THE IMPACT OF UNMODELED TIME SERIES PROCESSES IN WITHIN-SUBJECT RESIDUAL STRUCTURE IN CONDITIONAL LATENT GROWTH MODELING: A MONTE CARLO STUDY

By

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To my family in China

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As latent growth modeling is a popular method for analyzing longitudinal data, it is worthy of methodologists' attention to investigate the consequences of model misspecification. This study specifically investigated the impact of unmodeled time series processes in the within-person residual covariance structure on the parameter estimates and standard error estimates, as well as on the chi-square goodness of fit test and some commonly used fit indexes.

It was found that when the analysis model failed to include any type of time series process, all the fixed parameter estimates, together with their standard error estimates, were not affected. The variance components estimates were biased to different degrees under some conditions, depending on the type of within-person residual covariance structure. The standard error estimates of these variance components were not affected by model misspecification. Based on the results, it is recommended that applied researchers consider alternative covariance structures.

It was also found that when the within-person residual covariance structure is an AR (1) or a MA (1) process, the chi-square goodness of fit test and RMSEA can be used for model selection under many conditions. However TLI could be used to detect model misspecification for only one condition, while CFI and SRMR were not reliable in model differentiation. When

the within-person residual covariance structure was an ARMA (1, 1) process, only RMSEA could be used for model selection under certain conditions.

CHAPTER 1
INTRODUCTION

Longitudinal data, also called panel data, have been frequently encountered in social and behavioral sciences. A longitudinal data set contains observations of a number of subjects (individuals, firms, countries, etc.) measured over two or more time periods. For example, in educational research, a typical longitudinal data set contains the academic scores of a number of students measured at different time periods. Such data sets provide a large number of observations for a single individual subject and therefore greatly increase the degree of freedom in model estimation. The most important advantage of a longitudinal data set is that it allows researchers to investigate questions that could not be addressed by using just cross sectional data. For example, with a typical education longitudinal data set, researchers can measure the change or growth of the academic performance among students within a specified time period and can identify what factors affect their growth during this period. Such growth investigation could not be implemented with the cross-sectional data.

The popularity of studying change has been reflected in the availability of many large-scale national longitudinal data in social science. In the education field, widely used longitudinal data sets include the National Education Longitudinal Study (NELS), High School and Beyond (HSB), Early Childhood Longitudinal Study (ECLS), and National Longitudinal Study of Youth (NLSY). In economics fields, some prominent longitudinal data set such as the National Longitudinal Surveys of Labor Market Experience (NLS) and the University of Michigan's Panel Study of Income Dynamics (PSID) have been widely analyzed.

Accompanying the widely available data sets, a variety of methods for analyzing longitudinal data have emerged. The commonly used methods include analysis of variance (ANOVA), multivariate analysis of variance (MANOVA), hierarchical linear modeling (HLM), generalized linear model (GLM), fixed effects model, random effects model, and latent growth modeling. Each method has its own advantages and limitations. Their applicability depends on the actual research design and research questions that are of interest. Among all these models, latent growth model (LGM), also called latent curve model, growth curve model, emerged relatively recently but gained increasingly popularity. Moreover, with the recent development of more complex LGM, such as the mixed effect LGM, multilevel LGM, multivariate LGM, latent growth modeling becomes a powerful tool in various situations involving longitudinal data analysis. To see how popular the latent growth modeling method is in social science research, a search in Academic Search Premier, Business Source Premier, EconLit, Professional Development Collection, Psychology and Behavioral Sciences Collection, PsycINFO, Psychology and Behavioral Sciences Collection, Sociological Collection using the key word "latent growth model" in the peer review articles ranging from January 2000 to December 2008 resulted in 931 articles, which is sound proof of the popularity of this method.

LGM is composed of the trajectory equation (also called the within-subject or within-person equation) and the level and shape equation (also called the between-subject or between-person equation). The trajectory equation describes the growth trend of each individual. It contains an error term that captures all the unobserved characteristics for a single individual. The level and shape equation describes the latent level and latent shape respectively for all the individuals. In both the level and shape equation, a between-person error term is included to model the variation of growth level or growth trend between people.

The error in the within-person equation describes the difference between the value of observed outcome variable and the value predicted by the trajectory equation. It captures all the unmeasured factors for an individual, such as his/her ability, education level, health status or an event that might affect this person's growth. LGM, compared with traditional methods, such as ANOVA, MANOVA, gives substantial flexibility in specifying the within-person residual covariance structure. However, most applied researchers typically assume the within-person residuals are multivariate normally distributed with mean of zero and constant variance. That is, each individual has equal variance across time periods and the errors are independent across time. Under this assumption, the correlation of observed scores at any two time points is due solely to the presence of between-person variation. This simplification brings some concerns. First, some of the important aspects of change might be captured by the within-person residuals (Biesanz, West, & Kwok, 2003; Hedeker & Mermelstein, 2007). For example, Hedeker and Mermelstein (2007) showed that mood change in the smokers could be reflected in the within-person residual covariance structure rather than in the average change.

Second, as mentioned before, anything unmeasured but specific to an individual could be reflected in the within-person error term. If these characteristics remain approximately constant over the sample period, then the independence assumption of the within-person residual seems reasonable. If these characteristics vary over the sample period, the assumption is less realistic. It is not an unreasonable conjecture that some of the events might affect the individual over time. Consider evaluating the reading ability of kids in elementary school. The reading performance of a child might be increasing at a relatively constant rate, but individual observations might deviate from this general trend due to a number of factors in the individual's growth period (e.g., a health problem or a family crisis). Previous studies have shown that correlated measurement errors often exist in longitudinal data (e.g., Fitzmaurice, Laired, & Ware, 2004; Joreskog, 1979; Marsh, 1993; Rogosa, 1979; Sivo, 1997; Sivo & Willson, 1998). Therefore, the simple uncorrelated within-person error structure can not fully represent the data characteristics. A variety of more complex within-person residual structures have been identified, such as Toeplitz or moving average, autoregressive, compound symmetry and etc. (e.g. Goldstein, 1995; Wolfinger, 1993).

Third, when the within-person residual covariance structure is misspecified, the parameter estimate might be affected and the inference based on these estimates might be inaccurate. Various studies have been conducted on the impact of assumption violations in the within-person residual covariance structure on model parameters estimates (e.g., Yuan & Bentler, 2004; Ferron, Dailey, & Yi, 2002; Singer & Willett, 2003). See Chapter 2 for a presentation of results.

This study considers three time series within-person error structures: first-order autoregressive (AR) process, first-order moving average (MA) process and first order autoregressive and moving average (ARMA) process. The three times series are commonly encountered in time series analysis. The three kinds of residual covariance structures, although have been well discussed in fields like the econometrics, are relatively unpopular in education field.

LGM could be classified as unconditional LGM and conditional LGM. The two types of models differ in whether covariates are added in the model. In unconditional LGM, no time varying or time invariant covariate is added in the model, whereas in conditional LGM at least one covariate is included. Within applications of LGM, most applied research is conducted within the framework of conditional LGM, because conditional LGM enables researchers to include predictors and thus to capture the relationship between individual characteristics and growth parameter. However, most previous studies on model misspecification were conducted within the framework of unconditional LGM (e.g. Sivo, Fan & Witta, 2005; You, 2006). Although these studies with unconditional LGM shed some light on the possible consequence of model misspecification, whether those results could be generalized to conditional LGM is unknown. In unconditional LGM, the parameters of interest are mean, variance and covariance of the latent intercept and latent shape. With the inclusion of time varying and time invariant predictors, conditional LGM involves more parameters estimates, for instance, the direct effect of the predictor on latent factors. Therefore, the impact of model misspecification might be different from those occur in unconditional LGM. Moreover, even though the AR process has been well discussed in the context of LGM, up to now, very few studies include a systematic discussion of AR, MA and ARMA at the same time. Given their popularity and importance in time series analysis, they deserve a systematic application in longitudinal data analysis. Furthermore, no studies have been conducted to investigate the consequence of three unmodeled time series processes on conditional LGM. The three conditional LGMs investigated in this study are: LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process. These three types of LGMs are representatives of the typical conditional LGMs in applied research. They describe the standard way of including predictors and are commonly used.

The goal of this study is to investigate the impact of unmodeled time series processes in latent growth modeling through a Monte Carlo simulation study. To be specific, this study aims to evaluate how the model parameters estimates and standard errors, as well as GOF test and fit indices are affected when the within-person residual covariance structure demonstrates a time series process but the researchers fail to model these processes. This is an area less investigated in LGM. This study is believed to be an important contribution in empirically examining the impact of model misspecification and could provide researchers with better understanding of the consequence of assumption violation in growth modeling and provide useful information for handling these problems.

CHAPTER 2
LITERATURE REVIEW

This chapter is composed of six parts. The first part introduces the unconditional LGM and three types of conditional LGMs, with a general picture presented in the beginning of the first part, and the basic assumptions in LGM introduced at the end of the first part. Then the comparison between LGM with other methods is presented in the second part. The time series models are introduced in the third part, together with studies regarding modeling time series in the error structure in longitudinal data analysis. Followed in the fifth part are previous studies on the impact of model misspecifications. The sixth part presents the research questions and discusses the importance of this study.

Latent Growth Model

LGM can describe the individual change in a variety of ways: It can describe the individual initial status and growth trend, which can be linear, quadratic or other functional forms; It can estimate the variability across individuals in both initial level and trajectories, and can provide a means for testing the contribution of other predictors to the initial status and growth trajectories. Latent growth modeling methods accomplish these functions by analyzing not only the covariance structure but also the mean structure of variables. In other words, it can simultaneously estimate the changes in covariances, variances and means. The covariance structure contains information about individual differences while mean structure captures information at the aggregate level.

In LGM, there are three important latent factors: level, shape and error, which will be illustrated in the subsequent parts. The analytic interest in LGM is not specifically on the indicators but on the latent factors. Each outcome variable measured at any time is a function of these three latent factors. One of the advantages of LGM is that it allows the level and shape to vary across individuals under the assumption that the conceptualization is correct. The level represents the status of individuals in terms of the outcome variable at the measurement time set as a reference. If the first measurement time is taken as reference, the level can also be interpreted as the intercept (Muthén & Khoo, 1998). The level of an individual keeps constant across all measurement times. For different people, the level can be different from the beginning.

The shape factor, describes the rate of change across time. When the growth trend is linear, the shape is interpreted as a slope. The errors capture the deviation from the observed variables to the estimators obtained from the trajectory (within-person) model. The errors come from a variety of sources: it could be measurement error (e.g. the error caused by instrument or rater unreliability) or systematic error (e.g. the error due to unobserved variables or model misspecification of functional form).

Unconditional Latent Growth Model

As described in the introduction, the unconditional latent growth model refers to a model without predictors (See Figure 2-1). The trajectory equation (within-person equation) for this model is expressed as follows:

$$y_{it} = \alpha_i + \lambda_t \beta_i + \varepsilon_{it}, \tag{2-1}$$

where y_{it} is the outcome variable measured for the *i*th individual at time *t*. For a simple illustration, data are assumed to be collected in four equally spaced measurement times. All the subsequent introduced formulas would follow the four waves pattern. Therefore, t =1, 2, 3, 4. Parameter α_i refers to the level for the i th subject while parameter β_i is the shape for the i th subject. The α_i and β_i are considered latent factors. The parameters α_i and β_i are allowed to differ across individuals. The variable ε_{it} is the trajectory equation error of i th individual at time t with $E(\varepsilon_{it}) = 0$. More about the ε_{it} will be discussed later.

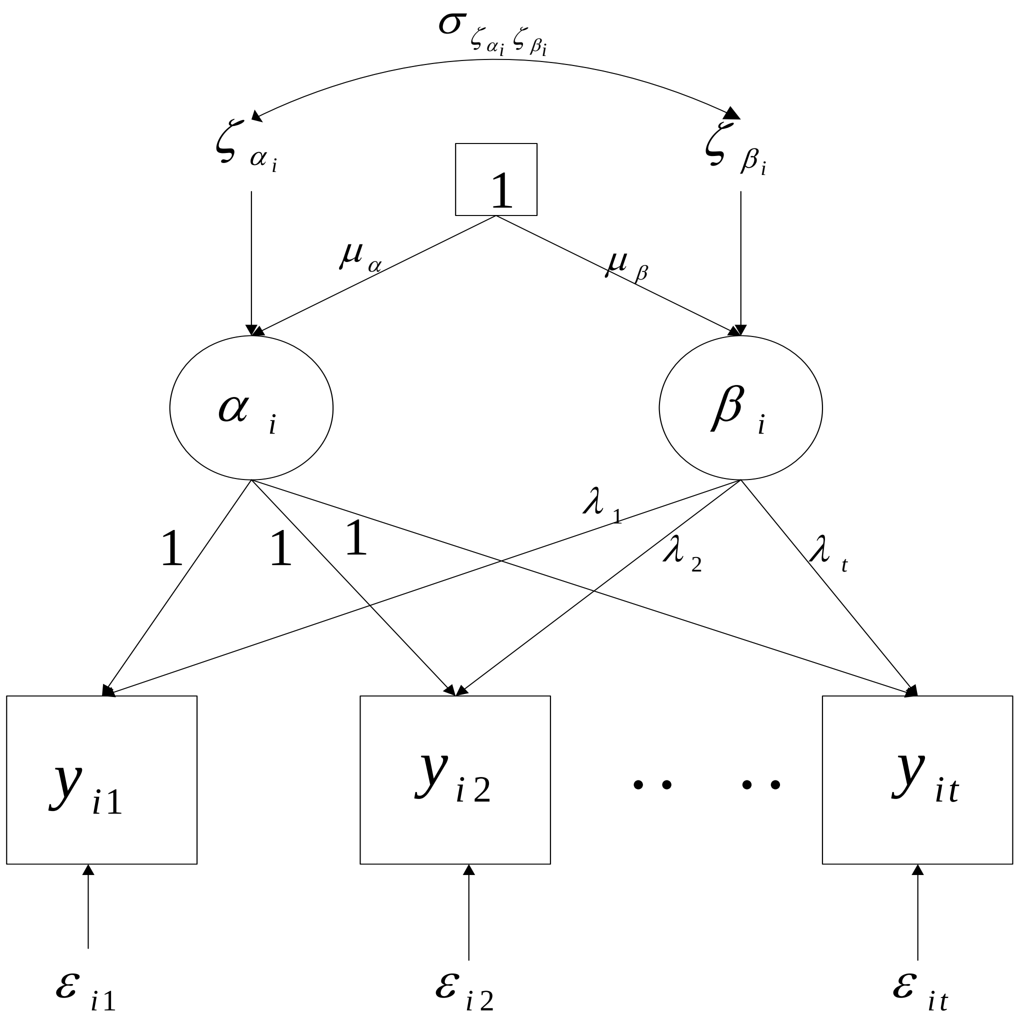

Figure 2-1. Unconditional latent growth model

Parameter λ_t refers to the factor loading of latent shape. The λ_t is fixed as $t-1$ across all measurement times. That is, λ_t =0, 1, 2, 3, which means all the measurements are taken at equally spaced time points. If the measurement is not taken at equal intervals, for instance, it is taken at month 1, month 2, month 3.5, and month 6, the λ_t are specified as 0, 2, 3.5, 6. When the loading is fixed to be zero, the time the zero loading represents is called reference point of development. In the above example, month 1 is considered as reference point. In this case, parameter λ_t represents the elapsed time from the reference point to time *t*. The functional form is linear, which means for equal time periods a given individual is growing by the same amount.

The individual level and shape can be decomposed into:

$$\begin{aligned}\alpha_i &= \mu_\alpha + \zeta_{\alpha i} \\ B_i &= \mu_B + \zeta_{Bi}\end{aligned}, \tag{2-2}$$

where μ_α and μ_B are the mean level and mean shape respectively. The mean level represents the average individual initial status. The mean shape represents the average growth rate across all sampled individuals. A positive μ_B indicates that on average individuals grow in the observed variable while a negative μ_B indicates a average growth decrease in the observed variable. The parameters $\zeta_{\alpha i}$ and ζ_{Bi} are the disturbances of level and shape respectively with mean of zero and variances of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$, as well as covariance of $\psi_{\alpha\beta}$. In unconditional models, the variances of these two disturbances (i.e., $\zeta_{\alpha i}$ and ζ_{Bi}) also represent the variance of the level and shape respectively. However, the interpretation is not the same when predictors are included in the level and shape equations. When predictors are included (see the subsequent introduction of conditional LGM), the variances of these two disturbances become residual variances, which are interpreted as the variability leftover in the level and shape factor after controlling the effects of predictors. A higher $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ indicate that sample subjects are more diverse. In the extreme case when $\zeta_{\alpha i}$ and ζ_{Bi} are all zero, there is no variability of level and shape across all people, which means all individuals have the same intercept and slope for their growth trajectories. A non-zero variance of $\zeta_{\alpha i}$ indicates that the sampled individuals differ from each other from the beginning of the study. A non-zero variance of ζ_{Bi} indicates that individuals grow at different rates. Hence, adding predictors in the model can help to account for the variability of individual growth (Willet & Keiley, 2000). Therefore, the level and shape equation describes the individual difference across the whole sample. The covariance between α_i and B_i represents the relationship between the level and growth trajectory.

The equation 2- 1 and equation 2-2 can be combined to a complete model:

$$y_{it} = \mu_\alpha + \lambda_t \mu_\beta + \zeta_{\alpha i} + \lambda_t \zeta_{\beta i} + \varepsilon_{it}. \tag{2-3}$$

This combined model is also called reduced form equation (Bollen & Curran, 2005) in that that the endogenous term α_i and β_i are replaced by their exogenous predictors and disturbances. The variable y_{it} is a combination of fixed component and random component, where the fixed component refers to the term $\mu_\alpha + \lambda_t \mu_\beta$, and random component refers to the term $\zeta_{\alpha i} + \lambda_t \zeta_{\beta i} + \varepsilon_{it}$. It should be noted that here the random component is heteroscedastic across time due to the effect of λ_t, which varies over time.

Equation 2-1 describes a linear trajectory relationship between the measurement time and individual growth change. If we want to extend this linear relationship to the broader class of nonlinear relationship, a simple way is to add higher-order polynomial terms. For example, a quadratic equation becomes:

$$y_{it} = \alpha_i + \lambda_t \beta_{1i} + \lambda_t^2 \beta_{2i} + \varepsilon_{it}, \tag{2-4}$$

where λ_t^2 is simply the squared value of time at measurement time t; β_{1i} is the slope for the linear term and β_{2i} is the slope for the quadratic term of the curve. The interpretations of other components of the equation remain the same. Similarly, we can incorporate cubic, quartic or other higher-power terms of time in this model. In equation 2-4, as the function does not describe a linear relationship anymore, the change of y is not the same for equal time passage. For instance, assuming measurement at equal intervals and the reference point is time 1, the change of y from time 1 to time 2 is equal to $\beta_{1i} + \beta_{2i}$, but the change of y from time 2 to time 3 is $\beta_{1i} + 5\beta_{2i}$. In the function describing linear relationship (equation 2-1), the change of y between any two time periods always equals β_i. The level and shape equation corresponding to the equation 2-4 is:

$$\alpha_i = \mu_\alpha + \zeta_{\alpha i}$$
$$\beta_{1i} = \mu_{\beta 1} + \zeta_{\beta 1i}$$
$$\beta_{2i} = \mu_{\beta 2} + \zeta_{\beta 2i}. \quad (2\text{-}5)$$

Equation 2-5 is similar to equation 2- 2 except the addition of the equation for the quadratic slope B_{2i}. The B_{2i}, similarly as α_i and B_{1i}, is randomly varying across individuals.

The structural equation form of the above linear trajectory equations could be expressed employing LISREL format (e.g. Muthén & Khoo, 1998; Singer & Willet, 2003, Bollen & Curran 2005). The LISREL formula is presented as follows:

$$y_i = \Lambda\eta_i + \varepsilon_i, \quad (2\text{-}6)$$

where y_i is a T x 1 vector of repeated measures, Λ is a T x m matrix of factor loadings, where m is the number of latent factors, η_i is an m x 1 vector of latent factors, and ε_i is a T x 1 vector of random errors.

The matrix format of each term in equation 2-6 can be illustrated as follows, assuming four repeated measures:

$$y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \eta_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}, \varepsilon_i = \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \varepsilon_{i3} \\ \varepsilon_{i4} \end{bmatrix}. \quad (2\text{-}7)$$

η_i can be expressed as:

$$\eta_i = \mu_\eta + \zeta_i, \tag{2-8}$$

where $\eta_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}, \mu_\eta = \begin{bmatrix} \mu_\alpha \\ \mu_B \end{bmatrix}, \zeta_\eta = \begin{bmatrix} \zeta_\alpha \\ \zeta_B \end{bmatrix}$.

The expression of y can be obtained by combining equation 2-7 and equation 2-8:

$$y = \Lambda\mu_\eta + (\Lambda\zeta + \varepsilon). \tag{2-9}$$

The model implied variance of the above equation is

$$\Sigma(\theta) = \Lambda\Psi\Lambda' + \Theta_\varepsilon, \tag{2-10}$$

where Ψ is the covariance matrix of ζ, and Θ_ε represents the variance and covariance matrix of the residuals of the outcome variable y.

The elements of Ψ and Θ_ε are:

$$\Psi = \begin{bmatrix} \psi_{\alpha\alpha} & \psi_{\alpha\beta} \\ \psi_{\beta\alpha} & \psi_{\beta\beta} \end{bmatrix}, \tag{2-11}$$

where $\psi_{\alpha\alpha} = \mathrm{var}(\zeta_{\alpha_i}), \psi_{\beta\beta} = \mathrm{var}(\zeta_{\beta_i})$ and $\psi_{\alpha\beta} = \mathrm{cov}(\zeta_{\alpha_i}, \zeta_{\beta_i})$ and (still assuming $T = 4$)

$$\Theta_\varepsilon = \begin{pmatrix} \sigma_{e_1} & 0 & 0 & 0 \\ 0 & \sigma_{e_2} & 0 & 0 \\ 0 & 0 & \sigma_{e_3} & 0 \\ 0 & 0 & 0 & \sigma_{e_4} \end{pmatrix}. \tag{2-12}$$

When the estimated model fits the data, the following equality holds:

$$\Sigma = \Sigma(\theta), \tag{2-13}$$

where Σ is the population covariance matrix of the y s, $\Sigma(\theta)$ is the model implied covariance matrix of the y's. The elements of Σ are

$$\Sigma = \begin{bmatrix} \sigma^2_{y_1} & \sigma^2_{y_1 y_2} & \dots & \dots\sigma^2_{y_1 y_4} \\ \sigma^2_{y_2 y_1} & \sigma^2_{y_2} & \dots & \dots\sigma^2_{y_2 y_4} \\ \sigma^2_{y_3 y_1} & \dots & \sigma^2_{y_3} & \dots\sigma^2_{y_3 y_4} \\ \sigma^2_{y_4 y_1} & \sigma^2_{y_4 y_2} & \dots & \dots\ \sigma^2_{y_4} \end{bmatrix}, \tag{2-14}$$

The model implied covariance matrix for the observed variables is

$$\Sigma(\theta) = \begin{bmatrix} \psi_{\alpha\alpha} + \lambda_1^2 \psi_{\beta\beta} + 2\lambda_1 \psi_{\alpha\beta} + \sigma^2_{e_1} & \cdots & \psi_{\alpha\alpha} + \lambda_1 \lambda_t \psi_{\beta\beta} + (\lambda_1 + \lambda_t)\psi_{\alpha\beta} \\ \psi_{\alpha\alpha} + \lambda_2 \lambda_1 \psi_{\beta\beta} + (\lambda_2 + \lambda_1)\psi_{\alpha\beta} & \cdots & \psi_{\alpha\alpha} + \lambda_2 \lambda_t \psi_{\beta\beta} + (\lambda_2 + \lambda_t)\psi_{\alpha\beta} \\ \vdots & \ddots & \vdots \\ \psi_{\alpha\alpha} + \lambda_t \lambda_1 \psi_{\beta\beta} + (\lambda_t + \lambda_1)\psi_{\alpha\beta} & \cdots & \psi_{\alpha\alpha} + \lambda_t^2 \psi_{\beta\beta} + 2\lambda_t \psi_{\alpha\beta} + \sigma^2_{e_t} \end{bmatrix}. \tag{2-15}$$

Finally, the expected value of the outcome variable equals

$$\mu_y = \Lambda \mu_\eta . \tag{2-16}$$

The model implied mean structure is $\mu(\theta)$. When the expected mean μ_y is equal to the model implied mean structure μ_y, the following equation should be obtained, in vector notation:

$$\begin{bmatrix} \mu_{y1} \\ \mu_{y2} \\ \vdots \\ \mu_{yT} \end{bmatrix} = \begin{bmatrix} \mu_\alpha + \lambda_1 \mu_\beta \\ \mu_\alpha + \lambda_2 \mu_\beta \\ \vdots \\ \mu_\alpha + \lambda_T \mu_\beta \end{bmatrix}. \tag{2-17}$$

The unconditional latent growth modeling is the simplest form of latent growth modeling. In practice, many researchers fit an unconditional growth model before fitting any type of more sophisticated LGM, such as conditional LGM, multilevel LGM, mixture LGM. The unconditional LGM could be used to establish the correct growth trajectory. Furthermore, the unconditional LGM describe the variability of the level and shape and serve as an assessment of whether adding predictors is justified. In general, the unconditional LGM is the first step in many LGM applied studies.

Conditional Latent Growth Model

As mentioned above, adding predictors in the model can help to account for the variability of individual growth. In many situations, researchers are interested in more complex research questions. The conditional growth model provides a convenient way to test various hypotheses. For instance, if we want to estimate the change of children's math skill by controlling their social economics status (SES), SES can be added as a predictor in the model. LGM allows us to incorporate predictors in the model in extremely flexible ways, which will be illustrated in the subsequent examples. Predictors could be time invariant or time varying. Time invariant predictors refer to variables that are constant across time, such as gender, nationality and ethnicity. Time varying predictors, on the contrary, refer to predictors that change as time passes by, such as students' test performance, marital status, individual's ability, and so on.

The conditional LGMs that were investigated in this study were LGM with a time invariant predictor, LGM with a time varying covariate and LGM with a parallel process. These are commonly used conditional LGM in applied research.

Latent growth model with a time-invariant covariate

For a simple illustration, only one predictor measured without error is included (See Figure 2-2). In real situations, more than one predictor can be incorporated into the model.

The trajectory equation is still the same as that in unconditional model:

$$y_{it} = \alpha_i + \lambda_t \beta_i + \varepsilon_{it} \tag{2-18}$$

The level and shape equation is different from that in unconditional model:

$$\begin{aligned} \alpha_i &= \mu_\alpha + \gamma_\alpha x_i + \zeta_{\alpha i} \\ B_i &= \mu_B + \gamma_B x_i + \zeta_{Bi} \end{aligned}, \tag{2-19}$$

where μ_α and μ_B are the mean level and mean slope respectively when predictor x is set to zero.. The parameters $\zeta_{\alpha i}$ and ζ_{Bi} are the disturbances of level and shape respectively after controlling the effect of the predictor x. As mentioned before, in unconditional models, the variances of these two disturbances also represent the variance of the level and shape respectively. However, once predictors are included in the level and shape equations, the $\zeta_{\alpha i}$ and ζ_{Bi} can not be simply interpreted as the variance of the level and shape respectively any more. The coefficients $\gamma_{\alpha 1}$ and γ_{B1} are the direct effects of x variable on level and shape respectively.

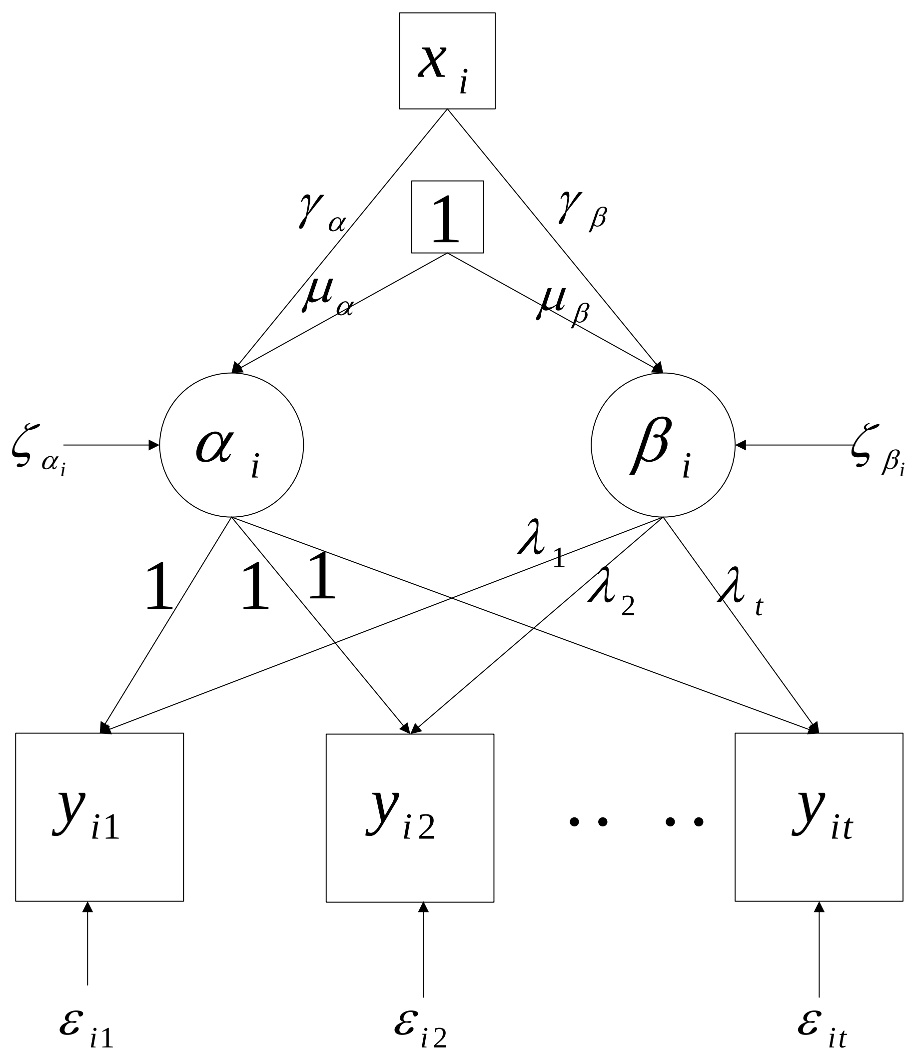

Figure 2-2. Latent growth model with a time invariant covariate

The structural equation form of the model is represented as follows (still assuming four waves):

$$y_i = \Lambda \eta_i + \varepsilon_i , \tag{2-20}$$

where $y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $\eta_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$, $\varepsilon_i = \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \varepsilon_{i3} \\ \varepsilon_{i4} \end{bmatrix}$

$$\eta_i = \mu_\eta + \Gamma x_i + \zeta_i , \tag{2-21}$$

where $\eta_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$, $\mu_\eta = \begin{bmatrix} \mu_\alpha \\ \mu_B \end{bmatrix}$, $\Gamma = \begin{bmatrix} \gamma_{\alpha 1} \\ \gamma_{\beta 1} \end{bmatrix}$, $\zeta_\eta = \begin{bmatrix} \zeta_{\alpha i} \\ \zeta_{\beta i} \end{bmatrix}$

The combined model is obtained by substituting η_i in equation 2-21 into equation 2-20:

$$y_i = \Lambda(\mu_\eta + \Gamma x_i) + \Lambda \zeta_i + \varepsilon_i . \tag{2-22}$$

The implied mean structure is

$$\mu_y = \Lambda(\mu_\eta + \Gamma \mu_\chi) . \tag{2-23}$$

The model implied covariance structure could be derived by using deviation score to simplify the analytical expression of the implied covariance matrix (Bollen and Curran, 2005). The deviation score formula is presented as follows:

$$y_i - \mu_y = [\Lambda(\mu_\eta + \Gamma \chi_i) + \Lambda \zeta_i + \varepsilon_i] - [\Lambda(\mu_\eta + \Gamma \mu_\chi)] = \Lambda(\Gamma(\chi_i - \mu_\chi) + \zeta_i) + \varepsilon_i . \tag{2-24}$$

The model implied covariance matrix is

$$
\begin{aligned}
\Sigma(\theta) &= \begin{bmatrix} \Sigma_{yy}(\theta) & \Sigma_{yx}(\theta) \\ \Sigma_{xy}(\theta) & \Sigma_{xx}(\theta) \end{bmatrix} = E\left(\begin{bmatrix} y_{i-}\mu_y \\ x_i - \mu_x \end{bmatrix} \begin{bmatrix} y_{i-}\mu_y \\ x_i - \mu_x \end{bmatrix}^{'} \right) \\
&= \begin{bmatrix} E\left[(y_{i-}\mu_y)(y_{i-}\mu_y)^{'}\right] & E\left[(y_{i-}\mu_y)(x_i - \mu_x)^{'}\right] \\ E\left[(x_i - \mu_x)(y_{i-}\mu_y)^{'}\right] & E\left[(x_i - \mu_x)(x_i - \mu_x)^{'}\right] \end{bmatrix}, \\
&= \begin{bmatrix} \Lambda(\Gamma \Sigma_{xx} \Gamma^{'} + \Psi)\Lambda^{'} + \Sigma_{\varepsilon\varepsilon} & \Lambda\Gamma\Sigma_{xx} \\ \Sigma_{xx}\Gamma^{'}\Lambda^{'} & \Sigma_{xx} \end{bmatrix}
\end{aligned}
\qquad (2\text{-}25)
$$

where $\sum_{xx}$ is the population covariance matrix of *x*s, and the meaning of the other symbols remain the same meanings as in the description of the unconditional model.

There are two ways to incorporate time invariant predictors. One way, as described above, is to let the predictor impose direct effect on latent curve factors but only has indirect effect on outcome variables. This model is also called growth predictor model by Stoel, R.D., Van den Wittenboer, D. & Hox, J. (2004). This is a widely used model in social science research. Among the 267 peer reviewed journal articles found by using key word “latent growth” searching in databases of Academic Search Premier, Business Source Premier, EconLit, Professional Development Collection, PsycINFO, and Sociological Collection from 2004 to 2008 more than 30% of studies employed this model.

Stoel, R.D., et al. (2004) argued that although this model had the distinctive advantage that the effect of time invariant covariate on growth parameters could be captured directly, the appropriateness of this model was based on the assumption of full mediation. That is, the direct effect of time invariant predictor on the outcome variable is equal to zero. If this assumption does not hold, the model is considered incorrect. Based on this argument, they proposed another way to incorporate time invariant predictors: regress predictors directly on outcome variables. This model was termed as direct effect model by Stoel, R.D., et al. (2004). The model trajectory equation is described as follows:

$$y_{it} = \alpha_i + \lambda_t \beta_i + \gamma_t x_i + \varepsilon_{it}, \tag{2-26}$$

where x_i is the time invariant covariate for each individual and γ_t is the regression coefficient between x_i and y_{it} .The subscript *t* for γ_t indicates that the effect of x_i on y_{it} changes at different time. The level and shape equation is the same as equation 2-2:

$$\begin{aligned} \alpha_i &= \mu_\alpha + \zeta_{\alpha i} \\ B_i &= \mu_B + \zeta_{Bi} \end{aligned}, \tag{2-27}$$

where all the symbols remain the same meaning as before.

Although this model is also widely used in applications, this study only focuses on growth predictor model.

Latent growth model with a time-varying covariate

The conditional model with a time varying covariate is more complex than model with a time invariant covariate in that the predictor varies with time (see Figure 2-3). The time varying covariate has to be added in the trajectory equation:

$$y_{it} = \alpha_i + \lambda_t \beta_i + \gamma_t x_{it} + \varepsilon_{it}, \tag{2-28}$$

where all the terms are the same as we specified in equation 2-26, except that x_{it} is a time varying covariate measured for individual *i* at time *t*, and its effect on outcome variable y_{it} is captured by coefficient γ_t. The variable y_{it} is now a function of level, shape, a time-specific influence of the covariate x_{it}, plus a random error.

The level and shape equation is the same as that in unconditional growth models:

$$\begin{aligned} \alpha_i &= \mu_\alpha + \zeta_{\alpha i} \\ B_i &= \mu_B + \zeta_{Bi} \end{aligned}, \tag{2-29}$$

where all the symbols remain the same meaning as before.

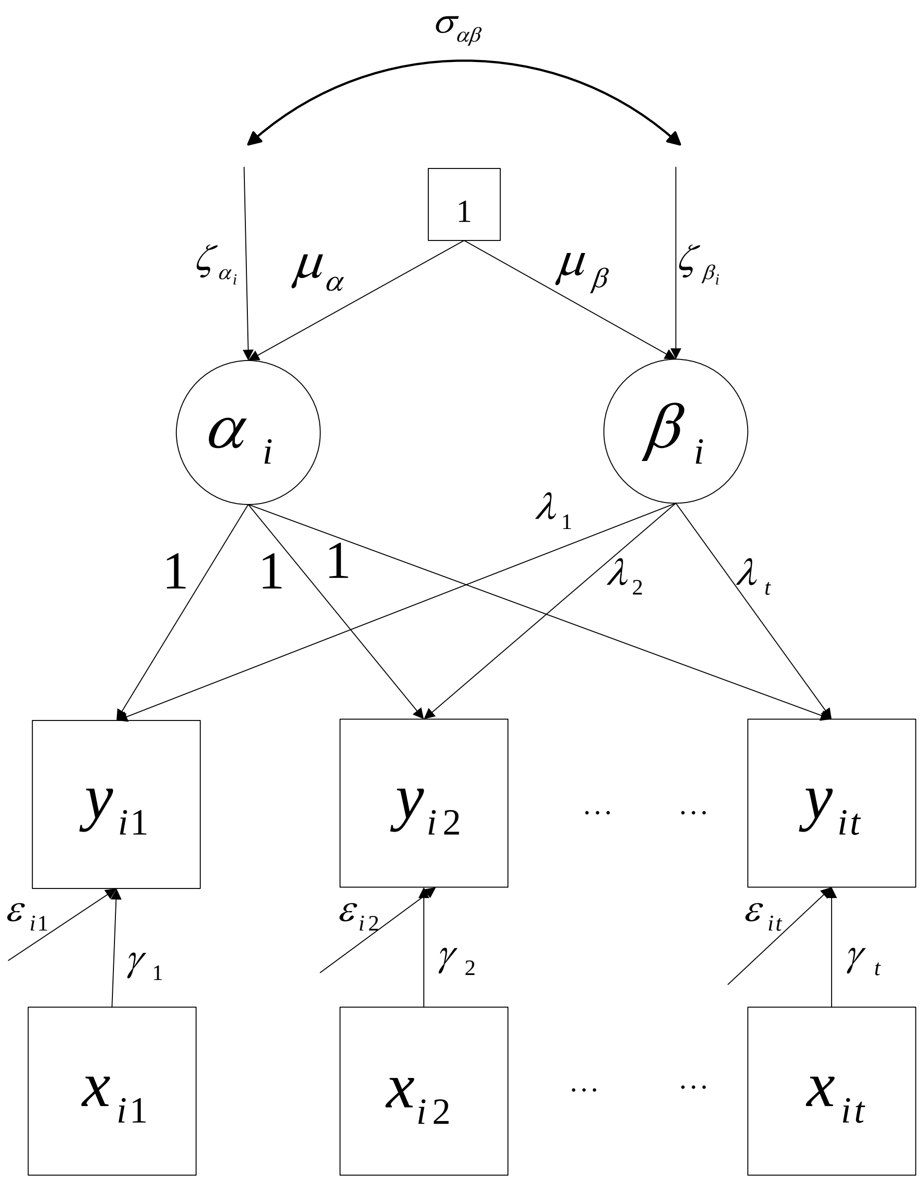

Figure 2-3. Latent growth model with a time varying covariate

The structural equation form of this model is represented as follows:

$$y_i = \Lambda \eta_i + \Gamma \chi_{it} + \varepsilon_i , \tag{2-30}$$

where $y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $\eta_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$, $\Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}$, $\chi_{it} = \begin{bmatrix} \chi_{i1} \\ \chi_{i2} \\ \chi_{i3} \\ \chi_{i4} \end{bmatrix}$ $\varepsilon_i = \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \varepsilon_{i3} \\ \varepsilon_{i4} \end{bmatrix}$.

$$\eta_i = \mu_\eta + \zeta_i, \qquad (2\text{-}31)$$

where $\eta_i = \begin{bmatrix} \alpha_i \\ \beta_i \end{bmatrix}$, $\mu_\eta = \begin{bmatrix} \mu_\alpha \\ \mu_B \end{bmatrix}$, $\zeta_\eta = \begin{bmatrix} \zeta_{\alpha i} \\ \zeta_{\beta i} \end{bmatrix}$.

According to this model, *y* is jointly affected by both the underlying random growth process and the time specific influences associated with the time varying covariate. A typical example of this model is the study conducted by Curran, Muthén and Hartford (1998), where they investigated time-specific impact of becoming married on heavy alcohol use. He tried to find out whether becoming married for the first time would affect heavy alcohol use controlling the normal development trend of alcohol use in early adulthood. This model is just appropriate for his research question.

Latent growth model with a parallel process

The previous two sections introduced two kinds of conditional LGMs that are also considered univariate LGM. That is, although there are multiple measurements on the outcome variable, they are multiple measures of one dependent variable. Sometimes we are interested in the analysis on more than one outcome variable. Suppose we have a dataset reflecting students' academic performance at school. We might be interested in not only the growth trend in both individual mathematics and reading achievement but also whether the individual concurrent changes in the two areas are mutually interrelated. This allows us to understand the change in several domains and how these domains relate to each other. When LGM includes the latent curve process on more than one outcome variable, this type of model is called multivariate LGM. In this paper, it is referred as LGM with a parallel process. An example of a parallel process model is presented in Figure 2-4.

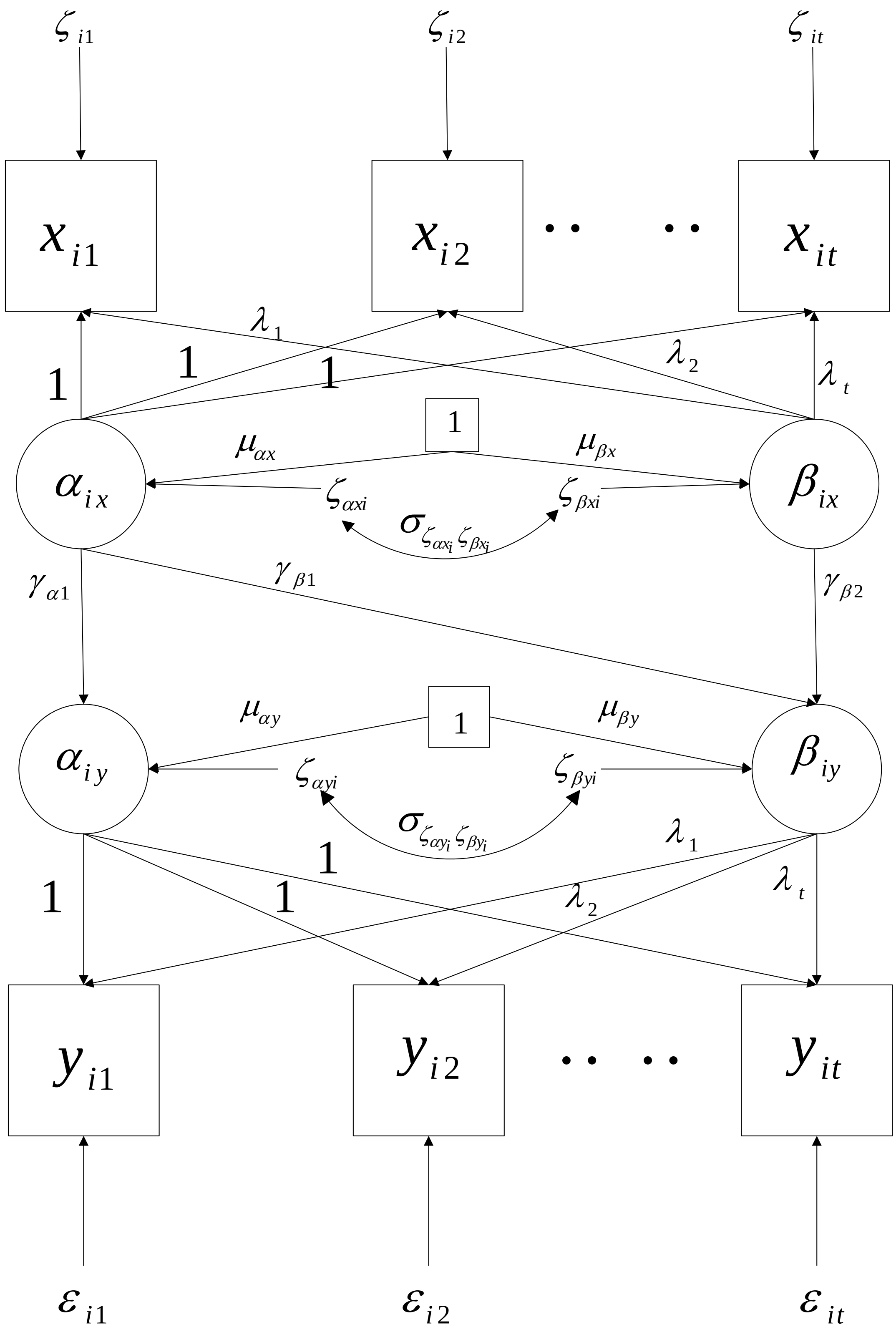

Figure 2-4. Latent growth model with a parallel process

For a simple illustration, only two outcome variables were included. The model equations for the variable y and for the variable x are described as:

$$y_{it} = \alpha_{iy} + \lambda_t \beta_{iy} + \varepsilon_{yit}, \tag{2-32}$$

$$x_{it} = \alpha_{ix} + \lambda_t \beta_{ix} + \varepsilon_{xit}, \tag{2-33}$$

$$\begin{aligned} \alpha_{iy} &= \mu_{\alpha y} + \gamma_{\alpha 1}\alpha_{ix} + \zeta_{\alpha yi} \\ B_{iy} &= \mu_{By} + \gamma_{\beta 1}\alpha_{ix} + \gamma_{\beta 2}\beta_{ix} + \zeta_{Byi}, \end{aligned} \tag{2-34}$$

and

$$\begin{aligned} \alpha_{ix} &= \mu_{\alpha x} + \zeta_{\alpha xi} \\ B_{ix} &= \mu_{Bx} + \zeta_{Bxi}, \end{aligned} \tag{2-35}$$

where α_{iy} and B_{iy} represent the level and shape factor respectively for the outcome variable *y*; parameters α_{ix} and B_{ix} represent the level and shape factor respectively for the variable *x*; parameters $\mu_{\alpha y}$ and μ_{By} are the mean level and mean slope respectively for the outcome variable y controlling all other terms in their separate equation; parameters $\mu_{\alpha x}$ and μ_{Bx} are the mean level and mean slope respectively for the outcome variable *x*; parameters $\zeta_{\alpha yi}$ and ζ_{Byi} are still the disturbance for the level α_{iy} and shape B_{iy} respectively and the parameters $\zeta_{\alpha xi}$ and ζ_{Bxi} are the disturbances of level and shape for the level α_{ix} and shape B_{ix} respectively. The coefficient $\gamma_{\alpha 1}$ indicates the effect of initial status of the *x* variable on the initial status of the *y* variable. If the $\gamma_{\alpha 1}$ is positive, higher growth status of the *x* would anticipate higher growth status of the *y* variable, after controlling the impact of the growth shape of the *x* variable. Parameter $\gamma_{\beta 1}$ captures the relationship between the level of the *x* variable and the shape of the *y* variable when the β_{ix} is controlled. Parameter $\gamma_{\beta 2}$ represents the effect of growth shape of the *x* variable on the growth shape of the *y* variable controlling the impact of α_{ix}. A positive value of $\gamma_{\beta 1}$ indicates that high growth status of the *x* variable would predict faster growth on the *y* variable. A positive value of $\gamma_{\beta 2}$ would indicate that individuals growing quickly on the *x* variable would also tend to grow quickly on the *y* variable. One difference between the univariate LGM and multivariate LGM is that the latent factors have to be subscripted with *y* or *x* to differentiate the repeated measure of interest. With two outcome variables, the relationship between latent factors of one variable and the other one becomes much more complex.

A point that is worthwhile to mention here is that there is no impact of the growth shape of the *x* variable on the level of the y variable. The rationale is obvious: the growth shape of the x variable is obtained later than the level of the y variable. Therefore, a future estimated variable can not be used to predict the current variable.

The structural form of the model could be represented as follows:

$$y_i = \Lambda \eta_{iy} + \varepsilon_i, \tag{2-36}$$

where $y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \\ y_{i4} \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $\eta_{iy} = \begin{bmatrix} \alpha_{iy} \\ \beta_{iy} \end{bmatrix}$, $\varepsilon_i = \begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \varepsilon_{i3} \\ \varepsilon_{i4} \end{bmatrix}$, and

$$\eta_{iy} = \mu_{\eta y} + \Gamma \xi + \zeta_i, \tag{2-37}$$

where $\eta_{iy} = \begin{bmatrix} \alpha_{iy} \\ \beta_{iy} \end{bmatrix}$, $\mu_{\eta y} = \begin{bmatrix} \mu_{\alpha y} \\ \mu_{\beta y} \end{bmatrix}$, $\Gamma = \begin{bmatrix} \gamma_{\alpha 1} & \gamma_{\alpha 2} \\ \gamma_{\beta 1} & \gamma_{\beta 2} \end{bmatrix}$, $\xi = \begin{bmatrix} \alpha_{ix} \\ \beta_{ix} \end{bmatrix}$, $\zeta_\eta = \begin{bmatrix} \zeta_\alpha \\ \zeta_B \end{bmatrix}$, and

$$\chi_i = \Lambda \xi + \delta_i, \tag{2-38}$$

where $\chi_i = \begin{bmatrix} \chi_{i1} \\ \chi_{i2} \\ \chi_{i3} \\ \chi_{i4} \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $\xi = \begin{bmatrix} \alpha_{ix} \\ \beta_{ix} \end{bmatrix}$, $\delta_i = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{bmatrix}$.

There are several variations of the parallel process model. In the model described above, only the level and shape of the *y* variable are predicted by the level and shape of *x* variable, not vice versa. In many studies, the level and shape of the *x* and *y* variables were predicted by each other in a variety of combinations. For example, the shape of the *x* variable can be predicted by the level of the outcome variable (e.g. Cheong, Mackinnon & Khoo, 2003; Curran, 2000). Therefore, the meaning of the outcome variable and predictor variable get blurred here. The key concept is that different domains are interrelated and are not independent of each other. All the variables must be assessed in the same measurement occasions.

As pointed out by Muthén (2002), one advantage of growth modeling in a latent variable framework was the ease with which to carry out analysis of multiple processes, both parallel in time and sequential. A variety of applications of this model have been discussed recently (e.g. Hudson, 2008; Simons, 2007; Mitchell, Kaufman, & Beals, 2005).

Assumptions of Growth Modeling

The assumptions of growth modeling can be summarized in three aspects: within-person residual covariance structure, measurement time and missing data, and functional form of growth.

Within-person residual covariance structure

When the outcome variable is continuous, it is commonly assumed that the within-person error ε_{it} is multivariate normally distributed with mean of zero and covariance matrix Θ_{ε}. If the outcome variable is categorical, alternative estimation method would be used, such as weighted least squares with corrected means and variance (Muthén & Khoo, 1998). Under the condition of categorical outcome variable, the assumption of multivariate normality should be relaxed.

In a fashion analogous to the assumption in regression analysis, all the variables in the right hand side of the trajectory equation are uncorrelated with the error. More formally, take equation 2-1 as an example, that is, $\operatorname{cov}(\varepsilon_{it}, \alpha_i) = 0$ and $\operatorname{cov}(\varepsilon_{it}, \beta_i) = 0$ for all i and t. The variance of ε_{it} could be constant or non constant, depending on the data characteristic and real situation. Although it is mentioned in the introduction part that LGM allows the measurement error to be correlated across different time, it is not a general assumption. Many studies assume that the errors are not correlated over time, i.e., $\operatorname{cov}(\varepsilon_{it}, \varepsilon_{i,t+s}) = 0$ for $s \neq 0$. It is also assumed that the errors of different individuals at different time are uncorrelated, that is $\operatorname{cov}(\varepsilon_{it}\varepsilon_{j,t+s}) = 0$ for $i \neq j$ and for $s \neq 0$. When the errors are assumed to be uncorrelated over time, the assumption about the residuals is expressed as the follows:

$$\begin{bmatrix} \varepsilon_{i1} \\ \varepsilon_{i2} \\ \vdots \\ \varepsilon_{it} \end{bmatrix} \sim N \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \sigma_{e1} & 0 & \cdots & 0 \\ 0 & \sigma_{e2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_{et} \end{bmatrix}. \tag{2-39}$$

Regarding the level and shape equation, the unconditional LGM was used for a simple illustration:

$$\begin{aligned} \alpha_i &= \mu_\alpha + \zeta_{\alpha i} \\ B_i &= \mu_B + \zeta_{Bi} \end{aligned}. \tag{2-40}$$

The disturbances $\zeta_{\alpha i}$ and ζ_{Bi} are normally distributed with mean of zero and variance of $\psi_{\alpha i}$ and ψ_{Bi}. They are also correlated with each other with covariance $\psi_{\alpha\beta}$. Furthermore, the two disturbances are assumed to be uncorrelated with the error ε_{it}.

Measurement time and missing data

It is commonly assumed in growth modeling that the repeated measures for individuals are equally numbered and equally spaced for all individuals and there is no missing data (Duncan, Duncan, Strycker, Li, & Alpert, 1999). This assumption is considered a serious limitation of LGM (e.g., Willett & Sayer, 1994). However, MacCallum, Kim, Malarkey, and Kiecolt-Glaser (1997) argued that the development of full information maximum likelihood can relax this assumption. This method defines the likelihood function using individual score instead of variance and covariance matrix. Therefore, even the measurement time is irregular and/or there is missing data, the estimation of LGM can still be accomplished by using full information maximum likelihood. However, this method is limited in its application by the available software.

Functional form of development

The fundamental assumption, also considered the most serious limitation of LGM, is that all subjects have to follow the same functional form of growth. That is, all individuals (firms, countries, etc.) have to keep the same linear, quadratic or other form of trend (Hertzog & Nesselroad, 2003; Lawrence & Hancock, 1998). Therefore, although LGM allows all subjects to have different growth trajectory, their basis functional form has to be the same.

With the development of multiple group SEM, individuals can be separated into different groups if sufficient information about the separation is known before. Then different groups can be described by different functional forms. However, within each separated group the trajectory equation has to take the same functional form for all individuals.

Comparisons with Other Methods

The fact that LGM can explicitly model measurement error is a potential advantage of SEM over other more traditional methods such as ANOVA and MANOVA. ANOVA has the most stringent constraints on covariance matrix for the observed variables. The covariance matrix has to meet the sphericity assumption: the variance of difference scores for each pair of time points are equal. MANOVA does not require the sphericity assumption but it has the same disadvantage of ANOVA: they treat the differences among individuals in their growth trajectory as error variance.

Multilevel modeling is another powerful tool in longitudinal data analysis. It also offers great flexibility in modeling covariance structure. The relationship between structural equation modeling and multilevel modeling has been extensively investigated (e.g. Curran, 2003; Raudenbush & Bryk, 2002). LGM, within the framework of SEM, is comparable with HLM in many aspects. It is believed that when repeated observations are nested within individuals, SEM and HLM are analytically equivalent methods (Curran, 2003).

In HLM, the level-one equation (also called the within-person equation), using notation from Raudenbush & Bryk (2002), is presented as follows:

$$y_{it} = \pi_{oi} + \pi_{1i} a_{it} + e_{it}, \tag{2-41}$$

Where y_{it} is the outcome variable for person i at time t; π_{oi} is the individual's initial status and π_{1i} is the growth rate; a_{it} represents the time of measurement and e_{it} is the error. Within SEM framework, π_{oi} and π_{1i} correspond to the level and shape factor respectively, and a_{it} corresponds to the factor loading λ_t.

The level-two equation (also called the between-person equation) describes the variability in initial status and growth rate across individuals. The equation is presented as follows:

$$\begin{aligned} \pi_{oi} &= \beta_{00} + \gamma_{oi} \\ \pi_{1i} &= \beta_{10} + \gamma_{1i} \end{aligned}, \tag{2-42}$$

where β_{00} is the mean initial status and β_{10} is the mean growth rate; γ_{0i} and γ_{1i} is the random variance component for level and shape respectively; the covariance of level and shape is captured by the covariance between γ_{0i} and γ_{1i}.

Note that equation 2-41 and 2-42 are similar with the unconditional LGM equation 2-1 and 2-2. Under general conditions, the two modeling methods, HLM and LGM, are approaching the same problem from a different perspective (Curran, 2003).

When a time invariant covariate is included in the model, it is added in the level-two equation as shown in equation 2-43, and the level-one equation is the same as equation 2-41

$$\begin{aligned}\pi_{oi} &= \beta_{00} + \beta_{01}X_{1i} + \gamma_{oi} \\ \pi_{1i} &= \beta_{10} + \beta_{11}X_{1i} + \gamma_{1i}\end{aligned} \quad , \tag{2-43}$$

where X_{1i} represents the time invariant covariate, β_{01} and β_{11} capture the effect of X_{1i} on initial status and growth rate respectively. This equation is comparable to equation 2-19 introduced in the section entitled "LGM with a Time Invariant Covariate".

Therefore, if we have only time invariant covariate, the HLM model equation is analytically the same as LGM with a time invariant covariate.

When time varying covariate is added in the model, the level-one equation becomes

$$y_{it} = \pi_{oi} + \pi_{1i}a_{ji} + \pi_{2i}x_{it} + e_{it} , \tag{2-44}$$

where x_{it} represents the time varying covariate and π_{2i} is the effect of time varying covariate on outcome variable controlling the influence of time.

In LGM, the trajectory equation with time varying covariate is presented in equation 2-28. The difference between equation 2-28 and equation 2-44 lies in the regression coefficient for the time varying covariate. In equation 2-44, the coefficient π_{2i} is a constant, which means the effect of the time varying covariate on the outcome variable remain the same across different time periods. In equation 2-28, as mentioned before, the regression coefficient γ_t has a subscript *t*, which shows that the effect of time varying covariate on outcome variable varies at different time period. A point worthwhile to mention is that the π_{2i} in equation 2-44 can also be allowed to change with time, although it is not introduced here. The level- two model for the HLM is still the same as equation 2-42. Therefore, under the situation when a time varying covariate is added, the LGM and HLM is comparable but the researchers should decide whether the regression coefficient of the time varying covariate varies with time.

According to Raudenbush and Bryk (2002), within the context of modeling change, the difference between HLM and SEM lies in the limitations of software rather than the real model difference. They recommended using SEM approach when correlated error existed because the available SEM software allows for easy specification and estimation of correlations between errors.

The above comparison of HLM and SEM focuses on the situation when repeated observations are nested within individuals. However, when data structure was individual level nested in group level, e.g. students nested in schools, using SEM to implement the analysis is a data management nightmare and error prone process (Curran, 2003). The interpretation of parameter estimates using SEM requires special care and attention. For example, the latent factor means in the SEM are regression coefficients in HLM. It is recommended by Curran (Curran, 2003) that using HLM would be a better approach when no other elements of the SEM are incorporated in a multilevel model.

The analysis of longitudinal data has also been well investigated in econometrics. The most commonly used models are fixed effects model and random effects model (also called variance components model) (e.g., Hsiao, 2003). For simplicity, only one independent variable is considered. The fixed effects model is given by:

$$y_{it} = \alpha_i + \beta x_{it} + u_{it}, \tag{2-45}$$

where i =1,…, N, t =1,…,T, y_{it} and x_{it} are the endogenous variable and exogenous variables respectively measured for the ith individual unit at time t. The α_i captures the specific effect for an individual unit and is assumed to be constant over time. The β is a constant parameter, representing the relationship between x and y, and is constant across all the individual units and time periods. The u_{it} is identically and independently distributed with mean of zero and variance σ_u^2. The u_{it} represents the effects of the omitted variables that are peculiar to both the individual units and time periods.

The fixed effects model assumes that both the α_i and β are non-random variables, whereas in LGM both α_i and β_i are random variables. The two models all assume that each individual unit has different intercepts (i.e., α_i is different across individuals). In fixed effects model, parameter β is a slope parameter and is assumed to be the same across all individuals, while in LGM β_i is different across individuals. The factor loading λ_t in LGM can be represented in fixed effects model by including time codes for each period as an extra explanatory variable. For example, an extra x_{it} which is a vector of (0,1,…,T) could be added for a linear time effect. The coefficient of the time varying covariate in LGM varies with time but remains constant in fixed effect model.

The random effects model formula is:

$$y_{it} = \beta x_{it} + v_{it} = \beta x_{it} + \alpha_i + d_t + u_{it}, \tag{2-46}$$

where $v_{it} = \alpha_i + d_t + u_{it}$, α_i, d_t and u_{it} are random variables with α_i : $iid(0, \sigma_\alpha^2)$,

d_t : $iid(0, \sigma_d^2)$, and u_{it} : $iid(0, \sigma_u^2)$, and the three random variables are jointly independent. Furthermore,

$$\begin{aligned} \text{cov}(v_{it}, v_{is}) &= \sigma_\alpha^2 + \sigma_d^2 + \sigma_u^2 \quad \text{for } t = s \\ &= \sigma_\alpha^2 \quad\quad\quad\quad\quad \text{for } t \neq s \end{aligned} , \tag{2-47}$$

and

$$\begin{aligned} \text{cov}(v_{it}, v_{js}) &= \sigma_d^2 \quad\quad \text{for } i \neq j, t = s \\ &= 0 \quad\quad\; \text{for } i \neq j, t \neq s \end{aligned} . \tag{2-48}$$

The α_i captures individual differences that endure over time and is a constant over time. Therefore the α_i is a time invariant individual effect. The d_t represents factors that are peculiar to specific time periods but affect individuals equally. So it is an individual invariant effect. The parameter α_i in random effects model is no longer a fixed variable as in fixed effects model. The α_i and d_t, like u_{it}, are treated as random variables in random effects model. Parameter α_i is termed as permanent component and u_{it} is a transitory disturbance. (MaCurdy, 1982).

To make the comparison between random effects model and LGM with a time varying covariate easier to understand, equation 2-28 and equation 2-29 for LGM with a time varying covariate is combined to the following equation.

$$y_{it} = \mu_\alpha + \gamma_t x_{it} + \lambda_t \mu_B + \zeta_{\alpha i} + \lambda_t \zeta_{Bi} + \varepsilon_{it} . \tag{2-49}$$

With two more explanatory variables x_{it} added, the random effects model is comparable to LGM with a time varying covariate. One x_{it} represents the time dummies for each period as described before, which represents the factor loading λ_t in LGM. The regression coefficient of this dummy variable is the mean slope (i.e., μ_B in equation 2-49). Another variable x_{it} is a vector of constant one, that is, x_{it} = (1,…1)'. The length of this vector equals to the total number of time periods. The coefficient of this constant variable is comparable to the mean level in the LGM (i.e., μ_α in equation 2-49). The α_i in the random effects model is comparable to the $\zeta_{\alpha i}$, and $\text{var}(\alpha_i) = \sigma_\alpha^2 = \psi_{\alpha\alpha}$. The d_t can represent the effect of the product of λ_t and ζ_{Bi}. Therefore, $\text{var}(d_t) = \sigma_d^2 = \lambda_t^2 \psi_{\beta\beta}$. The residual u_{it} is comparable to the within person equation residual ε_{it}. However, as in the fixed effects model, γ_t, the coefficient of the time varying covariate of x_{it} in LGM, is comparable to the β in random effects model except that γ_t changes with time but β remains constant.

The above two kinds of models all assume different intercepts for different individual units and the slope coefficient constant for either the time dummy or other time varying covariates. Therefore, both fixed effects model and random effects model belong to the category of variable intercept model. There are models in econometrics that assume coefficient to be random, that is, models that allow the coefficients to differ from unit to unit and/or from time to time. The general specification of the variable coefficient model is, assuming only cross-sectional differences are present:

$$y_{it} = \beta_i' x_{it} + u_{it}, \tag{2-50}$$

or assuming only time period differences are present:

$$y_{it} = \beta_t' x_{it} + u_{it}, \tag{2-51}$$

where β_i or β_t each is a K x 1 vector of parameters and x_{it} is a K x 1 vector of independent variables.

Because the variable coefficient model is not as widely used in empirical work as the variable intercept model due to the computational complexities (Hsiao, 2003), it is not introduced here.

Stationary Time Series Model

As mentioned before, correlated residuals often present in longitudinal data. Some of these correlations actually followed the stationary time series process (e.g. Sivo, 2001; Sivo and Willson, 2000).

A time series refers to "a set of observations generated sequentially in time" (Box and Jenkins, 1976). The time series could be strongly stationary or weakly stationary. In a strongly stationary time series, the joint probability distribution does not depend on time itself but on the difference of time points. In other words, those series whose statistical properties such as mean, variance, covariance, etc. do not depend on time t, that is, its statistical properties are all constant over time. Parameters such as the mean and variance of the outcome variable at time 1 should be equal to those at time 2, 3… and so forth. Furthermore, the covariance between any two of the observations, say, y_t and y_{t+s}, is assumed not dependent on time t, but only on the time periods between the two observations. A time series could be stationary in one statistics, e.g. the mean, (termed as mean stationary) but not stationary in another characteristic, e.g. variance. A time series is weakly stationary when it is both mean and variance stationary. With the stationarity assumption, one can simply predict that the statistical properties will be the same in the future as they have been in the past.

According to Box and Jenkins (1976), stationary time series data often may often be modeled by two distinct stochastic processes: autoregressive (AR) and moving average (MA). The autoregressive moving average (ARMA) process is a mixture of the two processes. Box and

Jenkins (1976) introduced three linear stationary models accordingly. They are AR model, MA model and ARMA model. These models would be introduced in the following sections.

Autoregressive (AR) Model

The idea of this model is that each measure at time *t* is a function of measures of previous time. The equation is as follows:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + ... + \rho_p y_{t-p} + \varepsilon_t, \quad (2\text{-}52)$$

where y_t is the outcome variable at time t, ρ is the correlation between two outcome at different time and $|\rho| < 1$. The variable ε_t is called "while noise" (Box and Jenkings,1976), which consists of a series independently distributed random shocks with $E(\varepsilon_t) = 0$ and $\text{var}(\varepsilon_t) = \sigma^2$.

The process defined by equation 2-52 is called autoregressive process of order p, also termed as AR(p) process. The first order autoregressive model AR(1) refers to the model in which the outcome variable in time *t* is only affected by its immediate previous variable at time *t*-1. Under AR(1) assumption, the equation 44 would be simplified as

$$y_t = \rho_1 y_{t-1} + \varepsilon_t. \quad (2\text{-}53)$$

For convenience in determining other properties of a time series process, there is no intercept included in the model, which means the mean of the y_t is equal to zero. The zero mean of an AR(1) process can be obtained by taking the expected value in equation 2-53:

$$E(y_t) = E(\rho_1 y_{t-1}) = \rho_1 \mu . \quad (2\text{-}54)$$

Since an AR process is stationary, the mean at all time periods are equal, that is, $E(y_t) = E(y_{t-1}) = ... = \mu$. The following equation could be obtained:

$$E(y_t) = E(\rho_1 y_{t-1}) = \rho_1 \mu = \mu. \quad (2\text{-}55)$$

As ρ is not equal to one, therefore, by equation 2-55, the μ is equal to zero. The variance of y_t is

$$\mathrm{var}(y_t) = \sigma_y^2 = \mathrm{var}(\rho_1 y_{t-1} + \varepsilon_t) = \rho_1^2 \,\mathrm{var}(y_{t-1}) + \mathrm{var}(\varepsilon_t)\,, \tag{2-56}$$

since y_{t-1} and ε_t are independent . The covariance is the same for any outcome variables with one period apart, which is

$$\begin{aligned} \mathrm{cov}(y_t, y_{t-1}) &= E[(y_t - \mu)(y_{t-1} - \mu)] = E[y_t y_{t-1}] = E[(\rho_1 y_{t-1} + \varepsilon_t) y_{t-1}] \\ &= \rho_1 E(y_{t-1}^2) + E(\varepsilon_t) E(y_{t-1}) = \rho_1 \sigma_y^2 \end{aligned}. \tag{2-57}$$

The covariance between y_t and y_{t-2} is given by

$$\begin{aligned} \mathrm{cov}(y_t, y_{t-2}) &= E[(y_t - \mu)(y_{t-2} - \mu)] = E(y_t y_{t-2}) = E[(\rho_1 y_{t-1} + \varepsilon_t) y_{t-2}] \\ &= \rho_1 E(y_{t-1} y_{t-2}) + E(\varepsilon_t y_{t-2}) = \rho_1^2 \sigma_y^2 \end{aligned}. \tag{2-58}$$

Similarly, the covariance between y_t and y_{t-k} could be derived as

$$\mathrm{cov}(y_t, y_{t-k}) = \rho_1^k \sigma_y^2 . \tag{2-59}$$

Therefore, the covariance matrix associated with an AR (1) model is defined as:

$$\sigma_y^2 \begin{bmatrix} 1 & \rho_1 & \rho_1^2 & \cdots & \rho_1^t \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_1^{t-1} \\ \rho_1^2 & \rho_1 & 1 & \cdots & \rho_1^{t-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_1^t & \rho_1^{t-1} & \rho_1^{t-2} & \cdots & 1 \end{bmatrix}. \tag{2-60}$$

This covariance matrix is symmetric with constant σ_y^2 in the diagonal. A matrix of this form is called autocovariance matrix and the corresponding correlation matrix is called autocorrelation matrix (Box & Jenkins, 1976). The AR (1) process is sometimes called the Markov process because the distribution of y_t given $y_{t-1}, y_{t-2}, y_{t-3}, \ldots$ is exactly the same as the distribution of y_t given y_{t-1}.

Similarly, the AR (2) model is defined as a model in which the outcome variable in time *t* is only affected by its immediate two previous variables:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t \,. \qquad (2\text{-}61)$$

In terms of practical importance, only an AR(1) or an AR(2) model are given considerable attention in application. For example, McCleary and Hay (1980) conducted a study to investigate the effect of community crime prevention program on the purse snatchings in Hyde Park, Chicago from January 1969 to September 1973. The number of purse snatchings followed an AR(2) time series.

Moving Average (MA) Model

In this model explains a construct measured at one time is affected by autocorrelated residuals:

$$y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - .... - \theta_q \varepsilon_{t-q} \,, \qquad (2\text{-}62)$$

where y_t is the outcome variable at time t , θ denotes the correlation between two residuals at some lag and $|\theta| < 1$. The sequence of residuals $\{\varepsilon_t\}$ is a "white noise" series with zero mean and constant variance σ^2.

The process defined in equation 2-62 is called moving average process of order q and could be abbreviated as MA(q). This process is useful in describing phenomena in which some random events introduce an immediate effect that only lasts for a short period of time.

The first order moving average model, MA (1) model, similar to an AR(1) model, refers to a model in which the outcome variable is only affected by its residual and the immediate previous residual. The formula is presented as follows:

$$y_t = \varepsilon_t - \theta_1 \varepsilon_{t-1} \,. \qquad (2\text{-}63)$$

The mean of the y_t for an MA (1) model is equal to 0 as $\{\varepsilon_t\}$ is a series with mean of zero and variance σ^2. The variance of y_t is given by the following equation:

$$\mathrm{var}(y_t) = \sigma_y^2 = E(y_t^2) = E(\varepsilon_t^2 - 2\theta_1\varepsilon_t\varepsilon_{t-1} + \theta_1^2\varepsilon_{t-1}^2) = \sigma^2(1+\theta_1^2)\,. \tag{2-64}$$

Therefore, by equation 2-64, the relationship between σ^2 and σ_y^2 is obtained as follows:

$$\sigma^2 = \sigma_y^2/(1+\theta_1^2)\,. \tag{2-65}$$

The covariance between y_t and y_{t-1} is derived as:

$$\begin{aligned}\mathrm{cov}(y_t, y_{t-1}) &= E(y_t y_{t-1}) = E[(\varepsilon_t - \theta_1\varepsilon_{t-1})(\varepsilon_{t-1} - \theta_1\varepsilon_{t-2})] \\ &= \theta_1 E(\varepsilon_{t-1})^2 = -\theta_1\sigma^2 = -\sigma_y^2\frac{\theta_1}{1+\theta_1^2}\,.\end{aligned} \tag{2-66}$$

The covariance between y_t and y_{t-2} is derived as:

$$\mathrm{cov}(y_t, y_{t-2}) = E(y_t y_{t-2}) = E[(\varepsilon_t - \theta_1\varepsilon_{t-1})(\varepsilon_{t-2} - \theta_1\varepsilon_{t-3})] = 0\,. \tag{2-67}$$

The covariance between y_t and y_{t-k} for $k > 2$ could be derived similarly and are all equal to zero. It seems that a MA (1) process has a ‘memory” only of one period while it is not true for an AR(1) process.

The covariance matrix associated with a MA (1) process is:

$$\sigma_y^2\begin{bmatrix} 1 & -\frac{\theta_1}{1+\theta_1^2} & 0 & \cdots & 0 \\ -\frac{\theta_1}{1+\theta_1^2} & 1 & -\frac{\theta_1}{1+\theta_1^2} & \cdots & 0 \\ 0 & -\frac{\theta_1}{1+\theta_1^2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{2-68}$$

The second order moving average process is defined as:

$$y_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}\,. \tag{2-69}$$

The MA (1) model and MA (2) model, just like the AR(1) model and AR(2) model, are particularly important in practice.

Autoregressive Moving Average (ARMA) Model

This model captures the process when the above two situations happen at the same time.

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + ... + \rho_p y_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - .... - \theta_q \varepsilon_{t-q}, \quad (2\text{-}70)$$

where all the symbols remain the same meaning as described above.

This process is referred as ARMA (p, q) process. It may be thought as a pth autoregressive process and a qth order moving average process.

The first order autoregressive moving average ARMA (1, 1) model is defined as follows:

$$y_t = \rho_1 y_{t-1} - \theta_1 \varepsilon_{t-1} + \varepsilon_t . \quad (2\text{-}71)$$

Taking the expected value for both sides of equation 2-71, the mean of y_t for ARMA (1,1) is equal to:

$$E(y_t) = \rho_1 E(y_{t-1}) = \rho_1 \mu . \quad (2\text{-}72)$$

Since $E(y_t) = E(y_{t-1}) = \mu$ and $|\rho| < 1$, equation 2-72 tells us that the value of μ is equal to zero, which is the same as the μ in an AR or an MA model.

The variance of the outcome variable in an ARMA process is:

$$\begin{aligned} \mathrm{var}(y_t) = \sigma_y^2 = E(y_t^2) &= E[(\rho_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1})^2] \\ &= \rho_1^2 \sigma_y^2 - 2\rho_1\theta_1 E(y_{t-1}\varepsilon_{t-1}) + \theta_1^2\sigma^2 + \sigma^2 , \\ &= \rho_1^2\sigma_y^2 - 2\rho_1\theta_1\sigma^2 + \theta_1^2\sigma^2 + \sigma^2 \end{aligned} \quad (2\text{-}73)$$

where $E(y_{t-1}\varepsilon_{t-1}) = E[(\rho_1 y_{t-2} + \varepsilon_{t-1} - \theta_1\varepsilon_{t-2})\varepsilon_{t-1}] = E(\varepsilon_{t-1}^2) = \sigma^2$ and ε_{t-1} is not correlated with y_{t-2} or ε_{t-2}.

According to equation 2-73,

$$\sigma^2 = \frac{(1-\rho_1^2)}{1+\theta_1^2 - 2\rho_1\theta_1}\sigma_y^2 . \quad (2\text{-}74)$$

The covariance between y_t and y_{t-1} could be derived as:

$$\begin{aligned}\operatorname{cov}(y_t, y_{t-1}) &= E[y_{t-1}(\rho_1 y_{t-1} + \varepsilon_t - \theta_1\varepsilon_{t-1})] \\ &= \rho_1 E(y_{t-1}^2) - \theta_1 E(y_{t-1}, \varepsilon_{t-1}) = \rho_1\sigma_y^2 - \theta_1\sigma^2 .\end{aligned} \quad (2\text{-}75)$$

Substituting σ^2 in equation 2-74 into equation 2-75 lead to the following result:

$$\operatorname{cov}(y_t, y_{t-1}) = \rho_1\sigma_y^2 - \theta_1\sigma^2 = \frac{(1 - \rho_1\theta_1)(\rho_1 - \theta_1)}{1 + \theta_1^2 - 2\rho_1\theta_1}\sigma_y^2 . \quad (2\text{-}76)$$

Similarly, the covariance between y_t and y_{t-2} could be derived as:

$$\begin{aligned}\operatorname{cov}(y_t, y_{t-2}) = E[y_{t-2}(\rho_1 y_{t-1} + \varepsilon_t - \theta_1\varepsilon_{t-1})] &= \rho_1 E(y_{t-1}, y_{t-2}) \\ &= \rho_1 E[y_{t-2}(\rho_1 y_{t-2} + \varepsilon_{t-1} - \theta_1\varepsilon_{t-2})] \\ &= \rho_1^2\sigma_y^2 - \rho_1\theta_1 E(y_{t-2}\varepsilon_{t-2}) \\ &= \rho_1^2\sigma_y^2 - \rho_1\theta_1\sigma^2 \\ &= \frac{\rho_1(1 - \rho_1\theta_1)(\rho_1 - \theta_1)}{1 + \theta_1^2 - 2\rho_1\theta_1}\sigma_y^2 ,\end{aligned} \quad (2\text{-}77)$$

and the covariance between y_t and y_{t-k} could be derived as:

$$\operatorname{cov}(y_t, y_{t-k}) = \rho_1^k\sigma_y^2 - \rho_1^{k-1}\theta_1^{k-1}\sigma^2 = \frac{\rho_1^{k-1}(1 - \rho_1\theta_1)(\rho_1 - \theta_1)}{1 + \theta_1^2 - 2\rho_1\theta_1}\sigma_y^2 . \quad (2\text{-}78)$$

The covariance matrix associated with an ARMA (1,1) process:

$$\sigma_y^2 \frac{(\rho_1 - \theta_1)(1 - \rho_1\theta_1)}{1 - 2\rho_1\theta_1 + \theta_1^2}\begin{bmatrix} \frac{1 - 2\rho_1\theta_1 + \theta_1^2}{(\rho_1 - \theta_1)(1 - \rho_1\theta_1)} & 1 & \rho_1 & \cdots & \rho_1^{t-2} \\ 1 & \frac{1 - 2\rho_1\theta_1 + \theta_1^2}{(\rho_1 - \theta_1)(1 - \rho_1\theta_1)} & 1 & \cdots & \rho_1^{t-3} \\ \rho_1 & \rho_1^2 & \frac{1 - 2\rho_1\theta_1 + \theta_1^2}{(\rho_1 - \theta_1)(1 - \rho_1\theta_1)} & \cdots & \rho_1^{t-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_1^{t-2} & \rho_1^{t-3} & \rho_1^{t-4} & \cdots & \frac{1 - 2\rho_1\theta_1 + \theta_1^2}{(\rho_1 - \theta_1)(1 - \rho_1\theta_1)} \end{bmatrix} . \quad (2\text{-}79)$$

Modeling Time Series in the Error Structure in Longitudinal Data Analysis

The use of time series process to model the error structure can be found in many longitudinal data analyses. For example, in econometrics literature, many studies estimated variance component models where transitory components followed time series structure (e.g., David, 1971; Hause 1977; Lillard &Willis, 1978; Lillard &Weiss, 1979; MaCurdy, 1982). As introduced in the model comparison part, the transitory component is comparable to the within-person residuals in LGM.

Although time series is relatively unpopular in educational research, it has gained increasing popularity lately. Researchers have attempted to either directly integrate time series model into growth model (e.g. Curran & Bollen, 2001; Sivo, et al., 2005) or capture the time series process in within-person residual covariance structure under the framework of HLM or SEM (e.g., Ferron, et al., 2002; You, 2006; Kwok, West and Green ,2007).

The most common time series process in a within-person residual structure is the AR (1) error structure (e.g., Wolfinger, 1993; Ferron, et al., 2002; Kwok, et al., 2007). It is taken as an alternative error structure in many studies investigating misspecification of within-person error covariance structure (e.g., Ferron, et al., 2002; Singer and Willet, 2003; Kwok, et al., 2007). Mplus, the commonly used SEM software even includes the AR (1) within-person residual structure in its demonstration examples. Although the use of MA or ARMA model is relatively less investigated than the AR model, they are not difficult to be located. For example, Sivo (1997) pointed out that if measurement error correlations were found in longitudinal data sets, these correlations usually were found at a particular lag nearest to the diagonal of the error covariance matrix, indicating a MA or an ARMA process. In other studies that investigated the effects of misspecifying the within-person residual structure, the MA (1) and/or ARMA(1,1) model were also selected as alternative error structures (e.g., Kwok, et al., 2007; Singer and Willet, 2003).

Studies on the Impact of Misspecifying the Within-Person Error Structure

As mentioned before, although researchers often assume the within- person error residuals are uncorrelated in many applied studies, previous literature indicated that this assumption was often violated. Failing to take account of the residual structures among repeated measures might bias the model estimates and lead to incorrect inferences (e.g., Fitzmaurice et al., 2004; Singer & Willett, 2003). Therefore, it deserves the methodologist's attention to investigate the impact of independence assumption violation on parameter estimates.

It is generally believed that in linear mixed models, the fixed effects estimates are consistent no matter whether the random effects part of the model is correctly specified (Verbeke & Molenberghs, 2000). However, when the random effects are not correct, the standard errors usually computed for the fixed effects estimates may no longer be appropriate.

Ferron, Dailey, and Yi (2002) investigated the impact of misspecifying the within-person residual structure under the framework of HLM through a series simulation studies. The models examined included multiple predictors in a level- two equation, non-linear growth curve, or missing or unequally spaced observations. It was found that when the residual covariance structure was simply assumed to be a diagonal matrix with constant variance, but it actually followed an AR (1) or a MA (1) process, under most conditions, except the nonlinear model with unbalanced design, the estimates of the fixed effect remain unbiased and the tests of the fixed effects were robust to the model misspecification. However, when model failed to include the AR (1) structure, intercept and slope variance estimates were inflated while their covariance were deflated. Model fit criteria frequently failed to identify the correct model when the length of measurement periods was short.

Based on the "opposites-naming" data, Singer and Willet (2003) compared the following six residual covariance structures in multilevel model: unstructured, compound symmetric, heterogeneous compound symmetric, autoregressive, heterogeneous autoregressive and toeplitz. They found that, except for the toeplitz and unstructured residual structure, other residual structure did not make a strong improvement in model fit. However, the precision of the fixed effect estimates improved for all the error structures except for the toeplitz, unstructured and standard residual structure. Their overall conclusion is consistent with those from Verbeke and Molenberghs (2000): Estimates of the fixed effect are unbiased regardless of the error structure, but the standard error of the fixed effect estimates would be affected by the selection of error structure. However, conclusions from this study might not be tenable due to the small sample size (35 participants) and short measurement periods (only 4 occasions were examined on each individual).

Yuan and Bentler (2004, 2006) analytically showed that the intercept and slope parameters in linear growth curve models could be estimated consistently even when the covariance structure was misspecified.

You (2006) evaluated how the growth model estimates were affected when both the homoscedasticity and independence assumption were violated. Her simulation design was conducted with a linear unconditional latent growth modeling. Results indicated that the misspecification of error structure had no impact on the estimates of the intercept and slope of the growth trajectory, which was consistent with those from Yuan and Bentler (2004, 2006). For the variance components estimates, You (2006) found that under most conditions, the variance estimates of the intercept and slope were generally inflated, while the covariance of intercept and slope was generally deflated.

Kwok, West and Green (2007) conducted a Monte Carlo study to investigate the impact of misspecifying the within-subject covariance structure in longitudinal multilevel models under the multilevel model framework. The multilevel model they employed is a random regression coefficient model with only time variable included in the level one equation. This model, as discussed before, is comparable to the unconditional latent growth model. It was found that when the within-subject covariance matrix was an AR(1) structure, misspecifying it as a diagonal matrix with constant variance resulted in over estimates of random effects (e.g., τ_{00}, τ_{01}). Regardless of the effect of misspecifying the within-subject error matrix, the fixed effect parameter estimates were unbiased but their standard error estimates were overestimated, which was consistent with those from Verbeke and Molenberghs (2000).

Significance of This Study

The above literature review has demonstrated the consequence of unmodeled time series processes in longitudinal study. It was shown that when the within-person error structure was misspecified, fixed effect parameter estimates were unbiased, the standard error estimates of fixed effects were possible affected, and the random effects were biased. The major focus of this study is to examine the effect of misspecification of the residual structure on the estimation and testing of the fixed effects and the random effects of the conditional LGMs. The motivation of doing this study is based on the following three reasons:

First, the above literature review has shown that latent growth modeling is a powerful tool in assessment of change, owing to its flexibility in including time varying and time invariant covariate, its less strict requirement on residual covariance structure assumption and its various model formats. Despite these advantages, if the model were misspecified, there would exist possibilities of biased estimates. Therefore, it is important to examine the extent to which latent growth modeling is robust to the model misspecification. However, there are limited simulation studies to examine this issue empirically, especially in the framework of LGM. Furthermore, although most applications have been conducted within the framework of conditional LGM since only conditional LGM provides a platform to test more complex research questions, previous studies about model misspecifications are mainly performed within the framework of unconditional LGM. Whether the results from unconditional LGM could be generalized to conditional LGM is unknown, given the much more complex nature of conditional LGM. Therefore, it is hoped that this study may contribute to the knowledge about the impact of model misspecification in LGM and make the results more generalizable. This study is hoped to result in recommendations to applied researchers in analyzing longitudinal data.

Second, although time series process has been extensively investigated in econometrics, it is still relatively unpopular in structural equation modeling. Due to the unique characteristic of longitudinal data, it is common to identify the presence of time series processes. Although AR process has been well discussed in the context of LGM, up to now, very few studies include a systematic discussion of the AR, MA and ARMA processes at the same time. Furthermore, there is no systematic study conducted to investigate the three unmodeled time series processes in the error structure of conditional LGM. Therefore, this study fills in the framework gap between the methodology issue and applied research, and aims to provide more insightful information to applied researchers.

Third, modeling time series process in latent growth modeling is a relatively new area. There are limited studies trying to integrate the time series process in latent growth modeling.. For one reason or another, some SEM researchers view the analysis of time series as something that uses fundamentally different concept and methods, which inhibits the interchange between the two ways of thinking. They might be mutually productive and beneficial to each other. This study hopes to contribute to the wider understanding and better application of time series analysis in educational and behavioral research.

Research Questions

This study aims to investigate how unmodeled time series processes in the error structure of latent growth curve models affect the parameter estimates and their standard error estimates, as well as model fit indexes and GOF test. The parameters that are of interest are: μ_α and μ_B, (i.e., the mean of the level and mean of the shape controlling all other terms in between-person equation), $\psi_{\alpha\alpha}$ (i.e the residual variance of level equation) , $\psi_{\beta\beta}$ (i.e., the residual variance of shape equation) and $\psi_{\alpha\beta}$ (i.e., the covariance of level and shape residuals) , as well as the following path coefficients depending on different models:

1. The direct effects of time invariant predictors on the level and shape factors in LGM with a time invariant predictors (i.e., γ_α and γ_β in equation 18);

2. The direct effect of time varying predictors on the outcome variables in LGM with a time varying predictors (i.e., γ_t in equation 27).

3. In LGM with a parallel process, the direct effects of the level and shape for the time varying predictor on the level and shape for the outcome variable (i.e., $\gamma_{\alpha 1}, \gamma_{\beta 1}$ and γ_{B2} in equation 32).

The fundamental research question this study aims to address is: are LGM parameter estimates and standard errors affected, when within-person residual covariance structure fails to include the time series process? Other research questions include whether commonly used fit indexes and GOF test can differentiate between two analysis models differing in within-person covariance structures, and whether the parameters and their standard error estimates are affected by design factors.

CHAPTER 3
METHOD

Monte Carlo simulations have been widely used in social science in investigating the possible effect of assumption violation. When an analytical approach is difficult or impossible to implement, Monte Carlo simulation offers researchers an alternative way to address research questions. In this study, the simulation was conducted through software R version 2.7.1 (R Development Core Team, 2008). A total of 5000 replications were simulated for each condition. The models investigated in this study were: 1. LGM with one time invariant covariate. 2. LGM with one time varying covariate. 3. LGM with a parallel process. This method section presents the following contents: (a) the simulation conditions, which include the design factors and the population parameters; (b) the data generation procedure; and (c) the data analysis criteria.

Design Factors

Number of Measurement Times

It is believed that the precision of parameter estimates tends to increase along with the number of observations for each individual (Duncan, et al., 1999). Kwok, et al. (2007) found that among the longitudinal studies published in *Developmental Psychology* in 2002, 52% of these studies collected three or four waves of data while 48% collected 8 waves. Among the 267 peer reviewed journal articles obtained by searching from Academic Search Premier, Business Source Premier, EconLit, Professional Development Collection, PsycINFO, Sociological Collection from 2004 to 2008, the number of waves ranged from three to eight. Around 50% of these studies had three or four waves of data. Around 30% had five waves of data and around 20% had more than six waves of data. For model identification purpose, a minimum of four measurement occasions are required in growth modeling assuming the errors are not identical (Muthén & Khoo, 1998). Hence, four was considered the minimum number of measurement periods in this simulation design. The rationale of using eight waves is that large number of measurement periods would make more obvious the parameter estimates difference if such difference exists. Hence, four waves, and eight waves were used in this study to represent a small, and a large number of repeated measures respectively.

Sample Size

Hamilton, Gagne and Hancock (2003) argued that a sample size between 100 and 200 was the minimum requirement for univariate LGM. It was also recommended by Anderson and Gerbing (1998), and Jackson (2003), that a sample size of 150 or 200 was necessary when maximum likelihood estimation method was used in growth modeling. Fan (2003b) recommended a minimum sample size of 150 for univariate growth modeling, together with 500 and 1000 representing the medium and large sample size respectively. Kline (1998) recommended a ratio of 10:1, that is, for each parameter estimated, there should be 10 observations. In our study, assuming eight waves, when time invariant covariate was included in the model, the total number of parameters was 15. When time varying covariate was included in the model, the total number of parameters was 23. In the model with a parallel process, the total number of parameters was 29. Hence, the sample size in this study should range from 150 to 290 according to the 10:1 ratio rule. Therefore, a sample size of 200, 500 and 2000 was to be simulated to represent a small, medium and large size.

Time Series Parameters

The time series parameters refer to the correlation coefficient in time series model. They are the autocorrelation coefficient ρ in AR (1) model, the MA parameter θ in MA(1) model and ρ and θ in ARMA(1,1) model. In this study, only data following an AR(1) or a MA (1) or an ARMA (1) process were simulated in that (a) the common characteristics of longitudinal data is that the variable is affected most by its immediately preceding variable and (b) assuming the lag one process can free a significant number of degrees of freedom. According to the range of values used in past simulation studies (Ferron, et al, 2002; Hamaker, Dolan, & Molenaar, 2002; Sivo & Willson, 2000), the value of AR/MA correlation coefficient was set as follows:

1. When the within-person residual covariance structure followed an AR (1) process, the AR parameter ρ was set to be 0.8 and 0.5 to represent high and moderate autocorrelation coefficient respectively.

2. When the within-person residual covariance structure followed a MA (1) process, the MA θ was set to be 0.8 and 0.5 to represent high and moderate moving average parameters respectively.

3. When the within-person residual covariance structure followed an ARMA (1, 1) process, the ρ and θ was set to be 0.2 and 0.8, or 0.5 and 0.45 respectively. The values of ρ and θ were chosen to be quite different from one another or quite close to one aother. The rationale for choosing the values is based on the following reasoning (McCleary & Hay, 1980): the ARMA model is an integration of AR model and MA model. If ρ and θ are not equal but are close to each other, the ARMA (1,1) model reduces approximately to an MA (2) model when ρ is not small, and reduces approximately to an MA(1) model when ρ is small. Therefore, it deserves out attention to investigate the results with two different types of ARMA parameter value.

Time Coding

The simulation study assumed linear conditional LGM with equally spaced time intervals for the dependent variable in all models and for the time varying covariate in the LGM with a time varying covariate and in the parallel process LGM. Therefore, the factor loadings for the level were all set equal to 1. The shape loadings were set from 0 to 1, 2, 3, 4… with the base time as the reference point.

Population Values

The population parameters were based on the analysis of data obtained from the Early Childhood Longitudinal Study – Kindergarten Cohort (ECLS-K) and values used in other simulation research. This data set provides descriptive information about the status of children from kindergarten to 8^{th} grade. It is the first large national study that followed a cohort of children from their kindergarten to middle school. Information was collected in a total of seven measurement periods: the fall and the spring of kindergarten (1998-99), the fall and spring of 1st grade (1999-2000), the spring of 3rd grade (2002), 5th grade (2004), and 8th (2007) grade, a total of seven measurement periods. Participants included children's teachers, schools and their parents. Information was collected on a variety of factors such as children's cognitive, social, emotional, and physical development. Its longitudinal nature and multifaceted character enables researchers to conduct various studies based on this data set (e.g., Bodovski & Farkas, 2007; Hong & Raudenbush, 2006; Kaplan, 2005).

The population parameter in this study was obtained by analyzing the following five waves data: The fall and the spring of kindergarten (1998-99), the spring of 1st grade (2000), the spring of 3rd grade (2002) and the spring of 5th grade (2004). The outcome variable was children's math performance in each of the five periods. The time invariant covariate was children's SES measured at kindergarten and the time varying covariate was children's reading score measured at the same time as the math score. As the measurement time in this study could be eight times, the additional population parameters were extrapolated according to the parameters obtained from the above five waves ECLS-K data.

Based on the analysis of the ECLS-K data, the population parameters were defined as follows:

Within-Person Residual Variance σ^2

To make the design simple, the within-person residual variance for the outcome variable measured at different periods was specified to be equal, with a constant value of 50.

Parameter μ_α and μ_B in Between-Person Equation

Parameter μ_α and μ_B refers to the mean of the level and mean of the shape controlling all other terms in between-person equation. Based on the parameter estimates obtained from ECLS-K, the μ_α was set to be 5 and μ_B was set to be 4.

Residual Variance of Level Equation (i.e., $\psi_{\alpha\alpha}$,) , Residual Variance of Shape Equation (i.e., $\psi_{\beta\beta}$), and Covariance of Level and Shape Residuals (i.e., $\psi_{\alpha\beta}$)

The parameter $\zeta_{\alpha i}$ and ζ_{Bi} are the disturbances of level and shape respectively with mean of zero and variances of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$, as well as covariance of $\psi_{\alpha\beta}$. The parameter $\psi_{\alpha\alpha}$ was set to be 80, the parameter $\psi_{\beta\beta}$ was set to be 60 and the parameter $\psi_{\alpha\beta}$ was set to be 35 respectively, based on the analysis of ECLS-K data.

Mean and Variance of Time Invariant Covariate

According to the data analysis of ECLS-K data, the time invariant covariate was generated to follow a normal distribution with a mean of 50 and standard deviation of 10.

Parameters of Time Varying Covariate

The time varying covariate was generated the same way as was the outcome variable. The time varying covariate was assumed to be measured at the same time as the outcome variable. The mean level and mean shape was specified to be 30 and 20 respectively. The $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$, and $\psi_{\alpha\beta}$ for the time varying covariate were 85, 30 and 23, respectively. The trajectory equation residual was assumed to be normally distributed with mean of zero and constant variance of 70.

Effect of Time Invariant Predictor on Latent Level and Latent Shape in Growth Predictor Model (i.e., γ_α and γ_β in Equation 2-18)

The regression coefficients for the time invariant predictor on latent level (i.e., γ_α) and latent shape (i.e., γ_β) were both set to be 0.5.

Effect of the Time Varying Predictor Variable on the Outcome Variable in LGM with a time varying Covariant (i.e., γ_t in Equation 2-27)

To simplify the design, all the regression coefficients between the outcome variable and the time varying covariate were set to be equal to each other. The regression coefficient was set as 0.4.

Effect of the Intercept and Slope of the Predictor on the Intercept and Slope of the Outcome Variable in LGM with a parallel process Model

The effect of latent intercept of the predictor on the intercept of the outcome variable (i.e., $\gamma_{\alpha 1}$ in equation2- 33) was set to be 0.6. The effect of the latent intercept of the predictor variable on latent intercept of the outcome variable (i.e., $\gamma_{\beta 1}$ in equation 2- 33) and the effect of the latent slope of the predictor on the latent slope of the outcome variable (i.e., γ_{B2} in equation 2-33) were set to be 0.5 and 0.6 respectively.

Summary of Population Values

The μ_α and μ_B in the between-person equation were 5 and 4 respectively. The residual variance of level equation (i.e., $\psi_{\alpha\alpha}$), the residual variance of shape equation (i.e., $\psi_{\beta\beta}$) and the covariance of level and shape residuals (i.e., $\psi_{\alpha\beta}$) were 80, 60 and 35 respectively. The within-person residual variance for the outcome variable was 50. These numbers were the same for all three LGMs. Other population values were presented as follows:

LGM with a Time Invariant Covariate

The time invariant covariant was normally distributed with mean of 50 and standard deviation of 10. The effects of the time invariant covariate on the level and shape of the outcome variable were both equal to 0.5.

LGM with a Time Varying Covariate

The effect of the time varying covariate on the outcome variable was 0.4. The mean level and mean shape of the time varying covariate was specified to be 30 and 20 respectively. The $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$, as well as covariance of $\psi_{\alpha\beta}$ of the time varying covariant was set to be 85, 30 and 23 respectively. The trajectory equation residual variance σ^2 was normally distributed with mean of zero and a constant variance of 70.

LGM with a parallel process

The predictor variable (i.e., the time varying predictor) was set the same way as was in LGM with a time varying covariate. That is: the mean level was 30 and the mean shape was 20; the variance of the level was 85 and the variance of the shape was 30; the covariance between the level and the shape was 23; the within-person residual for the predictor variable was normally distributed with mean of zero and a constant variance of 70. The effect of the intercept of the predictor on the intercept of the outcome variable was 0.6 while on the slope of the outcome variable was 0.5. The effect of the slope of the predictor variable on the slope of the outcome variable was 0.6.

Summary of Conditions

The conditions included three different LGMs, three types of within-person covariance structures, three different sample sizes, and two different number of measurement periods. Within each type of residual covariance structure there were two different time series parameters,. Above factors were fully crossed, resulting in a total of 108 (3x3x3x2x2) conditions. The time series parameters, the sample sizes and the measurement periods were design factors. The values of the three design factors are summarized as follows:

1. the sample size (200, 500 and 2000);

2. the length of waves (4 and 8);
3. the time series parameters: (a) the AR parameter ρ (0.8 and 0.5), (b) the MA parameter θ (0.8 and 0.5), and (c) the ARMA parameter ρ and θ (0.2 and 0.8 and 0.5 and 0.45).

Data Generation

The data were generated by R version 2.7.1 (R Development Core Team, 2008). The matrix equation of each LGM was used in the data simulation with population value filled in. The data sets were generated according to different model type and different within-person residual covariance structure. Under a certain kind of LGM, there were three different generating models, which had the same matrix format but differed in the within person residual covariance structure.

When the residuals follow an AR (1) process, that is,

$$\varepsilon_{it} = \rho\varepsilon_{i,t-1} + u_{it}, \qquad (3\text{-}1),$$

where ε_{it} and $\varepsilon_{i,t-1}$ are the within person residuals at time *t* and *t*-1 respectively with zero mean and constant variance σ^2, ρ is the autocorrelation coefficient, and u_{it} is the residual at a give time *t* with $E(u_{it}) = 0$ and $\text{var}(u_{it}) = \sigma_u^2$, the within -person residual covariance matrix is

$$\sigma^2 \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^t \\ \rho & 1 & \rho & \cdots & \rho^{t-1} \\ \rho^2 & \rho & 1 & \cdots & \rho^{t-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho^t & \rho^{t-1} & \rho^{t-2} & \cdots & 1 \end{bmatrix}, \qquad (3\text{-}2)$$

where σ^2 was equal to 50 as described before. The ρ was set equal to 0.8 or 0.5.

When the residuals followed an MA (1) process, that is,

$$\varepsilon_{it} = u_{it} - \theta u_{i,t-1}, \qquad (3\text{-}3)$$

where ε_{it} is still the within-person residual at time t, θ denotes the moving average parameter and $\{u_{it}\}$ is a zero mean series with constant variance σ_u^2. The within -person residual covariance matrix is

$$\sigma^2 \begin{bmatrix} 1 & -\frac{\theta}{1+\theta^2} & 0 & \cdots & 0 \\ -\frac{\theta}{1+\theta^2} & 1 & -\frac{\theta}{1+\theta^2} & \cdots & 0 \\ 0 & -\frac{\theta}{1+\theta^2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \quad (3\text{-}4)$$

The σ^2 was set to be 50 and the θ was set equal to 0.8 or 0.5

When the residual followed an ARMA (1, 1) process,

$$\varepsilon_{it} = \rho\varepsilon_{i,t-1} + u_{it} - \theta u_{i,t-1}, \quad (3\text{-}5)$$

all the terms are defined the same as in the AR (1) and MA (1) models. The accompanied within-person covariance structure is

$$\sigma^2 \frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2} \begin{bmatrix} \frac{1-2\rho\theta+\theta^2}{(\rho-\theta)(1-\rho\theta)} & 1 & \rho & \cdots & \rho^{t-2} \\ 1 & \frac{1-2\rho\theta+\theta^2}{(\rho-\theta)(1-\rho\theta)} & 1 & \cdots & \rho^{t-3} \\ \rho & \rho^2 & \frac{1-2\rho\theta+\theta^2}{(\rho-\theta)(1-\rho\theta)} & \cdots & \rho^{t-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{t-2} & \rho^{t-3} & \rho^{t-4} & \cdots & \frac{1-2\rho\theta+\theta^2}{(\rho-\theta)(1-\rho\theta)} \end{bmatrix}. \quad (3\text{-}6)$$

The within person residual σ^2 was 50. The ARMA parameter ρ and θ was set to be 0.2 and 0.8, or 0.5. and 0.45.

The data sets were generated according to the three LGMs and three different within person residual structures. A total of 9 (3x3) generating models with different values of design factors were formulated. Under each condition, a data set was generated and analyzed by an incorrect analysis model. Then another data set with the same condition was generated and then analyzed by a correct analysis model. The incorrect analysis model failed to consider the time series process in within- person level residual structure. The within-person residual covariance matrix under the incorrect analysis model was a diagonal matrix with non-constant variance. The within person residual covariance matrix under the correct analysis model was the same as in the above generating model. In Mplus 5.2, the default estimation covariance structure assumes uncorrelated errors. The time series process in the residual structure was modeled using constraint command.

A total of 5000 replications were simulated for each of 108 conditions. As two data sets were generated under the same condition for two analysis models, a total of 1,080,000 (5000x108x2) datasets were generated. The data were simulated in R and were saved to disk. The Mplus software then was used to fit the models on the generated data. Under each condition, the 5000 replications did not all converged. Therefore the non convergence rate was calculated and extra data sets were further simulated and analyzed until 5000 converged results were obtained. The Mplus output all parameter estimates, the accompanied standard error estimates, fit index, as well as warning messages. These outputs were saved for later analysis.

Data Analysis

There are several criteria to evaluate the performance of the latent growth models when the residual covariance structure is misspecified. In this simulation study, a high convergence rate was expected for estimating the LGMs. However, it is very likely that estimations will not converge for all replications of all conditions. Therefore, the convergence rate was calculated for each condition. Moreover, Mplus provides warning message regarding the occurrence of a non-positive definite latent variable covariance matrix. The occurrence of a non-positive definite matrix indicates "improper solutions", which includes a negative variance for a latent variable, a correlation greater than or equal to one between two latent variables. The percentage of occurrence of the non-positive definite matrix under each of 5000 replications was reported.

To evaluate the performance of the models with unmodeled time series process, the relative parameter bias and relative standard error bias were calculated. Relative bias was calculated for both parameter estimates and standard error estimates. The relative parameter bias was calculated by using the following formula (Hoogland & Boomsma, 1998):

$$B\left(\hat{\theta}\right)=\frac{Mean(\hat{\theta}_{ij})-\theta}{\theta}, \tag{3-7}$$

where $\hat{\theta}_{ij}$ is the average parameter estimate obtained for replication *i* of condition *j*, $Mean(\hat{\theta}_{ij})$ is the mean of estimates of $\hat{\theta}_{ij}$ under condition *j*, and θ is the population parameter.

To evaluate the standard errors, the estimated values were compared with empirical standard errors, which were obtained by computing the standard deviation of the parameter estimates from all the simulated datasets in a condition. The relative standard error bias was calculated using the following equation:

$$B\left(S_{\hat{\theta}_{ij}}\right)=\frac{\bar{SE}\left(\hat{\theta}_{ij}\right)-SD\left(\hat{\theta}_{ij}\right)}{SD\left(\hat{\theta}_{ij}\right)}, \tag{3-8}$$

where $\bar{SE}\left(\hat{\theta}_{ij}\right)$ is the average estimated standard error for $\hat{\theta}_{ij}$ across all 5000 replications under condition *j*, and $SD\left(\widehat{\theta_{ij}}\right)$ is the empirical standard error, calculated as the standard deviation of the 5000 estimates of $\hat{\theta}$ under condition *j*. According to Hoogland and Boomsma (1998), the acceptable cut off values for the relative parameter bias and relative standard error bias are 0.05 and 0.1 respectively. Values beyond this range would be considered unacceptable.

Results for chi-square goodness of fit (GOF) test were also reported. The percentage of *p* value that was below 0.05 under each selected condition would be reported. In SEM, a *p* value equal to or greater than 0.05 indicates adequate model fit. It is expected that *p* value is sensitive to model misspecification. That is, with the correct analysis model, *p* value should be at least equal to 0.05, and with the incorrect analysis model, *p* value is less than 0.05.

There exist many fit indexes in SEM, which are important criteria in evaluating whether the model fits the data adequately. In this study four commonly used fit indexes were selected for evaluation: the comparative fit index (CFI), the Tucker-Lewis index (TLI), the standardized root mean square residual (SRMR) and the root mean-square error of approximation (RMSEA). The criterion that suggests adequate model fit for each of the four fit index are (Hu & Bentler, 1999): CFI is greater than .95, TLI is greater than 0.95, SRMR is less than 0.08 and RMSEA is less than 0.06. In the results section, the percentage of replications that met each of the four criteria would be presented. That is, for each criterion and condition, the percentage of replications in which the criterion was met was calculated. It should be mentioned that the criteria used to suggest adequate model fit are not unique and are always controversial in the literature review (Marsh, Hau, & Grayson, 2005). The criteria used in this study were chosen simply to allow the examination of the effect of model misspecification.

Other model selection criteria such as Akaike's Information Criterion (AIC) or Schwartz's Bayesian Criterion (SBC) were not compared here. It has been shown in previous literature that these criteria do not always lead to the correct selection of the covariance structure (e.g. Keselman, Algina, Kowalchuk, & Wolfinger, 1998; Ferron et al., 2002).

CHAPTER 4
RESULTS

This chapter is composed of six sections. The first section presents the convergence rate and the occurrence rate for non-positive definite matrices. The second section through the sixth section reports results for fixed parameter estimates, standard error of fixed parameter estimates, variance components estimates, standard error of variance components estimates, and chi-square GOF test and GOF indexes, in this order. In the two sections for fixed parameter estimates and standard error estimates of fixed parameter, results are presented with latent growth model (LGM) with a time invariant covariate the first, with LGM with a time varying covariate the second, and with LGM with parallel process the third. From section four (variance components estimates) to section six (GOF test and GOF indexes), results are presented according to different within-person covariance structures, with an AR (1) error structure the first, a MA (1) error structure the second, and an ARMA (1, 1) error structure the last. Except for section one, a summary is presented at the end of each section for easy understanding.

Tables of relative biases are displayed according to combinations of conditions that show differences in the acceptability of the relative biases. The combinations of conditions are based on the following factors: sample size, time series parameters, analysis model type and number of waves. The four factors were fully crossed, resulting in a total number of 24 conditions (3x2x2x2). If there was any unacceptable bias under each of the 24 conditions, all the mean relative biases under the 24 conditions were reported. If the relative biases were all acceptable, the marginal mean relative biases aggregated under analysis model type would be reported.

Convergence Rate and Non-Positive Definite Covariance Matrix Occurrence Rate

The convergence rate for all misspecified analysis models was 100% for all LGMs (see table 4-1). The convergence rate for the correct analysis model depended on the within-person covariance structure and the four factors. With the correct analysis model, the convergence rate with a MA (1) error structure was the highest, with an ARMA (1, 1) the lowest, and with an AR (1) in between. This is the expected result as the ARMA (1, 1) covariance structure was the most complex, and the MA (1) error structure was the least complex among the three. The correct analysis models with a MA (1) error structure led to a convergence rate more than 99%. For the correct analysis model with an AR (1) structure, the convergence rate ranged from 74% to 87%, under conditions in which number of waves was four, the AR parameter was 0.8 and the sample size was 200 or 500; the convergence rate under all other conditions was more than 97%. With an AR (1) error structure, more measurement periods, a larger sample size, or a smaller AR parameter resulted in more converged solutions, holding other conditions constant. The analysis model with an ARMA (1, 1) error structure caused less convergence rate than with an AR (1) or a MA(1) error structure, especially under the conditions in which the ARMA parameter value was equal to 0.5 and 0.45, where the convergence rate ranged from 52% to 78%. With an ARMA (1, 1) error structure, a larger sample size did not necessarily lead to higher convergence rate but more measurement periods did, with other condition fixed. The convergence rate did not differ much across the three kinds of LGMs under the same condition.

Table 4-1. Convergence rate for all conditions

				LGM 1	LGM 2	LGM 3	LGM 1	LGM 2	LGM 3
Model		Size	Parameter		Wave = 4			Wave = 8	
Incorrect	AR(1)	200	0.5	100%	100%	100%	100%	100%	100%
		500	0.5	100%	100%	100%	100%	100%	100%
		2000	0.5	100%	100%	100%	100%	100%	100%
		200	0.8	100%	100%	100%	100%	100%	100%
		500	0.8	100%	100%	100%	100%	100%	100%
		2000	0.8	100%	100%	100%	100%	100%	100%
	MA(1)	200	0.5	100%	100%	100%	100%	100%	100%
		500	0.5	100%	100%	100%	100%	100%	100%
		2000	0.5	100%	100%	100%	100%	100%	100%

Table4-1. Continued.

Model		Size	Parameter	LGM 1	LGM 2	LGM 3	LGM 1	LGM 2	LGM 3
				Wave = 4			Wave = 8		
		200	0.8	100%	100%	100%	100%	100%	100%
		500	0.8	100%	100%	100%	100%	100%	100%
		2000	0.8	100%	100%	100%	100%	100%	100%
	ARMA	200	0.2, 0.8	100%	100%	100%	100%	100%	100%
	(1,1)	500	0.2, 0.8	100%	100%	100%	100%	100%	100%
		2000	0.2, 0.8	100%	100%	100%	100%	100%	100%
		200	0.5, 0.45	100%	100%	100%	100%	100%	100%
		500	0.5, 0.45	100%	100%	100%	100%	100%	100%
		2000	0.5, 0.45	100%	100%	100%	100%	100%	100%
Correct	AR(1)	200	0.5	99%	98%	98%	100%	100%	100%
		500	0.5	100%	100%	100%	100%	100%	100%
		2000	0.5	100%	100%	100%	100%	100%	100%
		200	0.8	76%	76%	74%	97%	97%	97%
		500	0.8	85%	87%	84%	100%	100%	100%
		2000	0.8	98%	98%	98%	100%	100%	100%
	MA(1)	200	0.5	99%	100%	100%	100%	100%	100%
		500	0.5	99%	100%	100%	100%	100%	100%
		2000	0.5	100%	100%	100%	100%	100%	100%
		200	0.8	99%	100%	100%	100%	100%	100%
		500	0.8	100%	100%	100%	100%	100%	100%
		2000	0.8	100%	100%	100%	100%	100%	100%
	ARMA	200	0.2, 0.8	85%	88%	85%	97%	97%	95%
	(1,1)	500	0.2, 0.8	90%	90%	87%	98%	97%	95%
		2000	0.2, 0.8	95%	91%	86%	97%	97%	94%
		200	0.5, 0.45	59%	63%	65%	67%	64%	73%
		500	0.5, 0.45	58%	60%	62%	66%	67%	72%
		2000	0.5, 0.45	56%	56%	52%	72%	78%	74%

Note: LGM 1, LGM 2 and LGM 3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Parameter refers to the time series parameter.

The occurrence rate of non-positive definite covariance matrices in each condition is presented in Table 4-2. Provided that the number of measurement waves was eight, failing to include the time series process in the analysis model did not result in any occurrences of non-positive definite matrices. When the number of waves was four and the time series process was not included in the analysis model, occurrence rates depended on the model used to generate the data. When the generating model was an AR (1), the occurrence rate was zero except when the sample size was 200 and the parameter vale was 0.8. Even then the occurrence rate did not exceed 2%. When the generating model was a MA (1), the occurrence rate was at least 20% when the parameter value was 0.8 but 24% or less when the parameter value was 0.5. In both cases occurrence of non-positive definite matrices decreased as the sample size increased. When the generating model was an ARMA (1, 1) there were non non-positive define matrices when the parameter values $(\rho \text{ and } \theta)$ were 0.5 and 0.45, respectively; when the parameter values were 0.2 and 0.8, non occurrence rates were less than 19% and declined as the sample size increased. When the analysis model was correct non-occurrence rates again depended on the time-series process. With the MA (1) model the occurrence rate was zero except in one condition: parallel process model, sample size of 200 and a parameter value of .8. Even then the occurrence rate was only 1%. For the AR (1) model the occurrence rates were less than 18% and were smaller when the number of waves was eight and the parameter value was 0.5, and tended to decline as the sample size increased. For the ARMA (1, 1) model occurrence rates were less than 20% and were smaller when the number of waves was larger and the parameter values $(\rho \text{ and } \theta)$ were 0.2 and 0.8, and the sample size was larger. Similar to the results for the convergence rate, the occurrence rate did not differ much across the three LGMs under the same condition.

Table 4-2. Rate of occurrence of non-positive definite matrix under all conditions

			LGM 1	LGM 2	LGM 3	LGM 1	LGM 2	LGM 3
Model	Size	Parameter		Wave = 4			Wave = 8	
Incorrect AR(1)	200	0.5	0%	0%	0%	0%	0%	0%
	500	0.5	0%	0%	0%	0%	0%	0%
	2000	0.5	0%	0%	0%	0%	0%	0%
	200	0.8	1%	1%	2%	0%	0%	0%
	500	0.8	0%	0%	0%	0%	0%	0%

Table 4-2. Continued.

Model		Size	Parameter	LGM 1	LGM 2	LGM 3	LGM 1	LGM 2	LGM 3
					Wave = 4			Wave = 8	
		2000	0.8	0%	0%	0%	0%	0%	0%
	MA(1)	200	0.5	20%	24%	22%	0%	0%	0%
		500	0.5	8%	8%	9%	0%	0%	0%
		2000	0.5	0%	0%	0%	0%	0%	0%
		200	0.8	43%	46%	44%	0%	0%	0%
		500	0.8	36%	37%	35%	0%	0%	0%
		2000	0.8	21%	21%	20%	0%	0%	0%
	ARMA	200	0.2, 0.8	15%	18%	16%	0%	0%	0%
	(1,1)	500	0.2, 0.8	5%	5%	5%	0%	0%	0%
		2000	0.2, 0.8	0%	0%	0%	0%	0%	0%
		200	0.5, 0.45	0%	0%	0%	0%	0%	0%
		500	0.5, 0.45	0%	0%	0%	0%	0%	0%
		2000	0.5, 0.45	0%	0%	0%	0%	0%	0%
Correct	AR(1)	200	0.5	13%	14%	14%	0%	0%	0%
		500	0.5	5%	4%	5%	0%	0%	0%
		2000	0.5	0%	0%	0%	0%	0%	0%
		200	0.8	16%	15%	15%	10%	11%	10%
		500	0.8	16%	17%	17%	3%	3%	3%
		2000	0.8	11%	12%	11%	0%	0%	0%
	MA(1)	200	0.5	0%	0%	0%	0%	0%	0%
		500	0.5	0%	0%	0%	0%	0%	0%
		2000	0.5	0%	0%	0%	0%	0%	0%
		200	0.8	0%	1%	0%	0%	0%	0%
		500	0.8	0%	0%	0%	0%	0%	0%
		2000	0.8	0%	0%	0%	0%	0%	0%
	ARMA	200	0.2, 0.8	5%	6%	7%	1%	0%	1%
	(1,1)	500	0.2, 0.8	1%	0%	1%	0%	0%	0%
		2000	0.2, 0.8	0%	0%	0%	0%	0%	0%
		200	0.5, 0.45	15%	13%	20%	6%	12%	14%
		500	0.5, 0.45	9%	8%	14%	3%	7%	9%
		2000	0.5, 0.45	5%	6%	10%	1%	2%	3%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Parameter refers to the time series parameter.

Fixed Parameter Estimates

LGM with a Time Invariant Covariate

The fixed parameters in LGM with a time invariant covariate refer to μ_α and μ_B (i.e., the mean of the level and mean of the shape in the between-person equation) and the direct effect of the time invariant predictors on the level and the shape factors (i.e., γ_α and γ_β in equation 2-19).

With each of the three kinds of residual covariance structures, the relative biases under all 24 conditions were acceptable. Therefore, only the marginal mean relative biases aggregated under the analysis model type are reported. Results in Table 4-3 indicate that regardless of the residual covariance structures, all the relative biases were trivial, ranging from -.006 to .003. The relative biases for γ_α and γ_β were almost zero, indicating that estimates of these two parameters were quite close to their respective population values.

Table 4-3. Marginal mean relative biases of fixed parameter estimates for LGM with a time invariant covariate

	Model	μ_α	μ_β	γ_α	γ_β
AR(1)	Incorrect	-0.004	-0.003	0.000	0.000
	Correct	-0.006	-0.001	0.000	0.000
MA(1)	Incorrect	0.003	0.000	0.000	0.000
	Correct	0.000	-0.001	0.000	0.000
ARMA(1,1)	Incorrect	-0.003	0.000	0.000	0.000
	Correct	-0.002	0.003	0.000	0.000

LGM with a Time Varying Covariate

The fixed parameters in the LGM with a time varying covariate refer to μ_α and μ_B, (i.e., the mean of the level and mean of the shape in the between-person equation), and the direct effect of a time varying predictor on outcome variables (i.e., γ_t in equation 2-28).

With each of the three kinds of residual covariance structures, the relative biases under each of the 24 conditions were acceptable. All the marginal mean relative biases were trivial, ranging from -0.001 to 0 (see Table 4-4). Most marginal mean relative biases were zero, indicating that the parameter estimates were quite close to their population values.

Table 4-4. Mean relative biases of fixed parameter estimates for LGM with a time varying covariate

	Model	μ_α	μ_B	γ_t
AR(1)	Incorrect	0.000	0.000	0.000
	Correct	-0.001	0.000	0.000
MA(1)	Incorrect	0.000	0.000	0.000
	Correct	-0.001	0.000	0.000
ARMA(1,1)	Incorrect	0.001	0.000	0.000
	Correct	0.000	0.000	0.000

LGM with a parallel process

The fixed parameters in LGM with a time varying covariate are μ_α and μ_B, (i.e., the mean of the level and mean of the shape controlling all other terms in the between-person equation), and the direct effects of the level and the shape of the time varying predictor on the level and shape for the outcome variable respectively (i.e., $\gamma_{\alpha 1}, \gamma_{\beta 1}$ and γ_{B2} in equation 2-33).

The relative bias under each of the 24 conditions was less than 0.05, and therefore only the marginal mean relative biases are reported. Results in Table 4-5 indicate that all the relative biases were acceptable. The absolute marginal mean relative biases of either μ_α or μ_B were larger than those of $\gamma_{\alpha 1}, \gamma_{\beta 1}$ and γ_{B2}. 75% of the absolute marginal mean relative biases for parameter μ_α or μ_B were greater than .01 while the marginal mean relative biases for the three regression coefficients (i.e., $\gamma_{\alpha 1}, \gamma_{\beta 1}$ and γ_{B2}) were trivial, ranging from -0.003 to 0.009.

Table 4-5. Mean relative biases of fixed parameter estimates for LGM with a parallel process

	Model	μ_α	μ_B	$\gamma_{\alpha 1}$	$\gamma_{\beta 1}$	γ_{B2}
AR(1)	Incorrect	-0.011	-0.009	0.003	0.005	-0.003
	Correct	-0.014	-0.012	0.004	0.005	-0.003
MA(1)	Incorrect	-0.012	-0.011	0.003	0.004	-0.002
	Correct	-0.008	-0.005	0.002	0.002	-0.001
ARMA (1,1)	Incorrect	-0.011	-0.013	0.003	0.005	-0.002
	Correct	-0.018	-0.015	0.009	0.008	0.000

Standard Error of the Fixed Parameter Estimates

LGM with a Time Invariant Covariate

The relative biases under each of the 24 conditions were less than 0.1 and therefore were all acceptable. Accordingly only the marginal mean relative biases were reported.

Under each of the three covariance structures, the marginal mean relative biases of standard error of the fixed parameter estimates were trivial, ranging from -0.012 to 0.003 (see Table 4-6). There were only 2 out of the 24 mean relative biases with absolute values great than 0.01, indicating that the standard error estimates of the fixed parameter were close to their respective empirical standard errors.

Table 4-6. Marginal mean relative biases of standard error estimates of fixed parameters for LGM with a time invariant covariate

	Model	μ_α	μ_β	γ_α	γ_β
AR(1)	Incorrect	-0.001	-0.011	0.000	-0.012
	Correct	-0.002	-0.002	-0.002	-0.002
MA(1)	Incorrect	-0.002	-0.002	-0.001	-0.003
	Correct	-0.004	0.003	-0.004	0.003
ARMA(1,1)	Incorrect	-0.003	-0.004	-0.003	-0.004
	Correct	-0.008	-0.006	-0.007	-0.004

LGM with a Time Varying Covariate

With each of the three residual covariance structures, all the absolute relative biases of the standard error estimates of the fixed parameter were less than 0.1, and therefore were all acceptable. The range of the marginal mean relative biases was from -0.009 to 0.002, indicating that the estimated standard errors were quite close to their respective empirical standard errors (see Table 4-7).

Table 4-7. Marginal mean relative biases of standard error estimates of fixed parameters for LGM with a time varying covariate

	Model	μ_α	μ_B	γ_t
AR(1)	Incorrect	-0.006	-0.009	0.002
	Correct	-0.008	-0.01	0.000
MA(1)	Incorrect	-0.002	-0.007	-0.001
	Correct	-0.001	0.000	0.001
ARMA(1,1)	Incorrect	-0.003	-0.006	0.001
	Correct	-0.005	-0.004	-0.006

LGM with a parallel process

The relative biases under each of the 24 conditions were all acceptable and therefore only the marginal mean relative biases were reported. Results in Table 4-8 indicate that the range of the marginal mean relative biases was from -0.018 to 0.003, with only 6 out 30 (20%) absolute marginal mean biases larger than 0.01. Based on the 0.10 criterion for the relative bias of standard error estimates, all the marginal mean relative biases were trivial, indicating that the estimated standard errors were quite close to their respective empirical standard errors.

Table 4-8. Marginal mean relative biases of standard error estimates of fixed parameters for LGM with a parallel process

	Model	μ_α	μ_B	$\gamma_{\alpha 1}$	$\gamma_{\beta 1}$	γ_{B2}
AR(1)	Incorrect	-0.007	0.003	-0.007	-0.002	0.000
	Correct	-0.004	-0.005	-0.004	-0.018	-0.016
MA(1)	Incorrect	-0.01	-0.003	-0.01	-0.012	-0.016
	Correct	-0.002	0.000	-0.004	-0.009	-0.009
ARMA(1,1)	Incorrect	-0.007	0.003	-0.007	-0.002	0.000
	Correct	-0.004	-0.005	-0.004	-0.018	-0.016

Summary of the Results for the Fixed Parameter Estimates together with Standard Error Estimates

When the generating model included each of the three types of within-person residual covariance structures, the relative biases of each fixed parameter estimates and the standard error estimates were acceptable under all conditions. None of the four factors had an impact on the acceptability of these biases. Many of the biases were trivial, indicating that even when the within-person covariance structure was misspecified, the estimation of the fixed parameters or tests of the fixed effects were not affected.

Variance Component Parameter Estimates

The variance components refer to the residual variance of the latent level (i.e., $\psi_{\alpha\alpha}$), the residual variance of the latent slope (i.e., $\psi_{\beta\beta}$) , as well as the covariance between the residual of latent intercept and the residual of latent slope (i.e., $\psi_{\alpha\beta}$). Beginning with the present section, the results are organized by the within-person residual covariance structures rather than by the LGM. Under each type of residual covariance structure, the results for $\psi_{\alpha\alpha}$ are presented first, followed by the results for $\psi_{\beta\beta}$, and the results for $\psi_{\alpha\beta}$ are presented the last.

AR (1) Within-Person Residual Covariance Matrix

When the generating model was an AR (1), not all the relative biases were acceptable, therefore the relative biases under each of the 24 conditions are reported (see Table 4-9). When the analysis model failed to include the AR (1) time series process, all the relative biases of $\psi_{\alpha\alpha}$ were inflated and none of the biases were acceptable. However, the analysis model that included the AR (1) process also resulted in some unacceptable negative biases. These unacceptable biases were observed under conditions in which the AR parameter value was 0.5, the number waves was four and the sample size was 200 or 500, or the AR parameter was 0.8, but excluding the condition when the number of waves was eight and the sample size was 2000. The magnitudes of these unacceptable biases were smaller than those obtained with the incorrect analysis model holding other factors constant.

Table 4-9. Mean relative biases of $\psi_{\alpha\alpha}$ estimates for three LGMs with an AR (1) within-person residual covariance matrix

Model	AR	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	**0.379**	**0.390**	**0.370**
	0.5	4	500	**0.387**	**0.390**	**0.380**
	0.5	4	2000	**0.390**	**0.390**	**0.390**
	0.5	8	200	**0.353**	**0.330**	**0.340**
	0.5	8	500	**0.360**	**0.330**	**0.360**
	0.5	8	2000	**0.363**	**0.340**	**0.370**
	0.8	4	200	**0.550**	**0.550**	**0.540**
	0.8	4	500	**0.553**	**0.560**	**0.550**
	0.8	4	2000	**0.559**	**0.560**	**0.560**
	0.8	8	200	**0.610**	**0.570**	**0.610**
	0.8	8	500	**0.620**	**0.580**	**0.620**
	0.8	8	2000	**0.624**	**0.580**	**0.620**
Correct	0.5	4	200	**-0.327**	**-0.280**	**-0.320**
	0.5	4	500	**-0.088**	**-0.070**	**-0.090**
	0.5	4	2000	-0.015	-0.010	-0.020
	0.5	8	200	-0.022	-0.010	-0.020
	0.5	8	500	-0.009	-0.010	-0.010
	0.5	8	2000	-0.001	0.000	0.000
	0.8	4	200	**-0.276**	**-0.250**	**-0.280**
	0.8	4	500	**-0.374**	**-0.380**	**-0.380**
	0.8	4	2000	**-0.232**	**-0.230**	**-0.240**
	0.8	8	200	**-0.248**	**-0.230**	**-0.250**
	0.8	8	500	**-0.081**	**-0.070**	**-0.080**
	0.8	8	2000	-0.013	-0.010	-0.020

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

Holding other factors constant, a higher value of AR parameter tended to result in larger biases than a lower value of AR parameter. The three LGMs did not differ much in the estimates of $\psi_{\alpha\alpha}$ in terms of the number of unacceptable mean relative biases and the magnitudes of these biases.

For the estimates of $\psi_{\beta\beta}$, results in Table 4-10 indicate that when the correct analysis model was used, all the mean relative biases of $\psi_{\beta\beta}$ were acceptable. When the incorrect analysis model was used, only biases observed with eight waves were acceptable, while none of the biases obtained with four waves was acceptable and these unacceptable biases were inflated. These unacceptable biases increased as the sample size increased. Similar to the results for $\psi_{\alpha\alpha}$, the biases of estimates of $\psi_{\beta\beta}$ did not differ much across the three LGMs.

Table 4-10. Mean relative biases of $\psi_{\beta\beta}$ estimates for three LGMs with an AR (1) within-person residual covariance matrix

Model	AR	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	**0.071**	**0.079**	**0.052**
	0.5	4	500	**0.081**	**0.083**	**0.075**
	0.5	4	2000	**0.083**	**0.085**	**0.083**
	0.5	8	200	0.017	0.016	0.004
	0.5	8	500	0.020	0.019	0.015
	0.5	8	2000	0.023	0.020	0.021
	0.8	4	200	**0.064**	**0.069**	**0.046**
	0.8	4	500	**0.069**	**0.072**	**0.065**
	0.8	4	2000	**0.074**	**0.073**	**0.072**
	0.8	8	200	0.018	0.022	0.011
	0.8	8	500	0.024	0.021	0.020
	0.8	8	2000	0.027	0.024	0.026
Correct	0.5	4	200	-0.02	-0.009	-0.039
	0.5	4	500	-0.009	-0.004	-0.017
	0.5	4	2000	-0.002	-0.001	-0.004
	0.5	8	200	-0.013	-0.003	-0.019
	0.5	8	500	-0.004	-0.003	-0.009
	0.5	8	2000	-0.001	0.000	-0.002
	0.8	4	200	0.001	0.003	-0.021
	0.8	4	500	0.001	0.002	-0.008
	0.8	4	2000	-0.001	-0.002	-0.003
	0.8	8	200	-0.011	-0.004	-0.018

Table 4-10. Continued.

Model	AR	Wave	Size	LGM 1	LGM 2	LGM 3
	0.8	8	500	-0.005	-0.002	-0.009
	0.8	8	2000	-0.002	0.000	-0.002

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

Results in Table 4-11 indicate that under each of the three LGMs, the incorrect analysis model resulted in unacceptable and negative biases of the estimates of $\psi_{\alpha\beta}$ while the correct

Table 4-11. Mean relative biases of $\psi_{\alpha\beta}$ estimates for three LGMs with an AR (1) within-person residual covariance matrix

Model	AR	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	**-0.213**	**-0.207**	**-0.215**
	0.5	4	500	**-0.205**	**-0.207**	**-0.206**
	0.5	4	2000	**-0.204**	**-0.205**	**-0.204**
	0.5	8	200	**-0.148**	**-0.124**	**-0.154**
	0.5	8	500	**-0.146**	**-0.127**	**-0.147**
	0.5	8	2000	**-0.143**	**-0.125**	**-0.142**
	0.8	4	200	**-0.158**	**-0.158**	**-0.165**
	0.8	4	500	**-0.158**	**-0.157**	**-0.154**
	0.8	4	2000	**-0.152**	**-0.153**	**-0.155**
	0.8	8	200	**-0.176**	**-0.149**	**-0.177**
	0.8	8	500	**-0.175**	**-0.147**	**-0.176**
	0.8	8	2000	**-0.171**	**-0.146**	**-0.172**
Correct	0.5	4	200	0.014	0.019	0.008
	0.5	4	500	0.002	0.004	0.002
	0.5	4	2000	0.002	0.003	0.001
	0.5	8	200	-0.006	-0.004	-0.007
	0.5	8	500	-0.002	0.000	-0.004
	0.5	8	2000	0.000	0.000	0.000
	0.8	4	200	-0.035	-0.035	-0.044
	0.8	4	500	-0.014	-0.013	-0.013
	0.8	4	2000	0.000	-0.001	-0.001
	0.8	8	200	-0.005	0.005	-0.010
	0.8	8	500	-0.001	0.001	0.000
	0.8	8	2000	-0.001	0.001	0.000

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

analysis model did not result in any unacceptable mean relative biases. The pattern and the magnitudes of mean relative biases did not differ much across the three LGMs.

MA (1) Within-Person Residual Covariance Matrix

Results in Table 4-12 indicate that all the relative biases of $\psi_{\alpha\alpha}$ estimates obtained with the incorrect analysis model were unacceptable and were negatively biased. The relative biases obtained with the correct analysis model were all acceptable, except those obtained under conditions in which the number of waves was four, the sample size was 200 and the MA parameter was 0.8. The biases obtained with the misspecified analysis model were larger than those obtained with the correct analysis model and the estimates observed with the four waves were more negatively biased than with the eight waves.

Table 4-12. Mean relative biases of $\psi_{\alpha\alpha}$ estimates for three LGMs with a MA (1) within-person residual covariance matrix

Model	MA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	**-0.362**	**-0.362**	**-0.368**
	0.5	4	500	**-0.357**	**-0.357**	**-0.360**
	0.5	4	2000	**-0.353**	**-0.353**	**-0.353**
	0.5	8	200	**-0.199**	**-0.199**	**-0.200**
	0.5	8	500	**-0.193**	**-0.193**	**-0.193**
	0.5	8	2000	**-0.188**	**-0.188**	**-0.188**
	0.8	4	200	**-0.448**	**-0.448**	**-0.449**
	0.8	4	500	**-0.439**	**-0.439**	**-0.440**
	0.8	4	2000	**-0.436**	**-0.436**	**-0.436**
	0.8	8	200	**-0.237**	**-0.237**	**-0.24**
	0.8	8	500	**-0.231**	**-0.231**	**-0.232**
	0.8	8	2000	**-0.228**	**-0.228**	**-0.227**
Correct	0.5	4	200	-0.047	-0.047	-0.042
	0.5	4	500	-0.02	-0.02	-0.019
	0.5	4	2000	-0.004	-0.004	-0.004
	0.5	8	200	-0.009	-0.009	-0.014
	0.5	8	500	-0.002	-0.002	-0.006
	0.5	8	2000	-0.001	-0.001	-0.003
	0.8	4	200	**-0.068**	**-0.068**	**-0.066**

Table 4-12. Continued.

Model	MA	Wave	Size	LGM 1	LGM 2	LGM 3
	0.8	4	500	-0.037	-0.037	-0.037
	0.8	4	2000	-0.016	-0.016	-0.015
	0.8	8	200	-0.016	-0.016	-0.018
	0.8	8	500	-0.006	-0.006	-0.009
	0.8	8	2000	-0.001	-0.001	-0.001

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

The relative biases obtained with the correct analysis model were all acceptable while the biases of $\psi_{\beta\beta}$ observed with the incorrect analysis model and four waves were unacceptable (see Table 4-13). These unacceptable biases were all negative.

Table 4-13. Mean relative biases of $\psi_{\beta\beta}$ estimates for three LGMs with a MA (1) within-person residual covariance matrix

Model	MA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	**-0.093**	**-0.093**	**-0.109**
	0.5	4	500	**-0.086**	**-0.086**	**-0.093**
	0.5	4	2000	**-0.083**	**-0.083**	**-0.083**
	0.5	8	200	-0.019	-0.019	-0.030
	0.5	8	500	-0.016	-0.016	-0.017
	0.5	8	2000	-0.012	-0.012	-0.013
	0.8	4	200	**-0.111**	**-0.111**	**-0.131**
	0.8	4	500	**-0.107**	**-0.107**	**-0.113**
	0.8	4	2000	**-0.105**	**-0.105**	**-0.105**
	0.8	8	200	-0.021	-0.021	-0.032
	0.8	8	500	-0.017	-0.017	-0.020
	0.8	8	2000	-0.013	-0.013	-0.014
Correct	0.5	4	200	-0.022	-0.022	-0.038
	0.5	4	500	-0.011	-0.011	-0.016
	0.5	4	2000	-0.002	-0.002	-0.004
	0.5	8	200	-0.009	-0.009	-0.017
	0.5	8	500	-0.003	-0.003	-0.007
	0.5	8	2000	-0.001	-0.001	-0.003
	0.8	4	200	-0.029	-0.029	-0.046
	0.8	4	500	-0.015	-0.015	-0.019
	0.8	4	2000	-0.005	-0.005	-0.006
	0.8	8	200	-0.006	-0.006	-0.018

Table 4-13. Continued.

Model	MA	Wave	Size	LGM 1	LGM 2	LGM 3
	0.8	8	500	-0.003	-0.003	-0.007
	0.8	8	2000	0.000	0.000	-0.002

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

All the biases of $\psi_{\alpha\beta}$ obtained with the incorrect analysis model were unacceptable and positive, while all the biases of $\psi_{\alpha\beta}$ obtained with the correct analysis model were acceptable (See Table 4-14). With the incorrect analysis model, the estimates observed with four waves were more positively biased than those with eight waves and the biases tended to increase with the increase of the sample size.

Table 4-14. Mean relative biases of $\psi_{\alpha\beta}$ estimates for three LGMs with a MA (1) within-person residual covariance matrix

Model	MA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	**0.255**	**0.255**	**0.250**
	0.5	4	500	**0.261**	**0.261**	**0.260**
	0.5	4	2000	**0.263**	**0.263**	**0.262**
	0.5	8	200	**0.062**	**0.062**	**0.060**
	0.5	8	500	**0.067**	**0.067**	**0.069**
	0.5	8	2000	**0.071**	**0.071**	**0.071**
	0.8	4	200	**0.323**	**0.323**	**0.310**
	0.8	4	500	**0.329**	**0.329**	**0.320**
	0.8	4	2000	**0.329**	**0.329**	**0.328**
	0.8	8	200	**0.077**	**0.077**	**0.072**
	0.8	8	500	**0.084**	**0.084**	**0.079**
	0.8	8	2000	**0.085**	**0.085**	**0.086**
Correct	0.5	4	200	0.021	0.021	0.008
	0.5	4	500	0.007	0.007	0.005
	0.5	4	2000	0.001	0.001	0.000
	0.5	8	200	-0.006	-0.006	-0.01
	0.5	8	500	0.000	0.000	-0.002
	0.5	8	2000	-0.001	-0.001	-0.002
	0.8	4	200	0.038	0.038	0.032
	0.8	4	500	0.021	0.021	0.022
	0.8	4	2000	0.011	0.011	0.010
	0.8	8	200	-0.004	-0.004	-0.009

Table 4-14. Continued.

Model	MA	Wave	Size	LGM 1	LGM 2	LGM 3
	0.8	8	500	-0.002	-0.002	-0.005
	0.8	8	2000	-0.001	-0.001	-0.001

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

ARMA (1, 1) Within-Person Residual Covariance Matrix

Results in Table 4-15 indicate that the magnitude of the relative biases of $\psi_{\alpha\alpha}$ was affected by the ARMA parameter value, the number of waves and the analysis model type. For the LGM with a time invariant covariate all the relative biases observed with an ARMA parameter of 0.2 and 0.8 were unacceptable regardless of the analysis model type. The biases observed with an ARMA parameter value of 0.5 and 0.45 and with the correct analysis model were unacceptable. For the LGM with a time varying covariate, the pattern of the acceptability of relative biases were similar to that for the LGM with a time invariant covariate, except that the relative biases observed with eight waves and the correct analysis model were acceptable, regardless of the value of the ARMA parameter. For the LGM with a parallel process, all the biases observed with an ARMA parameter of 0.2 and 0.8 were unacceptable while the biases observed with an ARMA parameter of 0.5 and 0.45 were all acceptable (except one bias that was barely acceptable). Across the three LGMs, with other factors holding constant, the absolute biases observed with an ARMA parameter of 0.2 and 0.8 were higher than those observed with an ARMA parameter of 0.5 and 0.45, and the absolute unacceptable biases observed with four waves were higher than those observed with eight waves,

Table 4-15. Mean relative biases of $\psi_{\alpha\alpha}$ estimates for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

ARMA	Model	Wave	Size	LGM 1	LGM 2	LGM 3
0.2,0.8	Incorrect	4	200	**-0.350**	**-0.335**	**-0.353**
	Incorrect	4	500	**-0.342**	**-0.340**	**-0.345**
	Incorrect	4	2000	**-0.339**	**-0.339**	**-0.338**

Table 4-15. Continued.

ARMA	Model	Wave	Size	LGM 1	LGM 2	LGM 3
	Incorrect	8	200	**-0.219**	**-0.219**	**-0.221**
	Incorrect	8	500	**-0.212**	**-0.213**	**-0.214**
	Incorrect	8	2000	**-0.209**	**-0.214**	**-0.210**
	Correct	4	200	**-0.214**	**-0.204**	**-0.263**
	Correct	4	500	**-0.202**	**-0.204**	**-0.251**
	Correct	4	2000	**-0.200**	**-0.207**	**-0.241**
	Correct	8	200	**-0.073**	-0.038	**-0.102**
	Correct	8	500	**-0.067**	-0.032	**-0.087**
	Correct	8	2000	**-0.062**	-0.025	**-0.071**
0.5,0.45	Incorrect	4	200	0.026	0.040	0.025
	Incorrect	4	500	0.038	0.038	0.037
	Incorrect	4	2000	0.040	0.041	0.041
	Incorrect	8	200	0.023	0.030	0.015
	Incorrect	8	500	0.030	0.032	0.027
	Incorrect	8	2000	0.032	0.034	0.031
	Correct	4	200	**-0.162**	**-0.124**	**-0.054**
	Correct	4	500	**-0.140**	**-0.099**	-0.021
	Correct	4	2000	**-0.130**	**-0.079**	-0.003
	Correct	8	200	**-0.053**	-0.039	-0.019
	Correct	8	500	**-0.050**	-0.035	-0.010
	Correct	8	2000	-0.046	-0.038	-0.007

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

Results in Table 4-16 indicate that for all the three LGMs, all the unacceptable biases were occurred when the ARMA parameter was equal to 0.2 and 0.8 and the number of waves was four. Under these conditions, the biases of $\psi_{\beta\beta}$ in the three LGMs were all unacceptable and negative when the analysis model was misspecified, but were acceptable when the analysis model was correct (a few relative biases for the LGM with a time invariant covariate and one relative bias for the LGM with a parallel process were barely unacceptable). The biases under other conditions were all acceptable.

Table 4-16. Mean relative biases of $\psi_{\beta\beta}$ estimates for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

ARMA	Model	Wave	Size	LGM 1	LGM 2	LGM 3
0.2 ,0.8	Incorrect	4	200	**-0.088**	**-0.079**	**-0.104**
	Incorrect	4	500	**-0.083**	**-0.077**	**-0.087**
	Incorrect	4	2000	**-0.078**	**-0.078**	**-0.079**
	Incorrect	8	200	-0.022	-0.017	-0.03
	Incorrect	8	500	-0.016	-0.015	-0.019
	Incorrect	8	2000	-0.013	-0.015	-0.014
	Correct	4	200	**-0.055**	-0.037	**-0.054**
	Correct	4	500	**-0.051**	-0.035	-0.029
	Correct	4	2000	**-0.054**	-0.039	-0.024
	Correct	8	200	-0.017	-0.007	-0.024
	Correct	8	500	-0.008	-0.004	-0.013
	Correct	8	2000	-0.005	-0.002	-0.007
0.5,0.45	Incorrect	4	200	-0.004	0.002	-0.024
	Incorrect	4	500	0.005	0.004	-0.003
	Incorrect	4	2000	0.007	0.008	0.005
	Incorrect	8	200	-0.009	0.000	-0.016
	Incorrect	8	500	-0.002	0.001	-0.004
	Incorrect	8	2000	0.001	0.002	0.000
	Correct	4	200	-0.021	-0.011	-0.039
	Correct	4	500	-0.011	-0.005	-0.014
	Correct	4	2000	-0.004	-0.003	-0.003
	Correct	8	200	-0.013	-0.006	-0.019
	Correct	8	500	-0.005	-0.001	-0.007
	Correct	8	2000	-0.001	-0.001	-0.001

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

Results in Table 4-17 indicated that unacceptable biases for $\psi_{\alpha\beta}$ were observed only when the ARMA parameter was equal to 0.2 and 0.8. With these parameter values, bias was unacceptable when the analysis model was incorrect, regardless of the number of waves, but the bias was larger when the number of waves was eight. With the 0.2 and 0.8 parameter values, bias was also unacceptable when the analysis model was correct and the number of waves was four, but only for the LGM with a time invariant covariate and a time varying covariate. It was noticed that the unacceptable biases increased with the increase of the sample size.

Table 4-17. Mean relative biases of $\psi_{\alpha\beta}$ estimates for three LGMs with an ARMA (1, 1) within-person residual matrix, collapsing across sample size

ARMA	Model	Wave	Size	LGM 1	LGM 2	LGM 3
0.2,0.8	Incorrect	4	200	**0.230**	**0.229**	**0.220**
	Incorrect	4	500	**0.232**	**0.238**	**0.229**
	Incorrect	4	2000	**0.235**	**0.235**	**0.235**
	Incorrect	8	200	**0.070**	**0.085**	**0.066**
	Incorrect	8	500	**0.076**	**0.090**	**0.075**
	Incorrect	8	2000	**0.079**	**0.088**	**0.076**
	Correct	4	200	**0.097**	**0.087**	0.031
	Correct	4	500	**0.112**	**0.088**	0.042
	Correct	4	2000	**0.134**	**0.101**	**0.054**
	Correct	8	200	0.017	0.013	0.021
	Correct	8	500	0.022	0.007	0.026
	Correct	8	2000	0.022	0.008	0.027
0.5,0.45	Incorrect	4	200	-0.034	-0.026	-0.032
	Incorrect	4	500	-0.024	-0.022	-0.023
	Incorrect	4	2000	-0.023	-0.022	-0.024
	Incorrect	8	200	-0.02	-0.016	-0.024
	Incorrect	8	500	-0.016	-0.013	-0.017
	Incorrect	8	2000	-0.014	-0.013	-0.015
	Correct	4	200	0.006	0.016	0.001
	Correct	4	500	0.011	0.009	-0.002
	Correct	4	2000	0.006	0.004	0.001
	Correct	8	200	-0.008	-0.005	-0.016
	Correct	8	500	-0.003	-0.002	-0.011
	Correct	8	2000	-0.001	0	-0.009

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

Summary of the Results for Variance Component Parameter Estimates

The above results indicated that unlike those of fixed parameter and accompanied standard error estimates, not all the relative biases of variance components estimates were acceptable. Whether the biases were acceptable depended on the type of within-person residual covariance structure. When the residual covariance structure was an AR (1) or a MA (1) process, the analysis model type had the most important impact on the mean relative biases. It was either the analysis model type, or the interaction of the analysis model type with other factors (e.g., the number of waves or the time series parameters) that had an impact on the relative biases. In comparison, when the residual covariance structure was an ARMA (1, 1) process, it was the ARMA parameter played an important role: all the relative biases were affected by either the ARMA parameter or the interaction of the ARMA parameter with other factors (i.e., the analysis model type or the number of waves, or both).

With an AR(1) or a MA (1) within-person residual covariance structure, in terms of the impact of the analysis model type, when the incorrect analysis model was used, all the estimates of $\psi_{\alpha\alpha}$ were biased and some estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ were biased; when the correct analysis model was used, all the biases of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ were acceptable but many biases of $\psi_{\alpha\alpha}$ were unexpectedly unacceptable. As some unacceptable biases were found with the correct analysis model, future investigations were conducted and were discussed later. The general pattern of the impact of the model type on variance components estimates was: under the same condition, the correct analysis model always led to better estimates than the incorrect analysis model.

With an AR (1) or a MA (1) within-person error structure, the sample size only had an impact on the acceptability of biases of estimate of $\psi_{\alpha\alpha}$ with a correct analysis model, not on the estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$. These unacceptable biases of $\psi_{\alpha\alpha}$ occurred with the correct analysis model were affected by the sample size only under conditions in which the AR parameter was 0.5, the number of waves was four and the sample size was 200 and 500, and under conditions in which the AR parameter was 0.8, the number of waves was eight and the sample size was 200 and 500. With these conditions, the unacceptable biases of $\psi_{\alpha\alpha}$ could be avoided by increasing sample size to 2000. With a MA (1) structure and with a correct analysis model, the only bias unacceptable occurred under the condition when the MA parameter was 0.8, the sample size was 200 and the number of waves was four. All other biases observed with the correct analysis model were acceptable. On average, the magnitude of the biases of the variance components decreased with the increase of sample size. However, exceptions were noted for the estimates of $\psi_{\beta\beta}$ with an AR (1) error structure and the estimates of $\psi_{\alpha\beta}$ with a MA (1) error structure: the biases observed with the incorrect analysis model were slightly increasing with the increase of the sample size.

With an AR (1) or a MA (1) within-person residual covariance structure, none of the magnitude of biases obtained with four waves were smaller than those obtained with eight waves, holding other conditions equal. The length of waves played an important role in the estimates of $\psi_{\beta\beta}$: the incorrect analysis model resulted in unacceptable biases only when the number of waves was four. When the number of waves was eight, all the biases of $\psi_{\beta\beta}$ were acceptable regardless of the analysis model type.

The AR parameter and the MA parameter only affected acceptability of the biases of the estimates of the $\psi_{\alpha\alpha}$. Holding other conditions constant, a higher value of AR or MA parameter resulted in higher absolute biases for the estimates of $\psi_{\alpha\alpha}$ than a lower value of AR or MA parameter did.

The pattern of biases with an ARMA (1, 1) within-person residual structure was different from that with an AR (1) or a MA (1) error structure. All the relative biases were affected by the value of ARMA parameter. It was found that controlling other factors, an ARMA parameter value of 0.2 and 0.8 always led to more biased estimates than its counterpart value of 0.5 and 0.45.

The analysis model type affected the relative biases through its interaction with the ARMA parameter. Without controlling the effect of the ARMA parameter, the relative biases observed with the correct analysis model were no better than those observed with the incorrect analysis model. Under the condition when the ARMA parameter was 0.2 and 0.8, the unacceptable biases caused by the incorrect analysis model were worse than those caused by the correct analysis model. For the estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$, regardless of the analysis model type, all the unacceptable relative biases were observed only when the ARMA parameter was 0.2 and 0.8, and all the relative biases observed when the ARMA parameter was 0.5 and 0.45 were acceptable. For the estimates of $\psi_{\alpha\alpha}$, unacceptable biases were observed with both the ARMA parameter values. However, when the ARMA parameter value was 0.2 and 0.8, the unacceptable biases were observed regardless of the analysis model type; when the ARMA parameter value was 0.5 and 0.45, only biases observed with the correct analysis model was unacceptable.

With an ARMA (1, 1) error structure, the unacceptable biases of estimates of $\psi_{\alpha\beta}$ increased with the increase of the sample size, which was unexpected. In terms of the impact of the length of waves, controlling other factors, biases observed with the four waves were always worse than biases observed with eight waves. Similar to the AR (1) or MA (1) error structure, the length of waves played an important role in the estimates of $\psi_{\beta\beta}$: When the ARMA parameter was 0.2 and 0.8, as long as the length of waves was eight, the biases of $\psi_{\beta\beta}$ were acceptable no matter what type the analysis model was.

Whenever the biases were unacceptable: 1.when the residual covariance structure was an AR (1) process, estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were inflated and estimates of $\psi_{\alpha\beta}$ were deflated; 2.when the residual covariance structure was a MA (1) or an ARMA (1, 1), estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were deflated and estimates of $\psi_{\alpha\beta}$ were inflated.

Based on the number of acceptable biases and the magnitude of these biases as a measure of sensitivity to model misspecification, under the same condition, $\psi_{\alpha\alpha}$ were more sensitive to model misspecification than $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$, and $\psi_{\beta\beta}$ was the least sensitive one among the three. Regarding the performance of the three LGMs, there was no substantial difference between these three LGMs.

Standard Error Estimates of Variance Components

The standard error estimates of the variance components refer to the standard error estimates for $\psi_{\alpha\alpha}$, $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$. The structure of this part follows the same pattern as that in previous section for the variance components estimates. Within each residual covariance structure, the results for the standard error of $\psi_{\alpha\alpha}$ will be presented first, followed by the results for the standard error of $\psi_{\beta\beta}$. The results of the standard error of $\psi_{\alpha\beta}$ will be presented last.

AR (1) Within-Person Residual Covariance Matrix

Results in Table 4-18 indicate that for the three LGMs, the relative biases of the standard error estimates of $\psi_{\alpha\alpha}$ were all acceptable when the analysis model was misspecified, but were not all acceptable when the analysis model was correct. All the unacceptable biases were observed with a sample size of 200 or 500. However, not all biases occurred with the sample size of 200 or 500 were unacceptable: when the AR parameter was 0.5 and the number of waves was eight, biases occurred with a sample size of 200 or 500 were still acceptable. The biases observed with a sample size of 2000 were all acceptable (except that one bias for the LGM with a parallel process was barely unacceptable). It was noticed that when the sample size increased from 200 to 500, the sign of some of the unacceptable biases changed from a positive value to a negative value, and the magnitude of the biases increased, which was out of expectation. It was noticed that there were some estimates that were substantially positively biased (inflated more than

100%) under the condition of an AR parameter of 0.8 and four waves. The magnitudes of the unacceptable biases observed with an AR value of 0.5 were larger than those observed with an AR value of 0.8.

Table 4-18. Mean relative biases of standard error estimates of $\psi_{\alpha\alpha}$ for three LGMs with an AR (1) within-person residual covariance matrix

Model	AR	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	-0.014	0.001	0.001
	0.5	4	500	-0.008	0.010	-0.004
	0.5	4	2000	0.016	-0.006	0.019
	0.5	8	200	-0.014	-0.006	-0.006
	0.5	8	500	0.016	-0.013	-0.020
	0.5	8	2000	0.007	-0.008	0.001
	0.8	4	200	-0.015	0.005	-0.015
	0.8	4	500	-0.009	-0.008	-0.010
	0.8	4	2000	-0.012	-0.002	-0.015
	0.8	8	200	-0.002	0.009	-0.021
	0.8	8	500	-0.018	-0.004	-0.004
	0.8	8	2000	0.010	-0.009	0.006
Correct	0.5	4	200	**0.146**	0.086	**0.125**
	0.5	4	500	**-0.435**	**-0.115**	**-0.407**
	0.5	4	2000	-0.025	-0.014	-0.004
	0.5	8	200	-0.013	0.002	-0.006
	0.5	8	500	-0.024	-0.001	0.007
	0.5	8	2000	-0.015	0.001	-0.004
	0.8	4	200	**2.245**	**2.061**	**2.167**
	0.8	4	500	**1.136**	**1.350**	**1.316**
	0.8	4	2000	-0.070	0.097	**0.131**
	0.8	8	200	**0.101**	**0.127**	**0.116**
	0.8	8	500	**-0.193**	**-0.223**	**-0.162**
	0.8	8	2000	0.004	-0.012	-0.011

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

For the three conditional LGMs, the biases of the standard error estimates of $\psi_{\beta\beta}$ or $\psi_{\alpha\beta}$ were all acceptable and therefore only the marginal means based on the analysis model type are reported (see Table 4-19). The range of the mean relative biases of standard error estimates of

$\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ was from -.004 to .009, indicating that the estimates were quite close to their empirical values.

Table 4-19. Mean relative biases of standard error estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ for three LGMs with an AR (1) within-person residual covariance matrix

	LGM 1		LGM 2		LGM 3	
Model	$\psi_{\beta\beta}$	$\psi_{\alpha\beta}$	$\psi_{\beta\beta}$	$\psi_{\alpha\beta}$	$\psi_{\beta\beta}$	$\psi_{\alpha\beta}$
Incorrect	-0.004	0.000	0.000	-0.001	-0.003	-0.004
Correct	0.009	0.005	0.007	0.011	0.008	0.005

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

MA (1) Within-Person Residual Covariance Matrix

The biases of estimates of standard error of all three variance components were acceptable and trivial, ranging from -0.006 to 0.001, indicating that the estimates were quite close to their respective empirical values (see Table 4-20).

Table 4-20. Mean relative biases of standard error estimates of variance components for three LGMs with a MA (1) within-person residual covariance matrix

	LGM 1			LGM 2			LGM 3		
Model	$\psi_{\alpha\alpha}$	$\psi_{\beta\beta}$	$\psi_{\alpha\beta}$	$\psi_{\alpha\alpha}$	$\psi_{\beta\beta}$	$\psi_{\alpha\beta}$	$\psi_{\alpha\alpha}$	$\psi_{\beta\beta}$	$\psi_{\alpha\beta}$
Incorrect	0.000	-0.003	-0.005	0.000	-0.003	-0.005	-0.005	-0.003	-0.006
Correct	0.001	0.001	-0.004	0.001	0.001	-0.004	-0.004	0.000	-0.004

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

ARMA (1, 1) Within-Person Residual Covariance Matrix

Results in Table 4-21 indicate that the misspecified model did not lead to any unacceptable biases. However, as long as the analysis model was correct, the biases were unacceptable (with one exception for the LGM with a time invariant covariant). The magnitudes of these unacceptable biases were much larger with an ARMA parameter value of 0.5 and 0.45 than those with an ARMA parameter value of 0.2 and 0.8. Many biases observed with a value of 0.5 and 0.45 and the correct analysis model were bigger than 1. The unacceptable biases observed with an ARMA parameter value of 0.2 and 0.8 demonstrated unexpected trends: the magnitude of these biases increased on average with the increase of sample size, and these biases observed with four waves changed sign when the sample size changed from 200 to 500.

Table 4-21. Mean relative biases of standard error estimates of $\psi_{\alpha\alpha}$ for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

Model	ARMA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.2,0.8	4	200	-0.002	0.000	-0.004
	0.2,0.8	4	500	-0.022	0.005	-0.006
	0.2,0.8	4	2000	-0.008	-0.002	0.017
	0.2,0.8	8	200	-0.026	0.002	-0.011
	0.2,0.8	8	500	-0.007	-0.005	0.000
	0.2,0.8	8	2000	-0.01	-0.012	0.011
	0.5,0.45	4	200	0.004	0.009	0.000
	0.5,0.45	4	500	-0.032	0.000	0.006
	0.5,0.45	4	2000	-0.011	0.011	-0.024
	0.5,0.45	8	200	-0.007	0.013	-0.016
	0.5,0.45	8	500	-0.005	-0.009	0.004
	0.5,0.45	8	2000	-0.003	0.012	-0.019
Correct	0.2,0.8	4	200	0.036	**0.229**	**0.130**
	0.2,0.8	4	500	**-0.266**	**-0.183**	**-0.251**
	0.2,0.8	4	2000	**-0.552**	**-0.562**	**-0.588**
	0.2,0.8	8	200	**-0.199**	**-0.113**	**-0.207**
	0.2,0.8	8	500	**-0.409**	**-0.220**	**-0.346**
	0.2,0.8	8	2000	**-0.673**	**-0.500**	**-0.604**
	0.5,0.45	4	200	**2.120**	**1.076**	**0.793**
	0.5,0.45	4	500	**1.749**	**1.062**	**0.891**
	0.5,0.45	4	2000	**1.522**	**0.995**	**0.548**
	0.5,0.45	8	200	**1.365**	**1.192**	**1.136**
	0.5,0.45	8	500	**1.748**	**1.317**	**1.103**
	0.5,0.45	8	2000	**1.260**	**1.018**	**0.791**

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

Results in Table 4-22 indicate that for the three LGMs, the only unacceptable biases occurred under the condition when the analysis model was correct, the ARMA parameter was equal to 0.2 and 0.8, the number of waves was four and the sample size was 2000. Biases obtained under all other conditions were acceptable.

Table 4-22. Mean relative biases of standard error estimates of $\psi_{\beta\beta}$ for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

Model	ARMA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.2,0.8	4	200	-0.016	-0.008	0.004
	0.2,0.8	4	500	-0.003	-0.003	0.011
	0.2,0.8	4	2000	-0.004	0.001	0.003
	0.2,0.8	8	200	-0.012	-0.003	0.006
	0.2,0.8	8	500	-0.007	0.000	0.019
	0.2,0.8	8	2000	-0.02	-0.003	0.006
	0.5,0.45	4	200	-0.003	-0.014	0.007
	0.5,0.45	4	500	0.000	-0.003	0.002
	0.5,0.45	4	2000	0.004	-0.002	-0.016
	0.5,0.45	8	200	0.01	-0.007	-0.007
	0.5,0.45	8	500	-0.025	-0.004	-0.003
	0.5,0.45	8	2000	0.006	0.006	0.031
Correct	0.2,0.8	4	200	-0.043	-0.039	-0.020
	0.2,0.8	4	500	-0.077	-0.058	-0.046
	0.2,0.8	4	2000	**-0.256**	**-0.242**	**-0.178**
	0.2,0.8	8	200	-0.001	-0.001	0.003
	0.2,0.8	8	500	-0.002	-0.001	-0.018
	0.2,0.8	8	2000	-0.027	-0.017	-0.044
	0.5,0.45	4	200	-0.013	-0.023	-0.024
	0.5,0.45	4	500	0.004	0.009	-0.030
	0.5,0.45	4	2000	0.017	-0.015	-0.043
	0.5,0.45	8	200	-0.010	0.009	-0.007
	0.5,0.45	8	500	-0.015	-0.007	-0.001
	0.5,0.45	8	2000	0.007	0.006	-0.020

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

The relative biases of standard error estimates of $\psi_{\alpha\beta}$ were all acceptable when the analysis model was misspecified (see Table 4-23). However, a few of unacceptable biases were observed under conditions in which the analysis model was correct, the ARMA parameter value was 0.2 and 0.8 and the sample size was 500 or 2000. These unacceptable biases were negative and the magnitude of these unacceptable biases increased with the increased sample size, which was out of expectation.

Table 4-23. Mean relative biases of standard error estimates of $\psi_{\alpha\beta}$ for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

Model	ARMA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.2, 0.8	4	200	-0.016	0.001	0.007
	0.2 ,0.8	4	500	0.004	0.011	-0.009
	0.2, 0.8	4	2000	0.022	0.015	-0.013
	0.5, 0.45	8	200	-0.001	-0.004	0.000
	0.5, 0.45	8	500	0.004	-0.015	0.005
	0.5, 0.45	8	2000	-0.010	0.008	0.001
	0.2, 0.8	4	200	-0.014	0.005	-0.012
	0.2 ,0.8	4	500	-0.014	0.000	0.004
	0.2, 0.8	4	2000	0.012	-0.013	-0.011
	0.5, 0.45	8	200	-0.014	-0.006	-0.020
	0.5, 0.45	8	500	-0.003	-0.003	-0.009
	0.5, 0.45	8	2000	0.002	-0.005	0.016
Correct	0.2, 0.8	4	200	-0.053	-0.068	-0.026
	0.2 ,0.8	4	500	**-0.135**	**-0.143**	-0.070
	0.2, 0.8	4	2000	**-0.405**	**-0.398**	**-0.262**
	0.5, 0.45	8	200	-0.094	-0.025	-0.039
	0.5, 0.45	8	500	**-0.164**	-0.031	-0.088
	0.5, 0.45	8	2000	**-0.376**	**-0.147**	**-0.240**
	0.2, 0.8	4	200	-0.018	-0.030	-0.085
	0.2 ,0.8	4	500	-0.013	-0.007	-0.084
	0.2, 0.8	4	2000	0.018	-0.026	-0.087
	0.5, 0.45	8	200	-0.013	-0.022	-0.034
	0.5, 0.45	8	500	-0.013	-0.010	-0.011
	0.5, 0.45	8	2000	-0.015	-0.018	-0.041

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively. Numbers in bold indicate unacceptable bias.

Summary of Standard Error Estimates of the Variance Components

It was found that across the three LGMs, the standard error estimates of the variance components were all acceptable when the analysis model failed to consider the time series process on the within-person residual covariance structure. Therefore, the model misspecification did not affect the standard error estimates of the variance components. However, some biases were found to be unacceptable with the correct analysis model with an AR (1) or an ARMA (1, 1) error structure. All the biases observed with a MA (1) error structure were acceptable, regardless of the analysis model type.

When the within-person residual covariance structure was AR (1) and the correct analysis model was employed, the unacceptable biases were observed only for the standard error estimates for $\psi_{\alpha\alpha}$, not for the standard error estimate for $\psi_{\beta\beta}$ or $\psi_{\alpha\beta}$. When the residual covariance structure was an ARMA (1, 1) and the correct analysis model was used, the unacceptable biases were observed for the standard error estimates of all three variance components. Furthermore, some suspicious problems were noticed. With an AR (1) error structure, the magnitudes of some of the unacceptable biases increased as the sample size increased and the sign of these unacceptable biases changed from positive to negative. With an ARMA (1, 1) error structure, the magnitude of the unacceptable biases observed with an ARMA parameter value of 0.2 and 0.8 increased with the increase of sample size. For the standard error estimates of $\psi_{\alpha\alpha}$, some of the unacceptable biases changed sign with the change of sample size. There were a few biases obtained with the correct analysis model were larger than 1. All the unacceptable biases of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ were observed only with a sample size of 500 or 2000, not with a sample size of 200. All these above unexpected findings would be investigated and discussed later.

With the correct analysis model, the absolute values of unacceptable biases observed with an AR parameter of 0.8 were higher than those observed with an AR parameter of 0.5, and the unacceptable biases of the standard error estimates for $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ were observed only with a value of ARMA parameter of 0.2 and 0.8, not with a value of ARMA parameter of 0.5 and 0.45, but for the standard error estimates of $\psi_{\alpha\alpha}$, both parameter values led to unacceptable biases, and these biases observed with a 0.5 and 0.45 value were larger than those observed with a 0.2 and 0.8 value.

On average, estimates obtained with four waves were more biased than those obtained with eight waves (i.e., 8).

The three LGMs showed little difference in the performance in terms of the pattern of the acceptability and the magnitude of the relative biases.

Chi-Square GOF Test and GOF Indexes

The chi-square GOF test and GOF indexes are used to detect model misspecification. Therefore, results reported in this section addressed whether the GOF test and GOF indexes could successfully differentiate the correct analysis model from the incorrect analysis model. Their performance was presented according to the following sequence: GOF test (*p* value), CFI and TLI, RMSEA and SRMR.

GOF Test

The null hypothesis in a GOF test is that the targeted model fits the data exactly. A Type I error rate is committed when the null hypothesis is rejected. Therefore, rejecting a correct model in GOF test results in a Type I error rate. As a common criterion is to control the Type I error rate within 5%, the following section reports the percentage of *p* value that is below 0.05 for each of the 24 conditions. Therefore the percentage reported here is the estimated Type I error when the null hypothesis is true and the estimated power when the null is false. The *p* value was said to be able to differentiate between the two types of analysis models when the Type I error rate was less than 5% and the power was larger than 80%.

AR (1) within-person residual covariance matrix

Results in Table 4-24 indicate that under conditions in which the length of waves was four and the sample size was 2000, or the length of waves was eight, the Type I error rate was 5% or slightly inflated, and the power was 100% for all three LGMs. For the LGM with a time invariant covariate, when the AR parameter was 0.5, the length of waves was four and the sample size was 500, the power was 82%. The power under other conditions were all less than 80%.

Table 4-24. Percentage of *p* value below 0.05 for three LGMs with an AR (1) within-person residual covariance matrix

Model	AR	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	39%	31%	22%
	0.5	4	500	82%	73%	56%
	0.5	4	2000	100%	100%	100%
	0.8	4	200	34%	28%	21%
	0.8	4	500	78%	70%	52%
	0.8	4	2000	100%	100%	100%
	0.5	8	200	100%	100%	100%
	0.5	8	500	100%	100%	100%
	0.5	8	2000	100%	100%	100%
	0.8	8	200	100%	100%	100%
	0.8	8	500	100%	100%	100%
	0.8	8	2000	100%	100%	100%
Correct	0.5	4	200	5%	6%	6%
	0.5	4	500	5%	5%	6%
	0.5	4	2000	5%	5%	6%
	0.8	4	200	5%	6%	6%
	0.8	4	500	5%	5%	6%
	0.8	4	2000	5%	5%	5%
	0.5	8	200	5%	8%	8%
	0.5	8	500	5%	6%	7%
	0.5	8	2000	5%	5%	5%
	0.8	8	200	6%	9%	9%
	0.8	8	500	5%	6%	6%
	0.8	8	2000	5%	6%	5%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

MA (1) within-person residual covariance matrix

Results for the *p* value were similar to those obtained with an AR (1) residual covariance structure (see Table 4-25). The power was 100% for every condition in which the number of waves was eight, or the number of waves was four and the sample size was 2000. All the Type I error rate was 5% or slightly inflated. For the LGM with a time invariant covariate, the power was more than 80% with four waves and a sample size of 500.

Table 4-25. Percentage of *p* value below 0.05 for three LGMs with a MA (1) within-person residual covariance matrix

Model	MA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.5	4	200	37%	31%	23%
	0.5	4	500	80%	72%	56%
	0.5	4	2000	100%	100%	100%
	0.8	4	200	50%	42%	31%
	0.8	4	500	92%	86%	74%
	0.8	4	2000	100%	100%	100%
	0.5	8	200	100%	100%	100%
	0.5	8	500	100%	100%	100%
	0.5	8	2000	100%	100%	100%
	0.8	8	200	100%	100%	100%
	0.8	8	500	100%	100%	100%
	0.8	8	2000	100%	100%	100%
Correct	0.5	4	200	7%	8%	7%
	0.5	4	500	6%	11%	5%
	0.5	4	2000	6%	10%	6%
	0.8	4	200	7%	8%	7%
	0.8	4	500	6%	8%	6%
	0.8	4	2000	6%	6%	6%
	0.5	8	200	6%	8%	10%
	0.5	8	500	6%	6%	7%
	0.5	8	2000	5%	5%	6%
	0.8	8	200	6%	9%	9%
	0.8	8	500	6%	6%	7%
	0.8	8	2000	5%	5%	5%

Note: Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

For the LGM with a time varying covariate, the power was 86% when the MA parameter was 0.8, the number of waves was four and the sample size was 500.

ARMA (1, 1) within-person residual covariance matrix

Results in Table 4-26 indicate that with an ARMA (1, 1) within-person residual covariance structure included in the generating model, the Type I error rates ranged from 6% to 9% with an ARMA parameter value of 0.5 and 0.45 but ranged from 9% to 44% with an ARMA parameter value of 0.2 and 0.8. The Type I error rate obtained with both four waves and eight waves and with the value of 0.2 and 0.8 were similar, but larger number of waves resulted in an increase of Type I error rate for the value of 0.5 and 0.45 and with a sample size of 200 and 500. With a value of 0.5 and 0.45, the Type I error rate tended to increase with the increase of sample size.

The power to detect the incorrect analysis model was more than 95% when the ARMA parameter was equal to 0.5 and 0.45 and the number of waves was eight, and the power was more than 88% when the number of waves was eight, the ARMA parameter value was equal to 0.5 and 0.45, and the sample size was 2000. The power under other conditions was less than 43%.

Table 4-26. Percentage of *p* value below 0.05 for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

Model	ARMA	Wave	Size	LGM 1	LGM 2	LGM 3
Incorrect	0.2, 0.8	4	200	6%	6%	6%
	0.2 ,0.8	4	500	5%	5%	5%
	0.2, 0.8	4	2000	6%	6%	5%
	0.5, 0.45	4	200	18%	16%	12%
	0.5, 0.45	4	500	43%	35%	26%
	0.5, 0.45	4	2000	98%	96%	88%
	0.2, 0.8	8	200	7%	10%	10%
	0.2 ,0.8	8	500	7%	8%	7%
	0.2, 0.8	8	2000	18%	15%	11%
	0.5, 0.45	8	200	100%	99%	95%
	0.5, 0.45	8	500	100%	100%	100%

Table 4-26. Continued.

Model	ARMA	Wave	Size	LGM 1	LGM 2	LGM 3
	0.5, 0.45	8	2000	100%	100%	100%
Correct	0.2, 0.8	4	200	7%	7%	7%
	0.2 ,0.8	4	500	6%	6%	6%
	0.2, 0.8	4	2000	6%	6%	6%
	0.5, 0.45	4	200	9%	10%	10%
	0.5, 0.45	4	500	13%	14%	13%
	0.5, 0.45	4	2000	34%	33%	40%
	0.2, 0.8	8	200	7%	9%	8%
	0.2 ,0.8	8	500	7%	6%	7%
	0.2, 0.8	8	2000	7%	6%	7%
	0.5, 0.45	8	200	26%	16%	33%
	0.5, 0.45	8	500	27%	16%	36%
	0.5, 0.45	8	2000	27%	16%	44%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

Summary of results for GOF test

The *p* value performed differently with the three types of error structures. When the generating model included an AR (1) or a MA (1) within-person residual covariance structure, the *P* value statistic can be used to differentiate between the two analysis models under conditions as long as the length of waves was eight or the sample size was 2000 based on the Type I error rate and power criterion for the three LGMs. For the LGM with a time invariant covariate and with a time varying covariate, a few more conditions met the above criterion.

The situation was different for an ARMA (1, 1) residual covariance structure. Under the conditions when the power was more than 80%, the Type I error rate was more than 16%. Under the conditions when the Type I error rate was less than 10%, the power was less than 18%. Therefore, the *p* value could not be used to differentiate between the two types of analysis models with an ARMA (1, 1) error structure. With a value of 0.5 and 0.45, the Type I error rates were large and tended to increase with the increase of sample size, and the eight waves led to higher Type I error rate than the four waves when the sample size was 200 or 500.

In summary, when the within-person residual covariance structure demonstrated an AR (1) or a MA (1) structure, GOF test can be used to differentiate between the two analysis models under conditions when the length of waves was eight or the sample size was 2000 for all three LGMs. When the within-person residual covariance structure demonstrated an ARMA (1, 1) structure, GOF test was of little use in model selection.

TLI and CFI

For TLI and CFI, criteria for adequate model fit are: CFI is greater than 0.95, TLI is greater than 0.95. The percentage of CFI and TLI that met the above criterion was reported in this section.

AR (1) within-person residual covariance matrix

The TLI and CFI suggested adequate model fit for all replications under all the conditions for the LGM with a parallel process and in at least 98% of the replications in all conditions for the LGM with a time invariant covariate (see Table 4-27). This was true for both the correct and the incorrect analysis models. For LGM with a time varying covariate, CFI and TLI suggested adequate model fit for all replications when the analysis model was correct. When the analysis model was incorrect CFI and TLI could not differentiate the correct analysis model from the incorrect analysis model if the number of waves was four. Differentiation was minimal with eight waves and a 0.5 parameter value, but more substantial with a .8 parameter value. Nevertheless, with eight waves perfect differentiation did not occur for CFI and occurred for TLI only when the sample size was 2000 and the AR parameter was 0.8.

Table 4-27. Percentage of TLI and CFI statistics that indicated adequate model fit for three LGMs with an AR (1) within-person residual covariance matrix

				LGM 1		LGM 2		LGM 3	
Model	AR	Wave	Size	TLI	CFI	TLI	CFI	CFI	TLI
Incorrect	0.5	4	200	100%	100%	100%	100%	100%	100%
	0.5	4	500	100%	100%	100%	100%	100%	100%
	0.5	4	2000	100%	100%	100%	100%	100%	100%
	0.8	4	200	100%	100%	100%	100%	100%	100%
	0.8	4	500	100%	100%	100%	100%	100%	100%
	0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5	8	200	100%	100%	97%	99%	100%	100%
	0.5	8	500	100%	100%	100%	100%	100%	100%
	0.5	8	2000	100%	100%	99%	100%	100%	100%
	0.8	8	200	99%	98%	32%	52%	100%	100%
	0.8	8	500	100%	100%	23%	56%	100%	100%
	0.8	8	2000	98%	100%	0%	61%	100%	100%
Correct	0.5	4	200	100%	100%	100%	100%	100%	100%
	0.5	4	500	100%	100%	100%	100%	100%	100%
	0.5	4	2000	100%	100%	100%	100%	100%	100%
	0.8	4	200	100%	100%	100%	100%	100%	100%
	0.8	4	500	100%	100%	100%	100%	100%	100%
	0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5	8	200	100%	100%	100%	100%	100%	100%
	0.5	8	500	100%	100%	100%	100%	100%	100%
	0.5	8	2000	100%	100%	100%	100%	100%	100%
	0.8	8	200	100%	100%	100%	100%	100%	100%
	0.8	8	500	100%	100%	100%	100%	100%	100%
	0.8	8	2000	100%	100%	100%	100%	100%	100%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

MA (1) within-person residual covariance matrix

Results obtained for TLI and CFI with a MA (1) within-person residual covariance structure were similar to those obtained with an AR (1) within-person residual covariance structure (see Table 4-28). The two statistics suggested adequate model fit in more than 99% of the replications under all the conditions for the LGM with a time invariant covariate and with a parallel process, even when the analysis model was incorrect. For the LGM a with time varying covariate, CFI and TLI suggested adequate model fit for all replications when the analysis model was correct. With the incorrect analysis model, CFI could not detect model misspecification when the number of waves was four, or when the number of waves was eight and the MA parameter was 0.5, and the differentiation was minimal when the number of waves was eight and the MA parameter was 0.8. With the incorrect analysis model, TLI detected model misspecification in 95% of the replications when the number of waves was eight, the sample size was 2000 and the AR parameter was 0.8. Its differentiation was minimal under all other conditions.

Table 4-28. Percentage of TLI and CFI statistics that indicated adequate model fit for three LGMs with a MA (1) within-person residual covariance matrix

				LGM 1		LGM 2		LGM 3	
Model	MA	Wave	Size	TLI	CFI	TLI	CFI	CFI	TLI
Incorrect	0.5	4	200	100%	100%	96%	100%	100%	100%
	0.5	4	500	100%	100%	100%	100%	100%	100%
	0.5	4	2000	100%	100%	100%	100%	100%	100%
	0.8	4	200	100%	100%	93%	100%	100%	100%
	0.8	4	500	100%	100%	99%	100%	100%	100%
	0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5	8	200	100%	99%	99%	100%	100%	100%
	0.5	8	500	100%	100%	100%	100%	100%	100%
	0.5	8	2000	100%	100%	100%	100%	100%	100%
	0.8	8	200	100%	100%	76%	88%	100%	100%
	0.8	8	500	100%	100%	92%	99%	100%	100%
	0.8	8	2000	100%	100%	5%	100%	100%	100%
Correct	0.5	4	200	100%	100%	100%	100%	100%	100%
	0.5	4	500	100%	100%	100%	100%	100%	100%
	0.5	4	2000	100%	100%	100%	100%	100%	100%
	0.8	4	200	100%	100%	100%	100%	100%	100%
	0.8	4	500	100%	100%	100%	100%	100%	100%
	0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5	8	200	100%	100%	100%	100%	100%	100%
	0.5	8	500	100%	100%	100%	100%	100%	100%
	0.5	8	2000	100%	100%	100%	100%	100%	100%
	0.8	8	200	100%	100%	100%	100%	100%	100%

Table 4-28. Continued.

Model	MA	Wave	Size	LGM 1 TLI	LGM 1 CFI	LGM 2 TLI	LGM 2 CFI	LGM 3 CFI	LGM 3 TLI
	0.8	8	500	100%	100%	100%	100%	100%	100%
	0.8	8	2000	100%	100%	100%	100%	100%	100%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

ARMA (1, 1) within-person residual covariance matrix

Results in Table 4-29 indicate that the two statistics suggested adequate model fit in almost 100% of the replications under all the conditions, even when the incorrect analysis model was used.

Table 4-29. Percentage of TLI and CFI statistics that indicated adequate model fit for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

Model	ARMA	Wave	Size	LGM 1 TLI	LGM 1 CFI	LGM 2 TLI	LGM 2 CFI	LGM 3 CFI	LGM 3 TLI
Incorrect	0.2, 0.8	4	200	100%	100%	99%	100%	100%	100%
	0.2 ,0.8	4	500	100%	100%	100%	100%	100%	100%
	0.2, 0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	200	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	500	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	2000	100%	100%	100%	100%	100%	100%
	0.2, 0.8	8	200	100%	100%	100%	100%	100%	100%
	0.2 ,0.8	8	500	100%	100%	100%	100%	100%	100%
	0.2, 0.8	8	2000	100%	100%	100%	100%	100%	100%
	0.5, 0.45	8	200	100%	100%	100%	100%	100%	100%
	0.5, 0.45	8	500	100%	100%	100%	100%	100%	100%
	0.5, 0.45	8	2000	100%	100%	100%	100%	100%	100%
Correct	0.2, 0.8	4	200	100%	100%	100%	100%	100%	100%
	0.2 ,0.8	4	500	100%	100%	100%	100%	100%	100%
	0.2, 0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	200	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	500	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	2000	100%	100%	100%	100%	100%	100%
	0.2, 0.8	8	200	100%	100%	100%	100%	100%	100%
	0.2 ,0.8	8	500	100%	100%	100%	100%	100%	100%
	0.2, 0.8	8	2000	100%	100%	100%	100%	100%	100%

Table 4-29. Continued.

				LGM 1		LGM 2		LGM 3	
Model	ARMA	Wave	Size	TLI	CFI	TLI	CFI	CFI	TLI
	0.5, 0.45	8	200	100%	100%	100%	100%	100%	100%
	0.5, 0.45	8	500	100%	100%	100%	100%	100%	100%
	0.5, 0.45	8	2000	100%	100%	100%	100%	100%	100%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

Summary of results for CFI and TLI

When the within-person residual covariance structure demonstrated each of the three time series processes, CFI and TLI could not be used to differentiate between the two types of analysis models under all the conditions, with the only one exception for TLI. When the generating model included an AR(1) or a MA(1) residual covariance structure, for the LGM with a time varying covariate, TLI differentiated in more than 95% of the replications under the condition when the number of waves was eight, sample size was 2000 and the AR or MA parameter was 0.8 and

RMSEA and SRMR

For RMSEA and SRMR criteria for adequate model fit are: SRMR is less than 0.08 and RMSEA is less than 0.06. The percentage of the two statistics that met the criteria is presented in the following subsections.

AR (1) within-person residual covariance matrix

Results in Table 4-30 indicate that the percentage of replications in which the fit of the model was considered acceptable by SRMR was 100% under all conditions for each of the three types of LGMs. When the analysis model was correct RMSEA indicated adequate model fit in more than 90% of the replications of each condition for each of the three LGMs. When the analysis model was incorrect, results depended on the type of LGMs. Under LGM with a time invariant covariate and LGM with a time varying covariate, RMSEA perfectly differentiated between the two analysis models when the number of waves was eight. Under LGM with a parallel process, RMSEA differentiated between the correct and incorrect models in 100% of the replications when the number of waves was eight and the MA parameters was 0.8 and in more than 97% of the replications when the number of waves was four, the MA parameter was 0.5 and the sample size was 500 or 2000.

One unexpected finding was noticed for the LGM with a time varying covariate and the LGM with a parallel process, with the incorrect analysis model and four waves, RMSEA became less capable of rejecting the model as the sample size got larger.

Table 4-30. Percentage of RMSEA and SRMR statistics that indicated adequate model fit for three LGMs with an AR (1) within-person residual covariance matrix

				LGM 1		LGM 2		LGM 3	
Model	AR	Wave	Size	RMSEA	SRMR	RMSEA	SRMR	RMSEA	SRMR
Incorrect	0.5	4	200	47%	100%	67%	100%	88%	100%
	0.5	4	500	43%	100%	79%	100%	99%	100%
	0.5	4	2000	29%	100%	94%	100%	100%	100%
	0.8	4	200	53%	100%	71%	100%	89%	100%
	0.8	4	500	50%	100%	82%	100%	99%	100%
	0.8	4	2000	41%	100%	97%	100%	100%	100%
	0.5	8	200	0%	100%	0%	100%	13%	100%
	0.5	8	500	0%	100%	0%	100%	3%	100%
	0.5	8	2000	0%	100%	0%	100%	0%	100%
	0.8	8	200	0%	100%	0%	100%	0%	100%
	0.8	8	500	0%	100%	0%	100%	0%	100%
	0.8	8	2000	0%	100%	0%	100%	0%	100%
Correct	0.5	4	200	91%	100%	94%	100%	98%	100%
	0.5	4	500	100%	100%	100%	100%	100%	100%
	0.5	4	2000	100%	100%	100%	100%	100%	100%
	0.8	4	200	92%	100%	95%	100%	98%	100%
	0.8	4	500	100%	100%	100%	100%	100%	100%
	0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5	8	200	100%	100%	100%	100%	100%	100%
	0.5	8	500	100%	100%	100%	100%	100%	100%
	0.5	8	2000	100%	100%	100%	100%	100%	100%
	0.8	8	200	100%	100%	100%	100%	100%	100%

Table 4-30. Continued.

Model	AR	Wave	Size	LGM 1		LGM 2		LGM 3	
				RMSEA	SRMR	RMSEA	SRMR	RMSEA	SRMR
	0.8	8	500	100%	100%	100%	100%	100%	100%
	0.8	8	2000	100%	100%	100%	100%	100%	100%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

MA (1) within-person residual covariance matrix

Results in Table 4-31 indicate that, for each LGM and every condition, SRMR indicated adequate model fit in 100% of the replications. Under LGM with a time invariant covariate and with a time varying covariate, when the analysis model was incorrect, RMSEA differentiated between the two analysis models in more than 94% of the replications when the number of waves was eight, and in 95% of the replications only for LGM with a time invariant covariate under the condition in which the number of waves was four, the MA parameter was 0.5 and the sample size was 2000. Under LGM with a parallel process, RMSEA differentiated between the two analysis models in more than 96% replications when the number of waves was eight and the MA parameter was 0.8. For the LGM with a time varying covariate and the LGM with a parallel process, with the incorrect analysis model and four waves, RMSEA became less capable of indicating inadequate model fit as the sample size became larger. For the LGM with a parallel process, the same problem also occurred when the MA parameter was 0.5 and the number of waves was eight.

Table 4-31. Percentage of RMSEA and SRMR statistics that indicated adequate model fit for three LGMs with a MA (1, 1) within-person residual covariance matrix

Model	MA	Wave	Size	LGM 1		LGM 2		LGM 3	
				RMSEA	SRMR	RMSEA	SRMR	RMSEA	SRMR
Incorrect	0.5	4	200	38%	100%	67%	100%	88%	100%
	0.5	4	500	24%	100%	78%	100%	99%	100%
	0.5	4	2000	5%	100%	95%	100%	100%	100%
	0.8	4	200	50%	100%	56%	100%	83%	100%

Table 4-31. Continued.

				LGM 1		LGM 2		LGM 3	
Model	MA	Wave	Size	RMSEA	SRMR	RMSEA	SRMR	RMSEA	SRMR
	0.8	4	500	45%	100%	61%	100%	96%	100%
	0.8	4	2000	32%	100%	67%	100%	100%	100%
	0.5	8	200	0%	100%	6%	100%	46%	100%
	0.5	8	500	0%	100%	0%	100%	55%	100%
	0.5	8	2000	0%	100%	0%	100%	62%	100%
	0.8	8	200	0%	100%	0%	100%	4%	100%
	0.8	8	500	0%	100%	0%	100%	0%	100%
	0.8	8	2000	0%	100%	0%	100%	0%	100%
Correct	0.5	4	200	89%	100%	92%	100%	98%	100%
	0.5	4	500	99%	100%	99%	100%	100%	100%
	0.5	4	2000	100%	100%	100%	100%	100%	100%
	0.8	4	200	90%	100%	93%	100%	98%	100%
	0.8	4	500	100%	100%	99%	100%	100%	100%
	0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5	8	200	100%	100%	100%	100%	99%	100%
	0.5	8	500	100%	100%	100%	100%	100%	100%
	0.5	8	2000	100%	100%	100%	100%	100%	100%
	0.8	8	200	99%	100%	100%	100%	100%	100%
	0.8	8	500	100%	100%	100%	100%	100%	100%
	0.8	8	2000	100%	100%	100%	100%	100%	100%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

ARMA (1, 1) Within-Person Residual Covariance Matrix

Results in Table 4-32 indicate that, for each LGM and every condition, SRMR indicated adequate model fit in 100% of the replications. For the LGM with a time varying covariate and the LGM with a parallel process, with the correct analysis model, RMSEA indicated adequate model fit in more than 93% of the replications. When the analysis model was incorrect, RMSEA failed to reject the model in more than 45% of the replications for the LGM with a time varying covariate and failed in more than 89% of the replications for the LGM with a parallel process. For the LGM with a time invariant covariate, when the analysis model was correct, RMSEA indicated adequate model fit in more than 78% of the replications. When the analysis model was incorrect, RMSEA could detect the model misspecifications only under conditions in which the number of waves was eight and the ARMA parameter value was 0.5 and 0.45, where RMSEA rejected the model in more than 98% of the replications.

Table 4-32. Percentage of RMSEA and SRMR statistics that indicated adequate model fit for three LGMs with an ARMA (1, 1) within-person residual covariance matrix

				LGM 1		LGM 2		LGM 3	
Model	ARMA	Wave	Size	RMSEA	SRMR	RMSEA	SRMR	RMSEA	SRMR
Incorrect	0.2, 0.8	4	200	89%	100%	83%	100%	94%	100%
	0.2, 0.8	4	500	99%	100%	96%	100%	100%	100%
	0.2, 0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	200	72%	100%	93%	100%	97%	100%
	0.5, 0.45	4	500	83%	100%	100%	100%	100%	100%
	0.5, 0.45	4	2000	97%	100%	100%	100%	100%	100%
	0.2, 0.8	8	200	99%	100%	45%	100%	89%	100%
	0.2, 0.8	8	500	100%	100%	50%	100%	100%	100%
	0.2, 0.8	8	2000	100%	100%	51%	100%	100%	100%
	0.5, 0.45	8	200	2%	100%	100%	100%	100%	100%
	0.5, 0.45	8	500	0%	100%	100%	100%	100%	100%
	0.5, 0.45	8	2000	0%	100%	100%	100%	100%	100%
Correct	0.2, 0.8	4	200	89%	100%	90%	100%	96%	100%
	0.2, 0.8	4	500	99%	100%	99%	100%	100%	100%
	0.2, 0.8	4	2000	100%	100%	100%	100%	100%	100%
	0.5, 0.45	4	200	85%	100%	93%	100%	98%	100%
	0.5, 0.45	4	500	98%	100%	100%	100%	100%	100%
	0.5, 0.45	4	2000	100%	100%	100%	100%	100%	100%
	0.2, 0.8	8	200	100%	100%	98%	100%	98%	100%
	0.2, 0.8	8	500	100%	100%	98%	100%	100%	100%
	0.2, 0.8	8	2000	100%	100%	99%	100%	100%	100%
	0.5, 0.45	8	200	80%	100%	100%	100%	100%	100%
	0.5, 0.45	8	500	79%	100%	100%	100%	100%	100%
	0.5, 0.45	8	2000	78%	100%	100%	100%	100%	100%

Note: LGM 1, LGM 2 and LGM3 represent LGM with a time invariant covariate, LGM with a time varying covariate and LGM with a parallel process respectively.

For each LGM, when the analysis model was incorrect, the ability of RMSEA to reject the model tended to decrease with the increase of the sample size for all conditions except when the ARMA parameter value was 0.5 and 0.45 and number of waves was eight.

Summary of results of SRMR and RMSEA

The above results indicate that SRMR non-discriminately suggested adequate model fit in 100% of the replications under all conditions for each of the three types of LGMs with each of the three within-person residual covariance structures.

The performance of RMSEA depended both on the type of within-person error structures and on the type of LGMs. When the generating model included an AR (1) error structure, for the LGM with a time invariant covariate and the LGM with a time varying covariate, RMSEA can perfectly differentiate between the two analysis models when the number of waves was eight and the AR parameter was 0.5 for the three LGMs but performed perfectly when the number of waves was four and the AR parameter was 0.8only for the LGM with a time invariant covariate and the LGM with a time varying covariate. When the generating model included a MA (1) error structure, for the LGM with a time invariant covariate and the LGM with a time varying covariate, RMSEA was able to differentiate between the two types of analysis models with good accuracy under conditions in which the number of waves was eight. For the LGM with a parallel process, RMSEA performed well in differentiation only under conditions in which the number of waves was eight and the MA parameter was 0.8.

When the within-person residual covariance structure followed an ARMA (1, 1) structure, RMSEA could be used to detect the model misspecification only under conditions in which the number of waves was eight and the ARMA parameter value was 0.5 and 0.45, and only for the LGM with a time invariant covariate.

It was noticed that with each type of within-person covariance structures, under certain conditions when the incorrect analysis model was employed, RMSEA became less capable of rejecting the model with the increase of the sample size.

Summary of GOF test and GOF indexes

Regarding whether the GOF test and fit indexes can be used to differentiate correct analysis model from the incorrect analysis model, CFI and SRMR was found not to be able to detect model misspecification under any conditions. For others, their performance depended on the type of within-person residual covariance structure included in the generating model and the type of LGMs.

When the within-person residual covariance structure was an AR (1) or a MA (1) process, the performance of the GOF test in model differentiation was perfect as long as the sample size was 2000 or the length of waves was eight for all LGMs. TLI could differentiate between the correct analysis model and incorrect analysis model only for the LGM with a time varying covariate under conditions in which the number of waves was eight, the sample size was 2000 and the AR or MA parameter was 0.8. RMSEA can be used in model selection with an AR (1) error structure for all three LGMs under conditions in which the number of waves was eight, although the performance was less well for the LGM with a parallel process than for the other two LGMs. With an MA (1) error structure, for the LGM with a time invariant covariate and with a time varying covariate, RMSEA could be used to differentiate between the two analysis models when the number of waves was eight. For the LGM with a parallel process, RMSEA could be used under conditions in which the number of waves was eight and the MA parameter was 0.8.

When the within-person residual covariance structure was an ARMA (1, 1), the *p* value could not be used for model selection. TLI suggested adequate model fit no matter what type of analysis model was used and therefore were not recommended. RMSEA could be used for model selection only for LGM with a time invariant covariate and under conditions in which the number of waves was eight and the ARMA parameter value was 0.5 and 0.45.

Some unexpected results were found. With an ARMA (1, 1) error structure, the Type I error rate tended to increase with the increase of the sample size when the parameter value was 0.5 and 0.45. For the RMSEA, it was noticed that under certain conditions for each type of within-person covariance structures, RMSEA became less capable of rejecting the misspecified model with the increase of the sample size.

CHAPTER 5
DISCUSSION AND CONCLUSION

Although correlated errors were often found in longitudinal data, current practices in LGM normally assume the within-person errors were uncorrelated. As the literature review has indicated, the error structure misspecification affected certain parameter estimates in LGM. Therefore it is worth methodologists' attention to investigate the consequence of model misspecification. Within the framework of SEM, there is no study systematically investigating the impact when the within-person residual covariance structure demonstrates one of the three commonly encountered time series process, but the analysis model fails to include these time series process. Furthermore, previous studies about investigating model misspecification within SEM category were mostly conducted on unconditional LGM. Therefore, this study specifically investigated the consequence of misspecification of the within-person error structure under three commonly used conditional LGMs with the aim to make the results more generalizable.

General Conclusions and Discussions

Results of this study has shown that when the within-person residual covariance structure failed to include one of the three types of time series process, the fixed parameters and their accompanied standard error estimates under any one of the three unconditional LGMs were unaffected. This conclusion is consistent with what was found in previous studies (e.g. Yuan & Bentler, 2004, 2006; Ferron et. al., 2002). Furthermore, the model misspecification did not bias the estimates of standard error of variance components, but did bias the estimates of variance components under some selected conditions. However, the variance components and their accompanied standard error estimates were unexpectedly biased when the analysis model was correct under some selected conditions. With an AR (1) or a MA (1) within-person error structure included in the generating model, some unexpected results were found only in the estimates of $\psi_{\alpha\alpha}$. It was shown that although all the biases of $\psi_{\alpha\alpha}$ obtained with the incorrect analysis model were unacceptable, some biases were unacceptable with the correct analysis model. For the standard error estimates of $\psi_{\alpha\alpha}$, it was found that with an AR(1) error structure, using the incorrect analysis model did not lead to any unacceptable biases but using the correct analysis model caused some unacceptable biases.

When the generating model included an ARMA (1, 1) process, whether the biases of the variance components were acceptable depended mainly on the ARMA parameters. It is difficult to interpret the results using the analysis model type alone. A value of ARMA parameter of 0.5 and 0.45 always led to acceptable biases of estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$, no matter what type of analysis model was used. However, for the estimates of $\psi_{\alpha\alpha}$, a value of ARMA parameter of 0.5 and 0.45 resulted in unacceptable bias only with the correct analysis model. With the correct analysis model, a value of ARMA parameter of 0.2 and 0.8 caused some unacceptable biases of all three variance components estimates. For the standard error estimates of the three variance components, a failure to include an ARMA(1,1) process in error structure did not lead to biased estimates, but including an ARMA(1,1) process in model specification caused some unacceptable biases for all three variance components estimates.

In summary, for the variance component estimates, although the incorrect analysis model caused some biased estimates, the correct analysis model also caused some biases. Under certain conditions it is only the correct analysis model that caused unacceptable bias. For the standard error estimates of variance components with an AR (1) or an ARMA (1, 1) residual structure, only the correct analysis model caused biased estimates. All the biases of standard error estimates of variance components were acceptable when the analysis model was misspecified. Therefore, it deserves our attention to investigate why the correct analysis model caused these unexpectedly biased estimates. As more unexpected biases were found with an ARMA (1, 1) error structure, estimates with an ARMA (1, 1) structure were examined and the findings also applied to an AR (1) or a MA (1) structure.

As the analysis model including the ARMA (1, 1) process resulted in many occurrence rate of non-positive definite covariance matrices (see Table 4-2), a first thought was to investigate whether the occurrence of non-positive definite matrix caused these unexpected estimates. Therefore new data sets were created by removing all the replications with a negative variance or a correlation greater than or equal to one, and supplementing data with replications without the non-positive definite covariance matrices. The biases of variance components and their standard error estimates under LGM with a parallel process, obtained separately with the original data sets and the new data sets were given as an illustration (see Table 5-1 to Table 5-6). Although only the results for LGM with a parallel process with an ARMA (1, 1) error structure are reported as an illustration. Similar results were found for the other two LGMs and therefore were not reported here.

Results displayed from Table 5-1 to Table 5-3 indicate that the estimates of variance components were almost the same with or without removing the non-positive definite covariance matrices.

Table 5-1. Biases of $\psi_{\alpha\alpha}$ obtained with three data sets for LGM with a parallel process with an ARMA (1, 1) within-person residual covariance matrix

ARMA	Model	Wave	Size	Data 1	Data 2	Data 3
0.2,0.8	Incorrect	4	200	**-0.353**	**-0.322**	**-0.353**
	Incorrect	4	500	**-0.345**	**-0.338**	**-0.345**
	Incorrect	4	2000	**-0.338**	**-0.338**	**-0.338**
	Incorrect	8	200	**-0.221**	**-0.221**	**-0.221**
	Incorrect	8	500	**-0.214**	**-0.214**	**-0.214**
	Incorrect	8	2000	**-0.210**	**-0.210**	**-0.210**
	Correct	4	200	**-0.263**	**-0.254**	**-0.263**
	Correct	4	500	**-0.251**	**-0.251**	**-0.251**

Table 5-1. Continued.

ARMA	Model	Wave	Size	Data 1	Data 2	Data 3
	Correct	4	2000	**-0.241**	**-0.241**	**-0.241**
	Correct	8	200	**-0.102**	**-0.102**	**-0.102**
	Correct	8	500	**-0.087**	**-0.087**	**-0.087**
	Correct	8	2000	**-0.071**	**-0.071**	**-0.071**
0.5,0.45	Incorrect	4	200	0.025	0.025	0.025
	Incorrect	4	500	0.037	0.037	0.037
	Incorrect	4	2000	0.041	0.041	0.041
	Incorrect	8	200	0.015	0.015	0.015
	Incorrect	8	500	0.027	0.027	0.027
	Incorrect	8	2000	0.031	0.031	0.031
	Correct	4	200	-0.054	0.006	-0.021
	Correct	4	500	-0.021	0.022	0.011
	Correct	4	2000	-0.003	0.020	0.019
	Correct	8	200	-0.019	-0.009	-0.010
	Correct	8	500	-0.010	-0.006	-0.004
	Correct	8	2000	-0.007	-0.007	-0.006

Note: Data 1 refers to original data. Data 2 refers to data deleting replications with non-positive definite covariance matrix. Data 3 refers to data with 0.4% extreme values trimmed.

Table 5-2. Biases of $\psi_{\beta\beta}$ obtained with three data sets for LGM with a parallel process with an ARMA (1, 1) within-person residual covariance matrix

ARMA	Model	Wave	Size	Data 1	Data 2	Data 3
0.2 ,0.8	Incorrect	4	200	**-0.104**	**-0.092**	**-0.104**
	Incorrect	4	500	**-0.087**	**-0.085**	**-0.087**
	Incorrect	4	2000	**-0.079**	**-0.079**	**-0.079**
	Incorrect	8	200	-0.030	-0.030	-0.030
	Incorrect	8	500	-0.019	-0.019	-0.019
	Incorrect	8	2000	-0.014	-0.014	-0.014
	Correct	4	200	**-0.054**	-0.048	**-0.054**
	Correct	4	500	-0.029	-0.028	-0.029
	Correct	4	2000	-0.024	-0.024	-0.024
	Correct	8	200	-0.024	-0.024	-0.024
	Correct	8	500	-0.013	-0.013	-0.013
	Correct	8	2000	-0.007	-0.007	-0.007
0.5,0.45	Incorrect	4	200	-0.024	-0.024	-0.024
	Incorrect	4	500	-0.003	-0.003	-0.003
	Incorrect	4	2000	0.005	0.005	0.005
	Incorrect	8	200	-0.016	-0.016	-0.016
	Incorrect	8	500	-0.004	-0.004	-0.004

Table 5-2. Continued.

ARMA	Model	Wave	Size	Data 1	Data 2	Data 3
	Incorrect	8	2000	0.000	0.000	0.000
	Correct	4	200	-0.039	-0.034	-0.038
	Correct	4	500	-0.014	-0.012	-0.013
	Correct	4	2000	-0.003	-0.003	-0.003
	Correct	8	200	-0.019	-0.019	-0.019
	Correct	8	500	-0.007	-0.007	-0.007
	Correct	8	2000	-0.001	-0.001	-0.001

Note: Data 1 refers to original data. Data 2 refers to data deleting replications with non-positive definite covariance matrix. Data 3 refers to data with 0.4% extreme values trimmed.

Table 5-3. Biases of $\psi_{\alpha\beta}$ obtained with three data sets for LGM with a parallel process with an ARMA (1, 1) within-person residual covariance matrix

ARMA	Model	Wave	Size	Data 1	Data 2	Data 3
0.2,0.8	Incorrect	4	200	**0.220**	**0.184**	**0.220**
	Incorrect	4	500	**0.229**	**0.219**	**0.229**
	Incorrect	4	2000	**0.235**	**0.235**	**0.235**
	Incorrect	8	200	**0.066**	**0.066**	**0.066**
	Incorrect	8	500	**0.075**	**0.075**	**0.075**
	Incorrect	8	2000	**0.076**	**0.076**	**0.076**
	Correct	4	200	0.031	0.015	0.031
	Correct	4	500	0.042	0.041	0.042
	Correct	4	2000	**0.054**	**0.054**	**0.054**
	Correct	8	200	0.021	0.021	0.021
	Correct	8	500	0.026	0.026	0.026
	Correct	8	2000	0.027	0.027	0.027
0.5,0.45	Incorrect	4	200	-0.032	-0.032	-0.032
	Incorrect	4	500	-0.023	-0.023	-0.023
	Incorrect	4	2000	-0.024	-0.024	-0.024
	Incorrect	8	200	-0.024	-0.024	-0.024
	Incorrect	8	500	-0.017	-0.017	-0.017
	Incorrect	8	2000	-0.015	-0.015	-0.015
	Correct	4	200	0.001	-0.014	-0.005
	Correct	4	500	-0.002	-0.005	-0.004
	Correct	4	2000	0.001	0.000	0.001
	Correct	8	200	-0.016	-0.017	-0.017
	Correct	8	500	-0.011	-0.011	-0.011
	Correct	8	2000	-0.009	-0.009	-0.009

Note: Data 1 refers to original data. Data 2 refers to data deleting replications with non-positive definite covariance matrix. Data 3 refers to data with 0.4% extreme values trimmed.

Results presented from Table 5-4 to Table 5-6 indicate that there was barely any difference for the two sets of biases for the standard error estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$. When the data was removed of the non-positive definite matrices, with the correct analysis model and with an ARMA parameter of 0.5 and 0.45, the relative biases of standard error of $\psi_{\alpha\alpha}$ estimates obtained were still unacceptable. Therefore, the occurrence of non-positive definite matrix could not explain the occurrence of those unexpected biases. Moreover, it was found that under some conditions (e.g., the sample size was 200, the ARMA parameter was 0.2 and 0.8 and the number of waves was 4), even though the rate of occurrence of non-positive definite matrices with the incorrect analysis model (16%) was higher than that with the correct analysis model (7%), biases obtained with the incorrect analysis model were still acceptable (see Table 4-2 and Table 4-17). Furthermore, Leite (2007) found that removing replications with non-positive definite matrix changes the normal distribution of the variance component estimates to a skewed distribution. Therefore the unexpected findings could not be attributed to the occurrence of non-positive definite covariance matrices.

Table 5-4. Biases of standard error estimates of $\psi_{\alpha\alpha}$ obtained with three data sets for LGM with a parallel process with an ARMA (1, 1) within-person residual matrix

Model	ARMA	Wave	Size	Data 1	Data 2	Data 3
Incorrect	0.2, 0.8	4	200	-0.004	0.092	-0.004
	0.2, 0.8	4	500	-0.006	0.031	-0.006
	0.2, 0.8	4	2000	0.017	0.018	0.017
	0.2, 0.8	8	200	-0.011	-0.011	-0.011
	0.2, 0.8	8	500	0.000	0.000	0.000
	0.2, 0.8	8	2000	0.011	0.011	0.011
	0.5, 0.45	4	200	0.000	0.000	0.000
	0.5, 0.45	4	500	0.006	0.006	0.006
	0.5, 0.45	4	2000	-0.024	-0.024	-0.024
	0.5, 0.45	8	200	-0.016	-0.016	-0.016
	0.5, 0.45	8	500	0.004	0.004	0.004
	0.5, 0.45	8	2000	-0.019	-0.019	-0.019

Table 5-4. Continued.

Model	ARMA	Wave	Size	Data 1	Data 2	Data 3
Correct	0.2, 0.8	4	200	**0.130**	**0.112**	0.063
	0.2, 0.8	4	500	**-0.251**	**-0.250**	**-0.251**
	0.2, 0.8	4	2000	**-0.588**	**-0.588**	**-0.588**
	0.2, 0.8	8	200	**-0.207**	**-0.207**	**-0.207**
	0.2, 0.8	8	500	**-0.346**	**-0.346**	**-0.346**
	0.2, 0.8	8	2000	**-0.604**	**-0.604**	**-0.604**
	0.5, 0.45	4	200	**0.793**	**0.860**	**0.561**
	0.5, 0.45	4	500	**0.891**	**0.892**	**0.827**
	0.5, 0.45	4	2000	**0.548**	**1.598**	**1.033**
	0.5, 0.45	8	200	**1.136**	**1.199**	**0.536**
	0.5, 0.45	8	500	**1.103**	**1.259**	**0.782**
	0.5, 0.45	8	2000	**0.791**	**0.797**	**0.625**

Note: Data 1 refers to original data. Data 2 refers to data deleting replications with non-positive definite covariance matrix. Data 3 refers to data with 0.4% extreme values trimmed.

Table 5-5. Biases of standard error estimates of $\psi_{\beta\beta}$ obtained with three data sets for LGM with a parallel process with an ARMA (1, 1) within-person residual covariance matrix

Model	ARMA	Wave	Size	Data 1	Data 2	Data 3
Incorrect	0.2, 0.8	4	200	0.004	0.004	0.004
	0.2, 0.8	4	500	0.011	0.011	0.011
	0.2, 0.8	4	2000	0.003	0.003	0.003
	0.2, 0.8	8	200	0.006	0.006	0.006
	0.2, 0.8	8	500	0.019	0.019	0.019
	0.2, 0.8	8	2000	0.006	0.006	0.006
	0.5, 0.45	4	200	0.007	0.007	0.007
	0.5, 0.45	4	500	0.002	0.002	0.002
	0.5, 0.45	4	2000	-0.016	-0.016	-0.016
	0.5, 0.45	8	200	-0.007	-0.007	-0.007
	0.5, 0.45	8	500	-0.003	-0.003	-0.003
	0.5, 0.45	8	2000	0.031	0.031	0.031
Correct	0.2, 0.8	4	200	-0.020	-0.019	-0.020
	0.2, 0.8	4	500	-0.046	-0.046	-0.046
	0.2, 0.8	4	2000	**-0.178**	**-0.178**	**-0.178**
	0.2, 0.8	8	200	0.003	0.003	0.003
	0.2, 0.8	8	500	-0.018	-0.018	-0.018
	0.2, 0.8	8	2000	-0.044	-0.044	-0.044
	0.5, 0.45	4	200	-0.024	-0.023	-0.025

Table 5-5. Continued

Model	ARMA	Wave	Size	Data 1	Data 2	Data 3
	0.5, 0.45	4	500	-0.030	-0.028	-0.029
	0.5, 0.45	4	2000	-0.043	-0.040	-0.043
	0.5, 0.45	8	200	-0.007	-0.006	-0.005
	0.5, 0.45	8	500	-0.001	0.000	-0.030
	0.5, 0.45	8	2000	-0.020	-0.020	-0.019

Note: Data 1 refers to original data. Data 2 refers to data deleting replications with non-positive definite covariance matrix. Data 3 refers to data with 0.4% extreme values trimmed.

Table 5-6. Biases of standard error estimates of $\psi_{\alpha\beta}$ obtained with three data sets for LGM with a parallel process with an ARMA (1, 1) within-person residual covariance matrix

Model	ARMA	Wave	Size	Data 1	Data 2	Data 3
Incorrect	0.2, 0.8	4	200	0.007	0.007	0.007
	0.2, 0.8	4	500	-0.009	-0.009	-0.009
	0.2, 0.8	4	2000	-0.013	-0.013	-0.013
	0.2, 0.8	8	200	0.000	0.000	0.000
	0.2, 0.8	8	500	0.005	0.005	0.005
	0.2, 0.8	8	2000	0.001	0.001	0.001
	0.5, 0.45	4	200	-0.012	-0.012	-0.012
	0.5, 0.45	4	500	0.004	0.004	0.004
	0.5, 0.45	4	2000	-0.011	-0.011	-0.011
	0.5, 0.45	8	200	-0.020	-0.020	-0.020
	0.5, 0.45	8	500	-0.009	-0.009	-0.009
	0.5, 0.45	8	2000	0.016	0.016	0.016
Correct	0.2, 0.8	4	200	-0.026	-0.021	-0.025
	0.2, 0.8	4	500	-0.070	-0.070	-0.070
	0.2, 0.8	4	2000	**-0.262**	**-0.262**	**-0.262**
	0.2, 0.8	8	200	-0.039	-0.040	-0.039
	0.2, 0.8	8	500	-0.088	-0.088	-0.088
	0.2, 0.8	8	2000	**-0.240**	**-0.240**	**-0.240**
	0.5, 0.45	4	200	-0.085	-0.075	-0.083
	0.5, 0.45	4	500	-0.084	-0.083	-0.084
	0.5, 0.45	4	2000	-0.087	-0.085	-0.089
	0.5, 0.45	8	200	-0.034	-0.036	-0.034
	0.5, 0.45	8	500	-0.011	-0.012	-0.011
	0.5, 0.45	8	2000	-0.041	-0.041	-0.041

Note: Data 1 refers to original data. Data 2 refers to data deleting replications with non-positive definite covariance matrix. Data 3 refers to data with 0.4% extreme values trimmed.

To find out why the correct analysis model resulted in worse estimates than the incorrect analysis model, further investigation was conducted. It was noticed that the range of the estimates of standard error of $\psi_{\alpha\alpha}$ was from 2.74 to 19799.94 in a total of 120,000 replications. The frequency table of the values of the estimates of standard error of $\psi_{\alpha\alpha}$ showed that a total of 119572 estimates fell in the range of 0 to 499, which was counted as 99.6% of total estimates (see Table 5-7). The remaining 428 estimates (counted as 0.4% of total estimates) were greater than 500 and varied substantially in value. It should be noticed that the largest empirical standard error of $\psi_{\alpha\alpha}$ was only 47.29 (under the condition when sample size was 200, number of waves was 4, ARMA parameter was 0.5 and 0.45 and the analysis model was correct). Therefore, around 0.4% estimates of standard error of $\psi_{\alpha\alpha}$ were more than 10 times larger than its empirical error. The 0.4% estimates included many extreme values. Among the 428 standard error estimates of $\psi_{\alpha\alpha}$, 42% were obtained with the occurrence of non-positive definite covariance matrix and 100% were obtained with the correct analysis model.

Table 5-7. The frequency table for the standard error estimates of $\psi_{\alpha\alpha}$ for LGM with a parallel process with an ARMA (1, 1) within-person residual covariance matrix

	Frequency
0-499	119572
500-1499	317
1500-2499	51
2500-3499	20
3500-4499	17
4500-5499	6
5500-5499	3
6500-7499	4
7500-8499	2
8500-9499	1
9500-10499	2
10500-11499	3
11500-12499	1
19500-20499	1

These observations indicated that all of the extreme values were associated with the correct analysis model. Therefore, it is suspected that the extreme values caused the unexpected findings. One thing worth to be mentioned is that among all the estimates greater than 1500 (a total of 111 estimates), 72% was accompanied with the occurrence of non-positive definite covariance matrix. These observations indicated that there might be a relationship among the occurrence of non-positive definite matrix, the extreme values and the correct analysis model, which deserves future investigations.

Regarding the standard error estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$, the estimates did not vary as much as the standard error estimates of $\psi_{\alpha\alpha}$. The variations of the standard error estimates of $\psi_{\alpha\alpha}$, $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$, measured by the variance of these standard error estimates, were 22506.37, 6.42 and 5.44 respectively. The range for the standard error estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ was from 1.83 to 313.81 and from 1.79 to 40.28 respectively. Therefore, although there were some unexpected biases for the standard error estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ with an ARMA (1, 1) error structure, the severity in terms of the number of unacceptable biases and the magnitude of the biases was much less than that for standard error estimates of $\psi_{\alpha\alpha}$. Based on the above observations regarding the extreme values and estimates variations, it is suspected that the unexpected findings were attributed to the extreme values. Therefore, the 0.4% estimates that were greater than 500 were trimmed from original data and biases obtained with the trimmed data were calculated. Results in Table 5-4 indicate that on average the standard error estimates of $\psi_{\alpha\alpha}$ obtained with the trimmed data was better than those obtained with the original data. The relative mean biases of standard error estimates for $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ obtained with the trimmed data were almost no different from those obtained with the original data (see Table 5-2 and Table 5-3). It is the expected result since the removal of the extreme values affected the range of the standard error estimates of $\psi_{\alpha\alpha}$ substantially but barely changed the range for the estimates of standard error estimates of $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$.

The results for the biases of variance component estimates with the trimmed data were also presented (see Table 5-1 to Table 5-3). The variance of the estimates of $\psi_{\alpha\alpha}$, $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ across all the replications were 407.82, 36.31 and 44.29. There were not as many extreme values as were for the estimates of $\psi_{\alpha\alpha}$. Therefore, the removal of extreme values barely changed the biases estimates.

Staring values were also imposed to see whether the estimates with complex covariance structures could be improved. Besides those unexpected biases occurring with the correct analysis model, other unexpected results were also found. Special interest were put on the estimates of $\psi_{\alpha\alpha}$, as generally more problems occurred with $\psi_{\alpha\alpha}$ than the other two variance components estimates. For example, with an AR (1) error structure, the biases of standard error estimates of $\psi_{\alpha\alpha}$ obtained with the correct analysis model fluctuated substantially under certain conditions when the sample size was 200 or 500, which changed from a positive value to a negative value or the magnitude of which increased with the increase of sample size (see Table 4-17). The population values of the three variance components were imposed as staring values to see whether the estimates were improved. Results for the biases of standard error of $\psi_{\alpha\alpha}$ with a sample size of 200 and 500, and with a four wave, under LGM with a time invariant covariate with an AR (1) covariance structure was presented in Table 5-8 as an illustration. Noted that the irregularity did not change, that is, the biases still changed from negative to positive under some conditions and the magnitude of the biases increased with an increase of sample size.

Table 5-8. Biases of standard error of $\psi_{\alpha\alpha}$ estimates obtained with and without imposing starting values for LGM with a time invariant covariate with an AR (1) within-person covariance matrix

Model	AR	Size	Wave	Original	New
Correct	0.5	200	4	**0.146**	**0.069**
	0.5	500	4	**-0.435**	**1.927**
	0.8	200	4	**2.245**	**-0.24**
	0.8	500	4	**1.136**	**1.193**

Note: original represents original data, while new represents results obtained with population value imposed as starting value.

Furthermore, there still existed many extreme values of the standard error estimates of $\psi_{\alpha\alpha}$. In the original data without the starting value imposed, the range of the standard error estimates of $\psi_{\alpha\alpha}$ was from 4.98 to 52599. In the new data set with the starting value imposed and without including estimates obtained with a sample size of 2000, the range was from 7.33 to 40348,. The variations of the standard error estimates of $\psi_{\alpha\alpha}$, $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ were 1044.32, 2.25 and 2.06 respectively in the original data, and 1084.27, 1.61, and 1.45 respectively in the new data. The variation for the estimates of standard error of $\psi_{\alpha\alpha}$ was still substantially larger than those for the estimates of the other two. To inspect whether removing the extreme value could remove the unexpected findings, based on the frequency table (see Table 5-9), estimates with a value greater than 500 was trimmed, which counted as a total of 1282 estimates and 1% estimates.

Table 5-9. The frequency table for the standard error estimates of $\psi_{\alpha\alpha}$ under LGM with a time invariant covariate with an AR (1) within-person covariance matrix

	Frequency
0-499	118718
500-1499	708
1500-2499	186
2500-3499	90
3500-4499	67
4500-5499	37
5500-5499	39

Table 5-9. Continued.

	Frequency
6500-7499	27
7500-8499	19
8500-9499	20
9500-10499	15
10500-11499	11
11500-12499	10
12500-13499	10
13500-14499	11
14500-15499	7
15500-16499	2
16500-17499	3
17500-27499	16
31500-33499	2
41500-42499	1
52500-53499	1

The estimates obtained with the trimmed data improved in many aspects (see Table 5-10). With the trimmed data, the number of unacceptable biases reduced, the average magnitude of the unacceptable biases reduced, there was no more unexpected change of the sign of the biases and no more bias increasing in magnitude with the increase of the sample size.

Table 5-10. Biases of standard error estimates of $\psi_{\alpha\alpha}$ obtained with two data sets for LGM with a time invariant covariate with an AR (1) within-person residual covariance matrix

Model	AR	Wave	Size	Original	Trimmed
Incorrect	0.5	4	200	-0.014	-0.014
	0.5	4	500	-0.008	-0.008
	0.5	4	2000	0.016	0.016
	0.5	8	200	-0.014	-0.014
	0.5	8	500	0.016	0.016
	0.5	8	2000	0.007	0.007
	0.8	4	200	-0.015	-0.015
	0.8	4	500	-0.009	-0.009
	0.8	4	2000	-0.012	-0.012
	0.8	8	200	-0.002	-0.002
	0.8	8	500	-0.018	-0.018
	0.8	8	2000	0.010	0.010
Correct	0.5	4	200	**0.146**	**0.114**

Table 5-10. Continued.

Model	AR	Wave	Size	Original	Trimmed
	0.5	4	500	**-0.435**	-0.066
	0.5	4	2000	-0.025	-0.025
	0.5	8	200	-0.013	-0.013
	0.5	8	500	-0.024	-0.024
	0.5	8	2000	-0.015	-0.015
	0.8	4	200	**2.245**	**1.021**
	0.8	4	500	**1.136**	**0.794**
	0.8	4	2000	-0.07	**0.206**
	0.8	8	200	**0.101**	**0.140**
	0.8	8	500	**-0.193**	-0.076
	0.8	8	2000	0.004	0.004

Note: original refers original data, while trimmed represents data with trimmed extreme values.

Based on the above findings, it is suspected that the extreme values caused unexpected biases, since removing the extreme values on average reduced the magnitude of the unacceptable biases associated with the correct analysis model. As the analysis model including an ARMA (1, 1) or an AR (1) within-person residual covariance structure is substantially more complex than the misspecified models, model complexity could be a factor determining the occurrence of extreme standard errors. The decreased convergence rate with the increasing complexity of analysis models was further evidence (see Table 4-1).

Although the inclusion of an ARMA(1,1) within-person residual covariance matrix within the framework of HLM analyzed by SAS was found to result in better estimates of random effect and smaller accompanied standard error estimates (Kwok, et. al., 2007), it is not true in the framework of LGM by this study. The different results might be due to the use of different software. As mentioned before, the ARMA process is less encountered in social science. Based on the above discussion, it is not recommended for the researchers to include the ARMA (1, 1) process in within-person residual covariance structure due to the estimation difficulty with the current SEM software.

Summary of Impact of Each Factor

Impact of Analysis Model Type

When the within-person residual covariance structure was an AR(1) or a MA(1) process, no impact of the analysis model type was found on the estimates of either the fixed parameters or their standard errors, but the impact on the variance components estimates was present. Under the same condition, the correct analysis model always led to better estimates than the incorrect analysis model. Except that the correct analysis model caused some unacceptable biases of $\psi_{\alpha\alpha}$, a correct analysis model led to unbiased estimates and biased estimates were observed with an incorrect analysis model. With an ARMA (1, 1) error structure, the effect of the analysis model type on variance components estimates could not be separated from the effect of the ARMA parameter value. With an ARMA parameter value of 0.2 and 0.8, the absolute biases occurred with the incorrect analysis model were higher than those occurred with the correct analysis model. However, some biases observed with the correct analysis model were also unacceptable, which deserved further investigation and was discussed above. Regarding the impact of the analysis model type on the estimates of standard error of variance components, estimates were not affected by the misspecification of the analysis model with each of the three types of error structures. However, unacceptable biases were observed only with the correct analysis model, which was also discussed in the previous section.

The convergence rate and the occurrence rate of non-positive definite matrix were affected by the analysis model type. As long as the analysis model was wrong, all the estimates converged. The correct analysis model resulted in a substantial low convergence rate with an ARMA (1, 1) error structure (as low as 52%), or with an AR (1) error structure (as low as 74%), but had little affect on the convergence rate with a MA (1) structure. The results were expected as among the three types of residual covariance structures, the most complex one was the ARMA (1, 1) error structure, and the least complex one was the MA (1) structure. The complexity of the residual covariance structure added the difficulty for estimation.

The impact of the analysis model type on the occurrence of non-positive definite matrix also depended on the number of waves. Provided the number of waves was eight, the misspecified model did not lead to any occurrence of non-positive definite covariance matrices, but led to some occurrences when the model included the AR (1) or ARMA (1, 1) error structure. When the number of waves was four, analytical models failing to include the AR (1) error structure resulted in much less occurrence rate than models included the AR (1) error structure. Analytical models failing to include the MA (1) error structure led to high occurrence rate while the occurrence rate was zero when models included the MA (1) error structure. For the ARMA (1, 1) error structure, a correct analysis model resulted in less occurrence rate than an incorrect analysis model under the condition when ARMA parameter value was 0.2 and 0.8. However, when ARMA parameter value was 0.5 and 0.45, the occurrence rate was zero with an incorrect analysis model but ranges from 5% to 20% with a correct analysis model.

Bollen and Curran (2005) pointed out that the possible causes that resulted in occurrence of non-positive definite matrix included sampling fluctuations, nonconvergence, outliers, model misspecification and empirical underidentification. In this study, as only converged and identified results were analyzed, it was suspected that it is the sampling fluctuation and the model complexity that caused the occurrence of non-positive definite matrices. However, this suspicion deserves further investigation.

Impact of Time Series Parameter

The impact of the time series parameter value was not found on the estimates of the fixed effect parameters and their standard error estimates. The AR parameter and the MA parameter only affected the acceptability of the biases of the estimates for $\psi_{\alpha\alpha}$. A higher value of AR or MA parameter tended to result in higher unacceptable biases for the estimates of $\psi_{\alpha\alpha}$ than a lower value of AR or MA parameter. The ARMA parameter value was critical in deciding whether the variance components estimates were biased under many situations. An ARMA parameter value of 0.2 and 0.8 always led to more biased estimates than its counterpart value of 0.5 and 0.45.

There was also an impact of the time series parameters on the standard error estimates of variance component with an AR(1) or an ARMA (1, 1) error structure, but not with a MA (1) structure. A higher value of an AR parameter led to more biased standard error estimates of $\psi_{\alpha\alpha}$ than a lower value of AR parameter. For the standard error estimates for $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$, unacceptable biases were observed only with a value of ARMA parameter of 0.2 and 0.8, not with a value of 0.5 and 0.45. However, for the standard error estimates of $\psi_{\alpha\alpha}$, unacceptable biases occurred with both parameter values and a 0.5 and 0.45 value resulted in more substantially biased estimates than a value of 0.2 and 0.8.

Additionally, with the correct analysis model, the convergence rate was lower with a higher AR parameter value than a lower AR parameter value, and the convergence rate was higher with an ARMA parameter value of 0.2 and 0.8 than with a value of 0.5 and 0.45.

With the incorrect analysis model, the occurrence rate of non-positive definite matrices was higher with a higher MA parameter, and the occurrence rate was zero for an ARMA parameter value of 0.5 and 0.45 but was high with a value of 0.2 and 0.8. With the correct analysis model, the occurrence rate of non-positive definite matrices was higher with a higher AR parameter, and the occurrence rate was high for an ARMA parameter value of 0.5 and 0.45 but not all zero with a value of 0.2 and 0.8.

As mentioned above, when the within-person residual covariance structure was an ARMA (1, 1), whether the biases of variance components estimates and their standard error estimates were acceptable, as well as the performance of some fit indexes were found to be related to the ARMA parameter. As mentioned in the method chapter, ARMA process is an integration of AR and MA process. If the AR parameter and MA parameter was close to each other and the AR parameter is not small (i.e., 0.8 in this study), the ARMA (1, 1) model reduces approximately to a MA (2) model. As found consistently in this study, the pattern of acceptability of biases and differentiating ability of fit indexes depended mainly on the type of within-person residual covariance structure, which explains why the value of time series parameter played an important role in the analysis of estimates obtained with an ARMA (1, 1) error structure.

Impact of Sample Size

It was found that sample size did not affect the parameter estimates of the fixed effects and their standard errors. This finding regarding the fixed effects estimates is consistent with previous studies. Ferron, et. al. (2002) found that sample size had no biased effect on the parameter estimates of fixed effects when the residual structure of level-one equation in HLM was misspecified. Regarding the effect of sample size on the acceptability of the biases of variance components in this study, with each of the three types of error structures, only the estimates of $\psi_{\alpha\alpha}$ was affected. Other variance components estimates were not affected by the sample size. However, the sample size was observed to be related to the unacceptable biases of $\psi_{\alpha\alpha}$ only because these unacceptable biases occurred with the correct analysis model. Based on the discussion above, it is suspected that without the unexpected findings, the sample size should not affect the variance component estimates. Some unexpected findings were also noted for the estimates of $\psi_{\beta\beta}$ with an AR (1) error structure and for the estimates for $\psi_{\alpha\beta}$ with a MA structure and an ARMA (1, 1) error structure. That is, the magnitude of the unacceptable biases increased with the increase of the sample size. These suspicious problems deserve further investigation.

The sample size affected the standard error estimates of variance components with an AR (1) or an ARMA (1, 1) error structure, and led to some suspicious problems such as the change of signs of the unacceptable biases and the increase of the magnitude of the unacceptable biases with the increase of the sample size. However, based on the discussions before, all these findings regarding the impact of sample size on the variance components and their standard error estimates were suspected to be tenable. Hamilton, et al. (2003) found that in linear latent growth modeling, the variance and covariance estimates of intercept and slope were not biased to a substantive degree by the sample size. The sample size in their study ranged from 25 to 1000. You (2006) found that sample size had no significant effect on the variance components estimates in growth modeling when the within-person residual structure was misspecified as homoscedastic and uncorrelated. Hamilton, et al. (2003) also found that sample size did not bias the standard error estimates of variance components. Based on these previous findings and discussions in this study, it is suspected that sample size may not affect the variance components and standard error estimates.

Hamilton, et al. (2003) found that a larger sample size usually reduced the improper estimates and the failure of convergence, and improved the model fit. In this study, the increase of sample size was found to increase the convergence rate with an AR (1) or an MA (1) error structure when the correct analysis model was used, but the result did not apply to an ARMA (1, 1) error structure. Regarding the improper estimates, consistent with what Hamilton, et al (2003) found, larger sample size tended to reduce the occurrence rate of non-positive definite matrix across the three types of residual structure. Large sample size also increased the differentiation ability of the *p* value: when the error structure was an AR (1) or a MA (1) process, the *p* value can differentiate between the correct analysis model and the incorrect model in 100% replications when the sample size was 2000.

Impact of Length of Waves

It is generally believed that increasing the number of measurement periods may improve the accuracy of parameter estimates in growth modeling (Duncan, *et al*., 1999; Fan 2003b). In this study, holding other conditions constant, on average, the shorter waves (i.e.4) resulted more biased estimates than the longer waves (i.e.,8), which is consistent with the general expectation. It was found that the length of waves played an important role in the estimates of $\psi_{\beta\beta}$. As long as the length of waves was eight, the biases of $\psi_{\beta\beta}$ were acceptable regardless of the analysis model type.

Results in this study indicated that longer waves substantially increased the convergence rate and reduced the number of occurrence rate of improper solutions, and RMSEA can be used for model differentiation only under certain conditions in which the length of waves was eight. Hamilton, et al. (2003) suggested that adding more number of measurement periods increased convergence rate and decreased the RMSEA upper bound. These findings were consistent with the conclusions from Hamilton, et al (2003). Furthermore, the number of waves affected the performance of *p* value. When the generating model included an AR (1) or a MA (1) process, the *p* value performed perfectly in differentiating between the incorrect and the correct analysis models under all conditions in which the number of waves was eight and can differentiate a misspecified ARMA (1, 1) error structure from a correct one only under certain conditions in which the number of waves was eight.

Analytic Results of Variance Components Estimates

As mentioned in the results part, whenever the variance components estimates were unacceptably biased, it was found that 1.the estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were inflated and the estimates of $\psi_{\alpha\beta}$ were deflated when the residual covariance structure followed an AR (1) process; 2. the estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were deflated and the estimates of $\psi_{\alpha\beta}$ were inflated when the residual covariance followed a MA (1) or an ARMA (1,1) process. These results could be analytically proved by examining the variance/covariance matrices under the two different analysis models. As mentioned in chapter 2, the implied variance/covariance matrix for a LGM is

$$\Sigma(\theta) = \Lambda\Psi\Lambda' + \Theta_{\varepsilon}, \qquad (5\text{-}1)$$

where all the symbols remain the same meaning as before. For simple illustration, the variance/covariance matrix of unconditional LGM was employed. When the within-person residual covariance structure is an AR (1) process, the model implied covariance matrix for the observed variables is

$$\Sigma(\theta) = \begin{bmatrix} \psi_{\alpha\alpha} + \lambda_1^2\psi_{\beta\beta} + 2\lambda_1\psi_{\alpha\beta} + \sigma_{e_1}^2 & \cdots & \psi_{\alpha\alpha} + \lambda_1\lambda_t\psi_{\beta\beta} + (\lambda_1 + \lambda_t)\psi_{\alpha\beta} + \sigma_e^2\rho^{t-1} \\ \psi_{\alpha\alpha} + \lambda_2\lambda_1\psi_{\beta\beta} + (\lambda_2 + \lambda_1)\psi_{\alpha\beta} + \sigma_e^2\rho & \cdots & \psi_{\alpha\alpha} + \lambda_2\lambda_t\psi_{\beta\beta} + (\lambda_2 + \lambda_t)\psi_{\alpha\beta} + \sigma_e^2\rho^{t-2} \\ \vdots & \ddots & \vdots \\ \psi_{\alpha\alpha} + \lambda_t\lambda_1\psi_{\beta\beta} + (\lambda_t + \lambda_1)\psi_{\alpha\beta} + \sigma_e^2\rho^{t-1} & \cdots & \psi_{\alpha\alpha} + \lambda_t^2\psi_{\beta\beta} + 2\lambda_t\psi_{\alpha\beta} + \sigma_{e_t}^2 \end{bmatrix}, \qquad (5\text{-}2)$$

where all the symbols remain the same meaning as before. When Θ_{ε} is misspecified as a diagonal matrix with uncorrelated errors, the model implied covariance matrix is as follows, which is the same as that in equation 2-15:

$$\Sigma(\theta)=$$

$$\begin{bmatrix} \psi_{\alpha\alpha}+\lambda_1^2\psi_{\beta\beta}+2\lambda_1\psi_{\alpha\beta}+\sigma_{e_1}^2 & \cdots & \psi_{\alpha\alpha}+\lambda_1\lambda_t\psi_{\beta\beta}+(\lambda_1+\lambda_t)\psi_{\alpha\beta} \\ \psi_{\alpha\alpha}+\lambda_2\lambda_1\psi_{\beta\beta}+(\lambda_2+\lambda_1)\psi_{\alpha\beta} & \cdots & \psi_{\alpha\alpha}+\lambda_2\lambda_t\psi_{\beta\beta}+(\lambda_2+\lambda_t)\psi_{\alpha\beta} \\ \vdots & \ddots & \vdots \\ \psi_{\alpha\alpha}+\lambda_t\lambda_1\psi_{\beta\beta}+(\lambda_t+\lambda_1)\psi_{\alpha\beta} & \cdots & \psi_{\alpha\alpha}+\lambda_t^2\psi_{\beta\beta}+2\lambda_t\psi_{\alpha\beta}+\sigma_{e_t}^2 \end{bmatrix}, \qquad (5\text{-}3)$$

where all the symbols remain the same meaning as described before. From equation 5-1, it should be noted that Θ_ε and Ψ compensate each other. To make the equation 5-2 and equation 5-3 equal to each other, each corresponding element of the two matrices should be equal. Therefore, the difference in Θ_ε has to be absorbed in the Ψ matrix. If the variance element in Θ_ε is increased, some of the variance elements in Ψ has to decrease to achieve equivalence, vice verse.

In our study, if four waves were assumed, the value of λ_t was defined to be 0, 1, 2, 3. Plugging these numbers into equation 5-2 led to

$$\Sigma(\theta)=$$

$$\begin{bmatrix} \psi_{\alpha\alpha}+A & \psi_{\alpha\alpha}+\psi_{\alpha\beta}+A_1 & \psi_{\alpha\alpha}+2\psi_{\alpha\beta}+A_2 & \psi_{\alpha\alpha}+3\psi_{\alpha\beta}+A_3 \\ \psi_{\alpha\alpha}+\psi_{\alpha\beta}+A_1 & \psi_{\alpha\alpha}+\psi_{\beta\beta}+2\psi_{\alpha\beta}+A & \psi_{\alpha\alpha}+2\psi_{\beta\beta}+3\psi_{\alpha\beta}+A_1 & \psi_{\alpha\alpha}+3\psi_{\beta\beta}+4\psi_{\alpha\beta}+A_2 \\ \psi_{\alpha\alpha}+2\psi_{\alpha\beta}+A_2 & \psi_{\alpha\alpha}+2\psi_{\beta\beta}+3\psi_{\alpha\beta}+A_1 & \psi_{\alpha\alpha}+4\psi_{\beta\beta}+4\psi_{\alpha\beta}+A & \psi_{\alpha\alpha}+6\psi_{\beta\beta}+5\psi_{\alpha\beta}+A_1 \\ \psi_{\alpha\alpha}+3\psi_{\alpha\beta}+A_3 & \psi_{\alpha\alpha}+3\psi_{\beta\beta}+4\psi_{\alpha\beta}+A_2 & \psi_{\alpha\alpha}+6\psi_{\beta\beta}+5\psi_{\alpha\beta}+A_1 & \psi_{\alpha\alpha}+9\psi_{\beta\beta}+6\psi_{\alpha\beta}+A \end{bmatrix}, \qquad (5\text{-}4)$$

where $A=\sigma_e^2$, $A_1=\sigma_e^2*\rho$, $A_2=\sigma_e^2*\rho^2$ and $A_3=\sigma_e^2*\rho^3$

Similarly, plugging the value of λ_t into equation 5-3 led to

$$\Sigma(\theta)=$$

$$\begin{bmatrix} \psi_{\alpha\alpha}+\sigma_{e_1}^2 & \psi_{\alpha\alpha}+\psi_{\alpha\beta} & \psi_{\alpha\alpha}+2\psi_{\alpha\beta} & \psi_{\alpha\alpha}+3\psi_{\alpha\beta} \\ \psi_{\alpha\alpha}+\psi_{\alpha\beta} & \psi_{\alpha\alpha}+\psi_{\beta\beta}+2\psi_{\alpha\beta}+\sigma_{e_2}^2 & \psi_{\alpha\alpha}+2\psi_{\beta\beta}+3\psi_{\alpha\beta} & \psi_{\alpha\alpha}+3\psi_{\beta\beta}+4\psi_{\alpha\beta} \\ \psi_{\alpha\alpha}+2\psi_{\alpha\beta} & \psi_{\alpha\alpha}+2\psi_{\beta\beta}+3\psi_{\alpha\beta} & \psi_{\alpha\alpha}+4\psi_{\beta\beta}+4\psi_{\alpha\beta}+\sigma_{e_3}^2 & \psi_{\alpha\alpha}+6\psi_{\beta\beta}+5\psi_{\alpha\beta} \\ \psi_{\alpha\alpha}+3\psi_{\alpha\beta} & \psi_{\alpha\alpha}+3\psi_{\beta\beta}+4\psi_{\alpha\beta} & \psi_{\alpha\alpha}+6\psi_{\beta\beta}+5\psi_{\alpha\beta} & \psi_{\alpha\alpha}+9\psi_{\beta\beta}+6\psi_{\alpha\beta}+\sigma_{e_4}^2 \end{bmatrix}. \qquad (5\text{-}5)$$

It is not difficult to prove when the analysis model fails to consider an AR (1) within-person residual covariance structure, the estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ are inflated and the estimates of $\psi_{\alpha\beta}$ are deflated. Suppose the Ψ matrix of the correct analysis model is

$$\Psi = \begin{bmatrix} \psi_{\alpha\alpha} & \psi_{\alpha\beta} \\ \psi_{\beta\alpha} & \psi_{\beta\beta} \end{bmatrix}, \qquad (5\text{-}6)$$

and the Ψ matrix of the incorrect analysis model is

$$\Psi = \begin{bmatrix} \psi_{\alpha\alpha 1} & \psi_{\alpha\beta 1} \\ \psi_{\alpha\beta 1} & \psi_{\beta\beta 1} \end{bmatrix}. \qquad (5\text{-}7)$$

The first step was to take the element in row one and column two, and the element in row one and column three in equation 5-4. Making each of the two elements equal to the corresponding element in equation 5-5 led to the following equation:

$$\begin{aligned} \psi_{\alpha\alpha} + \psi_{\alpha\beta} + \sigma_e^2 \rho &= \psi_{\alpha\alpha 1} + \psi_{\alpha\beta 1} \\ \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} + \sigma_e^2 \rho^2 &= \psi_{\alpha\alpha 1} + 2\psi_{\alpha\beta 1} \end{aligned}. \qquad (5\text{-}8)$$

As mentioned in the method part, the value of σ_e^2 was set to be 50 and ρ was assumed to be 0.8. Plugging these numbers into equation 5-8 led to the following equation:

$$\begin{aligned} \psi_{\alpha\alpha} + \psi_{\alpha\beta} + 40 &= \psi_{\alpha\alpha 1} + \psi_{\alpha\beta 1} \\ \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} + 32 &= \psi_{\alpha\alpha 1} + 2\psi_{\alpha\beta 1} \end{aligned}. \qquad (5\text{-}9)$$

The solution to equation 5-9 was

$$\begin{aligned} \psi_{\alpha\alpha} + 48 &= \psi_{\alpha\alpha 1} \\ \psi_{\alpha\beta} - 8 &= \psi_{\alpha\beta 1} \end{aligned}. \qquad (5\text{-}10)$$

Therefore, in misspecified model, estimates of $\psi_{\alpha\alpha}$ was inflated by a value of 48 and estimates of $\psi_{\alpha\beta}$ was deflated by a value of 8.

Then, taking the element in row two and column three in the two covariance matrices in equation 5-4 and equation 5-5 and making them equal to each other, led to the following equation:

$$\psi_{\alpha\alpha} + 2\psi_{\beta\beta} + 3\psi_{\alpha\beta} + 40 = \psi_{\alpha\alpha 1} + 2\psi_{\beta\beta 1} + 3\psi_{\alpha\beta 1}. \quad (5\text{-}11)$$

With results in equation 5-10 plugged into equation 5-11, it was shown that

$$\psi_{\beta\beta} + 8 = \psi_{\beta\beta 1}. \quad (5\text{-}12)$$

Therefore, estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were inflated and estimates of $\psi_{\alpha\beta}$ was deflated when the within-person error structure failed to include the AR (1) process. The population value for $\psi_{\alpha\alpha}$, $\psi_{\beta\beta}$ and $\psi_{\alpha\beta}$ were set to be 80 , 60 and 35 respectively. Therefore, the percentage of the inflation for estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ was 80% and 13.3% respectively, and the percentage of the deflation for the estimates of $\psi_{\alpha\beta}$ was 22.9%. All the relative deviations were substantially greater than 0.05. Therefore, when the analysis model failed to include an AR (1) process and the number of waves was 4, all the biases of the three variance components estimates were unacceptable and the magnitude of the biases of $\psi_{\alpha\alpha}$ was the largest. The above findings were consistent with what was found in this simulation study and previous studies (e.g., Ferron, et al., 2002; Kwok, et al., 2007)

It is worthwhile to point out that there is no single solution for the three variance components estimates in Ψ matrix to make equation 5-4 equal to equation 5-5 exactly. For example, if at the beginning the element in row one and column two and the element in row one and column four in equation 5-4 and 5-5 was used instead of using the element in row one and column two and the element in row one and column three, the final solution becomes

$$\begin{aligned} \psi_{\alpha\alpha1} &= \psi_{\alpha\alpha} + 47.2 \\ \psi_{\beta\beta1} &= \psi_{\beta\beta} + 7.2 \quad , \\ \psi_{\alpha\beta1} &= \psi_{\alpha\beta} - 7.2 \end{aligned} \tag{5-13}$$

which differs a little from what were obtained before (i.e., with the element in row one and column two and the element in row one and column three). However, any set of solutions would make the two covariance matrices in equation 5-4 and in equation 5-5 still close to each other.

To see the impact of the value of the AR parameter on the estimates of variance components estimate, the value of AR parameter was changed from 0.8 to 0.5. Similar calculation resulted in the following results:

$$\begin{aligned} \psi_{\alpha\alpha1} &= \psi_{\alpha\alpha} + 37.5 \\ \psi_{\beta\beta1} &= \psi_{\beta\beta} + 12.5 \; . \\ \psi_{\alpha\beta1} &= \psi_{\alpha\beta} - 12.5 \end{aligned} \tag{5-14}$$

The change of the value of the AR parameter from 0.8 to 0.5 resulted in an inflation of $\psi_{\alpha\alpha}$ estimate by 46.9%, an inflation of $\psi_{\beta\beta}$ estimate by 20.8% and a deflation of $\psi_{\alpha\beta}$ estimate by 35.7%. All the biases were still unacceptable. The magnitude of the bias was the least for estimates of $\psi_{\beta\beta}$ and the highest for estimates of $\psi_{\alpha\alpha}$. Compared with the results obtained with an AR parameter of 0.8, the estimates of $\psi_{\alpha\alpha}$ were inflated less but the estimates of $\psi_{\beta\beta}$ were inflated more and the estimates of $\psi_{\alpha\beta}$ were deflated more. These findings are consistent with what was found in this simulation study.

When the within-person residual covariance structure is a MA (1) process, the model implied covariance matrix for the observed variables becomes

$$\Sigma(\theta) = \begin{bmatrix} \psi_{\alpha\alpha} + \sigma_e^2 & \cdots & \psi_{\alpha\alpha} + \lambda_1\lambda_t\psi_{\beta\beta} + (\lambda_1 + \lambda_t)\psi_{\alpha\beta} \\ \psi_{\alpha\alpha} + \lambda_2\lambda_1\psi_{\beta\beta} + (\lambda_2 + \lambda_1)\psi_{\alpha\beta} - \sigma_e^2 * \theta / (1+\theta^2) & \cdots & \psi_{\alpha\alpha} + \lambda_2\lambda_t\psi_{\beta\beta} + (\lambda_2 + \lambda_t)\psi_{\alpha\beta} \\ \vdots & \ddots & \vdots \\ \psi_{\alpha\alpha} + \lambda_t\lambda_1\psi_{\beta\beta} + (\lambda_t + \lambda_1)\psi_{\alpha\beta} & \cdots & \psi_{\alpha\alpha} + \lambda_t^2\psi_{\beta\beta} + 2\lambda_t\psi_{\alpha\beta} + \sigma_e^2 \end{bmatrix}, \qquad (5\text{-}15)$$

where all the symbols remain the same meaning as before. Assuming four waves, we got the following covariance matrix:

$$\Sigma(\theta) = \begin{bmatrix} \psi_{\alpha\alpha} + \sigma_e^2 & \psi_{\alpha\alpha} + \psi_{\alpha\beta} + A & \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 3\psi_{\alpha\beta} \\ \psi_{\alpha\alpha} + \psi_{\alpha\beta} + A & \psi_{\alpha\alpha} + \psi_{\alpha\beta} + 2\psi_{\alpha\beta} + \sigma_e^2 & \psi_{\alpha\alpha} + 2\psi_{\beta\beta} + 3\psi_{\alpha\beta} + A & \psi_{\alpha\alpha} + 3\psi_{\beta\beta} + 4\psi_{\alpha\beta} \\ \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 2\psi_{\beta\beta} + 3\psi_{\alpha\beta} + A & \psi_{\alpha\alpha} + 4\psi_{\beta\beta} + 4\psi_{\alpha\beta} + \sigma_e^2 & \psi_{\alpha\alpha} + 6\psi_{\beta\beta} + 5\psi_{\alpha\beta} + A \\ \psi_{\alpha\alpha} + 3\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 3\psi_{\beta\beta} + 4\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 6\psi_{\beta\beta} + 5\psi_{\alpha\beta} + A & \psi_{\alpha\alpha} + 9\psi_{\beta\beta} + 6\psi_{\alpha\beta} + \sigma_e^2 \end{bmatrix}, \qquad (5\text{-}16)$$

where $A = -\sigma_e^2 * \theta / (1+\theta^2)$ and was the extra component caused by the MA (1) process.

If Θ_ε is misspecified as a diagonal matrix with uncorrelated error, the model implied covariance matrix for the observed variables is the same as that in equation 5-5:

$$\Sigma(\theta) = \begin{bmatrix} \psi_{\alpha\alpha} + \sigma_{e_1}^2 & \psi_{\alpha\alpha} + \psi_{\alpha\beta} & \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 3\psi_{\alpha\beta} \\ \psi_{\alpha\alpha} + \psi_{\alpha\beta} & \psi_{\alpha\alpha} + \psi_{\beta\beta} + 2\psi_{\alpha\beta} + \sigma_{e_2}^2 & \psi_{\alpha\alpha} + 2\psi_{\beta\beta} + 3\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 3\psi_{\beta\beta} + 4\psi_{\alpha\beta} \\ \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 2\psi_{\beta\beta} + 3\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 4\psi_{\beta\beta} + 4\psi_{\alpha\beta} + \sigma_{e_3}^2 & \psi_{\alpha\alpha} + 6\psi_{\beta\beta} + 5\psi_{\alpha\beta} \\ \psi_{\alpha\alpha} + 3\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 3\psi_{\beta\beta} + 4\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 6\psi_{\beta\beta} + 5\psi_{\alpha\beta} & \psi_{\alpha\alpha} + 9\psi_{\beta\beta} + 6\psi_{\alpha\beta} + \sigma_{e_4}^2 \end{bmatrix}. \qquad (5\text{-}17)$$

Applying the similar calculation as was did with an AR (1) covariance structure by taking the element in row one and column two and the element in row one and column three first, setting the MA parameter to be 0.8, led to the following equation:

$$\begin{aligned} \psi_{\alpha\alpha} + \psi_{\alpha\beta} - 24 &= \psi_{\alpha\alpha 1} + \psi_{\alpha\beta 1} \\ \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} &= \psi_{\alpha\alpha 1} + 2\psi_{\alpha\beta 1} \end{aligned}. \qquad (5\text{-}18)$$

The solution to equation 5-18 was

$$\begin{aligned} \psi_{\alpha\alpha} - 48 &= \psi_{\alpha\alpha 1} \\ \psi_{\alpha\beta} + 24 &= \psi_{\alpha\beta 1} \end{aligned}, \qquad (5\text{-}19)$$

where all the symbols remain the same meaning as before. Then making the element in row two and column three in the two matrices in equation 5-16 and equation 5-17 equal, and using the results in equation 5-19 led to the following solution:

$$\psi_{\beta\beta} - 24 = \psi_{\beta\beta 1}. \tag{5-20}$$

However, the solution is not unique. If the element in row two and column four was taken in previous step, the solution became

$$\psi_{\beta\beta} - 16 = \psi_{\beta\beta 1}. \tag{5-21}$$

Despite the multiple solutions, the general pattern of the estimates with the incorrect analysis model for the three variance components was: the estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were deflated and the estimates of $\psi_{\alpha\beta}$ were inflated when the within-person residual covariance structure failed to include a MA (1) process. The percentage of deflation of estimates of $\psi_{\alpha\alpha}$ was 60%, the percentage of deflation of estimates $\psi_{\beta\beta}$ was 26.7% (for a decrease of 16) or 40% (for a decrease of 24), and the percentage of inflation of estimates $\psi_{\alpha\beta}$ was 68.6%.

If the MA parameter was changed to 0.5 and the same calculation was followed, the solution to the estimates of the three variance components was

$$\begin{aligned} \psi_{\alpha\alpha} - 40 &= \psi_{\alpha\alpha 1} \\ \psi_{\alpha\beta} + 20 &= \psi_{\alpha\beta 1}, \\ \psi_{\beta\beta} - 20 &= \psi_{\beta\beta 1} \end{aligned} \tag{5-22}$$

or

$$\begin{aligned} \psi_{\alpha\alpha} - 40 &= \psi_{\alpha\alpha 1} \\ \psi_{\alpha\beta} + 20 &= \psi_{\alpha\beta 1} \quad . \\ \psi_{\beta\beta} - 13.3 &= \psi_{\beta\beta 1} \end{aligned} \tag{5-23}$$

The percentage of the deflation of estimates of $\psi_{\alpha\alpha}$ was 50%, the percentage of deflation of estimates of $\psi_{\beta\beta}$ was 22.2% (for a decrease of 13.3), or 33.3% (for a decrease of 20) and the percentage of inflation of estimates of $\psi_{\alpha\beta}$ was 57.1%. Therefore, the change of the value of the MA parameter did not change the acceptability of the biases and the direction of these biases. As long as the analysis model type was wrong and the number of waves was short, the biases of the three variance components were unacceptable and the estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were deflated while the estimates of $\psi_{\alpha\beta}$ were inflated. The magnitude of these biases were the least for the estimates of $\psi_{\beta\beta}$. Furthermore, a higher value of MA parameter resulted in more biased estimates than a lower value of MA parameter did. The above findings were consistent with what was found in the simulation study.

When the within-person residual covariance structure is an ARMA (1, 1) process, the model implied covariance matrix for the observed variables becomes (assuming four waves)

$$\Sigma(\theta) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}. \qquad (5\text{-}24)$$

Each element in the matrix in equation 5-24 is

$$A_{11} = \psi_{\alpha\alpha} + \sigma_e^2,$$
$$A_{22} = \psi_{\alpha\alpha} + \psi_{\beta\beta} + 2\psi_{\alpha\beta} + \sigma_e^2,$$
$$A_{33} = \psi_{\alpha\alpha} + 4\psi_{\beta\beta} + 4\psi_{\alpha\beta} + \sigma_e^2,$$
$$A_{44} = \psi_{\alpha\alpha} + 9\psi_{\beta\beta} + 6\psi_{\alpha\beta} + \sigma_e^2,$$
$$A_{12} = A_{21} = \psi_{\alpha\alpha} + \psi_{\alpha\beta} + \sigma_e^2 \frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2},$$
$$A_{13} = A_{31} = \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} + \sigma_e^2 \frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2}\rho,$$
$$A_{14} = A_{41} = \psi_{\alpha\alpha} + 3\psi_{\alpha\beta} + \sigma_e^2 \frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2}\rho^2,$$
$$A_{23} = A_{32} = \psi_{\alpha\alpha} + 2\psi_{\beta\beta} + 3\psi_{\alpha\beta} + \sigma_e^2 \frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2},$$
$$A_{24} = A_{42} = \psi_{\alpha\alpha} + 3\psi_{\beta\beta} + 4\psi_{\alpha\beta} + \sigma_e^2 \frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2}\rho, \text{ and}$$
$$A_{34} = A_{43} = \psi_{\alpha\alpha} + 6\psi_{\beta\beta} + 5\psi_{\alpha\beta} + \sigma_e^2 \frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2}.$$

If the analysis model failed to include the ARMA (1, 1) process in Θ_ε, the model implied covariance matrix for the observed variables is still the same as that in equation 5-5:

$$\Sigma(\theta) = \begin{bmatrix} \psi_{\alpha\alpha}+\sigma_{e_1}^2 & \psi_{\alpha\alpha}+\psi_{\alpha\beta} & \psi_{\alpha\alpha}+2\psi_{\alpha\beta} & \psi_{\alpha\alpha}+3\psi_{\alpha\beta} \\ \psi_{\alpha\alpha}+\psi_{\alpha\beta} & \psi_{\alpha\alpha}+\psi_{\beta\beta}+2\psi_{\alpha\beta}+\sigma_{e_2}^2 & \psi_{\alpha\alpha}+2\psi_{\beta\beta}+3\psi_{\alpha\beta} & \psi_{\alpha\alpha}+3\psi_{\beta\beta}+4\psi_{\alpha\beta} \\ \psi_{\alpha\alpha}+2\psi_{\alpha\beta} & \psi_{\alpha\alpha}+2\psi_{\beta\beta}+3\psi_{\alpha\beta} & \psi_{\alpha\alpha}+4\psi_{\beta\beta}+4\psi_{\alpha\beta}+\sigma_{e_3}^2 & \psi_{\alpha\alpha}+6\psi_{\beta\beta}+5\psi_{\alpha\beta} \\ \psi_{\alpha\alpha}+3\psi_{\alpha\beta} & \psi_{\alpha\alpha}+3\psi_{\beta\beta}+4\psi_{\alpha\beta} & \psi_{\alpha\alpha}+6\psi_{\beta\beta}+5\psi_{\alpha\beta} & \psi_{\alpha\alpha}+9\psi_{\beta\beta}+6\psi_{\alpha\beta}+\sigma_{e_4}^2 \end{bmatrix}, \quad (5\text{-}25)$$

where all the symbols remain the same meaning as before. Assume that the value of ARMA parameter was 0.5 and 0.45, then the value of $\frac{(\rho-\theta)(1-\rho\theta)}{1-2\rho\theta+\theta^2}$ was equal to 0.05. When the element in row one and column two and the element in row one and column three in equation 5-24 and in equation 5-25 were set to be equal to its corresponding one, the following equation was obtained

$$\begin{aligned}\psi_{\alpha\alpha} + \psi_{\alpha\beta} + 2.5 &= \psi_{\alpha\alpha 1} + \psi_{\alpha\beta 1} \\ \psi_{\alpha\alpha} + 2\psi_{\alpha\beta} + 1.25 &= \psi_{\alpha\alpha 1} + 2\psi_{\alpha\beta 1}\end{aligned} \quad (5\text{-}26)$$

The solution to equation 5-26 was

$$\begin{aligned}\psi_{\alpha\alpha} + 3.75 &= \psi_{\alpha\alpha 1} \\ \psi_{\alpha\beta} - 1.25 &= \psi_{\alpha\beta 1}\end{aligned} \quad (5\text{-}27)$$

Then equating the element in row two and column three in equation 5-24 and in equation 5-25 led to the following equation:

$$\psi_{\alpha\alpha} + 2\psi_{\beta\beta} + 3\psi_{\alpha\beta} + 2.5 = \psi_{\alpha\alpha 1} + 2\psi_{\beta\beta 1} + 3\psi_{\alpha\beta 1}. \quad (5\text{-}28)$$

Then with the results in equation 5-27 plugged into equation 5-28, the following equation was obtained:

$$\psi_{\beta\beta} + 1.25 = \psi_{\beta\beta 1}. \quad (5\text{-}29)$$

Based on the results shown in equation 5-27 and in equation 5-29, it was found that the estimates of $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ were inflated and the estimates of $\psi_{\alpha\beta}$ were deflated when the covariance structure did not include an ARMA(1,1) process. However, the percentage of the inflation of estimates of $\psi_{\alpha\alpha}$ was only 4.6%, the percentage of the inflation of estimates of $\psi_{\beta\beta}$ was 2.1%, and the percentage of the deflation of estimates of $\psi_{\alpha\beta}$ was 3.6%. Therefore, none of these variance estimates were biased using the 0.05 criterion. The result is consistent with what was found in the simulation study: when the ARMA parameter value was 0.5 and 0.45, model misspecification did not lead to biased estimates of all variance components.

When the ARMA parameter value was set to be 0.2 and 0.8, with the similar calculation, it was found that estimates of $\psi_{\alpha\alpha}$ was deflated by a value of 34.4 (a decrease of 43%), $\psi_{\beta\beta}$ are deflated by a value of 15.2(a decrease of 25.3%), and $\psi_{\alpha\beta}$ are inflated by a value of 15.2 (an increase of 43%). The directions of these biases were the same as obtained before. The magnitude of the biases was the least for the estimates of $\psi_{\beta\beta}$ among the three. The above findings are consistent with what was found in the simulation study: it explains why most unacceptable biases occurred when the ARMA parameter was 0.2 and 0.8.

GOF Test and GOF Indexes

It was found that when the actual within-person residual covariance structure was an AR (1) or a MA (1) process, the statistics that can reliably differentiate between the two types of analysis models were the *p* value and RMSEA under certain conditions in which the number of measurement periods was eight or the sample size was 2000 (only for adequate *p* value). TLI could be used only under a very restrictive condition and for only one type of LGM. When the within-person residual covariance structure was an ARMA (1, 1), only RMSEA could be used for model selection under certain conditions. As both the *p* value and RMSEA are based on chi-square statistics, their sensitivity to model selection is expected to be similar.

CFI and SRMR were not recommended to use, because with any type of analysis model CFI non-discriminately suggested adequate model fit for most of the conditions and SRMR could not detect model misspecification for all the conditions. You (2006) found that when the within-person error structure was misspecified as uncorrelated and homoscedastic, the fit index CFI was generally not sensitive to model misspecification and RMSEA was sensitive to model selection, which was consistent with our findings.

Some suspicious results were found with the *p* value and RMSEA. For the *p* value, it was found that under certain conditions the power did not depend on the sample size and the Type I error rate increased with the increase of sample size. For RMSEA, under certain conditions the ability to reject the misspecified model decreased as the sample size got larger, which deserves further investigation.

Suggestions to Applied Researchers

Suggestions are made according to different research plans. When researchers are only interested in fixed effects, the use of a simple diagonal within-person residual covariance structure would not produce problems in the parameter estimates or tests of fixed effects. Researchers interested in interpreting the variance parameters should consider the possibility of alternative error structures. When an AR (1) or a MA (1) process were found in within-person residual covariance structure, some variance component parameters were biased and the severity of the biases of variance component parameters increased for the incorrect analysis model than for the correct analysis model. Therefore, for better estimates of variance components, researchers should consider alternate covariance structures other than the simple diagonal covariance structure. However, this application is subject to the limitation of sample sizes and measurement periods. As discussed before, a small sample size and less measurement periods sometimes resulted in biased estimates even for the correct analysis model due to the complexity of covariance structure. Therefore, a larger sample size and more measurement periods are recommended for applied researchers. Larger sample size and more measurement periods also help to get unbiased standard error estimates of variance components and increase the sensibility of GOF test and fit indexes to model misspecification.

There existed some unexpected findings in this study, most of which were related to an ARMA (1, 1) covariance structure. It is believed that the complexity of the covariance structure makes it difficult for the current software to give the right estimation. However, as the ARMA process is rarely encountered in social science and under many situations it approximates an AR or MA process (McCleary & Hay, 1980), and the AR (1) and the MA (1) processes are the most commonly encountered time series, therefore, complex ARMA covariance structure should be considered only after an AR (1) or a MA (1) covariance structure is ruled out.

Limitations and Suggestions for Future Research

This study inevitably suffers the same limitations as many other simulation studies do: the scope of it is limited by the conditions that were examined. For example, more conditions can be included in the future. The AR and MA parameter was chosen to represent a medium or large effect to make the results more obvious. An AR parameter or a MA parameter of 0.8 may not be encountered quite often in educational research. It is worthwhile to test whether slightly misspecified model when an AR or a MA parameter is small would lead to the same results.

Another limitation is that only linear growth model was examined. It was shown that nonlinear growth model coupling with unequally spaced data caused problems for both the estimation and tests of fixed effects (Ferron, et al., 2002). As nonlinear growth models are common in applied researches, it deserves an investigation in future research.

It should be pointed out that in this simulation study, the population value of the within-person residual variance σ^2 was 50, which was not far away from the population value of $\psi_{\alpha\alpha}$ (i.e., 80) , the population value of $\psi_{\beta\beta}$ (i.e., 60) and the population value of $\psi_{\alpha\beta}$ (i.e., 35). As in LGM, most random effects come from between-person variation, it is unknown whether results would be different when the σ^2 is specified to be much smaller than the between-person variance components (e.g., change σ^2 from 50 to 1 and still keep values of between-person variance components the same). Furthermore, Muthén and Muthén (2002) pointed out that in most applied literature, the ratio between variance of the level and variance of the shape was 5 to 1. In this study, the ratio of the $\psi_{\alpha\alpha}$ and $\psi_{\beta\beta}$ was 4:3. Although the population value was obtained from the ECLS-K data, it is unknown whether a different ratio would lead to the same results.

The present study only examined the situation when within-person residual covariance structure was misspecified as a diagonal matrix. Although this is the most likely misspecification in practice, the possible other misspecifications also deserve examination. For example, a MA (1) residual covariance structure was used but actually it is an AR (1) structure operating; or an overly complex covariance structure was used when a simple structure can be the substitute. In this study, the choice of ARMA parameter 0.5 and 0.45 makes the ARMA (1, 1) process approximately reduce to an MA (2) process. Therefore, using ARMA (1, 1) process can serve as an example when an overly complex error structure was used. As no MA (2) error structure was examined in this study, further investigation can be conducted later.

APPENDIX
MPLUS CODE

Latent Growth Model with a Time Invariant Covariate with an AR (1) Process

```
DATA: FILE IS "c:\mplus\invar\data\filelist.txt";

Type=montecarlo;

VARIABLE:

NAMES ARE t1 t2 t3 t4 x;

USEVARIABLES ARE t1 t2 t3 t4 x;

ANALYSIS:

TYPE IS general;

iterations =5000;

estimator=ML;

MODEL:

i s| t1@0 t2@1 t3@2 t4@3;

[i* s*];

[t1-t4@0];

i* s*;

t1-t4(e);

i with s*;

i s on x ;

t1-t3 pwith t2-t4 (cov1);

t1-t2 pwith t3-t4 (cov2);

t1 with t4 (cov3);

MODEL CONSTRAINT:
```

```
new(lag);

cov1 = e*lag;

cov2=e*lag**2;

cov3=e*lag**3;

SAVEDATA:

results are results.txt;

output:  tech9;
```

Latent Growth Model with a Time Invariant Covariate with an MA (1) Process

```
DATA: FILE IS "c:\mplus\invar\data\filelist.txt";

Type = montecarlo;

VARIABLE:

NAMES ARE t1 t2 t3 t4 x ;

USEVARIABLES ARE t1 t2 t3 t4 x;

ANALYSIS:

TYPE IS general;

iterations =5000;

estimator=ML;

MODEL:

i s | t1@0 t2@1 t3@2 t4@3;

[i* s*];

[t1-t4@0];

i* s*;

t1-t4(e);

i with s*;
```

```
i s on x ;
t1-t3 pwith t2-t4 (cov);
MODEL CONSTRAINT:
new(lag);
cov = -lag*e/(1+lag^2);
SAVEDATA:
results are results.txt;
output: tech9;
```

Latent Growth Model with a Time Invariant Covariate with an ARMA (1, 1) Process

```
DATA: FILE IS "c:\mplus\invar\data\filelist.txt";
Type = montecarlo;
VARIABLE:
NAMES ARE t1 t2 t3 t4   x;
USEVARIABLES ARE t1 t2 t3 t4  x;
ANALYSIS:
TYPE IS general;
iterations =5000;
estimator=ML;
MODEL:
i s | t1@0 t2@1 t3@2 t4@3;
[i* s*];
[t1-t4@0];
i* s*;
t1-t4(e);
```

```
i with s*;
i s on x ;
t1-t3 pwith t2-t4 (cov1);
t1-t2 pwith t3-t4 (cov2);
t1 with t4 (cov3);
MODEL CONSTRAINT:
new(pho);
new(r);
cov1 =e*(1-pho*r)*(pho-r)/(1+r^2-2*pho*r);
cov2=e*(1-pho*r)*(pho-r)/(1+r^2-2*pho*r)*pho;
cov3=e*(1-pho*r)*(pho-r)/(1+r^2-2*pho*r)*pho^2;
output: tech4 tech9;
SAVEDATA:
results are results.txt;
```

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BIOGRAPHICAL SKETCH

Yuying Shi was born in China. She obtained her bachelor's degree in economics major in finance from Shanghai University of Finance and Economics. In 2003, she came to Kent State University for graduate study, where she received her master's degree in economics. She began her Ph.D. study in research and evaluation methodology at University of Florida in 2005 and received her doctorate in August 2009.