RANDOM MATRIX ENSEMBLES WITH SOFT-CONFINEMENT POTENTIAL

By

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To my family

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In this work, we study invariant-class of random matrix ensembles characterized by the asymptotic logarithmic soft-confinement potential $V(H) \sim [\ln H]^{(1+\lambda)}$ ($\lambda > 0$), named "λ-ensembles". The suggestion is inspired by the existing random matrix models such as the critical ensembles (λ=1), the free Levy matrices ($\lambda \rightarrow 0$ limit) and the Gaussian ensembles ($\lambda \rightarrow \infty$ limit) in an effort to investigate the novel universality associated with the fat-tail random matrix ensembles as well as the logarithmic soft-confinement potential within the framework of rotationally invariant random matrix theory. First of all, we show that the orthogonal polynomials with respect to the weight function $\exp[-(\ln x)^{1+\lambda}]$ belong to a novel orthogonal polynomial system, named "λ-generalization of q-polynomials". Second, we show that based on numerical construction of the "λ-generalization of q-polynomials", we can study the one-level and the two-level correlation functions as well as the level statistics of the λ-ensembles. Third, we show that the one-level correlation (eigenvalue density) has a power-law form $\rho(x) \propto [\ln x]^{\lambda-1}/x$ and the unfolded two-level correlation function possesses the normal/anomalous structure, characteristic of the critical ensembles. We further show that the anomalous part, so-called "ghost-correlation peak" is controlled by the parameter λ; decreasing λ increases the anomaly. Third, we also identify the two-level kernel of the λ-ensembles in the semi-classical regime, which can be written in a sinh-kernel form with more general argument that reduces to that of the critical

ensembles for $\lambda = 1$. Forth, we show that the number variance is linear in L for all λ and the slope (the level compressibility) is increasing as λ decreases, which is consistent with the λ-dependence of sum rule violation $0 < \chi(\lambda) < 1$. Finally, we will discuss the novel universality of the λ-ensembles, which interpolates the Gaussian ensembles ($\lambda \to \infty$ limit), the critical ensembles ($\lambda = 1$), the free Lévy matrices ($\lambda \to 0$ limit).

CHAPTER 1
MOTIVATION

1.1 Random Matrix Theory and Complex Systems

Random matrix theory (RMT) deals with eigenvalue and eigenvector correlations of random matrix ensembles drawn in a stochastic manner. The earlier application of RMT [1, 2] in physics dates back to the 1950s when Wigner introduced RMT to explain the fluctuation properties of heavy nuclei energy spectrum. In a slow nuclear reaction, incident nucleon interacts with the constituents of target nucleus in a complicated manner such that extra energy carried in by the an incident nucleon is shared non-trivially with the nucleons of the target nucleus while forming the excited states. However, the fluctuation properties of the excitation energy levels of the compound nucleus are not very well understood [3]. To explain the statistical nature of the energy spectrum of the compound nucleus, Wigner suggested to consider ensembles of Hamiltonians whose entries are randomly drawn from the Gaussian distribution. It turns out that the statistical properties of the eigenvalue correlations of such random Hamiltonian ensembles, e.g. the Gaussian ensembles, showed a good agreement with the statistical behaviors of the eigenvalue spectrum obtained from various heavy nuclei reaction experiments [3, 4]. The underlying idea of RMT is well explained by Dyson [5]:

What is required is a new kind of statistical mechanics, in which we renounce exact knowledge not of the state of the system but of the system itself. We picture a complex nucleus as a black box in which a large number of particles are interacting according to unknown laws. As in the orthodox statistical mechanics we shall consider an ensemble of Hamiltonians, each of which could describe a different nucleus. There is a strong logical expectation, though no rigorous mathematical proof, that an ensemble average will correctly describe the behavior of one particular system which is under observation. The expectation is strong, because the system might be one of a huge

variety of systems, and a very few of them will deviate much from a properly chosen ensemble average.

One of the surprising aspects of this "new kind of statistical mechanics" is that its utility is far reaching beyond its origination. Since its introduction to the many-body complex nuclei systems, RMT has been applied to a wide variety of systems in diverse areas, including many-body atoms and molecules, quantum chaos, mesoscopic disordered conductor, 2-D quantum gravity, conformal field theory, chiral phase transitions as well as zeros of Riemann zeta function, scale-free networks, biological networks, communication systems and financial markets [6–18]. This broad range of applicability of RMT in seemingly unrelated areas highlights the universal features of the correlations of the eigenvalues in RMT. Within the classical or Gaussian model pioneered by Wigner, these correlations are known as the Wigner-Dyson (WD) statistics of the Gaussian ensembles, which are qualitatively different from the statistical features of completely uncorrelated eigenvalues given by the Poisson statistics.

1.2 Complex Systems with Power-law Distributions

In the past decades, there have been growing interest of RMT applications to generic complex systems that appear in the study of natural, social, economical and biological systems. In these systems, the microscopic interactions among the constituents are so complicated that the statistical hypothesis made in RMT seems relevant. Namely, we are forced to "renounce exact knowledge not of the state of the system but of the system itself." Thus, it appears to be appropriate to apply the idea of RMT to understand certain statistical features of these systems.

One such application of RMT is found in the analysis of the empirical covariance matrices in a multivariate setting such as financial assets and climate systems [11, 12, 19] based on classical RM models. In these works, the application aims at decomposing genuine correlations, present in the eigenvalues and eigenvectors of the empirical covariance matrices from pure noise components, the RMT predictions. For example,

Refs. [11, 12] study the empirical cross-correlation matrices of the stock price returns of S&P 500 and the largest 1000 US companies for certain period time such that the elements of such matrices represent the correlations among the companies in a given period, e.g.,

$$C_{ij} \equiv \langle r_i(t) r_j(t) \rangle. \tag{1–1}$$

Here $< \ldots >$ denotes time average over the period studied and $r_i(t)$ represents a normalized time series of stock price return of a company i defined as

$$r_i(t) \equiv \frac{R_i(t) - \langle R_i(t) \rangle}{\sigma_i} \tag{1–2}$$

where the volatility $\sigma_i \equiv \sqrt{\langle R_i(t)^2 \rangle - \langle R_i(t) \rangle \langle R_i(t) \rangle}$ and $R_i(t)$ represents the logarithm of the price change,

$$R_i(t) \equiv \ln S_i(t + \Delta t) - lnS_i(t) \simeq \frac{S_i(t + \Delta t) - S_i(t)}{S_i(t)}. \tag{1–3}$$

Here, $S_i(t)$ is the time series of the stock price. Thus, the covariance matrix C_{ij} is by construction a real symmetric matrix. The statistical properties of the covariance matrix C_{ij} can be compared to the well known results of appropriate classical RM models, e.g. Gaussian orthogonal ensembles (GOE) or Wishart matrices (Laguerre ensembles). The study shows that most of the eigenvalues except some outliers are well within the RMT expectations in terms of the eigenvalue density, the nearest spacing distribution, the number variance and the inverse participation ratio, implying that the empirical covariance matrices are dominated by random Gaussian noise for the most part. The small fraction of the outliers carries some meaningful informations useful for the risk management.

This explanation, however, is not fully satisfactory because the *a priori* assumption that cross-correlations present in the stock price returns are dominated by the Gaussian noise is not in accordance with the observation that the distribution of stock price returns follows an asymptotic fat-tail distribution, the so called inverse-cubic law [20]. To be

precise, it is shown in Ref. [20] that the cumulative distribution of the stock price returns has an asymptotic fat-tail distribution, e.g. $P(r_i(t) \geq r) \sim r^{-\alpha}$ with $\alpha \simeq 3$. Thus, the application of the classical Gaussian RM models requires reconsideration in this example and more generally, in the complex systems where the fluctuation is better characterized by fat-tail noises. This situation motivates search for suitable RM models that can incorporate the fat-tail noises.

In this context, many attempts have been made to construct *generalized* random matrix ensembles that incorporate *power-law* or *fat-tail* distributions [21–28]. The significance of such generalization beyond the Gaussian ensembles is mainly two fold. First, there are numerous complex systems that exhibit fat-tail noises, notably, financial markets, earthquakes, scale-free networks etc. [29–32]. As pointed out earlier, the relevance of the classical RM models in these systems seems questionable. It is because the classical or Gaussian RM models are based on the assumption that the systems are characterized by the Gaussian noise, which is not appropriate for systems with fat-tail noise where the occurrence of extreme events are not as rare as expected from normal distributions. Second, it is conceived that the universality of classical or Gaussian RM models is closely linked to the prevalence of normal or Gaussian distributions in nature, a consequence of the central limit theorem [21]. It has, however, not been fully investigated if there is a counterpart of the Gaussian ensemble as implied by the structure of the *generalized* central limit theorem [21], comparing e.g, the Gaussian and the Lévy basins (refer to APPENDIX A for more details).

So far, fat-tail distributions in random matrix ensembles have been carefully incorporated in some limited cases and the calculation of the correlation functions of the eigenvalues have been carried out for certain special cases [21–28]. However, the question regarding the universality of the correlations of the eigenvalues remains unresolved. For Gaussian ensembles, it is the well known two-level sine kernel that establishes the universality of the correlations in the properly scaled large N (matrix

size) limit but it is not clear if there exists a similar universal two-level kernel for the power-law or the fat-tail ensembles as well.

1.3 Fat-tail RM Models

1.3.1 Lévy Matrices

The Lévy matrices [21] are introduced by P. Cizeau and J.P. Bouchaud in consideration of constructing a RM model based on the Lévy probability distribution. In particular, they considered a $N \times N$ real symmetric matrix ensemble of independent indentically-distributed (i.i.d.) random numbers of asymptotic Lévy distributions given by the following:

$$P(H_{ij}) \sim \frac{H_0^{\mu}}{|H_{ij}|^{1+\mu}}; \quad 0 < \mu < 2 \tag{1–4}$$

where H_{ij} denotes the matrix elements. By construction, $H_{ij} = H_{ji}$ and H_0 is set to be $N^{1/\mu}$ in order to ensure that the typical largest element of a row is $O(1)$. Since the matrix elements are drawn from the distributions with divergent moments, there is no direct way to construct an analytical model within the standard RMT techniques. Thus, they developed a novel analytical technique based on the cavity method [21] to evaluate the eigenvalue density and the inverse participation ratio in parallel with numerical simulations. The main results of their work are summarized in the following:

1. The eigenvalue density converges to a limiting form of distribution in the large N limit that has asymptotic power-law distribution. e.g. $\rho(x \to \infty) \sim \frac{1}{x^{1+\mu}}$, which has the same scaling power as that of the matrix elements.

2. From careful examinations of the inverse participation ratio, it is shown that the eigenstates undergo a non-trivial localization-delocalization transition within the eigenvalue spectrum at a certain critical value of $x = x_c$ depending on the parameter μ: for $\mu > 1$, there exists a critical value x_c that differentiate the extended states $(x < x_c)$ from the algebraically localized states $(x > x_c)$, which is an unusual type of localization since this type of localization allows non-zero conductivity. For $\mu < 1$, all the states are localized (finite participation ratio) except ground states. The states below x_c are localized in a usual sense. However, the finite fraction of the states above x_c are still extended over $O(N)$ site. In the limit $\mu \to 1$ or 2, the x_c diverges.

3. The numerically-obtained nearest neighbor spacing distribution is found to be non-universal [1] , which depends on the location of the eigenvalue spectrum considered.

1.3.2 Free Lévy Matrices

The free Lévy matrices, introduced by Z. Burda *et al*. [22], are constructed based on the theory of free random variable (FRV) that offers the probability theory of non-commuting variables as a generalization of the classical probability theory. The correspondence between the classical probability theory and the FRV theory is made in Ref. [33]:

1. Probability distribution $P(x)$: Spectral density $\rho(x) = -ImG(x + i0)/\pi$
2. Characteristic function $\hat{P}(k)$: Green function $G(z) \equiv < Tr(\frac{1}{z-M}) >$
3. Logarithm of characteristic function $\ln \hat{P}(k)$: R-transform $R(G(z)) \equiv z - \frac{1}{G(z)}$

In the classical probability theory, the logarithm of characteristic function is additive (the additivity of the cumulants) under convolution operation of two random variables. Analogously, in the FRV theory, the R-transform of two independent random matrix-valued numbers, M_1 and M_2, is additive, *i.e.*, $R_{M_1+M_2}(z) = R_{M_1}(z) + R_{M2}(z)$ [22, 33]. The free Lévy random matrix approach takes advantage of the fact that one can reconstruct the Green function, and the spectral density, and finally identify a particular probability measure that characterizes a random matrix ensemble from the known R-transforms that correspond to all free stable probability distributions under FRV calculus: *e.g.*, $R(z) = a + bz^{\alpha-1}$ where $0 < \alpha < 2$ $(\alpha \neq 1)$[2] which determines the asymptotic power-law behavior of the stable distributions. The parameters a and b are

[1] It should be emphasized that the numerical unfolding procedure is non-trivial in this model so that it requires further investigation regarding the universality of the spacing distribution. In fact, our numerical investigation shows possible universality of the spacing distribution.

[2] For $\alpha = 1$, $R(z) = a - i\gamma(1 + \beta) - \frac{2\beta\gamma}{\pi} \ln \gamma z$.

real valued and are associated with the shift, the slope (α) and the skewness (β) and the range (γ) of the stable distributions. Note that $R(z)$ is given without any particular realization of matrix ensembles.

In order to find a particular realization of a random matrix ensemble characterized by the probability measure of the form

$$e^{-NTr\ V(M)}dM \tag{1–5}$$

where Tr is the trace and $V(M)$ is specific to the realization of the ensembles and is related to the confining potential in the Coulomb gas analogy of the classical RM models developed by Dyson [34], Z. Burda *et al*. [22] first calculated the spectral densities [3] from the stable R-transforms for some exactly solvable cases of parameters. They showed that the asymptotic form of spectral densities displays power-law distributions with the exponents within the Lévy stability regime. They then showed that in all such cases, the asymptotic form of the confining potential $V(x)$ is given as $V(x) = ln\, x^2 + O(1/x)$ by using the relation between the potential and the spectral density arising in the Coulomb gas analogy.

After the identification of the confining potential and thus the matrix ensemble (free Lévy matrices), they further studied the eigenvalue correlations based on a standard RMT technique, namely the orthogonal polynomial method. This model successfully incorporates the asymptotic Lévy distributions and also allows the calculations of eigenvalue correlations within the standard RMT techniques due to the rotational invariance of the probability measure of the ensemble. It, however, suggests that the

[3] To give a quick illustration of the method, consider the trivial choice of $R(z) = a$, then Green function $G(z) = \frac{1}{z-a}$ and thus the spectral density $\rho(x) = \delta(x-a)$. Refer to Ref [22] for more details.

two-level correlations of the ensembles have a non-trivial N-dependence that cannot be simply scaled out.

1.3.3 Nonextensive q Ensembles

One way of constructing the classical RM models is to maximize the Shannon entropy S

$$S \equiv -\int dH P(H) \ln P(H), \tag{1–6}$$

subject to the condition that the probability is normalized to 1,

$$\int dH P(H) = 1. \tag{1–7}$$

Here dH is the measure associated with the matrix elements of H. In particular, requiring that the variance of the matrix elements be finite, (equivalently assuming that extremely large matrix elements are improbable)

$$\int dH \mathrm{Tr}(H^2) P(H) < \infty, \tag{1–8}$$

it can be shown by using the Lagrange multiplier (λ) method that the $P(H)$ that maximizes the entropy S subject to the the above conditions is given by

$$P(H)dH \sim e^{-\lambda \mathrm{Tr} H^2} dH. \tag{1–9}$$

Thus, the maximization of the entropy with the finite-variance constraint on $P(H)$ leads to the Gaussian ensembles.

In a similar manner, the non-extensive q ensembles [23] can be constructed from maximizing the non-extensive entropy S_q

$$S_q[P(H)] = \frac{1 - \int dH\ [P(H)]^q}{q-1} \tag{1–10}$$

with the usual normalization condition, $\int\ dH\ P(H) = 1$ and q is a parameter. Similar to the constraint that requires the finite variance of $P(H)$ leading to the Gaussian

ensembles, the non-extensive q ensembles require the following condition

$$\frac{\int\ dH\ trH^2\ [P(H)]^q}{\int\ dH\ [P(H)]^q} = \sigma^2 \tag{1–11}$$

with a constant σ. Using the Lagrange multiplier method, it can be shown that the $P(H)$ that maximizes the non-extensive entropy S_q subject to the above constraints is given by

$$P(H) \sim \exp_q(-\lambda trH^2) \tag{1–12}$$

where the q-exponential function $\exp_q(x)$ is defined as

$$\exp_q(x) \equiv \{[1 + (1 - q)x]_+\}^{\frac{1}{1-q}} \tag{1–13}$$

with

$$[...]_+ = max(..., 0). \tag{1–14}$$

Note that for q=1, the $\exp_q(x)$ reduces to the usual exponential function thus leading to the usual Gaussian ensembles. The study shows that the non-extensive q ensembles exhibit characteristically different behavior depending on the parameter q; for $q > 1$, the distributions of eigenvalue density show true long tails and for $q < 1$, the distributions have compact support. However, the variability of the parameter q in this case depends on the dimensionality N of the ensemble such that in the large N limit where universal behavior is expected, the maximum q allowed for the nonextensive ensembles approaches unity.

1.3.4 Ensembles Based on Superpositions of Classical or Gaussian Ensembles

There are several RM models categorized under this class of ensembles [24–28]. The main idea is to construct generalized ensembles with the superposition (or deformation) of the Gaussian or Wishart ensembles. To illustrate the details of the idea, the approach in Ref. [24] will be discussed in the following.

The underlying idea of the approach in the Ref. [24] is to work with an associated characteristic function, rather than a probability distribution, to discover a novel random

matrix ensemble. It is shown in this framework that the general n-point correlation function can be written down as

$$R_n(x_1, \cdots, x_n) = \int_0^\infty \frac{dbf(b)}{(4b)^{n/2}} det[K_N^G(\bar{x_i}, \bar{x_j})]_{i,j=1,2,\ldots n} \tag{1–15}$$

where $f(b)$ is called the spread function, defined as any non-negative normalizable function, $\int_0^\infty f(b)db = 1$ and $\bar{x_i} = x_i/2\sqrt{b}$ and K_N^G refers to the well known sine kernel of the Gaussian ensembles. In particular, the spectral density (1-point correlation function) can be given by

$$\sigma_N(x) = \frac{\sqrt{2N}}{2\pi} \int_0^\infty \frac{dbf(b)}{b^{1/2}} K_N^G(x, x) \tag{1–16}$$

and two point cluster function defined as $T_2(x_1, x_2) \equiv -R_2(x_1, x_2) + R_1(x_1)R_1(x_2)$ has the form $T_2 = T_2^0 - \delta T_2$, where

$$T_2^0(x_1, x_2) = \int_0^\infty \frac{dbf(b)}{4b} [K_N^G(\bar{x_1}, \bar{x_2})]^2 \tag{1–17}$$

and

$$\delta T = \int_0^\infty \frac{dbf(b)}{4b} \sigma_N^G(\bar{x_1})\sigma_N^G(\bar{x_2}) - \sigma_N(x_1)\sigma_N(x_2) \tag{1–18}$$

Thus, a variety of possible generalized ensembles can be obtained from a proper choice of superposition (a choice of the spread function $f(b)$) of the kernel of Gaussian ensemble. At the same time, it is possible to study the correlation functions for the choice of $f(b)$ immediately. For example, the Gaussian ensembles can be thought of as a trivial example corresponding to the choice of $f(b) = \delta(b - b_0)$. For a non-trivial choice of $f(b) = \frac{\mu^\lambda}{\Gamma(\lambda)} b^{-(\lambda+1)} e^{-\mu/b}$, it is shown that the spectral density exhibit asymptotic power-law distributions. Although this framework allows a successful realization of RM

model that can incorporate the desired fat-tail spectral density, it is not yet clear if the unfolded two-level cluster function can have N-independent asymptotic limit [4] .

[4] It turns out that the two-level correlation function carries non-trivial N dependence. The details will be provided in APPENDIX B.

CHAPTER 2
RANDOM MATRIX THEORY

2.1 Random Matrix Ensembles

The first step to construct a RME is to consider a set of $N \times N$ Hermitian matrices H with the following probability measure

$$P_N(H)dH \propto e^{-TrV(H)}dH, \tag{2–1}$$

where $V(H)$ is a suitably increasing function such that the probability measure is normalizable. Tr is the matrix trace and dH the invariant measure. Especially, the $TrV(H)$ ensures the rotational invariance under orthogonal, unitary and symplectic transformations. More explicitly, it means that under a transformation R,

$$H \to M = R^{\dagger}HR, \quad R^{\dagger}R = 1 \tag{2–2}$$

the probability measure remains the same

$$P_N(H)dH = P_N(M)dM. \tag{2–3}$$

Thus, by construction, all orientations of the eigenbasis are equally likely or in other words, there is no preferential basis. Each of the symmetries determines the structure of the hamiltonian systems, *e.g.*,

- Orthogonal symmetry (time-reversal invariant systems with rotational symmetry) $\to$ Real symmetric matrices
- Unitary symmetry (systems in which time-reversal symmetry is broken) $\to$ Complex Hermitian matrices
- Symplectic symmetry (time-reversal invariant systems with half-integer spin and broken rotational symmetry) $\to$ Self-dual quaternion matrices.

In order to find a specific realization of a RME, one needs to specify the probability measure $V(H)$ of a RME. One way to find $V(H)$ comes from the maximization of the entropy or the minimization of information content of the RME. Suppose that nothing is

known about the detailed dynamics of a system other than the fundamental symmetries, which gives no knowledge about the matrix elements of the hamiltonian systems other than the global symmetries. The information (I) contained in the $N \times N$ Hamiltonian matrix H can be defined as

$$I \equiv \int dH P(H) \ln P(H), \tag{2–4}$$

with the condition that the probability is normalized to 1,

$$\int dH P(H) = 1. \tag{2–5}$$

requiring the variance of the matrix elements be finite

$$\int dH \mathrm{Tr}(H^2) P(H) < \infty. \tag{2–6}$$

The information in Eq. 2–4 can be minimized subject to the constraints in Eq. 2–5 and Eq. 2–6 by using Lagrange multiplier method. For an arbitrary variation $\delta P(H)$ of $P(H)$,

$$\delta I = \int dH \delta P(H) \{1 + \ln P(H) + \lambda \mathrm{Tr}(H^2) P(H)\} = 0 \tag{2–7}$$

Thus,

$$P(H)dH \propto e^{-\lambda \mathrm{Tr}(H^2)} dH. \tag{2–8}$$

This is the probability measure of the well-known Gaussian ensembles. The Gaussian ensembles can also be obtained based on the two assumptions; invariance under transformation (Orthogonal, Unitary, Symplectic) and statistical independence of matrix elements [4]. The three different symmetries define three different classes of Gaussian ensembles: Gaussian Orthogonal Ensemble (GOE), Gaussian Unitary Ensemble (GUE), and Gaussian Symplectic Ensemble (GSE). If we only require the rotational invariance of $P(H)dH$, dropping the assumption of the statistical independence of the matrix elements, then $P(H)$ may be any function of the traces of powers of H. Thus, in general

the probability measure can be written down as

$$P(H)dH \propto e^{-\mathrm{Tr}V(H)}dH. \tag{2–9}$$

It is also possible to consider RMEs with non-inavarant form of probability measure under transformation, which is the so-called “Wigner class” ensembles [35]. In this class, each matrix element of the ensemble of $N \times N$ matrix are randomly taken from i.i.d. probability distributions so that the generic form of the probability measure of Wigner ensembles can be given by

$$P(M)dM \propto \prod_{i,j} f(M_{i,j}) \prod_{i,j} dM_{i,j} \tag{2–10}$$

where $f(x)$ is the probability distribution of each individual matrix element and the product $\prod_{i,j}$ is performed for all the degrees of the freedom of the ensemble under consideration. Some of the examples are:

- Wigner Guassian ensembles : Hermitian matrices whose entries are given by i.i.d. Gaussian distributions.
- Lévy matrices (Wigner Lévy matrices) : Real symmetric matrices whose entries are given by i.i.d. Lévy distributions.
- Banded matrices : Symmetric matrices whose entries m_{ij} are non-zero for $|i-j| < r$ and zero otherwise where r determines band width.
- Adjacency or Laplacian matrices of random graphs or scale-free networks.

In the following, we will mainly focus on the study of the invariant class ensembles and its universality.

2.2 Orthogonal Polynomial Method

The orthogonal polynomial method has been a fruitful tool in the development of RMT. The merit of the orthogonal polynomial method is that it allows to write down all the correlation functions of the eigenvalues in terms of orthogonal polynomials corresponding to the probability measure characterizing a RME. In particular, the two-level correlation function written in a properly scaled variable in the asymptotic large

N limit is central to establishing the *universality* of RMEs. Thus, in the following, we will review the orthogonal polynomial method and the results obtained for classical and Gaussian RMEs and the critical ensembles.

2.2.1 Joint Probability Distribution Function (JPDF)

In order to study the eigenvalue correlations, the probability measure $P(H)dH$ needs to be transformed from the matrix element basis to the eigenvalue/eigenvector basis,

$$H \rightarrow R^{\dagger} X R \quad \text{with} \quad R^{\dagger} R = 1. \tag{2–11}$$

Here, the matrix X is the diagonal matrix containing eigenvalues $(\{x_i\}, i = 1, ..., N)$ and the matrix R is the rotational transformation containing eigenvectors. After the transformation is considered, the invariant measure dH in the matrix element basis can be rewritten in the eigenvalue/eigenvector basis, *e.g.*,

$$dH = J(X, R)dXdR; \quad i = 1...N \tag{2–12}$$

where $J(X, R)$ is the Jacobian of the transform. The dX and dR refer to the measure associated with the eigenvalues and parameters of the rotational transformations respectively. Here, the Jacobian factor can be further calculated for a given symmetry class and it can be shown that [4]

$$J(X, R) = \prod_{i<j} |x_i - x_j|^{\beta} f(R) \tag{2–13}$$

where the β is the symmetry parameter of the rotational transformations such as Orthogonal ($\beta = 1$), Unitary ($\beta = 2$) and Symplectic ($\beta = 4$) symmetries. The $f(R)$ is the function of the parameters associated with the rotational transformations. Once the parameters associated with transformation is integrated out, we obtain the JPDF of N eigenvalues in the following form.

$$P_N(\{x_i\})dX \propto \prod_{i<j} |x_i - x_j|^{\beta} \prod_i w(\{x_i\})dx_1 \cdots dx_N \tag{2–14}$$

where $\prod_{i<j}|x_i - x_j|$ is the Vandermonde determinant and the $w(x)$ defines weight function. For Gaussian ensembles, $w(x) = \exp(-\frac{\beta}{2}x^2)$. For more general potential $V(x)$, $w(x) = \exp(-\frac{\beta}{2}V(x))$.

2.2.2 Determinant Form of JPDF and Kernel

The form of JPDF obtained in Eq. 2–14 can be greatly simplified by using matrix determinant. In particular, the unitary class ($\beta = 2$) is the simplest and thus, in the following we will focus on the unitary case of JPDF. To this end, we first observe that the Vandermonde determinant $\prod_{i<j}|x_i - x_j| = \det[x_i^{j-1}]$. More explicitly,

$$\det[x_i^{j-1}] = \prod_{i<j}|x_j - x_i| = \det\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{vmatrix} \tag{2–15}$$

After absorbing the weight function into the determinant above, JPDF can be written down as

$$P_N(\{x_i\}) = \frac{1}{Z_N}\det[Q^T Q] \tag{2–16}$$

where Z_N is a normalization constant and the matrix Q is defined as

$$Q = \begin{vmatrix} 1\sqrt{w(x_1)} & 1\sqrt{w(x_2)} & \dots & 1\sqrt{w(x_N)} \\ x_1\sqrt{w(x_1)} & x_2\sqrt{w(x_2)} & \dots & x_N\sqrt{w(x_N)} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1}\sqrt{w(x_1)} & x_2^{N-1}\sqrt{w(x_2)} & \dots & x_N^{N-1}\sqrt{w(x_N)} \end{vmatrix} \tag{2–17}$$

Since the determinant does not change by the adding to any one row or column to multiples of the other rows or columns, the matrix Q can be rewritten as

$$Q = \begin{vmatrix} p_0(x_1)\sqrt{w(x_1)} & p_0(x_2)\sqrt{w(x_2)} & \dots & p_0(x_N)\sqrt{w(x_N)} \\ p_1(x_1)\sqrt{w(x_1)} & p_1(x_2)\sqrt{w(x_2)} & \dots & p_N(x_N)\sqrt{w(x_N)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N-1}(x_1)\sqrt{w(x_1)} & p_{N-1}(x_2)\sqrt{w(x_2)} & \dots & p_{N-1}(x_N)\sqrt{w(x_N)} \end{vmatrix} \tag{2–18}$$

where $p_i(x)$ denotes an arbitrary polynomial of degree i. And after considering the matrix multiplication $Q^T Q$, we can obtain

$$P_N(\{x_i\}) = \frac{1}{N!}\det \begin{vmatrix} K_N(x_1,x_1) & K_N(x_1,x_2) & \dots & K_N(x_1,x_N) \\ K_N(x_2,x_1) & K_N(x_2,x_2) & \dots & K_N(x_2,x_N) \\ \vdots & \vdots & \ddots & \vdots \\ K_N(x_N,x_1) & K_N(x_N,x_2) & \dots & K_N(x_N,x_N) \end{vmatrix} \tag{2–19}$$

where the two-level kernel $K_N(x,y)$ is defined as

$$K_N(x,y) \equiv \sum_{i=0}^{N-1} \psi_i(x)\psi_i(y), \tag{2–20}$$

where the "wave functions" $\psi_i(x) \equiv p_i(x)\sqrt{w(x)} = p_i(x)e^{-V(x)/2}$.

Here the choice $p_i(x)$ is arbitrary but if $p_i(x)$ is chosen a system of the polynomials that are orthogonal with respect to the weight function $w(x) = e^{-V(x)}$, *i.e.*,

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)w(x)dx = \delta_{mn}. \tag{2–21}$$

The kernel $K_N(x,y)$ satisfies following important properties.

$$\int K_N(x,x)dx = N \tag{2–22}$$

and

$$K_N(x,z) = \int K_N(x,y)K_N(y,z)dy. \tag{2–23}$$

These imply that

$$\int \det[K_N(x_i, y_i)]_{i,j=1,\ldots,m} dx_m = (N - m + 1)\det[K_N(x_i, x_j)]_{i,j=1,\ldots,m-1}. \tag{2–24}$$

2.2.3 Correlation Functions and Cluster Functions

The probability of finding any n eigenvalues out of the available N eigenvalues in the intervals, $\{x_i \to x_i + dx_i; i = 1 \ldots n\}$ is given by n-level correlation function

$$R_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_{n+1} \ldots dx_N P_N(x_1, \ldots, x_N). \tag{2–25}$$

Using the determinant form of JPDF and the properties of the kernel, the n-level correlation function can be written in a compact form given by

$$R_n(x_1, x_2, \ldots., x_n) = \det[K_N(x_i, x_j)]_{\{i,j=1,\ldots,n\}}. \tag{2–26}$$

Note that the diagonal terms of the determinant are given by the one-level correlation function $R_1(x)$ (or the eigenvalue density $\sigma_N(x)$)

$$R_1(x) = K_N(x, x) = \sum_{i=0}^{N-1} \psi_i(x)\psi_i(y). \tag{2–27}$$

and the off-diagonal terms of the determinant are given by two-level kernel in terms of which the two-level correlation function can be written down as

$$R_2(x, y) = K_N(x, x)K_N(y, y) - K_N(x, y)K_N(y, x). \tag{2–28}$$

For practical purpose, it is useful to introduce the n-level cluster functions defined by

$$T_n = \sum_G (-1)^{n-l}(l-1)! \prod_{i=1}^{l} R_{G_j}(x_j, \text{with } j \text{ in } G_k) \tag{2–29}$$

Here, G stands for any division of the indices $(1, 2, \ldots.n)$ into m subgroups $(G_1, G_2, \ldots, G_m)$. For example, the one level cluster function

$$T_1(x) = R_1(x) \tag{2–30}$$

and two-level cluster function

$$T_2(x,y) = -R_2(x,y) + R_1(x,x)R_1(y,y) \quad (2\text{–}31)$$

$$= K_N(x,y)^2 \quad (2\text{–}32)$$

$$= \left(\sum_{n=0}^{N-1} \psi_n(x)\psi_n(y)\right)^2 . \quad (2\text{–}33)$$

2.2.4 Unfolding

Within the invariant class ensembles, the one-level correlation function (eigenvalue density) $\sigma_N(x) = K_N(x,x)$ is not universal and it is dependent on the specific form of the probability measure $dP(H)$. So in the limit of large matrix size, *e.g.*, $N \to \infty$, we need to introduce a proper scaling variable to study the universality of correlation function such that in the scaled variable the mean spacing of eigenvalues become unity. It is achieved by defining the unfolding variable

$$du \equiv \lim_{N\to\infty} K_N(x,x)dx. \quad (2\text{–}34)$$

The unfolded two-level kernel can be written in terms of the unfolding variables

$$\bar{K}(u,v) = \lim_{N\to\infty} \frac{K_N(x,y)}{\sqrt{K_N(x,x)K_N(y,y)}} \quad (2\text{–}35)$$

where u and v are the unfolding variables. Now all the correlation functions and the cluster functions can be written in terms of this unfolded kernel. For example, unfolded one-level cluster functions $Y(u) = 1$ and the two level cluster function

$$Y_2(u,v) = \bar{K}(u,v)^2 \quad (2\text{–}36)$$

Thus, the study of correlation functions requires understanding of two level kernel in the properly scaled variable which requires the knowledge of asymptotic behavior of the orthogonal polynomials.

Note that if the kernel satisfy Eq. 2–23 or in order words, the kernel is reproducing, the unfolded two-level kernel satisfies the normalization of sum rule.

$$\int dv Y(u,v) = \lim_{N\to\infty} \int K_N(y,y) \frac{K_N(x,y)^2}{K_N(x,x)K_N(y,y)} dy \tag{2–37}$$

$$= \lim_{N\to\infty} \frac{1}{K_N(x,x)} \int K_N(x,y)K_N(y,x)dy \tag{2–38}$$

$$= \lim_{N\to\infty} \frac{K_N(x,x)}{K_N(x,x)} = 1. \tag{2–39}$$

2.3 Gaussian Ensembles

2.3.1 Semi-Circle law

For the Gaussian ensembles $V(x) = x^2$, the eigenvalue density is given by the well-known semi-circle law.

$$\sigma_N^G(x) = K_N(x,x) = \sum_i \psi_i(x)\psi_i(x) \approx \frac{1}{2\pi}\sqrt{2N - x^2} \tag{2–40}$$

where the wave function $\psi_n(x) = h_n(x)e^{-x^2/2}$ satisfies

$$\int_{-\infty}^{\infty} \psi_n(x)\psi_m(y)e^{-x^2}dx = \sqrt{\pi}2^n n!\delta_{nm} \tag{2–41}$$

where $h_n(x)$ is the Hermite polynomial of degree n. The derivation of semi-circle law can be given in the following way. First of all, we recognize that Eq. 2–19, the determinant form of the JPDF, can be considered as the probability density of the ground state of the many-body wave function of non-interacting fermions, *e.g.*,

$$P(x_1, ..., x_N) \propto |\Psi(x_1, ..., x_N)|^2 \tag{2–42}$$

where

$$\Psi(x_1, ..., x_N) = \begin{vmatrix} \psi_0(x_1) & \psi_0(x_2) & ... & \psi_0(x_N) \\ \psi_1(x_1) & \psi_1(x_1) & ... & \psi_1(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{N-1}(x_1) & \psi_{N-1}(x_1) & ... & \psi_{N-1}(x_1) \end{vmatrix} \tag{2–43}$$

Since the wave function in the determinant is the solution of one-dimensional harmonic oscillator, it satisfies the Schrödinger equation Thus,

$$\left(-\hbar\frac{d^2}{dx^2}+\frac{1}{2}x^2\right)\psi_n(x)=(n+1/2)\hbar\psi_n(x). \tag{2–44}$$

In the large $n=N-1$ limit, the Fermi momentum can be read off locally at x

$$p_F=\sqrt{2N-1-x^2}. \tag{2–45}$$

The density of one dimensional fermions is related to the Fermi momentum p_F by

$$\sigma(x)\approx\frac{1}{2\pi\hbar}\int_{-p_F}^{p_F}dp=\frac{p_F}{\pi}\quad\hbar=1. \tag{2–46}$$

Combining Eq. 2–45 and Eq. 2–46, we get

$$\sigma(x)\approx\frac{1}{\pi}\sqrt{2N-x^2}. \tag{2–47}$$

2.3.2 Sine Kernel

In order to calculate the unfolded two level kernel, we consider the spectrum around the origin in the large N limit, so-called double scaling limit ($x\to 0$ and $N\to\infty$ limit) such that $x\sqrt{N}$ is finite. The proper unfolded variable in this limit $du=K_N(x,x)dx=\frac{\sqrt{2N}}{\pi}dx$. Using the definition of the kernel,

$$\bar{K}_N(u,v)=\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}\sum_{i=0}^{N-1}\phi_i(\frac{\pi}{\sqrt{2N}}u)\phi_i(\frac{\pi}{\sqrt{2N}}v). \tag{2–48}$$

Recalling the Christoffel-Darboux formula,

$$\sum_{j=0}^{N-1}\phi_j(x)\phi_j(y)=\sqrt{\frac{N}{2}}\left[\frac{\phi_N(x)\phi_{N-1}(y)-\phi_N(x)\phi_{N-1}(y)}{x-y}\right]. \tag{2–49}$$

$$\bar{K}_N(u,v)=\lim_{N\to\infty}\sqrt{\frac{N}{2}}\frac{\phi_N(\frac{\pi u}{\sqrt{2N}})\phi_{N-1}(\frac{\pi v}{\sqrt{2N}})-\phi_N(\frac{\pi v}{\sqrt{2N}})\phi_{N-1}(\frac{\pi u}{\sqrt{2N}})}{u-v} \tag{2–50}$$

Using asymptotic solution for harmonic oscillator wave function,

$$\lim_{m\to\infty}(-1)^m m^{1/4}\phi_{2m}(\frac{\pi u}{\sqrt{2N}}) = \pi^{-1/2}\cos(\pi u) \tag{2–51}$$

$$\lim_{m\to\infty}(-1)^m m^{1/4}\phi_{2m+1}(\frac{\pi u}{\sqrt{2N}}) = \pi^{-1/2}\sin(\pi u) \tag{2–52}$$

where $N = 2m$. We get the unfolded kernel

$$\bar{K}^G(u,v) = \frac{\sin(\pi(u-v))}{\pi(u-v)} \tag{2–53}$$

which is the well-known *sine* kernel.

2.3.3 Gap Probability and Spacing Distribution

The probability that there is no eigenvalue in the interval of s is called gap probability defined as a Fredholm determinant

$$E(s) = \det[1-\bar{K}^G] \tag{2–54}$$

where the $\bar{K}^G$ refers to the sine kernel over the interval of $-s/2$ and $s/2$. The spacing distribution is related to the gap probability by

$$p(s) = \frac{d^2E(s)}{ds^2} \tag{2–55}$$

which is commonly used to study short range fluctuations in the eigenvalue spectrum. This function measures the probability of finding two neighboring eigenvalues in the interval of s in the unfolded scale. The analytical calculation is highly non-trivial but the good approximation of the spacing distribution can be obtained from considering 2×2 matrix model suggested by Wigner. The general expression for the spacing distribution, Wigner's surmise or Wigner-Dyson statistics,

$$p(s) = c_\beta s^\beta e^{-d_\beta s^2} \tag{2–56}$$

where c_β and d_β are constants. The β refers to the symmetry parameter of the Gaussian ensembles. Note that in the small s limit, the spacing distribution is determined by s^β

that indicates that the two neighboring levels repel. The level repulsion is characteristic of the Gaussian ensembles. This is in contrast to the uncorrelated or Possion case, $p(s) = e^{-s}$, in which the level repulsion is absent and therefore the levels can be bunched up or separated far apart.

2.3.4 Number Variance

The number variance $\Sigma(s) = \langle n^2 \rangle - \langle n \rangle^2$ provides the measure of the long-range eigenvalue fluctuation. It is given by

$$\Sigma(s) = \int_{-s/2}^{s/2} \int_{-s/2}^{s/2} [R(u,v)] du dv \quad (2\text{–}57)$$

$$= \int_{-s/2}^{s/2} \int_{-s/2}^{s/2} [\delta(u-v) - Y(u,v)] du dv \quad (2\text{–}58)$$

where $R(u,v)$ is two-level correlation function defined by $R(u,v) \equiv \langle \rho(u)\rho(v) \rangle - 1$. Using the cluster function of GUE, $Y^G(u,v) = Y^G(u-v) = \left[\frac{\sin(\pi(u-v))}{\pi(u-v)}\right]^2$, the further calculation gives

$$\Sigma^G(s) = s - 2\int_0^s (s-r) Y(r) dr \quad (2\text{–}59)$$

$$= \frac{1}{\pi^2}(\ln(2\pi s) + \gamma + 1) + O(s^{-1}) \quad (2\text{–}60)$$

where γ is Euler's constant, $\gamma = 0.5772...$. Note that for large s, $\Sigma^G(s) \propto \ln s$, which displays another characteristic of the Gaussian Ensembles, the level rigidity. In contrast, for the Poisson or uncorrelated case, $R^P(u,v) = \delta(u-v)$ and thus $\Sigma^P(s) = s$.

2.4 Critical Ensembles

2.4.1 Anderson Transition and Critical Ensembles

The suggestion of a novel universality beyond the Gaussian ensembles comes from the study of the Anderson transition in the disordered electronic systems[8, 36–42]. In these systems, the Gaussian ensemble is only relevant in the metallic regime where all the eigenstates are extended across the entire system and correlations of the corresponding eigenvalues are well described by the WD statistics. As the disorder

is made strong enough, the eigenstates become localized and thus the eigenvalues become uncorrelated. Especially at the delocalization-localization transition, it has been established that the correlations of the eigenvectors exhibit novel features [8, 36–42] such as multi-fractality and the correlations of the eigenvalues lead to a level compressibility that is intermediate between WD and Poisson statistics. Similarly in the studies of quantum chaos, energy level statistics of systems that are intermediate between chaotic and regular states also require generalization beyond WD and Poisson statistics [3, 43–47].

In these contexts, extensive studies have been carried out to construct a parametric generalization of RM models that cross over from WD to Poisson [3, 8, 36–47] as a function of the parameter. Some of these generalizations indeed capture the essential features of the critical statistics, among which the family of q-RMEs [48–50] provides a particularly valuable insight. Within the common framework of rotationally invariant RM models [4] the q-RMEs show how the universality of the Gaussian ensemble characterized by the zero parameter two-level *sine* kernel breaks down and eventually gives rise to a different kind of universality for the critical ensembles, characterized by a one-parameter two-level *sinh* kernel. In particular, the rotationally invariant RM models are characterized by a "confining potential" which defines the weight function of a set of orthogonal polynomials; the key difference between the Gaussian and the critical ensembles comes from the fact that the corresponding orthogonal polynomials, namely classical vs. q-orthogonal polynomials, respectively, possess qualitatively different asymptotic properties [51, 52].

2.4.2 Critical Statistics

The conjecture of the novel universality at the Anderson transition is based on the studies of the spacing distribution at the critical point of the tight binding hamiltonian on

a cubic lattice ($L \times L \times L$) given by

$$H = \sum_i \epsilon_i c_i^\dagger c_i - \sum_{i,j} c_i^\dagger c_j \tag{2–61}$$

where $c_i^\dagger$ and c_i are the creation and annihilation operators at site i, and j denotes the neighbor sites of i thus $j = 1, ..., 6$ on the cubic lattice; ϵ_i is the random energy of the site i in unit of overlap energy of neighboring sites that is uniformly distributed in the range $[-W/2, W/2]$. In particular, Ref. [36] showed that the spacing distribution $P^C(s)$ is scale-invariant (does not depend on the system size L) at the transition point $W = W_c$ by investigating the quantity $\gamma(W, L) \equiv \frac{A-A_w}{A_p-A_w}$ [1] ; the sclae-invariant $P^C(s)$ is characterized by a linear slope in s for $s \ll 1$ and an exponential decay for $s \gg 1$. Thus the novel form of $P(s)$ is hybrid of the WD statistics and the Poission statistics.

Another characteristic of the critical statistics is the finite "level compressibility" or sub-Poissinan number variance, namely $\Sigma^C(s) = \chi s$ with $0 < \chi < 1$, which is intermediate between the WD statistics $\Sigma^G(s) \propto \ln s$ and Poisson one $\Sigma^P(s) \propto s$. The origin of this behavior is known to be the sum rule violation. *e.g.*,

$$\chi = \frac{d\langle\Sigma(s)\rangle}{ds} = \lim_{L\to\infty} \int_{-L/2}^{L/2} R(s)ds = 1 - \int_{-\infty}^{\infty} Y(s) \neq 0. \tag{2–62}$$

which is also related to the multi-fractal nature of the wave function correlation at the critical point [8, 36–42].

2.5 Coulomb Gas Analogy

Dyson suggested that the JPDF of the invariant ensembles can be written as a form of Gibbs distribution for a classical one-dimensional system of N particles described by

[1] where $A \equiv \int_{2\delta}^{\infty} P(s)ds$ and lower limit of the integral 2δ refers to the crossing point of $P_p(s)$ and $P_w(s)$ occurring at 2 $\delta \simeq 2.002$

the Hamiltonian $H(\{x_i\})$ [53] in the following form.

$$P(\{x_i\}) = \frac{1}{Z_{N,\beta}} \exp[-\beta H(\{x_i\})], \tag{2–63}$$

where

$$H(\{x_i\}) = -\sum_{i<j} \ln|x_i - x_j| + \sum_i V(x_i). \tag{2–64}$$

In this view point, the eigenvalues x_i can be considered fictitious particles interacting with each other through logarithmic repulsion at temperature $1/\beta$ while they are confined by the external potential $V(x)$. In particular, one can assume that the particle density $\rho(x) = \sum_i^N \delta(x - x_i)$ reaches a continuum in the large asymptotic system size $N \to \infty$ limit. In this limit, the Hamiltonian $H(\{x_i\})$ can be read as an functional of the density.

$$H[\rho] = -\frac{1}{2}\int_J\int_J dxdy\rho(x)\rho(y)\ln|x-y| + \int_J dx\rho(x)V(x) \tag{2–65}$$

where J denotes the support of the density. Using the saddle point approximation, we can obtain an integral equation for the average particle (mean-field) density of $\langle\rho(x)\rangle$ [2] .

$$\int_J dx\langle\rho(x)\rangle \ln|y-x| = V(y) + \text{const.} \tag{2–67}$$

where the additional constant term can be determined by the normalization condition of the density $\int \rho(x)dx = N$. The two level correlation function can be written in terms of the functional derivative of $\langle\rho(x)\rangle$ with respect to $V(x)$,

$$R(x,y) = -\frac{\beta^{-1}}{\langle\rho(x)\rangle\langle\rho(y)\rangle}\frac{\delta\langle\rho(x)\rangle}{\delta V(y)}. \tag{2–68}$$

---

[2]

$$\langle\rho(x)\rangle = \frac{\int_J \rho(x)e^{-\beta H[\rho]}D\rho}{\int_J e^{-\beta H[\rho]}D\rho} \tag{2–66}$$

Thus, by taking functional derivative $\delta/\delta V(y)$ in Eq. 2–67, we get

$$\int_J dx \langle\rho(x)\rangle\langle\rho(y)\rangle R(x,z)\ln|y-z| = -\beta^{-1}\delta(x-y). \tag{2–69}$$

In the large N limit, around the region under consideration, if $\langle\rho(x)\rangle$ scales as N and becomes a constant, we can introduce a new variable $u = x/\Delta$ scaled by the mean level spacing $\Delta = \rho^{-1}$ (*i.e.*, for the Gaussian ensembles, $\rho(x) \propto \sqrt{2N}$ in the $x \to 0$ limit) and rewrite Eq. 2–69, which reads

$$\int_J dw R(u-w)\ln|v-w| = -\beta^{-1}\delta(u-v). \tag{2–70}$$

This implies that the two-level correlation function does not depend on the specific form of $V(x)$, which carries microscopic information of the system. Therefore, the two-level kernel is universal. Within this framework, it is understood that for the soft confinement potential, the average density is not constant in the $N \to \infty$ limit so that it does not simply scale out. Therefore, the two-level correlation function is expected to be different from the WD universality [54].

CHAPTER 3
λ-ENSEMBLES

In the motivation to investigate the universality associated with fat-tail or power-law RMEs, we introduced a family of $U(N)$ invariant random matrix ensembles characterized by an asymptotic logarithmic potential $V(H) = A[\ln H]^{1+\lambda}$ with $\lambda > 0$ [55, 56], named "λ-ensembles" [1] . The reason for such suggestion is based on the following few observations. First, it is known that for $V(H) \propto [\ln H]^2$ corresponding to $\lambda = 1$ limit (the critical ensemble), the eigenvalue spectrum is given by inverse power-law distribution, which is known by the mean-field theoretic approach [57]. Second, for $V(H) \simeq N \ln H$ corresponding to the $\lambda \to 0$ limit with the constant A being order of N (free Lévy matrices), the spectral density is given by the fat-tail distributions. Third, in the limit $\lambda \gg 1$, it is expected that the confinement potential may grow sufficiently strong, thereby approaching the Gaussian limit. Therefore, we can speculate that the parameter λ is a controlling parameter of the power-law behavior. The fact that such parametric generalization (generalizing the power of the logarithm to arbitrary real value larger than 1) connects the existing RM models equipped with rotational invariance is interesting since the model allows us to explore any possible novel universality associated with fat-tail RMEs as well as the logarithmic soft-confinement potential within the framework of the rotationally invariant RMT.

The generic choice of the confining potential $V(x)$ that gives asymptotic logarithmic behavior is $V(x) = A[\ln x]^{1+\lambda}$. However, it has a unphysical singularity at the origin so that we need to regularize it in certain way. One possible way to do it is choosing, *i.e.*, $V(x) = A[\ln(1+x)]^{1+\lambda}$ but there are a variety of other forms that differ by the regularization behavior in the vicinity of origin, which will not change the characteristics

[1] The earlier name 'Lévy like ensembles' in Ref. [55] reflects the motivation of the study.

of the λ-ensembles. For our study, we particularly choose the following form of the potential,

$$V(x) = \frac{1}{\ln(1/q)}[\sinh^{-1} x]^{1+\lambda}; \quad \lambda > 0; \quad q < 1. \tag{3–1}$$

For simplicity, we will introduce $\gamma \equiv \ln(1/q)$ so that the model has two parameter λ and γ. The merit of choosing the Eq. 3–1 is that in the limit $\lambda = 1$, it coincides with the one possible form of the weight function of the q-RMEs so that we can compare our results with those of q-RMEs. For the q-RMEs, the mathematical properties of the corresponding orthogonal polynomials, known as "the Ismail-Masson q-polynomials" [52], are well established and accordingly, the two-level kernel (*sinh* kernel) and the all the spectral properties (the critical statistics) are very well known [48, 49].

In the following sections, we will show that for $\lambda \neq 1$, we can construct the corresponding orthogonal polynomials, "λ-generalization of q-polynomials" in a rigorous numerical method and thus we can study the one-level (spectral density) and the two-level kernel that is central to the test of the universality. We will also show that the spectral density and the two-level correlation function of the λ ensembles exhibit novel feature; the spectral density is given by a power-law form and two-level kernel possess the anomalous component which is considered one of the characteristics of the critical ensembles. In addition, we will suggest a novel form of two-level kernel of the λ-ensembles based on the unfolding analysis and discuss the details of its behavior as well as the properties of the level statistics that can be deduced from the kernel. Finally, we will discuss the implications and applications of the our results.

3.1 Method

The main difficulty of studying λ-ensembles is that the orthogonal polynomials corresponding to the weight function $w(x) = e^{-V(x)}$ for the arbitrary λ values (expect $\lambda = 1$) are not known. Thus, the first task is to obtain the orthogonal polynomials corresponding to the weight function of the λ-ensembles. In the following, we will

review the procedure to construct orthogonal polynomials for non-trivial arbitrary weight function.

To this end, we define the orthogonal polynomials $\phi(x)$ for arbitrary weight function $w(x)$, given by

$$\int_{-\infty}^{\infty} dx w(x)\phi_n(x)\phi_m(x) = \delta_{n,m}h_n \tag{3–2}$$

where h_n is the normalization constant. It is well known that every orthogonal system of real valued polynomials satisfy a three term recurrence relation [51] [2]

$$x\phi_n(x) = \phi_{n+1}(x) + S_n\phi_n(x) + R_n\phi_{n-1}(x), \tag{3–3}$$

where S_n and R_n are the real coefficients of the recurrence relation. In particular, R_n is related to the normalization constant h_n by $h_{n+1} = R_{n+1}h_n$ For example, the Hermite polynomials are determined by the recurrence relation of

$$\phi_{n+1}(x) = x\phi_n(x) - n\phi_{n-1}(x). \tag{3–4}$$

The recurrence relation of the q-polynomials is given by

$$\phi_{n+1}(x|q) = x\phi_n(x|q) - \frac{1}{4}q^{-n}(1-q^n)\phi_{n-1}(x|q), \quad 0 < q < 1 \tag{3–5}$$

where $q = e^{-\gamma}$, $\gamma > 0$. We point out that for these cases, the weight function is an even function, namely $V(-x) = V(x)$ and thus, all $S_n = 0$. Therefore, the R_n determines the properties of the orthogonal polynomials. In particular, by comparing the Hermite polynomials and q-polynomials, we observe that

$$R_n \propto n \quad (\text{Hermite polynomials}) \tag{3–6}$$

$$\propto e^{\gamma n} \quad (\text{q-polynomials}). \tag{3–7}$$

[2] Here, we consider that the $\phi_n(x)$ is a monic polynomial.

The significance of R_n is that it determines the upper bound of spectral density $D_N \propto \sqrt{R_n}$ and thus scaling behavior of the bulk of the spectrum in the large N limit. For example, for the Gaussian ensembles characterized by $V(x) = x^2$ (Hermite polynomials), the upper bound of the spectral density is $\sqrt{N}$ in the large N limit. Thus, the normalization condition of the spectral density requires the bulk of the spectrum to grow at an order of $\sqrt{N}$ as in the semi-circle law. While for the logarithmic soft-confinement potential, the spectral edge grows at an exponential rate $e^{\gamma n^{1/\lambda}}$ (which will be shown in the next section), the bulk of the spectrum does not scale as N. *i.e.*, for $V(x) \propto [\ln x]^2$, the spectral edge grows at $\sqrt{e^N}$, the bulk of the spectrum is given by $\rho(x) \propto 1/x$ which do not depend on N.

One way to determine the orthogonal polynomials for an arbitrary weight function is to use the Gram-Schmidt determinant formula, *e.g.*,

$$\phi_n(x) = \frac{1}{G_{n-1}} \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_n & \dots & a_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix} \tag{3–8}$$

where the G_n stands for the Gram-Schmidt determinant.

$$G_n = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \dots & a_{2n} \end{vmatrix} \tag{3–9}$$

and a_i are the moments given by

$$a_i = \int x^i w(x) dx \tag{3–10}$$

Then, the R_n can be determined in terms of G_n by [58]

$$R_{n+1} = \frac{G_n G_{n-2}}{G_{n-1}^2} \tag{3–11}$$

An alternative view of the above procedure is adopted in Ref. [59] to obtain the polynomials recursively. Following the Ref. [59], we define

$$Q_{n,m} \equiv \int_{-\infty}^{\infty} x^m \phi_n(x) w(x) dx \tag{3–12}$$

Using the fact that $x^n = \phi_n(x) + \sum_j^j a_j \phi_j(x)$, we can find

$$Q_{n,n} = h_n, \tag{3–13}$$

$$Q_{n,n+1} = h_n \sum_{j=0}^{n-1} S_j, \tag{3–14}$$

and

$$Q_{n,m} = Q_{n-1,m+1} - S_{n-1} Q_{n-1,m} - R_{n-1} Q_{n-2,m}. \tag{3–15}$$

In this case, the determination of R_n and S_n in order to calculate the polynomials of degree $n < N - 1$ requires only the knowledge of the $2N + 1$ integrals of $Q_{0,m}$ (or the moments)

$$Q_{0,m} = \int_{-\infty}^{\infty} x^m w(x) dx \tag{3–16}$$

for $m = 0, ..., 2N$.

3.2 Results

3.2.1 λ-generalization of q-polynomials

For the study of the λ-ensembles, we adopted the latter approach to construct the orthogonal polynomials. So first, we calculated $Q_{0,m}$ with Mathmatica

$$Q_{0,m} \equiv \int_{-\infty}^{\infty} x^m e^{-V(x)} dx, \tag{3–17}$$

with the $V(x)$ shown in the Eq. 3–1. Since we choose $V(x)$ to be symmetric around the origin, *e.g.*, $V(-x) = V(x)$, the $S_n = 0$ for all n. R_n alone defines the corresponding

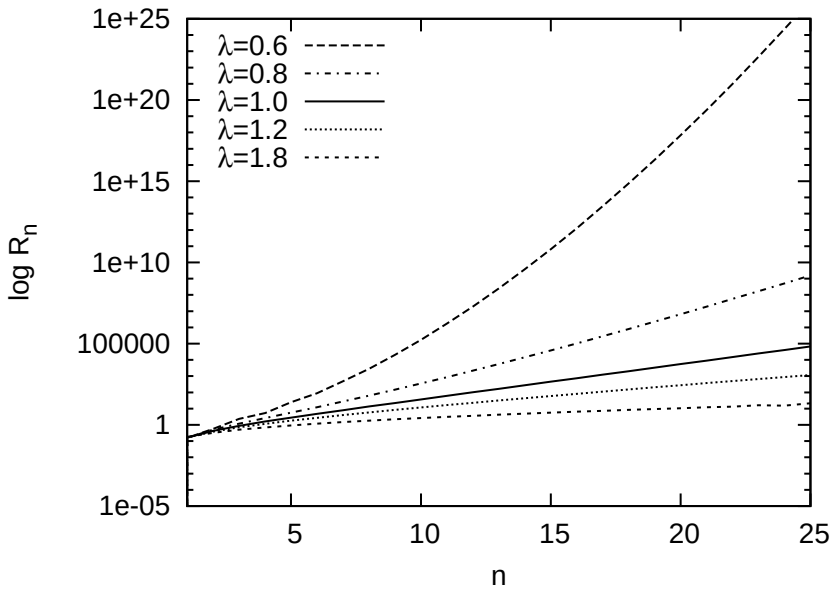

Figure 3-1. Log R_n as a function of n for different values of λ. Solid line corresponds to the critical ensemble $\lambda = 1$.

polynomials. As described earlier, we can determine R_n with $2N + 1$ integrals of $Q_{0,m}$ recursively.

It turns out that the R_n obtained in this way shows an intriguing behavior that is depicted in Fig 3-1. After careful examination, we found that the behavior of R_n for large n should be of the form

$$R_n \propto e^{\gamma n^{\alpha(\lambda)}}. \tag{3–18}$$

where

$$\alpha(\lambda) \simeq \frac{1}{\lambda} \tag{3–19}$$

The fitting result for $\alpha(\lambda)$ is provided in Fig. 3-2. As $\lambda = 1$, the $\alpha(\lambda)$ is equal to 1 and thus $\ln R_n$ grows linear in n, which coincides with the well-known R_n behavior for the q-polynomials (see Eq. 3–6) and for $\lambda \neq 1$, $\ln R_n$ grows in $n^{1/\lambda}$.

We recognized that this is a novel behavior thus named the orthogonal polynomials of the λ-ensembles as λ-generalizaton of q-polynomials, which is one of the central results of our work. Note that this is dramatically different from that for all Freud-like

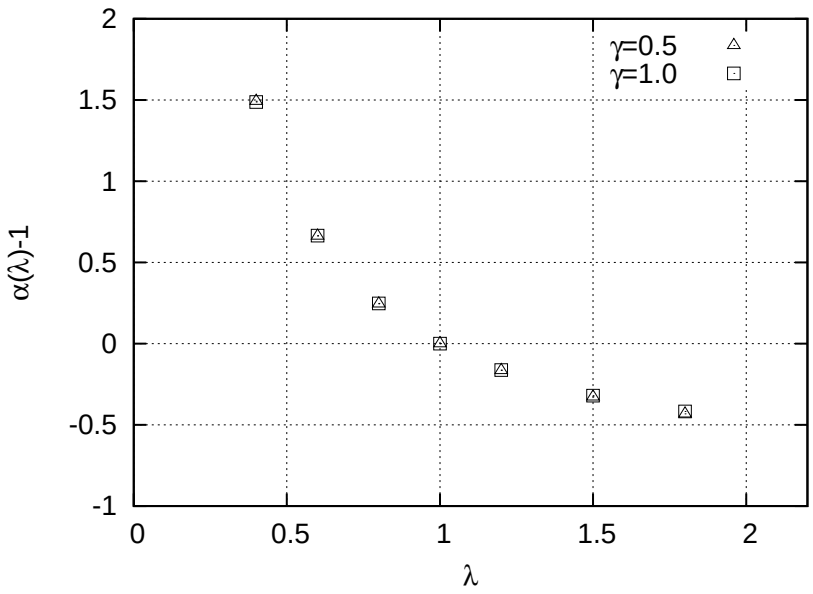

Figure 3-2. The exponent $\alpha(\lambda)$ as a function of λ for two different values of $\gamma \equiv \ln(1/q)$. $\lambda = 1$ corresponds to the q-polynomials describing the critical ensembles.

classical orthogonal polynomials whose weight function is given by $w(x) = e^{-x^m}$ with $m > 0$ where $R_n \propto n$.

3.2.2 Eigenvalue Density

The density of the eigenvalues $\rho(x) = K_N(x, x)$ can now be obtained for different values of λ from Eq. 2–20 by summing the products numerically. The results are shown in Figure 3-3. Earlier it was understood in Ref. [55] that the density of the λ-ensembles is given by a pure power-law, *e.g.*, $\rho(x) = \frac{1}{x^{1-\theta}}$. For $\lambda = 1$, $\theta = 0$ and for $\lambda > 1$ and $\lambda < 1$, $\theta > 0$ and $\theta < 0$.

However, careful investigation shows that the earlier interpretation is only approximate and more accurate form of the eigenvalue density is given by

$$\rho(x) \propto \frac{[\ln x]^{\lambda-1}}{x} \quad \text{for x} \gg \Lambda \tag{3–20}$$

where the lower cutoff Λ is dependent on the regularization of the confining potential $V(x)$. *i.e.*, for the choice of $V(x) \simeq [\ln x]^2$, the singularity, albeit unphysical, can be extended in the vicinity of $x \to 0$ limit [60]. For our choice Eq. 3–1, such singularity

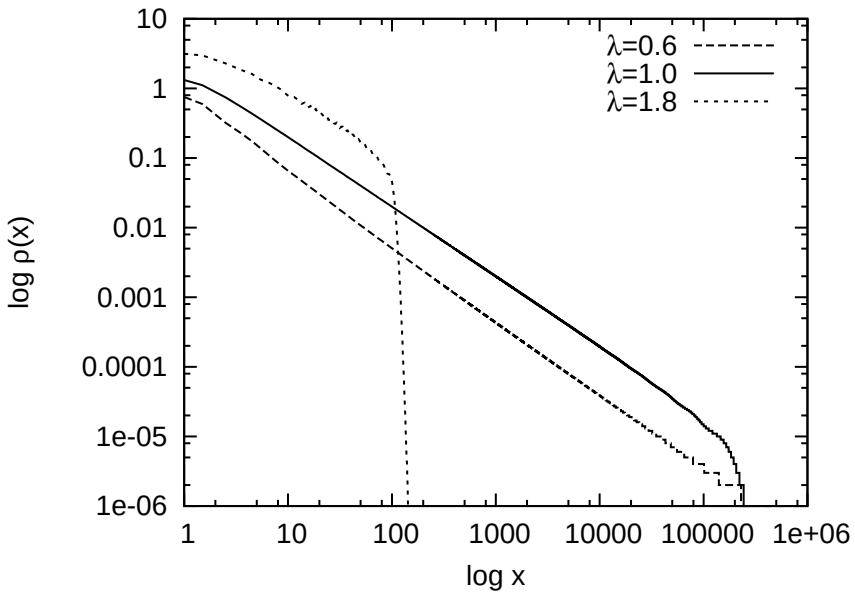

Figure 3-3. Density of eigenvalues for different values of λ.

doesn't exist. In general, it is expected that the regularization behavior (the cut-off Λ) is different for the choice of $V(x)$ sharing the same asymptotic logarithmic behavior.

The following facts convince us that the spectral density should be given by Eq 3–20. First of all, it can be shown [61] that for $\lambda > 1$, the density should be of the form 3–20 based on the mean-field approach, which reproduces the exact inverse power-law density as well as the *sinh* kernel of the critical ensembles ($\lambda = 1$.) Second, the validity of this form for all $\lambda > 0$ can be checked by considering the normalization condition of the spectral density

$$2\int_0^{D_N} \rho(x)dx = N. \tag{3–21}$$

Here the factor 2 comes from the fact that $\rho(x)$ is symmetric around origin. The upper bound D_N is given by the largest zero of the orthogonal polynomials of order N, namely $D_N \propto \sqrt{R_N}$. As pointed out, $R_n \propto \exp[n^{1/\lambda}]$. We notice that Ref. [62] studied the largest zeros of the orthogonal polynomials to the weight function of $\exp[-c(lnx)^m]$ for $c > 0$ and m a positive even integer and reported that it is of order $\exp(n^{\frac{1}{m-1}})$, which is the exact same behavior as the coefficient R_n of recurrence relation of the generalized

q-polyonomials in Eq. 3–18. Thus, our results seem to imply that the results of Ref. [62] can be extended to an arbitrary real $\lambda > 0$.

Eq. 3–20 can be verified in the numerical calculation as well. However, since our $V(x)$ is regularized near the origin unlike the choice $V(x) = [\ln x]^{1+\lambda}$ used in the mean-field approach [60] and in Ref. [62], we expect that the exact form of $\rho(x)$ will show agreement only for large x limit. To investigate this, we consider that the density is given by

$$\rho(x) = \frac{f(x;\lambda)}{x+\Lambda} \tag{3–22}$$

where $f(x;\lambda)$ is a logarithmically slowly varying function and Λ is a constant arising due to regularization of the density at the origin. Thus, we can expect $x\rho(x)$ to behave in the following way.

$$\begin{aligned} x\rho(x) &= \frac{x}{x+\Lambda} f(x;\lambda), && (3–23)\\ &\simeq \frac{x}{\Lambda} f(x;\lambda) \quad x \gg \Lambda, && (3–24)\\ &\simeq f(x;\lambda) \quad x \ll \Lambda. && (3–25) \end{aligned}$$

Fig. 3-4 and Fig. 3-5 show this behavior for $\lambda < 1$ ($\lambda = 0.5, 0.7, 0.9, \text{and } 1$) and the Fig. 3-6 and Fig. 3-7 for $\lambda > 1$ ($\lambda = 1.1, 1.3, 1.5, \text{and} 1$) respectively.

For all the cases, we chose $\gamma = O(1)$, which ensures the cut-off $\Lambda = O(1)$. To further investigate if $f(x;\lambda) \propto [\ln x]^{\lambda-1}$ for large x ($x \gg 1$), we plotted $\ln[x\rho(x)]$ vs. $\ln\ln x$ and fitted it in the range of $10 < x < 10^4$. Fig 3-8 and Fig. 3-9 show the expected linear behavior.

3.2.3 Two-level Correlation Function

The numerical calculations of the cluster function are performed based on

$$Y(u,v) \equiv [\bar{K}(u,v)]^2 \tag{3–26}$$

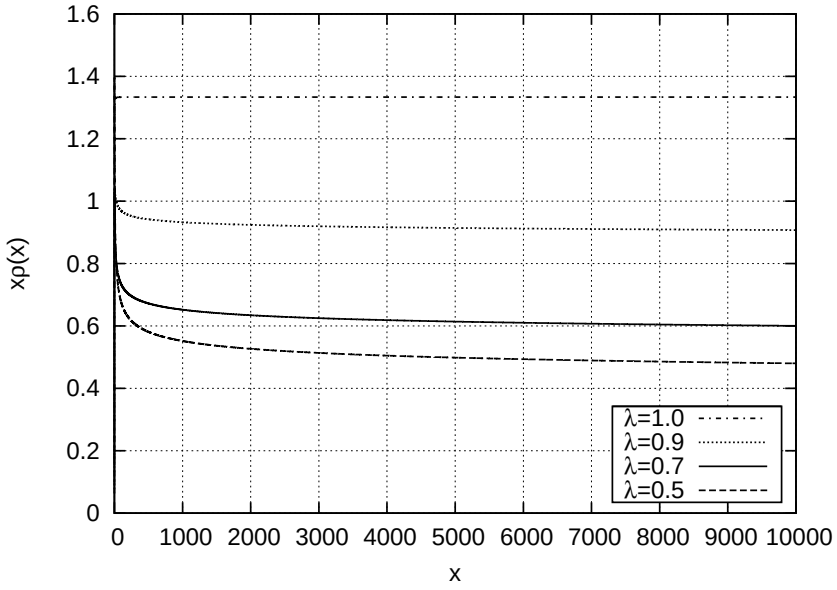

Figure 3-4. Eigenvalue density ($x = 0 - 10^4$) for λ-ensembles for $\lambda = 0.9$ (γ = 0.75), 0.7 (γ = 0.50), 0.5 (γ = 0.25) as well as 1 (γ = 0.75).

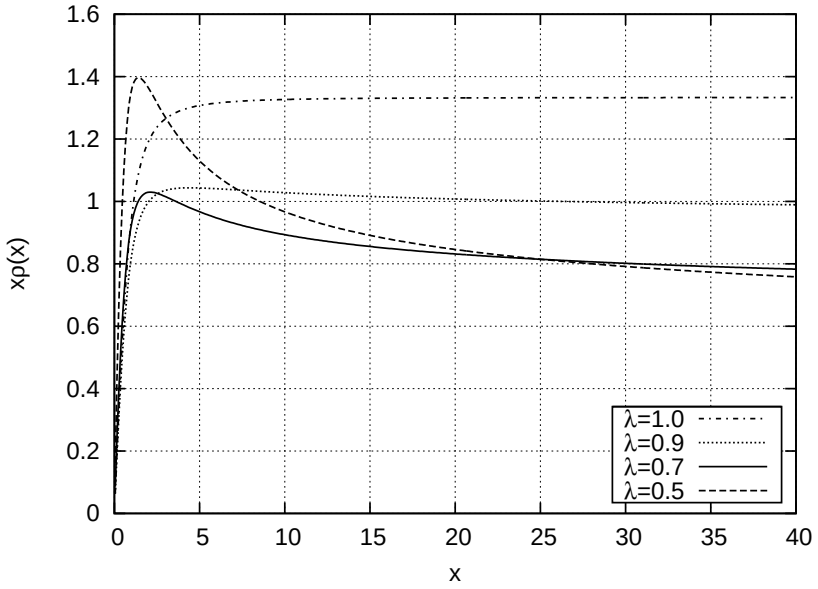

Figure 3-5. Eigenvalue density ($x = 0 - 40$) for λ-ensembles for $\lambda = 0.9$ (γ = 0.75), 0.7 (γ = 0.50), 0.5 (γ = 0.25) as well as 1 (γ = 0.75)

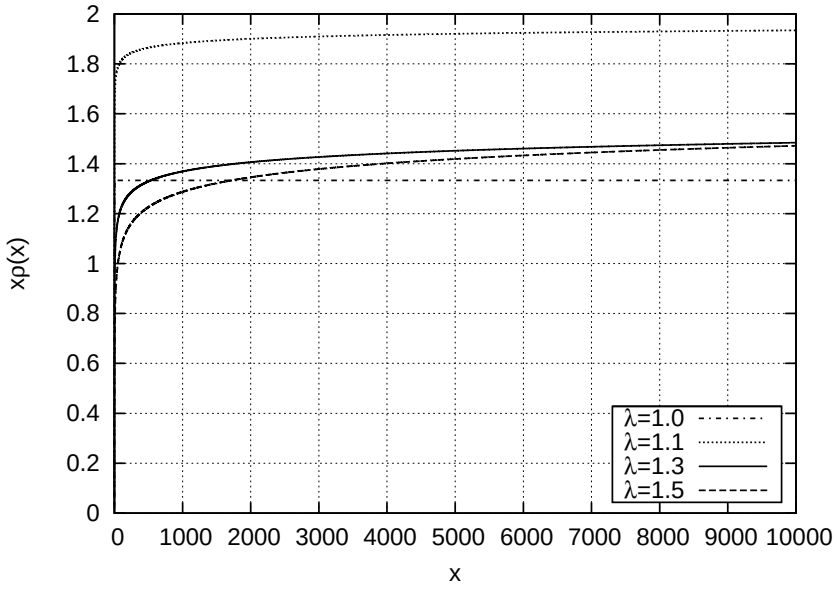

Figure 3-6. Eigenvalue density ($x = 0 - 10^4$) for λ-ensembles for $\lambda = 1.1$ (γ = 0.75), 1.3 (γ = 2.00), 1.5 (γ = 4.00) as well as 1 (γ = 0.75).

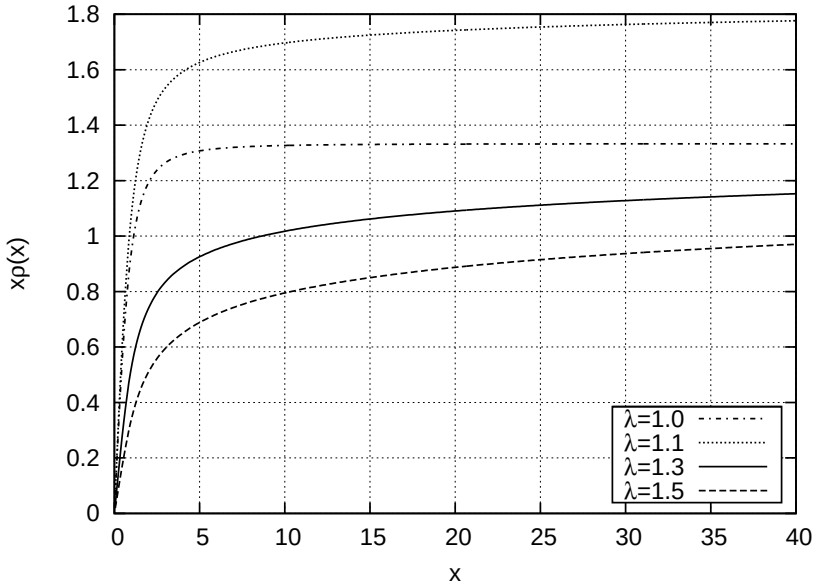

Figure 3-7. Eigenvalue density ($x = 0 - 40$) for λ-ensembles for $\lambda = 1.1$ (γ = 0.75), 1.3 (γ = 2.00), 1.5 (γ = 4.00) as well as 1 (γ = 0.75).

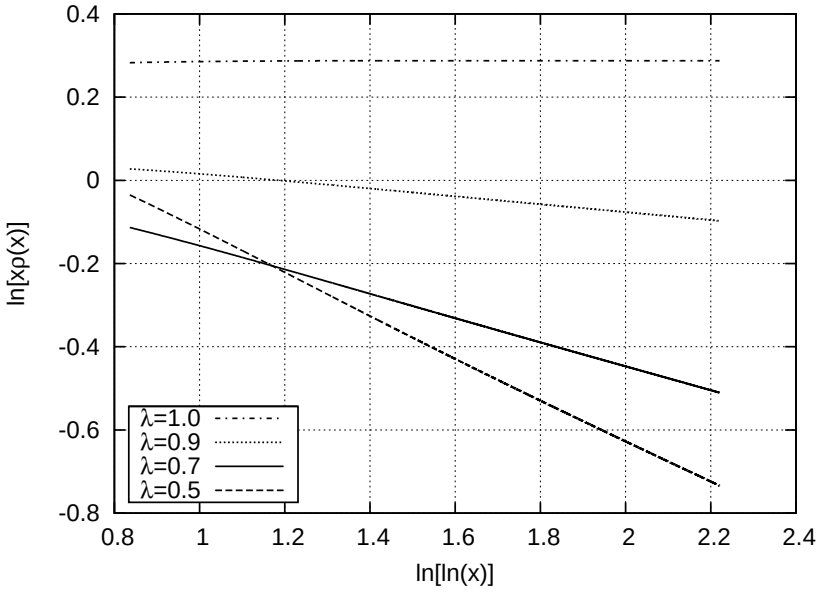

Figure 3-8. Eigenvalue density for λ-ensembles for $\lambda = 0.9$ (γ = 0.75), 0.7 (γ = 0.50), 0.5 (γ = 0.25) as well as 1 (γ = 0.75).

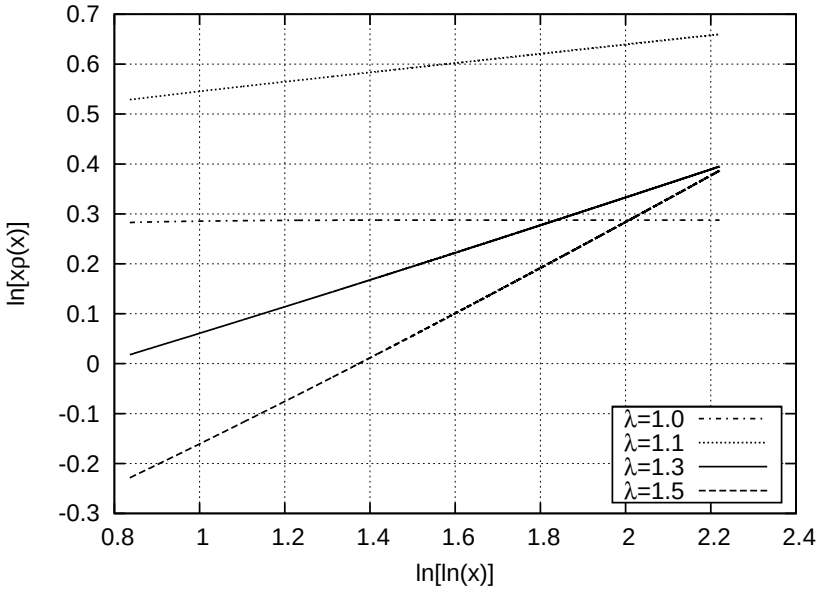

Figure 3-9. Eigenvalue density for λ-ensembles for $\lambda = 1.1$ (γ = 0.75), 1.3 (γ = 2.00), 1.5 (γ = 4.00) as well as 1 (γ = 0.75).

with the unfolding map, *e.g.*,

$$u(x) \equiv \int^{x} dx\rho(x). \tag{3–27}$$

3.2.3.1 Observation of Normal/Anomalous Structure of Two-level Kernel

It is well known that for λ =1, the two level cluster function has both normal and anomalous component. The normal part of the two-level cluster function for the critical ensembles for $uv > 0$ and $|u - v| << u$ is given, in the $\gamma \ll 2\pi^2$ limit, by the sinh kernel

$$Y_n^c(u, v) = \Big[\frac{\gamma}{2\pi}\frac{\sin[\pi(u-v)]}{\sinh[\frac{\gamma}{2}(u-v)]}\Big]^2, \tag{3–28}$$

The anomalous part of the cluster function, so called "ghost correlation peak" [54] for $uv < 0$ given by

$$Y_a^c(u, v) = \Big[\frac{\gamma}{2\pi}\frac{\sin[\pi(u-v)]}{\cosh[\frac{\gamma}{2}(u+v)]}\Big]^2. \tag{3–29}$$

The presence of the anomalous component occurring due to long range correlation is required by the normalization sum rule [57]

$$1 = \int_{-\infty}^{\infty} du\ [Y_n^c(u, u') + Y_a^c(u, u')]. \tag{3–30}$$

The deficiency of the sum rule

$$\chi \equiv 1 - \int_{-\infty}^{\infty} du Y_n^c(u, u') = \int_{-\infty}^{\infty} du Y_a^c(u, u') \tag{3–31}$$

is related to certain characteristics of the critical statistics [8, 36–42]: i) the level compressibility in the number variance and ii) the multi-fractality of eigenvectors. In particular,

$$\chi = \frac{d\Sigma(\langle L\rangle)}{d\langle L\rangle} = \frac{d - D_2}{2d}, \tag{3–32}$$

where the fractal dimensionality D_p determines the scaling behavior of the moments of the inverse participation ratio via

$$\langle \int d^d x |\phi(x)|^{2p} \rangle \propto L^{-D_p(p-1)}. \tag{3–33}$$

It turns out that for $\lambda \neq 1$, λ-ensembles also possess such normal/anomalous structure. Figures 3-10 and 3-11 show the comparison between the normal part of cluster function for $\lambda \neq 1$ and that for $\lambda = 1$ as well as that for the Gaussian. While the nodes of the cluster function remains the same (occurring at the integer value on the horizontal axis) for all λ values as well as the Gaussian case, the peak height and position between the nodes show an interesting behavior in the change of λ; as the λ value is decreased for a given γ value ($\gamma = 0.5$ in Fig. 3-10 and $\gamma = 2$ in Fig. 3-11), the peak height and position gradually reduces and shifts toward the node on the left. The corresponding behavior is observed in the anomalous part as well. The figures 3-12 and 3-13 show the numerical evaluation of 1- $Y_a(u, v)$ ($u > 0$ and $v < 0$) for a symmetric range around $v = -u$ for varying λ values for a fixed γ ($\gamma = 0.5$ in Fig. 3-12 and $\gamma = 2$ in Fig. 3-13). As the figure shows clearly, the magnitude of the ghost peak depends on λ in a significant way; for $\lambda < 1$, the peak is more pronounced than that for $\lambda = 1$ and for $\lambda > 1$ it is the opposite.

The observation that such long range correlation leading to the ghost peak is preserved for all $\lambda \neq 1$ seems to suggest that such features, to some degree, are common to all logarithmic confinement potentials. In other words, once the critical ensembles break the $U(N)$ symmetry of the Gaussian ensembles with the introduction of the parameter q, the λ ensembles remain in this broken symmetry family. The fact that as λ becomes large, the ghost peak shrinks seem to imply that the $U(N)$ symmetry might become fully restored in the limit of $\lambda \to \infty$. This expectation seems consistent with the asymptotic behavior of two-level correlation in the limit of $\lambda \to \infty$ that will be shown later.

The λ-ensembles are all "critical" in the sense that the two-level kernel violates the sum rule that can be associated with the characteristics of the critical statistics such as the level compressibility and the multi-fractality. In particular, the fact that the violation of the sum rule is controlled by the parameter λ, *e.g.*, $0 < \chi(\lambda) < 1$ is intriguing. As mentioned above, λ seems related to the degree of the $U(N)$ symmetry

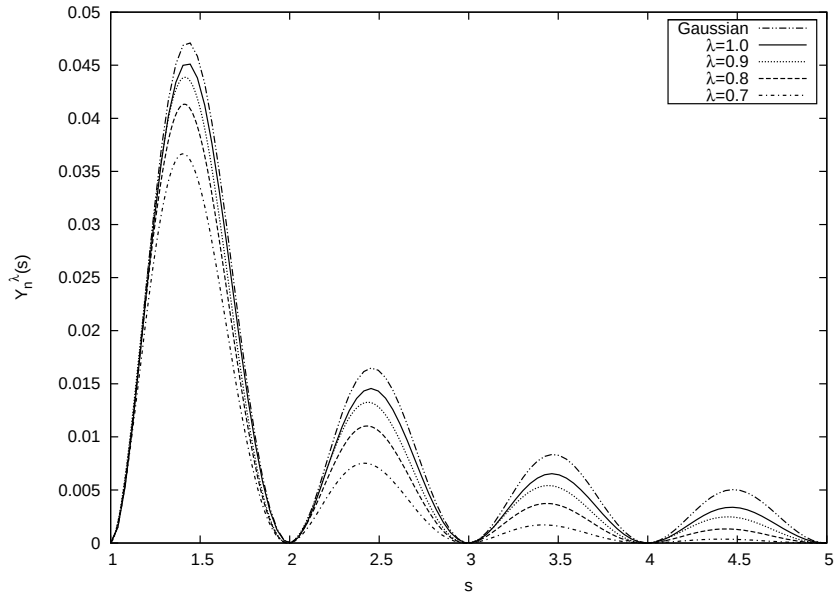

Figure 3-10. Normal component of the cluster function for λ-ensembles for $\lambda = 0.9$, 0.7, 0.5 as well as 1.0 (γ = 0.50).

breaking and thus, indicative of the non-trivial character of the eigenvector correlations, namely the multi-fractal dimensionality as can be seen immediately from the Eq. 3–32 and the λ-dependent sum-rule deficiency. In this regard, the study of the dimensional dependence of the critical statistics will be important to further understand the role of the parameter λ since the multi-fractal dimension of the eigenvector correlations at the critical states is dependent on the spacial dimension [63].

3.2.3.2 Two-level Kernel of λ-ensembles

It is known from rigorous results that for the soft-confinment potential of the critical ensembles, the translational-invariance is broken and the density does not depend on N. Thus the unfolding procedure is non-trivial [54, 64, 65]. In fact, the deformation from the Gaussian universality, the 'sine kernel', is the result of the non-trivial unfolding [54, 64, 65] In the semi-classical regime ($\gamma < 2\pi$ for the critical ensembles), the kernel for an arbitrary weight function can be written as [65–67]

$$\bar{K}(u,v) \sim \frac{\sin(\pi(u-v))}{x(u)-y(v)}, \tag{3–34}$$

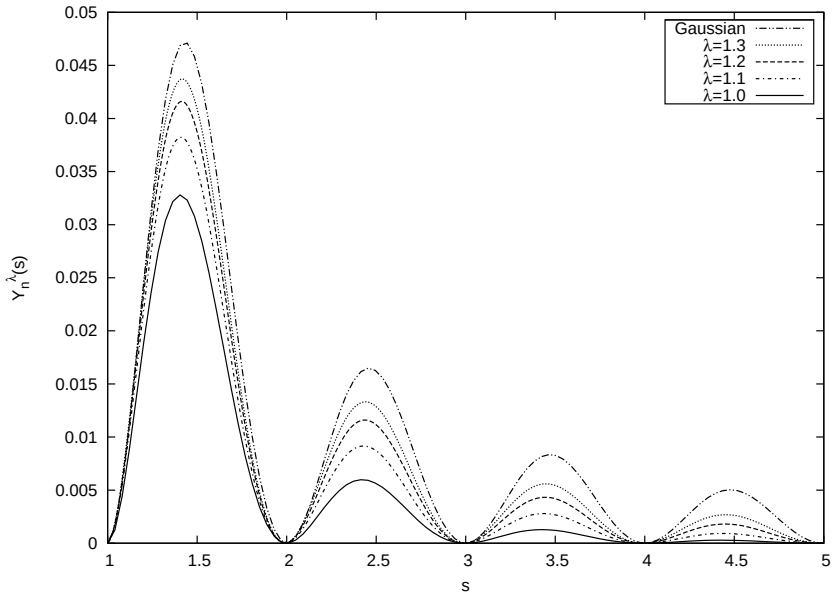

Figure 3-11. Normal component of the cluster function for λ-ensembles for $\lambda = 1.1$, 1.2, 1.3 as well as 1.0 (γ = 1.50).

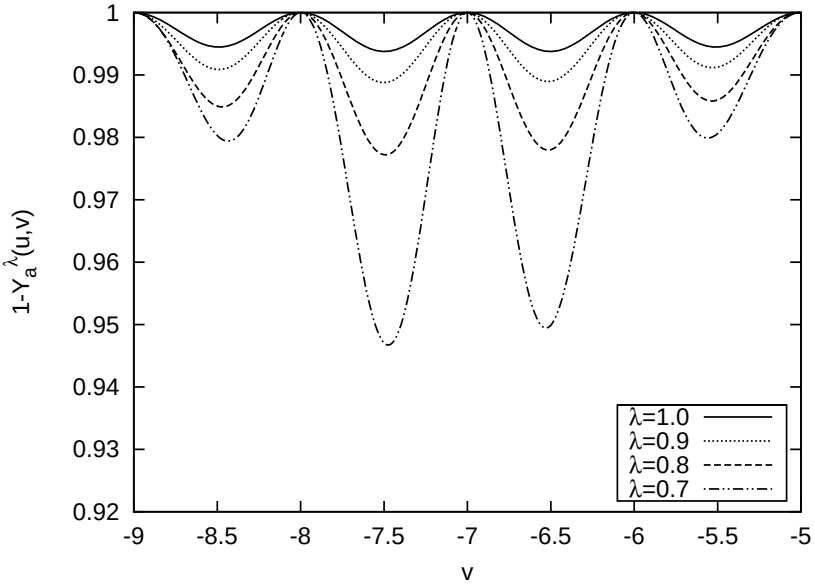

Figure 3-12. Anomalous component of the cluster function for λ-ensembles for λ =0.9, 0.8, 07 as well as 1.0 (γ = 0.50).

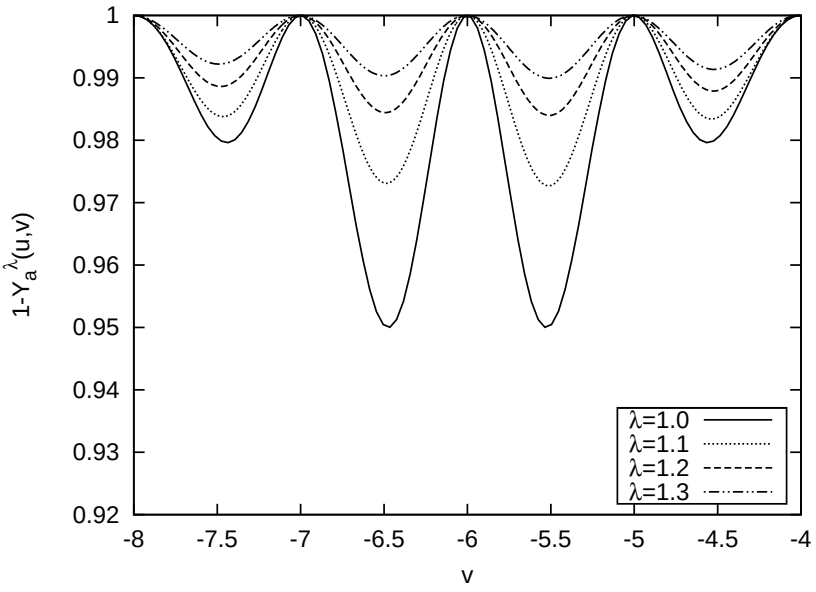

Figure 3-13. Anomalous component of the cluster function for λ-ensembles for λ =1.1, 1.2, 1.3 as well as 1.0 (γ = 1.50).

Thus we expect that in the same limit the difference between the λ-ensembles and the critical ensembles also arises from the difference in unfolding. To implement this, we define the unfolding variable for the critical ensembles in the following form,

$$u' = u - u_0 \equiv \int_{x_0}^{x} \frac{c}{t^1} dt = \ln x/x_0 \tag{3–35}$$

In the similar manner, we can define the unfolding variable for λ-ensembles for $\lambda \neq 1$,

$$u' = u - u_0 \equiv \int_{x_0}^{x} \frac{c[\ln t]^{\lambda-1}}{t} dt = \frac{c}{\lambda}([\ln x]^{\lambda} - [\ln x_0]^{\lambda}) \tag{3–36}$$

where $u_0 = \int_0^{x_0} \rho(x)dx$ and $x_0 >> \Lambda$, the cut off of the density that arises as a result of the regularization close to the origin. The constant $c \propto \rho(x = 0)$. Note that $\rho(x = 0)$ is a function of the parameters that do not depend on N. For the critical ensembles, $\rho(0) = \frac{1}{\gamma}$ and for the λ-ensembles, $\rho(0) = f(\lambda, \gamma)$ where f is some function of λ and γ. Rewriting the original variable x in terms of the unfolding variable, we obtain for both ensembles

$$x = \exp\left[\gamma^*(u' + a)^{1/\lambda}\right] \tag{3–37}$$

where $\gamma^* \equiv \left(\frac{\lambda}{c}\right)^{1/\lambda}$ and the $a \equiv \frac{c}{\lambda}[\ln x_0]^\lambda$. In the limit $\lambda = 1$, it reduces to

$$x = x_0 e^{\frac{u'}{c}}. \tag{3–38}$$

Thus for the critical ensembles we find the expected form of the kernel by using Eq. 3–34

$$x - y \to \sinh[\frac{1}{2c}(u - v)]. \tag{3–39}$$

For $\lambda \neq 1$, the same procedure results in

$$x - y \to \sinh[\frac{\gamma^*}{2}[(u' + a)^{1/\lambda} - (v' + a)^{1/\lambda}]]. \tag{3–40}$$

Thus if we redefine the unfolding variables by $u \to \tilde{u} = u' + a$ and $u \to \tilde{v} = v' + a$, we obtain a suggested form of the two-level kernel for the λ-ensembles

$$\bar{K}_n(\tilde{u}, \tilde{v}) = \frac{\Gamma}{2\pi} \frac{\sin[\pi(\tilde{u} - \tilde{v})]}{\sinh[\frac{\gamma^*}{2}(\tilde{u}^{1/\lambda} - \tilde{v}^{1/\lambda})]}, \tag{3–41}$$

where Γ is introduced to ensure that the kernel satisfy the condition $\sigma(\tilde{u}) = \lim_{\tilde{v}\to\tilde{u}} \bar{K}_n^\lambda(\tilde{u}, \tilde{v}) = 1$. To identify the necessary form of Γ, we consider the kernel in the limit of $\tilde{v} \to \tilde{u}$

$$\lim_{\tilde{v}\to\tilde{u}} \bar{K}_n^\lambda(\tilde{u}, \tilde{v}) = \frac{\Gamma}{\gamma^*} \frac{\tilde{u} - \tilde{v}}{\tilde{u}^{1/\lambda} - \tilde{v}^{1/\lambda}} = 1 \tag{3–42}$$

from which we obtain

$$\Gamma = \Gamma(\tilde{u}, \tilde{v}) \equiv \gamma^* g(\tilde{u}, \tilde{v}) \tag{3–43}$$

Here, we introduce a function $g(\tilde{u}, \tilde{v})$

$$g(\tilde{u}, \tilde{v}) \equiv \frac{\tilde{u}^{1/\lambda} - \tilde{v}^{1/\lambda}}{\tilde{u} - \tilde{v}} \tag{3–44}$$

Then, we finally obtain the proper form of the regular component of the two-level kernel ($\tilde{u}\tilde{v} > 0$) for the λ-ensembles as

$$\bar{K}_n(\tilde{u}, \tilde{v}) = \frac{\Gamma(\tilde{u}, \tilde{v})}{2\pi} \frac{\sin[\pi(\tilde{u} - \tilde{v})]}{\sinh[\frac{\Gamma(\tilde{u},\tilde{v})}{2}(\tilde{u} - \tilde{v})]}, \tag{3–45}$$

while the anomalous part of the kernel ($\tilde{u}\tilde{v} < 0$) for the λ-ensembles can be written as

$$\bar{K}_a(\tilde{u}, \tilde{v}) = \frac{\Gamma(\tilde{u}, \tilde{v})}{2\pi} \frac{\sin[\pi(\tilde{u} - \tilde{v})]}{\cosh[\frac{\Gamma(\tilde{u}, \tilde{v})}{2}(\tilde{u} + \tilde{v})]}, \tag{3–46}$$

We note that this kernel resembles the usual sinh kernel of the critical ensembles with more general argument. For $\lambda = 1$, $\Gamma(\tilde{u}, \tilde{v}) \to \gamma^* = \gamma$. Thus it reduces to the sinh kernel. However, in general for $\lambda \neq 1$, the $\Gamma(\tilde{u}, \tilde{v})$ is a non trivial function of $\tilde{u}$ and $\tilde{v}$. For example,

$$\Gamma(\tilde{u}, \tilde{v}) = \gamma^*(\tilde{u} + \tilde{v}) \quad \text{for } \lambda = 0.5 \tag{3–47}$$

and

$$\Gamma(\tilde{u}, \tilde{v}) = \frac{\gamma^*}{\sqrt{\tilde{u}} + \sqrt{\tilde{v}}} \quad \text{for } \lambda = 2.0 \tag{3–48}$$

It is obvious that the form of the function is not translationally invariant. However, if we choose $\tilde{v}$ to be a fixed value, i.e, $\tilde{v} = v - u_0 + a = a$, which is the same as choosing $v = u_0$, then $\tilde{u} = u - u_0 + a = u - v + a \equiv s + a$. In this way, the function $\Gamma(\tilde{u}, \tilde{v})$ and $g(\tilde{u}, \tilde{v})$ can be written in terms of a difference variable $s \equiv u - v = \tilde{u} - \tilde{v}$ alone with a constant $\tilde{v} = a$ that serves as a fixed reference point.

$$\Gamma(s + a, a) \equiv \Gamma(s, a) = \gamma^* \frac{(s + a)^{1/\lambda} - a^{1/\lambda}}{s} \tag{3–49}$$

Then

$$\bar{K}_n(s, a) = \frac{\Gamma(s, a)}{2\pi} \frac{\sin(\pi s)}{\sinh[\frac{\Gamma(s,a)}{2} s]}, \tag{3–50}$$

The figures 3-14, 3-15 shows the fitting results with the kernel given in Eq. 3–50 for different λ values. They show a fit with the numerically obtained two-level kernels with fit values $\gamma^* \simeq \gamma = 0.5$ and $a \simeq 2$ for $\lambda < 1$ (Fig. 3-14) and $\gamma^* \simeq \gamma = 1.5$ and $a \simeq 2$ for $\lambda > 1$. (Fig. 3-15) Finding the exact dependence of a and γ^* on the parameter λ and γ requires to know details of the eigenvalue density $\rho(x)$ in the vicinity of origin and the cutoff. This bears no importance in the discussion of the universal feature of the ensembles because $\rho(x)$ in the vicinity of the origin is simply a reflection of how one chooses V(x) to

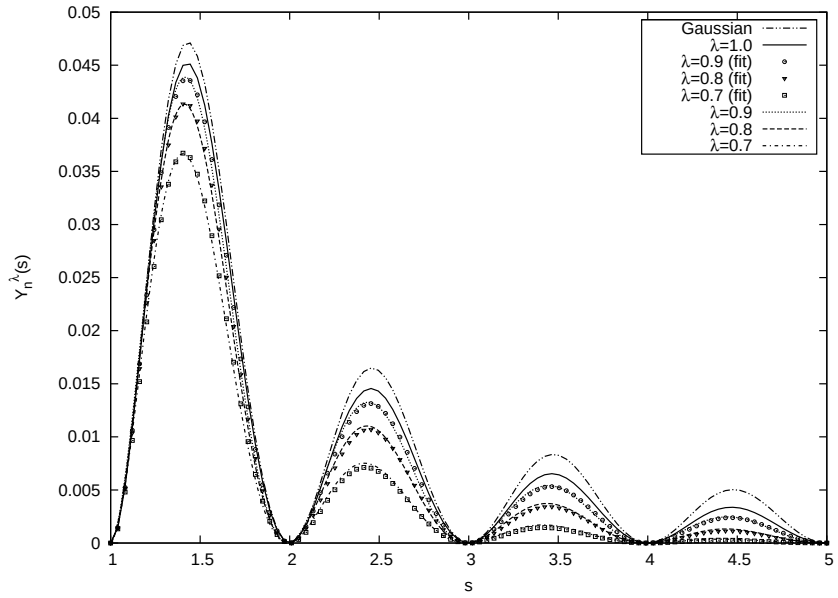

Figure 3-14. Fitting results for normal component of the cluster function for λ-ensembles for $\lambda = 0.9, 0.7, 0.5$ as well as 1.0 (γ = 0.50).

be regularized near the origin. We also show that in the similar range of the parameters, the anomalous components of the kernel fits well with the proposed form of the kernel Eq. 3–46 which are shown in the figures 3-16 and 3-17.

3.2.3.3 Universality of λ-ensembles

In the following, we will discuss novel asymptotic behavior of the two-level kernel of the λ-ensembles by further examining Eq. 3–50. First, we note that

$$\Gamma(s,a) \propto s^{1/\lambda-1} \quad \text{for } s \gg a \tag{3–51}$$

leads to

$$\bar{K}_n(s,a) \propto s^{1/\lambda-1} e^{-s^{1/\lambda}} \quad \text{for } s \gg a. \tag{3–52}$$

Looking at the λ-dependence, we observe that

$$\begin{aligned} \bar{K}_n(s,a) &\propto e^{-s^{\infty}} \quad \text{for } \lambda \to 0 \\ &\propto e^{-s} \quad \text{for } \lambda = 1 \\ &\propto s^{-1} \quad \text{for } \lambda \to \infty. \end{aligned} \tag{3–53}$$

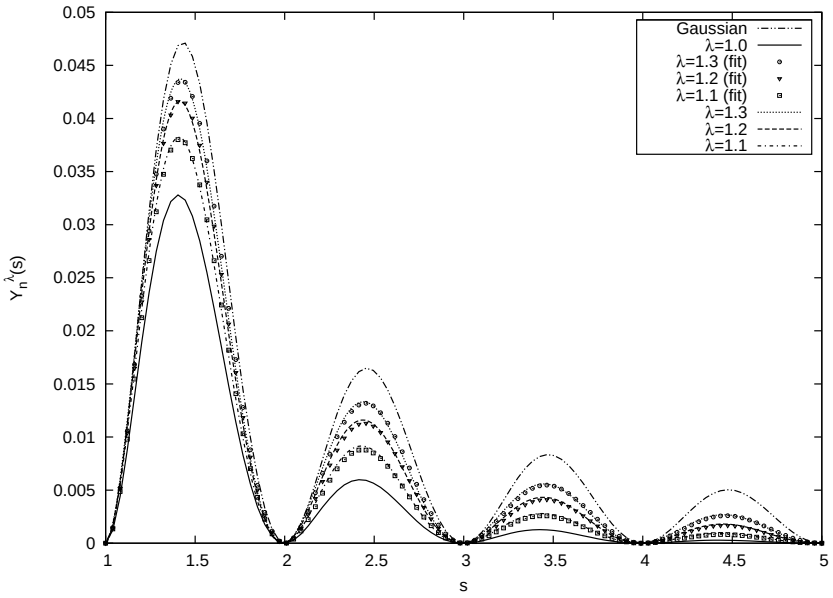

Figure 3-15. Fitting results for normal component of the cluster function for λ-ensembles for $\lambda = 1.1$, 1.2, 1.3 as well as 1.0 (γ = 1.50).

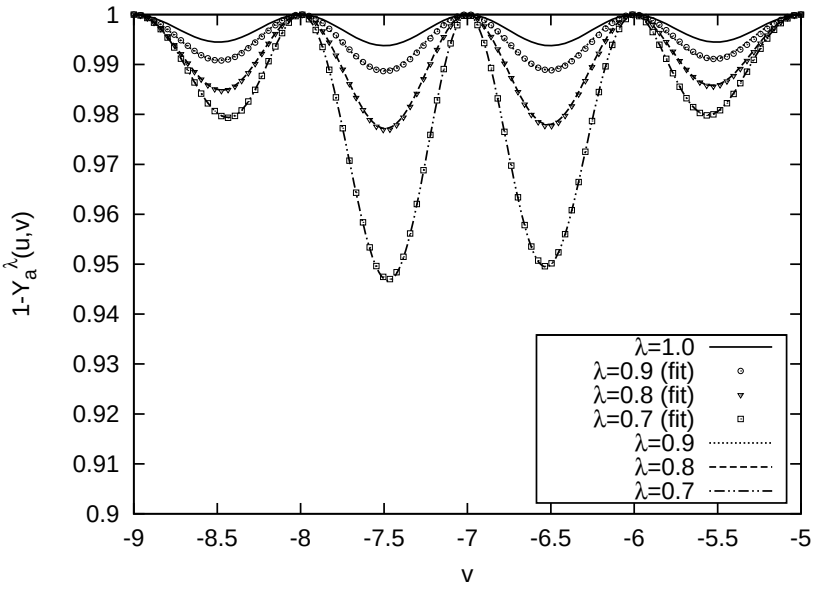

Figure 3-16. Fitting results for anomalous component of the cluster function for λ-ensembles for λ =0.9, 0.8, 07 as well as 1.0 (γ = 0.50).

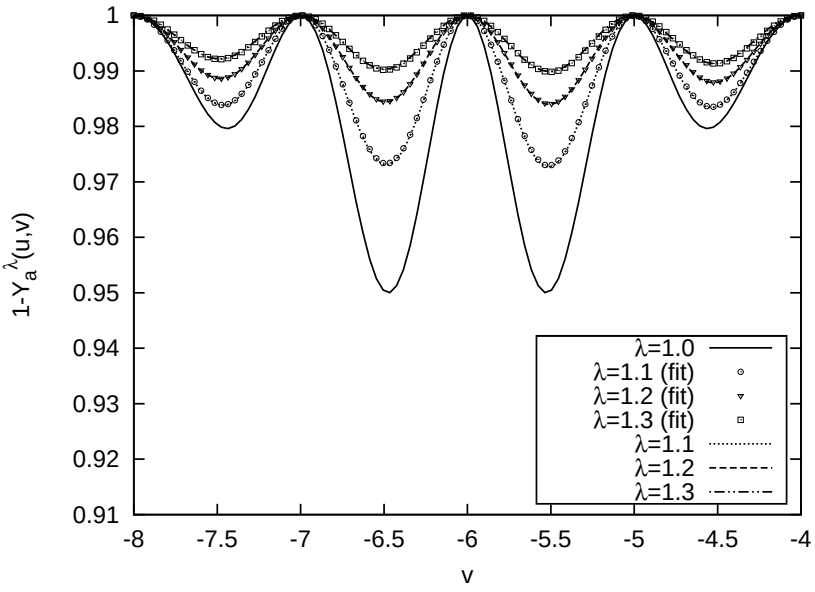

Figure 3-17. Fitting results for anomalous component of the cluster function for λ-ensembles for λ =1.1, 1.2, 1.3 as well as 1.0 (γ = 1.50).

Thus in the limit $\lambda \to \infty$, we get back the well-known asymptotic sine kernel decay. This is consistent with the fact that the magnitude of the ghost correlation peaks become smaller as λ increases, presumably disappearing in the Gaussian-ensemble limit of very large λ. For $\lambda = 1$, the kernel shows the expected exponential decay of the critical ensembles. For $\lambda \to 0$ limit, the asymptotic tail is given by the infinitely fast exponential decay, leading to the uncorrelated Poisson-like behavior. In general, for all values of λ, the large s behavior is governed by an exponential decay of $e^{-s^{1/\lambda}}$.

In the same token, we can study the two-level correlation function or density-density correlation function defined by

$$R(x,y) = \frac{\langle \rho(x)\rho(y)\rangle}{\langle \rho(x)\rangle\langle \rho(y)\rangle} - 1 \tag{3–54}$$

$$= \frac{\delta(x-y)}{\langle \rho(x)\rangle} + \frac{\langle \rho(x)\rho(y)\rangle}{\langle \rho(x)\rangle\langle \rho(y)\rangle}_{x\neq y} - 1 \tag{3–55}$$

In the unfolding scale $\rho(u) = \rho(x)dx$ where $\rho(u) = 1$, we can rewrite

$$R(u,v) = \delta(u-v) + \langle \rho(u)\rho(v)\rangle_{u\neq v} - 1 \equiv \delta(u-v) - Y(u,v) \tag{3–56}$$

In terms of the difference variable $s = u - v$, we define the two-level correlation function for the λ-ensembles as

$$R^{\lambda}(s) \equiv \delta(s) - Y^{\lambda}(s) \tag{3–57}$$

For $s \ll a$, $R^{\lambda}(s) \sim s^2$ for all values of λ, which is an expected feature due to the unitary symmetry of the ensembles. For large $s \gg a$, the results are simply obtained from Eq. 3–53

$$R^{\lambda}(s,a) \propto e^{-s^{\infty}} \quad \text{for } \lambda \to 0 \tag{3–58}$$

$$\propto e^{-s} \quad \text{for } \lambda = 1 \tag{3–59}$$

$$\propto s^{-2} \quad \text{for } \lambda \to \infty \tag{3–60}$$

We can see that the asymptotic behavior of the two-level correlation function interpolates that of Gaussian ensembles for $\lambda \to \infty$ limit, that of critical ensembles $\lambda = 1$ and the free Lévy matrices for $\lambda \to 0$ limit.

3.2.4 Number Variance

As an example of the level statistics obtained from the proposed form of the two-level kernel Eq. 3–50, we show the evaluation of the number variance $\Sigma(L)$ within a range L shown in Figure 3-18. It clearly shows that the number variance is linear in L the for all values of λ with a fixed $\gamma = 0.5$. As λ become smaller, the slope of the number variance increases. This is consistent with the fact that deficiency of sum rule rule $\chi(\lambda)$ increases as λ decreases since the slope of the number variance is directly related to the deficiency of the sum rule by $\chi(\lambda) = \frac{d\Sigma(\langle L \rangle)}{d\langle L \rangle}$.

3.3 Discussion

In this work, we further study an invariant-class of random matrix ensembles characterized by logarithmic soft-confinement potentials introduced in Ref. **??** which we refer to as the λ-ensembles. As a first step, we carefully reinvestigate the spectral density of the λ-ensembles and show that the spectral density is given more

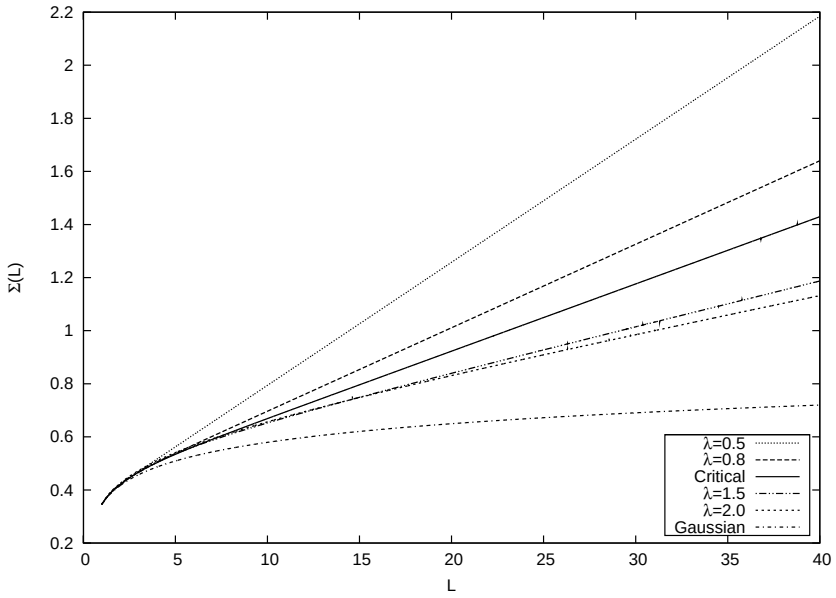

Figure 3-18. Number variance for λ-ensembles for λ =0.5, 0.8, 1,0, 1.5, 2.0 (γ = 0.5).

accurately by a power-law of the form $\rho(x) \propto [\ln x]^{\lambda-1}/x$. This result is suggested by the mean-filed approach and it can be checked by the normalization condition of the spectral density, the edge of which is determined by the coefficient R_n of the λ-generalization of q-polynomials.

Second, we show that the two-level kernel of the λ-ensembles has normal/anomalous structure, which is characteristic of the critical ensembles. The anomalous component arising due to the sum rule violation is dependent on the parameter λ for a fixed value of $\gamma \sim O(1)$; As the value of the λ decreases, the deficiency of the sum rule becomes larger.

Third, we identify the normal and anomalous components of the two-level kernel in the semi-classical regime, which are given by Eq. 3–45 and Eq. 3–46 that reduce to those of the critical ensembles for $\lambda = 1$. Further, we show that the two-level kernel of the λ-ensembles exhibit novel universal asymptotic behavior, shown by Eqs. 3–53, which includes the Gaussian ensembles ($\lambda \to \infty$ limit), the critical ensembles ($\lambda = 1$) as well as the free Lévy matrices ($\lambda \to 0$ limit). In particular, the large s behavior of the two-level kernel is governed by $\exp[-s^{1/\lambda}]$, which is a novel feature of the λ-ensembles.

It is expected that the asymptotic tail of the spacing distribution is also given by a similar exponential form.

Lastly, we show that the number variance is linear in L for large L. The slope of number variance or the level compressibility is dependent on the parameter λ. As λ decreases, the slope increases, which is consistent with that fact that anomaly or the deficiency of the sum rule increases as λ decrease.

These results seem to imply that the λ-ensembles in general are relevant to the description of the critical states of the localization-delocalization transition in the disordered systems. Since the critical level statistics are universal, depending only on the critical exponent and the dimensionality of space for a given symmetry class, it is conceivable that the parameter λ can be associated with these parameters. In particular, the study of the dimensional dependence of the critical statistics is interesting in this respect. At the same time, it is also interesting to see if λ-ensembles are also applicable in the study of quantum chaos. We note that recently the critical statistics has been found relevant in some cases of quantum chaos as well [68, 69]. It turns out that the two level kernel of chaotic systems with logarithmic singularity [70] have the exact same form as that of the critical ensembles.

CHAPTER 4
CONCLUSION

In this work, we study invariant-class of random matrix ensembles characterized by the asymptotic logarithmic soft-confinement potentials, named λ-ensembles. The suggestion is inspired by the existing RM models such as the critical ensembles (λ=1), the free levy matrices ($\lambda \to 0$ limit) and the Gaussian ensembles ($\lambda \to \infty$ limit) in an effort to investigate the novel universality associated with the fat-tail RMEs within the rotationally invariant RMT framework. The following is the summary of the main findings:

- The polynomials that are orthogonal with respect to the arbitrary power of the asymptotic logarithmic potential belong to a novel class of orthogonal polynomial system, named "λ-generalization of q-Hermite polynomials".
- The one-level correlation (the spectral density) of the λ-ensembles are given by a power-law form.
- The unfolded two-level correlation function shows the normal/anomalous structure, which is the characteristic of the critical ensembles. In particular, the sum rule violation is controlled by the parameter λ for a fixed value of $\gamma \sim O(1)$. As the value of λ decreases, the sum rule violation becomes greater.
- The asymptotic tail of the density-density correlations show a novel form shown in Eq. 3–58; for $1 < \lambda < \infty$, the asymptotic tail interpolates between the critical ensembles ($\lambda = 1$), the Gaussian ensembles ($\lambda \to \infty$), and the uncorrelated Poisson-like statistics ($\lambda \to 0$).

These results seem to have interesting applications/implications:

- It is of immediate interest whether these results are relevant in the context of the Anderson transition in disordered systems as well as in a broader context of the localization-delocation problems.
- It would also be interesting to see if the these results are applicable in the mixed states of the quantum chaos systems where the spectral properties are intermediate between WD and Poisson statistics.
- These results strongly impliy that the non-trivial N-dependence of two-level kernel of the fat-tail RMEs is a generic feature within the framework of the rotationally invariant RMT. In the $\lambda \to 0$ limit of the λ-ensemble, γ is required to be a N-dependent parameter to have the probability measure to be normalizable, which is the case for the free-levy matrices. Thus, the N-dependence of the two-level kernel in this limit can be understood as a consequence of the presence

of the N-dependent parameter in the model ensemble, which cannot be simply scaled out.

For the future, we further need to study the level statistics of the λ-ensembles such as the spacing distribution, the number variance, the spectral form factors, the inverse participation ratio etc. both analytically and numerically. These will allow much broader applications of our results. In addition, it is also important to explore if there exist a novel universality of random matrix ensembles associated with the fat-tail distribution in the framework of the Wigner-class (non-invriant) random matrix ensembles.

APPENDIX A
CENTRAL LIMIT THEOREM

It is understood that the universality of the Gaussian ensembles has some connection to the central limit theorem[1] . The fact that there are two kinds of stable probability distribution, e.g. the Gaussian/Lévy basin according to the central limit theorem[2] , inspires us that there would exist the counterpart structure in RMT, namely

- Gaussian distribution $\rightarrow$ Gaussian RMT
- Lévy distribution $\rightarrow$ Lévy RMT

Yet, this conjecture is not proven although there have been several attempts. Here, we will review the content of the central limit theorem.

The central limit theorem states that the distribution of the large sum of independent identically distributed (i.i.d) random numbers reach to certain limiting form of probability distributions. Suppose we consider a sum X of a large number (N) of i.i.d random numbers, x_i drown from a certain probability distribution,

$$X = x_1 + x_2 + x_3 + ... + x_N. \tag{A–1}$$

The sum X can be considered as the position of a particle undergoing a Brownian motion, each step size of which is given by a random number x_i. After the large sum is done or the large number of steps are taken, it is proven that the probability distribution of the sum X, or analogously that of the position of the particle under the Brownian motion converges to a limiting distribution, namely, the Gaussian distribution as long as

[1] In fact, it can be proven that the eigenvalue distribution of large random matrices becomes the well known-semi circle using the central limit theorem [71].

[2] To be precise, it should be referred to as “generalized central limit theorem”.

the each individual random variable x_i are sampled from the distributions with a finite variance regardless of other details. [3]

For a short proof, we define the characteristic function $\hat{P}(k)$ of a probability distribution as

$$\hat{P}(k) = \int dx\ e^{ikx}\ P(x) \tag{A–2}$$

After the large sum $X = \sum_i^N x_i$, the probability distribution of $P(X)$ is given as the inverse fourier transform of N times convolutions of $\hat{P}(k)$.

$$P(X) = \int \frac{dk}{2\pi}\ e^{-ikX}\ [\hat{P}(k)]^N \tag{A–3}$$

$\hat{P}(k)$ has the cumulant expansions around $k = 0$ in the following way,

$$\ln \hat{P}(k) = \sum_{n=0}^{\infty} \frac{c_n}{n!}(ik)^n \tag{A–4}$$

where c_i denotes the cumulants of distributions. The first few of them are named; c_1 is mean, c_2 is the variance, c_3 skewness and c_4 kurtosis etc. Then, we can write $P(X)$ as

$$P(X) = \int \frac{dk}{2\pi}\ e^{Nf(k,X)} \tag{A–5}$$

where $f(k, X)$ is given by

$$f(k, X) = -ik\frac{X}{N} + \sum_{n=0}^{\infty} \frac{c_n}{n!}(ik)^n \tag{A–6}$$

Using the method of steepest decent, we know that in the large N limit, the dominant contribution of $f(k, X)$ comes from the maximum of $f(k, X)$, which is at $k = 0$. Thus, in the rescaled variables $x = \frac{X - Nc_1}{c_2\sqrt{N}}$ and $w = c_2\sqrt{N}k$, we obtain the expansion of $P(x)$ in

[3] The existence of the higher order moments determine how fast it converges to the limiting form.

powers of $N^{-1/2}$. It is shown that

$$P(x) \approx \frac{e^{-w^2}}{\sqrt{2\pi}} [1 + \frac{h_3(x)}{\sqrt{N}} + \frac{h_4(x)}{N} + ...] \tag{A–7}$$

where $h_n(x)$ can be written in terms of the hermite polynomials of order n. This approximation breaks down when $h_3(x)/\sqrt{N}$ becomes $O(1)$. It means that the width of the Gaussian distribution approximation scales $x \approx O(N^{2/3})$.

In case the variance of the probability distribution does not exist, the central region collapses faster than $\sqrt{N \log N}$ and the probability distribution of large sum converges to a different limiting function, known as Lévy distribution. In particular, in the large x limit, the distribution shows a power-law tail of the form,

$$P(x \to \infty) \sim \frac{A}{x^{1+\alpha}}; \quad 0 < \alpha < 2 \tag{A–8}$$

APPENDIX B
N-DEPENDENCE PROBLEM IN SPREAD FUNCTION APPROACH

The "Spread function approach" is one of the generalization strategies of the Gaussian unitary ensemble [24]. The underlying idea of this approach is to work with associated characteristic function, not directly probability distribution itself, to discover a novel random matrix ensemble. In this framework that the general n-point correlation function are given by Eq. 1–15. Here we will show how the non-trivial N dependence arises when the spectral density designed to give a fat-tail distribution. As was observed in our numerical study [1] , we need to search for the novel universality in $x \to \infty$ and $N \to \infty$ limit where the fat-tail spectral density is expected to occur. We find [72] that there is an asymptotic expression for harmonic oscillator function $\Psi(x)$ in the limit where $x \to \infty$ and $N \to \infty$ so $u = \frac{x}{\sqrt{N}}$ finite, e.g., $x = \sqrt{2N}\cos\phi$.

$$\Psi_N(x) = \frac{2^{1/4}N^{-1/4}}{\sqrt{\pi \sin\phi}}\left\{ \sin\left[\left(\frac{N}{2}+\frac{1}{4}\right)(\sin 2\phi - 2\phi) + \frac{3\pi}{4}\right] + O\left(\frac{1}{N}\right)\right\} \tag{B–1}$$

Using the Christoffel-Darboux formula, we can write down the two-level kernel.

$$(x-y)K_N^G(x,y) = \frac{1}{2\pi\sqrt{\sin\phi \sin\psi}}H(x,y) \tag{B–2}$$

where $x = \sqrt{2N}\cos\phi$, $y = \sqrt{2N}\cos\psi$ and $u(\phi) = \sin(2\phi) - 2\phi$ and.

$$\begin{aligned} H(x,y) &= 2\sin\left[\frac{1}{4}(u(\psi)+u(\phi))\right]\sin\left[\frac{N}{2}(u(\psi)-u(\phi))\right] \\ &\quad +2\cos\left[\frac{N}{2}(u(\psi)+u(\phi))\right]\sin\left[\frac{1}{4}(u(\psi)-u(\phi))\right] \end{aligned}$$

[1] We investigated Lévy matrices discussed in Chapter 1 numerically. It turns out that the density $\sigma_N \sim \frac{N}{x^{1+\mu}}$ for large x. So it is expected that in a double scaling limit, *e.g.,* $x \to \infty$ and $N \to \infty$ such that $u \equiv \frac{N}{x^\mu}$ finite, the density can be unfolded $\bar{\sigma}(u) = 1$ and a novel universality may show up in terms of this variable u.

From the kernel, density of states reads directly as $\sigma_N^G(x) = \lim_{y\to x} K_N^G(x,y)$.

$$\sigma_N^G(x) \simeq \sqrt{2N}\sin(\phi - \sin\phi\cos\phi) \tag{B–3}$$

For the density Eq. B–3, it turns out that choosing the spread function $f(b) = \frac{\mu^\lambda}{\Gamma(\lambda)} b^{-(\lambda+1)} e^{-\mu/b}$ give rise to a fat-tail density, e.g.,

$$\sigma_N(x) \simeq \frac{\mu^\lambda}{\Gamma(\lambda)}\sqrt{2N}\left(\frac{8N}{x^2}\right)^{\lambda+1/2}\int_0^1 d\xi\xi^{2\lambda}e^{-\alpha\xi^2}\sin\left[\cos^{-1}\xi - \xi\sqrt{1-\xi^2}\right] \tag{B–4}$$

where $\xi = \frac{x}{\sqrt{8Nb}}$ finite in the limit $x \to \infty$, $N \to \infty$ and $b \sim N$ and $\alpha \equiv \frac{8\mu N}{x^2}$. So if one choose $\lambda = \frac{1}{2}$ and $\mu \sim 1/N$, then $\sigma_N(x)$ goes as $\frac{N}{x^2}$.

In order to study unfolded correlation function, we define a scaling variable $u = \frac{N}{x}$ such that $\bar{\sigma}(u) = 1$. In particular, the unfolding variables $\rho \equiv a\frac{N}{x_1}$ and $\chi \equiv a\frac{N}{x_2}$ where a is some constant. Then the unfolded cluster function

$$Y_2(\rho,\chi) \equiv \lim_{N\to\infty}\frac{dx}{d\rho}\frac{dy}{d\chi}T_2(x_1,x_2). \tag{B–5}$$

Since $T_2 = T_2^0 - \delta T_2$, similarly we define $Y_2 = Y_2^0 - \delta Y_2$ so that

$$Y_2^0(\rho,\chi) \equiv \lim_{N\to\infty}\frac{dx}{d\rho}\frac{dy}{d\chi}\int_0^\infty \frac{dbf(b)}{4b}[K_N^G(\bar{x_1},\bar{x_2})]^2 \tag{B–6}$$

and

$$\delta Y_2(\rho,\chi) \equiv \lim_{N\to\infty}\frac{dx}{d\rho}\frac{dy}{d\chi}\int_0^\infty \frac{dbf(b)}{4b}\sigma_N^G(\bar{x_1})\sigma_N^G(\bar{x_2}) - 1 \tag{B–7}$$

For further calculation, we begin with δY_2. First, we rewrite $\sigma_N^G(x)$ in terms of the unfolding variable ρ,

$$\sigma_N^G(\bar{x}) = \sigma_N^G\left(\frac{aN}{2\sqrt{b}\rho}\right) \simeq \sqrt{2N}\sin\left(\cos^{-1}\sqrt{\frac{a^2N}{8b}}\frac{1}{\rho} - \sqrt{\frac{a^2N}{8b}}\frac{1}{\rho}\sqrt{1-\frac{a^2N}{8b}\frac{1}{\rho^2}}\right) \tag{B–8}$$

Plugging this into eq. B–7, we get

$$\delta Y_2(\rho,\chi) \simeq \lim_{N\to\infty}\frac{a^2N^2}{\rho^2\chi^2}\int_0^\infty \frac{dbf(b)}{4b}\sigma_N^G\left(\frac{aN}{2\sqrt{b}\rho}\right)\sigma_N^G\left(\frac{aN}{2\sqrt{b}\chi}\right) - 1 \tag{B–9}$$

After substituting $f(b)$ and making a change of variable, $\xi = \sqrt{\frac{a^2N}{8b}}\frac{1}{\rho}$, N-dependence can be taken out from the integral. Thus we have,

$$\delta Y_2(\rho,\chi) \simeq \lim_{N\to\infty} \frac{8}{a^2}\frac{a^2N^2}{\rho^2\chi^2}\frac{\mu^\lambda}{\Gamma(\lambda)}\Big(\frac{8\rho}{a^2N}\Big)^\lambda I_2(\xi;\rho,\chi) - 1 \tag{B–10}$$

where

$$I_2(\xi;\rho,\xi) = \int_0^1 d\xi \xi^{2\lambda+1} e^{-\frac{8\lambda\rho^2}{a^2N}\xi^2} \sin\Big[\cos^{-1}\xi - \xi(1-\xi^2)\Big] \sin\Big[\cos^{-1}\xi\frac{\rho}{\chi} - \xi\frac{\rho}{\chi}(1-\xi^2\frac{\rho^2}{\chi^2})\Big] \tag{B–11}$$

In the limit $N \to \infty, \mu \sim N$, with other variables order of 1, the argument of exponential in Eq. B–11 can be made very small. So the leading order of the integral Eq. B–11 goes as 1. With this in mind, putting $\lambda = \frac{1}{2}$ into Eq. B–10, we can see that $\delta Y_2(\rho,\chi) \sim N$.

Next, we calculate the Y_2^0 in a similar way. After a little calculation, we get

$$Y_2^0(\rho,\chi) \simeq \lim_{N\to\infty} \frac{2}{a^2}\frac{a^2N^2}{\rho^2\chi^2}\rho^2\chi^2\frac{\mu^\lambda}{\Gamma(\lambda)}\Big(\frac{8\rho^2}{a^2N}\Big)^\lambda I_2^0(\xi;\rho,\chi) \tag{B–12}$$

where

$$I_2^0(\xi;\rho,\lambda) = \int_0^1 d\xi \frac{\xi^{2\lambda-1}e^{-\frac{8\mu\rho^2}{a^2N}\xi^2}\sin[(\frac{\bar{u}+\bar{v}}{4})]^2}{(1-\xi^2)^{1/2}(1-\frac{\rho^2}{\chi^2}\xi^2)^{1/2}} \cdot \frac{[\sin N(\frac{\bar{u}-\bar{v}}{2})]^2}{N^2(\rho-\chi)^2} \tag{B–13}$$

Here $\bar{u}$ and $\bar{v}$ are defined as

$$\frac{\bar{u}}{2} = \sqrt{1-\frac{\rho^2}{\chi^2}\xi^2}\cdot\frac{\rho}{\chi}\xi - \cos^{-1}\xi\frac{\rho}{\chi} \tag{B–14}$$

and

$$\frac{\bar{v}}{2} = \sqrt{1-\xi^2}\cdot\xi - \cos^{-1}\xi \tag{B–15}$$

When $N(\bar{u}-\bar{v}) \ll 1$, Eq. B–13 becomes N-independent so Y_2^0 goes as N with $\lambda = \frac{1}{2}$ and $\mu \sim N^{-1}$ otherwise, Y_2^0 goes to zero.

From above, it is shown that unfolded two-level cluster function carries non-trivial N dependence although one-level correlation function can be made to have power-law asymptotic behavior.

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