–32)$$

Equation (3–31) can be equivalently rewritten as

$$m_{j\ell(s)}^2(\alpha = 1) \equiv m_{j\ell\ell}^2 - m_{\ell\ell}^2. \quad (3–33)$$

To find the expression for its endpoint, one can set $\alpha = 1$ in Equation (3–29), and then realize that the logical condition for executing the upper line becomes $R_{AB} \leq 0$, which is impossible, since the mass ratios $R_{ij}$ in Equation (3–6) are always positive definite. Therefore, the endpoint $m_{j\ell(s)}^{\text{max}}(\alpha = 1)$ is always calculated according to the lower line in Equation (3–29), which results in [76]

$$\left( m_{j\ell(s)}^{\text{max}}(1) \right)^2 \equiv m_D^2(1 - R_{CD})(1 - R_{AC}). \quad (3–34)$$

Note that this endpoint is perfect for our purposes since the Equation (3–34) is always unique, i.e. it is independent of the parameter space region. The variable $m_{j\ell(s)}^2(\alpha = 1)$ will thus play a crucial role in our analysis below.
3.2.3.2 Kinematic Endpoints of \( m^2_{j\ell(s)}(\alpha) \) with \( \alpha < 1 \) and \( \alpha \neq 0 \)

Finally, in the case when \( \alpha < 1 \) but \( \alpha \neq 0 \), the \( m^2_{j\ell(s)}(\alpha) \) endpoint is given by the following expression

\[
(m^\text{max}_{j\ell(s)}(\alpha < 1))^2 \equiv \begin{cases} 
(m^\text{max}_{j\ell}(\alpha))^2, & R_{BC} \geq \left[ 1 + (1 - R_{AB})^{\frac{\alpha}{\alpha - 1}} \right]^{-1}, \\
 m^2_D(1 - R_{CD}) \left[ 1 + (1 - R_{AB})^{\frac{\alpha}{\alpha - 1}} \right]^{\frac{1}{\alpha - 1}}, & R_{BC} \leq \left[ 1 + (1 - R_{AB})^{\frac{\alpha}{\alpha - 1}} \right]^{-1},
\end{cases}
\]

(3–35)

where \( m^\text{max}_{j\ell}(\alpha) \) was already defined in Equation (3–30). Again as a cross-check, one can verify that in the limit \( \alpha \to -\infty \) the expression in Equation (3–35) reduces to Equation (3–9), in agreement with Equation (3–27). In the \( \alpha \to -\infty \) case, the upper line in Equation (3–35) corresponds to option \((-1, 1)\) in Equation (3–9), where \( m^\text{max}_{j\ell(lo)} = m^\text{max}_{j\ell} \), while the lower line in Equation (3–35) corresponds to options \((-1, 2)\) and \((-1, 3)\) in Equation (3–9), where \( m^\text{max}_{j\ell(lo)} = m^\text{max}_{j\ell(eq)} \). Unfortunately, the endpoint function in Equation (3–35) is again piecewise-defined, and does not meet one of our basic criteria spelled out in the introduction.

In passing, we note that the special case of \( \alpha = \frac{1}{2} \), which involves the linear sum of the two masses

\[
m^2_{j\ell(s)}(\alpha = \frac{1}{2}) \equiv (m_{j\ell_n} + m_{j\ell_f})^2,
\]

(3–36)

was previously explored in [76, 77]. In that case, from Equation (3–35) we find for its endpoint

\[
\left( m^\text{max}_{j\ell(s)}(\frac{1}{2}) \right)^2 \equiv \begin{cases} 
 m^2_D(1 - R_{CD}) \left( \sqrt{R_{BC}(1 - R_{AB})} + \sqrt{1 - R_{BC}} \right)^2, & R_{BC} \geq \frac{1 - R_{AB}}{2 - R_{AB}}, \\
 m^2_D(1 - R_{CD})(2 - R_{AB}), & R_{BC} \leq \frac{1 - R_{AB}}{2 - R_{AB}}.
\end{cases}
\]

(3–37)

3.2.4 The Difference \( |m^2_{j\ell_n} - m^2_{j\ell_f}| \)

Finally, one can also consider a set of variables which involve the absolute value of differences between \( m^2_{j\ell_n} \) and \( m^2_{j\ell_f} \). In analogy with Equation (3–26), we can define
another infinite set of variables

\[ m^2_{j\ell(d)}(\alpha) \equiv |m^2_{j\ell_n} - m^2_{j\ell_\ell}|^{\frac{1}{2}}. \] (3–38)

Once again, the user is free to consider arbitrary values of \( \alpha \). However, this freedom is redundant, when it comes to the issue of the kinematic endpoints of the variables in Equation (3–38). It is not difficult to see that the endpoints of \( m^2_{j\ell(d)}(\alpha) \) are always given by

\[ (m^\text{max}_{j\ell(d)}(\alpha))^2 \equiv (M^\text{max}_{j\ell(\ell)})^2 \] (3–39)

and are in fact independent of \( \alpha \)! Therefore, for the purposes of our discussion, it is sufficient to consider just one particular value of \( \alpha \). In the following we shall only use \( \alpha = 1 \):

\[ m^2_{j\ell(d)}(\alpha = 1) \equiv |m^2_{j\ell_n} - m^2_{j\ell_\ell}|, \] (3–40)

which is the analogue of \( m^2_{j\ell(s)}(\alpha = 1) \) defined in Equation (3–31).

The result of Equation (3–39) implies that the endpoint in Equation (3–40) does not contain any new amount of information, which was not already present in the two kinematic endpoints \( M^\text{max}_{j\ell(u)} \) and \( m^\text{max}_{j\ell(u)} \) discussed in Section 3.2.1. Nevertheless, the independent measurement of \((m^\text{max}_{j\ell(d)}(1))^2\) can still be very useful, since it will mark the location of \((M^\text{max}_{j\ell(u)})^2\) on the \( m^2_{j\ell(u)} \) distribution. Then one will be looking for the second endpoint \((m^\text{max}_{j\ell(u)})^2\) to the left, i.e. in the region of smaller \( m^2_{j\ell(u)} \) values.

This completes our discussion of the new invariant mass variables and their kinematic endpoints. For our basic proof-of-principle measurement technique presented in the next Section 3.3.1, we shall use only three of them, namely \( M^\text{max}_{j\ell(u)} \), \( m^\text{max}_{j\ell(u)} \), and \( m^\text{max}_{j\ell(s)}(\alpha = 1) \). However, the remaining variables are in principle just as good, their only disadvantage being that they failed our arbitrarily imposed condition at the beginning that the endpoint functions should all be region independent. Of course, one could, and in fact should, use all of the available kinematic endpoint information, which in a
global fit analysis can only increase the experimental precision of the sparticle mass
determination.

### 3.3 Theoretical Analysis

#### 3.3.1 Our Method And The Solution For The Mass Spectrum

Our starting point is the set of four measurements

\[ m_{\ell\ell}^{\text{max}}, M_{j\ell(u)}^{\text{max}}, m_{j\ell(u)}^{\text{max}}, m_{j\ell(s)}^{\text{max}}(\alpha = 1) \]  

(3–41)

in place of the conventional set in Equation (3–18). It is easy to verify that the measurements
in Equation (3–41) are always independent of each other, and thus never suffer from the
linear dependence problem discussed in Section 3.1.2.

Given the set of four measurements in Equation (3–41), it is easy to solve for the
mass spectrum. To simplify the notation, we introduce the following shorthand notation
for the endpoints of the mass squared distributions

\[ L \equiv (m_{\ell\ell}^{\text{max}})^2, \quad M \equiv (M_{j\ell(u)}^{\text{max}})^2, \quad m \equiv (m_{j\ell(u)}^{\text{max}})^2, \quad S \equiv (m_{j\ell(s)}^{\text{max}}(\alpha = 1))^2 \]  

(3–42)

The solution for the mass spectrum is then given by

\[ m_D^2 = \frac{Mm(L + M + m - S)}{(M + m - S)^2}, \]  

(3–43)

\[ m_C^2 = \frac{MmL}{(M + m - S)^2}, \]  

(3–44)

\[ m_B^2 = \begin{cases} 
\frac{ML(S-M)}{(M+m-S)^2}, & \text{if } R_{AB} \geq R_{BC}, \\
\frac{mL(S-m)}{(M+m-S)^2}, & \text{if } R_{AB} \leq R_{BC}; 
\end{cases} \]  

(3–45)

\[ m_A^2 = \frac{L(S-m)(S-M)}{(M + m - S)^2}. \]  

(3–46)

It is easy to verify that the right-hand side expressions in these equations are always
positive definite, so that one can safely take the square root and compute the linear
masses \( m_D, m_C, m_B \) and \( m_A \). Notice that in spite of the two-fold ambiguity in Equations
the solution for $m_D$, $m_C$ and $m_A$ is unique! Indeed, the expressions for $m_D$, $m_C$ and $m_A$ are symmetric under the interchange $M \leftrightarrow m$. The remaining two-fold ambiguity for $m_B$ is precisely the result of the ambiguous interpretation in Equations (3–22 and 3–23) of the two $m^2_{j\ell(u)}$ endpoints, and is related to the symmetry under Equation (3–13), or equivalently, under the interchange

$$R_{AB} \leftrightarrow R_{BC}. \quad (3–47)$$

In the next subsection we discuss several ways in which one can lift the remaining two-fold degeneracy for $m_B$ which is due to Equation (3–47).

Notice the great simplicity of this method. The expressions in Equations (3–43), (3–44) and (3–46) are region independent and therefore one does not have to go through the standard trial and error procedure involving the 9 parameter space regions $(N_{j\ell\ell}, N_{j\ell})$ [9, 17] associated with the various interpretations of the endpoints $m^\text{max}_{j\ell\ell}$, $m^\text{max}_{j\ell(lo)}$ and $m^\text{max}_{j\ell(hi)}$.

### 3.3.2 Disambiguation Of The Two Solutions For $m_B$

The method outlined in Section 3.3.1 allowed us to find the true masses of particles $A$, $C$ and $D$, but yields two separate possible solutions for the mass $m_B$ of particle $B$. We shall now discuss several ways of lifting the remaining two-fold degeneracy for $m_B$.

#### 3.3.2.1 Invariant mass endpoint method

One possibility is to use an additional measurement of an invariant mass endpoint. Indeed, as shown in Section 3.2, there are still quite a few one-dimensional invariant mass distributions at our disposal, which we have not used so far. Those include the conventional distributions of $m^2_{j\ell\ell}$, $m^2_{j\ell(lo)}$ and $m^2_{j\ell(hi)}$, as well as the new distributions $m^2_{j\ell(p)}$, $m^2_{j\ell(lo)}(\alpha)$ and $m^2_{j\ell(lo)}(1)$ which we introduced in Section 3.2. Which of them can be used for our purposes? Note that the duplication in Equation (3–45) arose due to the symmetry in Equation (3–47), so that any kinematic endpoint which violates this symmetry will be able to distinguish between the two solutions.
Figure 3-2. Comparison of the predictions for the kinematic endpoints $m_{j\ell}^{\text{max}}(\alpha)$ of the real and fake solutions, as a function of $\phi \equiv \arctan \alpha$ (in units of $\pi$), for the two examples discussed in detail in Section 3.4: (a) the LM1 CMS study point and (b) the LM6 CMS study point. In each panel, the prediction of the real (fake) solution is plotted in red (blue). The vertical dotted line indicates the case of $\phi = \pi/4$ ($\alpha = 1$), for which the two solutions give an identical answer, marked with a green dot. The horizontal dotted lines show the corresponding asymptotic values $m_{j\ell}^{\text{max}(\text{hi})}$ and $m_{j\ell}^{\text{max}(\text{lo})}$, obtained at $\alpha \to \pm \infty$ ($\phi \to \pm \pi/2$).

Let us begin with the conventional distributions $m_{j\ell\ell}^2$, $m_{j\ell\ell(\text{lo})}^2$, $m_{j\ell\ell(\text{hi})}^2$ and $m_{j\ell\ell(\theta > \pi/2)}^2$, whose endpoints we did not use in our analysis so far. It is easy to check that $m_{j\ell\ell}^{\text{max}}$, $m_{j\ell\ell(\text{hi})}^{\text{max}}$ and $m_{j\ell\ell(\theta > \pi/2)}^{\text{min}}$ are invariant under the interchange as in Equation (3–47) and cannot be used for discrimination. However, $m_{j\ell\ell(\text{lo})}^{\text{max}}$ is not symmetric under Equation (3–47) and can do the job. In fact, one can show that the two duplicate solutions for $m_B$ always\(^6\) give different predictions for $m_{j\ell\ell(\text{lo})}^{\text{max}}$.

More importantly, many of our new variables from Section 3.2 can provide an independent cross-check on the correct choice for the solution. For example, the kinematic endpoint in Equation (3–25) of the product variable $m_{j\ell(\rho)}^2$, also violates the symmetry of Equation (3–47) and distinguishes among the two solutions. The infinite set of variables $m_{j\ell(s)}^2(\alpha)$ can also be used, and for almost the whole range of $\alpha < 1$. To see

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\(^6\) The only exception is the trivial case of $R_{AB} = R_{BC}$, but then the two solutions for $m_B$ coincide, and $m_B$ is again uniquely determined.
this, in Figure 3-2 we compare the predictions for the kinematic endpoints $m_{j\ell(s)}^{\text{max}}(\alpha)$ of the real and fake solutions, for the two examples discussed in detail in Section 3.4: (a) the LM1 CMS study point and (b) the LM6 CMS study point. The corresponding mass spectra are listed in Table 3-1 below. For convenience, we plot versus the parameter

$$\phi \equiv \arctan \alpha,$$

(3–48)

which allows us to map the whole definition domain $(-\infty, \infty)$ for $\alpha$ into the finite region $(-\frac{\pi}{2}, \frac{\pi}{2})$ for $\phi$. Figure 3-2 shows that for most of the allowed $\phi$ range, the two solutions predict different values for the kinematic endpoints $m_{j\ell(s)}^{\text{max}}(\alpha)$. In fact, for $\phi < \frac{\pi}{4}$, the two predictions are always different, apart from the trivial case of $\phi = 0$ ($\alpha = 0$). Even for $\phi > \frac{\pi}{4}$, there still exists a range of $\phi$, for which, at least theoretically, a discrimination can be made. The predictions are guaranteed to coincide only for $\phi = \frac{\pi}{4}$ ($\alpha = 1$) (as they should, see Equation (3–41)), and for a certain range of the largest possible values of $\phi$.

3.3.2.2 Invariant mass correlations

Another way to resolve the twofold ambiguity in our solution in Equation (3–45) is to simply go back to the original measurements of $M_{j\ell(u)}^{\text{max}}$ and $m_{j\ell(u)}^{\text{max}}$ and already at that point try to decide which of the two measured $m_{j\ell(u)}$ endpoints is $m_{j\ell u}^{\text{max}}$ and which one is $m_{j\ell f}^{\text{max}}$. As already discussed in [21, 76], this identification is in principle possible, if one considers the correlations which are present in the two-dimensional distribution $m_{j\ell(u)}^2$ versus $m_{j\ell(u)}^2$. The basic idea is illustrated in Figure 3-3, where we show scatter plots of $m_{j\ell(u)}$ versus $m_{j\ell}$, for the two examples used in Figure 3-2 and discussed in detail later in Section 3.4. Figure 3-3(a) (Figure 3-3(b)) shows the result for the real (fake) solution corresponding to the LM1 study point, while Figures 3-3(c) and 3-3(d) show the analogous results for the LM6 study point. In each plot we used 10,000 entries, which roughly corresponds to 20 fb$^{-1}$ (200 fb$^{-1}$) of data for the actual LM1 (LM6) SUSY study point. Here and below we show the ideal case where we neglect smearing effects due to the finite detector resolution, finite particle widths and combinatorial backgrounds.
Figure 3-3. Predicted scatter plots of $m_{j(u)}$ versus $m_{\ell\ell}$, for the case of the real and fake solutions for each of the two study points LM1 and LM6: (a) the real solution LM1; (b) the fake solution LM1'; (c) the real solution LM6; and (d) the fake solution LM6'. The red solid horizontal (blue dashed inclined) line indicates the conditional maximum $m_{j(u)}^{\text{max}}(m_{\ell\ell})$ given by Equation (3–49) (Equation (3–50)). Each panel contains 10,000 entries. The results shown here are idealized in the sense that we neglect smearing effects due to the finite detector resolution, finite particle widths and combinatorial backgrounds. Notice the use of quadratic power scale on the two axes, which preserves the simple shapes of the scatter plots, even when plotted versus the linear masses $m_{j(u)}$ and $m_{\ell\ell}$.

All of our plots are at the parton level (using our own Monte-Carlo phase space generator) and without any cuts. Notice that in order to avoid dealing with the large numerical values of the squared masses, we use a quadratic power scale on both axes, which allows us to preserve the simple shapes of the scatter plots when plotting versus the linear masses themselves.
Figure 3-3 shows that the combined distribution \( m^2_{j\ell(u)} \) is simply composed of the two separate distributions \( m^2_{j\ell_a} \) and \( m^2_{j\ell_f} \), but they are correlated differently with the dilepton distribution \( m^2_{\ell\ell} \). In particular, let us concentrate on the conditional maxima \( m^\text{max}_{j\ell_a}(m_{\ell\ell}) \) and \( m^\text{max}_{j\ell_f}(m_{\ell\ell}) \), i.e. the maximum allowed values of \( m_{j\ell_a} \) and \( m_{j\ell_f} \), respectively, for a given fixed value of \( m_{\ell\ell} \) [21, 76]. A close inspection of Figure 3-3 shows that the values of \( m^2_{j\ell_a} \) and \( m^2_{\ell\ell} \) are uncorrelated, and as a result, the conditional maximum \( m^\text{max}_{j\ell_a}(m_{\ell\ell}) \) does not depend on \( m_{\ell\ell} \). In turn, this implies that the endpoint value \((m^\text{max}_{j\ell_a})^2\) given in Equation (3–3) can be obtained for any \( m^2_{\ell\ell} \):

\[
n \equiv (m^\text{max}_{j\ell_a})^2 = [m^\text{max}_{j\ell_a}(m_{\ell\ell})]^2 = m^2_D (1 - R_{CD}) (1 - R_{BC}). \quad \forall m_{\ell\ell} \in [0, m^\text{max}_{\ell\ell}]. \tag{3–49}
\]

Because of Equation (3–49), the shape of the \( m^2_{j\ell_a} \) versus \( m^2_{\ell\ell} \) scatter plot is a simple rectangle [21, 76]. This is confirmed by the plots in Figure 3-3, where the (red) horizontal solid line indicates the constant value as in Equation (3–49) for the conditional maximum \( m^\text{max}_{j\ell_a}(m_{\ell\ell}) \).

In contrast, the values of \( m^2_{j\ell_f} \) and \( m^2_{\ell\ell} \) are correlated. The conditional maximum \( m^\text{max}_{j\ell_f}(m_{\ell\ell}) \) does depend on the value of \( m_{\ell\ell} \) as follows:

\[
(m^\text{max}_{j\ell_f}(m_{\ell\ell}))^2 = p + \frac{f - p}{L} m^2_{\ell\ell}, \tag{3–50}
\]

where we introduce the shorthand notation used in [9]

\[
f \equiv (m^\text{max}_{j\ell_f})^2 = m^2_D (1 - R_{CD}) (1 - R_{AB}), \tag{3–51}
\]

\[
p \equiv R_{BC} f = m^2_D (1 - R_{CD}) R_{BC} (1 - R_{AB}). \tag{3–52}
\]

The absolute maximum of \( m^2_{j\ell_f} \), which is given by Equation (3–4) and denoted here by \( f \), can only be obtained when \( m^2_{\ell\ell} \) itself is at a maximum [21, 76]:

\[
f \equiv [m^\text{max}_{j\ell_f}(m^\text{max}_{\ell\ell})]^2. \tag{3–53}
\]
On the other hand, the conditional maximum $m^\text{max}_{\ell\ell}(m_{\ell\ell})$ obtains its minimum value at $m^2_{\ell\ell} = 0$ and corresponds to [21, 76]

$$p \equiv [m^\text{max}_{\ell\ell}(0)]^2 \leq f. \quad (3-54)$$

Equations (3–53 and 3–54) imply that the shape of the $m^2_{\ell\ell}$ versus $m^2_{\ell\ell}$ scatter plot is a right-angle trapezoid. This is confirmed by the plots in Figure 3-3, where we mark with a (blue) dashed line the conditional maximum in Equation (3–50). With sufficient statistics, this difference in the kinematic boundaries may be observable, and would reveal the identity of $m^\text{max}_{\ell\ell}$ and $m^\text{max}_{\ell\ell}$ [21, 76]. Once the individual $m^\text{max}_{\ell\ell}$ and $m^\text{max}_{\ell\ell}$ are known, the solution for the mass spectrum is unique – see e.g. Appendix A in [9]. Of course, in cases where $p \sim f$, namely $R_{BC} \sim 1$, it may be difficult in practice to tell which of the two boundaries in the scatter plot is inclined and which one is horizontal. One example of this sort is offered by point LM6, which has $R_{BC} = 0.91$ and leads to a rather flat $m^\text{max}_{\ell\ell}(m_{\ell\ell})$ function, as seen in Figure 3-3(c).

An alternative and somewhat related method will be to investigate the shapes of the one-dimensional distributions themselves [78]. In Appendix A we provide the analytical expressions for the shapes of the four invariant mass distributions $m^2_{\ell\ell}$, $m^2_{\ell\ell}(u)$, $m^2_{\ell\ell}(s)(1)$ and $m^2_{\ell\ell}(d)(1)$ used in our basic analysis from Section 3.3.1. Given what we have already seen in Figure 3-3, it is not surprising that the true and the fake solutions predict different shapes for the one-dimensional distributions as well. In the LM1 and LM6 examples considered below in Section 3.4, this difference is particularly noticeable for the $m^2_{\ell\ell}(u)$ and $m^2_{\ell\ell}(d)(1)$ distributions (see Figures 3-4(b), 3-4(d), 3-5(b) and 3-5(d)), and can be tested experimentally.

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7 A separate problem, which arises in the case of $p \sim f$, will be discussed below in Section 3.4.1.
Table 3-1. The relevant part of the SUSY mass spectrum for the LM1 and LM6 study points. The corresponding duplicated solutions LM1’ and LM6’ are obtained by interchanging $R_{BC} \leftrightarrow R_{AB}$ as in Equation (3–47). In the table we also list the corresponding values for various invariant mass endpoints. The first four of those represent our basic set of measurements of Equation (3–41) discussed in detail in Section 3.4.1, while the last two ($m_{\ell\ell}^{\text{max}}$ and $m_{\ell\ell}^{\text{min}}$) are not directly observable. The remaining invariant mass endpoints are considered in Section 3.4.2. In the case of $m_{\ell\ell}^{\text{max}}(\alpha)$, we show several representative values for $\alpha$. For the complete $\alpha$ variation, refer to Figure 3-2. Recall that $m_{\ell\ell}^{\text{max}}(+\infty) = m_{\ell\ell}^{\text{max}}(hi)$ and $m_{\ell\ell}^{\text{max}}(-\infty) = m_{\ell\ell}^{\text{max}}(lo)$.

<table>
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<th>Variable</th>
<th>LM1</th>
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<th>LM6</th>
<th>LM6'</th>
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</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\alpha = 1)$ (GeV)</td>
<td>451.8</td>
<td>689.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\alpha)$ (GeV)</td>
<td>451.8</td>
<td>689.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{min}}(\theta &gt; \frac{\pi}{2})$ (GeV)</td>
<td>215.2</td>
<td>176.4</td>
<td></td>
<td></td>
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<tr>
<td>$m_{\ell\ell}^{\text{max}}(hi)$ (GeV)</td>
<td>398.8</td>
<td>676.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\alpha = 2)$ (GeV)</td>
<td>406.6</td>
<td>398.8</td>
<td>676.8</td>
<td>677.0</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\alpha = 1.5)$ (GeV)</td>
<td>417.9</td>
<td>402.5</td>
<td>676.8</td>
<td>678.4</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\alpha = 0.5)$ (GeV)</td>
<td>611.0</td>
<td>638.9</td>
<td>886.0</td>
<td>807.1</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\alpha = -0.5)$ (GeV)</td>
<td>142.9</td>
<td>159.7</td>
<td>174.9</td>
<td>138.0</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\alpha = -1)$ (GeV)</td>
<td>200.1</td>
<td>225.9</td>
<td>224.8</td>
<td>184.8</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(lo)$ (GeV)</td>
<td>274.6</td>
<td>319.1</td>
<td>239.8</td>
<td>229.9</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(lo)$ (GeV)</td>
<td>292.0</td>
<td>319.4</td>
<td>393.7</td>
<td>310.9</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\rho)$ (GeV)</td>
<td>398.8</td>
<td>320.6</td>
<td>239.8</td>
<td>676.8</td>
</tr>
<tr>
<td>$m_{\ell\ell}^{\text{max}}(\rho)$ (GeV)</td>
<td>320.6</td>
<td>398.8</td>
<td>676.8</td>
<td>239.8</td>
</tr>
</tbody>
</table>
3.4 Numerical Examples

We shall now illustrate the ideas of the previous section with two specific numerical examples: the LM1 and LM6 SUSY study points in CMS [75]. The mass spectra at LM1 and LM6 are listed in Table 3-1. Point LM1 is similar to benchmark point A (A’) in Reference [79] (Reference [80]) and to benchmark point SPS1a in Reference [81]. Point LM6 is similar to benchmark point C (C’) in Reference [79] (Reference [80]). The Table 3-1 also lists the corresponding duplicate solutions LM1’ and LM6’, which are obtained by interchanging $R_{BC} \leftrightarrow R_{AB}$, or equivalently, by replacing the mass of $B$ via

$$m_B \rightarrow m_B' = \frac{m_A m_C}{m_B}. \quad (3–55)$$

It is interesting to note that LM1 and LM6 represent both sides of the ambiguity in Equation (3–47): at LM1, we have $R_{AB} > R_{BC}$ and correspondingly, $m_{j\ell_{n}}^{\text{max}} > m_{j\ell_{f}}^{\text{max}}$ and Equation (3–22) applies. On the other hand, at LM6 we have $R_{AB} < R_{BC}$ and $m_{j\ell_{n}}^{\text{max}} < m_{j\ell_{f}}^{\text{max}}$, so that Equation (3–23) applies. Another interesting difference is that at LM1 particle $B$ is the right-handed slepton $\tilde{\ell}_R$, while at LM6 the role of particle $B$ is played by the left-handed slepton $\tilde{\ell}_L$. To the extent that we are interested in kinematical features, this difference is not relevant.

3.4.1 Mass Measurements At Study Points LM1 and LM6

Given the mass spectra in Table 3-1, it is straightforward to construct and investigate the relevant invariant mass distributions. For the purposes of illustration, we shall ignore spin correlations, referring the readers interested in those effects to Reference [82–84]. We are justified to do so for several reasons. First, our method relies on the measurement of kinematic endpoints, whose location is unaffected by the presence of spin correlations. Second, in the case of supersymmetry (which is really what we have

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8 Although the right-handed slepton $\tilde{\ell}_R$ is also kinematically accessible at point LM6, the wino-like neutralino $\tilde{\chi}_2^0$ decays much more often to $\tilde{\ell}_L$ as opposed to $\tilde{\ell}_R$. 

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in mind here), particle \( B \) is a scalar, which automatically washes out any spin effects in the \( m_{\ell\ell}^2 \) and \( m_{j\ell}^2 \) distributions. Furthermore, if particles \( D \) and their antiparticles \( \bar{D} \) are produced in equal numbers, as would be the case if the dominant production is from \( gg \) and/or \( q\bar{q} \) initial state, any spin correlations in the \( m_{j\ell}^2 \) distribution are also washed out. Under those circumstances, therefore, the pure phase space distributions shown here are in fact the correct answer.

We begin our discussion with the four invariant mass distributions \( m_{\ell\ell}^2 \), \( m_{j\ell(\bar{u})}^2 \), \( m_{j\ell(s)}^2(\alpha = 1) \) and \( m_{j\ell(d)}^2(\alpha = 1) \), which form the basis of our method outlined in Section 3.3.1. Figure 3-4 (Figure 3-5) shows those four distributions for the case of study point LM1 (LM6). In each panel, the red (solid) histogram corresponds to the nominal spectrum (LM1 or LM6), while the blue (dotted) histogram corresponds to the “fake” solution (LM1’ or LM6’), which is obtained through the replacement (3–55). For all figures in this section, we use the same 4 samples of 10,000 events each, which were already used to make Figure 3-3. Notice our somewhat unconventional way of filling and then plotting the histograms in this section. First, we show differential distributions in the corresponding mass squared, i.e. \( dN/dm^2 \). This is done in order to preserve the connection to the analytical results in Appendix A, which are written the same way. More importantly, the shapes of the one-dimensional histograms are much simpler in the case of \( dN/dm^2 \) as opposed to \( dN/dm \) \[82–84\]. In the next step, however, we choose to plot the thus obtained histogram versus the mass itself rather than the mass squared. This allows one to read off immediately the corresponding endpoint and compare directly to the values listed in Table 3-1. It also keeps the \( x \)-axis range within a manageable range. However, since the histograms were binned on a mass squared scale, if we were to use a linear scale on the \( x \)-axis, we would get bins with varying size. This would be rather inconvenient and more importantly, would distort the nice simple shapes of the \( dN/dm^2 \) distributions. Therefore, we use a quadratic scale on the \( x \)-axis, which preserves the nice shapes and leads to a constant bin size on each plot.
Figure 3-4. One-dimensional invariant mass distributions for the case of LM1 (red solid lines) and LM1’ (blue dotted lines) spectra. The kinematic endpoints in Equation (3–41) used in our analysis in Section 3.3.1 can be observed from these distributions as follows: $m_{\ell\ell}^{\text{max}}$ is the upper kinematic endpoint of the $m_{\ell\ell}$ distribution in panel (a); $M_{j(\ell)}^{\text{max}}$ is the absolute upper kinematic endpoint seen in both the combined $m_{j(\ell)}$ distribution in panel (b), or the difference distribution $m_{j(\ell)(d)}(1)$ in panel (d); $m_{j(\ell)}^{\text{max}}$ is the intermediate kinematic endpoint seen in panel (b); and $m_{j(\ell)(s)}^{\text{max}}(\alpha = 1)$ is the upper kinematic endpoint of the $m_{j(\ell)(s)}(\alpha = 1)$ distribution in panel (c).
Figure 3-5. One-dimensional invariant mass distributions for the LM6 mass spectrum (red solid lines) and the LM6’ mass spectrum (blue dotted lines).
Figures 3-4 and 3-5 illustrate how each one of the measurements in Equation (3–41) can be obtained. For example, $m_{\ell\ell}^{\text{max}}$ is the classic upper kinematic endpoint of the $m_{\ell\ell}$ distributions in Figures 3-4(a) and 3-5(a). This endpoint is very sharp and should be easily observable. $M_{j\ell(u)}^{\text{max}}$ is the absolute upper kinematic endpoint seen in the combined $m_{j\ell(u)}$ distribution in Figures 3-4(b) and 3-5(b). Notice that the same endpoint can independently also be observed as the absolute upper kinematic limit of the difference distributions $m_{j\ell(d)}(1)$ shown in Figures 3-4(d) and 3-5(d). The fact that there are two independent ways of getting to the endpoint $M_{j\ell(u)}^{\text{max}}$ should allow for a reasonable accuracy of its measurement. Upon closer inspection of the combined $m_{j\ell(u)}$ distribution in Figures 3-4(b) and 3-5(b), we also notice the intermediate kinematic endpoint $m_{j\ell(u)}^{\text{max}}$ seen around 320 GeV in Figure 3-4(b) and around 240 GeV in Figure 3-5(b). Finally, $m_{j\ell(s)}^{\text{max}}(\alpha = 1)$ is the upper kinematic endpoint of the $m_{j\ell(s)}(\alpha = 1)$ distribution shown in Figures 3-4(c) and 3-5(c). It is also rather well defined, and should be well measured in the real data. At this point we would like to comment on one potential problem which is not immediately obvious, but nevertheless has been encountered in practical applications of the invariant mass technique for SUSY mass determinations [78]. It has been noted that in the case of $\rho \sim f$ (Equations (3–51,3–52)), the numerical fit for the mass spectrum becomes rather unstable. Given our analytical results in Section 3.3.1, we are now able to trace the root of the problem. Notice that $\rho \sim f$ implies that $R_{BC} \sim 1$. In this limit, from Equations (3–2), (3–3), (3–4) and (3–34) we find

$$\lim_{R_{BC} \to 1} (L) = 0, \quad \lim_{R_{BC} \to 1} (n) = 0, \quad \lim_{R_{BC} \to 1} (M + m - S) = 0.$$  

(3–56)

This means that the functions in Equations (3–43 through 3–46) giving the solution for the mass spectrum will all behave as $\frac{a}{a^2}$, and, given the statistical fluctuations in an actual analysis, will have very poor convergence properties. We note that this problem is not limited to our preferred set of measurements (3–41) and is rather generic, but has been missed in most previous studies simply because the case of $R_{BC} \sim 1$ was
rarely considered. Figures 3-4 and 3-5 reveal that, as expected, the real (red solid lines) and fake (blue dotted lines) solutions always give identical results for our basic set of four endpoint measurements in Equation (3–41). This is by design, and in order to discriminate among the real and the fake solution, we need additional experimental input, as discussed in Section 3.3.2. Before we proceed with the disambiguation analysis in the next subsection, we should stress once again that the real and fake solutions agree on 75% of the relevant mass spectrum, i.e. they give the same values for the masses of particles $D$, $C$ and $A$ (Table 3-1). The only question mark at this point is, what is the mass of particle $B$. This issue is addressed in the following subsection.

3.4.2 Eliminating The Fake Solution for $m_B$

As already discussed in Section 3.3.2, there are several handles which could discriminate among the two alternative values of $m_B$ in the real and the fake solution. One possibility is to use additional independent measurements of $M_{\ell\ell}$ kinematic endpoints. Another possibility, discussed in Section 3.3.2.2 and demonstrated explicitly with Figure 3-3, is to use the different correlations in the 2-dimensional invariant mass distributions $(m_{\ell\ell}, m_{f_{\alpha}})$ and $(m_{\ell\ell}, m_{f_{\ell}})$. The near-far lepton ambiguity is avoided by studying the scatter plot of $(m_{\ell\ell}, m_{f_{(\ell)}})$, shown in Figure 3-3, which should be in principle sufficient to discriminate among the two alternatives.

In keeping with the main theme of this paper, in this subsection we shall concentrate on the third possibility, already suggested in Section 3.3.2.1. We shall simply explore additional invariant mass endpoint measurements, which would hopefully discriminate among the two solutions for $m_B$. Figures 3-6 and 3-7 show several invariant mass distributions which have already been mentioned at one point or another in the course of our previous discussion. Figure 3-6 shows the following 6 distributions: (a) $m_{f_{\ell\ell}}^2$, (b) $m_{f_{(hi)}}^2$; (c) $m_{f_{(p)}}^2$; (d) $m_{f_{(lo)}}^2$; (e) $m_{f_{(s)}}^2(\alpha = -1)$ and (f) $m_{f_{(s)}}^2(\alpha = \frac{1}{2})$, for the LM1 mass spectrum (red solid lines) and its LM1’ counterpart (blue dotted lines). Figure 3-7 shows the same 6 distributions, but for the LM6 and LM6’ mass spectra. In Figures 3-6 and 3-7,
we follow the same plotting conventions as in Figures 3-4 and 3-5: we form the mass squared distribution $dN/dm^2$, and then plot versus the corresponding linear mass $m$ using a quadratic scale on the $x$-axis. Notice that the sum of the $m^2_{j\ell(h)}$ distribution in Figure 3-6(b) (Figure 3-7(b)) and the $m^2_{j\ell(lo)}$ distribution in Figure 3-6(d) (Figure 3-7(d)) precisely equals the combined distribution $m^2_{j\ell(u)}$ in Figure 3-4(b) (Figure 3-5(b)). In order to be able to see this by the naked eye, we have kept the same $x$ and $y$ ranges on the corresponding plots.

As seen in Figures 3-6 and 3-7, not all of the remaining invariant mass distributions are able to discriminate among the two $m_B$ solutions. As explained in Section 3.3.2.1, the suitable distributions are those whose endpoints violate the symmetry in Equation (3–47), which caused the $m_B$ ambiguity in the first place. For example, Figures 3-6(a) and 3-7(a) show that the endpoint of the $m^2_{j\ell}$ distribution is the same for the real and the fake solution. This is to be expected, since the defining Equation (3–5) for $m^\text{max}_{j\ell}$ is symmetric under Equation (3–47). Figures 3-6(a) and 3-7(a) also show that even the shapes of the $m^2_{j\ell}$ distributions for the real and fake solution are very similar. In spite of this, the observation of the $m^2_{j\ell}$ endpoint can still be very useful, e.g. in reducing the experimental error on the mass determination.

Similar comments apply to the $m^2_{j\ell(lo)}$ distributions shown in Figures 3-6(b) and 3-7(b). Here again the endpoint is a symmetric function of $R_{AB}$ and $R_{BC}$, and the real and fake solutions predict identical endpoints. However, while the endpoints are the same, this time the shapes are not. The shape difference is more pronounced in the case of LM1 shown in Figure 3-6(b), and less visible in the case of LM6 shown in Figure 3-7(b).

The remaining four distributions shown in Figures 3-6(c-f) and 3-7(c-f) already have different endpoints and can thus be used for discrimination among the real and fake solution for $m_B$. All of the endpoints in Figures 3-6(c-f) and 3-7(c-f) are relatively sharp and should be measured rather well. One should not forget that in Figures 3-6 and 3-7
we show $m_{j\ell(s)}^2(\alpha)$ distributions for only three representative values of $\alpha$: $\alpha = -\infty$ in panels (d), $\alpha = -1$ in panels (e), and $\alpha = 0.5$ in panels (f). As seen in Figure 3-2, there are infinitely many other choices for $\alpha$, which would still exhibit different endpoints for the real and fake $m_B$ solutions. Our conclusion is that through a suitable combination of additional endpoint measurements one would be able to tell apart the real solution for $m_B$ from its fake cousin.
Figure 3-6. Some other one-dimensional invariant mass distributions of interest, for the case of the LM1 mass spectrum (red solid lines) and LM1' mass spectrum (blue dotted lines): (a) $m_{j\ell\ell}^2$ distribution; (b) $m_{j\ell(h)}^2$ distribution; (c) $m_{j\ell(p)}^2$ distribution; (d) $m_{j\ell(lo)}^2$ distribution; (e) $m_{j\ell(s)}^2(\alpha = -1)$ distribution; (f) $m_{j\ell(s)}^2(\alpha = \frac{1}{2})$ distribution. All distributions are then plotted versus the corresponding mass, on a quadratic scale for the $x$-axis.
Figure 3-7. Some other one-dimensional invariant mass distributions of interest for the LM6 mass spectrum (red solid lines) and the LM6' mass spectrum (blue dotted lines).
CHAPTER 4
SUBSYSTEM $M_{T2}$ METHOD

The idea for a subsystem $M_{T2}$ was first discussed in [85] and applied in [86] for a specific supersymmetry example (associated squark-gluino production and decay). Here we shall generalize that concept for a completely general decay chain. The subsystem $M_{T2}$ variable will be defined for the subchain inside the blue (yellow-shaded) box in Figure 4-1. Before we give a formal definition of the subsystem $M_{T2}$ variables, let us first introduce some terminology for the BSM particles appearing in the decay chain. We shall find it convenient to distinguish the following types of BSM particles:

- “Grandparents”. Those are the two BSM particles $X_n$ at the very top of the decay chains in Figure 4-1. Since we have assumed symmetric events, the two grandparents in each event are identical, and carry the same index $n$. Of course, one may relax this assumption, and consider asymmetric events, as was done in [86, 87]. Then, the two “grandparents” will be different, and one would simply need to keep track of two separate grandparent indices $n^{(1)}$ and $n^{(2)}$.

- “Parents”. Those are the two BSM particles $X_p$ at the top of the subchain used to define the subsystem $M_{T2}$ variable. In Figure 4-1 this subchain is identified by the blue (yellow-shaded) rectangular box. The idea behind the subsystem $M_{T2}$ is simply to apply the usual $M_{T2}$ definition for the subchain inside this box. Notice that the $M_{T2}$ concept usually requires the parents to be identical, therefore here we will characterize them by a single “parent” index $p$.

- “Children”. Those are the two BSM particles $X_c$ at the very end of the subchain used to define the subsystem $M_{T2}$ variable, as indicated by the blue (yellow-shaded) rectangular box in Figure 4-1. The children are also characterized by a single index $c$. In general, the true mass $M_c$ of the two children is unknown. As usual, when calculating the value of the $M_{T2}$ variable, one needs to choose a child “test” mass, which we shall denote with a tilde, $\tilde{M}_c$, in order to distinguish it from the true mass $M_c$ of $X_c$.

- Dark matter candidates. Those are the two stable neutral particles $X_0$ appearing at the very end of the cascade chain. We see that while those are the particles responsible for the measured missing momentum in the event, they are relevant for $M_{T2}$ only in the special case of $c = 0$. 

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Figure 4-1. Illustration of the subsystem $M_{T2}^{(n,p,c)}$ variable defined in Equation (4–3).

With those definitions, we are now ready to generalize the conventional $M_{T2}$ definition [10, 32]. From Figure 4-1 we see that any subchain is specified by the parent index $p$ and the child index $c$, while the total length of the whole chain (and thus the type of event) is given by the grandparent index $n$. Therefore, the subsystem $M_{T2}$ variable will have to carry those three indices as well, and we shall use the notation $M_{T2}^{(n,p,c)}$. In the following we shall refer to this generalized quantity as either “subsystem” or “subchain” $M_{T2}$. It is clear that the set of three indices $(n, p, c)$ must be ordered as follows:

$$n \geq p > c \geq 0 .$$  \hspace{1cm} (4–1)

We shall now give a formal definition of the quantity $M_{T2}^{(n,p,c)}$, generalizing the original idea of $M_{T2}$ [10, 32]. The parent and child indices $p$ and $c$ uniquely define a subchain, within which one can form the transverse masses $M_{T}^{(1)}$ and $M_{T}^{(2)}$ of the two parents:

$$M_{T}^{(k)}(p_{p}^{(k)}, p_{p-1}^{(k)}, \ldots, p_{c+1}^{(k)}, \tilde{P}_{cT}^{(k)}; \tilde{M}_{c}) , \hspace{1cm} k = 1, 2 .$$  \hspace{1cm} (4–2)

Here $p_{i}^{(k)}$, $c + 1 \leq i \leq p$, are the measured 4-momenta of the SM particles within the subchain, $\tilde{P}_{cT}^{(k)}$ are the unknown transverse momenta of the children, while $\tilde{M}_{c}$ is their unknown (test) mass. Then, the subsystem $M_{T2}^{(n,p,c)}$ is defined by minimizing the larger of the two transverse masses as in Equation (4–2) over the allowed values of the children’s
transverse momenta $\vec{P}^{(k)}$:

$$M^{(n,p,c)}_{T^2}(\tilde{M}_c) = \min_{\sum_{k=1}^2 \vec{P}^{(k)}_{cT} - \sum_{k=1}^{n} \sum_{j=1}^c \vec{P}^{(k)}_{jT} - \vec{p}_T} \left\{ \max \left\{ M^{(1)}_{T^2}, M^{(2)}_{T^2} \right\} \right\}, \quad (4-3)$$

where $\vec{p}_T$ indicates any additional transverse momentum due to initial state radiation (ISR) (Figures 4-1). Notice that in this definition, the dependence on the grandparent index $n$ enters only through the restriction on the children’s transverse momenta $\vec{P}^{(k)}_{cT}$.

Using momentum conservation in the transverse plane

$$\sum_{k=1}^2 \vec{P}^{(k)}_{0T} + \sum_{k=1}^{n} \sum_{j=1}^c \vec{P}^{(k)}_{jT} + \vec{p}_T = 0, \quad (4-4)$$

Furthermore, the measurement of the missing transverse momentum $\vec{p}_{T,miss}$ provides two additional constraints

$$\vec{P}^{(1)}_{0T} + \vec{P}^{(2)}_{0T} = \vec{p}_{T,miss} \quad (4-5)$$

on the unknown transverse momentum components $\vec{P}^{(k)}_{0T}$.

Now we can rewrite the restriction on the children’s transverse momenta $\vec{P}^{(k)}_{cT}$ as

$$\sum_{k=1}^2 \vec{P}^{(k)}_{cT} = \sum_{k=1}^2 \vec{P}^{(k)}_{0T} + \sum_{k=1}^{n} \sum_{j=1}^c \vec{P}^{(k)}_{jT} = \vec{p}_{T,miss} + \sum_{k=1}^2 \sum_{j=1}^c \vec{P}^{(k)}_{jT}, \quad (4-6)$$

where in the last step we used Equation (4-5). Equation (4-6) allows us to rewrite the subsystem $M^{(n,p,c)}_{T^2}$ definition of Equation (4-3) in a form which does not manifestly depend on the grandparent index $n$:

$$M^{(n,p,c)}_{T^2}(\tilde{M}_c) = \min_{\sum_{k=1}^2 \vec{P}^{(k)}_{cT} = \vec{p}_{T,miss} + \sum_{k=1}^2 \sum_{j=1}^c \vec{P}^{(k)}_{jT}} \left\{ \max \left\{ M^{(1)}_{T^2}, M^{(2)}_{T^2} \right\} \right\}. \quad (4-7)$$

However, the grandparent index $n$ is still implicitly present through the global quantity $\vec{p}_{T,miss}$, which knows about the whole event. We shall see below that the interpretation of the experimentally observable endpoints, kinks, etc., for the so defined subsystem $M^{(n,p,c)}_{T^2}$ quantity, does depend on the grandparent index $n$, which justifies our notation.
We are now in a position to compare our subsystem $M_{T2}^{(n,p,c)}$ quantity to the conventional $M_{T2}$ variable. The latter is nothing but the special case of $n = p$ and $c = 0$:

$$M_{T2} \equiv M_{T2}^{(n,p,0)},$$

(4–8)
i.e. the conventional $M_{T2}$ is simply characterized by a single integer $n$, which indicates the length of the decay chain. We see that we are generalizing the conventional $M_{T2}$ variable in two different aspects: first, we are allowing the parents $X_p$ to be different from the particles $X_n$ originally produced in the event (the grandparents), and second, we are allowing the children $X_c$ to be different from the dark matter particles $X_0$ appearing at the end of the cascade chain and responsible for the missing energy. The benefits of this generalization will become apparent in the next section, where we shall discuss the available measurements from the different subsystem $M_{T2}^{(n,p,c)}$ variables.

4.1 A Short Decay Chain $X_2 \rightarrow X_1 \rightarrow X_0$

A relatively long ($n \geq 3$) new physics decay chain can be handled by a variety of mass measurement methods, and in principle a complete determination of the mass spectrum in that case is possible at a hadron collider. We also showed that a relatively short ($n = 1$ or $n = 2$) decay chain would present a major challenge, and a complete mass determination might be possible only through $M_{T2}$ methods. From now on we shall therefore concentrate only on this most problematic case of $n \leq 2$.

First let us summarize what types of subsystem $M_{T2}^{(n,p,c)}$ measurements are available in the case of $n \leq 2$. As illustrated in Figure 4-2, there exist a total of 4 different $M_{T2}^{(n,p,c)}$ quantities.

Each $M_{T2}^{(n,p,c)}$ distribution would exhibit an upper endpoint $M_{T2,max}^{(n,p,c)}$, whose measurement would provide one constraint on the physical masses. In order to be able to invert and solve for the masses of the new particles in terms of the measured endpoints, we need to know the analytical expressions relating the endpoints $M_{T2,max}^{(n,p,c)}$ to the physical masses $M_i$. 
In this section we summarize those relations for each $M_{T2}^{(n,p,c)}$ quantity with $n \leq 2$. Some of these results (e.g. portions of Sections 4.1.1 and Sections 4.1.3) have already appeared in the literature, and we include them here for completeness. The discussion in Sections 4.1.2 and Sections 4.1.4, on the other hand, is new. In all cases, we shall allow for the presence of an arbitrary transverse momentum $p_T$ due to ISR. This represents a generalization of all existing results in the literature, which have been derived in the two special cases $p_T = 0$ [37] or $p_T = \infty$ [36].

We shall find it convenient to write the formulas for the endpoints $M_{T2,\text{max}}^{(n,p,c)}$ not in terms of the actual masses, but in terms of the mass parameters

$$
\mu_{(n,p,c)} \equiv \frac{M_n}{2} \left( 1 - \frac{M_c^2}{M_p^2} \right).
$$

(4–9)

The advantage of using this shorthand notation will become apparent very shortly.

4.1.1 The Subsystem Variable $M_{T2}^{(1,1,0)}$

We start with the simplest case of $n = 1$ shown in Figure 4-2(a). Here $M_{T2}^{(1,1,0)}$ is the only possibility, and it coincides with the conventional $M_{T2}$ variable, as indicated by Equation (4–8). Therefore, the previous results in the literature which have been derived
for the conventional $M_{T_2}$ variable in Equation (4–8), would still apply. In particular, in the
limit of $p_T = 0$, the upper endpoint $M_{T_{2,\text{max}}}^{(1,1,0)}$ depends on the test mass $\tilde{M}_0$ as follows [37]

$$M_{T_{2,\text{max}}}^{(1,1,0)} (\tilde{M}_0, p_T = 0) = \mu_{(1,1,0)} + \sqrt{\mu_{(1,1,0)}^2 + \tilde{M}_0^2}, \quad (4–10)$$

where the parameter $\mu_{(1,1,0)}$ is defined in terms of the physical masses $M_1$ and $M_0$
according to Equation (4–9):

$$\mu_{(1,1,0)} \equiv \frac{M_1}{2} \left( 1 - \frac{M_0^2}{M_1^2} \right) = \frac{M_1^2 - M_0^2}{2M_1}. \quad (4–11)$$

As usual, the endpoint in Equation (4–10) can be interpreted as the mass $M_1$ of the
parent particle $X_1$, so that Equation (4–10) provides a relation between the masses of
$X_0$ and $X_1$. In the early literature on $M_{T_2}$, this relation had to be derived numerically,
by building the $M_{T_2}$ distributions for different values of the test mass $\tilde{M}_0$, and reading
off their endpoints. Nowadays, with the work of Cho et al. [“Measuring superparticle
masses at hadron collider using the transverse mass kink, JHEP 0802, 035 (2008)] [37],
the relation is known analytically, and, as seen from Equation (4–10), is parameterized
by a single parameter $\mu_{(1,1,0)}$. Therefore, in order to extract the value of this parameter,
we only need to perform a single measurement, i.e. we only need to study the $M_{T_2}$
distribution for one particular choice of the test mass $\tilde{M}_0$. We shall find it convenient to
choose $\tilde{M}_0 = 0$, in which case Equations (4–10) and (4–11) give

$$M_{T_{2,\text{max}}}^{(1,1,0)} (\tilde{M}_0 = 0, p_T = 0) = 2 \mu_{(1,1,0)} = \frac{M_1^2 - M_0^2}{M_1}, \quad (4–12)$$

providing the required measurement of the parameter $\mu_{(1,1,0)}$ demonstrates the
usefulness of the $M_{T_2}$ concept – just a single measurement of the endpoint of the
$M_{T_2}$ distribution for a single fixed value of the test mass $\tilde{M}_0$ is sufficient to provide us
with one constraint among the unknown masses ($M_1$ and $M_0$ in this case).

Unfortunately, one single measurement in Equation (4–12) is not enough to
pin down two different masses. In order to measure both $M_0$ and $M_1$, without any
theoretical assumptions or prejudice, we obviously need additional experimental input. From the general expression as Equation (4–10) it is clear that measuring other $M_{T2,\text{max}}^{(1,1,0)}$ endpoints, for different values of the test mass $\tilde{M}_0$, will not help, since we will simply be measuring the same combination of masses $\mu_{(1,1,0)}$ over and over again, obtaining no new information. Another possibility might be to consider events with the next longest decay chain $(n = 2)$, which, as advertised in the Introduction and shown below in Section 4.2, will be able to provide enough information for a complete mass determination of all particles $X_0$, $X_1$ and $X_2$. However, the existence and the observation of the $n = 2$ decay chain is certainly not guaranteed – to begin with, the particles $X_2$ may not exist, or they may have too low cross-sections. It is therefore of particular importance to ask the question whether the $n = 1$ process in Figure 4-2(a) alone can allow a determination of both $M_0$ and $M_1$. The answer to this question, at least in principle, is “Yes” [36], and what is more, one can achieve this using the very same $M_{T2}$ variable $M_{T2}^{(1,1,0)}$.

The key is to realize that in reality at any collider, and especially at hadron colliders like the Tevatron and the LHC, there will be sizable contributions from initial state radiation (ISR) with nonzero $p_T$, where one or more jets are radiated off the initial state, before the hard scattering interaction. (In Figures 4-1 and 4-2 the green ellipse represents the hard scattering, while “ISR” stands for a generic ISR jet.). This effect leads to a drastic change in the behavior of the $M_{T2,\text{max}}^{(1,1,0)}(\tilde{M}_0, p_T)$ function, which starts to exhibit a kink at the true location of the child mass $\tilde{M}_0 = M_0$:

$$\left(\frac{\partial M_{T2,\text{max}}^{(1,1,0)}}{\partial \tilde{M}_0}(\tilde{M}_0, p_T)\right)_{\tilde{M}_0=M_0-\epsilon} \neq \left(\frac{\partial M_{T2,\text{max}}^{(1,1,0)}}{\partial \tilde{M}_0}(\tilde{M}_0, p_T)\right)_{\tilde{M}_0=M_0+\epsilon}, \quad (4–13)$$

and furthermore, the value of $M_{T2,\text{max}}^{(1,1,0)}$ at that point reveals the true mass of the parent as well:

$$M_{T2,\text{max}}^{(1,1,0)}(\tilde{M}_0 = M_0, p_T) = M_1. \quad (4–14)$$
This kink feature in Equations (4–13, 4–14) was observed and illustrated in A. J. Barr et al. [Weighing Wimps with Kinks at Colliders: Invisible Particle Mass Measurements from Endpoints, JHEP 0802, 014 (2008)] Section 4.4 [36]. We find that it can also be understood analytically, by generalizing the result of Equation (4–10) to account for the additional ISR transverse momentum \( \vec{p}_T \). Recall that Equation (4–10) was derived in Ref. [37] under the assumption that the missing transverse momentum due to the two escaping particles \( X_0 \) is exactly balanced by the transverse momenta of the two visible particles \( x_1 \) used to form \( M_{T2}^{(1,1,0)} \):

\[
\vec{P}^{(1)}_{0T} + \vec{P}^{(2)}_{0T} + \vec{P}^{(1)}_{1T} + \vec{P}^{(2)}_{1T} = 0 .
\]

(4–15)

We may sometimes refer to this situation as a “balanced” momentum configuration\(^1\). In the presence of ISR with some non-zero transverse momentum \( \vec{p}_T \), Equation (4–15) in general ceases to be valid, and is modified to

\[
\vec{P}^{(1)}_{0T} + \vec{P}^{(2)}_{0T} + \vec{P}^{(1)}_{1T} + \vec{P}^{(2)}_{1T} = -\vec{p}_T .
\]

(4–16)

in accordance with (4–4). Including the ISR effects, we find that the expression (4–10) for the \( M_{T2,max}^{(1,1,0)} \) endpoint splits into two branches

\[
M_{T2,max}^{(1,1,0)}(\tilde{M}_0, \rho_T) = \begin{cases} 
F^{(1,1,0)}_L(\tilde{M}_0, \rho_T), & \text{if } \tilde{M}_0 \leq M_0 , \\
F^{(1,1,0)}_R(\tilde{M}_0, \rho_T), & \text{if } \tilde{M}_0 \geq M_0 ,
\end{cases}
\]

(4–17)

\(^1\) This should not be confused with the term “balanced” used for the analytic \( M_{T2} \) solutions discussed in [33, 37].
where

\[
F^{(1,1,0)}_L(\tilde{M}_0, p_T) = \begin{cases} 
\left[ \mu_{(1,1,0)}(p_T) + \sqrt{\left( \mu_{(1,1,0)}(p_T) + \frac{p_T^2}{2} + \tilde{M}_0^2 \right)^2 - \frac{p_T^2}{4}} \right]^\frac{1}{2}, \\
\end{cases}
\]

\[
F^{(1,1,0)}_R(\tilde{M}_0, p_T) = \begin{cases} 
\left[ \mu_{(1,1,0)}(-p_T) + \sqrt{\left( \mu_{(1,1,0)}(-p_T) - \frac{p_T^2}{2} + \tilde{M}_0^2 \right)^2 - \frac{p_T^2}{4}} \right]^\frac{1}{2}, \\
\end{cases}
\]

and the \(p_T\)-dependent parameter \(\mu_{(1,1,0)}(p_T)\) is defined as

\[
\mu_{(1,1,0)}(p_T) = \mu_{(1,1,0)} \left( \sqrt{1 + \left( \frac{p_T}{2M_1} \right)^2 - \frac{p_T^2}{2M_1}} \right). \tag{4-20}
\]

Both branches correspond to extreme momentum configurations in which all three transverse vectors \(\vec{p}_{1T}^{(1)}, \vec{p}_{1T}^{(2)}\) and \(\vec{p}_T\) are collinear. The difference is that the left branch \(F^{(1,1,0)}_L\) corresponds to the configuration \(\vec{p}_{1T}^{(1)} \uparrow \uparrow \vec{p}_{1T}^{(2)} \uparrow \uparrow \vec{p}_T\), while the right branch \(F^{(1,1,0)}_R\) corresponds to \(\vec{p}_{1T}^{(1)} \uparrow \downarrow \vec{p}_{1T}^{(2)} \uparrow \downarrow \vec{p}_T\). Therefore, the two branches are simply related as

\[
F^{(1,1,0)}_R(\tilde{M}_0, p_T) = F^{(1,1,0)}_L(\tilde{M}_0, -p_T). \tag{4-21}
\]

It is easy to verify that in the absence of ISR, (i.e. for \(p_T = 0\)) our general result \(4-17\) reduces to the previous formula \(4-10\).

Our result \(4-17\) for the \(M^{(1,1,0)}_{T2,max}\) upper kinematic endpoint as a function of the test mass \(\tilde{M}_0\) is illustrated in Figure 4-3(a). We consider a single ISR jet and show results for several different values of its transverse momentum \(p_T\), starting from \(p_T = 0\) (the green solid line) and increasing the value of \(p_T\) in increments of \(\Delta p_T = 100\) GeV. The uppermost solid line corresponds to the limiting case \(p_T \to \infty\). The true value of the parent (child) mass is marked by the horizontal (vertical) dotted line. The red (blue) lines correspond to the function \(F^{(1,1,0)}_L, F^{(1,1,0)}_R\). The solid portions of those lines correspond to the true \(M^{(1,1,0)}_{T2,max}\) endpoint, while the dashed segments are simply the extension of \(F^{(1,1,0)}_L, F^{(1,1,0)}_R\) into the “wrong” region for \(\tilde{M}_0\), giving a false endpoint.
Figure 4-3. (a) Dependence of the $M_{T2,max}^{(1,1,0)}$ upper kinematic endpoint (solid lines) on the value of the test mass $\tilde{M}_0$, for $M_1 = 300$ GeV, and $M_0 = 100$ GeV, and for different values of the transverse momentum $p_T$ of the ISR jet, starting from $p_T = 0$ (green line), and increasing up to $p_T = 3$ TeV in increments of $\Delta p_T = 100$ GeV, from bottom to top. The uppermost line corresponds to the limiting case $p_T \to \infty$. The horizontal (vertical) dotted line denotes the true value of the parent (child) mass. Solid (dashed) lines indicate true (false) endpoints. The red lines correspond to the function $F_L^{(1,1,0)}$ defined in Equation (4–18), while the blue lines correspond to the function $F_L^{(1,1,0)}$ defined in Equation (4–19). (b) The value of the kink $\Delta \Theta^{(1,1,0)}$ defined in (4–28), as a function of the dimensionless ratios $\frac{p_T}{M_1}$ and $\frac{\tilde{M}_0}{M_1}$.

Figure 4-3(a) reveals that the two branches in Equations (4–18) and (4–19) always cross at the point $(M_0, M_1)$, in agreement with Equation (4–14). Interestingly, the sharpness of the resulting kink at $\tilde{M}_0 = M_0$ depends on the hardness of the ISR jet, as can be seen directly from (4–17). For small $p_T$, the kink is barely visible, and in the limit $p_T \to 0$ we obtain the old result (4–10) for the “balanced” momentum configuration, shown with the green solid line, which does not exhibit any kink. In the other extreme, at very large $p_T$, we see a pronounced kink, which has a well-defined limit as $p_T \to \infty$.

The $M_{T2,max}^{(1,1,0)}$ kink exhibited in Equation (4–17) and in Figure 4-3(a) is our first, but not last, encounter with a kink feature in an $M_{T2}^{(n,p,c)}$ variable. Below we shall see that the $M_{T2}$ kinks are rather common phenomena, and we shall encounter at least two other kink types by the end of Section 4.1. Therefore, we find it convenient to quantify the
sharpness of any such kink as follows. Consider a generic subsystem $M_{T2}^{(n,p,c)}$ variable whose endpoint $M_{T2,max}(\tilde{M}_c, p_T)$ exhibits a kink:

$$M_{T2,max}^{(n,p,c)}(\tilde{M}_c, p_T) = \begin{cases} F_L^{(n,p,c)}(\tilde{M}_c, p_T), & \text{if } \tilde{M}_c \leq M_c, \\ F_R^{(n,p,c)}(\tilde{M}_c, p_T), & \text{if } \tilde{M}_c \geq M_c. \end{cases} \quad (4–22)$$

The kink appears because $M_{T2,max}^{(n,p,c)}(\tilde{M}_c, p_T)$ is not given by a single function, but has two separate branches. The first (“low”) branch applies for $\tilde{M}_c \leq M_c$, and is given by some function $F_L^{(n,p,c)}(\tilde{M}_c, p_T)$, while the second (“high”) branch is valid for $\tilde{M}_c \geq M_c$, and is given by a different function, $F_R^{(n,p,c)}(\tilde{M}_c, p_T)$. The function $M_{T2,max}^{(n,p,c)}(\tilde{M}_c, p_T)$ itself is continuous and the two branches coincide at $\tilde{M}_c = M_c$:

$$F_L^{(n,p,c)}(M_c, p_T) = F_R^{(n,p,c)}(M_c, p_T), \quad (4–23)$$

but their derivatives do not match:

$$\left(\frac{\partial F_L^{(n,p,c)}}{\partial \tilde{M}_c}\right)_{\tilde{M}_c = M_c} \neq \left(\frac{\partial F_R^{(n,p,c)}}{\partial \tilde{M}_c}\right)_{\tilde{M}_c = M_c}, \quad (4–24)$$

leading to the appearance of the kink. Let us define the left and right slope of the $M_{T2,max}^{(n,p,c)}(\tilde{M}_c, p_T)$ function at $\tilde{M}_c = M_c$ in terms of two angles $\Theta_L^{(n,p,c)}$ and $\Theta_R^{(n,p,c)}$, correspondingly:

$$\tan \Theta_L^{(n,p,c)} \equiv \left(\frac{\partial F_L^{(n,p,c)}(\tilde{M}_c)}{\partial \tilde{M}_c}\right)_{\tilde{M}_c = M_c},$$

$$\tan \Theta_R^{(n,p,c)} \equiv \left(\frac{\partial F_R^{(n,p,c)}(\tilde{M}_c)}{\partial \tilde{M}_c}\right)_{\tilde{M}_c = M_c}. \quad (4–25)$$

Now we shall define the amount of kink as the angular difference $\Delta \Theta^{(n,p,c)}$ between the two branches:

$$\Delta \Theta^{(n,p,c)} \equiv \Theta_R^{(n,p,c)} - \Theta_L^{(n,p,c)} = \arctan\left(\frac{\tan \Theta_R^{(n,p,c)} - \tan \Theta_L^{(n,p,c)}}{1 + \tan \Theta_R^{(n,p,c)} \tan \Theta_L^{(n,p,c)}}\right). \quad (4–27)$$
A large value of $\Delta \Theta^{(n,p,c)}$ implies that the relative angle between the low and high branches at the point of their junction $\tilde{M}_c = M_c$ is also large, and in that sense the kink would be more pronounced and relatively easier to see.

This definition can be immediately applied to the $M^{(1,1,0)}_{T2,max}$ kink that we just discussed. Substituting the formulas (4–18) and (4–19) for the two branches $F^{(1,1,0)}_L$ and $F^{(1,1,0)}_R$ into the definitions (4–25,4–26) and subsequently into (4–27), we obtain an expression for the size $\Delta \Theta^{(1,1,0)}$ of the $M^{(1,1,0)}_{T2,max}$ kink:

$$
\Delta \Theta^{(1,1,0)} = \arctan \left( \frac{M_0 (M_1^2 - M_0^2) p_T \sqrt{4M_1^2 + p_T^2}}{M_1 (M_1^2 - M_0^2)^2 + 2M_0^2 M_1 (4M_1^2 + p_T^2)} \right). 
$$

(4–28)

The result (4–28) is illustrated numerically in Figure 4-3(b). As can be seen from (4–28), $\Delta \Theta^{(1,1,0)}$ depends on the two masses $M_0$ and $M_1$, as well as the size of the ISR $p_T$. However, since $\Delta \Theta^{(1,1,0)}$ is a dimensionless quantity, its dependence on those three parameters can be simply illustrated in terms of the dimensionless ratios $\frac{p_T}{M_1}$ and $\frac{M_0}{M_1}$. This is why in Figure 4-3(b) we plot $\Delta \Theta^{(1,1,0)}$ (in degrees) as a function of $\frac{p_T}{M_1}$ and $\frac{M_0}{M_1}$.

Figure 4-3(b) confirms that the kink develops at large $p_T$, and is completely absent at $p_T = 0$, a result which may have already been anticipated on the basis of Figure 4-3(a). For any given mass ratio $\frac{M_0}{M_1}$, the kink is largest for the hardest possible $p_T$. In the limit $p_T \to \infty$ we obtain

$$
\lim_{p_T \to \infty} \Delta \Theta^{(1,1,0)} = \arctan \left( \frac{M_1^2 - M_0^2}{2M_0 M_1} \right). 
$$

(4–29)

From Figure 4-3(b) one can see that at sufficiently large $p_T$, the $\Delta \Theta^{(1,1,0)}$ contours become almost horizontal, i.e. the size of the kink $\Delta \Theta^{(1,1,0)}$ becomes very weakly dependent on $p_T$. A careful examination of Figure 4-3(b) reveals that the asymptotic behavior at $p_T \to \infty$ is in agreement with the analytical result (4–29). Notice that the maximum possible value of any kink of the type (4–22) is $\Delta \Theta^{(n,p,c)}_{max} = 90^\circ$. According to Figure 4-3(b) and Equation (4–29), in the case of $\Delta \Theta^{(1,1,0)}$ the absolute maximum can be obtained only in the $p_T \to \infty$ and $M_0 \to 0$ limit. The former condition will never be
realized in a realistic experiment, while the latter condition makes the observation of the kink rather problematic, since the “low” branch $F_L$ of the $M_{T_{2,\max}}^{(1,1,0)}(\tilde{M}_0, p_T)$ function is too short to be observed experimentally. Therefore, under realistic circumstances, we would expect the size of the kink $\Delta\Theta^{(1,1,0)}$ to be only on the order of a few tens of degrees, which are the more typical values seen in Figure 4-3(b).

According to Figure 4-3(b), for a given fixed $p_T$, the sharpness of the $\Delta\Theta^{(1,1,0)}$ kink depends on the mass hierarchy of the particles $X_1$ and $X_0$. When they are relatively degenerate, i.e. their mass ratio $M_0/M_1$ is large, the kink is relatively small. Conversely, when $X_0$ is much lighter than $X_1$, the kink is more pronounced. The optimum mass ratio $M_0/M_1$ which maximizes the kink for a given $p_T$, is rather weakly dependent on the $p_T$, and for $p_T \to \infty$ eventually goes to zero, in agreement with Equation (4–29). However, for more reasonable values of $p_T$ as the ones shown on the left half of the plot, the optimal ratio $M_0/M_1$ varies between 0.3 (at $p_T \sim 0$) to 0.1 (at $p_T \sim 5M_1$). In this sense, the value of $M_0/M_1 = \frac{1}{3}$ which was chosen for the illustration in Figure 4-3(a).

In conclusion of this subsection, it is worth summarizing the main points from it. The good news is that the $\Delta\Theta^{(1,1,0)}$ kink in principle offers a second, independent piece of information about the masses of the particles $X_0$ and $X_1$. When taken together with the $M_{T_{2,\max}}^{(1,1,0)}$ endpoint measurement (4–12), it will allow us to determine both masses $M_0$ and $M_1$, in a completely model-independent way. Our analytical results regarding the $\Delta\Theta^{(1,1,0)}$ kink complement the study of A. J. Barr et al. [“Weighing Wimps with Kinks at Colliders: Invisible Particle Mass Measurements from Endpoints, JHEP 0802, 014 (2008)]. [36], where this kink was first discovered. However, on the down side, we should mention that much of our discussion regarding the $\Delta\Theta^{(1,1,0)}$ kink may be of limited practical interest, for several reasons. First, as seen in Figure 4-3, the kink becomes visible only for sufficiently large values of the $p_T$. Since the ISR $p_T$ spectrum is falling rather steeply, one would need to collect relatively large amounts of data, in order to guarantee the presence of events with sufficiently hard ISR jets. Even then,
the collected events may not contain the momentum configuration required to give the maximum value of $M_{T2}^{(1,1,0)}$. An alternative approach to make use of the kink structure would be to measure the endpoint function $M_{T2,\text{max}}^{(1,1,0)}(\tilde{M}_0, \rho_T)$ for several different $\rho_T$ ranges, and then fit it to the analytical formula (4–17). Whether and how well this can work in practice, remains to be seen, but the results of [36] from a toy exercise in the absence of any backgrounds and detector resolution effects do not appear very encouraging. Nevertheless, while the kink structure $\Delta \Theta^{(1,1,0)}$ may be difficult to observe, the measurement (4–12) of the endpoint $M_{T2,\text{max}}^{(1,1,0)}(\tilde{M}_0 = 0, \rho_T = 0)$ should be relatively straightforward. In Sections 4.2.1 and 4.2.3 we shall see that the additional $M_{T2}$ information from events with $n = 2$ decay chains will eventually allow us to determine all the unknown masses.

4.1.2 The Subsystem Variable $M_{T2}^{(2,2,1)}$

The subsystem variable $M_{T2}^{(2,2,1)}$ is illustrated in Figure 4-2(b), where we use the subchain within the smaller rectangle on the left. $M_{T2}^{(2,2,1)}$ is a genuine subchain variable in the sense that we only use the SM decay products $x_2$, and ignore any remaining objects arising from the two $x_1$'s. In the absence of ISR ($\rho_T = 0$) one can adapt the results from [37] and show that the formula for the $M_{T2}^{(2,2,1)}$ endpoint is

$$M_{T2,\text{max}}^{(2,2,1)}(\tilde{M}_1, \rho_T = 0) = \mu_{(2,2,1)} + \sqrt{\mu_{(2,2,1)}^2 + \tilde{M}_1^2}, \quad (4–30)$$

where the parameter $\mu_{(2,2,1)}$ was defined in Equation (4–9):

$$\mu_{(2,2,1)} \equiv \frac{M_2}{2} \left(1 - \frac{M_1^2}{M_2^2}\right) = \frac{M_2^2 - M_1^2}{2M_2}. \quad (4–31)$$

Almost all of our discussion from the previous Section 4.1.1 can be directly applied here as well. For example, in order to measure the parameter $\mu_{(2,2,1)}$, we only need to extract the endpoint of a single distribution, for a single fixed value of the test mass $\tilde{M}_1$. As
before, we choose to use $\tilde{M}_1 = 0$. The resulting endpoint measurement
\[ M_{T2,\text{max}}^{(2,2,1)}(\tilde{M}_1 = 0, \rho_T = 0) = 2 \mu_{(2,2,1)} = \frac{M_2^2 - M_1^2}{M_2} \] (4–32)
provides the required measurement of the parameter $\mu_{(2,2,1)}$ appearing in Equation (4–30), as well as one constraint on the masses $M_1$ and $M_2$ involved in the problem. More importantly, the new constraint in Equation (4–32) is independent of the relation in Equation (4–12) found previously in Section 4.1.1.

The new variable $M_{T2}^{(2,2,1)}$ will also exhibit a kink in the plot of its endpoint $M_{T2,\text{max}}^{(2,2,1)}$ as a function of the test mass $\tilde{M}_1$. This is the same type of kink as the one discussed in the previous subsection, therefore all of our previous results would apply here as well. In particular, the analytical expression for the kink is given by
\[ M_{T2,\text{max}}^{(2,2,1)}(\tilde{M}_1, \rho_T) = \begin{cases} F_L^{(2,2,1)}(\tilde{M}_1, \rho_T), & \text{if } \tilde{M}_1 \leq M_1, \\ F_R^{(2,2,1)}(\tilde{M}_1, \rho_T), & \text{if } \tilde{M}_1 \geq M_1, \end{cases} \] (4–33)
where
\[ F_L^{(2,2,1)}(\tilde{M}_1, \rho_T) = \left\{ \left[ \mu_{(2,2,1)}(\rho_T) + \sqrt{\left( \mu_{(2,2,1)}(\rho_T) + \frac{\rho_T}{2} \right)^2 + \tilde{M}_1^2} \right]^2 - \frac{\rho_T^2}{4} \right\}^{\frac{1}{2}}, \] (4–34)
\[ F_R^{(2,2,1)}(\tilde{M}_1, \rho_T) = \left\{ \left[ \mu_{(2,2,1)}(-\rho_T) + \sqrt{\left( \mu_{(2,2,1)}(-\rho_T) - \frac{\rho_T}{2} \right)^2 + \tilde{M}_1^2} \right]^2 - \frac{\rho_T^2}{4} \right\}^{\frac{1}{2}}, \] (4–35)
and the $\rho_T$-dependent parameter $\mu_{(2,2,1)}(\rho_T)$ is defined in analogy to Equation (4–20)
\[ \mu_{(2,2,1)}(\rho_T) = \mu_{(2,2,1)} \left( \sqrt{1 + \left( \frac{\rho_T}{2M_2} \right)^2} - \frac{\rho_T}{2M_2} \right). \] (4–36)
The size of the new kink $\Delta \Theta^{(2,2,1)}$ can be easily read off from Equation (4–28), where one should make the obvious replacements $M_0 \rightarrow M_1$ and $M_1 \rightarrow M_2$.  

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We can now generalize the two examples discussed so far \( M_{T^2}^{(1,1,0)} \) and \( M_{T^2}^{(2,2,1)} \) to the case of an arbitrary grandparent index \( n \), with \( p = n \) and \( c = n - 1 \). We get

\[
M_{T^2,\text{max}}^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T) = \begin{cases} 
F_L^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T), & \text{if } \tilde{M}_{n-1} \leq M_{n-1}, \\
F_R^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T), & \text{if } \tilde{M}_{n-1} \geq M_{n-1}.
\end{cases}
\]  

(4–37)

where

\[
F_L^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T) = \left\{ \left[ \mu_{(n,n,n-1)}(p_T) + \sqrt{\left( \mu_{(n,n,n-1)}(p_T) + \frac{p_T^2}{2} \right)^2 + \tilde{M}_{n-1}^2} \right]^2 - \frac{p_T^2}{4} \right\}^{\frac{1}{2}},
\]

(4–38)

\[
F_R^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T) = \left\{ \left[ \mu_{(n,n,n-1)}(-p_T) + \sqrt{\left( \mu_{(n,n,n-1)}(-p_T) - \frac{p_T^2}{2} \right)^2 + \tilde{M}_{n-1}^2} \right]^2 - \frac{p_T^2}{4} \right\}^{\frac{1}{2}},
\]

(4–39)

and the \( p_T \)-dependent parameter \( \mu_{(n,n,n-1)}(p_T) \) is simply the generalization of Equations (4–20) and (4–36):

\[
\mu_{(n,n,n-1)}(p_T) = \mu_{(n,n,n-1)} \left( \sqrt{1 + \left( \frac{p_T}{2M_n} \right)^2} - \frac{p_T}{2M_n} \right). \]

(4–40)

For \( n = 1 \) or \( n = 2 \), the general formula of Equation (4–37) reproduces our previous results in Equations (4–17) and (4–33), correspondingly.

### 4.1.3 The Subsystem Variable \( M_{T^2}^{(2,2,0)} \)

The variable \( M_{T^2}^{(2,2,0)} \) is illustrated in Figure 4-2(b), where we use the whole chain within the larger rectangle. As long as we ignore the effects of any ISR, we have a balanced\(^2\) momentum configuration and the analytical results from Ref. [37] would apply. In particular, the endpoint \( M_{T^2,\text{max}}^{(2,2,0)}(\tilde{M}_0, p_T = 0) \) is given by [37]

\[
M_{T^2,\text{max}}^{(2,2,0)}(\tilde{M}_0, p_T = 0) = \begin{cases} 
F_L^{(2,2,0)}(\tilde{M}_0, p_T = 0), & \text{if } \tilde{M}_0 \leq M_0, \\
F_R^{(2,2,0)}(\tilde{M}_0, p_T = 0), & \text{if } \tilde{M}_0 \geq M_0.
\end{cases}
\]  

(4–41)

\(^2\) In the sense of Equation (4–15). See the discussion following Equation (4–15).
where

\[
F_L^{(2,2,0)}(\tilde{M}_0, p_T = 0) = \mu_{(2,2,0)} + \sqrt{\mu_{(2,2,0)}^2 + \tilde{M}_0^2}, \quad (4-42)
\]

\[
F_R^{(2,2,0)}(\tilde{M}_0, p_T = 0) = \mu_{(2,2,1)} + \mu_{(2,1,0)} + \sqrt{(\mu_{(2,2,1)} - \mu_{(2,1,0)})^2 + \tilde{M}_0^2}, \quad (4-43)
\]

and the various parameters \(\mu_{(n,p,c)}\) are defined in (4–9). Notice that these expressions are valid only for \(p_T = 0\). We have also derived the corresponding generalized expression for \(M_T^{(2,2,0)}(\tilde{M}_0, p_T)\) for arbitrary values of \(p_T\), which we list in Appendix B.

The most striking feature of the endpoint function (4–41) is that it will also exhibit a kink \(\Delta \Theta^{(2,2,0)}\) at the true value of the test mass \(\tilde{M}_0 = M_0\). However, as emphasized in [36], the physical origin of this kink is different from the kinks \(\Delta \Theta^{(1,1,0)}\) and \(\Delta \Theta^{(2,2,1)}\) which we encountered previously in Sections 4.1.1 and 4.1.2. This is easy to understand – in Sections 4.1.1 and 4.1.2 we saw that the kinks \(\Delta \Theta^{(1,1,0)}\) and \(\Delta \Theta^{(2,2,1)}\) arise due to ISR effects, while Equation (4–41) holds in the absence of any ISR. The explanation for the \(\Delta \Theta^{(2,2,0)}\) kink has actually already been provided in [37]. In essence, one can treat the SM decay products \(x_1\) and \(x_2\) in each chain as a composite particle of variable mass, and the two branches \(F_L^{(2,2,0)}\) and \(F_R^{(2,2,0)}\) correspond to the two extreme values for the mass of this composite particle.

In spite of its different origin, the kink in the function (4–41) shares many of the same properties. Let us use a specific example as an illustration. Consider a popular example from supersymmetry, such as gluino pair-production, followed by sequential two-body decays to squarks and the lightest neutralinos. This is precisely a cascade of the type \(n = 2\), in which \(X_2\) is the gluino \(\tilde{g}\), \(X_1\) is a squark \(\tilde{q}\), and \(X_0\) is the lightest neutralino \(\tilde{\chi}_1^0\). Let us choose the superpartner masses according to the SPS1a mass spectrum, which was also used in [37]:

\[
M_2 = 613 \text{ GeV}, \quad M_1 = 525 \text{ GeV}, \quad M_0 = 99 \text{ GeV}. \quad (4–44)
\]
Figure 4-4. Dependence of the $M_{T2,max}^{(2,2,0)}$ and $M_{T2,max}^{(2,1,0)}$ upper kinematic endpoints on the value of the test mass $\tilde{M}_0$, for (a) the SPS1a parameter point in MSUGRA: $M_2 = 613$ GeV, $M_1 = 525$ GeV, and $M_0 = 99$ GeV; or (b) a split spectrum with $M_2 = 2000$ GeV, $M_1 = 200$ GeV, and $M_0 = 100$ GeV. The horizontal (vertical) dotted lines denote the true value of the parent (child) mass for each case. Solid (dashed) lines indicate true (false) endpoints, while red (blue) lines correspond to $F_L^{(n,p,c)}(F_R^{(n,p,c)})$ branches.

The resulting function $M_{T2,max}^{(2,2,0)}(\tilde{M}_0, p_T = 0)$ is plotted in Figure 4-4(a) with the upper set of lines. There are several noteworthy features of $M_{T2,max}^{(2,2,0)}(\tilde{M}_0, p_T = 0)$ which are evident from Figure 4-4(a). First, when the test mass $\tilde{M}_0$ is equal to the true child mass $M_0$, the $M_{T2}$ endpoint yields the true parent mass, in this case $M_2$:

$$M_{T2,max}^{(2,2,0)}(\tilde{M}_0 = M_0, p_T = 0) = M_2. \quad (4–45)$$

This property of $M_{T2}$ is true by design, and is confirmed by the dotted lines in Figure 4-4(a).

Second, as seen from Equation (4–41), $M_{T2,max}^{(2,2,0)}(\tilde{M}_0, p_T = 0)$ is not given by a single function, but has two separate branches. The first (“low”) branch $F_L^{(2,2,0)}$ applies for $\tilde{M}_0 \leq M_0$, and is shown in Figure 4-4(a) with red lines. The second (“high”) branch $F_R^{(2,2,0)}$ is valid for $\tilde{M}_0 \geq M_0$ and is shown in blue in Figure 4-4(a). While the two branches coincide at $\tilde{M}_0 = M_0$:

$$F_L^{(2,2,0)}(M_0, p_T = 0) = F_R^{(2,2,0)}(M_0, p_T = 0). \quad (4–46)$$
Figure 4-5. The amount of kink: (a) $\Delta \Theta^{(2,2,0)}$ and (b) $\Delta \Theta^{(2,1,0)}$ in degrees, as a function of the mass ratios $\sqrt{y}$ and $\sqrt{z}$. The white dot and the white asterisk denote the locations in this ($\sqrt{y}$, $\sqrt{z}$) parameter space of the two sample spectra (4–44) and (4–62) used for Figures 4-4(a) and 4-4(b), correspondingly.

Their derivatives do not match:

$$
\left( \frac{\partial F_{L}^{(2,2,0)}}{\partial \tilde{M}_{0}} \right)_{\tilde{M}_{0}=M_{0}} \neq \left( \frac{\partial F_{R}^{(2,2,0)}}{\partial \tilde{M}_{0}} \right)_{\tilde{M}_{0}=M_{0}},
$$

leading to a kink $\Delta \Theta^{(2,2,0)}$ in the function $M_{T2,max}(\tilde{M}_{0}, p_{T} = 0) [34–37]$. Applying the general definition (4–27), we obtain the size of this kink quantitatively,

$$
\Delta \Theta^{(2,2,0)} = \arctan \left( \frac{2(1-y)(1-z)\sqrt{yz}}{(y+z)(1+yz)+4yz} \right),
$$

where we have defined the squared mass ratios

$$
y \equiv \frac{M_{1}^{2}}{M_{2}^{2}}, \quad z \equiv \frac{M_{0}^{2}}{M_{1}^{2}}.
$$

The result (4–48) is plotted in Figure 4-5(a) as a function of the mass ratios $\sqrt{y}$ and $\sqrt{z}$.

Figure 4-5(a) demonstrates that as both $y$ and $z$ become small, the kink $\Delta \Theta^{(2,2,0)}$ gets more pronounced. Figure 4-5(a) also shows that the kink $\Delta \Theta^{(2,2,0)}$ is a symmetric function of $y$ and $z$, as can also be seen directly from Equation (4–48). Therefore, the kink $\Delta \Theta^{(2,2,0)}$ will be best observable in those cases where $y$ and $z$ are both small, and in addition, the mass spectrum happens to obey the relation $y = z$, i.e. $M_{1} = \ldots$
For this special value of \( M_1 = \sqrt{M_0 M_2} \), the upper endpoint of the invariant mass distribution \( M_{x_1 x_2} \) is the same as in the case when the intermediate particle \( x_1 \) is off-shell, i.e. when \( M_1 > M_2 \). Then we find that the \( M^{(2,2,0)}_{T_2,max} \) formulas and corresponding kink structures are identical in the on-shell and off-shell cases. We provide more details in Appendix B. Unfortunately, the SPS1a study point is rather far from this category – the spectrum (4–44) corresponds to the values \( \sqrt{y} = 0.856 \) and \( \sqrt{z} = 0.189 \), which are indicated in Figure 4-5(a) by a white dot. This conclusion is also supported by Figure 4-4(a), which shows a rather mild kink in the SPS1a case.

We shall be rather ambivalent in our attitude toward the \( \Delta \Theta^{(2,2,0)} \) kink as well. While the interpretation of the kink is straightforward, its observation in the actual experiment is again an open issue. On the one hand, the experimental precision would depend on the particular signature, i.e. the type of the SM particles \( x_1 \) and \( x_2 \). If those are leptons, their 4-momenta \( p^{(k)}_1 \) and \( p^{(k)}_2 \) will be measured relatively well and the kink might be observable. However, when \( x_1 \) and \( x_2 \) are jets, the experimental resolution may not be sufficient. Secondly, as seen in Figure 4-4(a), the kink itself may not be very pronounced, and its observability will in fact depend on the particular mass spectrum.

The main lesson from the above discussion is that while the existence of the kink is without a doubt, its actual observation is by no means guaranteed. Therefore, our main mass measurement method, described later in Section 4.2.1, will not use any information related to the kink. In fact in Section 4.2.1 we shall show that one can completely reconstruct the mass spectrum of the new particles, using just measurements of \( M_{T_2} \) endpoints, each done at a single fixed value of the corresponding test mass. It is worth noting that, in general, an endpoint in a spectrum is a sharper feature than a kink of the type (4–27). Therefore, we would expect that the experimental precision on the extracted endpoints will be much better than the corresponding precision on the kink location. The kink will also not play any role in our hybrid method,
described in Section 4.2.3. Only for the method described in Section 4.2.2, we shall try to make use of the kink information.

Let us now return to our original discussion of the \( \tilde{M}_{T_2}^{(2,2,0)} \) endpoint (4–41).

Following our previous approach from Sections 4.1.1 and 4.1.2, we would choose a fixed value of the test mass \( \tilde{M}_0 \) and measure the corresponding \( M_{T_2} \) endpoint. However, the presence of two branches in Equations (4–42) and (4–43) leads to a slight complication: for a randomly chosen value of \( \tilde{M}_0 \), we will not know whether we should use Equations (4–42) or (4–43) when interpreting the endpoint measurement. This requires us to make very special choices for the fixed value of \( \tilde{M}_0 \), which would remove this ambiguity. It is easy to see that by choosing \( \tilde{M}_0 = 0 \), we can ensure that the endpoint is always described by the “low” branch in Equation (4–42), and the \( \tilde{M}_{T_2,max}^{(2,2,0)} \) measurement can then be uniquely interpreted as

\[
\tilde{M}_{T_2,max}^{(2,2,0)}(\tilde{M}_0 = 0, p_T = 0) = 2 \mu_{(2,2,0)} = \frac{M_2^2 - M_0^2}{M_2}.
\] (4–50)

However, we could also design a special choice of \( \tilde{M}_0 \), which would select the “high” branch in Equation (4–42) and again uniquely remove the branch ambiguity. For this purpose, we must choose a value for the test mass \( \tilde{M}_0 \) which is sufficiently large, in order to safely guarantee that it is well beyond the true mass \( M_0 \). Since the true mass \( M_0 \) can never exceed the beam energy \( E_b \), one obvious safe and rather conservative choice for \( \tilde{M}_0 \) could be \( \tilde{M}_0 = E_b \), in which case from (4–41) we get

\[
\tilde{M}_{T_2,max}^{(2,2,0)}(\tilde{M}_0 = E_b, p_T = 0) = \mu_{(2,2,1)} + \mu_{(2,1,0)} + \sqrt{(\mu_{(2,2,1)} - \mu_{(2,1,0)})^2 + E_b^2}.
\] (4–51)

Notice that the high branch function \( F_{R}^{(2,2,0)} \) in Equation (4–43) is rather unique in one very important aspect: it depends not just on one, but on two mass parameters, namely the combinations \( \mu_{(2,2,1)} + \mu_{(2,1,0)} \) and \( \mu_{(2,2,1)} - \mu_{(2,1,0)} \). In contrast, the “low” branch \( F_{L}^{(2,2,0)} \), as well as the previously discussed endpoint functions \( \tilde{M}_{T_2,max}^{(n,n,n-1)}(\tilde{M}_0, p_T = 0) \),
each contained a single $\mu$ parameter. As a result, in those cases we did not benefit from any extra measurements for different values of the test mass $\tilde{M}_0$ – had we done that, we would have been measuring the same $\mu$ parameter over and over again.

However, the situation with $F_R^{(2,2,0)}$ is different, and here we will benefit from an additional measurement for a different value of $\tilde{M}_0$. For example, let us choose $\tilde{M}_0 = \tilde{E}_b$, with $\tilde{E}_b > E_b$, which will still keep us on the high branch. We obtain another constraint

$$M_{T2,max}^{(2,2,0)}(\tilde{M}_0 = \tilde{E}_b, p_T = 0) = \mu_{(2,2,1)} + \mu_{(2,1,0)} + \sqrt{(\mu_{(2,2,1)} - \mu_{(2,1,0)})^2 + \tilde{E}_b^2}$$

(4–52)

It is easy to check that the constraints in Equations (4–50 through 4–52) are all independent, thus providing three independent equations$^3$ for the three unknown masses $M_0$, $M_1$ and $M_2$. These three Equations (4–50 through 4–52) can be solved rather easily$^4$, and one obtains the proper solution for the masses $M_0$, $M_1$ and $M_2$, up to a two-fold ambiguity:

$$M_2 \rightarrow M_2, \quad M_1 \rightarrow \frac{M_0}{M_1} M_2, \quad M_0 \rightarrow M_0,$$

(4–53)

which is nothing but the interchange $y \leftrightarrow z$ at a fixed $M_2$. The ambiguity arises because the expression (4–41) for the endpoint $M_{T2,max}^{(2,2,0)}$ (and consequently, the set of constraints

$^3$ In practice, instead of relying on individual endpoint measurements for three different values of $\tilde{M}_0$, one may prefer to use the experimental information for the whole function $M_{T2,max}^{(2,2,0)}(\tilde{M}_0, p_T = 0)$ and simply fit to it the analytical expression (4–41) for the three floating parameters $M_0$, $M_1$ and $M_2$, as was done [37]. As we shall see shortly, this method does not lead to any new information, and may only improve the statistical error on the mass determination. Therefore, to keep our discussion as simple as possible, we prefer to talk about the three individual measurements in Equations (4–50 through 4–52) as opposed to fitting the whole distribution (4–41).

$^4$ The general solution for $M_2$, $M_1$ and $M_0$ in terms of the measured endpoints (4–50–4–52) is rather messy and not very illuminating, therefore we do not list it here.
(4–50-4–52)) is invariant under the transformation (4–53). Because of this ambiguity, in addition to the original SPS1a input values (4–44) for the mass spectrum, we obtain a second solution:

\[
M_2 = 613 \text{ GeV}, \quad M_1 = 115.6 \text{ GeV}, \quad M_0 = 99 \text{ GeV}.
\]  

(4–54)

This second solution was missed in the analysis of Ref. [37]. It is easy to check that the alternative mass spectrum (4–54) gives an identical \(M^{(2,2,0)}_{T_2,\text{max}}(\tilde{M}_0, p_T = 0)\) distribution as the one shown in Figure 4-4(a), so that it is impossible to rule it out on the basis of \(M^{(2,2,0)}_{T_2,\text{max}}\) measurements alone.

The previous discussion reveals an important and somewhat overlooked benefit from the existence of the kink – one can make not one, not two, but three independent endpoint measurements from a single \(M^{(n,p,c)}_{T_2}\) distribution! In fact, we shall argue that the three measurements in Equations (4–50 through 4–52) are much more robust than the kink measurement (4–27). For example, when the child mass is relatively small, the lower branch \(F^{(2,2,0)}_L\) is relatively short and the kink will be difficult to see, even under ideal experimental conditions. An extreme example of this sort is presented in Section 4.2, where we discuss top quark events, in which the child (neutrino) mass \(M_0\) is practically zero and the kink cannot be seen at all. However, even under those circumstances, the endpoint measurements in Equations (4–50 through 4–52) are still available. More importantly, the constraints in Equations (4–50 through 4–52) are independent of the previously found relations in Equations (4–12) and (4–32), so that the latter can be used to resolve the two-fold ambiguity in Equation (4–53).

Before we move on to a discussion of the last remaining subsystem \(M_{T_2}\) quantity in the next Section 4.1.4, let us recap our main result derived in this subsection. We showed that the \(M^{(2,2,0)}_{T_2}\) variable yields three independent endpoint measurements in Equations (4–50 through 4–52), and possibly a kink measurement in Equation (4–27). The \(M^{(2,2,0)}_{T_2}\) endpoint measurements by themselves are sufficient to determine all three
masses $M_0$, $M_1$, and $M_2$, up to the two-fold ambiguity in Equation (4–53). This represents a pure $M_{T2}$-based mass measurement method, which does not use any kink or invariant mass information.

4.1.4 The Subsystem Variable $M_{T2}^{(2,1,0)}$

The variable $M_{T2}^{(2,1,0)}$ is illustrated in Figure 4-2(b), where we use the subchain within the smaller rectangle on the right. This is another genuine subsystem quantity, since we only use the SM decay products $x_1$ and ignore the upstream objects $x_2$. However, the upstream objects $x_2$ are important in the sense that they have some non-zero transverse momentum, and as a result, the sum of the transverse momenta $\vec{p}^{(k)}_{0T}$ of the children $X_0$ is not balanced by the sum of the transverse momenta of the SM objects $x_1$ used in the $M_{T2}$ calculation:

$$\vec{p}^{(1)}_{0T} + \vec{p}^{(2)}_{0T} + \vec{P}_{1T}^{(1)} + \vec{P}_{1T}^{(2)} = -\vec{p}^{(1)}_{2T} - \vec{p}^{(2)}_{2T} - \vec{p}_T \neq 0. \quad (4–55)$$

Notice that even in the absence of any ISR $p_T$, this is still an unbalanced configuration, due to the transverse momenta $\vec{p}^{(1)}_{2T}$ and $\vec{p}^{(2)}_{2T}$ of the upstream objects $x_2$. Therefore, we cannot use the existing analytical results on $M_{T2}$, since previous studies always assumed that the right-hand side of Equation (4–55) is exactly zero, due to the lack of any particles upstream. We therefore need to generalize the previous treatments of $M_{T2}$ and obtain the corresponding endpoint formulas for our new subsystem $M_{T2}^{(2,1,0)}$ variable. In particular, in the absence of any intrinsic ISR (i.e., for $p_T = 0$), we find that the endpoint of the $M_{T2}^{(2,1,0)}$ distribution is given by

$$M_{T2,max}^{(2,1,0)}(\tilde{M}_0, p_T = 0) = \begin{cases} F_{L}^{(2,1,0)}(\tilde{M}_0, p_T = 0), & \text{if } \tilde{M}_0 \leq M_0, \\ F_{R}^{(2,1,0)}(\tilde{M}_0, p_T = 0), & \text{if } \tilde{M}_0 \geq M_0. \end{cases} \quad (4–56)$$
where

\[
F^{(2,1,0)}_L(\tilde{M}_0, p_T = 0) = \left\{ \left[ \mu_{(2,2,0)} - \mu_{(2,2,1)} + \sqrt{\mu_{(2,2,0)}^2 + \tilde{M}_0^2} \right]^2 - \mu_{(2,2,1)}^2 \right\}^{\frac{1}{2}}, \quad (4–57)
\]

\[
F^{(2,1,0)}_R(\tilde{M}_0, p_T = 0) = \left\{ \left[ \mu_{(2,1,0)} + \sqrt{(\mu_{(2,2,1)} - \mu_{(2,1,0)})^2 + \tilde{M}_0^2} \right]^2 - \mu_{(2,2,1)}^2 \right\}^{\frac{1}{2}}, \quad (4–58)
\]

and the various parameters \(\mu_{(n,p,c)}\) are defined in (4–9). The corresponding expressions for general \(p_T\) (i.e., arbitrary intrinsic ISR) are listed in Appendix B.

From Equation (4–56) we see that, once again, the endpoint function \(M^{(2,1,0)}_{\max}(\tilde{M}_0, p_T = 0)\) would exhibit a kink \(\Delta\Theta^{(2,1,0)}\) at the true value of the test mass \(\tilde{M}_0 = M_0\):

\[
\left( \frac{\partial F^{(2,1,0)}_L}{\partial \tilde{M}_0} \right)_{\tilde{M}_0=M_0} \neq \left( \frac{\partial F^{(2,1,0)}_R}{\partial \tilde{M}_0} \right)_{\tilde{M}_0=M_0}. \quad (4–59)
\]

The existence of this kink should come as no surprise — Ref. [36] showed (in the \(p_T \to \infty\) limit) that any type of upstream momentum will generate a kink in an otherwise smooth \(M^{\max}_{\text{T2}}\) function. As before, the value of the \(M_{\text{T2}}\) endpoint \(M^{(2,1,0)}_{\text{T2,max}}\) at the kink location reveals the true mass of the parent:

\[
M^{(2,1,0)}_{\text{T2,max}}(\tilde{M}_0 = M_0, p_T = 0) = M_1. \quad (4–60)
\]

At the same time, the physical origin of this kink is different from either of the two kink types \((\Delta\Theta^{(1,1,0)}\) and \(\Delta\Theta^{(2,2,0)}\)) discussed earlier. Clearly, the new kink is different from \(\Delta\Theta^{(2,2,0)}\), which was due to the varying invariant mass of the \(\{x_1, x_2\}\) system. Here we are using a single SM particle \(x_1\) whose mass is constant. Furthermore, the new kink \(\Delta\Theta^{(2,1,0)}\) cannot be due to any ISR like in the case of \(\Delta\Theta^{(1,1,0)}\), since Equation (4–56) does not account for any ISR effects. The real reason for this new \(\Delta\Theta^{(2,1,0)}\) kink is a third one, namely, the kinematical restrictions placed by the decays of the upstream particles (in this case, the grandparents \(X_2\)).

We now proceed to investigate the new kink \(\Delta\Theta^{(2,1,0)}\) quantitatively. Using the same example of gluino pair-production for the SPS1a mass spectrum (4–44), we plot...
the function (4–56) in Figure 4-4(a). Comparing the lower and the upper set of lines in Figure 4-4(a), we notice that the $M_{T2,max}^{(2,1,0)}$ and $M_{T2,max}^{(2,2,0)}$ variables share several common characteristics. They both exhibit a kink at the true location of the child mass $\tilde{M}_0 = M_0$, while their values at that point reveal the true parent mass in each case: $M_1$ for $M_{T2,max}^{(2,1,0)}$ and $M_2$ for $M_{T2,max}^{(2,2,0)}$. Using the definition (4–27), we find that the size of the $\Delta \Theta^{(2,1,0)}$ kink is given by

$$\Delta \Theta^{(2,1,0)} = \arctan \left( \frac{(1 - y^2)(1 - z)\sqrt{z}}{2z(1 + y^2) + y(1 + z^2) + 2yz} \right),$$

where the parameters $y$ and $z$ were already defined in (4–49). The kink $\Delta \Theta^{(2,1,0)}$ is plotted in Figure 4-5(b) as a function of $\sqrt{y}$ and $\sqrt{z}$. We notice that the kink structure becomes more pronounced for relatively small $y$ and $z$. Comparing Figures 4-5(a) and 4-5(b), we see that for any given set of values for $y$ and $z$, the $\Delta \Theta^{(2,1,0)}$ kink discussed here is more pronounced than the $\Delta \Theta^{(2,2,0)}$ kink from the previous subsection. The difference is particularly noticeable in the region of $\sqrt{y} \sim 0$ and $\sqrt{z} \sim 0.2$. The SPS1a mass spectrum (4–44) in our previous example was rather far away from this region, as indicated by the white dots in Figure 4-5. Now let us choose a different mass spectrum, which is closer to the region where the difference between the two kinks becomes more noticeable, for example

$$M_2 = 2000 \text{ GeV}, \quad M_1 = 200 \text{ GeV}, \quad M_0 = 100 \text{ GeV},$$

(4–62)

corresponding to the point marked with the white asterisk in Figures 4-5(a) and 4-5(b). The resulting endpoint functions $M_{T2,max}^{(2,2,0)}$ and $M_{T2,max}^{(2,1,0)}$ are plotted in Figure 4-4(b). Indeed we see that with this new spectrum the kink in the $M_{T2,max}^{(2,1,0)}$ function is much more noticeable than the kink in the $M_{T2,max}^{(2,2,0)}$ function. Therefore, our first conclusion regarding the $M_{T2}^{(2,1,0)}$ variable is that its kink is in general sharper and appears to

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5 This statement can also be verified using the analytical formulas (4–48) and (4–61).
be more promising than the previously discussed kink in the \( M_{T2}^{(2,2,0)} \) variable from Section 4.1.3.

Following our previous strategy, we shall not dwell too long on the kink, but instead we shall discuss the available endpoint measurements for various values of \( \tilde{M}_0 \). Again, the presence of two branches in Equation (4–56) can be used to our advantage. As in Section 4.1.3, we first choose a test mass value \( \tilde{M}_0 = 0 \), which would “select” the low branch (4–57) and result in an endpoint measurement

\[
M_{T2,max}^{(2,1,0)}(\tilde{M}_0 = 0, p_T = 0) = 2 \sqrt{\mu_{(2,2,0)} (\mu_{(2,2,0)} - \mu_{(2,2,1)})}.
\]  

(4–63)

Just as before, we could also choose a rather large value for \( \tilde{M}_0 = E_b \), which would select the high branch (4–58) and result in the measurement

\[
M_{T2,max}^{(2,1,0)}(\tilde{M}_0 = E_b, p_T = 0) = \left\{ \left[ \mu_{(2,1,0)} + \sqrt{\left( \mu_{(2,2,1)} - \mu_{(2,1,0)} \right)^2 + E_b^2} \right]^2 - \mu_{(2,2,1)}^2 \right\}^{\frac{1}{2}}.
\]  

(4–64)

A third choice, \( \tilde{M}_0 = \tilde{E}_b \), with \( \tilde{E}_b > E_b \), would yield yet another endpoint measurement

\[
M_{T2,max}^{(2,1,0)}(\tilde{M}_0 = \tilde{E}_b, p_T = 0) = \left\{ \left[ \mu_{(2,1,0)} + \sqrt{\left( \mu_{(2,2,1)} - \mu_{(2,1,0)} \right)^2 + \tilde{E}_b^2} \right]^2 - \mu_{(2,2,1)}^2 \right\}^{\frac{1}{2}}.
\]  

(4–65)

Again we obtained three Equations (4–63 through 4–65) for the three unknown \( \mu \)-parameters \( \mu_{(2,2,0)}, \mu_{(2,2,1)} \) and \( \mu_{(2,1,0)} \), or equivalently, for the three unknown masses \( M_0, M_1 \) and \( M_2 \). Equations (4–63 through 4–65) are all independent and can be easily solved, giving a total of four solutions. However, three of the solutions are always unphysical, so that we end up with a single unique solution. This represents an important advantage of the \( M_{T2,max}^{(2,1,0)} \) variable in comparison with the \( M_{T2,max}^{(2,2,0)} \) variable discussed in Section 4.1.3. There we found that \( M_{T2,max}^{(2,2,0)} \) always gives rise two a two-fold ambiguity in the mass spectrum, while now we see that \( M_{T2,max}^{(2,1,0)} \) does not suffer from this problem and already by itself allows for a complete and unambiguous determination of the mass spectrum.
4.2 $M_{T2}$-based Mass Measurement methods

In this section we use the analytical results derived in the previous section to propose three different strategies for determining the masses in $n \leq 2$ decay chains. We shall illustrate each of our methods with a specific example, for which we choose to consider the dilepton samples from $W^+W^-$ and $t\bar{t}$ events. The former is an example of the $n = 1$ decay chain exhibited in Figure 4-2(a), while the latter is an example of the $n = 2$ decay chain in Figure 4-2(b). Most importantly, these samples already exist in the Tevatron data and will also be among the first to be studied at the LHC. Correspondingly, throughout this section we shall use the following mass spectrum

$$
M_2 = m_t = 173 \text{ GeV},
M_1 = m_W = 80 \text{ GeV},
M_0 = m_\nu = 0 \text{ GeV}.
$$

(4–66)

Before we begin, let us review the four different $M_{T2}^{(n,p,c)}$ variables which are in principle available in that case. Each one of them is plotted in Figure 4-6 for five different values of the corresponding test mass (0, 100, 200, 300 and 400 GeV). In Figure 4-6(a) we show the $M_{T2}^{(1,1,0)}$ variable from $W^+W^-$ pair production events, while in Figures 4-6(b-d) we correspondingly show the $M_{T2}^{(2,2,1)}$, $M_{T2}^{(2,2,0)}$ and $M_{T2}^{(2,1,0)}$ variables from $t\bar{t}$ events. We used PYTHIA [54] for event generation and did not impose any selection cuts, since they will not affect the location of the $M_{T2}$ endpoint$^6$. The plots are made for the Tevatron (a $p\bar{p}$ collider with a 2 TeV center-of-mass energy), where the relevant data is already available. The corresponding analysis for the LHC is very similar. All of our plots in this Section have the full ISR effects.

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$^6$ The cuts would have an impact on the overall acceptance and efficiency. This effect is not relevant here, since we are showing unit-normalized distributions. The cuts may also distort the shape of each distribution, but should preserve the location of the upper endpoint.
Figure 4-6. Unit-normalized distributions of $M_{T2}^{(n,p,c)}$ variables in dilepton events from (a) $W^+W^-$ pair production and (b-d) $t\bar{t}$ pair production. Each panel shows results for five different values (0, 100, 200, 300 and 400 GeV) of the corresponding test mass. The methods of Sections 4.2.1 and 4.2.3 only make use of the $M_{T2}$ endpoint at zero test mass, $M_{T2,max}(\tilde{M}_c = 0)$, which is indicated by the vertical red arrow. In panel (c), the two dotted line $M_{T2}^{(2,2,0)}$ distributions correspond to the correct and the wrong pairing of the two $b$-jets with the leptons, while the solid line distribution is the average of these two.

As discussed in Section 4.1, the presence of ISR with nonzero $p_T$ will increase the nominal $M_{T2}^{(n,p,c)}$ endpoints:

$$M_{T2,max}(\tilde{M}_c, p_T) \geq M_{T2,max}(\tilde{M}_c, 0),$$

(4–67)

where the equality is obtained only when $\tilde{M}_c = M_c$. ISR will therefore introduce some systematic error when one is trying to measure $M_{T2,max}(\tilde{M}_c, 0)$. The size of this
error depends on the ISR $p_T$ spectrum, which in turn depends on the type of collider (Tevatron or LHC). At the Tevatron, this will not be such a serious issue, as evidenced from Figure 4-6, where the observed endpoints in the presence of ISR match pretty well with their expected values for the $p_T = 0$ case. On the other hand, at the LHC this may become a problem, which can be handled in one of two ways. First, depending on the particular signature, one may be able to select a sample with $p_T \approx 0$ (at a certain cost in statistics), by imposing a suitably designed jet veto to remove jets from ISR. Alternatively, one can use the full event sample (which would include ISR jets), and make use of our general formulas in Appendix B, which contain the explicit $p_T$ dependence of $M^{(n,p,c)}_{T_{2,max}}$.

In the previous Section 4.1 we derived that in the case of $n \leq 2$ cascades, there are 8 different $M_{T_2}$ endpoint measurements: one for $M^{(1,1,0)}_{T_{2,max}}$ (see Equation (4–12) and Section 4.1.1), one for $M^{(2,2,1)}_{T_{2,max}}$ (see Equation (4–32) and Section 4.1.2), three for $M^{(2,2,0)}_{T_{2,max}}$ (see Equations (4–50 through 4–52) and Section 4.1.3), and three for $M^{(2,1,0)}_{T_{2,max}}$ (see Equations (4–63 through 4–65) and Section 4.1.4). Given that we are trying to determine only three masses $M_0, M_1$ and $M_2$, it is clear that these 8 measurements should be sufficient to completely determine the spectrum. The number of available measurements is in fact much larger than the number of $M^{(n,p,c)}_{T_{2,max}}$ variables. Indeed, as shown in Sections 4.1.3 and 4.1.4, there are cases where we might be able to obtain more than one mass constraint from a given $M^{(n,p,c)}_{T_{2,max}}$ variable. Of course, the 8 measurements cannot all be independent among themselves, as they only depend on three parameters. Our three methods below will be distinguished based on which subset of these measurements we are using.

4.2.1 Pure $M_{T_2}$ Endpoint Method

With this method, we use $M_{T_2}$ endpoint measurements $E_{npc}$ at a single fixed value of the test mass, which for convenience we take to be $\tilde{M}_c = 0$:

$$E_{npc} \equiv M^{(n,p,c)}_{T_{2,max}}(\tilde{M}_c = 0, p_T = 0).$$
The corresponding formulas interpreting those measurements in terms of the physical masses \( M_0, M_1 \) and \( M_2 \) were derived in Section 4.1:

\[
E_{110} \equiv M_{T^2,\text{max}}^{(1,1,0)}(0,0) = \frac{M_1^2 - M_0^2}{M_1} = M_2 \sqrt{y} (1 - z), \tag{4–69}
\]

\[
E_{221} \equiv M_{T^2,\text{max}}^{(2,2,1)}(0,0) = \frac{M_2^2 - M_1^2}{M_2} = M_2 (1 - y), \tag{4–70}
\]

\[
E_{220} \equiv M_{T^2,\text{max}}^{(2,2,0)}(0,0) = \frac{M_2^2 - M_0^2}{M_2} = M_2 (1 - yz), \tag{4–71}
\]

\[
E_{210} \equiv M_{T^2,\text{max}}^{(2,1,0)}(0,0) = \frac{1}{M_2} \sqrt{(M_2^2 - M_0^2)(M_1^2 - M_0^2)} = M_2 \sqrt{y(1 - z)(1 - yz)}, \tag{4–72}
\]

Using the mass spectrum (4–66), the predicted locations of these four \( M_{T^2} \) endpoints are

\[
E_{110} = 80 \text{ GeV}, \tag{4–73}
\]

\[
E_{221} = 136 \text{ GeV}, \tag{4–74}
\]

\[
E_{220} = 173 \text{ GeV}, \tag{4–75}
\]

\[
E_{210} = 80 \text{ GeV}, \tag{4–76}
\]

which are marked with the vertical red arrows in Figure 4-6. Given that we have four measurements (4–69 through 4–72) for only three parameters \( M_0, M_1 \) and \( M_2 \), one should be able to uniquely determine all three of the unknown parameters. Naively, it seems that using just three of the measurements (4–69 through 4–72) should be sufficient for this purpose, and furthermore, that any three of the measurements (4–69 through 4–72) will do the job. However, one should exercise caution, since not all four measurements (4–69 through 4–72) are independent. It is easy to check that \( E_{221}, E_{220} \) and \( E_{210} \) obey the following relation

\[
E_{210}^2 = E_{220} (E_{220} - E_{221}). \tag{4–77}
\]

This means that in order to be able to solve for the masses from Equations (4-69 through 4–72), we must always make use of the \( E_{110} \) measurement in Equation (4–69),
and then we have the freedom to choose any two out of the remaining three measurements (4–70 through 4–72). For example, using the set of three measurements \( \{E_{110}, E_{221}, E_{220}\} \) (i.e. Equations (4–69 through 4–71)), the masses are uniquely determined as

\[
M_0 = \frac{E_{110} \left( E_{221} (E_{220} - E_{221}) \left[ E_{220} (E_{220} - E_{210}) - E_{110}^2 \right] \right)^{1/2}}{E_{110}^2 - (E_{220} - E_{221})^2},
\]

(4–78)

\[
M_1 = \frac{E_{110} E_{221} (E_{220} - E_{221})}{E_{110}^2 - (E_{220} - E_{221})^2},
\]

(4–79)

\[
M_2 = \frac{E_{110}^2 E_{221}}{E_{110}^2 - (E_{220} - E_{221})^2}.
\]

(4–80)

Similarly, one can solve for \( M_0, M_1 \) and \( M_2 \) using the set of measurements \( \{E_{110}, E_{220}, E_{210}\} \), or alternatively, the set of measurements \( \{E_{110}, E_{221}, E_{210}\} \). In each case, the remaining fourth unused measurement provides a useful consistency check on the mass determination.

**4.2.2 \( M_{T2} \) Endpoint Shapes And Kinks**

The method proposed in Section 4.2.1 uses the measured endpoints from several different \( M_{T2}^{(n,p,c)} \) variables. Now we discuss an alternative method which makes use of a single \( M_{T2}^{(n,p,c)} \) variable.

Let us begin with the simplest case of \( n = 1 \) as shown in Figure 4-2(a). In that case, we have only one \( M_{T2} \) variable at our disposal, namely \( M_{T2}^{(1,1,0)} \). Its properties were discussed in Section 4.1.1, where we showed that its endpoint \( M_{T2,\text{max}}^{(1,1,0)} \) can allow the determination of both masses \( M_0 \) and \( M_1 \), at least as a matter of principle. Indeed, the endpoint measurement (4–69) at zero test mass provides one relation among \( M_0 \) and \( M_1 \). The key observation in Section 4.1.1 (which was first done in [36]) was that with the inclusion of ISR effects, the endpoint function \( M_{T2,\text{max}}^{(1,1,0)}(\tilde{M}_0, p_T) \) exhibits a kink at \( \tilde{M}_0 = M_0 \), which can then be used to determine both masses \( M_0 \) and \( M_1 \). The method can be readily applied to the existing dilepton event sample from \( W^+W^- \) pair production, which will allow an independent measurement of the \( W \) mass \( m_W \) and the neutrino mass \( m_\nu \). While the precision of this measurement will not be competitive.
with existing $W$ and neutrino mass determinations, it is nevertheless useful to test the viability of this approach with real data.

Now let us discuss the more complicated case of $n = 2$, which in our example corresponds to $t\bar{t}$ pair production with both tops decaying leptonically. As discussed in Sections 4.1.2, 4.1.3 and 4.1.4, here we have a choice of three different $M_{T2}$ variables: $M_{T2}^{(2,2,1)}$, $M_{T2}^{(2,2,0)}$, and $M_{T2}^{(2,1,0)}$. Because of the larger $t\bar{t}$ cross-section, we expect that the statistical precision on each one of those three variables will be better than the $M_{T2}^{(1,1,0)}$ variable of the $n = 1$ case. As shown in Sections 4.1.3 and 4.1.4, each of the two variables $M_{T2}^{(2,2,0)}$ and $M_{T2}^{(2,1,0)}$ exhibits a kink in its endpoint $M_{T2,\text{max}}^{(n,p,c)}$ when considered as a function of the test mass $\tilde{M}_0$, even when the transverse momentum of the intrinsic ISR in the event is zero, $p_T = 0$. Then, which of these two variables is better suited for a mass determination? The case of $M_{T2,\text{max}}^{(2,2,0)}(\tilde{M}_0, p_T = 0)$ was already discussed in [34, 36, 37]. Here we would like to propose the alternative measurement of $M_{T2,\text{max}}^{(2,1,0)}(\tilde{M}_0, p_T = 0)$. What is more, we would like to emphasize that our function $M_{T2,\text{max}}^{(2,1,0)}(\tilde{M}_0, p_T = 0)$ offers several unique advantages over the previously considered case of $M_{T2,\text{max}}^{(2,2,0)}(\tilde{M}_0, p_T = 0)$:

1. The subsystem variable $M_{T2}^{(2,1,0)}$ does not suffer from the combinatorics problem which is present for $M_{T2}^{(2,2,0)}$. Indeed, when constructing the $M_{T2}^{(2,2,0)}$ distribution, one has to decide how to pair up the $b$-jets with the two leptons. Because it is difficult to distinguish between a $b$ and a $\bar{b}$, there is a two-fold ambiguity which is quite difficult to resolve by other means. In contrast, our subsystem variable $M_{T2}^{(2,1,0)}$ does not make direct use of the $b$-jets, and is therefore free of such combinatorics issues.

2. As we already saw in Section 4.1.3, even under perfect experimental conditions, the fit to the $M_{T2,\text{max}}^{(2,2,0)}$ endpoint results in two separate solutions for the mass spectrum: one solution (Equation (4–44)) is given by the true values of the input masses, while the second solution (Equation (4–54)) is obtained by the transformation in Equation (4–53). Using $M_{T2,\text{max}}^{(2,2,0)}$ alone, there is no way to tell the difference between these two mass spectra. In contrast, our variable $M_{T2}^{(2,1,0)}$ does not suffer from this ambiguity, and according to our results from Section 4.1.4 the solution is always unique.
3. The third advantage of the subsystem variable \( M_{T2}^{(2,1,0)} \) is related to the expected precision on the determination of the masses. As we pointed out in Section 4.1.4 and illustrated explicitly in Figure 4-5, the kink \( \Delta \Theta^{(2,1,0)} \) in the \( M_{T2,\text{max}}^{(2,1,0)}(\tilde{M}_0, p_T = 0) \) function is much sharper than the corresponding kink \( \Delta \Theta^{(2,2,0)} \) in the \( M_{T2,\text{max}}^{(2,2,0)}(\tilde{M}_0, p_T = 0) \) function. This can also be seen explicitly from the two examples shown in Figure 4-4. As a result, we expect that the kink structure can be better identified in the case of \( M_{T2,\text{max}}^{(2,1,0)} \), which would lead to smaller errors on the mass determination.

Of course, one could (and in fact should) use the experimental information from both \( M_{T2,\text{max}}^{(2,2,0)} \) and \( M_{T2,\text{max}}^{(2,1,0)} \), if available. Our main goal here is simply to point out the obvious advantages of the subsystem variable \( M_{T2}^{(2,1,0)} \), which so far has not been used in the literature.

4.2.3 Hybrid Method: \( M_{T2} \) Endpoints Plus An Invariant Mass Endpoint

Any cascade with \( n \geq 2 \) will provide a certain number of measurements like as invariant mass endpoints in addition to the \( M_{T2} \) measurements discussed so far. In particular, for the \( n = 2 \) example considered here, there will be one measurement of the endpoint of the \( M_{x_1x_2} \) invariant mass distribution. The formula for the endpoint \( M_{x_1x_2,\text{max}} \) in terms of the unknown physical masses \( M_0, M_1 \) and \( M_2 \) is in general given by

\[
E_{im} \equiv M_{x_1x_2,\text{max}} = \frac{1}{M_1} \sqrt{(M_2 - M_1^2)(M_1^2 - M_0^2)} = M_2 \sqrt{(1 - y)(1 - z)}. \tag{4–81}
\]

In the case of \( t\bar{t} \) events considered here, this is simply the endpoint of the invariant mass distribution \( m_{bl} \) of each lepton and its corresponding \( b \)-jet. This distribution (unit-normalized) is shown in Figure 4-7. Unfortunately, here one is facing the same combinatorial problem as with the \( M_{T2}^{(2,2,0)} \) variable – we cannot easily tell the charge of the \( b \)-jet, therefore a priori it is not clear which \( b \)-jet goes with which lepton. Fortunately, there are only two possibilities: the result from the correct (wrong) pairing is shown in Figure 4-7 with the green (blue) dotted line. We see that the green histogram with the correct pairing has an endpoint at the expected location

\[
E_{im} = \sqrt{\frac{(m_t^2 - m_W^2)(m_W^2 - m_b^2)}{m_W^2}} = 153.4 \text{ GeV}. \tag{4–82}
\]
with a relatively small tail due to the finite width effects. More importantly, the (blue) distribution from the wrong pairings is relatively smooth, and as a result the endpoint as in Equation (4–82) is preserved in the experimentally observable (red) distribution, which includes all possible $b\ell$ pairings. Now we can add the new measurement (4–81) to the previously discussed set of measurements in Equations (4–69 through 4–72). We obtain a total of five measurements for the three underlying parameters $M_0$, $M_1$ and $M_2$, therefore there exist two relations among the measurements. The first relation is already given by Equation (4–77) and does not involve the invariant mass endpoint in Equation (4–81). The second relation is given by

$$E_{im}^2 = \frac{E_{211}^2 E_{110}^2}{E_{220} - E_{221}}.$$  

We can now consider a hybrid method, which would make use of the invariant mass endpoint (4–81), plus any two of the $M_{T2}$ measurements in Equations (4–69 through 4–72). In principle, one again needs to be careful and make sure that the three used measurements are independent. Fortunately, as seen from Equations (4–77,4–83), the invariant mass endpoint $E_{im}$ is independent from any pair of $M_{T2}$ measurements. There are 6 possible pairs among the $M_{T2}$ measurements (4–69 through 4–72), and in principle each one can be used in combination with the invariant mass endpoint (4–81). What is the best choice? We find that in all 6 of those cases one obtains a unique solution for the masses $M_0$, $M_1$ and $M_2$. Therefore, the optimal choice is dictated by the experimental precision on each of the measurements (4–69 through 4–72). We expect that the measurement (4–10) of $M_{T2,max}^{(1,1,0)}$ will be less precise due to the smaller cross-section for $W^+W^-$ production. Similarly, $M_{T2,max}^{(2,2,0)}$ suffers from the combinatorial problem already mentioned earlier. Therefore for our illustration of the hybrid method we choose to use the $M_{T2}^{(2,2,1)}$ endpoint (4–70), the $M_{T2}^{(2,1,0)}$ endpoint (4–72), and the invariant mass endpoint (4–81).
Figure 4-7. Unit-normalized $m_{b\ell}$ invariant mass-squared distributions in dilepton $t\bar{t}$ events. The green (blue) dotted line corresponds to the correct (wrong) pairing of the leptons and the $b$-jets, while the red solid line is the average of those two distributions. The endpoint (4–81) of the $m_{b\ell}$ distribution is marked by the vertical red arrow.

The solution for the masses in terms of those three measurements is given by

$$M_0 = \frac{\sqrt{2} E_{221} E_{im} \left(2E_{221}E_{210}^2 + E_{221}E_{im}^2 - E_{im}^2 \sqrt{E_{221}^2 + 4E_{210}^2}\right)^{\frac{1}{2}}}{E_{221}^2 + 2E_{im}^2 - E_{221}\sqrt{E_{221}^2 + 4E_{210}^2}}, \quad (4–84)$$

$$M_1 = \frac{\sqrt{2} E_{221} E_{im} \left(E_{221}\sqrt{E_{221}^2 + 4E_{210}^2} - E_{im}^2\right)^{\frac{1}{2}}}{E_{221}^2 + 2E_{im}^2 - E_{221}\sqrt{E_{221}^2 + 4E_{210}^2}}, \quad (4–85)$$

$$M_2 = \frac{2E_{221} E_{im}^2}{E_{221}^2 + 2E_{im}^2 - E_{221}\sqrt{E_{221}^2 + 4E_{210}^2}}. \quad (4–86)$$

It is easy to check that substituting the measured values of the endpoints $E_{221}$, $E_{210}$ and $E_{im}$ from Equations (4–74), (4–76) and (4–82), into Equations (4–86) above yields the values for the neutrino, $W$ and top quark mass, correspondingly.
CHAPTER 5
ONE DIMENSIONAL PROJECTION METHOD

Most of mass reconstruction methods rely on exclusive channels where a sufficiently long decay chain can be properly identified. Unfortunately, this almost inevitably requires the use of hadronic jets in some form in the analysis – in most SUSY models, the main LHC signal is due to the strong production of colored superpartners, whose cascade decays to the neutral LSP necessarily involve hadronic jets. For many reasons, jets are notoriously difficult to deal with, especially in a hadron collider environment. Because of the high jet multiplicity in SUSY signal events, any jet-based analysis is bound to face a severe combinatorial problem and is unlikely to achieve any good precision. Thus it is imperative to have alternative methods which avoid the direct use of jets and instead rely only on the well measured momenta of any (isolated) leptons in the event.

In this chapter, we describe methods, which are free of the jet combinatorial problem. In the first two sections, we will use $M_{T2}$ and the last section, we will illustrate how to use full phase space information with contravese variable $M_{CT}$ as an example.

5.1 Detailed Study On $M_{T2}$’s Characteristics

For illustration in two sections, we shall use the standard example of $R$-parity conserving supersymmetry with a $\tilde{\chi}_0^0$ LSP. Its collider signatures have been extensively studied, and typically involve jets, leptons and missing transverse energy [89]. Among those, the inclusive same-sign dilepton channel has already been identified as a unique opportunity for an early SUSY discovery at the LHC [75, 90]. The two leptons of the same charge can be easily triggered on, and provide a good handle for suppressing the SM background. In our analysis we use the LM6 CMS study point [75], whose relevant mass spectrum is given in Table 5-1. At point LM6, signal events with two isolated same-sign leptons typically arise from the SUSY event topology in Figure 5-1.
Table 5-1. Selected sparticle masses (in GeV) at point LM6. We list the average $\tilde{q}_L$ mass $M_{\tilde{q}_L} = \frac{1}{2}(M_{\tilde{u}_L} + M_{\tilde{d}_L})$.

<table>
<thead>
<tr>
<th></th>
<th>$M_{\tilde{g}}$</th>
<th>$M_{\tilde{q}_L}$</th>
<th>$M_{\tilde{\chi}^+_1}$</th>
<th>$M_{\tilde{\ell}_L}$</th>
<th>$M_{\tilde{\nu}_L}$</th>
<th>$M_{\tilde{\chi}_0^1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>939.8</td>
<td>862</td>
<td>305.3</td>
<td>291.0</td>
<td>275.7</td>
<td>158.1</td>
</tr>
</tbody>
</table>

Figure 5-1. The typical SUSY event topology producing two isolated same-sign leptons at point LM6 (see text for details). The diagram for a pair of negatively charged leptons $\ell^-\ell^-$ is analogous.

Consider the inclusive production of same-sign charginos, which decay leptonically as shown in the yellow-shaded box in Figure 5-1. The resulting sneutrino ($\tilde{\nu}_L$) could be the LSP itself, or, as in the case of LM6, may further decay invisibly to a neutrino $\nu$ and the true LSP $\tilde{\chi}_0^1$. Such same-sign chargino pairs typically result from squark decays, as indicated in Figure 5-1. In turn, the squarks may be produced directly through a $t$-channel gluino exchange, or indirectly in gluino decays. Note that the two same-sign leptons in Figure 5-1 are accompanied by a number of upstream objects (typically jets) which may originate from various sources, e.g. initial state radiation, squark decays, or decays of even heavier particles up the decay chain. In order to stay clear of jet combinatorial issues, we shall adopt a fully inclusive approach to the same-sign dilepton signature, by treating all the upstream objects within the black rectangular frame in Figure 5-1 as a single entity of total transverse momentum $\vec{P}_T$.

Given this very general setup, we now pose the following question: assuming that a SUSY discovery is made in the inclusive same-sign dilepton channel, is it possible to measure the individual sparticle masses $M_p$ and $M_c$ involved in the leptonic decays
of Figure 5-1, using only the transverse momenta of the two leptons $\vec{p}_{\ell T}^{(1)}$ and $\vec{p}_{\ell T}^{(2)}$, and the total upstream transverse momentum $\vec{P}_T$? Although it may appear that those three vectors do not provide a lot of information to go on, we shall show that this is possible. We discuss three different approaches.

**Method I.** Let us concentrate directly on the observed lepton momenta $\vec{p}_{\ell T}^{(i)}$. Consider the two collinear momentum configurations illustrated in Figure 5-2 and defined as follows. In each configuration, the lepton momenta are the same: $\vec{p}_{\ell T}^{(1)} = \vec{p}_{\ell T}^{(2)}$, and then they can be either parallel or anti-parallel to the measured upstream $\vec{P}_T$:

$$s = +1 \Rightarrow \vec{p}_{\ell T}^{(1)} = \vec{p}_{\ell T}^{(2)} \uparrow \uparrow \vec{P}_T;$$

$$s = -1 \Rightarrow \vec{p}_{\ell T}^{(1)} = \vec{p}_{\ell T}^{(2)} \uparrow \downarrow \vec{P}_T.$$  

(5–1)

(5–2)

In what follows we shall use the integer $s = +1 (s = -1)$ to refer to the parallel (anti-parallel) configuration: $s \equiv \cos(\vec{p}_{\ell T}^{(1)}; \vec{P}_T) = \cos(\vec{p}_{\ell T}^{(2)}; \vec{P}_T)$. Now let us measure the maximum lepton momentum in each configuration:

$$p_{\ell T}(sP_T) \equiv \max_{\vec{p}_{\ell T}^{(i)} = \vec{p}_{\ell T}^{(2)} \wedge \cos(\vec{p}_{\ell T}^{(i)}; \vec{P}_T) = s} \left\{ p_{\ell T}^{(i)} \right\}. \quad (5–3)$$

Observe that both $p_{\ell T}(+P_T)$ and $p_{\ell T}(-P_T)$ can be directly measured from the lepton $\rho_T$ distributions. For example, construct a 2D scatter plot $\{x, y\}$ of

$$x = \cos(\vec{p}_{\ell T}^{(1)} + \vec{p}_{\ell T}^{(2)}, \vec{P}_T), \quad y = |\vec{p}_{\ell T}^{(1)} + \vec{p}_{\ell T}^{(2)}|,$$

(5–4)

with the cut $|\vec{p}_{\ell T}^{(1)} - \vec{p}_{\ell T}^{(2)}| < \epsilon (\sim 0)$, and take the limit

$$p_{\ell T}(sP_T) = \lim_{x \to s} \left( \frac{y}{2} \right). \quad (5–5)$$

Armed with the two measurements $p_{\ell T}(+P_T)$ and $p_{\ell T}(-P_T)$, we can now directly solve for the masses $M_p$ and $M_c$. The formula for $p_{\ell T}(sP_T)$ is

$$p_{\ell T}(sP_T) = \frac{M_p^2 - M_c^2}{4M_p^2} \left( \sqrt{4M_p^2 + (sP_T)^2} - sP_T \right). \quad (5–6)$$

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Inverting Equation (5–6), we get

\[ M_p = \frac{\sqrt{p_{\ell T}(-P_T) + p_{\ell T}(+P_T)}}{p_{\ell T}(-P_T) - p_{\ell T}(+P_T)} P_T, \]  

(5–7)

thus fixing the absolute mass scale in the problem. Once the parent mass \( M_p \) is known, the child mass \( M_c \) is

\[ M_c = M_p \sqrt{1 - 2 \frac{p_{\ell T}(-P_T) - p_{\ell T}(+P_T)}{P_T}}. \]  

(5–8)

Thus we found the true sparticle masses \( M_p \) and \( M_c \) directly in terms of the measured lepton momenta \( p_{\ell T}(\pm P_T) \) and upstream momentum \( P_T \). Note that the choice of the value for \( P_T \) in Equations (5–7) and (5–8) is arbitrary, which can be used to our advantage, e.g. to select the most populated \( P_T \) bin, minimizing the statistical error.

**Method II.** In our previous method, the lepton momenta \( p_{\ell T}(\pm P_T) \) were measured directly from the data as implied by Equation (5–5). Alternatively, we can obtain them indirectly from the endpoint of the Cambridge \( M_{T2} \) variable. To be more precise, we apply the “subsystem” \( M_{T2} \) variable introduced in [23] to the purely leptonic subsystem in the yellow-shaded box of Figure 5-1. Following the generic notation of Reference [23], we denote the input (test) mass of the sneutrino child as \( \tilde{M}_c \). The subsystem \( M_{T2} \) variable is now defined as follows. First form the transverse mass \( M_T \) for each (chargino) parent

\[ M_T^{(i)} = \sqrt{\tilde{M}_c^2 + 2 \left( |\vec{p}_{\ell T}^{(i)}| \sqrt{\tilde{M}_c^2 + |\vec{p}_{cT}^{(i)}|^2} - \vec{p}_{\ell T}^{(i)} \cdot \vec{p}_{cT}^{(i)} \right)}, \]  

Figure 5-2. The two special momentum configurations defined in Equations (5–1,5–2).
in terms of the assumed test mass $\tilde{M}_c$ and transverse momentum $\vec{p}_{cT}^{(i)}$ for each (sneutrino) child. Just like the traditional $M_{T2}$, the leptonic subsystem $M_{T2}$ variable [23] is defined through a minimization procedure over all possible partitions of the unknown children momenta $\vec{p}_{cT}^{(k)}$, consistent with transverse momentum conservation
\[
\sum_k (\vec{p}_{cT}^{(k)} + \vec{p}_{\ell T}^{(k)}) + \vec{p}_T = 0
\]

The $M_{T2}$ distribution has an upper kinematic endpoint

\[
M_{T2}^{\text{max}}(\tilde{M}_c, P_T) \equiv \max \{ M_{T2}(\tilde{M}_c, \vec{p}_T, \vec{p}_{\ell T}^{(i)}) \}, \quad (5-10)
\]

which can be experimentally measured and subsequently interpreted as the corresponding parent mass $\tilde{M}_p$

\[
\tilde{M}_p(\tilde{M}_c, P_T) \equiv M_{T2}^{\text{max}}(\tilde{M}_c, P_T), \quad (5-11)
\]

providing one functional relationship among $\tilde{M}_p$ and $\tilde{M}_c$, but leaving the individual masses still to be determined.

For us the importance of the $M_{T2}$ variable (5–9) is that the momentum configurations in Figure 5–2 are precisely the ones which determine its endpoint $M_{T2}^{\text{max}}$. The complete analytical dependence of the $M_{T2}$ endpoint $\tilde{M}_p(\tilde{M}_c, P_T)$ on both of its arguments $\tilde{M}_c$ and $P_T$ is now known [23]:

\[
\tilde{M}_p(\tilde{M}_c, P_T) = \begin{cases} 
\tilde{M}_p(\tilde{M}_c, +P_T), & \text{if } \tilde{M}_c \leq M_c, \\
\tilde{M}_p(\tilde{M}_c, -P_T), & \text{if } \tilde{M}_c \geq M_c,
\end{cases} \quad (5–12)
\]

where

\[
\tilde{M}_p(\tilde{M}_c, sP_T) = \left\{ \left[ p_{\ell T}(sP_T) + \sqrt{\left( p_{\ell T}(sP_T) + sP_{T} \right)^2 + \tilde{M}_c^2} \right]^2 - (sP_T)^2 \right\}^{\frac{1}{2}}. \quad (5–13)
\]

Thus we can alternatively obtain the sparticle masses by measuring just two $M_{T2}$ kinematic endpoints, with arbitrary choices for the test mass $\tilde{M}_c$ and the upstream
For concreteness, let us pick some fixed \( \tilde{M}_c' \) and \( P_T' \), form the corresponding \( M_{T2} \) distribution \( (5–9) \) and measure its endpoint \( \tilde{M}_p' \), also making a note of the configuration \( s' \):

\[
\left\{ \tilde{M}_c', P_T' \right\} \xrightarrow{\text{measure}} \left\{ \tilde{M}_p', s' \right\}.
\quad (5–14)
\]

Now perform a second such measurement

\[
\left\{ \tilde{M}_c''', P_T'' \right\} \xrightarrow{\text{measure}} \left\{ \tilde{M}_p'', s'' \right\}.
\quad (5–15)
\]

By inverting Equation \( (5–13) \), these two measurements allow the experimental determination of

\[
p_{\ell T}(s' P_T') = \frac{\tilde{M}_p^2 - \tilde{M}_c^2}{4 \tilde{M}_p^2} \left( \sqrt{4 \tilde{M}_p^2 + (s' P_T')^2} - s' P_T' \right)
\quad (5–16)
\]

and similarly for \( p_{\ell T}(s'' P_T'') \). Now taking the ratio

\[
r \equiv \frac{p_{\ell T}(s' P_T')}{p_{\ell T}(s'' P_T'')} = \frac{\sqrt{4 \tilde{M}_p^2 + (s' P_T')^2} - s' P_T'}{\sqrt{4 \tilde{M}_p^2 + (s'' P_T'')^2} - s'' P_T''},
\quad (5–17)
\]

where in the second step we used Equation \( (5–6) \), we can solve Equation \( (5–17) \) for the true parent mass \( M_p \) in terms of measured quantities:

\[
M_p = \left\{ \frac{-r s' P_T' s'' P_T''}{(1 - r)^2} \left( r - \frac{s' P_T'}{s'' P_T''} \right) \left( r - \frac{s'' P_T''}{s' P_T'} \right) \right\}^{\frac{1}{2}},
\quad (5–18)
\]

and then find the true child mass \( M_c \) from Equation \( (5–6) \) as

\[
M_c = M_p \left[ 1 - \left( 1 - \frac{\tilde{M}_c^2}{\tilde{M}_c^2} \right) \frac{\sqrt{4 \tilde{M}_p^2 + (s' P_T')^2} - s' P_T'}{\sqrt{4 \tilde{M}_p^2 + (s'' P_T'')^2} - s'' P_T''} \right]^{\frac{1}{2}}
\quad (5–19)
\]

with \( M_p \) already given by Equation \( (5–18) \). Note than in this method, the values of \( \tilde{M}_c' \), \( \tilde{M}_c'' \), \( P_T' \) and \( P_T'' \) can be chosen at will, allowing for repeated measurements of \( M_p \) and \( M_c \).

**Method III.** The third and final method for extracting the two masses \( M_p \) and \( M_c \) will make use of the celebrated “kink” in the \( M_{T2} \) endpoint function \( (5–12) \).
Figure 5-3. $M_{T2}^{\text{max}}$ versus the test mass $\tilde{M}_c$, as obtained in our simulations (data points) from a sample with $P_T = 420 \pm 50$ GeV, or theoretically from Equation (5–12) (blue solid line), as well as their difference (lower panel).

Since $\tilde{M}_p(\tilde{M}_c, +P_T)$ and $\tilde{M}_p(\tilde{M}_c, -P_T)$ have different slopes at the crossover point $\tilde{M}_c = M_c$, the function $\tilde{M}_p(\tilde{M}_c, P_T)$ has a slope discontinuity precisely at the correct value $M_c$ of the child mass, providing an alternative measurement of the absolute mass scale. The procedure is illustrated in Figure 5-3 for the LM6 study point of Table 5-1.

The blue solid line shows the theoretically expected shape from Equation (5–12), for $P_T = 420$ GeV, which is roughly the mean of the $P_T$ distribution at point LM6. In the LM6 case the kink is very mild, only 3.3° [23].

In order to test the precision of the three methods, we perform event simulations using the PYTHIA event generator [54] and PGS detector simulation [55]. We consider the LHC at its nominal energy of 14 TeV and 100 fb$^{-1}$ of data. To ensure discovery, we use standard CMS cuts as follows [75, 91]: exactly two isolated leptons with $p_T > 10$ GeV, at least three jets with $p_T > (175, 130, 55)$ GeV, $E_T > 200$ GeV and a veto on tau jets. With those cuts, in the dimuon channel alone, the remaining SM background cross-section is rather negligible (0.15 fb), while the SUSY signal is 14 fb, already
leading to a $22\sigma$ discovery with just $10 \text{ fb}^{-1}$ of data [75, 91]. In order to compare to the theoretical result in Figure 5-3, we select a $\pm 50 \text{ GeV} \ P_T$ bin around $P_T = 420 \text{ GeV}$ and construct a series of $M_{T2}$ distributions, for different input values of $\tilde{M}_c$. For each case, we include all SM and SUSY combinatorial backgrounds, and extract the $M_{T2}^{\text{max}}$ endpoint by a linear unbinned maximum likelihood fit, obtaining the data points shown in Figure 5-3. We see that the $M_{T2}$ endpoint can be determined rather well ($\delta \tilde{M}_p \lesssim 3 \text{ GeV}$), but only on the right branch $\tilde{M}_c \geq M_c$. In contrast, the $M_{T2}$ endpoints on the left branch $\tilde{M}_c \leq M_c$ are considerably underestimated, washing out the expected kink.

There are two separate reasons behind this effect. Recall that the $M_{T2}$ endpoint on the left branch is obtained in the configuration $s = +1$ of Figure 5-2, which requires the lepton to be emitted in the backward direction. As a result, the parent boost favors configurations with $s \simeq -1$ over $s \simeq +1$. Another consequence is that leptons with $s \simeq +1$ are softer and more easily rejected by the offline $p_T$ cuts. We conclude that $M_{T2}^{\text{max}}$ measurements on the left branch are in general not very reliable, and tend to jeopardize the traditional kink method. For example, using Method III to fit the data in Figure 5-3 (green dotted line), we find best fit values of only $M_{p(\text{fit})} = 212 \text{ GeV}$ and $M_{c(\text{fit})} = 188 \text{ GeV}$. Method I has a similar problem, since $p_{\ell T}(+P_T)$ is measured from events in the $s = +1$ configuration. Using the $\tilde{M}_p$ measurements from Figure 5-3 at $\tilde{M}_c = 0$ and $\tilde{M}_c = 1 \text{ TeV}$, we find from Equation (5–16) that $p_{\ell T}(+420 \text{ GeV}) = 8.8 \text{ GeV}$ and $p_{\ell T}(-420 \text{ GeV}) = 50.6 \text{ GeV}$ (compare to the nominal values of 14.8 GeV and 53.6 GeV, correspondingly). The resulting mass determination via Equations (5–7, 5–8) is $M_{p(\text{fit})} = 212 \text{ GeV}$ and $M_{c(\text{fit})} = 190 \text{ GeV}$. We see that in both Method I and Method III, the masses are underestimated due to the systematic underestimation of the left $M_{T2}^{\text{max}}$ branch in Figure 5-3. It is therefore of great interest to have an alternative method, which relies on the right $M_{T2}^{\text{max}}$ branch alone.
Figure 5-4. Scaling factors relating the error $\delta \tilde{M}_p$ in the extraction of the $M_{T2}$ endpoint to the resulting uncertainties $\delta M_p$ and $\delta M_c$ on the parent and child masses calculated from (5–18) and (5–19), as a function of the true input masses $M_c$ and $M_p$.

This is where the available freedom in Method II comes into play, since both test masses $\tilde{M}_c'$ and $\tilde{M}_c''$ can be chosen on the right branch. Taking $P'_T = 350 \pm 50$ GeV and $P''_T = 500 \pm 50$ GeV and repeating our earlier analysis, we find that $\delta \tilde{M}_p$ on the right branch is still on the order of 3 GeV, as in Figure 5-3. The resulting error $\delta M_p$ ($\delta M_c$) on the measured parent (child) mass can be easily propagated from Equations (5–18,5–19). The two ratios $\delta M_p/\delta \tilde{M}_p$ and $\delta M_c/\delta \tilde{M}_p$ are shown in Figure 5-4, where for concreteness we have taken $\tilde{M}_c' = \tilde{M}_c'' = 1000$ GeV. Figure 5-4 reveals that the LM6 input values of $M_c$ and $M_p$ are rather unlucky, since the error $\delta \tilde{M}_p$ on the $M_{T2}$ endpoint is then amplified by a factor of almost 70. However, if $M_c$ and $M_p$ happened to be different, with the rest of the spectrum the same, the precision quickly improves. For example, with $\delta \tilde{M}_p = \pm 3$ GeV, the masses can be determined to within $\pm 30$ GeV ($\pm 75$ GeV) within the yellow (orange) region. One should keep in mind that the dominant
uncertainty on $\delta \tilde{M}_p$ is due to the SUSY combinatorial background. We have verified that in the absence of such combinatorial background, $\delta \tilde{M}_p \lesssim 1$ GeV and the typical precision on $M_p$ and $M_c$ from Figure 5-4 is then at the level of 10%.

5.2 Using the $P_T$ of the Upstream Jet with $M_{T2}$ Method

Unfortunately, in order to apply previous method, one must work with a subset of events within a relatively narrow fixed $P_T$ range of upstream objects (including jets), incurring some loss in statistics. To be more general, we treat all upstream particles into a single sector as “upstream object $U$”, and denote its total transverse momentum $\tilde{U}_T$ as in Figure 5-5. In this section, we propose a new method which uses the full data set. As in the previous section, we define the endpoint $M_{T2}^{\text{max}}$ of $M_{T2}$ distribution as the parent mass $\tilde{M}_p$ with the trial child mass $\tilde{M}_c$ for a given $U_T$, :

$$\tilde{M}_p(\tilde{M}_c, U_T) \equiv M_{T2}^{\text{max}}(\tilde{M}_c, U_T).$$

(5–20)

Here we propose to obtain a relation by using the property that the function $\tilde{M}_p(\tilde{M}_c, U_T)$ is independent of $U_T$ at the true child mass $M_c$:

$$\tilde{M}_p(M_c, U_T + \Delta U_T) - \tilde{M}_p(M_c, U_T) = 0, \forall \Delta U_T,$$

(5–21)

which we can rewrite more informatively as

$$\tilde{M}_p(\tilde{M}_c, U_T) - \tilde{M}_p(\tilde{M}_c, 0) \geq 0,$$

(5–22)

with equality being achieved only for $\tilde{M}_c = M_c$. Equation (5–22) implies that, for any given $\tilde{M}_c$, there will always be a certain number of events whose $M_{T2}$ values will exceed the reference value $\tilde{M}_p(\tilde{M}_c, 0)$, unless the trial mass $\tilde{M}_c$ happens to coincide with the true child mass $M_c$. In order to quantify this effect, we define the function

$$N(\tilde{M}_c) \equiv \sum_{\text{all events}} H \left( M_{T2} - \tilde{M}_p(\tilde{M}_c, 0) \right).$$

(5–23)
Figure 5-5. The generic event topology under consideration. All particles visible in the detector are clustered into three groups: upstream objects $U$ with total transverse momentum $\vec{U}_T$, and two composite visible particles $V_i$ ($i = 1, 2$), each with invariant mass $m_i$ and total transverse momentum $\vec{p}_iT$.

$H(x)$ is the Heaviside step function. From the definition of $N(\tilde{M}_c)$ it is clear that it is minimized at $\tilde{M}_c = M_c$, where in theory we would expect

$$N_{\text{min}} \equiv \min\{N(\tilde{M}_c)\} = N(M_c) = 0.$$  \hspace{1cm} (5–24)

In reality, the value of $N_{\text{min}}$ will be lifted from 0, due to finite particle width effects, detector resolution, etc. Nevertheless we expect that the location of the $N(\tilde{M}_c)$ minimum will still be at $\tilde{M}_c = M_c$, allowing a direct measurement of the child mass $M_c$:

$$M_c = \left\{ \tilde{M}_c \mid N(\tilde{M}_c) = N_{\text{min}} \right\},$$  \hspace{1cm} (5–25)

which is our first main result. Once the child mass $M_c$ is found from Equation (5–25), the true parent mass $M_p$ is obtained as usual from Equation (5–20) as $M_p = \tilde{M}_p(M_c, U_T)$.

At this point it is not clear whether we have gained anything statistics-wise, since the reference quantity $\tilde{M}_p(\tilde{M}_c, 0)$ appearing in the definition of Equation (5–23) has to be measured at a fixed $U_T = 0$ anyway. Our second main result is that $\tilde{M}_p(\tilde{M}_c, 0)$ can in fact be measured from the full data set with no loss in statistics as follows.
Step I. Orthogonal decomposition of the observed transverse momenta with respect to the $\vec{U}_T$ direction. The Tevatron and LHC collaborations currently use fixed axes coordinate systems to describe their data. Instead, we propose to rotate the coordinate system from one event to another, so that the transverse axes are always aligned with the direction $T_\parallel$ selected by the measured upstream transverse momentum vector $\vec{U}_T$ and the direction $T_\perp$ orthogonal to it (Figure 5-6). The visible transverse momentum vectors from Figure 5-5 are then decomposed as

$$\vec{p}_T = -\vec{p}_{1T} - \vec{p}_{2T} - \vec{U}_T$$

Step II. Constructing the transverse and longitudinal contransverse masses $M_{T2\parallel}$ and $M_{T2\perp}$.

Now we define 1D $M_{T2}$ decompositions in complete analogy with the standard $M_{T2}$ definition of Equation (5–9):

$$M_{T2\parallel} \equiv \min_{\vec{p}_{1T\parallel} + \vec{p}_{2T\parallel} = \vec{p}_T} \left\{ \max\left\{ M_{1T\parallel}, M_{2T\parallel} \right\} \right\}, \quad (5–28)$$

$$M_{T2\perp} \equiv \min_{\vec{p}_{1T\perp} + \vec{p}_{2T\perp} = \vec{p}_T} \left\{ \max\left\{ M_{1T\perp}, M_{2T\perp} \right\} \right\}. \quad (5–29)$$
These decompositions are extremely useful. For once, the 1D variables in Equations (5–28, 5–29) can be calculated via simple analytic expressions as shown below. In contrast, a general formula for the original $M_{T2}$ variable in Equation (5–9) in the presence of arbitrary $U_T$ is unknown and one still has to compute $M_{T2}$ numerically [38]. More importantly, $M_{T2\perp}$ allows us to measure the reference quantity $\tilde{M}_p(\tilde{M}_c, 0)$ in Equation (5–23) from the full data set, using events with any value of $U_T$.

To understand the basic idea, it is sufficient to consider the simplest, yet most challenging case of a single step decay chain. Let $V_i$ be a single, (approximately) massless SM particle: $m_1 = m_2 = 0$. (The discussion for the massive case proceeds analogously.) In what follows, for illustration we shall use the same-sign dilepton channel in supersymmetry, where each $V_i$ is a lepton resulting from a chargino decay to a sneutrino [39]. The charginos themselves are produced indirectly in the decays of squarks and gluinos. For concreteness we shall use a SUSY spectrum given by the LM6 CMS study point as in Table 5-1.

In our simulations we use the PYTHIA event generator [54] and the PGS detector simulation program [55].

The variable $M_{T2\perp}$ has several unique properties. Eventwise, it can be calculated analytically as

$$M_{T2\perp} = \sqrt{A_{T\perp}^2 + \sqrt{A_{T\perp}^2 + \tilde{M}_c^2}},$$

$$A_{T\perp} = \frac{1}{2} (|\vec{p}_1 T\perp| |\vec{p}_2 T\perp| + \vec{p}_1 T\perp \cdot \vec{p}_2 T\perp).$$

The endpoint of the $M_{T2\perp}$ distribution is given by

$$M_{T2\perp}^{max}(\tilde{M}_c) = \mu + \sqrt{\mu^2 + \tilde{M}_c^2},$$

in terms of the parameter $\mu$ introduced in [23]

$$\mu = \frac{M_p}{2} \left( 1 - \frac{M_c^2}{M_p^2} \right).$$
Figure 5-7. The unit-normalized $M_{T2\perp}$ distribution (5–53) for the same-sign dilepton channel in a SUSY model with LM6 CMS mass spectrum and a choice of test mass $\tilde{M}_c = 100$ GeV. The yellow shaded distribution shows the theoretically predicted shape (5–53), matching very well the parton level result from PYTHIA with no cuts (red histogram). The green (blue) histogram is the corresponding result after PGS detector simulation with mild (hard) cuts as explained in the text. The endpoint expected from Equation (5–31) is 132.1 GeV and is marked with the vertical arrow.

Equation (5–31) reveals perhaps the most important feature of the $M_{T2\perp}$ variable: its endpoint is independent of the upstream $P_T$ and can thus be measured with the whole data sample. We can even predict analytically the shape of the (unit-normalized) differential $M_{T2\perp}$ distribution

$$
\frac{dN}{dM_{T2\perp}} = N_{0\perp}(M_{T2\perp} - \tilde{M}_c) + (1 - N_{0\perp}) \frac{d\tilde{N}}{dM_{T2\perp}},
$$

(5–33)

where $N_{0\perp}$ is the fraction of events in the lowest $\tilde{M}_c$ bin $M_{T2\perp} = \tilde{M}_c$, while the shape of the remaining (unit-normalized) $M_{T2\perp}$ distribution is given by (Figure 5-7)

$$
\frac{d\tilde{N}}{dM_{T2\perp}} = \frac{M_{T2\perp}^4}{\mu^2 M_{T2\perp}^3} \ln \left( \frac{2\mu M_{T2\perp}}{M_{T2\perp}^2 - \tilde{M}_c^2} \right).
$$

(5–34)
Figure 5-8. Observable $M_{T2\perp}$ distribution after hard cuts for 100 fb$^{-1}$ of LHC data. The total stacked distribution consists of the SUSY signal (red) and the SM background (blue). The solid line is the result of a simple linear fit, revealing endpoints at 134.4 GeV and 172.4 GeV.

Notice that this shape does not depend on any unknown kinematic parameters, such as the unknown center-of-mass energy or longitudinal momentum of the initial hard scattering. It is also insensitive to spin correlation effects, whenever the upstream momentum results from production and/or decay processes involving scalar particles (e.g. squarks) or vectorlike couplings (e.g. the QCD gauge coupling). It is even independent of the actual value of the upstream momentum $P_T$. Thus we are not restricted to a particular $P_T$ range and can use the whole event sample in the $M_{T2\perp}$ analysis. For any choice of $\tilde{M}_c$ (in Figure 5-7 we used $\tilde{M}_c = 100$ GeV), Equation (5–53) is a one-parameter curve which can be fitted to the data to obtain the parameter $\mu$ and from there the $M_{T2\perp}$ endpoint (5–31).

As always, there are practical limitations to the use of such shape fitting. First, the shape in Equation (5–53) is modified in the presence of “mild” cuts, which are required for lepton identification in PGS (green histogram in Figure 5-7), and more importantly,
for the discovery of the same-sign dilepton SUSY signal over the SM backgrounds. To ensure discovery, we use “hard” cuts as follows [75, 91]: exactly two isolated leptons with $p_T > 10$ GeV, at least three jets with $p_T > (175, 130, 55)$ GeV, $P_T > 200$ GeV and a veto on tau jets. With those cuts, in the dimuon channel alone, the remaining SM background cross-section is dominated by $t\bar{t}$ and is just 0.15 fb, while the SUSY signal is 14 fb, leading to a $22\sigma$ discovery with just $10 \text{ fb}^{-1}$ of data [75, 91]. The distortion of the $M_{T2\perp}$ shape with these hard offline cuts is illustrated by the blue (rightmost) histogram in Figure 5-7. The actual $M_{T2\perp}$ distribution which we expect to observe with $100 \text{ fb}^{-1}$ of data, is shown in Figure 5-8 and is comprised of a relatively small SM background component (blue) and a dominant SUSY signal component (red). In spite of the presence of a sizable SUSY combinatorial background, the $M_{T2\perp}$ endpoint expected from Figure 5-7 is clearly visible and its location from a simple linear fit is obtained as 134.4 GeV, which is very close to the nominal value of 132.1 GeV. (Interestingly, the data reveals a second endpoint at 172.4 GeV, which is due to events in which one chargino decays through a charged slepton: $\tilde{\chi}_1^+ \rightarrow \tilde{\ell}_L^\pm \rightarrow \tilde{\chi}_1^0$ [39]. Its nominal value is 169.2 GeV.)

Our final key observation is that

$$\tilde{M}_p(\tilde{M}_c, 0) = M^{\text{max}}_{T2}(\tilde{M}_c, 0) = M^{\text{max}}_{T2\perp}(\tilde{M}_c). \quad (5–35)$$

which allows to rewrite the function $N(\tilde{M}_c)$ of Equation (5–23) as

$$N(\tilde{M}_c) \equiv \sum_{\text{all events}} H\left( M_{T2} - M^{\text{max}}_{T2\perp}(\tilde{M}_c) \right). \quad (5–36)$$

The $M_{T2\perp}$ analysis just described allows a very precise measurement of the benchmark quantity $M^{\text{max}}_{T2\perp}(\tilde{M}_c)$ appearing in Equation (5–36), so that the function $N(\tilde{M}_c)$ itself can be reliably reconstructed, using the whole event sample all the way throughout the analysis, without any loss in statistics.
Figure 5-9. The function $\hat{N}(\tilde{M}_c)$ defined in Equation (5–37). The blue (red) set of measurements are with (without) SUSY combinatorial background. The error bars shown are purely statistical.

We show our result in Figure 5-9, where for convenience we unit-normalize the function $N(\tilde{M}_c)$ as

$$\hat{N}(\tilde{M}_c) = N(\tilde{M}_c)/\langle N(\tilde{M}_c) \rangle,$$

where the averaging is performed over the plotted range of $\tilde{M}_c$. As expected, the function $\hat{N}(\tilde{M}_c)$ exhibits a minimum in the vicinity of the true sneutrino mass $\tilde{M}_c = M_c = 275.7$ GeV. Ignoring the SUSY combinatorial background, this measurement (red data points) is quite precise, at the level of a few percent. In order to reduce the combinatorial background, we select events with $\tilde{M}_c < M_{T2\perp} < M_{T2\perp}^{\text{max}}$ and veto very hard leptons with $p_T > 60$ GeV. The resulting $M_c$ measurement (blue data points) is at the level of 10%. This precision is clearly sufficient to exclude SM neutrinos as the source of the missing energy, hinting at a potential dark matter discovery at the LHC. Note that the traditional

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1 The measured value of $M_{T2\perp}^{\text{max}}$ in Figure 5-8 already implies that the mass splitting $M_\rho - M_c$ is on the order of 30 GeV, resulting in a rather soft lepton $p_T$ spectrum.
“kink” for the LM6 study point is only a few degrees and appears difficult to observe experimentally [39], unlike the result in Figure 5-9.

For completeness, we now discuss the properties of the variable $M_{T2\parallel}$. In any given event, it can be calculated as follows. If $\vec{p}_{1\parallel} \cdot \vec{p}_{2\parallel} \leq 0$, then simply $M_{T2\parallel} = \tilde{M}_c$, and the event falls in the lowest $\tilde{M}_c$ bin. In the alternative case of $\vec{p}_{1\parallel} \cdot \vec{p}_{2\parallel} > 0$, $M_{T2\parallel}$ can be found from

$$M^2_{T2\parallel} = \left( |\vec{p}_{1\parallel}| + \sqrt{\tilde{M}_c^2 + |\vec{p}_{cT\parallel}|^2} \right)^2 - |\vec{p}_{1\parallel} + \vec{p}_{cT\parallel}|^2,$$

where the child test momentum is

$$\vec{p}_{cT\parallel}^{(1)} = -\frac{1}{2} \left[ \vec{U}_T + (1 - \alpha) \vec{p}_{1\parallel} + (1 + \alpha) \vec{p}_{2\parallel} \right],$$

$$\alpha \equiv \sqrt{\frac{\tilde{M}_c^2}{|\vec{p}_{1\parallel} \cdot \vec{p}_{2\parallel}|} + \frac{|\vec{U}_T + \vec{p}_{1\parallel} + \vec{p}_{2\parallel}|^2}{|\vec{p}_{1\parallel} + \vec{p}_{2\parallel}|^2}}.$$

The endpoint $M^\text{max}_{T2\parallel}$ of the $M_{T2\parallel}$ distribution is identical to the endpoint (5–20) of the $M_{T2}$ distribution itself:

$$M^\text{max}_{T2\parallel} = M^\text{max}_{T2}(\tilde{M}_c, U_T) = \tilde{M}_p(\tilde{M}_c, U_T), \quad (5–38)$$

and is thus known as a function of $U_T$ [23]

$$M^\text{max}_{T2\parallel} = \left\{ \left[ \mu(sU_T) + \sqrt{\left( \mu(sU_T) + \frac{sU_T}{2} \right)^2 + \tilde{M}_c^2} \right]^2 - \frac{(sU_T)^2}{4} \right\}^{1/2}, \quad (5–39)$$

where

$$\mu(sU_T) \equiv \mu \left( \sqrt{1 + \left( \frac{sU_T}{2M_p} \right)^2} - \frac{sU_T}{2M_p} \right),$$

and $s = \pm 1$ is an integer defined as $s \equiv \text{sgn}(\tilde{M}_c - \tilde{M}_c)$. Unfortunately, the shape of the $M_{T2\parallel}$ distribution cannot be predicted in a model-independent way, since it is sensitive to the underlying $\sqrt{\xi}$, as well as to the measured $U_T$. Nevertheless, one can imagine several useful applications of $M_{T2\parallel}$. For example, one can study the boundary of the two-dimensional \{ $M_{T2\perp}, M_{T2\parallel}$ \} distribution as we will do with one dimensional projected variables of $M_{CT}$. One could also consider an analogue of (5–36), defined in terms of
\[ M_{T2∥} : \]
\[ N_{∥}(\tilde{M}_c) \equiv \sum_{\text{all events}} H \left( M_{T2∥} - M_{T2∥}^{\max}(\tilde{M}_c) \right), \quad (5-40) \]

whose minimum will also mark the location of the true child mass \( M_c \).

5.3 Using Full Phase Space Information With \( M_{CT} \)

In this section, we show how to get more information from the boundary of phase space, with 1D-decomposed \( M_{CT} \) variable. Compared to \( M_{T2} \) approaches, \( M_{CT} \) method has two advantages. First, it is simpler – it uses only the observed objects \( U, V_1 \) and \( V_2 \) in the event and makes no reference to the missing particle kinematics (or mass). Second, it is more precise, since it utilizes the whole kinematic boundary of the relevant two-dimensional distribution and not just the kinematic endpoint of its one-dimensional projection.

**Step I.** Constructing the transverse and longitudinal contransverse masses \( M_{CT⊥} \) and \( M_{CT∥} \). Our starting point is the original contransverse mass variable \[ M_{CT} = \sqrt{m_1^2 + m_2^2 + 2(e_{1T}e_{2T} + \vec{p}_{1T} \cdot \vec{p}_{2T})}, \quad (5-41) \]

where \( e_{iT} \) is the “transverse energy” of \( V_i \)

\[ e_{iT} = \sqrt{m_i^2 + |\vec{p}_{iT}|^2}. \quad (5-42) \]

For events with \( U_T = 0 \), \( M_{CT} \) has an upper endpoint which is insensitive to the unknown \( \sqrt{s} \), providing one relation among \( M_p \) and \( M_c \) \[ M_{CT}^{\max}(U_T = 0) = \sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cosh(\zeta_1 + \zeta_2)}, \quad (5-43) \]

where

\[ \sinh \zeta_i = \frac{\lambda \left( M_p^2, M_c^2, m_i^2 \right)}{2M_pm_i}, \quad (5-44) \]

\[ \lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \quad (5-45) \]
Unfortunately, the $U_T = 0$ limit is not particularly interesting at hadron colliders (especially for inclusive studies), since a significant amount of upstream $U_T$ is typically generated by ISR (and other) jets. One possible fix is to use all events, but modify the definition (5–41) to approximately compensate for the transverse $\bar{U}_T$ boost [43]. One then recovers a distribution whose endpoint is still given by Equation (5–43).

Alternatively, one could stick to the original $M_{CT}$ variable, and simply account for the $U_T$ dependence of its endpoint as

$$M_{CT}^{max}(U_T) = \sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cosh(2\eta + \zeta_1 + \zeta_2)} \tag{5–46}$$

where $\zeta_i$ were already defined in Equation (5–44), and

$$\sinh \eta \equiv \frac{U_T}{2M_p}, \quad \cosh \eta \equiv \sqrt{1 + \frac{U_T^2}{4M_p^2}}. \tag{5–47}$$

Our approach here is to utilize the one-dimensional projections from Equations (5–26,5–27) and construct one-dimensional analogues of the $M_{CT}$ variable

$$M_{CT\perp} \equiv \sqrt{m_1^2 + m_2^2 + 2(e_1^T e_2^T + \vec{p}_1^T \cdot \vec{p}_2^T)}, \tag{5–48}$$
$$M_{CT\parallel} \equiv \sqrt{m_1^2 + m_2^2 + 2(e_1^\parallel e_2^\parallel + \vec{p}_1^\parallel \cdot \vec{p}_2^\parallel)}, \tag{5–49}$$

where the corresponding “transverse energies” are

$$e_{iT\perp} \equiv \sqrt{m_i^2 + |\vec{p}_{iT\perp}|^2}, \quad e_{iT\parallel} \equiv \sqrt{m_i^2 + |\vec{p}_{iT\parallel}|^2}. \tag{5–50}$$

The benefit of the decomposition as Equations (5–48,5–49) is that one gets “two for the price of one”, i.e. two independent and complementary variables instead of the single variable in Equation (5–41).

The variable $M_{CT\perp}$ in particular is very useful for our purposes. To illustrate the basic idea, it is sufficient to consider the most common case, where $V_i$ is approximately massless ($m_i = 0$), as the leptons in our $t\bar{t}$ example. A crucial property of $M_{CT\perp}$ is that
its endpoint is independent of $U_T$:

$$M^\text{max}_{\rm CT} = \frac{M_p^2 - M_c^2}{M_p}, \quad \forall U_T. \quad (5-51)$$

In fact the whole $M_{\text{CT}}$ distribution is insensitive to $U_T$:

$$\frac{dN}{dM_{\text{CT}}} = N_0 \delta(M_{\text{CT}}) + (N_{\text{tot}} - N_0) \frac{d\tilde{N}}{dM_{\text{CT}}}, \quad (5-52)$$

where $N_0$ is the number of events in the zero bin $M_{\text{CT}} = 0$. Using phase space kinematics, we find that the shape of the remaining (unit-normalized) zero-bin-subtracted distribution is simply given by

$$\frac{d\tilde{N}}{d\hat{M}_{\text{CT}}} \equiv -4 \hat{M}_{\text{CT}} \ln \hat{M}_{\text{CT}} \quad (5-53)$$

in terms of the unit-normalized $M_{\text{CT}}$ variable

$$\hat{M}_{\text{CT}} \equiv \frac{M_{\text{CT}}}{M_{\text{CT}}^\text{max}}. \quad (5-54)$$

The observable $M_{\text{CT}}$ distribution for our $t\bar{t}$ example is shown in Figure 5-10, for 10 fb$^{-1}$ of LHC data at 7 TeV. Events were generated with PYTHIA [54] and processed with the PGS detector simulator [55]. We apply standard background rejection cuts as follows [92]: we require two isolated, opposite sign leptons with $p_{T} > 20$ GeV, $m_{\ell^+\ell^-} > 12$ GeV, and passing a $Z$-veto $|m_{\ell^+\ell^-} - M_Z| > 15$ GeV; at least two central jets with $p_{T} > 30$ GeV and $|\eta| < 2.4$; and a $E_T$ cut of $E_T > 30$ GeV ($E_T > 20$ GeV) for events with same flavor (opposite flavor) leptons. We also demand at least two $b$-tagged jets, assuming a flat $b$-tagging efficiency of 60%. With those cuts, the SM background from other processes is negligible [92].

Figure 5-10 demonstrates that the $M_{\text{CT}}$ endpoint can be measured quite well. Since the theoretically predicted shape of Equation (5–53) is distorted by the cuts, we use a linear slope with Gaussian smearing, and fit for the endpoint and the resolution parameter.
Figure 5-10. Zero-bin subtracted $M_{CT_{\perp}}$ distribution after cuts, for $t\bar{t}$ dilepton events. The yellow (lower) portion is our signal, while the blue (upper) portion shows $t\bar{t}$ combinatorial background with isolated leptons arising from $\tau$ or $b$ decays.

We find $M_{CT_{\perp}}^{\text{max}} = 80.9$ GeV (compare to the true value $M_{CT_{\perp}}^{\text{max}} = 80.4$ GeV), which gives one constraint (5–51) among $M_{p}$ and $M_{c}$. At this point, a second, independent constraint can in principle be obtained from an analogous measurement of the $M_{CT_{\perp}}^{\text{max}}$ endpoint in Equation (5–46) at a fixed value of $U_{T}$ (resulting in loss in statistics), after which the two masses can be found from

\[
M_{p} = \frac{U_{T} M_{CT_{\perp}}^{\text{max}}(U_{T})}{(M_{CT_{\perp}}^{\text{max}}(U_{T}))^2 - (M_{CT_{\perp}}^{\text{max}})^2},
\]

\[
M_{c} = \sqrt{M_{p} (M_{p} - M_{CT_{\perp}}^{\text{max}})}.
\]

However, the orthogonal decomposition as Equations (5–48,5–49) offers another approach, which we pursue in the last step.

**Step II.** Fitting to kinematic boundary lines. It is known that two-dimensional correlation plots reveal a lot more information than one-dimensional projected histograms [22]. To this end, consider the scatter plot of $M_{CT_{\perp}}$ vs $M_{CT_{\parallel}}$ in Figure 5-11(a), where for illustration we used 10,000 events at the parton level.
Figure 5-11. Scatter plots of (a) $M_{CT \perp}$ versus $M_{CT \parallel}$ and (b) $M_{CT \perp}$ versus $M_{CT}$, for a fixed representative value $U_T = 75$ GeV. The solid lines show the corresponding boundaries defined in (5–58) and (5–61), for the correct value of $M_{CT \perp}^{\text{max}}$ and several different values of $M_p$ as shown.

For a given value of $M_{CT \perp}$, the allowed values of $M_{CT \parallel}$ are bounded by

$$M_{CT \parallel}^{(\text{lo})}(M_{CT \perp}) \leq M_{CT \parallel} \leq M_{CT \parallel}^{(\text{hi})}(M_{CT \perp}), \quad (5-57)$$

where $M_{CT \parallel}^{(\text{lo})}(M_{CT \perp}) = 0$ and

$$M_{CT \parallel}^{(\text{hi})}(M_{CT \perp}) = M_{CT \parallel}^{\text{max}} \left( \sqrt{1 - \frac{M_{CT \perp}^2}{M_p^2}} \cosh \eta + \sinh \eta \right). \quad (5-58)$$

Figure 5-11(a) reveals that the endpoint $M_{CT \parallel}^{\text{max}}$ of the one-dimensional $M_{CT \parallel}$ distribution is obtained at $M_{CT \perp} = 0$

$$M_{CT \parallel}^{\text{max}} = M_{CT \parallel}^{(\text{hi})}(0) = M_{CT \parallel}^{\text{max}}(\cosh \eta + \sinh \eta) = \frac{1}{2} \left( 1 - \frac{M_e^2}{M_p^2} \right) \left( \sqrt{4M_p^2 + U_T^2} + U_T \right). \quad (5-59)$$

Notice that events in the zero bins $M_{CT \perp} = 0$ and $M_{CT \parallel} = 0$ fall on one of the axes and cannot be distinguished on the plot.
Now consider the scatter plot of $M_{CT\perp}$ vs $M_{CT}$ shown in Figure 5-11(b). $M_{CT}$ is similarly bounded by

$$M_{CT}^{(lo)}(M_{CT\perp}) \leq M_{CT} \leq M_{CT}^{(hi)}(M_{CT\perp}), \quad (5-60)$$

where this time $M_{CT}^{(lo)}(M_{CT\perp}) = M_{CT\perp}$ and

$$M_{CT}^{(hi)}(M_{CT\perp}) = M_{CT\perp}^{\max} \left( \cosh \eta + \sqrt{1 - \hat{M}_{CT\perp}^2 \sinh \eta} \right). \quad (5-61)$$

We see that the endpoint $M_{CT}^{\max}$ of the one-dimensional $M_{CT}$ distribution is also obtained for $M_{CT\perp} = 0$:

$$M_{CT}^{\max} = M_{CT}^{(hi)}(0) = M_{CT\perp}^{\max}(\cosh \eta + \sinh \eta) = M_{CT\parallel}^{\max}. \quad (5-62)$$

Figure 5-11 reveals a conceptual problem with one-dimensional projections. While all points in the vicinity of the boundary lines in Equations (5–58) and (5–61) are sensitive to the masses, the $M_{CT\perp}^{\max}$ endpoint is extracted mostly from events with $M_{CT\perp} \sim M_{CT\perp}^{\max}$, while the $M_{CT\parallel}^{\max}$ and $M_{CT}^{\max}$ endpoints are extracted mostly from the events with $M_{CT\perp} \sim 0$. The events near the boundary, but with intermediate values of $M_{CT\perp}$, will not enter efficiently either one of these endpoint determinations.

So how can one do better, given the knowledge of the boundary line of Equation (5–61)? In the spirit of [93], we define the signed distance to the corresponding boundary, e.g.

$$D_{CT}(M_p, M_c) \equiv M_{CT}^{(hi)}(M_{CT\perp}, U_T, M_p, M_c) - M_{CT}$$

and similarly for $D_{CT\parallel}$. The key property of this variable is that for the correct values of $M_p$ and $M_c$, its lower endpoint $D_{CT}^{\min}$ is exactly zero (see Figure 5-12(b)):

$$D_{CT}^{\min}(M_p, M_c) = 0. \quad (5–63)$$

In that case the boundary line provides a perfectly snug fit to the scatter plot — notice the green boundary line marked “80” in Figure 5-11(b).
Figure 5-12. $D_{\text{CT}}$ distributions for four different values of $M_p$ (and $M_c$ given from (5–56)). The yellow (light shaded) histograms use only events in the zero bin $M_{\text{CT} \perp} = 0$. The red solid lines show linear binned maximum likelihood fits.

While in general Equation (5–63) represents a two-dimensional fit to $M_p$ and $M_c$, in practice one can already use the $M_{\text{CT} \perp}^{\text{max}}$ measurement to reduce the problem to a single degree of freedom, e.g. the parent mass $M_p$, as presented in Figures 5-11 and 5-12. We see that the correct parent mass $M_p = 80$ GeV provides a perfect envelope, for which $D_{\text{CT}}^{\text{min}} = 0$. If, on the other hand, $M_p$ is too low, a gap develops between the outlying points in the scatter plot and their expected boundary, which results in $D_{\text{CT}}^{\text{min}} > 0$. 

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Figure 5-13. Fitted values of $D_{CT}^{\text{min}}$ as a function of $M_p$.

Conversely, if $M_p$ is too high, some of the outlying points from the scatter plot fall outside the boundary and have $D_{CT} < 0$, leading to $D_{CT}^{\text{min}} < 0$, as seen in Figure 5-12(c,d). The resulting fit for $D_{CT}^{\text{min}}$ as a function of $M_p$ from our PGS data sample is shown in Figure 5-13, which suggests that a $W$ mass measurement at the level of a few percent might be viable.
In this chapter, we study asymmetric events based on following reasons.

- **Single dark matter component.** A common assumption throughout the collider phenomenology literature is that colliders are probing only one dark matter species at a time, i.e. that the missing energy signal at colliders is due to the production of one and only one type of dark matter particles. Of course, there is no astrophysical evidence that the dark matter is made up of a single particle species: it may very well be that the dark matter world has a rich structure, just like ours \[95\]. Consequently, if there exist several types of dark matter particles, each contributing some fraction to the total relic density, a priori there is no reason why they cannot all be produced in high energy collisions. Theoretical models with multiple dark matter candidates have also been proposed \[45–47, 96–101\].

- **Identical missing particles in each event.** A separate assumption, common to most previous studies, is that the two missing particles in each event are identical. This assumption could in principle be violated as well, even if the single dark matter component hypothesis is true. The point is that one of the missing particles in the event may not be a dark matter particle, but simply some heavier cousin which decays invisibly. An invisibly decaying heavy neutralino \( \tilde{\chi}_i^0 \rightarrow \nu \bar{\nu} \tilde{\chi}_1^0 \) with \( i > 1 \) and an invisibly decaying sneutrino \( \tilde{\nu} \rightarrow \nu \tilde{\chi}_1^0 \) are two such examples from supersymmetry. As far as the event kinematics is concerned, the mass of the heavier cousin is a relevant parameter and approximating it with the mass of the dark matter particle will simply give nonsensical results. Another relevant example is provided by models in which the SUSY cascade may terminate in any one of several light neutral particles \[102\].

Given our utter ignorance about the structure of the dark matter sector, in this chapter we set out to develop the necessary formalism for carrying out missing energy studies at hadron colliders in a very general and model-independent way, without relying on any assumptions about the nature of the missing particles. In particular, we shall not assume that the two missing particles in each event are the same. We shall also allow for the simultaneous production of several dark matter species, or alternatively, for the production of a dark matter candidate in association with a heavier, invisibly decaying particle. Under these very general circumstances, we shall try to develop a method for measuring the individual masses of all relevant particles - the various
missing particles which are responsible for the missing energy, as well as their parents which were originally produced in the event.

### 6.1 Generalizing $M_{T2}$ To Asymmetric Event Topologies

In general, by now there is a wide variety of techniques available for mass measurements in SUSY-like missing energy events. Such events are characterized by the pair production of two new particles, each of which undergoes a sequence of cascade decays ending up in a particle which is invisible in the detector. Each technique has its own advantages and disadvantages\(^1\). For our purposes, we chose to revamp the method of the Cambridge $M_{T2}$ variable [10] and adapt it to the more general case of an asymmetric event topology shown in Figure 6-1.

Consider the inclusive production of two identical\(^2\) parents of mass $M_p$ as shown in Figure 6-1. The parent particles may be accompanied by any number of “upstream” objects, such as jets from initial state radiation [35, 36, 103], or visible decay products of even heavier (grandparent) particles [23]. The exact origin and nature of the upstream objects will be of no particular importance to us, and the only information about them that we shall use will be their total transverse momentum $\vec{P}_{UTM}$. In turn, each parent particle initiates a decay chain (shown in red) which produces a certain number $n^{(\lambda)}$ of Standard Model (SM) particles (shown in gray) and an intermediate “child” particle of mass $M^{(\lambda)}_C$. Throughout this chapter we shall use the index $\lambda$ to classify various objects as belonging to the upper ($\lambda = a$) or lower ($\lambda = b$) branch in Figure 6-1. The child particle may or may not be a dark matter candidate: in general, it may decay further as shown by the dashed lines in Figure 6-1.

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\(^1\) For a comparative review of the three main techniques, [23].

\(^2\) In principle, the assumption of identical parents can also be relaxed, by a suitable generalization of the $M_{T2}$ variable, in which the mass ratio of the two parents is treated as an additional input parameter [40].
Figure 6-1. The generic event topology under consideration in this chapter. We consider the inclusive pair-production of two “parent” particles with identical masses $M_p$. The parents may be accompanied by “upstream” objects, e.g. jets from initial state radiation, visible decay products of even heavier particles, etc. The transverse momentum of all upstream objects is measured and denoted by $\vec{P}_{UTM}$. In turn, each parent particle initiates a decay chain (shown in red) which produces a certain number $n^{(\lambda)}$ of SM particles (shown in gray) and an intermediate “child” particle of mass $M_c^{(\lambda)}$, where $\lambda = a$ ($\lambda = b$) for the branch above (below). In general, the child particle does not have to be the dark matter candidate, and may decay further as shown by the dashed lines. The $M_{T2}$ variable is defined for the subsystem inside the blue box and is defined in terms of two arbitrary children “test” masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$. The $n^{(\lambda)}$ SM particles from each branch form a composite particle of transverse momentum $\vec{p}_{T}^{(\lambda)}$ and invariant mass $m^{(\lambda)}$, correspondingly. The trial transverse momenta $\vec{q}_T^{(\lambda)}$ of the children obey the transverse momentum conservation relation shown inside the green box. In general, the number $n^{(\lambda)}$, as well as the type of SM decay products in each branch do not have to be the same.
We shall apply the “subsystem” $M_{T2}$ concept \cite{23, 85} to the subsystem within the blue rectangular frame. The SM particles from each branch within the subsystem form a composite particle of known\footnote{We assume that there are no neutrinos among the SM decay products in each branch.} transverse momentum $\vec{p}_T^{(\lambda)}$ and invariant mass $m_\lambda$. Since the children masses $M_c^{(a)}$ and $M_c^{(b)}$ are a priori unknown, the subsystem $M_{T2}$ will be defined in terms of two “test” masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$. In Figure 6-1, $\vec{q}_T^{(\lambda)}$ are the trial transverse momenta of the two children. The individual momenta $\vec{q}_T^{(\lambda)}$ are also a priori unknown, but they are constrained by transverse momentum conservation:

$$\vec{q}_T^{(a)} + \vec{q}_T^{(b)} \equiv \vec{Q}_{tot} = - (\vec{p}_T^{(a)} + \vec{p}_T^{(b)} + \vec{P}_{UTM}). \quad (6-1)$$

Given this very general setup, in Section 6.3 we shall consider a generalization of the usual $M_{T2}$ variable which can apply to the asymmetric event topology of Figure 6-1. There will be two different aspects of the asymmetry:

- First and foremost, we shall avoid the common assumption that the two children have the same mass. This will be important for two reasons. On the one hand, it will allow us to study events in which there are indeed two different types of missing particles. We shall give several such examples in the subsequent sections. More importantly, the endpoint of the asymmetric $M_{T2}$ variable will allow us to measure the two children masses separately. Therefore, even when the events contain identical missing particles, as is usually assumed throughout the literature, one would be able to establish this fact experimentally from the data, instead of relying on an ad hoc theoretical assumption.

- As can be seen from Figure 6-1, in general, the number as well as the types of SM decay products in each branch may be different as well. Once we allow for the children to be different, and given the fact that we start from identical parents, the two branches of the subsystem will naturally involve different sets of SM particles.

In what follows, when referring to the more general $M_{T2}$ variable defined in Section 6.3, we shall interchangeably use the terms “asymmetric” or “generalized” $M_{T2}$. In contrast,
we shall use the term “symmetric” when referring to the more conventional $M_{T2}$ definition with identical children.

The traditional $M_{T2}$ approach assumes that the children have a common test mass $	ilde{M}_c \equiv \tilde{M}_c^{(a)} = \tilde{M}_c^{(b)}$ and then proceeds to find one functional relation between the true child mass $M_c$ and the true parent mass $M_p$ as follows [10]. Construct several $M_{T2}$ distributions for different input values of the test children mass $\tilde{M}_c$ and then read off their upper kinematic endpoints $M_{T2(max)}(\tilde{M}_c)$. These endpoint measurements are then interpreted as an output parent mass $\tilde{M}_p$, which is a function of the input test mass $\tilde{M}_c$:

$$\tilde{M}_p(\tilde{M}_c) \equiv M_{T2(max)}(\tilde{M}_c).$$  \hspace{1cm} (6–2)

The importance of this functional relation is that it is automatically satisfied for the true values $M_p$ and $M_c$ of the parent and child masses:

$$M_p = M_{T2(max)}(M_c).$$  \hspace{1cm} (6–3)

In other words, if we could somehow guess the correct value $M_c$ of the child mass, the function (6–2) will provide the correct value $M_p$ of the parent mass. However, since the true child mass $M_c$ is a priori unknown, the individual masses $M_p$ and $M_c$ still remain undetermined and must be extracted by some other means.

At this point, it may seem that by considering the asymmetric $M_{T2}$ variable with non-identical children particles, we have regressed to some extent. Indeed, we are introducing an additional degree of freedom in Equation (6–2), which now reads

$$\tilde{M}_p(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \equiv M_{T2(max)}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}).$$  \hspace{1cm} (6–4)

The standard $M_{T2}$ endpoint method will still allow us to find the parent mass $\tilde{M}_p$, but now it is a function of two input parameters $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$ which are completely unknown. Of course, if one knew the correct values of the two children masses $M_c^{(a)}$ and $M_c^{(b)}$ entering Equation (6–4), the true parent mass $M_p$ will be given in a manner analogous to
Equation (6–3):

\[ M_p = M_{T2}^{(max)}(M_c^{(a)}, M_c^{(b)}). \]  

(6–5)

Our main result is that in spite of the apparent remaining arbitrariness in Equation (6–4), one can nevertheless uniquely determine all three masses \( M_p, M_c^{(a)} \) and \( M_c^{(b)} \), just by studying the behavior of the measured function \( \tilde{M}_p(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \). More importantly, this determination can actually be done in two different ways! Our first method is simply a generalization of the observation made in References. [23, 34–37] that under certain circumstances (varying \( m_\lambda \) or nonvanishing upstream momentum \( P_{UTM} \)), the function (6–2) develops a “kink” precisely at the correct value \( M_c \) of the child mass:

\[
\begin{align*}
\left( \frac{\partial \tilde{M}_p(\tilde{M}_c)}{\partial \tilde{M}_c} \right)_{\tilde{M}_c + \epsilon} - \left( \frac{\partial \tilde{M}_p(\tilde{M}_c)}{\partial \tilde{M}_c} \right)_{\tilde{M}_c - \epsilon} &= 0, \quad \text{if} \quad \tilde{M}_c \neq M_c, \\
&= 0, \quad \text{if} \quad \tilde{M}_c = M_c.
\end{align*}
\]  

(6–6)

In other words, the function (6–2) is continuous, but not differentiable at the point \( \tilde{M}_c = M_c \). In the asymmetric \( M_{T2} \) case, we find that the function (6–4) is similarly non-differentiable at a set of points \( \{ (\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \} \), so that the kink of Equation (6–6) is generalized to a “ridge” on the 2-dimensional hypersurface defined by Equation (6–4) in the three-dimensional parameter space of \( \{ \tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, \tilde{M}_p \} \). Interestingly enough, the ridge often (albeit not always) exhibits a special point which marks the exact location of the true values \( (M_c^{(a)}, M_c^{(b)}) \).

Our second method for determining the two children masses \( \tilde{M}_c^{(a)} \) and \( \tilde{M}_c^{(b)} \) is even more general and is applicable under any circumstances. The main starting point is that just like the endpoint of the symmetric \( M_{T2} \), the endpoint of the asymmetric \( M_{T2} \) also depends on the value of the upstream transverse momentum \( P_{UTM} \), so that

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4 Reference [40] studied the orthogonal scenario of different parents \( (M_p^{(a)} \neq M_p^{(b)}) \) and identical children \( (M_c^{(a)} = M_c^{(b)}) \) and found a similar non-differentiable feature, called a “crease”, on the corresponding two-dimensional hypersurface within the three-dimensional parameter space \( \{ \tilde{M}_c, \tilde{M}_p^{(a)}, \tilde{M}_p^{(b)} \} \).
Equation (6–4) is more properly written as

$$\tilde{M}_p(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}) = M_{T2(\text{max})}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}).$$  \hspace{1cm} (6–7)

Now we can explore the $P_{UTM}$ dependence in Equation (6–7) and note that it is absent for precisely the right values of $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$:

$$\left. \frac{\partial M_{T2(\text{max})}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM})}{\partial P_{UTM}} \right|_{\tilde{M}_c^{(a)}=\tilde{M}_c^{(b)}, \tilde{M}_c^{(a)}=\tilde{M}_c^{(b)}} = 0.$$  \hspace{1cm} (6–8)

While this property has been known, it was rarely used in the case of the symmetric $M_{T2}$, since it offers redundant information: once the correct child mass $M_c$ is found through the $M_{T2}$ kink as Equation (6–6), the parent mass $M_p$ is given by Equation (6–2) and there are no remaining unknowns, thus there is no need to further investigate the $P_{UTM}$ dependence. In the case of the asymmetric $M_{T2}$, however, we start with one additional unknown parameter, which cannot always be determined from the “ridge” information alone. Therefore, in order to pin down the complete spectrum, we are forced to make use of Equation (6–8). The nice feature of the $P_{UTM}$ method is that it always allows us to determine both children masses $M_c^{(a)}$ and $M_c^{(b)}$, without relying on the “ridge” information at all. In this sense, our two methods are complementary and each can be used to cross-check the results obtained by the other.

### 6.2 The Conventional Symmetric $M_{T2}$

#### 6.2.1 Definition

We begin our discussion by revisiting the conventional definition of the symmetric $M_{T2}$ variable with identical daughters, following the general notation introduced in Figure 6-1. Let us consider the inclusive production of two parent particles with common mass $M_p$. Each parent initiates a decay chain producing a certain number $n^{(\lambda)}$ of SM particles. In this section we assume that the two chains terminate in children particles of the same mass: $M_c^{(a)} = M_c^{(b)} = M_c$. (From Section 6.3 on we shall remove this assumption.) In most applications of $M_{T2}$ in the literature, the children particles
are identified with the very last particles in the decay chains, i.e. the dark matter candidates. However, the symmetric $M_{T_2}$ can also be usefully applied to a subsystem of the original event topology, where the children are some other pair of (identical) particles appearing further up the decay chain [23, 85]. The $M_{T_2}$ variable is defined in terms of the measured invariant mass $m(\lambda)$ and transverse momentum $\vec{p}_T(\lambda)$ of the visible particles on each side (Figure 6-1). With the assumption of identical children, the transverse mass of each parent is

$$M_T^{(\lambda)}(\vec{p}_T(\lambda); \vec{q}_T(\lambda); m(\lambda); \tilde{M}_c) = \sqrt{m^2(\lambda) + \tilde{M}_c^2 + 2 \left( e(\lambda) \tilde{e}(\lambda) - \vec{p}_T(\lambda) \cdot \vec{q}_T(\lambda) \right)} , \tag{6–9}$$

where $\tilde{M}_c$ is the common test mass for the children, which is an input to the $M_{T_2}$ calculation, while $\vec{q}_T(\lambda)$ is the unknown transverse momentum of the child particle in the $\lambda$-th chain. In Equation (6–9) we have also introduced shorthand notation for the transverse energy of the composite particle made from the visible SM particles in the $\lambda$-th chain

$$e(\lambda) = \sqrt{m^2(\lambda) + \vec{p}_T^2(\lambda)} \tag{6–10}$$

and for the transverse energy of the corresponding child particle in the $\lambda$-th chain

$$\tilde{e}(\lambda) = \sqrt{\tilde{M}_c^2 + \vec{q}_T^2(\lambda)} \cdot \vec{q}_T(\lambda) \tag{6–11}$$

Then the event-by-event symmetric $M_{T_2}$ variable is defined through a minimization procedure over all possible partitions of the two children momenta $\vec{q}_T(\lambda)$ [10]

$$M_{T_2} \left( \vec{p}_T^{(a)}, \vec{p}_T^{(b)}; m^{(a)}, m^{(b)}; \tilde{M}_c, P_{UTM} \right) = \min_{\vec{q}_T^{(a)}, \vec{q}_T^{(b)}-\vec{q}_{tot}} \left[ \max \left\{ M_T^{(a)} \left( \vec{p}_T^{(a)}, \vec{q}_T^{(a)}; m^{(a)}; \tilde{M}_c \right), \ M_T^{(b)} \left( \vec{p}_T^{(b)}, \vec{q}_T^{(b)}; m^{(b)}; \tilde{M}_c \right) \right\} \right] \tag{6–12}$$

consistent with the momentum conservation constraint (6–1) in the transverse plane.

### 6.2.2 Computation

The standard definition as in Equation (6–12) of the $M_{T_2}$ variable is sufficient to compute the value of $M_{T_2}$ numerically, given a set of input values for its arguments.
The right-hand side of Equation (6–12) represents a simple minimization problem in two variables, which can be easily handled by a computer. In fact, there are publicly available computer codes for computing $M_{T2}$ [104, 105]. The public codes have even been optimized for speed [38] and give results consistent with each other (as well as with our own code)\(^5\). Nevertheless, it is useful to have an analytical formula for calculating the event-by-event $M_{T2}$ for several reasons. First, an analytical formula is extremely valuable when it comes to understanding the properties and behavior of complex mathematical functions like in Equation (6–12). Second, computing $M_{T2}$ from a formula will be faster than any numerical scanning algorithm. The computing speed becomes an issue especially when one considers variations of $M_{T2}$ like $M_{T2gen}$, where in addition one needs to scan over all possible partitions of the visible objects into two decay chains [33]. Therefore we shall pay special attention to the availability of analytical formulas and we shall quote such formulas whenever they are available.

In the symmetric case with identical children, an analytical formula for the event-by-event $M_{T2}$ exists only in the special case $P_{UTM} = 0$. It was derived in [33] and we provide it here for completeness. (In the next section we shall present its generalization for the asymmetric case of different children.) The symmetric $M_{T2}$ is known to have two types of solutions: “balanced” and “unbalanced” [32, 33]. The balanced solution is achieved when the minimization procedure in Equation (6–12) selects a momentum configuration for $q_T^{(\lambda)}$ in which the transverse masses of the two parents are the same: $M_T^{(a)} = M_T^{(b)}$. In that case, typically neither $M_T^{(a)}$ nor $M_T^{(b)}$ is at its global (unconstrained) minimum. In what follows, we shall use a superscript $B$ to refer to such balanced-type solutions. The formula for the balanced solution $M_B^{T2}$ of the

\(^5\) Unfortunately, the assumption of identical children is hardwired in the public codes and they cannot be used to calculate the asymmetric $M_{T2}$ introduced below in Section 6.3 without additional hacking. We shall return to this point in Section 6.3.
symmetric $M_{T2}$ variable is given by [33, 37]

$$
\left[ M_{T2}^{B}(\vec{p}_T^{(a)}, \vec{p}_T^{(b)}; m_{(a)}, m_{(b)}; \tilde{M}_c) \right]^2 = \tilde{M}_c^2 + A_T + \sqrt{\left( 1 + \frac{4\tilde{M}_c^2}{2A_T - m_{(a)}^2 - m_{(b)}^2} \right) \left( A_T^2 - m_{(a)}^2 m_{(b)}^2 \right)},
$$

(6–13)

where $A_T$ is a convenient shorthand notation introduced in [37]

$$
A_T = c^{(a)} e^{(b)} + \vec{p}_T^{(a)} \cdot \vec{p}_T^{(b)}
$$

(6–14)

and $c^{(\lambda)}$ was already defined in Equation (6–10).

On the other hand, unbalanced solutions arise when one of the two parent transverse masses ($M_T^{(a)}$ or $M_T^{(b)}$, as the case may be) is at its global (unconstrained) minimum. Denoting the two unbalanced solutions with a superscript $U\lambda$, we have [32]

$$
M_{T2}^{Ua}(\vec{p}_T^{(a)}, \vec{p}_T^{(b)}; m_{(a)}, m_{(b)}; \tilde{M}_c) = m_{(a)} + \tilde{M}_c,
$$

(6–15)

$$
M_{T2}^{Ub}(\vec{p}_T^{(a)}, \vec{p}_T^{(b)}; m_{(a)}, m_{(b)}; \tilde{M}_c) = m_{(b)} + \tilde{M}_c.
$$

(6–16)

Given the three possible options for $M_{T2}$, Equations (6–13), (6–15) and (6–16), it remains to specify which one actually takes place for a given set of values for $\vec{p}_T^{(a)}$, $\vec{p}_T^{(b)}$, $m_{(a)}$, $m_{(b)}$, $\tilde{M}_c$ and $P_{UTM} = 0$ in the event$^6$. The balanced solution (6–13) applies when the following two conditions are simultaneously satisfied:

$$
M_T^{(b)}(\vec{p}_T^{(b)}; \tilde{q}_T^{(b)} = -\tilde{q}_T^{(b)}; m_{(a)}; \tilde{M}_c) \geq M_T^{(a)}(\vec{p}_T^{(a)}; \tilde{q}_T^{(a)} = \tilde{q}_T^{(a)}; m_{(a)}; \tilde{M}_c) = m_{(a)} + \tilde{M}_c,
$$

(6–17)

$$
M_T^{(a)}(\vec{p}_T^{(a)}; \tilde{q}_T^{(a)} = -\tilde{q}_T^{(a)}; m_{(b)}; \tilde{M}_c) \geq M_T^{(b)}(\vec{p}_T^{(b)}; \tilde{q}_T^{(b)} = \tilde{q}_T^{(b)}; m_{(b)}; \tilde{M}_c) = m_{(b)} + \tilde{M}_c,
$$

(6–18)

where

$$
\tilde{q}_T^{(\lambda)} = \frac{\tilde{M}_c}{m_{(\lambda)}} \vec{p}_T^{(\lambda)}, \quad (\lambda = a, b),
$$

(6–19)

$^6$ Recall that Equation (6–13) only applies for $P_{UTM} = 0$. 

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gives the global (unconstrained) minimum of the corresponding parent transverse mass \( M_T^{(\lambda)} \). The unbalanced solution \( M_{T2}^{Ua} \) applies when the condition (6–17) is false and condition in Equation (6–18) is true, while the unbalanced solution \( M_{T2}^{Ub} \) applies when the condition (6–17) is true and condition in Equation (6–18) is false. It is easy to see that conditions in Equations (6–17) and (6–18) cannot be simultaneously violated, so these three cases exhaust all possibilities.

6.2.3 Properties

Given its definition (6–12), one can readily form and study the differential \( M_{T2} \) distribution. Although its shape in general does carry some information about the underlying process, it has become customary to focus on the upper endpoint \( M_{T2(max)} \), which is simply the maximum value of \( M_{T2} \) found in the event sample:

\[
M_{T2(max)}(\tilde{M}_c, P_{UTM}) = \max_{\text{all events}} \left[ M_{T2}(\tilde{p}_T^{(a)}, \tilde{p}_T^{(b)}; m(a), m(b); \tilde{m}_c) \right]. \tag{6–20}
\]

Notice that in the process of maximizing over all events, the dependence on \( \tilde{p}_T^{(a)}, \tilde{p}_T^{(b)}, \) \( m(a) \) and \( m(b) \) disappears, and \( M_{T2(max)} \) depends only on two input parameters: \( \tilde{M}_c \) and \( P_{UTM} \), the latter entering through \( \vec{Q}_{tot} \) in the momentum conservation constraint of Equation (6–1). The measured function in Equation (6–20) is the starting point of any \( M_{T2} \)-based mass determination analysis. We shall now review its three basic properties which make it suitable for such studies [12].

6.2.3.1 Property I: Knowledge Of \( M_p \) As A Function of \( M_c \)

This property was already identified in the original papers and served as the main motivation for introducing the \( M_{T2} \) variable in the first place [10, 32]. Mathematically it can be expressed as

\[
\tilde{M}_p(\tilde{M}_c, P_{UTM}) \equiv M_{T2(max)}(\tilde{M}_c, P_{UTM}). \tag{6–21}
\]

This is the same as Equation (6–2), but now we have been careful to include the explicit dependence on \( P_{UTM} \), which will be important in our subsequent discussion.
Figure 6-2. Plots of (a) the $M_{T2}$ endpoint $M_{T2(max)}(\tilde{M}_c, P_{UTM})$ defined in Equation (6–20), and (b) the function $\Delta M_{T2(max)}(\tilde{M}_c, P_{UTM})$ defined in Equation (6–24) as a function of the test child mass $\tilde{M}_c$, for several fixed values of $P_{UTM}$: $P_{UTM} = 0$ GeV (solid, green), $P_{UTM} = 500$ GeV (dot-dashed, black), $P_{UTM} = 1$ TeV (dashed, red), and $P_{UTM} = 2$ TeV (dotted, blue). The process under consideration is pair production of sleptons of mass $M_p = 300$ GeV, which decay directly to the lightest neutralino $\tilde{\chi}_1^0$ of mass $M_c = 100$ GeV.

As indicated in Equation (6–21), the function $\tilde{M}_p(\tilde{M}_c, P_{UTM})$ can be experimentally measured from the $M_{T2}$ endpoint of Equation (6–20). The crucial point now is that the relation in Equation (6–21) is satisfied by the true values $M_p$ and $M_c$ of the parent and child mass, correspondingly:

$$M_p = M_{T2(max)}(M_c, P_{UTM}).$$  \hspace{1cm} (6–22)

Notice that Equation (6–22) holds for any value of $P_{UTM}$, so in practical applications of this method one could choose the most populated $P_{UTM}$ bin to reduce the statistical error. On the other hand, since a priori we do not know the true mass $M_c$ of the missing particle, Equation (6–22) gives only one relation between the masses of the mother and the child. This is illustrated in Figure 6-2(a), where we consider the simple example of
direct slepton pair production\(^7\), where each slepton (\(\tilde{\ell}\)) decays to the lightest neutralino (\(\chi^0_1\)) by emitting a single lepton \(\ell: \tilde{\ell} \rightarrow \ell + \chi^0_1\). Here the slepton is the parent and the neutralino is the child. Their masses were chosen to be \(M_p = 300\) GeV and \(M_c = 100\) GeV, correspondingly, as indicated with the black dotted lines in Figure 6-2(a). In this example, the upstream transverse momentum \(P_{UTM}\) is provided by jets from initial state radiation. In Figure 6-2(a) we plot the function in Equation (6–21) versus \(\tilde{M}_c\), for several fixed values of \(P_{UTM}\). The green solid line represents the case of no upstream momentum \(P_{UTM} = 0\). In agreement with Equation (6–22), this line passes through the point \((M_c, M_p)\) corresponding to the true values of the mass parameters. Notice that the property of Equation (6–22) continues to hold for other values of \(P_{UTM}\). Figure 6-2(a) shows three more cases: \(P_{UTM} = 500\) GeV (dotdashed black line), \(P_{UTM} = 1\) TeV (dashed red line) and \(P_{UTM} = 2\) TeV (dotted blue line). All those curves still pass through the point \((M_c, M_p)\) with the correct values of the masses, illustrating the robustness of the property of Equation (6–22) with respect to variations in \(P_{UTM}\).

**6.2.3.2 Property II: Kink In \(M_{T2(max)}\) At The True \(M_c\)**

The second important property of the \(M_{T2}\) variable was identified rather recently [23, 34–37]. Interestingly, the \(M_{T2}\) endpoint \(M_{T2(max)}\), when considered as a function of the unknown input test mass \(\tilde{M}_c\), often develops a kink as Equation (6–6) at precisely the correct value \(\tilde{M}_c = M_c\) of the child mass. The appearance of the kink is a rather general phenomenon and occurs under various circumstances. It was originally noticed in event topologies with composite visible particles, whose invariant mass \(m_\lambda\) is a variable parameter [34, 37]. Later it was realised that a kink also occurs in the presence of non-zero upstream momentum \(P_{UTM}\) [23, 35, 36], as in the example of Figure 6-2(a), where \(P_{UTM}\) arises due to initial state radiation. As can be seen in Figure 6-2(a), the

\(^7\) The corresponding event topology is shown in Figure 6-3(a) below with \(M_c^{(a)} = M_c^{(b)} = M_c\).
kink is absent for $P_{UTM} = 0$, but as soon as there is some non-vanishing $P_{UTM}$, the kink becomes readily apparent. As expected, the kink location (marked by the vertical dotted line) is at the true child mass ($M_c = 100$ GeV), where the corresponding value of $M_{T2(max)}$ (marked by the horizontal dotted line) is at the true parent mass ($M_p = 300$ GeV). Figure 6-2(a) also demonstrates that with the increase in $P_{UTM}$, the kink becomes more pronounced, thus the most favorable situations for the observation of the kink are cases with large $P_{UTM}$, e.g. when the upstream momentum is due to the decays of heavier (grandparent) particles [23].

In Section 6.3.3 we shall see how the kink feature (6–6) of the symmetric $M_{T2}$ endpoint $\tilde{M}_p(\tilde{M}_c)$ defined by Equation (6–2) is generalized to a “ridge” feature on the asymmetric $M_{T2}$ endpoint $\tilde{M}_p(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$ defined in Equation (6–4).

### 6.2.3.3 Property III: $P_{UTM}$ Invariance Of $M_{T2(max)}$ At The True $M_c$

This property is the one which has been least emphasized in the literature. Notice that the $M_{T2}$ endpoint function of Equation (6–21) in general depends on the value of $P_{UTM}$. However, the first property of Equation (6–22) implies that the $P_{UTM}$ dependence disappears at the correct value $M_c$ of the child mass:

$$\left.\frac{\partial M_{T2(max)}(\tilde{M}_c, P_{UTM})}{\partial P_{UTM}}\right|_{\tilde{M}_c = M_c} = 0. \tag{6–23}$$

In order to quantify this feature, let us define the function

$$\Delta M_{T2(max)}(\tilde{M}_c, P_{UTM}) \equiv M_{T2(max)}(\tilde{M}_c, P_{UTM}) - M_{T2(max)}(\tilde{M}_c, 0), \tag{6–24}$$

which measures the shift of the $M_{T2}$ endpoint due to variations in $P_{UTM}$. The function $\Delta M_{T2(max)}(\tilde{M}_c, P_{UTM})$ can be measured experimentally: the first term on the right-hand side of Equation (6–24) is simply the $M_{T2}$ endpoint observed in a subsample of events with a given (preferably the most common) value of $P_{UTM}$, while the second term on the right-hand side of Equation (6–24) contains the endpoint $M_{T2\perp}^{(max)}$ of the 1-dimensional
$M_{T2\perp}$ variable introduced in [12]:

$$M_{T2(max)}(\tilde{M}_c, 0) = M_{T2\perp}^{(max)}(\tilde{M}_c). \quad (6–25)$$

Given the definition (6–24), the third property of Equation (6–23) can be rewritten as

$$\Delta M_{T2(max)}(\tilde{M}_c, P_{UTM}) \geq 0, \quad (6–26)$$

where the equality holds only for $\tilde{M}_c = M_c$:

$$\Delta M_{T2(max)}(M_c, P_{UTM}) = 0, \; \forall P_{UTM}. \quad (6–27)$$

Equations (6–26) and (6–27) provide an alternative way to determine the true child mass $M_c$: simply find the value of $\tilde{M}_c$ which minimizes the function $\Delta M_{T2(max)}(\tilde{M}_c, P_{UTM})$. This procedure is illustrated in Figure 6-2(b), where we revisit the slepton pair production example of Figure 6-2(a) and plot the function $\Delta M_{T2(max)}(\tilde{M}_c, P_{UTM})$ defined in Equation (6–24) versus the test mass $\tilde{M}_c$, for the same set of (fixed) values of $P_{UTM}$.

Clearly, the zero of the function as Equation (6–24) occurs at the true child mass $\tilde{M}_c = M_c = 100 \text{ GeV}$, in agreement with Equation (6–27). In our studies of the asymmetric $M_{T2}$ case in the next sections, we shall find that the third property in Equation (6–27) is extremely important, since it will always allow us the complete determination of the mass spectrum, including both children masses $M_c^{(a)}$ and $M_c^{(b)}$.

6.3 The Generalized Asymmetric $M_{T2}$

After this short review of the basic properties of the conventional symmetric $M_{T2}$ variable in Equation (6–12), we now turn our attention to the less trivial case of $\tilde{M}_c^{(a)} \neq \tilde{M}_c^{(b)}$. Following the logic of Section 6.2, in Section 6.3.1 we first introduce the asymmetric $M_{T2}$ variable and then in Sections 6.3.2 and 6.3.3 we discuss its computation and mathematical properties, correspondingly.
6.3.1 Definition

The generalization of the usual definition (6–12) to the asymmetric case of $\tilde{M}_c^{(a)} \neq \tilde{M}_c^{(b)}$ is straightforward [40]. We continue to follow the conventions and notation of Figure 6-1, but now we simply avoid the assumption that the children masses are equal, and we let each one be an independent input parameter $\tilde{M}_c^{(\lambda)}$. Without loss of generality, in what follows we assume $M_c^{(b)} \geq M_c^{(a)}$. The transverse mass of each parent in Equation (6–9) is now a function of the corresponding child mass $\tilde{M}_c^{(\lambda)}$:

$$M_T^{(\lambda)}(\vec{p}_T^{(\lambda)}; \vec{q}_T^{(\lambda)}; m; \tilde{M}_c^{(\lambda)}) = \sqrt{m^2 + \left(\tilde{M}_c^{(\lambda)}\right)^2 + 2\left(e^{(\lambda)}\tilde{e}^{(\lambda)} - \vec{p}_T^{(\lambda)} \cdot \vec{q}_T^{(\lambda)}\right)}, \quad (6–28)$$

where the transverse energy $e^{(\lambda)}$ of the composite SM particle on the $\lambda$-th side of the event was already defined in Equation (6–10), while the transverse energy $\tilde{e}^{(\lambda)}$ of the child is now generalized from Equation (6–11) to

$$\tilde{e}^{(\lambda)} = \sqrt{\left(\tilde{M}_c^{(\lambda)}\right)^2 + \vec{q}_T^{(\lambda)} \cdot \vec{q}_T^{(\lambda)}}. \quad (6–29)$$

The event-by-event asymmetric $M_{T2}$ variable is defined in analogy to Equation (6–12) and is given by [40]

$$M_{T2}(\vec{p}_T^{(\alpha)}, \vec{p}_T^{(\beta)}, m^{(a)}, m^{(b)}, \tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}) = \min_{\tilde{M}_c^{(a)} + \tilde{M}_c^{(b)} = \vec{q}_T^{(\alpha)} + \vec{q}_T^{(\beta)}} \left[ \max \left\{ M_T^{(a)}(\vec{p}_T^{(a)}; \vec{q}_T^{(a)}; m^{(a)}; \tilde{M}_c^{(a)}), M_T^{(b)}(\vec{p}_T^{(b)}; \vec{q}_T^{(b)}; m^{(b)}; \tilde{M}_c^{(b)}) \right\} \right], \quad (6–30)$$

which is now a function of two input test children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$. In the special case of $\tilde{M}_c^{(a)} = \tilde{M}_c^{(b)} \equiv \tilde{M}_c$, the asymmetric $M_{T2}$ variable defined in Equation (6–30) reduces to the conventional symmetric $M_{T2}$ variable (6–12).

6.3.2 Computation

In this subsection we generalize the discussion in Section 6.2.2 and present an analytical formula for computing the event-by-event asymmetric $M_{T2}$ variable (6–30).

Just like the formula (6–13) for the symmetric case, our formula will hold only in the special case of $P_{UTM} = 0$. As before, the asymmetric $M_{T2}$ variable has two types of
solutions – balanced and unbalanced. The balanced solution occurs when the following two conditions are simultaneously satisfied (compare to the analogous conditions (Equations 6–17) and (6–18) for the symmetric case)

\[ M_{T}^{(b)}\left(\tilde{\beta}_{T}^{(b)};\tilde{q}_{T}^{(b)}=\tilde{q}_{T(0)}^{(b)}+\tilde{\beta}_{T};m_{(a)};\tilde{\lambda}_{c}^{(b)}\right) \geq M_{T}^{(a)}\left(\tilde{\beta}_{T}^{(a)};\tilde{q}_{T}^{(a)}=\tilde{q}_{T(0)}^{(a)};m_{(a)};\tilde{\lambda}_{c}^{(a)}\right) = m_{(a)} + \tilde{M}_{c}^{(a)}, \quad (6–31) \]

\[ M_{T}^{(a)}\left(\tilde{\beta}_{T}^{(a)};\tilde{q}_{T}^{(a)}=\tilde{q}_{T(0)}^{(a)}+\tilde{\beta}_{T};m_{(a)};\tilde{\lambda}_{c}^{(a)}\right) \geq M_{T}^{(b)}\left(\tilde{\beta}_{T}^{(b)};\tilde{q}_{T}^{(b)}=\tilde{q}_{T(0)}^{(b)};m_{(b)};\tilde{\lambda}_{c}^{(b)}\right) = m_{(b)} + \tilde{M}_{c}^{(b)}, \quad (6–32) \]

where, in analogy to Equation (6–19),

\[ \tilde{q}_{T(0)}^{(\lambda)} = \frac{\tilde{M}_{c}^{(\lambda)}}{m_{(\lambda)}} \tilde{\beta}_{T}^{(\lambda)}, \quad (\lambda = a, b), \quad (6–33) \]

is the test child momentum at the global unconstrained minimum of \( M_{T}^{(\lambda)} \). The balanced solution for \( M_{T2} \) is now given by

\[
\left[ M_{T2}^{B}\left(\tilde{\beta}_{T}^{(a)};\tilde{q}_{T}^{(a)};m_{(a)};\tilde{\lambda}_{c}^{(a)};\tilde{\lambda}_{c}^{(b)}\right) \right]^{2} = \tilde{M}_{+}^{2} + A_{T} + \left( \frac{m_{(b)}^{2} - m_{(a)}^{2}}{2A_{T} - m_{(a)}^{2} - m_{(b)}^{2}} \right) \tilde{M}_{-}^{2} \\
\pm \sqrt{1 + \frac{4\tilde{M}_{c}^{2}}{2A_{T} - m_{(a)}^{2} - m_{(b)}^{2}}} + \left( \frac{2\tilde{M}_{c}^{2}}{2A_{T} - m_{(a)}^{2} - m_{(b)}^{2}} \right) ^{2} \times \sqrt{A_{T}^{2} - m_{(a)}^{2} m_{(b)}^{2}} \quad (6–34)
\]

where \( A_{T} \) was defined in Equation (6–14). For convenience, in Equation (6–34) we have introduced two alternative mass parameters

\[
\tilde{M}_{+}^{2} = \frac{1}{2} \left\{ (\tilde{M}_{c}^{(a)})^{2} + (\tilde{M}_{c}^{(a)})^{2} \right\}, \quad (6–35) \\
\tilde{M}_{-}^{2} = \frac{1}{2} \left\{ (\tilde{M}_{c}^{(a)})^{2} - (\tilde{M}_{c}^{(a)})^{2} \right\}, \quad (6–36)
\]

in place of the original trial masses \( \tilde{M}_{c}^{(a)} \) and \( \tilde{M}_{c}^{(b)} \). The new parameters \( \tilde{M}_{+} \) and \( \tilde{M}_{-} \) are simply a different parametrization of the two degrees of freedom corresponding to the unknown child masses \( \tilde{M}_{c}^{(a)} \) and \( \tilde{M}_{c}^{(b)} \) entering the definition of the asymmetric \( M_{T2} \).

The parameters \( \tilde{M}_{+} \) and \( \tilde{M}_{-} \) allow us to write formula (6–34) in a more compact form.

More importantly, they also allow to make easy contact with the known results from
Section 6.2 by taking the symmetric limit $\tilde{M}_c^{(a)} = \tilde{M}_c^{(b)} \equiv \tilde{M}_c$ as

$$\tilde{M}_+ \to \tilde{M}_c, \quad \tilde{M}_- \to 0.$$  \hfill (6–37)

It is easy to see that in the symmetric limit as Equation (6–37) our balanced solution in Equation (6–34) for the asymmetric $M_{T2}$ reduces to the known result of Equation (6–13) for the symmetric $M_{T2}$.

An interesting feature of the asymmetric balanced solution is the appearance of a $\pm$ sign on the second line of Equation (6–34). In principle, this sign ambiguity is present in the symmetric case as well, but there the minus sign always turns out to be unphysical and the sign issue does not arise [33]. However, in the asymmetric case, both signs can be physical sometimes and one must make the proper sign choice in Equation (6–34) as follows. For the given set of test masses $(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$, calculate the transverse center-of-mass energy

$$\sqrt{\tilde{s}^+} = e^{(a)} + e^{(b)} + \frac{2(e^{(b)} - e^{(a)})\tilde{M}_2^2}{2A_T - m_{(a)}^2 - m_{(b)}^2} \pm \frac{(e^{(b)} + e^{(a)})A_T - (e^{(b)}m_{(a)}^2 + e^{(a)}m_{(b)}^2)}{\sqrt{A_T^2 - m_{(a)}^2m_{(b)}^2}} \times \sqrt{1 + \frac{4\tilde{M}_c^2}{2A_T - m_{(a)}^2 - m_{(b)}^2} + \left(\frac{2\tilde{M}_c}{2A_T - m_{(a)}^2 - m_{(b)}^2}\right)^2},$$  \hfill (6–38)

corresponding to each sign choice in Equation (6–34), and compare the result to the minimum allowed value of $\sqrt{\tilde{s}}$

$$\sqrt{\tilde{s}}_{(\text{min})} = e^{(a)} + e^{(b)} + \sqrt{Q_{\text{tot}}^2 + \left(\tilde{M}_c^{(a)} + \tilde{M}_c^{(b)}\right)^2}.$$  \hfill (6–39)

The minus sign in Equation (6–34) takes precedence and applies whenever it is physical, i.e. whenever $\sqrt{\tilde{s}} > \sqrt{\tilde{s}}_{(\text{min})}$. In the remaining cases when $\sqrt{\tilde{s}} < \sqrt{\tilde{s}}_{(\text{min})}$ and the minus sign is unphysical, the plus sign in Equation (6–34) applies. If one of the conditions in Equations (6–31), (6–32) is not satisfied, the asymmetric $M_{T2}$ is given by
an unbalanced solution, in analogy to Equations (6–15) and (6–16):

\[ M_{T2}^{Ua}(\bar{\mu}_T^{(a)}, \bar{\mu}_T^{(b)}; \bar{m}_1(a), \bar{m}_1(b); \bar{\tilde{M}}_c^{(a)}, \bar{\tilde{M}}_c^{(b)}) = m_1(a) + \tilde{M}_c^{(a)} , \quad (6–40) \]

\[ M_{T2}^{Ub}(\bar{\mu}_T^{(a)}, \bar{\mu}_T^{(b)}; \bar{m}_1(a), \bar{m}_1(b); \bar{\tilde{M}}_c^{(a)}, \bar{\tilde{M}}_c^{(b)}) = m_1(b) + \tilde{M}_c^{(b)} . \quad (6–41) \]

The unbalanced solution \( M_{T2}^{Ua} \) of Equation (6–40) applies when the condition (6–31) is false and condition (6–32) is true, while the unbalanced solution \( M_{T2}^{Ub} \) of Equation (6–41) applies when the condition of Equation (6–31) is true and condition of Equation (6–32) is false.

Equations (6–34), (6–40) and (6–41) represent one of our main results. They generalize the analytical results of References. [33, 37] and allow the direct computation of the asymmetric \( M_{T2} \) variable without the need for scanning and numerical minimizations. This is an important benefit, since the existing public codes for \( M_{T2} \) [104, 105] only apply in the symmetric case \( M_c^{(a)} = M_c^{(b)} \).

6.3.3 Properties

All three properties of the symmetric \( M_{T2} \) discussed in Section 6.2.3 readily generalize to the asymmetric case.

6.3.3.1 Property I: Knowledge Of \( M_p \) As A Function Of \( M_c^{(a)} \) And \( M_c^{(b)} \)

In the asymmetric case, the endpoint \( M_{T2(max)} \) of the \( M_{T2} \) distribution still gives the mass of the parent, only this time it is a function of two input test masses for the children:

\[ \tilde{M}_p(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}) = M_{T2(max)}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}) . \quad (6–42) \]

The important property is that this relation is satisfied by the true values of the children and parent masses:

\[ M_p = M_{T2(max)}(M_c^{(a)}, M_c^{(b)}, P_{UTM}) . \quad (6–43) \]

Thus the true parent mass \( M_p \) will be known once we determine the two children masses \( M_c^{(a)} \) and \( M_c^{(b)} \).
6.3.3.2 Property II: Ridge In $M_{T2(max)}$ Through The True $M_c^{(a)}$ And $M_c^{(b)}$

In the symmetric $M_{T2}$ case, the endpoint function in Equation (6–21) is not continuously differentiable and has a “kink” at the true child mass $\tilde{M}_c = M_c$. In the asymmetric $M_{T2}$ case, the endpoint function in Equation (6–42) is similarly non-differentiable at a set of points

$$\left\{ \left( \tilde{M}_c^{(a)}(\theta), \tilde{M}_c^{(b)}(\theta) \right) \right\}$$  \hspace{1cm} (6–44)

parametrized by a single continuous parameter $\theta$. The gradient of the endpoint function in Equation (6–42) suffers a discontinuity as we cross the curve defined by Equation (6–44). Since Equation (6–42) represents a hypersurface in the three-dimensional parameter space of $\{ \tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, \tilde{M}_p \}$, the gradient discontinuity will appear as a “ridge” (sometimes also referred to as a “crease” [40]) on our three-dimensional plots below.

The important property of the ridge is that it passes through the correct values for the children masses, even when they are different:

$$M_c^{(a)} = \tilde{M}_c^{(a)}(\theta_0),$$  \hspace{1cm} (6–45)

$$M_c^{(b)} = \tilde{M}_c^{(b)}(\theta_0),$$  \hspace{1cm} (6–46)

for some $\theta_0$. Thus the ridge information provides a relation among the two children masses and leaves us with just a single unknown degree of freedom — the parameter $\theta$ in Equation (6–44).

Interestingly, the shape of the ridge provides a quick test whether the two missing particles are identical or not\(^8\). If the shape of the ridge in the $(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$ plane is symmetric with respect to the interchange $\tilde{M}_c^{(a)} \leftrightarrow \tilde{M}_c^{(b)}$, i.e. under a mirror reflection with respect to the $45^\circ$ line $\tilde{M}_c^{(a)} = \tilde{M}_c^{(b)}$, then the two missing particles are the same.

\(^8\) To be more precise, the ridge shape tests whether the two missing particles have the same mass or not.
Conversely, when the shape of the ridge is not symmetric under $\tilde{M}_c^{(a)} \leftrightarrow \tilde{M}_c^{(b)}$, the missing particles are in general expected to have different masses.

### 6.3.3.3 Property III: $P_{UTM}$ Invariance Of $M_{T2\text{(max)}}$ At The True $M_c^{(a)}$ And $M_c^{(b)}$

The third $M_{T2}$ property, which was discussed in Section 6.2.3.3, is readily generalized to the asymmetric case as well. Note that Equation (6–43) implies that the $P_{UTM}$ dependence of the asymmetric $M_{T2}$ endpoint in Equation (6–42) disappears at the true values of the children masses:

$$\frac{\partial M_{T2\text{(max)}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM})}{\partial P_{UTM}}\bigg|_{\tilde{M}_c^{(a)}=M_c^{(a)}, \tilde{M}_c^{(b)}=M_c^{(b)}} = 0.$$  \hfill (6–47)

This equation is the asymmetric analogue of Equation (6–23). Proceeding as in Section 6.2.3.3, let us define the function

$$\Delta M_{T2\text{(max)}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}) \equiv M_{T2\text{(max)}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}) - M_{T2\text{(max)}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, 0),$$  \hfill (6–48)

which quantifies the shift of the asymmetric $M_{T2}$ endpoint as Equation (6–42) in the presence of $P_{UTM}$. By definition,

$$\Delta M_{T2\text{(max)}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM}) \geq 0,$$  \hfill (6–49)

with equality being achieved only for the correct values of the children masses:

$$\Delta M_{T2\text{(max)}}(M_c^{(a)}, M_c^{(b)}, P_{UTM}) = 0, \quad \forall P_{UTM}.$$  \hfill (6–50)

The Equation (6–50) reveals the power of the $P_{UTM}$ invariance method. Unlike the kink method discussed in Section 6.3.3.2, which was only able to find a relation between the two children masses $M_c^{(a)}$ and $M_c^{(b)}$, the $P_{UTM}$ invariance implied by Equation (6–50) allows us to determine each individual children mass, without any theoretical assumptions, and even in the case when the two children masses happen to be different ($M_c^{(a)} \neq M_c^{(b)}$).
Figure 6-3. *The three different event-topologies under consideration in this chapter. In each case, two parents with mass \( M_p \) are produced onshell and decay into two daughters of (generally different) masses \( M_z^{(a)} \) and \( M_z^{(b)} \). Case (a), which is the subject of Section 6.4, has a single massless visible SM particle in each leg and some arbitrary upstream transverse momentum \( \vec{P}_{UTM} \). In the remaining two cases (b) and (c), which are discussed in Section 6.5, there are two massless visible particles in each leg, which form a composite visible particle with varying invariant mass \( m_{(\lambda)} \). The intermediate particle of mass \( M_i^{(\lambda)} \) is (b) heavy and off-shell \( (M_i^{(\lambda)} > M_p) \), or (c) on-shell \( (M_p > M_i^{(\lambda)} > M_c^{(\lambda)}) \). For simplicity, we do not consider any upstream momentum in cases (b) and (c).*

### 6.3.4 Examples

In the next two sections we shall illustrate the three properties discussed so far in Section 6.3.3 with some concrete examples. Instead of the most general event topology depicted Figure 6-1, here we limit ourselves to the three simple examples shown in Figure 6-3.

The simplest possible case is when \( n^{(\lambda)} = 1 \), i.e. when each cascade decay contains a single SM particle, as in Figure 6-3(a). In this example, \( m_{(\lambda)} \) is constant. For simplicity, we shall take \( m_{(\lambda)} \approx 0 \), which is the case for a lepton or a light flavor jet. If the SM particle is a \( Z \)-boson or a top quark, its mass cannot be neglected, and one must keep the proper value of \( m_{(\lambda)} \). This, however, is only a technical detail, which does not affect our main conclusions below. In spite of its simplicity, the topology of Figure 6-3(a) is actually the most challenging case, due to the limited number of available measurements [23]. In order to be able to determine all individual masses
in that case, one must consider events with upstream momentum $\vec{P}_{UTM}$, as illustrated in Figure 6-3(a). This is not a particularly restrictive assumption, since there is always a certain amount of $P_{UTM}$ in the event (at the very least, from initial state radiation).

In Section 6.4 the topology of Figure 6-3(a) will be extensively studied - first for the asymmetric case of $M_c^{(a)} \neq M_c^{(b)}$ in Section 6.4.1, and then for the symmetric case of $M_c^{(a)} = M_c^{(b)}$ in Section 6.4.2.

Another simple situation arises when there are two massless visible SM particles in each leg, as illustrated in Figures 6-3(b) and 6-3(c). In either case, the invariant mass $m_{(\lambda)}$ is not constant any more, but varies within a certain range $m_{(\lambda)}^{\text{min}} \leq m_{(\lambda)} \leq m_{(\lambda)}^{\text{max}}$, where $m_{(\lambda)}^{\text{min}} = 0$, while the value of $m_{(\lambda)}^{\text{max}}$ depends on the mass $M_{i(\lambda)}$ of the corresponding intermediate particle. In Figure 6-3(b) we assume $M_{i(\lambda)} > M_p$, so that the intermediate particle is off-shell and

$$m_{(\lambda)}^{\text{max}} = M_p - M_{i(\lambda)}.$$  \hspace{1cm} (6–51)

The “off-shell” case of Figure 6-3(b) will be discussed in Section 6.5.1.

In contrast, in Figure 6-3(c) we take $M_p > M_{i(\lambda)} > M_{\xi(\lambda)}$, in which case the intermediate particle is on-shell and the range for $m_{(\lambda)}$ is now limited from above by

$$m_{(\lambda)}^{\text{max}} = M_p \sqrt{\left[1 - \left(\frac{M_{i(\lambda)}}{M_p}\right)^2\right] \left[1 - \left(\frac{M_{\xi(\lambda)}}{M_p}\right)^2\right]}.$$ \hspace{1cm} (6–52)

We shall discuss the “on-shell” case of Figure 6-3(c) in Section 6.5.2.

In the event topologies of Figures 6-3(b) and 6-3(c), the mass $m_{(\lambda)}$ is varying and the ridge of Equation (6–44) will appear even if there were no upstream transverse momentum in the event. Therefore, in our discussion of Figures 6-3(b) and 6-3(c) in Section 6.5 below we shall assume $P_{UTM} = 0$ for simplicity. The presence of non-zero $P_{UTM}$ will only additionally enhance the ridge feature.
6.3.5 Combinatorial Issues

Before going on to the actual examples in the next two sections, we need to discuss one minor complication, which is unique to the asymmetric $M_{T2}$ variable and was not present in the case of the symmetric $M_{T2}$ variable. The question is, how does one associate the visible decay products observed in the detector with a particular decay chain $\lambda = a$ or $\lambda = b$. This is the usual combinatorics problem, which now has two different aspects:

- The first issue is also present in the symmetric case, where one has to decide how to partition the SM particles observed in the detector into two disjoint sets, one for each cascade. In the traditional approach, where the children particles are assumed to be identical, the two sets are indistinguishable and it does not matter which one is first and which one is second. This particular aspect of the combinatorial problem will also be present in the asymmetric case.

- In the asymmetric case, however, there is an additional aspect to the combinatorial problem: now the two cascades are distinguishable (by the masses of the child particles), so even if we correctly divide the visible objects into the proper subsets, we still do not know which subset goes together with $M_{T2}^{(a)}$ and thus gets a label $\lambda = a$, and which goes together with $M_{T2}^{(b)}$ and gets labelled by $\lambda = b$. This leads to an additional combinatorial factor of 2 which is absent in the symmetric case with identical children.

The severity of these two combinatorial problems depends on the event topology, as well as the type of signature objects. For example, there are cases where the first combinatorial problem is easily resolved, or even absent altogether. Consider the event topology of Figure 6-3(a) with a lepton as the SM particle on each side. In this case, the partition is unique, and the upstream objects are jets, which can be easily identified [39]. Now consider the event topologies of Figures 6-3(b) and 6-3(c), with two opposite sign, same flavor leptons on each side. Such events result from inclusive pair production of heavier neutralinos in supersymmetry. By selecting events with different lepton flavors: $e^+ e^- \mu^+ \mu^-$, we can overcome the first combinatorial problem above and uniquely associate the $e^+ e^-$ pair with one cascade and the $\mu^+ \mu^-$ pair with the other. However, the second combinatorial problem remains, as we still have to decide which of
the two lepton pairs to associate with $\lambda = a$ and which to associate with $\lambda = b$. Recall that the labels $\lambda = a$ and $\lambda = b$ are already attached to the child particles, which are distinguishable in the asymmetric case. We use the convention that $\lambda = a$ is attached to the lighter child particle:

$$\tilde{M}_\xi^{(a)} \leq \tilde{M}_\xi^{(b)}, \quad (6-53)$$

which also ensures that the $\tilde{M}_-$ parameter defined in Equation (6–36) is real.

We can put this discussion in more formal terms as follows. The correct association of the visible particles with the corresponding children will yield

$$M_{T2}(\vec{p}_{T}^{(a)},\vec{p}_{T}^{(b)};m_{(a)},m_{(b)};\tilde{M}_{\xi}^{(a)},\tilde{M}_{\xi}^{(b)}), \quad (6-54)$$

while the other, wrong association will give simply

$$M_{T2}(\vec{p}_{T}^{(a)},\vec{p}_{T}^{(b)};m_{(a)},m_{(b)};\tilde{M}_{\xi}^{(b)},\tilde{M}_{\xi}^{(a)}). \quad (6-55)$$

Both of these two $M_{T2}$ values can be computed from the data, but a priori we do not know which one corresponds to the correct association. The solution to this problem is however already known [23, 33]: one can conservatively use the smaller of the two

$$M_{T2}^{(c)} \equiv \min \left\{ M_{T2}(\vec{p}_{T}^{(a)},\vec{p}_{T}^{(b)};m_{(a)},m_{(b)};\tilde{M}_{\xi}^{(a)},\tilde{M}_{\xi}^{(b)}), M_{T2}(\vec{p}_{T}^{(a)},\vec{p}_{T}^{(b)};m_{(a)},m_{(b)};\tilde{M}_{\xi}^{(b)},\tilde{M}_{\xi}^{(a)}) \right\} \quad (6-56)$$

in order to preserve the location of the upper $M_{T2}$ endpoint. This is illustrated in Figure 6-4, where we show results for the event topology of Figure 6-3(b) with a mass spectrum as follows: $M_{\xi}^{(a)} = 100$ GeV, $M_{\xi}^{(b)} = 200$ GeV and $M_p = 600$ GeV. The test children masses are taken to be the true masses: $\tilde{M}_\xi^{(a)} = M_\xi^{(a)}$ and $\tilde{M}_\xi^{(b)} = M_\xi^{(b)}$. The dotted black distribution is the unit-normalized true $M_{T2}$ distribution, where one ignores the combinatorial problem and uses the Monte Carlo information to make the correct association. The red histogram shows the unit-normalized distribution of the $M_{T2}^{(c)}$ variable defined in Equation (6–56).
Figure 6-4. Unit-normalized $M_{T2}$ distributions for the event topology of Figure 6-3(b). The mass spectrum is chosen as $M^{(a)}_c = 100$ GeV, $M^{(b)}_c = 200$ GeV and $M_p = 600$ GeV. The test children masses are taken to be the true masses: $\tilde{M}^{(a)}_c = M^{(a)}_c$ and $\tilde{M}^{(b)}_c = M^{(b)}_c$. The dotted black distribution is the true $M_{T2}$ distribution, ignoring the combinatorial problem. The red histogram shows the distribution of the $M^{(<)}_{T2}$ variable defined in Equation (6–56) while the blue histogram shows the distribution of the $M^{(>)}_{T2}$ variable defined in Equation (6–58).

We see that the definition in Equation (6–56) preserves the corresponding endpoint:

$$M^{(<)}_{T2(max)} = M_{T2(max)}.$$  \hspace{1cm} (6–57)

Of course, we can also consider the alternative combination

$$M^{(>)}_{T2} \equiv \max \left\{ M_{T2} \left( \beta^{(a)}, \beta^{(b)}; m_\alpha, m_\beta; \tilde{M}^{(a)}_c, \tilde{M}^{(b)}_c \right), M_{T2} \left( \beta^{(a)}, \beta^{(b)}; m_\alpha, m_\beta; \tilde{M}^{(b)}_c, \tilde{M}^{(a)}_c \right) \right\},$$  \hspace{1cm} (6–58)

whose unit-normalized distribution is shown in Figure 6-4 with the blue histogram. One can see that some of the wrong combination entries in the $M^{(>)}_{T2}$ histogram violate the original endpoint $M_{T2(max)}$, yet there is still a well defined $M^{(>)}_{T2}$ endpoint

$$M^{(>)}_{T2(max)} \geq M_{T2(max)}.$$  \hspace{1cm} (6–59)
Strictly speaking, in our analysis in the next sections, we only need to study the \( M_{T2}^{(c)} \) endpoint in Equation (6–57), which contains the relevant information about the physical \( M_{T2} \) endpoint. At the same time, with our convention as Equation (6–53) for the children masses, we only need to concentrate on the upper half \( \tilde{M}_c^{(b)} \geq \tilde{M}_c^{(a)} \) of the \((\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})\) plane. However, for completeness we shall also present results for the \( M_{T2}^{(c)} \) endpoint in Equation (6–59), and we shall use the lower \((\tilde{M}_c^{(b)} < \tilde{M}_c^{(a)})\) half of the \((\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})\) plane to show those. Thus the \( M_{T2} \) endpoint shown in our plots below should be interpreted as follows

\[
M_{T2(\text{max})} = \begin{cases} 
M_{T2(\text{max})}^{(c)} , & \text{if } \tilde{M}_c^{(a)} \leq \tilde{M}_c^{(b)}, \\
M_{T2(\text{max})}^{(c)} , & \text{if } \tilde{M}_c^{(a)} > \tilde{M}_c^{(b)}. 
\end{cases}
\]

(6–60)

6.4 The Simplest Event Topology: One Standard Model Particle On Each Side

In this section, we consider the simplest topology with a single visible particle on each side of the event. We already introduced this example in Section 6.3.4, along with its event topology in Figure 6-3(a). In Section 6.4.1 below we first discuss an asymmetric case with different children. Later in Section 6.4.2 we consider a symmetric situation with identical children masses. The mass spectra for these two study points are listed in Table 6-1.

### 6.4.1 Asymmetric Case

Before we present our numerical results, it will be useful to derive an analytical expression for the asymmetric \( M_{T2} \) endpoint in Equation (6–42) in terms of the corresponding physical spectrum of Table 6-1 and the two test children masses \( \tilde{M}_c^{(a)} \) and \( \tilde{M}_c^{(b)} \). Our result will generalize the corresponding formula derived in [37] for the
symmetric case of $\tilde{M}_c^{(a)} = \tilde{M}_c^{(b)} \equiv \tilde{M}_c$ and no upstream momentum ($P_{UTM} = 0$). For the event topology of Figure 6-3(a) the $M_{T2}$ endpoint is always obtained from the balanced solution and is given by [37]

$$M_{T2(max)}(\tilde{M}_c, P_{UTM} = 0) = \mu_{ppc} + \sqrt{\mu_{ppc}^2 + \tilde{M}_c^2}.$$  \hspace{1cm} (6–61)

Here we made use of the convenient shorthand notation introduced in [23] for the relevant combination of physical masses

$$\mu_{npc} \equiv \frac{M_n}{2} \left\{ 1 - \left( \frac{M_c}{M_p} \right)^2 \right\}.$$  \hspace{1cm} (6–62)

The $\mu$ parameter defined in Equation (6–62) is simply the transverse momentum of the (massless) visible particle in those events which give the maximum value of $M_{T2}$ [39]. Squaring Equation (6–61), we can equivalently rewrite it as

$$M_{T2(max)}^2(\tilde{M}_c, P_{UTM} = 0) = 2\mu_{ppc}^2 + \tilde{M}_c^2 + \sqrt{4\mu_{ppc}^2 (\mu_{ppc}^2 + \tilde{M}_c^2)}.$$  \hspace{1cm} (6–63)

Now let us derive the analogous expressions for the asymmetric case $M_c^{(a)} \neq M_c^{(b)}$. Just like the symmetric case, the asymmetric endpoint $M_{T2(max)}$ also comes from a balanced solution and is given by

$$M_{T2(max)}^2(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM} = 0) = 2\tilde{\mu}_{ppc}^2 + \tilde{M}_c^2 + \sqrt{4\tilde{\mu}_{ppc}^2 (\tilde{\mu}_{ppc}^2 + \tilde{M}_c^2)} + \tilde{M}^4.$$  \hspace{1cm} (6–64)

where the parameters $\tilde{M}_c^2$ and $\tilde{M}_c^2$ were already defined in Equation (6–35) and (6–36), while $\tilde{\mu}_{ppc}$ is now the geometric average of the corresponding individual $\mu_{ppc}$ parameters

$$\tilde{\mu}_{ppc} \equiv \mu_{ppc_a} \mu_{ppc_b} \equiv \frac{(M_p^2 - (M_c^{(a)})^2)(M_p^2 - (M_c^{(b)})^2)}{4M_p^2}.$$  \hspace{1cm} (6–65)

It is easy to check that in the symmetric limit

$$\tilde{M}_c^{(b)} \rightarrow \tilde{M}_c^{(a)} \implies \tilde{\mu}_{ppc} \rightarrow \mu_{ppc}, \quad \tilde{M}_+ \rightarrow \tilde{M}_c, \quad \tilde{M}_- \rightarrow 0,$$  \hspace{1cm} (6–66)

Equation (6–64) reduces to its symmetric counterpart of Equation (6–63), as it should.
Figure 6-5. $M_{T2}^{(max)}$ as a function of the two test children masses, $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$, for the event topology of Figure 6-3(a) with no upstream momentum ($P_{UTM} = 0$), and the asymmetric mass spectrum I from Table 6-1: $(M_c^{(a)}, M_c^{(b)}, M_D) = (250, 500, 600)$ GeV. We show (a) a three dimensional view and (b) contour plot projection on the $(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$ plane (red contour lines). The green dot marks the true values of the children masses. Panel (b) also shows a gradient plot, where longer (shorter) arrows imply steeper (gentler) slope. A kink structure is absent in this case. The symmetric endpoint $M_{T2}^{(max)}(\tilde{M}_c)$ of Equation (6–61) can be obtained by going along the diagonal orange line $\tilde{M}_c^{(b)} = \tilde{M}_c^{(a)}$.

We are now ready to present our numerical results for the event topology of Figure 6-3(a). We first take the asymmetric mass spectrum I from Table 6-1 and consider the case with no upstream momentum, when formula (6–64) applies.

Figure 6-5 shows the corresponding $M_{T2}$ endpoint as a function of the two test children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$. In panel (a) we present a three dimensional view, while in panel (b) we show a contour plot projection on the $(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$ plane (red contour lines). On either panel, the green dot marks the true values of the children masses, $M_c^{(a)}$ and $M_c^{(b)}$. Panel (b) also shows a gradient plot, where longer (shorter) arrows imply steeper (gentler) slope. The symmetric endpoint $M_{T2}^{(max)}(\tilde{M}_c, P_{UTM} = 0)$ of Equation (6–61) can be obtained by going along the diagonal orange line $\tilde{M}_c^{(b)} = \tilde{M}_c^{(a)}$ in Figure 6-5(b). We remind the reader that the endpoint $M_{T2}^{(max)}$ plotted in Figure 6-5 should be interpreted as in Equation (6–60).
Figure 6-5 illustrates the first basic property of the asymmetric $M_{T2}$ variable, which was discussed in Section 6.3.3.1. The $M_{T2}$ endpoint allows us to find one relation between the two children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$ and the parent mass $\tilde{M}_p = M_{T2(max)}$, and in order to do so, we do not have to assume equality of the children masses, as is always done in the literature. The crucial advantage of our approach, in which we allow the two children masses to be arbitrary, is its generality and model-independence. It allows us to extract the basic information contained in the $M_{T2}$ endpoint, without muddling it up with additional theoretical (and unproven) assumptions.

Unfortunately, to go any further and determine each individual mass, we must make use of the additional properties discussed in Sections 6.3.3.2 and 6.3.3.3. In the case of the simplest event topology of Figure 6-3(a) considered here, they both require the presence of some upstream momentum [23, 36]. As a proof of concept, we now reconsider the same type of events, but with a fixed upstream momentum of $P_{UTM} = 1$ TeV. (The upstream momentum may be due to initial state radiation, or decays of heavier particles upstream.) The corresponding results are shown in Figure 6-6.

Figure 6-6 demonstrates the second basic property of the asymmetric $M_{T2}$ variable discussed in Section 6.3.3.2. Unlike the result shown in Figure 6-5(a), which was perfectly smooth, this time the $M_{T2(max)}$ function in Figure 6-6(a) shows a ridge, corresponding to the slope discontinuity marked with the black solid line in Figure 6-6(b). The most important feature of the ridge is the fact that it passes through the green dot marking the true values of the children masses. Notice that applying the traditional symmetric $M_{T2}$ approach in this case will give a completely wrong result. If we were to assume equal children masses from the very beginning, we will be constrained to the diagonal orange line in Figure 6-6(b). The $M_{T2}$ endpoint will then still exhibit a kink, but the kink will be in the wrong location. In the example shown in Figure 6-6(b), we will underestimate the parent mass, while for the child mass we will find a value which is somewhere in between the two true masses $M_c^{(a)}$ and $M_c^{(b)}$. 

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Figure 6-6. $M_{T2(\text{max})}$ for the event topology of Figure 6-3(a) with fixed upstream momentum of $P_{UTM} = 1$ TeV. The ridge structure (shown as the black solid line) is revealed by the sudden increase in the slope (gradient) in panel (b). Notice that the ridge goes through the true values of the children masses marked by the green dot.

Using the ridge information, we now know an additional relation among the children masses, which allows us to express all three masses in terms of a single unknown parameter $\theta$, as illustrated in Figure 6-7(a). Let us choose to parametrize the ridge by the polar angle in the $(\tilde{M}_{c}^{(a)}, \tilde{M}_{c}^{(b)})$ plane:

$$\theta = \tan^{-1}\left(\frac{\tilde{M}_{c}^{(b)}}{\tilde{M}_{c}^{(a)}}\right). \quad (6–67)$$

Using the ridge information from Figure 6-6, we can then find all three masses as a function of $\theta$. The result is shown in Figure 6-7(a). The mass $\tilde{M}_{c}^{(a)}$ of the lighter child is plotted in red, the mass $\tilde{M}_{c}^{(b)}$ of the heavier child is plotted in blue, while the parent mass $\tilde{M}_{p}$ is plotted in black. With our convention (6–53) for the children masses, only values of $\theta \geq 45^\circ$ are physical, and the corresponding masses are shown with solid lines. The dotted lines in Figure 6-7(a) show the extrapolation into the unphysical region $\theta < 45^\circ$.

Figure 6-7(a) has some important and far reaching implications. For example, one may now start asking the question: Are there really any massive invisible particles in those events, or is the missing energy simply due to neutrino production [94]?
Figure 6-7. (a) Particle masses obtained along the $M_{T2(max)}$ ridge seen in Figure 6-6. The ridge is parametrized by the angle $\theta$ defined in Equation (6–67). The two children masses $\tilde{M}_c^{(a)}(\theta)$ (in red) and $\tilde{M}_c^{(b)}(\theta)$ (in blue) as well as the parent mass $\tilde{M}_p$ (in black) are then plotted as a function of $\theta$. In our convention (6–53) only values of $\theta \geq 45^\circ$ are physical, and the corresponding masses are shown with solid lines. Dotted lines show the extrapolation for $\theta < 45^\circ$. (b) Contour plot of the quantity $\Delta M_{T2(max)}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM} = 1 \text{ TeV})$ defined in Equation (6–48), in the ($\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}$) plane. This plot is obtained simply by taking the difference between Figure 6-6(a) and Figure 6-5(a). The solid black curve indicates the location of the $M_{T2(max)}$ ridge. Only the point corresponding to the true children masses (the green dot) satisfies the $P_{UTM}$ invariance condition $\Delta M_{T2(max)} = 0$ from Equation (6–50).

The ridge results shown in Figure 6-7(a) begin to provide the answer to that quite fundamental question. According to Figure 6-7(a), for any value of the (still unknown) parameter $\theta$, the two children particles cannot be simultaneously massless. This means that the missing energy cannot be simply due to neutrinos, i.e. there is at least one new, massive invisible particle produced in the missing energy events. At this point, we cannot be certain that this is a dark matter particle, but establishing the production of a WIMP candidate at a collider is by itself a tremendously important result. Notice that while we cannot be sure about the masses of the children, the parent mass $M_p$ is
determined with a very good precision from Figure 6-7(a): the function $\tilde{M}_p(\theta)$ is almost flat and rather insensitive to the particular value of $\theta^9$.

Once we have proved that some kind of WIMP production is going on, the next immediate question is: how many such WIMP particles are present in the data – one or two? Unfortunately, the ridge analysis of Figure 6-7(a) alone cannot provide the answer to this question, since the value of $\theta$ is still undetermined. If $\theta = 90^\circ$, one of the missing particles is massless, which is consistent with a SM neutrino. Therefore, if $\theta$ were indeed $90^\circ$, the most plausible explanation of this scenario would be that only one of the missing particles is a genuine WIMP, while the other is a SM neutrino. On the other hand, almost any other value of $\theta < 90^\circ$ would guarantee that there are two WIMP candidates in each event. In that case, the next immediate question is: are they the same or are they different? Fortunately, our asymmetric approach will allow answering this question in a model-independent way. If $\theta$ is determined to be $45^\circ$, the two WIMP particles are the same, i.e. we are producing a single species of dark matter. On the other hand, if $45^\circ < \theta < 90^\circ$, then we can be certain that there are not one, but two different WIMP particles being produced.

We see that in order to completely understand the physics behind the missing energy signal, we must determine the value of $\theta$, i.e. we must find the exact location of the true children masses along the ridge. One of our main results in this chapter is that this can be done by using the third $M_{T2}$ property discussed in Section 6.3.3.3. The idea is illustrated in Figure 6-7(b), where we show a contour plot in the $(\tilde{M}_{\ell}^{(a)}, \tilde{M}_{\ell}^{(b)})$ plane of the quantity $\Delta M_{T2(max)}(\tilde{M}_{\ell}^{(a)}, \tilde{M}_{\ell}^{(b)}, P_{UTM})$ defined in Equation (6–48), for a fixed $P_{UTM} = 1$ TeV. This plot is obtained simply by taking the difference between

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9 Interestingly, for the example in Figure 6-7(a), the maximum value of $\tilde{M}_p(\theta)$ happens to give the true parent mass $M_p$, but we have checked that this is a coincidence and does not hold in general for other examples which we have studied.
Figure 6-6(a) and Figure 6-5(a). (A more practical method for obtaining this information was proposed in [12].) Recall that the function $\Delta M_{T2(\text{max})}$ was introduced in order to quantify the $P_{\text{UTM}}$ invariance of the $M_{T2}$ endpoint, and it is expected that $\Delta M_{T2(\text{max})}$ vanishes at the correct values of the children masses (see Equation (6–50)). This expectation is confirmed in Figure 6-7(b), where we find the minimum (zero) of the $\Delta M_{T2(\text{max})}$ function exactly at the right spot (marked with the green dot) along the $M_{T2(\text{max})}$ ridge. Thus the $\Delta M_{T2(\text{max})}$ function in Figure 6-7(b) completely pins down the spectrum, and in this case would reveal the presence of two different WIMP particles, with unequal masses $M_c^{(a)} \neq M_c^{(b)}$. Our analysis thus shows that colliders can not only produce a WIMP dark matter candidate and measure its mass, as discussed in the existing literature, but they can do a much more elaborate dark matter particle spectroscopy, as advertized in the title. In particular, they can probe the number and type of missing particles, including particles from subdominant dark matter species, which are otherwise unlikely to be discovered experimentally in the usual dark matter searches.

6.4.2 Symmetric Case

While in our approach the two children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$ are treated as independent inputs, this, of course, does not mean that the approach is only valid in cases when the children masses are different to begin with. The techniques discussed in the previous subsection remain applicable also in the more conventional case when the children are identical, i.e. when colliders produce a single dark matter component. In order to illustrate how our method works in that case, we shall now work out an example with equal children masses. We still consider the simplest event topology of Figure 6-3(a), but with the symmetric mass spectrum II from Table 6-1. We then repeat the analysis done in Figures 6-5, 6-6, and 6-7 and show the corresponding results in Figures 6-8, 6-9 and 6-10.
Figure 6-8. $M_{T2(\text{max})}$ for the event topology of Figure 6-3(a) with no upstream momentum. Particles have the symmetric mass spectrum II from Table 6-1, i.e. $(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, \tilde{M}_p) = (100, 100, 300)$ GeV.

Figure 6-9. $M_{T2(\text{max})}$ for the event topology of Figure 6-3(a) with fixed upstream momentum $P_{UTM} = 1$ TeV. Particles have the symmetric mass spectrum II from Table 6-1, i.e. $(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, \tilde{M}_p) = (100, 100, 300)$ GeV.

The conclusions from this exercise are very similar to what we found earlier in Section 6.4.1 for the asymmetric case. The $M_{T2}$ endpoint still provides one relation among the two children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$ and the parent mass $\tilde{M}_p = M_{T2(\text{max})}$. This relation is shown in Figure 6-8 (Figure 6-9) for the case without (with) upstream momentum $P_{UTM}$. As seen in Figure 6-8, in the absence of any upstream $P_{UTM}$, the function $\tilde{M}_p(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$ is smooth and reveals nothing about the children masses.
Figure 6-10. The same as in Figure 6-7 but for the symmetric mass spectrum II from Table 6-1, i.e. \((M_c^{(a)}, M_c^{(b)}, M_P) = (100, 100, 300)\) GeV. Notice that, in contrast to Figure 6-7, the minimum of the \(\Delta M_{T2_{\text{max}}}\) function is now obtained at \(\tilde{M}_c^{(a)} = \tilde{M}_c^{(b)}\), implying that the two missing particles are the same.

However, the presence of upstream momentum significantly changes the picture and the function \(\tilde{M}_P(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})\) again develops a ridge, which is clearly visible\(^{10}\) in both the three-dimensional view of Figure 6-9(a), as well as the gradient plot in Figure 6-9(b).

The ridge information now further constrains the children masses to the black solid line in Figure 6-9(b), leaving only one unknown degree of freedom. Parametrizing it with the polar angle \(\theta\) as in \((6-67)\), we obtain the spectrum as a function of \(\theta\), as shown in Figure 6-10(a). Once again we find the fortuitous result that in spite of the remaining arbitrariness in the value of \(\theta\), the parent mass \(M_P\) is very well determined, since \(\tilde{M}_P(\theta)\) is a very weakly varying function of \(\theta\). Furthermore, both Figure 6-9(a) and Figure 6-9(b)

\[^{10}\text{We caution the reader that here we are presenting only a proof of concept. In the actual analysis the ridge may be rather difficult to see, for a variety of reasons - detector resolution, finite statistics, combinatorial and SM backgrounds, etc. Nevertheless, we expect that the ridge will be just as easily observable as the traditional kink in the symmetric \(M_{T2}\) endpoint. If the kink can be seen in the data, the ridge can be seen too, and there is no reason to make the assumption of equal children masses. Conversely, if the kink is too difficult to see, the ridge will remain hidden as well.}\]
exhibit a high degree of symmetry under $\tilde{M}_c^{(a)} \leftrightarrow \tilde{M}_c^{(b)}$, which is a good hint that the children are in fact identical. This suspicion is confirmed in Figure 6-10(b), where we find that the $P_{UTM}$ dependence disappears at the symmetric point $\tilde{M}_c^{(a)} = \tilde{M}_c^{(b)} = 100$ GeV, revealing the true masses of the two children.

In the two examples considered so far in Sections 6.4.1 and 6.4.2, we used a fixed finite value of the upstream transverse momentum $P_{UTM} = 1$ TeV, which is probably rather extreme — in realistic models, one might expect typical values of $P_{UTM}$ on the order of several hundred GeV. However, things begin to get much more interesting if one were to consider even larger values of $P_{UTM}$. On the one hand, the ridge feature becomes sharper and easier to observe. More importantly, the ridge structure itself is modified, and a second set of ridgelines appears at sufficiently large $P_{UTM}$. All ridgelines intersect precisely at the point marking the true values of the children masses, thus allowing the complete determination of the mass spectrum by the ridge method alone. This procedure was demonstrated explicitly in Reference [40], which investigated the extreme case of $P_{UTM} = \infty$ for a study point with different parents and identical children. The assumption of $P_{UTM} = \infty$ justified the use of a “decoupling argument”, in which the two branches $\lambda = a$ and $\lambda = b$ are treated independently, allowing the derivation of simple analytical expressions for the $M_{T2}$ endpoint [40]. In Appendix C we reproduce the analogous analytical results at $P_{UTM} \to \infty$ for the case of interest here (identical parents and different children) and study in detail the $P_{UTM}$ dependence of the ridgelines. Unfortunately, we find that the values of $P_{UTM}$ necessary to reveal the additional ridge structure, are too large to be of any interest experimentally. On the positive side, the $P_{UTM}$ invariance method discussed in Section 6.2.3.3 does not require

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11 A keen observer may have already noticed a hint of those in Figures 6-7(b) and 6-10(b).
such extremely large values of $P_{UTM}$ and can in principle be tested in more realistic experimental conditions.

### 6.4.3 Mixed Case

For simplicity, so far in our discussion we have been studying only one type of missing energy events at a time. In reality, the missing energy sample may contain several different types of events, and the corresponding $M_{T2}$ measurements will first need to be disentangled from each other.

For concreteness, consider the inclusive pair production of some parent particle $\chi_p$, which can decay either to a child particle $\chi_a$ of mass $M_c^{(a)}$, or a different child particle $\chi_b$ of mass $M_c^{(b)}$. Let the corresponding branching fractions be $B_a$ and $B_b$, i.e. $B_a \equiv B(\chi_p \rightarrow \chi_a)$ and $B_b \equiv B(\chi_p \rightarrow \chi_b)$. Furthermore, let $\chi_b$ decay invisibly\(^{12}\) to $\chi_a$. Such a situation can be easily realized in supersymmetry, for example, with the parent being a squark, a slepton, or a gluino, the heavier child $\chi_b$ being a Wino-like neutralino $\tilde{\chi}_2^0$ and the lighter child $\chi_a$ being a Bino-like neutralino $\tilde{\chi}_1^0$. The heavier neutralino has a large invisible decay mode $\tilde{\chi}_2^0 \rightarrow \tilde{\chi}_1^0 \nu \bar{\nu}$, if its mass happens to fall between the sneutrino mass and the left-handed slepton mass: $M_{\tilde{\nu}} < M_{\tilde{\chi}_2^0} < M_{\tilde{\ell}_L}$.

Let us start with a certain total number of events $N_{pp}$ in which two parent particles $\chi_p$ have been produced. Then the missing energy sample will contain $N_{bb} = N_{pp} B_b^2$ symmetric events where the two children are $\chi_b$ and $\chi_b$, $N_{aa} = N_{pp} B_a^2$ symmetric events where the two children are $\chi_a$ and $\chi_a$, and $N_{ab} = 2N_{pp} B_a B_b$ asymmetric events where the two children are $\chi_a$ and $\chi_b$. How can one analyze such a mixed event sample with a single $M_{T2}$ variable?

The black histogram in Figure 6-11 shows the unit-normalized $M_{T2}$ distribution for the whole (mixed) event sample (for convenience, we do not show the zero bin [12]).

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\(^{12}\) If $\chi_b$ decays visibly, then the respective types of events can in principle be sorted by their signature.
Figure 6-11. Unit-normalized, zero-bin subtracted $M_{T2}$ distribution (black histogram) for the full mixed event sample, as well as the individual components $\chi_a\chi_a$ (red), $\chi_a\chi_b$ (blue) and $\chi_b\chi_b$ (green). We took zero test masses for the children $\tilde{M}^{(a)}_c = \tilde{M}^{(b)}_c = 0$ and equal branching fraction for the parents $B_a = B_b = 50\%$. The mass spectrum is taken from the asymmetric study point I in Table 6-1 with $M^{(a)}_c = 250$ GeV, $M^{(b)}_c = 500$ GeV and $M_p = 600$ GeV. The three arrows indicate the expected endpoints for each individual component in the sample.

For this plot, we used the asymmetric mass spectrum I from Table 6-1: $M^{(a)}_c = 250$ GeV, $M^{(b)}_c = 500$ GeV and $M_p = 600$ GeV, and chose zero test masses for the children $\tilde{M}^{(a)}_c = \tilde{M}^{(b)}_c = 0$. For definiteness, we fixed equal branching fractions $B_a = B_b = 50\%$, so that the relative normalization of the three individual samples is $N_{aa} : N_{bb} : N_{ab} = 1 : 1 : 2$.

Figure 6-11 shows that the observable $M_{T2}$ distribution is simply a superposition of the $M_{T2}$ distributions of the three individual samples $\chi_a\chi_a$, $\chi_a\chi_b$ and $\chi_b\chi_b$, which are shown with the red, blue and green histograms, correspondingly. Each individual sample exhibits its own $M_{T2}$ endpoint, marked with a vertical arrow, which can also be seen in the combined $M_{T2}$ distribution. Using Equation (6–64), the three endpoints are found to
be

\[ \chi_a \chi_a \rightarrow M_{T2(\text{max})}^{(aa)}(0, 0, 0) = M_p \left[ 1 - \left( \frac{M_c^{(a)}}{M_p} \right)^2 \right] = 496 \text{ GeV}, \quad (6–68) \]

\[ \chi_a \chi_b \rightarrow M_{T2(\text{max})}^{(ab)}(0, 0, 0) = M_p \left[ 1 - \left( \frac{M_c^{(a)}}{M_p} \right)^2 \right] \left[ 1 - \left( \frac{M_c^{(b)}}{M_p} \right)^2 \right] = 301 \text{ GeV} \quad (6–69) \]

\[ \chi_b \chi_b \rightarrow M_{T2(\text{max})}^{(bb)}(0, 0, 0) = M_p \left[ 1 - \left( \frac{M_c^{(b)}}{M_p} \right)^2 \right] = 183 \text{ GeV}. \quad (6–70) \]

Now suppose that all three endpoints as in Equations (6–68 through 6–70) are seen in the data. Their interpretation is far from obvious, and in fact, there will be different competing explanations. If one insists on the single missing particle hypothesis, there can be only one type of child particle, and the only way to get three different endpoints in Figure 6-11 is to have production of three different pairs of parent particles, each of which decays in exactly the same way. Since the three parent masses are a priori unrelated, one does not expect any particular correlation among the three observed endpoints in Equations (6–68 through 6–70). Now consider an alternative explanation where we produce a single type of parents, but have two different children types. This situation also gives rise to three different event topologies, with three different \( M_{T2} \) endpoints, as we just discussed. However, now there is a predicted relation among the three \( M_{T2} \) endpoints, which follows simply from Equations (6–68 through 6–70):

\[ M_{T2(\text{max})}^{(ab)}(0, 0, 0) = \sqrt{M_{T2(\text{max})}^{(aa)}(0, 0, 0) M_{T2(\text{max})}^{(bb)}(0, 0, 0)}. \quad (6–71) \]

If the parents are the same and the children are different, this relation must be satisfied. If the parents are different and the children are the same, a priori there is no reason why Equation (6–71) should hold, and if it does, it must be by pure coincidence.

The prediction of Equation (6–71) therefore is a direct test of the number of children
particles. Another test can be performed if we could estimate the individual event counts \( N_{aa}, N_{ab} \) and \( N_{bb} \), although this appears rather difficult, due to the unknown shape of the \( M_{T2} \) distributions in Figure 6-11. In the asymmetric example discussed here, we have another prediction, namely

\[
N_{ab} = 2\sqrt{N_{aa}N_{bb}},
\]

(6–72)

which is another test of the different children hypothesis. Notice that Equation (6–72) holds regardless of the branching fractions \( B_a \) and \( B_b \), although if one of them dominates, the two endpoints which require the other (rare) decay may be too difficult to observe.

Of course, the ultimate test of the single missing particle hypothesis is the behavior of the intermediate \( M_{T2} \) endpoint in Figure 6-11 corresponding to the asymmetric events of type \( \chi_a\chi_b \). Applying either one of the two mass determination methods discussed earlier in Figures 6-7 and 6-10, we should find that \( M_{T2(max)}^{(ab)} \) is a result of asymmetric events, indicating the simultaneous presence of two different invisible particles in the data.

6.5 A More Complex Event Topology: Two Visible Particles On Each Side

In this section, we consider two more examples: the off-shell event topology of Figure 6-3(b) is discussed in Section 6.5.1, while the on-shell event topology of Figure 6-3(c) is discussed in Section 6.5.2. (For simplicity, we do not consider any \( P_{UTM} \) in this section.) Now there are two visible particles in each leg, which form a composite visible particle of varying mass \( m_{(\lambda)} \). In general, by studying the invariant mass distribution of \( m_{(\lambda)} \), one should be able to observe two different invariant mass endpoints, suggesting some type of an asymmetric scenario.

6.5.1 Off-shell Intermediate Particle

Here we concentrate on the example of Figure 6-3(b). Since the intermediate particle is offshell, the maximum kinematically allowed value for \( m_{(\lambda)} \) is given by Equation (6–51).
Recall that for the simple topology of Figure 6-3(a) discussed in the previous section, the $M_{T2}$ endpoint (6–64) always corresponded to a balanced solution. More precisely, the $M_{T2}$ variable was maximized for a momentum configuration $\vec{p}_T^{(\lambda)}$ in which $M_{T2}$ was given by the balanced solution (6–34). However, in this section we shall find that for the more complex topologies of Figures 6-3(b) and 6-3(c), the $M_{T2}$ endpoint may result from one of four different cases altogether: two different balanced solutions, which we shall label as $B$ and $B'$, or the unbalanced solutions $U_a$ and $U_b$ discussed in Section
6.3.2. Depending on the type of solution giving the endpoint $M_{T2(max)}$, the $(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$ parameter plane divides into the three regions\(^{13}\) shown in Figure 6-12.

The green dot in Figure 6-12 denotes the true children masses in this parameter space. Within each region, we show the relevant momentum configuration for the visible particles (red arrows) and the children particles (blue arrows) in each leg ($a$ or $b$). The momenta are quoted in the “back-to-back boosted” frame [37], in which the two parents are at rest. The length of an arrow is indicative of the magnitude of the momentum. A blue dot implies that the corresponding daughter is at rest and therefore the two visible particles are emitted back-to-back. The two balanced solutions are denoted as $B$ and $B'$, while the two unbalanced solutions are $Ua$ and $Ub$. The black solid lines represent phase changes between different solution types and delineate the expected locations of the ridges in the $M_{T2(max)}$ function shown in Figure 6-13 below. Perhaps the most striking feature of Figure 6-12 is that the three (in fact, all four) regions come together precisely at the green dot marking the true values of the two children masses. The boundaries of the regions shown in Figure 6-12 will manifest themselves as the locations of the ridges (i.e. gradient discontinuities) in the $M_{T2(max)}$ function. Therefore, we expect that by studying the ridge structure and finding its “triple” point, one will be able to completely determine the mass spectrum.

We shall now give analytical formulas for the $M_{T2}$ endpoint in each of the four regions of Figure 6-12. We begin with the two balanced solutions $B$ and $B'$, for which the event-by-event balanced solution for $M_{T2}$ is given by Equation (6–34). In the parameter space region of Figure 6-12 which is adjacent to the origin, we find the balanced configuration $B$, in which all visible particles have the same direction in the

\(^{13}\) The fourth case of the $B'$ balanced solution happens to coincide with the two unbalanced solutions along the boundary between $Ua$ and $Ub$. 
“back-to-back boosted” frame. As a result, we have

\[ m(a) = m(b) = 0 \]  

and

\[ A_T = \frac{(M_p^2 - (M_c^{(a)})^2)(M_p^2 - (M_c^{(b)})^2)}{2M_p^2} \].

Substituting Equations (6–73) and (6–74) in the balanced \( M_{T2} \) solution (6–34), where we should take the plus sign, we obtain

\[ \left[ M_{T2_{\text{max}}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \right]^2 = 2\tilde{\mu}_{ppc}^2 + \tilde{M}_+^2 + \sqrt{4\tilde{\mu}_{ppc}^2(\tilde{\mu}_{ppc}^2 + \tilde{M}_+^2) + \tilde{M}_-^2}, \]

which we recognize as the balanced solution (6–64) found for the decay topology of Figure 6-3(a).

Moving away from the origin in Figure 6-12, we find a second balanced solution \( B' \) along the boundary of the unbalanced regions \( U_a \) and \( U_b \). In this case the visible particles are back-to-back, and their invariant mass is maximized:

\[ m(\lambda) = M_p - M_c^{(\lambda)} \],

and correspondingly

\[ A_T = (M_p - M_c^{(a)}) (M_p - M_c^{(b)}) \].

Substituting Equations (6–76) and (6–77) in the balanced \( M_{T2} \) solution (6–34), we obtain the \( B' \)-type \( M_{T2} \) endpoint as

\[ \left[ M_{T2_{\text{max}}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \right]^2 = (M_p - M_c^{(a)}) (M_p - M_c^{(b)}) + \tilde{M}_+^2 + \frac{2M_p - M_c^{(a)} - M_c^{(b)}}{M_c^{(b)} - M_c^{(a)}} \tilde{M}_-^2. \]

The corresponding formulas for the unbalanced cases \( U_a \) and \( U_b \) are obtained by taking the maximum value for the invariant mass of the visible particles in the
corresponding decay chain:

\[ m_{(a)} = m_{(a)}^{\text{max}} = M_p - M_{c}^{(a)} \quad \text{for region (Ua)}, \quad (6–79) \]

\[ m_{(b)} = m_{(b)}^{\text{max}} = M_p - M_{c}^{(b)} \quad \text{for region (Ub)}. \quad (6–80) \]

The corresponding formula for \( M_{T2(\text{max})} \) is then given by

\[ M_{T2(\text{max})}^{Ua}(\tilde{M}_c^{(a)}) = M_p - M_{c}^{(a)} + \tilde{M}_c^{(a)}, \quad (6–81) \]

\[ M_{T2(\text{max})}^{Ub}(\tilde{M}_c^{(b)}) = M_p - M_{c}^{(b)} + \tilde{M}_c^{(b)}. \quad (6–82) \]

One can now use the analytical results of Equations (6–75), (6–78), (6–81) and (6–82) to understand the ridge structure shown in Figure 6-12. For example, the boundary between the \( B \) and \( Ua \) regions is parametrically given by the condition

\[ M_{T2(\text{max})}^{B}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) = M_{T2(\text{max})}^{Ua}(\tilde{M}_c^{(a)}), \quad (6–83) \]

while the boundary between the \( B \) and \( Ub \) regions is parametrically given by

\[ M_{T2(\text{max})}^{B}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) = M_{T2(\text{max})}^{Ub}(\tilde{M}_c^{(b)}). \quad (6–84) \]

On the other hand, the boundary

\[ M_{T2(\text{max})}^{Ua}(\tilde{M}_c^{(a)}) = M_{T2(\text{max})}^{Ub}(\tilde{M}_c^{(b)}) \quad (6–85) \]

between the two unbalanced regions \( Ua \) and \( Ub \) is quite interesting. The parametric equation (6–85) is nothing but a straight line in the \((\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})\) plane:

\[ \tilde{M}_c^{(b)} = M_{c}^{(b)} - M_{c}^{(a)} + \tilde{M}_c^{(a)}, \quad (6–86) \]

as seen in Figure 6-12.
Figure 6-13. $M_{T2(\text{max})}$ as a function of the two test children masses, $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$, for the off-shell event topology of Figure 6-3(b). We use the mass spectrum from the example in Figure 6-4: $M_c^{(a)} = 100$ GeV, $M_c^{(b)} = 200$ GeV and $M_p = 600$ GeV and for simplicity consider only events with $P_{UTM} = 0$.

It is now easy to understand the triple point structure in Figure 6-12. The triple point is obtained by the merging of all three boundaries as in Equations (6–83), (6–84) and (6–85), i.e.

$$M_{T2(\text{max})}^B(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) = M_{T2(\text{max})}^{B'}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) = M_{T2(\text{max})}^{Ua}(\tilde{M}_c^{(a)}) = M_{T2(\text{max})}^{Ub}(\tilde{M}_c^{(b)}).$$

(6–87)

It is easy to check that $\tilde{M}_c^{(a)} = M_c^{(a)}$ and $\tilde{M}_c^{(b)} = M_c^{(b)}$ identically satisfy these equations, thereby proving that the triple intersection of the boundaries seen in Figure 6-12 indeed takes place at the true values of the children masses. These results are confirmed in our numerical simulations. In Figure 6-13 we present (a) a three dimensional view and (b) a gradient plot of the ridge structure found in events with the off-shell topology of Figure 6-3(b). The mass spectrum for this study point was fixed as in Figure 6-4, namely $M_c^{(a)} = 100$ GeV, $M_c^{(b)} = 200$ GeV and $M_p = 600$ GeV. Since the ridge structure for this topology does not require the presence of upstream momentum, for simplicity we consider only events with $P_{UTM} = 0$. The ridge pattern is clearly evident in Figure 6-13(a), which shows a three-dimensional view of the $M_{T2}$ endpoint function $M_{T2(\text{max})}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$. It is even more apparent in Figure 6-13(b), where one can see a
sharp gradient change along the ridge lines: in regions $Ua$ and $Ub$, the corresponding gradient vectors point in trivial directions (either horizontally or vertically), in accord with Equations (6–81) and (6–82). On the other hand, the gradient in region $B$ is very small, and the $M_{T2}$ endpoint function is rather flat. The green dot marks the location of the true children masses ($M_{c}^{(a)} = 100$ GeV, $M_{c}^{(b)} = 200$ GeV) and is indeed the intersection point of the three ridgelines. As expected, the corresponding $M_{T2(max)}$ at that point is the true parent particle mass $M_{p} = 600$ GeV.

At this point, it is interesting to ask the question, what would be the outcome of this exercise if one were to make the usual assumption of identical children, and apply the traditional symmetric $M_{T2}$ to this situation. The answer can be deduced from Figure 6-13(b), where the diagonal orange dotdashed line corresponds to the usual assumption of $\tilde{M}_{c}^{(a)} = \tilde{M}_{c}^{(b)}$. In that case, one still finds a kink, but at the wrong location: in Figure 6-13(b) the intersection of the diagonal orange line and the solid black ridgeline occurs at $\tilde{M}_{c}^{(a)} = \tilde{M}_{c}^{(b)} = 65.3$ GeV and the corresponding parent mass is $\tilde{M}_{p} = 565.3$ GeV. Therefore, the traditional kink method can easily lead to a wrong mass measurement. Then the only way to know that there was something wrong with the measurement would be to study the effect of the upstream momentum and see that the observed kink is not invariant under $P_{UTM}$.

We should note that, depending on the actual mass spectrum, the two-dimensional ridge pattern seen in Figures 6-12 and 6-13(b) may look very differently. For example, the balanced region $B$ may or may not include the origin. One can show that if

$$M_{p} < \frac{M_{c}^{(b)}}{4M_{c}^{(a)}} \left( M_{c}^{(b)} + \sqrt{8(M_{c}^{(a)})^{2} + (M_{c}^{(b)})^{2}} \right),$$  

(6–88)

the boundary between $B$ and $Ua$ does not cross the $\tilde{M}_{c}^{(a)}$ axis. In this case the diagonal line in Figure 6-13(b) does not cross any ridgelines and the traditional $M_{T2}$ approach will not produce any kink structure, in contradiction with one's expectations.
Figure 6-14. The four regions in the \((\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})\) parameter plane leading to the four different types of solutions for the \(M_{T2}\) endpoint for the onshell scenario illustrated in Figure 6-3(c).

This exercise teaches us that the failsafe approach to measuring the masses in missing energy events is to apply from the very beginning the asymmetric \(M_{T2}\) concept.

### 6.5.2 On-shell Intermediate Particle

Our final example is the on-shell event topology illustrated in Figure 6-3(c). Now there is an additional parameter which enters the game — the mass \(M_i^{(\lambda)}\) of the intermediate particle in the \(\lambda\)-th decay chain. As a result, the allowed range of invariant masses for the visible particle pair on each side is limited from above by Equation (6–52).

In this case we find that the \(M_{T2}\) endpoint exhibits a similar phase structure as the one shown in Figure 6-12. One particular pattern is illustrated in Figure 6-14, which exhibits the same four regions \(B, B', U_a\) and \(U_b\) seen in Figure 6-12. The difference now is that region \(B'\) is considerably expanded, and as a result, region \(B\) does not have a common border with regions \(U_a\) and \(U_b\) any more. The triple point of Figure 6-12 has now disappeared and the correct values of the children masses now lie somewhere on
the border between regions $B$ and $B'$, but their exact location along this ridgeline is at this point unknown.

Just like we did for the off-shell case in Section 6.5.1, we shall now present analytical formulas for the $M_{T2}$ endpoint in each region of Figure 6-14. In the balanced region $B$, we find the same results as Equations (6–73 through 6–75) as in the off-shell case considered in the previous Section 6.5.1. The other balanced region $B'$ is characterized by

$$m_{(\lambda)} = m_{(\lambda)}^{\text{max}},$$

where $m_{(\lambda)}^{\text{max}}$ is given by Equation (6–52), and

$$A_T = \frac{M_p^2}{4} \left[ 2 - \left( \frac{M_{(a)}^{(a)}}{M_p} \right)^2 - \left( \frac{M_{(a)}^{(a)}}{M_i} \right)^2 \right] \left[ 2 - \left( \frac{M_{(b)}^{(b)}}{M_p} \right)^2 - \left( \frac{M_{(b)}^{(b)}}{M_i} \right)^2 \right]$$

$$+ \frac{M_p^2}{4} \left[ \left( \frac{M_{(a)}^{(a)}}{M_i} \right)^2 - \left( \frac{M_{(a)}^{(a)}}{M_p} \right)^2 \right] \left[ \left( \frac{M_{(b)}^{(b)}}{M_i} \right)^2 - \left( \frac{M_{(b)}^{(b)}}{M_p} \right)^2 \right].$$

(6–90)

The formula for the endpoint $M_{T2}^{B'}(\text{max})$ in region $B'$ is then simply obtained by substituting Equations (6–89) and (6–90) into the balanced solution (6–34).

Finally, the $M_{T2}$ endpoint in the unbalanced regions $Ua$ and $Ub$ is given by

$$M_{T2}^{Ua}(\tilde{M}_{(a)}) = m_{(a)}^{\text{max}} + \tilde{M}_{(a)}^{(a)},$$

(6–91)

$$M_{T2}^{Ub}(\tilde{M}_{(b)}) = m_{(b)}^{\text{max}} + \tilde{M}_{(b)}^{(b)},$$

(6–92)

where $m_{(a)}^{\text{max}}$ and $m_{(b)}^{\text{max}}$ are given by Equation (6–52).

In Figure 6-15 we present our numerical results in this on-shell scenario. The mass spectrum is fixed as: $M_{(a)}^{(a)} = 100$ GeV, $M_{(b)}^{(b)} = 200$ GeV, $M_{(a)}^{(a)} = M_{(b)}^{(b)} = 550$ GeV and $M_p = 1$ TeV, and we still do not include the effects of any upstream momentum. Figure 6-15(a) shows the three-dimensional view of the $M_{T2}$ endpoint function $M_{T2}^{Ua}(\tilde{M}_{(a)}, \tilde{M}_{(b)})$, which exhibits three different sets of ridges, which are more easily seen in the gradient plot of Figure 6-15(b).
As usual, the green dot marks the true children masses. Figure 6-15(b) shows that the ridgeline separating the two balanced regions $B$ and $B'$ does go through the green dot and thus reveals a relationship between the two children masses, leaving the ridgeline parameter $\theta$ as the only remaining unknown degree of freedom. However, unlike the off-shell case of Section 6.5.1, now there is no special point on this ridgeline, and we cannot completely pin down the masses by the ridge method. Thus, in order to determine all masses in the problem, one must use an additional piece of information, for example the visible invariant mass endpoint (6–52) or the $P_{UTM}$ invariance method suggested in Section 6.3.3.3.

6.6 Application To More General Cases

In this section we discuss a few other possible applications of the asymmetric $M_{T2}$ idea, besides the examples already considered.

1. Invisible decays of the next-to-lightest particle. Most new physics models introduce some new massive and neutral particle which plays the role of a dark matter candidate. Often the very same models also contain other, heavier particles, which for collider purposes behave just like a dark matter candidate: they decay invisibly and result in missing energy in the detector. For example, in supersymmetry one may find an invisibly decaying sneutrino $\tilde{\nu}_\ell \rightarrow \nu_\ell \tilde{\chi}_1^0$, in UED one finds an invisibly
Figure 6-16. Event topology for the effectively different missing particles. The black solid lines represent SM particles which are visible in the detector while red solid lines represent particles at intermediate sages. The missing particles are denoted by dotted lines. (a) Squark pair production with decay chains terminating in two different invisible particles ($\tilde{\chi}^0_1$ and $\tilde{\nu}_\ell$, correspondingly). In this case $\tilde{\nu}_\ell$ decays invisibly. (b) The subsystem $M_{T2}$ variable applied to $t\bar{t}$ events. The $W$-boson in the lower leg is treated as a child particle and can decay either hadronically or leptonically.

decaying KK neutrino $\nu_1 \to \nu_1^{\gamma}$, etc. These scenarios can easily generate an asymmetric event topology. For example, consider the strong production of a squark ($\tilde{q}$) pair, as illustrated in Figure 6-16(a). One of the squarks subsequently decays to the second lightest neutralino $\tilde{\chi}^0_2$, which in turn decays to the lightest neutralino $\tilde{\chi}^0_1$ by emitting two SM fermions $\chi^0_2 \to \ell^+ \ell^- \tilde{\chi}^0_1$ (or $\chi^0_2 \to j \tilde{\chi}^0_1$). The other squark decays to a chargino $\tilde{\chi}^\pm_1$, which then decays to a sneutrino as $\tilde{\chi}^\pm_1 \to \ell^\pm \tilde{\nu}_\ell$. Since $\tilde{\nu}_\ell$ can only decay invisibly, we obtain the asymmetric event topology outlined with the blue box in Figure 6-16(a). The two squarks are the parents, the lightest neutralino $\tilde{\chi}^0_1$ is the first child, and the sneutrino $\tilde{\nu}_\ell$ is the second child.

2. Applying $M_{T2}$ to an asymmetric subsystem. One can also apply the $M_{T2}$ idea even to events in which there is only one (or even no) missing particles to begin with. Such an example is shown in Figure 6-16(b), where we consider $t\bar{t}$ production in the dilepton or semi-leptonic channel. In the first leg we can take $b\ell$ as our visible system and the neutrino $\nu_\ell$ as the invisible particle, while in the other leg we can treat the $b$-jet as the visible system and the $W$-boson as the child particle. In this case, there still should be a ridge structure revealing the true $t$, $W$ and $\nu$ masses.

3. Multi-component dark matter. Of course, the model may contain two (or more) different genuine dark matter particles [45–47, 96–101], whose production in various combinations will inevitably lead at times to asymmetric event topologies.
CHAPTER 7
CONCLUSIONS

We have proposed methods for mass measurements in missing energy events at hadron colliders. In this chapter, we will summarize all proposed variables.

7.1 $\sqrt{s}_{\text{min}}$

We proposed $\sqrt{s}_{\text{min}}^{(\text{reco})}$ and $\sqrt{s}_{\text{min}}^{(\text{sub})}$, which have a clear physical meaning: the minimum CM energy in the (sub)system, which is required in order to explain the observed signal in the detector.

The first variant, the RECO-level variable $\sqrt{s}_{\text{min}}^{(\text{reco})}$ is basically a modification of the prescription for computing the original $\sqrt{s}_{\text{min}}$ variable: instead of using (muon-corrected) calorimeter deposits, as was done in [50, 51], one could instead calculate $\sqrt{s}_{\text{min}}$ with the help of the reconstructed objects (jets and isolated photons, electrons and muons). Our examples in Sections 2.4, 2.5 and 2.6 showed that this procedure tends to automatically subtract out the bulk of the UE contributions, rendering the $\sqrt{s}_{\text{min}}^{(\text{reco})}$ variable safe.

Our second suggestion was to apply $\sqrt{s}_{\text{min}}$ to a subsystem of the observed event, which is suitably defined so that it does not include the contributions from the underlying event. The easiest way to do this is to veto jets from entering the definition of the subsystem. In this case, the subsystem variable $\sqrt{s}_{\text{min}}^{(\text{sub})}$ is completely unaffected by the underlying event. However, depending on the particular scenario, in principle one could also allow (certain kinds of) jets to enter the subsystem. As long as there is an efficient method (through cuts) of selecting jets which (most likely) did not originate from the UE, this should work as well, as demonstrated in Fig. 2-6 with our $t\bar{t}$ example.

$\sqrt{s}_{\text{min}}^{(\text{reco})}$ (and to some extent $\sqrt{s}_{\text{min}}^{(\text{sub})}$) is a general, global, and inclusive variable, which can be applied to any type of events, regardless of the event topology, number or type of reconstructed objects, number or type of missing particles, etc. For example, all of the arbitrariness associated with the number and type of missing particles is encoded by a single parameter $\mathcal{M}$. 

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The most important property of both \( \sqrt{s}_{\text{min}}^{(\text{reco})} \) and \( \sqrt{s}_{\text{min}}^{(\text{sub})} \) is that they exhibit a peak in their distributions, which directly correlates with the mass scale \( M_p \) of the parent particles. Compared to a kinematic endpoint, a peak is a feature which is much easier to observe and subsequently measure precisely over the SM backgrounds.

### 7.2 Invariant Mass Endpoint Method

With our new proposed sets of invariant mass endpoint, the precision of the BSM mass determination is expected to improve. We provide the analytical expressions for all differential invariant mass distributions used in our basic analysis: \( m_{\ell\ell}^2 \), \( m_{j\ell(u)}^2 \) and \( m_{j\ell(s)}^2(1) \). We also provide the corresponding expression for the \( m_{j\ell(d)}^2(1) \) distribution, whose upper endpoint offers an independent measurement of \( M_{j\ell(u)}^{\text{max}} \). Finally, we also list the formula for the differential distribution of \( m_{j\ell(p)}^2 \), whose endpoint can be used for selecting the correct \( m_B \) solution. The knowledge of the shape of the whole distribution is indispensable and greatly improves the accuracy of the endpoint extraction. In the absence of any analytical results like those in Appendix A, one would be forced to use simple linear extrapolations, which would lead to a significant systematic error.

Clearly, not all invariant mass variables will have their endpoints measured with exactly the same precision – some endpoints will be measured better than others. This difference can be due to many factors, e.g. the slope of the distribution near the endpoint, the shape (convex versus concave) of the distribution near the endpoint, the actual location of the endpoint, the level of SM and SUSY combinatorial background near the endpoint, etc. We provide a number of available measurements tremendously exceeds the number of unknown mass parameters. Thus, we can choose the best invariant mass endpoint variables for specific application.

All of the new variables exhibit milder sensitivity to the parameter space region, in comparison to the conventional endpoint \( m_{j\ell\ell}^{\text{max}} \). The endpoint for each of our variables is given by at most two different expressions, as opposed to four in the case of \( m_{j\ell\ell}^{\text{max}} \). A notable exception is the variable \( m_{j\ell(s)}^2(1) \), whose endpoint is actually uniquely predicted,
and is independent of the parameter space region. We therefore strongly encourage the use of \( m_{j\ell}(1) \) in future analyses of SUSY mass determinations.

We can already uniquely determine three out of the four masses involved in the problem. Then, the addition of a fifth measurement, as discussed in Sections 3.3.2.1 and 3.4.2, is sufficient to pin down all four of the BSM masses. In contrast, with the conventional approach, one also starts with four measurements as in (3–12), but in the worst case scenario this results in infinitely many solutions, due to the linear dependence problem (3–14) discussed in Section 3.1.2. Adding a fifth measurement as in (3–18) helps, but once again, the worst case scenario leads to two alternative solutions [9]. In order to resolve the remaining duplication, and thus guarantee uniqueness of the solution under any circumstances, one needs at least 6 measurements.

### 7.3 Subsystem \( M_{T2} \) Method

We showed that the \( M_{T2} \) method by itself is sufficient for a complete mass spectrum determination, even in the problematic cases of \( N_{\text{cascade}} = 1 \) or \( N_{\text{cascade}} = 2 \). We backed our claim with two explicit examples: \( W^+W^- \) pair production, which is an example of an \( n = 1 \) chain, and \( t\bar{t} \) pair production, which is an example of an \( n = 2 \) chain. We showed that the \( M_{T2} \) method in principle provides more than enough measurements for the unambiguous determination of the complete mass spectrum.

When applying the \( M_{T2} \) method, we generalized the concept of \( M_{T2} \) by introducing various subsystem (or subchain) \( M_{T2}^{(n,p,c)} \) variables. The latter are defined similarly to the conventional \( M_{T2} \) variable, but are labelled by three integers \( n, p, \) and \( c \), whose meaning is as follows. The integer \( n \) labels the “grandparent” particle originally produced in the hard scattering and initiating the decay chain. We then apply the usual \( M_{T2} \) concept to the subchain starting at the “parent” particle labelled by \( p \) and terminating at the “child” particle labelled by \( c \). In general, the “child” particle does not have to be the very last (i.e. the missing) particle in the decay chain, just like the “parent” particle does not have to be the very first particle produced in the event. The introduction of the \( M_{T2}^{(n,p,c)} \...
subchain variables greatly proliferates the number of available $M_{T2}$-type measurements, and allows us to make full use of the power of the $M_{T2}$ concept.

### 7.4 One Dimensional Projection Method

We proposed one dimensional projections of the kinematic variables $M_{T2}$ and $M_{CT}$ with respect to $\vec{P}_T$ of upstream objects. By doing this decomposition, we can measure BSM particles’ mass spectra in a very short decay where pair produced particles in a hard collision are decaying into missing particles and one visible particle $(N_{\text{cascade}} = 1)$. To the extent that the definition of $M_{T2\perp}$ relies only on the direction and not the magnitude of the upstream $\vec{P}_T$, our method is insensitive to the jet energy scale error \[11\]. We have also provided exact analytical formulas for the computation of the 1D decomposed $M_{T2\perp}$, $M_{T2\parallel}$ and the shape of the $M_{T2\perp}$ distribution.

We show how the perpendicular and the parallel projected variables are related with each other. By studying the maximum allowed boundary of this co-relation, we can increase statistics and correspondingly get more precise measurements.

### 7.5 Asymmetric Event Topology

The dark matter signatures at colliders always involve missing transverse energy. Such events will be quite challenging to fully reconstruct and/or interpret. All previous studies have made (either explicitly or implicitly) the assumption that each event has two identical missing particles. Our main point is that this assumption is unnecessary, and by suitable modifications of the existing analysis techniques one can in principle test both the number and the type of missing particles in the data. Our proposal here was to modify the Cambridge $M_{T2}$ variable \[10\] by treating each children mass as an independent input parameter. In this approach, one obtains the $M_{T2\text{endpoint}}$ as a function of the two children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$, and proceeds to study its properties.

The function $M_{T2\text{endpoint}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)})$ exhibits a ridge structure (i.e. a gradient discontinuity). The point corresponding to the correct children masses always lies
on a ridgeline, thus the ridgelines provide a model-independent constraint among the children masses, just like the $M_{T2}$ endpoint provides a model-independent constraint on the masses of the child(ren) and the parent.

In general, the $M_{T2}$ endpoint function also depends on the value of the upstream transverse momentum in the event. $M_{T2(\text{max})}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, P_{UTM})$. However, the $P_{UTM}$ dependence disappears completely for precisely the right values of the children masses. This provides a second, quite general and model-independent, method for measuring the individual particle masses in such missing energy events.
APPENDIX A

ANALYTICAL EXPRESSIONS FOR THE SHAPES OF THE INVARIANT MASS DISTRIBUTIONS

This appendix A provides the analytical expressions for the shapes of the invariant mass distributions \( m^2_{\ell\ell}, m^2_{j\ell} \equiv m^2_{j\ell_n} \cup m^2_{j\ell_r} \), \( m^2_{j\ell(n)}(1) \equiv m^2_{j\ell_n} + m^2_{j\ell_r} \), \( m^2_{j\ell(d)}(1) \equiv |m^2_{j\ell_n} - m^2_{j\ell_r}| \), and \( m^2_{j\ell(p)} \). To simplify the expressions, we introduce the shorthand notation for the corresponding endpoints, which was already introduced in Equations (3–42), (3–49), (3–51) and (3–52):

\[
L \equiv (m^{\text{max}}_{\ell\ell})^2 = m^2_D (1 - R_{BC}) (1 - R_{AB}), \quad (A-1)
\]
\[
n \equiv (m^{\text{max}}_{\ell_n})^2 = m^2_D (1 - R_{CD}) (1 - R_{BC}), \quad (A-2)
\]
\[
f \equiv (m^{\text{max}}_{\ell_r})^2 = m^2_D (1 - R_{CD}) (1 - R_{AB}), \quad (A-3)
\]
\[
p \equiv R_{BC} f = m^2_D (1 - R_{CD}) R_{BC} (1 - R_{AB}). \quad (A-4)
\]

In this appendix, we shall ignore spin correlations and consider only pure phase space decays. General results including spin correlations for \( m^2_{\ell\ell}, m^2_{j\ell_n} \) and \( m^2_{j\ell_r} \) exist and can be found in [84]. We shall unit-normalize the \( m^2_{\ell\ell}, m^2_{j\ell_n}, m^2_{j\ell(d)} \) and \( m^2_{j\ell(p)} \) distributions, to which each event contributes a single entry. In contrast, the union distribution \( m^2_{j\ell(u)} \) has two entries per event, so it will be normalized to 2 instead. It is also convenient to write the distributions in terms of masses squared instead of linear masses. Of course, the two are trivially related by

\[
\frac{dN}{dm} = 2m \frac{dN}{dm^2}. \quad (A-5)
\]

A.1 Dilepton Mass Distribution \( m^2_{\ell\ell} \)

The differential dilepton invariant mass distribution is given by

\[
\frac{dN}{dm^2_{\ell\ell}} = \frac{1}{L}. \quad (A-6)
\]
which is unit-normalized:
\[
\int_0^L \, dm_{\ell\ell} \left( \frac{dN}{dm_{\ell\ell}^2} \right) = 1 . \tag{A-7}
\]

### A.2 Combined Jet-lepton Mass Distribution \( m_{j\ell(u)}^2 \)

The differential distribution for \( u \equiv m_{j\ell(u)}^2 \) is given by
\[
\frac{dN}{du} = \theta(n - u) \, \theta(u) \left( \frac{1}{n} + \theta(p - u) \, \theta(u) \, \frac{\ln(f/p)}{f - p} + \theta(f - u) \, \theta(u - p) \, \frac{\ln(f/u)}{f - p} \right), \tag{A-8}
\]
where \( \theta(x) \) is the usual Heaviside step function
\[
\theta(x) \equiv \begin{cases} 
1, & x \geq 0, \\
0, & x < 0. \end{cases} \tag{A-9}
\]
It is easy to verify the normalization condition
\[
\int_0^M \, du \left( \frac{dN}{du} \right) = 2, \tag{A-10}
\]
where \( M \equiv (M_{j\ell(u)}^{\text{max}})^2 \) was already defined in Equation. (3–42).

In Figure A-1(a) we cross-check the prediction of Equation. (A–8) (blue dashed line) with the numerically obtained \( m_{j\ell(u)}^2 \) distribution in Figure 3-4(b) (red solid line), for the case of study point LM1. We see that within the statistical errors, our formula is in perfect agreement with the numerical result.

### A.3 Distribution of the sum \( m_{j\ell(s)}^2(\alpha = 1) \)

The differential distribution for \( \sigma \equiv m_{j\ell(s)}^2(\alpha = 1) \) is given by
\[
\frac{dN}{d\sigma} = \frac{1}{f - p} \left\{ \theta(m - \sigma) \, \theta(\sigma) \ln \left( \frac{fn}{fn - \sigma(f - p)} \right) \\
+ \theta(M - \sigma) \, \theta(\sigma - m) \ln \left( \frac{M}{M - (f - p)} \right) \\
+ \theta(n + p - \sigma) \, \theta(\sigma - M) \ln \left( \frac{fn - \sigma(f - p)}{p(n + p - f)} \right) \right\}, \tag{A-11}
\]
Figure A-1. Comparison of the numerically obtained differential invariant mass distributions for study point LM1 (red solid lines) with the analytical results presented in this appendix (blue dashed lines): (a) the distribution of the combined jet-lepton mass \( u \equiv m^2_{\ell j(u)} \) from Figure 3-4(b) versus the analytical prediction of Equation. (A–8); (b) the distribution of the sum \( \sigma \equiv m^2_{\ell j(s)}(\alpha = 1) \) from Figure 3-4(c) versus the analytical prediction of Equation. (A–11); (c) the distribution of the difference \( \Delta \equiv m^2_{\ell j(d)}(\alpha = 1) \) from Figure 3-4(d) versus the analytical prediction of Equations (A–15 through A–19); (d) the distribution of the product \( \rho \equiv m^2_{\ell j(p)(\rho)} \) from Figure 3-6(c) versus the analytical prediction of Equations (A–22 through A–23).

where \( m \equiv (m_{j_{\ell(u)}}^{\text{max}})^2 \) was defined in (3–42), and \( n, f \) and \( \rho \) were defined in Equations (A–2 through A–4). The normalization condition for Equation. (A–11) reads

\[
\int_0^S d\sigma \left( \frac{dN}{d\sigma} \right) = 1, \quad (A–12)
\]

where \( S \) is defined in Equation. (3–42).

As a cross-check, Figure A-1(b) shows that our analytical formula in Equation. (A–11) agrees with the numerical result from Figure 3-4(c) for the LM1 study point.
A.4 Distribution Of The Difference $m^2_{j}(d)(\alpha = 1)$

The differential distribution for the difference $\Delta \equiv m^2_{j}(d)(\alpha = 1)$ depends on the values of $R_{BC}$ and $R_{AB}$. To simplify the notation, we define an antisymmetric function

$$L(x, y) = -L(y, x) \equiv \ln \left( \frac{nf + x(f - p)}{nf + y(f - p)} \right).$$

(A–13)

which we heavily use in writing down the result for the differential $\Delta$ distribution.

Notice that there are various equivalent ways to write down these formulas, due to the transitivity property

$$L(x, y) + L(y, z) = L(x, z).$$

(A–14)

For $\Delta \equiv m^2_{j}(d)(\alpha = 1)$ one needs to consider five separate cases:

If $\frac{2}{3 - R_{AB}} \leq R_{BC} < 1$, then

$$\frac{dN}{d\Delta} = \frac{1}{f - p} \left\{ \theta(n - \Delta) \theta(\Delta) \left[ L(0, -n) + L(-\Delta, -n) \right] 
+ \theta(p - n - \Delta) \theta(\Delta - n) L(0, -n) 
+ \theta(f - \Delta) \theta(\Delta - (p - n)) L(f, \Delta) \right\}.\tag{A–15}$$

If $\frac{1}{2 - R_{AB}} \leq R_{BC} < \frac{2}{3 - R_{AB}}$, then

$$\frac{dN}{d\Delta} = \frac{1}{f - p} \left\{ \theta(p - n - \Delta) \theta(\Delta) \left[ L(0, -n) + L(-\Delta, -n) \right] 
+ \theta(n - \Delta) \theta(\Delta - (p - n)) \left[ L(f, \Delta) + L(-\Delta, -n) \right] 
+ \theta(f - \Delta) \theta(\Delta - n) L(f, \Delta) \right\}.\tag{A–16}$$
If $R_{AB} \leq R_{BC} < \frac{1}{2 - R_{AB}}$, then
\[
\frac{dN}{d\Delta} = \frac{1}{f - p} \left\{ \theta(n - p - \Delta) \theta(\Delta) \left[ L(f, \Delta) + L(f, 0) \right] \\
+ \theta(n - \Delta) \theta(\Delta - (n - p)) \left[ L(f, \Delta) + L(-\Delta, -n) \right] \\
+ \theta(f - \Delta) \theta(\Delta - n) L(f, \Delta) \right\} .
\] (A–17)

If $\frac{R_{AB}}{2 - R_{AB}} \leq R_{BC} < R_{AB}$, then
\[
\frac{dN}{d\Delta} = \frac{1}{f - p} \left\{ \theta(n - p - \Delta) \theta(\Delta) \left[ L(f, \Delta) + L(f, 0) \right] \\
+ \theta(f - \Delta) \theta(\Delta - (n - p)) \left[ L(f, \Delta) + L(-\Delta, -n) \right] \\
+ \theta(n - \Delta) \theta(\Delta - f) L(-\Delta, -n) \right\} .
\] (A–18)

If $0 \leq R_{BC} < \frac{R_{AB}}{2 - R_{AB}}$, then
\[
\frac{dN}{d\Delta} = \frac{1}{f - p} \left\{ \theta(f - \Delta) \theta(\Delta) \left[ L(f, \Delta) + L(f, 0) \right] \\
+ \theta(n - p - \Delta) \theta(\Delta - f) L(f, 0) \\
+ \theta(n - \Delta) \theta(\Delta - (n - p)) L(-\Delta, -n) \right\} .
\] (A–19)

The normalization condition now reads
\[
\int_0^M d\Delta \left( \frac{dN}{d\Delta} \right) = 1 .
\] (A–20)

As before, in Figure A-1(c) we compare the prediction of our analytical formula in Equations (A–15 through A–19) to the numerical result obtained earlier in Figure 3-4(d) for the LM1 study point, and we find very good agreement.
A.5 Distribution Of The Product $m_{jl(p)}^2$

Finally, for completeness we also list the differential distribution for the product variable in Equation (3–24), for which here we shall use the shorthand notation $\rho \equiv m_{jl(p)}^2$. To further simplify the notation, we define the function

$$X_\pm(\rho) \equiv \frac{\sqrt{n}}{2(f - p)} \left( \sqrt{n} f \pm \sqrt{f^2 n + 4(p - f) \rho^2} \right). \quad (A–21)$$

where $n$, $f$ and $\rho$ are defined as before in Equations (A–2 through A–4). There are two separate cases:

If $R_{BC} \leq 0.5$, the $\rho$ distribution is made up of two branches joining at $\rho = \sqrt{n} p$ (see, for example the LM1 distribution in Figure 3-6(c) and the LM6’ distribution in Figure 3-7(c))

$$\frac{dN}{d\rho} = \frac{2\rho}{nf} \left\{ \theta(\sqrt{n} p - \rho) \theta(\rho) \left[ \ln \left( \frac{n}{\rho} \right) + 2 \ln \left( \frac{\rho}{X_- (\rho)} \right) \right] + \theta \left( \frac{f \sqrt{n}}{2\sqrt{f - p}} - \rho \right) \theta(\rho - \sqrt{n} p) 2 \ln \left( \frac{X_+ (\rho)}{X_- (\rho)} \right) \right\}. \quad (A–22)$$

If $R_{BC} \geq 0.5$, there is a single branch, as illustrated by the LM1’ distribution in Figure 3-6(c) and the LM6 distribution in Figure 3-7(c):

$$\frac{dN}{d\rho} = \frac{2\rho}{nf} \theta(\sqrt{n} p - \rho) \theta(\rho) \left\{ \ln \left( \frac{n}{\rho} \right) + 2 \ln \left( \frac{\rho}{X_- (\rho)} \right) \right\}. \quad (A–23)$$

In both of those cases, the normalization condition is

$$\int_0^{\rho_{\text{max}}} d\rho \left( \frac{dN}{d\rho} \right) = 1, \quad (A–24)$$

where $\rho_{\text{max}}$ is the corresponding $m_{jl(p)}^2$ endpoint defined in Equation (3–25).

Figure A-1(d) demonstrates that our analytical result of Equation (A–22) agrees well with the numerically derived $m_{jl(p)}^2$ distribution in Figure 3-6(c) for the LM1 study point.
APPENDIX B
ANALYTICAL EXPRESSIONS FOR $M_{T^2, \text{MAX}}^{(N,P,C)}(\tilde{M}_c, p_T)$

The purpose of this Appendix B is to collect in one place all relevant formulas for the various subsystem $M_{T^2}$ endpoints $M_{T^2, \text{MAX}}^{(n,p,c)}(\tilde{M}_c, p_T)$ in the presence of initial state radiation (ISR) with arbitrary transverse momentum $p_T$. In all cases, we will find that $M_{T^2, \text{MAX}}^{(n,p,c)}(\tilde{M}_c, p_T)$ is given by two branches:

$$
M_{T^2, \text{MAX}}^{(n,p,c)}(\tilde{M}_c, p_T) = \begin{cases} 
F_L^{(n,p,c)}(\tilde{M}_c, p_T), & \text{if } \tilde{M}_c \leq M_c, \\
F_R^{(n,p,c)}(\tilde{M}_c, p_T), & \text{if } \tilde{M}_c \geq M_c.
\end{cases}
$$

(B–1)

In what follows we shall list the analytic expressions for each branch $F_L^{(n,p,c)}$ and $F_R^{(n,p,c)}$, for all possible $(n, p, c)$ cases with $n - c \leq 2$. The grandparents $X_n$, the parents $X_p$ and the children $X_c$ are always assumed to be on-shell. However, any intermediate particles $X_m$ with $n > m > p$ or $p > m > c$ may or may not be on-shell, and the two cases will have to be treated differently. Such an example is provided by the endpoint function $M_{T^2, \text{MAX}}^{(n,n,n-2)}(\tilde{M}_{n-2}, p_T)$ discussed below in Section B.2. For convenience, our results will be written in terms of the mass parameters $\mu_{(n,p,c)}$ defined in Equation (4–9)

$$
\mu_{(n,p,c)} \equiv \frac{M_n}{2} \left(1 - \frac{M_c^2}{M_p^2}\right).
$$

(B–2)

These parameters represent certain combinations of the masses of the grandparents $(M_n)$, parents $(M_p)$ and children $(M_c)$, and do not contain any dependence on the ISR transverse momentum $p_T$. As we discussed in Sections 4.1 and 4.2, these are generally the quantities which are directly measured by experiment. Therefore, with the $M_{T^2}$ method, the goal of any experiment would be to perform a sufficient number of $\mu$-parameter measurements and then from those to determine the particle masses themselves.
In some special cases, namely \( n = p \), we shall also define \( p_T \)-dependent \( \mu \) parameters, where the \( p_T \) dependence is explicitly shown as an argument:

\[
\mu_{(n,n,c)}(p_T) = \mu_{(n,n,c)} \left( \sqrt{1 + \left( \frac{p_T}{2M_n} \right)^2 - \frac{p_T}{2M_n}} \right). \tag{B–3}
\]

When \( p_T = 0 \), the \( p_T \)-dependent parameters (B–3) simply reduce to the \( p_T \)-independent ones (B–2):

\[
\mu_{(n,n,c)}(p_T = 0) = \mu_{(n,n,c)}. \tag{B–4}
\]

We also remind the reader that test masses for the children are denoted with a tilde: \( \tilde{M}_c \), while the true mass of any particle does not carry a tilde sign.

**B.1 The Subsystem Variable** \( M_{n,n-1}^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T) \)

The corresponding expressions were already given in Equations \( (4–38) \) and \( (4–39) \) and we list them here for completeness:

\[
F_L^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T) = \left\{ \left[ \mu_{(n,n,n-1)}(p_T) + \sqrt{\left( \mu_{(n,n,n-1)}(p_T) + \frac{p_T}{2} \right)^2 + \vec{\tilde{M}}_{n-1}^2} \right]^2 - \frac{p_T^2}{4} \right\}^{\frac{1}{2}}, \tag{B–5}
\]

\[
F_R^{(n,n,n-1)}(\tilde{M}_{n-1}, p_T) = \left\{ \left[ \mu_{(n,n,n-1)}(-p_T) + \sqrt{\left( \mu_{(n,n,n-1)}(-p_T) - \frac{p_T}{2} \right)^2 + \vec{\tilde{M}}_{n-1}^2} \right]^2 - \frac{p_T^2}{4} \right\}^{\frac{1}{2}}, \tag{B–6}
\]

where the \( p_T \)-dependent parameter \( \mu_{(n,n,n-1)}(p_T) \) was already defined in (B–3):

\[
\mu_{(n,n,n-1)}(p_T) = \mu_{(n,n,n-1)} \left( \sqrt{1 + \left( \frac{p_T}{2M_n} \right)^2 - \frac{p_T}{2M_n}} \right). \tag{B–7}
\]

As already mentioned in Section 4.1.1, the left branch \( F_L^{(n,n,n-1)} \) corresponds to the momentum configuration \( (\vec{p}^{(1)}_{nT} \uparrow \uparrow \vec{p}^{(2)}_{nT}) \uparrow \uparrow \vec{p}_T \), while the right branch \( F_R^{(n,n,n-1)} \) corresponds to \( (\vec{p}^{(1)}_{nT} \uparrow \uparrow \vec{p}^{(2)}_{nT}) \uparrow \downarrow \vec{p}_T \).
B.2 The Subsystem Variable $M_{T2,max}^{(n,n,n-2)}(\tilde{M}_{n-2}, p_T)$

In this case there is an intermediate particle $X_{n-1}$ between the parent $X_n$ and the child $X_{n-2}$ (Figures 4-1). Our formulas below are written in such a way that they can be applied both in the case when the intermediate particle $X_{n-1}$ is on shell ($M_n > M_{n-1}$) and in the case when $X_{n-1}$ is off-shell ($M_{n-1} \geq M_n$).

In both cases (off-shell or on-shell) we find that the left branch of $M_{T2,max}^{(n,n,n-2)}(\tilde{M}_{n-2}, p_T)$ is given by

$$F_{L}^{(n,n,n-2)}(\tilde{M}_{n-2}, p_T) = \left\{ \left[ \mu_{(n,n,n-2)}(p_T) + \sqrt{\left( \mu_{(n,n,n-2)}(p_T) + \frac{p_T}{2} \right)^2 + \tilde{M}_{n-2}^2} \right]^2 - \frac{p_T^2}{4} \right\}^{\frac{1}{2}},$$

(B–8)

where the $p_T$-dependent parameter $\mu_{(n,n,n-2)}(p_T)$ was already defined in (B–3):

$$\mu_{(n,n,n-2)}(p_T) = \mu_{(n,n,n-2)} \left( \sqrt{1 + \left( \frac{p_T}{2M_n} \right)^2} - \frac{p_T}{2M_n} \right).$$

(B–9)

The right branch $F_{R}^{(n,n,n-2)}$ is given by three different expressions, depending on the mass spectrum and the size of the ISR $p_T$:

$$F_{R}^{(n,n,n-2)}(\tilde{M}_{n-2}, p_T) =$$

$$= \begin{cases} 
F_{L}^{(n,n,n-2)}(\tilde{M}_{n-2}, -p_T), & \text{if } p_T > \frac{M_n^2 - M_{n-2}^2}{M_{n-2}}, \\
F_{R, off}^{(n,n,n-2)}(\tilde{M}_{n-2}, p_T), & \text{if } p_T \leq \frac{M_n^2 - M_{n-2}^2}{M_{n-2}} \text{ and } \Delta M_{n,n-2}(p_T) \leq M_{n-1}x_{n,max}, \\
F_{R, on}^{(n,n,n-2)}(\tilde{M}_{n-2}, p_T), & \text{if } p_T \leq \frac{M_n^2 - M_{n-2}^2}{M_{n-2}} \text{ and } \Delta M_{n,n-2}(p_T) \geq M_{n-1}x_{n,max}. 
\end{cases}$$

(B–10)

Here $\Delta M_{n,n-2}(p_T)$ is a $p_T$-dependent mass parameter defined as

$$\Delta M_{n,n-2}(p_T) \equiv \left\{ \left[ \sqrt{M_n^2 + \frac{p_T^2}{4}} - M_{n-2} \right]^2 - \frac{p_T^2}{4} \right\}^{\frac{1}{2}},$$

(B–11)

which in the limit $p_T \to 0$ reduces to

$$\Delta M_{n,n-2}(p_T = 0) = M_n - M_{n-2},$$

(B–12)
Now returning to the logic of Equation (B–10), one would always use the expression \( F \) and never its alternative \( X \). Correspondingly, the expression \( F \) is understood as follows. When the intermediate particle \( X_{n-1} \) is off-shell and \( M_{n-1} < M_n \), Equation (B–13) reduces to Equation (4–81). The two expressions \( F^{(n,n,n-2)}_{R,\text{off}} \) and \( F^{(n,n,n-2)}_{R,\text{on}} \) appearing in Equation (B–10) are given by

\[
F^{(n,n,n-2)}_{R,\text{off}}(\tilde{M}_{n-2}, p_T) = \left\{ \tilde{M}_{n-2} + \sqrt{\Delta M_{n,n-2}^2(p_T) + \frac{p_T^2}{4}}} - \frac{p_T^2}{4} \right\}^{\frac{1}{2}},
\]

\[
F^{(n,n,n-2)}_{R,\text{on}}(\tilde{M}_{n-2}, p_T) = \left\{ \sqrt{\tilde{M}_{n-2}^2 + p^2_{\text{vis}}(p_T) + \tilde{M}_{n-2}^2 + \left( p_{\text{vis}}(p_T) - \frac{p_T}{2} \right)^2} - \frac{p_T^2}{4} \right\}^{\frac{1}{2}},
\]

where \( \Delta M_{n,n-2}(p_T) \) and \( M_{n-1 \times n, \max} \) were already defined in (B–11) and (B–13), correspondingly. The subscripts “off” and “on” in Equations (B–14) and (B–15) can be understood as follows. When the intermediate particle \( X_{n-1} \) is off-shell and \( M_{n-1} \geq M_n \), from Equations (B–11) and (B–13) we get

\[
\Delta M_{n,n-2}^2(p_T) = \frac{1}{4} \left( M_n - M_{n-2} \right)^2 \leq \frac{p_T^2}{4}. \tag{B–16}
\]

Now returning to the logic of Equation (B–10), we see that in the off-shell case at low \( p_T \) one would always use the expression \( F^{(n,n,n-2)}_{R,\text{off}}(\tilde{M}_{n-2}, p_T) \) defined in Equation (B–14), and never its alternative \( F^{(n,n,n-2)}_{R,\text{on}}(\tilde{M}_{n-2}, p_T) \) from Equation (B–15). To put it another way, the expression \( F^{(n,n,n-2)}_{R,\text{on}}(\tilde{M}_{n-2}, p_T) \) in Equation (B–15) is only relevant when the intermediate particle \( X_{n-1} \) is on-shell.
Finally, the quantity \( p_{\text{vis}}(\rho_T) \) appearing in Equation (B–15) is a shorthand notation for the total transverse momentum of the visible particles \( x_n \) and \( x_{n-1} \) in each leg:

\[
p_{\text{vis}} \equiv |\vec{p}_{nT}^{(k)} + \vec{p}_{(n-1)T}^{(k)}|.
\]

In the case relevant for \( F_{R,\text{on}}^{(n,n,n-2)} \), the value of \( p_{\text{vis}} \) is given by

\[
p_{\text{vis}}(\rho_T) \equiv (\mu_{(n,n,n-1)} + \mu_{(n,n-1,n-2)}) \frac{\rho_T}{2M_n} + |\mu_{(n,n,n-1)} - \mu_{(n,n-1,n-2)}| \sqrt{1 + \frac{\rho_T^2}{4M_n^2}} . \tag{B–17}
\]

It is easy to check that in the limit of \( \rho_T \to 0 \) our Equations (B–8) and (B–10) reduce to the known results for the case of no ISR (Equations (70) and (74) in W. S. Cho et al. [“Measuring superparticle masses at hadron collider using the transverse mass kink, JHEP 0802, 035 (2008)] [37]).

The left branch \( F_{L}^{(n,n,n-2)} \) in Equation (B–8) corresponds to the momentum configuration

\[
\left( \vec{p}_{nT}^{(k)} + \vec{p}_{(n-1)T}^{(k)} \right) \uparrow \uparrow \vec{\rho}_T ,
\]

while the right branch \( F_{R}^{(n,n,n-2)} \) in Equation (B–10) corresponds to

\[
\left( \vec{p}_{nT}^{(k)} + \vec{p}_{(n-1)T}^{(k)} \right) \uparrow \downarrow \vec{\rho}_T .
\]

In the latter case, \( F_{R,\text{off}}^{(n,n,n-2)} \) is obtained when \( X_{n-2} \) is at rest: \( p_{(n-2)T}^{(k)} = 0 \), while \( F_{R,\text{on}}^{(n,n,n-2)} \) corresponds to the case when \( p_{(n-2)T}^{(k)} = \frac{1}{2}\rho_T - p_{\text{vis}}(\rho_T) \).
B.3 The Subsystem Variable $M_{T2, \max}^{(n,n-1,n-2)}(\tilde{M}_{n-2}, \rho_T)$

Here we generalize our $\rho_T = 0$ result of Equation (4–56) from Section 4.1.4 to the case of arbitrary ISR $\rho_T$:

\[ F_{L}^{(n,n-1,n-2)}(\tilde{M}_{n-2}, \rho_T) = \]
\[ = \left\{ \begin{array}{l}
\mu_{(n-1,n-1,n-2)}(\hat{\rho}_T) + \sqrt{\left( \mu_{(n-1,n-1,n-2)}(\hat{\rho}_T + \frac{\hat{\rho}_T}{2} \right)^2 + \tilde{M}_{n-2}^2} - \frac{\hat{\rho}_T^2}{4} \right\}^{\frac{1}{2}}, \quad (B–18)
\]

\[ F_{R}^{(n,n-1,n-2)}(\tilde{M}_{n-2}, \rho_T) = \]
\[ = \left\{ \begin{array}{l}
\mu_{(n-1,n-1,n-2)}(-\hat{\rho}_T) + \sqrt{\left( \mu_{(n-1,n-1,n-2)}(-\hat{\rho}_T + \frac{\hat{\rho}_T}{2} \right)^2 + \tilde{M}_{n-2}^2} - \frac{\hat{\rho}_T^2}{4} \right\}^{\frac{1}{2}} \quad (B–19)
\]

where we have introduced the shorthand notation

\[ \hat{\rho}_T \equiv \rho_T + 2\mu_{(n,n,n-1)}(\rho_T) . \quad (B–20) \]

Notice that the second term on the right-hand side contains the $\rho_T$-dependent $\mu$ parameter defined in Equation (B–7).

The left branch $F_{L}^{(n,n-1,n-2)}$ in Equation (B–18) corresponds to the momentum configuration

\[ \vec{p}^{(k)}_{(n-1)T} \uparrow\uparrow \left( \vec{p}^{(k)}_{nT} \uparrow\uparrow \vec{p}_T \right) , \]

while the right branch $F_{R}^{(n,n-1,n-2)}$ in Equation (B–19) corresponds to

\[ \vec{p}^{(k)}_{(n-1)T} \uparrow\downarrow \left( \vec{p}^{(k)}_{nT} \uparrow\uparrow \vec{p}_T \right) . \]

It is worth checking that our general $\rho_T$-dependent results in Equations (B–18) and (B–19) reduce to our previous formulas in Equations (4–57) and (4–58) in the $\rho_T \to 0$ limit and in the special case of $n = 2$. First taking the limit $\rho_T \to 0$ from Equations (B–20) and (B–7) we get

\[ \lim_{\rho_T \to 0} \hat{\rho}_T = 2\mu_{(n,n,n-1)} , \quad (B–21) \]
\[
\lim_{p_T \to 0} \mu_{(n-1,n-1,n-2)}(\hat{p}_T) = \mu_{(n-1,n-1,n-2)}(2\mu_{(n,n-1)}) = \mu_{(n,n,n-2)} - \mu_{(n,n-1)}, \quad (B-22)
\]
\[
\lim_{p_T \to 0} \mu_{(n-1,n-1,n-2)}(-\hat{p}_T) = \mu_{(n-1,n-1,n-2)}(-2\mu_{(n,n-1)}) = \mu_{(n,n-1,n-2)}. \quad (B-23)
\]

Substituting Equations (B–21 through B–23) into Equations (B–18) and (B–19), we get

\[
F_{L}^{(n,n-1,n-2)}(\tilde{M}_{n-2}, p_T = 0) =
\]
\[
= \left\{ \left[ \mu_{(n,n,n-2)} - \mu_{(n,n,n-1)} + \sqrt{\mu_{(n,n,n-2)}^2 + \tilde{M}_{n-2}^2} \right]^2 - \mu_{(n,n,n-1)}^2 \right\}^{\frac{1}{2}}, \quad (B-24)
\]
\[
F_{R}^{(n,n-1,n-2)}(\tilde{M}_{n-2}, p_T = 0) =
\]
\[
= \left\{ \left[ \mu_{(n,n-1,n-2)} + \sqrt{\left( \mu_{(n,n,n-1)} - \mu_{(n,n-1,n-2)} \right)^2 + \tilde{M}_{n-2}^2} \right]^2 - \mu_{(n,n,n-1)}^2 \right\}^{\frac{1}{2}}. \quad (B-25)
\]

which are nothing but the generalizations of Equations (4–57) and (4–58) for arbitrary \( n \).
In this Appendix C, we revisit our previous two examples from Sections 6.4.1 and 6.4.2, this time considering the infinitely large $P_{UTM}$ limit \[40\]. While this situation is impossible to achieve in a real experiment, its advantage is that it can be treated by analytical means. In the $P_{UTM} \to \infty$ limit, the “decoupling argument” holds \[40\], and one finds the following analytical expression for the $M_{T2}$ endpoint as a function of the two test children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$:

$$M_{T2\text{\,max}}(\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}, \infty) = \begin{cases} 
\sqrt{M_p^2 - (\tilde{M}_c^{(a)})^2 + (\tilde{M}_c^{(b)})^2}, & \text{if } (\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \in \mathcal{R}_1, \\
\sqrt{M_p^2 - (\tilde{M}_c^{(b)})^2 + (\tilde{M}_c^{(a)})^2}, & \text{if } (\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \in \mathcal{R}_2, \\
\frac{\tilde{M}_c^{(a)}}{\tilde{M}_c^{(b)}} M_p, & \text{if } (\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \in \mathcal{R}_3, \\
\frac{\tilde{M}_c^{(b)}}{\tilde{M}_c^{(a)}} M_p, & \text{if } (\tilde{M}_c^{(a)}, \tilde{M}_c^{(b)}) \in \mathcal{R}_4,
\end{cases} \tag{C-1}$$

where the four defining regions $\mathcal{R}_i$, ($i = 1, \ldots, 4$) are shown in Figure C-1 and are defined as follows:

$$\begin{align*}
\mathcal{R}_1 : \tilde{M}_c^{(b)} &< \sqrt{(M_c^{(b)})^2 - (M_c^{(a)})^2 + (\tilde{M}_c^{(a)})^2} \land \tilde{M}_c^{(a)} < M_c^{(a)}, \\
\mathcal{R}_2 : \sqrt{(M_c^{(b)})^2 - (M_c^{(a)})^2 + (\tilde{M}_c^{(a)})^2} &< \tilde{M}_c^{(b)} < M_c^{(b)}, \\
\mathcal{R}_3 : M_c^{(b)} &< \tilde{M}_c^{(b)} \land \tilde{M}_c^{(a)} < \left(\frac{M_c^{(a)}}{M_c^{(b)}}\right) \tilde{M}_c^{(b)}, \\
\mathcal{R}_4 : M_c^{(a)} &< \tilde{M}_c^{(a)} \land \tilde{M}_c^{(b)} < \left(\frac{M_c^{(b)}}{M_c^{(a)}}\right) \tilde{M}_c^{(a)}. \tag{C-2, C-3, C-4, C-5}
\end{align*}$$

Since the functional expression for $M_{T2\text{\,max}}$ within each region $\mathcal{R}_j$ is different, there is in general a gradient discontinuity when crossing from one region into the next. Therefore, the ridges on the $M_{T2\text{\,max}}$ hypersurface will appear along the common boundaries of the four regions $\mathcal{R}_i$. Let us denote by $L_{ij}$ the boundary between regions $\mathcal{R}_i$ and $\mathcal{R}_j$. 229
The parameter plane of test children masses squared, divided into the four different regions $R_i$ used to define the $M_{T2}$ endpoint function \((C-1)\). Their common boundaries $L_{ij}$ are parametrically defined in Equations \((C-6-C-9)\). The black dot corresponds to the true values of the children masses.

As indicated in Figure \(C-1\), each $L_{ij}$ is a straight line in the parameter space of the children test masses squared and is given by

\[ L_{12} : (\tilde{M}_c^{(b)})^2 = (M_c^{(b)})^2 - (M_c^{(a)})^2 ; \]
\[ L_{23} : \tilde{M}_c^{(b)} = M_c^{(b)} , \tilde{M}_c^{(a)} \leq M_c^{(a)} ; \]
\[ L_{34} : \tilde{M}_c^{(b)} = \frac{M_c^{(b)}}{M_c^{(a)}} \tilde{M}_c^{(a)} , \tilde{M}_c^{(a)} \geq M_c^{(a)} ; \]
\[ L_{14} : \tilde{M}_c^{(a)} = M_c^{(a)} , \tilde{M}_c^{(b)} \leq M_c^{(b)} . \]

As seen in Figure \(C-1\), all four lines $L_{ij}$ meet at the true children mass point $\tilde{M}_c^{(a)} = M_c^{(a)}$, $\tilde{M}_c^{(b)} = M_c^{(b)}$, where in turn the $M_{T2}$ endpoint $M_{T2(max)}$ gives the true parent mass $M_p$, in accordance with Equation \((6-43)\).

With those preliminaries, we are now in a position to revisit our two examples from Sections 6.4.1 and 6.4.2. Figures C-2 and C-3 are the corresponding analogues of Figures 6-6 and 6-9 in the case of infinite $\mathcal{P}_{UTM}$. Comparing with our earlier results, we notice both quantitative and qualitative changes in the ridge structure. First, the smooth
ridge in Figure 6-6(b) (Figure 6-9(b)) has now been deformed into two straight line
segments, one horizontal ($L_{23}$) and the other vertical ($L_{14}$), which meet at an angle of $90^\circ$
precisely at the true values of the children masses. More importantly, Figures C-2 and
C-3 now exhibit another pair of ridges $L_{12}$ and $L_{34}$ (plotted in red in Figures C-2(b) and
C-3(b)), which were absent from the earlier figures in Section 6.4. The system of four
ridges seen in Figures C-2(a) and C-3(a) is very similar to the crease structure observed
in A. J. Barr et al. [“Transverse masses and kinematic constraints: from the boundary to
the crease, JHEP 0911, 096 (2009)] [40]. We thus confirm the result of Reference. [40]
that in the infinite $P_{UTM}$ limit there exist four different ridges, whose common intersection
point reveals the true masses of the parent and children particles.

At this point it is instructive to contrast the two sets of ridgelines: $L_{23}$ and $L_{14}$ (shown
in Figures C-2(b) and C-3(b) in black) versus $L_{12}$ and $L_{34}$ (shown in Figures C-2(b) and
C-3(b) in red). The boundaries $L_{23}$ and $L_{14}$ separate the union of regions $R_1$ and $R_2$
from the union of regions $R_3$ and $R_4$. Along those boundaries, we observe a transition in
the configuration of visible momenta which yields the maximum possible value of $M_{T2}$.
More precisely, in regions $R_1$ and $R_2$ we find that the visible momenta $\vec{p}^{(\lambda)}_T$ for $M_{T2(\text{max})}$
are parallel to the direction of the upstream momentum $\vec{P}_{UTM}$, while in regions $R_3$ and
$R_4$ we find that $\vec{p}^{(\lambda)}_T$ are anti-parallel to $\vec{P}_{UTM}$. This fact remains true even at finite values
of $P_{UTM}$, which is why the ridgelines $L_{23}$ and $L_{14}$ could also be seen in the earlier plots
from Section 6.4 at finite $P_{UTM} = 1$ TeV.

On the other hand, the ridgelines $L_{12}$ and $L_{34}$ shown in red in Figures C-2(b) and
C-3(b) are due to the “decoupling argument” [40], which is strictly valid only in the infinite
$P_{UTM}$ limit. This is why these ridges become apparent only at very large values of $P_{UTM}$,
and are gradually smeared out at smaller $P_{UTM}$. 

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Figure C-2. $M_{T2(max)}$ for the event topology of Figure 6-3(a) with fixed upstream momentum of $P_{UTM} \to \infty$.

Figure C-3. $M_{T2(max)}$ for the event topology of Figure 6-3(a) with the symmetric mass spectrum II from Table 6-1 with upstream momentum $P_{UTM} \to \infty$. 

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Figure C-4. A study of the sharpness of the $M_{T2}$ ridge for the example considered in Section 6.4.1. The event topology is that of Figure 6-3(a) and the mass spectrum is $M_c^{(a)} = 250$ GeV, $M_c^{(b)} = 500$ GeV and $M_p = 600$ GeV. We plot the asymmetric $M_{T2}$ endpoint $M_{T2(max)}(\tilde{M}_c^{(a)}(\phi), \tilde{M}_c^{(b)}(\phi), P_{UTM})$, as a function of the angular variable $\phi$ parameterizing the circle of radius $R$ defined in Equations (C–10,C–11). The radius $R$ of the circle is taken to be $R = 50$ GeV in panel (a) and $R = 5$ GeV in panel (b). We present results for four different choices of the upstream momentum $P_{UTM}$ as labelled in the plot.

Figure C-5. A study of the sharpness of the $M_{T2}$ ridge for the case when missing particles are the same where the input mass spectrum is fixed as $M_c^{(a)} = 100$ GeV, $M_c^{(b)} = 100$ GeV and $M_p = 300$ GeV.
The evolution of the ridge structure as a function of $P_{UTM}$ is shown in Figures C-4 and C-5. In order to compare the sharpness of the four ridges, we choose to vary the test children masses $\tilde{M}_c^{(a)}$ and $\tilde{M}_c^{(b)}$ along a circle centered on their true values and with a fixed radius $R$. Such a circle is guaranteed to cross all four ridges, and can be parameterized in terms of an angular coordinate $\phi$ as follows
\begin{align}
\tilde{M}_c^{(a)}(\phi) &= M_c^{(a)} + R \cos \phi, \\
\tilde{M}_c^{(b)}(\phi) &= M_c^{(b)} + R \sin \phi.
\end{align}

Then in Figure C-4 (Figure C-5) we plot the asymmetric $M_{T2}$ endpoint $M_{T2(\text{max})}(\tilde{M}_c^{(a)}(\phi), \tilde{M}_c^{(b)}(\phi), P_{UTM})$, as a function of the angular variable $\phi$, for the case of mass spectrum I studied in Section 6.4.1 (mass spectrum II studied in Section 6.4.2). The radius $R$ is taken to be $R = 50$ GeV in panels (a) and $R = 5$ GeV in panels (b). We present results for four different choices of the upstream momentum: $P_{UTM} = 100$ GeV (black lines), $P_{UTM} = 1$ TeV (blue lines), $P_{UTM} = 4$ TeV (magenta lines), and $P_{UTM} = \infty$ (red lines). Notice that the red lines at $P_{UTM} = \infty$ in Figures C-4 and C-5 are directly correlated to the three-dimensional plots of Figures C-2 and C-3, while the blue lines at $P_{UTM} = 1$ TeV in Figures C-4 and C-5 are directly correlated to the three-dimensional plots of Figures 6-6 and 6-9.

Each one of the previously discussed ridges manifests itself as a kink in Figures C-4 and C-5. Indeed, the red lines for $P_{UTM} = \infty$ reveal four clear kinks, which (from left to right) correspond to the ridgelines $L_{34}$, $L_{23}$, $L_{12}$, and $L_{14}$. Using Equations (C–6–C–9), it is easy to find the expected location of each kink in the $P_{UTM} \to \infty$ limit: $\phi = \{63.4^\circ, 180^\circ, 204.9^\circ, 270^\circ\}$ for Figure C-4(a), $\phi = \{63.4^\circ, 180^\circ, 206.4^\circ, 270^\circ\}$ for Figure C-4(b), and $\phi = \{45^\circ, 180^\circ, 225^\circ, 270^\circ\}$ for Figures C-5(a) and C-5(b). However, as the upstream momentum is lowered to more realistic values, the kinks gradually wash out, albeit to a different degree. As anticipated from our earlier results, the smearing effect is quite severe for $L_{34}$ and $L_{12}$, and by the time we reach $P_{UTM} = 1$ TeV, those
two kinks have completely disappeared. On the other hand, $L_{23}$ and $L_{14}$ are affected to a lesser degree by the smearing effect and are still visible at $P_{UTM} = 1$ TeV, but by $P_{UTM} = 100$ GeV they are essentially gone as well. Notice that the variation in $P_{UTM}$ affects not only the sharpness of the kinks, but also their location. This was to be expected, since we already saw that the shape of the ridge is different at $P_{UTM} = 1$ TeV and $P_{UTM} = \infty$: compare the black ridge lines in Figures 6-6(b) and 6-9(b) to those in Figures C-2(b) and C-3(b). Finally, as a curious fact we notice that the results shown in panels (a) and panels (b) of Figures C-4 and C-5 are approximately related by a simple scaling with a constant factor.
REFERENCES


[59] https://twiki.cern.ch/twiki/bin/view/Main/SezenSekmen


[76] Luc Pape, “Reconstruction of sparticle masses from endpoints (and others) at LHC”, CMS Internal Note CMS IN-2006/12.


[78] Georgia Karapostoli, “Observation and measurement of the supersymmetric process $\tilde{\chi}^0_2 \rightarrow \tilde{\chi}^0_1 \ell \ell$ with the CMS experiment at LHC”, CMS thesis CMS TS-2009/007.


BIOGRAPHICAL SKETCH

Myeonghun Park was born in Chung-Ju, South Korea. He has one older sister and two younger brothers. He grew up mostly in Jeon-Ju City, graduating from Dong-Am High School in 1995. He earned his B.S. in Physics and Mathematics in Korea Advanced Institute of Science and Technology in 2000. He finished M.S. degree in Physics from Seoul National University on 2002. After graduation he fulfilled military service as a navy officer from 2002 to 2005. For two years, he had been a engineering officer in combat ships, and spent his last service year in Navy warehouses. He received his Ph.D. from the University of Florida in the spring of 2011. He will go to the European Organization for Nuclear Research (CERN) in Geneva Switzerland for a postdoctoral fellowship from 2011 to 2013. Myeonghun Park married Kelly Chung in 2009.