EXTENSIONS TO THE ECONOMIC LOT SIZING PROBLEM

By

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I dedicate this dissertation to my parents Hacı Mehmet Önal and Hülya Önal

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We present several variants of the basic economic lot sizing model. In one of our models, we integrate pricing, procurement, transportation, and inventory decisions for a two echelon supply chain. We propose polynomial time dynamic programming algorithms to the problem under various procurement and inventory holding cost functions. In another model, we incorporate pricing decisions to a capacitated production environment where production setups consume a fixed amount of available resource time. We propose Dantzig–Wolfe formulations and a branch-and-price algorithm to solve the problem. We also consider economic lot sizing models, where items can be procured from a number of suppliers in each period and they perish after a certain number of periods. We propose solution algorithms to solve some special cases of the problem.

CHAPTER 1
INTRODUCTION

Dynamic economic lot sizing problem (ELS) as described by Wagner and Whitin (39) is as follows. There are demands for an item over a finite and discrete planning horizon of T periods. The demands have to be satisfied by producing in a facility with no capacity restrictions. An item produced in a period can only satisfy demands in that period and the following periods. Whenever there is positive production, a setup has to take place, which entails a fixed setup cost. Any item produced incurs a unit production cost and any item carried to the next period as inventories incurs a unit inventory holding cost. The objective of the (ELS) is to find the minimum cost production plan given the production and inventory holding costs in each period. Wagner and Whitin showed that there exists an optimal solution to the (ELS) that possesses the zero inventory order property (ZIO), where if there is positive production in a period, then there is no inventory carried to that period. This basic model proposed by Wagner and Whitin has been generalized in several ways. For instance, Zangwill (40) allowed backlogging and assumed general concave cost functions. Baker et al. (1), Bitran and Yanasse (3) and Florian et al. (11) extended the (ELS) by integrating production capacities.

In some cases, items may go through several intermediate steps before satisfying any demand. This happens in serial supply chains where value is added in different production facilities to a certain product. In that case items that have been processed in one facility need to be transported to another facility to go through other processes. In another example, items can be procured from the suppliers and transported to and stored in several warehouses before they are being consumed by the customers. Items that are procured are first sent to the first level warehouses and then to second and third level warehouses (if they exist) until they are finally transported to retailers. This type of serial supply chains can be represented by a two level lot sizing model. Such multiple level lot sizing problems were analyzed by Zangwill (41), Kaminsky and Simchi-Levi (27), Van Hoesel et al. (23).

The above models assume that demands are given in advance and hence they are exogenous to the model. However, through pricing decisions, demand levels may be affectively controlled and optimal demand levels may be chosen in each period. Thomas (36), Kunreuther and Scrage (28) and van den Heuvel and Wagelmans (20) developed models that integrate pricing decisions to the (ELS). Later, Geunes et al. ((16),(14)) extended these pricing models by incorporating production capacities. There are two pricing strategies that have been studied extensively in the literature. One is the dynamic pricing strategy, where it is assumed that different prices can be set to items in each period. The other one is the constant pricing strategy, where it is assumed that a single price should be set for the item over the planning horizon.

Perishability of the items is another issue to consider while making production plans. For instance, dairy products and agricultural products can not be held in the inventories indefinitely, which implies that not all production plans are feasible if the items are perishable. There is a rich literature in production planning problems with perishable items. Majority of this work is continuous time models or discrete time models with either stochastic demands or stochastic lifetimes. Nahmias (33) has an extensive review on such inventory planning models. Friedman and Hoch (12) and Hsu (25) consider economic lot sizing models with deterministic demands where a known fraction of items deteriorate in each period as the items get older.

In this dissertation, we investigate further extensions of the basic lot sizing models. In Chapter 2, we consider an uncapacitated lot sizing problem where production, inventory carrying, transportation, and pricing decisions are integrated to maximize total profits in a two echelon supply chain. Items are produced (or procured) in the fist echelon and then transported to the second echelon to satisfy the demands. We assume that as demands are satisfied, revenues are realized. We show how this problem, under many different revenue functions and production, inventory holding, and transportation cost structures can be solved in polynomial time. As a byproduct, we develop polynomial-time algorithms for generalizations of single-level lot-sizing problems with pricing as well.

In Chapter 3, we study the multi-item capacitated lot sizing problem with setup times and pricing (CLSTP) over a finite and discrete planning horizon. The (CLSTP) is a generalization of the well-known capacitated lot sizing problem with setup times (CLST). In this class of problems, the demand for each independent item in each time period is affected by pricing decisions. The corresponding demands are then satisfied through production in a single capacitated facility or from inventory, and the goal is to set prices and determine a production plan that maximizes total profit. In contrast with many traditional lot sizing problems with fixed demands, we cannot without loss of generality restrict ourselves to instances without initial inventories, which greatly complicates the analysis of the (CLSTP). We develop two alternative Dantzig–Wolfe decomposition formulations of the problem, and propose to solve their relaxations using column generation and the overall problem using branch-and-price algorithm. The associated pricing sub–problem is studied under both dynamic and constant pricing strategies. Efficient algorithms to solve the resulting sub–problems to optimality exist when there are no initial inventories. However, they become more challenging when initial inventories are present. We develop polynomial time solution algorithms for these problems. Through a computational study we analyze both the efficacy of our algorithms and the benefits of allowing item prices to vary over time.

In Chapter 4, we introduce the economic lot sizing problem with perishable items (ELS-PI) where we assume that there are multiple suppliers available to procure the items in each period. We assume that items deteriorate after a certain number of periods, which is a function of the supplier the items are procured from. We assume that items are good for consumption and no deterioration occurs until they reach the end of their lifetimes or their expiration dates. We explore the structural properties of the optimal solutions and propose polynomial time algorithms to the problem under various cost function structures. We show that the optimal solution does not posses the ZIO property. This implies that two items procured from two different suppliers (and possibly two different expiration dates) can be found in the inventories together. In such a case, the consumers may aggressively search for items that expire later if they are aware that the store has fresher items in the inventories. We show that lower cost procurement plans can be achieved if the customer can be manipulated such that they buy early expiring items even if there are items in the inventories that expire later. One way to manipulate customers' consumption is to design the inventory system as a queue so that customers always pick the item at the front of the queue. Although this dictates a certain consumption order for the items, it does not result in much improvement in procurement costs if the items have to be inserted either always to the back of the queue or to the front of the queue. We show that, in general, consumer preference and the inventory system puts restrictions on the inventory manager such that not all procurement plans can be achieved. Some procurement plans, although much less costly than the others, may turn out to be infeasible given the consumer preferences or the inventory system. We also present several extensions of the problem in this chapter. In one of the extensions that we consider, we assume that there are procurement capacities associated with each supplier. We prove that, unlike the capacitated lot size models with non–deteriorating items, the ELS-PI with constant procurement capacities is NP-hard even in the case where there is a single supplier available in each period. Other extensions that we consider include backlogging and pricing decisions.

One of the main results of Chapter 4 is that it is the store manager's best interest to sell the early expiring items first. Consumers, on the other hand, always look for the later expiring items. Given consumer's tendency to the later expiring items, in Chapter 5, we consider two models where the store manager can manipulate item consumption. In the first model, items are stored and displayed in different locations. The store manager can now procure items from any supplier in any period. Since the display location is separated from the storage location, he displays only the items he wants to sell to the customers. This allows him to sell early expiring products although there are later expiring products in the inventories. However, he now has to pay for the transportation costs between the display and the storage areas. In the other model, the items are displayed in a queue and the customers are restricted to purchase from the front end of the queue. To impose a certain consumption order, the store manager is allowed to insert items to any place in the queue whenever they are procured. However, there is some cost associated with this process because each insertion of the items may require a considerable amount of rearrangement of the inventories.

CHAPTER 2
TWO-ECHELON REQUIREMENTS PLANNING WITH PRICING DECISIONS

2.1 Introduction

We consider the deterministic requirements planning and pricing problem in a two-echelon supply chain nominally consisting of a supplier and a retailer. In particular, our model integrates procurement, transportation, and inventory holding decisions in both echelons with pricing (or, equivalently, demand) decisions in order to maximize total profit over a finite horizon. Such models have been very well-studied for the case where demands are fixed, in which case the objective becomes the minimization of total cost since total revenues are constant; see Wagner and Whitin (39) and Zangwill (41) for the earliest studies of single-level and multi-echelon lot-sizing problems, and Kaminsky and Simchi-Levi (27), Van Hoesel et al. (23), and Sargut and Romeijn (35) for extensions that allow for finite procurement capacities. In general, costs are assumed to be concave as a result of economies of scale, which makes the problem of when to procure and transport nontrivial.

Classical lot-sizing problems such as the ones mentioned above assume that demands are known in advance. This is a reasonable assumption if firms make pricing decisions for the items before making procurement/transportation plans for those items. However, since the price of an item may affect the demand for that item, significant profit increases can be observed if we can integrate pricing and procurement planning decisions. Thomas (36) was the first to incorporate pricing decisions into a single-echelon lot-sizing model. More recently, this area of research has gained significant attention from researchers via extensions that incorporate a finite procurement capacity (see, e.g., Biller et al. (2), Deng and Yano (8), and Geunes et al. (16)) or the constraint that a single price must be set for the duration of the planning horizon (see, e.g., Kunreuther and Schrage (28), Gilbert (18; 19), Van den Heuvel and Wagelmans (20), and Geunes et al. (14)). In this chapter, we extend the well-studied single-echelon model with a second echelon representing (for example) transportation to a different location or a second stage in a production process.

Throughout this chapter, we will assume that demand decreases as the price of the item increases, and that there is a one to one relation between price and demand. This implies that we can find optimal prices, which correspond to optimal demand levels in each period and potentially obtain higher profits. With the inclusion of pricing in the model, the objective function becomes to find the optimal demands that maximize profits when the revenues are assumed to be concave functions of demand satisfied in each period. As is common in procurement planning problems, we assume concave cost structures.

The remainder of the chapter is organized as follows. In Section 2.2 we present a mathematical formulation of our integrated requirement planning model and describe our dynamic programming approach. In particular, we show that this approach requires the repeated solution of generalized single-echelon requirements planning problems with pricing decisions. In Section 2.3, we therefore develop polynomial-time algorithms for solving such problems under various forms of the cost and revenue functions. Section 2.4 then analyzes the resulting running time of the algorithm for the two-echelon problem as well as develops significant efficiency improvements that are obtained by considering the similarity between the subproblems that need to be solved. In Section 2.5 we analyze further efficiency improvements that can be obtained if some or all of the cost functions are linear. Finally, we conclude the chapter in Section 2.6 with a discussion of future research opportunities.

2.2 Model Formulation and Algorithmic Framework

For convenience, we will throughout this chapter refer to the first echelon as the supplier level, while we will refer to the second level as the retailer level. The cost of procuring a quantity y_t at the supplier level in period t is given by an associated concave procurement cost function p_t, while the cost of transporting a quantity x_t from the supplier level to the retailer level in period t is given by a concave transportation cost function c_t ($t = 1, \ldots, T$). The cost of carrying a quantity $I_t^{(\ell)}$ in inventory from period t to period $t+1$ at the supplier level (echelon $\ell = 1$) and the retailer level (echelon $\ell = 2$) are given by concave inventory holding cost functions $h_t^{(\ell)}$ ($t = 1, \ldots, T$; $\ell = 1, 2$). Finally, the revenue earned in period t is a concave function R_t of the amount D_t of demand served in period t ($t = 1, \ldots, T$).

2.2.1 Model Formulation

Our two-level requirements planning and pricing problem can now be formulated as follows:

$$\text{maximize} \sum_{t=1}^{T} R_t(D_t) - \sum_{t=1}^{T} \Big(p_t(y_t) + c_t(x_t) + h_t^{(1)}(I_t^{(1)}) + h_t^{(2)}(I_t^{(2)}) \Big)$$

subject to (RPP-2L)

$$I_{t-1}^{(1)} + y_t = x_t + I_t^{(1)} \qquad t = 1, \ldots, T \qquad (2\text{–}1)$$

$$I_{t-1}^{(2)} + x_t = D_t + I_t^{(2)} \qquad t = 1, \ldots, T \qquad (2\text{–}2)$$

$$I_0^{(\ell)} = 0 \qquad \ell = 1, 2 \qquad (2\text{–}3)$$

$$y_t, x_t, D_t \geq 0 \qquad t = 1, \ldots, T$$

$$I_t^{(\ell)} \geq 0 \qquad t = 1, \ldots, T;\ \ell = 1, 2.$$

The objective function represents the difference between revenues and costs. Constraints (2–1) and (2–2) are simply the flow balance constraints at the manufacturer and retailer levels in each period. Constraints (2–3) ensure that the initial inventory at each level is zero. Unfortunately, since the feasible region is unbounded it is not guaranteed that an optimal solution to (RPP-2L) exists. However, existence of an optimal solution may be guaranteed under mild additional conditions, e.g., the existence of finite demand values $\bar{D}_t$ beyond which revenues no longer increase, i.e., with the property that $R_t(D_t) = R_t(\bar{D}_t)$ for all $D_t \geq \bar{D}_t$. We will therefore, in the remainder of this chapter, simply assume that an optimal solution to (RPP-2L) exists.

It is easy to see that we can represent (RPP-2L) as a minimum cost network flow problem as shown in Figure 2-1. In this network, node 0 is the source node, nodes $(1, t)$

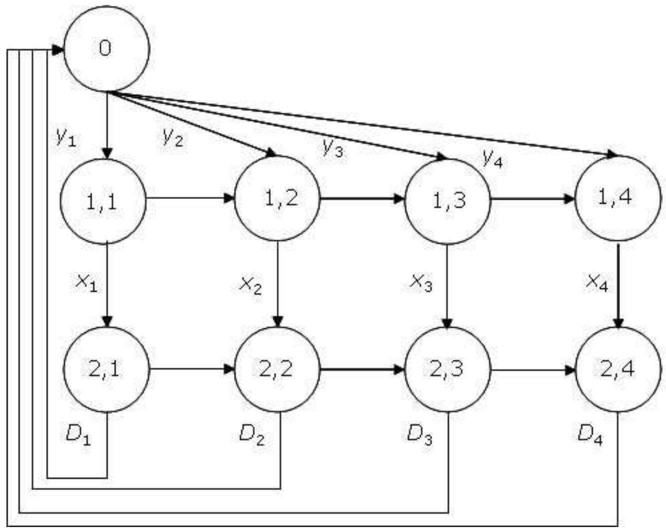

Figure 2-1: Network representation of RPP-2L with 4 periods.

$(t = 1, \ldots, T)$ represent the nodes at the manufacturer level, and nodes $(2, t)$ $(t = 1, \ldots, T)$ represent the nodes at the retailer level. The directed arcs connecting the source node 0 to the nodes $(1, t)$ represent procurement, while the directed arcs connecting nodes $(1, t)$ and $(2, t)$ represent transportation. The directed arcs connecting nodes $(1, t)$ to $(1, t+1)$ and nodes $(2, t)$ to $(2, t+1)$ represent inventory holding. Finally, the directed arcs connecting nodes $(2, t)$ to the source node 0 represent demand. Since the cost associated with the demand arcs corresponding to period t are given by $-R_t(D_t)$, which is a convex function of the flow D_t on this arcs, we unfortunately do not have a concave cost network flow problem. This means, in particular, that we cannot conclude that an extreme point optimal solution exists to the problem. However, if the demand values D_t were *given* we could eliminate the demand arcs and associate a supply of $\sum_{t=1}^{T} D_t$ with the source node and a demand of D_t with node $(2, t)$ $(t = 1, \ldots, T)$. The resulting problem is then the standard two-level lot-sizing problem studied by Zangwill (41), and it is well-known that an extreme point optimal solution exists. Such an extreme point can be characterized by a spanning tree in the network, where only arcs in the spanning tree may carry a positive flow. But this immediately implies that an optimal solution to our problem exists that enjoys the spanning tree structure in the reduced network by simply considering a set of optimal demand values. Our solution approaches in the remainder of this chapter will use this observation extensively. With a slight abuse of terminology, we will refer to solutions that correspond to a spanning tree in the reduced network as "extreme point solutions" or "spanning tree solutions" despite the fact that these solutions are generally not extreme points of the feasible region of (RPP-2L).

2.2.2 Solution Structure

We can decompose any spanning tree solution to (RPP-2L) into components, each of which is characterized by an arc emanating from the source node 0. These components, which we will refer to as *subplans*, have the property that (i) procurement only takes place in one period in the subplan, and (ii) no inventory transfer takes place between subplans. The notion of a subplan resembles the regeneration interval concept of Wagner and Whitin (39) for single-level lot-sizing problems, and was also used by Van Hoesel et al. (23) in the context of multi-level lot-sizing problems with procurement capacities. Figure 2-2 shows an example of a spanning tree solution decomposed in to subplans. Each subplan can be

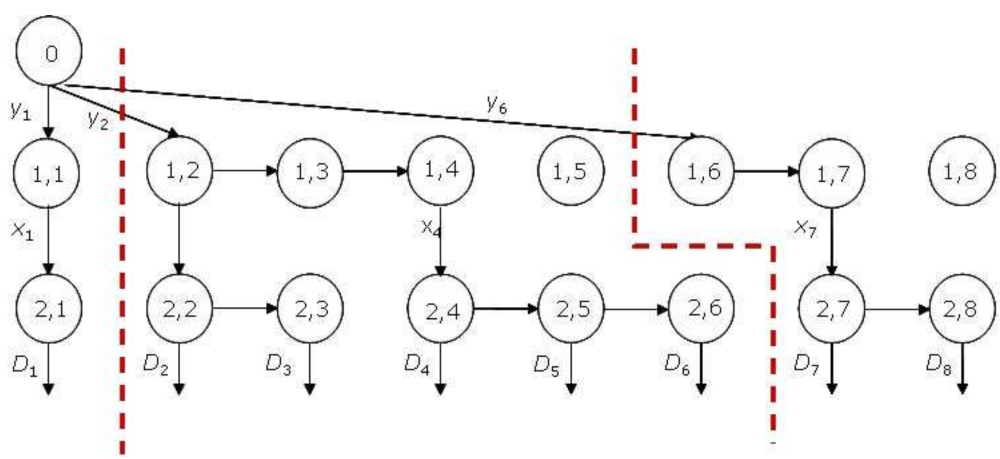

Figure 2-2: Network for (RPP-2L) with dashed lines indicating subplan $(2, 5, 2, 6)$.

characterized by the first and last periods in each of the two levels. That is, a subplan can be represented as $(t_1, t_2, \tau_1, \tau_2)$, where (t_1, t_2) are the first and last periods of the subplan at the manufacturer level while (τ_1, τ_2) are the first and last periods of the subplan at the retailer level. Clearly, $(t_1, t_2, \tau_1, \tau_2)$ is only a valid subplan if $1 \leq t_1 \leq \tau_1 \leq t_2 \leq \tau_2 \leq T$. Note that, without loss of generality, we may assume that the first period in a subplan at the supplier level is the unique procurement period in the subplan.

2.2.3 Dynamic Programming Approach

The observation that (RPP-2L) always contains an optimal solution that can be decomposed into subplans gives rise to a dynamic programming approach. In particular, let $\Pi'(t_1, t_2, \tau_1, \tau_2)$ denote the maximum profit associated with the subplan $(t_1, t_2, \tau_1, \tau_2)$. That is, the function Π determines the maximum profit that can be obtained by serving the retailer in periods $\tau_1, \ldots, \tau_2$ using procurement in period t_1 and allowing transportation to take place only in periods $\tau_1, \ldots, t_2$. Moreover, let $F'(t_1, \tau_1)$ denote the maximum profit that can be obtained by serving the retailer in periods $\tau_1, \ldots, T$ when procurement is allowed in periods $t_1, \ldots, T$ and transportation in periods $\tau_1, \ldots, T$. Then the optimal solution to (RPP-2L) is given by the solution to the following dynamic programming recursion:

$$
\begin{aligned}
F'(t_1, \tau_1) &= \max_{(t_2,\tau_2):\tau_2 \geq t_2 > \tau_1} \{\Pi'(t_1, t_2, \tau_1, \tau_2) + F'(t_2 + 1, \tau_2 + 1)\} && 0 \leq t_1 \leq \tau_1 \leq T \\
F'(t_1, T + 1) &= 0 && 0 \leq t_1 \leq T + 1.
\end{aligned}
$$

This immediately implies that, if the values of Π' are given, the problem can be solved to optimality in $O(T^4)$ time, and the overall running time of the algorithm depends on the time required to find the optimal profit values Π' for each of the $O(T^4)$ subplans.

We can increase the efficiency of the approach if we observe that we do not need to restrict ourselves to extreme point solutions, as long as we can guarantee that we consider at least one optimal solution. This means, in particular, that we may consider *major regeneration intervals* (τ_1, τ_2) at the retailer level (as opposed to the minor regeneration intervals that we will introduce later), and let $\Pi(\tau_1, \tau_2)$ denote the optimal profit that can be obtained from serving the retailer in periods $\tau_1, \ldots, \tau_2$. If we then define $F(\tau_1)$ denote the maximum profit that can be obtained by serving the retailer in periods $\tau_1, \ldots, T$, we have the following result:

Theorem 2.1. *The following dynamic programming recursion solves (RPP-2L):*

$$F(\tau_1) = \max_{\tau_2 : \tau_2 \geq \tau_1} \{\Pi(\tau_1, \tau_2) + F(\tau_2 + 1)\} \qquad \tau_1 = 1, \ldots, T$$

$$F(T + 1) = 0.$$

Proof. Note that $\Pi(\tau_1, \tau_2)$ denotes the optimal profit that can be obtained from serving the retailer in periods $\tau_1, \ldots, \tau_2$ using procurement in one of the periods $1, \ldots, \tau_1$ and allowing transportation to take place in periods $1, \ldots, t_2$. It therefore follows immediately that $\Pi(\tau_1, \tau_2) \geq \Pi'(t_1, t_2, \tau_1, \tau_2)$ for all $1 \leq t_1 \leq \tau_1 \leq t_2 \leq \tau_2 \leq T$. This, in turn, implies that any solution that is considered by the dynamic programming recursion for F' is implicitly considered in the dynamic programming recursion for F as well.

Next, consider any sequence of major regeneration intervals obtained by solving the recursion for F. Fixing the corresponding demands, each of these major regeneration intervals has an associated partial procurement plan in (RPP-2L). These plans can then be aggregated to obtain a feasible solution to the (RPP-2L). If the corresponding flows in the network for the (RPP-2L) are disjoint this is an extreme point solution and the length of the corresponding path in the dynamic programming network for F is equal to the profit of this solution. Otherwise, we do not have an extreme point solution and, by the concavity of the cost functions, the length of the corresponding path in the dynamic programming network for F may fall short of the profit of this solution. This, together with the earlier part of the proof, implies that solving the dynamic programming recursion for F yields an optimal solution to (RPP-2L). □

This result now implies that, if the values of Π are given, the problem can be solved to optimality in $O(T^2)$ time, and the overall running time of the algorithm depends on the time required to find the optimal profit values Π for each of the $O(T^2)$ major regeneration intervals. Much of the remainder of our work will therefore focus on these subproblems.

2.2.4 Dynamic Programming Subproblems

Consider a particular major regeneration interval (τ_1, τ_2). In order to determine the value of $\Pi(\tau_1, \tau_2)$, we first consider the maximum profit that can be obtained by supplying demand in the periods in the major regeneration interval from procurement in some fixed period $t_1 \le \tau_1$ with transportation allowed in periods $t_1, \dots, \tau_2$. This value is the optimal solution value to the following optimization subproblem:

$$\text{maximize} \sum_{t=\tau_1}^{\tau_2} R_t(D_t) - p_{t_1}\left(\sum_{t=\tau_1}^{\tau_2} D_t\right) - \sum_{t=t_1}^{\tau_2}\left[c_t(x_t) + h_t^{(1)}\left(\sum_{t'=\tau_1}^{\tau_2} D_{t'} - \sum_{t'=\tau_1}^{t} x_{t'}\right) + h_t^{(2)}(I_t^{(2)})\right]$$

subject to $\qquad\qquad (\text{SP}^{(\tau_1,\tau_2;t_1)})$

$$\begin{aligned}
I_{t-1}^{(2)} + x_t &= I_t^{(2)} && t = t_1, \dots, \tau_1 - 1 \\
I_{t-1}^{(2)} + x_t &= D_t + I_t^{(2)} && t = \tau_1, \dots, \tau_2 \\
I_{t_1-1}^{(2)} &= 0 && \\
x_t, I_t^{(2)} &\ge 0 && t = t_1, \dots, \tau_2 \\
D_t &\ge 0 && t = \tau_1, \dots, \tau_2.
\end{aligned}$$

We will denote the optimal value to this optimization problem by $\Pi(\tau_1, \tau_2; t_1)$. Then, noting that it is of course feasible to not satisfy any demands in a major regeneration interval, it follows that

$$\Pi(\tau_1, \tau_2) = \max\left\{0, \max_{t_1=1,\dots,\tau_1} \Pi(\tau_1, \tau_2; t_1)\right\}.$$

In the next section we will therefore focus on efficient algorithms for finding the values $\Pi(\tau_1, \tau_2; t_1)$.

2.3 Generalized Single-Level Requirements Planning Problem

For notational convenience, we will study the following single-level requirements planning problem with pricing:

$$\text{maximize } \sum_{t=1}^{T} R_t(D_t) - p\left(\sum_{t=1}^{T} D_t\right) - \sum_{t=1}^{T}\left[c_t(x_t) + g_t\left(\sum_{t'=1}^{T} D_{t'} - \sum_{t'=1}^{t} x_{t'}\right) + h_t(I_t)\right]$$

subject to (RPP)

$$\begin{aligned} I_{t-1} + x_t &= D_t + I_t && t = 1, \ldots, T \\ I_0 &= 0 \\ x_t, D_t, I_t &\geq 0 && t = 1, \ldots, T. \end{aligned}$$

Note that $(\mathrm{SP}^{(\tau_1,\tau_2;t_1)})$ is obtained as an instance of (RPP) by relabeling the planning periods $1, \ldots, T$ as $t_1, \ldots, \tau_2$; choosing the functions $p = p_{t_1}$, $g_t = h_t^{(1)}$, and $h_t = h_t^{(2)}$; and replacing the revenue functions R_t for $t = t_1, \ldots, \tau_1 - 1$ by a function that is identically zero. (Note that setting the revenue functions from the production period through the period preceding the first demand period ensures that, without loss of optimality, no demand is satisfied in these earlier periods.) This problem is of independent interest, since it generalizes the single-level requirements planning problems with pricing as studied by Thomas (36) and Geunes et al. (16). The generalizations lie in the incorporation of a procurement function p and inventory holding cost functions g_t $(t = 1, \ldots, T)$, as well as allowing for nonlinearity of the holding cost functions h_t $(t = 1, \ldots, T)$. Note that, in this context, p and g_t could be interpreted as procurement and holding cost functions for raw materials, respectively.

If all revenue and cost functions are general concave functions, the (RPP) is a global optimization problem (see, e.g., Horst and Tuy (24)), so that we cannot expect to be able to efficiently solve it in general. In this chapter, we will focus on several particular structures for the revenue and cost functions that allow for solution algorithms for the subproblems that have a running time that is polynomial in the planning horizon T.

These can then be applied to the subproblems ($\mathrm{SP}^{(\tau_1,\tau_2;t_1)}$) to obtain polynomial-time algorithms for solving the (RPP-2L) itself.

2.3.1 Linear Procurement and Holding Cost Functions

For ease of exposition, we will start by investigating problems where the procurement cost function p as well as the inventory holding cost functions g_t and h_t are linear:

$$\begin{aligned} p(y) &= \rho y & & \\ g_t(I) &= \xi_t I & & t = 1, \ldots, T \\ h_t(I) &= \zeta_t I & & t = 1, \ldots, T. \end{aligned}$$

If we then define

$$\begin{aligned} \bar{R}_t(D) &= R_t(D) - \left(\rho + \sum_{t'=1}^{T} \xi_{t'}\right) D & & t = 1, \ldots, T \\ \bar{c}_t(x) &= c_t(x) - \left(\sum_{t'=t}^{T} \xi_{t'}\right) x & & t = 1, \ldots, T \end{aligned}$$

the objective function of (RPP) reduces to

$$\sum_{t=1}^{T} \bar{R}_t(D_t) - \sum_{t=1}^{T} \left[\bar{c}_t(x_t) + \zeta_t I_t\right].$$

The problem can then be solved by using an observation similar to the one we noted and exploited for the (RPP-2L): we only need to consider solutions to the (RPP) that have an extreme point structure for a given set of demands. In other words, we only need to consider consecutive sequences of demands served by a single shipment, i.e., sequences of regeneration intervals of periods $1 \leq \tau_1, \ldots, \tau_2 \leq T$, which we will denote by $[\tau_1, \tau_2]$ and refer to as *minor regeneration intervals*. Now denote the optimal solution value to the following problem by $\bar{\pi}[\tau_1, \tau_2]$:

$$\max_{D_{\tau_1}, \ldots, D_{\tau_2}} \sum_{t=\tau_1}^{\tau_2} \left[\bar{R}_t(D_t) - \left(\sum_{t'=\tau_1}^{t-1} \zeta_{t'}\right) D_t\right] - \bar{c}_{\tau_1}\left(\sum_{t'=\tau_1}^{\tau_2} D_{t'}\right). \qquad (2\text{–}4)$$

The optimal profit in a minor regeneration interval $[\tau_1, \tau_2]$ is then given by $\pi[\tau_1, \tau_2] = \max\{0, \bar{\pi}[\tau_1, \tau_2]\}$, where it is interesting to note that the optimal total demand satisfied in minor regeneration interval $[\tau_1, \tau_2]$ is positive if $\pi[\tau_1, \tau_2] > 0$. Given these optimal minor regeneration interval profits, the (RPP) can be solved in $O(T^2)$ time using a standard dynamic programming recursion.

2.3.2 Fixed-Charge Procurement and Inventory Holding Cost Functions

We now generalize our results to allow for fixed setup costs in both procurement and inventory holding cost functions:

$$p(y) = \begin{cases} 0 & \text{if } y = 0 \\ P + \rho y & \text{if } y > 0 \end{cases}$$

$$h_t^{(\ell)}(I^{(\ell)}) = \begin{cases} 0 & \text{if } I^{(\ell)} = 0 \\ H_t^{(\ell)} + \zeta_t^{(\ell)} I^{(\ell)} & \text{if } I^{(\ell)} > 0 \end{cases} \qquad t = 1, \ldots, T;\ \ell = 1, 2.$$

It is easy to see that the inclusion of a fixed-charge in the procurement cost function is a trivial extension of the results in the previous section: we simply subtract the fixed cost P from the optimal profit, and decide to not satisfy any demand if the resulting value is nonpositive. However, more care is required to account for the fixed-charge structure in the inventory holding cost functions. First, it is immediate that we again only need to consider solutions to the (RPP) that consist of a sequence of minor regeneration intervals $[\tau_1, \tau_2]$. Provided that (i) the demand in the last period of a minor regeneration interval is positive ($D_{\tau_2} > 0$), and (ii) the total demand in a minor regeneration interval with $\tau_2 = T$ is positive ($\sum_{t=\tau_1}^{T} > 0$), the fixed charges corresponding to the inventory costs at either level are *independent* of the actual magnitudes of the demands satisfied in the minor regeneration interval. Now observe that we may without loss of generality assume that the first provision holds, i.e., that $D_{\tau_2} > 0$, as long as we explicitly consider the possibility of satisfying no demand at all in a minor regeneration interval. This means that, if the inventory holding costs at the supplier level are *linear* (i.e., $H_t^{(1)} = 0$ for $t = \tau_1, \ldots, \tau_2$), we then have that the optimal profit in a minor regeneration interval $[\tau_1, \tau_2]$ is given by

$$\pi[\tau_1, \tau_2] = \max \left\{ 0, \bar{\pi}[\tau_1, \tau_2] - \sum_{t=\tau_1}^{\tau_2 - 1} H_t^{(2)} \right\}$$

where $\bar{\pi}[\tau_1, \tau_2]$ is as in Section 2.3.1. Note that again the optimal set of demands in minor regeneration interval $[\tau_1, \tau_2]$ is positive if $\pi[\tau_1, \tau_2] > 0$.

The situation with respect any fixed charges at the supplier level is more complicated. If we assume that the total demand satisfied in the last minor regeneration interval is strictly positive, we have that the fixed inventory holding cost charges at the supplier level are independent of the magnitudes of the demands, so that we can incorporate these by defining:

$$\pi[\tau_1, \tau_2] = \begin{cases} \max \left\{ 0, \bar{\pi}[\tau_1, \tau_2] - \sum_{t=\tau_1}^{\tau_2 - 1} H_t^{(2)} \right\} & \text{if } \tau_2 < T \\ \max \left\{ 0, \bar{\pi}[\tau_1, \tau_2] - \sum_{t=\tau_1}^{\tau_2 - 1} H_t^{(2)} \right\} - \sum_{t=1}^{\tau_1 - 1} H_t^{(1)} & \text{if } \tau_2 = T. \end{cases}$$

Noting that all partial sums of fixed holding cost charges can be determined in $O(T^2)$ time, we immediately conclude that the running time for the (RPP) in the presence of fixed-charge procurement and holding cost functions under the assumption above is the same as for the linear cost cases discussed in Section 2.3.1.

If we solve the (RPP) as a subproblem of the (RPP-2L), we can assume without loss of generality that the total demand satisfied in the last minor regeneration interval is indeed positive since we explicitly account for major regeneration intervals in which all demands are zero. However, when solving a stand-alone instance of (RPP) we have to otherwise allow for the possibility that the last minor regeneration interval has zero demands. We could achieve that by solving the (RPP) with any planning horizon $t = 1, \ldots, T$ and choosing the planning horizon that achieves the maximum profit, which would increase the running time by a factor of $O(T)$. A computationally more efficient approach is to redefine the optimal profit for minor regeneration intervals of the form $[\tau_1, T]$ by allowing for not satisfying any demands in periods $\tau_2 + 1, \ldots, T$ for some

$\tau_2 \in \{\tau_1, \ldots, T\}$. Summarizing, this means that the optimal minor regeneration interval profits can be defined as:

$$\pi[\tau_1, \tau_2] = \max\left\{0, \bar{\pi}[\tau_1, \tau_2] - \sum_{t=\tau_1}^{\tau_2-1} H_t^{(2)}\right\} \qquad \tau_2 = 1, \ldots, T-1$$

$$\pi[\tau_1, T] = \max_{\tau_2=\tau_1,\ldots,T}\left\{\max\left\{0, \bar{\pi}[\tau_1, \tau_2] - \sum_{t=\tau_1}^{\tau_2-1} H_t^{(2)}\right\} - \sum_{t=1}^{\tau_1-1} H_t^{(1)}\right\}.$$

2.3.3 Running Time Analysis for the (RPP)

While the (RPP) can be solved in $O(T^2)$ time when given the optimal minor regeneration interval profits $\pi[\tau_1, \tau_2]$, the actual running time of our approach to the (RPP) of course depends on the ability to efficiently determine these minor regeneration interval profits. First, note that *all* values

$$\sum_{t=t'}^{T} \xi_t \qquad t' = 1, \ldots, T$$

$$\sum_{t=t'}^{t''} \zeta_t \qquad t'' = t', \ldots, T;\ t' = 1, \ldots, T$$

$$\sum_{t=t'}^{t''} H_t^{(\ell)} \qquad t'' = t', \ldots, T;\ t' = 1, \ldots, T;\ \ell = 1, 2$$

can be determined in a preprocessing step in $O(T^2)$ time. In the remainder of this section, we will study the time required to solve (RPP) under some important classes of revenue and cost functions.

2.3.3.1 Piecewise-linear concave transportation cost functions

The variant of the (RPP) with piecewise-linear concave transportation cost functions was studied in Geunes et al. (15; 16) (and also in Thomas (36) for the special case of fixed charge transportation costs). In particular, consider the expression of each of these cost functions as the minimum of no more than K fixed-charge functions as follows:

$$\bar{c}_t(x) = \begin{cases} 0 & \text{if } x = 0 \\ \min_{k=1,\ldots,K}\{C_{tk} + \gamma_{tk}x\} & \text{if } x > 0 \end{cases} \qquad t = 1, \ldots, T.$$

Solving the optimization problem (2–4) then becomes

$$\max_{k=1,\dots,K} \max_{D_{\tau_1},\dots,D_{\tau_2}} \sum_{t=\tau_1}^{\tau_2} \left[\bar{R}_t(D_t) - \left(\gamma_{\tau_1 k} + \sum_{t'=\tau_1}^{t-1} \zeta_{t'} \right) D_t \right].$$

Solving this problem thus reduces to solving the following collection of independent one-dimensional concave maximization problems

$$\max_{D_t} \quad \bar{R}_t(D_t) - \left(\gamma_{\tau_1 k} + \sum_{t'=\tau_1}^{t-1} \zeta_{t'} \right) D_t \qquad t = \tau_1, \dots, \tau_2;\ k = 1, \dots, K.$$

If we assume that all $O(KT)$ of these problems can be solved in constant time, the values of $\bar{\pi}[\tau_1, \tau_2]$ can be found in $O(KT^2)$ time. This implies that the corresponding variant of the (RPP) is solvable in $O(KT^2 + T^2) = O(KT^2)$ time, which generalizes the result from Thomas (36) to the case of $K > 1$.

2.3.3.2 Piecewise-linear concave revenue functions

To analyze problem instances with piecewise-linear concave revenue functions, it is convenient to express these functions through their segments rather than as the minimum of fixed-charge functions as we did in Section 2.3.3.1. If the revenue functions have no more than J (positive-slope) segments, they can be written as

$$\bar{R}_t(D) = \sum_{j=1}^{k-1} r_{jt} d_{jt} + r_{kt} \left(D - \sum_{j=1}^{k-1} d_{jt} \right) \qquad \sum_{j=1}^{k-1} d_{jt} \leq D < \sum_{j=1}^{k} d_{jt};\ k = 1, \dots, J$$

where $0 \leq r_{1t} \leq \cdots \leq r_{Jt}$. We then introduce decision variables $z_{jt} \in [0, 1]$ that represent the fraction of demand segment j in period t that is satisfied. Problem (2–4) can now be reformulated as follows:

$$\max_{z_{jt} \in [0,1]:\, j=1,\dots,J;\, t=\tau_1,\dots,\tau_2} \sum_{t=\tau_1}^{\tau_2} \sum_{j=1}^{J} \left(r_{jt} - \sum_{t'=\tau_1}^{t-1} \zeta_{t'} \right) d_{jt} z_{jt} - \bar{c}_{\tau_1} \left(\sum_{t'=\tau_1}^{\tau_2} \sum_{j=1}^{J} d_{jt'} z_{jt} \right). \qquad (2--4')$$

Due to the concavity of $\bar{c}_{\tau_1}$, we apply a result from Huang et al. (26) to conclude that a binary optimal solution exists to this problem with the property that segment/period pairs

(j, t) are satisfied in nonincreasing order of

$$\mathcal{R}_{jt;\tau_1} \equiv \frac{\left(r_{jt} - \sum_{t'=\tau_1}^{t-1} \zeta_{t'}\right) d_{jt}}{d_{jt}} = r_{jt} - \sum_{t'=\tau_1}^{t-1} \zeta_{t'}.$$

This means that an optimal solution to each of the $O(T^2)$ problems of the form (2–4′) can be found in $O(JT)$ time *given* that the relevant pairs (j, t) are sorted in nonincreasing order of $\mathcal{R}_{jt;\tau_1}$, for a total time of $O(JT^3)$. Now consider all $O(JT^2)$ values $\mathcal{R}_{jt;\tau_1}$ and sort each of the $O(T)$ sets of such values grouped by a common value of τ_1, the total time required for sorting for *all* $O(T^2)$ problems of the form (2–4′) is $O\left(JT^2 \log(JT)\right)$, so that this variant of the (RPP) is solvable in $O\left(JT^3 + JT^2 \log(JT) + T^2\right) = O\left(JT^3 + JT^2 \log(JT)\right)$ time. We also refer to Deng and Yano (8) and Geunes et al. (15; 16), who studied a variant of this problem with time-invariant transportation capacities, no procurement costs, fixed-charge transportation cost functions, and linear inventory holding cost functions.

2.4 Running Time Analysis for the (RPP-2L)

In the previous sections we presented solution algorithms and their running times for the (RPP) and noted that running time of our solution approach for the (RPP-2L) depends heavily on the running times of the subproblems (RPP). As noted before, we can solve the (RPP-2L) in $O(T^2)$ time if all values of $\Pi(\tau_1, \tau_2; t)$ are given. Since there are $O(T^3)$ values of $\Pi(\tau_1, \tau_2; t)$, and each of these involves solving an instance of the (RPP), it is likely that computing these values will form the bottleneck of the solution approach. A *straightforward* implementation of our solution approach therefore yields the following bounds on the worst-case running times, where we emphasize the different components of the algorithm:

(i) *Piecewise-linear concave transportation cost functions*
In this case, the (RPP-2L) can be solved in $O(KT^5 + T^5)$ time, where $O(KT^5)$ corresponds to the time required to find the minor regeneration interval profits for each of the $O(T^3)$ corresponding instances of the (RPP) that needs to be solved, and $O(T^5)$ corresponds to the time required to solve all $O(T^3)$ instances of the (RPP) given these minor regeneration interval profits.

(ii) *Piecewise-linear concave revenue functions*
In this case, the (RPP-2L) can be solved in $O\left(JT^6 + JT^5 \log(JT) + T^5\right)$ time, where $O\left(JT^6 + \;\; JT^5 \log(JT)\right)$ corresponds to the time required to find the minor regeneration interval profits for each of the $O(T^3)$ corresponding instances of the (RPP) that needs to be solved, and $O(T^5)$ corresponds to the time required to solve all $O(T^3)$ instances of the (RPP) given these minor regeneration interval profits.

However, it is easy to see that the instances of (RPP) that need to be solved are very closely related. We can obtain tighter running time bounds by exploiting this relationship.

Our first two observations apply to all instances of the (RPP-2L). First, note that in the instance of (RPP) that corresponds to $\Pi(\tau_1, \tau_2; t_1)$, the revenue functions for the first $\tau_1 - t_1$ periods have been set to zero. Due to the definition of the minor regeneration intervals, we can assume without loss of optimality that the first transportation does not occur before period τ_1. This then means that we only need to consider the values of $\bar{\pi}[\tau_1', \tau_2']$ with $\tau_1' \geq \tau_1$ thereby ignoring the possibility of having 0 revenue functions in $[\tau_1', \tau_2']$. This makes the value of $\overline{\pi}[\tau_1', \tau_2']$ only a function of procurement period for the (RPP). Now suppose that we start by computing the value of $\Pi(t, T; t)$. In doing so, we in fact end up determining all relevant values $\bar{\pi}[\tau_1', \tau_2']$ that are required to compute the values of $\Pi(\tau_1, \tau_2; t)$ for $\tau_1 = t, \ldots, T$ $\tau_2 = \tau_1, \ldots, T$ as well. That is, we obtain all minor regeneration interval profits to be used in the computation of (RPP) problems that share the same procurement period. This implies that the time required to find all the minor regeneration interval profits is reduced by a factor of T.

Next, note that determining the value of $\Pi(\tau_1, \tau_2; t)$ via a forward dynamic programming recursion yields, at no additional cost, the values of $\Pi(\tau_1, \tau; t)$ for all $\tau_1 \leq \tau \leq \tau_2$. Therefore, it suffices to solve the $O(T^2)$ dynamic programming problems associated with $\Pi(\tau_1, T; t)$ $(t = 1, \ldots, T;\ \tau_1 = t, \ldots, T)$. This reduces the time required to find all major regeneration interval profits *given* the minor regeneration interval profits by a factor of T.

Our final observation only pertains to the case where the revenue functions are piecewise-linear. Recall that, when solving the associated instances of (RPP), we need to order values $\mathcal{R}_{j\tau;\tau_1'}$ for each $\tau_1' = 1, \ldots, T$ (where the instance of the (RPP) also depends on t, the production period specified in the underlying value of $\Pi(\tau_1, \tau_2; t)$). Instead of ordering these values for each instance of the (RPP), we can instead find all $O(T^2)$ such orderings in a preprocessing stage, which takes $O\left(JT^3 \log(JT)\right)$ time.

Taking these savings into account shows that our solution approach to the (RPP-2L) can be implemented with the following worst-case running times for the two cost and revenue structures considered:

(i) *Piecewise-linear concave transportation cost functions*
In this case, the (RPP-2L) can be solved in $O(T^4 + KT^3)$ time.

(ii) *Piecewise-linear concave revenue functions*
In this case, the (RPP-2L) can be solved in $O\left(JT^4 + JT^3 \log(JT)\right)$ time.

2.5 Linear Cost Functions

In the remainder of this chapter we will investigate more efficient implementations of our solution approach when some or all cost functions are linear.

2.5.1 Linear Procurement and Supplier Inventory Holding Cost Functions

In the two cases that we analyzed above, we allowed for the presence of fixed charges in procurement and inventory holding cost functions. Due to the resulting concavity of the cost functions, we had to search for the optimal procurement period t for every major regeneration interval (τ_1, τ_2). If, instead, procurement and supplier inventory holding cost functions are linear we can obtain significant running time savings. Using similar notation as in Section 2.3.1, we let

$$p_t(y) = \rho_t y \qquad t = 1, \ldots, T$$

$$h_t^{(1)}(I) = \xi_t I \qquad t = 1, \ldots, T.$$

In that case, given that a unit of product is transported in, say, period τ, we can determine the optimal procurement period for that unit *independently* of the procurement quantities in any period. In particular, the procurement and inventory holding costs at the supplier level for each unit transported in period τ is given by

$$\min_{t=1,\ldots,\tau} \; \rho_t + \sum_{t'=t}^{\tau-1} \xi_{t'}$$

where, of course, a unit transported in period τ is procured in period $q_\tau = \arg\min_{t=1,\ldots,\tau} \rho_t + \sum_{t'=t}^{\tau-1} \xi_{t'}$. This means that, in this case, the (RPP-2L) actually reduces to an instance of the (RPP) with

$$\begin{aligned} p(y) &= 0 & & \\ g_t(I) &= 0 & & t = 1,\ldots,T \\ h_t(I) &= \zeta_t I & & t = 1,\ldots,T \end{aligned}$$

and transportation cost functions

$$\hat{c}_t(x) = \left(\rho_{q_t} + \sum_{t'=t}^{\tau-1} \xi_{t'}\right) x + c_t(x) \qquad t = 1,\ldots,T.$$

Now observe from the analysis in Section 2.3 that the fact that p and g_t $(t = 1,\ldots,T)$ are identically zero does not impact the running time of our approach. Moreover, the transportation cost functions can be transformed in $O(T)$ time, so that this instance of the (RPP-2L) is of the same difficult as the associated instance of the (RPP).

2.5.2 Linear Procurement, Transportation, Holding Cost Functions

Now suppose that, in addition, the transportation and retailer inventory holding cost functions are linear as well:

$$\begin{aligned} c_t(x) &= \gamma_t x & & t = 1,\ldots,T \\ h_t^{(2)}(I) &= \zeta_t I & & t = 1,\ldots,T. \end{aligned}$$

In that case, we can determine the shortest paths from node 0 to all nodes of the form $(2,t)$ $(t = 1,\ldots,T)$ in the graph in Figure 2-1 in $O(T)$ time since the network is acyclic and has $O(T)$ arcs. Denoting the lengths of these shortest paths by f_t $(t = 1,\ldots,T)$, the

(RPP-2L) then simply reduces to

$$\max_{D_1,\ldots,D_T} \quad \sum_{t=1}^{T} \left[R_t(D_t) - f_t D_t\right]$$

which decomposes into the following T independent maximization problems:

$$\max_{D_t} \quad R_t(D_t) - f_t D_t \qquad\qquad t = 1, \ldots, T.$$

If each of these problems can be solved in constant time, the variant of the (RPP-2L) with concave revenue functions and linear cost functions is solvable in $O(T)$ time.

2.5.3 Linear Transportation and Inventory Holding Cost Functions

Finally, we consider the case where *only* the transportation and inventory holding cost functions are linear, with the same notation as in the sections above. In this case we have that, for all procurement periods $t = 1, \ldots, T$, the unit transportation and inventory holding costs for satisfying demand in period τ can be denoted by $f(t, \tau)$. These values are the all-to-all shortest path distances in a network with $O(T)$ arcs and edges, so that they can be determined in $O(T^2)$ time. Given these values, determining $\Pi(\tau_1, \tau_2; t_1)$ amounts to solving the optimization problem

$$\max_{D_{\tau_1},\ldots,D_{\tau_2}} \sum_{t=\tau_1}^{\tau_2} \left[R_t(D_t) - f(t_1, t) D_t\right] - p_{t_1}\left(\sum_{t'=\tau_1}^{\tau_2} D_{t'}\right).$$

For general concave revenue and procurement cost functions this is a global optimization problem. However, if the revenue functions are piecewise-linear concave as in Section 2.3.3.2 this problem can be formulated as

$$\max_{z_{jt}\in[0,1]:\, j=1,\ldots,J;\, t=\tau_1,\ldots,\tau_2} \sum_{t=\tau_1}^{\tau_2} \sum_{j=1}^{J} \left(r_{jt} - f(t_1, t)\right) d_{jt} z_{jt} - p_{t_1}\left(\sum_{t'=\tau_1}^{\tau_2} \sum_{j=1}^{J} d_{jt'} z_{jt}\right)$$

and can therefore be solved in $O(JT)$ time *given* the ordering of segment/period pairs (j, t) in nonincreasing order of

$$\mathcal{R}_{jt;\tau_1} \equiv \frac{\left(r_{jt} - f(t_1, t)\right) d_{jt}}{d_{jt}} = r_{jt} - f(t_1, t)$$

which, for a fixed procurement period t_1, can be found in $O(JT\log(JT))$ time. Since there are $O(T^3)$ values of $\Pi(\tau_1, \tau_2; t_1)$ that need to be determined, we conclude that the variant of the (RPP-2L) with piecewise-linear revenue functions, concave procurement functions, and linear transportation and inventory holding cost functions can be solved in $O(JT^4 + JT^2\log(JT))$ time.

2.6 Conclusion and Future Research

In this chapter we considered a class of problems that generalizes both single-echelon requirements planning and pricing problems as well as two-echelon requirements planning problems by considering demand flexibility through pricing decisions as well as procurement, transportation, and inventory decisions in an integrated model. We developed a dynamic programming framework for solving this class of problems under concave cost functions and proposed polynomial time algorithms to the problem under a number of different and practically relevant revenue and cost function structures.

Future research in this area would include analyzing situations where the price is restricted to be a constant over the entire planning horizon. In many applications, procurement capacities are relevant as well, and incorporating such constraints will be an important extension of the work in this chapter. Although the problem is known to be NP-hard in general under procurement capacities, it would be interesting to identify cases that can be solved in polynomial time. Another important generalization would simultaneously consider multiple items. Finally, a study of the structure of a mixed-integer or global optimization formulation of the problem may lead to effective branch-and-price algorithms for the problem studied in this chapter as well as extensions.

CHAPTER 3
MULTI-ITEM CAPACITATED LOT-SIZING PROBLEM WITH SETUP TIMES AND PRICING DECISIONS

3.1 Introduction

We consider the capacitated lot-sizing problem with setup times and pricing decisions (CLSTP). This problem generalizes the well-known capacitated lot-sizing problem with setup times (CLST) by incorporating pricing decisions for all items, thereby making demand endogenous to the model rather than exogenous. In the CLST, demands for multiple items over a finite and discrete planning horizon are given in advance and they are satisfied by producing in a single common facility with limited available resource time. The problem is to find the production plan that minimizes total costs of production and inventory carriage to satisfy the given demands. In contrast, in the CLSTP demands are not given but, through pricing decisions, the demand levels to satisfy for each item in each period can effectively be chosen. Since the demand levels influence both revenues and costs, the goal becomes to maximize profit, which is defined to be total revenues minus total cost of satisfying the resulting demands.

Single-item uncapacitated lot-sizing problems were first studied in the seminal work of Wagner and Whitin (39). Thomas (36) was the first to incorporate pricing decisions in an uncapacitated lot-sizing model, assuming a dynamic pricing strategy (i.e., prices can vary over time). Kunreuther and Schrage (28) developed a heuristic approach to solve this problem for the case where a constant price should be set for the item over the planning horizon by restricting the form of the demand or revenue functions. While Gilbert (18) was the first to develop an exact algorithm for this problem that runs in polynomial time under further restricted demand functions, Van den Heuvel and Wagelmans (2006) proposed a polynomial-time algorithm to the original problem posed by Kunreuther and Schrage (28). Single-item lot-sizing problems with finite but stationary production capacities under constant and dynamic pricing strategies were studied by Geunes et al. (14; 16). Finally, Gilbert (19) considered a multi-product planning problem with constant-priced goods that share procurement capacity under linear procurement cost functions.

We will next formally describe the CLST and review earlier work on this problem. Assume there are demands for N items to be satisfied over a planing horizon of T periods. Demand for item i in period t is denoted by d_{it}. The amount of resource time available in the production facility in period t is C_t. (In the remainder of this chapter we will assume that $C_t > 0$ for $t = 1, \ldots, T$. If this is not the case we reduce the model by eliminating the corresponding setup and production variables and capacity constraints from the model.) Production for an item i in period t can only take place after a setup is performed, which incurs a fixed cost S_{it} and consumes a fixed amount of the available resource time b_{it}. In addition, for each unit item produced a variable production cost c_{it} is incurred and variable resource time a_{it} is consumed. Let x_{it} be the amount of production for item i in period t. Moreover, let the setup indicator variable y_{it} equal 1 if a setup takes place for item i in period t and 0 otherwise. Inventory for item i at the end of period t, I_{it}, is carried to the next period incurring a cost of h_{it} per unit carried, and no back-orders are allowed. With this setting, the CLST is formulated as follows.

$$\text{minimize} \sum_{i=1}^{n} \sum_{t=1}^{T} (S_{it} y_{it} + c_{it} x_{it} + h_{it} I_{it}) \tag{3–1}$$

subject to

$$\sum_{i=1}^{N} (b_{it} y_{it} + a_{it} x_{it}) \leq C_t \qquad t = 1, \ldots, T \tag{3–2}$$

$$I_{i,t-1} + x_{it} = d_{it} + I_{it} \qquad t = 1, \ldots, T;\ i = 1, \ldots, N \tag{3–3}$$

$$x_{it} \leq M_{it} y_{it} \qquad t = 1, \ldots, T;\ i = 1, \ldots, N \tag{3–4}$$

$$y_{it} \in \{0, 1\} \qquad t = 1, \ldots, T;\ i = 1, \ldots, N \tag{3–5}$$

$$I_{it}, x_{it} \geq 0 \qquad t = 1, \ldots, T;\ i = 1, \ldots, N \tag{3–6}$$

$$I_{iT} = I_{i0} = 0 \qquad i = 1, \ldots, N. \tag{3–7}$$

In this formulation, objective function (3–1) minimizes the total production and inventory carrying costs. Constraints (3–2) are the capacity constraints; constraints (3–3) are the flow balance constraints; constraints (3–4) are the setup forcing constraints (where M_{it} is an upper bound on the production of item i in period t that is satisfied in without loss of optimality); and constraints (3–5) and (3–6) ensure integrality and nonnegativity of the decision variables. Constraints (3–7) ensure that both initial and final inventories are zero. Note that nonzero fixed initial and terminal inventory levels can be incorporated by appropriately modifying the demand pattern since, without loss of generality, we can assume that demand will be satisfied according to a first-in first-out policy; moreover, if the terminal inventory levels are decision variables they can also be assumed equal to zero without loss of generality due to the problem's cost structure.

Much research has been done to reduce the integrality gap of formulation (3–1)–(3–7) for the CLST. Lagrangean relaxation and Dantzig-Wolfe decomposition are two of the techniques that have been studied to find improved lower bounds. The two methods are equivalent in that one is the dual of the other and they both exploit the structure of the problem such that when tying capacity constraints (3–2) are removed, we are left with independent single item lot-size problems. Trigeiro et al. (37) and Hindi et al. (21) are two examples of heuristic solution techniques that rely on the Lagrangean relaxation and subgradient optimization of the problem obtained by relaxing the capacity constraints.

The idea behind applying Dantzig-Wolfe decomposition to the CLST is to write feasible solutions as a convex combination of extreme points of convex hulls of lot-sizing polytopes, which results in a tighter formulation since the subproblems, which are single-item lot-sizing problems, do not have the integrality property. In an early attempt, Manne (30) proposed an LP formulation of the CLST by defining the concept of dominant production schedule for individual items, which are otherwise known as schedules that have the so-called zero-inventory ordering (ZIO) property: $I_{i,t-1}x_{it} = 0$ for $i = 1, \dots, N$ and $t = 1, \dots, T$ (see Wagner and Whitin (39)). In other words, dominant production schedules are ones in which demand for an item in a period is satisfied by production in the most recent period in which a set up is carried out. Manne then only considers production schedules for the CLST that are convex combinations of dominant production plans. It is easy to see that the number of dominant schedules is $O(2^T)$ for each item. Later, Dzielinski and Gomory (9) described a Dantzig-Wolfe decomposition approach and a column generation method to generate dominant production plans when they are needed. This solved the problem of prior generation of large number of production schedules to consider in Manne's formulation. Unfortunately, however, Manne's formulation is *not* equivalent to the CLST since, by construction, any integral solution will consist only of ZIO production plans for each item, while it is well-known that, in the presence of capacity constraints, an optimal production plan may produce even in periods in which the starting inventory is nonzero (see Florian and Klein (10)). Nevertheless, Manne shows that the LP-relaxation of his formulation will give a good lower bound to the CLST since, in any basic feasible solution, the number of dominant production schedules with fractional weights is no more than the number of capacity constraints. Recently, Degraeve and Jans (7) developed a correct Dantzig-Wolfe formulation of the CLST (along with a corresponding branch-and-price algorithm) by showing that it is sufficient to extend the collection of dominant production plans with periods in which a setup takes place but no production, leading to a total of $O(3^T)$ production plans for each item.

In this chapter, we develop two alternative Dantzig-Wolfe decomposition formulations for the CLSTP along with corresponding branch-and-price algorithms. The column generation approach that is used to solve a relaxation of the problem formulation at each node of the branch-and-bound tree requires the solution of a pricing problem which is shown to decompose into appropriate uncapacitated single-item lot-sizing problems with pricing decisions. As mentioned above, efficient algorithms to solve such problems to optimality exist in the absence of initial inventories. However, they become more challenging when initial inventories are present, and we develop effective polynomial-time algorithms for these problems that may also be of independent interest for solving uncapacitated single-item lot-sizing problems with pricing decisions.

The remainder of the chapter is organized as follows. In Section 3.2, we state our assumptions and present the formulation of the CLSTP. In Section 3.3 we develop our two Dantzig-Wolfe formulations and the associated column generation methods. In Section 3.4 we discuss algorithms for solving the resulting pricing problems under dynamic and static pricing strategies. In Section 3.5 we provide implementation details of our branch-and-price algorithm. We present computational results in Section 3.6 and conclude the chapter in Section 3.7.

3.2 The CLSTP Model and Assumptions

Let p_{it} be the price set for item i in period t and suppose that demand d_{it} changes with price p_{it} according to the relation $d_{it} = D_{it}(p_{it}) = \alpha_{it} - \beta_{it}\Pi_{it}(p_{it})$, where α_{it} and $\beta_{it} > 0$ are parameters and $\pi_{it} \equiv \Pi_{it}(p_{it})$ is a price-dependent parameter that is called the *price effect* on demand. Note that we could of course simply set $\alpha_{it} = 0$, $\beta_{it} = -1$, and $\Pi_{it}(p_{it}) = D_{it}(p_{it})$. However, as we will see later, the more general representation allows us to more effectively study, and develop efficient algorithms for, models that incorporate constraints on price patterns (such as the constraint that prices should be constant over the time horizon) while still allowing for a broad class of demand relations. We assume that the functions Π_{it} are strictly increasing, continuous, and convex. Notice that with these assumptions, Π_{it} has an inverse function P_{it} and there exists a one to one relationship between p_{it} and π_{it} given by

$$P_{it}(\pi_{it}) = \Pi_{it}^{-1}(\pi_{it}) = p_{it} \text{ or, equivalently, } \Pi_{it}(p_{it}) = P_{it}^{-1}(p_{it}) = \pi_{it}.$$

Therefore, throughout the chapter, when we are referring to a certain price effect π_{it} we will be implicitly referring to a certain price p_{it}. This allows us to formulate the CLSTP in terms of π_{it} instead of p_{it}. We will represent the set of vectors of feasible prices for item i

by Γ_i and assume that it is a nonempty polytope. We next define the revenue in period t, R_{it}, as a function of the price effect π_{it} as

$$R_{it}(\pi_{it}) = p_{it}d_{it} = P_i(\pi_{it})\left(\alpha_{it} - \beta_{it}\pi_{it}\right).$$

Given the assumptions on Π_{it}, it is easy to show that the revenue function R_{it} is continuous and concave. We further assume that this concave revenue function has no more than J points at which it is nondifferentiable and achieves its supremum at a finite value $\pi'_{it} \equiv \arg\max_{\pi_{it}} R_{it}(\pi_{it})$. We can now formulate the CLSTP as follows:

$$\text{maximize} \sum_{i=1}^{N}\sum_{t=1}^{T}\left(R_{it}(\pi_{it}) - S_{it}y_{it} - c_{it}x_{it} - h_{it}I_{it}\right) \tag{3–8}$$

subject to

$$\sum_{i=1}^{N}(b_{it}y_{it} + a_{it}x_{it}) \le C_t \qquad t = 1,\dots,T \tag{3–9}$$

$$I_{i,t-1} + x_{it} = \alpha_{it} - \beta_{it}\pi_{it} + I_{it} \qquad t = 1,\dots,T;\ i = 1,\dots,N \tag{3–10}$$

$$x_{it} \le M_{it}y_{it} \qquad t = 1,\dots,T;\ i = 1,\dots,N \tag{3–11}$$

$$\{\pi_{i1},\dots,\pi_{iT}\} \in \Gamma_i \qquad i = 1,\dots,N \tag{3–12}$$

$$y_{it} \in \{0,1\} \qquad t = 1,\dots,T;\ i = 1,\dots,N \tag{3–13}$$

$$I_{it}, x_{it} \ge 0 \qquad t = 1,\dots,T;\ i = 1,\dots,N \tag{3–14}$$

$$I_{i0} = \bar{I}_{i0} \qquad i = 1,\dots,N. \tag{3–15}$$

The objective function (3–8) maximizes the difference between the total revenues and the total costs. In contrast with the CLST, we can now not assume without loss of generality that the item inventories are equal to zero at the start of the planning period and represent the initial inventories by $\bar{I}_{i0} \ge 0$ in constraint (3–15). As in the CLST, M_{it} in constraint (3–11) is an upper bound on the production of item i in period t that is satisfied in without loss of optimality; for example, we may set $M_{it} = \min\left\{\sum_{s=t}^{T}\left(\alpha_{is} - \beta_{is}\pi'_{is}\right), (C_t - b_{it})/a_{it}\right\}$.

The set Γ_i in constraint (3–12) represents the desired pricing strategy, and in this chapter we study two particular pricing strategies in detail: a dynamic and a constant pricing strategy. The first strategy is one where the price of an item can vary from period to period. In that case, Γ_i is defined by lower and upper bound constraints only:

$$\left\{(\pi_{i1}, \ldots, \pi_{it}) : \pi_{it}^L \leq \pi_{it} \leq \pi_{it}^U,\ t = 1, \ldots, T\right\}$$

where we may choose $\pi_{it}^L = \Pi_{it}^{-1}(0)$ to ensure that the price of an item is always nonnegative, and $\pi_{it}^U = \alpha_{it}/\beta_{it}$ to ensure that the resulting demand is always nonnegative. In fact, in the absence of initial inventories (i.e., $\bar{I}_{i0} = 0$), we can potentially tighten the lower bound constraints by recalling that the revenue function attains it maximum at π'_{it}, and that demands are nonincreasing in the price effect. Therefore, when $\pi_{it} < \pi'_{it}$ any increase in cost resulting from the increased demands when π_{it} is further decreased is not offset by a large enough increase in revenues, so that such values of π_{it} are not profitable. This means that we can then set $\pi_{it}^L = \max\{\Pi_{it}^{-1}(0), \pi'_{it}\}$. In the second pricing strategy that we consider, the price of an item should be constant over the planning horizon. We follow earlier studies that consider this pricing strategy by restricting ourselves to demand functions for which the functions Π_{it} are stationary, i.e., $\Pi_{it} = \Pi_i$ for $t = 1, \ldots, T$ and $i = 1, \ldots, N$. In that case, Γ_i is given by:

$$\left\{(\pi_{i1}, \ldots, \pi_{it}) : \pi_{it} = \pi_{i1},\ t = 2, \ldots, T;\ \pi_{i1}^L \leq \pi_{i1} \leq \pi_{i1}^U\right\}$$

where, similarly to the dynamic pricing case, we may set $\pi_{i1}^L = \Pi_i^{-1}(0)$ to ensure that the price of an item is always nonnegative; and $\pi_{i1}^U = \min_{1 \leq t \leq T} \{\alpha_{it}/\beta_{it}\}$ to ensure that the resulting demands are nonnegative in each period. Interestingly, under this strategy and demand relation we have that $\pi_{it} = \pi_{i1}$ for $t = 2, \ldots, T$, so that we may have that, in the optimal solution, $\pi_{it} < \pi'_{i1}$ in one or more periods t even when there are no initial inventories. However, when there are no initial inventories for item i we can still find a potentially tighter lower bound on π_{i1} by considering the total revenue curve $R_i = \sum_{t=1}^{T} R_{it}$. Since each of the revenue functions R_{it} attains its maximum at a finite price effect π'_{it}, the function R_i will attain its maximum at a finite value $\pi'_i \equiv \arg\max_{\pi_i} R_i(\pi_i)$ as well. This means that this value provides a lower bound on π_{i1}^L *provided* it is no larger than π_{i1}^U. Therefore, without loss of optimality, we can set $\pi_{i1}^L = \max\left\{\Pi_i^{-1}(0), \min\{\pi'_i, \pi_{i1}^U\}\right\}$.

3.3 Dantzig-Wolfe Decompositions of the CLSTP

3.3.1 Framework Development

In this section, we will develop two alternative Dantzig-Wolfe decompositions of the CLSTP. As in the traditional CLST, the capacity constraints (3–9) are the the tying constraints and if they are removed, the problem reduces to a collection of independent uncapacitated single item lot-sizing problems with pricing decisions for each item i. Now let X_i be the region defined by constraints (3–10)–(3–15) for item i. Note that since Γ^i is bounded, X_i is a polytope as well. We can therefore build a Dantzig-Wolfe decomposition formulation of the CLSTP by writing any feasible solution to the problem as a convex combination of the finite set of extreme points E_i of conv(X_i). For ease of exposition, we will refer to a typical element of E_i as either $(y_{i1}^j, \ldots, y_{iT}^j, x_{i1}^j, \ldots, x_{iT}^j, I_{i1}^j, \ldots, I_{iT}^j, \pi_{i1}^j, \ldots, \pi_{iT}^j)$ or, for short, j, with associated cost $\kappa_i^j = \sum_{t=1}^{T}\left(S_{it}y_{it}^j + c_{it}x_{it}^j + h_{it}I_{it}^j\right)$ and capacity consumption $\rho_{it}^j = a_{it}y_{it}^j + b_{it}x_{it}^j$. Letting the decision variable λ_i^j represent the weight of production and demand plan $j \in E_i$ in the convex combination, we obtain the following formulation of the CLSTP:

$$\text{maximize} \sum_{i=1}^{N}\sum_{t=1}^{T} R_{it}\left(\sum_{j\in E_i}\pi_{it}^j\lambda_i^j\right) - \sum_{i=1}^{N}\sum_{j\in E_i}\kappa_i^j\lambda_i^j$$

subject to

$$\sum_{i=1}^{N}\sum_{j\in E_i}\rho_{it}^j\lambda_i^j \le C_t \qquad t = 1, \ldots, T \qquad (3\text{–}16)$$

$$\sum_{j\in E_i} y_{it}^j\lambda_i^j \in \{0,1\} \qquad i = 1, \ldots, N;\ t = 1, \ldots, T \qquad (3\text{–}17)$$

$$\sum_{j \in E_i} \lambda_i^j - 1 = 0 \qquad i = 1, \ldots, N \tag{3–18}$$

$$-\lambda_i^j \leq 0 \qquad j \in E_i;\ i = 1, \ldots, N. \tag{3–19}$$

Constraints (3–16) ensure that the weighted average of the capacity requirements of the extreme plans in period t should not exceed the available capacity in that period, while constraints (3–17) relate to the original binary setup indicator variables y_{it} and state that the weighted averages of the extreme point setup variables should represent a valid production plan and therefore be binary. Constraints (3–18) and (3–19) ensure that we indeed consider convex combinations of extreme point plans only. This reformulation leads to a branch-and-price algorithm in which we use the continuous relaxation of this formulation to determine upper bounds. This continuous relaxation itself will be solved through a column generation approach in which we repeatedly solve a so-called restricted master problem containing only a (relatively small) subset of the decision variables (columns) and determine whether additional columns should be added through an associated pricing problem. Since the formulation of the CLSTP above contains only a finite number of decision variables this column generation algorithm will converge finitely. We will therefore refer to this formulation as the *Finite Formulation*.

Although the finiteness of the formulation is certainly a major advantage, this approach suffers from a major drawback as well: the master problems to be solved in the column generation phases are concave maximization problems. While effective algorithms for solving such nonlinear optimization problems exist, it is nevertheless questionable whether the need to repeatedly solve such a problem will yield an effective solution approach to the CLSTP. Our alternative approach is based on the following reformulation of the CLSTP:

$$\text{maximize} \ \sum_{i=1}^{N} \sum_{t=1}^{T} (r_{it} - S_{it} y_{it} - c_{it} x_{it} - h_{it} I_{it}) \tag{3–20}$$

subject to

$$\sum_{i=1}^{N} (b_{it} y_{it} + a_{it} x_{it}) \leq C_t \qquad t = 1, \ldots, T \tag{3–21}$$

$$r_{it} \leq R_{it}(\pi_{it}) \qquad t = 1, \ldots, T;\ i = 1, \ldots, N \tag{3–22}$$

$$(x_i, y_i, I_i, \pi_i) \in X_i \qquad i = 1, \ldots, N \tag{3–23}$$

where we note that, without loss of optimality, constraints (3–22) will be satisfied at equality. Letting $\hat{X}_i$ be the region defined by constraints (3–22)–(3–23) for item i, it is not hard to see that any optimal solution to the CLSTP can be expressed as a convex combination of a finite number of extreme points of conv($\hat{X}_i$). However, this set of extreme points, say $\hat{E}_i$, may contain an uncountably infinite number of points, making a direct analogon of the earlier approach impossible since the full master problem would then contain an uncountably infinite number of decision variables. However, any finite subset of points from $\hat{E}_i$ would define a restricted master problem whose continuous relaxation is a linear program. We will show that this leads to a branch-and-price algorithm in which we solve a continuous relaxation of the CLSTP using column generation, where the restricted master problems are linear programs. However, the price we have to pay for this is that the column generation phases may not terminate finitely. We will therefore refer to this formulation as the *Infinite Formulation*.

In the remainder of this section we will develop pricing problems that are used to generate promising columns in the column generation phase of both the finite and infinite formulations.

3.3.2 Pricing Problem

3.3.2.1 Finite formulation

Although we have not yet explicitly characterized the set of extreme points E_i, we are now able to derive the general form of the pricing problem that, in the column generation phase of our algorithm, identifies one or more additional columns that should be added to the master problem. To this end, consider the continuous relaxation of the binary constraints (3–17):

$$\sum_{j \in E_i} y_{it}^j \lambda_i^j \leq 1 \qquad\qquad i = 1, \ldots, N;\ t = 1, \ldots, T.$$

Since in any production schedule $j \in E_i$ we have that $y_{it}^j \in \{0, 1\}$ for all $t = 1, \ldots, T$, constraints (3–18) and (3–19) imply that this constraint is redundant and we may remove it from the problem to obtain a continuous relaxation of the CLSTP. Since the objective function is concave and the feasible region is a polytope, a feasible solution vector $\bar{\lambda}$ is optimal to the relaxation problem if and only if there exists a solution to the following KKT conditions, where the dual variables θ_t are associated with constraints (3–16), ϕ_i with constraints (3–18), and γ_i^j with constraints (3–19):

$$\begin{aligned}
-\sum_{t=1}^{T} \partial R_{it}\left(\sum_{j \in E_i} \pi_{it}^j \bar{\lambda}_i^j\right) + \kappa_i^j + \sum_{t=1}^{T} \theta_t \rho_{it}^j - \gamma_i^j + \phi_i \ni 0 & \qquad j \in E_i;\ i = 1, \ldots, N \\
\theta_t \left(\sum_{i=1}^{N} \sum_{j \in E_i} \rho_{it}^j \bar{\lambda}_i^j - C_t\right) = 0 & \qquad t = 1, \ldots, T \\
\phi_i \left(\sum_{j \in E_i} \bar{\lambda}_i^j - 1\right) = 0 & \qquad i = 1, \ldots, N \\
-\bar{\lambda}_i^j \gamma_i^j = 0 & \qquad j \in E_i;\ i = 1, \ldots, N \\
\gamma_i^j \geq 0 & \qquad j \in E_i;\ i = 1, \ldots, N \\
\theta_t \geq 0 & \qquad t = 1, \ldots, T.
\end{aligned}$$

Using the definitions of κ_i^j and ρ_{it}^j, the first set of KKT conditions is equivalent to

$$-\sum_{t=1}^{T} u_{it} \pi_{it}^j + \sum_{t=1}^{T} \left(S'_{it} y_{it}^j + c'_{it} x_{it}^j + h_{it} I_{it}^j\right) + \phi_i = \gamma_i^j \geq 0 \qquad j \in E_i \qquad (3\text{–}24)$$

where

$$\bar{u}_{it} \in \partial R_{it}\left(\sum_{j \in E_i} \pi_{it}^j \bar{\lambda}_i^j\right) \qquad\qquad t = 1, \ldots, T;\ i = 1, \ldots, N$$

$$S'_{it} = S_{it} + \theta_t b_{it} \qquad t = 1, \ldots, T;\ i = 1, \ldots, N$$

and

$$c'_{it} = c_{it} + \theta_t a_{it} \qquad t = 1, \ldots, T;\ i = 1, \ldots, N.$$

Now suppose that $\bar{\lambda}$ is actually the optimal solution to a *restricted* relaxed master problem with only a subset of the extreme points in E_i $(i = 1, \ldots, N)$ along with a dual solution $(\bar{\theta}, \bar{\phi}, \bar{\gamma})$. If (3–24) happens to be satisfied then we can conclude that we have found the optimal solution to the relaxation. Otherwise, we can add to the master problem any extreme plans $j \in E_i$ for which (3–24) is violated. In particular, we may choose to add, for each item i, the extreme plan for which (3–24) is most violated (if any). We can find that plan by finding an extreme point optimal solution to the following pricing problem for each item i:

$$\text{maximize} \sum_{t=1}^{T} \bar{u}_{it}\pi_{it} - \sum_{t=1}^{T} \left(\bar{S}'_{it} y_{it} + \bar{c}'_{it} x_{it} + h_{it} I_{it}\right) - \bar{\phi}_i$$

subject to (PP-F)

$$(x_i, y_i, I_i, \pi_i) \in X_i$$

where $\bar{S}'_{it} = S_{it} + \bar{\theta}_t b_{it}$ and $\bar{c}'_{it} = c_{it} + \bar{\theta}_t a_{it}$ $(t = 1, \ldots, T;\ i = 1, \ldots, N)$. Clearly, if the optimal solution to this problem is negative the optimal solution corresponds to the constraint in (3–24) that is most violated. Now (PP-F) is an uncapacitated single-item lot-sizing problem with pricing decisions, linear revenue functions, and initial inventories, and we will focus on solution approaches to this problem in Section 3.4.

3.3.2.2 Infinite formulation

As mentioned above, the finiteness of the approach developed in Section 3.3.2.1 comes at the expense of a master's problem that is a concave optimization problem. We could address this fact by restricting ourselves to piecewise-linear revenue functions, which would allow for the reformulation of the relaxation of the finite formulation as a linear programming problem. In general, however, we propose to use a different approach which, as it will turn out, essentially constructs a sequence of ever more accurate piecewise-linear concave lower approximations to the revenue functions. In light of the reformulation (3–20)–(3–23) of the CLSTP, intuition suggests that the pricing problem for generating additional columns should be

$$\text{maximize} \sum_{t=1}^{T} R_{it}(\pi_{it}) - \sum_{t=1}^{T} \left(\bar{S}'_{it} y_{it} + \bar{c}'_{it} x_{it} + h_{it} I_{it}\right) - \bar{\phi}_i$$

subject to (PP-I)

$$(x_i, y_i, I_i, \pi_i) \in X_i.$$

Now (PP-I) is an uncapacitated single-item lot-sizing problem with pricing decisions, concave revenue functions, and initial inventories, and we will focus on solution approaches to this problem in Section 3.4. It is clear that this approach suffers from two potential complications: (i) the column generation algorithm could potentially generate an infinite number of columns; and (ii) it is not obvious that the procedure converges to an optimal solution to the relaxation.

However, the correctness of the above algorithm follows from the fact that it is an application of the algorithm proposed by Dantzig (6) for convex programming problems. Dantzig (5) proved that this column generation algorithm either finds the optimum in a finite number of iterations or converges to the optimal solution as long as there exists a non-degenerate basic solution to the master problem. (In Section 3.5 we will address the problem of finding a non-degenerate basic solution to the master problem.) Since each iteration of this algorithm adds a new breakpoint to a piecewise-linear concave under-approximation to the concave revenue functions, this algorithm is sometimes also called *grid linearization* (See Lasdon (29)). This method for creating a piecewise-linear approximation is more efficacious than creating a linear approximation to the function in advance since, with this approach, we only introduce breakpoints as needed and make an accurate approximation to the function only in the neighborhood of promising solutions.

3.4 Solving the Pricing Problem

For both the finite and the infinite formulation, the pricing problem for a given item is of the following generic form:

$$\text{maximize } \sum_{t=1}^{T} R_t(\pi_t) - \sum_{t=1}^{T} (S_t y_t + c_t x_t + h_t I_t)$$

subject to (PP)

$$
\begin{aligned}
I_{t-1} + x_t &= \alpha_t - \beta_t \pi_t + I_t && t = 1, \dots, T && \text{(3–25)} \\
x_t &\le M_t y_t && t = 1, \dots, T \\
\{\pi_1, \dots, \pi_T\} &\in \Gamma \\
y_t &\in \{0, 1\} && t = 1, \dots, T \\
I_t, x_t &\ge 0 && t = 1, \dots, T \\
I_0 &= \bar{I}_0.
\end{aligned}
$$

It will be convenient to eliminate the inventory variables using the inventory balance constraints (3–25):

$$I_{it} = \bar{I}_{i0} + \sum_{j=1}^{t} x_{it} - \sum_{j=1}^{t} (\alpha_{it} - \beta_{it}\pi_{it}) \qquad t = 1, \dots, T.$$

Then, if we define

$$
\begin{aligned}
\tilde{R}_t(\pi_t) &= R_t(\pi_t) - \pi_t \beta_t \sum_{\tau=t}^{T} h_\tau && t = 1, \dots, T \\
\tilde{c}_t &= c_t + \sum_{\tau=t}^{T} h_\tau && t = 1, \dots, T
\end{aligned}
$$

we can formulate (PP) as

$$\text{maximize } \sum_{t=1}^{T} \tilde{R}_t(\pi_t) - \sum_{t=1}^{T} (S_t y_t + \tilde{c}_t x_t)$$

subject to

$$\begin{aligned}
\bar{I}_0 + \sum_{s=1}^{t} x_s - \sum_{s=1}^{t} (\alpha_s - \beta_s \pi_s) &\geq 0 && t = 1, \ldots, T \\
x_t &\leq M y_t && t = 1, \ldots, T \\
\{\pi_1, \ldots, \pi_T\} &\in \Gamma && \\
y_t &\in \{0, 1\} && t = 1, \ldots, T \\
x_t &\geq 0 && t = 1, \ldots, T
\end{aligned}$$

(where we have omitted the constant $-\sum_{t=1}^{T} h_t \bar{I}_0$ from the objective).

3.4.1 Dynamic Pricing Strategy

Recall that, under a dynamic pricing strategy, the set Γ contains only bound constraints. For this case, Thomas (36) proposes a dynamic programming algorithm that solves the problem in $O(T^2)$ time when no initial inventory is present and the inverse of the derivative of the revenue functions can be evaluated in constant time. This algorithm is based on the observation that there exists an optimal production and pricing plan that possesses the zero-inventory-ordering (ZIO) property and can therefore, like for the standard economic lot-sizing problem with fixed demands (see Wagner and Whitin (39)), be decomposed into a sequence of regeneration intervals. (We will, with a slight abuse of notation, refer to such solutions as *extreme point solutions*.) The problem can then be solved by determining the optimal price effects (or, equivalently, demands or prices) for the periods in each regeneration interval. Letting $f(s,t)$ be the maximum profit obtainable in the regeneration interval (s,t) (i.e., in periods $s, \ldots, t$) and $F(t)$ the maximum profit obtainable in periods 1 through t, the following backward dynamic recursion solves the problem:

$$\begin{aligned}
F(s) &= \max_{t:t \geq s} \{f(s,t) + F(t+1)\} && s = 1, \ldots, T \\
F(T+1) &= 0.
\end{aligned}$$

To generalize this result to the case with initial inventory, we go back to the underlying core observation that, for fixed price effects (i.e., fixed demands) the resulting concave cost minimization problem can be formulated as a shortest path problem in a network with a demand node for each period, in addition to two supply nodes corresponding to production and initial inventory. It is well-known that there exists an optimal solution to the problem in which the set of arcs carrying positive flow is acyclic. If we then, similarly to the problem without initial inventories, define a regeneration interval as starting in either the first period or a period with no incoming inventory, we can immediately conclude that we may again limit ourselves to solutions that decompose into regeneration intervals. Moreover, for any regeneration interval (s, t) with $s > 1$ we can determine the associated optimal profit $f(s, t)$ in precisely the same way as for the problem without initial inventories. For regeneration intervals of the form $(1, t)$, however, the situation is different. In the first regeneration interval of an optimal extreme point solution, production will take place in no more than one period, so that we should consider two different situations: (i) no production period; (ii) exactly one production period in $\{1, \ldots, t\}$. We will therefore, for each candidate first regeneration interval $(1, t)$, consider both options and set $f(1, t)$ to the largest corresponding profit. Interestingly, since we know that there exists an extreme point optimal solution to the overall problem, in doing so we may ignore any nonnegativity constraints on the inventories in the regeneration interval and discard any candidate solutions in which these are violated. Next, we consider in more detail the two situations mentioned above.

(i) If there is no production in the regeneration interval $(1, t)$, the total demand satisfied in periods 1 through t is equal to the initial inventory $\bar{I}_0$ if $t < T$ and will not exceed $\bar{I}_0$ if $t = T$. In other words, ignoring the nonnegativity constraints on the inventory levels as discussed above we should solve the following optimization problem if the entire initial inventory is used up in regeneration interval $(1, t)$:

$$\text{maximize} \sum_{s=1}^{t} \tilde{R}_s(\pi_s)$$

subject to

$$\sum_{s=1}^{t} (\alpha_s - \beta_s \pi_s) = \bar{I}_0$$

$$\pi_s^L \le \pi_s \le \pi_s^U \qquad s = 1, \ldots, t.$$

Appropriately redefining $\tilde{R}$ outside the bound constraints we obtain that an optimal solution to this problem is given by values of r and π_t $(t = 1, \ldots, t)$ such that

$$\tilde{R}_s'^{+}(\pi_s) \le r \le \tilde{R}_s'^{-}(\pi_s) \qquad s = 1, \ldots, t \tag{3–26}$$

$$\sum_{t=1}^{s} (\alpha_s - \beta_s \pi_s) = \bar{I}_{i0}$$

where $\tilde{R}_s'^{+}(\pi_s)$ and $\tilde{R}_s'^{-}(\pi_s)$ are, respectively, the right and left derivatives of $\tilde{R}_s$, respectively. (Note that, due to the bound constraints, we set $\tilde{R}_s'^{-}(\pi_s^L) = 0$ and $\tilde{R}_s'^{+}(\pi_s^U) = -\infty$.) Interestingly, this is exactly the same setting studied in Geunes et al. (14) for the lot-sizing problem with constant capacities and dynamic pricing decisions in a regeneration interval, where the first period is the only production period and the production capacity is equal to $\bar{I}_{i0}$. The results in that paper then imply that this subproblem can be solved in $O\left((J + 1 + R \log \nu)T\right)$ time, where

$$\nu = \max_{s=1,\ldots,t} \tilde{R}_s'^{+}(\pi_s^U) - \min_{s=1,\ldots,t} \tilde{R}_s'^{-}(\pi_s^L)$$

and $O(R)$ is the time required to find a value of π_s satisfying (3–26) for some value of r.
In case $t = T$ we also have to account for the possibility that some of the initial inventories remain at the end of the planning horizon. This will only happen if there is no marginal revenue associated with satisfying any demand from these remaining inventories. In other words, this solution should satisfy $\tilde{R}_s'^{+}(\pi_s) = 0$ for $s = 1, \ldots, T$ and $\sum_{t=1}^{s} (\alpha_s - \beta_s \pi_s) \le \bar{I}_{i0}$. It is easy to see that such a solution, if one exists, can be found in $O(RT)$ time.

(ii) Next, suppose that production in regeneration interval $(1, t)$ takes place in period τ. We again ignore the nonnegativity constraints on the inventory levels as discussed above. Moreover, we will have that the production quantity in period τ satisfies

$$x_\tau = \bar{I}_0 - \sum_{s=1}^{t} (\alpha_s - \beta_s \pi_s)$$

so that we should solve the following optimization problem:

$$\text{maximize} \sum_{s=1}^{t} \left(\tilde{R}_s(\pi_s) - \tilde{c}_\tau \beta_s \pi_s\right)$$

subject to

$$\pi_s^L \leq \pi_s \leq \pi_s^U \qquad s = 1, \ldots, t$$

(where we have omitted the constant $-S_\tau - \tilde{c}_\tau \bar{I}_0 + \tilde{c}_\tau \sum_{s=1}^{t} \alpha_s$ from the objective). It is easy to see that an optimal solution to this subproblem satisfies

$$\tilde{R}_s'^{+}(\pi_s) \leq \tilde{c}_\tau \beta_s \leq \tilde{R}_s'^{-}(\pi_s) \qquad s = 1, \ldots, T$$

and this solution can be found in $O(RT)$ time. This result is consistent with a similar result in Geunes et al. (14) for regeneration intervals with a single fractional procurement period in the lot-sizing problem with constant capacities and dynamic pricing decisions.

Now observe that there are $O(T^2)$ regeneration intervals with $s > 1$, and the profits of these can be found in $O(RT^2)$ time. Moreover, for the $O(T)$ initial regeneration intervals, we can find the profit in case (i) in $O\left((J + 1 + R \log \nu)T\right)$, and the profit in case (ii) for each of the $O(T)$ fixed production periods in $O(RT)$ time. This means that (PP) under a dynamic pricing strategy can be solved in $O\left((J + 1 + R \log \nu)T^2\right)$ time. In particular, this immediately implies that, for the special case where the revenue functions are linear (which is relevant in our algorithm for the Finite Formulation), (PP) under a dynamic pricing strategy can be solved in $O\left((1 + \log \nu)T^2\right)$ time.

3.4.2 Constant Pricing Strategy

Under a constant pricing strategy we add the constraints $\pi_t = \pi_1$ for $t = 2, \ldots, T$ to the definition of the set Γ. For convenience, we will in this section simply denote the single price effect variable by π, and the corresponding bounds by π^L and π^U. Van den Heuvel and Wagelmans (20) study this problem in the case of zero initial inventories and under the assumption that period 1 is a production period and develop an exact algorithm with a running time of $O(T^3 \log T)$.

In general, suppose that τ is the first production period. Noting that we will again restrict ourselves to extreme point solutions, this period is either (i) the first period of the second regeneration interval (and demand in the first regeneration interval is satisfied precisely by initial inventory only); or (ii) in the first regeneration interval (and demand in the first regeneration interval is satisfied by initial inventory and production in period τ). We will consider these two cases separately (where, when appropriate, we will use the notation from Section 3.4.1):

(i) In this case, it is easy to see that this can only happen if $\tau > 1$ and

$$I_0 = \sum_{s=1}^{\tau-1} (\alpha_s - \beta_s \pi)$$

or, equivalently,

$$\pi = \pi^{(\tau)} \equiv \frac{\sum_{s=1}^{\tau-1} \alpha_s - I_0}{\sum_{s=1}^{\tau-1} \beta_s}.$$

If, in the latter case, $\pi^{(\tau)} \in \left[\pi^L, \pi^U\right]$ we can then determine all demand levels and solve a standard economic lot-sizing problem over periods $\tau, \ldots, T$ in $O(T \log T)$ time using the algorithm of Wagelmans et al. (38).

(ii) In this case, a particular price effect is only valid if the initial inventories are sufficiently high to satisfy all demands up to the first production period, so that we should ensure that

$$I_0 \geq \sum_{s=1}^{\tau-1} (\alpha_s - \beta_s \pi)$$

or, equivalently, we should restrict ourselves to values

$$\pi \in \left[\max\{\pi^L, \pi^{(\tau)}\}, \pi^U\right].$$

We next follow Van den Heuvel and Wagelmans (20) and write the total lot-sizing costs as a function of the price effect π for a given set $S \subseteq \{\tau, \ldots, T\}$ of production periods (with, of course, $\tau \in S$). Letting $\tau_t(S)$ denote the first production period in $S \cap \{t, \ldots, T\}$, this cost function reads as follows:

$$\begin{aligned} C^{(\tau,S)}(\pi) &= \sum_{t \in S} S_t - \tilde{c}_\tau \bar{I}_0 + \sum_{t=\tau}^{T} \tilde{c}_{\tau_t(S)} \left(\alpha_t - \beta_t \pi\right) \\ &= \sum_{t \in S} S_t - \tilde{c}_\tau \bar{I}_0 + \sum_{t=\tau}^{T} \tilde{c}_{\tau_t(S)} \alpha_t - \sum_{t=\tau}^{T} \tilde{c}_{\tau_t(S)} \beta_t \pi \\ &= A^{(\tau,S)} - B^{(\tau,S)} \pi \end{aligned}$$

where $A^{(\tau,S)}$ and $B^{(\tau,S)}$ are defined appropriately. This means that $C^{(\tau,S)}$ is piecewise-linear and convex and, moreover, has the same structure as the cost function in Van den Heuvel and Wagelmans (20). We can therefore find the optimal price effect for this case in $O(T^3 \log T)$ time.

Combining both results above we immediately obtain that (PP) under a constant pricing strategy can be found in $O(T^4 \log T)$ time. It is interesting to note that, for the special case where the revenue functions are linear (which is relevant in our algorithm for the Finite Formulation), we only need to consider the $O(T)$ (valid) price effects in $\{\pi^L, \pi^U\} \cup \{\pi^{(\tau)} : \tau = 2, \ldots, T\}$. Therefore, (PP) with linear revenue functions and under a constant pricing strategy can be solved in $O(T^2 \log T)$ time.

3.5 Branch-and-Price Algorithm

In this section we discuss some implementation details of our branch-and-price algorithm. In particular, we will address how we guarantee convergence of the column generation procedure for solving the infinite formulation. However, before we proceed it is important to note that, since finite termination of the column generation algorithm for the infinite formulation cannot be guaranteed, we have to allow for a finite pre-specified tolerance; in particular, we will consider the problem solved as soon as we find a solution that is within $\epsilon > 0$ in value of the optimal solution.

3.5.1 Initial Columns and Convergence

To start the column generation process at a particular node with either the finite or the infinite formulation we need to find an initial basic feasible solution to the master problem at that node. When such a feasible solution can be found, convergence with finite formulation is guaranteed because the number of extreme points of the formulation is finite. However, in order to guarantee convergence with the infinite formulation Dantzig (5; 6) shows that at least one non-degenerate basic feasible solution to the master problem at a node should exist. In our implementation of this method, we will make sure that we start the algorithm with an initial non-degenerate basic feasible solution.

3.5.1.1 Dynamic pricing strategy

In the case of dynamic prices, we start with an initial solution in which no production takes place for any of the items. Furthermore, in these production plans we set the prices as high as possible, i.e., $\pi_{it} = \pi_{it}^U$ $(t = 1, \ldots, T)$, so that the demands in all periods are zero. This means that, if there are positive initial inventories these are carried over the end of the planning horizon without being consumed. This solution can be represented through a column for each item, each of which represents a "production plan" for the corresponding item. A basis for this initial solution then consists of the slack variables of the capacity constraints (3–16) and the λ_i variables corresponding to the initial production plans for items $i = 1, \ldots, N$. Since capacities are strictly positive this basis is non-degenerate so that convergence of the column generation algorithm for both the finite and the infinite formulation is guaranteed.

3.5.1.2 Constant pricing strategy

Under the assumption of a constant price for each item it may not be possible to find a feasible solution in which all item demands are zero in each period. In particular, setting the prices to their upper bounds, i.e., $\pi_{i1} = \pi_{i1}^U = \min_t\{\frac{\alpha_{it}}{\beta_{it}}\}$ does not guarantee that the corresponding demands are all equal to zero, so that positive production may be required to satisfy these demands. In fact finding a feasible integral solution to the master problem with the prices that correspond to the lowest possible demands (or recognizing that such a solution does not exist) is NP-hard (since it is the feasibility question for an instance of the CLST problem). We therefore propose to use a two-phase column generation approach in which we first find a (non-degenerate) basic feasible solution to the problem (if one exists), and then solve our optimization problem from this starting solution. With the prices at their upper bounds as above, we start the first phase by attempting to satisfy the corresponding demands through initial inventories. If these initial inventories suffice to satisfy all demands then we have obtained an initial non-degenerate basic feasible solution as in Section 3.5.1.1. However, if any demands remain unsatisfied we solve an auxiliary problem where we relax the capacity constraints and maximize the minimum amount of unused capacity (which may be negative!) over periods $t = 1, \ldots, T$. If the optimal objective function value of this problem is negative we conclude that there is no feasible solution to our problem (at the current node in the branch-and-bound tree). Otherwise, if the optimal objective function value is nonnegative the optimal solution corresponds to a non-degenerate basic feasible solution which can be used to start column generation at that node.

3.5.2 Bounding

At each iteration of the column generation process we calculate upper and lower bounds on the optimal objective function value of the relaxed problem. Let v^k_{RMP} be the optimal value of the restricted master problem at iteration k. Clearly, this provides a lower bound on the optimal solution value of the relaxed master problem. Moreover, letting $v^k_{\text{SP}_i}$ be the reduced costs of the columns that we have generated for items $i = 1, \ldots, N$ at iteration k of the column generation process an upper bound on the optimal solution value of the relaxed master problem is given by

$$v^k_{\text{RMP}} + \sum_{i=1}^{N} v^k_{\text{SP}_i}$$

(see Dantzig (6) and Lasdon (29)). Especially when using the infinite formulation we stop whenever the difference between these upper and lower bounds is below the tolerance $\epsilon > 0$.

3.5.3 Branching

When the column generation stops at the root node, we check whether the binary constraints are satisfied for the y_{it} variables. If none of the y_{it} variables are fractional, we conclude that we have found an integral solution at the root node (that is within the desired tolerance). Otherwise, we branch on the fractional y_{it} variable closest to 1. Preliminary experiments on a subset of test instances showed that this results in a slight improvement in the final solutions as compared to alternative branching strategies where we branch first on the y_{it} variable that is closest to 0 or closest to 0.5. We branch down by imposing the constraint $y_{it} = 0$ and branch up by imposing the constraint $y_{it} = 1$ on the child nodes.

3.5.4 Search Strategy and Heuristics

We perform a *hybrid depth-first search strategy* where, after the two child nodes of the current node are generated, linear relaxations in both child nodes are solved to optimality. Then we first investigate the child node with the higher upper bound on the objective value of the relaxed problem.

After we complete the column generation process at a node, we perform a simple rounding heuristic in an attempt to find a feasible solution. We set a *cutoff* value between 0 and 1, and round y_{it} variables that exceed the cutoff up to 1 and down to 0 otherwise. We then solve for the optimal values of the remaining decision variables to obtain an integer solution. This procedure is repeated for different values of the cutoff parameter: increasing from 0.1 to 1 at 0.1 increments. Similar to what was observed by Degraeve and Jans (7), we empirically find that the objective values obtained by this repeated rounding heuristic appear to follow a unimodal pattern as the cutoff value is increased. Therefore, we stop this heuristic as soon as the objective value decreases.

3.6 Computational Results

3.6.1 Creating Problem Instances

For our computational tests, we modified a collection of 540 widely used problem instances created by Trigeiro et al. (37) for the CLST. These problem instances for the CLST were generated using a full factorial design on five problem characteristics: number of items, coefficient of variation of demand across periods, time between orders, average setup times and capacity utilization, while the number of periods was fixed to $T = 20$. Since, in our problem instances, we do not have a fixed demand pattern the second characteristic was not utilized. Instead, we created parameter values for the demand functions as follows: α_{it} fixed at 250 and β_{it} randomly generated from the uniform distribution on the set $\{2.0, 2.5, 3.0\}$. This yields a range of demand values from 0 to 250. Furthermore, we chose $\Pi_{it}(p_{it}) = p_{it}$, corresponding to a commonly used linear relationship between demands and prices. This, in turn, means that the revenue functions are quadratic: $R_{it}(\pi_{it}) = \alpha_{it}\pi_{it} - \beta_{it}\pi_{it}^2$, so that our instances of the CLSTP are mixed-integer quadratic optimization problems.

Our problem instances have the following properties with respect to the remaining four characteristics. First, we consider instances for which the number of items is $N \in \{10, 20, 30\}$. With respect to the time between orders (TBO), capacities, and setup times, we first solved all problem instances without capacity constraints but with Trigeiro's values for the setup times. For each instance, the ratio between setup and holding costs determines the average TBO. We then classified all instances according to an average TBO of 1 (low), 2 (medium), and 4 (high) periods. Next, the capacity in each period was set to 50% (low), 60% (medium), and 70% (high) of the average capacity consumption over the planning horizon on the unconstrained solution. Finally, we further grouped all instances according to setup times consuming 10% (low) or 30% (high) of total capacity.

3.6.2 Computational Tests

We tested both of our algorithms on each of the 540 problem instances under both dynamic and constant pricing strategies (for a total of 1080 problem instances). The linear and quadratic mixed-integer problems were all solved using CPLEX 11.1. In addition, we solved the formulation of the CLSTP from Section 3.2 of each instance directly using CPLEX as well. All experiments were performed on a 3.4 Ghz Pentium IV System (MEMORY) under Windows XP. Due to the difficulty of solving especially large-scale instances to optimality we imposed a global upper bound on the solution time of 1200 seconds for all instances and all solution approaches.

We group the results of all of our results by pricing strategy (constant, dynamic), capacity values (low, medium, high), setup time values (low, high), and TBO (low, medium, high) in order to assess the effect of the corresponding parameters on the algorithms' performance. We define the error gap as (UB − LB)/LB, where LB is a lower bound, i.e., the value of the best integral solution found by the algorithm within the given time limit, and UB is the best upper bound on the optimal integer solution value to the problem instance. Tables 3-1 and 3-2 show the average percent error gaps using the Finite and Infinite Formulation, respectively, while Table 3-3 shows the average percent error gap using CPLEX, where in each case the best LB and UB found with the corresponding method was used to determine the gap. We observed that, consistently, the UB obtained by the Infinite Formulation was tighter than that obtained with the other methods. We therefore also report an improved bound on the actual gap achieved with the Finite Formulation and CPLEX that uses the best UB found using the Infinite Formulation; these are provided in parentheses in Tables 3-1 and 3-3).

Table 3-1: Average percent error gaps obtained with the Finite Formulation using its own upper bound (and using the upper bound found with the Infinite Formulation).

	pricing strategy	**constant**			**dynamic**		
	# items	**10**	**20**	**30**	**10**	**20**	**30**
capacity	**low**	5.13 (1.81)	5.08 (1.47)	4.71 (1.32)	5.57 (1.61)	5.95 (1.81)	6.72 (2.55)
	medium	4.84 (1.51)	4.53 (1.28)	4.60 (1.33)	4.25 (1.20)	4.37 (1.31)	5.17 (2.04)
	high	3.72 (1.07)	3.34 (0.85)	3.44 (0.92)	2.97 (0.77)	3.07 (0.91)	3.55 (1.37)
setup time	**low**	2.31 (0.81)	2.16 (0.62)	2.11 (0.60)	3.09 (0.68)	3.20 (0.73)	3.33 (0.86)
	high	6.82 (2.12)	6.48 (1.78)	6.39 (1.79)	5.44 (1.70)	5.72 (1.96)	6.96 (3.11)
TBO	**low**	3.28 (1.07)	3.19 (0.91)	3.12 (1.01)	2.80 (0.71)	3.11 (0.99)	3.24 (1.14)
	medium	4.27 (1.35)	3.91 (1.13)	3.94 (1.16)	3.92 (1.06)	4.08 (1.21)	4.84 (1.94)
	high	6.14 (1.98)	5.86 (1.56)	5.68 (1.40)	6.07 (1.80)	6.21 (1.83)	7.28 (2.84)

From these tables we may immediately conclude that the performance of the Infinite Formulation is far superior to that of the Finite Formulation and CPLEX in terms of solution quality obtained within the specified time limit. In fact, CPLEX was able to find a better integral solution than the Infinite Formulation in only 8 out of the 540 instances with a dynamic pricing strategy (all with 10 items) and none of the 540 instances with a static pricing strategy. In contrast, CPLEX was able to find a better integral solution than

Table 3-2: Average percent error gaps obtained with the Infinite Formulation using its own upper bound.

	pricing strategy	constant			dynamic		
	# items	**10**	**20**	**30**	**10**	**20**	**30**
	low	0.91	0.37	0.24	0.53	0.18	0.11
capacity	**medium**	0.75	0.29	0.18	0.44	0.16	0.08
	high	0.62	0.24	0.15	0.33	0.13	0.07
setup time	**low**	0.50	0.19	0.12	0.25	0.09	0.05
	high	1.02	0.41	0.26	0.62	0.22	0.12
	low	0.57	0.22	0.13	0.28	0.10	0.06
TBO	**medium**	0.65	0.28	0.17	0.37	0.15	0.08
	high	1.05	0.41	0.27	0.64	0.22	0.13

Table 3-3: Average percent error gaps obtained with CPLEX using its own upper bounds (and using the upper bound found with the Infinite Formulation).

	pricing strategy	constant			dynamic		
	# items	**10**	**20**	**30**	**10**	**20**	**30**
	low	5.98	9.04	10.35	6.82	8.84	9.25
		(1.50)	(1.80)	(2.13)	(0.77)	(0.80)	(0.75)
capacity	**medium**	5.53	7.25	8.73	6.47	7.85	8.31
		(1.29)	(1.39)	(1.64)	(0.67)	(0.65)	(0.65)
	high	4.32	5.63	6.60	5.83	6.77	7.12
		(1.07)	(1.10)	(1.16)	(0.61)	(0.54)	(0.57)
	low	3.85	4.72	5.13	4.89	6.01	6.30
setup time		(0.73)	(0.83)	(0.97)	(0.47)	(0.48)	(0.49)
	high	6.70	9.91	12.01	7.85	9.63	10.15
		(1.85)	(2.03)	(2.32)	(0.89)	(0.85)	(0.82)
	low	3.31	4.22	5.13	5.14	5.90	6.09
	low	(0.87)	(0.91)	(1.09)	(0.53)	(0.49)	(0.50)
TBO	**medium**	4.63	6.24	7.53	5.90	7.13	7.53
		(1.10)	(1.24)	(1.47)	(0.61)	(0.59)	(0.59)
	high	7.88	11.54	13.12	8.08	10.42	11.05
		(1.89)	(2.14)	(2.38)	(0.90)	(0.91)	(0.88)

the Finite Formulation in 430 of the 540 instances with a dynamic pricing strategy and 208 of the 540 instances with a constant pricing strategy.

Moreover, as already mentioned above, the upper bound obtained using the Infinite Formulation is superior to the one obtained with the other methods, which has the additional advantage that a much more accurate assessment of solution quality is obtained. In most cases (except with the Finite Formulation under dynamic prices) we see that the error gap tends to decrease as the number of items increases. Similar observations were made for the CLST by Degraeve and Jans (7) and Trigeiro et al. (37). While for the CLST that is expected since Manne (30) showed that the linear relaxation of the Dantzig-Wolfe formulation of the CLST is a good approximation to the integer problem whenever the number of items is large as compared to the number of capacity constraints, it is interesting that this result, at least empirically, extends to the CLSTP.

In general, the error gap increases as capacities get tighter, average setup times increase, and TBO increases. We also observe that the effects of capacity, setup time, and TBO are more pronounced when the number of items is small. It is also interesting to note that the error gaps are lower under dynamic prices than under constant prices when using the Infinite Formulation while the reverse is true when using the Finite Formulation. Finally, we observe that the advantage of our branch-and-price algorithm with the Infinite Formulation over CPLEX increases as problems become more difficult to solve (i.e., as capacities get tighter, average setup times increase, and TBO increases). This is particularly apparent under a constant pricing strategy, high TBO values, and a larger number of items.

The results above all used a fixed running time of 1200 seconds. To assess the rate of convergence of our branch-and-price algorithms and CPLEX we have, for several representative instances, created a plot that tracks the upper and lower bounds found by the algorithms as time progresses. Figures 3-1–3-4 show the plots on a representative set of test instances with 10 and 30 items respectively. The test instances for the figures were selected to show the behavior of the solution approaches on a range of combinations of TBO, setup time, and capacity values. The caption with each table indicates the number of items, low/high TBO, and the pricing strategy used. Moreover, each figure is labeled with the remaining problem characteristics; e.g., "low ST, high C" represents an instance with low setup times and high capacities. In all figures, the solid lines represent lower bounds and the dashed lines upper bounds. The bounds for the branch-and-price algorithm with the Infinite Formulation (IF) and the Finite Formulation are drawn in red and green, respectively, while the bounds for CPLEX are drawn in blue. We see from the figures that branch-and-price algorithm using the Infinite Formulation finds high-quality integer solutions fast and tends to dominate CPLEX in terms of lower and upper bounds over time. These figures also suggest that the branch-and-price algorithm performs much better than CPLEX on instances with a larger number of items.

Finally, to quantify the effect of the choice of pricing strategy on the optimal profit we compared the lower bounds obtained by the Infinite Formulation with both a constant and a dynamic pricing strategy. In our test problems, the dynamic pricing strategy enjoyed an overall average additional profit of 4.7% over the profit obtained with a constant pricing strategy. Table 3-4 breaks this profit increase down by the different characteristics of the problem instances. In particular, we conclude that the dynamic pricing strategy is particularly profitable when capacities are low and setup times and TBO are high. This corresponds with what one might intuitively expect: for any fixed demand vector, total costs tend to be larger when setup times are larger because some of the available limited capacity is used for setups. Moreover, due to this use of capacity for setup time reduces the number of feasible production plans. A similar argument can be made for the costliness of reduced capacities and larger TBO. In these cases, the ability to control prices and hence affecting the demands provides a great deal of flexibility to the producer. Particularly dynamic prices can serve to better match demands to variable and limited capacities. Since constant prices provide a smaller degree of flexibility the difference between the profit under the different pricing strategies is amplified when capacity is smaller and setup times and TBO are larger.

3.7 Conclusion and Future Research

In this chapter we considered the capacitated lot-sizing problem with setup times where we allow demand flexibility through pricing decisions. We developed two Dantzig-Wolfe formulations to the problem which lead to an exact branch-and-price

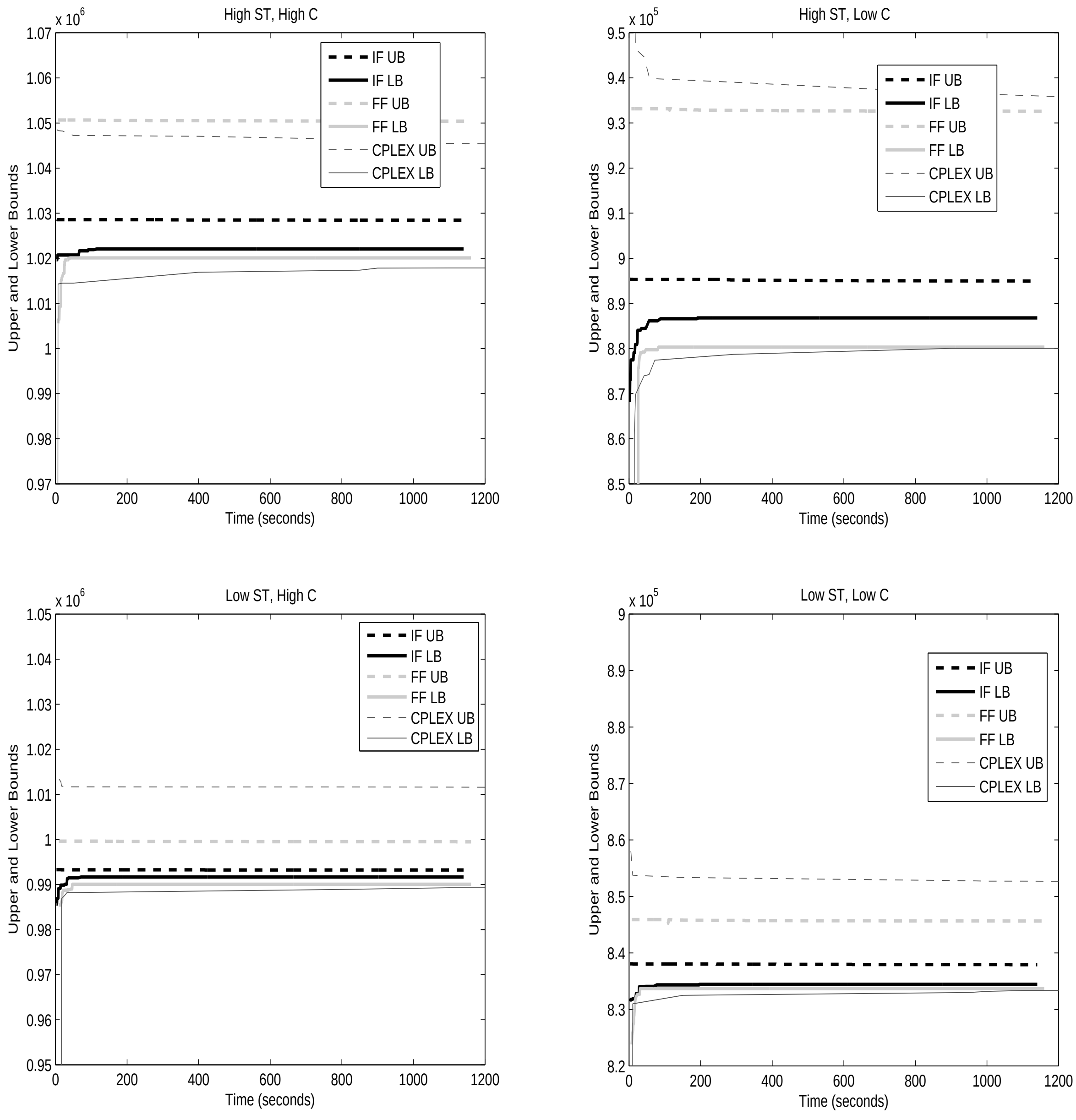

Figure 3-1: Comparison of branch-and-price and CPLEX performance on a representative set of test problems with 10 items, low TBO, and constant pricing.

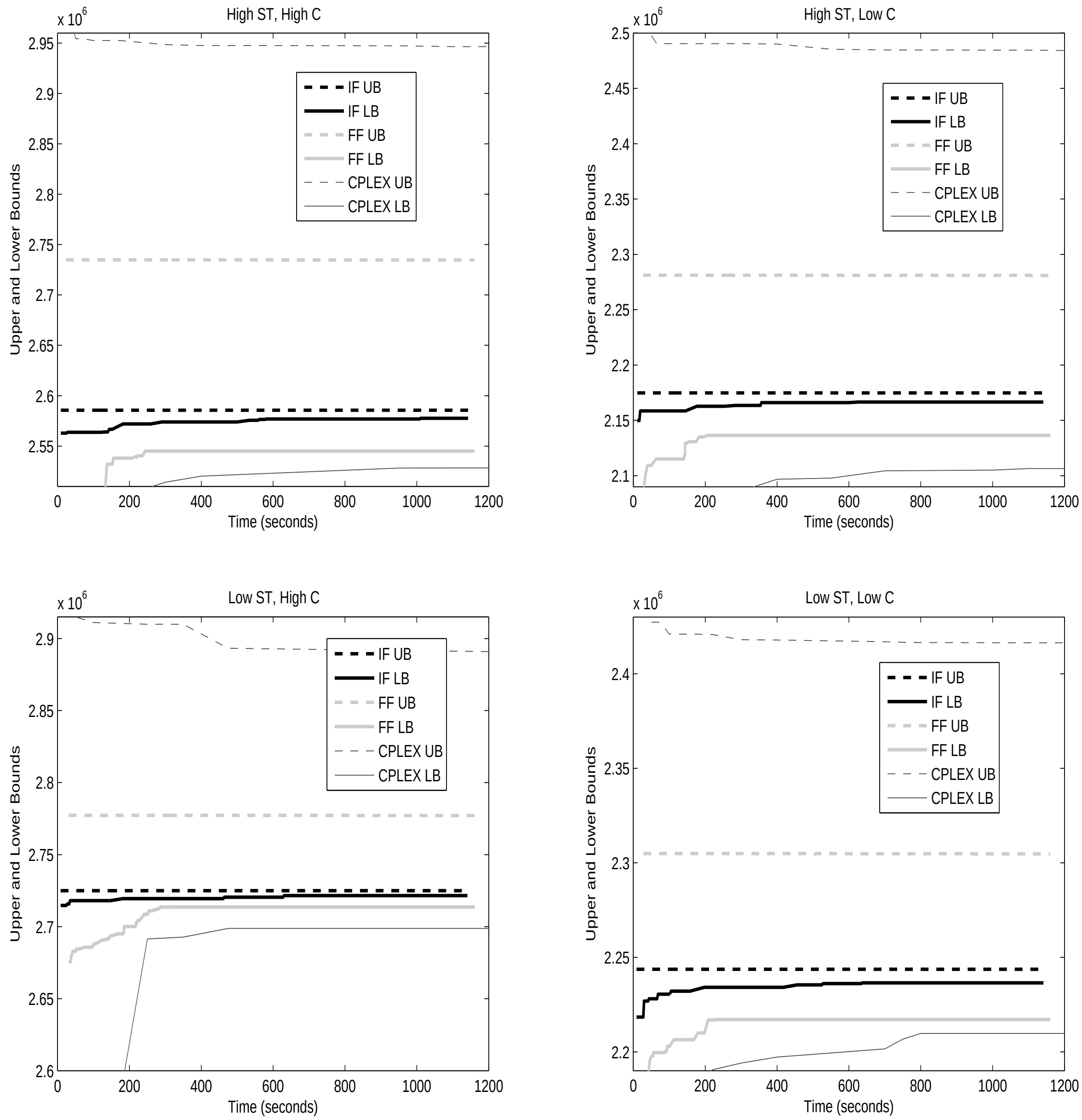

Figure 3-2: Comparison of branch-and-price and CPLEX performance on a representative set of test problems with 30 items, high TBO, and constant pricing.

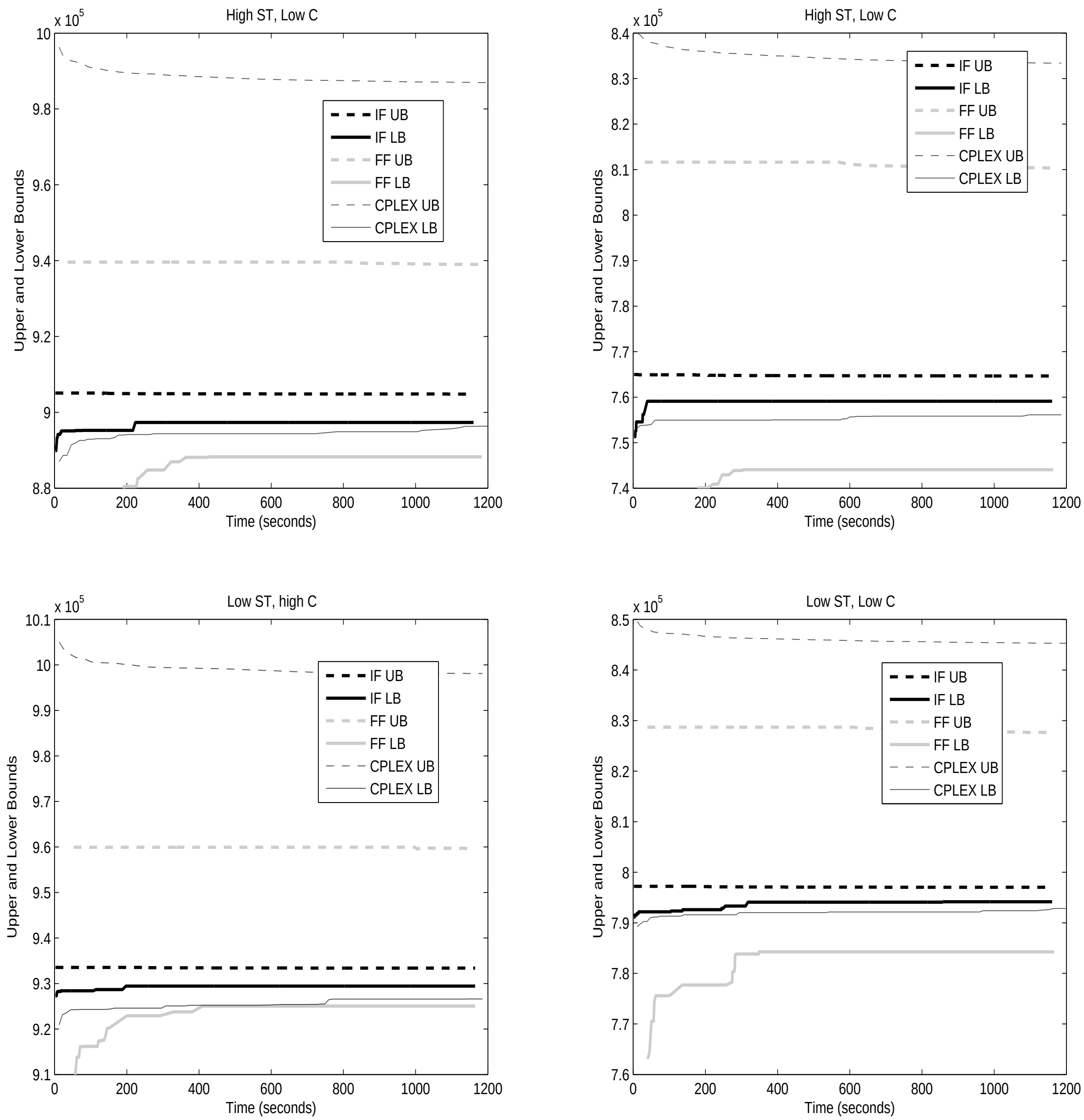

Figure 3-3: Comparison of branch-and-price and CPLEX performance on a representative set of test problems with 10 items, high TBO, and dynamic pricing.

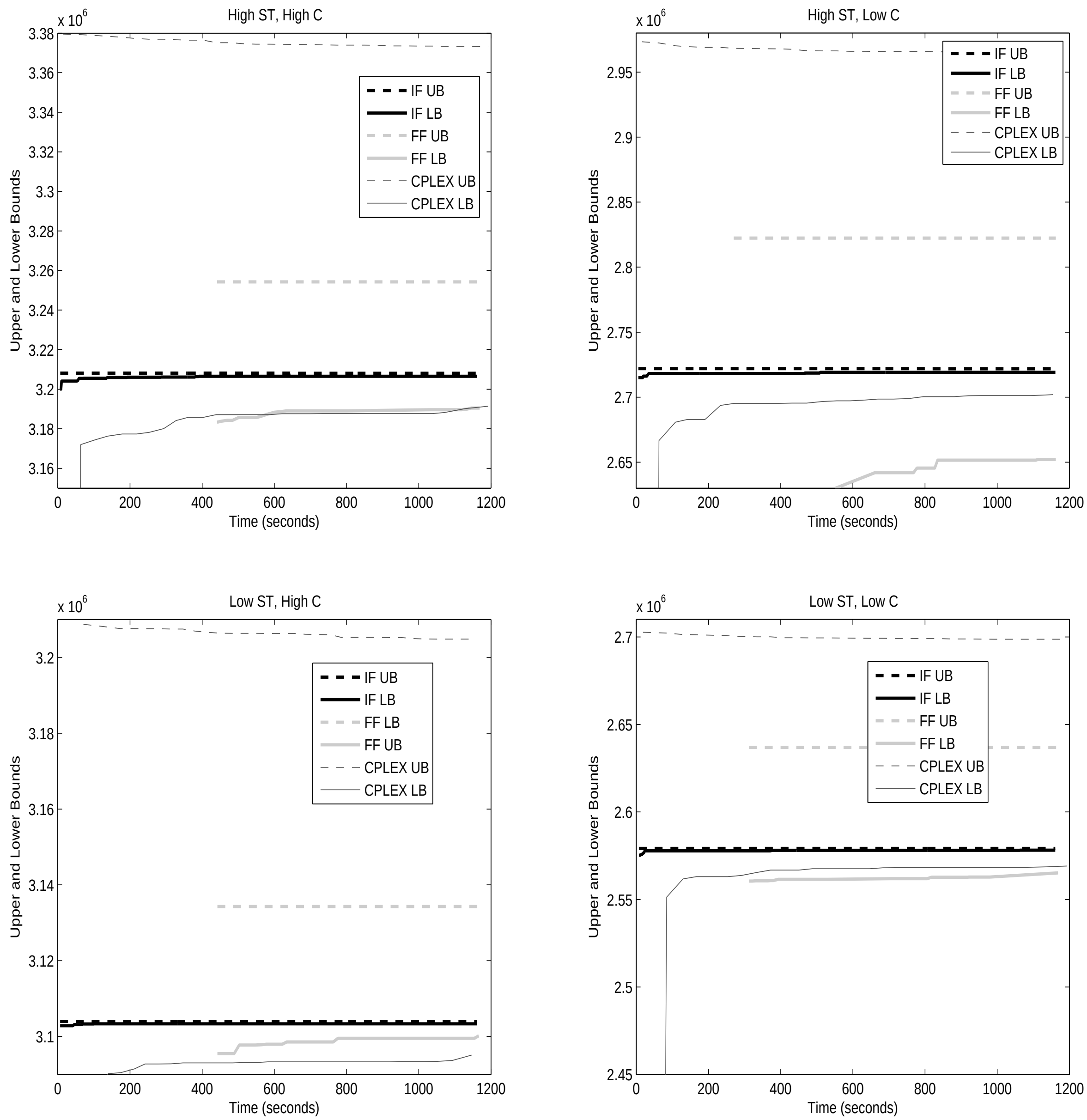

Figure 3-4: Comparison of branch-and-price and CPLEX performance on a representative set of test problems with 30 items, low TBO, and dynamic pricing.

Table 3-4: Percentage increase in profit when a dynamic pricing strategy is used instead of a constant pricing strategy.

	# items	10	20	30
capacity	**low**	5.36	5.30	5.04
	medium	5.12	4.75	4.71
	high	4.44	4.11	4.11
setup time	**low**	4.15	4.05	3.88
	high	6.37	5.98	5.81
TBO	**low**	4.15	4.05	3.88
	medium	4.77	4.32	4.35
	high	6.06	5.82	5.69

algorithm. The first formulation resulted in a nonlinear master and a finite column generation algorithm whereas the second formulation resulted in a linear master problem and an infinite column generation algorithm. With both formulations, columns were generated by solving single item lot-sizing subproblems with pricing decisions. There are polynomial time algorithms to the subproblems in the literature assuming dynamic and constant price strategies with no initial inventories. We developed solution algorithms to the more general case where a given amount of initial inventory exists at the beginning of the planning horizon.

For ease of implementation, we performed our computational experiments by assuming no initial inventories. The test results showed that branch-and-price algorithm with the second formulation results in very low integrality gaps. This helps to find high quality integer solutions very early in the algorithm compared to CPLEX.

Future research would include working on the polyhedral properties of the problem to reduce the integrality gap of the mixed integer programming formulation. Other decomposition methods leading to different and maybe tighter formulations may also be considered. Single item lot sizing subproblems are also interesting to work on; polynomial time solution algorithms may be found for various pricing strategies.

CHAPTER 4
THE ECONOMIC LOT SIZING PROBLEM WITH PERISHABLE ITEMS

4.1 Introduction

All the economic lot sizing (ELS) models cited and studied in previous chapters assumed that the items remain intact and therefore can be kept in inventories indefinitely to meet future demands. However, some items such as agricultural products and dairy products may spoil in time and those models can not be applied anymore. Nahmias (33) has an extensive review on production and inventory models that deal with such perishable items. Continuous time EOQ type models with deterministic demands include the works of Ghare and Schrader (17) and Cohen (4). On the other hand, there is a rich stream of literature on discrete time models with stochastic demand and fixed lifetimes. For instance, Nahmias and Pierskalla (34), Fries (13), Nahmias (31; 32) develop models where demand is stochastic and the items deteriorate after a fixed lifetime.

Friedman and Hoch (12) consider an extension of the (ELS) where the inventories spoil in each period at a rate as a function of their ages. They assume that if there are I_{it} units of items of age i at the end of period t, then to the period $t+1$ only $r_i I_{it}$ of those items will be transferred. Here, $(1 - r_i)$ is the deterioration rate of an item at age i where $0 \leq r_i \leq 1$. Friedman and Hoch show that if the items are perishable, optimal solutions may not satisfy the *zero inventory ordering* (ZIO) property. However, they demonstrate that, if the deterioration rate is assumed to increase as the items get older, that is, if $r_i \geq r_{i+1}$, then there exists an optimal solution where production in a period satisfies some set of consecutive demand periods.

The deterioration rate in Friedman and Hoch (12) depends only on the age of the item and is independent of the period the item is produced. This may not be realistic considering that food products tend to deteriorate faster in summer than in winter. Given this, Hsu (25) proposes an (ELS) model where stock deterioration rates depend both on the age and the period the item is procured. In his model, if in period t there are y_{it} items left in stock from the items that were procured in period i, then $(1 - \alpha_{it})y_{it}$ items will be carried over to period $t + 1$. Here, α_{it} is the deterioration rate for an item in period t, that was procured in period i. Similar to Friedman and Hoch, he assumes that as the items get older, they deteriorate faster. That is $\alpha_{it} \geq \alpha_{jt}$ for $i \leq j \leq t$. Hsu proposes his most general model assuming general concave procurement and inventory holding cost functions with the assumption that the longer a unit is carried, the higher its inventory carrying cost is. With these assumptions, optimal solution structure is the same as the one proposed in Friedman and Hoch; procurement in a period satisfies a set of consecutive demand periods.

In this chapter, we study the economic lot sizing problem with perishable items (ELS-PI), where items are assumed to deteriorate completely after a deterministic lifetime and can not be sold thereafter. Unlike the previous (ELS) models where a certain ratio of items may deteriorate in each period, we assume that items remain undamaged and available for consumption until the end of their lifetimes. We assume that there are a number of suppliers for the store manager to order the items in each period. The procurement costs for each supplier might be different and the lifetimes of items procured from different suppliers might be different as well. Therefore, it is possible that an item procured later deteriorate earlier and hence we can no longer claim the existence of optimal solutions where procurement from a supplier satisfies a set of consecutive periods. In fact, we can observe optimal solutions where two distant demand periods are satisfied through procurement from a supplier in a particular period whereas the demand periods in between are satisfied by procurement in a different period from another supplier. For such a solution to be achievable, the store manager might need to sell early expiring items even if there are items in the inventories that expire later. But that may not be possible because consumers have a natural tendency to purchase the items that have a later expiration date.

To be able to sell early expiring items to the consumer, the store manager should have absolute control on which item the consumer will purchase. In some cases, the shelves the items are displayed are in the form of a queue. In such exhibits of the items, the consumer is affectively forced to purchase the item at the front end of the shelves regardless of their preferences. Although consumer behavior is regulated to some extent with this kind of an inventory system, it does not grant the inventory manager full control on the allocation of the items if the store manager is also restricted to place the newly procured items either always to the back of the queue or to the front of the queue. We call the restrictions imposed on the distribution of the items to demands *item consumption order constraints* and demonstrate that such constraints significantly influence the optimal solution structure. In particular, we show that if the items are allocated to the customers in such a manner that early expiring items are consumed earlier, lowest cost production plans are achieved.

This chapter is organized as follows. In Section 4.2, we present the ELS-PI model and discuss optimal solution structures under different consumption order constraints. These observations then lead us to polynomial time dynamic programming algorithms, which we discuss in Section 4.3. In Section 4.4 we investigate the problem under procurement capacities. In section 4.5, we consider two extensions to the problem: backlogging and pricing. Section 4.6 concludes this chapter by stating possible further research directions.

4.2 Model Formulation and Analysis

4.2.1 The Model

We present the (ELS-PI) under various item consumption order constraints. Particularly we focus on consumption orders that are caused by either the consumer behavior or the physical constraints of the inventory system. The consumer behavior may become important when items of two different ages are exhibited together on the shelves. Whenever this happens, consumers usually have a preference for the items that have a longer life remaining for obvious reasons. Whether they are able to exercise their preferences or not, however, depends on the allocative mechanism. If the consumers are allowed to choose the items themselves, which is usually the case in retail stores, they will choose the ones with the longest life remaining. In such a mechanism, we say that the inventory is consumed in a *Last-Expired, First-Out* (LEFO) manner. On the other hand, if the inventory manager has the complete control on the consumption of the items, he may allocate the items to each demand to minimize spoilage. With this mechanism, the inventory is consumed in a *First-Expired, First-Out* (FEFO) manner, which we show, results in the minimum cost production plan.

The LEFO and FEFO are not the only orders in which the inventory is consumed. Depending on the way the items are exhibited and placed in the store, the inventory may also be consumed in a *First-In, First-Out* (FIFO) or *Last-In, First-Out* (LIFO) manner. FIFO consumption order results if the inventory system is designed as a queue such that as the items are procured, they are placed at the end of the queue. With this inventory system, the consumption of items procured in a particular period does not start unless all the items that were procured in earlier periods have been depleted. In case the store manager orders items from several suppliers in a given period, we assume that the items with sooner expiration dates are inserted to the queue earlier so that they are consumed earlier. On the other hand, LIFO consumption order occurs if the inventory system is designed as a stack such that the newly procured items are always put in front of the stack and they are the first to be consumed out of this stack. In case the store manager orders items from several suppliers in a given period, we assume that the items with sooner expiration dates are inserted to the stack later so that they are consumed earlier.

We assume a discrete and finite planning horizon consisting of T periods. The demand in each period t is completely known and is denoted by D_t $(t = 1, \ldots, T)$. Further, the demand in any period is satisfied on time, and backlogging is not allowed. In each period, there are a finite number of suppliers available to order the items from. We let K_t represent the number of available suppliers in period t and let $K = \max_t K_t$. Without loss of generality, we assume that $K_t \geq 1$ because we can always assume that there is a supplier with infinite procurement cost. We use the period–supplier pairs (t, k)

$(t = 1, \ldots, T)$, $(k = 1, \ldots, K_t)$ to refer to the k^{th} supplier available in period t and assume that each supplier is an independent entity. We say that supplier (t_1, k_1) precedes supplier (t_2, k_2) if (i) $t_1 = t_2$ and $k_1 < k_2$ or (ii) $t_1 < t_2$. To indicate such precedence, we write $(t_1, k_1) < (t_2, k_2)$ and assume that items procured from the supplier (t_1, k_1) are placed in the storage before the items procured from the supplier (t_2, k_2). We assume that if the items are procured from multiple suppliers in a single period, they are placed in the storage respecting the precedence relation among them and demand is realized after all the items that are procured in that period are all available in the inventories. We say suppliers (t_1, k_1) and (t_2, k_2) are consecutive if the supplier (t_1, k_1) is the first supplier that immediately precedes supplier (t_2, k_2).

We assume that items have finite and completely known lifetimes depending on the supplier they are procured. An item ordered from the supplier (t, k) is good for consumption until the end of period v_{tk}, which we call the expiration date of the item. We assume that there is no deterioration in the quality of the item as long as the expiration date of the item has not passed. For each supplier (t, k) we define $F(t, k) = \{t, \ldots, v_{tk}\}$ to be the set of periods that an item procured from the supplier (t, k) can satisfy and define $F^{-1}(i) = \{(t, k) : 1 \leq k \leq K_t, \quad t \leq i \leq v_{tk}\}$ to be the set of suppliers (t, k) that can satisfy the demand in period i.

The precedence relation between the suppliers is particularly important when we are investigating optimal solutions under FIFO and LIFO consumption order constraints. Recall that FIFO consumption order occurs if the inventory system is designed as a queue. Due to our definition, we assume that the suppliers in period t are indexed in increasing order of the expiration dates of the items purchased from them such that $v_{t,k+1} \geq v_{tk}$ for $k = 1, \ldots, K_t - 1$. Likewise, LIFO consumption order occurs if the inventory system is designed as a stack. In such a case, we assume that the suppliers in each period t are indexed in decreasing order of the expiration dates of the items purchased from them such that $v_{t,k+1} \leq v_{tk}$ for $k = 1, \ldots, K_t - 1$.

We let x_{tki} be the amount procured from the supplier (t,k) $(t = 1,\ldots,T)$ $(k = 1,\ldots,K_t)$ and allocated to satisfy the demand of period i $(i = t,\ldots v_{t_k})$ and define $x_{tk} = \sum_{i=t}^{v_{tk}} x_{tki}$ to be the total procurement from the supplier (t,k). We denote the procurement cost function from the supplier (t,k) by P_{tk}. We define I_t to be the inventory carried at the end of period t and define H_t to be the cost of inventory carriage in period t. We assume that P_{tk} and H_t are both concave. With this setting, the ELS-PI is formulated as follows.

$$\text{Minimize} \quad \sum_{t=1}^{T}\left(\sum_{k=1}^{K_t} P_{tk}\left(x_{tk}\right) + H_t\left(I_t\right)\right)$$

subject to

$$I_t = \sum_{i=1}^{t}\sum_{k=1}^{K_i} x_{ik} - \sum_{i=1}^{t} D_i \qquad t = 1,\ldots,T \tag{4–1}$$

$$x_{tk} = \sum_{i\in F(t,k)} x_{tki} \qquad t = 1,\ldots,T;\ k = 1,\ldots,K_t \tag{4–2}$$

$$D_i = \sum_{(t,k)\in F^{-1}(i)} x_{tki} \qquad i = 1,\ldots,T \tag{4–3}$$

$$\{x_{tki}\} \in ICO_i \tag{4–4}$$

$$I_0 = 0, I_t \geq 0, x_{tki} \geq 0 \qquad t = 1,\ldots,T;\ k = 1,\ldots,K_t;\ i = t,\ldots,v_{tk}$$

In the formulation above, objective function minimizes total procurement and inventory holding costs. Constraints (4–1) are the inventory balance constraints. Constraints (4–2) and (4–3) guarantee that items procured from a certain supplier can only be allocated to satisfy the demands of periods before the expiration date of the items. On the other hand, constraints (4–4) guarantee that the item allocation variables x_{tki} obey the assumed item consumption order i (ICO_i) where $i \in \{LEFO, FEFO, LIFO, FIFO\}$.

4.2.2 Structural Properties of Optimal Solutions

As mentioned earlier, if the lifetimes are independent of one another, then an optimal solution is possible where items purchased from a single supplier in a particular period satisfies two distant demand periods while the demand periods in between are satisfied through other suppliers in other periods. Example 4.1 shows how such an optimal solution might be realized.

Example 4.1. *Consider the 5 period problem where we assume that there is exactly one supplier available in each period. We assume that* $H_t(x) = 0$ $(t = 1, \ldots, 5)$ *and that the procurement cost function for the single supplier available in period* t *is given by*

$$P_t(x) = \begin{cases} S_t + p_t x & x > 0 \\ 0 & x = 0 \end{cases}$$

Table 4-1: Data for example 4.1.

t	**1**	**2**	**3**	**4**	**5**
S_t	50	50	50	50	50
p_t	0	5	10	0	10
v_t	2	5	5	4	5
D_t	20	20	20	20	20

Table 4-1 lists relevant data for the example. We assume that there are no constraints on the inventory consumption order. That is, the inventory manager is free to distribute any item in the inventories to the consumers. In that case, he would procure 40 units in periods 1 and 2 and 20 units in period 4. He would distribute the items procured in period 1 to satisfy the demands in periods 1 and 2; the items procured in period 2 to satisfy the demands in periods 3 and 5; the items procured in period 4 to satisfy the demand in period 4. The total cost of this procurement plan is 350.

The first structural property we present relates to the distribution of items to demands and is a generalization of the ZIO property that incorporates the perishability of the items and multiple suppliers available in each period.

Theorem 4.1 (Assignment of Demands). *There exists an optimal solution to the ELS-PI such that for any two suppliers* (t_1, k_1) *and* (t_2, k_2)*, either* $x_{t_1 k_1 \tau} = 0$ *or* $x_{t_2 k_2 \tau} = 0$ *for all* $\tau \in F(t_1, k_1) \cap F(t_2, k_2)$*.*

Proof. Observe that any feasible solution to the ELS-PI corresponds to a flow over a network with K_t source nodes in every period t and T demand nodes as drawn in Figure 2 for a 3 period problem. Amount of flow on the arc connecting the source node (t, k) to the demand node t represents x_{tk}, the amount of procurement in period t. The flow on arc between demand nodes t and $t + 1$ represents I_t, the inventory carried between periods t and $t + 1$.

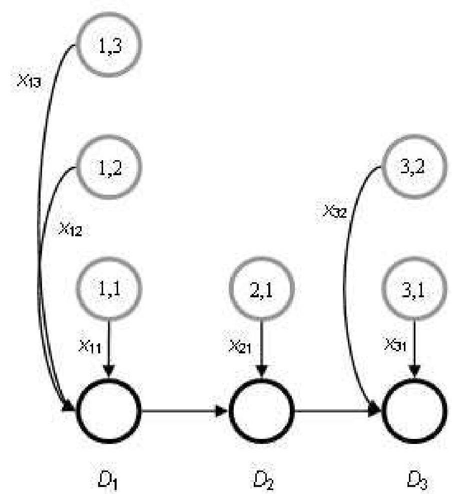

Figure 4-1. Network representation of the ELS-PI for a 3–period problem

Now, assume that in an optimal solution, there are two suppliers (t_1, k_1) and (t_2, k_2) such that $\Delta_1 = \sum_{\tau \in F(t_1,k_1) \cap F_(t_2,k_2)} x_{t_1 k_1 \tau} > 0$ and $\Delta_2 = \sum_{\tau \in F(t_1,k_1) \cap F(t_2,k_2)} x_{t_2 k_2 \tau} > 0$. That is, supplier (t_1, k_1) satisfies a total demand of Δ_1 and supplier (t_2, k_2) satisfies a total demand of Δ_2 in periods $F(t_1, k_1) \cap F(t_2, k_2)$. Let this solution correspond to the flow $X = (x_{tk}, I_t)$ in the network and consider generating two other solutions and corresponding network flows as follows:

(i) Decrease procurement from the supplier (t_2, k_2) by Δ_2 and increase procurement from the supplier (t_1, k_1) by the same amount to obtain $X' = (x'_{tk}, I'_t)$, where $x'_{t_1k_1} = x_{t_1k_1} + \Delta_2$, $x'_{t_2k_2} = x_{t_2k-2} - \Delta_2$. Assuming no items deteriorate before being consumed, after this change we have $I'_t = I_t + \Delta_2$ for $t = \min\{t_1, t_2\}, \ldots, \max\{t_1, t_2\} - 1$ while amount of items procured from the other suppliers and inventory carried in other periods remain the same as in X.

(ii) Decrease procurement from the supplier (t_1, k_1) by Δ_1 and increase procurement from the supplier (t_2, k_2) by the same amount to obtain $X'' = (x''_{tk}, I''_t)$, where $x''_{t_1k_1} = x_{t_1k_1} - \Delta_1$, $x''_{t_2k_2} = x_{t_2k_2} + \Delta_1$. Assuming no items deteriorate before being consumed, after this change we have $I'_t = I_t - \Delta_1$ for $t = \min\{t_1, t_2\}, \ldots, \max\{t_1, t_2\} - 1$ while the amount of items procured from the other suppliers and inventory carried in other periods remain the same as in X.

Observe that $X = \lambda X' + (1 - \lambda) X''$ for some $\lambda \in (0, 1)$. Since the costs are concave, X' and X'' should also be optimal and in neither of the solutions, suppliers (t_1, k_1) and (t_2, k_2) both satisfy the demands of periods in $F(t_1, k_1) \cap F(t_2, k_2)$. Continuing in this manner, an optimal solution can be reached where the property is satisfied. What remains is to show that after the changes in X, order of consumption does not cause any item to perish before being consumed. This will then imply that both X' and X'' are feasible.

The changes in X lead to feasible solutions if there are no constraints on the item consumption order since any allocation scheme is possible. To show that X' and X'' are both feasible under any of the four (i.e., FIFO, LIFO, FEFO and LEFO) item consumption orders, we will investigate each consumption order separately. basically, we will show that the above changes in the procurement plan do not delay consumption of any item beyond its expiration date.

FIFO case: Choose $(t_1, k_1) < (t_2, k_2)$ to be two consecutive suppliers that violates the property. This choice of suppliers will guarantee that items procured from the supplier (t_2, k_2) are placed to the queue right after the items procured from the supplier (t_1, k_1). Then, a decrease of $\Delta_2 > 0$ in procurement from the supplier (t_2, k_2) can be immediately compensated by an equivalent increase in procurement from the supplier (t_1, k_1). Likewise, a decrease of $\Delta_1 > 0$ in procurement from the supplier (t_1, k_1) can be compensated by an equivalent increase in procurement from the supplier (t_2, k_2). Because no other item is consumed between the time consumers first start purchasing the items procured from the supplier (t_1, k_1), until the time they deplete all the items procured from the supplier (t_2, k_2), these changes can not result in any delay in the consumption of the items procured from other suppliers in other periods and hence, no items perish. Given that $F(t_1, k_1) \cap F(t_2, k2) = t_2, \ldots, \min\{v_{t_1,k_1}, v_{t_2k_2}\}$, the changes will result in one of the following outcomes: (i) procurement from one of the supplier decreases to zero, (ii) items procured from the supplier (t_2, k_2) do not satisfy any demand before period $v_{t_1k_1}$ or (iii) items procured from the supplier (t_1, k_1) do not satisfy any demand after period $t_2 - 1$ and hence they are not carried in the inventories after period $t_2 - 1$. This ensures that if the property is satisfied between every two consecutive suppliers, then it is satisfied between each supplier pair. Therefore, selecting the pairs in this manner, we obtain an optimal solution that satisfies the property stated in the theorem.

LIFO case: For the supplier (t_2, k_2), choose (t_1, k_1) to be the first supplier with positive procurement preceding the supplier (t_1, k_1) that violates the stated property. Note that, due to the assumed consumption order, all the items procured from the supplier (t_2, k_2) must have depleted before the items procured from the supplier (t_1, k_1) could satisfy any demand in $F(t_1, k_1) \cap F(t_2, k_2)$. Suppose also that some of the demands in $F(t_1, k_1) \cap F(t_2, k_2)$ is satisfied through procurement from other suppliers and let S denote the set of such suppliers. That is,

$$S = \{(i, j) \neq (t_1, k_1) | x_{ij\tau} > 0 \text{ for } \tau \in F(t_1, k_1) \cap F(t_2, k_2)\}.$$

Since (t_1, k_1) is the first supplier that precedes (t_2, k_2) which violates the property, we should have that $(t_1, k_1) < (i, j)$ and $(t_2, k_2) < (i, j)$ for all $(i, j) \in S$ and as such items procured from the suppliers $(i, j) \in S$ have higher consumption priority than the items procured from the suppliers (t_1, k_1) and (t_2, k_2). Given this, any increase (decrease) in procurement from the supplier (t_2, k_2) and an equivalent decrease (increase) in procurement from the supplier (t_1, k_1) can not affect the consumption of items procured from $(i, j) \in S$. Therefore, no item perishes due to a possible delay in their consumption.

FEFO case: Let (t_1, k_1) and (t_2, k_2) be two suppliers with positive procurement and assume, without loss of generality, that $v_{t_1k_1} \leq v_{t_2k_2}$ and consider the demands in $F(t_1, k_1) \cap F(t_2, k_2)$. In case $v_{t_1k_1} = v_{t_2k_2}$, since consumers are indifferent between the two items, a redistribution of the items to the demands is readily available without interfering with the consumption of the other items and the feasibility of the solution is maintained. As a result, we can safely assume that no two suppliers (t_1, k_1) and (t_2, k_2) with $v_{t_1k_1} = v_{t_2k_2}$ violate the property and hence items from those suppliers do not exist together in the inventories. Therefore, we assume $v_{t_1k_1} < v_{t_2k_2}$.

Assume that the last demand in $F(t_1, k_1) \cap F(t_2, k_2)$ that is satisfied by the items procured from the supplier (t_2, k_2) is the L^{th} demand in $F(t_1, k_1) \cap F(t_2, k_2)$. Assume that items procured from other suppliers also satisfy some of the demands in $F(t_1, k_1) \cap F(t_2, k_2)$ and that they are totally consumed before the L^{th} demand in $F(t_1, k_1) \cap F(t_2, k_2)$ is satisfied. Letting S be the set of such suppliers, we should have that $v_{ij} < v_{t_2k_2}$ for all $(i, j) \in S$. Otherwise, they wouldn't be consumed before the items procured from (t_2, k_2) were depleted. Then, we have that

$$L = \Delta_1 + \Delta_2 + A$$

where, $A = \sum_{(i,j) \in S} \sum_{\tau \in F(t_1,k_1) \cap F(t_2,k_2)} x_{ij\tau}$ is the total demand until the L^{th} demand in $F(t_1, k_1) \cap F(t_2, k_2)$ that were satisfied by the suppliers $(i, j) \in S$. Assume that there exists a subset $S' \subseteq S$ such that $v_{ij} < v_{t_1k_1}$ for $(i, j) \in S'$ and $v_{ij} \geq v_{t_1k_1}$ for all $(i, j) \in S - S'$. Observe that any increase (decrease) in production in t_2 and an equivalent decrease (increase) in production in period t_1 can not delay the consumption of items procured from $(i, j) \in S'$ because those items have sooner expiration dates than the ones procured from (t_1, k_1) and (t_2, k_2). On the other hand, if the amount of items procured from (t_1, k_1) is increased by Δ_2 (and the amount of items produced in period t_2 is decreased by the same amount), the consumption of items procured in $(i,j) \in S - S'$ will be delayed. However, they will be totally consumed before or at the time the L^{th} demand occurs in $F(t_1,k_1) \cap F(t_2,k_2)$. Since $v_{ij} \leq v_{t_2k_2}$ for $(i,j) \in S$, this guarantees that those items will not deteriorate before being consumed. Decrease in the amount of items procured from (t_1,k_1) can not result in a solution where some of the items deteriorate due to consumption order constraints.

LEFO case: Let (t_1,k_1) and (t_2,k_2) be two suppliers with positive procurement such that $v_{t_1k_1} \geq v_{t_2k_2}$ and consider the demands in $F(t_1,k_1) \cap F(t_2,k_2)$. In case $v_{t_1k_1} = v_{t_2k_2}$, a redistribution of items to the demands is readily available without violating the consumption order constraints and the feasibility of the solution. As a result, we can safely assume that no two suppliers (t_1,k_1) and (t_2,k_2) with $v_{t_1k_1} = v_{t_2k_2}$ violate the property and hence they can not exist together in the inventories. Therefore, we assume $v_{t_1k_1} < v_{t_2k_2}$. This implies that consumption of the items procured from the supplier (t_2,k_2) can not start in $F(t_1,k_1) \cap F(t_2,k_2)$ before the items procured from the supplier (t_1,k_1) are totally exhausted. Assume that the last demand in $F(t_1,k_1) \cap F(t_2,k_2)$ that is satisfied by the items procured from (t_2,k_2) is the L^{th} demand in $F(t_1,k_1) \cap F(t_2,k_2)$. Assume also that some of the demands in $F(t_1,k_1) \cap F(t_2,k_2)$ are satisfied by procurement from other suppliers and they are depleted before the items procured from the supplier (t_2,k_2) are depleted. Letting S be the set of such suppliers, we should have that $v_{ij} > v_{t_2k_2}$ for $(i,j) \in S$ and we have

$$L = \Delta_1 + \Delta_2 + A$$

where, $A = \sum_{(i,j)\in S} \sum_{\tau \in F(t_1,k_1)\cap F(t_2,k_2)} x_{ij\tau}$ is the total demand until the L^{th} demand in $F(t_1,k_1) \cap F(t_2,k_2)$ that were satisfied by procurement from the suppliers $(i,j) \in S$. Assume there exists a subset $S' \subseteq S$ such that $v_{ij} > v_{t_1k_1}$ for $(i,j) \in S'$ and $v_{ij} \leq v_{t_1,k_1}$ for $(i,j) \in S - S'$. Observe that any increase (decrease) in procurement from the supplier (t_1,k_1) and an equivalent decrease (increase) in procurement from the supplier (t_2,k_2) can not delay the assignment of items procured from $(i,j) \in S'$ because those items have later expiration dates than the ones procured from the suppliers (t_1, k_1) and (t_2, k_2). Now assume that we increase procurement from the supplier (t_1, k_1) by Δ_2 and decrease procurement from (t_2, k_2) by the same amount. The increase in procurement from (t_1, k_1) will not affect the consumption of items procured from $(i, j) \in S'$ since those items have higher consumption priority. But the items procured from $(i, j) \in S - S'$ will be consumed only after all the items procured from (t_1, k_1) are depleted. This will delay their consumption. But, as long as the increase in procurement from (t_1, k_1) is not greater than Δ_2, they will be totally consumed before or at the time the L^{th} demand in $F_{t_1} \cap F_{t_2}$ is satisfied. Those items will not deteriorate since $v_{ij} \geq v_{t_2k_2}$ for all $(i, j) \in S$ and the solution will still be feasible. □

Theorem 4.1 further implies that, demand of a period is not satisfied by more than one supplier in an optimal solution. Therefore, as a corollary to this theorem, we claim that an optimal solution exists where a unique supplier fully satisfies the demand of a period.

Corollary 4.1 (Unique Supplier Property)**.** *There exists an optimal solution to the ELS-PI in which the demand in period i is fulfilled by a unique supplier $(t, k) \in F^{-1}(i)$.*

Item consumption order constraints have a significant effect on the nature of optimal solutions to the ELS-PI. We first prove that if the consumption order constraints are dropped, the store manager will distribute the early expiring items first. In other words, FEFO constraints are never restricting.

Corollary 4.2. *There exists an optimal solution to the ELS-PI with no consumption order constraints, where the items are distributed in FEFO manner.*

Proof. Consider two suppliers $(t_1, k_1) < (t_2, k_2)$ and assume that $v_{t_1k_1} > v_{t_2k_2}$. Then, $F(t_1, k_1) \cap F(t_2, k_2) = \{t_2, \ldots, \min\{v_{t_1k_1}, v_{t_2k_2}\} = v_{t_2k_2}\}$. If $x_{t_2k_2\tau} = 0$ for all $\tau \in F(t_1, k_1) \cap F(t_2, k_2)$, then $x_{t_2k_2} = \sum_{\tau=t_2}^{v_{t_2k_2}} x_{t_2k_2\tau} = 0$, i.e., there is no procurement from the supplier (t_2, k_2). On the other hand, if $x_{t_1k_1\tau} = 0$ for all $\tau \in F(t_1, k_1) \cap F(t_2, k_2)$, then this implies that items procured from the supplier (t_1, k_1) are not consumed while the items procured from the supplier (t_2, k_2) can be present in the inventories. In all cases, items that expire earlier are consumed earlier.

Now assume that $v_{t_1k_1} \leq v_{t_2k_2}$ such that $F(t_1, k_1) \cap F(t_2, k_2) = \{t_2, \ldots, v_{t_1k_1}\}$. If $x_{t_1k_1\tau} = 0$ for all $\tau \in F(t_1, k_1) \cap F(t_2, k_2)$ then items procured from the supplier (t_1, k_1) are never carried in inventories to period t_2, hence they are not displayed together with the items procured from the supplier (t_2, k_2). On the other hand, if $x_{t_2k_2\tau} = 0$ for all $\tau \in F(t_1, k_1) \cap F(t_2, k_2)$, then either $x_{t_2k_2} = 0$ (in case $v_{t_1} = v_{t_2}$) or $x_{t_2k_2\tau} > 0$ only for $\tau > v_{t_1k_1}$, that is, items procured from supplier (t_2, k_2) are consumed only after the ones procured from supplier (t_1, k_1) are depleted. In all cases, we observe that items that expire earlier are consumed earlier. □

Although Corollary 4.2 states that it is best for the store manager to first sell the items that expire earlier, when the items are displayed on the shelves and when the consumers are free to choose among the displayed items, they will buy the one that expires later leading to a LEFO consumption order. In such a case, in an optimal procurement plan, two items from different suppliers are not carried in the inventories together. This immediately implies that items are ordered from at most one supplier in each period and that ZIO property holds in optimal solutions.

Theorem 4.2. *There exists an optimal solution to the ELS-PI with the LEFO consumption order constraints where, items procured from different suppliers are never stored in the inventories together.*

Proof. Consider two suppliers $(t_1, k_1) < (t_2, k_2)$.

(i) Assume $v_{t_1k_1} = v_{t_2k_2}$. Due to Theorem 4.1, either $x_{t_1k_1\tau=0}$ or $x_{t_2k_2\tau} = 0$ for all $\tau \in F(t_1, k_1) \cap F(t_2, k_2) = \{t_2, \ldots, v_{t_1k_1} = v_{t_2k_2}\}$. In the first case, items procured from (t_1, k_1) are not carried over to period t_2 and hence they are not in the inventories with the items procured from (t_2, k_2). In the latter case, $x_{t_2k_2} = \sum_{\tau=t_2}^{v_{t_2k_2}} x_{t_2k_2\tau} = 0$, which implies that there is no procurement from the supplier (t_2, k_2). In neither of the cases items procured from two different suppliers are displayed together in the inventories.

(ii) Assume $v_{t_1k_1} > v_{t_2k_2}$. Due to Theorem 4.1, either $x_{t_1k_1\tau} = 0$ or $x_{t_2k_2\tau} = 0$ for all $\tau \in F(t_1,k_1) \cap F(t_2,k_2) = \{t_2, \ldots, \min\{v_{t_1k_1}, v_{t_2k_2}\} = v_{t_2k_2}\}$. If $x_{t_2k_2\tau} = 0$ for all $\tau \in F(t_1,k_1) \cap F(t_2,k_2)$, then $x_{t_2k_2} = \sum_{\tau=t_2}^{v_{t_2k_2}} x_{t_2k_2\tau} = 0$ which implies that there is no procurement from the supplier (t_2,k_2). Therefore consider the case where $x_{t_1k_1\tau} = 0$ for all $\tau \in F(t_1,k_1) \cap F(t_2,k_2)$. Due to LEFO consumption order constraints, we can not have $x_{t_2k_2\tau} > 0$ for any $\tau \in F(t_1,k_1) \cap F(t_2,k_2)$ while $x_{t_1k_1\tau} > 0$ for some $\tau > v_{t_2k_2}$. Therefore, no item procured from the supplier (t_1,k_1) is carried in the inventories along with the items procured from the supplier (t_2,k_2).

(iii) Assume $v_{t_1k_1} < v_{t_2k_2}$. In that case, if $x_{t_2k_2} > 0$, then customers will prefer the items procured from (t_2,k_2) instead of the items procured from (t_1,k_1) in periods $F(t_1,k_1) \cap F(t_2,k_2)$. Together with Theorem 4.1, this implies that if $x_{t_2k_2} > 0$, $x_{t_1k_1\tau} = 0$ for all $\tau \in F(t_1,k_1) \cap F(t_2,k_2) = \{t_2, \ldots, \min\{v_{t_1k_1}, v_{t_2k_2}\} = v_{t_1k_1}\}$, which further implies that no items exist in the inventories that were procured from the supplier (t_1,k_1) when the items from the supplier (t_2,k_2) are procured and placed in the inventories. Therefore we either have $x_{t_2k_2} = 0$ or we do not carry the items procured from the supplier (t_1,k_1) in the inventories to period t_2. Both cases imply that items procured from (t_1,k_1) are not displayed together with the items procured from the supplier (t_2,k_2).

□

Corollary 4.3. *There exists an optimal solution to the ELS-PI with the LEFO consumption order constraints such that items are procured from at most one supplier in each period. Moreover, this solution obeys the ZIO property.*

Structural properties of optimal solutions under FIFO and LIFO consumption order constraints are readily achievable using Theorem 4.1 and through observation of the inventory system that leads to those consumption orders. Consider first the ELS-PI with the LIFO consumption order constraints. LIFO consumption order is realized if the inventory system is designed as a stack, where the items are always inserted to and consumed from the front of the stack. Recall that, if the items are procured from several suppliers in a single period, items procured from the supplier with lower index number are placed in the storage earlier than the ones procured from the supplier with higher index number. Then, using Theorem 4.1, it is easy to prove the following corollary.

Corollary 4.4. *Suppose* $(t_1, k_1) < (t_2, k_2)$ *are two suppliers with positive procurement in an optimal solution to the ELS-PI with LIFO consumption order constraints. Then,* $x_{t_1k_1\tau} = 0$ *for all* $\tau \in F(t_1, k_1) \cap F(t_2, k_2)$.

FIFO consumption order is realized if the inventory system is a queue where the items are placed always to the back of the queue and consumed from the front of the queue. Optimal solutions might not posses the ZIO property but since the items are always ranked in the queue such that early procured items are always at the front of the queue, there exists an optimal solution where a supplier always satisfies a group of consecutive demand periods.

Theorem 4.3. *Suppose* $(1, 1) \leq (t_1, k_1) < \ldots < (t_N, k_N) \leq (T, K_T)$ *are N suppliers with positive procurement in an optimal solution to the ELS-PI with FIFO consumption order constraints. Then, there are N+1 indices* $t_1 = j_1 < \ldots < j_{N+1} = T$ *such that (i)* $j_i \in \{t_i, v_{t_{i-1},k_{i-1}} + 1, T\}$ *and that (ii) demands in periods* j_i *through* j_{i+1} *are solely satisfied by procurement from the supplier* (t_i, k_i).

Proof. The claim in (ii) is due to the nature of the consumption order and the Corollary 4.1. To prove that $j_i \in \{t_i, v_{t_{i-1},k_{i-1}} + 1, T\}$, consider two consecutive suppliers with positive procurement: (t_{i-1}, k_{i-1}) and (t_i, k_i).

If $t_{i-1} < t_i$ and there are no inventories carried over to period t_i, then it is trivial that $j_i = t_i$. Otherwise, i.e., if there are items carried over to period t_i that were procured in period t_{i-1}, due to Theorem 4.1, procurement from the supplier (t_i, k_i) does not satisfy the demands in periods $F(t_{i-1}, k_{i-1}) \cap F(t_i, k_i) = \{t_i, \ldots, \min\{v_{t_{i-1}k_{i-1}}, v_{t_ik_i}\} = v_{t_{i-1}k_{i-1}}\}$, which implies that $j_i = v_{t_{i-1}k_{i-1}} + 1$.

If $t_{i-1} = t_i$, then since items procured from the supplier (t_{i-1}, k_{i-1}) are placed in the queue first, they are consumed first. Due to Theorem 4.1, this implies that $x_{t_ik_i} = 0$ for $\tau \in F(t_{i-1}, k_{i-1}) \cap F(t_i, k_i) = \{t_i, \ldots, \min\{v_{t_{i-1}k_{i-1}}, v_{t_ik_i}\} = v_{t_{i-1}k_{i-1}}\}$, i.e., procurement from the supplier (t_i, k_i) does not satisfy the demands in periods t_i through $v_{t_{i-1}k_{i-1}}$ and hence $j_i = v_{t_{i-1}k_{i-1}} + 1$. Since no inventory is carried after the planning horizon, $j_{N+1} = T$. □

Since the consumption order constraints affect the nature of optimal solutions, they ultimately affect the optimal objective values. Let ϕ_i be the objective function value of the $ELS-PI$ under item consumption order i, $i \in \{LEFO, FEFO, LIFO, FIFO\}$ and let ϕ_{ZIO} be the objective value when the solution is restricted to obey the ZIO property, i.e., when items procured from two different suppliers are never stored together in the inventories. Moreover let ϕ_0 be the objective function value when no item consumption order constraints are present, i.e., when the store manager has full control on the distribution of items to the demands.

Theorem 4.4. *The following relations exist between ϕ_{ZIO}, ϕ_0 and ϕ_i, $i \in \{LEFO, FEFO, LIFO, FIFO\}$.*

i $\phi_{ZIO} = \phi_{LEFO} \geq \phi_{FIFO} \geq \phi_{FEFO} = \phi_0$

ii $\phi_{ZIO} = \phi_{LEFO} \geq \phi_{LIFO} \geq \phi_{FEFO} = \phi_0$

Proof. Due to Corollary 4.3, an optimal solution that obeys the ZIO property exists when the items are consumed in LEFO manner. Therefore $\phi_{ZIO} = \phi_{LEFO}$. Any procurement plan that obeys the ZIO property also obeys any of the four consumption orders by nature since no items from different supplier are displayed together in a solution that obeys the ZIO property. Therefore $\phi_{ZIO} \geq \phi_{LIFO}$ and $\phi_{ZIO} \geq \phi_{FIFO}$. Due to Corollary 4.2 there exists an optimal solution to the ELS-PI with no consumption order constraints which obeys the FEFO consumption order. Therefore $\phi_{LIFO} \geq \phi_{FEFO}$ and $\phi_{FIFO} \geq \phi_{FEFO}$.

The above argument does not leave out the possibility that the objective values will be equal under all consumption order constraints. However, Example 4.1 can be used to show that strict inequalities can be realized under different consumption orders. For the problem in Example 4.1, it is easy to see that $\phi_{ZIO} = \phi_{LEFO} = 500$, $\phi_{LIFO} = 450$, $\phi_{FIFO} = 400$ and $\phi_{FEFO} = 350$. Observe that the objective function value with the $FEFO$ consumption order constraints is equal to the objective function value without any consumption order constraints. That is $\phi_{FEFO} = \phi_0$ as stated in the theorem. □

4.3 Polynomial Time Algorithms

4.3.1 LEFO and FIFO Consumption Orders

Due to Corollary 4.3, an optimal solution to the ELS-PI with LEFO consumption order constraints can be decomposed into the so called *regeneration intervals*. A regeneration interval (τ_1, τ_2) is a set of consecutive periods $\tau - 1, \tau_1 + 1, \ldots, \tau_2$ such that $I_{\tau_1 - 1} = I_{\tau_2} = 0$ and $I_\tau > 0$ for $\tau = \tau_1, ..., \tau_2 - 1$ and a single supplier in period period τ_1 satisfies the demand of periods τ_1 through τ_2. Letting $K = \max_t \{K_t\}$, an $O(KT^2)$ algorithm to the problem is then straightforward to achieve even under general concave cost functions where we search for the optimal supplier for each possible regeneration interval.

If the consumption order is FIFO, ZIO property might be violated but optimal solutions have the property that a supplier always satisfies a group of consecutive demand periods. To solve the ELS-PI with FIFO consumption order constraints, we define a network where the shortest distance between the source node and one of the sink nodes will be equivalent to the cost of the optimal procurement plan over the planning horizon. In this network, every node corresponds to a triple (X, t, k), where $t \in \{0, \ldots, T\}$, $k \in \{1, \ldots, K\}$ and X is the cumulative procurement up to and including the supplier (t, k). Due to Theorem 4.3, $X \in \{D_{11}, \ldots, D_{1T}\}$, where $D_{uv} = \sum_{i=u}^{v} D_i$. The source node is $(0, 0, 0)$ and the sink nodes are of the form (D_{1T}, T, k), where $k = 1, \ldots, K_T$. Observe that there are $O(KT^2)$ nodes in this network.

Arcs in the network represent the inventory holding and procurement decisions. They exist between the nodes of the form (X_1, t, k_1) and $(X_2, t + 1, k_2)$ and between the nodes of the form (X_1, t, k_1) and (X_2, t, k_2) where $k_2 > k_1$. If there is an arc between the nodes (X_1, t, k_1) and $(X_2, t + 1, k_2)$, this implies that the amount of items procured from the supplier $(t + 1, k_2)$ is $X_2 - X_1$ and the amount of items carried in the inventories between period t and $t + 1$ is $X_1 - D_{1,t}$. The cost of this arc is given by

$$P_{t+1,k_2}(X_2 - X_1) + H_t(X_1 - D_{1t}).$$

If there is an arc between the nodes (X_1, t, k_1) and (X_2, t, k_2), this implies that the amount of items procured from the supplier (t, k_2) is $X_2 - X_1$. This arc indicates that besides the supplier (t, k_1), the items are procured from the supplier (t, k_2) in period t. The cost of this arc is therefore given by

$$P_{tk_2}\left(X_2 - X_1\right).$$

An arc between the nodes (X_1, t_1, k_1) and (X_2, t_2, k_2) is possible if $X_1 \leq X_2 \leq D_{1,v_{t_2 k_2}}$ so that procurement is not negative and items procured from supplier (t_2, k_2) do not satisfy the demands occurring after their expiration dates. There is no arc from the node (X_1, t_1, k_1) to (X_2, t_2, k_2) if $X_1 \geq D_{1,v_{t_2,k_2}}$ and $X_2 > X_1$. Otherwise, items procured from the supplier (t_2, k_2) deteriorate before being consumed. If $X_2 = X_1$, we have that $t_2 = t_1 + 1$ and $k_2 = 1$ for convenience. Also, an arc between the nodes (X_1, t_1, k_1) and (X_2, t_2, k_2) is possible if $X_1 \geq D_{1t}$ so that inventory carried in period t is not negative.

Observe that there are $O(KT)$ arcs emanating from each node. Therefore, the network has $O(K^2T^3)$ arcs in total and the shortest path between the source and the sink nodes can be found in $O(K^2T^3)$ time. This further implies that the ELS-PI with FIFO consumption order constraints can be solved in $O(K^2T^3)$ time under general concave cost functions.

Reduced complexities can be achieved if we assume that holding cost functions are linear, i.e., $H_t(x) = h_t x$, and that procurement cost functions are concave with a fixed charge structure such that

$$P_{tk}(x) = \begin{cases} S_{tk} + p_{tk}x & x > 0 \\ 0 & x = 0 \end{cases} \tag{4–5}$$

Observe that, if the procurement and holding costs are defined as above, we can substitute $I_t = \sum_{i=1}^{t} \sum_{k=1}^{K_t} x_{ik} - \sum_{i=1}^{t} D_i$ in the objective function of the ELS-PI and drop the inventory decision variables. We can then set procurement cost functions to

$$P'_{tk}(x) = \begin{cases} S_{tk} + c_{tk}x & x > 0 \\ 0 & x = 0 \end{cases} \quad (4\text{–}6)$$

where $c_{tk} = p_{tk} + \sum_{j=t}^{T} h_j$. In this manner, we can generate an equivalent problem where inventory holding costs are zero.

Then, a slight modification of the algorithm proposed in Hsu (25) can be applied to solve the problem. Following his procedure, we define $\phi(t, k; \tau)$ to be the optimal cost of the ELS-PI where demands to be satisfied are restricted to be from period 1 through period τ ($1 \leq \tau \leq T$) and the last supplier with positive procurement is (t, k), which completely satisfies the demand of period τ. We set $\phi(1, k; 1) = S_{1k} + c_{1k}D_1$ for $k = 1, \ldots, K_1$ and define

$$\phi(t, k; t) = \min_{(i,j): i<t, (i,j) \in F^{-1}(t-1)} \{\phi(i, j; t-1) + S_{tk} + c_{tk}D_t\} \;\; t = 1, \ldots, T;\; k = 1, \ldots, K_t. \quad (4\text{–}7)$$

To compute $\phi(t, k; \tau)$ for $\tau = 2, \ldots, T$, $t = 1, \ldots, \tau - 1$, $(t, k) \in F^{-1}(\tau)$, we solve the following set of dynamic recursions

$$\phi(t, k; \tau) = \min \begin{cases} \phi(t, k; \tau - 1) + c_{tk}D_\tau \\ \min\limits_{(i,j) \in \kappa(t,k,\tau)} \phi(i, j; \tau - 1) + S_{tk} + c_{tk}D_\tau \end{cases} \quad (4\text{–}8)$$

where $\kappa(t, k, \tau) = \{(i, j) | (i, j) < (t, k) \text{ and } v_{ij} = \tau - 1\}$. Observe that in (4–8), we can restrict our search to $(i, j) \in \kappa(t, k; \tau)$ due to Theorem 4.3. The order of processes is as follows. We first calculate $\phi(1, k; 1)$ for $k = 1, \ldots, K_1$. Then, using recursion (4–8) we compute $\phi(1, k; \tau)$ for $k = 1, \ldots, K_1$ and $\tau = 2, \ldots, v_{1k}$. Then we compute $\phi(2, k; 2)$ for $k = 1, \ldots, K_2$ using (4–7) and then using recursion (4–8) we compute $\phi(2, k; \tau)$ for $k = 1, \ldots, K_2$ and $\tau = 3, \ldots, v_{2k}$. The algorithm continues in this manner until all possible

$\phi(t,k;\tau)$ values are calculated. Clearly, the optimal objective function value is given by $\min\limits_{(t,k)\in F^{-1}(T)} \phi(t,k;T)$.

It takes $O(1)$ time to compute $\phi(t,k;t)$ for a fixed (t,k) pair using recursion (4–7). On the other hand, for any fixed (t,k,τ) triple with $\tau > t$, it takes $O(|\kappa(t,k,\tau)|)$ time to compute $\phi(t,k;\tau)$ given that the value of $\phi(t,k;\tau-1)$ is known. Then, the complexity to compute all $\phi(t,k;\tau)$ values, and also the complexity of the overall algorithm, is given by

$$O\left(KT + \sum_{(t,k)\leq(T,K_T)} \sum_{\tau=t+1}^{T} |\kappa(t,k,\tau)|\right).$$

Items from the supplier (t,k) have a unique expiration date v_{tk}. Therefore, for a fixed (t,k) pair, any particular supplier (i,j) is investigated only once while computing $\phi(t,k;\tau)$ for $\tau = t+1,\ldots,v_{tk}$. As a result, for a fixed (t,k) pair, we have

$$\sum_{\tau=t+1}^{T} |\kappa(t,k,\tau)| \leq KT.$$

Therefore, the overall complexity of the algorithm is $O(KT + K^2T^2)$.

4.3.2 FEFO and LIFO Consumption Orders

We present our algorithm to solve the ELS-PI with FEFO consumption order constraints assuming linear inventory holding costs and fixed charge production cost functions defined as in (4–5). Figure 4-2 shows a network representation of a possible optimal solution to the ELS-PI with FEFO consumption order constraints over a planing horizon of 7 periods. In the figure, it is assumed that there is a single supplier available in each period. Demand nodes with the same color are satisfied by the same supplier. Items procured in period 1 satisfy the demands of periods 1,2 and 7; items procured in period 3 satisfy the demands of periods 3 and 4 whereas items procured in period 3 satisfy the demands of periods 5 and 6.

We define a *block* $(\tau_1,\tau_2;t,k)$ to be a set of consecutive demand periods $\tau_1,\ldots,\tau_2$ such that a setup has already been carried out to procure items from the supplier (t,k) and that procurement from that supplier satisfies the demand in period t_2 and can satisfy

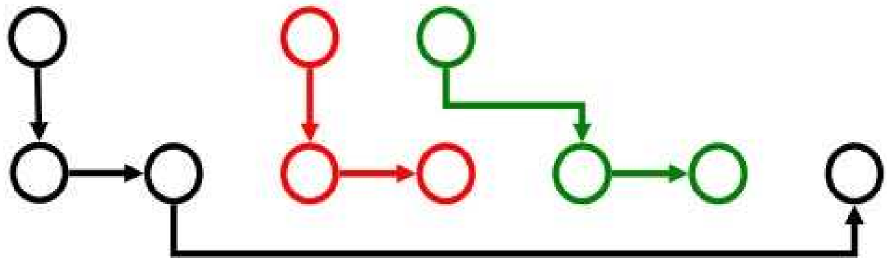

Figure 4-2: An optimal solution to the problem with FEFO consumption order constraints

the demands of periods τ_1 through τ_2. Clearly, for the block $(\tau_1, \tau_2; t, k)$ to be feasible, we should have that $t \leq \tau_1 \leq \tau_2 \leq v_{tk}$. Note that this definition allows for satisfying the demands of some of the periods between τ_1 and τ_2 by procuring from suppliers other than (t, k) where new setups have to be carried out, as in the example drawn in Figure 4-2.

We define $\pi(\tau_1, \tau_2; t, k)$ to be the minimum cost to satisfy the demands of periods in the block $(\tau_1, \tau_2; t, k)$ excluding the setup cost of the supplier (t, k). We assume that $\pi(\tau_1, \tau_2; t, k) = \infty$ if that block is infeasible. To satisfy the demands between periods τ' and τ'', where $\tau_1 \leq \tau' \leq \tau'' < \tau_2$, the store manager either (i) orders items from the supplier (t, k) or (ii) orders items from another supplier. The latter option incurs additional setup costs. Since, in an optimal solution, the block $(\tau_1, \tau_2; t, k)$ can be in another block and a block may contain several other blocks within itself (as in Figure 4-2), we carry out the following approach. We first compute the costs of the smallest blocks that contain a single demand period that are satisfied by a single supplier, i.e., we compute $\pi(\tau, \tau; t, k)$ for $\tau = 1, \ldots, T$ and $(t, k) \in F^{-1}(\tau)$ such that

$$\pi(\tau, \tau; t, k) = c_{tk} D_\tau.$$

We then compute the costs of the blocks with two demand periods, i.e., $\pi(\tau, \tau + 1; t, k)$ for $\tau = 1, \ldots, T - 1$ and $(t, k) \in F^{-1}(\tau) \cap F^{-1}(\tau + 1)$. We continue this procedure by calculating $\pi(\tau, \tau+j; t, k)$ $(\tau = 1, \ldots, T-j)$ where we set j to 2,3 and so on. We use the following recursions to calculate the cost of the block $(\tau_1, \tau_2; t, k)$

$$\pi(\tau_1, \tau_2; t, k) = \min \begin{cases} \min\limits_{0 \le \tau' < \tau_2} \{\pi(\tau_1, \tau') + \pi(\tau'+1, \tau_2; t, k)\} \\ \min\limits_{0 \le \tau' < \tau_2} \left\{ c_{tk} \sum\limits_{i=\tau_1}^{\tau'} D_i + \pi(\tau'+1, \tau_2; t, k) \right\} \end{cases}$$

where,

$$\pi(\tau_1, \tau_2) = \min_{(t,k) \in F^{-1}(\tau_1) \cap F^{-1}(\tau_2)} \{S_{tk} + \pi(\tau_1, \tau_2; t, k)\}.$$

Observe that while computing the block cost $\pi(\tau_1, \tau_2; t, k)$, we carry out a first step analysis where we investigate whether or not the demands in periods τ_1 through τ', for some $\tau' < \tau_2$, should be satisfied through a procurement from a supplier other than (t, k). The cost of the block is then computed accordingly. Due to the order of processes, when calculating $\pi(\tau_1, \tau_2)$ we know the values of $\pi(\tau_1, \tau')$ and $\pi(\tau'+1, \tau_2; t, k)$ for $\tau' < \tau_2$. Given those values, it takes $O(T)$ time to compute the cost of a single block $\pi(\tau_1, \tau_2; t, k)$. We then need to search over feasible (t, k) pairs to compute $\pi(\tau_1, \tau_2)$, which takes $O(KT)$ time. This implies that the complexity to calculate $\pi(\tau_1, \tau_2)$ for a single (τ_1, τ_2) pair is $O(KT^2)$. The values of $\pi(\tau_1, \tau_2)$ for all (τ_1, τ_2) pairs can therefore be found in $O(KT^4)$ time. Finally, we solve the following recursion to find the production plan over the planning horizon:

$$\phi(\tau_1) = \max_{\tau_2 : \tau_2 \ge \tau_1} \{\pi(\tau_1, \tau_2) + \phi(\tau_2 + 1)\} \qquad \tau_1 = 1, \ldots, T \tag{4–9}$$

$$\phi(T+1) = 0$$

where, $\phi(\tau)$ is the minimum cost of satisfying the demands of periods τ through T. Clearly, $\phi(1)$ is the optimal objective function value of the problem. Recursion (4–9) is solved in $O(T^2)$ time, which implies that the total complexity of this algorithm is $O(KT^4)$.

This algorithm can also be used to solve the ELS-PI with LIFO consumption order constraints. When the items are consumed in LIFO manner, if there is positive procurement in a period t, then the demand in period t is satisfied through one of the suppliers in period t. Therefore, $\pi(\tau_1, \tau_2) = \min_{1 \le k \le K_{\tau_1}} \pi(\tau_1, \tau_2; \tau_1, k)$ and hence no search is necessary to find the optimal period of procurement. This reduces the complexity of the algorithm to $O(KT^3)$.

4.4 The ELS-PI with Procurement Capacities

In this section, we extend the ELS-PI by introducing procurement capacities. Specifically, we assume that there is a finite procurement capacity C_{tk} for $t = 1, \ldots, T$ and $k = 1, \ldots, K_t$ such that amount of items procured from the supplier (t, k) can not exceed C_{tk}. It is well known that the economic lot sizing problem with procurement capacities is NP-hard even under various special cost structures (see Florian et al. (11) and Bitran and Yanasse (3)). ELS-PI generalizes the economic lot sizing problem by incorporating item deterioration and multiple suppliers available in each period to procure the items. Then, it immediately follows that the ELS-PI with procurement capacities is also NP-hard under all those special cases. On the other hand, the economic lot sizing problem with constant procurement capacities can be solved in polynomial time under general concave cost functions (see Florian and Klein (10) and van Hoesel and Wagelmans (22)). We, therefore, try to identify special cases of the ELS-PI that can be solved in polynomial time under constant procurement capacities where we assume that $C_{tk} = C$ for each supplier (t, k). We first prove that although the economic lot sizing problem with constant procurement capacities is polynomially solvable, the ELS-PI with constant procurement capacities is NP-hard even if there is a single supplier available in each period.

Theorem 4.5. *The ELS-PI with constant procurement capacities is NP-hard.*

Proof. Consider a special case of the ELS-PI where there is exactly one supplier available in each period. Procurement capacity for the unique supplier available in each period t are such that $C_t = C$ for $t = 1, \ldots, T$, where C is finite. The expiration dates of the items procured in period t are such that $v_t = t$ for $t = 2, \ldots, T$ and $v_1 = T$. In other words, the special case of the ELS-PI has the following properties:

(i) There is exactly one supplier available in each period

(ii) Items procured in period t deteriorate after period t for $t = 2, \ldots, T$ while the items procured in period 1 can last until the end of period T

Let's call this problem the ELS-PI(C1). Assume that $H_t(I_t) = 0$ for $t = 1, \ldots, T$ and that the procurement cost function of the unique supplier in period t is given by

$$P_t(x) = \begin{cases} S_t & x > 0 \\ 0 & x = 0 \end{cases} \qquad t = 1, \ldots, T$$

Let $Q = \{t | 2 \leq t \leq T-1, \quad D_t > C_t\}$, be the set of periods 2 through $T-1$ where demand exceeds capacity. Note that a total demand of $D_t - C_t$ in all periods $t \in Q$ should be satisfied through items procured in period 1 whether or not there is a setup to procure items in period t. Therefore we can generate an equivalent problem where there is a new set of demand vector D' such that $D'_t = \min\{D_t, C_t\}$ for $t = 2, \ldots, T$ and $D'_1 = D_1 + \sum_{t \in Q}\{D_t - C_t\}$. In this problem $D'_t \leq C_t$ for $t = 2, \ldots, T$ and hence without loss of generality we can assume that there are no procurement capacities in periods 2 through T. In other words, for any ELS-PI(C1) instance with constant procurement capacities in each period, there is an equivalent ELS-PI(C1) instance where procurement is capacitated only in the first period. Then it is sufficient to prove NP-hardness of the ELS-PI(C1) assuming that $C_1 = C$ and $C_t = \infty$ for $t = 2, \ldots, T$.

The decision problem for the KNAPSACK is as follows. Given a finite set $\{1, \ldots, N\}$, a size $s(t) \in \mathbf{Z}^+$ and a value $v(t) \in \mathbf{Z}^+$ for each $t \in \{1, \ldots, N\}$ and positive integers B and K, is there is a subset $T' \subseteq \{1, \ldots, N\}$ such that $\sum_{t \in T'} s(t) \geq B$ and such that $\sum_{t \in T'} v(t) \leq K$.

In polynomial time, we can transform the KNAPSACK into an ELS-PI(C1) instance with procurement capacities only in the first period. Letting $T = N + 1$, we set $S_t = v(t-1)$, $D_t = s(t-1)$ for $t = 2, \ldots, T$ and set $S_1 = 0$ and $D_1 = 0$. We then set the procurement capacity in period 1 to $C = \sum_{t=1}^{N} s(t) - B$.

Since procurement cost is zero in period 1, it is least costly to procure in that period to satisfy the demands over the whole planning horizon. However, procurement is restricted such that items procured in period 1 can satisfy only a total demand of $C_1 = C$. Therefore, the remaining demands have to be satisfied by procurement in periods 2 through T, which incurs additional setup costs. The setup periods 2 through T can only satisfy the demands in those periods since the items procured in those periods can not be carried in inventories before they deteriorate. Then, there is a solution with costs less than K if and only if there exists a set $T' \subseteq \{2, \ldots, T\}$ of procurement periods such that

$$\sum_{t \in T'} D_t \geq \sum_{t \in T} D_t - C = B$$

and

$$\sum_{t \in T'} S_t \leq K.$$

In other words, ELS-PI(C1) is equivalent to KNAPSACK and hence the ELS-PI(C1) is NP-hard. Since ELS-PI(C1) is a special case of the ELS-PI, we conclude that the ELS-PI with constant procurement capacities is NP-hard. □

Since the special case of the ELS-PI constructed in the proof of Theorem 4.5 is also a special case of the ELS-PI with LIFO and FEFO constraints, we can immediately conclude that the ELS-PI with constant capacities remain NP-complete with those consumption order constraints. This holds even when the holding costs are zero and production cost functions have a fixed charge structure where the variable part is zero.

Corollary 4.5. *The ELS-PI with constant capacities and LIFO consumption order constraints is NP-hard.*

Corollary 4.6. *The ELS-PI with constant capacities and FEFO consumption order constraints is NP-hard.*

When there are FIFO consumption order constraints, the fact that the items procured earlier are consumed earlier leads to optimal solutions that can be decomposed into consecutive sub–plans, which are denoted by $(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$. This notation implies that items to satisfy the demands of periods τ_1 through τ_2 can be procured only from the suppliers (i, j) such that $(t_1, k_1) \leq (i, j) \leq (t_2, k_2)$ and the last supplier that satisfies the demand of period τ_2 is (t_2, k_2). Obviously, we should have that $1 \leq t_1 \leq \tau_1 \leq t_2 \leq \tau_2$. The sub–plan $(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$ has the property that

$$\sum_{(t_1,k_1)\leq(i,j)\leq(t_2,k_2)} x_{ij} = \sum_{i=\tau_1}^{\tau_2} D_i$$

and for any $\tau \geq \tau_1$,

$$\sum_{(t_1,k_1)\leq(i,j)\leq(t,k)} x_{ij} \neq \sum_{i=\tau_1}^{\tau} D_i \text{ for } (t, k, \tau) \neq (t_2, k_2, \tau_2)$$

Letting $(t_1, k_1) \leq (i_1, j_1) < \ldots < (i_n, j_n) = (t_2, k_2)$ be n suppliers with positive procurement in the sub–plan $(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$, the above property implies that for every pair of consecutive suppliers $(i_k, j_k) < (i_{k+1}, j_{k+1})$ with positive procurement, there corresponds a period s_k $(k = 1, \ldots, n-1)$ where the following holds

$$\sum_{j=\tau_1}^{s_k-1} D_j < \sum_{(i_1,j_1)\leq(i,j)\leq(i_k,j_k)} x_{ij} < \sum_{j=\tau_1}^{s_k} D_j \text{ for } k = 1, \ldots, n-1.$$

In other words, every two consecutive suppliers with positive procurement partially satisfy exactly one demand period in the sub–plan. Note that this definition does not exclude the possibility that $s_k = s_{k+1}$ for some $k = 1, \ldots, n-1$.

Following the notation in the literature, we call supplier (t, k) fractional if procurement from that supplier is strictly between 0 and C_{tk}, i.e., $0 < x_{tk} < C_{tk}$. Then, it can be shown that, in an optimal solution, in the sub–plan $(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$, there is at most one fractional supplier. Procurement is either zero or at capacity from all other suppliers that can satisfy the demand in the sub–plan. This is a generalization of the so called *fractional procurement property* stated for the economic lot sizing problem with procurement capacities.

Theorem 4.6. *There exists an optimal solution to the ELS-PI with constant capacities that can be decomposed into consecutive sub–plans such that there is at most one fractional period in each sub–plan.*

Proof. Consider a sub–plan $(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$ in an optimal solution $X = (x_{tk}, I_t)$ and assume that there are n suppliers $(t_1, k_1) \leq (i_1, j_1) < \ldots < (i_n, j_n) = (t_2, k_2)$ that satisfy the demand of the sub–plan.

Now, assume on the contrary that there are more than one fractional periods in the sub–plan. In particular, assume that (i_ℓ, j_ℓ) and (i_m, j_m) $(1 \leq \ell < m \leq n)$ are two such fractional periods. Then set

$$\epsilon_1 = \min\left\{ \min_{k=\ell,\ldots,m-1} \left\{ \sum_{j=\tau_1}^{s_k} D_j - \sum_{(i_1,j_1)\leq(i,j)\leq(i_k,j_k)} x_{ij} \right\}, C_{i_\ell j_\ell} - x_{i_\ell j_\ell}, x_{i_m, j_m} \right\}$$

and

$$\epsilon_2 = \min\left\{ \min_{k=\ell,\ldots,m-1} \left\{ \sum_{(i_1,j_1)\leq(i,j)\leq(i_k,j_k)} x_{ij} - \sum_{j=\tau_1}^{s_k-1} D_j \right\}, C_{i_m j_m} - x_{i_m j_m}, x_{i_\ell j_\ell} \right\}.$$

Consider two other solutions obtained as follows.

(i) Increase procurement from the supplier (i_ℓ, j_ℓ) by ϵ_1 and decrease procurement from the supplier (i_m, j_m) by the same amount to obtain $X' = (x'_t, I'_t)$ where $x'_{i_\ell j_\ell} = x_{i_\ell j_\ell} + \epsilon_1$ and $x'_{i_m j_m} = x_{i_m j_m} - \epsilon_1$ while every variable remains the same as in X. This change may result in three possible consequences: (i) procurement from the supplier (i_ℓ, j_ℓ) raises to capacity; (ii) procurement from the supplier (i_m, j_m) decreases to zero; and/ or (iii) we will have $\sum_{(i_\ell,j_\ell)\leq(i,j)\leq(i_\nu,j_\nu)} x_{ij} = \sum_{j=\tau_1}^{s_\nu} D_j$ for some ν that satisfies $\ell \leq \nu < m$. If (i) or (ii) happens, in the new solution we will have one less fractional supplier in the sub–plan. If (iii) happens, the fractional suppliers are separated to two different sub–plans: $(\tau_1, s_\nu; t_1, k_1; i_\nu, j_\nu)$ and $(s_\nu + 1, \tau_2; i_{\nu+1}, j_{\nu+1}; t_2, k_2)$.

(ii) Decrease procurement from the supplier (i_ℓ, j_ℓ) by ϵ_2 and increase procurement from the supplier (i_m, j_m) by the same amount to obtain $X'' = (x''_t, I''_t)$ where $x''_{i_\ell j_\ell} = x_{i_\ell j_\ell} - \epsilon_2$ and $x''_{i_m j_m} = x_{i_m j_m} + \epsilon_2$ while every variable remains the same as in X. This change may result in three possible consequences: (i) procurement from the supplier (i_m, j_m) raises to capacity; (ii) procurement in period (i_ℓ, j_ℓ) decreases to zero; and/ or (iii) we will have $\sum_{(i_\ell,j_\ell)\leq(i,j)\leq(i_\nu,j_\nu)} x_{ij} = \sum_{j=\tau_1}^{s_\nu-1} D_j$ for some ν that satisfies $\ell \leq \nu < m$. If (i) or (ii) happens, in the new solution we will have one less fractional supplier in the sub–plan. If (iii) happens, the fractional suppliers are separated to two different sub–plans: $(\tau_1, s_\nu - 1; t_1, k_1; i_{\nu-1}, j_{\nu-1})$ and $(s_\nu, \tau_2; i_\nu, j_\nu; t_2, k_2)$.

Note that $X = \lambda X' + (1-\lambda)X''$ for some $\lambda \in (0,1)$. Since the costs are concave, we have that X' and X'' are both optimal as well. If we continue in this manner, we get an optimal solution where sub–plans have exactly one fractional supplier. □

Given the property in Theorem 4.6, we propose the following approach to solve the ELS-PI with constant procurement capacities assuming general concave procurement costs and linear inventory holding costs such that $H_t(x) = h_t x$ for $t = 1, \ldots, T$. We define $\pi(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$ to be the minimum cost of the sub–plan $(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$. Then, we define a network where the shortest distance between the source node and the sink node will be equivalent to the minimum cost for the sub–plan $\pi(\tau_1, \tau_2; t_1, k_1; t_2, k_2)$. In this network, every node corresponds to a triple (X, t, k), where $t \in \{t_1, \ldots, \tau_2\}$, $k \in \{1, \ldots, K\}$ and X is the cumulative procurement in the sub–plan from the suppliers (i,j) that satisfy $(t_1, k_1) \leq (i,j) \leq (t,k)$. We define $D_{uv} = \sum_{i=u}^{v} D_i$ and assume $D_{uv} = 0$ for $v < u$. Due to Theorem 4.6, we have that

$$X \in \bigcup_{j=0}^{J} \{jC, jC + \epsilon\},$$

where $J = \lfloor D_{\tau_1\tau_2}/C \rfloor$ and $\epsilon = D_{\tau_1\tau_2} - JC$. The source node is $(0, t_1 - 1, 0)$ and the sink node is $(D_{\tau_1,\tau_2}, t_2, k_2)$. Observe that there are $O(KT^2)$ nodes in this network.

Arcs in the network represent the inventory holding and procurement decisions. They exist between the nodes of the form (X_1, t, k_1) and $(X_2, t+1, k_2)$, and between the nodes of the form (X_1, t, k_1) and (X_2, t, k_2) where $k_2 > k_1$. If there is an arc between the nodes (X_1, t, k_1) and $(X_2, t+1, k_2)$, this implies that the amount of items procured from the supplier $(t+1, k_2)$ is $X_2 - X_1$ and the amount of items carried in the inventories between period t and $t+1$ is $X_1 - D_{1,t}$. The cost of this arc is given by

$$P_{t+1,k_2}(X_2 - X_1) + h_t(X_1 - D_{\tau_1,t}).$$

If there is an arc between the nodes (X_1, t, k_1) and (X_2, t, k_2), this implies that the amount of items procured from the supplier (t, k_2) is $X_2 - X_1$. This arc indicates that besides the supplier (t, k_1), the items are procured from the supplier (t, k_2) in period t. The cost of this arc is therefore given by

$$P_{tk_2}(X_2 - X_1).$$

An arc between the nodes (X_1, t_1, k_1) and (X_2, t_2, k_2) is possible if $X_2 \leq D_{\tau_1, v_{t_2 k_2}}$ so that items procured from supplier (t_2, k_2) do not satisfy the demands occurring after their expiration dates. There is no arc from the node (X_1, t_1, k_1) to (X_2, t_2, k_2) if $X_1 \geq D_{1, v_{t_2,k_2}}$ and $X_2 > X_1$. Otherwise items procured from the supplier (t_2, k_2) deteriorate before being consumed. If $X_2 = X_1$, we have that $t_2 = t_1 + 1$ and $k_2 = 1$ for convenience. An arc between the nodes (X_1, t, k_1) and $(X_2, t+1, k_2)$ is possible if $X_1 \geq D_{\tau_1,t}$. That is, inventory carried in a period can not be negative. In all cases, we should have that $X_2 - X_1 \in \{0, \epsilon, C\}$ such that $X_2 - X_1 = \epsilon$ only if $X_1 = jC$ for some $j = 1, \ldots, J$. This guarantees that there is only one fractional procurement in the sub–plan.

Note that $O(K)$ arcs emanate from each node, which implies that there are a total of $O(K^2T^2)$ arcs in the network. This further implies that the minimum cost of the sub–plan can be found in $O(K^2T^2)$ time. Given that there are $O(K^2T^4)$ sub–plans to investigate, total complexity of computing sub–plan costs is $O(K^4T^6)$.

We define $\phi(t_1, k_1, \tau_1)$ to denote the minimum cost to satisfy demands of periods τ_1 through T where procurement from the suppliers $(i, j) < (t_1, k_1)$ is not allowed and set

$$\phi(t_1, 1, T+1) = 0 \quad 1 \leq t_1 \leq T+1.$$

Then, $\phi(t_1, k_1, \tau_1)$ for $(t_1, k_1) \leq (\tau_1, K_{\tau_1}) \leq (T, K_T)$ can be found by solving the following recursions.

$$\phi(t_1, k_1, \tau_1) = \max_{(t_2,k_2,\tau_2):k_2=K_{t_2}} \{\pi(\tau_1, \tau_2; t_1, k_1; t_2, k_2) + \phi(t_2 + 1, 1, \tau_2 + 1)\}$$

$$\phi(t_1, k_1, \tau_1) = \max_{(t_2,k_2,\tau_2):k_2<K_{t_2}} \{\pi(\tau_1, \tau_2; t_1, k_1; t_2, k_2) + \phi(t_2, k_2 + 1, \tau_2 + 1)\}$$

The optimal solution value is equal to $\phi(1, 1, 1)$.

The recursions above can be solved in $O(KT^2)$ time for a single $\phi(t_1, k_1, \tau_1)$, which implies that they can be solved in $O(K^2T^4)$ time in total to find the value of $\phi(1, 1, 1)$. Since the highest complexity is $O(K^4T^5)$, we conclude that the ELS-PI with general concave procurement costs and linear holding cots can be solved in $O(K^4T^5)$ if the consumption order is FIFO and there are constant procurement capacities in each period.

4.5 Further Extensions

4.5.1 Backlogging

In this section, we investigate an extension of the ELS-PI where backlogging is allowed. In other words, we allow the demand in a period τ to be satisfied later, through procurement in some period $t > \tau$. We let u_t denote the total amount of demand that has been backlogged and let B_t be the cost of backlogging in period t. We assume that B_t is a concave function of u_t and formulate the ELS-PI with backlogging as follows.

$$\text{Minimize} \quad \sum_{t=1}^{T} \left(\sum_{k=1}^{K_t} P_{tk}(x_{tk}) + H_t(I_t) + B_t(u_t) \right)$$

subject to

$$I_t - u_t - \sum_{i=1}^{t} \sum_{k=1}^{K_i} x_{ik} + \sum_{i=1}^{t} D_i = 0 \qquad t = 1, \ldots, T$$

$$x_{tk} = \sum_{i \in \{1,\ldots,t-1\} \cup F(t,k)} x_{tki} \qquad t = 1, \ldots, T;\ k = 1, \ldots, K_t$$

$$D_i = \sum_{(t,k):i \leq v_{tk}\}} x_{tki} \qquad i = 1, \ldots, T$$

$\{x_{tki}\} \in ICO_i$

$I_0 = 0, x_{tki}, I_t, u_t \geq 0 \qquad t = 1, \ldots, T;\ k = 1, \ldots, K_t;\ i = 1, \ldots, v_{tk}$

We assume the following order of processes in any period t. The items are ordered from the supplier and placed in the inventories in the increasing order of the index number of the supplier. If there are demands that were not satisfied in earlier periods, they are satisfied first. After the backlogged demands are satisfied, demand of period t is satisfied. The left over items are carried in inventories to satisfy future demands. Together with the concavity of the cost functions, this implies that, in an optimal solution the supplier that satisfies the demand in a period t satisfies the total amount of backlogged demand until period t as stated below.

Property 4.1. *There is an optimal solution to the (ELS-PI) with backlogging, such that if demand in a period τ is satisfied from a supplier (t, k) ($\tau < t$), then demand in periods i, $\tau \leq i \leq t$, are also satisfied from supplier (t, k).*

4.5.1.1 LEFO and FIFO consumption orders

When the items are consumed in LEFO manner, an optimal solution has the property that items procured from two different suppliers can not exist together in the inventories. This implies that there exists an optimal solution to the ELS-PI with backlogging that can be decomposed into regeneration intervals $(\tau_1, \tau_2; t, k)$ such that demands in periods τ_1 through τ_2 are satisfied by procurement from a unique supplier (t, k), where $t = \tau_1, \ldots, \tau_2$ and $k = 1, \ldots, K_t$ and $\tau_2 \leq v_{tk}$. Then, an $O(KT^3)$ algorithm can be achieved under general concave cost functions where we search over all possible suppliers (t, k) for each possible (τ_1, τ_2) pairs that can constitute a regeneration interval.

If the items are consumed in FIFO manner, an optimal solution can be decomposed into consecutive and independent sub–plans (τ_1, τ_2), such that $I_{\tau_1 - 1} = I_{\tau_2} = 0$ and that $i_\tau > 0$ for $\tau = \tau_1, \ldots, \tau_2 - 1$. Demands between periods τ_1 and τ_2 are satisfied through some consecutive suppliers $(i_1, j_1) < \ldots < (i_n, j_n)$ such that $\tau_1 \leq i_k \leq \tau_2$ for $k = 1, \ldots, n$

and (i_1, j_1) satisfies the backlogged demands from periods τ_1 through $i_1 - 1$. In that case, the algorithm we proposed in Section 4.3.1 to the ELS-PI with general concave cost functions can be modified to account for the backlogging case as follows. We create a network such that the shortest distance between the source node and one of the sink nodes corresponds to the minimum cost for a particular sub–plan. The nodes of the network are represented by the triple (X, t, k), such that to solve for the sub–plan (τ_1, τ_2), we have that $t \in \{\tau_1 - 1, \ldots, \tau_2\}$, $k \in \{1, \ldots, K\}$ and $X \in \{D_{\tau_1\tau_1}, \ldots, D_{\tau_1\tau_2}\}$. The source node is $(0, \tau_1 - 1, 0)$ and the sink nodes are of the form $(D_{\tau_1\tau_2}, t, k)$, where $t = \tau_1, \ldots, \tau_2$ and $k = 1, \ldots, K_t$. Observe that we can set the expiration dates of items procured from the supplier (t, k) to τ_2 if $v_{tk} > \tau_2$ because no items are carried after period τ_2 in the sub–plan (τ_1, τ_2). Then, due to Theorem 4.3, in this network, an arc between the nodes (X_1, t_1, k_1) and (X_2, t_2, k_2) is possible if

$$X_2 \in \{X_1, D_{\tau_1, v_{t_2 k_2}}\}.$$

There is no arc from the node (X_1, t_1, k_1) to (X_2, t_2, k_2) if $X_1 \geq D_{1, v_{t_2, k_2}}$ and $X_2 > X_1$, so that no item procured from the supplier (t_2, k_2) deteriorates before being consumed. If $X_2 = X_1$, we have that $t_2 = t_1 + 1$ and $k_2 = 1$ for convenience. An arc between the nodes (X_1, t, k_1) and $(X_2, t + 1, k_2)$ is possible if $X_1 \geq D_{\tau_1, t}$. That is, inventory carried in a period can not be negative.

Unlike the network defined in Section 4.3.1, there are arcs between the source node and the nodes (X, t, k) for $t = \tau_1, \ldots, \tau_2$ and $k = 1, \ldots, K_t$. These arcs indicate that a total demand of $D_{\tau_1, t-1}$ were backlogged from the supplier (t, k). Therefore, the costs of these arcs are given by

$$\sum_{i=\tau_1}^{t-1} B_i \left(D_{i,t-1}\right) + P_{tk}(X).$$

The costs of the remaining arcs are computed as in the network in Section 4.3.1.

There are $O(KT^2)$ nodes in this network. $O(KT)$ arcs emanate from the source node. Observe that since the procurement costs are concave, if there are two suppliers,

(t, k_1) and (t, k_2), available in period t, with the same expiration date, we can replace these two suppliers by a virtual single supplier (t, k_3) with the same expiration date and procurement function defined as

$$P_{tk_3}(x) = \min\{P_{tk_1}(x), P_{tk_2}(x)\}.$$

Therefore, we can assume that $v_{tk_1} \neq v_{tk_2}$ if $k_1 \neq k_2$. This implies that, in period t there is at most one supplier k that satisfies that $X = v_{tk}$, which further implies that at most three arcs emanate from the node (X_1, t, k_1). The first one is to the node (X_2, t, k_2), where $k_2 > k_1$ and $X_2 = v_{tk_2}$ if such k_2 exists. The second one is to the node $(X_1, t+1, 1)$ if $X_1 \geq D_{\tau_1, t+1}$. The third one is to $(X_2, t+1, k_2)$, where $k_2 = 1, \ldots, K_{t_2}$ and $X_2 = v_{t+1,k_2}$ if such k_2 exists. Therefore, there are $O(KT + KT^2)$ arcs in total in this network, which implies that the optimal cost for a single sub–plan can be computed in $O(KT^2)$ time. Since there are $O(T^2)$ sub–plans, the ELS-PI with backlogging and general concave cost functions can be solved in $O(KT^4)$ time.

4.5.1.2 FEFO and LIFO consumption orders

Using Property 4.1, the algorithm we proposed in Section 4.3.2 can be easily extended to include backlogging where we assume that production costs have a fixed charge structure as in (4–5), holding costs are linear such that $H_t(x) = h_t x$ and backlogging costs are linear such that $B_t(x) = b_t(x)$ for $t = 1, \ldots, T$. We can then, substitute $I_t = \sum_{i=1}^{t} \sum_{k=1}^{K_i} x_{ik} - \sum_{i=1}^{t} D_i + u_t$ in the formulation. Doing that, we eliminate the inventory variables, modify the production cost functions as in (4–6) and modify the unit backlogging costs to $b'_t = b_t + h_t$. To allow for backlogging, we redefine the definition of the block $(\tau_1, \tau_2; t, k)$ where we let the supplier (t, k) be such that $t > \tau_1$. We still assume that supplier (t, k) satisfies the demand in period τ_2 and hence we should have $(t, k) \in F^{-1}(\tau_2)$ for the block $(\tau_1, \tau_2; t, k)$ to be feasible. Calculation of $\pi(\tau_1, \tau_2; t, k)$ for $t \leq \tau_1$ is the same as in section 4.3.2. For $t > \tau_1$, we make the following adjustments.

$$\pi(\tau_1, \tau_2; t, k) = \sum_{i=t_1}^{\tau} \left(c_\tau + \sum_{j=i}^{\tau-1} b'_j \right) D_i + \pi\left(\tau + 1, t_2; \tau\right)$$

We then define $\pi(\tau_1, \tau_2) = \min_{(t,k)\in F^{-1}(t_2)} \{S_{tk} + \pi(\tau_1, \tau_2; t, k)\}$ and the algorithm follows as in Section 4.3.2. Observe that these changes do not add to the complexity in the computation of the block costs. Therefore, the ELS-PI with backlogging can be solved in $O(KT^4)$ time as well.

If the items are consumed in LIFO manner, Corollary 4.4 still holds. Unlike Section 4.3.2, due to backlogging, we still need to search for the best procurement period for each block and hence the complexity remains at $O(KT^4)$.

4.5.2 Pricing

Another extension to the basic ELS-PI model that we consider in this chapter is the case where demands are not fixed over the planning horizon but they can be manipulated through pricing decisions so that demands can be set to optimal levels in each period. We assume that backlogging is allowed. We assume concave revenues R_t ($t = 1, \ldots, T$) as functions of demands satisfied in each period and try to find the optimal demand vector that results in the highest profit. With the introduction of pricing, demands are now decision variables and the problem becomes a maximization problem:

$$\text{Maximize} \quad \sum_{t=1}^{T} \left(R_t(D_t) - \sum_{k=1}^{K_t} P_{tk}(x_{tk}) - H_t(I_t) - B_t(u_t) \right)$$

subject to

$$I_t - u_t - \sum_{i=1}^{t} \sum_{k=1}^{K_i} x_{ik} + \sum_{i=1}^{t} D_i = 0 \qquad t = 1, \ldots, T$$

$$x_{tk} = \sum_{i\in\{1,\ldots,t-1\}\cup F(t,k)} x_{tki} \qquad t = 1, \ldots, T;\ k = 1, \ldots, K_t$$

$$D_i = \sum_{(t,k):i\leq v_{tk}\}} x_{tki} \qquad i = 1, \ldots, T$$

$$\{x_{tki}\} \in ICO_i$$

$$I_0 = 0, x_{tki}, I_t, u_t \geq 0 \qquad t = 1, \ldots, T;\ k = 1, \ldots, K_t;\ i = 1, \ldots, v_{tk}$$

We assume that production costs have a fixed charge structure as in (4–5), holding costs are linear such that $H_t(x) = h_t x$ and backlogging costs are linear such that $B_t(x) = b_t(x)$ for $t = 1, \ldots, T$. We can then, substitute $I_t = \sum_{i=1}^{t} \sum_{k=1}^{K_i} x_{ik} - \sum_{i=1}^{t} D_i + u_t$ in the formulation. Doing that, we eliminate the inventory variables, modify the production cost functions as in (4–6), modify the unit backlogging costs to $b'_t = b_t + h_t$ and change the revenue functions to $\widetilde{R}_t(D_t) = R_t(D_t) - D_t \sum_{i=t}^{T} h_i$.

We assume dynamic pricing strategy, where prices are allowed to change in each period. Our main observation is that the composition of the optimal solution structure specified in Section 4.2 holds for any demand vector. Therefore, it should hold for the optimal demand vector as well. Therefore, we can still restrict our search to the solutions with the same characteristics stated in Section 4.5.1.

Let $D_\tau(t, k)$ be the optimal demand level to satisfy in period τ given that demand in period τ is satisfied by procurement from the supplier (t, k). Then, $D_\tau(t, k)$ satisfies that

$$\widetilde{R}'^{+}_{\tau}(D_\tau(t,k)) \leq c_{tk} \leq \widetilde{R}^{-}_{\tau'}(D_\tau(t,k)) \quad \text{for } t \leq \tau \leq v_{tk}$$

$$\widetilde{R}^{+}_{\tau'}(D_\tau(t,k)) \leq c_{tk} + \sum_{i=\tau}^{t-1} b_{\tau'} \leq \widetilde{R}^{-}_{\tau'}(D_\tau(t,k)) \quad \text{for } \tau < t$$

where, $\widetilde{R}^{+}_{\tau'}$ and $\widetilde{R}^{-}_{\tau'}$ are the right and left derivatives of the function R_τ respectively. The above inequalities simply state that optimal demand values are such that at those values, revenues gained (lost) through marginal increase (decrease) in the demands is higher than the resulting increase (decrease) in the costs to satisfy those demands.

Since, this is a maximization problem, instead of block cost, we compute block profits. We define block profits $\phi(\tau_1, \tau_2; t, k)$ to refer to the maximum profit obtainable by satisfying the demands in periods τ_1 through τ_2 such that a setup has already carried out to procure items from the supplier (t, k). Our solution approach assuming FEFO consumption order is similar to the one proposed for the backlogging case where the recursions are modified as follows:

$$\phi(\tau_1, \tau_2; t, k) = \sum_{i=\tau_1}^{t} \left(c_{tk} + \sum_{j=i}^{t-1} b'_j \right) D_i(t,k) + \phi\left(t+1, \tau_2; t, k\right) \text{ for } \tau_1 < t$$

$$\phi(\tau_1, \tau_2; t, k) = \max \begin{cases} \max\limits_{0 \le \tau' < \tau_2} \{\phi(\tau_1, \tau') + \phi(\tau'+1, \tau_2; t, k)\} \\ \max\limits_{0 \le \tau' < \tau_2} \left\{ \widetilde{R}_i(D_i(t,k)) - c_{tk} \sum\limits_{i=\tau_1}^{\tau'} D_i(t,k) + \phi(\tau'+1, \tau_2; t, k) \right\} \end{cases} \text{for } t \le \tau_1$$

where, $\phi(\tau_1, \tau_2) = \max\limits_{(t,k) \in F^{-1}(\tau_2)} \{\phi(\tau_1, \tau_2; t, k) - S_{tk}\}$. Similar to the procedure we followed in Section 4.3.2, we first calculate $\phi(\tau, \tau)$ for ($\tau = 1, \ldots, T$) and then we move on to the larger blocks to calculate $\phi(\tau, \tau + j)$ for each possible (τ, j) pair to finally solve the following recursions:

$$G(\tau_1) = \max_{\tau_2 : \tau_2 \ge \tau_1} \{\phi(\tau_1, \tau_2) + G(\tau_2 + 1)\} \qquad \tau_1 = 1, \ldots, T$$

$$G(T+1) = 0.$$

where we assume that $\phi(\tau_1, \tau_2) = -\infty$ if there exists no supplier $(t,k) \in F^{-1}(\tau_2)$.

Observe that pricing decisions add no additional complexity with the assumed cost function structures under FEFO consumption case. Hence, the algorithm runs in $O(KT^4)$ time. Under different item consumption order, the complexity of the algorithm changes as follows. If the items consumption order is LEFO, then a single supplier satisfies the demand in a single block. Therefore for each set of consecutive demand periods (τ_1, τ_2), we search for a supplier (t,k), $(\tau_1 \le t \le \tau_2)$, that can fully satisfy demands in those periods. Note that all block profits can be found in $O(KT^3)$ time and hence the complexity is $O(KT^3)$. The above complexity result also holds if the item consumption order is FIFO. Finally, if the item consumption order is LIFO, unlike Section 4.3.2, due to backlogging, we still need to search for the best procurement period for each block and hence the complexity remains at $O(KT^4)$.

4.6 Conclusion and Future Research

In this chapter, we considered the economic lot sizing problem with perishable items where we assumed that there are several suppliers available in each period to procure the items from. We assume that items are good for consumption until the end of their lifetimes and that lifetimes of the items depend on the supplier they are procured.

The manner items are consumed has a significant effect on the optimal solution of the problem. We proposed dynamic programming solution algorithms to the problem under different item consumption orders, which may result from the consumer behavior or the physical constraints of the inventory system. When the lifetimes of items in the inventories are different, consumers have a tendency to buy the items that expire later. We showed that this poses a big constraint on the store manager and that it is store manager's best interest to be able to sell the early expiring items first.

In this chapter, we assumed that suppliers available in each period are independent entities. It would be interesting to work on problem variants where a given supplier, is available in all periods, and in which the expiration dates of the items from a given supplier are non-decreasing. Future research may also include working on polyhedral properties of the problem.

CHAPTER 5
INVENTORY MANIPULATION UNDER CONSUMER PREFERENCE

5.1 Introduction

One of the main results in Chapter 4 was that if the items are perishable and if the store manager can control the consumption of items in the inventories, he first sells the items that expire earlier, which leads to a FEFO consumption order and obtains minimum cost procurement plans. On the other hand, customers always look for the items that expire later. Therefore, if the store manager can not control the consumption of the items in the inventories, they will be consumed in LEFO manner. In that case, an optimal plan exists that obeys the ZIO property. As we claimed in Theorem 4.4 and showed later in Example 4.1, this results in higher procurement costs. In this chapter we investigate two strategies where the store manager can influence the inventory consumption order to develop production plans with costs close to the costs of the procurement plans that could be achieved under FEFO consumption order constraints although the consumers are always looking for the items that expire later, which enforces LEFO consumption order.

The first strategy is to separate the display and the storage areas of the items. Instead of displaying the items as soon as they are procured, inventory manager holds back some of the items in a separate storage area and transfers them to the display area whenever they are needed. With the existence of a storage area, the store manager does not need to stick to a ZIO policy; he can posses items with different lifetimes in the inventories and can manipulate the order they are consumed by displaying them when necessary. However, he now has to pay for the transfer of items between the storage and the display areas and for owning a separate storage location.

The second strategy is to manipulate the order in which the items are consumed by displaying the items in a queue and restricting the consumers to purchase the items only from the front of the queue. Vendor machines and milk racks in the grocery stores are the simplest examples for this type of queue displays. Although they are effective in controlling consumer behavior, as we showed in Chapter 4, such inventory systems may also restrict the store manager if the items can be inserted in the queue only through the back of the queue or through the front of the queue which lead to the FIFO and the LIFO item consumption orders respectively. In this chapter, we will introduce a more general queuing system for the inventories where we will assume that it is possible to insert the items into the queue at any position. However, we will assume that these insertions incur certain costs since certain amount of rearrangement of items in the queue is necessary.

In the remaining of this chapter, we will consider these two inventory models. In both, the main motivation is to control the order of inventory consumption when consumers have a tendency towards the later expiring item. As in all the economic lot size problems, the main question in both models is "when and how much to procure". In the first model, the customer is free to choose any of the items displayed on the shelves. The question to be answered is "when and which items to display". We will refer to the problem of finding optimal procurement plans in the existence of a storage area to hold the procured items and to transfer them when necessary "the item hold back problem". In the second model, consumer is restricted to buy only from the front of the queue but the inventory manager can insert items in any place in the queue thereby affecting the consumption order. In this model, the question is "where to insert the newly procured items". We will refer to the procurement planning problem under such a queuing model "the item insertion problem".

This chapter is organized as follows. In Sections 5.2 and 5.3, we introduce and analyze the item hold back and the item insertion problems respectively. In Section 5.4 we propose polynomial time solution algorithms to some special cases of those problems and in Section 5.5, we conclude this chapter.

5.2 The Item Hold Back Problem

As we mentioned earlier, if the items are stored and displayed in the same location and the customers are free to choose any one of them, they will be consumed in LEFO manner. The store manager will then have to follow a procurement plan with the ZIO property, where two items procured from two different suppliers do not exist together in the inventories. However, lower production costs can be achieved if the items from different suppliers can be kept together in the inventories and consumption order can be controlled. In our first model, to achieve control over the consumption order, we propose to separate the display and the storage areas for the items. Motivation in separating these two locations in the procurement planning of the perishable items is due to the following scenario. In a particular period, the cost of procurement from a particular supplier is low but lifetimes of the items procured from that supplier are shorter than the lifetimes of items already available in the inventories. Given the consumer behavior towards the later expiring items, if there are already enough items to satisfy the demands until the expiration dates of the newly procured items in the inventories, the new items will not be consumed at all and will deteriorate before all the items in the inventories with later expiration dates are depleted. In that case, the inventory manager will naturally choose not to procure from that supplier. However, he could enjoy the low procurement costs if he had a different storage location where the items with later expiration dates could be held back while the items with earlier expiration dates were displayed on the shelves. With the use of a storage location he could control the order of inventory consumption. However, there might be costs associated with holding the items in the storage area and transferring them to the display area.

We formulate the item hold back problem as a two level lot size problem with perishable items. Multi level lot size problems where items do not deteriorate in time have been studied by Zangwill (41), Kaminsky and Simchi-Levi (27) and Van Hoesel et al. (23). In the multi level lot size models, each level may represent one of the steps in the manufacturing and distribution of a product. In the item hold back problem, the first level represents the storage area, i.e., the area where the procured items are first brought, whereas the second level represents the display area, i.e., the area where the demands are satisfied.

We introduce the following additional notation to the ELS-PI model introduced in Chapter 4. We define $I_t^{(i)}$ $(i = 1, 2)$ to be the inventory carried from period t to period $t+1$ and let $H_t^{(i)}$ $(i = 1, 2)$ be the holding cost function in period t in the storage and display areas respectively. We let x_{tkji} denote the amount of items procured from the supplier (t, k) transferred to the display area in period j and allocated to satisfy the demand in period i and we let x_{tk} denote the total procurement from the supplier (t, k). We define z_t to be the amount of items transferred from the storage area to the display area and let G_t be the transfer cost function in period t $(t = 1, \ldots, T)$. Then, the problem is formulated as follows.

$$\text{Minimize} \sum_{t=1}^{T} \left(\sum_{k=1}^{K_t} P_{tk}\left(x_{tk}\right) + G_t\left(z_t\right) + H_t^{(1)}\left(I_t^{(1)}\right) + H_t^{(2)}\left(I_t^{(2)}\right) \right) \tag{5–1}$$

subject to

$$x_{tk} = \sum_{i \in F(t,k)} \sum_{j=t}^{i} x_{tkji} \qquad t = 1, \ldots, T;\ k = 1, \ldots, K_t \tag{5–2}$$

$$D_i = \sum_{(t,k) \in F^{-1}(i)} \sum_{j=t}^{i} x_{tkji} \qquad i = 1, \ldots, T \tag{5–3}$$

$$I_t^{(1)} = \sum_{j=1}^{t} \sum_{k=1}^{K_j} x_{jk} - \sum_{j=1}^{t} z_j \qquad t = 1, \ldots, T \tag{5–4}$$

$$I_t^{(2)} = I_{t-1}^{(2)} + z_t - D_t \qquad t = 1, \ldots, T \tag{5–5}$$

$$z_t = \sum_{j=1}^{t} \sum_{k=1}^{K_j} \sum_{i=t}^{v_{jk}} x_{jkti} \qquad t = 1, \ldots, T \tag{5–6}$$

$$\{x_{tkji}\} \in ICO_{LEFO} \tag{5–7}$$

$$I_0^{(1)} = I_0^{(2)} = 0, I_t^{(1)}, I_t^{(2)}, z_t, x_{tkji} \geq 0 \qquad t = 1, \ldots, T;\ k = 1, \ldots, K_t$$

$$i = t, \ldots, v_{tk};\ j = t, \ldots, i \tag{5–8}$$

Objective function (5–1) minimizes the sum of procurement, transfer and the inventory holding costs in the display and the storage areas. Constraints (5–4) and (5–5) are inventory balance equations in the display and the storage areas respectively. Constraints (5–7) states that items are consumed in LEFO manner. Constraints (5–8) guarantee nonnegativity of the variables and state that initial inventories in both the display and the storage areas are zero.

Our fist observation regarding this model is that, in an optimal solution, items procured from two different suppliers are not displayed together on the shelves.

Theorem 5.1. *In an optimal solution to the item hold back problem, no items that were procured from different suppliers are displayed together.*

Proof. Any feasible solution to the problem corresponds to a flow over a two level network with concave costs as shown in Figure 5-1 for a 3 period problem. In the first level, items are procured and placed in the storage area. In the second level,items are displayed on the shelves and they are consumed. Flow on arcs between a supplier node and a node in the first level represents the amount of procurement from a particular supplier. Flow on vertical arcs between the first level and the second level represents the amount of items transferred in a period from the storage area to the diplay area. The flow on the horizontal arcs represents the amount of inventory carried between two periods in the storage and the display areas.

Let $X = \left(x_{tk}, z_t, I_t^{(1)}, I_t^{(2)}\right)$ represent the flow of the items in the network corresponding to an optimal solution. Consider two suppliers (t_1, k_1) and (t_2, k_2) and let τ be the first period where the items procured from those suppliers appear on the shelves together. Let $\Delta_1 > 0$ and $\Delta_2 > 0$ be the amount of items on the shelves procured from the suppliers (t_1, k_1) and (t_2, k_2) respectively that become available before demand occurs in period τ. Note that we do not exclude the possibility that there may be items from other periods available on the shelves as well.

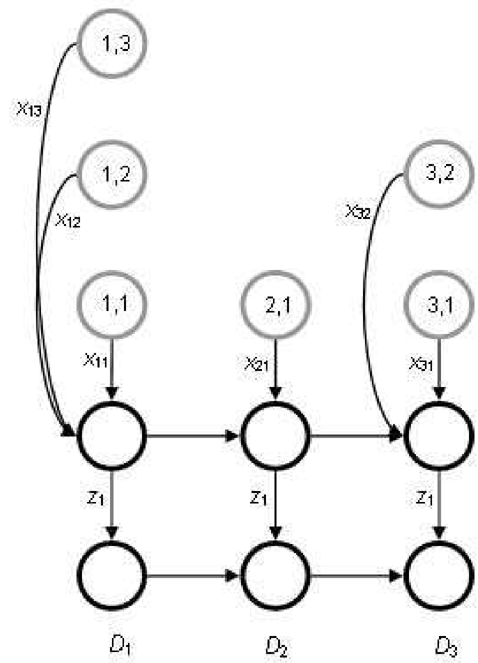

Figure 5-1. A two level network for a 3 period problem.

Due to the concavity of the costs, we can assume that Δ_1 items procured from (t_1, k_1) and Δ_2 items procured from (t_2, k_2) are all transferred in a unique period. If this is not the case, a reconfiguration of the flow is possible with an equivalent or lower flow cost where this assumption is satisfied. Let $j_1 \geq t_1$ and $j_2 \geq t_2$ be the transfer periods of the items procured from the suppliers (t_1, k_1) and (t_2, k_2) respectively. We can generate two other solutions and corresponding flows as follows.

(i) Decrease the number of items procured from (t_2, k_2) and transferred in period j_2 by Δ_2 and increase the number of items procured from (t_1, k_1) and transferred in period j_1 by the same amount. Assuming no items deteriorate due to this change in the procurement plan, we obtain $X' = \left(x'_{tk}, z'_t, I_t^{(1)'}, I_t^{(2)'}\right)$, where

$$
\begin{aligned}
x'_{t_2k_2} &= x_{t_2k_2} - \Delta_2 \\
x'_{t_1k_1} &= x_{t_1k_1} + \Delta_2 \\
I_t^{(1)'} &= \begin{cases} I_t^{(1)} + \Delta_2 & t = t_1, \ldots, j_1 \\ I_t^{(1)} - \Delta_2 & t = t_2, \ldots, j_2 \end{cases} \qquad \text{if } j_1 < t_2
\end{aligned}
$$

$$
\begin{aligned}
I_t^{(1)'} &= \begin{cases} I_t^{(1)} + \Delta_2 & t = t_1, \ldots, t_2 \\ I_t^{(1)} & t = t_2, \ldots, j_1 \\ I_t^{(1)} - \Delta_2 & t = j_1, \ldots, j_2 \end{cases} \qquad \text{if } j_1 \geq t_2 \\
z'_{j_1} &= \begin{cases} z_{j_1} + \Delta_2 & j_1 < j_2 \\ z_j & j_1 = j_2 = j \end{cases} \\
z'_{j_2} &= \begin{cases} z_{j_2} - \Delta_2 & j_1 < j_2 \\ z_j & j_1 = j_2 = j \end{cases} \\
I_t^{(2)'} &= I_t^{(2)} + \Delta_2 \qquad t = j_1, \ldots, j_2
\end{aligned}
$$

(ii) Decrease the number of items procured from (t_1, k_1) and transferred in period j_1 by Δ_1 and increase the number of items procured from (t_2, k_2) and transferred in period j_2 by the same amount. Assuming no items deteriorate after this change in the procurement plan, we obtain $X'' = \left(x''_{tk}, z''_t, I_t^{(1)''}, I_t^{(2)''}\right)$, where

$$
\begin{aligned}
x''_{t_2k_2} &= x_{t_2k_2} + \Delta_1 \\
x''_{t_1k_1} &= x_{t_1k_1} - \Delta_1 \\
I_t^{(1)''} &= \begin{cases} I_t^{(1)} - \Delta_1 & t = t_1, \ldots, j_1 \\ I_t^{(1)} + \Delta_1 & t = t_2, \ldots, j_2 \end{cases} \qquad \text{if } j_1 < t_2 \\
I_t^{(1)''} &= \begin{cases} I_t^{(1)} - \Delta_1 & t = t_1, \ldots, t_2 \\ I_t^{(1)} & t = t_2, \ldots, j_1 \\ I_t^{(1)} + \Delta_1 & t = j_1, \ldots, j_2 \end{cases} \qquad \text{if } j_1 \geq t_2 \\
z''_{j_1} &= \begin{cases} z_{j_1} - \Delta_1 & j_1 < j_2 \\ z_j & j_1 = j_2 = j \end{cases} \\
z''_{j_2} &= \begin{cases} z_{j_2} + \Delta_1 & j_1 < j_2 \\ z_j & j_1 = j_2 = j \end{cases} \\
I_t^{(2)''} &= I_t^{(2)} - \Delta_1 \qquad t = j_1, \ldots, j_2
\end{aligned}
$$

Observe that $X = \lambda X' + (1 - \lambda) X''$ for some $\lambda \in (0, 1)$. Since the costs are concave, if both X' and X'' are feasible, we have that X' and X'' are both optimal as well. It remains to show that X' and X'' are both feasible after the changes in X. It is sufficient to show that the delay in consumption of items procured from other suppliers due to changes in X to obtain X' and X'' do not cause any item to deteriorate before being consumed.

If $v_{t_1k_1} = v_{t_2k_2}$, customers are indifferent between those two items. Therefore, the changes in the procurement plan do not delay the consumption of other items and the feasibility is maintained. As a result, we can safely assume that no two items from two different suppliers with the same expiration dates exist together in the inventories. Now assume $v_{t_1k_1} > v_{t_2k_2}$. Since consumers prefer the items that expire later, Δ_1 items from (t_1, k_1) will be consumed before any of the Δ_2 items procured from (t_2, k_2) will be consumed. Assume that the last demand that the items procured from (t_2, k_2) and transferred in period j_2 satisfy is the L^{th} demand from the beginning of period τ. Also, assume that there are items procured from other suppliers $(i_1, j_1), \ldots, (i_n, j_n)$ that are available in the display area together with the items from (t_1, k_1) and (t_2, k_2). Let $A \geq 0$ be the total number of demands until the L^{th} demand that were satisfied by suppliers (i_ℓ, j_ℓ) $(\ell = 1, \ldots, n)$ such that

$$L = \Delta_1 + \Delta_2 + A.$$

Note that we should have $v_{i_\ell j_\ell} > v_{t_2k_2}$ for $\ell = 1, \ldots, n$ since they were consumed before the items procured from (t_2, k_2) have been depleted.

If the amount of items procured from (t_1, k_1) and transferred in period j_1 is increased by Δ_2 and the amount of items procured from (t_2, k_2) and transferred in period j_2 is decreased by the same amount, consumption of the items procured from (i_ℓ, j_ℓ) $(\ell = 1, \ldots, n)$ is not delayed later than the L^{th} demand. Since $v_{i_\ell j_\ell} > v_{t_2k_2}$, they do not deteriorate and feasibility is maintained in both X' and X''. Continuing in this manner, we obtain optimal solutions where items procured from two different suppliers are never seen on the shelves together. $\square$

Following Theorem 5.1, it is straightforward to prove the Unique Production Period Property for the item hold back problem as stated in Corollary 5.1. It is also easy to conclude that items transferred in a period share the same supplier and hence the same expiration date as stated in Corollary 5.2. The proof of Theorem 5.1 also implies that in an optimal solution no transfer takes place before all the items in the display area are depleted as stated in Corollary 5.3.

Corollary 5.1 (Unique Production Period Property)**.** *There exists an optimal solution to the "Inventory Hold-back" problem in which the demand in a period is fulfilled by a unique production period.*

Corollary 5.2. *In an optimal solution to the item hold back problem, the items transferred in a period have a unique supplier and thus share the same expiration date.*

Corollary 5.3 (Zero Inventory Transfer Property)**.** *In an optimal solution to the item hold back problem, items are transferred when no inventory is left in the display area. That is,* $z_t I_{t-1}^{(2)} = 0$.

As a generalization of Corollary 4.2, we prove in Theorem 5.2 that in an optimal solution to the item hold back problem, among the items in the storage location, the store manager transfers to the display area the items that expire earlier first.

Theorem 5.2. *In an optimal solution to the item hold back problem, items with earlier expiration dates are transferred to the display area first.*

Proof. Assume that $X = \left(x_{tk}, I_t^{(1)}, I_t^{(2)}, z_t\right)$ is an optimal solution and that there are two suppliers (t_1, k_1) and (t_2, k_2) with $v_{t_1k_1} \geq v_{t_2k_2}$. Assume that transfer of items occurs in periods τ_1 and τ_2 where $\max\{t_1, t_2\} \leq \tau_1 < \tau_2$. Contrary to the property stated in the theorem, assume that items procured from (t_1, k_1) are transferred in period τ_1 and they satisfy the demands of periods τ_1 through τ_1' for $\tau_1 \leq \tau_1' < \tau_2$, whereas items procured from (t_2, k_2) are transferred in period τ_2 to satisfy the demands of τ_2 through τ_2' for $\tau_2 \leq \tau_2' \leq T$. Now, consider two other feasible solutions $X' = \left(x'_{tk}, I_t^{(1)'}, I_t^{(2)'}, z'_t\right)$ and $X'' = \left(x''_{tk}, I_t^{(1)''}, I_t^{(2)''}, z''_t\right)$, where

$$\begin{aligned}
x'_{t_1k_1} &= x_{t_1k_1} + \sum_{i=\tau_2}^{\tau_2'} D_i \\
x'_{t_2k_2} &= x_{t_2k_2} - \sum_{i=\tau_2}^{\tau_2'} D_i \\
I_t^{(1)'} &= I_t^{(1)} + \sum_{i=\tau_2}^{\tau_2'} D_i \quad t = \min\{t_1, t_2\}, \ldots, \max\{t_1, t_2\} - 1
\end{aligned}$$

$$
\begin{aligned}
x''_{t_1k_1} &= x_{t_1k_1} - \sum_{i=\tau_1}^{\tau_1'} D_i \\
x''_{t_2k_2} &= x_{t_2k_2} + \sum_{i=\tau_1}^{\tau_1'} D_i \\
I_t^{(1)''} &= I_t^{(1)} - \sum_{i=\tau_1}^{\tau_1'} D_i \quad t = \min\{t_1, t_2\}, \ldots, \max\{t_1, t_2\} - 1
\end{aligned}
$$

while all the variables in X' and X'' are kept as in X. Clearly $X = \lambda X' + (1-\lambda)X''$ for some $\lambda \in (0,1)$. Since the costs are concave, X' and X'' should both be optimal as well. Continuing in this manner, we generate two solutions where either early expiring items are used to satisfy immediate demands or they are not procured at all. □

5.3 The Item Insertion Problem

One of the methods to control the inventory consumption order is to display the items in a queue such that consumer always sees and acquires the item at the front of the queue. If the items are not perishable, the manner in which the items are put in such a queue would not make any difference; overall procurement and holding costs will be the same if the inventory manager always places the items at the back or the front of the queue. However, if the items are subject to deterioration before the end of the planning horizon, then, where the items are placed in the queue would make quite a difference. To see this, consider a queuing system where items are always placed to the back of the queue and assume that in some period t, procurement costs are extremely low but that the items procured in that period have very short lifetimes. If the inventory manager places those items far back in the queue, they will deteriorate in the time the consumers buy all the items placed before them. In that case, the inventory manager, will either cancel the procurement decision in that period or would update the whole procurement plan so that there are fewer or no other items in the queue at the time those fast expiring items are to be procured.

We therefore propose a queuing system where the items can be inserted at any place in the queue after they are procured from the suppliers. This allows the store manager to enforce any consumption order for the items as long as the items are arranged in the queue in the right order. Any insertion in to the queue may require rearrangement of the items already available in the queue and may incur additional costs. We assume that the cost of inserting items into the queue is a function of the supplier the items are procured, amount of items to be inserted and the total amount of items in the inventories at the time of insertion. Note that, the amount of items in the inventories at the time the items from the supplier (t, k) is inserted into the queue is given by

$$\sum_{(i,j)<(t,k)} x_{ij} - \sum_{i=1}^{t-1} D_i$$

due to our convention that items procured from lower indexed suppliers are placed in the storage earlier. Letting $G_{tk} : R^2 \rightarrow R$ be the concave insertion cost function for the items procured from the supplier (t, k), the problem is formulated as the ELS-PI model described in Chapter 4. However, since the consumption order is defined by the order of the items in the queue, the inventory consumption order constraints are dropped and the objective function of the ELS-PI is changed to

$$\text{Minimize} \quad \sum_{t=1}^{T} \left(\sum_{k=1}^{K_t} \left(P_{tk}\left(x_{tk}\right) + G_{tk} \left(x_{tk}, \sum_{(i,j)<(t,k)} x_{ij} - \sum_{i=1}^{t-1} D_i \right) \right) + H_t\left(I_t\right) \right)$$

to account for the cost of insertions into the queue. That is, the store manager now has the power to control the consumption order by organizing the items in the queue but has to pay whenever the items in the queue are rearranged. Since the cost functions are still concave, and the constraint set is the same as the ELS-PI (except that consumption order constraints are removed), Theorem 4.1 and Corollary 4.1 of Chapter 4 can be extended to this model. Therefore optimal solution structure is the same as the ELS-PI with no consumption order constraints.

5.4 Solution Algorithms

In this section, we present solution algorithms to some special cases of the item hold back and the item insertion problems. In all cases, we assume that procurement cost functions P_{tk} are concave with a fixed charge structure such that

$$P_{tk}(x) = \begin{cases} S_{tk} + p_{tk}x & x > 0 \\ 0 & x = 0 \end{cases} \quad t = 1, \ldots, T;\ k = 1, \ldots, K_t.$$

Holding cost functions in the storage area for the item hold back problem and holding cost functions in the item insertion problem are assumed to be linear such that $H_t^{(1)}(x) = h_t^{(1)}x$ and $H_t(x) = h_t x$ $(t = 1, \ldots, T)$. Unless otherwise stated, all the other related cost functions are assumed to be general concave.

5.4.1 The Item Hold Back Problem

Due to Corollary 5.2, the transferred items share the same supplier and due to Theorem 5.2, items that expire earlier are always transferred earlier. These properties result in a certain optimal solution structure. As in Chapter 4, we define a block $(\tau_1, \tau_2; t, k)$ to be a set of consecutive demand periods $\tau_1, \ldots, \tau_2$ where a setup has already been carried out for the supplier (t, k) such that procurement from that supplier satisfies the demand in period τ_2 and can potentially satisfy the demand of any period between τ_1 and τ_2. An optimal solution to the item hold back problem is decomposed into such blocks, where these blocks can subsume other blocks. Figure 5-2 shows a possible optimal solution to a seven period problem. In the figure, different colors on arcs represent the flow of items procured from different suppliers. Observe that although there are items that were procured from the suppliers (1,1) and (2,1) in the storage, new items are procured from supplier (5,1) to satisfy the demand of period 5. There are three blocks in the figure; block (5,5;5,1) is in block (3,6;2,1), which is in block (1,7;1,1).

We define $SP(\tau_1, \tau_2; t, k)$ to be the cost of block $(\tau_1, \tau_2; t, k)$. It is also the solution to the two level lot sizing problem defined between periods t and τ_2 such that (i) demands

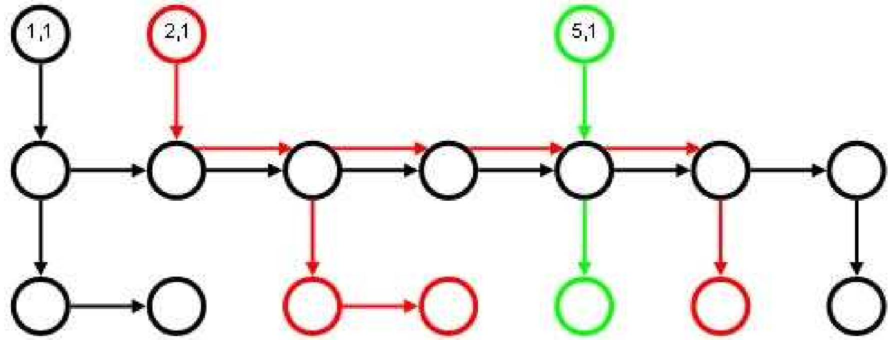

Figure 5-2. An optimal solution for a 7 period problem

between τ_1 and τ_2 are all satisfied by procurement from the supplier (t, k), (ii) a setup for that supplier has already been carried out and, (iii) transfer of items can not occur before period τ_1. For fixed values of τ_1, τ_2 and a fixed supplier (t, k), $SP\,(\tau_1, \tau_2; t, k)$ can be computed in $O(T^2)$ time because it can be reduced to a single level economic lot size problem (see Chapter 2 and Zangwill (41) for the solution algorithms to two level lot size problems with general concave costs). Moreover, when we are solving for $SP\,(\tau_1, T; t, k)$, we find all the values $SP\,(\tau_1, \tau_2; t, k)$ for $\tau_1 \le \tau_2 \le T$. Therefore the values for all $SP\,(\tau_1, \tau_2; t, k)$ can be computed in $O(KT^4)$ time in total. We then follow a similar approach as in Chapter 4. We define $\pi(\tau_1, \tau_2; t, k)$ to be the minimum cost to satisfy demand of the block $(\tau_1, \tau_2; t, k)$ and compute it by the following recursion.

$$\pi(\tau_1, \tau_2; t, k) = \min \begin{cases} \min\limits_{0 \le \tau' < \tau_2} \{SP\,(\tau_1, \tau'; t, k) + \pi\,(\tau' + 1, \tau_2; t, k)\} \\ \min\limits_{0 \le \tau' < \tau_2} \{\pi(\tau_1, \tau') + \pi(\tau' + 1, \tau_2; t, k)\} \end{cases}$$

where,

$$\pi(\tau_1, \tau_2) = \min_{(t,k) \in F^{-1}(\tau_1) \cap F^{-1}(\tau_2)} \{S_{tk} + \pi(\tau_1, \tau_2; t, k)\}.$$

Due to the order of processes, when calculating $\pi(\tau_1, \tau_2)$ for a particular (τ_1, τ_2) pair, we know all $\pi(\tau_1, \tau')$, $SP\,(\tau_1, \tau'; t, k)$ and all $\pi(\tau' + 1, \tau_2; t, k)$ for $\tau' < \tau_2$. Once those values are known, it takes $O(T)$ time to calculate a single block cost. Letting $K = \max_t K_t$, this implies that it takes $O(KT^4)$ time to compute $\pi(\tau_1, \tau_2)$ for $\tau_1 = 1, \ldots, T$ and $\tau_2 = \tau_1, \ldots, T$. Finally, we solve the following recursion to find the procurement plan over the planning horizon:

$$\phi(t_1) = \max_{t_2 : t_2 \geq t_1} \{\pi(t_1, t_2) + \phi(t_2 + 1)\} \qquad t_1 = 1, \ldots, T \tag{5–9}$$

$$\phi(T+1) = 0.$$

where, $\phi(t)$ is the minimum cost of satisfying the demands of periods t through T. The above recursion is solved in $O(T^2)$ time once all $\pi(\tau_1, \tau_2)$ values are known. Since the highest complexity is $O(KT^4)$, total complexity of this algorithm is $O(KT^4)$.

5.4.2 The Item Insertion Problem

Observe that we can always substitute $I_t = \sum_{i=1}^{t} \sum_{k=1}^{K_i} x_{ik} - \sum_{i=1}^{t} D_i$ in the objective function and drop the inventory decision variables out of the formulation. We can then set procurement cost functions to

$$P'_{tk}(x) = \begin{cases} S_{tk} + c_{tk} x & x > 0 \\ 0 & x = 0 \end{cases}$$

where $c_{tk} = p_{tk} + \sum_{j=t}^{T} h_j$. In this manner, we can generate an equivalent formulation of the problem where inventory holding costs are zero. For convenience, we will assume that procurement costs are of the form P'_{tk} and that there is no charge for carrying inventories. In the following sections, we present polynomial time solution algorithms under certain insertion cost function structures.

5.4.2.1 Constant Insertion Costs

As previously, we define a block $(\tau_1, \tau_2; t, k)$ to be a set of consecutive demand periods $\tau_1, \ldots, \tau_2$ where the supplier (t, k) has already carried out a setup such that procurement from (t, k) satisfies the demand in period τ_2 and can potentially satisfy any demand between periods τ_1 and τ_2. We assume that the insertion costs are of the following form

$$G_{tk}(x,y) \;=\; \begin{cases} B_{tk} & (x,y) > (0,0) \\ 0 & x = 0 \end{cases}$$

and carry out a similar approach as in section 4.3.2 where we slightly modify the definition of the block cost $\pi(\tau_1, \tau_2; t, k)$ as follows

$$\pi(\tau_1, \tau_2; t, k) = \min \begin{cases} \min\limits_{0 \le \tau' < \tau_2} \{\pi'(\tau_1, \tau') + \pi(\tau' + 1, \tau_2; t, k)\} \\ \min\limits_{0 \le \tau' < \tau} \left\{ c_{tk} \sum\limits_{i=\tau_1}^{\tau'} D_i + \pi(\tau' + 1, \tau_2; t, k) \right\} \end{cases}$$

where,

$$\pi'(\tau_1, \tau_2) = \min_{(t,k) \in F^{-1}(\tau_1) \cap F^{-1}(\tau_2)} \{S_{tk} + B_{tk+} \pi(\tau_1, \tau_2; t, k)\}.$$

Observe that in computing the block cost $\pi(\tau_1, \tau_2; t, k)$, we check whether or not the demands in periods τ_1 through τ', for some $\tau' < \tau_2$, should be satisfied by a supplier other than (t, k). If that is the case, the insertion cost is added to the block cost. Otherwise the demands between periods τ_1 through τ' are satisfied by the items procured from the supplier (t, k). Due to the order of processes, when calculating $\pi(\tau_1, \tau_2; t, k)$ for fixed values of τ_1 and τ_2 and a fixed supplier (t, k), we know all $\pi(\tau' + 1, \tau_2; t, k)$and $\pi'(\tau_1, \tau')$ for any $\tau' < \tau_2$. Once those values are known, it takes $O(T)$ time to compute a single block cost, which implies that $\pi(\tau_1, \tau_2; t, k)$ for $\tau_1 = 1, \ldots, T$, $\tau_2 = \tau_1, \ldots, T$ and $((1, 1) \le (t, k) \le (T, K_T))$ can be computed in $O(KT^4)$ time. Finally, we compute $\pi(\tau_1, \tau_2)$, which is given by

$$\pi(\tau_1, \tau_2) = \min_{(t,k) \in F^{-1}(\tau_1) \cap F^{-1}(\tau_2)} \{S_{tk} + \pi(\tau_1, \tau_2; t, k)\}.$$

We then solve recursion (5–9) to find the production plan over the planning horizon. Since the highest complexity is $O(KT^4)$, total complexity of this algorithm is $O(KT^4)$.

5.4.2.2 Concave Insertion Costs with Fixed Charges

The complexity of the Insertion problem with linear and time variant insertions costs is not known. However, the problem is solvable in polynomial time if we assume the following conditions:

1. The insertion cost functions are concave with fixed charge structure with time invariant continuous components. That is,

$$G_{tk}(x,y) \quad = \quad \begin{cases} B_{tk} + g(x+y) & (x,y) > (0,0) \\ 0 & x = 0 \end{cases}$$

2. The insertions can be made only to the beginning of the queue. That is, the queue is a LIFO queue but there is a cost for inserting any item as a function of the items already in the queue (and had to be moved towards the back) at the time of insertion

In that case, we solve the item insertion problem as follows. First, we transform it to a problem with a single supplier in each period as follows. For each $t = 1, \ldots, T$, we associate $i_t = \sum_{j=1}^{t} K_j$ and derive an equivalent problem defined over a planning horizon of $T' = i_T = \sum_{j=1}^{T} K_j$ periods with the demand vector D' such that

$$D'_{i_t} = D_t \text{ for } t = 1, \ldots, T$$

$$D'_{i_t - j + 1} = 0 \text{ for } t = 1, \ldots, T, \quad j = 2, \ldots, K_t.$$

Since we have exactly one supplier available in each period, we drop the subscript for the supplier in our notation and set the cost functions and expiration dates as follows

$$S'_{i_t - k + 1} = S_{tk} \text{ for } t = 1, \ldots, T, \quad k = 1, \ldots, K_t$$

$$c'_{i_t - k + 1} = c_{tk} \text{ for } t = 1, \ldots, T, \quad k = 1, \ldots, K_t$$

$$B'_{i_t - k + 1} = B_{tk} \text{ for } t = 1, \ldots, T, \quad k = 1, \ldots, K_t$$

$$v'_{i_t - k + 1} = i_{v_{tk}} \text{ for } t = 1, \ldots, T, \quad j = 1, \ldots, K_t$$

We apply the following algorithm to the transformed problem. We define a block by $(\tau_1, \tau_2; t; n, m)$ such that a setup has already been carried out for the supplier available in period t and that exactly n insertions, $0 \leq n < (\tau_2 - \tau_1)$, in total are to be carried out between periods τ_1 and τ_2. Moreover there have been m insertions, $0 \leq m < (\tau_1 - t)$, between periods t and τ_1. We let $\pi(\tau_1, \tau_2; t; n, m)$ to be the cost of that block (excluding the setup cost of the supplier in period t). Notice that, in block $(\tau_1, \tau_2; t; n, m)$, all the items procured from the supplier in period t, that are carried over to period τ_1, have been rearranged at least m times since there have been m insertions before those items become available in period τ_1. This implies that for each unit procured from the supplier in period t that become available in the inventories in period τ_1, an additional cost of mg has been incurred. Therefore, we account for the continuous part of the insertion cost by setting the variable component of the procurement cost to $c'_t + mg$ for the block $(\tau_1, \tau_2; t; n, m)$. Then for $t < \tau_1$ we solve the following recursion to compute block cost $\pi(\tau_1, \tau_2; t; n, m)$:

$$\pi(\tau_1, \tau_2; t; n, m) = \min \begin{cases} \min\limits_{0 \leq \tau' < \tau_2} \min\limits_{0 \leq n_1 \leq n} \begin{cases} S'_{\tau_1} + B'_{\tau_1} + \pi(\tau_1, \tau'; t, n_1, 0) + \\ g \sum\limits_{i=\tau_1}^{\tau'} D_i + \pi(\tau' + 1, \tau_2; t; n - n_1, m + n_1 + 1) \end{cases} \\ \min\limits_{0 \leq \tau' < \tau_2} \left\{ (c'_t + mg) \sum\limits_{i=\tau_1}^{\tau'} D_i + \pi(\tau' + 1, \tau_2; t; n, m) \right\} \end{cases}$$

and for $\tau = t$,

$$\pi(\tau_1, \tau_2; \tau_1; n, 0) = \min_{0 \leq \tau' < \tau_2} \left\{ c'_t \sum_{i=\tau_1}^{\tau'} D_i + \pi(\tau' + 1, \tau_2; t; n, 0) \right\}$$

Blocks of the form $(\tau_1, \tau_2; \tau_1; n, 0)$ indicate that the items procured in period τ_1 are inserted at the front of the queue and hence they satisfy the demand in period τ_1. After computing $\pi(\tau_1, \tau_2; \tau_1; n, 0)$, we compute

$$\pi(\tau_1, \tau_2) = \min_{0 \leq n < \tau_2 - \tau_1} \left\{ S'_{\tau_1} + \pi(\tau_1, \tau_2; \tau_1, n, 0) \right\}$$

Algorithm then follows as in previous sections where we compute the costs of larger blocks.

We always keep a record of how many insertions are done in a block. Starting with the smallest block, it is easy to find the minimum cost in a block when the number of insertions in that block is restricted to a certain number. When calculating $\pi(\tau_1, \tau_2; t; n, m)$, we investigate all possible combinations such that the number of insertions in the inserted block plus the insertions in the following block is equal to n.

The values for $\pi(\tau_1, \tau_2; t; n, m)$ can be computed in $O(T'^2)$ time as long as $\pi(\tau_1, \tau'; t; n, m)$ for $\tau' < \tau_2$ are known. Since, τ_1, τ_2, t, n and m can assume $O(T)$ possible values, all $\pi(\tau_1, \tau_2)$ can be found in $O(T'^7)$ time. Since $T' = O(T^7)$, the complexity of the overall algorithm for the item insertion problem with multiple suppliers is $O(K^7T^7)$.

5.5 Conclusion and Future Research

When items of two different ages are exhibited together on the shelves, consumers have a tendency to buy the item that expires later. However, as we showed in Chapter 4, lower cost procurement plans can be achieved if the store manager can manipulate item consumption to sell early expiring items first. In this chapter, we considered two models where the item consumption order can be controlled to promote the sale of early expiring items.

In the item hold back model, we assumed that new items are always placed in the storage area first and then transported to the display area. We did not alow any item transfer from the display area back to the storage area. It would be interesting to work on a variant of the model where items are allowed to be transferred both ways. This will then allow the store manager to optimize on holding cost as well as to promote the sale of early expiring items. It would also be interesting to work on the complexity of the item insertion problem and identifying polynomially solvable special cases.

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BIOGRAPHICAL SKETCH

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