To my wife Linda
ACKNOWLEDGMENTS

I gratefully acknowledge a number of individuals who have been instrumental in allowing me to continue my lifelong pursuit of education. I am most profoundly grateful to my professors at the University of Florida’s Civil and Coastal Department, some of whom have been teaching me since 1973. In particular, I am indebted to Drs. Ashish Mehta, Robert Dean and Max Sheppard who have each sustained the quality of the educational experience at the University of Florida for me. I am also grateful to Dr. Robert Thieke, an exceptional teacher who recognizes the imperative of learning by getting your hands wet, challenges his students and guarantees a continuing level of educational excellence in the department. I am also grateful to Dr. Tian-Jian Hsu for his guidance and suggestions and to Dr. John Jaeger for his willing participation on my committee in spite of my distance learning over the past several years.

I am especially grateful to Dr. Mehta for his tolerance of an older student set in his ways. I appreciate his constant willingness to accommodate a wandering path to this culmination. In particular, I acknowledge the value of his guidance and council through the procedural maze of the University.

I acknowledge the management of the Coastal and Hydraulics Laboratory (CHL) at the U.S. Army Engineer Research and Development Center (ERDC) for their financial support and commitment to higher education and an innovative work environment that encourages higher learning. I am grateful to my coworkers at ERDC for their friendship and encouragement. Special thanks are offered to the field data collection staff for their assistance in analysis of the San Francisco data.

I am especially grateful to my parents, Joseph and Helen Letter, for their unfailing love and support throughout my life. Nothing is impossible for me in their eyes. I especially
acknowledge their patience and understanding of my educational priorities at a time in their lives when moments spent with them are so very precious.

I am profoundly grateful to my wife, Linda, for her love, support and patience and for mowing the lawn by herself on far too many occasions during the pursuit of my education.

Finally, I acknowledge the patience and understanding of the rest of my family at all the times when I missed those family gatherings because I was in my “cave.”
## TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>4</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>11</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>13</td>
</tr>
<tr>
<td>NOTATION</td>
<td>23</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>35</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>38</td>
</tr>
<tr>
<td>1.1 Need for Research</td>
<td>38</td>
</tr>
<tr>
<td>1.2 Objectives and Tasks</td>
<td>43</td>
</tr>
<tr>
<td>1.3 Approach</td>
<td>44</td>
</tr>
<tr>
<td>1.3.1 Examination of Governing Equations</td>
<td>45</td>
</tr>
<tr>
<td>1.3.2 Revision of Aggregation/Disaggregation Model for Probabilistic Fluctuations</td>
<td>45</td>
</tr>
<tr>
<td>1.3.3 Effects of Suspended Sediment on Turbulence</td>
<td>46</td>
</tr>
<tr>
<td>1.3.4 Development of a Probabilistic Bed-Exchange Model</td>
<td>46</td>
</tr>
<tr>
<td>1.3.5 Application of the Probabilistic Formulation to Selected Test Cases</td>
<td>46</td>
</tr>
<tr>
<td>1.4 Scope</td>
<td>46</td>
</tr>
<tr>
<td>1.5 Presentation Outline</td>
<td>47</td>
</tr>
<tr>
<td>COHESIVE SEDIMENT TRANSPORT</td>
<td>48</td>
</tr>
<tr>
<td>2.1 Estuarine Cohesive Sediment Properties</td>
<td>48</td>
</tr>
<tr>
<td>2.1.1 Estuarine Sediment</td>
<td>48</td>
</tr>
<tr>
<td>2.1.2 Fine Sediment Classification</td>
<td>49</td>
</tr>
<tr>
<td>2.1.2.1 By size</td>
<td>49</td>
</tr>
<tr>
<td>2.1.2.2 By shape</td>
<td>50</td>
</tr>
<tr>
<td>2.1.2.3 By composition</td>
<td>50</td>
</tr>
<tr>
<td>2.1.2.4 By electrochemical properties</td>
<td>51</td>
</tr>
<tr>
<td>2.1.3 Characterizing Aggregates</td>
<td>52</td>
</tr>
<tr>
<td>2.1.3.1 Primary particle distribution</td>
<td>52</td>
</tr>
<tr>
<td>2.1.3.2 Order of aggregation</td>
<td>54</td>
</tr>
<tr>
<td>2.1.3.3 Floc size spectra</td>
<td>55</td>
</tr>
<tr>
<td>2.1.3.4 Fall velocity</td>
<td>55</td>
</tr>
<tr>
<td>2.1.3.5 Floc density</td>
<td>55</td>
</tr>
<tr>
<td>2.1.3.6 Floc shearing strength</td>
<td>56</td>
</tr>
<tr>
<td>2.1.3.7 Fractal dimension</td>
<td>58</td>
</tr>
<tr>
<td>2.1.4 Bulk Properties of Cohesive Sediments</td>
<td>61</td>
</tr>
<tr>
<td>2.2 Cohesive Sediment Transport in Steady Uniform Flow</td>
<td>61</td>
</tr>
<tr>
<td>2.2.1 Aggregation Processes</td>
<td>62</td>
</tr>
<tr>
<td>2.2.1.1 Brownian motion</td>
<td>63</td>
</tr>
</tbody>
</table>
SEDIMENT TRANSPORT AND DEPOSITION EXPERIMENTS .......................................................... 230

5.1 Kynch (1952) Sedimentation Theory .................................................................................. 230
5.2 Krone (1962) Flume Deposition Experiments ..................................................................... 235
  5.2.1 Settling Tests with Variable Shear ............................................................................. 235
  5.2.2 Settling Test with Tagged Sediments ......................................................................... 235
5.3 Mehta 1973 Flume Results ................................................................................................. 236
5.4 Parchure and Mehta (1985) Dilution Test ......................................................................... 237
5.5 Parchure and Mehta (1985) Erosion Test .......................................................................... 237
5.6 Sanford and Halka (1993) Data Set ................................................................................... 238

METHOD APPLICATION ............................................................................................................. 252

6.1 Preamble ............................................................................................................................ 252
6.1 Kynch (1952) Quiescent Deposition Test ......................................................................... 254
6.2 Mehta 1973 Flume Deposition Tests ................................................................................ 256
6.3 Krone (1962) Flume Deposition Tests ............................................................................. 263
6.4 Parchure and Mehta (1985) Dilution Test ......................................................................... 264
6.5 Parchure and Mehta (1985) Erosion Test ......................................................................... 270
6.6 Sanford and Halka (1993) Field Datasets ......................................................................... 271
6.7 Krone (1962) Tagged-Sediment Settling Test ................................................................... 274

SUMMARY AND CONCLUSIONS ............................................................................................... 320

7.1 Summary ............................................................................................................................ 320
7.2 Conclusions ...................................................................................................................... 324
7.3 Recommendations ............................................................................................................ 325

DEVELOPMENT OF GOVERNING EQUATIONS FOR UNSTEADY AND NONUNIFORM SEDIMENT TRANSPORT ................................................................. 327

A.1 Hydrodynamics .................................................................................................................. 327
  A.1.1 Continuity Equation ..................................................................................................... 328
  A.1.2 Momentum Equations ............................................................................................... 328
A.2 Sediment Transport Equation .......................................................................................... 329
  A.2.1 Continuity Equation for Sediment Laden Fluid ........................................................... 332
  A.2.2 Continuity Equation with Differential Sediment Particle Velocity ............................. 333
A.3 Turbulence ........................................................................................................................ 335
  A.3.1 Continuity Equation ..................................................................................................... 337
    A 3.1.1 Variable density case with CTD ............................................................................ 337
    A 3.1.2 Homogeneous case with CTD ............................................................................. 338
    A 3.1.3 Variable density case with MTD ......................................................................... 338
  A.3.2 Turbulence Effects on Continuity Equation .................................................................. 339
    A.3.2.1 CTD method ........................................................................................................ 340
    A.3.2.2 Homogeneous flow .............................................................................................. 341
    A.3.2.3 MTD method ....................................................................................................... 341
  A.3.3 Viscous Stresses ......................................................................................................... 342
OUTLINE OF KEY SEDIMENT SUBROUTINES IN COMPUTATIONAL MODEL ...........407

D.1 AGGFLUX ....................................................................................................................407
D.2 FALLVEL......................................................................................................................409
D.3 BEDXCHG..................................................................................................................410

LIST OF REFERENCES.............................................................................................................414

BIOGRAPHICAL SKETCH .......................................................................................................428
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 2-1. Size classification and general cohesive characteristics</td>
<td>50</td>
</tr>
<tr>
<td>Table 2-2. Properties of example clay minerals</td>
<td>51</td>
</tr>
<tr>
<td>Table 2-3 Comparison of shear strength versus excess density for locales</td>
<td>58</td>
</tr>
<tr>
<td>Table 3-1. Variables important to cohesive sediment transport processes</td>
<td>132</td>
</tr>
<tr>
<td>Table 3-2. Parameters in the shear stress distributions shown in Figure 3-3</td>
<td>133</td>
</tr>
<tr>
<td>Table 3-3. Summary of Monte Carlo simulation of the Krone flume deposition experiment shear stress distributions</td>
<td>133</td>
</tr>
<tr>
<td>Table 3-4. Statistical parameters for example probability distribution curves in Figures 3-8, 3-9 and 3-10</td>
<td>133</td>
</tr>
<tr>
<td>Table 3-5. Coefficients of Equation 3-12 for fits to data sets in Figure 2-7 and the normalized deviation between the fit and the data</td>
<td>134</td>
</tr>
<tr>
<td>Table 3-6. Probabilistic treatment of significant CST variables</td>
<td>134</td>
</tr>
<tr>
<td>Table 4-1. Standard $k-\varepsilon$ model coefficients for high Reynolds number flow</td>
<td>200</td>
</tr>
<tr>
<td>Table 4-2. Summary of example flocculation model</td>
<td>200</td>
</tr>
<tr>
<td>Table 4-3. Summary of boundary condition specifications</td>
<td>200</td>
</tr>
<tr>
<td>Table 4-4. Simulation conditions for special laminar flow problems</td>
<td>200</td>
</tr>
<tr>
<td>Table 4-5. Error measures for the Stokes first problem for varying time and number of cells</td>
<td>201</td>
</tr>
<tr>
<td>Table 4-6. Error measures for the Couette problem for varying time and number of cells</td>
<td>201</td>
</tr>
<tr>
<td>Table 4-7. Error measures for the Stokes second problem for varying time and number of cells</td>
<td>201</td>
</tr>
<tr>
<td>Table 5-1. Summary of Kynch settling column data and graphical analysis (after Mehta, 2007)</td>
<td>239</td>
</tr>
<tr>
<td>Table 5-2. Estimation of the concentration profiles based on the intersection of characteristic lines with vertical profiles at specific times (after Mehta, 2007)</td>
<td>239</td>
</tr>
<tr>
<td>Table 6-1. Parameters common to all simulated tests</td>
<td>276</td>
</tr>
<tr>
<td>Table 6-2. Parameters used in the Kynch (1952) deposition tests</td>
<td>276</td>
</tr>
</tbody>
</table>
### LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 2-1</td>
<td>Example particle size distributions, showing the MIT size classification. The flocculated San Francisco Bay sediment has a size distribution comparable to beach sand.</td>
<td>92</td>
</tr>
<tr>
<td>Figure 2-2</td>
<td>Example of size distributions for dispersed particles and for flocculated sediments in suspension expressed as a volume fraction. The symbols are the measurements for San Francisco Bay sediments. The lines are the fits of Equation 2-3 distributions (adapted from Kranck and Milligan, 1992).</td>
<td>93</td>
</tr>
<tr>
<td>Figure 2-3</td>
<td>Conceptual model of order of aggregate flocculation processes (adapted from Krone, 1963).</td>
<td>94</td>
</tr>
<tr>
<td>Figure 2-4</td>
<td>Effects of the settling decay term, K, on the particle size distribution spectra in volume fraction (adapted from Kranck and Milligan, 1992).</td>
<td>95</td>
</tr>
<tr>
<td>Figure 2-5</td>
<td>Measured particle size distributions in San Francisco Bay: a) dispersed grain distributions and b) flocculated distribution (from Kranck and Milligan, 1992; reprinted with permission).</td>
<td>96</td>
</tr>
<tr>
<td>Figure 2-6</td>
<td>Dispersed particle distributions for nearby bed samples for San Francisco Bay site (solid) and for suspended sediments in San Pablo Strait. All distributions had a slope, m, near zero (from Kranck and Milligan, 1992; reprinted with permission).</td>
<td>96</td>
</tr>
<tr>
<td>Figure 2-7</td>
<td>Relationship of shear strength of flocs to the excess density of the flocs (data from Krone, 1963). The data values are from multiple harbors; curve fits are shown for each harbor, for all of the data combined. A curve fit with the exponent on the density of 2.5 (Partheniades, 1993) is shown in red.</td>
<td>97</td>
</tr>
<tr>
<td>Figure 2-8</td>
<td>Effect of salinity on settling velocity (adapted from Krone, 1962). Final peak floc size was estimated from Stokes Law using the final peak settling velocity.</td>
<td>98</td>
</tr>
<tr>
<td>Figure 2-9</td>
<td>Comparison of Equations 2-17 and 2-18 for the drag coefficient as a function of the particle Reynolds number.</td>
<td>99</td>
</tr>
<tr>
<td>Figure 2-10</td>
<td>Comparison of Equations 2-26 and 2-31 for settling velocity versus floc diameter (adapted from Winterwerp, 1999).</td>
<td>100</td>
</tr>
<tr>
<td>Figure 2-11</td>
<td>Settling velocity versus floc diameter from Chesapeake Bay and Tamar (UK) estuary (adapted from Winterwerp, 1999).</td>
<td>101</td>
</tr>
<tr>
<td>Figure 2-12</td>
<td>Settling velocity versus floc diameter from VIS, Ems ’89 and Ems ’90 (adapted from Winterwerp, 1999).</td>
<td>102</td>
</tr>
</tbody>
</table>
Figure 2-13. Comparison of fit curves for individual data sets of fall velocity with a single fit to all data. ..................................................................................................................................................103

Figure 2-14. Fractal dimension from Equation 2-19 used in Equation 2-31 as plotted in Figure 2-10; \( d_{fc} = 8000 \) microns, \( D_{fc}= 2.6 \), \( d_{50} = 2 \) microns. .................................................................104

Figure 2-15. Data comparison of model for settling velocity with a power law for the fractal dimension (from Khelifa and Hill, 2006, reprinted with permission). .........................................................105

Figure 2-16. Effects of variable fractal dimension on the effective (excess) density as a function of floc diameter (from Khelifa and Hill, 2006). ..............................................................106

Figure 2-17. Variation of settling velocity with suspended sediment concentration. Results based on field and laboratory tests using sediment from Cleveland Bay, Australia (adapted from Wolanski et al., 1992). ...........................................................................................................107

Figure 2-18. Schematic diagram showing the dependence of settling velocity on floc size in the flocculation settling range (adapted from Teeter, 2001)...108

Figure 2-19. Comparison of Equation 2-46 with settling data for varying initial concentration and shear rate (adapted from Teeter, 2001). .................................................................108

Figure 2-20. Paradox of simultaneous versus exclusive erosion and deposition: a) bed deposition, b) bed erosion. ..............................................................................................................109

Figure 2-21. Number of hours required to settle a distance of one meter as a function of particle size, based on the Equation 2-29 curve in Figure 2-10.................................110

Figure 2-22. Conceptual model of the feedback between aggregation and disaggregation with flow conditions (from Maggi, 2005; reprinted with permission). Top: shows the temporal variation in the number of flocs within a unit volume. As aggregation occurs the number of particles is reduced. With disaggregation the number of particles increases. Bottom: shows the floc size distribution (FSD), which reflects the aggregation/disaggregation cycling. .................................................................111

Figure 2-23. Relationship between the modal floc diameter and shear stress and concentration (from Dyer, 1989). ..............................................................................................................112

Figure 3-1. Conceptual view of CST processes.................................................................135

Figure 3-2. Fit of Equation 3-5 to sample data set (data from Obi, et al., 1996)...............136

Figure 3-3. Shear stress distributions used by Winterwerp and van Kesteren (2004) developed from analysis of Petit (1999). Parameters for the fits were developed by a two-parameter error minimization algorithm in applying Equation 3-7.........................137
Figure 3-4. Effect of normally distributed velocity distribution on the shear stress distribution based on a Monte Carlo simulation of Equation 3-10. The stochastic velocity distribution is confirmed against the analytical normal distribution.................138

Figure 3-5. Transformation of a mean 0.113 m/s normally distributed velocity with a standard deviation of 0.022 m/s to the shear stress distribution of Winterwerp and van Kesteren.....................................................................................................................139

Figure 3-6. Transformation of a mean 0.134 m/s normally distributed velocity with a standard deviation of 0.0243 m/s to the shear stress distribution of Winterwerp and van Kesteren.....................................................................................................................140

Figure 3-7. Transformation of a mean 0.152 m/s normally distributed velocity with a standard deviation of 0.026 m/s to the shear stress distribution of Winterwerp and van Kesteren.....................................................................................................................141

Figure 3-8. Transformation of 0.5 m/s mean velocity to shear stress for varying standard deviation in the normal velocity distribution. The solid symbols are the velocity distributions. The open symbols are the shear stress distributions with the same symbol as the associated velocity distribution.................................................................142

Figure 3-9. Transformation of 1.0 m/s mean velocity to shear stress for varying standard deviation in the normal velocity distribution. The solid symbols are the velocity distributions. The open symbols are the shear stress distributions, with the same symbol as the associated velocity distribution.............................................................................143

Figure 3-10. Transformation of 2.0 m/s mean velocity to shear stress for varying standard deviation in the normal velocity distribution. The solid symbols are the velocity distributions. The open symbols are the shear stress distributions, with the same symbol as the associated velocity distribution.............................................................................144

Figure 3-11. Ratio of the standard deviation of velocity to the mean velocity plotted against the ratio of the standard deviation of shear stress to the mean shear stress..................145

Figure 3-12. Data from Krone (1963) relating the floc strength to the order of aggregation of sediments from a number of harbors. ..................................................................................................................146

Figure 3-13. Data from Krone (1963) showing the effects of concentration on shear strength of flocs from a variety of harbors. ..................................................................................................................147

Figure 3-14. Variation in shear strength of naturally deposited cohesive bed as a function of the mean shear strength (data from Arulanandan, et al., 1980). The line is a regression fit to the log-transformed variables. .................................................................148

Figure 3-15. The probability of erosion when both the shear stress and the bed shear strength with respect to erosion are represented probabilistically. The CDF for the shear strength is used in the integral in Equation 3-18 in conjunction with the PDF for the shear stress............................................................................................................149
Figure 3-16. Integration of a probability distribution of shear stress for the case of a single-valued bed shear strength with respect to erosion. The PDF for the shear strength is a delta function and the CDF is a heaviside function.

Figure 3-17. Shear stress standard deviation, $\sigma_b$, effect on the probability of erosion for the case of $\tau_s = 1$ Pa and $\sigma_s = 0.25$ Pa.

Figure 3-18. Shear strength standard deviation, $\sigma_s$, effect on the probability of erosion for the case of $\tau_s = 1$ Pa and $\sigma_b / \tau_b = 0.30$.

Figure 3-19. Effect of the variable $\tau_s$ on the probability of erosion for the case of $\sigma_s = 25\%$ and $\sigma_b / \tau_b = 0.30$.

Figure 3-20. Variation of erosion flux with shear stress for data in Long Island Sound (after Wang, 2003). Variation is in part due to differing tidal conditions and wave energy, with the largest erosion associated with storms.

Figure 3-21. The variation of the probability of erosion and deposition with the mean shear stress for $\tau_s = 0.5$ Pa, $\sigma_s = 25\%$ and $\sigma_b = 30\%$.

Figure 3-22. Probability threshold greater than 0.5 (0.75) used for definition of critical shear stresses leads to a critical shear stress or erosion greater than the critical shear stress for deposition.

Figure 3-23. Probability threshold less than 0.5 (0.33) used for definition of critical shear stresses leads to a critical shear stress or erosion less than the critical shear stress for deposition.

Figure 4-1. Comparison of Equations 4-32 and 4-33 for erosion rate with data from Partheniades (1965), using the coefficient $C_b = 0.8459$ kg/m$^3$ in Equation 4-32, $C_b = 0.112$ kg/m$^3$ in Equation 4-33, $\rho_w = 1030$ kg/m$^3$, $f_s = 1$, and the variables developing the probability of erosion $\tau_s = 0.55$ Pa, $\sigma_s = 0.25$ Pa and $\sigma_b = 0.3$ Pa.

Figure 4-2. Collision frequency for a particle diameter of 10.6 microns with variable second particle diameter. The flow conditions for this case are a flow depth of 0.3048 m, with a depth-averaged velocity of 0.142 m/s. The probabilistic settling velocity cases assumed a 30 percent standard deviation in the settling velocity.

Figure 4-3. Computer program “COHPROB” flow chart (MAIN) for phase 1.

Figure 4-4. Phase 2 computer program flow chart for “COHPROB” (MAIN); spin up of the hydrodynamic model.

Figure 4-5. Phase 3 (sediment transport) flow chart of “COHPROB” (MAIN).

Figure 4-6. Self-similar velocity distribution solution for the Stokes first problem.

Figure 4-7. Results of 80-cell resolution over domain for simulation of Stokes first problem.
Figure 4-8. Results of 40-cell resolution over domain for simulation of Stokes first problem. 208

Figure 4-9. Results of 20-cell resolution over domain for simulation of Stokes first problem. 208

Figure 4-10. Results of 10-cell resolution over domain for simulation of Stokes first problem. 209

Figure 4-11. Nondimensional velocity distribution for 80 cell simulation of Stokes first problem. 209

Figure 4-12. Nondimensional velocity distribution for 40-cell simulation of Stokes first problem. 210

Figure 4-13. Nondimensional velocity distribution for 20-cell simulation of Stokes first problem. 210

Figure 4-14. Nondimensional velocity distribution for 10-cell simulation of Stokes first problem. 211

Figure 4-15. Effects of suspended sediment concentration of 20 kg/m³ on Stokes first problem solution. The clear symbols are for no sediment and the blackened symbols are for the sediment-laden case. 212

Figure 4-16. Effects of suspended sediment concentration of 100 kg/m³ on Stokes first problem solution. The clear symbols are for no sediment and the blackened symbols are for a sediment concentration of 100 kg/m³. 213

Figure 4-17. Analytical velocity distribution solution of the Couette flow problem for various nondimensional time scales, tₚ. 214

Figure 4-18. Comparison of simulation of Couette flow problem with 80 cells to the analytical solution. 215

Figure 4-19. Comparison of simulation of Couette flow problem with 40 cells to the analytical solution. 216

Figure 4-20. Comparison of simulation of Couette flow problem with 20 cells to the analytical solution. 217

Figure 4-21. Comparison of simulation of Couette flow problem with 10 cells to the analytical solution. 218

Figure 4-22. Effects of 20 kg/m³ suspended sediment concentration on the Couette flow problem. The time scale for the sediment laden flow was computed with the clear water viscosity to show the effects. 219
Figure 4-23. Effects of 100 kg/m³ suspended sediment concentration on the Couette flow problem. The time scale for the sediment laden flow was computed with the clear water viscosity to show the effects.

Figure 4-24. Analytical solution of Stokes second problem (Equation 4-91). The dashed red envelopes are the bounding curve for the amplitude of the damped harmonic oscillation.

Figure 4-25. Simulation with 80 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.

Figure 4-26. Simulation with 40 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.

Figure 4-27. Simulation with 20 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.

Figure 4-28. Simulation with 10 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.

Figure 4-29. Effects of 20 kg/m³ suspended sediment concentration on the results of Stokes second problem for the 40-cell test case. The value of $\eta$ for the sediment laden flow was computed with the clear water viscosity to show the effects.

Figure 4-30. Effects of 100 kg/m³ suspended sediment concentration on the results of Stokes second problem for the 40-cell test case. The value of $\eta$ for the sediment laden flow was computed with the clear water viscosity to show the effects.

Figure 4-31. Temporal development of velocity profile using von Karman mixing length test case.

Figure 4-32. Comparison of simulated fully developed velocity profile to the analytical solution, using 80 cells over water column. Also plotted is the shear stress distribution over the water column from the simulation.

Figure 5-1. Kynch (1952) settling test development. a) initial uniform dilute suspension in a quiescent setting column, b) the secondary lutocline (S) settles at a rate $w_s$ while the isopycnal interface defining the primary lutocline (P) rises from the bed at a rate $w_p$, c) isopycnal primary lutocline meets the secondary lutocline, and d) the final deposit concentration $C_f$ and height $h_f$ are reached after a period of hindered settling.

Figure 5-3. Kynch (1952) settling test evolution of the secondary lutocline elevation. Application of the method of characteristics to estimate the suspended sediment concentrations (after Mehta, 2007).

Figure 5-4. Estimated concentration profiles using the Kynch graphical method based on the method of characteristics.
Figure 5-5. Deposition test results from Krone (1962)...............................................................244

Figure 5-6. Results of the tagged sediment experiment of Krone (1962).................................245

Figure 5-7. Results from Mehta (1973) showing the effects of shear stress on the relative concentration. Initial concentrations were 1.0 kg/m³.................................................................246

Figure 5-8. Results of the flow volume replacement experiment by Parchure (1985)..............247

Figure 5-10. Suspended sediment concentration and shear stress during monitoring exercise on 5 January 1989 (from Sanford and Halka, 1993)......................................................249

Figure 5-11. Suspended sediment concentration and shear stress during monitoring exercise on 2 February 1990 (from Sanford and Halka, 1993)......................................................250

Figure 5-12. Suspended sediment concentration and shear stress during monitoring exercise on 15 January 1991 (from Sanford and Halka, 1993)......................................................251

Figure 6-1. Example of fall velocity estimates from video analysis of in situ sediments in San Francisco Bay (data from Smith, 2007).........................................................................286

Figure 6-2. Results of simulation of the Kynch (1952) test case using a hindered settling exponent of $m=1$ in Equation 2-40. ......................................................................................287

Figure 6-3. Results of simulation of the Kynch (1952) test case using a hindered settling exponent of $m=2$ in Equation 2-40. ......................................................................................288

Figure 6-5. Example representation of shear strength of flocs, critical shear stress for erosion and critical shear stress for deposition as functions of particle size. Shear strength defined by Equation 3-11, with $B_f = 1200$ Pa and $D_f = 2.6$. Critical shear stresses for erosion and deposition are defined by Equation 2-53. The critical shear stress for deposition is based on $\tau_{d0} = 0.01$ Pa, $d_{ref} = 0.1$ microns and $\delta = 0.5$.................290

Figure 6-7. Specification development for Mehta (1973) for use of the mean values in the classical erosion and deposition exclusive paradigm.......................................................292

Figure 6-9. Simulations of Mehta (1973) experimental tests for shear stresses of 0.25 Pa, 0.40 Pa, 0.60 Pa and 0.85 Pa, and using classical erosion probabilities with simultaneous deposition.........................................................294

Figure 6-11. Simulations of Mehta (1973) test cases for shear stress of 0.40 Pa using combinations of probabilistic versus mean-valued depositional and erosion treatment, with either simultaneous or exclusive erosion/deposition. ............................................296

Figure 6-12. Schematic representation of a probabilistic representation for the Mehta 1973 0.25 Pa test. The shaded area is the zone of deposition from the average value analysis. The differences are conceptual only since the displayed range of values is only +/- 1 standard deviation for each variable ($\sigma_{ce}$, $\sigma_{cd}$, $\sigma_b$). .................................................................297
Figure 6-13. Particle size distribution comparison between the mean value simulation and the probabilistic simulation for the 0.25 Pa Mehta (1973) test. Both simulations used the same sediment characteristics and critical shear stresses and the exclusive paradigm. ........................................................................................................................298

Figure 6-14. Simulation using the classical exclusive bed exchange processes using mean values for the Krone (1962) deposition test with a shear stress of 0.0305 Pa. One sensitivity simulation was made with an added supplemental internal shear of 100 Hz.....................................................................................................................................299

Figure 6-15. Simulation using the classical bed exchange processes using mean values for the Krone (1962) deposition test with a shear stress of 0.0415 Pa. .........................................................................................300

Figure 6-16. Simulation using the classical bed exchange processes using mean values for the Krone (1962) deposition test with a shear stress of 0.0515 Pa. .........................................................................................301

Figure 6-17. Illustration of the change in depositional response in the Krone (1962) recirculating flume tests when concentrations fall below approximately 0.3 kg/m$^3$. .........................................................302

Figure 6-18. Model simulation to test the dilution rate for the Parchure and Mehta (1985) dilution test.................................................................303

Figure 6-19. Initial particle size concentration distribution for bed initialization for the Parchure and Mehta (1985) dilution experiment. The simulation started with no sediment in suspension and then eroded the bed to an equilibrium concentration..............304

Figure 6-20. Parchure and Mehta (1985) dilution test results with bed exchange included, with the classical excess shear stress exclusive formulation and an exclusive simulation using probabilistic treatment of the key parameters. ........................................................................305

Figure 6-21. Variation of floc size distribution during the Parchure and Mehta (1985) dilution test......................................................................................................................................................306

Figure 6-22. Initial cohesive bed particle concentration for the Parchure and Mehta (1985) erosion test.................................................................307

Figure 6-23. Results of simulation of an erosion test (Parchure and Mehta, 1985) with a progressive increase in shear stress using the exclusive erosion/deposition and mean values. .................................................................................................................................308

Figure 6-24. Effects of switching from classically exclusive mean-value calibrated bed exchange to an exclusive/probabilistic treatment without parameter adjustments. ...............309

Figure 6-25. Variation in water depth during Sanford and Halka (1993) field test.................................................................310

Figure 6-26. Results of calibrating continuous deposition for use on an average shear stress and using the probabilistic model of bed exchange: 5 January 1989. ..................................................311
Figure 6-27. Results of calibrating continuous deposition for use on an average shear stress and using the probabilistic model of bed exchange: 2 February 1990. ..........................312

Figure 6-28. Results of calibrating continuous deposition for use on an average shear stress and using the probabilistic model of bed exchange: 15 January 1991. ..........................313

Figure 6-29. Evolution of floc size distribution during numerical simulation of the Sanford and Halka (1993) data set of 5 January 1989. Both tests used the continuous deposition bed model. The black distributions are for the use of the average bottom shear stress, while the red curves are for the probabilistic shear stress formulation. (hours refer to Figure 6-26)..............................................................................................314

Figure 6-30. Evolution of floc size distribution during numerical simulation of the Sanford and Halka (1993) data set of 5 January 1989. Both tests used the exclusive erosion-deposition bed model. The black distributions are for the use of the average bottom shear stress, while the red curves are for the probabilistic shear stress formulation. (hours refer to Figure 6-26)..............................................................................................315

Figure 6-31. Results of simulations showing the effects of the continuous deposition and probabilistic bed exchange treatment 5 January 1989. All other model variables are held the same....................................................................................................................316

Figure 6-32. Results of simulations showing the effects of the continuous deposition and probabilistic bed exchange treatment 2 February 1990. All other model variables are held the same....................................................................................................................317

Figure 6-33. Results of simulations showing the effects of the continuous deposition and probabilistic bed exchange treatment 15 January 1991. All other model variables are held the same....................................................................................................................318

Figure 6-34. Results of simulation of the Krone gold-tagged sediment experiment using the simultaneous erosion and deposition treatment of the mean variables.................................319

Figure D-1. Particle size distribution for example video analysis. .........................................................406
NOTATION

\(a_f\) = coefficient in settling velocity equation \((\text{m/s})\)

\(a_n\) = scale factor for exponent in the hindered settling equation \((-\))

\(A\) = coefficient on diameter squared in Stokes’ settling law \((\text{m}^{-1}\text{s}^{-1})\)

\(A\) = a generic variable or constant (varies)

\(b_f\) = hindered settling coefficient in settling velocity equation \((\text{kg/m}^3)\)

\(b_n\) = exponent in the equation for exponent in the hindered settling equation \((-\))

\(B_f\) = floc strength scale factor \((\text{Pa})\)

\(c_i\) = concentration of indexed cohesive sediment class \(i\) \((\text{kg/m}^3)\)

\(c_{si}\) = concentration of indexed silt sediment class \(i\) \((\text{kg/m}^3)\)

\(c_{1\varepsilon}\) = coefficient on production term in turbulent dissipation equation \((-\))

\(c_{2\varepsilon}\) = coefficient on dissipation term in turbulent dissipation equation \((-\))

\(c_{3\varepsilon}\) = coefficient on buoyancy term in turbulent dissipation equation \((-\))

\(c_{\mu}\) = scaling coefficient on relationship of eddy viscosity to turbulence parameters \((-\))

\(C\) = total sediment concentration, mass per unit volume \((\text{kg/m}^3, \text{mg/l})\)

\(C_D\) = coefficient of drag \((-\))

\(C_L\) = concentration of sediment below the lutocline \((\text{kg/m}^3)\)

\(C_0\) = initial or reference concentration \((\text{kg/m}^3)\)

\(d\) = sediment or particle diameter \((\text{m, or } \mu\text{m})\)

\(d_f\) = floc diameter \((\text{m, or } \mu\text{m})\)

\(d_f\) = floc diameter for class \(i\) \((\text{m, or } \mu\text{m})\)

\(d_c\) = reference floc diameter \((\text{m, or } \mu\text{m})\)

\(d_i\) = indexed diameter \((\text{m, or } \mu\text{m})\)
\(d_p\) = primary grain diameter (m, or \(\mu m\))

\(d_{50}\) = median grain diameter (m, or \(\mu m\))

\(D\) = general collection term for dissipative terms in a generic balance equation (varies)

\(D\) = used as indicator of total derivative (-)

\(D\) = depositional flux (kg m\(^{-2}\)s\(^{-1}\))

\(D_B\) = diffusion associated with Brownian motion (m\(^3\)/s)

\(D_f\) = fractal dimension (-)

\(D_{fc}\) = reference fractal dimension (-)

\(D_i\) = dispersion coefficient in tensor notation (m\(^2\)/s)

\(D_i\) = depositional flux for sediment class \(i\) (kg m\(^{-2}\)s\(^{-1}\))

\(D_m\) = molecular diffusivity (m\(^2\)/s)

\(D_t\) = turbulent diffusivity (m\(^2\)/s)

\(D_{ti}\) = sediment class \(i\) dependent turbulent dispersion coefficient (m\(^2\)/s)

\(D_x\) = dispersion coefficient in x-direction (m\(^2\)/s)

\(D_y\) = dispersion coefficient in y-direction (m\(^2\)/s)

\(D_z\) = dispersion coefficient in z-direction (m\(^2\)/s)

\(e\) = exponential constant = 2.718281828… (-)

\(E\) = expected value of any variable (varies)

\(E\) = erosion flux (kg m\(^{-2}\)s\(^{-1}\))

\(E_i\) = erosion flux for class \(i\) (kg m\(^{-2}\)s\(^{-1}\))

\(E_f\) = floc erosion flux (kg m\(^{-2}\)s\(^{-1}\))

\(E_v\) = energy required to disperse unit volume of aggregate (N-m, joules)

\(f\) = probability density function of variable A (units of A\(^{-1}\))
$f_c =$ friction coefficient (-)

$f_c =$ damping factor for rate of turbulent kinetic energy dissipation near bottom (-)

$f_\mu =$ damping factor for turbulent viscosity near bottom (-)

$F_A =$ cumulative probability function of the variable A (-)

$F_C =$ combined buoyancy and cohesive forces in mobility analysis (N)

$F_C =$ collision diameter function (-)

$F_i =$ applied forces per unit volume on the fluid parcel (N/m$^3$)

$F_i =$ depositional flux for sediment class $i$ (kg/m-s)

$F_{i,v} =$ applied viscous forces on the fluid parcel (Pa)

$F_L =$ hydrodynamic lift (N)

$F_P =$ coefficient of relative depth of penetration during particle collision (-)

$F_s =$ force of shear (Pa)

$F_s =$ local settling flux (kg m$^{-2}$s$^{-1}$)

$g =$ acceleration of gravity (m/s$^2$)

$g_i =$ acceleration of gravity tensor (m/s$^2$)

$G =$ internal shear (Hz)

$G_{iA} =$ gain of flocs to class $i$ due to aggregation (-)

$G_{iB} =$ gain of flocs to class $i$ due to shear breaking of flocs (-)

$G_{iC} =$ gain of flocs to class $i$ due to collision breaking of flocs (-)

$h =$ local water depth (m)

$h_L =$ height of lutocline (m)

$H =$ Heaviside function (-)

$K =$ decay coefficient for settling in partial size distribution model (s/m)
$K$ = relational coefficient for floc shear strength (varies)

$K_f$ = relational coefficient for shear strength as a function of excess density (varies)

$k$ = counting index (-)

$k$ = turbulent kinetic energy (m$^2$/s$^2$)

$k_b$ = Boltzmann’s constant (ergs/oK)

$k_f$ = correlation constant for computing shear strength based on concentration (Pa kg$^{-5/2}$m$^{15/2}$)

$k_s$ = roughness height (m)

$L_{iA}$ = loss of flocs from class $i$ due to aggregation (-)

$L_{iB}$ = loss of flocs from class $i$ due to shear breaking of flocs (-)

$L_{iC}$ = loss of flocs from class $i$ due to collision breaking of flocs (-)

$m$ = exponent in particle size distribution model (-)

$m$ = modal factor in the shear stress distribution model (Pa)

$m_{eq}$ = milliequivalents

$m_f$ = hindered settling exponent (-)

$m_i$ = mass of particles in class $i$ (kg)

$m_i$ =i-th moment of the shear stress distribution (m$^1$)

$m_1$ = number of primary grains in primary floc (-)

$m_2$ = size increase on flocculation order increase (-)

$M$ = erosion rate constant (kg/m$^2$/s)

$M_{class}$ = number of cohesive size classes (-)

$M_i$ = i-th moment of floc distribution (m$^3$)

$M_L$ = mass of suspended sediment below lutocline (kg/m$^2$)

$n$ = fluid porosity, volume of fluid per unit volume (-)

$n$ = aggregation number for higher floc, order of aggregate = n-1 (-)
\( n_i = \) number concentration of particles of class \( i \) (m\(^{-3}\))

\( n_f = \) exponent for concentration effects on flocculation settling (-)

\( N = \) number of aggregate bonds (-)

\( N_{\text{class}} = \) number of noncohesive size classes (-)

\( N_f = \) number of primary grains in a floc (-)

\( N_{ij} = \) number of new flocs created by aggregation (-)

\( N_z = \) number of cells in the vertical (-)

\( p = \) pressure, force per unit area (Pa)

\( P = \) probability (-)

\( P = \) general collection term for production terms in a generic balance equation (varies)

\( P_d = \) probability of deposition (-)

\( P_e = \) probability of erosion (-)

\( q = \) extraction rate (m\(^3\)/s/m\(^2\))

\( Q = \) concentration scale for particle size distribution (ppm)

\( R\text{-squared} = \) sum of regression squared residuals (-)

\( Re_{ws} = \) particle size Reynolds number (-)

\( Re_t = \) turbulence Reynolds number (-)

\( R_{ij} = \) Reynolds stresses (m\(^2\)/s\(^2\))

\( R_{ijk} = \) third order moment Reynolds stresses (m\(^3\)/s\(^3\))

\( s = \) skewness factor in the shear stress distribution function (Pa)

\( s = \) specific gravity of sediment particles (-)

\( S = \) salinity of the fluid, practical salinity units (psu)

\( S = \) slope of energy grade line (-)
$S_i$ = net source of sediment into size class $i$ (kg/m$^3$/s)

$S_{ij}$ = deviator stress tensor (Pa)

$S_0$ = Reference salinity (psu)

$t$ = time (s)

$t_{ij}$ = kinematic Reynolds mean stress tensor (m$^2$/s$^2$)

$t_s$ = dimensionless time scale (-)

$t_d$ = hydrodynamic mean flow time scale (s)

$T$ = temperature, degrees Celsius unless otherwise specified ($^\circ$C)

$T_L$ = turbulence eddy timescale (s)

$T_P$ = particle response to turbulence timescale (s)

$T_{ss}$ = process time scale (s)

$u$, $v$, $w$ = Cartesian components of water velocity in $x$, $y$, $z$ directions, respectively (m/s)

$u_i$ = Cartesian components of water velocity using tensor notation (m/s)

$u_{ip}$ = effective sediment velocity for total sediment concentration for tensor direction $i$ (m/s)

$u_{id}$ = differential sediment velocity relative to fluid velocity for total sediment concentration for tensor direction $i$ (m/s)

$u_{mi}$ = sediment particle velocity for sediment class $m$ for tensor direction $i$ (m/s)

$u_*$ = shear velocity (m/s)

$V_i$ = volume of particle in class $i$ (m$^3$ or $\mu$m$^3$)

$w_p$ = rate of rise of primary lutocline (m/s)

$w_s$ = sediment particle fall velocity (m/s)

$w_{seff}$ = effective settling velocity (m/s)

$w_{sm}$ = peak flocculation settling velocity at the beginning of hindered settling (m/s)
$x, y, z =$ Cartesian coordinates, $z$ vertical (m)

$x_i =$ Cartesian coordinates in tensor notation (m)

$y =$ probability density distribution of shear stress (Pa$^{-1}$)

Greek Symbols

$\alpha =$ generic scaling coefficient (varies)

$\alpha =$ scale factor for fractal dimension (-)

$\alpha_a =$ aggregation efficiency (-)

$\alpha_c =$ efficiency of collision-induced floc breakage (-)

$\alpha_d =$ disaggregation efficiency (-)

$\alpha_t =$ time scale scaling factor (-)

$\alpha_1 =$ coefficient in flux coupling function (-)

$\alpha_1 =$ mass factor coefficient (kg/m$^3$)

$\alpha_2 =$ flux limiting concentration in flux coupling function (kg/m$^3$)

$\alpha_2 =$ area shape factor

$\beta =$ class size progression factor (-)

$\beta =$ fractal dimension exponent (-)

$\beta =$ drag force asymmetry factor (-)

$\beta_B =$ collision frequency due to Brownian motion (-)

$\beta_D =$ collision frequency due to differential settling (-)

$\beta_d =$ class size progression factor for particle diameter (-)

$\beta_m =$ class size progression factor for particle mass (-)

$\beta_T =$ collision frequency due to shear (-)
\( \gamma \) = natural logarithm of the Reynolds number, \( ln(Re) \) (-)  

\( \delta \) = boundary layer thickness (m)  

\( \delta \) = exponent in critical shear stress for deposition relationship (-)  

\( \delta_{ij} \) = Kronecker delta, equals 1 when \( i = j \); zero otherwise (-)  

\( \delta_i \) = scaling coefficient for depositional flux coupling (-)  

\( \delta_2 \) = bounding concentration control for depositional flux coupling (kg/m³)  

\( \Delta \rho_f \) = excess floc density (kg/m³)  

\( \Delta \rho_s \) = excess sediment grain density (kg/m³)  

\( \Delta t \) = time step of numerical simulation (s)  

\( \Delta z \) = computational discretization cell depth (m)  

\( \varepsilon \) = dissipation rate of turbulent kinetic energy (m²/s³)  

\( \varepsilon \) = erosion flux (kg/m²/s)  

\( \varepsilon_i \) = voids ratio of floc (-)  

\( \zeta \) = kinematic bulk viscosity of the fluid (m²/s)  

\( \eta \) = dimensionless distance (-)  

\( \theta \) = generic variable (varies)  

\( \kappa \) = dynamic bulk viscosity of the fluid (kg/m/s²)  

\( \kappa \) = von Karman coefficient (-)  

\( \lambda \) = Taylor microscale (m)  

\( \lambda_i \) = aggregation mass distribution factor (-)  

\( \lambda_0 \) = Kolmogorov eddy length scale (m)  

\( \lambda_i \) = coefficient of flocculation effects of shear on settling velocity (s)
\( \lambda_2 \) = coefficient of disaggregation effects of shear on settling velocity \((s^2)\)

\( \lambda_3 \) = decay coefficient of shear effects on settling velocity with increasing concentration \((-)\)

\( \mu \) = dynamic shear viscosity of the fluid \((kg/m/s^2)\)

\( \mu \) = mean shear stress in shear stress distribution function \((Pa)\)

\( \nu \) = kinematic fluid viscosity \((m^2/s)\)

\( \nu_t \) = kinematic turbulent fluid viscosity \((m^2/s)\)

\( \xi \) = representation of generic turbulent variable

\( \pi \) = ratio of circular circumference to its diameter \((-)\)

\( \rho \) = fluid composite density, mass per unit volume \((kg/m^3)\)

\( \rho_f \) = floc density, mass per unit volume \((kg/m^3)\)

\( \rho_s \) = density of sediment particles or flocs, mass per unit volume \((kg/m^3)\)

\( \rho_w \) = clear fluid density, mass per unit volume \((kg/m^3)\)

\( \rho_{\Delta r} \) = normalized autocorrelation of the turbulent velocity perturbation \((-)\)

\( \sigma_b \) = standard deviation of the bottom shear \((Pa)\)

\( \sigma_c \) = correlation scale between turbulent viscosity and concentration diffusivity \((-)\)

\( \sigma_{ce} \) = standard deviation of the critical shear stress for erosion \((Pa)\)

\( \sigma_{cd} \) = standard deviation of the critical shear stress for deposition \((Pa)\)

\( \sigma_k, \sigma_\varepsilon \) = standard deviations of turbulence variables \((m^2/s^2, m^2/s^3)\)

\( \sigma_y \) = viscous stress tensor, force per unit area \((Pa)\)

\( \sigma_L \) = standard deviation of the lift force on a particle \((N)\)

\( \sigma_T \) = correlation scale between turbulent viscosity and concentration diffusivity \((-)\)
\( \sigma_v \) = standard deviation of the velocity (m/s)

\( \sigma_e \) = correlation scale between turbulent viscosity and diffusivity of dissipation rate (-)

\( \sigma_t \) = standard deviation of the shear stress (Pa)

\( \tau \) = shear stress, force per unit area (Pa)

\( \tau_b \) = bottom shear stress, force per unit area (Pa)

\( \tau_d \) = critical shear stress for deposition, force per unit area (Pa)

\( \tau_{d0} \) = reference critical shear stress for deposition (Pa)

\( \tau_e \) = critical shear stress for erosion, force per unit area (Pa)

\( \tau_f \) = shear strength of flocs, force per unit area (Pa)

\( \tau_{ij} \) = shear stress in the i-normal face acting in the j direction, (Pa)

\( \tau_{ijk} \) = shear stress in the i-normal face acting in the j direction (Pa)

\( \tau_y \) = yield stress of flocs (Pa)

\( \phi \) = volumetric concentration (-)

\( \phi_f \) = volumetric concentration of floc (-)

\( \phi_s \) = volumetric concentration of sediment grains (-)

\( \phi_p \) = volumetric concentration of primary aggregates (-)

\( \phi_{fma} \) = volumetric concentration of first order flocs (-)

\( \phi_s \) = space filling volumetric concentration (-)

\( \psi \) = floc density factor for nonuniform primary particles (-)

\( \omega \) = angular frequency (s\(^{-1}\))

\( \omega \) = dummy of integration (varies)
Subscripting

\(i, j, k, \) or \(m\) = Cartesian direction tensor indices (-)
\(i, j, k\) = particle size class indices (-)
\(k\) = spatial discretization reference (-)
\(\xi\) = representation of generic turbulent variable
\(\bar{\xi}\) = temporal average of generic variable
\(\bar{\xi'}\) = mass-weighted temporal average of generic variable
\(\xi'\) = turbulent perturbation from the conventional temporal average
\(\xi''\) = turbulent perturbation from the mass-weighted temporal average
\(\bar{\xi}\) = non-dimensional generic variable

ACRONYMS

ASM algebraic second-moment closure model
CCA cluster-cluster aggregation
CEC cation exchange capacity
CST cohesive sediment transport
DLA diffusion-limited aggregation
DLCCA diffusion limited cluster-cluster aggregation
DSM differential second-moment closure model
EVM eddy viscosity/diffusivity model
FSD floc size distribution
MIT Massachusetts Institute of Technology
RANS Reynolds-averaged Navier-Stokes
RLCCA reaction limited cluster-cluster aggregation
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>SAR</td>
<td>sodium adsorption ratio</td>
</tr>
<tr>
<td>SMC</td>
<td>second-moment closure model</td>
</tr>
<tr>
<td>TKE</td>
<td>turbulent kinetic energy</td>
</tr>
<tr>
<td>USACE</td>
<td>United States Army Corps of Engineers</td>
</tr>
</tbody>
</table>
The primary issue addressed in this study is whether the decades old paradigm of exclusive erosion or deposition in turbulent flow has legitimacy based on physical principles within cohesive sediment dynamics. The exclusive paradigm assumes that sediment exchange condition at the bed-water interface is either erosion, deposition or neither, but never both. In contrast, the more recently espoused simultaneous exchange paradigm admits the possibility of erosion and deposition of cohesive sediment occurring at the same time.

The exclusive paradigm is, in part, the result of early attempts to understand basic cohesive sediment transport behavior based on inferred data in laboratory apparatuses such as flumes averaged over time and space. The time scale of averaging is longer than the time scale of turbulence and the spatial dimension is scaled by water depth in the apparatus. Bed sediment exchange has been deduced primarily from the increase or reduction in the suspended sediment concentration within the water column, rather then from difficult to record observations of particle movement very close to the bed surface. The net result of averaging will be positive, negative or zero sediment flux at the bed surface, but not both positive and negative.
With the inclusion of greater details in newer mathematical models, such as particle size
distributions and flocculation sub-models, the bed exchange algorithms have required revision.
Numerical modelers have found the need to use the simultaneous approach to replicate observed
sedimentation rates in the field environment.

The numerical sediment transport tool developed for this research has been shown to be
capable of simulating several processes critical for simulation of bed exchange. These processes
include aggregation and disaggregation dynamics, stochastic effects in bed exchange and
aggregation/disaggregation, hindered settling, attainment of a depositional or erosional
equilibrium concentration for a fixed shear stress, and floc spectrum features documented by
field experimentation.

Observations made during development and application of the numerical tool are:

- The effects of a probabilistic treatment of the key variables are more pronounced for
  erosion than for deposition. These variables include current velocity, bottom shear stress,
  floc shear strength, critical shear stresses for erosion and deposition, internal shear and
  settling velocity.

- Probabilistic effects are amplified through the flocculation model over the effects that
  occur through bed exchange alone.

- For a given shear stress the flocculation model will tend toward an equilibrium distribution
  of particle sizes.

- The probabilistic treatment results in a broader floc distribution spectrum than occurs with
  use of mean-valued variables.

- Deposition or erosion will be initiated sooner and transition from one to the other more
  gradual in response to changing shear stress when a probabilistic treatment is used
  compared to a mean-valued treatment. The differential timing will be a function of the
  standard deviations of the probabilistic variables and the rate of change of the shear stress.

- The use of the exclusive paradigm with a floc size distribution can perform as well as a
  simultaneous treatment with a single particle size.

- A simulation was performed of a flume test by Parchure and Mehta (1985) designed to
  evaluate the exclusive versus simultaneous paradigm by diluting the concentration of a
  flume suspension that had achieved an equilibrium concentration from bed erosion. If the
exclusive paradigm was valid, the concentration at the end of dilution should remain constant. If the concentration began to rise after dilution was ceased, then the simultaneous paradigm would be an explanation. The flume concentration did rise after the dilution stopped, but at a very low rate of erosion. The numerical model was able to replicate the flume behavior with the correct rate of rise after the end of dilution by using the exclusive paradigm with a probabilistic treatment of the variables.

- The appropriate use of either the exclusive or continuous paradigm appears to be dictated by the level of temporal and spatial averaging used in the development of empirical data and in the formulation of the variables in the analysis.

Empirical coefficients developed for mean-valued analysis may require adjustment when used in a probabilistic treatment.
CHAPTER 1
INTRODUCTION

1.1 Need for Research

The state of knowledge in cohesive sediment transport (CST) is the result of an evolving philosophical approach to CST research. The primary motivation for research has been driven by the need to better evaluate impacts of cohesive sediments on maintenance of estuarine harbors and on sensitive environmental areas (US Army Engineer Committee on Tidal Hydraulics, 1963). Research needs have expanded in the last several decades to include cohesive sediments as a valuable resource in coastal restoration and erosion control. Coastal Louisiana is nearing a crisis over accelerating land loss associated with global eustatic sea level rise (Day, et. al., 1995), aggravated by local subsidence of land mass due to subsurface sediment compaction, tectonic down-warping and oil and gas withdrawals (Gagliano, 1981; Boesch, et. al., 1994; Day, et. al., 2000). Management of sediment supplies of the Mississippi and Atchafalaya Rivers has been identified as critical to mitigating land loss (DeLaune, et. al., 1992; USACE, 2004).

Estuarine sedimentation is complex on a number of levels (Dyer, 1989). Estuaries are characterized by significant variability in sediment supply and local transport and deposition. A typical estuary may have a wide range of sediment types, from coarse sand to clay, depending on the local sediment supply and the hydrodynamic environment. These sediment mixing zones vary depending on the temporal variability of hydrodynamics, and therefore in sediment supply. As river discharge and tidal conditions vary, the character of the local sediment supply and transport processes may vary significantly. A river delta is a vivid example of fluctuating spatial deposition characteristics between clays at low flows that deposit a broad expanse of shallow pro-delta clays to sands at high river discharges resulting in the formation of deltaic lobes.
(Roberts, 1997). Strong variation in tide range during the spring neap monthly cycle may result in substantial variation in sediment mobility, dominating the morphology of the estuary.

Cohesive sediment processes often dominate estuarine sedimentation. Much of the research in sediment transport for non-cohesive sediment has focused on transport under steady flows, attempting to develop equilibrium transport relationships for each flow condition. Much of estuarine cohesive research has focused on the rates of erosion or deposition for a given flow condition (Krone, 1962; Partheniades 1962) rather than equilibrium conditions, which are elusive for cohesive transport. Most cohesive sediment research has also been under uniform steady flow conditions. Research has attempted to define macroscale hydrodynamic conditions, characterized by current velocity and bottom shear stresses under which erosion and deposition will occur.

This paradigm of seeking conditions of either erosion or deposition has had a strong influence on the conceptual model of cohesive bed exchange with suspended sediments. As long as the mathematical treatment of the processes remained macroscale in spatial and temporal discretization, for example focusing on the total suspended concentration, the macroscale incorporation of a mean settling velocity is consistent with a bulk treatment of erosion and deposition. Within steady uniform flume test conditions these bulk representations of bed exchange fit the macroscale variables well and reinforce the conceptual model of exclusive erosion and deposition.

As treatment of the cohesive sedimentation has evolved to the discretization of the particle size distribution and the associated variation in the settling velocities within complex flocculation models, the paradigm of exclusive erosion or deposition has been challenged. If the differential response of varying particle and floc sizes to hydrodynamics within a flocculation
model is to be consistently formulated with the bottom bed exchange, then it may require the admisssibility of simultaneous erosion and deposition. This need is further stressed by the differential response of size classes to temporal and spatial variations in hydrodynamics in estuaries.

The general processes that need to be addressed further in order to better deal with estuarine cohesive sediment transport are:

- Sediment aggregation (floc growth)
- Sediment disaggregation (floc breakup)
- Sediment settling velocity
- Deposited sediment consolidation
- Fluid mud formation
- Sediment resuspension
- Shear strength of settled bed

The growth and breakup of flocs directly impact settling velocities through changes in both size and density of the flocs. This extends to the bed consolidation rate by affecting the initial structure of the bed and the rate of dewatering. Resuspension of sediment from an unconsolidated bed is partially due to breakup of flocs in the bed as well as simple reentrainment. The application of modest shear to a partially consolidated bed can strengthen the bed by shifting the flocs in a manner that reinforces inter-floc bonds within the bed.

When the further complexities of unsteady and non-uniform flows are added these additional processes may need to be addressed:

- Sediment sorting due to spatial variability in hydrodynamics
- Differential mobility of various sediment classes
- Fluid mud flows
- Wave-induced effects
- Modifications to each of the above processes due to temporal and spatial gradients in both hydrodynamics and sediment concentration

Field observations have provided the motivation for historical research. Each field data set, however, is constrained in its usefulness to the processes that were monitored and the accuracy of those measurements. The complexity of environmental conditions experienced in field monitoring programs makes comparisons between data sets highly complex.

Laboratory testing can simplify data set comparisons by controlling the conditions of the tests. Laboratory data are also constrained by the accuracy of measurements and by the processes that were monitored. Extremely accurate data sets monitored historically will not, however, be useful for evaluating process interactions if one of the critical processes was not monitored.

The basic connections between the real world and the ability to scientifically study CST processes are the conceptual models that have been developed for each of the processes. These models are typically cast in mathematical terms developed from first principles and basic analyses of the monitored data, from both the field and laboratory. The conceptual models are then useful in evaluating engineering aspects of CST and in the design of new field or laboratory testing.

Early numerical models of cohesive sediment transport were relatively simplistic in conceptual framework of CST processes. These simplifications were in part the result of limitations placed on these models of the temporal and spatial scales that could reasonably be addressed and efficiently incorporated into numerical models. For the last few decades a combination of increased computational power and sophistication in computational techniques
has lead to fewer simplifications in the formulation of estuarine CST models (conceptual and numerical). Each of the processes can now be represented in a more descriptive and interactive manner at smaller spatial and temporal scales (McAnally and Mehta, 2001).

These more complex numerical models now play a significant role in guiding the direction of new research. New research is typically aimed at improving specific process conceptual models. The numerical models can be tested to identify the sensitivity of the model primary results to each of the component process descriptions. The trade-offs between model sensitivity and critical knowledge gaps can then be used to optimize additional research. If a particular process can only be estimated within a given accuracy, but the sensitivity of other primary processes is small, then improving that secondary process may by given a lower priority. Toorman (1993) refers to this class of numerical models as “virtual laboratories.”

Even with strides in computational capabilities, engineering applications of CST numerical models must still utilize parameterizations of the basic processes to a fair degree. When regional spatial scales and seasonal to annual hydrological time scales are required to evaluate morphodynamic issues, the CST processes must be bulked to more manageable relationships.

Within classical steady-state flume experiments, the macroscale bulking paradigm dealt with the mean flow conditions, averaging out turbulence. Erosion and deposition rates were often estimated by the change in the suspended sediment concentration, which was essentially an averaging technique. The measurement of turbulent fluctuations in velocity and bottom shear stresses within a flume show that erosion or deposition with time scales less than the turbulent frequency may both occur for nominally steady-state conditions.

Measurements are generally incomplete to fully define all processes for a given data set, whether field or laboratory data. This study intends to make use of existing data sets that
incorporate monitoring of as many of the processes as reasonable and limited additional field observations in a highly complex estuarine setting.

The earliest conceptual models for cohesive sediment erosion and/or deposition have been simplistic, but have served the science well (Krone, 1962; Ariathurai, 1974). These involve threshold values for both erosion and deposition based on the bulk properties of the sediment; a shear stress threshold for deposition above which deposition will not occur, and a critical shear stress for erosion based on the bulk strength of the sediment in the bed, below which erosion will not occur. The fact that these bulk thresholds naturally tend to separate, with the deposition threshold less than the erosional threshold, supports the simple conceptual model and has been used by many to argue that erosion and deposition should not occur simultaneously (Partheniades, 1971). This inference is arguably valid when discussing the processes using bulk sediment properties, particularly when compared against the time-averaged bottom shear stress. However, there has been a recurring debate on the issue that stems from a basic intuitive sense that the real world is just not that simple (Sanford and Halka, 1993; Lau and Krishnappan, 1994; Winterwerp, 2007). The exclusive process (erosion or deposition) view evolved partly from the use of these bulk sediment parameters and mean shear stresses derived from mean velocity values. Now that most research sediment transport models for CST incorporate multiple floc sizes, many of these bulking assumptions are no longer seen as viable (Winterwerp, 2007).

Based on the limitations and research needs described above, the following objectives and tasks were defined.

1.2 Objectives and Tasks

This research investigates the importance of stochastic properties to cohesive sediment transport processes, including aggregation and disaggregation, settling, deposition, erosion and turbulent mixing. The incorporation of a probabilistic treatment of the primary variables is
targeted to study the influence of turbulence and heterogeneity of sediment properties on bed exchange.

The objectives of this research are to:

- Develop and test probabilistic representations of selected hydrodynamic variables and sediment properties.
- Evaluate their significance to aggregation and disaggregation processes along with bed interaction.
- Investigate their sensitivity and interactions.

The primary tasks performed to achieve these objectives are:

- Identify significant variables for incorporation into a probability function, based on a thorough review of the literature.
- Develop a conceptual model of cohesive sediment transport processes that incorporates probabilistic representations of the variables.
- Define the applicability of the new conceptual model by numerical testing against field and laboratory data with sensitivity testing to identify the most critical variables for full probabilistic specification.
- Determine the appropriate means and value of parameterization of the new conceptual models of aggregation and disaggregation into engineering scale numerical models.
- Assess the needs for future research.

1.3 Approach

To accomplish the objectives, a careful and thorough examination was carried out of the governing equations for flow and sediment transport. Terms included in the equations are interaction between turbulence closure in hydrodynamics and sediment concentration, and aggregation processes in a multiple particle-size-class CST model. A literature review was conducted to define the current state of the science in cohesive sedimentation. The current state of the art in discrete particle aggregation modeling was modified to incorporate additional effects arising from interaction of probabilistic terms. The first step was to evaluate the importance of
differential particle response to a probabilistic parameterization on sediment aggregation and disaggregation. Once aggregation has proceeded and increased concentrations evolve near the bottom, the influence of sediment concentration on damping vertical mixing could be addressed. A bed exchange formulation was developed based on integration of joint probability functions weighting the erosion threshold.

Research was performed to compile field and laboratory data appropriate for testing the revised algorithms. In general, the data of interest for this research include particle and floc size distributions, settling velocity distribution by floc size, total suspended sediment concentration, turbulent velocity measurements, water depth, shear strength of sediment deposits, erosion rate estimates, salinity of the fluid, organic content and bulk densities of the bed material.

1.3.1 Examination of Governing Equations

The governing equations for the hydrodynamics are the Navier-Stokes equations. The turbulence closure for the momentum equations when time averaging is required was handled in two ways: use of Boussinesq approximation of the Reynolds stresses and the turbulent kinetic energy (TKE) transport equation ($\kappa$) with a damping equation for the TKE ($\varepsilon$). The sediment transport equation is the advective-diffusion equation with turbulent mixing and source/sink terms due to deposition and associated with aggregation processes. These equations were examined using revised interaction terms incorporating probabilistic variable interactions.

1.3.2 Revision of Aggregation/Disaggregation Model for Probabilistic Fluctuations

The effects of the probabilistic representation of specific variables of importance to the flocculation model of aggregation and disaggregation were incorporated and tested.

A sediment particle suspended within a turbulent hydrodynamic flow field will be buffeted around by turbulent fluctuations in the velocities in three dimensions. The response of the particle to these fluctuations will depend on the frequency and magnitude of the turbulence, the
size and density of the sediment aggregates and the viscosity of the fluid. When the sediment suspension is made up of a variety of aggregate sizes and densities, the responses will vary and the potential exists for differential turbulent particle responses that can enhance particle collisions.

1.3.3 Effects of Suspended Sediment on Turbulence

Particle collisions increase with sediment concentration and can eventually reach the point where they significantly extract momentum from the turbulence. This effect combined with the feedback effects of flow blockage, similar to hindered settling, can significantly dampen vertical mixing due to the turbulence. These effects are handled by the incorporation of a two-parameter turbulence closure model.

1.3.4 Development of a Probabilistic Bed-Exchange Model

The conventional bed exchange model was modified to incorporate the effects of probabilistic variables on the erosion and deposition. The sensitivity of the probabilistic formulation in conjunction with the discretized particle size distribution was evaluated using both the exclusive and simultaneous conceptual models of erosion/deposition.

1.3.5 Application of the Probabilistic Formulation to Selected Test Cases

The probabilistic model was applied to select test cases in an attempt to provide insight into when the use of probabilistic parameterization of certain variables provides added performance.

1.4 Scope

The goal of the research described here will be to attempt development of improvements in the conceptual models of cohesive sediment aggregation and bed exchange processes as typically encountered within estuarine waters. The development of numerical modeling tools will be performed simply to provide a mechanism to test the revised algorithms.
1.5 Presentation Outline

The research is reported in the following presentation outline.

Chapter 1 has presented the introduction including motivation, research needs, objectives of the present research, approach and the scope of present research. Chapter 2 presents an overview of the current state of CST processes in the context of both steady uniform flow assumptions and discusses the specific constraints imposed by those assumptions, as well as the limited knowledge available for complex hydrodynamic situations. Chapter 3 presents the development of a probabilistic description for sediment properties and hydrodynamics for CST in fully generalized unsteady nonuniform hydrodynamics along with the mathematical approach for incorporating the effects. Chapter 4 presents the implementation of the probabilistic treatment and details of the numerical model. The effects of new terms on multi-class deposition with aggregation are presented in Chapter 4. Chapter 5 details field and laboratory test cases that have been compiled for evaluation. Applications of the new methodology to the test cases are presented in Chapter 6. Finally, the conclusions and recommendations of further research are discussed in Chapter 7.
CHAPTER 2
COHESIVE SEDIMENT TRANSPORT

2.1 Estuarine Cohesive Sediment Properties

The general properties of estuarine sediments will be discussed to a level appropriate for the research topic presented.

2.1.1 Estuarine Sediment

Cohesive sediments in estuarine waters are unique in their ability to morph from single mineral particles to flocs in appropriate flow conditions. These flocs have a distribution of sizes, from the smallest tightly packed zero-order units to the largest units limited by the turbulent flow shear. When torn apart by higher shear or collisions, the larger flocs may morph back to their zero-order constituents. If a suitable chemical dispersing agent is now added and the suspension is centrifuged, the zero-order flocs will revert to the primary particle state. They are easily mobilized into suspension by current and wave energy levels found in estuaries. Cohesive sediments may transition back and forth between the zero-order flocs and various larger floc sizes innumerable times before becoming a more permanent bed deposit when transported out of the higher energy reaches of the estuary. Cohesive sediments ultimately find their way to quiescent areas and settle out of the water column to create more permanent deposits. Fine particles tend to migrate to the recesses of estuaries, along the shorelines and in wetlands, making them very critical to many ecological systems.

Cohesive sediments have historically been described by bulk properties of the deposited sediment, such as bulk density, water content and plasticity, which serve as indicators of erosion resistance. General sediment size fractions of sand, silt and clay within the bottom sediments can provide a strong indication of the expected transport behavior of the bulk sediment.
Sea salinity provides the electrolytes necessary to restrict the electrostatic repulsive forces sufficiently to allow the van der Waal forces to create a net attraction between cohesive sediment particles. The saline environment of estuaries is a catalyst for aggregation and can lead to dramatic sedimentation within the estuary.

2.1.2 Fine Sediment Classification

There are several ways fine sediments are classified. These include sediment size, shape, mineral composition, rheologic properties, electrochemical properties, and by the sediment transport behavior. Floc properties are generally a reflection of hydrodynamic flow conditions. Therefore, most investigators focus on the dispersed particle distribution when classifying the sediment.

2.1.2.1 By size

The size distribution of fine cohesive sediments has been most effectively classified by the MIT scale as shown in Table 2-1. Particles in suspension less than a tenth of a micron in size comprise a “sol” and when concentrations are sufficiently high are essentially dominated by particle cohesion. Fine, medium and coarse clays lie between 0.1 and 2 microns, and cohesion is very important in their behavior. As the size classes become increasingly larger, the importance of cohesion diminishes. Coarse silt is practically cohesionless and sands are cohesionless.

Size, as discussed in Table 2-1, refers to the unflocculated primary particle size distribution, because the floc size distribution can be a function of the flow field. Example distributions of cohesive sediments are shown in Figure 2-1. The figure presents for comparison the dispersed particles distributions for San Francisco Bay mud, Maracaibo Bay mud, and kaolinite (Mehta, 1973). San Francisco Bay dispersed grain and floc distributions are also shown. The suspended floc size distribution is comparable to that of beach sand. Figure 2-1 also presents the MIT size classification boundaries.
One notable feature of Figure 2-1 is the variation in the size distribution both for the dispersed and the flocculated sediment. These distributions will need to be explicitly addressed in analysis of the basic CST processes.

<table>
<thead>
<tr>
<th>Size (μm)</th>
<th>Classification</th>
<th>Level of Cohesion</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;0.1</td>
<td>Sol</td>
<td>Cohesion dominates</td>
</tr>
<tr>
<td>0.1 to 2</td>
<td>Fine, medium, and coarse clay</td>
<td>Cohesion very important</td>
</tr>
<tr>
<td>2 to 20</td>
<td>Fine and medium silt</td>
<td>Cohesion important</td>
</tr>
<tr>
<td>20 to 40</td>
<td>Coarse silt</td>
<td>Cohesion increasingly important with decreasing size</td>
</tr>
<tr>
<td>40 to 62.5</td>
<td>Coarse silt</td>
<td>Practically cohesionless</td>
</tr>
<tr>
<td>&gt; 62.5</td>
<td>Sand and coarser</td>
<td>Cohesionless</td>
</tr>
</tbody>
</table>

2.1.2.2 By shape

The shape of the primary particles has a strong influence on the cohesive properties of the sediment. Clay minerals are plate-like structures, which result in a very large surface area to volume (or mass) ratio, amplifying the influence of electrochemical forces relative to gravitational effects. Shape is a property of the basic mineralogy of the sediment grains. Some clay minerals form needles and tubes. The shape of the minerals can affect the degree of cohesion through the impact on the surface to volume ratio and electrochemical forces. Shape has a role in interactions with flow for noncohesive particles. However, for cohesive sediments, since their dispersed particles are incorporated within flocs, the shape of the flocs, not the dispersed particles, influence settling and deposition.

2.1.2.3 By composition

The mineral composition of fine cohesive sediments influences their degree of cohesion. The crystalline structure of various clay minerals is a combination of layers of silica, oxygen, hydroxyls, and alumina (or iron or magnesium), with possible adsorbed water layers. The specific arrangement defines the class of clay mineral. The general properties of the primary
clay minerals discussed in the literature are shown in Table 2-2. The variation in the mineral content of the sediments within a particular estuary is not considered in the current research. In addition, the content of organic material can significantly affect the cohesion of the sediment mixture.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>kaolinite</th>
<th>Montmorillonite</th>
<th>Illite</th>
<th>chlorite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specific gravity</td>
<td>2.60-2.68</td>
<td>2.20-2.70</td>
<td>2.64-3.10</td>
<td>2.60-2.96</td>
</tr>
<tr>
<td>Plate diameter ((\mu m))</td>
<td>0.1-4</td>
<td>0.01-0.1</td>
<td>0.003-0.3</td>
<td>1.0-4.0</td>
</tr>
<tr>
<td>Plate thickness ((\mu m))</td>
<td>0.05-2</td>
<td>(\leq 0.01)</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>Specific surface area</td>
<td>(10^3-10^4)</td>
<td>(10^4-10^5)</td>
<td>(10^4-10^5)</td>
<td>(10^4)</td>
</tr>
<tr>
<td>Cation exchange capacity (mEq/100g)</td>
<td>3.0-15.0</td>
<td>80-120</td>
<td>10.0-40.0</td>
<td>20-50</td>
</tr>
<tr>
<td>Source</td>
<td>Mehta, 2007</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.1.2.4 By electrochemical properties

The degree of cohesion is related to the strength of the electrochemical interactions of the mineral grains. A measure of the strength is the cation exchange capacity (CEC), which reflects the affinity of the clay minerals to exchange ions with the solute. The CEC of clay sediment grains varies with the mineral composition and a number of other factors, including the size of the particles, availability of ions, valence of the ions and organic content. The range of CEC for four example clays is presented in Table 2-2.

The electrostatic charges on the surface of the primary cohesive particles in dionized water will generally cause repulsion of the grains when they come into close proximity. Natural fresh water contains sufficient ions in solution to alter the electrochemical potential field around the particles and allow some particles to get close enough for the van der Waals attractive forces to cause cohesion. When the sediment enters the estuarine environment, there are significantly more ions available from the ocean salinity to dramatically increase the probability of cohesion. Adhesion occurs when particles that collide are bonded together by attractive van der Waals
forces or biological material. The efficiency of adhesion is the fraction of colliding particles that are bonded. The efficiency of adhesion in estuaries is normally taken as unity.

Clay mineral particle plate-like surfaces are negatively charged. The positive ions in the solution are attracted to the surface of the mineral and create the “double layer” of water around the particle. An inner layer of water containing positively charged ions adheres to the particle and moves with the particle. The edge of the inner layer is called the slipping plane, since when in motion through the fluid the particle will have a shear along that surface. The double layer of water contains preferential positive ions, so that the mineral double layer acts like a flat plate condenser.

The concentration of ions in the solution influences the thickness of the double layer. At higher ionic concentrations, the double layer is compressed reducing the strength of the repulsive forces, resulting in a greater chance that the van der Waals attraction will dominate and result in particle cohesion.

2.1.3 Characterizing Aggregates

2.1.3.1 Primary particle distribution

The primary particle distribution of the sediments in suspension is the building block for the cohesive aggregates. The primary particle distribution has been proposed as a self-similar distribution for a particular system (Kranck and Milligan, 1992), whereby the basic shape of the dispersed or flocculated size spectra is retained as the total concentration varies and the mean floc size changes with time. Kranck and Milligan proposed a spectral shape that is a combination of a distribution for the fine end of the spectra and a decay shape at the coarser end of the spectrum. The shape for the fine end of the spectrum is shown in Equation 2-1.

\[ C_o(d) = Qd^m \] (2-1)
\( C_o(d) \) is the particle size distribution (measured as volume of particles per volume of suspension), \( Q \) is a coefficient scaled with the total suspended concentration, \( d \) is the particle diameter and \( m \) is an exponential coefficient. This was combined with the analytical time-dependent solution for a well-mixed suspension with settling (Equation 2-2).

\[
C(t) = C_o e^{-\frac{w_s}{h} t} \tag{2-2}
\]

The variables are: \( t \) is time, \( w_s \) is the settling velocity, and \( h \) is the water depth. The expression for fall velocity was substituted and the term \( t/h \) was viewed as a settling decay term, \( K \), to yield Equation 2-3.

\[
C(d) = Qd^m \exp\left(-\frac{KAd^2}{h}\right) \tag{2-3}
\]

The term \( Ad^2 \) represents the fall velocity, where \( A \) includes all of the terms from Stokes settling law, \( A = \frac{g(\rho_f - \rho_w)}{18\nu\rho_w} \), \( g \) is gravitational acceleration, \( \rho_f \) is the floc density, \( \rho_w \) is the fluid density, and \( \nu \) is the kinematic viscosity of the fluid. This basic spectral shape agreed with the observed distribution from data collected in San Francisco Bay for both the dispersed and flocculated sediment (see Figure 2-2). The dispersed spectra were developed from a Coulter counter analysis of the disaggregated sediment and the flocculated spectra from the analysis of in-situ photographs taken by a plankton camera. The curve fit to the dispersed spectrum of Equation 2-3 was reported by Kranck and Milligan to have the coefficients \( m = 0.022, K = 3.53 \) and \( Q = 1.32 \). They reported the flocculated spectrum curve fit has \( m = 2.72, K = 0.081 \) and \( Q = 0.0001 \). Careful evaluation of the equation and their data suggest two issues. First, they stated that the units of the coefficient \( K \) are \( \text{s/cm} \) and fitting of the curves to their data suggests that they used the diameter of the particles in microns for the initial concentration. Within the exponential function, the diameter needs to be in centimeters in order for the value of \( K \) indicated. Secondly,
in the fitting of the curves to the flocs and the grains, they would have used the full grain density, in error, for the floc density curve in order for the stated value of \( K \) (0.081) to match the data. For each of these curves they apparently used the grain density of 2.65 g/cm\(^3\) and 1.0 g/cm\(^3\) for the fluid, giving a value of \( A = 8029 \) 1/(gs), since with the \( K \) values cannot match with a lower relative density ratio. The modal floc size for the flocculated sediments is around 400 microns.

2.1.3.2 Order of aggregation

Krone (1963) developed a conceptual model of flocculation based on the order of aggregation. He hypothesized that when primary mineral particles aggregate, their bonds and mineral arrangement are established and will not significantly change on successive higher order aggregations. Primary aggregates are formed from the cohesion of mineral grains, forming strong bonds. Primary aggregates do not compress, retaining their density. Primary or zeroth-order aggregates (pa), collide and combine to form first order aggregates (p2a). In the notation (p\( n \)a), the order of aggregation is (n-1). First order aggregates then combine to create second order aggregates (p3a) and so on. The strength of the flocs gets weaker as the order of aggregation increases. The progression of orders of aggregation is shown schematically in Figure 2-3. It was assumed that aggregation beyond primary aggregates requires intermeshing of the lower order aggregates. Krone then made the assumption that the ratio of the increment in voids to the volumes of the preceding aggregation is the same as the ratio of voids created during the preceding aggregation to the preceding aggregate volume (\( \varepsilon_{i+1} = \varepsilon_i/(\varepsilon_i + 1) \)), where \( \varepsilon_i \) is the voids ratio for flocs of \((i-1)\)-th order of aggregation (pia). This fractal assumption leads to a relationship for the volume fraction as a function of the primary aggregate volume fraction, the order of aggregation and the void ration of the primary (first order) flocs (See Equation 2-4).

\[
\phi_{pia} = \phi_p \left[ 1 + (n - 1)\varepsilon_i \right]
\]  

(2-4)
The volume fraction of primary aggregates is \(\phi_p\), the voids ratio of a primary aggregate is \(\varepsilon_1\), and the order of aggregation is \((n-1)\).

### 2.1.3.3 Floc size spectra

Figure 2-4 illustrates the effects of the parameter \(K\) in Equation 2-3 on the size distribution. Increasing the value of \(K\) causes the spectra to fall off more quickly as the particle size increases. The value of \(Q\) is scaled by the total concentration.

The basic slope, \(m\), of the spectra on the fine end was consistently near zero for all of the San Francisco Bay samples collected (see Figure 2-5). The consistency in the grain spectra is believed to be characteristic of the primary sediments in the entire bay system. The grain spectra for bottom sediments nearby the profiling location and in the San Pablo Strait south of the deployment showed a similar shape (Figure 2-6).

### 2.1.3.4 Fall velocity

Fall velocity is another way to characterize cohesive sediment, since it is readily measurable. It can be measured either on site for in situ conditions or in a laboratory environment for the dispersed particle conditions. The size distribution can then be inferred from the fall velocity based on Stokes law.

### 2.1.3.5 Floc density

Floc density is inversely related to floc size. The relationship is consistent with Krone’s order of aggregation and has been shown to be fractal in nature (Kranenburg, 1994). Primary sediment grains have the basic mineral density. As primary sediment grains flocculate into small flocs, if the floc size is defined as some enclosing volume, the density of the volume will be lower than the mineral density because water will fill the voids between the primary grains. As smaller flocs aggregate into larger flocs, the relative fraction of voids within the now larger
enclosing volume increases and the floc density is further reduced. The fractal relationship between floc excess density and floc dimension is shown in Equation 2-5 (Kranenburg, 1994).

\[
\Delta \rho_f = \Delta \rho_s \left[ \frac{d_p}{d_f} \right]^{3-D_f}
\]  

(2-5)

The variable \( \Delta \rho_f \) is excess density of the floc, \((\rho_f - \rho_w)\), \( \Delta \rho_s \) is excess density of the primary particles \((\rho_s - \rho_w)\), \( \rho_s \) is sediment mineral density, \( d_p \) is the primary sediment grain diameter, \( d_f \) is the floc diameter and \( D_f \) is the fractal dimension. The fractal dimension for excess density varies with the flocculation environment and the controls on flocculation.

**2.1.3.6 Floc shearing strength**

The shearing strength of flocs is also inversely proportional to the floc size; with the floc structure becoming more fragile as the floc size increases. Krone (1963) developed a rotating concentric cylinder viscometer and related an inferred Bingham yield (shear) strength of the suspended flocs, \( \tau_f \), to concentration for San Francisco Bay sediment by Equation 2-6.

\[
\tau_f = KC^{5/2}
\]

(2-6)

The units-dependent coefficient \( K \) was related to the order of aggregation and the suspended sediment concentration (see Equation 2-7).

\[
K = \frac{N}{\left[ 1 + \frac{(n-2)e_i}{\phi_{pna}} \right] E_v \left[ \phi_{pna} \phi_s \rho_s \right]^{5/2}}
\]

(2-7)

\( N \) is the number of simultaneous particle aggregate bonds ruptured, \( E_v \) is the energy required to disperse a unit volume of particle aggregate, and the other variables as previously defined.

Kranenburg (1944) argued that because the shear strength and breaking of flocs is controlled by the weakest bonds within the floc, the force required to break a floc should be
independent of floc size. Since the force on the floc at rupture is proportional to the shear strength times the exposed area of the floc, it can be assumed that the shear strength is proportional to the inverse of the particle dimension squared (Equation 2-8).

\[ F_s \approx \tau_f d_f^2 = \text{constant} \quad \Rightarrow \quad \frac{\tau_f}{d_f^2} \approx \text{constant} \quad (2-8) \]

Combining Equations 2-5 and 2-8 results in Equation 2-9, expressing the shear strength based on the excess floc density.

\[ \tau_f = \text{constant} \left( \frac{\Delta \rho_f^{\frac{2}{3-D_f}}}{d_s^2} \right) \Delta \rho_f^{\frac{2}{3-D_f}} \quad (2-9) \]

Based on the data of Krone, Partheniades (1993) proposed that the floc strength is related to the excess density of the flocs (Equation 2-10).

\[ \tau_f = K_f \Delta \rho_f^{\frac{5}{2}} \quad (2-10) \]

Since the terms in the parentheses in Equation 2-9 equals a constant, we can equate the exponents on the excess floc density in Equations 2-9 and 2-10: \(5/2 = 2/(3-D_f)\) implies that the fractal dimension is \(D_f = 2.2\).

Shear strength data from Krone for various harbors are presented in Figure 2-7 against the excess density of the flocs. The figure shows the reasonableness of Partheniades 5/2 exponent \((D_f = 2.2)\). Linear regression of the log-transformed data sets shows a variation in the exponent and the fractal dimension. These are summarized in Table 2-3.

Regressions for each of the individual harbor data sets have a very good fit (high R-squares), with a minimum R-squared value of 0.952. Within any of the data sets, the relationship between excess density and floc strength is predictable, but there remains a large scatter in the data when handled as one dataset, with an R-squared value of only 0.559. This means that for an
accurate estimation of the shear strength for a specific system, data collection and analysis may be required.

Table 2-3 Comparison of shear strength versus excess density for locales

<table>
<thead>
<tr>
<th>Data set</th>
<th>Regression exponent</th>
<th>R-squared</th>
<th>Fractal dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>All data</td>
<td>1.91</td>
<td>0.559</td>
<td>1.95</td>
</tr>
<tr>
<td>Wilmington</td>
<td>1.89</td>
<td>0.952</td>
<td>1.94</td>
</tr>
<tr>
<td>Brunswick</td>
<td>2.70</td>
<td>0.998</td>
<td>2.26</td>
</tr>
<tr>
<td>Gulfport</td>
<td>2.05</td>
<td>0.976</td>
<td>2.02</td>
</tr>
<tr>
<td>San Francisco</td>
<td>2.86</td>
<td>0.967</td>
<td>2.30</td>
</tr>
<tr>
<td>White River</td>
<td>2.02</td>
<td>0.974</td>
<td>2.01</td>
</tr>
</tbody>
</table>

2.1.3.7 Fractal dimension

The fractal dimension of cohesive sediments enters relationships between several descriptive variables as a function of a spatial scale, typically the particle or floc size. The fractal description of aggregation starts with the consideration of the primary aggregation of a number, \(m_1\), of primary mineral grains into a primary aggregate. Then it is assumed that \(m_1\) of these primary aggregates combine, structurally similar in arrangement to the arrangement of the mineral grains in the primary aggregate, to form a first order aggregate. The number of primary grains in the first order aggregate will then be \(m_1m_1\). If the size of the newly formed first order aggregate is a factor, \(m_2\), larger in size than the size of a primary aggregate, and subsequent levels of aggregation repeat the same structural similarity and size scaling, we will have the total number of primary grains in a floc related to the order of aggregation or its size. The number of primary particles in a floc, \(N_f\), will be \((m_1)^n\), where \(n\) is the order of aggregation. The size of the floc compared to the size of the primary floc will be \(d_f/d_p = (m_2)^n\). The fractal dimension, \(D_f\), can be introduced as shown in Equation 2-11.

\[
N_f = \left[ \frac{d_f}{d_p} \right]^{D_f}
\]

(2-11)
This can be expressed as a relationship between \( m_1 \) and \( m_2 \) as in Equation 2-12.

\[
N_f = m_1^n \left[ \frac{d_f}{d_p} \right]^{D_f} = \left[ \frac{(d_p m_2^n)}{d_p} \right]^{D_f} = [m_2^n]^{D_f} \tag{2-12}
\]

Equation 2-12 can be converted to a definition of the fractal dimension (Equation 2-13).

\[
D_f = \frac{\ln(m_1)}{\ln(m_2)} = \frac{\ln(N_f)}{\ln \left( \frac{d_f}{d_p} \right)} \tag{2-13}
\]

This fractal dimension is associated with the linear scale of the flocs. Other fractal dimensions could be developed for surface area or volume. Alternatively, other relationships can be expressed as functions of the linear fractal dimension. The ratio of the volume fraction of flocs to the volume fraction of the mineral grains in the floc can be derived (Equation 2-14).

\[
\frac{\phi_f}{\phi_s} = \left[ \frac{d_f}{d_p} \right]^{3-D_f} \tag{2-14}
\]

The ratio of the floc densities to the primary grain densities is expressed in Equation 2-5, scaling at the power of \( (3-D_f) \). The fractal relationship for the floc strength is shown in Equation 2-9, scaling with the excess density to the power \( 2/(3-D_f) \), or to the inverse square of the floc diameter. As will be shown later, the floc settling velocity scales to the power of \( (D_f - 1) \).

Khelifa and Hill proposed (2006) that it may not be reasonable to expect that the fractal dimension remains a constant over the full spectrum of floc dimensions experienced in the literature. Their derivation will be summarized here. The definition of floc density is extended (Equation 2-15) to acknowledge that the primary sediment grains are polysized, with a primary particle distribution defined by \( k \) sizes \( d_i \) \((i=1,k)\).

\[
\rho_f = \rho_s \sum_{i=1}^{k} \frac{d_i^3}{d_f^3} \tag{2-15}
\]
The equivalent spherical diameter of the floc, \( d_f \), for monosized primary particles of diameter \( d \) is defined in Equation 2-16.

\[
d_f = d \left( \frac{k}{D_f} \right)^{1/D_f}
\]  

(2-16)

They extended this definition to polysized primary particles as the summation over each of the grain sizes raised to the fractal dimension, and then taking the fractal root of the sum (see Equation 2-17).

\[
d_f = \left[ \sum_{i=1}^{k} d_i^{D_f} \right]^{1/D_f}
\]  

(2-17)

A fractal dimension that varies with floc size was proposed in the form of a power law as shown in Equation 2-18. This takes into account the observed changing structure of the flocs as they grow.

\[
D_f = \alpha \left( \frac{d_f}{d} \right)^\beta
\]  

(2-18)

The coefficients \( \alpha \) and \( \beta \) were defined from two limiting cases. First, the fractal dimension should approach a maximum value of 3 when the floc size approaches the primary particle size. The fractal dimension should reach a lower value \( D_{fc} \) when the floc size reaches some characteristic value \( d_{fc} \). Applying these end values to Equation 2-18 we get Equation 2-19, where the primary particle, \( d_p \), replaces \( d \).

\[
D_f = 3 \left( \frac{d_f}{d_p} \right)^{\frac{\log(D_{fc}/3)}{\log(d_{fc}/d_p)}}
\]  

(2-19)

This variation in fractal dimension will be incorporated into the density and settling velocity later. It generally improves the range of agreement of the models with observations.
2.1.4 Bulk Properties of Cohesive Sediments

Bulk properties of cohesive sediments are useful in the ability to infer other characteristics of the sediment that can be used in engineering applications, including development of parameters for more applied numerical methods. These include general size class percentages (sand, silt, and clay percentages), organic content, mineral content, fall velocity and bulk erosion rates in conjunction with shear strength of bed deposits. These parameters do not provide information useful in evaluating aggregation processes, but provide general guidance for expected behavior. The bulk bed exchange properties will be discussed later.

2.2 Cohesive Sediment Transport in Steady Uniform Flow

The current state of the knowledge in CST is applicable to steady uniform flow. The sedimentation processes will be discussed in this section in the context of their interaction with macroscale hydrodynamics that are steady and uniform.

Steady flow conditions are loosely defined as flows that change very slowly with time. But “slowly” is a relative term. A time scale can be defined that reflects processes over the dominant spatial scale of the system of interest. That time scale may be defined differently, depending on the specific processes of interest. For CST, the logical time scale, $T_{ss}$, is a factor times the ratio of the depth to a representative settling velocity, $\alpha h / \omega_s$. An indicator for steady state would be that $\frac{t_A}{T_{ss}} > 1$, where $t_A$ is a measure of how slowly the flow conditions are changing. Ultimately, a tolerance for change needs to be defined for the primary forcing. For example, if the mean value of a critical variable, $\zeta^*$, has a tolerance of $\Delta \zeta^*$ before the flow conditions are considered unsteady, then the time scale $t_A$ can be defined as the tolerance divided by the time derivative of the critical variable (see Equation 2-20).
\[ t_a \equiv \frac{\Delta \zeta}{\left( \frac{\partial \zeta}{\partial t} \right)} \]  

(2-20)

The steady-state criterion can then be expressed as Equation 2-21.

\[ \frac{w_i \Delta \zeta}{\alpha h} > \left( \frac{\partial \zeta}{\partial t} \right) \]  

(2-21)

The tolerance of steady-state fluctuations is associated with an acceptable change in the conditions of interest. The value of \( \alpha \) should reflect how stringent the need for a steady-state condition. For example, assuming \( \alpha \) is 10 and the depth is 1 m with a fall velocity of \( 10^{-4} \) m/s, \( T_{ss} \) would be approximately 28 hours. That would be the time scale for comparing the observed changes in the system, for example, in a large test flume. For a field scale of 10 m deep the value of \( T_{ss} \) would be 277 hours.

### 2.2.1 Aggregation Processes

Flocculation results from electrostatic cohesion and electrochemical adhesion between sediment grains, small flocs and organic matter. Flocculation processes are affected by temperature, salinity, ionic content, pH, mineral content, organic content, the total suspended concentration and turbulence in the water column.

Aggregation processes can be divided into two general categories (Winterwerp, 1999, Vicsek, 1992); diffusion limited aggregation (DLA) or cluster-cluster aggregation (CCA). DLA occurs in extremely dilute concentrations, limited by the number of aggregates to interact. CCA occurs when there are many clusters of aggregates available to adhere. CCA can be further divided into two classes. In diffusion limited CCA (DLCCA), the probability of cluster adhesion approaches unity, so the only factor to limit the aggregation is the number of collisions, assumed to be controlled by diffusion. When the probability of aggregation is much less than unity, the
process is reaction limited (RLCCA). The particles can bounce into one another very frequently, but the probability of adhesion limits the aggregation rate. The fractal dimension for DLCCA typically ranges from 1.7 to 1.8. RLCCA has a typical fractal dimension between 1.9 and 2.1.

The presence of organic material, which increases the probability of adhesion, will effectively lower the fractal dimension of the flocs. When very high-suspended concentrations are encountered, the interaction between particles affects the fractal dimension. The distance between flocs becomes comparable to the floc dimension itself and clusters pack very closely together at a fractal dimension approaching 3. But the floc clusters themselves have a structure of fractal dimension about 2.0, resulting in an overall macroscale fractal dimension of 2.6 to 2.8. Marine snow has a fractal dimension around 1.4, suggesting they are formed in a DLA environment.

The processes that bring particles close enough to create cohesion are related to the interactions of the particles with the fluid. These are the modes of aggregation. The modes for steady uniform flow are Brownian motion, differential settling, and turbulent shear. Biogenic aggregation can also be an important factor in certain environments (Andersen, 2001; Hill, 1998); on the continental shelf or in wetlands, areas where turbulence is not as great a factor.

2.2.1.1 Brownian motion

Brownian motion is the collision of water molecules with sediment particles as the water molecules move erratically due to thermal energy. Brownian motion is only of importance at the smallest particle sizes (less than 0.5 μm) (Hunt, 1982). Water molecule collisions with the sediment become less significant as the size of the particle increases because the mass of the particle increases, making the momentum exchange small relative to the inertia of the sediment. In addition, at larger scales, the number of molecular collisions increases and tends to average to
a zero net impact because of their random nature. Brownian motion may be important in the
initiation of flocculation at small scales in quiescent waters. Brownian motion is insignificant
for aggregation in most estuarine waters and is often neglected in aggregation models (Maggi,
2005).

2.2.1.2 Differential settling
Differential settling creates differential particle velocities that allow larger particles to
overtake the smaller particles. The effects of larger spherical particles on fluid flow lines in the
return flow helps to move the smaller particles away from the falling larger particle. However,
large voids within the open structure of flocs diminish the effects on the flow, increasing the
probability of a collision with the slower settling smaller flocs compared to the case of solid
spherical particles. As the concentration increases, the influence of differential settling increases
due to the increased probability of encounters and because the effects of the particles on the
return flow are constricted by the proximity of other particles. Even without aggregation, larger
particles can pull smaller particles down with them as they settle (Teeter, 2001). Hunt (1982)
found that differential settling is the most common mechanism for collisions for floc sizes
greater than 50-60 microns.

2.2.1.3 Shear
Either laminar or turbulent shear can induce particle collisions by the differential transport
perpendicular to the velocity gradient. In laminar flow, the differential velocity is the component
of velocity in the primary flow direction. Collisions occur because faster moving particles in the
positive velocity gradient direction overtake slower moving particles down gradient. In turbulent
flows, the differential velocities can be three-dimensional due to differences in inertial response
of different size particles to the turbulent fluctuations as well as the shear effects from the mean
flow. Because the magnitude of the velocity differential is a function of the size of the particles, shear has a significant impact on collisions of flocs in the 2 to 10 micron range (Hunt, 1982).

The relative importance of the modes of aggregation varies with the sediment size spectrum and the hydrodynamic conditions. In a quiescent settling column, shear will not be important, while in a highly turbulent tidal bore, Brownian motion will not be important. Very large flocs in relatively quiescent waters, such as the deep ocean, tend to be formed by differential settling (Lick, et. al., 1993; Hill, et. al., 2001).

2.2.1.4 Salinity

Salinity in estuaries provides the ions to compress the double layer, allowing for increased efficiency of aggregation. Krone (1962) investigated the effects of increasing salinity on flocculation in one-liter graduated cylinders at four different sediment concentrations. Figure 2-8 summarizes his findings, with the settling velocity showing dependence both on salinity and on the initial suspended concentration. Between 1 and 2 ppt, the fall velocity increases rapidly for all concentration levels. Median floc size increased for the final equilibrium settling velocity with the initial concentration. Above a salinity of 10 ppt the effects of salinity are minor and are fully realized above 15 ppt where the maximum fall velocity has been realized.

2.2.2 Fall Velocity

Flocs settle and deposit at rates several orders of magnitude faster than the constituent particles in the flocs. The settling velocity of flocs is influenced by the floc size, density, concentration and the level of turbulence. Free settling velocity is assumed at concentrations low enough that the effect of individual flocs falling through the water does not impact adjacent particle trajectories. Free settling is estimated from Stokes settling law, developed from a balance between the drag force and the buoyant weight of the particle, presented in Equation 2-22, for quadratic drag.
\[ w_s = \sqrt{\frac{4g \Delta \rho}{3C_D \rho} d} \] (2-22)

The drag coefficient is a function of the particle fall velocity Reynolds number, defined by Equation 2-23.

\[ \text{Re}_{w_s} = \frac{w_s d}{\nu} \] (2-23)

It is assumed that the drag coefficient for spheres is appropriate for flocs, since there are no data for flocs. This is the reason why Stokes diameter is used. The variation of the drag coefficient for a spherical particle was developed by Clift, et. al. (1978) over a series of ranges of \( \text{Re}_{w_s} \). These are shown in Equation 2-24.

\[
C_D = \begin{cases} 
\frac{24}{\text{Re}_{w_s}} \left( 1 + \frac{3}{16} \text{Re}_{w_s} \right); & \text{Re}_{w_s} \leq 0.01 \\
\frac{24}{\text{Re}_{w_s}} \left( 1 + 0.1315 \text{Re}_{w_s}^{0.82 - 0.5 \Upsilon} \right); & 0.01 < \text{Re}_{w_s} \leq 20 \\
\frac{24}{\text{Re}_{w_s}} \left( 1 + 0.1935 \text{Re}_{w_s}^{0.6305} \right); & 20 < \text{Re}_{w_s} \leq 260 \\
10^{(1.6435 - 1.1242 \Upsilon + 0.1558 \Upsilon^2)}; & 260 < \text{Re}_{w_s} \leq 1500 
\end{cases}
\] (2-24)

The variable \( \Upsilon \) is defined as the natural logarithm of the Reynolds number, \( \Upsilon = \ln(\text{Re}_{w_s}) \).

An alternative single equation recommended by Graf (1971) is shown in Equation 2-25. The comparison between Equations 2-24 and 2-25 is presented in Figure 2-9, indicating that the single equation gives reasonable agreement.

\[ C_D = \frac{24}{\text{Re}_{w_s}} \left( 1 + 0.15 \text{Re}_{w_s}^{0.687} \right) \quad \text{for } \text{Re}_{w_s} \leq 800 \] (2-25)
When the effects of flocculation on particle density (Equation 2-5) are included in Equation 2-24 and the drag coefficient from Equation 2-25 is used we get Equation 2-26.

\[
w_s = \frac{\alpha g}{18 \beta \mu} \left(1 + 0.15 \text{Re}^{0.687}_{\text{avg}}\right) d_p^{3-D_f} d_f^{D_f-1}
\]  

(2-26)

This was also developed by Winterwerp (1999) who added the weighting factor \( \alpha/\beta \), where \( \alpha \) is a shape factor for non-spherical particle effects in the gravitational force and \( \beta \) in the drag force. This fall velocity is for free settling with a variable drag coefficient and with the effects of floc density included.

The effect of concentration on the fall velocity is indirect. As the concentration of all sizes increases, the probability of flocculation increases. As flocculation progresses, the mean fall velocity of the sediment increases through the dependence of the fall velocity with particle size in Equation 2-26.

Khelifa and Hill (2006) developed a polydisperse version of Equation 2-5, still based on a constant fractal dimension (see Equation 2-27) for a floc comprised of \( k \) primary particles of arbitrary dimension \( d_i \).

\[
\Delta \rho_f = \Delta \rho_s \frac{\sum_{i=1}^{k} d_i^3}{\left(\sum_{i=1}^{k} d_i^{D_f}\right)^{3/D_f}}
\]  

(2-27)

Mean variables were then introduced (Equation 2-28).

\[
m_s = \frac{\sum_{i=1}^{k} d_i^3}{k} \quad \text{and} \quad m_f = \frac{\sum_{i=1}^{k} d_i^{D_f}}{k}
\]  

(2-28)

These can be introduced into Equation (2-27) to yield Equation (2-29).

\[
\Delta \rho_f = \Delta \rho_s k^{(D_f-3)/D_f} \psi
\]  

(2-29)
The variable $\psi$ is defined as $\psi = m_x / m_f \cdot \psi$. Khelifa and Hill showed that for monosized particles $\psi = 1$ and this reduced back to Equation 2-5. The formulation of Equation 2-29 is intractable since the number of particles within a floc is not generally known. It was therefore proposed that a median primary particle diameter be used as representative of the primary particles, so that a more usable form is obtained (Equation 2-30).

$$\Delta \rho_f = \Delta \rho_s \left( \frac{d_f}{d_{50}} \right)^{(D_f - 3)} \psi$$

(2-30)

This equation is their proposed model for the excess density of flocs that takes into account the distribution of polydisperse primary particles and, with the use of Equation 2-19, a variable fractal dimension, with $d$ replaced by $d_{50}$.

Incorporating the revised excess density of Equation 2-29 into Equation 2-26 results in the settling velocity relationship is shown in Equation 2-31.

$$w_s = \frac{\alpha g}{18 \beta \mu} \frac{\Delta \rho_s}{(1 + 0.15 \text{Re}_s^{0.687})^2} d_{50}^{3-D_f} d_f^{D_f-1} \psi$$

(2-31)

Khelifa and Hill recommend that when using $d_{50}$ to represent the primary particle distribution that the value of $\psi$ should be unity. Sensitivity of the $\psi$ can be included with the shape factor terms $\alpha$ and $\beta$ into a single coefficient for implementation and calibration to observed data, particularly since the effects of particle shape was not incorporated into Equations 2-28, which define $\psi$.

Data summarized by Winterwerp (1999) are presented in Figure 2-10 with fits of Equations 2-26 and 2-30. There is significant scatter in the data, and in order to make a reasonable fit using Equation 2-26 a fractal dimension of 2.6 is needed. The curve in the figure from Equation 2-31 used the Khelifa & Hill parameters: $D_f = 2.1$, $d_f = 8000$ microns and $d_{50} = 2$
microns. Because of the scatter, it is unrealistic to expect a single set of parameters for a theoretical relationship to be universally applicable and provide a good fit to a composite dataset as shown in Figure 2-10. Separate datasets from four of the estuaries in Figure 2-10 are presented in Figures 2-11 and 2-12 with their own independent curve fits. Most of the individual datasets can be reasonably fit with a theoretical curve, when the coefficients are adjusted to that estuary. The variation in the fit curves is presented in Figure 2-13 along with the composite curve fit for comparison. The composite curve is essentially the same curve as used to fit the Ems estuary (The Netherlands) 1990 dataset. The spread of the data values biases the settling velocity curve to a steeper slope (higher fractal dimension). It is of particular interest that the Ems estuary has such a significant difference between the two datasets.

The variation in the fractal dimension of Equation 2-31 fit in Figure 2-10 is shown in Figure 2-14. The fractal dimension is 3 at the primary \(d_{50}\) diameter of 2 microns and decreases by the power of -0.043 (\(\beta\) in Equation 2-18) until the fractal dimension is 2.1 at a floc size of 8,000 microns.

Differences in the primary particle distributions, as well as salinity regime and tidal energy, may explain the wide scatter in the data. The data reinforce the point that accurate representation of the CST processes in a particular estuarine system requires site specific data collection and analysis. The motivation behind the variable fractal dimension was the tendency for the settling velocity to be over-predicted at larger floc sizes with a constant fractal dimension (see Figure 2-15). There may be a correlation between the larger floc sizes and higher concentrations, for which hindered settling may have a role. The effect of the variable fractal dimension on the excess density as a function of floc size is illustrated in Figure 2-16. The behavior above 100 microns has been observed by several investigators who have displayed
constant fractal curves for multiple data sets that essentially have a pivot in the slope of those data just above 100 microns.

### 2.2.3 Hindered Settling

As the suspended sediment concentration becomes large, the settling particles effect on the fluid being displaced becomes significant. The displaced fluid begins to interact with the displaced fluid from nearby particles as they settle. Wolanski, et. al.(1989) proposed an empirical formula for hindered settling as shown in Equation 2-32.

\[ w_s = a_f \frac{C^{n_f}}{(C^2 + b_f)}^{m_f} \]  

(2-32)

The coefficient \( a_f \) is a scaling factor, \( n_f \) is the flocculation settling exponent, \( b_f \) the hindered settling coefficient and \( m_f \) the hindered settling exponent. The coefficients fit to the Krone (1962) San Francisco Bay data are: \( a_f = 0.048, n_f = 0.40, b_f = 25 \text{ kg/m}^3 \) and \( m_f = 1.0 \). These values were, however, generally outliers for the coefficients developed for 24 data sets from various systems around the world. It is noted that if the exponents are not selected such that \( n_f = 2m_f \), then the units of \( a_f \) will not be the same as \( w_s \). For the Krone values above the units of \( a_f \) are \((\text{m/s}) (\text{kg/m}^3)^{1.6}\).

Consider deposition in a quiescent fluid, for example a settling column. Applying the results from Equation (A-26) for the continuity equation for a sediment-laden fluid to sediment depositing, we have Equation 2-33.

\[ \frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho_f} \frac{\partial u_i C}{\partial x_i} \]  

(2-33)

This equation states that the divergence of the velocity field is in balance with the changing volume of sediment in suspension, the divergence in the sediment volume. The divergence in the sediment volume has the same meaning as for the fluid. Both the sediment and the fluid are
incompressible, so the divergence of one is offset by the divergence in the other with the opposite sign. A settling column initially mixed uniformly with sediment will upon initiation of settling have a downward volume flux of sediment that must be balanced by an upward water volume flux.

This equation takes the same form as a compressible flow equation because with a change in the sediment concentration there is a change in the volume of fluid in a unit volume. When the water flux is balanced around the control volume the gain or loss in water volume will be matched by the loss or gain of sediment volume. The differential sediment velocity \( u_{id} \) is measured relative to the water. This is assumed to be the reference particle settling velocity relative to a static fluid (see Equation 2-34).

\[
\begin{align*}
    u_{id} &= u_p - u_i = -|w_s| \\
    (2-34)
\end{align*}
\]

The variable \( u_p \) is the particle velocity. The fluid velocity \( u_i \) is referenced to the sides of the fixed reference frame (settling column). The fall velocity relative to the fluid is \( w_s \). Inserting Equation 2-34 into 2-33 yields Equation 2-35.

\[
\begin{align*}
    \frac{\partial u_i}{\partial x_i} &= \frac{1}{\rho_f} \frac{\partial w_s C}{\partial x_i} \\
    (2-35)
\end{align*}
\]

If the fall velocity is a property of the sediment and does not change with \( x_i \) then we have Equation 2-36.

\[
\begin{align*}
    \frac{\partial u_i}{\partial x_i} &= \frac{w_s}{\rho_f} \frac{\partial C}{\partial x_i} \\
    (2-36)
\end{align*}
\]

Integrating with respect to \( x_i \) we get Equation 2-37.

\[
\begin{align*}
    u_i &= \frac{C}{\rho_f} w_s \\
    (2-37)
\end{align*}
\]
The constant of integration is zero so that in the limit as \( C \) approaches zero the return flow of the water approaches zero. Equation 2-37 can now be combined with Equation 2-34 to solve for the particle velocity relative to the fixed reference frame (Equation 2-38).

\[
u_p = -w_s + u_i = -w_s \left( 1 - \frac{C}{\rho_f} \right)
\]  

(2-38)

The particle velocity is positive upward and it is an effective settling velocity, \(-w_{\text{seff}}\). The relative reduction in fall velocity can be expressed as given in Equation 2-39.

\[
\frac{w_{\text{seff}}}{w_s} = \left( 1 - \frac{C}{\rho_f} \right)
\]  

(2-39)

The ratio \( C/\rho_f \) is the volume fraction of the flocs \( \phi_f \), so the correction for the return flow is \((1 - \phi_f)\). Winterwerp (1999) derived a similar expression and included the effects of buoyancy and viscosity (see Equation 2-40).

\[
\frac{w_{\text{seff}}}{w_s} = \frac{(1-\phi_i)^m (1-\phi_f)}{(1+2.5\phi_f)}
\]  

(2-40)

The variable \((1-\phi_i)^m\) is the return flow effect, where \(\phi_i\) is set to the \(\min\{1, \phi_f\}\), to account for cases of consolidating fluid mud where \(\phi_f\) can exceed one. The exponent \(m\) is introduced to account for the nonlinear effects of return flow interaction between particles (pressure distributions, drag coefficients and added mass), particle collisions and interactions. Winterwerp advocates using \(m = 1\). The term \((1-\phi_f)\) is a correction for buoyancy effects and \((1+2.5\phi_f)\) is the correction for viscosity (Einstein, 1911).

The effects of shear on settling velocity were developed by Winterwerp, et. al. (2006) taking into account aggregation and floc breakup (see Equation 2-41).
The coefficients $k_2, k_3, k_4, \alpha''$ and $q$ are sediment dependent coefficients, $C$ is the total concentration, $h$ is the water depth and $\tau$ is the shear stress. Within Equation 2-41 units of the coefficients are a function of the exponents, making comparisons between specific data sets difficult.

Teeter (2001) found that in order to simulate the particle size distribution observed by Kranck and Milligan (1992), it was necessary to couple the interactions between grain classes during settling. When discretizing the sediment sizes into successive classes, the depositional flux was coupled to the next higher size class when the probability for deposition for the class, $P_{dis}$ was less than 0.05 (see Equation 2-42).

$$F_i = \frac{\alpha_i C_i F_{i+1}}{C_{i+1} + \alpha_2} \quad P_{dis} < 0.05 \quad (2-42)$$

$F_i$ is the depositional flux for the $i$-th sediment class, $C_i$ is the suspended mass concentration of the $i$-th class, $\alpha_i$ is a coefficient that controls the proportion of the flux and $\alpha_2$ limits the flux of smaller grains as $C_{i+1}$ tends toward zero. This effect is analogous to a sinking ship drawing huge masses of water down with it.

Wolanski et al. (1992) determined the settling velocity $w_s$ of suspended sediment in Cleveland, Australia, in two ways. The first method involved measurements of vertical profiles of concentration at different times during a dredged material disposal operation from a hopper dredger. In the second instance the same sediment tested in a laboratory settling column in which vertically oscillating rings (Figure 2-17 inset), nearly flush with the inner wall, were used to generate flow shear during the settling process. The depth-mean shear rate can be controlled
by setting the amplitude (and frequency) of oscillation. Concentration \( C \) profiles at different times were measured. Equation 2-43 along with an estimate of the diffusion coefficient \( D_z \), initial condition of constant concentration at the start of the test, surface condition of zero net sediment flux and bed condition of zero resuspension were solved numerically to obtain the plots shown in Figure 2-17. The settling velocity in Equation 2-43 is taken as a positive value with the downward flux associated with the negative sign on the term.

\[
\frac{\partial C}{\partial t} - w_z \frac{\partial C}{\partial z} - \frac{\partial}{\partial z} \left( D_z \frac{\partial C}{\partial z} \right) = 0
\]  

Equation (2-43)

Two noteworthy effects are the difference between observations under different levels of agitation in water, and the difference between field and laboratory measurements. Field data indicate higher settling velocities when the weather was calm (minor wave activity) than under higher wave action (with a swell period of about 15 s and sea period of 3.5 s). The slowing of settling by waves was attributed to the formation of a lutocline that contained large, more slowly settling aggregates than when the conditions were calm. A similar behavior is seen in the laboratory tests, in which settling was more hindered when the rings were oscillated than under quiescent condition. Differences between field and laboratory results indicate the effects of aggregation processes that are not easily scaled, and in turn highlight the importance of field observations.

In Figure 2-17 the effect of shear rate on the settling velocity appears to be important over the entire range of concentrations, although above \(~10 \text{ kg m}^{-3}\) there is the onset of a trend of convergence of the curves. This trend is the result of the role of high concentration, which damps turbulence and leads to increasingly congruent hindered settling of the sediment mass. Schematically, one would expect the actual trends to appear as shown by the dotted curves in Figure 2-18. Teeter (2001), however, assumed a simpler model in the flocculation settling range.
in which the exponent \( n_f(d) \) decreases with increasing floc diameter \( d_f \) in the cumulative floc-size distribution, as the effect of cohesion also decreases with increasing size.

The settling velocity was defined as shown in Equation 2-44.

\[
w_s(d) = w_{sm} \left( \frac{C}{C_h} \right)^{n_f(d)}
\]

(2-44)

Works of van Leussen (1994), Malcherek and Zielke (1996) and Teisson (1997) indicate the relationship in Equation 2-45, relating the settling velocity to a representative turbulence-mean flow shear rate \( \overline{G} \) (Hz or s\(^{-1}\)). The no-shear settling velocity, \( w_{s\overline{G}=0} \), is obtained from a concentration dependent settling velocity function such as Equation 2-32, and \( \lambda_1 \) and \( \lambda_2 \) are sediment-specific coefficients.

\[
w_s = \left( \frac{1 + \lambda_1 \overline{G}}{1 + \lambda_2 \overline{G}^2} \right) w_{s\overline{G}=0}
\]

(2-45)

The shear rate \( \overline{G} = v / \lambda_0^2 \), where the Kolmogorov eddy length scale is defined \( \lambda_0 = \left( \frac{v^3}{\varepsilon} \right)^{1/4} \),

\( v \) is the kinematic viscosity of the fluid (water) and \( \varepsilon \) is the turbulent energy dissipation rate.

Teeter (2001) combined Equations 2-44 and 2-45 to propose Equation 2-46.

\[
w_s = w_{sm} \left( \frac{C}{C_h} \right)^{n_f(d)} \left[ \left( \frac{1 + \lambda_1 \overline{G}}{1 + \lambda_2 \overline{G}^2} \right) e^{-\lambda_3 \frac{C}{C_f}} + 1 \right]
\]

(2-46)

From laboratory work on mud from San Francisco Bay, Teeter (2001) obtained values of \( \lambda_1 \) and \( \lambda_2 \) were found to be 266 and 9, respectively. Figure 2-19 shows a comparison between Equation 2-44 and data on the settling velocity of a natural mud over the range of concentration \( C_f = 0.06 \text{ kg m}^{-3} \) to \( C_h = 0.660 \text{ kg m}^{-3} \). With increasing (initial test) concentration the effect of shear rate is gradually subsumed by the effect of concentration, which eventually becomes nearly the sole determinant of the settling velocity.
Teeter found that the values of $n_f$ decreased with increasing floc size. The relationship as shown in Equation 2-47 is used here, with the exponent considered negative to agree with Teeter.

$$n_f = a_n \left( \frac{d}{d_p} \right)^{b_n} \quad (2-47)$$

### 2.2.4 Deposition and Erosion

The exchange of cohesive sediment at the water-bed interface is a function of the settling velocity of particles in suspension and the turbulent mixing that resuspends particles upward that have either reached the near bed by settling or by erosion from the bed by the bed shear stress. The level of the turbulent mixing can be characterized by the mean shear stress on the bottom.

Erosion has been observed in several modes in laboratory and field experiments (Mehta and Partheniades, 1982; Amos, et al., 1993), controlled by the shear stress and the vertical structure of the bed sediment. The strength of the bed to erosion is a reflection of the history of previous bed exchange. A recently deposited bed will tend to have a stratified structure because of the lack of sufficient time for consolidation. Older beds that may have been exposed by previous erosion will be compacted and generally stronger, with a more uniform vertical structure.

At low shear stresses particle surface erosion can occur when the inter-particle bond strength for primary particles or flocs adhered to the bed are broken. The response for a stratified bed, given sufficient time, will be asymptotic in suspended concentration as the bed erodes down to an equilibrium between the shear stress and the bed strength (Mehta and Partheniades, 1982). Amos et al. (1993) further defined supply limited asymptotic type I surface erosion as type Ia surface erosion. At intermediate shear stresses when the asymptotic behavior is unclear whether due to limited supply or a balance between deposition and erosion Amos et.
al. (1993) classified it as *type Ib* surface erosion, acknowledging the possibility of simultaneous erosion and deposition. Mehta and Partheniades (1992) classified a sediment response with linear increase in concentration, with no apparent limit in supply as would accompany an unstratified bed, as *type II* surface erosion.

At higher shear stresses the erosion patterns can become more chaotic and nonlinear without any clear trends toward equilibrium. Tolhurst et. al. (2009) refer to this as *transitional* erosion. At even higher shear stresses the shear load may be transferred into the bed and a weaker layer in the bed can fail causing the mass of sediment above it to become entrained in the flow. This mode of erosion has been called *type II bulk erosion*. Given sufficient time Tolhurst et. al. argue that even type II surface erosion would eventually reach an equilibrium.

In special conditions, particularly in wave environments, sediment can be eroded through the pulsing action of the waves to create a fluid mud layer near the bed. When the waves attenuate the fluid mud layer will generally redeposit. Another mode on erosion is the entrainment of fluid mud upward into suspension. Fluid mud has the ability to absorb wave energy through increased viscosity of the fluid mud layer itself, which can very efficiently dissipate the oscillatory wave energy. The present research focuses primarily on surface particle erosion, which is the most common mode of erosion within the estuarine environment.

Tolhurst et. al. (2009) discuss the fact that there is essentially no true critical entrainment shear stress, given that evidence of winnowing of small particles from the bed has been observed for very low shear stresses (Mehta and Partheniades, 1982; Black, 1991). This opens the question as whether such entrainment can occur during deposition. The assignment of a critical shear stress for erosion, therefore, requires the specification of a minimum sediment entrainment rate. No standardization of the threshold for “significant” bed flux has been identified.
There is usually a stark contrast between observed erosion in field experiments compared to laboratory experiments, where the flume beds are carefully constructed through controlled deposition or reforming of poured slurry. The field is inherently heterogeneous in bed structure and biological stabilization. Tolhurst, et. al. (2000) conducted comparisons on four different in-situ erosion devices in a field experiment and documented orders of magnitude differences in estimated erosion rate between devices. They attributed the variance to the differences in device size, the heterogeneity of the bed and differences in the device deployment time.

The simplified governing equation for the vertical distribution of suspended sediment in uniform flow can be expressed as in Equation 2-43. The boundary condition at the bottom of the water column can be written as in Equation 2-48.

\[ -w_s C - D \frac{\partial C}{\partial z} = -D + E \]  
(2-48)

The variable \( E \) is erosion and \( D \) is deposition. The sign of \( D \) is positive because \( w_s \) is defined to have a positive value. The erosion, \( E \), is also positive, since the concentration gradient is typically negative. Many investigators have asserted there is exclusively either erosion or deposition (Mehta and Partheniades, 1975; Lau and Krishnappan, 1993). The classical cohesive model for bed interaction treats erosion and deposition as mutually exclusive processes. The decision for whether deposition, erosion or no exchange occurs is based on the threshold shear stress, \( \tau_e \) and \( \tau_d \), the critical shear stresses for erosion and deposition, respectively. It is generally acknowledged that \( \tau_e > \tau_d \). When the bottom shear stress, \( \tau_b \), is less than \( \tau_d \) deposition will occur at a rate of the settling flux times a probability of deposition (Equation 2-49).

\[ D = P_d w_s C \]  
(2-49)

The probability of deposition was estimated by Krone (1962) as given in Equation (2-50).
When the bottom shear stress lies between the critical shear stress for deposition and erosion, then the classical cohesive bed model has no bed exchange.

When the bottom shear stress exceeds the critical shear stress for erosion, the erosion is estimated by Equation (2-51) (Kandiah, 1974; Ariathurai, 1974).

\[
P_D = 1 - \frac{\tau_b}{\tau_d} \quad \text{for} \quad \tau_b < \tau_d
\]

\[
P_D = 0 \quad \text{for} \quad \tau_b > \tau_d
\]

\[E = M \left( \frac{\tau_b}{\tau_e} - 1 \right)^\delta \quad \text{for} \quad \tau_b > \tau_e\]  \hspace{1cm} (2-51)

\(M\) is the rate of erosion coefficient, in units of \(\text{kg/m}^2/\text{s}\) and \(\delta\) is an empirical exponent. \(M\) is normally developed from analysis of bottom samples from the study site. An alternative formulation (Mehta and Parchure, 2000), which works well for stratified sediments characteristic of newly deposited beds (type I surface erosion; Mehta and Partheniades, 1982) is shown in Equation 2-52, which differentiates between a floc erosion rate, \(E_f\), and the bulk erosion rate, \(E\).

\[
\ln \left( \frac{E}{E_f} \right) = \alpha \left( \tau_b - \tau_e \right)^\beta \quad \text{for} \quad \tau_b > \tau_e
\]

The linear form of Equation 2-51 has been reported to work better for compacted beds (type II surface erosion) and the exponential form of Equation 2-52 works better for type I surface erosion (Tolhurst, et al., 2009).

The critical shear stress for erosion has been related to the shear strength of the flocs. The floc strength is the strength required to break the floc apart, while the critical shear stress for erosion is that required to dislodge the particle from the bed. (Mehta, 2007) relates the critical shear stress for erosion to the yield strength, which can be approximated as the floc strength.
(Equation 2-53) and the critical shear stress for deposition is assumed to be a power function of the relative floc size (class-referenced as $i$).

$$
\tau_{ei} = 0.256 \tau_{yi} \quad \text{for } \tau_{yi} > 1.6 \text{ Pa}
$$

$$
\tau_{ei} = 0.289 \sqrt[1.6]{\tau_{yi}} \quad \text{for } \tau_{yi} \leq 1.6 \text{ Pa}
$$

$$
\tau_{di} = \tau_{d0} \left( \frac{d_i}{d_p} \right)^{0.256}
$$

Otsubo and Muraoka (1988) developed a similar relationship for $\tau_{ei}$ with a single equation to the power of 0.6 and a coefficient of 0.27, which is close to the two-part equations in Equation 2-53. The relationship for the critical shear stress for deposition in Equation 2-53 is similar to a form developed by Mehta and Lott (1987), based on the relative fall velocities. If the fall velocities are computed as a function of the floc sizes, then these are functionally equivalent.

Partheniades (1962) developed a probabilistic estimate of the erosion flux, $\varepsilon$, of a cohesive bed, modifying the stochastic approach of Einstein (1950) for noncohesive sediments using the statistics of the velocity fluctuations. He derived an integral equation method that he could evaluate after assuming that the lift force is normally distributed, leading to the evaluation of error functions (Equation 2-54).

$$
\varepsilon = k_1 \left\{ 1 - \frac{1}{2} \left[ \text{erf} \left( \frac{1}{\sqrt{2\pi}} \left( \frac{F_C}{k \tau_y \sigma_L} - \frac{1}{\sigma_L} \right) \right) - \text{erf} \left( \frac{1}{\sqrt{2\pi}} \left( - \frac{F_C}{k \tau_y \sigma_L} - \frac{1}{\sigma_L} \right) \right) \right] \right\}
$$

Winterwerp (2007) advocated using a highly complex asymmetric shear stress distribution based on Petit (1999) arguing that the normal distribution assumed by Partheniades overlooked the importance of the skewness on erosion. The formulation of Petit, however, is cumbersome to use.
The conventional bed exchange theory for noncohesive sediments permits simultaneous erosion and deposition at the bed. This approach has been advocated by several investigators for use with cohesive sediments as well (Winterwerp, 1999; Sanford and Halka, 1993). Winterwerp advocates the inclusion of the settling flux as a continuous process with a probability of one applied. In order to account for the rates of sedimentation in a number of European estuaries, the depositional flux was required without constraint by a threshold shear stress.

Net deposition in the simultaneous models is a balance between settling flux and vertical turbulent mixing near the bed. As larger and weaker flocs settle toward the high shear zone near the bed, they may be torn apart and reentrained into the upper water column. The argument for exclusiveness in the processes is partly based on the logic that if a floc is strong enough to settle through the shear layer near the bed and become attached to the bed, then it will be strong enough to withstand the shear stresses to erode it until the shear increases.

The seeming paradox of simultaneous or exclusive erosion and/or deposition is illustrated in Figure 2-20. The question can be posed; if the net flux across an arbitrary boundary some distance, \( \delta \), above the permanent water bed interface is either a net downward (deposition) or a net upward (erosion), is the interaction at the bed interface either all erosion or all deposition or can it be both erosion and deposition with the net at the water bed interface equaling the net at the arbitrary boundary at \( \delta \)? The logical step is to allow \( \delta \) to approach zero and argue that in the limit the two conditions should match, implying that the exclusive model should fail. But the argument can still be made that as \( \delta \) approaches zero the net flux over the upper boundary should asymptote to the exclusive erosion or deposition. The argument for exclusivity would be that as \( \delta \) approaches zero, one or the other of the erosion or deposition terms will vanish at the upper surface as appropriate to the overall net flux at the bed interface.
A more pragmatic line of logic is to consider the temporal variation in the bottom shear stress exerted on the bed. Winterwerp (2004) proposed the use of a probability distribution of the shear stress because of the influence of near-bed turbulent bursts and sweeps that are common in estuarine scale hydrodynamics. Even in relatively homogeneous “steady” flow conditions, some significant variability remains in the turbulence at time scales shorter than what would be required to average to a “steady-state” hydrodynamic condition. In addition, he argues that the asymmetry in the shear stress distribution can be critical to the estimation of either erosion or deposition potential. This will be discussed further in the next chapter.

2.3 Cohesive Sediment Transport in Unsteady Nonuniform Flow

In unsteady nonuniform flow, all of the processes discussed previously will continue to be important. The CST processes will now likely be put into a continuous state of being out of equilibrium. Unsteady and nonuniform hydrodynamics will now introduce additional factors into the processes. The importance of these additional effects will vary for different cases. Most studies assume the time and space scales of tidal variations are large compared to the time scales for CST processes. Figure 2-21 presents the inverse of the fall velocity, converted to hours, based on the settling velocity curve for Equation 2-31 in Figure 2-10. The parameters used in Equation 2-31 were $d_{50} = 2$ microns, $D_{fc} = 2.1$ and $d_{fc} = 8$ mm. This represents the length of time required for the particle of a given size to settle one meter. Assuming that one meter is an appropriate scale for CST processes in the vertical, and that the tidal time scales of significant changes in currents and water depth is three hours, then the settling of particle diameters smaller than 25 microns will be significantly affected by tidal energy. The minimum in the curve corresponds to the maximum fall velocity.

Because the time step involved in most numerical CST calculations is controlled by the fastest process, the time steps are on the order of seconds to minutes. As long as the parameters
that are affected by the changing hydrodynamics are updated in each time step, the changes in
the impacts will be realized. However, additional terms may need to be added to the
calculations.

2.3.1 Aggregation Processes

Aggregation processes from steady uniform flow conditions will continue to be important
in an unsteady nonuniform environment. Unsteady flows include inertial effects associated with
accelerating or decelerating flows and changes in water depth. The larger more massive particles
will be slower to respond to accelerating than the smaller less massive particles, creating
additional differential velocities. These could augment either aggregation or disaggregation.

Maggi (2005) presents a conceptual description of an example time evolution of
aggregation and disaggregation for the case of a fixed steady state shear condition. Figure 2-22
shows the case of steady-state hydrodynamics, but serves to conceptualize the feedback between
the aggregation, disaggregation and the flow conditions. The initial state has the floc size
distribution (FSD) that is well out of equilibrium with the shear condition. The FSD has a
maximum floc size that can generally withstand the shear condition. Flocs larger than that size
have a high likelihood of eventual disaggregation. Flocs smaller have a high likelihood of
aggregation. The steady-state shear condition is an average condition, with some degree of
variability in shear. The plot at the top in the figure shows the temporal variation in the number
of flocs within a unit volume. As aggregation occurs, the number of particles is reduced. With
disaggregation the number of particles increases. The plot at the bottom shows the concurrent
FSD, which reflects the aggregation/disaggregation cycling. It could be possible that steady-
state equilibrium may never be achieved.

The relationship between the modal floc diameter within the FSD, the shear stress and the
concentration was presented by Dyer (1989) and is shown in Figure 2-23. At very low shear
stresses the modal floc diameter increases dramatically with increasing concentration. The lower shear has a positive effect on flocculation. As the shear stress increases, the destructive effects of shearing begins to control the maximum floc size. At higher shear stresses, an increase in concentration increases the likelihood of particles collisions that create disaggregation and the modal floc diameter is reduced even further. McAnally (1999) developed an aggregation model that included three-body collisions (as recommended by Burban, et. al., 1989). In an attempt to replicate Dyer’s Figure (2-23), the model significantly over predicted floc sizes for low-shear conditions. The over-prediction of floc sizes at low shear stresses could arise from either over-predicting the aggregation or under-predicting the disaggregation. At low shear rates it is more likely that the aggregation is over-predicted. The balance between aggregation associated with differential settling and with shear could be a factor in the overprediction, since at the no-shear boundary the model should be using mainly differential settling. The inclusion of three-body collisions could also be a factor.

Burban et. al. (1989) tested the effects of shear and concentration and concluded that the mean floc diameter decreased with increasing concentration. The shears tested were all greater than 0.1 Pa, assuming that \( \tau_b \approx \mu G \), all above the low shear peak in Figure 2-23.

With the conceptual picture in Figure 2-22 of the FSD and the aggregation/disaggregation processes oscillating about steady-state equilibrium, consider the impact of continually moving the target equilibrium associated with varying tidal conditions in an estuary. The target equilibrium condition as characterized by the shear will be constantly changing.

### 2.3.2 Settling Velocity

The particle settling velocity is normally assumed to be independent of the hydrodynamics. The concept of settling velocity is based on equilibrium between the gravitational and drag
forces on the particle as it falls through quiescent water, reaching a terminal velocity. It varies with the fluid density, particle size, particle density and viscosity. The assumption is made throughout the science that the settling velocity can be linearly superimposed on the current velocity field as a vector addition. Because of that assumption, the settling velocity, as used, will not be affected by either unsteady or nonuniform hydrodynamics. However, variations in the shape of the particles and effective density for flocs leads to some variability in the settling velocity for a nominally defined particle size.

2.3.3 Bed Exchange

Effects of unsteady and nonuniform hydrodynamics on bed exchange will only be potentially seen in the erosion rates, based on the arguments above about the setting velocity. Because the theory behind whether a particle or floc is disengaged from the sediment bed is based on the instantaneous hydrodynamic lift and drag and the properties of the particle, it is reasonable to assert that the only way that the erosion can be affected is through the turbulence characteristics within the bottom boundary layer. If changing hydrodynamics has an impact on the frequency and intensity of turbulent bursts and sweeps then it may influence that erosion rate. That influence would have to be handled in a statistical manner to be tractable (Butterfield, 1993).

The other mechanism for having variable hydrodynamics influence bed exchange is through the cumulative effects of gradients in space and time to create localized perturbations in either hydrodynamics or the suspended sediment concentration field, which can then have second order effects on the CST processes. Such an effect is the phenomenon of a turbidity maximum.

Tidal sorting leads to mixtures of cohesive and noncohesive sediment for which the mobility of the noncohesive sediments becomes related to the fraction of cohesive sediment
present in the bed. Mehta and Lee (1994) developed a relationship (Equation 2-55), taking the
effects of cohesion into account.

\[
\frac{\tau_{cs}}{g(\rho_s - \rho_w)d} = \frac{\alpha_3 \tan \phi_a}{(\alpha_1 + \alpha_2 \tan \phi_a)} + \frac{F_c \tan \phi_a}{g(\rho_s - \rho_w)d^3}
\] (2-55)

The force of cohesion, \(F_c\), adds an additional component to be overcome to have sediment
mobility. If \(F_c\) vanishes, then Equation 2-55 takes on the form of the original Shields
relationship. The coefficients \(\alpha_1\), \(\alpha_2\) and \(\alpha_3\) are related to shape factors and ratios of lift to drag
coefficients and \(\phi_a\) is the angle to the critical contact point for pivoting for particle entrainment.

For the case of no cohesive force the angle \(\phi_a\) can be estimated form Equation 2-56.

\[
\phi_a = \cos^{-1}\left[ \frac{d + z_*}{k_s} \right]
\] (2-56)

where \(z_*\) is the average level of the bottom of the almost moving particles. A value of -0.02 for
\(z_*\) is recommended.

2.4 Governing Equations

The governing equations for CST in unsteady and non-uniform flows is presented in
Equation 2-57n for the suspended sediment concentration of a single sediment class (class
number \(n\)). The derivation of this equation in tensor notation can be found in Appendix A
(Equation A-164n). The subscript ‘\(n\)’ is placed on the equation number to denote that there are
multiple equations, one for each size class. Equation 2-57n is time averaged after turbulent
decomposition into the mean and fluctuating components.

\[
\frac{\partial \bar{c}_n}{\partial t} + \frac{\partial \bar{u}_n \bar{c}_n}{\partial x_j} - \frac{\partial}{\partial x_j} \left( \delta_{j3} \frac{\bar{w}_n \bar{c}_n}{\bar{c}^2_n} \right) - \frac{\partial}{\partial x_j} \left( \delta_{j3} \frac{\bar{w}_m \bar{c}_n}{\bar{c}^2_n} \right) = \bar{S}_n + \frac{\partial}{\partial x_j} \left( D_m + D_y \right) \bar{c}_n
\] (2-57n)
The index ‘j’ indicates the coordinate components 1 through 3, with 3 being the vertical coordinate. The velocity component, \( u_{jn} \), admits the possibility that the particle velocity may be different from the fluid velocity. \( \delta_{j3} \) is the Kronecker delta function which equals one when \( j = 3 \) and zero otherwise. This equation incorporates the temporal change in sediment class concentration, the spatial advective gradients, particle settling, turbulent diffusion, a turbulent correlation term between the settling velocity and the concentration and a source/sink term. The source/sink term incorporates the aggregation and disaggregation processes. The diffusion term \( D_m \) is molecular diffusivity and \( D_j \) turbulent diffusivity.

The governing equation for the total concentration is presented in Equation (2-58)

\[
\frac{\partial \tilde{C}}{\partial t} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \tilde{u}_n \tilde{c}_n - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} (\delta_{j3} \tilde{w}_n \tilde{c}_n) - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} (\delta_{j3} \tilde{w}_n \tilde{c}_n') = \sum_{n=1}^{N} \tilde{S}_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left( D_m + D_j \right) \frac{\partial \tilde{c}_n}{\partial x_j}
\]

(2-58)

Strictly speaking, the total concentration equation is redundant. If all of the individual classes are conserved by virtue of the proper specification of the source/sink terms, then the total will have to be conserved. The total concentration equation imposes that balance in the interaction terms, including interaction with the bed as a boundary condition. The boundary condition at the water surface is a no-flux condition. These boundary conditions are shown in Equation 2-59n.

\[
\begin{align*}
\left\{ (u_3 - w_n) c_n - (D_m + D_{i3}) \frac{\partial \tilde{c}_n}{\partial x_3} \right\}_{z = z_{ms}} &= 0 \\
\left\{ (u_3 - w_n) c_n - (D_m + D_{i3}) \frac{\partial \tilde{c}_n}{\partial x_3} \right\}_{z = z_{s}} &= E_n - D_n
\end{align*}
\]

(2-59n)
The bed boundary condition is applied to each size class. The issue of the simultaneous versus the exclusive erosion and deposition remains unresolved. However, the formulation of the size distribution of the bed exchange is affected by whether the exclusive or simultaneous theory is used due to required resolution of inconsistencies in the relationships between floc strength and the implementation of bulk critical shear stresses.

2.5 Applicability of Existing Knowledge to Field Conditions

Steady-state CST processes have been applied directly to real world problems with dramatic unsteady and nonuniform features with reasonable success. Some difficulties have been observed in obtaining sufficient vertical mixing of suspended sediment into the upper water column when applying the steady-state developed processes (Czernuszenko, 1998).

Extreme variability in the documented CST variables for specific sites indicates that the use of existing theories requires site-specific data for calibration of the models.

2.6 Biological Influences

Most sediments in natural waters are influenced by biological activity. Organisms, either living or deceased, can have an influence on sediment processes. Deceased organisms, both plant and animal, decay and provide natural organic matter (NOM) that becomes mixed with mineral sediment particles to form complex structures (Paterson, 1997). A wide range of biota produces cell exudates, which are of great importance in sediment stabilization. These exudates, as well as fecal pellets, are also NOM. Live organisms also disturb the settled bed, which is their habitat, resulting in bioturbation. The effects of bioturbation can be either beneficial or detrimental to stability of the bed with regard to erosion.

Adsorbed organic matter can affect clay particles by influencing interparticle bonds (Bennett, et. al., 1991). Flume tests by Dennett et. al. (1998) with kaolinite showed that increasing organic matter levels (% carbon) from 0.0 to 0.12% caused increasing resistance to
erosion. This was explained by a change in the predominant type of associations between plate-like clay particles. Without organics the particles are mainly edge-to-face (E-F) bonds in settled suspensions, resulting in a loose structure with large water content. The positively charged edges are preferentially associated with the negatively charged faces of the clay minerals. With increased organic content the large negatively charged macromolecules of NOM associate mainly with the positively charged particle edges. The number of face-to-face (F-F) particle associations increases, with a reduction in the porosity and water content of the flocs. Initial erosion rates, defined from the slope of the concentration curve at initiation of erosion, and critical shear stresses were lowest for the no-organic tests and highest for intermediate organic levels (0.006 to 0.009% carbon) where the structure was a mixture of face-to-face and edge-to-face, creating a braced structure. At higher organic contents the edge-to-face structures began to break down and the initial erosion rate and the critical shear stress both increased.

Duck and McManus (1991) conducted laboratory experiments confirming the influence of organic matter on settling velocity via enhanced aggregation. Settling tests were replicated after removal of organic matter from the samples by oxidation with hydrogen peroxide. Settling tests were performed in natural water and dispersing solution for samples with and without organics. The results showed that without organics the accelerated settling associated with aggregation was not observed as it was in the sediments with organics.

Organic matter also may contribute to more than 60% of the volume of mud particles in situ (Greiser, et. al., 1996). Jarvis, et. al. (2005) conducted experiments on the growth, breakup and regrowth of flocs with NOM. They found that flocs that contain polymer chains, which were broken during floc breakup, did not reform to the previous floc sizes since polymer destruction is not reversible. However, flocs formed from physical/chemical bonds were found to replicate
floc sizes in growth, breakup and regrowth. This finding implies that organic contributions to floc size will be more significant in low shear environments, such as lakes, reservoirs and the continental shelf than in high energy coastal and estuarine waters.

The presence of living organisms in the benthic zone leads to impacts on both cohesive and noncohesive sediment processes. A biofilm coating on noncohesive mineral grains can make them behave cohesively and contribute to aggregation.

Biological activity of living organisms can have a negative or positive impact on the stability of the bed sediments. Biogenic stabilization results in a decrease in erodibility of sediment from the bed surface (Paterson and Daborn, 1991). As mentioned bioturbation of the bed can also have either a positive or negative impact on stability (Paterson, 1997). Bioturbation creates challenges in analysis of depositional events for geologists when the biodiffusivity and organism density are sufficient to destroy the stratigraphic record (Wheatcraft and Drake, 2003).

Benthic diatoms, which produce large amounts of polymer (mucilage), are particularly important (Holland et. al., 1974) in cohesive sediment processes. These polymers form a stabilizing cohesive framework bridging between sediment particles and diatoms (Paterson, 1997). Cyanobacteria and blue-green algae are also of major importance to bed stability (Wetherbee et al., 1998). Growth of bacteria, algae and diatoms at the bed surface, filling voids and reducing hydraulic roughness can result in biostabilization of the bed. The net effect is a reduction in the shear stress at the bed and reducing erodibility.

Olafsson and Paterson (2004) found in laboratory testing that the effects of biogenesis can be both physical and chemical. They showed that shear strength in the upper centimeter of the bed increased with the bed density of tube-building chironomid larvae, which build their tubes with silk that spreads over the bed surface adjacent to the tubes.
Tolhurst, et. al. (2008) investigated the formation of microphytobenthic biofilm in the laboratory over a 45-day test period. They found that the erosion threshold was positively correlated with the water content, chlorophyll a and colloidal carbohydrate within the upper 2 mm of the bed. However, the bulk density was highly variable.

Lumborg, et. al. (2006) studied biogenic effects on an intertidal mud flat, finding that the cohesive sediment dynamics was more controlled by benthic biology than by physical parameters. Haubois, et. al. (2005) describe the significance of motile epipelic diatoms to move vertically within the bed layer in intertidal mud flats, effectively distributing the biological influence over a deeper zone.

Clay particles with organic coatings have a high adsorption capacity for metals and hydrophobic organic contaminants (Dennett, et al., 1998) and play an important role in their transport and fate.

The added variability of the degree of influence of biota and organics as a stabilizing effect on erosion and an influence on aggregation of both cohesive and noncohesive mineral particles makes cohesive sediment processes presently more unpredictable and variable, both in space and time. The variation in time is most significant on a seasonal scale. However, the spatial variability makes the use of bulk parameters to characterize the behavior of such complex interactions a crude approximation.
Figure 2-1. Example particle size distributions, showing the MIT size classification. The flocculated San Francisco Bay sediment has a size distribution comparable to beach sand.
Figure 2-2. Example of size distributions for dispersed particles and for flocculated sediments in suspension expressed as a volume fraction. The symbols are the measurements for San Francisco Bay sediments. The lines are the fits of Equation 2-3 distributions (adapted from Kranck and Milligan, 1992).
Figure 2-3. Conceptual model of order of aggregate flocculation processes (adapted from Krone, 1963).
Figure 2-4. Effects of the settling decay term, $K$, on the particle size distribution spectra in volume fraction (adapted from Kranck and Milligan, 1992).
Figure 2-5. Measured particle size distributions in San Francisco Bay: a) dispersed grain distributions and b) flocculated distribution (from Kranck and Milligan, 1992; reprinted with permission).

Figure 2-6. Dispersed particle distributions for nearby bed samples for San Francisco Bay site (solid) and for suspended sediments in San Pablo Strait. All distributions had a slope, $m$, near zero (from Kranck and Milligan, 1992; reprinted with permission).
Figure 2-7. Relationship of shear strength of flocs to the excess density of the flocs (data from Krone, 1963). The data values are from multiple harbors; curve fits are shown for each harbor, for all of the data combined. A curve fit with the exponent on the density of 2.5 (Partheniades, 1993) is shown in red.
Figure 2-8. Effect of salinity on settling velocity (adapted from Krone, 1962). Final peak floc size was estimated from Stokes Law using the final peak settling velocity.
Figure 2-9. Comparison of Equations 2-17 and 2-18 for the drag coefficient as a function of the particle Reynolds number.
Figure 2-10. Comparison of Equations 2-26 and 2-31 for settling velocity versus floc diameter (adapted from Winterwerp, 1999).
Figure 2-11. Settling velocity versus floc diameter from Chesapeake Bay and Tamar (UK) estuary (adapted from Winterwerp, 1999).
Figure 2-12. Settling velocity versus floc diameter from VIS, Ems '89 and Ems '90 (adapted from Winterwerp, 1999).
Figure 2-13. Comparison of fit curves for individual data sets of fall velocity with a single fit to all data.
Figure 2-14. Fractal dimension from Equation 2-19 used in Equation 2-31 as plotted in Figure 2-10; $d_k = 8000$ microns, $D_k = 2.6$, $d_{50} = 2$ microns.
Figure 2-15. Data comparison of model for settling velocity with a power law for the fractal dimension (from Khelifa and Hill, 2006, reprinted with permission).
Figure 2-16. Effects of variable fractal dimension on the effective (excess) density as a function of floc diameter (from Khelifa and Hill, 2006).
Figure 2-17. Variation of settling velocity with suspended sediment concentration. Results based on field and laboratory tests using sediment from Cleveland Bay, Australia (adapted from Wolanski et al., 1992).
Figure 2-18. Schematic diagram showing the dependence of settling velocity on floc size in the flocculation settling range (adapted from Teeter, 2001).

Figure 2-19. Comparison of Equation 2-46 with settling data for varying initial concentration and shear rate (adapted from Teeter, 2001).
Figure 2-20. Paradox of simultaneous versus exclusive erosion and deposition: a) bed deposition, b) bed erosion.
Figure 2-21. Number of hours required to settle a distance of one meter as a function of particle size, based on the Equation 2-29 curve in Figure 2-10.
Figure 2-22. Conceptual model of the feedback between aggregation and disaggregation with flow conditions (from Maggi, 2005; reprinted with permission). Top: shows the temporal variation in the number of flocs within a unit volume. As aggregation occurs the number of particles is reduced. With disaggregation the number of particles increases. Bottom: shows the floc size distribution (FSD), which reflects the aggregation/disaggregation cycling.
Figure 2-23. Relationship between the modal floc diameter and shear stress and concentration (from Dyer, 1989).
CHAPTER 3
PROBABILISTIC DESCRIPTION OF COHESIVE SEDIMENT TRANSPORT

3.1 Conceptual Framework

This chapter identifies those variables for consideration in the evaluation of the influence of probabilistic representation within CST processes related to bed exchange.

One way to view CST processes is from the degree of uncertainty in the processes. Uncertainty can result from lack of adequate knowledge of the processes or ability to measure them accurately (epistemic uncertainty), or from the random nature of the processes themselves (natural uncertainty) (Merz and Thieken, 2005). Epistemic uncertainty can be reduced through research and improvements in data collection techniques. Natural uncertainty cannot be reduced, just better sensed. Improving the estimate of the natural variability, by definition, is epistemic uncertainty reduction.

For CST processes, uncertainty in the size distribution of the primary particles would be classified as epistemic uncertainty. The actual size distribution is a real entity that we are limited in estimating by measurement and analysis procedures. Uncertainty in bottom shear stress is clearly controlled by natural uncertainty. The best possible instrumentation will not change the fact that turbulence is a random process.

“Hybrid” uncertainty arises from the interaction of epistemic uncertainty with natural uncertainty. An example is the settling velocity of a particular particle size. The size of the particle is something that we could define with acceptable uncertainty, but the process of its falling through a fluid introduces some level of natural uncertainty, associated uncertainty in the drag coefficient for higher particle Reynolds numbers and interference between particles. The level of uncertainty in the fall velocity for a spherical grain is relatively small. However, variation in the shape increases the uncertainty. Flocs of different density, shape and fractal
dimension introduce even more uncertainty. The configuration in which cohesive particles combine in flocs is influenced by the details of the double layer around each particle. The variation in the size and shape of the particles introduces variability in the cohesive bonds. The fractal dimension of sediment is a reflection of the statistical tendency seen in a large number of sediment flocs. The details for an individual floc will deviate about the mean tendency. The uncertainty could be quantified as the variance in observed fall velocities for a particular equivalent size particle. Settling velocity contains both epistemic and natural uncertainty.

In resolving the complexity of CST processes, a closer look at uncertainty in each component process is warranted. Figure 3-1 presents a simplified view of sediment transport processes, depicting the pathways of an individual sediment grain as it transits through various states and locations. The sediment encounters various processes during the transitions between floc conditions in suspension, deposition to and erosion from the bed.

Table 3-1 lists the concepts of importance to sediment transport. The entities of interest are the water and the sediment, both having their associated properties. Properties are further divided into the basic properties: size shape and density for the sediment and temperature, conductivity and salinity for water. Density could be considered a basic property of water, but it is defined by the other properties (salinity, temperature and suspended matter). A much longer list of water quality parameters could be listed, but are not pertinent here. Any uncertainty in each of these basic properties is epistemic, since these properties are definitive tangible values.

Table 3-1 also indicates where sediment and water may be located as they factor into CST. Water can be significant when its properties within the floc interstices vary from the ambient conditions within the flow. The sediment properties of significance depend on whether the sediment is in suspension or in the bed.
Water and sediment take on additional properties based on state properties, in the sense that they are in a condition that creates the property. The fall velocity of the sediment comes from the state of settling, while the internal shear of water is created by the state of turbulent flow. State properties, it is assumed, are the result of stochastic processes and their uncertainty is natural.

Primary processes within sediment transport are also listed in Table 3-1. With the exception of cases of laminar settling in quiescent conditions, all of the other processes are driven by the turbulence of the flow. These processes are therefore considered stochastic.

Bulk properties of both water and sediment are included in Table 3-1. Water can be characterized by its density and viscosity. The flow can be characterized by the level of turbulence (turbulent kinetic energy and turbulent dissipation). The sediment is characterized by critical shear stress for transport for sediment and the bed shear strength with respect to erosion, the floc strength, a threshold shear stress for deposition and the fractal dimension of cohesive sediment. The source of uncertainty for bulk properties can be both natural and epistemic. How to best deal with these variables within CST analysis, in part, depends on the origins of the uncertainty. The primary cohesive sediment transport variables will be considered in greater detail.

It is of value to clarify the definitions used herein of probabilistic variables and stochastic variables. Probabilistic representation of variables defines the range and frequency of occurrence of specific values of the variable. The distribution of those probabilities may represent fixed known values, such as the grain size distribution. A probabilistic distribution does not necessarily provide any insight into the processes that contribute to that distribution. Stochastic variables have a random characteristic. Their statistical (probability) distribution may
resemble the distribution of non-stochastic variables, but only when the duration of temporal averaging is sufficiently long to reach a stationary distribution. Stochastic variables may have significant variability when the time span is relatively short.

In this chapter the focus is on use of probabilistic frequency distributions for both stochastic and non-stochastic variables.

The variables can be represented in one of several ways in the analysis:

- The variable can be assumed to be a constant.
- The variable can be assumed to have a stationary statistical distribution for fixed discretized values.
- The variable can be assumed to have a stationary statistical distribution, with a continuous analytical specification that is easily integrated.
- The variable can have a non-stationary statistical distribution, with explicitly discretized values, whose probability changes with time.

### 3.2 Particle Definitions

The particle size distribution is a basic property of the sediment and requires explicit discretization for CST analysis. By incorporating size classes into the analysis, much of the probabilistic character is handled. If there is significant variability of the properties within a single size class, it can be addressed by increasing the number of size classes.

The particle size classes need not distinguish between flocculated and non-flocculated sediments, primarily because their distributions do not normally overlap significantly (see Figure 2-5). Larger particles and smaller flocs within the same size range exhibit significantly different densities. Some thought was given to the possibility of differentiating between the two. Teeter (2001) included separate classes for cohesive and noncohesive sediments, without interaction within aggregation and disaggregation. The definition of two classes of sediment is straightforward. However, in order to keep a distinction between particles and flocs the particle
size distributions within the flocs would have to be tracked through the aggregation and
disaggregation processes. That effort is not deemed feasible at this time. Aggregation and
disaggregation conceptual models used today can move sediment mass between size classes as
appropriate, but do not track the variable particle sizes making up the flocs.

3.3 Shear Stress

The shear stress to which sediment is exposed is the primary property affecting bed
exchange and overall CST processes. It controls both erosion and deposition and, through its
relation to the internal shear, influences flocculation. Furthermore, turbulence in the flow field is
the most significant stochastic property among the variables in Table 3-1. As mentioned in
Chapter 2, Partheniades (1965) developed an approach that handled the bottom lift force with a
Gaussian probability distribution. The erosion flux was developed by integrating the probability
distribution over limits defined by a balance between the lift force on the sediment particles and
the restraining forces of buoyant weight and cohesion in an analogous approach to threshold
motion in noncohesive sediment. The derivation followed the approach of Einstein (1950) and a
probability of erosion, \( P_e \), was introduced as an integral of the normally distributed lift force (or
shear stress) fluctuation. The erosion flux, \( \varepsilon \), was estimated as shown in Equation 3-1.

\[
\varepsilon = MP_e = M \left\{ 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{\tilde{F}_L}{\sigma_e^2}} d\omega \right\}
\]

The erosion rate constant, \( M \), is usually developed from laboratory erosion experiments for
specific sediment and has units of kg/m²/s. The combined forces of buoyant weight and
cohesion are included in the force variable \( F_c \). The hydrodynamic lift is \( F_L \), and the variable \( \sigma_c \) is
the standard deviation of the normalized lift fluctuation variable \( F'_L / F_L \), the ratio of the
fluctuating lift force to the time-averaged mean lift. The value of $\sigma_L$ was estimated from measurements to be approximately 0.5.

Sharma (1973) reported the variation in the relative turbulent intensity for air (defined in Equation 3-2), the ratio of the root mean square (RMS) turbulent horizontal velocity fluctuations to the mean velocity, to range between 14% to 16%. The relative turbulent intensity is a normalized standard deviation.

$$Relative \ turbulent \ intensity = \frac{\sqrt{u'^2}}{\bar{u}} \quad (3-2)$$

Sharma also reported the normalized boundary shear stress standard deviation (replacing velocities with shear stresses in Equation 3-2) to have a range of 18% to 30% for air. He performed a numerical evaluation of the shear stress distribution, wherein he assumed that the equation for the mean shear stress as a power function of the mean velocity was also applicable to the instantaneous velocity fluctuations to estimate the instantaneous shear stress. He showed that the estimated shear stress fluctuations were 43% to 93% higher than the shear stress variations measured by a flush mounted hot film probe. From this he concluded that the shear stress response to velocity fluctuations (at 1 mm from the boundary) is either not instantaneous or that there may be a better correlation closer to the boundary.

The mean shear stress is computed by Equation 3-3, given the probability distribution function, $y(\tau)$, of the instantaneous bed shear stress $\tau$.

$$\bar{\tau} = \int_{-\infty}^{\infty} \tau \ y(\tau) \ d\tau \quad (3-3)$$

The normalized standard deviation is computed via Equation 3-4, with the coefficients of skewness and kurtosis, via Equations 3-5 and 3-6, respectively.
rather than using the Gaussian distribution, Winterwerp and van Kesteren (2004) used an asymmetric probability distribution of the shear stress, $y(\tau)$, developed by Petit (1999) as shown in Equation 3-7. They argued that the process of turbulent ejections of vortices and subsequent sweeps downward into the bed create asymmetric shear stress distributions. The importance of these bursts of shear to bed interaction may be significant.

$$y(\tau_b; \mu, m, s) = \frac{1}{2m} \left[ 1 + \text{erf} \left( \frac{m(\tau + \mu) - s^2}{ms\sqrt{2}} \right) \right] e^{-\left( \frac{2m(\tau + \mu) - s^2}{2m^2} \right)}$$  \hspace{1cm} (3-7)$$

The variable $\mu$ is the mean value and $m$ and $s$ are shape variables for the distribution. The higher moments of this asymmetric distribution are computed by Equation 3-8.

$$M_n = \int_{-\infty}^{\infty} \tau^n \cdot y(\tau; \mu, m, s) d\tau \hspace{1cm} (3-8)$$

The parameters in the distribution function are defined in Equation 3-9.

$$\mu = M_1$$

$$m = \frac{1}{2} \left( 8M_1^3 - 12M_1 M_2 + 4M_3 \right)^{1/3} \hspace{1cm} (3-9)$$

$$s = \frac{1}{2} \sqrt{-2^{3/5} \left( 2M_1^3 - 3M_1 M_2 + M_3 \right)^{2/3} - 4M_1^2 + 4M_2}$$

An example asymmetric distribution used by Winterwerp and van Kesteren (2004) was fit to a distribution of Equation 3-7 using $\mu = 0.03$, $m=1.01$ and $s=0.56$ (presented in Figure 3-2, data from Obi, et al., 1996). In addition, they developed shear stress distributions for application
to three sedimentation experiments of Krone (1962) as shown in Figure 3-3. The distribution curves were fit to Equation 3-7 using a multi-parameter optimization algorithm to minimize the error in the fit to determine the parameters \( m \) and \( s \) given the mean, \( \mu \). The parameters used in the fitting of Equation 3-7 to these data are presented in Table 3-2.

A skewed shear stress distribution can be attributed to the nonlinear transformation of the velocity to a shear stress. A normally distributed velocity record will yield a skewed shear stress. Assume a normal distribution for the velocity. To illustrate the transformation a Monte Carlo simulation was performed using the velocity cumulative distribution function (CDF). The CDF spans the full range of velocity values from accumulative probability of 0 for the minimum to a cumulative probability of 1 for the maximum. Random numbers were generated between 0 and 1 to select an associated velocity value from the velocity CDF. The shear stress was then computed for that velocity based on Equation 3-10, assuming, as Sharma (1973) did, that it is valid for the fluctuating shear stresses from the fluctuating velocity.

\[
\tau = f_c \rho_f \frac{\bar{u}^2}{2}
\]  

(3-10)

The variables in Equation 3-10 are the friction coefficient, \( f_c \), the fluid density, \( \rho_f \), and the velocity, \( u \). The simulation used ten million random velocity values and 200 partitions of both velocity and shear stress. The results of one simulation are presented in Figure 3-4 for a velocity distribution with a 0.5 m/s mean velocity and a standard deviation of 0.09 m/s. The variables used in Equation 3-10 were selected as 0.005 for \( f_c \) and 1,027 kg/m\(^3\) for \( \rho_f \). The Monte Carlo reconstitution of the input normal distribution is plotted against the analytical normal distribution function to confirm the velocity distribution. The resulting shear stress distribution has a mean value of 0.650 Pa, and a standard deviation of 0.232 Pa. The normalized shear stress standard deviation is 35.4%. The coefficient of skewness is 0.535 and the coefficient of kurtosis is 3.372.
For comparison, a normal distribution has a skewness of 0 and a kurtosis of 3.0. The shear stress distribution is skewed toward higher shear stresses and the distribution is more peaked than a normal distribution.

Monte Carlo simulations were repeated for the three experiments of Krone (1962) for which Winterwerp and van Kesteren (2004) developed shear stress distributions. Results of these simulations are presented in Figures 3-5, 3-6 and 3-7 for the flume tests with mean velocities of 0.113 m/s, 0.134 m/s and 0.152 m/s, respectively. The figures compare the assumed normal velocity distributions for each test with the stochastic velocity PDF generated by the Monte Carlo simulation and compare the Monte Carlo simulation shear stress distribution with the shear stress distributions presented by Winterwerp and van Kesteren. Standard deviations of the velocity distributions were adjusted in the Monte Carlo simulations to get a match of the shear stress distribution curves of Winterwerp and van Kesteren. The parameters for each of the Krone test curves are summarized in Table 3-3.

The effects of the standard deviation of the velocity distribution on the shear stress distribution are summarized in Figures 3-8, 3-9 and 3-10 for mean velocities of 0.5 m/s, 1.0 m/s and 2.0 m/s, respectively. The statistics of these sample distributions are presented in Table 3-4. As expected, increasing the velocity standard deviation increases the standard deviation of the shear stress. The modal shear stress is reduced with increasing velocity standard deviation. The normalized standard deviation for the shear stress is amplified compared to the normalized standard deviation of the velocity distribution as shown in Figure 3-11. The slope of the trend in the Monte Carlo simulation data matches the exponent (2) on the shear stress-velocity relationship in Equation 3-10. Using Equation 3-10 for the transformation correlates the velocity and the shear stress and results in greater amplification of the variance measured by Sharma.
(1973). However, as Sharma concluded the spatial separation in his measurements may account for the reduced correlation.

This analysis illustrates that a specific shear stress distribution need not be explicitly specified if the velocity distribution can be estimated and the shear stress relationship to the velocity is known. The bottom shear stress will be tested for the importance of probability distribution on the accuracy and sensitivity of simulations of CST processes.

**3.4 Aggregation and Disaggregation**

Variables that enter into the aggregation processes include the concentration of the individual size classes, the internal turbulent shear and the settling velocity of each floc size class. The aggregation model accounts for the variation in floc size and concentration explicitly. Internal shear can be developed as a probabilistic distribution if the bed shear is treated as a distribution. Collision probabilities are also related to the number concentration of particles and their sizes.

The settling velocity of flocs for a specific equivalent spherical diameter may vary with floc density and shape, resulting in differences in their response to the hydrodynamics. Specific measurements of settling will include both natural variations as well as errors in the sampling process itself. When large numbers of sediment particles are tested, the natural variability will tend to average itself out. Variations could be important when addressing differential settling related aggregation. However, if fluctuations in apparent settling velocities are associated with turbulence, then the effects can be included into the shear-induced aggregation. Hwang (1985) showed that sediment response to oscillating flows was to reduce the effective fall velocity, through inhibiting the attainment of terminal settling velocity.

Variation in settling velocity may contribute to the complexity of bed exchange. The settling velocity will be tested probabilistically.
3.5 Floc Density

Floc density, for the majority of studies, is an inferred parameter. For example, it is often estimated from the particle size and observed settling velocity using Stokes law. This parameter is best handled as a static fractal property of the basic flocculation model and will not be treated probabilistically, in part, because much of its impact is realized through the fall velocity.

3.6 Floc Strength

Floc strength is a function of the mineral characteristics and environmental factors that influence the fractal structure of the flocs. Conditions under which the flocs form influence the strength of the flocs.

Jarvis, et al. (2005) offers a review of methods for estimating floc strength. Methods are either macroscopic or microscopic, with historical methods focusing on the macroscopic relationship between applied shear and the shear rate. New techniques measure the energy required to break individual flocs by tensile or compressive forces. However, many microscopic techniques damage the flocs in the process of measurement. Relating the maximum floc size to floc strength based on fractal characteristics of flocs is a common assumption in microscopic methods.

The primary macroscopic method assumption is that the strength of the flocs is manifested in the viscosity of the water-sediment mixture. Measurements of the viscosity can be related to the strength of flocs through documenting the thresholds for transitions between the consistent slopes (of the stress versus shear rate) with increasing shear rate. Reductions in the slope of the stress-shear rate plots have been assumed to be due to the progressive breaking of the highest order of aggregation flocs. The data have been interpreted as indicating the shear strength based on the order of aggregation as shown in Figure 3-12. Data from Krone (1963) based on applied
shear through a concentric cylinder viscometer showed a consistent relationship between the concentration and the floc strength (see Equation 2-6).

The $5/2$ exponent in Equation 2-6 was found to be consistent among different sediment types, while the proportionality coefficient, $k_f$, varied from site to site. The data are presented in Figure 3-13 along with Equation 2-6 using a value of $k_f = 2.83 \times 10^{-6} \text{ Pa/(kg/m}^3\text{)}^{5/2}$. This relationship is awkward to use because of the units of $k_f$.

Arulanandan, et al. (1980) developed the relationship between the erosion flux constant and the bed shear strength with respect to erosion for undisturbed settled beds within a flume. In the development of the relationship they documented the range in the estimate of the critical shear stresses for each test. Those data provide insight into the variability in the shear stresses. Those ranges are assumed to be approximately two standard deviations, and a coefficient of variability is estimated and plotted against the mean shear stress in Figure 3-14. As the shear strength increases the variability decreases. The trend is a reflection of effects of consolidation. As sediments consolidate the floc structures collapse and the weakest primary particle bonds are broken, resulting in higher strength. Consolidation essentially progressively lowers the order of aggregation of the flocs that settle to the bed. It is inferred that the largest magnitude of variability, associated with the lower shear strengths are a result of a greater range in the strengths of the bonds within the flocs.

The floc strength was defined by McAnally (1999) as a modification of the development of Kranenburg (1994) as shown in Equation 3-11.

$$\tau_f = B_f \left( \frac{\Delta \rho_f}{\rho_w} \right)^{2/(3-D_f)}$$

(3-11)
The coefficients in this equation used in fitting the data of Figure 2-7 are presented in Table 3-5. The table includes the average normalized residuals of the data values for each project site. The differences between the fit and the data were normalized by the fit value. The mean and standard deviations of those normalized residuals are listed in Table 3-5. The standard deviations ranged between 50% and 100% of the mean normalized residuals.

The floc strength exhibits considerable variation and uncertainty. It will be evaluated as a probabilistic variable in the numerical evaluation.

3.7 Bed Exchange

The estimation of the probability of erosion using a probability distribution for both the shear stress and the sediment strength is presented in Figure 3-15. For multiple variables with probability distributions, the probability that a particular combination of values of the variables will occur can be estimated by the products of the individual probabilities for their values, defining their joint probability. The validity of the joint probability approach is based on the assumption that the variables are statistically independent. Figure 3-15 shows a probability distribution of shear stress that includes negative values in the lower tail of the distribution. Negative shear is associated with negative turbulent velocity fluctuations due to complex turbulent processes. The potential for erosion exists for both positive and negative shear. The negative shear can be evaluated by taking the shear strength distribution and folding it over the origin and integrating the negative shear tail. The modified form for the probability of erosion is presented in Equation 3-12, assuming that the probability of erosion is the probability that the bed shear stress is greater than the shear strength. This assumes that sediment mobility will result if the instantaneous shear stress is greater than the shear strength of the bed. If the instantaneous shear stress is less than the shear strength of the bed no erosion will occur. The probability distribution of the shear strength can be viewed as the variation in strength over a unit
surface area of the bed. Each level of instantaneous shear stress, if assumed uniform over the unit area of bed, may cause sediment mobility over some fraction of the bed area based on the fractional area for which the shear stress exceeds the distributed shear strength. As the shear stress increases a larger fraction of the bed will experience erosion, until when the minimum shear stress is greater than the maximum shear strength area of erosion will be the entire bed surface, or the probability of erosion is 1. Note that the probability of erosion does not address the magnitude of the erosive flux, only whether erosion is likely to occur. If the shear stresses are dramatically reduced the probability of erosion will approach zero. If the shear stress distribution is bounded, it will reach 0 when the maximum shear stress is less than the minimum shear strength.

\[
p_e = p_e^+ + p_e^- = \int_{0}^{\infty} \int_{0}^{\infty} f_{\tau_s}(\tau_s) d\tau_s f_{\tau_b}(\tau_b) d\tau_b + \int_{-\infty}^{0} \int_{-\infty}^{\tau_s} f_{\tau_s}(\tau_s) d\tau_s f_{\tau_b}(\tau_b) d\tau_b
\]

(3-12)

Alternatively, Equation 3-12 could be written as shown in Equation 3-13.

\[
p_e = p_e^+ + p_e^- = \int_{\tau_s}^{\infty} \int_{\tau_b}^{\infty} f_{\tau_s}(\tau_s) d\tau_s f_{\tau_b}(\tau_b) d\tau_b + \int_{-\infty}^{0} \int_{-\infty}^{\tau_s} f_{\tau_s}(\tau_s) d\tau_s f_{\tau_b}(\tau_b) d\tau_b
\]

(3-13)

Both of these equations can be expressed as shown in Equation 3-14, using a Heaviside step function. A Heaviside function takes a value of 0 if the argument is negative and a value of 1 if the argument is positive and a value of 0.5 for an argument of zero.
The most convenient form for the probability of erosion comes from the recognition that the integral over the shear strength within the brackets in Equation 3-12 is the CDF function for the shear strength. This simplifies the probability of erosion to Equation 3-15.

\[
P_e = P_e^+ + P_e^- = \int_0^\infty \int_0^\infty f_{\tau_s}(\tau_h) f_{\tau_s}(\tau_s) H(\tau_h - \tau_s) \, d\tau_s \, d\tau_h + \int_{-\infty}^0 \int_{-\infty}^0 f_{\tau_s}(\tau_h) f_{\tau_s}(\tau_s) H(\tau_h - \tau_s) \, d\tau_s \, d\tau_h
\]

(3-14)

This form is presented graphically in Figure 3-15. The CDF for the shear strength (which only includes \( \tau_s \geq 0 \) values) is applied in a symmetric manner about the origin to facilitate integrating both the positive and negative shear stress contributions to the erosion probability. The PDF for the bottom shear stress is taken as its signed value on each side of the origin to be consistent with the folding of the shear strength distribution. The negative contribution to the probability is usually insignificant because it must exceed the (hypothetically) negative value of the minimum shear strength before it makes any contribution and negative shear stresses contributions tend to be in the tails of both distributions, where the probabilities are both low. Physically, that means it does not happen that often and, when it does, the erosive flux will be small.

The probability is made up of positive and negative shear stress contributions associated with turbulent velocity fluctuations that change direction between downstream and upstream. The area under the shear stress probability distribution that contributes to the probability of erosion is the product of the probability density distribution for the shear stress and the cumulative probability distribution for the shear strength. This is compared to the equivalent
integral for a single-valued shear strength in Figure 3-16, where the CDF of the shear strength is a heaviside step function as shown in Equation 3-16. The PDF of a single-valued shear strength is a Dirac delta function at the value of \( \tau_s \).

\[
P_e = P^+ + P^- = \int_0^\infty f(\tau_b)H(\tau_b - \tau_s) d\tau_s + \int_{-\infty}^0 f(\tau_b)[1 - H(\tau_b - \tau_s)] d\tau_s d\tau_b
\]

As the mean shear stress increases, the integrals of the probability functions will approach the value 1. The sensitivity of the varying shear stress to the changes in the variables \( \sigma_b \), the standard deviation of the bed shear stress, \( \sigma_s \), the standard deviation of the bed shear strength, and \( \tau_s \) are presented in Figures 3-17, 3-18 and 3-19, respectively. The selection of the standard deviation of the shear stress was based on the range of the ratio the standard deviation to the mean shear stress observed in Table 3-4. The mean shear strength and standard deviation values were selected based on the data from Figure 3-14, with the standard deviation of the higher side so as to facilitate the demonstration of the effects. The greatest influence is the change in the mean shear strength. The sensitivity to changes in the standard deviation of the shear stress is a variation in the spread of the probability distribution about the fixed 0.5 probability at a fixed shear stress. The greater the variability in the shear stress the greater the spread in the probability of erosion. The variation of the probability with either of the standard deviations is for a steeper probability transition for the smaller standard deviations. With a larger variation in either of the variables there is an increased region of overlap in their distributions. This results in a reduction in the probability of erosion for mean shear stresses greater than the mean strength and an increase in the probability for mean shear stresses lower than the mean strength. With small standard deviations, the onset of erosion occurs over a narrow range of shear stress. In the extreme case of \( \sigma_l = \sigma_s = 0 \) the onset of erosion would be a dramatic bed failure as the shear stress
reached the shear strength. The larger standard deviation would result in a gradual increase in bed erosion as the shear stress increased through the range of bed strength.

Figure 3-20 presents in situ erosion flux data versus shear stresses collected in Long Island Sound (Wang, 2003) during a twelve day period in December 1983 covering a variety of tidal and wave conditions. Hydrographic and suspended sediment data were collected from a bottom-mounted instrument array with sensors positioned at 1 m above the bottom. The array was equipped with a two-axis electromagnetic current meter, a compass, two water temperature sensors, a conductivity probe, and two red-light transmissometers with a 10-cm path length. The tidal conditions at the site have a mean tide range of approximately 2 m, with peak tidal velocities between 0.25 and 0.5 m/s. These data illustrate the scatter in field data, with the erosion flux varying by an order of magnitude for a given shear stress. Wang considered the temporal variation in the processes as contributing to the variability because there was little horizontal variation in the suspended sediment concentration. The presence of loose recently deposited sediment from the previous slack water biased the erosion flux during subsequent erosion events. These data are shown primarily to illustrate a case where temporal variability leads to significant uncertainty in the processes when horizontal variability is small.

The application of the reverse logic of the probability of erosion to the probability of deposition results in Equation 3-17.

$$\begin{align*}
P_d &= P_d^+ + P_d^- = \int_0^\infty \left[1 - F_{\tau_e}(\tau_b)\right] f_{\tau_e}(\tau_b) d\tau_b + \int_{-\infty}^0 \left[1 - F_{\tau_e}(\tau_b)\right] f_{\tau_e}(\tau_b) d\tau_b \quad (3-17)
\end{align*}$$

The integral in Equation 3-17 yields the logical finding that $P_d = 1 - P_e$. The probability that the shear stress is greater than the bed strength plus the probability that it is less than it equals one. An example plot of the two curves for the case of $\sigma_e=0.25$, $\sigma_b=0.3$, $\tau_e = 0.5$ Pa is shown in Figure 3-21. The bed shear strength with respect to erosion and the shear threshold for
deposition are normally viewed as single-valued shear stress levels for a given sediment, with the value for erosion greater than the value for deposition (Krone, 1962; Partheniades, 1962; Ariathurai, 1974). If the criterion for defining a threshold value is a 50% probability for either erosion or deposition, then the thresholds for erosion and deposition based on Figure 3-21 would be the same shear stress (approximately 0.57 Pa). If the same probability is to be applied to both erosion and deposition in estimating the threshold, then the deposition threshold will be less than the erosion threshold only if the probability criteria are greater than 0.5 (see Figure 3-22). If it were less than 0.5, then the erosion threshold would be less than the deposition threshold (see Figure 3-23).

The processes model will be tested for insights and sensitivity to a probabilistic formulation of the shear stress and the erosion strength with regard to bed exchange. This research will evaluate the response of the model to use of probability distributions for the shear stress and the erosion strength with regard to bed exchange separately and in combination.

**3.8 Summary of Probabilistic Treatment**

The approach used in the model for the treatment of probabilistic variables is summarized in Table 3-6. The nature of the primary variables is summarized along with the method of treatment in the model. The discretization of the variables is also listed. Variables that are affected by the local hydrodynamics and concentration will require computation for each cell in the water column. Variables associated with particle sizes require development by class. The mineral density is a specified uniform constant. The current velocity, floc shear strength and the settling velocity have a normalized stochastic standard deviation specified, then the stochastic probability distribution developed for a normal distribution with the specified mean and standard deviation. Those distributions maybe associated with cells in the vertical over the water column or for size classes. The bottom shear stress stochastic probability distribution is developed from
a transformation of the velocity distribution. The TKE, its dissipation rate and the vertical velocity distribution are solved from their governing equations. Many of the remaining variables are probabilistic as a result of dependence on the specified probabilistic variables. However, the expected values are the reported values.
<table>
<thead>
<tr>
<th>Entity</th>
<th>Basic property</th>
<th>Location</th>
<th>State property</th>
<th>Processes</th>
<th>Bulk property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>Temperature</td>
<td>Flow</td>
<td>Internal shear</td>
<td>Momentum transfer</td>
<td>Density</td>
</tr>
<tr>
<td></td>
<td>Conductivity</td>
<td>Floc interstices</td>
<td>Velocity</td>
<td></td>
<td>Viscosity</td>
</tr>
<tr>
<td></td>
<td>Salinity</td>
<td>Bed</td>
<td></td>
<td></td>
<td>TKE</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>TKE Dissipation</td>
</tr>
<tr>
<td>Cohesive Sediment</td>
<td>Grain size</td>
<td>Suspension</td>
<td>Floc size</td>
<td>Aggregation</td>
<td>$D_f$</td>
</tr>
<tr>
<td></td>
<td>Grain density</td>
<td>Bed</td>
<td>Floc density</td>
<td>Disaggregation</td>
<td>$\tau_d$</td>
</tr>
<tr>
<td></td>
<td>Grain shape</td>
<td></td>
<td></td>
<td>Settling</td>
<td>$\tau_e$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Hindered settling</td>
<td>Shear strength</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Erosion</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Entrainment</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Deposition</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Consolidation</td>
<td></td>
</tr>
</tbody>
</table>
### Table 3-2. Parameters in the shear stress distributions shown in Figure 3-3

<table>
<thead>
<tr>
<th>Velocity (m/s)</th>
<th>Mean shear stress (Pa)</th>
<th>M</th>
<th>s</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.113</td>
<td>0.0305</td>
<td>0.00828</td>
<td>0.00797</td>
</tr>
<tr>
<td>0.134</td>
<td>0.0415</td>
<td>0.01146</td>
<td>0.01069</td>
</tr>
<tr>
<td>0.152</td>
<td>0.0515</td>
<td>0.01459</td>
<td>0.01320</td>
</tr>
</tbody>
</table>

### Table 3-3. Summary of Monte Carlo simulation of the Krone flume deposition experiment shear stress distributions

<table>
<thead>
<tr>
<th>Mean velocity (m/s)</th>
<th>Velocity Standard deviation (m/s)</th>
<th>Reported mean shear stress (Pa)</th>
<th>Simulated mean shear stress (Pa)</th>
<th>Shear stress standard deviation (Pa)</th>
<th>Coefficient of skewness</th>
<th>Coefficient of kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.113</td>
<td>0.0220</td>
<td>0.0305</td>
<td>0.0293</td>
<td>0.0111</td>
<td>0.570</td>
<td>3.43</td>
</tr>
<tr>
<td>0.134</td>
<td>0.0243</td>
<td>0.0415</td>
<td>0.0410</td>
<td>0.0145</td>
<td>0.533</td>
<td>3.38</td>
</tr>
<tr>
<td>0.152</td>
<td>0.0260</td>
<td>0.0515</td>
<td>0.0526</td>
<td>0.0176</td>
<td>0.502</td>
<td>3.32</td>
</tr>
</tbody>
</table>

### Table 3-4. Statistical parameters for example probability distribution curves in Figures 3-8, 3-9 and 3-10

<table>
<thead>
<tr>
<th>Velocity (m/s)</th>
<th>Shear stress (Pa)</th>
<th>Normalized</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \bar{u} )</td>
<td>( \sigma_u )</td>
</tr>
<tr>
<td>0.5</td>
<td>0.06</td>
<td>0.638</td>
</tr>
<tr>
<td>0.5</td>
<td>0.07</td>
<td>0.642</td>
</tr>
<tr>
<td>0.5</td>
<td>0.08</td>
<td>0.646</td>
</tr>
<tr>
<td>0.5</td>
<td>0.09</td>
<td>0.655</td>
</tr>
<tr>
<td>1.0</td>
<td>0.12</td>
<td>2.514</td>
</tr>
<tr>
<td>1.0</td>
<td>0.14</td>
<td>2.527</td>
</tr>
<tr>
<td>1.0</td>
<td>0.16</td>
<td>2.542</td>
</tr>
<tr>
<td>1.0</td>
<td>0.18</td>
<td>2.560</td>
</tr>
<tr>
<td>2.0</td>
<td>0.24</td>
<td>10.258</td>
</tr>
<tr>
<td>2.0</td>
<td>0.28</td>
<td>10.293</td>
</tr>
<tr>
<td>2.0</td>
<td>0.32</td>
<td>10.309</td>
</tr>
<tr>
<td>2.0</td>
<td>0.36</td>
<td>10.297</td>
</tr>
</tbody>
</table>
Table 3-5. Coefficients of Equation 3-12 for fits to data sets in Figure 2-7 and the normalized deviation between the fit and the data

<table>
<thead>
<tr>
<th>Project</th>
<th>$B_f$ (Pa)</th>
<th>$D_f$</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wilmington District</td>
<td>45.3</td>
<td>1.94</td>
<td>0.216</td>
<td>0.199</td>
</tr>
<tr>
<td>Brunswick Harbor</td>
<td>726.6</td>
<td>2.26</td>
<td>0.056</td>
<td>0.028</td>
</tr>
<tr>
<td>Gulfport Channel</td>
<td>155.0</td>
<td>2.02</td>
<td>0.165</td>
<td>0.116</td>
</tr>
<tr>
<td>San Francisco Bay</td>
<td>113.1</td>
<td>2.30</td>
<td>0.244</td>
<td>0.133</td>
</tr>
<tr>
<td>White River</td>
<td>139.1</td>
<td>2.01</td>
<td>0.165</td>
<td>0.111</td>
</tr>
</tbody>
</table>

Table 3-6. Probabilistic treatment of significant CST variables

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nature of variable</th>
<th>Treated by code</th>
<th>Discretization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mineral density</td>
<td>Constant</td>
<td>Defined constant</td>
<td>uniform</td>
</tr>
<tr>
<td>Particle/floc size</td>
<td>Probabilistic</td>
<td>Explicit probabilistic</td>
<td>by class</td>
</tr>
<tr>
<td>Floc density</td>
<td>Fractal</td>
<td>Fractally defined</td>
<td>by class</td>
</tr>
<tr>
<td>Settling velocity</td>
<td>Probabilistic stochastic</td>
<td>Specified stochastic</td>
<td>by class and cell</td>
</tr>
<tr>
<td>Concentration</td>
<td>Deterministic</td>
<td>Solved governing equations</td>
<td>by class and cell</td>
</tr>
<tr>
<td>Floc shear strength</td>
<td>Probabilistic</td>
<td>Specified stochastic</td>
<td>by class</td>
</tr>
<tr>
<td>Shear strength of bed</td>
<td>Probabilistic</td>
<td>Derived stochastic</td>
<td>by class</td>
</tr>
<tr>
<td>Velocity</td>
<td>Probabilistic stochastic</td>
<td>Specified stochastic</td>
<td>by cell</td>
</tr>
<tr>
<td>TKE</td>
<td>Deterministic</td>
<td>Solved governing equations</td>
<td>by cell</td>
</tr>
<tr>
<td>TKE dissipation</td>
<td>Deterministic</td>
<td>Solved governing equations</td>
<td>by cell</td>
</tr>
<tr>
<td>Eddy viscosity</td>
<td>Probabilistic stochastic</td>
<td>Resultant stochastic</td>
<td>by cell</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Probabilistic stochastic</td>
<td>Derived stochastic</td>
<td>by bottom</td>
</tr>
<tr>
<td>Local shear</td>
<td>Probabilistic stochastic</td>
<td>Derived stochastic</td>
<td>by cell</td>
</tr>
<tr>
<td>Internal shear rate</td>
<td>Probabilistic stochastic</td>
<td>Derived stochastic</td>
<td>by cell</td>
</tr>
<tr>
<td>Aggregation</td>
<td>Probabilistic stochastic</td>
<td>Resultant stochastic</td>
<td>by class, cell</td>
</tr>
<tr>
<td>Disaggregation</td>
<td>Probabilistic stochastic</td>
<td>Resultant stochastic</td>
<td>by class, cell</td>
</tr>
</tbody>
</table>
Figure 3-1. Conceptual view of CST processes.
Figure 3-2. Fit of Equation 3-5 to sample data set (data from Obi, et al., 1996).
Figure 3-3. Shear stress distributions used by Winterwerp and van Kesteren (2004) developed from analysis of Petit (1999). Parameters for the fits were developed by a two-parameter error minimization algorithm in applying Equation 3-7.
Figure 3-4. Effect of normally distributed velocity distribution on the shear stress distribution based on a Monte Carlo simulation of Equation 3-10. The stochastic velocity distribution is confirmed against the analytical normal distribution.
Figure 3-5. Transformation of a mean 0.113 m/s normally distributed velocity with a standard deviation of 0.022 m/s to the shear stress distribution of Winterwerp and van Kesteren.
Figure 3-6. Transformation of a mean 0.134 m/s normally distributed velocity with a standard deviation of 0.0243 m/s to the shear stress distribution of Winterwerp and van Kesteren.
Figure 3-7. Transformation of a mean 0.152 m/s normally distributed velocity with a standard deviation of 0.026 m/s to the shear stress distribution of Winterwerp and van Kesteren.
Figure 3-8. Transformation of 0.5 m/s mean velocity to shear stress for varying standard deviation in the normal velocity distribution. The solid symbols are the velocity distributions. The open symbols are the shear stress distributions with the same symbol as the associated velocity distribution.
Figure 3-9. Transformation of 1.0 m/s mean velocity to shear stress for varying standard deviation in the normal velocity distribution. The solid symbols are the velocity distributions. The open symbols are the shear stress distributions, with the same symbol as the associated velocity distribution.
Figure 3-10. Transformation of 2.0 m/s mean velocity to shear stress for varying standard deviation in the normal velocity distribution. The solid symbols are the velocity distributions. The open symbols are the shear stress distributions, with the same symbol as the associated velocity distribution.
Figure 3-11. Ratio of the standard deviation of velocity to the mean velocity plotted against the ratio of the standard deviation of shear stress to the mean shear stress.
Figure 3-12. Data from Krone (1963) relating the floc strength to the order of aggregation of sediments from a number of harbors.
Figure 3-13. Data from Krone (1963) showing the effects of concentration on shear strength of flocs from a variety of harbors.
Figure 3-14. Variation in shear strength of naturally deposited cohesive bed as a function of the mean shear strength (data from Arulanandan, et al., 1980). The line is a regression fit to the log-transformed variables.
Figure 3-15. The probability of erosion when both the shear stress and the bed shear strength with respect to erosion are represented probabilistically. The CDF for the shear strength is used in the integral in Equation 3-18 in conjunction with the PDF for the shear stress.
Figure 3-16. Integration of a probability distribution of shear stress for the case of a single-valued bed shear strength with respect to erosion. The PDF for the shear strength is a delta function and the CDF is a heaviside function.
Figure 3-17. Shear stress standard deviation, $\sigma_b$, effect on the probability of erosion for the case of $\tau_s = 1$ Pa and $\sigma_s = 0.25$ Pa.
Figure 3-18. Shear strength standard deviation, $\sigma_s$, effect on the probability of erosion for the case of $\tau_s = 1$ Pa and $\sigma_b / \tau_b = 0.30$. 
Figure 3-19. Effect of the variable $\tau_s$ on the probability of erosion for the case of $\sigma_s = 25\%$ and $\sigma_b/\tau_b = 0.30$. 

Effect of $\tau_s$: given $\sigma_s = 0.25$, $\sigma_b/\tau_b = 0.3$.
Figure 3-20. Variation of erosion flux with shear stress for data in Long Island Sound (after Wang, 2003). Variation is in part due to differing tidal conditions and wave energy, with the largest erosion associated with storms.
Figure 3-21. The variation of the probability of erosion and deposition with the mean shear stress for $\tau_s = 0.5$ Pa, $\sigma_s = 25\%$ and $\sigma_b = 30\%$. 
Figure 3-22. Probability threshold greater than 0.5 (0.75) used for definition of critical shear stresses leads to a critical shear stress or erosion greater than the critical shear stress for deposition.
Figure 3-23. Probability threshold less than 0.5 (0.33) used for definition of critical shear stresses leads to a critical shear stress or erosion less than the critical shear stress for deposition.
This chapter details the overall implementation of the basic concepts from Chapter 2 and the special features associated with the probabilistic treatment developed in Chapter 3 into a numerical procedure. The first topics addressed within the chapter address the sediment transport model. These include:

- Sediment size class definition
- Discretization of the sediment transport equation
- Settling velocities by class
- Vertical mixing of sediment
- Bed exchange
- Aggregation
- Disaggregation

This chapter also discusses the implementation of the hydrodynamics within the numerical model. A general description of the numerical solution method and probabilistic integration is discussed. A general description of the numerical program is outlined.

A series on analytical test cases is presented to close this chapter that illustrates the accuracy and consistency of the numerical implementation of the governing equations. These test cases are presented here because each one has an analytical solution that can be compared with the model results.
4.2 Sediment Transport

4.2.1 Sediment Size Classes

The model has optional specification of the sediment classes either by particle diameter or by particle mass. If sediment interaction is viewed as a pure deterministic calculation for single sized particles for each class with precise mass balances during aggregation and disaggregation, then specifying the sediment classes by mass would be the best method of size class specification. Defining the particle sizes such that the particle masses are integer multiples of the smallest mass class will lead to results that guarantee precise mass conservation. However, if the class concentrations are viewed as the result of a distribution of particles within the size range for the class, then explicit accounting for individual floc interactions becomes inconsistent and the mass conservation must be associated with the concentrations rather than numbers of particles. The classes can be defined by particle diameter and mass conservation incorporated directly into the flocculation model. If the floc size is used to define size classes fractional flocs can be processed by the aggregation model. If the deterministic approach is used, then the calculations are essentially dealing with integer math for the individual flocs and flocs cannot get combined precisely unless they create masses for which a larger class exists.

Therefore, sediment classes were defined by a logarithmic progression of grain/floc diameter as in Equation 4-1. The ratio of successive size classes is a constant, $\beta$, and the difference between the logarithms of successive size classes is a constant. The subscript “$i$” refers to the sediment size class, so that the $i$-th class size will be a factor $\beta$ times the previous size class, $i-1$.

$$d_i = \beta_d \ d_{i-1}$$  

$$or \quad m_i = \beta_m \ m_{i-1}$$  

(4-1)
The coefficient of the progression, $\beta$, is defined by Equation 4-2 based on the minimum particle size, $d_{\text{min}}$, maximum particle size, $d_{\text{max}}$, and the number of sediment size classes, $M_{\text{class}}$. If the size classes are defined based on the mass of the particles, then $d_{\text{min}}$ and $d_{\text{max}}$ are the sizes associated with particle masses $m_{\text{min}}$ and $m_{\text{max}}$, the range of mass to be represented. Equations 4-1 and 4-2 give the appearance of a fractal representation. If the number of size classes is large enough, careful selection of the size range can result in a value of $\beta$ that provides advantageous features, taking on a fractal character. The difficulty arises from the differing character of aggregation and disaggregation. If disaggregation was simply the reverse process of aggregation, the selection would be trivial and $\beta = 2$ would lead to reasonable results. However, aggregation is not a simple reversible process, with collisional disaggregation resulting in fractional breakage products on the order of $3/16$ the original mass (see Section 4.6.2). Note that for nontrivial specifications of the particle size classes, $M_{\text{class}}$ will always be greater than 1.

$$
\beta_d = \left( \frac{d_{\text{max}}}{d_{\text{min}}} \right)^{1/(M_{\text{class}}-1)} ; \quad \beta_m = \left( \frac{m_{\text{max}}}{m_{\text{min}}} \right)^{1/(M_{\text{class}}-1)}
$$

(4-2)

As long as a reasonable number of sediment classes is used, the method of defining the sediment classes should not have an impact on the results. For example, if the fractional masses associated with collisional breakage of flocs are to be accommodated in the size distribution we would require that $\beta_m$ is on the order of $1/(1-3/16) = 1.23$. Note that to use Equation 4-2 the number of classes must be greater than one. The size classes will span the combined range of disaggregated grains and aggregates, allowing the aggregation model to rearrange the distribution based on the processes. To span a size range of 1 to 1000 microns would span a particle mass of approximately $1.8 \times 10^{-15}$ kg to $5.4 \times 10^{-6}$ kg. This would require approximately 100 size classes for $\beta_m = 1.23$. The differentiation between the dispersed grain size distribution
and small flocs composed of the smallest of grains cannot be directly addressed within a single 
size class progression as described above.

The total suspended sediment concentration is defined as the sum of the concentrations of 
the individual size classes (Equation 4-3).

\[
C = \sum_{i=1}^{N_{class}} c_i + \sum_{i=1}^{N_{class}} c_i 
\]  

(4-3)

Separate \(N_{class}\) silt classes are included in Equation 4-3, which would have its own size 
distribution. The silt sizes (2 to 62 microns) are assumed to be noncohesive. In subsequent 
equations silt classes will not be explicitly separated.

4.2.2 Sediment Transport Equation

The starting point for development of the sediment transport model is the conservation 
equation for the mean concentration of each sediment size class (Equation 4-4i).

\[
\frac{\partial \bar{c}_i}{\partial t} + \frac{\partial (\bar{u}_j \bar{c}_i)}{\partial x_j} - \frac{\partial \left( \delta_{ij} \bar{w}_{mn} \bar{c}_i \right)}{\partial x_j} - \frac{\partial \left( \delta_{ij} \bar{w}_i' \bar{c}_i' \right)}{\partial x_j} = \bar{S}_i + \frac{\partial}{\partial x_j} \left( D_m + D_{ij} \right) \frac{\partial \bar{c}_i}{\partial x_j} 
\]  

(4-4i)

All variables have been previously defined. Equation 4-4i is three-dimensional and applies 
to any point in the flow system in its present form. The advective velocity, \(\bar{u}_j\), is subscripted 
with the particle class \(i\) as well as the coordinate \(j\), admitting the possibility of differential 
particle and fluid velocities. Both the mean and turbulent fluctuating settling velocities, \(\bar{w}_{si}\) and 
\(w_i'\), are functions of the size class. The diffusion is at this point separated into molecular and 
turbulent mixing coefficients. The vertical dimension is defined by the \(x_3\) coordinate. The 
boundary conditions are mass fluxes along all boundaries. The conventional water surface \((z = 
z_{ws})\) boundary condition balances vertical diffusion with the net vertical advective movement of
sediment. At the bottom of the water column \((z = z_b)\) the difference in that balance equals the net erosion from the bed. These are expressed in Equation 4-5.

\[
\left\{ \left( u_3 - \bar{w}_{sl} \right) \bar{c}_i - \left( D_m + D_i \right) \frac{\partial \bar{c}_i}{\partial z} \right\}_{z = z_w} = 0
\]

\[
\left\{ \left( u_3 - \bar{w}_{sl} \right) \bar{c}_i - \left( D_m + D_i \right) \frac{\partial \bar{c}_i}{\partial z} \right\}_{z = z_b} = E_i - D_i
\]

(4-5)

If the vertical water velocity is assumed to be zero then the surface boundary condition matches the settling flux with the diffusive flux. The lateral inflow/outflow boundaries must have either an inflow (or outflow) flux or a diffusive flux. The most common diffusive flux boundary is zero, achieved by setting the concentration gradient to zero. Lateral walls have a total flux set to zero, since the advective flux will be zero, then the diffusive flux will also be zero.

The source/sink term, \(S_i\), in Equation 4-4, includes interactions between sediment classes. Size class “\(i\)” can gain particles from aggregation of smaller particles into this class or from disaggregation of larger flocs that have class “\(i\)” particles as a remnant of floc breakup. Class “\(i\)” can lose particles due to disaggregation by shear breaking of flocs or from collisions of flocs. These categories of interactions are delineated as shown in Equation 4-6.

\[
S_i = G_{ia} - L_{ia} + G_{ib} - L_{ib} + G_{ic} - L_{ic}
\]

(4-6)

The terms in Equation 4-6 represent gains \((G)\) and losses \((L)\) of particles in class “\(i\)” associated with the processes of aggregation \((A)\), floc breakage \((B)\) due to shear and due to collisions \((C)\). Each of these terms will be discussed in section 4.2.7.

Collision terms are not present in all flocculation models (Maggi, 2005). In estuarine environments many assume that all collisions result in aggregation (Friedlander, 2000). McAnally (1999) and Lick and Lick (1988) include collisions as a disaggregation potential.
Burban, et al. (1998) found that fluid shear had no significant influence on disaggregation in the range of 100 to 600 $s^{-1}$. They also concluded that three-body collisions were necessary to account for the reduced median floc size as the concentration increases. They acknowledged that floc breakage from collisions may be an indirect result of the shear.

The governing equation for the total suspended concentration, $C$, comes from the summation of the $M_{\text{class}}$ equations to obtain Equation 4-7.

\[
\frac{\partial}{\partial t} \sum_{i=1}^{M_{\text{disp}}} \bar{c}_i + \frac{\partial}{\partial x_j} \left[ \bar{u}_j \sum_{i=1}^{M_{\text{disp}}} \bar{c}_i \right] - \delta_{j3} \frac{\partial}{\partial x_j} \left( \sum_{i=1}^{M_{\text{disp}}} \bar{w}_{si} \bar{c}_i \right) - \frac{\partial}{\partial x_j} \left( \sum_{i=1}^{M_{\text{disp}}} \bar{w}_{si} \bar{c}^\prime_i \right) = \frac{\partial}{\partial x_j} \sum_{i=1}^{M_{\text{disp}}} \bar{S}_i + \frac{\partial}{\partial x_j} \left( \bar{D}_m + \bar{D}_d \right) \frac{\partial}{\partial x_j} \sum_{i=1}^{M_{\text{disp}}} \bar{c}_i
\]

(4-7)

\(\delta_{j3}\) is the Dirac delta function (= 0 for $j=1,2$, and =1 for $j=3$). The advective velocity has been assumed to be the same for the fluid and sediment so that the velocity moves outside the summation on the advective term. A representative bulk settling velocity can be defined based on concentration-weighted class settling velocities as shown in Equation 4-8.

\[
\bar{w}_s \equiv \frac{\sum_{i=1}^{M_{\text{disp}}} \bar{w}_s \bar{c}_i}{\sum_{i=1}^{M_{\text{disp}}} \bar{c}_i} = \frac{\sum_{n=1}^{M_{\text{disp}}} \bar{w}_s \bar{c}_i}{C}
\]

(4-8)

It is asserted that the net effect of aggregation and disaggregation over all floc size classes has to balance, so that $\sum_{i=1}^{M_{\text{disp}}} \bar{S}_i = 0$. This and Equation 4-8 can be substituted into Equation 4-7 to yield Equation 4-9.

\[
\frac{\partial}{\partial t} \bar{c} + \frac{\partial}{\partial x_j} \bar{c} - \delta_{j3} \frac{\partial}{\partial x_j} \left( \bar{w}_s \bar{c} \right) - \frac{\partial}{\partial x_j} \left( \sum_{i=1}^{M_{\text{disp}}} \bar{w}_{si} \bar{c}^\prime_i \right) = \frac{\partial}{\partial x_j} \left( \bar{D}_m + \bar{D}_d \right) \frac{\partial}{\partial x_j} \bar{c}
\]

(4-9)
The remaining summation term in Equation 4-9 will vanish if the settling velocity is taken as a constant, with no variability \( w'_{si} = 0 \), which is not a reasonable assumption. If the sediment concentration is to be treated only as an expected value the effects of this remaining term can be incorporated into the turbulent diffusion coefficient (Equation 4-10).

\[
D_t \frac{\partial \bar{C}}{\partial x_j} = \sum_{n=1}^{N_{\text{size}}} \left( \delta_{ij} w'_{si} c_{n} - u' c_{n} \right) \tag{4-10}
\]

Many of these assumptions may never be invoked because the settling and diffusion are simulated for each separate size class rather than for the total concentration, which is basically a summation step once the individual size class concentrations are known.

The primary transport mechanisms for sediment movement in the vertical are through gravitational settling and vertical diffusion. In some extreme cases of complex flows, such as around bridge piers or in macrotidal estuaries, vertical accelerations of flow can move sediments vertically through the vertical component of advective velocity.

4.2.3 Settling Velocities

Settling velocities have been discussed in Chapter 2. The progression of estimates of settling velocities begins in low concentrations where the “free” settling velocity is a function of particle size and density. As the concentration increases above the limit for free settling (around 0.05 to 0.3 kg/m\(^3\)), flocculation increases because of the larger numbers of particles in suspension, causing larger flocs with increased settling velocity up to the upper limit of flocculation settling (8-13 kg/m\(^3\)). Beyond this peak concentration the rate of settling is progressively hindered with increasing concentration through fluid mud concentrations.

Particle fall velocities are coded in the model based on a 3-step computation. These steps are:
• Free settling velocity is computed based on Equation 2-22, which is the Stokes fall velocity using a coefficient of drag from Equation 2-25.

• Effects of flocculation are then included based on Equation 2-26, which accounts for the density effects through the fractal dimension and the floc size.

• Effects of concentration and internal shear are then included through Equations 2-40, 2-45 and 2-46.

4.2.4 Vertical Mixing

Vertical mixing in the sediment model is handled as a turbulent mixing coefficient proportional to the turbulent eddy viscosity. The estimation of turbulent eddy viscosity is from either an assumed logarithmic profile or by application of a $k$-$\varepsilon$ turbulence model over the water column.

Vertical distribution of eddy viscosity for a logarithmic profile is estimated from the shear velocity, as in Equation 4-11.

$$v_r = \kappa u_z \left(1 - \frac{z}{h}\right)$$  \hspace{1cm} (4-11)

An alternative estimation of the mixing was developed by the application of a $k$-$\varepsilon$ model. The model was based on the standard high-Reynolds number $k$-$\varepsilon$ model (Hanjalic, 2004; Speziale, 1998; Winterwerp, 1999) with adjustments for the effects of suspended sediment (Hsu, et. al., 2007) and for localized low Reynolds number flows (Hanjalic, 2004).

The governing equations for the turbulent kinetic energy per unit mass, $k$, and the turbulent kinetic energy dissipation rate, $\varepsilon$, are given in Equations 4-12 and 4-13.

$$\begin{align*}
\frac{\partial k}{\partial t} + \frac{\partial \overline{u}_j k}{\partial x_j} - \frac{\partial}{\partial x_j} \left[ \nu_T \frac{\partial k}{\partial x_j} \right] &= \nu_T \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) \frac{\partial \overline{u}_i}{\partial x_j} + \delta_{ij} \frac{g}{\rho} \frac{\partial \rho}{\partial x_j} - \varepsilon \\
&= \nu_T \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) \frac{\partial \overline{u}_i}{\partial x_j} + \delta_{ij} \frac{g}{\rho} \frac{\partial \rho}{\partial x_j} - \varepsilon
\end{align*}$$  \hspace{1cm} (4-12)
\[
\frac{\partial \varepsilon}{\partial t} + \frac{\partial \bar{u}_j \varepsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left\{ \left( \frac{\nu_T}{\sigma_\varepsilon} \right) \frac{\partial \varepsilon}{\partial x_j} + \frac{c_{1e} \nu_T}{k} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} \right\} + \delta_{ji} \left( 1 - c_{3e} \right) \frac{\varepsilon}{k} \frac{g}{\bar{\rho} \sigma_T} \frac{\partial \bar{\rho}}{\partial x_j} - c_{2e} \frac{\varepsilon^2}{k}
\]

(4-13)

These comprise the standard high-Reynolds number form of the equations with coefficients as provided in Table 4-1.

When the \( k - \varepsilon \) model is used the turbulent eddy viscosity is defined (Equation 4-14) as a function of \( k \) and \( \varepsilon \).

\[
\nu_T = \frac{k^2}{\varepsilon}
\]

(4-14)

In uniform flow Equations 4-12 and 4-13 simplify to Equations 4-15 and 4-16.

\[
\frac{\partial k}{\partial t} - \frac{\partial}{\partial z} \left\{ \nu + \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial k}{\partial z} \right\} = \nu_T \left( \frac{\partial \bar{u}_i}{\partial z} \right)^2 + \frac{g}{\bar{\rho} \sigma_T} \frac{\partial \bar{\rho}}{\partial z} - \varepsilon
\]

(4-15)

\[
\frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial z} \left\{ \nu + \frac{\nu_T}{\sigma_k} \frac{\partial \varepsilon}{\partial z} \right\} = c_{1e} \nu_T \frac{\varepsilon}{k} \left( \frac{\partial \bar{u}_i}{\partial z} \right)^2 + \left( 1 - c_{3e} \right) \frac{\varepsilon}{k} \frac{g}{\bar{\rho} \sigma_T} \frac{\partial \bar{\rho}}{\partial z} - c_{2e} \frac{\varepsilon^2}{k}
\]

(4-16)

If the flow is at steady state, where the vertical turbulence structure is stabilized, the time terms are omitted and the equations reduce to Equations 4-17 and 4-18.

\[
- \frac{\partial}{\partial z} \left\{ \nu + \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial k}{\partial z} \right\} = \nu_T \left( \frac{\partial \bar{u}_i}{\partial z} \right)^2 + \frac{g}{\bar{\rho} \sigma_T} \frac{\partial \bar{\rho}}{\partial z} - \varepsilon
\]

(4-17)

\[
- \frac{\partial}{\partial z} \left\{ \nu + \frac{\nu_T}{\sigma_k} \frac{\partial \varepsilon}{\partial z} \right\} = c_{1e} \nu_T \frac{\varepsilon}{k} \left( \frac{\partial \bar{u}_i}{\partial z} \right)^2 + \left( 1 - c_{3e} \right) \frac{\varepsilon}{k} \frac{g}{\bar{\rho} \sigma_T} \frac{\partial \bar{\rho}}{\partial z} - c_{2e} \frac{\varepsilon^2}{k}
\]

(4-18)

Boundary conditions for the equations are given in Equation (4-19).
\[ k|_{z=z_b} = \frac{u_r^2}{\varepsilon^{1/3}} \quad \quad \varepsilon|_{z=z_b} = \frac{u_r^3}{kz_0} \]

\[ k|_{z=z_s} = 0; or \quad \frac{\partial k}{\partial z} \bigg|_{z=z_s} = 0 \quad \quad \quad \varepsilon|_{z=z_s} = 0; or \quad \frac{\partial \varepsilon}{\partial z} \bigg|_{z=z_s} = 0 \] (4-19)

A correction for the presence of suspended sediment in Equation 4-15 was proposed by Hsu, et al. (2007), as shown in Equation 4-20. This form ignores the advective transport of \( k \).

\[
(1 - \phi) \frac{\partial k}{\partial t} - \frac{\partial}{\partial z} \left\{ v \frac{v_T}{\sigma_k} \right\} \frac{\partial (1 - \phi) k}{\partial z} = \varepsilon = v_T \left( \frac{\partial u}{\partial z} \right)^2 + (s-1) \frac{v_T}{\sigma_c} \frac{\partial \phi}{\partial z} - (1 - \phi) \varepsilon - \frac{2\phi sk}{T_p + T_L} \] (4-20)

The volume concentration of sediment, \( \phi \), and the specific gravity of the sediment, \( s \), are used to approximate the effects of two-phase flow features. The corresponding dissipation equation is given in Equation 4-21.

\[
(1 - \phi) \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial z} \left\{ v + \frac{v_T}{\sigma_k} \right\} \frac{\partial (1 - \phi) \varepsilon}{\partial z} = \varepsilon = c_{1\varepsilon} v_T \varepsilon \left( \frac{\partial u}{\partial z} \right)^2 - c_{2\varepsilon} \frac{\varepsilon^2}{k} (1 - \phi) - c_{3\varepsilon} \frac{\varepsilon}{k} \left[ (s-1) \frac{v_T}{\sigma_c} \frac{\partial \phi}{\partial z} + \frac{2\phi sk}{T_p + T_L} \right] \] (4-21)

Hsu, et al. (2007) modified the bottom boundary condition for the \( k \)-equation (Equation 4-20) from a Dirichlet (specified \( k \)) to a Neumann condition \( \frac{\partial k}{\partial z} \bigg|_{z=z_b} = 0 \) due to the complexities of the sediment-turbulence interactions near the bottom, which make the definition of a concentration an over-constraint. The bottom boundary condition for the dissipation rate was specified as a Dirichlet condition \( \varepsilon \bigg|_{z=z_b} = \frac{C_{3/4}^{3/2}k^{3/2}}{\kappa z} \). The specific gravity, \( s \), is computed via Equation 4-22, taking into account the density of the flocs as well as the silt concentration.
\[ s_j = \frac{\sum_{i=1,M_{class}} c_{f,ij} \rho_{f,ij} + \rho_s \sum_{i=1,N_{silt}} c_{s,ij}}{\rho_w C_j} \]  

(4-22)

The index \( j \) in Equation 4-22 corresponds to the \( j \)-th cell in the vertical, indicating that the specific gravity of the composite sediment mixture will be a function of the mix of floc sizes and silt concentrations.

In the estuarine environment, there are periods close to slack water for which the high-Reynolds number assumption used for Equations 4-12 through 4-21 may not be satisfied. The coefficients in Table 4-1 are assumed valid for wall Reynolds numbers above 100. The wall Reynolds number may be defined as in Equation 4-23, using the distance above the bottom, \( z \), as the length scale.

\[ \text{Re}_i = \frac{k^{1/2}z}{\nu} \]  

(4-23)

Corrections for low Reynolds numbers within the \( k-\varepsilon \) model are described by Hanjalic (2004), which involve adding damping terms in Equation 4-14 (Equation 4-24).

\[ \nu_T = \frac{c_\mu f_\mu}{\varepsilon} k^2 \]  

(4-24)

The damping term, \( f_\mu \), is defined by Equation 4-25.

\[ f_\mu = \exp \left\{ \frac{-3.4}{\left(1 + \text{Re}_i/50\right)^2} \right\} \]  

(4-25)

In addition, a damping term, \( f_\varepsilon \), is added to the dissipation term in the turbulent dissipation equation, as shown in Equation 4-26.
\[
(1 - \phi) \frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial z} \left( \nu + \frac{v_T}{\sigma_k} \right) \frac{\partial (1 - \phi) \varepsilon}{\partial z} = 
\]

\[
c_{1e}v_T \varepsilon \left( \frac{\partial \bar{u}}{\partial z} \right)^2 - c_{2e}f_{\varepsilon} \varepsilon^2 (1 - \phi) - c_{3e} \varepsilon \left[ (s-1) g \frac{v_T}{\sigma_c} \frac{\partial \phi}{\partial z} + \frac{2\varepsilon}{T_p + T_L} \right]
\]

(4-26)

The dissipation damping factor is defined by Equation 4-27.

\[
f_{\varepsilon} = 1 - 0.3 \exp \left\{ -\text{Re}_{i}^{2} \right\}
\]

(4-27)

The variable \( T_p \) is the particle response time and was estimated by Hsu, et. al. (2007) as in Equation 4-28, using Stokes law with a correction for hindered settling.

\[
T_p = \frac{\rho_s d^2 (1 - \phi)^3}{18 \rho_w v}
\]

(4-28)

The variable \( T_L \) is the turbulent eddy time scale and is estimated from Equation 4-29.

\[
T_L = \frac{k}{6 \varepsilon}
\]

(4-29)

The conventional argument is that these corrections are needed in the near bottom layer where viscous effects become more significant. However, Hanjalic (2004) points out that the need arises more from the inadequacies of the linear eddy viscosity model used in the standard \( k-\varepsilon \) model and the lack of isotropy in the turbulence near a wall.

4.2.5 Bed Exchange

The simulation model incorporates options for the specification of the deposition and erosion fluxes at the bed interface. These include decisions on whether erosion and deposition are allowed to occur simultaneously or exclusively one or the other, whether to incorporate the probabilistic variables in the estimations of probabilities of erosion and/or deposition and which variables to treat probabilistically.
When deposition and erosion at the bed are handled in the model as simultaneous processes, they are treated as described in Section 3.7, based on the comparison of the sediment shear strength and the shear stress at the bed. Each individual sediment class is treated separately. The depositional flux is straightforward for individual floc size classes from the class settling velocity and concentration. The erosive flux associated with individual floc sizes is more of a challenge. When flocs settle into contact with the bed, they start to lose their characteristics and begin to coalesce into more permanent contact with other sediment within the bed. Immediately after deposition their erosive resistance is likely very similar to the floc strength of resistance to shearing disaggregation. Krone (1963) suggested that if flocs deposit to the bed without breakage by the shear stresses near the bottom, the bed surface will initially have flocs of one order of aggregation higher than the settling flocs. With continued deposition the weight of flocs over the deposited layer can reach the point where the stress exceeds the shear strength of the deposited flocs and the layer will begin to crush at the lower layer. Krone reasoned that this crushing of flocs into the bed is a key mechanism in development of erosion resistant shoal material in variable flows.

The treatment of the erosive flux by Partheniades (1965) was partly based on concepts used by Einstein for noncohesive sediments. The primary development starts with the statement (Equation 4-30) that the number of sediment particles eroded per unit time per unit area of the bed, \( N \), can be expressed as the probability of erosion divided by the area of the bed occupied by a single particle and by a characteristic time scale of erosion, \( t_b \).

\[
N = \frac{P_e}{\alpha_2 d^2 t_b} \quad (4-30)
\]

The coefficient \( \alpha_2 \) is an area shape factor. Einstein assumed that the time scale is proportional to \( d/w_s \), the ratio of the particle size to the settling velocity of the particle. For
noncohesive particles the focus is in the near-bed zone, where the majority of mobilized particles fall quickly back to the bed. He related the bedload transport to the probability of how quickly the particles return to the bed, correlated to the distance traveled while saltating along the bed. For cohesive transport much of the resistance to mobility comes from the cohesive bonds. Once overcome, the particles are likely to travel significant distances before returning to the bed. The distances above and along the bed take on the scale of the water depth. The focus of the time scale for bed interaction in cohesive transport becomes the time scale on the particle entrainment itself. For most cohesive transport models that address the vertical variation the analogous particle saltation is modeled explicitly by the simulation.

The expression in Equation 4-30 was developed for either uniform particle size or a representative size. If a variation in particle sizes is considered, every term in the equation will become particle size dependent (except for $\alpha_2$ if spheres are considered.). Equation 4-31 is a modification of Equation 4-30 for a particle size among others that occupies a fraction of the bed, $f_i$. That fraction is applied to the number rather than the area of the particle.

$$N_i = f_i \frac{P_{ei}}{\alpha_2 d_i^2 t_{pi}}$$ \hspace{1cm} (4-31)

The mass flux is essentially the number for each class multiplied by the mass of each size class particle. It is reasonable to expect that the time response will be dependent on the intensity of the erosive force in addition to the particle size. An alternative choice for a time scale would be $t_{bi} = \alpha_i d_i / u_*$, where $\alpha_i$ is a scaling factor. If the particle mass, $m_i = \alpha_i d_i^3$, is incorporated into Equation 4-30 we obtain an estimate for the erosive mass flux for a single class (Equation 4-32).

$$\epsilon_i = m_i N_i = f_i \frac{\alpha_i u_*}{\alpha_2 \alpha_t} P_{ei} \approx C_h u_* f_i P_{ei}$$ \hspace{1cm} (4-32)
The result of this expression is that the basic erosion rate coefficient, \( C_b \), is independent of the particle size if the scaling factor \( \alpha \) is independent size. However, with \( u_* \), the erosion flux would be nonlinear in \( \tau_b \). This retains an empirical coefficient \( C_b \), with units of concentration which is still likely to be dependent on the cohesive properties of the sediment (e.g. mineral composition, cation exchange capacity, organic material). If \( \alpha \) is dependent on the particle density, then the coefficient \( C_b \) would be different for each size class (\( C_{bi} \)).

Prooijen and Winterwerp (2009, in publication) pointed out that if the erosion rate constant is a function of the shear stress, then it needs to be incorporated within the integral of Equation 3-16. With this approach the integral is no longer the development of a probability of erosion, but an integral of the incremental contributions to the total erosion. This modification is shown in Equation 4-33.

\[
E_i = \frac{C_b f_i}{\sqrt{\rho}} \left[ \int_{-\infty}^{\infty} \int_{0}^{\infty} \tau_b^{1/2} f(\tau_b) H(\tau_b - \tau_s) d\tau_s \ d\tau_b + \int_{-\infty}^{0} \int_{0}^{\infty} \tau_b^{1/2} f(\tau_b)[1-H(\tau_b - \tau_s)] d\tau_s \ d\tau_b \right]
\]  

(4-33)

When the shear stress and shear strength in Equation 4-33 are both represented probabilistically the erosion efficiency will involve a double integral of the probabilities after expressing both \( u_* \) and \( \tau \) as functions of \( \tau_b \).

This functional relationship is presented against the erosion data of Partheniades (1965) in Figure 4-1, assuming a class independent erosion concentration, \( C_{bi} \), of 0.859 kg/m\(^3\), using the coefficient \( C_b = 0.8585 \) kg/m\(^3\), \( \rho_w = 1000 \) kg/m\(^3\), and \( f_i = 1 \). The variables used in developing the probability of erosion distribution (via Equation 3-16) were \( \tau_s = 0.55 \) Pa, \( \sigma_s = 0.25 \) Pa and \( \sigma_b = 0.3 \) Pa. Also included in Figure 4-1 is the linear excess shear stress formulation of Ariathurai (1974),
using a value of $\tau_{ce}=0.32$ Pa and an erosion rate constant $M=0.03125$ kg/m$^2$/hr/Pa (Equation 2-51). The derivation of Partheniades essentially assumed that $t_b$ is independent of the flow conditions, which leads to an expression that places an upper bound on the erosive flux at $M$ (see Equation 3-1), since the probability of erosion should never be greater than unity if the probability integration approach is used. The form of Equation 4-32 continues to monotonically increase erosion flux with increasing shear stress. Also plotted in Figure 4-1 is the form of the equation if the probability of erosion is set to 1 over the full range of shear stress. This provides the asymptote toward which the erosion converges as the probability increases toward one. The effect of taking the time constant inside the integral on the erosion rate (Equation 4-33) is also plotted in Figure 4-1, using a seawater density (1030 kg/m$^3$) and a value of $C_b = 0.112$ kg/m$^3$.

When the probability of deposition becomes small for a particular size class, but the next larger size class is experiencing deposition, there is a tendency for coupling of the smaller sediment class to the larger because of wake effects of the larger flocs (Teeter, 2001; Dent, 1999). The coupling of size classes used by Teeter (2001) is incorporated as an option (Equation 4-34).

\[
F_i = \frac{\delta_1 c_i F_{i+1}}{c_{i+1} + \delta_2} \tag{4-34}
\]

The depositional flux for the “i”th sediment class is $F_i$, the concentration $c_i$ and the coefficients $\delta_1$ and $\delta_2$ are controlling parameters for the coupling. Note that there is an implicit assumption that $d_{i+1} > d_i$. Equation 4-34 was applied when the probability of deposition for the size class was less than 0.05. The preliminary development here was to apply the equation in a cascading manner from the largest size class down to the next higher class. However, that method would have no flux correction if the concentration of the next higher class were small, but
there could be significant flux from even larger sizes. Therefore, the application of Teeter’s flux correction has been revised (Equation 4-35) to apply the summation of fluxes of larger classes. In addition, the correction has been revised to include the fluxes of both the cohesive sediments and the silt classes into the correction.

\[
F_{c,j} = \delta_c c_{c,i} \left[ \sum_{k=i+1}^{M_{class}} F_{c,k} + \sum_{k=j}^{N_{silt}} F_{s,k} \right] + \delta_2 \sum_{k=j}^{M_{class}} c_{c,k} + \sum_{k=j}^{N_{silt}} c_{s,k}
\]

\[
F_{s,j} = \delta_s c_{s,i} \left[ \sum_{k=j}^{M_{class}} F_{c,k} + \sum_{k=i+1}^{N_{silt}} F_{s,k} \right] + \delta_2 \sum_{k=j}^{M_{class}} c_{c,k} + \sum_{k=i+1}^{N_{silt}} c_{s,k}
\]  

(4-35)

The size class “j” in the summations of Equation 4-35 refers to the smallest size class of other sediment type (cohesive or silt) with a greater settling velocity.

For the assumption of simultaneous deposition and erosion, the depositional flux is always included. Erosion is only evident when it exceeds the depositional flux.

4.2.6 Aggregation Processes

The aggregation module incorporated within the model is based on the standard collision frequency approach utilized by a number of researchers (Saffman & Turner, 1956; Broadway, 1978; Hunt, 1980; McCave, 1984; Tsai & Hwang, 1995, McAnally, 1999; Parshukov, 2001, Maggi, 2005, etc.). The basis for the aggregation module is that the frequency of collision is a function of the number of sediment particles within the suspension and forces that tend to bring the particles close enough to collide. Once a collision occurs there is an assumed efficiency of aggregation, a probability the particles will adhere to one another to form a larger floc. In addition, a probability is developed for disaggregation, whereby the collision causes the flocs to break apart into smaller flocs. The number of particles within a particular size class, defined from Equations 4-1 and 4-2, will be the class concentration divided by the estimated mass of the
particles within the class. The mass of the particles within size class “i”, $m_i$, is defined by

Equation 4-36, using Equation 2-5 for the floc size, $d_{fi}$, and an equivalent spherical volume, $V_i$.

$$m_i = \rho_i V_i = \left(\rho_w + \Delta \rho_{fi}\right) \frac{\pi d_{fi}^3}{6} = \left(\rho_w + \Delta \rho_{fi}\right) \frac{d_{fi}^3}{d_{fi}^{3-D_{fi}}} \frac{\pi d_{fi}^3}{6}$$  \hspace{1cm} (4-36)

The number concentration of particles, $n_i$, in suspension size class “i” will be the concentration divided by the particle mass (Equation 4-37).

$$n_i = \frac{c_i}{m_i}$$  \hspace{1cm} (4-37)

Contributors to collisions of sediment particles and flocs include Brownian motion, laminar and turbulent shear and differential settling. The assumption is made that these effects are linear and can be added together (Maggi, 2005). The number of new flocs formed per unit volume by collisions between the two mass classes $i$ and $j$ is defined by an efficiency term times the cumulative probabilities of collision from Brownian motion, velocity shear and differential settling, times the product of the numbers of particles in each class per unit volume. This is expressed in Equation 4-36, with the laminar and turbulent shear contributions lumped together.

$$N_{ijk} = \alpha_a \left(\beta_{B,ij} + \beta_{T,ijk} + \beta_{D,ijk}\right) n_{ik} n_{jk}$$  \hspace{1cm} (4-38)

The collision frequencies, $\beta$, are indexed to a spatial location by the “k”-index and the particle classes by “i” and “j”. The subscripts $B$, $T$ and $D$ correspond to Brownian motion, turbulence and differential settling, respectively. The collision frequency for Brownian motion is not indexed to a spatial location because it is assumed to be uniform throughout the flow field.

The relative significance of the three mechanisms will be evaluated in connection with the influence of the probabilistic variables on these contributors to aggregation processes.
The efficiency, $\alpha_a$, is the fraction of collisions that will result in aggregation. Its value ranges between 0 and 1. It has been proposed that above a reference salinity of approximately 2 psu (practical salinity units) the aggregation efficiency equals 1, since in estuarine waters it has been observed that essentially all collisions result in aggregation (Krone, 1962; McAnally, 1999). McAnally proposed a functional relationship for $\alpha_a$ as presented in Equation 4-39.

$$\alpha_a = \begin{cases} \frac{S}{S_0} & \text{for } S < S_0 \\ 1 & \text{for } S \geq S_0 \end{cases} \quad (4-39)$$

It was recommended that $S_0$ is approximately 2 psu, so for most areas within an estuary $\alpha_a$ is 1.

The collision frequencies are computed as shown in Equations 4-40, 4-41 and 4-42:

Brownian motion:

$$\beta_{B,ij} = \left(\frac{2 k_B \bar{T} F_c}{3 \mu} \right) \left(\frac{d_i + d_j}{d_i d_j}\right)^2$$  \quad (4-40)

Turbulent shear:

$$\beta_{T,ijk} = \left(\frac{4 G_k}{3} \right) \left(\frac{d_i + d_j}{d_i d_j}\right)^3$$  \quad (4-41)

Differential settling:

$$\beta_{D,ijk} = \left(\frac{\pi F_c^2}{4} \right) \left(\frac{d_i + d_j}{d_i d_j}\right)^2 \left|w_{ijk} - w_{jik}\right|$$  \quad (4-42)

Terms in Equation 4-40 are Boltzmann’s constant, $k_B = 1.38054 \times 10^{-16} \text{ erg/}^\circ\text{K}$, the average temperature, $\bar{T}$ in $^\circ\text{K}$, an empirical collision diameter function, $F_c$, the dynamic viscosity, $\mu$, and the class particle diameters, $d_i$ and $d_j$. The collision diameter function, $F_c$, varies between 0 and 1 and is a correction factor to account for particle collisions having some meshing together before aggregation can occur; a slight glancing may not result in aggregation. The turbulent shear collision efficiency is a function of the internal shear rate, $G$, and the particle
diameters. For the differential settling frequency the settling velocities are associated with locations in space because of the dependence of the settling velocity on concentration and shear.

An example comparison of the significance of each process for a “first particle” of 10.6 microns relative to a “second particle” of varying size is presented in Figure 4.2. The first particle size class at 10.6 microns was the closest to 10 microns from a size class distribution spanning 0.1 microns to 1000 microns using 60 size classes. That first particle class illustrates the relative importance of the contributions to the aggregation frequency over the defined range. The differential settling collision frequency is relatively constant for second particle sizes smaller than 10 microns, but increases by eight orders of magnitude between 10 and 1000 microns. For a second particle size near 10 microns the collision frequency for differential settling becomes very small, approaching zero for 10.6 microns, where both particles settle at the same rate. The collision frequency for Brownian motion is small over the full range of second particle diameters, with the smallest frequency at the matching size of 10.6 microns. It is only important relative to the other modes of collision below 0.5 microns. For the example case shown, turbulence is significant at all size classes relative to Brownian motion and differential settling. For a different flow condition (e.g., quiescent settling) the turbulence might not be as dominant.

The effects of using a probability distribution for the settling velocities are to eliminate the zero contribution well at 10.6 microns. This implies that particles of the same “effective” spherical diameter will likely make a contribution to aggregation from differential settling. This may be interpreted as a result of the lack of perfect association between particle mass, water content, particle density, effective size and ultimately settling velocity. For large differences in particle size there is no difference in the contribution between using the mean settling rate and a probabilistic representation.
The computation of the internal shear is dependent on whether the $k$-$\varepsilon$ model is invoked or a profile assumed. For using the $k$-$\varepsilon$ model the shear is computed via Equation 4-43.

$$G = \frac{\varepsilon}{\sqrt{\nu}}$$  \hspace{1cm} (4-43)

This is applied locally from the solved field of $\varepsilon$. When a logarithmic profile is assumed the shear is estimated based on the computed shear distribution from the profile Equation 4-44.

$$G = \frac{u_*^3}{\nu \kappa z} \left( 1 - \frac{z}{h} \right)$$  \hspace{1cm} (4-44)

“Aggregation fluxes” are the mass transfer rates between sediment size classes as a result of aggregation of particles. These fluxes for an arbitrary size class, $i$, can be either a mass loss ($L_{ij, A}$) due to flocs in class $i$ combining with other particles of size class $j$ to form larger flocs, or mass gains ($G_{j, \zeta, i, A}$) from the aggregation of smaller particle sizes ($j$ and $\zeta$) that combine to form flocs of size class $i$. The total aggregation fluxes to and from size class $i$ is the summation over all particle interactions within the flocculation model. The total aggregation gains for class $i$ is the result of the summation shown in Equation 4-45.

$$G_{i, A} = \sum_j \sum_{\zeta} G_{j, \zeta, i, A}$$  \hspace{1cm} (4-45)

The specific aggregation fluxes ($G_{j, \zeta, i, A}$) are determined from the rate of particle aggregation, based on Equation 4-38, and the particle masses. This relationship is presented in Equation 4-46.

$$G_{j, \zeta, i, A} = N_{j, \zeta} \left( m_j + m_\zeta \right) \lambda_i \quad \Rightarrow \quad m_i \approx \left( m_j + m_\zeta \right)$$  \hspace{1cm} (4-46)

The combined masses of particles of class $j$ and $\zeta$ result in approximately the mass of size class $i$. The parameter $\lambda_i$ is the distribution factor for the combined mass to size class $i$. The distribution factor is developed from linear interpolation between log-transformed particle masses as shown in Equation 4-47.
The loss of mass due to aggregation, \( L_{iA} \), from class \( i \) is also based on the rate of particle aggregation and the class \( i \) particle mass (see Equation 4-48), the summation of flux loses to all other size classes that have experienced aggregation with class \( i \).

\[
L_{iA} = \sum_j L_{ijA} = \sum_j N_j m_i
\]  

(4-48)

The results of particle collisions can be either aggregation or disaggregation of the colliding particles. When two particles of known size combine to form a new larger aggregate, the combined size may not precisely match the mass of a larger size class. This introduces the need to deal with the number concentrations as real numbers, interpreted as an expected value for the integer number of particles per unit volume. This frees the numerical analysis from dealing with special cases of integer mathematics. It does require the use of “breakup distribution functions” (Lick and Lick, 1988) to distribute the products of the disaggregation to lower size classes in a manner that both conserves mass and is based on observed behavior.

**4.2.7 Disaggregation Processes**

**4.2.7.1 Shear-Induced Disaggregation**

When the local shear stress in the water column exceeds the shear strength of the floc, the floc is broken into smaller flocs based on a breakup distribution function such that the mass of the original floc is conserved. The shear strength of the flocs is assumed to be a function of the density as defined in Equation 3-11.
The local shear stress is assumed to be linear from a maximum at the bottom to zero at the surface. The relationship is given in Equation 4-49 for the case of using an assumed velocity profile.

\[
\tau = \rho \, u^2 \left( 1 - \frac{z}{h} \right) = \tau_b \left( 1 - \frac{z}{h} \right) \quad (4-49)
\]

The value \( u_* \) can be estimated via Equation 4-50, from the depth-averaged velocity, \( \bar{u}_{av} \), and a Darcy-Weisbach friction factor, \( f \).

\[
u_* = \left( \frac{f}{8} \right) \bar{u}_{av} \quad (4-50)
\]

If the \( k-\varepsilon \) model is used the shear stress can be computed from the local dissipation rate and the vertical turbulent-mean velocity gradient as shown in Equation (4-51).

\[
\tau = \frac{\varepsilon}{\partial \bar{u}} \quad (4-51)
\]

The “breakage fluxes” can also be either a loss or again for a specific size class. Mass loss from a class results when flocs within that size class are broken apart. The remnants of that breakage become mass gains to the appropriate smaller size class. This analysis assumes that floc breakage results in only two remnant particles, of masses \( \xi \, m_j \) and \( (1 - \xi) \, m_j \), where \( \xi \) varies between 0 and 1, but logically would be closer to 0.5. McAnally (1999) recommends a value of 3/16. The resulting remnant mass is then distributed to the closest existing mass classes consistently with Equation 4-47, using \( \xi \, m_j \) and \( (1 - \xi) \, m_j \), rather than \( m_j + m_k \). The shear breakage loss fluxes can then be expressed as shown in Equation 4-52.

\[
L_{sb} = \frac{\alpha_{dc}}{t} \, C_i \quad (4-52)
\]
The efficiency for shear induced breakage is assumed to apply to all sediments in suspension, since no collisions are needed to have breakage. The time scale, $t$, is assumed to be defined as the ratio $\kappa z / u_\ast$.

The efficiency of the disaggregation has been assumed to be proportional to the excess shear stress, with a time constant proportional to the shear velocity (see Equation 4-53).

$$\alpha_d = A \frac{d_f}{\lambda_0} \left( \frac{\tau - \tau_f}{\tau} \right)$$  (4-53)

The variable $A$ is an adjustment coefficient and $\lambda_0$ is the Kolmogorov eddy length scale. When the shear stress, $\tau$, is below the strength, $\tau_f$, of the floc no disaggregation will occur and as the shear stress increases beyond the strength of the floc, the rate of disaggregation will increase proportionally.

The mass flux gains to size class $i$ resulting from shear breaking of flocs can be expressed as the summation of all classes broken that yield remnants of mass in class $i$ (see Equation 4-54).

$$G_{ib} = \sum_j G_{ijb} = \sum_j \xi_j \frac{\alpha_d}{\alpha_{\sigma}} N_{ij} m_j$$  (4-54)

The variable $\xi_j$ in Equation 4-54 represents all fractions that yield breakage byproducts within size class $i$ (i.e, both $\xi$ and 1- $\xi$ terms).

### 4.2.7.2 Collision-Induced Disaggregation

Disaggregation as a result of collisions between particles is defined based on a modification of the approach of McAnally (1999). The decision for breakage due to collision is based on the estimation of the stress of the collisions according to Equation 4-55, which estimates the shear stress on particle $k$ during a two-body collision with particle $i$. McAnally (1999) extended Equation 4-55 to three-body collisions, arguing that three-body collisions more
frequently result in breakage. The hydrodynamic interaction of a three-body collision was schematized by Clercx and Schram (1992) as a sequence of two two-body collisions, enabling consideration of three-body collisions as implicitly accounted for over time intervals larger than the breakup time scale.

\[
\tau_{sk,k} = \frac{8u^2 m_i m_k}{\pi F_p d_k^2 \left( d_i + d_k \right) \left( m_i + m_k \right)}
\]  

(4-55)

The variable \( F_p \) is a coefficient of the relative depth of inter-particle collision, which acknowledges that flocs are not hard spheres, but rather can deform upon collision, its value ranging from 0 to 0.5, with a value of 0.1 used. The collision velocity, \( u_i \), is estimated by McAnally (1999) based on Equation (4-56).

\[
u_i = \begin{cases} \frac{k_b T}{3\pi \mu d_i d_k} & \text{for Brownian motion} \\ \frac{\left( d_i + d_k \right)}{2} \sqrt{\frac{2\varepsilon}{15\pi \nu}} & \text{for internal shear} \\ |w_{si} - w_{sk}| & \text{for differential settling} \end{cases}
\]  

(4-56)

For isotropic turbulence the definition of the turbulence dissipation can be simplified (Equation 4-57). The Taylor microscale length, \( \lambda \), is defined as shown in Equation 4-58 (Tennekes and Lumley, 1973), where \( S_{ij} \) is the deviator stress tensor.

\[
\varepsilon = 2\nu S_{ij} S_{ij} = 15\nu \left( \frac{\partial u'}{\partial z} \right)^2 
\]  

(4-57)

\[
\left( \frac{\partial u'}{\partial z} \right)^2 = \frac{u_z^2}{\lambda^2} 
\]  

(4-58)
This scale length is representative of the energy transfer from large to small scales. The Taylor microscale can be used with the shear velocity to estimate the dissipation rate as shown in Equation 4-59.

\[
\varepsilon \approx 15\nu \left( \frac{\partial u'}{\partial z} \right)^2 \approx 15\nu \left( \frac{u_\ast}{\lambda} \right)^2
\]  

(4-59)

If the shear, \(G\), is assumed to be approximated by \(\left( u_\ast / \lambda \right)\) we obtain Equation 4-60.

\[
G \equiv \sqrt{\frac{\varepsilon}{15\nu}}
\]  

(4-60)

If the factor of 15 is ignored in Equation 4-60, then Equation 4-43 is obtained. The shear term in Equation 4-60 will underestimate the shear if the additional \(2/\pi\) is also incorporated under the radical, since the conventional approach has been to ignore the \(1/\sqrt{15}\), to obtain 4-43 (Maggi, 2005).

Floc breakage logic is based on the relative values of the individual particle shear strength and the collision shear. If the shear strengths of both particles involved in the collision are greater than the collision shear stress, aggregation may occur based on the efficiency of aggregation (Equation 4-39). If only one of the two particles is stronger than the collision shear, the stronger floc will “steal” a fraction of the weaker floc. McAnally (1999) argued that the most likely fraction to transfer was 3/16 based on the angle of collision. If both of the particles are weaker than the collision shear, then the outcome will be three particles, with essentially the two stolen parts of the two original flocs combining to form the third floc. Again, the most likely fractions were assumed to be 3/16 from each original floc.

The collision breakage mass fluxes can be expressed similarly to the shearing fluxes. The collision disaggregation efficiency is defined by Equation 4-61.
The collision breakage mass flux losses from class $i$ due to collisions with particles of class $j$ is based on the dissaggregation efficiency applied to those collisions that do not result in aggregation (see Equation 4-62).

\[
\alpha_{Ci} = A \frac{d_{ji}}{\lambda_0} \left( \tau_{j,j} - \tau_{j,j} \right) 
\]

(4-61)

This suggests that when the aggregation efficiency approaches one, as suggested for estuaries, that the significance of collision-based disaggregation vanishes.

The disaggregation gains for size class $i$ from collision disaggregation when particles of classes $j$ and $\zeta$ collide is defined by Equation 4-63.

\[
L_{ic} = \sum_j L_{ijc} = \sum_j \frac{\alpha_{Ci}}{\alpha_A} (1 - \alpha_A) N_i m_i
\]

(4-62)

\[
G_{ic} = \sum_j G_{ijc} = \sum_j \frac{\alpha_{Ci}}{\alpha_A} (1 - \alpha_A) N_{ij} \lambda_i \left( \xi_j m_j + \xi_\zeta m_\zeta \right) \approx \xi_j m_j + \xi_\zeta m_\zeta \approx m_i
\]

(4-63)

The variables take the same meaning as described previously. The fractions of the original flocs, $\xi_j$ and $\xi_\zeta$, are the appropriate values associated with smaller or larger byproducts of collisions as necessary to result in the mass of the class $i$ flocs. The distribution fractions $\lambda_i$ are based on the linear interpolation in log-transformed space (see Equation 4-47).

For the calculation of the gains and losses of sediment mass for aggregation and disaggregation, the gains and losses for each process must balance. That is,

\[
\sum_i (G_{ia} + L_{ia}) = 0; \quad \sum_i (G_{jb} + L_{jb}) = 0; \quad \sum_i (G_{ic} + L_{ic}) = 0
\]

An example accounting of the net balances in the computational cells over the vertical is presented in Table 4-2. The breakage balances precisely, while the net aggregation is accurate to six orders of magnitude smaller ($O(10^9)$) than the aggregation flux itself ($O(10^3)$)).
4.3 Hydrodynamics

Hydrodynamics are evaluated by the use of the Reynolds-averaged Navier-Stokes momentum equations (Equation 4-64).

\[
\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_j u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + g_i + \frac{\partial}{\partial x_i} \left[ \bar{p}(\nu+\nu_t) \frac{\partial \bar{u}_i}{\partial x_j} \right] \tag{4-64}
\]

Adding corrections for effects of sediment concentration on the momentum equations emulating the effects form two-phase flow, again based on Hsu, et. al. (2007), results in Equation 4-65.

\[
\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_j u_i}{\partial x_j} = -\frac{1}{\rho_w (1-\phi)} \frac{\partial \bar{p}}{\partial x_i} + g_i \left[ \frac{(s-1)\phi}{(1-\phi)} \right] + \frac{1}{\rho_w (1-\phi)} \frac{\partial}{\partial x_i} \left\{ \rho_w (\nu+\nu_t+\nu_r) \left( e^{-2.34\phi} \frac{\partial \bar{u}_i}{\partial x_j} + 1.89 \frac{\phi^3}{(1-\phi)^3} \frac{\partial \bar{u}_i}{\partial x_j} \right) \right\} \tag{4-65}
\]

The volume concentration at the space-filling threshold, \( \phi_0 \), is included in a manner that greatly amplifies the effects of viscosity as \( \phi \rightarrow \phi_0 \). For the case of uniform flow Equation 4-65 simplifies to Equation 4-66.

\[
\frac{\partial \bar{u}}{\partial t} = -\frac{1}{\rho_w (1-\phi)} \frac{\partial \bar{p}}{\partial x} + \frac{1}{\rho_w (1-\phi)} \frac{\partial}{\partial z} \left\{ \bar{p}(\nu+\nu_t+\nu_r) \frac{\partial \bar{u}}{\partial z} \right\} \tag{4-66}
\]

The turbulent viscosity is estimated by the revised form of Equation 4-24, taking the sediment concentration into account, as shown in Equation 4-67,

\[
\nu_T = c_\mu f_\mu \frac{k^2}{s} (1-\phi) \tag{4-67}
\]
In addition, Hsu, et. al. also included the effects of the sediment stresses on the momentum through an added relative viscosity term, defined by Equation 4-68.

\[ \nu_r = \nu \frac{e^{-2.34\phi}}{\left(1 - \frac{\phi}{\phi_0}\right)^3} \]  

(4-68)

The boundary condition for the momentum equation at the water surface is a no-stress boundary, which results in a zero gradient boundary condition (Equation 4-69).

\[ \frac{\partial u}{\partial z} \bigg|_{z=s} = 0 \]  

(4-69)

The bottom boundary condition for the mean-velocity momentum equation is the bottom shear stress, which leads to a velocity gradient condition (Equation 4-70).

\[ \frac{\partial u}{\partial z} \bigg|_{z=0} = \frac{\tau_b}{\rho \nu_t} \]  

(4-70)

In applying Equation 4-66 the pressure term is converted, using the hydrostatic pressure assumption into Equation 4-71.

\[ \frac{\partial \bar{p}}{\partial x} = -\rho_w g \frac{\partial h}{\partial x} = -\rho_w g S = -\rho_w g \left(\frac{\bar{u}^2 n^2}{R^{4/3}}\right) \]  

(4-71)

The new variables in Equation 4-71 are the bed slope, \( S \), the depth-averaged mean velocity, \( \bar{u} \), and the hydraulic radius, \( R \). This formulation can be viewed as either the case of a horizontal bed with a pressure gradient applied or as a uniform flow case with no pressure gradient but with a gravitational force component along the bed slope of the system.

### 4.4 Solution Method

The equations for the flow and sediment transport are solved by a simple finite difference discretization of each of the equations over evenly spaced cells through the water column. The
vertical dimension of each cell is therefore the water depth divided by the number of cells,

\[ \Delta z = \frac{h}{N_z}. \]

The general form of most of the governing equations in the uniform flow case takes the form of Equation 4-72.

\[ \frac{\partial \theta}{\partial t} - A \frac{\partial}{\partial x_j} \frac{\partial \theta}{\partial x_j} = P - D \]  \hspace{1cm} (4-72)

The time rate of change of a generic variable \( \theta \) less the diffusion of the variable balances the production and dissipation, or other forcing. Other transformations of the variable can be lumped into the production or dissipation.

The governing equation is discretized as shown in Equation 4-73. The temporal time step is a forward difference for the temporal derivative and the remaining terms in the equation are averaged over the time step.

\[ \frac{\theta^n_{i+1} - \theta^n_i}{\Delta t} = \frac{L^n + L^n_i}{2} \]  \hspace{1cm} (4-73)

The loading terms, \( L \), are expressed as shown in Equation 4-74.

\[
L^n_i = L^n_i \left( \theta^n_{i-1}, \theta^n_i, \theta^n_{i+1}, V^n \right)
\]
\[
L^n_{i+1} = L^n_{i+1} \left( \theta^n_{i-1}, \theta^n_i, \theta^n_{i+1}, V^n, V^{n+1} \right)
\]  \hspace{1cm} (4-74)

The right hand side of Equation 4-73 at the time level \( n \) involves the tri-diagonal variables of \( \theta^n \) and other generic variables, \( V^n \), at time level \( n \) variables which contribute to terms in the right-hand-side of the matrix equations. The loading term above at the new time level \( n+1 \) will involve the implicit variables for the \( \theta \) variable as the unknowns, but with a mixture of other variables at both the time levels \( n \) and \( n+1 \), depending on whether those variables have been
solved for at the new time level. Equation 4-75 presents the diffusion term for the new time level, with the coefficients $A$ evaluated with the most currently available information (time level $n$ for the predictor step and time level $n+1$ for the corrector step). Each of the primary variables $u_i$, $k_i$, $\varepsilon_i$, and $c_{ij}$ are solved in succession and then an additional “corrector” sweep of solutions is performed to update the new time level solutions.

$$L_{k_{n+1}} = \frac{1}{\Delta z} \left[ \left( A_{k+1}^{n\prime} + A_k^{n\prime} \right) \left( \theta_{k+1}^{n+1} - \theta_k^{n+1} \right) - \frac{1}{2} \left( A_{k+1}^{n\prime} + A_{k-1}^{n\prime} \right) \left( \theta_k^{n+1} - \theta_{k-1}^{n+1} \right) \right]$$  \hspace{1cm} (4-75)

The coefficient matrix of Equation 4-73 is tri-diagonal and a standard tri-diagonal matrix inversion algorithm was used to solve the matrix. The boundary conditions applied at the bottom and the surface were either Dirichlet or a Neumann condition, depending on the specific variable being solved. The boundary conditions applied for each of the governing equations are summarized in Table 4-3.

4.5 Probabilistic Representation

The representation of variables probabilistically leads to the issue of how variables are combined within the governing equations. The probability of one variable exceeding another was addressed in Chapter 3 (Equations 3-13 through 3-16). When two variables are multiplied together, the products can be developed based on rules of integration. If both variables are expressed probabilistically then the product can also be represented as a probability distribution. For most of the calculations developed within the model the goal from the products of the variables is the “expected value” of the product.

Assume that two variables $A$ and $B$ are to be incorporated as a product, when each is represented by a probability distribution. The expected value of the product is shown in Equation 4-76 for both double and triple products.
\[ p(A) = f_A(A) \]
\[ p(B) = f_B(B) \]
\[ E(A^n B^m) = \int \int f_A(\tilde{A}) f_B(\tilde{B}) \tilde{A}^n \tilde{B}^m d\tilde{A} d\tilde{B} \]
\[ E(A^n B^m C^p) = \int \int \int f_A(\tilde{A}) f_B(\tilde{B}) f_C(\tilde{C}) \tilde{A}^n \tilde{B}^m \tilde{C}^p d\tilde{A} d\tilde{B} d\tilde{C} \]
\[ P(A < B) = \int_{-\infty}^{B} \left\{ \int_{-\infty}^{d} f_A(\tilde{A}) d\tilde{A} \right\} f_B(\tilde{B}) d\tilde{B} = \int_{-\infty}^{\infty} \left\{ F_A(B) \right\} f_B(\tilde{B}) d\tilde{B} \]

The variables \( A, B \) and \( C \) are arbitrary. The probability distribution functions (PDFs) of those variables are \( f_A, f_B \) and \( f_C \). The cumulative distribution function (CDF) for \( A \) is \( F_A \). In products of many variables, if one of the variables is not represented probabilistically, it is treated as a constant and moves outside the integrals. All the variables with a tilde are dummies of integration.

### 4.6 Program Outline

The overall flow logic of the computer program is presented in Figures 4-3 through 4-5. The flow logic for the MAIN portion of program “COHPROB” is presented in Figure 4-3 for the preliminary initialization phase, in Figure 4-4 for the hydrodynamic spin-up phase and in Figure 4-5 for the simulation phase for sediment transport. The details of selected critical subroutines are flow-charted in Appendix D.

The first phase of the program includes data initialization, specification of constants, data input and preliminary setup calls to certain subroutines to fully develop parametric input data specification. The first phase also includes several special problem subroutines to perform simulations of analytical test cases.

### 4.7 Analytical Test Cases

A series of analytical cases were simulated within the computational program to provide validation of the numerical procedures used. These cases are separate from the sediment cases described in Chapter 5, which are based on measured data. The laminar flow test cases described
here each have analytical solutions that can be used to calculate the accuracy of the simulation using error estimates. Table 4-4 summarizes the laminar flow problems tested and pertinent simulation parameters.

### 4.7.1 Stokes first problem

Stokes first problem (Schlichting, 1968) involves the instantaneous acceleration of an infinite flat plate from rest to a fixed velocity, $u_0$, in the plane of the plate. The gradients parallel to the plate vanish and the pressure is assumed uniform within the domain. The governing equation for the movement simplifies to a one-dimensional equation (Equation 4-77) with the $z$-direction taken perpendicular to the plate.

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}$$  \hspace{1cm} (4-77)

The boundary conditions are:

$$\begin{align*}
  u &= 0 \quad \text{for all } z, \quad \text{for } t \leq 0 \\
  u &= u_0 \quad \text{for } z = 0, \quad \text{for } t > 0
\end{align*}$$  \hspace{1cm} (4-78)

The viscosity of the fluid is assumed to be uniform within the domain. Equation 4-77 is reduced to an ordinary differential equation by the substitution of a nondimensional distance shown in Equation 4-79.

$$\eta = \frac{z}{2\sqrt{vt}}$$  \hspace{1cm} (4-79)

Using the assumption that $u$ is a function of $\eta$ given in Equation 4-80, the ordinary differential Equation 4-81 is derived.

$$u = u_0 f(\eta)$$  \hspace{1cm} (4-80)

$$f'' + 2\eta f' = 0$$  \hspace{1cm} (4-81)
The boundary conditions for Equation 4-81 are \( f = 1 \) at \( \eta = 0 \) and \( f = 0 \) at \( \eta = \infty \). The solution is given in Equation 4-82, which involves the complimentary error function.

\[
u = u_0 \text{erfc} (\eta)
\]  \hspace{1cm} (4-82)

The complimentary error function is given in Equation 4-83.

\[
\text{erfc}(\eta) = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} \exp(-\eta^2) d\eta
\]  \hspace{1cm} (4-83)

The non-dimensionalization of the distance from the wall via equation 4-68 results in the velocity distribution being self-similar. The boundary layer grows proportionally with the square root of the product of the kinematic viscosity and time. The analytical solution (Equation 4-82) is presented in Figure 4-6 as the nondimensional velocity, \( \nu/\nu_0 \), versus the nondimensional distance, \( \eta \).

Stokes first problem was simulated within the model for spatial dimensions and velocity of the plate that would result in a low Reynolds number (below 500), making the laminar flow assumption reasonable. Strictly speaking laminar flow assumption is valid when the Reynolds number is 0.01. The velocity distribution and the boundary layer evolve with time. The domain of the analytical problem is an semi-infinite half space; however, the numerical model must be bounded. The boundary condition applied at the limit of the model domain away from the moving wall was chosen to be a zero gradient boundary. A zero flow specification was also tested. The limit of the validity of the simulation is reached when the velocity at the boundary becomes nontrivial, and the zero flux condition makes interpretation of when that limit occurs more straightforward. The depth of the domain was chosen as 0.1 m, with a wall velocity of 0.01 m/s. The kinematic viscosity for all of the laminar flow problems was \( 1.12 \times 10^{-6} \) m\(^2\)/s. The model used a 0.1 second time step.
The boundary conditions for the discretized domain are given in Equation 4-84. The wall boundary condition at the moving wall is a Dirichlet specification for the velocity of the wall converted to a stress condition. The boundary condition at the open boundary is a no-flux condition.

\[
\tau_0(t) = \nu \frac{\partial u}{\partial z}(t) \bigg|_{z=0} \approx \nu \frac{u_0 - u(1,t)}{\Delta z / 2} \quad \text{at } z = 0
\]

\[
\frac{\partial u}{\partial z} \bigg|_{z=h} = 0 \quad \text{for all } t
\]

(4-84)

The variable \( h \) is the size of the domain simulated in the model and has no significance to the problem other than to limit the duration of the validity of the approximate boundary condition there. The number of cells, \( N \), and the domain size, \( h \), define the grid cell size (\( \Delta z = h/N \)).

The model was simulated with a number of spatial resolutions to evaluate the sensitivity of the accuracy of the solution. The results of the simulations using 80, 40, 20 and 10 cells across the domain, respectively, are presented in Figures 4-7 through 4-10, presenting the normalized velocity versus the distance from the wall for three times after the initiation of the wall movement. These figures illustrate that the relative resolution becomes a function of the boundary layer thickness, which is proportional to \( \sqrt{vt} \). The resolution for the 40-cell case after 90 seconds is comparable to the 80-cell case at 24 seconds. The 10-cell case resolution at 198 seconds is comparable to the 20-cell case after 24 seconds. These observations are a direct result of the self-similarity of the evolving velocity distribution. The times 24, 90 and 198 seconds were selected for comparison because they result in uniform incremental boundary layer thicknesses.
Although the model simulated the domain 0.1 m across, the useable domain was limited to about half that distance to minimize the effects of the finite domain and the approximate no-flux boundary condition at \( z=h \). The limitation on the domain essentially translates into a limit on the time length of a valid simulation.

The varying resolutions are presented using the nondimensional distance from the wall in Figures 4-11, 4-12 and 4-13. These figures confirm the self-similarity of the velocity profiles generated by the model. The 80-cell and 40-cell simulations very accurately reproduced the analytical distribution. The 20-cell simulation slightly underestimated the velocities for the 24 second profile. The 10-cell simulation underpredicted the velocities at all three analyzed time levels, but with the error becoming smaller with time as the effective resolution increases.

The errors in the nondimensional velocity distribution were estimated by differencing the simulated velocity (divided by \( u_0 \)) and the analytical solution. These error estimates were then analyzed to obtain a standard deviation of the errors. These error standard deviations are summarized in Table 4-5 for the Stokes first problem. Several observations can be made of the relative errors. The 80-cell results are consistently more accurate than the lower resolution simulations. However, as time progresses, the error increases for the 80-cell simulation. This may be associated with the finite bound on the domain and the artificial boundary condition. The lower resolution simulations had the error decrease as the simulation progressed, primarily due to the poor representation with less relative resolution early in the simulation when the boundary layer was small relative to the grid cell size. These error estimates are only comparable within the context of this approximation of this analytical problem and have little meaning for comparing between test problems.
The effects of sediment concentration on the solution were tested by defining a uniform concentration and defining the viscosity by Equation 4-68. The simulation results are presented in Figures 4-15 and 4-16, for concentrations of 20 and 100 kg/m³ corresponding to the fluid mud density range. The presence of the sediment increases the effective viscosity and accelerates the evolution of the boundary layer. The 20 kg/m³ concentration increased the boundary layer growth rate by approximately 4 percent, while the 100 kg/m³ concentration increased the growth rate on the order of 16 percent.

4.7.2 Couette flow problem

Couette flow is a modification of the Stokes first problem by imposing a stationary boundary at a distance, \( h \), from the moving wall. The moving wall is instantaneously accelerated to the constant speed, \( u_0 \), but the velocity at the opposite wall is zero. The governing equation remains Equation 4-77. The boundary conditions are:

\[
\begin{align*}
    u &= 0 & \text{for all } z, \text{ for } t \leq 0 \\
    u &= u_0 & \text{for } z = 0, \text{ for } t > 0 \\
    u &= 0 & \text{for } z = h, \text{ for all } t
\end{align*}
\]

The analytical solution for the Couette flow problem is given in Equation 4-86, a series solution based on a Laplace transform.

\[
\frac{u}{u_0} = \sum_{n=0}^{\infty} \text{erfc}[2n\eta_i + \eta] - \sum_{n=0}^{\infty} \text{erfc}[2(n+1)\eta_i + \eta]
\]

\[
= \text{erfc}(\eta) - \text{erfc}(2\eta_i - \eta) + \text{erfc}(2\eta_i + \eta) - \text{erfc}(4\eta_i - \eta) + \text{erfc}(4\eta_i + \eta) - \ldots + \ldots
\]

(4-86)

where \( \eta_i \) is the dimensionless distance between the two walls (Equation 4-87).

\[
\eta_i = \frac{h}{2\sqrt{\nu t}}
\]
The analytical solution (Equation 4-86) is presented in Figure 4-17 for a series of nondimensional times, \( t_s \), defined by Equation 4-88.

\[
t_s = \frac{4\sqrt{\nu t}}{h} = \frac{2}{\eta}
\]  

(4-88)

The steady-state solution for Couette flow is a linear variation of velocity between the two plates.

The numerical model hydrodynamic component was applied to the Couette problem with varying level of spatial resolution. The boundary condition at the moving wall is the same as used in Stokes first problem (Equation 4-84). The zero-flow specification at the opposite wall was converted into an applied stress, dependent on the velocity in the cell adjacent to the wall (Equation 4-89).

\[
\tau_0(t) = -\nu \frac{\partial u}{\partial z}(t) \bigg|_{z=0} \approx -\nu \frac{u_0-u(1,t)}{\Delta z / 2} \quad \text{at } z = 0
\]

\[
u \bigg|_{z=h} = 0 \implies \tau_{z=h}(t) = \nu \frac{\partial u}{\partial z}(t) \bigg|_{z=h} \approx \nu \frac{u(N,t)}{\Delta z / 2}
\]  

(4-89)

At steady-state conditions \( (t = \infty) \) the stress will be the same magnitude at the walls but with opposite sign. The viscosity and time step were the same as for the Stokes first problem.

The results of the application of the model to the Couette problem are presented in Figures 4-18 through 4-21 for numbers of cells of 80, 40, 20 and 10, respectively. The numerical model results are plotted as symbols against the analytical solution, which are the lines in the plots, for several nondimensional time scales. The 80-cell and 40-cell results very accurately replicate the analytical results. The 20-cell resolution shows some deviation from the analytical solution at the earlier time scales, but the accuracy is good for larger time scales. The 10-cell results show even greater error for the earlier time scales, but agree well for dimensionless time scales of 1.0 and greater.
It is interesting to note that for the earlier time scales of the Couette flow problem, before the fixed opposite wall has a significant stress, the simulation is essentially Stokes first problem. The conclusions about the relative resolution near the moving wall apply during the early portion of the Couette flow problem.

The standard deviations of the error between the simulated and analytical results are summarized in Table 4-6 for the Couette flow problem. These error measures show that as the resolution increases the error is reduced and as the solution evolves toward steady state the errors are generally reduced.

The effects of suspended sediment concentration on the Couette flow problem were simulated with concentrations of 20 and 100 kg/m³. The results are presented in Figures 4-22 and 4-23. The effective time scale was computed for the sediment simulations using clear water viscosity in order to see the effects of the sediment. If the effective viscosity with the sediment were used, the curves would be the same. This also shows that the presence of sediment accelerates the development of the velocity profile.

4.7.3 Stokes second problem

Stokes second problem is an extension of the first problem to include a harmonic oscillation of the moving plate. The governing equation remains Equation 4-77 and the boundary conditions are given in Equation 4-90: The effect of suspended sediment is greatest early in the simulation before the velocities reach the opposite wall, during which the solution is essentially the Stokes first problem.

\[
\begin{align*}
  u(0,t) &= u_0 \cos(\omega t) \quad \text{at } z = 0 \text{ for time } t \\
  u(\infty,t) &= 0 \quad \text{at } z = \infty \text{ for all time } t
\end{align*}
\]

(4-90)

The solution of Equation 4-77 with the boundary conditions of Equation 4-90 is given in Equation 4-91.
\[ u(z,t) = u_0 \exp(-kz) \cos(\omega t - kz) \]  

(4-91)

The dispersion equation, relating \( k \) and \( \omega \) for Equation 4-90 as a solution of Equation 4-26 is given in Equation 4-92.

\[ k = \sqrt{\frac{\omega}{2\nu}} \]  

(4-92)

The analytical solution (Equation 4-91) is presented in Figure 4-24. The solution is a damped harmonic oscillation, with the bounding amplitude of the harmonic plotted as the dashed red lines in Figure 4-24.

In applying the numerical model to Stokes second problem the no-slip boundary condition at the moving wall is again converted to a stress boundary condition as given in Equation 4-93.

\[ \tau_0(t) = -\nu \frac{\partial u}{\partial z}(t) \bigg|_{z=0} \approx -\nu \frac{u_0(t) - u(\Delta z/2,t)}{\Delta z/2} \text{ at } z = 0 \]  

(4-93)

where \( u_0(t) \) is the oscillating velocity at the wall and \( u(\Delta z/2,t) \) is the velocity in the center of the cell adjacent to the wall. The boundary condition at the open boundary is the same as for the Stokes first problem, a no-stress condition (see Equation 4-84).

The frequency of the oscillation used in simulating Stokes second problem was carefully selected in conjunction with the size of the simulated domain such that the solution is substantially damped at the open boundary. The period of oscillation was chosen to be 288 seconds, giving an angular frequency of 0.021816616 s\(^{-1}\). The size of the simulated domain was retained at 0.1 m, as for the Stokes first problem. The time step for the simulation was 0.1 s and the clear water kinematic viscosity 1.12 x 10\(^{-6}\) m\(^2\)/s.

The results of the Stokes second problem simulations are presented in Figures 4-25 through 4-28 for spatial discretization of 80, 40, 20 and 10 cells, respectively. Ignoring the 10-cell case, the agreement is very good with the exception of the 20-cell case very close to the
oscillating wall. The case with only 10 cells was simulated and the results presented but the results were so poorly resolved that they should be discounted. The effective resolution is practically about half of the stated resolution because of the need to put the open boundary far from the oscillating wall.

The sensitivity of the Stokes second problem to sediment concentrations was tested by simulations with 20 and 100 kg/m$^3$ suspended sediment concentrations. Those tests used 40-cell resolution. The results are presented in Figures 4-29 and 4-30. For these plots the values of $\eta$ for the sediment runs were computed using the clear water viscosity to see the impact of the suspended sediment. The effect of the sediment, through increased viscosity, is to stretch the velocity distribution away from the oscillating wall by a factor of the square root of the ratio of the viscosities. The relative effects are the same as seen in the other laminar flow test cases.

4.7.4 von Karman Mixing Length Velocity Profile for Fully Rough Flow

The hydrodynamic setting for most of the sedimentation analyses addressed within this work are open channel flows in a fully rough condition ($Re_\kappa = k u_\kappa / \nu > 70$). A model simulation was made to evaluate the velocity profile generated using the von Karman mixing length which results in the turbulent eddy viscosity distribution shown in Equation 4-94.

$$v_t = \kappa u_\kappa z \left( 1 - \frac{z}{h} \right)$$

(4-94)

The hydrodynamic equation (Equation 4-66) was solved using a pressure gradient calculated via Equation 4-71. The depth-averaged velocity used in Equation 4-71 is the final equilibrium velocity, not the time evolving actual mean velocity. The initial condition was set at no flow [$u(z,0) = 0$]. The velocity profile evolved over time until it reached an equilibrium profile. The evolution of the profile over nondimensional time (Equation 4-88) is presented in Figure 4-31. After a nondimensional time $t_e = 0.73$, the velocity distribution remained
unchanged. The equilibrium velocity distribution from the model is compared (Figure 4-32) with the classical analytical logarithmic profile given in Equation 4-95 to evaluate the performance of the hydrodynamic component. The simulation used a water depth of 1 m, a depth-averaged velocity of 0.5 m/s and the roughness height was 0.1 m. The estimated shear velocity was 0.0424 Pa.

\[
\frac{\tau}{u_*} = \frac{1}{\kappa} \ln \left( \frac{z}{z_0} + 1 \right) = \frac{1}{\kappa} \ln \left( \frac{29.7z}{k_s} + 1 \right) \tag{4-95}
\]

The differences between the model results and the analytical profile give a standard deviation of the error of 0.00267 m/s. Also presented in Figure 4-32 is the vertical distribution of the local shear stress at equilibrium. The shear stress profile is linear, as expected.
Table 4-1. Standard $k-\varepsilon$ model coefficients for high Reynolds number flow.

<table>
<thead>
<tr>
<th>$C_{\mu}$</th>
<th>$C_{\varepsilon 1}$</th>
<th>$C_{\varepsilon 2}$</th>
<th>$C_{\varepsilon 3}$</th>
<th>$\sigma_k$</th>
<th>$\sigma_{\varepsilon}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.09</td>
<td>1.44</td>
<td>1.92</td>
<td>1.2</td>
<td>1</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Table 4-2. Summary of example flocculation model.

<table>
<thead>
<tr>
<th>Vertical cell (m)</th>
<th>Elevation (m)</th>
<th>Aggregation (kg/m$^3$/s) Gains</th>
<th>Losses</th>
<th>Net</th>
<th>Breakage (kg/m$^3$/s) Gains</th>
<th>Losses</th>
<th>Net</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.015</td>
<td>7.54E-03</td>
<td>7.54E-03</td>
<td>9.31E-10</td>
<td>4.04E-05</td>
<td>4.04E-05</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.046</td>
<td>6.96E-03</td>
<td>6.96E-03</td>
<td>-4.19E-09</td>
<td>1.33E-05</td>
<td>1.33E-05</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.076</td>
<td>6.76E-03</td>
<td>6.76E-03</td>
<td>-1.86E-09</td>
<td>7.89E-06</td>
<td>7.89E-06</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.107</td>
<td>6.64E-03</td>
<td>6.64E-03</td>
<td>-4.19E-09</td>
<td>5.55E-06</td>
<td>5.55E-06</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0.137</td>
<td>6.56E-03</td>
<td>6.56E-03</td>
<td>-1.86E-09</td>
<td>4.24E-06</td>
<td>4.24E-06</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0.168</td>
<td>6.49E-03</td>
<td>6.49E-03</td>
<td>5.12E-09</td>
<td>3.39E-06</td>
<td>3.39E-06</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0.198</td>
<td>6.44E-03</td>
<td>6.44E-03</td>
<td>0.00E+00</td>
<td>2.79E-06</td>
<td>2.79E-06</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0.229</td>
<td>6.38E-03</td>
<td>6.38E-03</td>
<td>-2.79E-09</td>
<td>2.32E-06</td>
<td>2.32E-06</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0.259</td>
<td>6.32E-03</td>
<td>6.32E-03</td>
<td>-8.38E-09</td>
<td>1.92E-06</td>
<td>1.92E-06</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>0.29</td>
<td>6.25E-03</td>
<td>6.25E-03</td>
<td>0.00E+00</td>
<td>1.48E-06</td>
<td>1.48E-06</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4-3. Summary of boundary condition specifications.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Bottom boundary condition</th>
<th>Surface boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrodynamics: Stokes first problem</td>
<td>$u(0) = u_0$</td>
<td>no stress</td>
</tr>
<tr>
<td>Hydrodynamics: Couette flow problem</td>
<td>$u(0) = u_0$</td>
<td>$u(h) = 0$</td>
</tr>
<tr>
<td>Hydrodynamics: Stokes second problem</td>
<td>$u(0) = u_0 \cos(\omega t)$</td>
<td>no stress</td>
</tr>
<tr>
<td>Hydrodynamics: Logarithmic profile open channel flow</td>
<td>$u(0) = 0$</td>
<td>no stress</td>
</tr>
<tr>
<td>Turbulent kinetic energy</td>
<td>$k(0) = \frac{u_s^2}{\sqrt{C_{\mu}}}$</td>
<td>No flux</td>
</tr>
<tr>
<td>Turbulent kinetic energy dissipation</td>
<td>$\varepsilon(0) = \frac{u_s^3}{kz_0}$</td>
<td>No flux</td>
</tr>
<tr>
<td>Suspended sediment concentration</td>
<td>specified flux</td>
<td>no flux</td>
</tr>
</tbody>
</table>

Table 4-4. Simulation conditions for special laminar flow problems.

<table>
<thead>
<tr>
<th>Special Problem</th>
<th>Domain size (m)</th>
<th>$u_0$ (m/s)</th>
<th>Frequency (1/s)</th>
<th>time step (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stokes first problem</td>
<td>0.1</td>
<td>0.01</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Couette flow problem</td>
<td>0.1</td>
<td>0.01</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Stokes second problem</td>
<td>0.1</td>
<td>0.01</td>
<td>0.0218166616</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Table 4-5. Error measures for the Stokes first problem for varying time and number of cells.

<table>
<thead>
<tr>
<th>Number of cells</th>
<th>24 ($\eta =0.01$)</th>
<th>90 ($\eta =0.02$)</th>
<th>198 ($\eta =0.03$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0385</td>
<td>0.0121</td>
<td>0.00538</td>
</tr>
<tr>
<td>20</td>
<td>0.00823</td>
<td>0.00202</td>
<td>0.000933</td>
</tr>
<tr>
<td>40</td>
<td>0.00146</td>
<td>0.000313</td>
<td>0.000538</td>
</tr>
<tr>
<td>80</td>
<td>6.02 E-10</td>
<td>8.27 E-6</td>
<td>0.00014</td>
</tr>
</tbody>
</table>

Table 4-6. Error measures for the Couette problem for varying time and number of cells.

<table>
<thead>
<tr>
<th>Time scale, $t_s$</th>
<th>80</th>
<th>40</th>
<th>20</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nondimensional error standard deviation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Number of cells</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.25</td>
<td>0.000414</td>
<td>0.000906</td>
<td>0.005432</td>
<td>0.031066</td>
</tr>
<tr>
<td>0.5</td>
<td>0.001245</td>
<td>0.001528</td>
<td>0.002894</td>
<td>0.009549</td>
</tr>
<tr>
<td>1.0</td>
<td>0.000697</td>
<td>0.000649</td>
<td>0.000408</td>
<td>0.001633</td>
</tr>
<tr>
<td>1.5</td>
<td>0.000943</td>
<td>0.000921</td>
<td>0.000688</td>
<td>0.000278</td>
</tr>
<tr>
<td>2.0</td>
<td>0.000454</td>
<td>0.000394</td>
<td>0.000366</td>
<td>0.000176</td>
</tr>
</tbody>
</table>

Table 4-7. Error measures for the Stokes second problem for varying time and number of cells.

<table>
<thead>
<tr>
<th>Number of cells</th>
<th>0</th>
<th>$\pi/2$</th>
<th>$\pi$</th>
<th>$3\pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nondimensional error standard deviation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Phase, $\omega t$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.03193</td>
<td>0.09237</td>
<td>0.31791</td>
<td>0.28498</td>
</tr>
<tr>
<td>20</td>
<td>0.00702</td>
<td>0.00950</td>
<td>0.00587</td>
<td>0.00910</td>
</tr>
<tr>
<td>40</td>
<td>0.00300</td>
<td>0.00282</td>
<td>0.00160</td>
<td>0.00243</td>
</tr>
<tr>
<td>80</td>
<td>0.00265</td>
<td>0.00164</td>
<td>0.00164</td>
<td>0.00152</td>
</tr>
</tbody>
</table>
Figure 4-1. Comparison of Equations 4-32 and 4-33 for erosion rate with data from Partheniades (1965), using the coefficient \( C_b = 0.8459 \) kg/m\(^3\) in Equation 4-32, \( C_b = 0.112 \) kg/m\(^3\) in Equation 4-33, \( \rho_w = 1030 \) kg/m\(^3\), \( f_i = 1 \), and the variables developing the probability of erosion \( \tau_s = 0.55 \) Pa, \( \sigma_s = 0.25 \) Pa and \( \sigma_b = 0.3 \) Pa.
Figure 4.2. Collision frequency for a particle diameter of 10.6 microns with variable second particle diameter. The flow conditions for this case are a flow depth of 0.3048 m, with a depth-averaged velocity of 0.142 m/s. The probabilistic settling velocity cases assumed a 30 percent standard deviation in the settling velocity.
Figure 4-3. Computer program “COHPROB” flow chart (MAIN) for phase 1.
Figure 4-4. Phase 2 computer program flow chart for “COHPROB” (MAIN); spin up of the hydrodynamic model.
Figure 4-5. Phase 3 (sediment transport) flow chart of “COHPROB” (MAIN).
Figure 4-6. Self-similar velocity distribution solution for the Stokes first problem.

Figure 4-7. Results of 80-cell resolution over domain for simulation of Stokes first problem.
Figure 4-8. Results of 40-cell resolution over domain for simulation of Stokes first problem.

Figure 4-9. Results of 20-cell resolution over domain for simulation of Stokes first problem.
Figure 4-10. Results of 10-cell resolution over domain for simulation of Stokes first problem.

Figure 4-11. Nondimensional velocity distribution for 80 cell simulation of Stokes first problem.
Figure 4-12. Nondimensional velocity distribution for 40-cell simulation of Stokes first problem.

Figure 4-13. Nondimensional velocity distribution for 20-cell simulation of Stokes first problem.
Figure 4-14. Nondimensional velocity distribution for 10-cell simulation of Stokes first problem.
Figure 4-15. Effects of suspended sediment concentration of 20 kg/m$^3$ on Stokes first problem solution. The clear symbols are for no sediment and the blackened symbols are for the sediment-laden case.
Figure 4-16. Effects of suspended sediment concentration of 100 kg/m³ on Stokes first problem solution. The clear symbols are for no sediment and the blackened symbols are for a sediment concentration of 100 kg/m³.
Figure 4-17. Analytical velocity distribution solution of the Couette flow problem for various nondimensional time scales, $t_s$. 
Figure 4-18. Comparison of simulation of Couette flow problem with 80 cells to the analytical solution.
Figure 4-19. Comparison of simulation of Couette flow problem with 40 cells to the analytical solution.
Figure 4-20. Comparison of simulation of Couette flow problem with 20 cells to the analytical solution.
Figure 4-21. Comparison of simulation of Couette flow problem with 10 cells to the analytical solution.
Figure 4-22. Effects of 20 kg/m³ suspended sediment concentration on the Couette flow problem. The time scale for the sediment laden flow was computed with the clear water viscosity to show the effects.
Figure 4-23. Effects of 100 kg/m³ suspended sediment concentration on the Couette flow problem. The time scale for the sediment laden flow was computed with the clear water viscosity to show the effects.
Figure 4-24. Analytical solution of Stokes second problem (Equation 4-91). The dashed red envelopes are the bounding curve for the amplitude of the damped harmonic oscillation.
Figure 4-25. Simulation with 80 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.
Figure 4-26. Simulation with 40 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.
Figure 4-27. Simulation with 20 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.
Figure 4-28. Simulation with 10 cells of Stokes second problem. Symbols are the model results and the lines are the analytical solution at the appropriate phases.
Figure 4-29. Effects of 20 kg/m³ suspended sediment concentration on the results of Stokes second problem for the 40-cell test case. The value of $\eta$ for the sediment laden flow was computed with the clear water viscosity to show the effects.
Figure 4-30. Effects of 100 kg/m$^3$ suspended sediment concentration on the results of Stokes second problem for the 40-cell test case. The value of $\eta$ for the sediment laden flow was computed with the clear water viscosity to show the effects.
Figure 4-31. Temporal development of velocity profile using von Karman mixing length test case.
Figure 4-32. Comparison of simulated fully developed velocity profile to the analytical solution, using 80 cells over water column. Also plotted is the shear stress distribution over the water column from the simulation.
CHAPTER 5
SEDIMENT TRANSPORT AND DEPOSITION EXPERIMENTS

There are several laboratory flume experiments by various investigators that have been recognized as key to the development of CST theory. In addition, because of their frequent use as benchmark cases for testing new developments in the literature they remain valuable data sets for use in new research. Some of these test cases are briefly documented and used in simulations documented in Chapter 6.

5.1 Kynch (1952) Sedimentation Theory

The sedimentation theory of Kynch (1952) provides a general description of the development of a sediment deposit during settling in quiescent conditions. Conservation of sediment mass is combined with the method of characteristics to provide insights into densification during settling. The theory provides a good test case for the simulation model. The description here is based on lecture notes of Mehta (2007). The theory deals with the case of a quiescent settling column that initially has a uniformly mixed dilute sediment suspension at a concentration below the level of hindered settling (see Figure 5-1). As the sediment begins to settle at the free settling rate, sediment near the bottom of the column will accumulate and begin to affect, through hindered settling, the subsequent sediment settling rate approaching the bed. The transition location within the water column where free settling transitions to hindered settling will begin to rise upward from the bed at a speed designated as \( w_p \), creating the primary lutocline (labeled P in Figure 5-1).

Water at the surface will become sediment free as the settling sediment cannot be replaced from above. The interface between the clear water and the settling suspension is a secondary lutocline, which falls downward at the free settling rate, \( w_s \). This secondary lutocline settling
rate is assumed to be equal to the free settling rate of the sediment that is located at the lutocline. Figure 5-2 presents a schematized representation of the time evolution of the sediment deposit.

The governing equation for the conservation of sediment mass is given in Equation 5-1.

\[
\frac{\partial c}{\partial t} + \frac{\partial F_s}{\partial z} = 0
\]  (5-1)

The settling flux can be defined as \( F_s = -w_s c \). Equation 5-1 can be revised using the chain rule for differentiation as shown in Equation 5-2.

\[
\frac{\partial c}{\partial t} + \frac{\partial F_s}{\partial c} \frac{\partial c}{\partial z} = 0
\]  (5-2)

At the primary lutocline the rate of rise can be defined as \( w_p(c) = \frac{\partial F_s}{\partial c} \), leading to Equation (5-3), which is a statement that the total derivative of the sediment concentration is zero.

\[
\frac{\partial c}{\partial t} + w_p \frac{\partial c}{\partial z} = \frac{Dc}{Dt} = 0
\]  (5-3)

The conservation of the sediment concentration within the z-t domain can be expressed as shown in Equation 5-4, using the fundamental statement of the total derivative.

\[
\frac{\partial c}{\partial t} + \frac{\partial z}{\partial t} \frac{\partial c}{\partial z} = \frac{Dc}{Dt} = 0
\]  (5-4)

Comparison of Equations 5-3 and 5-4 leads to the definition \( w_p = \frac{\partial z}{\partial t} \), which is the characteristic velocity of the rising primary lutocline, \( P \). Below the primary lutocline settling is hindered, but it is unclear how the concentration may vary. However, if the rate of rise of the primary lutocline were constant, the average concentration below the lutocline would need to remain constant. During the unhindered settling phase, the supply of sediment to the sediment deposit below the lutocline can be estimated as the sum of the free settling flux \( (w_s c_o) \) and the
sediment overtaken by the rising lutocline \( w_p c_0 \). This is based on the observation that if the upper portion of the suspension between the primary and secondary lutoclines is assumed to settle at a constant velocity, then the concentration will remain constant at \( c_0 \) until it encounters the primary lutocline. Therefore, the rate of change in the total mass of sediment below the primary lutocline, \( M_L \), can be expressed as shown in Equation 5-5.

\[
\frac{\partial M_L}{\partial t} = (w_p + w_s) c_0
\]  

(5-5)

The total mass below the lutocline can be defined as the average concentration times the height of the lutocline, \( h_L \). Equation 5-5 can be expressed as Equation 5-6.

\[
\frac{\partial \bar{c} h_L}{\partial t} = h_L \frac{\partial \bar{c}}{\partial t} + \bar{c}_L \frac{\partial h_L}{\partial t} = (w_p + w_s) c_0
\]  

(5-6)

If \( w_p \) is assumed to be a constant during the unhindered settling phase, then \( h_L = w_p t \) and by definition \( \frac{\partial h_L}{\partial t} = w_p \). With these equalities, Equation 5-6 can be converted to Equation 5-7.

\[
\bar{c}_L + t \frac{\partial \bar{c}_L}{\partial t} = \frac{(w_p + w_s)}{w_p} c_0
\]  

(5-7)

The right hand side of Equation 5-7 is constant if it is assumed that \( w_p \) is constant. Therefore, the only way for this to hold is if \( \bar{c}_L \) is constant and equal to the right hand side of Equation 5-7. Equation 5-6 is valid even if \( w_p \) is not a constant, however, only when the concentration above the primary lutocline is constant at \( c_0 \). The mean concentration is constant below the primary lutocline if the characteristics for all concentrations below the lutocline originate at the bottom \( (z=0) \) at the initiation of settling \( (t=0) \) and each has a constant slope to the
characteristic for that concentration, $\frac{\partial z}{\partial t}$, assuring the proportionality among the concentrations.

If any arbitrary characteristic is extended to the point of intersection with the secondary lutocline, at a height of $h$, it will define a time, $t_*$, at point Q in Figure 5-2. The concentration between the primary and secondary lutoclines is assumed to be constant, $c$, which leads to an approximate conservation statement shown in Equation 5-8. This is approximate because it would be exact if the two lutoclines moved linear until they met and the concentration above the primary lutocline remained at the initial concentration.

$$\frac{M_s}{A} = c(w_p + w_s)t_* = c_0h_0$$  \hspace{1cm} (5-8)

This is a statement that if the two lutoclines transited through the water column at fixed rates, then the two transit heights would equal the depth of the column. The total sediment mass is the initial concentration times that water depth.

Since the equalities shown in Equation 5-9 hold, they can be introduced into Equation 5-8 to yield Equation 5-10.

$$w_s = -\frac{dh}{dt}; \quad w_p = \frac{h}{t_*}$$  \hspace{1cm} (5-9)

$$c_0h_0 = c\left(h - t_*\frac{dh}{dt}\right) = ch_*$$  \hspace{1cm} (5-10)

The boundary between unhindered and hindered settling is assumed to occur when the falling secondary lutocline is no longer linear. The sediment below the primary lutocline begins the consolidation process soon after the deposit has formed and will affect the nonlinearity of the evolution of the primary lutocline such that the rise is not linear.
The conditions for each of the settling zones delineated in Figure 5-1 can be summarized as follows:

**Unhindered settling zone:**

\[ w_s = -\frac{dh}{dt}; \quad c = c_0 \]  
(5-11)

**Hindered settling zone:**

\[ c(t) = \frac{c_0 h_0}{h_*} \]  
(5-12)

**Settled zone:**

\[
\begin{align*}
  c(t) &= c_f = \frac{c_0 h_0}{h_f} \\
  h &= h_f \\
  w_s &= -\frac{dh}{dt} = 0
\end{align*}
\]  
(5-13)

From the settling column test data of Kynch (1952), the estimated settling velocity as a function of the lutocline elevation was developed by Mehta (2007) and is summarized in Table 5-1. The settling velocity is taken as the slope of the secondary lutocline. The estimated concentration profiles based on the method of characteristics are summarized in Figure 5-3 and Table 5-2. At each of the data values defining the falling secondary lutocline in Figure 5-3, the characteristic lines were connected back to the origin. The tangent lines at each of the data values define \(dh/dt\), which then define \(h^*\) via Equation 5-10 and the concentration for the characteristic line is defined by Equation 5-12. Figure 5-3 has the concentrations computed at each data point on the lutocline curve, which then apply along the associated characteristic line. The estimated concentration profiles at hours 5, 15 and 40 are summarized in Table 5-2 for the elevations where the characteristics cross the profiles. The resulting vertical profiles are plotted.
in Figure 5-4. The profiles were integrated to estimate the total mass in suspension, yielding 1.257, 1.263 and 1.243 kg/m², for the hours 5, 15 and 40 profiles. These compare well with the initial condition mass of 1.25 kg/m².

5.2 Krone (1962) Flume Deposition Experiments

Krone (1962) reported results of investigations in a recirculating flume 30.5 m long and 0.9 m wide with a level bed. Flow was generated by a propeller pump attached to a variable speed motor. The flume had an active depositional length of 27.4 m, with the upper 3.1 m containing baffling to dissipate entrance turbulence. Measurements performed in the flume were the temporal evolution of suspended sediment concentrations from the return flow at the upstream end of the flume and inferred flow conditions.

5.2.1 Settling Tests with Variable Shear

The primary tests of interest from the flume study were the deposition tests in flowing water. The tests were conducted in water with a configured sodium chloride and calcium chloride solution to the proportion of monovalent and divalent cations found in seawater, at a concentration of 17 ppt. The focus of these tests was initial concentrations below 0.3 kg/m³, typical of those found in San Francisco Bay. Tests were run with variable flows and initial concentrations. Some of the test results are presented in Figure 5-5. Of particular interest is the change in the slopes of the concentration curves at around 0.3 kg/m³. Above that concentration the deposition is more rapid, and below the slope with time is constant, representing exponential decay. This behavior has been explained as the result of flocculation effects on the settling velocity at the higher concentrations.

5.2.2 Settling Test with Tagged Sediments

Another valuable test conducted by Krone was an experiment that tagged a portion of the suspended sediment with gold-198. This test was conducted in a 0.3 m wide and 10.7 m long
flume. The mean velocity in the test was reported to be 0.085 m/s, and the bottom shear stress was estimated to be 0.02 Pa. The results of this test are presented in Figure 5-6. The figure suggests that the tagged sediment had deposited at a more rapid rate than the total concentration of sediment. Krone concluded that there had to be an interchange between the bed and suspension during transport, even in a depositional environment.

Krone indicated that only a fraction of the suspended sediment was “labeled.” Figure 5-6, reproduced after Krone (1962), indicates that at hour 0 the “tagged” sediment had a concentration of 1.3 kg/m³ and shortly after the initiation of the test (at approximately 0.1 hour) the total suspended sediment concentration was 1.4 kg/m³. This apparent discrepancy is explained if the initial concentration of gold-“labeled” sediment in suspension, although not 100 percent of the sediment, serves to “tag” the initial sediment mixture. For example, if only one percent of the sediment was initially “labeled” at the beginning of the test and after some time the concentration of “labeled” sediment drops to one half percent, then the initial tagged sediment concentration dropped to 50 percent.

5.3 Mehta 1973 Flume Results

The deposition tests of Mehta (1973) were conducted in a rotating annular flume consisting of a circular channel, 0.2 m wide and 0.45 m deep with a mean diameter of 1.5 m. The solution was driven by a rotating upper lid, with the ability for the flume itself to rotate in the opposite direction to minimize secondary currents. Experiments were carried out on a variety of mud, including San Francisco Bay mud. Mud was added to the flume with a high flow velocity to mix it thoroughly before the shear stress was reduced below the critical level for deposition. Testing evaluated the relationship between shear stress and deposition. Some of the results are summarized in Figure 5-7. The results show an equilibrium concentration during deposition that is a fixed percentage of the initial concentration that is related to the shear stress. This is counter
to what Krone (1962) had found, where no equilibrium was ever reached. Mehta explained this as an effect of the particle size distribution of the source mud. If the same mud is used for all tests, then if a fraction of the sediment is a very fine non-cohesive that resists deposition for a given shear stress level, then the same percentage will remain in suspension for the same shear stress, independent of the initial concentration.

5.4 Parchure and Mehta (1985) Dilution Test

Parchure and Mehta (1985) conducted an experiment in the same flume as the Mehta experiments to help evaluate the question of simultaneous versus exclusive erosion and/or deposition. A deposited bed of commercial kaolinite was flocculated in tap water of a total salt concentration of 278 ppm. The settled bed was then eroded for a period of 120 hours with a bottom shear stress of 0.2 Pa in a flow depth of 0.26 m. The rate of erosion decreased rapidly during the first few hours as softer layers were eroded exposing deeper deposited sediments with greater shear strength. The concentration reached an equilibrium value of 3.85 kg/m³ early and remained constant throughout the remainder of the test. After 120 hours, the volume of the flume was slowly replaced with sediment-free water over a period of 4 hours without disturbing the bed. At the end of the 4 hours of sediment removal the concentration within the flume had dropped to 0.03 kg/m³. Flow conditions were maintained for an additional 24 hours during which the concentration slightly increased to 0.1 kg/m³. The results of the variation in concentration during the fluid replacement are shown in Figure 5-8. Parchure discounted the small increase in concentration at the end of the test and concluded that, for the conditions tested, the erosion was independent of suspended sediment concentration.

5.5 Parchure and Mehta (1985) Erosion Test

Parchure and Mehta (1985) also conducted an erosion test on a deposited bed subjected to progressive increases in the shear stress. The goal of the testing was to evaluate the vertical
structure of the bed shear strength. The initial concentration was zero, with an initial shear stress of 0.1 Pa. Subsequent shear stresses were increased by 20% over the previous shear stress. Each shear stress was maintained for an hour, before the next increase. The progression of shear stresses was: 0.1, 0.12, 0.144, 0.173, 0.207, 0.249, 0.299, and 0.358 Pa. The time evolution of the suspended sediment concentration was monitored. The results of one of the test series are presented in Figure 5-9. Within the first five steps in the shear stress the erosion had reached an approximate equilibrium by the end of the hour. For the higher shear stresses one hour may not have been sufficient to fully reach equilibrium.

5.6 Sanford and Halka (1993) Data Set

Sanford and Halka (1993) presented a series of field deployments to measure the resuspension and deposition of placed dredged material at several disposal sites in the Chesapeake Bay. Their attempts to numerically simulate the observed data led them to conclude that for using a single grain size of sediment that it was required to use the simultaneous erosion and deposition paradigm. The data sets were collected at three consecutive years, 1989-1991, with a bottom-moored tripod equipped with a profiling capability to monitor currents, salinity, temperature and turbidity. From the collected data, shear stresses were estimated and suspended sediment loads calculated over single semidiurnal tidal periods. Also collected was the water surface elevation, from which the local water depth was estimated.

The sediment loads were converted to average concentrations and are presented in Figures 5-10 through 5-12. The data from the 1989 and 1991 monitoring were following a dredging disposal, while the 1990 monitoring was at a disposal site where the material had consolidated for about a year.

These field sites provide a means of testing the probabilistic formulation in a dynamic mode when transient conditions are important.
Table 5-1. Summary of Kynch settling column data and graphical analysis (after Mehta, 2007).

<table>
<thead>
<tr>
<th>Time, hours</th>
<th>Secondary lutocline elevation, m</th>
<th>Settling velocity, m/s $h_*$, m</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.25</td>
<td>9.30E-06 0.25</td>
</tr>
<tr>
<td>1</td>
<td>0.22</td>
<td>3.80E-06 0.232</td>
</tr>
<tr>
<td>2</td>
<td>0.205</td>
<td>3.70E-06 0.231</td>
</tr>
<tr>
<td>3</td>
<td>0.192</td>
<td>3.40E-06 0.229</td>
</tr>
<tr>
<td>4</td>
<td>0.18</td>
<td>3.20E-06 0.226</td>
</tr>
<tr>
<td>5</td>
<td>0.17</td>
<td>3.00E-06 0.224</td>
</tr>
<tr>
<td>6</td>
<td>0.16</td>
<td>2.60E-06 0.216</td>
</tr>
<tr>
<td>8</td>
<td>0.146</td>
<td>2.00E-06 0.203</td>
</tr>
<tr>
<td>10</td>
<td>0.134</td>
<td>1.40E-06 0.181</td>
</tr>
<tr>
<td>15</td>
<td>0.116</td>
<td>7.00E-07 0.152</td>
</tr>
<tr>
<td>20</td>
<td>0.105</td>
<td>3.90E-07 0.133</td>
</tr>
<tr>
<td>25</td>
<td>0.099</td>
<td>1.70E-07 0.114</td>
</tr>
<tr>
<td>30</td>
<td>0.097</td>
<td>1.10E-07 0.109</td>
</tr>
<tr>
<td>35</td>
<td>0.096</td>
<td>7.80E-08 0.106</td>
</tr>
<tr>
<td>40</td>
<td>0.095</td>
<td>5.60E-08 0.104</td>
</tr>
</tbody>
</table>

Table 5-2. Estimation of the concentration profiles based on the intersection of characteristic lines with vertical profiles at specific times (after Mehta, 2007).

<table>
<thead>
<tr>
<th>Time = 5 hours</th>
<th>Time = 15 hours</th>
<th>Time = 40 hours</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elevation (m)</td>
<td>$h^*$ (m)</td>
<td>$C$ (kg/m³)</td>
</tr>
<tr>
<td>0.170</td>
<td>0.224</td>
<td>5.6</td>
</tr>
<tr>
<td>0.130</td>
<td>0.216</td>
<td>5.8</td>
</tr>
<tr>
<td>0.090</td>
<td>0.203</td>
<td>6.2</td>
</tr>
<tr>
<td>0.068</td>
<td>0.181</td>
<td>6.9</td>
</tr>
<tr>
<td>0.039</td>
<td>0.152</td>
<td>8.2</td>
</tr>
<tr>
<td>0.027</td>
<td>0.133</td>
<td>9.4</td>
</tr>
<tr>
<td>0.016</td>
<td>0.109</td>
<td>11.6</td>
</tr>
<tr>
<td>0.014</td>
<td>0.106</td>
<td>11.8</td>
</tr>
<tr>
<td>0.012</td>
<td>0.104</td>
<td>12</td>
</tr>
<tr>
<td>0.009</td>
<td>0.095</td>
<td>13.2</td>
</tr>
<tr>
<td>0</td>
<td>0.095</td>
<td>13.2</td>
</tr>
<tr>
<td>0</td>
<td>0.095</td>
<td>13.2</td>
</tr>
</tbody>
</table>
Figure 5-1. Kynch (1952) settling test development. a) initial uniform dilute suspension in a quiescent settling column, b) the secondary lutocline (S) settles at a rate $w_s$ while the isopycnal interface defining the primary lutocline (P) rises from the bed at a rate $w_p$, c) isopycnal primary lutocline meets the secondary lutocline, and d) the final deposit concentration $C_f$ and height $h_f$ are reached after a period of hindered settling.
Figure 5-2. Schematic representation of the Kynch (1952) settling test with the use of the method of characteristics.
Figure 5-3. Kynch (1952) settling test evolution of the secondary lutocline elevation. Application of the method of characteristics to estimate the suspended sediment concentrations (after Mehta, 2007).
Figure 5-4. Estimated concentration profiles using the Kynch graphical method based on the method of characteristics.
Figure 5-5. Deposition test results from Krone (1962).
Figure 5-6. Results of the tagged sediment experiment of Krone (1962).
Figure 5-7. Results from Mehta (1973) showing the effects of shear stress on the relative concentration. Initial concentrations were 1.0 kg/m$^3$. 
Figure 5-8. Results of the flow volume replacement experiment by Parchure (1985).
Figure 5-9. Step erosion test series of Parchure and Mehta (1985)
Figure 5-10. Suspended sediment concentration and shear stress during monitoring exercise on 5 January 1989 (from Sanford and Halka, 1993).
Figure 5-11. Suspended sediment concentration and shear stress during monitoring exercise on 2 February 1990 (from Sanford and Halka, 1993).
Figure 5-12. Suspended sediment concentration and shear stress during monitoring exercise on 15 January 1991 (from Sanford and Halka, 1993)
6.1 Preamble

This chapter describes the application of the computational procedures described in Chapter 4 to the sediment transport test cases identified in Chapter 5. The overall program outline was presented in section 4.6 of Chapter 4. A detailed description of the primary sediment transport subroutines is presented in Appendix D, with identification of the equations used for each of the calculations. The hydrodynamic components of the model were tested in Chapter 4 on the classical analytical cases.

With the exception of the Kynch deposition test, all simulations performed incorporated the flocculation model. Because the concentration levels are relatively low for most of the test cases, sensitivity testing showed that the TKE model added limited value to these simulations and complicated the interpretation of the primary testing. In addition, the computational burden of the probabilistic bed exchange, the flocculation model and the TKE model proved to be unmanageable. Therefore, the simulations reported here do not incorporate the TKE model.

The probabilistic treatment, when invoked, includes the representation of the bottom shear stress, the floc strength, the critical shear stress for erosion, the critical shear stress for deposition, the local shear in the water column, the internal shear and the fall velocity as probability distributions. The probability density for velocity is assumed to be normally distributed, with a standard deviation assumed to be 20 percent of the mean velocity, which is based on a typical ratio of turbulence intensity to mean velocity (Sharma, 1973). The probability distribution for the bottom shear stress is developed from the transformation of the velocity distribution as discussed in section 3.3. The mean shear stress versus mean velocity relationship has been assumed to be valid for the instantaneous shear stress versus the instantaneous velocity.
The relationship is assumed to be of the form $\tau = Au^B$. The coefficients $A$ and $B$ were varied for each experiment simulated based on information from the original studies when available.

Floc shear strengths were also assumed to be normally distributed about the mean values for each size class defined by Equation 3-11. The standard deviations for the shear strengths were assumed to be 20 percent of the mean values, based on the variability in the shear strength data of Krone (1963).

Settling velocities were also assumed to be normally distributed about the size class mean settling velocity developed from the equations in sections 2.2.2, 2.2.3 and detailed in section D.2 of Appendix D. The standard deviation of the settling velocities was assumed to be 30 percent of the mean value. This value is consistent with estimates of distribution of settling velocities developed from analysis of in situ video images collected in San Francisco Bay (see Figure 6-1). A standard deviation of 30 percent fits the data well in the middle floc sizes between 80 to 150 microns, where there are sufficient particles analyzed for statistical accuracy and less effects of scatter due to residual fluid motion within the settling chamber, which causes scatter at smaller sizes.

The probability density distributions for the local shear stress in the water column and the internal shear rate were assumed to be proportional to the probability distribution for the bottom shear stress, since each of these variables is derived from the nonlinear dependence on velocity. Any arbitrary variable, $R$, whose distribution is assumed to be proportional to the bottom shear stress distribution would satisfy the relation: 
$$pdf(R) = pdf(\tau_b)\frac{\tau_b}{\bar{R}}.$$ 

The distributions all used a numerical integration based on a discretization of the PDFs, using 101 values for discretization, with truncation of the tails of the distributions at plus and
minus three standard deviations from the mean value, which will represent 99.8 percent of the total probability of a normal distribution.

Model parameters common to all simulations conducted are presented in Table 6-1. The specific variables developed for each test will be presented in Tables 6-2 through 6-12

6.1 Kynch (1952) Quiescent Deposition Test

The Kynch (1952) deposition test described in section 5.1 was simulated as a test of settling in quiescent conditions and for the ability to replicate the theoretical treatment of hindered settling derived from a laboratory test case. No information was available from the literature on the particle size distribution of the physical test data set. A single size silt particle of 4.31 microns was selected to match the free settling velocity observed in the data. The flow in the model was quiescent and the initial suspended concentration was 5 kg/m³. The simulation duration was 40 hours. The simulation used a space-filling concentration of 13.2 kg/m³, which was the maximum concentration in the physical data. For quiescent conditions, vertical diffusion was set to Brownian diffusion (Equation 6-1) in the absence of flow.

\[ D_B = \frac{k_B T}{3\pi \mu d} \]  

(6-1)

Processes of importance to the Kynch test are accumulation of near bed material and the upward movement of the primary lutocline. In order to capture these processes, bed exchange was turned off within the model. Neither deposition downward out of the bottom cell, nor reentrainment from the bed (outside the model domain) was permitted. This allowed the high concentration of suspended sediment to experience hindered settling, resulting in the development of the primary lutocline.

The model simulation coefficients for the Kynch test simulations are presented in Table 6-2. The model used 40 uniform cells over the water depth, with no depth-averaged velocity or
shear stress. The number of cells in the vertical was set based on sensitivity testing to insure adequate resolution to resolve the lutoclines. The concentration effect on settling velocity used the coefficients from Equation 2-46 of $\lambda_1 = 320$ s, $\lambda_2 = 75$ s$^2$, and $\lambda_3 = 0.8$. These values were used by Teeter (2001; personal communication). Because silt was simulated the flocculation model was turned off, primarily because the only contribution to the test would have been from Brownian motion, and this would have been minor.

The results of a test using the exponent $n=1$ in the hindered settling equation of Winterwerp (1999; Equation 2-40) are presented in Figure 6-2. After 5 hours, the model suspended sediment in the upper portion of the water column had dropped nearly uniformly downward, forming a diffuse secondary lutocline that lacked a sharp interface. This was apparently the result of numerical diffusion, since the assigned numerical term was computed to be the Brownian diffusion (order $10^{-8}$ m$^2$/s). Below the diffuse upper lutocline the model concentration remained constant at the initial concentration over almost half of the water column. Near the bottom of the column the primary lutocline had begun to form as a sharp interface in the model results.

After 15 hours the general location of both the primary and the secondary lutocline were in approximately the same location as in the test data. However, the model secondary upper lutocline had been further diffused. After 40 hours the model generally matched the final concentration profile, but retained some diffusion at the merged primary and secondary lutoclines.

One significant feature of the test data is that the concentration at the secondary lutocline increased above the original concentration early in the test (hour 5). This observation suggests that the suspension is experiencing hindered settling even at the initial concentration. In order to
increase the hindered settling within the model the hindered settling exponent $m$ was increased to 2. The results of that test simulation are presented in Figure 6-3. The model results show increased effects of hindered settling, with the location raised in the water column where at hour 5 the concentration is increased above the initial value (from 0.09 m for the $m=1$ simulation to 0.13m above the bed for the $m=2$ simulation). For the $m=2$ test the orientation of the concentration gradient above the primary lutocline aligns much more closely with the test data.

The results of these simulations demonstrated that the model is capable of simulating the hindered settling process with qualitative agreement. Without more precise information about the actual sediment size distribution of the settling column, further numerical testing was not performed. Furthermore, for the quiescent conditions of this test case it was not felt to be of significant value to test the probabilistic formulation of the model. The probabilistic features of the model are derived from the influence of turbulence on the processes, which is missing in this test case.

### 6.2 Mehta 1973 Flume Deposition Tests

Mehta (1973) documented that the percentage of sediment remaining in suspension at equilibrium for a given bulk suspended sediment sample is independent of the initial concentration for a fixed shear stress. When the sediment concentrations are relatively dispersed, such that they do not impact the level of turbulence, then the flow’s ability to sustain the particles in suspension is related to the individual particle characteristics rather than the properties of the total suspension. So if the initial concentration is doubled, but the fraction of a particular size particle is fixed, then the final percentage of the particles capable of remaining in suspension will be fixed compared to the total sediment initially in suspension. When the shear stress changes, the fraction of sediment capable of staying in suspension will change based on the size distribution of the sediment and the critical shear stress for deposition. If the initial
concentration is sufficiently high, then the sediment concentration will begin to affect turbulence and alter the suspending capacity of the flow.

The model was first simulated using the classical excess shear stress formulas for both deposition and erosion, as described by Equations 2-50 and 2-51 in section 2.2.4. Model simulation specifications for the Mehta (1973) simulations are presented in Tables 6-3 and 6-4. The testing was performed on four different shear stresses: 0.25, 0.40, 0.60 and 0.85 Pa. The flow depth was assumed to be 0.305 m. The conversion of the mean velocity specified for the flow was converted to a shear stress by the relationship developed by Mehta (1973) for the original flume (Equation 6-2, in units of Pa and m/s)).

\[
\tau_b \approx 0.956 \bar{u}^{0.94}
\] (6-2)

The over-bar on the velocity refers to the depth-averaged velocity. The calibration procedure used by Mehta (1973) for Equation 6-2 involved profiling the velocity in the flume with a miniature propeller probe attached to the upper ring of the rotating flume. With the known differential velocity, \( \Delta V \), between the top and bottom flume rotating surfaces and the probe velocity, \( u_p \), the velocity relative to the bed at the measurement elevation is the differential velocity less the probe velocity, \( u = \Delta V - u_p \). The velocity profiling was performed without sediment in suspension due to interference of sediment particles with the velocity probe. Profiling was performed for two flume depths (0.16 m and 0.23 m) and for multiple differential velocities. The bed shear stress was developed from a regression of direct measurements of shear stress with strain gages to the differential velocity. Shear stress was regressed against the computed depth-averaged velocities from the measured profiles to obtain Equation 6-2. The mean velocities that gave the four shear stresses above were 0.24, 0.40, 0.61, and 0.88 m/s.
Model simulations used only 5 cells over the water depth because the concentrations were vertically almost uniform in the model. The shear stress profile was developed using a Karman mixing length linear profile. Sediment classes were defined for 60 cohesive classes between 0.1 and 2000 microns and for 10 silt classes between 4 and 60 microns. The initialization of the floc distribution was limited to the first 40 size classes (0.1 to 70 microns). A fractal dimension of 2.2 was used, which corresponds to strong estuarine flocs (Winterwerp, 1999). As discussed in section 2.1.3.6, analysis by Partheniades (1963) of Krone’s data showed the shear strength to vary with the excess density to the $5/2$ power, which is effectively a fractal dimension of 2.2 ($5/2=2/(3-D_f)$).

Figure 6-4 presents a comparison of the shear strength developed from Equation 3-11 with the Krone (1963) data. Two curves for Equation 3-11 are presented; one using a scaling coefficient, $B_f = 600$ Pa with a fractal dimension of 2.2 and another with a scaling coefficient $B_f = 1200$ Pa and a fractal dimension of 2.5. These show that varying the fractal dimension requires rescaling the equation to fit Krone’s data.

In the laboratory tests the same sediment was used in each test case with varying shear stress. Therefore, the sediment properties need to be established and not changed between simulations for each test case. Calibrating the model required careful evaluation of the defined critical shear stresses for erosion and deposition (by Equations 2-53 in section 2.2.4) relative to the range of tested shear stresses. Figure 6-5 presents an example variation of the mean floc shear strength, and mean critical shear stresses for erosion and deposition. When the functional value of the critical shear stress for deposition becomes larger than the critical shear stress for erosion, it should take on the erosion threshold value.
If a 0.085 Pa shear stress is assumed, (presented in Figure 6-6 for the example in Figure 6-5), then for use of a classical probability of deposition (Equation 2-50), deposition would occur over a floc size range from approximately 2 to 70 microns. Above 70 microns erosion would occur.

The critical shear stresses and shear strength used in simulation of the Mehta (1973) deposition experiments for the use of mean value treatment of the processes is presented in Figure 6-7, with the two extreme bottom shear stresses denoted by the horizontal lines at the appropriate shear stress values. The critical shear stress for erosion was specified from the floc shear strength with $B_f = 1800$ Pa and a fractal dimension of 2.2. The critical shear stress for deposition is from Equation 2-53, with $\tau_{d0} = 0.03$ Pa, $d_{ref} = 0.1$ microns and exponent $\delta = 0.8$. In implementing Equation 2-53, it was discovered that the two equations intersect at a yield strength of 1.27 Pa, rather than at 1.6 Pa. In order to avoid a discontinuity in the critical shear stress for erosion the threshold of 1.27 Pa was used between the two parts of Equation 2-53.

At the lower shear stress (0.25 Pa) of the four Mehta tests the mean-value exclusive paradigm using a classical probability of deposition indicates deposition could occur over the floc size range of 1.5 to 15 microns. Erosion of flocs would occur for flocs larger than 15 microns. At a shear stress of 0.85 Pa there is essentially no opportunity for deposition, and flocs larger than 8 microns would potentially erode. For intermediate shear stresses between 0.25 and 0.85 Pa, the floc size window for deposition shrinks with increasing shear stress above 0.25 Pa, and the minimum floc-size for erosion becomes smaller with increasing shear stress.

The initial silt fraction in suspension was set at 40 percent, which was the percent of the silt fraction in the commercial kaolinite used in the original flume investigation. The flocculation model used an aggregation efficiency of 1.0, disaggregation efficiency of 0.75 and
collision disaggregation efficiency of 0.5. McAnally (1999) reported a range of effective aggregation efficiencies of 0.005 to 0.70 from numerical collision model experiments. Winterwerp (1999) proposed that these coefficients are basically empirical.

The initial suspended sediment concentration was specified for each size class as shown in Figure 6-8. The model was not very sensitive to the initial specification of the sediment size distribution. The model makes no distinction between initially present primary particles and flocs, essentially assuming a single primary particle size. Consequently, the numerical flocculation model redistributes the size to approach equilibrium between aggregation and disaggregation for the given shear stress.

The results of the model for a classical shear stress treatment are presented in Figure 6-9, with use of the simultaneous deposition paradigm. The model sediment size concentration distribution in suspension initially was adjusted, along with the critical shear stresses for deposition and erosion to obtain reasonable agreement with the original flume data. Attempts at calibration of the model using the exclusive paradigm were not very successful, with difficulty in obtaining equilibrium concentrations as low as the data.

The model reproduces the asymptotic trend toward an apparent equilibrium concentration, dependent on the shear stress. The initial sediment reductions within the first hour were not precisely replicated, particularly for the two intermediate shear stress cases. The complexity of the processes and the number of degrees of freedom in the model specification make precise agreement difficult. The analysis of sediment size concentration distribution prior to initiation of settling in light of the composite effective settling velocity could provide some guidance on improving the replication of the results.
Figure 6-10 provides a comparison of the exclusive and simultaneous paradigm simulations for the 0.25 Pa test. Simultaneous deposition reduces the equilibrium concentration compared to the exclusive simulation by increasing the depositional flux to better match the flume data.

The sensitivity of probabilistic versus mean-value representation and the exclusive versus simultaneous bed exchange are illustrated in Figure 6-11 for the 0.60 Pa test. The same coefficients used in the replication of the results in Figure 6-9 were used with the probabilistic treatment of the primary variables. The results clearly show that the increased variability introduced by the probabilistic treatment leads to increased erosion potential. The use of simultaneous deposition sets the equilibrium concentration lower for both the mean value and the probabilistic approach. If calibration has been performed using one choice of averaging/deposition paradigm combination and the combination is changed, then to replicate the results will require recalibration of the empirical coefficients.

Figure 6-12 presents a conceptual illustration of the impact of using probabilistic variables in the analysis. The variations plotted about the mean shear values represent one standard deviation of each variable. The model performs the calculation on three standard deviations. The critical thresholds for deposition will be the intersection of the minimum bottom shear stress with the largest critical shear stress for deposition. The effect of the variance in each variable is to widen the span of floc sizes for which deposition may occur, while allowing an overlap in the deposition and erosion sizes when the exclusive paradigm in used in conjunction with probabilistic variables. The span of deposition for the mean-value analysis is shown as the shaded area. Figure 6-12 offers the concept using only a single standard deviation.
The evaluation of the effect of using probabilistic variables by inspection of Figure 6-12 would initially imply that additional deposition should occur. However, with the expansion of the variability in the variables to three standard deviations, the overlap between erosion and deposition covers a wider range of floc sizes. Over time, erosion will dominate deposition because of the effects of aggregation, which creates a flux of sediment mass toward larger flocs, out of the deposition range of floc sizes.

Figure 6-13 presents a comparison of the floc size distribution at the end of simulations using the exclusive paradigm for the mean value and probabilistic treatment of the variables. The distributions are potted for both surface and bottom cells in the model for each simulation. The results show that the probabilistic method yields a broader floc spectrum, which results in higher overall suspended sediment concentration than the mean value simulation. The mean value simulation has an abrupt drop in concentration on the lower side of the spectrum associated with the single-valued floc size threshold for deposition. The probabilistic spectrum has a more gradual a transition. In comparing the two spectrums in Figure 6-13 with the characteristics spectra developed by Kranck and Milligan (1992) in Figures 2-2 and 2-5, the probabilistic spectrum shape has the same skewed spectrum with the modal floc size larger than the mean floc size. The simulated spectrum with use of average values of the variables is more symmetric. The relative variation between surface and bottom distributions is maintained for both averaging methods, implying that the differences are controlled by the mean stresses in the water column.

The primary observation from the simulation of the Mehta 1973 tests is that the simultaneous deposition paradigm seems to work better than the exclusive erosion or deposition method. One advantage of the simultaneous paradigm is the simplification of parameter specification, as the critical shear stress for deposition no longer plays a role in the model. The
probabilistic treatment of variables complicates the exclusive paradigm by making the influence of the aggregation processes different relative to the mean valued approach. This makes the development of general guidelines for parameter adjustment between the two methods more difficult.

6.3 Krone (1962) Flume Deposition Tests

The flume deposition tests of Krone (1962), as described in section 5.2.1, were simulated for three shear stress experiments. The model specifications for these simulations are presented in Table 6-5. The simulations used averaged variables (shear stresses, shear strengths and settling velocities) and the classical probabilities of erosion and deposition based on excess shear stress (Equations 2-50 and 2-51). Simulation results are presented in Figures 6-14 through 6-16 for the three shear stress tests, with 0.0305, 0.0415 and 0.0515 Pa.

The model is able to track the settling for each of the test cases for the first 50 hours of the simulations. However, the flume data has a long-term trend that was not satisfactorily replicated in the numerical simulations. After about fifty hours in the flume tests there is a change in the depositional trend. The trend can be seen in Figure 6-17. The plot shows that the linear trends on a log plot indicate exponential decay with a break in the decay rate at around 50 hours into each test.

The inability of the numerical model to replicate the trend may be associated with the effects of the recirculating pumps altering the floc size distribution in the flume. Adding additional shear in the numerical model tended to increase flocculation and increase deposition rather than decrease it. Figure 6-14 presents the effects of adding 100 Hz supplemental shear to the 0.0305 Pa Krone deposition test. Krone did not measure floc sizes, but estimated the maximum floc size to be between 20 and 30 microns based on the estimated thickness of the laminar boundary layer near the bed. Krone had estimated the shearing within the pumping
system of the flume to be in excess of 220 to 950 Hz in the walls of the return lines for the range of shear tests being considered here. The peak floc size within the distribution in the numerical experiments for the flume experiments was in the range of about 40 to 50 microns.

Winterwerp (2004) also had difficulty in replicating the Krone depositional experiments. He incorporated a time varying critical shear stress for erosion in combination with the continuous deposition paradigm that could not fully account for the persistence of the flume sediments to remain in suspension. He questioned the possible accuracy of the bottom shear stress measurement estimates.

The primary change in depositional behavior when the concentration falls below 0.3 kg/m³, suggests that some transition occurred in the aggregation. That change could be related to the interaction of the suspended sediments within the pumping system causing some lubricating effect that changed as the concentration fell.

One aspect of the Krone depositional experiments that makes them different from other laboratory tests is the basic time scale of the tests. While most of the annular flume experiments for deposition at flow depths and shear stresses comparable to the Krone tests have test durations on the order of 10 hours, the Krone tests have time scales of tens to hundreds of hours. The primary differences are in the recirculating system for the flume.

The primary conclusion from the simulations of the Krone deposition tests is that extensive sensitivity analysis of probabilistic treatment and the exclusive versus continuous deposition paradigm are not warranted.

6.4 Parchure and Mehta (1985) Dilution Test

The simulation of the volume removal test by Parchure and Mehta (1985), as described in section 5.4, was performed by assuming that the sediment mass extracted over the four-hour
period is taken uniformly over the flume. The expression for the time rate of change of the depth-averaged concentration in a unit horizontal area of the flume due to an extraction rate per unit area of the flume, $q$, is presented in Equation 6-3. This expression assumes that the effects of the extraction on the suspended sediment concentration are uniform within the flume, both laterally and vertically. Longitudinal uniformity is assured in the annular flume.

$$\frac{\partial \bar{C}}{\partial t} = -\frac{q \bar{C}}{h}$$  \hspace{1cm} (6-3)

An analytical solution for Equation 6-3 is known, as shown in Equation 6-4. With the water flow depth known (0.26 m), the observed dilution of 0.00789 (0.03/3.8) and the duration of the test, the approximate withdrawal discharge can be estimated as 0.00008765 m$^3$/s/m$^2$.

$$\bar{C} = \bar{C}_0 \exp \left[ -\frac{qt}{h} \right]$$  \hspace{1cm} (6-4)

The model was tested with no bed exchange to confirm the extraction dilution formulation and prove that it can match the solution given in Equation 6-4. Results of the dilution rate test are presented in Figure 6-18. For this test the initial concentration was prescribed as the equilibrium (3.85 kg/m$^3$) developed in the original flume test by eroding a settled bed prior to initiation of extraction of water/sediment mixture. With no bed exchange the numerical solution should match the analytical solution of Equation 6-3, which is also included in Figure 6-18. This shows that the analytical solution describes the flume data very well and also that the numerical model very precisely matches the analytical solution.

The model was then simulated with bed exchange turned on. The specifications for the Parchure and Mehta dilution simulations are presented in Table 6-6. For this test the initial suspended sediment concentration was set to zero and an initial particle size distribution was specified within the bed. The initial bed concentration distribution is presented in Figure 6-19,
which shows a uniform concentration below 70 microns, with no sediments larger than 70 microns. The total bed mass was adjusted until the erosion of the bed during the initial spin up of the model approached the desired initial concentration (3.85 kg/m³) as an equilibrium. The model was run for 4 hours, having reached near equilibrium in less than an hour, and then the dilution begun at hour 4 and continued until hour 8. The model simulation continued for a total simulation period of 15 hours. This test was simulated with the classical exclusive deposition or erosion model based on excess shear stress (Equations 2-50 and 2-51) and also with the probabilistic treatment of the bed exchange described in Chapter 4. The results of these tests are presented in Figure 6-20.

Both of the model simulations quickly reached the desired initial equilibrium concentration. During the first 2 hours of the dilution phase the two model tests follow the test data very closely. However, near the end of the dilution, the classical treatment continues to follow the analytical solution, while the probabilistic treatment deviates slightly, with some apparent additional erosion from the bed. After the cessation of dilution the classical simulation remains nearly constant at the final concentration after dilution of 0.03 kg/m³. The concentration does rise slightly to 0.0304 kg/m³ by the end of the simulation at hour 15. The probabilistic treatment, however, exhibits more significant erosion after the end of the dilution phase, increasing the suspended concentration to 0.066 kg/m³ by the end of the simulation. This relatively minor erosion is in agreement with an increase in the flume test to 0.1 kg/m³ after an additional 24 hours after the end of the dilution phase. The probabilistic treatment of the bed exchange is in both qualitative and quantitative agreement with the flume data with regard to the persisting slight erosion. The classical treatment does not capture that feature.
Lau and Krishnappan (1994) also conducted experiments in a rotating annular flume designed to evaluate the question of simultaneous erosion and deposition. Instead of conducting erosion to equilibrium, as Parchure and Mehta did, they conducted deposition experiments that reached an apparent equilibrium concentration. After equilibrium was reached they replicated the Parchure and Mehta experiment (1985), slowly removing sediment-laden water from the flume, replacing it with clear water. They monitored the floc size distribution periodically during each phase of the test. They identified three features of the experimental results that they reasoned could be used to judge the question of simultaneous erosion and deposition. First was the change in particle size distribution during the depositional period between initiation of the deposition and when equilibrium concentration was reached. They argued that changes in the floc size distribution that showed reductions in all size classes means that all size classes must have experienced settling. They argued that this is contradicts the conceptual model of Lick (1982) that fine fractions never settle, largest fractions deposit quickly and intermediate fractions experience both deposition and reentrainment. The Lick model was interpreted to lead to a shifting of the particle size distribution to the smaller fractions. Another explanation could be that smaller size flocs can undergo aggregation, converting to larger flocs before settling, thus maintaining a proportional distribution.

The second feature of their results was the change in particle size distribution during dilution. The Partheniades, et al. (1968) conceptual model would expect that no further deposition will occur after equilibrium has been reached and that no further erosion should occur upon dilution. The Partheniades et al. model for the process relies on the condition that equilibrium concentrations are reached because the sediment supply from the bed for erodible sediments has been exhausted and that all material capable of depositing has settled out. The
particle size distribution should simply dilute, maintaining its proportionality. The Lick model would again result in a redistribution of the sizes, as the finer fractions would not be replaced, while the intermediate sizes could be replaced by erosion, resulting in a coarsening of the distribution. Their size distribution evolution supported the Partheniades et al. model, with the $D_{50}$ remaining essentially constant and the class concentrations each essentially diluted.

The final feature that was used to judge the processes was the evolution of the total concentration during dilution. If the simultaneous erosion and deposition model is correct then the diluted concentration should be replaced progressively by erosion that would result in a new equilibrium concentration. The Partheniades et al. model would simply dilute the concentration; which was also born out in their results.

The tests by Lau and Krishnappan were an improvement with the monitoring of the size class distribution but still relied on inference of the net deposition from the suspended concentration. Their logic in judging the simultaneous question relies, in part, on the assumption of a static equilibrium condition for aggregation and disaggregation during their tests. In addition, their conclusions are based on the assumption of a constant reservoir of all particles sizes from the bed for potential erosion, without any winnowing during the test. The Lau and Krishnappan particle size distribution evolution during the initial dilution is not consistent with the observations of Kranck and Milligan that the distribution varies primarily on the large floc size end of the distribution, resulting in the mean and modal flocs size being dependent on the total concentration. The deposition phase prior to dilution, when the shear stress was reduced should have experienced a reduction in the mean floc size, according to the Kranck and Milligan model.
The changes in the floc size distribution during the present numerical simulation are presented in Figure 6-21. Comparison between the average shear stress simulation and the probabilistic shear stress simulation at the initiation of dilution (hour 4) shows sharply different distributions. The probabilistic simulation results in significantly larger flocs, with a modal size of around 300 microns, compared to approximately 70 microns for the average shear stress simulation. The initial floc size distribution during the Lau and Krishnappan depositional equilibrium test had a modal floc size of only 5 microns. The modal floc size at the end of dilution in the present simulations dropped dramatically in both the average and the probabilistic treatments, with the probabilistic peak size (15 microns) slightly smaller than the peak for the average shear stress simulation (18 microns). However, after dilution the probabilistic distribution retained its broader distribution compared to the average shear stress. The distribution at the end of the simulation (hour 15) had not changed for the average shear stress method, plotted as points to show that the distributions overlay one another. The probabilistic distribution, however, shows an expansion of the distribution into larger floc sizes. Note that this behavior is consistent with the Krank and Milligan (1992) model of floc distribution evolution.

The simulations of the Parchure and Mehta (1985) dilution experiment showed that the probabilistic treatment of the primary variables was able to replicate the concentration rebound after the dilution was stopped. This winnowing erosion was not seen in the simulation using the mean values of the variables. The comparison of these numerical results with the results of the Lau and Krishnappan laboratory floc size data suggest that the floc size distribution evolution may depend on the nature of the process that led to the initial concentration prior to dilution. The Parchure and Mehta test approached equilibrium by erosion prior to dilution, while the Lau and Krishnappan test approached equilibrium by deposition prior to dilution. The Lau and
Krishnappan tests used higher shear stresses (0.25 to 0.37 Pa) than the Parchure and Mehta test (0.2 Pa). The numerical model floc size distribution evolution follows the observations of the Kranck and Milligan conceptual model.

### 6.5 Parchure and Mehta (1985) Erosion Test

The step-erosion test of Parchure and Mehta (1985) described in section 5.5 was chosen to evaluate the models ability to address progressive erosion for stepped increases in shear stress. Table 6-7 presents the specifications for these simulations. The model specification for the bed was a particle size distribution of the sediment in the bed combined with the total mass of sediment in the erodible bed layer. The distribution for the cohesive sediment mass in the initial bed is presented in Figure 6-22. The silt fraction was 10 percent of the total mass and was uniformly distributed over the silt classes. The model initialization of the bed was the primary calibration tool to match the incremental erosion for each shear stress. Because the shear strength and critical shear stress for erosion are determined in the model as a function of particle size, the distribution of erosion resistance in the bed is directly correlated to the size distribution. The shear strength and critical shear stress for erosion are inversely proportional to the particle size (Equations 2-9 and 2-53). The results of the model simulation using average shear stresses are presented in Figure 6-23. The simulation was able to capture the incremental erosion and asymptotic behavior by the adjustment of the erosion rate constant.

To illustrate the difference in the response between the classical treatment of the erosion and the probabilistic, again the simulation was repeated, keeping the erosion rate constants fixed. A comparison of the two simulations is included in Figure 6-24. This clearly illustrates the need for development of erosion rate information that is consistent with the shear stress treatment.

The dramatic initial jump in the concentration during the first shear level (0.1 Pa) hour is an indication that the first improvement when applying the probabilistic approach would be the
analysis of the bottom shear stress distribution plotted against the critical shear stress levels for each shear level. Note that after the first major erosion event in the first hour, the progression of erosion for the probabilistic approach is not dramatically different from the data, if the initial offset is removed.

6.6 Sanford and Halka (1993) Field Datasets

The three data sets from Sanford and Halka (1993) described in section 5.6 were simulated within the process model to reevaluate the conclusion of the authors regarding the need for using a continuous deposition model to accurately simulate the observed erosion and deposition in a tidal estuarine environment. The simulations conducted in the evaluation of these datasets are summarized in Table 6-8 through 6-11. Tables 6-8 through 6-10 present the simulations for the three years of data collection used to identify sensitivity of bed exchange treatment. Table 6-11 presents the adjustments in the basic coefficients to obtain the calibrations presented for each data period simulation.

The simulations performed differed from the original Sanford and Halka numerical application in the number of size classes used. They used a single size class. Rather than specifying particle size, they used settling velocity, ranging from 0.00009 to 0.00014 m/s for their three experiments. The current modeling application used 40 size classes for cohesive sediment ranging from 0.1 to 1000 microns, with a fractal dimension of 2.2. The model had 10 silt size classes between 2 and 60 microns, comprised 40 percent of the bed material and of the initial suspended concentrations. The water depths during each of the three deployments are as shown in Figure 6-25. The number of size classes was limited for these tests because of the computational burden of the probabilistic approach and the flocculation model.
The Sanford and Halka numerical analysis evaluated three bed exchange models. One was the classical exclusive linear treatment, using Equations 2-50 and 2-51. Sanford and Halka referred to the exclusive treatment as the cohesive model. They also considered the continuous deposition model combined with the linear classical erosion model. The third model they evaluated was continuous deposition with a power law representation for erosion (Equation 6-5)

\[ E = \alpha |\tau_b|^{\eta} \]  

(6-5)

The coefficients were developed empirically during model application. The values developed for \( \alpha \) were 5.9, 0.69 and 7.5 mg/cm\(^2\)/h, for the 1989, 1990 and 1991 data, respectively. The corresponding values of the exponent, \( \eta \), were 1.4, 4 and 1.3. Sanford and Halka came to the conclusion that the continuous paradigm is required to model the field concentrations accurately, but that either the linear or power law erosion equations worked well over the range of conditions they evaluated.

Model simulations in the current research were performed separately for the use of the classical exclusive erosion or deposition model and the continuous deposition model. Each of these bed models was also tested with both the average shear stress in the bed exchange equations and with the probabilistic treatment of shear stresses and settling velocities. It was confirmed from the tests that the best performance of the models in matching the observed suspended sediment variation was with the continuous deposition model. This was true for both the classical average shear stresses and for the probabilistic bed exchange. The quality of the calibrations is compared for the two shear stress treatments with the continuous deposition model in Figures 6-26 through 6-28 for the consecutive years of deployment. The parameter adjustments that were made for calibration are presented in Table 6-11. These adjustments were as simple as changing the initial concentration or adjusting the erosion rate constant.
The comparisons show that the probabilistic treatment captures the variability in the concentrations much better than the average shear stresses. In addition, the timing of the occurrence of extremes in the concentration variation is more in phase for the probabilistic method.

The relative response of the probabilistic versus the average shear stress method can be better understood by inspecting the changes in the floc size distribution during the simulation. Figure 6-29 presents the floc distributions for the two shear methods for continuous deposition at the times of the extremes in the concentration variability for the 1989 case. The distributions show three characteristic regions of the distribution, generally the same for both formulations. Below 1 micron the rate of loss of the finest particles is associated with aggregation to larger sizes, since the settling velocity of the finest particles is very low. At sizes between 1 and 10 microns the initial distribution is progressively being reduced for both methods. This is associated with the continuous deposition, in a range of floc sizes where the shear stresses are not sufficient to re-entrain the sediment. Above 10 microns the distribution rises and falls in conjunction with erosion and deposition of the larger flocs. The concentration is at a minimum near hour 14, a maximum at hour 17 and then at another minimum at hour 21.

Another feature of the floc size distribution comparison was that the probabilistic approach results in higher modal volume fractions, with the modal fraction at larger floc sizes. In addition, the range of floc sizes associated with the modal rise is broader for the probabilistic method. All of these features are a direct result of the higher shear stresses associated with the upper tail of the shear stress distribution.

The floc size distribution evolution for the simulations using the exclusive erosion-deposition model is presented in Figure 6-30. The exclusive model shows that for both the
average and the probabilistic methods the floculation end of the distribution is more prominent. Concentrations in the middle range of floc sizes (1-10 microns) are reduced compared to the simultaneous distributions (Figure 6-29). The net effect is that the exclusive treatment results in a shift to larger median particle size over the duration of the simulation for both shear stress treatments.

To illustrate the relative significance of the probabilistic treatment of the processes and of the use of continuous deposition, a group of simulations were made where the only changes in the model specifications were the choice of continuous deposition versus exclusive erosion or deposition and whether the probabilistic method is used. This gave four simulations for each of the Sanford and Halka test cases. These are presented in Figures 6-31 through 6-33.

Note that the simultaneous-probabilistic simulations in these tests were not the same as those presented in Figures 6-26 through 6-28 where other parameters were calibrated to improve the comparison to the observations. The simulations on each of the test cases reflect the same overall results. The lowest concentrations are associated with the simultaneous/average simulations. This follows from the less variable shear stresses and the continuous deposition. In contrast, the highest concentrations are associated with the exclusive treatment combined with probabilistic shear stresses. That combination maximizes erosion and minimizes deposition.

6.7 Krone (1962) Tagged-Sediment Settling Test

Attempts were made to design a numerical simulation that could address the exclusive versus simultaneous paradigm question raised by the Krone (1962) gold-labeled experiment described in section 5.2.2. A separate class of cohesive sediment was included in the numerical code, with identical cohesive properties as the primary cohesive sediment. The interactions of the two sediments within the aggregation model were handled by simple mass ratios between interacting classes. A sample simulation using the simultaneous deposition and erosion is
presented in Figure 6-34. The results have not been calibrated well to the flume data. However, there is a slight difference between the apparent deposition rate of the tagged sediment and the total concentration. That difference is associated with initial bed erosion differences at the beginning of the simulation. These results provided no insights relative to the paradigm.

6.9 Concluding Comments

These simulations reiterate the often proclaimed need for development of sediment transport parameters that are site specific. In addition, it is apparent that the methods applied for the analysis may have to be taken into account when field and laboratory observations are to be utilized. This is particularly true for the probabilistic treatment of the variables, which most standardized field and laboratory analyses are not designed to support.

Specific data that would improve the understanding of the bed exchange processes, which are not currently available, are:

- Erosive fluxes from the bed by specific particles size distribution. This would improve the bed exchange models directly and would provide information on the similarity of the form of that distribution for various sites. If self-similar patterns are observed, then historical bulk erosion estimates could be used in a distributed manner.

- Measurements or indicators of spatial heterogeneity of bed fluxes. This data would help to develop statistical models of variability that can be incorporated into the models of probabilistic interaction.

- Floc spectra that differentiate between primary mineral particles and flocs, with the ability to estimate the order of aggregation of the flocs. Much of the variability in settling velocities may be related to the ratio of primary mineral grains to floc characteristics.

- Process monitoring in the near-bed zone to indentify the gross and net bed fluxes over some relatively large domain. This would be the final determination of the exclusive versus simultaneous paradigm. However, it is recognized that measuring exclusive deposition or erosion once is a necessary condition to argue in favor of the exclusive paradigm, but it is not a sufficient condition to exclude the possibility of simultaneous bed exchange.
Table 6-1. Parameters common to all simulated tests.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Units</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water density</td>
<td>kg/m³</td>
<td>1025</td>
</tr>
<tr>
<td>Cohesive mineral density</td>
<td>kg/m³</td>
<td>2650</td>
</tr>
<tr>
<td>Silt density</td>
<td>kg/m³</td>
<td>2650</td>
</tr>
<tr>
<td>Shear effects coefficient, $\lambda_1$</td>
<td>s</td>
<td>320</td>
</tr>
<tr>
<td>Shear effects coefficient, $\lambda_2$</td>
<td>s²</td>
<td>75</td>
</tr>
<tr>
<td>Shear effects coefficient, $\lambda_3$</td>
<td>-</td>
<td>0.8</td>
</tr>
<tr>
<td>Free settling concentration, $C_f$</td>
<td>kg/m³</td>
<td>0.06</td>
</tr>
<tr>
<td>Hindered settling concentration, $C_h$</td>
<td>kg/m³</td>
<td>0.66</td>
</tr>
</tbody>
</table>

Table 6-2. Parameters used in the Kynch (1952) deposition tests.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Kynch-1</td>
</tr>
<tr>
<td>Water depth</td>
<td>m</td>
<td>0.25</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>40</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>4.3</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>4.32</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>100</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.0000164</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>4.3</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m³</td>
<td>5.0</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>0.0</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s⁻¹</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>no</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>none</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m³</td>
<td>0.0</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

276
Table 6-3 Parameters used in the Mehta (1973) deposition tests.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>M25-C1</td>
</tr>
<tr>
<td>Water depth</td>
<td>m</td>
<td>0.305</td>
</tr>
<tr>
<td>Number of cells</td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>2000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>4</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>60</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>1800</td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td></td>
<td>1.05</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td></td>
<td>0.03</td>
</tr>
<tr>
<td>scale factor, $\tau_{cd0}$</td>
<td>Pa</td>
<td>0.8</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td></td>
<td>0.0002</td>
</tr>
<tr>
<td>exponent, $\delta$</td>
<td></td>
<td>4.0</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.0002</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td></td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td></td>
<td>0.75</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m³</td>
<td>5.0</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>0.25</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s⁻¹</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td></td>
<td>No</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td></td>
<td>Sᵃ</td>
</tr>
<tr>
<td>Silt deposition</td>
<td></td>
<td>Sᵃ</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m³</td>
<td>1.0</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m³</td>
<td>1.0</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>

ᵃ S = simultaneous;
Table 6-4  Parameters used in the Mehta (1973) deposition sensitivity tests.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>M25-C2</td>
</tr>
<tr>
<td>Water depth</td>
<td>m</td>
<td>0.305</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>5</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>60</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>2000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>4</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>60</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td>0.2</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>1800</td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td>1.05</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td>Pa</td>
<td>0.03</td>
</tr>
<tr>
<td>scale factor, $\tau_{cd0}$</td>
<td>-</td>
<td>0.8</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td>-</td>
<td>0.8</td>
</tr>
<tr>
<td>exponent, $\delta$</td>
<td></td>
<td>0.8</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.0002</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>4.0</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0.75</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0.25</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m³</td>
<td>5.0</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td>0.4</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>0.25</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s⁻¹</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>No</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m³</td>
<td>1.0</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m³</td>
<td>1.0</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

<sup>a</sup> S = simultaneous;
Table 6-5  Parameters used in the Krone (1962) deposition tests.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>K305-C1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K415-C1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K515-C1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>K305-C2</td>
</tr>
<tr>
<td>Water depth</td>
<td>m</td>
<td>0.305</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.305</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.305</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.305</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>2000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.2</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>1200</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1200</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1200</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1200</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td></td>
<td>Pa</td>
</tr>
<tr>
<td>scale factor, $\tau_{cd0}$</td>
<td></td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.02</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td></td>
<td>Pa</td>
</tr>
<tr>
<td>exponent, $\delta$</td>
<td></td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.45</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.45</td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.05</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.05</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.00001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00001</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.00001</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2.6</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.75</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.75</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.25</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m³</td>
<td>0.72</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.93</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.72</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>0.0305</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.40</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0305</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s⁻¹</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td></td>
<td>No</td>
</tr>
<tr>
<td></td>
<td></td>
<td>No</td>
</tr>
<tr>
<td></td>
<td></td>
<td>No</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>Exclusive</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exclusive</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exclusive</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exclusive</td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td>Exclusive</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exclusive</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exclusive</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exclusive</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m³</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m³</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
Table 6-6  Parameters used in the Parchure and Mehta (1985) dilution tests.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
<th>PMD-C1</th>
<th>PMD-P1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water depth</td>
<td>m</td>
<td></td>
<td>0.26</td>
<td>0.26</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td></td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td></td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td></td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td></td>
<td>2000</td>
<td>2000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td></td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td></td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td></td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td></td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td></td>
<td>1200</td>
<td>1200</td>
</tr>
<tr>
<td>Critical shear stress for deposition scale factor, $\tau_{cd0}$</td>
<td>Pa</td>
<td>0.1</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Critical shear stress for deposition exponent, $\delta$</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td></td>
<td>1.05</td>
<td>1.05</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td></td>
<td>0.0002</td>
<td>0.0002</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td></td>
<td>4.0</td>
<td>4.0</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td></td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td></td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td></td>
<td>2.2</td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td></td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td></td>
<td>0.75</td>
<td>0.75</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td></td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m³</td>
<td></td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td></td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s⁻¹</td>
<td></td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td></td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td></td>
<td>Exclusive</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td></td>
<td>Exclusive</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m³</td>
<td></td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m³</td>
<td></td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 6-7 Parameters used in the Parchure and Mehta (1985) erosion tests.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>PME-C1</td>
</tr>
<tr>
<td>Water depth</td>
<td>m</td>
<td>0.26</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>60</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>2000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>2</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>20</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td>0.1</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>100</td>
</tr>
<tr>
<td>Critical shear stress for deposition scale factor, $\tau_{cd0}$</td>
<td>Pa</td>
<td>0.02</td>
</tr>
<tr>
<td>Critical shear stress for deposition exponent, $\delta$</td>
<td>-</td>
<td>0.14</td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td>1.1</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.004</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0.75</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0.25</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m$^3$</td>
<td>0.0</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>0.2</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s$^{-1}$</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>No</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m$^3$</td>
<td>1.0</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m$^3$</td>
<td>1.0</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 6-8  Parameters used in the Sanford and Halka (1993) tidal tests for 1989.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>SH-SA1</th>
<th>SH-SP1</th>
<th>SH-EA1</th>
<th>SH-EP1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water depth</td>
<td>m</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Critical shear stress for deposition scale factor, $\tau_{cd0}$</td>
<td>Pa</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>Critical shear stress for deposition exponent, $\delta$</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td>1.05</td>
<td>1.05</td>
<td>1.05</td>
<td>1.05</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.2</td>
<td>2.2</td>
<td>2.2</td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m$^3$</td>
<td>0.015</td>
<td>0.015</td>
<td>0.015</td>
<td>0.015</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s$^{-1}$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td>Exclusive</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td>Exclusive</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m$^3$</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m$^3$</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$^a$ $S$ = simultaneous;
<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SH90-SA1</td>
</tr>
<tr>
<td>Water depth</td>
<td>m</td>
<td>Variable</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>40</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>1000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>2</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>60</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td>0.4</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>1200</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td>scale factor, $\tau_{cd0}$</td>
<td>Pa</td>
</tr>
<tr>
<td>Critical shear stress for deposition</td>
<td>exponent, $\delta$</td>
<td>-</td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td>1.05</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.001</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>5.0</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td>2.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>1.0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0.5</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0.5</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m$^3$</td>
<td>0.009</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td>0.4</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>Variable</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s$^{-1}$</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>No</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>S$^a$</td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td>S$^a$</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m$^3$</td>
<td>0.01</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m$^3$</td>
<td>0.01</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

$^a$ S = simultaneous;
Table 6-10  Parameters used in the Sanford and Halka (1993) tidal tests for 1991.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>SH91-SA1</th>
<th>SH91-SP1</th>
<th>SH91-EA1</th>
<th>SH91-EP1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water depth</td>
<td>m</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{min}$</td>
<td>micron</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{max}$</td>
<td>micron</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td>60</td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>1800</td>
<td>1800</td>
<td>1800</td>
<td>1800</td>
</tr>
<tr>
<td>Critical shear stress for deposition scale factor, $\tau_{cd0}$</td>
<td>Pa</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>Critical shear stress for deposition exponent, $\delta$</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td>1.05</td>
<td>1.05</td>
<td>1.05</td>
<td>1.05</td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>Reference particle size, $D_{ref}$</td>
<td>micron</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.2</td>
<td>2.2</td>
<td>2.2</td>
<td>2.2</td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m³</td>
<td>0.009</td>
<td>0.009</td>
<td>0.009</td>
<td>0.009</td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s⁻¹</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td>Exclusive</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td>Exclusive</td>
<td>Exclusive</td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m³</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m³</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

* $S^a$ = simultaneous;
Table 6-11 Parameters used in the Sanford and Halka (1993) tidal tests for calibration.

<table>
<thead>
<tr>
<th>Variable or parameter</th>
<th>Units</th>
<th>Test simulation</th>
<th>SH89-SP2</th>
<th>SH91-SP2</th>
<th>SH91-SP2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water depth</td>
<td>m</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td>Number of cells</td>
<td>-</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Number of noncohesive classes</td>
<td>-</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>Minimum cohesive size, $D_{\text{min}}$</td>
<td>micron</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Maximum cohesive size, $D_{\text{max}}$</td>
<td>micron</td>
<td>1000</td>
<td>1000</td>
<td>1000</td>
<td></td>
</tr>
<tr>
<td>Number of silt classes</td>
<td>-</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>Minimum silt size</td>
<td>micron</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Maximum silt size</td>
<td>micron</td>
<td>60</td>
<td>60</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>Initial silt fraction in bed</td>
<td>-</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>Floc strength coefficient, $B_f$</td>
<td>Pa</td>
<td>1800</td>
<td>1800</td>
<td>1800</td>
<td></td>
</tr>
<tr>
<td>Critical shear stress for deposition scale factor, $\tau_{\text{cd0}}$</td>
<td>Pa</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
<td></td>
</tr>
<tr>
<td>Critical shear stress for deposition exponent, $\delta$</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Silt mobility coefficient</td>
<td>-</td>
<td>1.05</td>
<td>1.05</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>Maximum settling velocity</td>
<td>m/s</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Cohesive settling velocity factor</td>
<td>-</td>
<td>5.0</td>
<td>5.0</td>
<td>5.0</td>
<td></td>
</tr>
<tr>
<td>Silt settling velocity factor</td>
<td>-</td>
<td>2.0</td>
<td>2.0</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Reference particle size, $D_{\text{ref}}$</td>
<td>micron</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Fractal dimension, $D_f$</td>
<td>-</td>
<td>2.2</td>
<td>2.2</td>
<td>2.2</td>
<td></td>
</tr>
<tr>
<td>Aggregation efficiency</td>
<td>-</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Disaggregation efficiency</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Collision breakage efficiency</td>
<td>-</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Initial concentration</td>
<td>kg/m$^3$</td>
<td>0.01</td>
<td>0.024</td>
<td>0.009</td>
<td></td>
</tr>
<tr>
<td>Initial silt fraction in suspension</td>
<td>-</td>
<td>0.4</td>
<td>0.4</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>Bottom shear stress</td>
<td>Pa</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
<td>Variable</td>
</tr>
<tr>
<td>Supplemental internal shear</td>
<td>s$^{-1}$</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>Probabilistic treatment</td>
<td>-</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>Cohesive deposition</td>
<td>-</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td></td>
</tr>
<tr>
<td>Silt deposition</td>
<td>-</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td>$S^a$</td>
<td></td>
</tr>
<tr>
<td>Clay erosion scale factor</td>
<td>kg/m$^3$</td>
<td>0.004</td>
<td>0.004</td>
<td>0.008</td>
<td></td>
</tr>
<tr>
<td>Silt entrainment scale factor</td>
<td>kg/m$^3$</td>
<td>0.004</td>
<td>0.004</td>
<td>0.01</td>
<td></td>
</tr>
<tr>
<td>Hindered settling exponent</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

$^a$ S = simultaneous;
Figure 6-1. Example of fall velocity estimates from video analysis of in situ sediments in San Francisco Bay (data from Smith, 2007).
Figure 6-2. Results of simulation of the Kynch (1952) test case using a hindered settling exponent of $m=1$ in Equation 2-40.
Figure 6-3. Results of simulation of the Kynch (1952) test case using a hindered settling exponent of $m=2$ in Equation 2-40.
Figure 6-4. Effects of fractal dimension and shear strength coefficient on density difference versus floc shear strength (based on Equation 3-11), with comparison with data from Krone (1963).
Figure 6-5. Example representation of shear strength of flocs, critical shear stress for erosion and critical shear stress for deposition as functions of particle size. Shear strength defined by Equation 3-11, with $B_f = 1200$ Pa and $D_f = 2.6$. Critical shear stresses for erosion and deposition are defined by Equation 2-53. The critical shear stress for deposition is based on $\tau_{d0} = 0.01$ Pa, $d_{ref} = 0.1$ microns and $\delta = 0.5$. 
Figure 6-6. Relationship between mean shear strength, critical shear stresses for erosion and deposition, and bottom shear stress as a function of particle size.
Figure 6-7. Specification development for Mehta (1973) for use of the mean values in the classical erosion and deposition exclusive paradigm.
Figure 6-8. Distribution of the initial suspended sediment concentration for the Mehta (1973) deposition tests.
Figure 6-9. Simulations of Mehta (1973) experimental tests for shear stresses of 0.25 Pa, 0.40 Pa, 0.60 Pa and 0.85 Pa, and using classical erosion probabilities with simultaneous deposition.
Figure 6-10. Comparison of simulations using the simultaneous paradigm with the exclusive paradigm for the 0.25 Pa test using the mean values in calculations.
Figure 6-11. Simulations of Mehta (1973) test cases for shear stress of 0.40 Pa using combinations of probabilistic versus mean-valued depositional and erosion treatment, with either simultaneous or exclusive erosion/deposition.
Figure 6-12. Schematic representation of a probabilistic representation for the Mehta 1973 0.25 Pa test. The shaded area is the zone of deposition from the average value analysis. The differences are conceptual only since the displayed range of values is only +/- 1 standard deviation for each variable ($\sigma_{ce}$, $\sigma_{cd}$, $\sigma_b$).
Figure 6-13. Particle size distribution comparison between the mean value simulation and the probabilistic simulation for the 0.25 Pa Mehta (1973) test. Both simulations used the same sediment characteristics and critical shear stresses and the exclusive paradigm.
Figure 6-14  Simulation using the classical exclusive bed exchange processes using mean values for the Krone (1962) deposition test with a shear stress of 0.0305 Pa. One sensitivity simulation was made with an added supplemental internal shear of 100 Hz.
Figure 6-15. Simulation using the classical bed exchange processes using mean values for the Krone (1962) deposition test with a shear stress of 0.0415 Pa.
Figure 6-16. Simulation using the classical bed exchange processes using mean values for the Krone (1962) deposition test with a shear stress of 0.0515 Pa.
Figure 6-17. Illustration of the change in depositional response in the Krone (1962) recirculating flume tests when concentrations fall below approximately 0.3 kg/m$^3$. 
Figure 6-18. Model simulation to test the dilution rate for the Parchure and Mehta (1985) dilution test.
Figure 6-19. Initial particle size concentration distribution for bed initialization for the Parchure and Mehta (1985) dilution experiment. The simulation started with no sediment in suspension and then eroded the bed to an equilibrium concentration.
Figure 6-20. Parchure and Mehta (1985) dilution test results with bed exchange included, with the classical excess shear stress exclusive formulation and an exclusive simulation using probabilistic treatment of the key parameters.
Figure 6-21. Variation of floc size distribution during the Parchure and Mehta (1985) dilution test.
Figure 6-22. Initial cohesive bed particle concentration for the Parchure and Mehta (1985) erosion test.
Figure 6-23. Results of simulation of an erosion test (Parchure and Mehta, 1985) with a progressive increase in shear stress using the exclusive erosion/deposition and mean values.
Figure 6-24. Effects of switching from classically exclusive mean-value calibrated bed exchange to an exclusive/probabilistic treatment without parameter adjustments.
Figure 6-25. Variation in water depth during Sanford and Halka (1993) field test.
Figure 6-26. Results of calibrating continuous deposition for use on an average shear stress and using the probabilistic model of bed exchange: 5 January 1989.
Figure 6-27. Results of calibrating continuous deposition for use on an average shear stress and using the probabilistic model of bed exchange: 2 February 1990.
Figure 6-28. Results of calibrating continuous deposition for use on an average shear stress and using the probabilistic model of bed exchange: 15 January 1991.
Figure 6-29. Evolution of floc size distribution during numerical simulation of the Sanford and Halka (1993) data set of 5 January 1989. Both tests used the continuous deposition bed model. The black distributions are for the use of the average bottom shear stress, while the red curves are for the probabilistic shear stress formulation. (hours refer to Figure 6-26)
Figure 6-30. Evolution of floc size distribution during numerical simulation of the Sanford and Halka (1993) data set of 5 January 1989. Both tests used the exclusive erosion-deposition bed model. The black distributions are for the use of the average bottom shear stress, while the red curves are for the probabilistic shear stress formulation. (hours refer to Figure 6-26).
Figure 6-31. Results of simulations showing the effects of the continuous deposition and probabilistic bed exchange treatment 5 January 1989. All other model variables are held the same.
Figure 6-32. Results of simulations showing the effects of the continuous deposition and probabilistic bed exchange treatment 2 February 1990. All other model variables are held the same.
Figure 6-33. Results of simulations showing the effects of the continuous deposition and probabilistic bed exchange treatment 15 January 1991. All other model variables are held the same.
Figure 6-34. Results of simulation of the Krone gold-tagged sediment experiment using the simultaneous erosion and deposition treatment of the mean variables.
7.1 Summary

The primary issue addressed within the research here is whether the paradigm of exclusive erosion or deposition has scientific legitimacy within the physics of CST. The exclusive paradigm assumes that bed exchange condition is either erosion, deposition or neither, but never both. In contrast, the simultaneous paradigm admits the possibility of both occurring at the same time. Data supporting the exclusive paradigm tend to come from laboratory experiments (Ariathurai, et al., 1977; Partheniades, 1965; Lau and Krishnappan, 1994), while data supporting the simultaneous model come primarily from field experiments (Lick, 1982; Bedford, et al., 1987; Sanford and Halka, 1993; Winterwerp and van Kesteren, 2004).

The original exclusive paradigm is, in part, the result of early attempts to understand basic cohesive sediment transport combined with limitations of data collection and analysis procedures. Early research collected sediment deposition and erosion information by inference. The analysis procedures have been highly innovative in developing basic cohesive sediment processes information from other measurements (for example, floc shear strength, densities and orders of aggregation from viscometers). However, a lack of technology to measure erosion and/or deposition at the turbulence time scale leaves analysis methods still with significant level of inference based on averaging over longer time scales. Bed exchange has been inferred primarily from the increase or reduction in the suspended sediment concentration within the flume, which is an averaging over time (and over the domain of the measuring apparatus such as a flume). The net result of averaging will be positive, negative or zero, but not both positive and
negative. Settling velocities are also normally inferred from depositional devices that provide averaged data.

Mathematical conceptual models developed of CST were based on correlations between the time-averaged inferred data and bulk properties of the sediment. The result of these conceptual models is that averaged variables predict averaged responses. Until recently, this has not been a problem since most numerical models are also based on time-averaged equations applied over discretized spatial domains that also involve averaging (e.g., cell face averaged in finite difference). The earliest conceptual mathematical models of bed exchange have generally been appropriate for the numerical treatment (Ariathurai, 1974).

With the development of greater details in conceptual models, such as particle size distributions and flocculation sub-models, the bed exchange algorithms have required revision. The first logical step has been to extend the bulk equations to the individual particle size classes, which has shown the need for recalibration of the coefficients characterizing bulk equations. The next logical step (used herein) is to define the relative behavior between sediment classes and apply bulk scaling over all classes. There is a knowledge gap between the theoretical framework of the processes and the data collection and monitoring technology to support the theory, still largely related to averaging issues. The present modeling effort has shown that the time step required to support the flocculation model can be no longer than 1 second. Use of larger times steps generally resulted in the condition where the fraction of a class concentration being transferred to other class sizes by the aggregation/disaggregation fluxes approaches 1.0 and the numerical scheme becomes unstable.

The variability of conditions in field experiments typically takes empirical relationships derived from highly controlled laboratory experiments well outside the limits of calibration.
Numerical modelers have found the need to use the simultaneous approach to replicate real world depositional behavior (Sanford and Halka, 1994; Winterwerp, 2007). In response to this, the next level of complexity to be incorporated into the conceptual and numerical models is the explicit incorporation of stochastic effects into the cohesive processes (Prooijen and Winterwerp, 2009).

The logical arguments for use of the exclusive paradigm are strong within the appropriate context. When the question of simultaneous exchange is applied to a single sediment particle the exclusive paradigm obviously must be applied. However, the numbers of particles within a suspension require the application of statistical measures (even if just the mean). If the scales of concern are at an engineering level, for example seasonal or yearly sedimentation within a harbor, then the exclusive view may be appropriate simply based on the averaging involved.

The characteristic difference between values of the critical shear stress for erosion and the critical stress for deposition has been one strong argument for the exclusive paradigm. The difference is the result of the disparity between the binding cohesive force needed to be overcome to dislodge cohesive particles from the bed and the shear force required to retain the same size particle in suspension (Partheniades, 1965, 1971). When a wide range of particle sizes is considered the difference in stresses for the bulk behavior of cohesive sediment becomes less defined (Tolhurst, 2009).

Arguments for the simultaneous erosion and deposition approach are all associated with the statistical variation in the variables. Sediment properties within the bed are usually heterogeneous and the erosive forces are stochastic. The particle size distribution combined with turbulence fluctuations offers a clear conceptual visualization of the potential for simultaneous erosion and deposition in terms of some sizes undergoing erosion while others are depositing. A
visualization of a unit area of the bed with a variety of irregularities exposed to turbulent flow should lead one to admit the probability of both erosion and deposition within the unit area at some time. The conclusion, however, may change with varying the units of the area. A single particle will truly either erode or deposit or stay where it is. But engineering analysis cannot be performed at the single grain scale, so simultaneous erosion and deposition is almost unavoidable, unless the defined variables are the result of averaging over time and space. Winnowing of fine sediments is an example of a simultaneous effect. Even at very low flow velocities, there are very weak primary particle bonds in the bed that can eventually be broken and particles eroded flake by flake. This is a result of interaction of the very weak end of the floc strength spectrum with the high end of the shear stress spectrum.

The probabilistic treatment includes a distribution of most of the primary variables in cohesive sediment transport. These include current velocity, bottom shear stress, floc shear strength, critical shear stresses for erosion and deposition, internal (flow) shear and settling velocity. In the present analysis the current velocity, shear strength and settling velocities are assumed to be normally distributed. The shear stress variables are generated from transformations of the current velocity distribution. The primary effects of the implementation are realized in the aggregation model and in bed exchange. The application to bed exchange is accomplished through numerical integration of the appropriate products of probability density functions and their interactions. This allows for the investigation of the effects of various bed exchange paradigms explicitly within the numerical model.

The numerical sediment transport tool developed for this research has been shown to be capable of:

- Simulating aggregation and disaggregation, related to interparticle collisions due to Brownian motion, internal shearing, and differential settling.
Incorporating stochastic effects into bed exchange and aggregation/disaggregation through effects on differential settling velocities and interparticle collision-related floc breakage associated with the probabilistic treatment of the floc strength.

Addressing hindered settling by using previously developed theory (Winterwerp, 1999) within the formulation of probabilistic settling velocity.

Reaching a depositional equilibrium concentration for a fixed shear stress. This is not a significant new finding, but it provides some validation that the probabilistic treatment produces certain fundamental behavior.

Reaching an erosional equilibrium concentration for a fixed shear stress. This is a validation of another fundamental behavior for layered type I surface erosion (Mehta and Partheniades, 1982).

Exhibiting the floc spectrum features documented by Kranck and Milligan (1992). The numerical results show that erosion and deposition occur primarily on the larger end of the particle size spectra and that the shape of the spectra at the lowest particles sizes tends to remain consistent during tidal variation (see Figures 6-29 and 6-30).

7.2 Conclusions

A number of observations have been made during the development and application of the research numerical tool. Some observations of interest are:

The effects of a probabilistic treatment of the key variables are more pronounced for erosion than for deposition. The erosion rate is controlled by the interaction between the shear stress distribution and the spectrum of critical shear stress for erosion. The deposition rate is controlled by the settling velocity spectrum, with the probability of reentrainment, essentially erosion. The erosion effects, involving the interaction of two spectra, can be viewed as an amplification of the probabilistic effect.

Probabilistic effects are amplified through the flocculation model. The probabilistic variables when incorporated into the existing collision frequency formulations, result in a greater collision frequency for differential setting and particle collision breakage. This increases the interparticle size-class mass fluxes, which assists in providing a sediment supply to a wider range of size classes for bed exchange.

For a given shear stress the flocculation model will tend toward an equilibrium distribution, confirming a basic CST behavior (Kranck, 1973; Kranck and Milligan, 1992).

The probabilistic treatment results in a broader floc distribution spectrum. This broadening of the spectrum is the result of increased inter-class mass exchanges associated with non-discrete values in the aggregation model. The effect is analogous to adding diffusion to the floc spectrum. Without the probabilistic treatment, the resulting spectra normally have a
sharp boundary at the size classes bounding those discrete thresholds. The probabilistic results are more intuitively acceptable.

- A probabilistic treatment facilitates a faster response in the erosion and deposition to changes in the hydrodynamics. This is associated with the interactions of the tails of the probability distributions. If the shear stress is clearly below the threshold for erosion with an oncoming increase in shear stress due to changing hydrodynamics, the mean-value analysis will not exhibit erosion until the mean value of the shear stress rises above the mean shear strength of the bed. With a probabilistic treatment, when the upper tail of the shear stress distribution interacts with the lower tail of the shear strength distribution erosion will be initiated. Similar logic can be used for deposition if the mean threshold criteria are used.

- The use of the exclusive paradigm with a floc size distribution can perform as well as a simultaneous treatment with a single particle size. The use of size distribution will, by definition, add a distribution of settling velocities and critical shear stresses. So within the distribution, there can be erosion of larger flocs while deposition occurs at intermediate floc sizes (see Figure 6-12).

- The rebound of the concentration at the end of the dilution test (Parchure and Mehta, 1985) can be replicated using the probabilistic treatment. This implies that the equilibrium that was achieved prior to dilution may have been the result of a balance between small residual erosion and deposition.

- The time step required for accurate simulation of the aggregation/disaggregation sub-model is limited by the magnitude of the inter-class mass fluxes compared to the concentration of the size class.

- Changing from an application using averaged variables to one that incorporates a probabilistic treatment may require recalibration of the empirical coefficients. The calibration changes for the Sanford and Halka (1994) field tests included changing the erosion rate coefficient by a factor of 2 in one test, and by simply changing the initial condition for the starting concentration in the other two tests.

- Empirical coefficients developed based on mean values of the parameters will require some adjustment when applied with a probabilistic treatment. It is expected that the variability between the empirical values will be not more severe than the natural variability of the mean-valued coefficients between specific sites.

### 7.3 Recommendations

Many of the key features of cohesive processes, such as variability in settling rates and the transition to erosion from deposition and from erosion to deposition over a range of shear stresses, can be captured with a properly resolved particle size distribution combined with an
aggregation model. If however, rapid changes in sediment response are important, then use of a probabilistic treatment can improve the model performance. The definition of rapid change is relative to the ability of the analysis to adequately track changes in the sediment transport. The Sanford and Halka (1993) example showed that even with the simultaneous deposition paradigm, the concentrations in the field observations fell more rapidly than could be simulated with the mean value of the variables. The probabilistic treatment allows for the largest fall velocities within the probability distribution to initiate deposition (or not be reentrained if the continuous paradigm is used) sooner. The initiation of erosion will, similarly, occur sooner on increasing shear stress.

The following future research is recommended for consideration:

- Development of more rigorous methods for application of existing empirical coefficients such as erosion rate constants and threshold shear stresses to analysis that addresses individual size classes. Research to investigate the development of class based empirical coefficients is needed.

- Evaluation of the effects of a probabilistic treatment of variables within the aggregation model. A more thorough sensitivity analysis is needed for the changes in the performance of the aggregation model when the probabilistic treatment is used. The current application impacts the performance through differential settling and interparticle collision disaggregation.

- Effects of probabilistic treatment in the hindered settling region. When hindered settling begins to become important, it is reasonable to expect that the probability distribution of settling velocities will become much narrower. The influence in the current application is a linear scaling with reduction in the mean settling rate. The nonlinear effects need further research.

- Development of more efficient approach to incorporate probabilistic effects without explicit integration of the probability distributions. This is needed to make the effects available to engineering tools. This research was recently addressed by van Prooijen and Winterwerp (2009), with an analytical treatment of the probability densities, which they then simplified into a polynomial. Further research is needed in this area.
APPENDIX A
DEVELOPMENT OF GOVERNING EQUATIONS FOR UNSTEADY AND NONUNIFORM SEDIMENT TRANSPORT

The motivation for this dissertation is the evaluation of nonlinear effects introduced by unsteady and nonuniform hydrodynamics on the sediment transport processes. The governing equations for sediment transport are linear with regard to the sediment concentration in the transport and diffusion terms within the formulation. Any nonlinearity in the sediment transport equation is introduced through the dependence of the effective diffusion coefficient on turbulent mixing and the dependence of the fall velocity on the sediment concentration. Careful investigation of the development of the turbulent momentum exchange and mixing processes in the presence of suspended sediment is needed to deal with these nonlinearities in the sediment transport equation.

The temporal and spatial variations in density associated with the suspended sediment concentration influence the balance of momentum. As the density of a composite sediment-laden fluid parcel increases, the momentum will increase when the velocity remains constant. In order to develop the governing equations for sediment transport the mixing process driven by the hydrodynamics must first be addressed. Therefore, the derivation of the governing hydrodynamic equations will start with what appears to be compressible flow conditions, allowing to evaluate the influence of the changing density on the overall momentum balance. The rigorous development for compressible flow will be a valuable tool in understanding the sediment transport processes.

A.1 Hydrodynamics

We will develop the basic governing hydrodynamic equations in the presence of suspended sediment and then impose the conventional constraints to simplify the derivation to the case
without suspended sediment within the fluid. This will help in developing an understanding of the interpretation of specific terms in the system of equations.

The governing equations for compressible hydrodynamics in three dimensions in the conservative form using tensor notation are:

**A.1.1 Continuity Equation**

The basic universally applicable continuity equation is Equation A-1.

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \tag{A-1}
\]

In Equation A-1 the density is a function of temperature, pressure and all constituent variables.

**A.1.2 Momentum Equations**

\[
\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_j u_i}{\partial x_j} = F_i \tag{A-2_i}
\]

where

\[
F_i = -\frac{\partial p}{\partial x_j} + \rho g_i + F_{i}^{\text{vis}}
\]

\[
F_{i}^{\text{vis}} = \frac{\partial \sigma_{ij}}{\partial x_j}
\]

\(\sigma_{ij}\) is the viscous stress tensor and the viscous forces are derived from the gradients in the stresses. The stress tensor is

\[
\sigma_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \delta_{ij} \frac{\partial u_k}{\partial x_k} \tag{A-3_{ij}}
\]

Note that the subscript “i” in the equation label for Equations A-2_i corresponds to the i-index for the equation and is in essence three equations, \(i=1,2,3\) for the three Cartesian coordinates. In Equations A-3_{ij} both \(i\) and \(j\) span 1 to 3, independently, resulting in nine
equations. The $k$ indices in Equations A-3$_{ij}$ imply a summation of paired values of $k$ from 1 to 3. The last partial derivative in Equation A-3 with dual $k$ indices is the divergence of the velocity field. Also note that the viscous stress tensor is dependent on the dynamic viscosity. If we represent the dynamic viscosity as the product of the density and the kinematic viscosity, then $F_i^\nu$ will include terms involving the density gradients.

**A.2 Sediment Transport Equation**

If the sediment in suspension can be characterized by a series of size or mass classes, then the transport conservation equation can be developed for each class. The sediment transport for a sediment class “$m$” is given generally by

$$\frac{\partial c_m}{\partial t} + \frac{\partial}{\partial x_j}\left((u_{mj} - \delta_{i3}w_{sm})c_m\right) = S_m + \frac{\partial}{\partial x_j}\left(D_{mj} + D_{mj}\right)\frac{\partial c_m}{\partial x_j}$$  \hspace{1cm} (A-4$_m$)

where there are $m=1,M$ equations for each of $M$ sediment size classes. The equations are written at this point with the particle advective velocity $u_{mj}$ separate for each size class, and with diffusion coefficients $D_{mj}$ for each size class. Also, the generalized source term $S_m$ is by size class. The total source term for sediment class $m$ will be the summation of the interactions with all other size classes.

$$S_m = \sum_{i=1}^{M} S_{mi}(1 - \delta_{im})$$  \hspace{1cm} (A-5$_m$)

At this point we need to define the total suspended sediment concentration as a summation of the concentrations of the sediment classes as in Equation A-6.

$$C = \sum_{i=1}^{M} c_m$$  \hspace{1cm} (A-6)
The questions to be evaluated are how does $D_m$ interact with equations (A-2) and C with Equation A-1 and Equations A-2i.

The equations for conservation of momentum are a statement that the changes in the momentum are affected by changes in the fluid velocity and density in response to pressure gradients, gravity and viscous transfer of momentum with adjacent fluid of different momentum. The focus of attention here is that the fluid momentum is dependent on changes in the composite density of the fluid, regardless of the contribution, whether from temperature, salinity, or suspended sediment gradients. For sediment transport problems the composite fluid density is primarily affected by the suspended sediment. So the density $\rho$ in the equations above must be a combination of fluid and sediment densities weighted by the porosity as shown in Equation A-7.

$$\rho = n\rho_w + (1-n)\rho_s$$  \hspace{1cm} (A-7)

where $n$ is the porosity of the water parcel (volume of water per unit volume of water-sediment mixture). This can be developed further by defining an equation of state for the fluid that related $\rho_w$ to water temperature, salinity and pressure, although the pressure correction is negligible for shallow estuarine conditions. The sediment density is assumed to be independent of temperature. This assumption may not be completely accurate if the sediment particles are defined as flocs that are made up of minerals and interstitial water, which will be affected by temperature and salinity. The porosity of the fluid mixture for a given sediment mass concentration will be dependent on the effective density of the collection of sediment particles and flocs. If the mixture is all sand then $\rho_s$ will be the mineral density, the specific gravity of sand times the density of water. If the sediment is cohesive, the size and density of the flocs will vary. The porosity is defined form the sediment concentration and sediment density in Equation A-8.
\[ n = \left(1 - \frac{C}{\rho_s}\right) \]  \hspace{1cm} (A-8)

where \( \rho_s \) is the effective sediment density. The effective density is defined as the density that when used with the total sediment concentration yields the correct total effective volume of sediment in the mixture. Thus

\[ \rho_s = \frac{C}{\sum_{m=1}^{M} \frac{c_m}{\rho_{sm}}} \]  \hspace{1cm} (A-9)

where \( \rho_{sm} \) is the density associated with the sediment size class \( m \).

Inserting equation (A-8) into equation (A-7) yields

\[ \rho = \left(1 - \frac{C}{\rho_s}\right) \rho_w + C \]  \hspace{1cm} (A-10)

or alternatively

\[ \rho = \rho_w + C \left(1 - \frac{\rho_w}{\rho_s}\right) \]  \hspace{1cm} (A-11)

In the computation of the composite fluid density with the effects of the suspended sediment it is more convenient to work directly from the mass concentration of the suspended sediment rather than to attempt to address the size of the flocs and the effective floc density. That attention can be deferred to the particle interaction analysis. At this point it is sufficient to state that the composite density is a function of salinity, temperature, pressure and suspended sediment (Equation A-12).

\[ \rho = \rho(S, T, p, \rho_s, C_s) \]  \hspace{1cm} (A-12)

This density can be developed from any appropriate equation of state (e.g., EOS 90).
A.2.1 Continuity Equation for Sediment Laden Fluid

Now we will derive the continuity equation for the fluid in the presence of suspended sediment mass based on the following assumptions. The volume of fluid within the unit volume of fluid and sediment is “n”, the porosity. Therefore, the mass of fluid within the control volume is \((n \rho_w)\). The flux of fluid mass across a control volume face perpendicular to the \(x_i\)-axis is \((nu_i \rho_w)\). Following a conventional control volume development we obtain for the fluid mass in the control volume the governing equation.

\[
\frac{\partial n \rho_w}{\partial t} + \frac{\partial n u_i \rho_w}{\partial x_i} = 0
\]  
\((A-13)\)

This equation is universally valid for any sediment-laden fluid. Expanding these terms yields

\[
n \frac{\partial \rho_w}{\partial t} + \frac{\partial n}{\partial t} + n u_i \frac{\partial \rho_w}{\partial x_i} + \rho_w n \frac{\partial u_i}{\partial x_i} + \rho_w u_i \frac{\partial n}{\partial x_i} = 0
\]  
\((A-14)\)

If the incompressible assumption is invoked for the clear fluid density, then \(\frac{\partial \rho_w}{\partial t} = \frac{\partial \rho_w}{\partial x_i} = 0\), and the fluid continuity equation becomes

\[
\rho_w \frac{\partial n}{\partial t} + \rho_w n \frac{\partial u_i}{\partial x_i} + \rho_w u_i \frac{\partial n}{\partial x_i} = 0
\]  
\((A-15)\)

Returning to the variable density case, inserting equation (A-8) into equation (A-14) results after reorganizing

\[
\left(1 - \frac{C}{\rho_s}\right) \left\{ \frac{\partial \rho_w}{\partial t} + \frac{\partial u_i \rho_w}{\partial x_i} \right\} = \frac{\rho_w}{\rho_s} \left\{ \frac{\partial C}{\partial t} + \frac{\partial u_i C}{\partial x_i} \right\} - \frac{\rho_w}{\rho_s^2} C \left\{ \frac{\partial \rho_s}{\partial t} + \frac{\partial u_i \rho_s}{\partial x_i} \right\}
\]  
\((A-16)\)

This statement acknowledges that with time the relative volume of water and the volume and density of the sediment within the control volume are changing. This change is either a source or
sink of water or sediment and they are directly and inversely related. Assuming that the sediment mineral density is constant, then equation (A-16) becomes

$$\left(1 - \frac{C}{\rho_s}\right)\left[\frac{\partial \rho_s}{\partial t} + \frac{\partial u_i \rho_s}{\partial x_i}\right] = \frac{\rho_w}{\rho_s} \left[\frac{\partial C}{\partial t} + \frac{\partial u_i C}{\partial x_i}\right] - \frac{\rho_w}{\rho_s} C \frac{\partial u_i}{\partial x_i} \quad (A-17)$$

As the concentration of sediment in the control volume rises, the total volume and mass of water will be reduced by a proportion of approximately \( \frac{\rho_s}{\rho_w} \left(1 - \frac{C}{\rho_s}\right) \). If the sediment concentration is negligible compared to the sediment density then \( n \approx 1 \) and the fluid continuity equation becomes equation (A-1) with \( \rho \) replaced by \( \rho_w \).

If the clear fluid density is assumed to be constant, then equation (A-17) reduces further to

$$\frac{\partial u_i}{\partial x_i} = \frac{1}{\rho_s} \left[\frac{\partial C}{\partial t} + \frac{\partial u_i C}{\partial x_i}\right] \quad (A-18)$$

This states that the divergence of the velocity field vanishes only if the conservation of sediment mass in the control volume is attained when using the fluid velocity field.

### A.2.2 Continuity Equation with Differential Sediment Particle Velocity

For the suspended sediment within the control volume we will now acknowledge that the sediment particle velocity need not be precisely the same as the fluid velocity. This is may be particularly true in an accelerating flow field or a gravitational field. Define the sediment particle velocity components as

$$u_{sp} = u_i + u_{id} \quad (A-19i)$$

This defines the sediment velocity as the fluid velocity plus a differential component. The characteristics of this difference will be dealt with in the momentum equations. For now we will address the impacts on the sediment continuity equation.
The mass of sediment within the unit control volume is by definition \( C \). The unit flux of sediment across the \( x_i \)-coordinate face is \( u_{ip} C \). Using similar logic for the sediment fraction of the control volume fluxes and storage we get the sediment mass conservation equation, ignoring diffusion, as
\[
\frac{\partial C}{\partial t} + \frac{\partial u_{ip} C}{\partial x_i} = 0 \quad (A-20)
\]
Expanding the particle velocity we get the sediment equation
\[
\frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} + C \frac{\partial u_i}{\partial x_i} + u_{id} \frac{\partial C}{\partial x_i} + C \frac{\partial u_{id}}{\partial x_i} = 0
\quad (A-21)
\]
We can obtain an alternate form of the fluid continuity equation by combining equation (A-21) and equation (A-17) to give
\[
\left(1 - \frac{C}{\rho_s}\right) \left\{ \frac{\partial \rho_w}{\partial t} + \frac{\partial u_i \rho_w}{\partial x_i} \right\} + \rho_s \left\{ C \frac{\partial u_i}{\partial x_i} + \frac{\partial u_{id}}{\partial x_i} \right\} = 0
\quad (A-22)
\]
This final fluid continuity equation accounts for the conventional fluid continuity equation for the fraction of the control volume occupied by the fluid and the change in the fractional volume occupied by the fluid due to the net divergence of the sediment from the control volume. The two individual contributions are not each zero because the fraction of the fluid in the control volume changes in time as the sediment concentration changes.

Notice that the first grouping in equation (A-22), the conventional continuity equation, will only manifest itself alone when there is no sediment in the control volume. When \( C = 0 \) equation (A-22) reduces to equation (A-1) with \( \rho_w \) replacing \( \rho \).

The basic density of the fluid, \( \rho_w \), changes with temperature and salinity in estuarine waters. If it is assumed that the temporal changes in the density are negligible the continuity equation becomes
\[
\left(1 - \frac{C}{\rho_i}\right)\frac{\partial u_i}{\partial x_i} + \frac{P_w}{\rho_i} \left\{ C \frac{\partial u_i}{\partial x_i} + \frac{\partial u_{id}}{\partial x_i} \right\} = 0
\]  
(A-23)

Next, if the spatial gradients in clear fluid density are small compared to the changes due to sediment effects, with using some identities we can simplify to

\[
\frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho_i} \frac{\partial u_{id}}{\partial x_i} \quad (A-24)
\]

If the particle velocities in the horizontal are then assumed to be equal to the fluid velocity, then \(u_{id} = 0\) and several terms can be canceled and the continuity equation reduces to

\[
\frac{\partial u_i}{\partial x_i} = 0 \quad (A-25)
\]

So the conclusion is that only if the clear fluid is assumed incompressible and the sediment particles are transported by the fluid velocity exactly does the continuity equation reduce to the statement that the divergence of the velocity field vanishes.

**A.3 Turbulence**

The practical impact of turbulence on both hydrodynamics and sediment transport is that the temporal evolution of the details of flow and sediment concentration fields becomes unpredictable after a relatively short period of time. There are a variety of explanations for the lack of predictability. Most of these relate to the lack of the proper physics of the initiation of turbulence and eddy formation. However, from a pragmatic point of view, even if the physics of turbulence were known because the scale of initiation and given the current computational capability, it would be impractical to simulate details of turbulence for typical estuarine problems.

Consequently, turbulence has classically been addressed from a statistical perspective. The philosophy (Bernard, et. al. 1998) with regard to turbulence is the development of statistical
measures from a number (an ensemble) of turbulent time series associated with the same macroscale conditions (discharge, depth, pressure gradient, etc.). The main dependent parameters are the mean values of each turbulent variable.

The turbulent fluctuations of all variables are normally represented in the classical manner, where the parameters are represented as the sum of their mean and fluctuating parts, Reynolds’ decomposition.

\[
\begin{align*}
    u_i &= \bar{u}_i + u'_i \\
    p &= \bar{p} + p' \\
    \tau_i &= \bar{\tau}_i + \tau'_i \\
    T &= \bar{T} + T' \\
    S &= \bar{S} + S' \\
    c &= \bar{c} + c' \\
    \rho &= \bar{\rho} + \rho'
\end{align*}
\]

(A-26)

The classical method can be called conventional turbulent decomposition (CTD), where the variables are directly averaged or integrated. The overbar is the standard averaging of the ensemble and the single prime the perturbation from that mean.

In the development of the governing equations for turbulent compressible flow Canuto (1997) presented an ensemble averaging based on Farve filtering for the velocity component that is a mass-weighted averaging yielding unique properties. The generic chaotic variable is decomposed either by CTD into \( \xi = \bar{\xi} + \xi' \) or into a mass-weighted average and its perturbation.

\[
\xi = \bar{\xi} + \xi''
\]

(A-27)

The mass weighted ensemble average is written

\[
\bar{\xi} = \{ \xi \} = \frac{\{ \rho \xi \}}{\{ \rho \}} = \frac{\bar{\rho} \bar{\xi} + \rho' \bar{\xi}'}{\bar{\rho}}
\]

(A-28)

This definition leads to the following relations:

\[
\begin{align*}
    \bar{\xi} &\equiv \{ \xi \} \\
    \{ \xi' \} &\equiv \bar{\xi}' = 0 \\
    \{ \rho \xi'' \} &= \bar{\rho} \bar{\xi''} = 0 \\
    \{ \rho' \xi'' \} &= \{ \rho' \xi'' \}
\end{align*}
\]

(A-29a)

(A-29b)

(A-29c)
Alternative approaches to the derivation of the governing equations for turbulent flow will now be explored. Three approaches are of interest. First is the conventional turbulence decomposition with variable density in space and time with a chaotic turbulent component. Secondly is the simplification of the CTD approach for homogeneous flow, where the density is assumed constant. Finally, a mass-weighted turbulent decomposition (MTD) averaging of the velocity components in a variable density situation is considered. Each of these derivations will be presented for the purpose of better understanding the contributions of each of the terms in the resulting equations for turbulent sediment-laden flow.

A.3.1 Continuity Equation

The continuity equation will first be addressed for variable density without explicit consideration of the sediment concentration.

A 3.1.1 Variable density case with CTD

With variable density the continuity equation when using CTD results in the continuity equation

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho' u_j}{\partial x_j} + \frac{\partial \rho' u_j'}{\partial x_j} + \frac{\partial \rho' u_j'}{\partial x_j} = 0
\]  

(A-30)

Averaging results in an equation for the mean density.
\[
\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial \overline{\rho u_j}}{\partial x_j} + \frac{\partial \rho'u'_j}{\partial x_j} = 0
\]  
(A-31)

or alternatively

\[
\frac{D\overline{\rho}}{Dt} + \overline{\rho} \frac{\partial \overline{u_j}}{\partial x_j} + \frac{\partial \rho'u'_j}{\partial x_j} = 0, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \overline{u_j} \frac{\partial}{\partial x_j}
\]  
(A-32)

and for the fluctuating density, subtracting equation (A-31) from (A-30) gives

\[
\frac{\partial \rho'}{\partial t} + \frac{\partial \rho'u'_j}{\partial x_j} + \frac{\partial \rho'u'_j}{\partial x_j} + \frac{\partial \rho'u'_j}{\partial x_j} = 0
\]  
(A-33)

This can be converted to an equation for \( \rho' / \overline{\rho} \)

\[
\overline{\rho} \frac{D}{Dt} \left( \frac{\rho'}{\overline{\rho}} \right) + \frac{\partial \rho'u'_j}{\partial x_j} - \frac{\partial \rho'u'_j}{\partial x_j} = 0
\]  
(A-34)

**A 3.1.2 Homogeneous case with CTD**

The continuity equation for homogeneous turbulent flow reduces to two equations, one for the mean velocity components and one for the turbulent components:

\[
\frac{\partial \overline{u_i}}{\partial x_i} = 0
\]  
(A-35a)

\[
\frac{\partial u'_i}{\partial x_i} = 0
\]  
(A-35b)

**A 3.1.3 Variable density case with MTD**

Revisiting the continuity equation using the mass-weighted turbulence decomposition for the velocity with conventional decomposition for density yields

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = \frac{\partial \overline{\rho}}{\partial t} + \overline{\rho} \frac{\partial \overline{u_j}}{\partial x_j} + \frac{\partial \rho'u_j}{\partial x_j} + \frac{\partial \rho'u_j}{\partial x_j} = 0
\]  
(A-36)

Temporal averaging yields
\[
\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho} \tilde{u}_j}{\partial x_j} = 0 \quad (A-37)
\]

Or alternatively
\[
\frac{\tilde{D} \tilde{\rho}}{Dt} + \tilde{\rho} \frac{\partial \tilde{u}_j}{\partial x_j} = 0, \quad \frac{\tilde{D}}{Dt} \equiv \frac{\partial}{\partial t} + \tilde{u}_j \frac{\partial}{\partial x_j} \quad (A-38)
\]

For steady-state conditions this expresses the conservation of the mass flux \( \tilde{\rho} \tilde{u}_j \). Subtracting equation (A-37) from (A-36) yields an equation for \( \rho' \)
\[
\frac{\tilde{D} \rho'}{Dt} + \rho' \frac{\partial \tilde{u}_j}{\partial x_j} + \frac{\partial \rho u''_j}{\partial x_j} = 0 \quad (A-39)
\]

This can also be converted to an equation for \( \rho' / \tilde{\rho} \)
\[
\frac{\tilde{\rho}}{\tilde{D}} \left( \frac{\rho'}{\tilde{\rho}} \right) + \frac{\partial \rho u''_j}{\partial x_j} = 0 \quad (A-40)
\]

The MTD equations are somewhat more compact, with fewer terms.

**A.3.2 Turbulence Effects on Continuity Equation**

Now the effects of sediment suspension in turbulent flow on the continuity equation are addressed. Define the primary variable turbulent decompositions as follows

\[
\begin{align*}
    u_{ip} &= u_i + u_{ip} \\
    u_i &= \bar{u}_i + u'_i \\
    u_{id} &= \bar{u}_{ip} - \bar{u}_i + u'_{ip} - u'_i \\
    \bar{u}_{id} &= \bar{u}_{ip} - \bar{u}_i \\
    C &= \bar{C} + C'
\end{align*} \quad (A-41_i)
\]

The turbulent perturbation in the density of the fluid is assumed to be primarily dependent on the suspended sediment concentration. The temperature and salinity fluctuations are assumed to be negligible at the time and space scales of the turbulence.
Also note that it can be shown that the turbulent decomposition of the sediment concentration passes linearly to the class sizes as follows. The mean sediment concentration can be defined by averaging equation (A-6)

\[ \bar{C} = \sum_{i=1}^{M} c_m = \sum_{i=1}^{M} \bar{c}_m \]  

(A-42)

So by definition

\[ C' = C - \bar{C} = \sum_{i=1}^{M} c_m - \sum_{i=1}^{M} \bar{c}_m = \sum_{i=1}^{M} (c_m - \bar{c}_m) = \sum_{i=1}^{M} c'_m \]  

(A-43)

Therefore, assuming that \( \rho_w \) and \( \rho_s \) are constants, the density perturbations associated with suspended sediment are defined as

\[ \rho = \bar{\rho} + \rho' \]

\[ \bar{\rho} = \rho_w + \bar{C} \left( 1 - \frac{\rho_w}{\rho_s} \right) \]  

(A-44)

\[ \rho' = C' \left( 1 - \frac{\rho_w}{\rho_s} \right) \]

A.3.2.1 CTD method

If the variations in the clear fluid density are ignored the sediment water mixture continuity Equation A-24 with conventional turbulent decomposition results in the instantaneous equation

\[ \frac{\partial (x_i + u'_i)}{\partial x_i} + \frac{1}{\rho_s} \left[ \frac{\partial \bar{u}_i}{\partial x_i} \bar{C} + \frac{\partial u'_i}{\partial x_i} \bar{C} + \frac{\partial \bar{u}_i}{\partial x_i} \bar{C}' + \frac{\partial u'_i}{\partial x_i} \bar{C}' \right] = 0 \]  

(A-45)

On averaging we get

\[ \frac{\partial \bar{u}_i}{\partial x_i} + \frac{1}{\rho_s} \left[ \frac{\partial \bar{u}_i}{\partial x_i} \bar{C} + \frac{\partial u'_i}{\partial x_i} \bar{C}' \right] = 0 \]  

(A-46)

Subtracting Equation A-46 from Equation A-45 yields the instantaneous equation
\[
\frac{\partial (u'_i)}{\partial x_i} + \frac{1}{\rho_s} \left\{ \frac{\partial u'_i C}{\partial x_i} + \frac{\partial \bar{u}'_i C'}{\partial x_i} + \frac{\partial u'_i C'}{\partial x_i} - \frac{\partial \bar{u}'_i C'}{\partial x_i} \right\} = 0
\]  
(A-47)

### A.3.2.2 Homogeneous flow

The homogeneous case will essentially be the case of \( C = 0 \), in which case equation (A-46) reverts back to equation (A-36a) and (A-47) to (A-36b).

### A.3.2.3 MTD method

Applying the mass-weighted decomposition to the velocity field gives

\[
\begin{align*}
    u_i &= \bar{u}_i + u''_i \\
    \bar{u}_i &= \frac{\langle \rho u_i \rangle}{\bar{\rho}} \\
    u_{ip} &= \bar{u}_{ip} + u''_{ip} \\
    \bar{u}_{ip} &= \frac{\langle \rho u_{ip} \rangle}{\bar{\rho}} \\
    u_i &= u_i + u'_{ip} \\
    u_{id} &= \bar{u}_{ip} - \bar{u}_i + u''_{ip} - u''_i \\
    \bar{u}_{id} &= \bar{u}_{ip} - \bar{u}_i = \frac{\langle \rho u_{id} \rangle}{\bar{\rho}} \\
    u''_{id} &= u''_{ip} - u''_i \\
    C &= \bar{C} + C'
\end{align*}
\]  
(A-48)

Equation A-24 is independent of the method of turbulence decomposition. This becomes on the use of the MTD method

\[
\frac{\partial \bar{u}}{\partial x_i} + \frac{\partial u''_i}{\partial x_i} = -\frac{1}{\rho_s} \frac{\partial (u_{ip} - \bar{u}_i) C}{\partial x_i} - \frac{1}{\rho_s} \frac{\partial (u''_{ip} - u''_i) C}{\partial x_i}
\]  
(A-49)

Time-averaging results in

\[
\frac{\partial \bar{u}}{\partial x_i} + \frac{\partial u''_i}{\partial x_i} = -\frac{1}{\rho_s} \left\{ \frac{\partial \bar{u}_{id} C}{\partial x_i} + \frac{\partial u''_{ip} C}{\partial x_i} \right\}
\]  
(A-50)

This is similar to the CTD method result (Equation A-46) except that the concentration perturbation in the averaged term with the velocity fluctuation difference is replaced by the full concentration, \( C \).
These various forms of the continuity equations are summarized in Table A-1

A.3.3 Viscous Stresses

Expanding the viscous term in equation (A-2i)

\[ F_{ij}^{vis} = \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \left\{ \rho \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \left( \frac{2}{3} \rho \nu - \kappa \right) \delta_{ij} \frac{\partial u_k}{\partial x_k} \right\} \]  

(A-51i)

where \( \kappa \) is the bulk viscosity. This is normally only important sediment (two-phase flow) the term can become important when the divergence is significant (Brady, Khair and Swaroop, 2006). The difficulty is that the bulk viscosity is difficult to measure.

The bulk viscosity is hypothesized to be similarly dependent on the density as the shear viscosity. Define the kinematic bulk viscosity \( \zeta = \frac{\kappa}{\rho} \).

For example,

\[
\sigma_{11} = \mu \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) - \left( \frac{2}{3} \mu - \kappa \right) \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = \mu \left( 2 \frac{\partial u_1}{\partial x_1} - \left( \frac{2}{3} - \frac{\kappa}{\mu} \right) \nabla \cdot \mathbf{u} \right) 
\]

\[
\sigma_{12} = \mu \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) 
\]

\[
\sigma_{13} = \mu \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) 
\]

The viscous stress comes from the gradients in these stresses

\[ F_{ij}^{vis} = \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} \]

expanding for the \( u_i \) equation gives for constant density
\[ F_{\text{vis}}^{i} = \frac{\partial \sigma_{11}}{\partial x_i} + \frac{\partial \sigma_{12}}{\partial x_j} + \frac{\partial \sigma_{13}}{\partial x_k} = \mu \left( 2 \frac{\partial^2 u_1}{\partial x_1^2} - \frac{2 - \kappa}{\mu} \left( \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + \frac{\partial^2 u_2}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_1 \partial x_3} \right) \right) + \mu \left( \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_2 \partial x_3} \right) + \mu \left( \frac{\partial^2 u_1}{\partial x_3^2} \right) = \mu \left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \right) + \left( \frac{\mu \kappa}{3} + \kappa \right) \frac{\partial}{\partial x_i} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \right) + \left( \frac{\mu \kappa}{3} + \kappa \right) \frac{\partial}{\partial x_i} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \right) + 2 \nu \frac{\partial \rho}{\partial x_1} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \right) \]

If the flow is incompressible with constant density the dilatation term vanishes, then we get

\[ F_{\text{vis}}^{i} = \mu \left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \right) \]

If the density is not constant then we get

\[ F_{\text{vis}}^{i} = \mu \left( \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2} \right) + \left( \frac{\mu \kappa}{3} + \kappa \right) \frac{\partial}{\partial x_i} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \right) + 2 \nu \frac{\partial \rho}{\partial x_1} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) \right) \]

A.3.4 Turbulent Interaction with Viscosity

A.3.4.1 CTD method

Inserting the conventional turbulence decomposition into the viscous stress term yields, upon averaging

\[ \overline{F_{\text{vis}}^{i}} = \frac{\partial \overline{\sigma}_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ \nu \overline{\partial u_i \partial u_j} + \nu \overline{\rho \partial u_i \partial u_j} \right] + \left( \frac{2}{3} \nu - \zeta \right) \delta_{ij} \left( \overline{\rho \partial u_k \partial u_k} + \overline{\rho' \partial u_k' \partial u_k'} \right) \right) \]
A.3.4.2 Homogeneous case

If the flow is assumed homogeneous, then equation (A-52_i) reduces back to the form of equation (A-51_i) above with the averaged values replacing the full variables

$$\overline{F_{i\text{ss}}} = \nu \frac{\partial}{\partial x_j} \left\{ \rho \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) - \left( \frac{2}{3} \frac{\kappa}{\mu} \right) \delta_{ij} \rho \frac{\partial \overline{u}_k}{\partial x_k} \right\}$$  \hspace{1cm} (A-53_i)

The implication is that the turbulence has no direct effect on the viscous dissipation unless the fluid density also has a turbulent component; then the nonlinear effects make a contribution.

A.3.4.3 MTD method

If the mass-weighted turbulent decomposition is used we obtain

$$\overline{F_{i\text{ss}}} = \nu \frac{\partial}{\partial x_j} \left\{ \rho \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) + \rho \frac{\partial \bar{u}_i^\prime}{\partial x_j} + \rho \frac{\partial \bar{u}_j^\prime}{\partial x_i} \right\} \left( \frac{2}{3} \frac{\kappa}{\mu} \right) \delta_{ij} \rho \frac{\partial \bar{u}_k^\prime}{\partial x_k}$$  \hspace{1cm} (A-54_i)

This is the same form as the one above for the CTD approach except that the full density is used in the product of the turbulent velocity gradients rather than the density perturbation. With the MTD approach, the equation reduces to the same homogeneous form as above because if the density is constant, since it can be taken inside the spatial derivatives of the velocity perturbations and then terms will vanish by definition for the MTD averaging method.

A.3.5 Mean Momentum Equations

A.3.5.1 CTD approach

The instantaneous equation from the conventional turbulence decomposition and variable density is developed by inserting the equations (A-7) into equations (A-2_i) to yield
With the conventional turbulence decomposition we obtain the Reynold’s form of the Navier–Stokes equation with variable density by averaging

\[
\frac{\partial \rho \bar{u}_i}{\partial t} + \frac{\partial \rho' \bar{u}_i}{\partial t} + \frac{\partial \rho' u'_i}{\partial t} + \frac{\partial \rho \bar{u}_j \bar{u}_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j \bar{u}_i}{\partial x_j} + \frac{\partial \rho' u'_j \bar{u}_i}{\partial x_j} + \frac{\partial \rho \bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j u'_i}{\partial x_j} \nonumber
\]

\[
+ \frac{\partial \rho' \bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j u'_i}{\partial x_j} \nonumber
\]

\[
= -\frac{\partial \rho'}{\partial x_i} + \bar{\rho} g_i + \rho' g_i + F_{i\text{vis}} \tag{A-55i}
\]

The viscous contribution for the variable density case using CTD was addressed by equation (A-56i).

\[
\bar{F}_i = -\frac{\partial \rho'}{\partial x_i} + \bar{\rho} g_i + F_{i\text{vis}} \tag{A-57i}
\]

The viscous contribution for the variable density case using CTD was addressed by equation (A-52i).

### A.3.5.2 Homogeneous case

Assuming constant density results in the following variables to vanish:

\[
\rho' = \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial x_j} = \frac{\partial \rho}{\partial x_j} = \frac{\partial u'_i}{\partial x_j} = 0
\]

The constant density momentum equation for the homogeneous case reduces to:

\[
\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial \bar{u}_j \bar{u}_i}{\partial x_j} + \frac{\partial u'_j \bar{u}_i}{\partial x_j} = -\frac{1}{\bar{\rho}} \frac{\partial \bar{p}}{\partial x_i} + g_i + \frac{1}{\bar{\rho}} F_{i\text{vis}} \tag{A-58i}
\]

The viscous contribution for the constant density case using CTD was addressed by equation (A-53i).
A.3.5.3 MTD approach

Introducing the mass-weighted decomposition variables for the velocity and the conventional decomposition for the remaining variables into the momentum equations results in the instantaneous turbulent momentum equation (Equation A-59i).

\[
\begin{align*}
&\frac{\partial \bar{p} \bar{u}_i}{\partial t} + \frac{\partial \rho' \bar{u}_i}{\partial t} + \frac{\partial \rho' \bar{u}_i''}{\partial t} + \frac{\partial \bar{p} \bar{u}_j \bar{u}_i}{\partial x_j} + \frac{\partial \rho' \bar{u}_j \bar{u}_i}{\partial x_j} + \frac{\partial \bar{p} \bar{u}_j u_i''}{\partial x_j} + \\
&\frac{\partial \rho' \bar{u}_j u_i''}{\partial x_j} + \frac{\partial \bar{p} u_j'' \bar{u}_i}{\partial x_j} + \frac{\partial \rho' u_j'' \bar{u}_i}{\partial x_j} + \frac{\partial \bar{p} u_j'' u_i''}{\partial x_j} + \frac{\partial \rho' u_j'' u_i''}{\partial x_j} + \\
&\frac{\partial \rho' u_j'' u_i''}{\partial x_j} + \frac{\partial \bar{p} u_j'' u_i''}{\partial x_j} + \frac{\partial \rho' u_j'' u_i''}{\partial x_j} + \frac{\partial \bar{p} u_j'' u_i''}{\partial x_j} + \frac{\partial \rho' u_j'' u_i''}{\partial x_j} + \frac{\partial \bar{p} u_j'' u_i''}{\partial x_j} + \frac{\partial \rho' u_j'' u_i''}{\partial x_j} + \\
&=-\frac{\bar{p}}{\partial x_i} - \frac{\bar{p}'}{\partial x_i} + \bar{p} g_i + \rho' g_i + F_i^{\text{vis}}
\end{align*}
\] (A-59i)

For getting the most simplification from the MTD equations the density is retained within the derivatives. Some terms will then vanish on the temporal averaging. If the momentum equations are averaged over the time scale used to define the averaged values defined above we obtain for the MTD

\[
\begin{align*}
&\frac{\partial \bar{p} \bar{u}_i}{\partial t} + \frac{\partial \bar{p} \bar{u}_j \bar{u}_i}{\partial x_j} + \frac{\partial \bar{p} u_j'' \bar{u}_i}{\partial x_j} = -\frac{\bar{p}}{\partial x_i} + \bar{p} g_i + F_i^{\text{vis}}
\end{align*}
\] (A-60i)

The viscous contribution for the variable density case using MTD was addressed by Equation A-54i.

A.3.6 Instantaneous Fluctuating Momentum Equations

A. 3.6.1 CTD approach

Now subtracting Equation A-56i from Equation A-55i we obtain an equation for the instantaneous turbulent fluctuations using the CTD approach
\[
\frac{\partial \rho\bar{u}_i}{\partial t} + \frac{\partial \rho u'_i}{\partial t} + \frac{\partial \rho' u'_i}{\partial t} + \frac{\partial \rho\bar{u}_j\bar{u}_i}{\partial x_j} + \frac{\partial \rho u'_j\bar{u}_i}{\partial x_j} + \frac{\partial \rho' u'_j\bar{u}_i}{\partial x_j} + \frac{\partial \rho\bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho u'_j u'_i}{\partial x_j} + \frac{\partial \rho' u'_j u'_i}{\partial x_j} = \frac{\partial \rho' u'_i}{\partial x_j} + \frac{\partial \rho u'_j u'_i}{\partial x_j} + \frac{\partial \rho' u'_j u'_i}{\partial x_j} - \frac{\partial \rho' u'_i}{\partial t}
\]

(A-61i)

with

\[
F_i - \bar{F}_i = -\frac{\partial \rho'}{\partial x_i} + \rho' g_i + F_{i^{\text{vis}}} - \bar{F}_{i^{\text{vis}}}
\]

(A-62i)

and

\[
F_{i^{\text{vis}}} - \bar{F}_{i^{\text{vis}}} = \nu \frac{\partial}{\partial x_j} \left[ \bar{\rho} \left( \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_i} \right) + \rho' \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_j}{\partial x_i} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \right] - \left( 2 - \frac{\kappa}{\mu} \right) \bar{\rho} \delta_j^i \frac{\partial u'_k}{\partial x_k} - \left( 2 - \frac{\kappa}{\mu} \right) \rho' \delta_j^i \frac{\partial u'_k}{\partial x_k} - \left( 2 - \frac{\kappa}{\mu} \right) \rho' \delta_j^i \frac{\partial u'_k}{\partial x_k} + \left( 2 - \frac{\kappa}{\mu} \right) \delta_j^i \rho' \frac{\partial u'_k}{\partial x_k}
\]

(A-63i)

A.3.6.2 Homogeneous Case

Invoking constant density in equations (A-61i) yields the homogeneous case equation for the fluctuating momentum

\[
\frac{\partial u'_i}{\partial t} + \frac{\partial u'_i}{\partial x_j} + \frac{\partial \bar{u}_j \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j u'_i}{\partial x_j} + \frac{\partial u'_j u'_i}{\partial x_j} = F_i - \bar{F}_i
\]

(A-64i)

A 3.6.3 MTD approach

Subtracting Equation (A-60i) from equation (A-59i) we obtain an equation for the instantaneous turbulent fluctuations using the MTD approach.
\[ \frac{\partial \rho' \bar{u}_i}{\partial t} + \frac{\partial \rho u_i'}{\partial t} + \frac{\partial \rho u_j u_i'}{\partial x_j} + \frac{\partial \rho u_j u_i}{\partial x_j} + \frac{\partial \rho u_j ' u_i}{\partial x_j} + \frac{\partial \rho u_i'}{\partial x_j} + \frac{\partial \rho' u_i u_j }{\partial x_j} + \frac{\partial \rho' u_i u_j }{\partial x_j} \]  

(A-65i)

\[ + \frac{\partial \rho'' u_i u_j ''}{\partial x_j} + \frac{\partial \rho' u_i u_j }{\partial x_j} - \frac{\partial \rho'' u_i u_j ''}{\partial x_j} = -\frac{\partial \rho'}{\partial x_i} + \rho' g_i + F_i^{\text{vis}} - F_i^{\text{vis}} \]

A.3.7 Large Scale Velocity Field Stresses

A.3.7.1 CTD method

Using the CTD method the large-scale velocity field \( \bar{u} \) kinematic Reynolds stress tensor can be defined as \( t_{ij} = \bar{u}_i \bar{u}_j \). Multiplying Equation A-56, for \( \bar{u}_i \) by \( \bar{u}_j \) and again by reversing the i and j we obtain after summing

\[ \bar{u}_j \frac{\partial \rho \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \rho' u_i'}{\partial t} + \bar{u}_j \frac{\partial \rho u_i u_j }{\partial x_k} + \bar{u}_j \frac{\partial \rho u_i u_j }{\partial x_k} + \bar{u}_j \frac{\partial \rho u_i u_j '}{\partial x_k} + \bar{u}_j \frac{\partial \rho u_i u_j '}{\partial x_k} + \bar{u}_j \frac{\partial \rho' u_i u_j }{\partial x_k} + \bar{u}_j \frac{\partial \rho' u_i u_j }{\partial x_k} \]

\[ \bar{u}_i \frac{\partial \rho \bar{u}_j}{\partial t} + \bar{u}_i \frac{\partial \rho' u_j '}{\partial t} + \bar{u}_i \frac{\partial \rho u_i u_j }{\partial x_k} + \bar{u}_i \frac{\partial \rho u_i u_j '}{\partial x_k} + \bar{u}_i \frac{\partial \rho u_i u_j '}{\partial x_k} + \bar{u}_i \frac{\partial \rho u_i u_j '}{\partial x_k} + \bar{u}_i \frac{\partial \rho' u_i u_j '}{\partial x_k} + \bar{u}_i \frac{\partial \rho' u_i u_j '}{\partial x_k} \]  

(A-66ij)

\[ = -\bar{u}_j \frac{\partial \rho}{\partial x_i} + \bar{u}_j \rho g_i + \bar{u}_j F_i^{\text{vis}} - \bar{u}_i \frac{\partial \rho}{\partial x_j} + \bar{u}_i \rho g_j + \bar{u}_i F_j^{\text{vis}} \]

In the “large-scale” equations it is common at this point to ignore the viscous terms (Canuto, 1997). However, because of the treatment of the effects of molecular viscosity on the dissipation of energy these terms are retained.

Simplifying we obtain

\[ \bar{p} \frac{\partial \bar{u}_i \bar{u}_j}{\partial t} + \bar{p} \frac{\partial \bar{u}_i \bar{u}_j}{\partial x_k} + \bar{u}_i \left\{ \frac{\partial \rho' u_j '}{\partial t} + \frac{\partial \bar{u}_i \rho' u_j '}{\partial x_k} \right\} + \bar{u}_j \left\{ \frac{\partial \rho' u_i '}{\partial t} + \frac{\partial \bar{u}_i \rho' u_i '}{\partial x_k} \right\} \]  

(A-67ij)

\[ + \bar{u}_j \frac{\partial \rho' u_i '}{\partial x_k} + \bar{u}_i \frac{\partial \rho' u_i '}{\partial x_k} = - \left\{ \bar{u}_j \frac{\partial \tau_{ik} }{\partial x_k} + \bar{u}_i \frac{\partial \tau_{jk} }{\partial x_k} \right\} + \bar{u}_i F_i + \bar{u}_j F_j \]

In this equation the viscous terms are retained within the “F” terms. If the homogeneous assumption is made this reduces to the constant density case:
\[
\frac{\partial \tilde{u}_i \tilde{u}_j}{\partial t} + \rho \frac{\partial \tilde{u}_i}{\partial x_k} \frac{\partial \tilde{u}_j}{\partial x_k} = -\left\{ \tilde{u}_j \frac{\partial \tau_{ik}}{\partial x_k} + \tilde{u}_i \frac{\partial \tau_{jk}}{\partial x_k} \right\} + \tilde{u}_j \tilde{F}_i + \tilde{u}_i \tilde{F}_j
\]  
(A-68_{ij})

### A.3.7.2 MTD Method

Utilizing the mass-weighted turbulence decomposition yields a somewhat simpler expression. Multiplying Equation A-60, for \( \tilde{u}_i \) by \( \tilde{u}_j \) and again by reversing the i and j we obtain after summing

\[
\frac{\partial \tilde{u}_i \tilde{u}_j}{\partial t} + \rho \frac{\partial \tilde{u}_i}{\partial x_k} \frac{\partial \tilde{u}_j}{\partial x_k} = -\left\{ \tilde{u}_j \frac{\partial \tau_{ik}}{\partial x_k} + \tilde{u}_i \frac{\partial \tau_{jk}}{\partial x_k} \right\} - \\
\quad -\tilde{u}_j \rho \frac{\partial}{\partial x_j} + \tilde{u}_j \rho g_i + \tilde{u}_j F_{ii}^{vis} \\
\quad -\tilde{u}_i \rho \frac{\partial}{\partial x_j} + \tilde{u}_i \rho g_j + \tilde{u}_i F_{jj}^{vis}
\]  
(A-69_{ij})

Defining

\[
\bar{F}_i = -\frac{\partial \bar{p}}{\partial x_i} + \rho g_i + F_{ii}^{vis}
\]  
(A-70_{i})

we get

\[
\frac{\partial \tilde{u}_i \tilde{u}_j}{\partial t} + \rho \frac{\partial \tilde{u}_i}{\partial x_k} \frac{\partial \tilde{u}_j}{\partial x_k} = -\left\{ \tilde{u}_j \frac{\partial \tau_{ik}}{\partial x_k} + \tilde{u}_i \frac{\partial \tau_{jk}}{\partial x_k} \right\} + \tilde{u}_j \bar{F}_i + \tilde{u}_i \bar{F}_j
\]  
(A-71_{ij})

### A.3.8 Mean Velocity Reynolds Stresses

#### A.3.8.1 CTD approach

If we construct and equation by taking the sum of \( u_j \times \) Equations A-2i for \( u_i \) and \( u_i \times \) Equations A-2j for \( u_j \) we obtain

\[
u_j \frac{\partial \rho u_i}{\partial t} + u_i \frac{\partial \rho u_j}{\partial t} + u_j \frac{\partial \rho u_i u_i}{\partial x_k} + u_i \frac{\partial \rho u_j u_k}{\partial x_k} = u_j \bar{F}_i + u_i \bar{F}_j
\]

or reorganizing

349
\[
\frac{\partial \rho u_i u_j}{\partial t} + \frac{\partial \rho u_i u_k}{\partial x_k} + u_j u_k \left[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_k}{\partial x_k} \right] = u_j F_i + u_i F_j
\]

The brackets contain the full continuity equation, which vanishes to yield

\[
\frac{\partial \rho u_i u_j}{\partial t} + \frac{\partial \rho u_i u_k}{\partial x_k} = u_j F_i + u_i F_j
\]  \hspace{1cm} \text{(A-72)}

This equation can now be averaged to yield

\[
\frac{\partial \rho u_i u_j}{\partial t} + \frac{\partial \rho u_i u_k}{\partial x_k} + \frac{\partial \rho u'_i u'_j}{\partial t} + \frac{\partial \rho u'_i u'_k}{\partial x_k} + \frac{\partial \rho u'_i u'_j}{\partial t} + \frac{\partial \rho u'_i u'_k}{\partial x_k} = u_j F_i + u_i F_j
\]

Consolidating the equation and using the dynamic form of the Reynolds stress tensor definitions

\[
\tau_{ij} = \rho u'_i u'_j \text{ and } \tau_{ijk} = \rho u'_i u'_j u'_k
\]

we obtain

\[
\frac{\partial \tau_{ij}}{\partial t} + \frac{\partial \tau_{ij}}{\partial x_j} + \frac{\partial \rho u'_i u'_j}{\partial t} + \frac{\partial \rho u'_i u'_j}{\partial x_j} = u_j F_i + u_i F_j
\]  \hspace{1cm} \text{(A-73)}

Using equation (A-71) the terms within the bracket in equation (A-73) can be replaced to yield
\[
\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} + \frac{\partial \tau_{ijk}}{\partial x_k} + \\
\frac{\partial \bar{u}_j \rho' u'_j}{\partial t} + \frac{\partial \bar{u}_j \rho' u'_j}{\partial x_k} + \frac{\partial \bar{u}_i \rho' u'_j}{\partial x_k} + \frac{\partial \bar{u}_i \rho' u'_j}{\partial x_k} = \text{(A-74ij)}
\]

\[
= - \left[ \tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} + \tau_{ik} \frac{\partial \bar{u}_i}{\partial x_k} \right] + u_i F_i + \bar{u}_i F_j - \bar{u}_j F_i - \bar{u}_i F_j
\]

**A.3.8.2 Homogeneous form**

If the constant density assumption is invoked equation (A-74ij) simplifies to

\[
\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} + \frac{\partial \tau_{ijk}}{\partial x_k} = \text{(A-75ij)}
\]

\[
= - \left[ \tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} + \tau_{ik} \frac{\partial \bar{u}_i}{\partial x_k} \right] + u_i F_i + \bar{u}_i F_j - \bar{u}_j F_i - \bar{u}_i F_j
\]

**A.3.8.3 MTD method**

Utilizing the mass-weighted turbulence decomposition yields a somewhat simpler expression. Expanding equation (A-72ij) using the mass-weighted turbulence decomposition yields

\[
\frac{\partial \bar{p} u_j u_j}{\partial t} + \frac{\partial \rho u_j u_j}{\partial t} + \frac{\partial \rho u_j u_j}{\partial t} + \frac{\partial \rho u_j u_j}{\partial t} = \text{(A-76ij)}
\]

Using equation (A-70ij) we obtain

\[
\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k \frac{\partial \tau_{ij}}{\partial x_k} + \frac{\partial \tau_{ijk}}{\partial x_k} = \text{(A-77ij)}
\]

\[
= - \left\{ \tau_{ij} \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{ik} \frac{\partial \bar{u}_j}{\partial x_k} + \tau_{jk} \frac{\partial \bar{u}_i}{\partial x_k} \right\} + u_i F_i + \bar{u}_i F_j - \bar{u}_j F_i - \bar{u}_i F_j
\]
The form of the MTD equation above is the same form as the homogeneous case for CTD method of decomposition (A-74ij)

A.3.9 Reynolds Stresses

A.3.9.1 CTD method

The form of the equations for the Reynolds stresses as velocity perturbation correlations can be developed from the CTD equations for shear stresses (equations A-74ij). Defining the Reynolds stresses as \( R_{ij} = \frac{1}{\rho} \tau_{ij} \) and \( R_{ijk} = \frac{1}{\rho} \tau_{ijk} \)

\[
\begin{align*}
\frac{\overline{\rho}}{\partial t} + \overline{\rho u_i} \frac{\partial \overline{R_{ij}}}{\partial x_j} + \frac{\partial \overline{\rho R_{ijk}}}{\partial x_k} + \\
\{ \frac{\partial \overline{u_i \rho u_j}}{\partial t} + \frac{\partial \overline{u_i \rho u_j'}}{\partial x_k} + \frac{\partial \overline{u_i u_j \rho u_k'}}{\partial x_k} + \frac{\partial \overline{u_i u_j \rho u_k'}}{\partial x_k} + \frac{\partial \overline{u_i u_j \rho u_k'}}{\partial x_k} \} = \\
= -\left\{ \overline{\rho} R_{ij} \frac{\partial \overline{u_i}}{\partial x_k} + \overline{\rho} R_{ijk} \frac{\partial \overline{u_i}}{\partial x_k} \right\} + u_i F_j + u_i F_j - \overline{u_i F_j} - \overline{u_i F_j}
\end{align*}
\]

(A-78ij)

A.3.9.2 Homogeneous flow

The Reynolds stress equation with the constant density assumption reduces to

\[
\frac{\overline{\rho}}{\partial t} + \overline{\rho u_i} \frac{\partial \overline{R_{ij}}}{\partial x_j} + \frac{\partial \overline{\rho R_{ijk}}}{\partial x_k} = \\
= -\left\{ \overline{\rho} R_{ij} \frac{\partial \overline{u_i}}{\partial x_k} + \overline{\rho} R_{ijk} \frac{\partial \overline{u_i}}{\partial x_k} \right\} + u_i F_j + u_i F_j - \overline{u_i F_j} - \overline{u_i F_j}
\]

(A-79ij)

A.3.9.3 MTD method

The form of the equations for the Reynolds stresses as velocity perturbation correlations can be developed from the MTD equations for shear stresses (equations A-77ij).

\[
\begin{align*}
\frac{\overline{\rho}}{\partial t} + \overline{\rho u_i} \frac{\partial \overline{R_{ij}}}{\partial x_j} + \frac{\partial \overline{\rho R_{ijk}}}{\partial x_k} = \\
= -\left\{ \overline{\rho} R_{ij} \frac{\partial \overline{u_i}}{\partial x_k} + \overline{\rho} R_{ijk} \frac{\partial \overline{u_i}}{\partial x_k} \right\} + u_i F_j + u_i F_j - \overline{u_i F_j} - \overline{u_i F_j}
\end{align*}
\]

(A-80ij)
The MTD method yields a Reynolds stress equation that has the same form as the homogeneous equation for the CTD method.

A.3.10 Forcing Terms in Reynolds Stress Equations

A.3.10.1 CTD method

The forcing terms, involving $F$, on the right hand side of equation (A-78$_{ij}$) can be expanded as

$$
\overline{u_j F_i} + \overline{u_i F_j} - \overline{u_i F_i} = -\left\{u'_j \frac{\partial \rho'}{\partial x_j} + u'_i \frac{\partial \rho'}{\partial x_i}\right\} + \rho u'_j g_i + \rho u'_i g_j + u_j \frac{\partial \sigma_{ik}}{\partial x_k} + u_i \frac{\partial \sigma_{jk}}{\partial x_k} - \overline{u_j} \frac{\partial \overline{\sigma}_{ik}}{\partial x_k} - \overline{u_i} \frac{\partial \overline{\sigma}_{jk}}{\partial x_k} 
$$

(A-81$_{ij}$)

A.3.10.2 Homogeneous flow

With the constant density assumption equation (A-81$_{ij}$) reduces to

$$
\overline{u_j F_i} + \overline{u_i F_j} - \overline{u_i F_i} = -\left\{u'_j \frac{\partial \rho'}{\partial x_j} + u'_i \frac{\partial \rho'}{\partial x_i}\right\} + \rho u'_j g_i + \rho u'_i g_j + u_i \frac{\partial \sigma_{jk}}{\partial x_k} - \overline{u_j} \frac{\partial \overline{\sigma}_{ik}}{\partial x_k} - \overline{u_i} \frac{\partial \overline{\sigma}_{jk}}{\partial x_k} 
$$

(A-82$_{ij}$)

A.3.10.3 MTD method

With MTD averaging

$$
\overline{u_j F_i} + \overline{u_i F_j} - \overline{u_i F_i} = \frac{1}{\rho} \left\{u'_j \frac{\partial \rho}{\partial x_j} + u'_i \frac{\partial \rho}{\partial x_i}\right\} - \left\{u'_j \frac{\partial \rho'}{\partial x_j} + u'_i \frac{\partial \rho'}{\partial x_i}\right\} + \rho u'_j g_i + \rho u'_i g_j + u_j \frac{\partial \sigma_{ik}}{\partial x_k} + u_i \frac{\partial \sigma_{jk}}{\partial x_k} - \overline{u_j} \frac{\partial \overline{\sigma}_{ik}}{\partial x_k} - \overline{u_i} \frac{\partial \overline{\sigma}_{jk}}{\partial x_k} 
$$

(A-83$_{ij}$)

Notice that the form of the MTD method does not have the gravitational buoyancy terms, but rather additional terms involving the mean pressure gradients. These represent fluctuations in mass due to the mass averaging of the velocity field. There will be differences in the viscous
terms as well, but the buoyancy terms are important for high-concentration sediment suspensions that create density stratification.

### A.3.11 Reynolds Stresses Revisited

The insertion of the forcings expansions yields revised Reynolds equations as follows

#### A.3.11.1 CTD method

\[
\frac{\partial R_{ij}}{\partial t} + \bar{\rho} \bar{u}_k \frac{\partial R_{ij}}{\partial x_k} + \frac{\partial \bar{p} R_{ijk}}{\partial x_k} + \left\{ \frac{\partial \bar{u}_i \rho' u_{j}'}{\partial t} + \frac{\partial \bar{u}_i \rho' u_{j}'}{\partial x_k} \right\} = \frac{\partial \bar{p}'}{\partial x_k} \left( u_{j}' \frac{\partial \rho'}{\partial x_i} + u_{i}' \frac{\partial \rho'}{\partial x_j} \right)
\]

\[= - \left\{ \frac{\partial \bar{p} R_{ik}}{\partial x_k} + \frac{\partial \bar{p} R_{jk}}{\partial x_k} \right\} - \left\{ \frac{u_{j}' \partial \rho'}{\partial x_i} + \frac{u_{i}' \partial \rho'}{\partial x_j} \right\} + \rho' u_{j}' g_i + \rho' u_{i}' g_j
\]

This equation can be rewritten as

\[
\bar{p} \left[ \frac{DR_{ij}}{Dt} + D_{ij} \right] = \sum_{ij} + B_{ij} - \pi_{ij} + \delta_{ij} PD - T_T - T_G - \varepsilon_{ij}
\]

(A-85)

Where the terms are defined as:

**total derivative**

\[
\frac{DR_{ij}}{Dt} = \frac{\partial R_{ij}}{\partial t} + \bar{u}_k \frac{\partial R_{ij}}{\partial x_k}
\]

\[D_{ij} = \text{diffusion tensor}
\]

\[
D_{ij} = \frac{1}{\bar{p}} \frac{\partial}{\partial x_k} \left\{ \bar{p} R_{ik} + \frac{2}{3} \delta_{ij} u_k' p' - \bar{\sigma}_{ik} u_j + \bar{\sigma}_{jk} u_i + \bar{\sigma}_{ik} \bar{u}_j + \bar{\sigma}_{jk} \bar{u}_i \right\}
\]

(A-86)

**temporal transformations**

\[
T_T = \left\{ \frac{\partial \bar{u}_i \rho' u_{j}'}{\partial t} + \frac{\partial \bar{u}_j \rho' u_{j}'}{\partial t} \right\}
\]

(A-87)

**transformation from spatial gradients in density fluctuations**

\[T_G = \text{transformation from spatial gradients in density fluctuations}
\]

354
\[ T_o = \left\{ \frac{\partial \bar{u}_i \partial \bar{u}_j \rho' u'_k}{\partial \chi_k} + \frac{\partial \bar{u}_i \partial \bar{u}_j \rho' u'_l}{\partial \chi_l} + \frac{\partial \bar{u}_i \partial \bar{u}_j \rho' u'_m}{\partial \chi_m} \right\} \]  

(A-88ij)

\[ \Sigma_{ij} = \text{production from mean velocity gradients} \]

\[ \Sigma_{ij} = -\left\{ \bar{p} R_{ik} \frac{\partial \bar{u}_j}{\partial \chi_k} + \bar{p} R_{jk} \frac{\partial \bar{u}_i}{\partial \chi_k} \right\} \]  

(A-89ij)

\[ B_{ij} = \text{buoyancy production} \]

\[ B_{ij} = \rho' u'_i g_i + \rho' u'_i g_j \]  

(A-90ij)

\[ \pi_{ij} = \text{pressure-velocity correlation transformations} \]

\[ \pi_{ij} = \left\{ u'_j \frac{\partial \rho'}{\partial \chi_i} + u'_i \frac{\partial \rho'}{\partial \chi_j} \right\} - \frac{2}{3} \delta_{ij} u'_k \frac{\partial \rho'}{\partial \chi_k} \]  

(A-91ij)

\[ PD = \text{pressure dilatation term} \]

\[ PD = \frac{2}{3} p' \frac{\partial u'_i}{\partial \chi_k} = \frac{2}{3} p'd \]  

(A-92ij)

where the dilatation is \( d \equiv \frac{\partial u'_k}{\partial \chi_k} \)

\[ \varepsilon_{ij} = \text{dissipation tensor} \]

\[ \varepsilon_{ij} = \sigma_{ik} \frac{\partial u_j}{\partial \chi_k} + \sigma_{jk} \frac{\partial u_i}{\partial \chi_k} \]  

(A-93ij)

**A.3.11.2 Homogeneous flow**

With the constant density assumption the temporal and gradient transformation terms vanish as well as the pressure-dilatation term, to simplify to

\[ \bar{p} \left[ \frac{D R_{ij}}{D t} + D_{ij} \right] = \Sigma_{ij} - \pi_{ij} - \varepsilon_{ij} \]  

(A-94ij)
A.3.11.3 MTD method

The equation becomes by inserting Equation A-83 into Equation A-80

$$
\frac{\rho}{\partial t} \frac{\partial R_{ij}}{} + \rho \partial u_k \frac{\partial R_{ij}}{} + \rho \frac{\partial R_{ij}}{} = - \left\{ \rho \frac{\partial u_j}{\partial x_k} + \rho \frac{\partial R_{ij}}{} \right\} \\
+ \left\{ \frac{1}{\rho} \left[ \rho' \frac{\partial u_j}{\partial x_i} + \rho' \frac{\partial u_i}{\partial x_j} \right] - \left\{ u_j' \frac{\partial p'}{\partial x_i} + u_i' \frac{\partial p'}{\partial x_j} \right\} \right\} \\
+ \left\{ u_j \frac{\partial \sigma_{jk}}{\partial x_k} + u_i \frac{\partial \sigma_{jk}}{\partial x_j} - \ddot{u}_j \frac{\partial \sigma_{jk}}{\partial x_k} - \ddot{u}_i \frac{\partial \sigma_{jk}}{\partial x_k} \right\} \\
(A-95_{ij})
$$

This can be generalized as

$$
\rho \left[\frac{\partial D_{ij}}{\partial t} + D_{ij}\right] = \Sigma_{ij} + B_{ij} - \pi_{ij} + \delta_{ij} PD - \varepsilon_{ij} \\
(A-96_{ij})
$$

Where the terms are now defined as

- **total derivative**
  \[ \frac{\partial D_{ij}}{\partial t} = \frac{\partial R_{ij}}{\partial t} + u_k \frac{\partial R_{ij}}{\partial x_k} \]  \hspace{1cm} (A-97_{ij})

- **diffusion tensor**
  \[ D_{ij} = \frac{1}{\rho} \frac{\partial}{\partial x_k} \left\{ \rho R_{jk} + \frac{2}{3} \delta_{ij} u_k p' - \sigma_{jk} u_j + \sigma_{jk} u_j + \sigma_{jk} u_i \right\} \]  \hspace{1cm} (A-98_{ij})

- **production from mean velocity gradients**
  \[ \Sigma_{ij} = - \left\{ \rho \frac{\partial u_j}{\partial x_k} + \rho \frac{\partial u_i}{\partial x_k} \right\} \]  \hspace{1cm} (A-99_{ij})

- **buoyancy production**
  \[ B_{ij} = \frac{1}{\rho} \left\{ \rho' u_j' \frac{\partial \rho}{\partial x_i} + \rho' u_i' \frac{\partial \rho}{\partial x_j} \right\} \]  \hspace{1cm} (A-100_{ij})

- **\pi_{ij}** = pressure-velocity correlation transformations
\[ \pi_{ij} = \left\{u_i' \frac{\partial}{\partial x_i} + u_j' \frac{\partial}{\partial x_j}\right\} - \frac{2}{3} \delta_{ij} u_k' \frac{\partial p'}{\partial x_k} \]  \hspace{1cm} (A-101_{ij})

\[ PD = \text{pressure dilatation term} \]

\[ PD = \frac{2}{3} p' \frac{\partial u_k'}{\partial x_k} \equiv \frac{2}{3} p'd \]  \hspace{1cm} (A-102)

where the dilatation is now \( d \equiv \frac{\partial u_k'}{\partial x_k} \)  \hspace{1cm} (A-103)

\[ \varepsilon_{ij} = \text{dissipation tensor} \]

\[ \varepsilon_{ij} = \sigma_{ik} \frac{\partial u_j'}{\partial x_k} + \sigma_{jk} \frac{\partial u_i'}{\partial x_k} \]  \hspace{1cm} (A-104_{ij})

The MTD method results in the elimination of the temporal and gradient transformation terms. The buoyancy term has changed to an equivalent buoyancy that is a function of the mean pressure gradient. When stratification develops, the buoyancy can be manifested in the gradients of pressure. If the hydrostatic assumption is made the MTD buoyancy term matches that of the CTD method directly.

**A.4 Turbulent Kinetic Energy**

The turbulent kinetic energy equation is obtained from the trace of the Reynolds stress equation, that is for \( i=j \).

**A.4.1 CTD Method**

The turbulent kinetic energy for the CTD method is defined as

\[ k = \frac{1}{2} \left(\frac{\sum_{i=1}^{3} u_i' u_i'}{\rho}\right) = \frac{1}{2} R_{ii} \]

Taking one half of the trace of equation (A-84_{ii}) yields the following equation (avoiding confusion by not using \( k \) as an index)
\[
\begin{align*}
\frac{\partial k}{\partial t} + \rho u_j \frac{\partial k}{\partial x_j} + \frac{1}{2} \frac{\partial \rho R_{ij}}{\partial x_j} + \frac{1}{2} \frac{\partial \rho}{\partial t} + \frac{1}{2} \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_j} \rho u_j' = \\
= - \left( \bar{p} R_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \left[ u_i' \frac{\partial p'}{\partial x_i} + \rho u_i' g_i + u_i \frac{\partial \sigma_y}{\partial x_j} - \bar{u}_i \frac{\partial \bar{\sigma}_y}{\partial x_j} \right] \right)
\end{align*}
\]

This can be abbreviated as

\[
\bar{p} \left[ \frac{Dk}{Dt} + D(k) \right] = \frac{1}{2} \Sigma_{ij} + \frac{1}{2} B_{ii} - \pi_{ii} + PD - T_T - T_G - \bar{\rho} \varepsilon
\]

With the terms defined as

- **Total derivative**
  \[
  \frac{Dk}{Dt} = \frac{\partial k}{\partial t} + \bar{u}_j \frac{\partial k}{\partial x_j}
  \]

- **Diffusion tensor**
  \[
  D(k) = \frac{1}{\bar{\rho}} \frac{\partial}{\partial x_j} \left[ \frac{1}{2} \bar{p} R_{ij} + \frac{1}{3} u_i' p' - \sigma_y u_i + \bar{\sigma}_y \bar{u}_i \right]
  \]

- **Temporal transformations**
  \[
  T_T = \frac{\partial \bar{u}_i \rho u_j'}{\partial t}
  \]

- **Transformation from spatial gradients in density fluctuations**
  \[
  T_G = \left\{ \frac{1}{2} \frac{\partial \bar{u}_i \bar{u}_j \rho u_j'}{\partial x_j} + \frac{\partial \bar{u}_i \bar{u}_j \rho u_j'}{\partial x_j} \right\}
  \]

- **Production from mean velocity gradients**
  \[
  \Sigma_{ij} = -2 \bar{p} R_{ij} \frac{\partial \bar{u}_i}{\partial x_j}
  \]

- **Buoyancy production**
  \[
  B_{ii} = 2 \rho u_i g_i
  \]

- **Pressure-velocity correlation transformations**
  \[
  \pi_{ii} = \rho \varepsilon
  \]
\[ \pi_ii = \frac{2}{3} u'_i \frac{\partial p'}{\partial x_i} \quad (A-112) \]

**PD = pressure dilatation term**

\[ PD = \frac{1}{3} p' \frac{\partial u_i'}{\partial x_j} \equiv \frac{1}{3} p' \delta_{ij} \quad (A-113) \]

where the dilatation is \( d \equiv \frac{\partial u_i'}{\partial x_j} \)

\[ \varepsilon = \text{rate of dissipation of TKE} \]

\[ \varepsilon = \frac{1}{2 \rho} \varepsilon_{ii} = \frac{1}{\rho} \sigma_{ij} \frac{\partial u_j}{\partial x_j} \quad (A-114) \]

**A.4.2 Homogeneous Flow**

For the case of homogeneous flow the turbulent kinetic energy equation becomes

\[
\frac{\partial k}{\partial t} + \frac{u_j}{\partial x_j} \frac{\partial k}{\partial x_j} + \frac{1}{2} \frac{\partial R_{ij}}{\partial x_j} = -R_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{1}{\rho} \left\{ u'_i \frac{\partial p'}{\partial x_i} + u_i \frac{\partial \sigma_{ij}}{\partial x_j} - \bar{u}_i \frac{\partial \sigma_{ij}}{\partial x_j} \right\} \quad (A-115)
\]

Or summarizing

\[
\bar{\rho} \left[ \frac{Dk}{Dt} + D(k) \right] = \frac{1}{2} \sum_{ii} -\pi_{ii} + PD - \bar{\rho} \varepsilon \quad (A-116)
\]

**A.4.3 MTD Method**

The TKE equation using the MTD method of turbulence decomposition gives

\[
\bar{\rho} \frac{\partial k}{\partial t} + \bar{\rho} \bar{u}_j \frac{\partial k}{\partial x_j} + \frac{1}{2} \frac{\partial \bar{\rho} R_{ij}}{\partial x_j} = -\left\{ \bar{\rho} R_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \right\} + \frac{1}{\bar{\rho}} \left\{ \bar{\rho} u'_i \frac{\partial p'}{\partial x_i} \right\} - \left\{ u'_i \frac{\partial p'}{\partial x_i} \right\} + u_i \frac{\partial \sigma_{ij}}{\partial x_j} - \bar{u}_i \frac{\partial \sigma_{ij}}{\partial x_j} \quad (A-117)
\]

This can be summarized as
\[
\bar{\rho} \left[ \frac{\partial k}{\partial t} + D(k) \right] = \frac{1}{2} \sum_{ij}^2 + \frac{1}{2} B_{ij} - \pi_{ij} + PD - \bar{\rho} \varepsilon \quad (A-118)
\]

\[
D(k) = \frac{1}{\bar{\rho}} \frac{\partial}{\partial x_j} \left\{ \frac{1}{2} \bar{\rho} R_{ij} + \frac{1}{3} u_i u_j' - \sigma_{ij} u_i + \bar{\sigma}_{ij} \bar{u}_i \right\} \quad (A-119)
\]

\[\Sigma_{ij} = \text{production from mean velocity gradients} \]

\[\Sigma_{ij} = -2\bar{\rho} R_{ik} \frac{\partial \bar{u}_i}{\partial x_k} \quad (A-120_{ij})\]

\[B_{ij} = \text{buoyancy production} \]

\[B_{ij} = \frac{2}{\bar{\rho}} \bar{\rho} u_i' \frac{\partial \bar{p}}{\partial x_i} \quad (A-121)\]

\[\pi_{ij} = \text{pressure-velocity correlation transformations} \]

\[\pi_{ij} = u_i' \frac{\partial \bar{p}}{\partial x_i} - \frac{1}{3} u_j' \frac{\partial \bar{p}}{\partial x_j} \quad (A-122_{ij})\]

\[PD = \text{pressure dilatation term} \]

\[PD = \frac{1}{3} \bar{p} \frac{\partial u_k''}{\partial x_k} \equiv \frac{1}{3} \bar{p}' d \quad (A-123)\]

where the dilatation is now \[d \equiv \frac{\partial u_k''}{\partial x_k} \quad (A-124)\]

\[\varepsilon_{ij} = \text{trace of the dissipation tensor} \]

\[\varepsilon_{ii} = 2\sigma_{ij} \frac{\partial \bar{u}_i}{\partial x_j} \quad (A-125_{ij})\]

**A.5 Rate of Turbulent Energy Dissipation**

The rate of turbulent energy dissipation developed in the governing equation for the transport of turbulent kinetic energy was defined as
The conventional definition of the dissipation rate developed for incompressible flow is

\[ \varepsilon = \frac{1}{2 \rho} \sigma_{ij} = \frac{1}{\rho} \frac{\partial u_i}{\partial x_j} = \frac{1}{\rho} \left[ \rho \nu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \left( \frac{2}{3} \rho \nu - \kappa \right) \delta_{ij} \frac{\partial u_k}{\partial x_k} \right] \frac{\partial u_i}{\partial x_j} = \]

\[ = \frac{\nu}{\rho} \left\{ \frac{\partial^2 u_i'}{\partial x_j \partial x_i} + \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_i} + \rho' \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_i} + 2 \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_i}{\partial x_i} \right\} - \frac{1}{\rho} \left( \frac{2}{3} \nu - \xi \right) \delta_{ij} \left\{ \frac{\partial^2 \tilde{u}_k}{\partial x_j \partial x_k} - \rho' \frac{\partial \tilde{u}_k}{\partial x_j} \frac{\partial \tilde{u}_k}{\partial x_k} - \frac{\partial \tilde{u}_k}{\partial x_j} \frac{\partial \tilde{u}_k}{\partial x_k} - \frac{\partial \tilde{u}_k}{\partial x_j} \frac{\partial \tilde{u}_k}{\partial x_k} \right\} \]

(A-126)

The terms are the solenoidal, dilatative and the density fluctuation contributions. The density is subscripted “c” because we are considering the density variations of the fluid parcel associated with the sediment concentration.

The governing equation for the transport of the rate of dissipation can be developed by assuming that equation A-61 is a nonlinear operator \( N(u_i') \) and constructing the equation (Speziale and So, 1998)

\[ \varepsilon_s = \nu \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \]

(A-127)
\[ 2\nu \frac{\partial u_i'}{\partial x_j} \frac{\partial}{\partial x_j} N(u_i') = 0 \]  

(A-129)

The challenge is whether the additional dissipative terms will manifest themselves from this constructed equation.

Inspecting equation (A-61i) will lead to some simplification. The previously averaged terms that were subtracted from the instantaneous equation to form the equation above will average out when multiplied by a turbulent variable, so remove them now and combine the density decomposition terms to remain compact. This leaves

\[ \frac{\partial \rho'\bar{u}_i}{\partial t} + \frac{\partial \rho u'_i}{\partial x_j} + \frac{\partial \rho \bar{u}_j u'_i}{\partial x_j} + \frac{\partial \rho u'_j \bar{u}_i}{\partial x_j} + \frac{\partial \rho u'_j u'_i}{\partial x_j} = -\frac{\partial \rho'}{\partial x_i} + \rho' g_i + F_{i}^{\text{vis}} - F_{i}^{\text{vis}} \]  

(A-130_i)

\[ F_i - \overline{F}_i = -\frac{\partial \rho'}{\partial x_i} + \rho' g_i + F_{i}^{\text{vis}} - F_{i}^{\text{vis}} \]  

(A-131_i)

and similarly with the viscous terms, remove terms that will average out.

\[ F_i^{\text{vis}} - \overline{F}_i^{\text{vis}} = \nu \frac{\partial}{\partial x_j} \left\{ \rho' \left[ \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_j} \right] + \rho \left[ \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} \right] \right\} \]  

(A-132_i)

\[ \left\{ -\frac{2}{3} \rho' \frac{\partial \bar{u}_i}{\partial x_j} - \frac{2}{3} \rho' \frac{\partial \bar{u}_j}{\partial x_i} - \frac{\partial \rho'}{\partial x_i} \frac{\partial g_i}{\partial x_j} - \frac{\partial \rho'}{\partial x_j} \frac{\partial g_j}{\partial x_i} \right\} \]

So we will start with the nonlinear equation \( N(u_i') \) defined as
The dissipation equation will be formed as

\[ 2\nu \frac{\partial u_i'}{\partial x_m} \frac{\partial}{\partial x_m} N\left(u_i'\right) \]

This results in

\[ 2\nu \frac{\partial u_i'}{\partial x_m} \frac{\partial}{\partial x_m} \left( \begin{array}{cc}
\frac{\partial \rho' \overline{u}_i}{\partial t} + \frac{\partial \rho u_i'}{\partial t} + \frac{\partial \rho' \overline{u}_j u_i'}{\partial x_j} + \frac{\partial \rho u_i' \overline{u}_j}{\partial x_j} + \frac{\partial \rho u_i' \overline{u}_i}{\partial x_i} + \frac{\partial \rho u_i' u_i'}{\partial x_i}
\end{array} \right) \]

\[ = -\frac{\partial p'}{\partial x_i} + \rho' g_i + \nu \frac{\partial}{\partial x_j} \left( \begin{array}{cc}
\rho' \left[ \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right] + \rho \left[ \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right] \end{array} \right) \]

\[ - \frac{2}{3} \rho \delta_{ij} \frac{\partial u_k'}{\partial x_k} - \frac{2}{3} \rho' \delta_{ij} \frac{\partial u_k}{\partial x_k} \]  

\[ \text{(A-134)} \]

The first term on the LHS

\[ = 2\nu \frac{\partial u_i'}{\partial x_m} \frac{\partial}{\partial x_m} \left( \rho' \frac{\partial \overline{u}_i}{\partial t} + \overline{u}_i \frac{\partial \rho'}{\partial t} \right) = \]

\[ = 2\nu \frac{\partial u_i'}{\partial x_m} \left( \frac{\partial \rho' \overline{u}_i}{\partial x_m} + \rho' \frac{\partial^2 \overline{u}_i}{\partial x_m \partial t} + \frac{\partial \rho' \overline{u}_i}{\partial x_m} + \frac{\partial \rho \overline{u}_i}{\partial x_m} + \frac{\partial \rho \overline{u}_i}{\partial x_m} + \frac{\partial \rho \overline{u}_i}{\partial x_m} \right) = \]

\[ = 2\nu \left\{ \frac{\partial \overline{u}_i \frac{\partial u_i'}{\partial t}}{\partial x_m} + \frac{\partial^2 \overline{u}_i}{\partial x_m \partial t} + \frac{\partial \overline{u}_i \frac{\partial u_i'}{\partial t}}{\partial x_m} + \frac{\partial \overline{u}_i \frac{\partial u_i'}{\partial t}}{\partial x_m} + \frac{\partial \overline{u}_i \frac{\partial u_i'}{\partial t}}{\partial x_m} + \frac{\partial \overline{u}_i \frac{\partial u_i'}{\partial t}}{\partial x_m} \right\} \]  

\[ \text{(A-135)} \]
The second term on the LHS

\[
2v \frac{\partial u'_i}{\partial x_m} \frac{\partial}{\partial t} \frac{\partial p u'_i}{\partial x_m} = 2v \frac{\partial u'_i}{\partial x_m} \frac{\partial}{\partial t} \left( \frac{\partial u'_i}{\partial t} + u'_i \frac{\partial \rho}{\partial t} \right) = \\
2v \frac{\partial u'_i}{\partial x_m} \left( \frac{\partial \rho}{\partial x_m} \frac{\partial u'_i}{\partial t} + u'_i \frac{\partial^2 u'_i}{\partial x_m \partial t} + u'_i \frac{\partial \rho}{\partial x_m} \frac{\partial u'_i}{\partial t} + u'_i \frac{\partial^2 \rho}{\partial x_m \partial t} \right) = \\
2v \frac{\partial u'_i}{\partial x_m} \left( \frac{\partial \rho}{\partial x_m} \frac{\partial u'_i}{\partial t} + \frac{\partial^2 u'_i}{\partial x_m \partial t} + 2v \frac{\partial u'_i}{\partial x_m} \frac{\partial \rho}{\partial x_m} \frac{\partial u'_i}{\partial t} + 2v \frac{\partial u'_i}{\partial x_m} \frac{\partial^2 \rho}{\partial x_m \partial t} \right) = (A-136)
\]

\[
= 2v \frac{\partial \rho}{\partial x_m} \frac{\partial u'_i}{\partial x_m} \frac{\partial u'_i}{\partial t} + 2v \frac{\partial \rho'}{\partial x_m} \frac{\partial u'_i}{\partial x_m} \frac{\partial u'_i}{\partial t} + \bar{p} \frac{\partial \varepsilon}{\partial t} + 2\varepsilon \frac{\partial \bar{\rho}}{\partial t} + 2v \frac{\partial u'_i}{\partial x_m} \frac{\partial^2 \rho}{\partial x_m \partial t} + 2vu_i \frac{\partial^2 \rho}{\partial x_m \partial t}
\]
The third term on LHS

\[
2v \left( \frac{\partial u_i'}{\partial x_m} \frac{\partial \rho u_j'}{\partial x_j} \right) = 2v \frac{\partial u_i'}{\partial x_m} \left[ \frac{\partial u_i'}{\partial x_j} + \rho u_i' \frac{\partial \bar{u}_i}{\partial x_j} + \rho \bar{u}_i' \frac{\partial u_i'}{\partial x_j} \right] =
\]

\[
= 2v \frac{\partial u_i'}{\partial x_m} \left( \frac{\partial \rho}{\partial x_j} \frac{\partial u_i'}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_j} + \rho u_i' \frac{\partial \bar{u}_i}{\partial x_j} + \rho \bar{u}_i' \frac{\partial u_i'}{\partial x_j} \right) +
\]

\[
= 2v \frac{\partial \rho}{\partial x_j} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial u_i'}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m} +
\]

\[
+2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m} +
\]

\[
+2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m} +
\]

\[
+2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m} +
\]

\[
+2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} + 2v \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m} +
\]

\[
(A-137)
\]
The fourth term on the LHS

\[2\nu \frac{\partial u'_i}{\partial x_m} \frac{\partial \rho u'_j}{\partial x_j} = 2\nu \frac{\partial u'_i}{\partial x_m} \left[ \frac{\partial \rho}{\partial x_j} + \frac{\partial u'_j}{\partial x_j} + \frac{\partial u'_i}{\partial x_j} \right] =
\]

\[= 2\nu \frac{\partial u'_i}{\partial x_m} \left[ \frac{\partial \rho}{\partial x_j} + \frac{\partial u'}{\partial x_j} + \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'}{\partial x_j} \right] =
\]

\[= 2\nu \frac{\partial u'_i}{\partial x_m} \left[ \frac{\partial \rho}{\partial x_j} + \frac{\partial u'}{\partial x_j} + \frac{\partial u'_i}{\partial x_j} + \frac{\partial u'}{\partial x_j} \right] =
\]

\[= 2\nu \frac{\partial u'_i}{\partial x_m} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial \rho}{\partial x_j} +
\]

\[+ 2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'}{\partial x_j} \frac{\partial \rho}{\partial x_j} +
\]

\[+ 2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'_i}{\partial x_j} \frac{\partial \rho}{\partial x_j} + 2\nu \frac{\partial u'}{\partial x_j} \frac{\partial \rho}{\partial x_j}.
\]
The fifth term on the LHS
The sixth term on the LHS

\[
2v \frac{\partial u_i'}{\partial x_m} \frac{\partial}{\partial x_j} \frac{\partial p u_j'}{\partial x_j} = 2v \frac{\partial u_i'}{\partial x_m} \left[ u_j' \frac{\partial p}{\partial x_j} + \rho u_j' \frac{\partial u_j'}{\partial x_j} + \rho u_j' \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} \right]
\]

\[
= 2v \frac{\partial u_i'}{\partial x_m} \left[ \frac{\partial u_j'}{\partial x_j} + u_j' \frac{\partial u_j'}{\partial x_j} + \rho u_j' \frac{\partial u_j'}{\partial x_j} + \rho u_j' \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} \right]
\]

\[
= 2v \frac{\partial u_i'}{\partial x_m} \left[ \frac{\partial u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} \right]
\]

\[
+ 2v \frac{\partial u_i'}{\partial x_m} \left[ \frac{\partial u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} \right]
\]

\[
= 2v \frac{\partial u_i'}{\partial x_m} \left[ \frac{\partial u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} + \frac{\partial u_j'}{\partial x_m} \frac{\partial \nu u_j'}{\partial x_j} \right]
\]

\[
+ 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j}
\]

\[
+ 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j}
\]

\[
+ 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j}
\]

\[
+ 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j} + 2v \frac{\partial \nu u_j'}{\partial x_j}
\]

\[
(A-140)
\]
Now the RHS

\[
RHS = 2v \frac{\partial u'}{\partial x_m} \frac{\partial}{\partial x_m} \left[ -\frac{\partial p'}{\partial x_i} + \rho' g_i \right] + v \frac{\partial}{\partial x_j} \left[ \rho' \left( \frac{\partial u'}{\partial x_j} + \frac{\partial p}{\partial x_j} \right) + \rho \left( \frac{\partial u'}{\partial x_j} + \frac{\partial u'}{\partial x_i} \right) \right]
\]

\[
= -2v \frac{\partial u'}{\partial x_m} \frac{\partial^2 p'}{\partial x_m \partial x_i} + 2v \frac{\partial u'}{\partial x_m} \frac{\partial \rho'}{\partial x_m} g_i
\]

\[
+ 2v \frac{\partial u'}{\partial x_m} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} \left[ \rho' \left( \frac{\partial u'}{\partial x_j} + \frac{\partial p}{\partial x_j} \right) + \rho \left( \frac{\partial u'}{\partial x_j} + \frac{\partial u'}{\partial x_i} \right) \right]
\]

\[
= -2v \frac{\partial u'}{\partial x_m} \frac{\partial^2 p'}{\partial x_m \partial x_i} + 2v \frac{\partial u'}{\partial x_m} \frac{\partial \rho'}{\partial x_m} g_i
\]

\[
+ 2v \frac{\partial u'}{\partial x_m} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} \left[ \rho' \left( \frac{\partial u'}{\partial x_j} + \frac{\partial p}{\partial x_j} \right) + \rho \left( \frac{\partial u'}{\partial x_j} + \frac{\partial u'}{\partial x_i} \right) \right]
\]

\[
= -2v \frac{\partial u'}{\partial x_m} \frac{\partial^2 p'}{\partial x_m \partial x_i} + 2v \frac{\partial u'}{\partial x_m} \frac{\partial \rho'}{\partial x_m} g_i
\]

\[
+ 2v \frac{\partial u'}{\partial x_m} \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_j} \left[ \rho' \left( \frac{\partial u'}{\partial x_j} + \frac{\partial p}{\partial x_j} \right) + \rho \left( \frac{\partial u'}{\partial x_j} + \frac{\partial u'}{\partial x_i} \right) \right]
\]

\[
= -2v \frac{\partial u'}{\partial x_m} \frac{\partial^2 p'}{\partial x_m \partial x_i} + 2v \frac{\partial u'}{\partial x_m} \frac{\partial \rho'}{\partial x_m} g_i
\]

\[
\text{(A-141)}
\]
\[
RHS = -2\nu \frac{\partial u'_i}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_m \partial x_j} + 2\nu \frac{\partial u'_i}{\partial x_m} \frac{\partial \rho'}{\partial x_m} g_i + \\
\left[ \frac{\partial^2 \rho'}{\partial x_m \partial x_j} \frac{\partial u'_i}{\partial x_j} + \frac{\partial \rho'}{\partial x_m} \frac{\partial^2 u'_i}{\partial x_j^2} + \frac{\partial^2 \rho'}{\partial x_m \partial x_i} \frac{\partial u'_j}{\partial x_i} + \frac{\partial \rho'}{\partial x_m} \frac{\partial^2 u'_j}{\partial x_i^2} + \frac{\partial^2 \rho}{\partial x_m} \frac{\partial u'_j}{\partial x_i} + \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'_j}{\partial x_i^2} + \right]
\]
\[
+ \left[ \frac{\partial^2 \rho}{\partial x_m} \frac{\partial u'_i}{\partial x_j} + \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'_i}{\partial x_j^2} + \frac{\partial^2 \rho}{\partial x_m} \frac{\partial u'_j}{\partial x_i} + \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'_j}{\partial x_i^2} + \right]
\]
\[
+ \left[ -2\frac{\kappa}{3} \frac{\partial^2 \rho}{\partial x_m} \frac{\partial u'_i}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \left( \frac{2}{3} \frac{\kappa}{\mu} \right) \frac{\partial \rho}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \right]
\]
\[
+ \left[ -2\frac{\kappa}{3} \frac{\partial^2 \rho}{\partial x_m} \frac{\partial u'_i}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \left( \frac{2}{3} \frac{\kappa}{\mu} \right) \frac{\partial \rho}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \right]
\]
\[
+ \left[ -2\frac{\kappa}{3} \frac{\partial^2 \rho}{\partial x_m} \frac{\partial u'_i}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \left( \frac{2}{3} \frac{\kappa}{\mu} \right) \frac{\partial \rho}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \right]
\]
\[
+ \left[ -2\frac{\kappa}{3} \frac{\partial^2 \rho}{\partial x_m} \frac{\partial u'_i}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \left( \frac{2}{3} \frac{\kappa}{\mu} \right) \frac{\partial \rho}{\partial x_j} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} \right]
\]
\[
(A-142)
\]
RHS = \(-2v \frac{\partial u'_i}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_m \partial x_j} + 2v \frac{\partial u'_i}{\partial x_m} \frac{\partial \rho'}{\partial x_j} + 2v \frac{\partial^2 \rho'}{\partial x_m \partial x_j} \)
RHS = −2ν

∂ui′ ∂ 2 p′
∂u ′ ∂ρ ′
∂u ∂u ′ ∂ 2 ρ ′
gi + 2ν 2 i i
+ 2ν i
∂xm ∂xm ∂xi
∂xm ∂xm
∂x j ∂xm ∂xm ∂x j

+2ν 2

∂u ∂u ′ ∂ 2 ρ ′
∂ 2u j ∂ui′ ∂ρ ′
∂ 2ui ∂ui′ ∂ρ ′
+ 2ν 2 j i
+ 2ν 2
+
∂xm ∂x j ∂xm ∂x j
∂xi ∂xm ∂xm ∂x j
∂xm ∂xi ∂xm ∂x j

+2ν 2

∂ 2u j ∂ui′ ∂ρ ′
∂ 2ui ∂ui′ ∂ρ ′
∂ 3ui ∂ui′
2
2
′
2
2
ν
ν
ρ
+
+
∂x j 2 ∂xm ∂xm
∂xm∂x j 2 ∂xm
∂x j ∂xi ∂xm ∂xm

+2ν 2

∂ui′
∂ 2 ρ ∂ui′ ∂ui′
∂ 2 ρ ′ ∂ui′ ∂ui′
ρ ′ + 2ν 2
+ 2ν 2
∂xm ∂x j ∂xi ∂xm
∂xm∂x j ∂xm ∂x j
∂xm ∂x j ∂xm ∂x j

+ν

∂ 3u j

∂ρ ∂ ⎛ ∂ui′ ∂ 2ui′
ν
⎜2
∂x j ∂x j ⎜⎝ ∂xm ∂xm ∂x j

⎞
∂ρ ′ ⎛ ∂ui′ ∂ 2ui′
⎟ +ν
⎜ 2ν
⎟
∂x j ⎜⎝ ∂xm ∂xm∂x j
⎠

⎞
∂ 2 ρ ∂ui′ ∂u′j
⎟ + 2ν 2
⎟
∂xm∂x j ∂xm ∂xi
⎠

+2ν 2

2
2
∂ 2 ρ ′ ∂ui′ ∂u′j
∂ρ ∂ui′ ∂ u′j
∂ρ ′ ∂ui′ ∂ u′j
+ 2ν 2
+ 2ν 2
∂xm ∂x j ∂xm ∂xi
∂x j ∂xm ∂xm∂xi
∂x j ∂xm ∂xm∂xi

+2ν 2

⎛ ∂u ′ ∂ 3u′ ⎞
′ 2
∂ρ ∂ui′ ∂ 2ui′
2 ∂ρ ′ ∂ui ∂ ui′
i
2
+
+
ν
νρ
⎜ 2ν i
⎟
⎜ ∂xm ∂xm ∂x j 2 ⎟
∂xm ∂xm ∂x j 2
∂xm ∂xm ∂x j 2
⎝
⎠

⎛ ∂u ′ ∂ 3u′ ⎞
′ ∂ 2u′j
′ ∂ 2u′j
2 ∂ρ ∂ui
2 ∂ρ ′ ∂ui
i
2
2
ν
ν
+νρ ′ ⎜ 2ν i
+
+
⎟
⎜ ∂xm ∂xm∂x j 2 ⎟
∂xm ∂xm ∂x j ∂xi
∂xm ∂xm ∂x j ∂xi
⎝
⎠
+2ν 2 ρ

3
3
∂ui′ ∂ u′j
∂u ′ ∂ u′j
+ 2ν 2 ρ ′ i
∂xm ∂xm ∂x j ∂xi
∂xm ∂xm ∂x j ∂xi

⎛2 κ ⎞
∂ 2 ρ ∂ui′ ∂uk′ ⎛ 2 κ ⎞ 2 ∂ρ ∂ui′ ∂ 2uk′
2ν δ ij
− ⎜ − ⎟ 2ν 2δ ij
−
−
∂xm ∂x j ∂xm ∂xk ⎜⎝ 3 μ ⎟⎠
∂x j ∂xm ∂xm ∂xk
⎝3 μ⎠
⎛2 κ ⎞
∂ρ ∂ui′ ∂ ∂uk′ ⎛ 2 κ ⎞ 2
∂u ′ ∂ ∂ 2uk′
− ⎜ − ⎟ 2ν δ ij ρ i
− ⎜ − ⎟ 2ν 2δ ij
∂xm ∂xm ∂x j ∂xk ⎝ 3 μ ⎠
∂xm ∂x j ∂xm∂xk
⎝3 μ⎠
⎛2 κ ⎞
∂u ′ ∂ 2 ρ ′ ∂uk ⎛ 2 κ ⎞ 2 ∂ui′ ∂ρ ′ ∂ 2uk
− ⎜ − ⎟ 2ν 2δ ij i
−
−
2ν δ ij
∂xm ∂xm∂x j ∂xk ⎜⎝ 3 μ ⎟⎠
∂xm ∂x j ∂xm ∂xk
⎝3 μ⎠
⎛2 κ ⎞
∂u ′ ∂ 3uk
∂ρ ′ ∂ui′ ∂ ∂uk ⎛ 2 κ ⎞ 2
− ⎜ − ⎟ 2ν δ ij ρ ′ i
− ⎜ − ⎟ 2ν 2δ ij
∂xm ∂xm∂x j ∂xk
∂xm ∂xm ∂x j ∂xk ⎝ 3 μ ⎠
⎝3 μ⎠

372

(A-144)


$$RHS = -2v^2 \frac{du'_i}{dx} \frac{d^2\rho'}{dx_m dx_j} + 2v \frac{du'_i}{dx} \frac{d\rho'}{dx_m dx_j} g_i + 2v^2 \frac{\partial^2 \Pi}{\partial x_m \partial x_j} + 2v^2 \frac{\partial u'_i}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_j dx_m dx_j}$$
$$+ 2v^2 \frac{\partial^2 \Pi}{\partial x_m dx_j} \frac{d u'_i}{dx} + 2v^2 \frac{\partial^2 \Pi}{\partial x_m dx_j} \frac{d^2 \rho'}{\partial x_j dx_m dx_j} + 2v^2 \frac{\partial u'_i}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_j dx_m dx_j} + 2v^2 \frac{\partial^2 \Pi}{\partial x_m dx_j} \frac{d u'_i}{dx}$$
$$+ 2v^2 \frac{\partial^2 \Pi}{\partial x_m dx_j} \frac{d^2 u'_i}{dx} + 2v^2 \rho \frac{\partial^2 u'_i}{\partial x_m dx_j} + 2v^2 \frac{\partial^2 \rho}{\partial x_m dx_j} \frac{d^2 u'_i}{dx}$$
$$+ 2v^2 \rho' \frac{d u'_i}{dx} \frac{d^2 u'_i}{dx_m dx_j} + 2v^2 \frac{\partial^2 \rho}{\partial x_m dx_j} \frac{d u'_i}{dx} \frac{d^2 u'_i}{dx_m dx_j}$$
$$+ 2v^2 \rho' \frac{d u'_i}{dx} \frac{d^2 u'_i}{dx_m dx_j} + 2v^2 \frac{\partial^2 \rho}{\partial x_m dx_j} \frac{d u'_i}{dx} \frac{d^2 u'_i}{dx_m dx_j} + 2v^2 \frac{\partial^2 \rho}{\partial x_m dx_j} \frac{d u'_i}{dx}$$

\[(A-145)\]
Or reorganizing

\[ \text{RHS} = -2v \left( \frac{\partial^2 u'}{\partial x_m \partial x_i} \right) + 2v \left( \frac{\partial u'}{\partial x_m} \right) \left( g_j + v \frac{\partial^2 \rho}{\partial x_j \partial x_i} \right) + \frac{\partial^3 E_j}{\partial x_m} + v \frac{\partial \rho}{\partial x_j} \frac{\partial e}{\partial x_i} \]

\[ -2v^2 \rho \frac{\partial^2 u'}{\partial x_m \partial x_j} + 2v^2 \frac{\partial u'}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_j \partial x_i} + 2v^2 \left( \frac{\partial u'}{\partial x_m} \frac{\partial \rho'}{\partial x_i} \right) \left( \frac{\partial \rho}{\partial x_j} \frac{\partial e}{\partial x_i} \right) \]

\[ +2v^2 \frac{\partial u'}{\partial x_m} \frac{\partial \rho'}{\partial x_i} \frac{\partial}{\partial x_j} \left( \frac{\partial \rho}{\partial x_i} \frac{\partial e}{\partial x_i} \right) + 2v^2 \frac{\partial^2 \rho}{\partial x_m \partial x_j} \left( \frac{\partial u'}{\partial x_m} \frac{\partial u'}{\partial x_j} + \frac{\partial u'}{\partial x_j} \frac{\partial u'}{\partial x_m} \right) \]

\[ +2v^2 \frac{\partial^2 \rho'}{\partial x_m \partial x_j} \frac{\partial u'}{\partial x_m} \frac{\partial u'}{\partial x_j} + 2v^2 \frac{\partial \rho'}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_i} + 2v^2 \frac{\partial \rho'}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_i} \]

\[ -2v^2 \frac{\partial^2 \rho}{\partial x_m \partial x_j} \frac{\partial u'}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_i} + 2v^2 \frac{\partial \rho'}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_i} + 2v^2 \frac{\partial \rho'}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_i} \]

\[ -2v^2 \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_i} \]

\[ - \left( \frac{2}{3} - \frac{\kappa}{\mu} \right) 2v^2 \delta_{ij} \left[ \begin{array}{ccc}
\frac{\partial^3 \rho}{\partial x_m \partial x_j \partial x_k} & \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_k} & \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'}{\partial x_k \partial x_j} \\
\frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'}{\partial x_j \partial x_k} & \frac{\partial^3 \rho}{\partial x_m \partial x_j \partial x_k} & \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'}{\partial x_k \partial x_j} \\
\frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'}{\partial x_k \partial x_j} & \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u'}{\partial x_k \partial x_j} & \frac{\partial^3 \rho}{\partial x_m \partial x_j \partial x_k}
\end{array} \right] \]

(A-146)
Combining the density terms on the RHS for equation compactness

\[
RHS = -2v \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 p'}{\partial x_m \partial x_i} + 2v \frac{\partial u_i'}{\partial x_m} \frac{\partial \rho'}{\partial x_m} g_i + v \rho \frac{\partial^2 \varepsilon_i}{\partial x_i} + v \rho' \frac{\partial^2 \varepsilon_i}{\partial x_i} + v \rho \frac{\partial \varepsilon_i}{\partial x_i} + v \rho' \frac{\partial \varepsilon_i}{\partial x_i}
\]

\[
+ 2v^2 \frac{\partial^2 \rho}{\partial x_m \partial x_j} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial \rho'}{\partial x_j} \right) + 2v^2 \frac{\partial^2 \rho}{\partial x_m \partial x_j} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial \rho'}{\partial x_j} \right)
\]

\[
+ 2v^2 \frac{\partial^2 \rho}{\partial x_m \partial x_j} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial \rho'}{\partial x_j} \right) + 2v^2 \frac{\partial^2 \rho}{\partial x_m \partial x_j} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial \rho'}{\partial x_j} \right)
\]

\[
- 2v^2 \frac{\partial^2 u_i'}{\partial x_m \partial x_j} - 2v^2 \frac{\partial^2 u_i'}{\partial x_m \partial x_j} - 2v^2 \frac{\partial^2 u_i'}{\partial x_m \partial x_j}
\]

\[
\left(- \frac{2}{3} - \frac{\kappa}{\mu}\right) 2v^2 \delta_j = \left\{ \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_m \partial x_k} + \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_m \partial x_k} + \frac{\partial^2 \rho}{\partial x_m \partial x_m} \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_m \partial x_k} \right\}
\]

(A-147)

For homogeneous incompressible conditions this simplifies to

\[
RHS = -2v \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 p'}{\partial x_m \partial x_i} + v \rho \frac{\partial^2 \varepsilon_i}{\partial x_i} - v \rho' 2v \frac{\partial^2 u_i'}{\partial x_m \partial x_i} \frac{\partial^2 u_i'}{\partial x_m \partial x_i}
\]

(A-148)
Assembling the pieces of the dissipation equation using the CTD method

\[
\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} + \frac{\partial \rho' \rho'_e}{\partial x_j} + \epsilon \left( \frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} \right) + 2\nu \frac{\partial^2 \rho}{\partial x_m^2} \left\{ u_j' u_j' + u_j' u_j' \right\} + 2\nu \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial u_j'}{\partial x_m} + u_j' \frac{\partial u_j'}{\partial x_m} \right\}
\]

\[
+ 2\nu \left\{ \frac{\partial \rho}{\partial x_m} \left( u_j' \frac{\partial u_j'}{\partial x_m} + u_j' \frac{\partial u_j'}{\partial x_m} \right) + 2\nu \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial u_j'}{\partial x_m} + u_j' \frac{\partial u_j'}{\partial x_m} \right\} \right\} + 2\nu \left\{ \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial^2 u_j'}{\partial x_m^2} + \frac{\partial^2 u_j'}{\partial x_m^2} \right\} \right\}
\]

\[
+ 2\nu \left\{ \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial^2 u_j'}{\partial x_m^2} + \frac{\partial^2 u_j'}{\partial x_m^2} \right\} \right\} + 2\nu \left\{ \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial^2 u_j'}{\partial x_m^2} + \frac{\partial^2 u_j'}{\partial x_m^2} \right\} \right\}
\]

\[
+ 2\nu \left\{ \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial^2 u_j'}{\partial x_m^2} + \frac{\partial^2 u_j'}{\partial x_m^2} \right\} \right\} + 2\nu \left\{ \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial^2 u_j'}{\partial x_m^2} + \frac{\partial^2 u_j'}{\partial x_m^2} \right\} \right\}
\]

\[
+ \left\{ \frac{\partial^2 \rho}{\partial x_m^2} \left\{ \frac{\partial^2 u_j'}{\partial x_m^2} + \frac{\partial^2 u_j'}{\partial x_m^2} \right\} \right\} = \text{RHS}
\]

(A-149)
Now assuming homogeneous incompressible flow the dissipation equation can be greatly simplified. All terms with a $\rho'$, $\frac{\partial \rho}{\partial t}$, $\frac{\partial u_i}{\partial x_i}$, $\frac{\partial u_i'}{\partial x_i}$ or $\frac{\partial u_i'}{\partial x_i}$ vanish, leaving

$$p \frac{\partial e}{\partial t} + \bar{\rho} \frac{\partial e}{\partial x_j} = -2v \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_j} - 2v \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_j} - 2v \frac{\partial^2 u_i}{\partial x_m \partial x_j} - 2v^2 \frac{\partial^2 u_i'}{\partial x_m \partial x_j} \frac{\partial u_i'}{\partial x_m}$$

(A-150)

Dividing through by $\bar{\rho}$

$$\frac{\partial e}{\partial t} + \bar{u} \frac{\partial e}{\partial x_j} = -2v \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_j} - 2v \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_j} - 2v \frac{\partial^2 u_i}{\partial x_m \partial x_j} - 2v^2 \frac{\partial^2 u_i'}{\partial x_m \partial x_j} \frac{\partial u_i'}{\partial x_m}$$

(A-151)

This equation is equivalent to the equation presented for incompressible flow in Speziale and So (1998).

$$\frac{\partial e}{\partial t} + \bar{u} \frac{\partial e}{\partial x_i} = -2v \frac{\partial u_i'}{\partial x_i} \frac{\partial u_i'}{\partial x_i} - 2v \frac{\partial u_i'}{\partial x_i} \frac{\partial u_i'}{\partial x_i}$$

$$-2v \frac{\partial u_i'}{\partial x_i} \frac{\partial u_i'}{\partial x_i} - 2v^2 \frac{\partial^2 u_i'}{\partial x_i \partial x_i} \frac{\partial u_i'}{\partial x_i}$$

(A-152)
Expanding the Speziale derivatives in parentheses

\[
\frac{\partial \varepsilon}{\partial t} + u_i \frac{\partial \varepsilon}{\partial x_i} = -2\nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - 2\nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - 2\nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - 2\nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - \frac{2\nu}{\rho} \frac{\partial p'}{\partial x_i} \frac{\partial u'_i}{\partial x_i} + \frac{2\nu}{\rho} \frac{\partial p'}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - \nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - \nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - \nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - \nu \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} - \frac{2\nu^2}{\partial x_i} \frac{\partial u'_i}{\partial x_i} \frac{\partial u'_i}{\partial x_i} + \nu \frac{\partial^2 \varepsilon}{\partial x_i^2}
\]

(A-153)

This is exactly the same equation. These results were confirmed by developing the equation from the original homogeneous equation.

The various forms of the momentum equation and its components are summarized in Table A-2. The terms in the TKE equation are summarized in Table A-3.

### A.6 Sediment Transport Equation

The sediment conservation equation for sediment within size class \( n \) is provided in Equation A-154n.

\[
\frac{\partial c_n}{\partial t} + \frac{\partial (u_n - \delta \omega_n w_m)}{\partial x_j} c_n = S_n + \frac{\partial}{\partial x_j} D_m \frac{\partial c_n}{\partial x_j}
\]

(A-154n)

The diffusion coefficient \( D_m \) is molecular diffusion at this stage of development, and is coordinate independent. If we insert the conventional turbulent decomposition into the equation

\[
\frac{\partial \tau_n}{\partial t} + \frac{\partial \tau'_n}{\partial x_j} + \frac{\partial \omega_n \tau_n}{\partial x_j} + \frac{\partial \omega'_n \tau'_n}{\partial x_j} + \frac{\partial \omega'_n \tau'_n}{\partial x_j} - \frac{\partial \omega'_n \tau'_n}{\partial x_j} - \frac{\partial \omega'_n \tau'_n}{\partial x_j} - \frac{\partial \omega'_n \tau'_n}{\partial x_j} - \frac{\partial \omega'_n \tau'_n}{\partial x_j} - \frac{\partial \omega'_n \tau'_n}{\partial x_j} - \frac{\partial \omega'_n \tau'_n}{\partial x_j} - \frac{\partial \omega'_n \tau'_n}{\partial x_j}
\]

(A-155n)
If the total sediment concentration is the sum of the individual class concentrations

\[ C = \sum_{n=1}^{M} c_n \]  

(A-156)

And we decompose the concentrations into the mean and fluctuating components

\[ c_n = \bar{c}_n + c'_n \]  

(A-157)

Then the average of the total concentration will be

\[ \bar{C} = \sum_{n=1}^{M} c_n = \sum_{n=1}^{M} \bar{c}_n \]  

(A-158)

And the turbulent fluctuation in the total concentration will be

\[ C' = C - \bar{C} = \sum_{n=1}^{M} c_n - \sum_{n=1}^{M} \bar{c}_n = \sum_{n=1}^{M} (c_n - \bar{c}_n) = \sum_{n=1}^{M} c'_n \]  

(A-159)

The identities in (A-153) and (A-155) allows the summation of equation (A-152n) for all classes to yield

\[
\begin{align*}
\frac{\partial C}{\partial t} + \frac{\partial C'}{\partial t} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \bar{u}_{j,n} \bar{c}_n &+ \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \bar{u}_{j,n} c'_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} u'_{j,n} c'_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} u'_{j,n} \bar{c}_n \\
- \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \bar{w}_{sn} \bar{c}_n - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} w'_{sn} c'_n - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} w'_{sn} \bar{c}_n = &+ \frac{\partial}{\partial x_j} \sum_{n=1}^{N} w'_{sn} \bar{c}_n \] \\
= \sum_{n=1}^{N} \bar{S}_n + \sum_{n=1}^{N} S'_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left\{ D_m \left[ \frac{\partial c_n}{\partial x_j} + \frac{\partial c'_n}{\partial x_j} \right] \right\} 
\end{align*}
\]  

(A-160)

The summation on the advective terms does not pass through to the concentrations because of the size dependent particle velocities. Similarly, the size dependent diffusion coefficients keep the summation from operating on the sediment class concentrations in the diffusion terms.

The temporal averaged version of Equation A-155n is presented in Equation A161n.
\[ \frac{\partial \bar{c}_n}{\partial t} + \frac{\partial \bar{u}_n \bar{c}_n}{\partial x_j} + \frac{\partial \bar{u}_n' \bar{c}_n'}{\partial x_j} - \frac{\partial}{\partial x_j} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n \right) - \frac{\partial}{\partial x_j} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n' \right) = \]  
\[ = \bar{S}_n + \frac{\partial}{\partial x_j} D_m \frac{\partial \bar{c}_n}{\partial x_j} \]

(A-161)

The temporally averaged total concentration equation (A-160) becomes Equation A-162.

\[ \frac{\partial \bar{C}}{\partial t} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \bar{u}_n \bar{c}_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \bar{u}_n' \bar{c}_n' - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n \right) \]
\[ - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n' \right) = \sum_{n=1}^{N} \bar{S}_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left( D_m \frac{\partial \bar{c}_n}{\partial x_j} \right) \]

(A-162)

The turbulence correlation term between the velocity and the concentration can be expressed as being proportional to the mean gradients (see Equation A-163)

\[ \frac{\partial \bar{u}_n' \bar{c}_n'}{\partial x_j} \approx \frac{\partial}{\partial x_j} \left[ D_g \frac{\partial \bar{c}_n}{\partial x_j} \right] \]

(A-163)

The turbulent diffusivity, \( D_g \), may be different in the coordinate directions. However, it is assumed to be independent of sediment size class. The class time averaged equation becomes Equation (A-164n)

\[ \frac{\partial \bar{c}_n}{\partial t} + \frac{\partial \bar{u}_n \bar{c}_n}{\partial x_j} - \frac{\partial}{\partial x_j} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n \right) - \frac{\partial}{\partial x_j} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n' \right) = \]
\[ = \bar{S}_n + \frac{\partial}{\partial x_j} \left( D_m + D_g \right) \frac{\partial \bar{c}_n}{\partial x_j} \]

(A-164)

The total concentration conservation equation becomes Equation (A-165)

\[ \frac{\partial \bar{C}}{\partial t} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \bar{u}_n \bar{c}_n - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n \right) \]
\[ - \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left( \bar{\delta}_{ij} \bar{w}_{sn} \bar{c}_n' \right) = \sum_{n=1}^{N} \bar{S}_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left( \left( D_m + D_g \right) \frac{\partial \bar{c}_n}{\partial x_j} \right) \]

(A-165)
<table>
<thead>
<tr>
<th>Equation</th>
<th>Homogeneous case</th>
<th>Conventional turbulence decomposition</th>
<th>Mass-weighted turbulence decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Instantaneous continuity</td>
<td>$\frac{\partial \bar{u}_i}{\partial x_i} + \frac{\partial u'_i}{\partial x_i} = 0$</td>
<td>$\frac{D\bar{\rho}}{Dt} + \frac{\partial \bar{\rho} u'_j}{\partial x_j} + \frac{\partial \rho' u'_j}{\partial x_j} + \frac{\partial \rho u'_j}{\partial x_j} + \frac{\partial \rho u'}{\partial x_j} = 0$</td>
<td>$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho} u_j}{\partial x_j} + \frac{\partial \rho' u_j}{\partial x_j} + \frac{\partial \rho u_j}{\partial x_j} = 0$</td>
</tr>
<tr>
<td>Temporally averaged continuity</td>
<td>$\frac{\partial \bar{u}_i}{\partial x_i} = 0$</td>
<td>$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \bar{\rho} u'_j}{\partial x_j} + \frac{\partial \rho' u'_j}{\partial x_j} = 0$</td>
<td>$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho u'_j}{\partial x_j} = 0$</td>
</tr>
<tr>
<td>Relative density fluctuation</td>
<td>$\frac{\partial u'_i}{\partial x_i} = 0$</td>
<td>$\bar{\rho} \frac{D}{Dt} \left( \frac{\rho'}{\bar{\rho}} \right) + \frac{\partial \rho' u'_j}{\partial x_j} - \frac{\partial \rho u'_j}{\partial x_j} = 0$</td>
<td>$\bar{\rho} \frac{D}{Dt} \left( \frac{\rho'}{\bar{\rho}} \right) + \frac{\partial \rho u''_j}{\partial x_j} = 0$</td>
</tr>
<tr>
<td>Instantaneous sediment-laden</td>
<td>$\frac{\partial \bar{u}_i}{\partial x_i} + \frac{\partial u'_i}{\partial x_i} = 0$</td>
<td>$\frac{\partial (\bar{u}<em>i + u'<em>i)}{\partial x_i} + \frac{1}{\rho_s} \left{ \frac{\partial \bar{u}</em>{id} C}{\partial x_i} + \frac{\partial \bar{u}'</em>{id} C}{\partial x_i} + \frac{\partial \bar{u}<em>{id} C'}{\partial x_i} + \frac{\partial \bar{u}'</em>{id} C'}{\partial x_i} \right} = 0$</td>
<td>$\frac{\partial \bar{u}<em>i}{\partial x_i} + \frac{\partial u'<em>i}{\partial x_i} = -\frac{1}{\rho_s} \frac{\partial (\bar{u}</em>{id} - \bar{u}) C}{\partial x_i} - \frac{1}{\rho_s} \frac{\partial (u''</em>{id} - u^*) C}{\partial x_i}$</td>
</tr>
<tr>
<td>Sediment-laden averaged</td>
<td>$\frac{\partial \bar{u}_i}{\partial x_i} = 0$</td>
<td>$\frac{\partial \bar{u}<em>i}{\partial x_i} + \frac{1}{\rho_s} \left{ \frac{\partial \bar{u}</em>{id} C}{\partial x_i} + \frac{\partial \bar{u}'_{id} C'}{\partial x_i} \right} = 0$</td>
<td>$\frac{\partial \bar{u}<em>i}{\partial x_i} + \frac{\partial u'<em>i}{\partial x_i} = -\frac{1}{\rho_s} \left{ \frac{\partial \bar{u}</em>{id} C}{\partial x_i} + \frac{\partial \bar{u}'</em>{id} C}{\partial x_i} \right}$</td>
</tr>
</tbody>
</table>
Table A-2. Comparison of momentum equation forms for method of turbulence decomposition

<table>
<thead>
<tr>
<th>Equation</th>
<th>Homogeneous case</th>
<th>Conventional turbulence decomposition</th>
<th>Mass-weighted turbulence decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean viscous stresses</td>
<td>[ F_{i}^{\text{vis}} = \frac{\partial}{\partial x_j} \left{ \rho \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}<em>j}{\partial x_i} \right) - \left( \frac{2}{3} - \kappa \right) \delta</em>{ij} \rho \frac{\partial \overline{\mu}}{\partial x_k} \right} ]</td>
<td>[ F_{i}^{\text{vis}} = \frac{\partial}{\partial x_j} \left{ \rho \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) + v \rho' \frac{\partial \overline{u}_i}{\partial x_j} + v \rho' \frac{\partial \overline{u}_j}{\partial x_i} \right} ]</td>
<td>[ F_{i}^{\text{vis}} = \frac{\partial}{\partial x_j} \left{ \rho \left( \frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i} \right) + \rho \frac{\partial \overline{u}_i}{\partial x_j} + \rho \frac{\partial \overline{u}_j}{\partial x_i} \right} ]</td>
</tr>
<tr>
<td>Momentum (time-averaged)</td>
<td>[ \frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_j \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j' \overline{u}<em>i'}{\partial x_j} = - \frac{1}{\rho} \frac{\partial \rho'}{\partial x_i} + g_i + \frac{1}{\rho} F</em>{i}^{\text{vis}} ]</td>
<td>[ \frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_j \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j' \overline{u}_i'}{\partial x_j} + \frac{\partial \rho' \overline{u}_i}{\partial x_j} + \frac{\partial \rho' \overline{u}_j \overline{u}_i}{\partial x_j} + \frac{\partial \rho' \overline{u}_j' \overline{u}<em>i'}{\partial x_j} = - \frac{1}{\rho} \frac{\partial \rho'}{\partial x_i} + g_i + F</em>{i}^{\text{vis}} ]</td>
<td>[ \frac{\partial \rho' \overline{u}_i}{\partial t} + \frac{\partial \rho' \overline{u}_j \overline{u}_i}{\partial x_j} + \frac{\partial \rho' \overline{u}_j' \overline{u}<em>i'}{\partial x_j} = - \frac{1}{\rho} \frac{\partial \rho'}{\partial x_i} + \rho \frac{g_i}{\rho} + \rho' \frac{g_i}{\rho} + F</em>{i}^{\text{vis}} ]</td>
</tr>
</tbody>
</table>
Table A-2. Comparison of momentum equation forms for method of turbulence decomposition (continued)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Equation</th>
<th>Equation</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turbulent Velocity</td>
<td>Turbulent Velocity</td>
<td>Turbulent Velocity</td>
<td>Turbulent Velocity</td>
</tr>
<tr>
<td>( \frac{\partial u_i'}{\partial t} + \frac{\partial u_i u_j'}{\partial x_j} + \frac{\partial u_i u_j'}{\partial x_j} ) + ( \frac{\partial u_i u_j'}{\partial x_j} = F_i - F_j )</td>
<td>( \frac{\partial \rho u_i'}{\partial t} + \frac{\partial \rho u_i u_j'}{\partial x_j} + \frac{\partial \rho u_i u_j'}{\partial x_j} ) + ( \frac{\partial \rho u_i u_j'}{\partial x_j} = F_i - F_j )</td>
<td>( \frac{\partial \rho u_i'}{\partial t} + \frac{\partial \rho u_i u_j'}{\partial x_j} + \frac{\partial \rho u_i u_j'}{\partial x_j} ) + ( \frac{\partial \rho u_i u_j'}{\partial x_j} = F_i - F_j )</td>
<td>( \frac{\partial \rho u_i'}{\partial t} + \frac{\partial \rho u_i u_j'}{\partial x_j} + \frac{\partial \rho u_i u_j'}{\partial x_j} ) + ( \frac{\partial \rho u_i u_j'}{\partial x_j} = F_i - F_j )</td>
</tr>
<tr>
<td>Mean velocity Reynolds stresses</td>
<td>Mean velocity Reynolds stresses</td>
<td>Mean velocity Reynolds stresses</td>
<td>Mean velocity Reynolds stresses</td>
</tr>
<tr>
<td>( \frac{\partial \tau_{ij}}{\partial t} + \bar{u}<em>k \frac{\partial \tau</em>{ij}}{\partial x_k} + \frac{\partial \tau_{ij}}{\partial x_k} ) = ( -\tau_{ik} \frac{\partial \bar{u}<em>i}{\partial x_k} - \tau</em>{jk} \frac{\partial \bar{u}<em>j}{\partial x_k} - \tau</em>{ij} \frac{\partial \bar{u}_k}{\partial x_k} ) + ( u_j F_i + u_i F_j - \bar{u}_j F_i - \bar{u}_i F_j )</td>
<td>( \frac{\partial \tau_{ij}}{\partial t} + \bar{u}<em>k \frac{\partial \tau</em>{ij}}{\partial x_k} + \tau_{jk} \frac{\partial \bar{u}<em>j}{\partial x_k} + \tau</em>{ik} \frac{\partial \bar{u}<em>i}{\partial x_k} ) = ( -\tau</em>{ij} \frac{\partial \bar{u}<em>j}{\partial x_k} - \tau</em>{ik} \frac{\partial \bar{u}<em>i}{\partial x_k} + \tau</em>{jk} \frac{\partial \bar{u}_k}{\partial x_k} ) + ( u_j F_i + u_i F_j - \bar{u}_j F_i - \bar{u}_i F_j )</td>
<td>( \frac{\partial \tau_{ij}}{\partial t} + \bar{u}<em>k \frac{\partial \tau</em>{ij}}{\partial x_k} + \tau_{jk} \frac{\partial \bar{u}<em>j}{\partial x_k} + \tau</em>{ik} \frac{\partial \bar{u}<em>i}{\partial x_k} ) = ( -\tau</em>{ij} \frac{\partial \bar{u}<em>j}{\partial x_k} - \tau</em>{ik} \frac{\partial \bar{u}<em>i}{\partial x_k} + \tau</em>{jk} \frac{\partial \bar{u}_k}{\partial x_k} ) + ( u_j F_i + u_i F_j - \bar{u}_j F_i - \bar{u}_i F_j )</td>
<td>( \frac{\partial \tau_{ij}}{\partial t} + \bar{u}<em>k \frac{\partial \tau</em>{ij}}{\partial x_k} + \tau_{jk} \frac{\partial \bar{u}<em>j}{\partial x_k} + \tau</em>{ik} \frac{\partial \bar{u}<em>i}{\partial x_k} ) = ( -\tau</em>{ij} \frac{\partial \bar{u}<em>j}{\partial x_k} - \tau</em>{ik} \frac{\partial \bar{u}<em>i}{\partial x_k} + \tau</em>{jk} \frac{\partial \bar{u}_k}{\partial x_k} ) + ( u_j F_i + u_i F_j - \bar{u}_j F_i - \bar{u}_i F_j )</td>
</tr>
</tbody>
</table>
Table A-2. Comparison of momentum equation forms for method of turbulence decomposition (concluded)

<table>
<thead>
<tr>
<th>Equation</th>
<th>Homogeneous case</th>
<th>Conventional turbulence decomposition</th>
<th>Mass-weighted turbulence decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turbulent Reynolds Stresses</td>
<td>( \bar{\rho} \left[ \frac{\partial R_{ij}}{\partial t} + D_{ij} \right] = \sum_{ij} - \pi_{ij} - \varepsilon_{ij} )</td>
<td>( \bar{\rho} \left[ \frac{\partial R_{ij}}{\partial t} + D_{ij} \right] = \sum_{ij} + B_{ij} - \pi_{ij} + \delta_{ij}PD - T_{k} - T_{G} - \varepsilon_{ij} )</td>
<td>( \bar{\rho} \left[ \frac{\partial R_{ij}}{\partial t} + D_{ij} \right] = \sum_{ij} + B_{ij} - \pi_{ij} + \delta_{ij}PD - \varepsilon_{ij} )</td>
</tr>
<tr>
<td>Turbulent kinetic energy</td>
<td>( \bar{\rho} \left[ \frac{\partial k}{\partial t} + D(k) \right] = \frac{1}{2} \sum_{ii} - \pi_{ii} + PD - \bar{\rho} \varepsilon )</td>
<td>( \bar{\rho} \left[ \frac{\partial k}{\partial t} + D(k) \right] = \frac{1}{2} \sum_{ii} + \frac{1}{2} B_{ii} - \pi_{ij} + PD - T_{k} - T_{G} - \bar{\rho} \varepsilon )</td>
<td>( \bar{\rho} \left[ \frac{\partial k}{\partial t} + D(k) \right] = \frac{1}{2} \sum_{ii} + \frac{1}{2} B_{ii} - \pi_{ij} + PD - \bar{\rho} \varepsilon )</td>
</tr>
</tbody>
</table>
**Table A-3  Terms in the TKE equation**

<table>
<thead>
<tr>
<th>Term</th>
<th>Normal Incompressible development</th>
<th>Variable density development</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of TKE dissipation</td>
<td>( \nu \frac{\partial u_i'}{\partial x_i} )</td>
<td>( \nu \frac{\partial u_i'}{\partial x_i} + \frac{1}{\rho} \nu \rho' \frac{\partial u_i'}{\partial x_i} = \frac{1}{\rho} \rho e_h )</td>
</tr>
<tr>
<td>Mechanical production due to mean velocity gradients</td>
<td>(-u_j' u_i' \frac{\partial u_i}{\partial x_j} )</td>
<td>(-u_j' u_i' \frac{\partial u_i}{\partial x_j} - \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} + \rho' u_i' u_i' \frac{\partial u_i}{\partial x_j} + u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} \right) )</td>
</tr>
<tr>
<td>Turbulent diffusion of TKE</td>
<td>(-u_j' u_i' \frac{\partial u_i}{\partial x_j} )</td>
<td>(-u_j' u_i' \frac{\partial u_i}{\partial x_j} - \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} + \rho' u_i' u_i' \frac{\partial u_i}{\partial x_j} + u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} \right) )</td>
</tr>
<tr>
<td>Molecular diffusion of TKE</td>
<td>( \nu \frac{\partial^2 k}{\partial x_j^2} )</td>
<td>( \nu \frac{\partial k}{\partial x_j} )</td>
</tr>
<tr>
<td>Diffusion from pressure fluctuations</td>
<td>(- \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} )</td>
<td>(- \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} )</td>
</tr>
<tr>
<td>Temporal transformations</td>
<td>(- \frac{1}{\rho} \frac{\partial}{\partial t} \left( u_j \rho' \frac{\partial u_i}{\partial x_i} + u_i \rho' \frac{\partial u_i}{\partial x_i} + 2k \frac{\partial u_i}{\partial x_i} + \rho' u_i' \frac{\partial u_i}{\partial x_i} + u_j \frac{\partial \rho'}{\partial x_i} \right) )</td>
<td>(- \frac{1}{\rho} \frac{\partial}{\partial t} \left( u_j \rho' \frac{\partial u_i}{\partial x_i} + u_i \rho' \frac{\partial u_i}{\partial x_i} + 2k \frac{\partial u_i}{\partial x_i} + \rho' u_i' \frac{\partial u_i}{\partial x_i} + u_j \frac{\partial \rho'}{\partial x_i} \right) )</td>
</tr>
<tr>
<td>Buoyancy production</td>
<td>( \frac{1}{\rho} \frac{\partial u_i \rho g_i}{\partial t} )</td>
<td>( \frac{1}{\rho} \frac{\partial u_i \rho g_i}{\partial t} )</td>
</tr>
<tr>
<td>Mean dilatation transformation</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_i \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} \right) )</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_i \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} \right) )</td>
</tr>
<tr>
<td>Transformation from turbulent dilatation</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_i \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} \right) )</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_i \rho' u_i' \frac{\partial u_i}{\partial x_j} + u_j \rho' u_i' \frac{\partial u_i}{\partial x_j} \right) )</td>
</tr>
<tr>
<td>Production from mean density gradients</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_i \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} \right) )</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_i \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} \right) )</td>
</tr>
<tr>
<td>Transformation from turbulent density gradients</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_i \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} \right) )</td>
<td>(- \frac{1}{\rho} \left( u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_i \rho' u_i' \frac{\partial \rho}{\partial x_j} + u_j \rho' u_i' \frac{\partial \rho}{\partial x_j} \right) )</td>
</tr>
</tbody>
</table>
APPENDIX B: 
DIMENSIONLESS ANALYSIS

Dimensionless analysis defines representative scales for each of the variables in the governing equation so that the relative significance of each term in the equation can be weighed against the other terms in the equation. The intent of defining nondimensional scales is to result in scaled terms in the equation that involve groupings of nondimensional variables that approach a scale of unity. The significance of each differential term is then defined in the resulting nondimensional coefficient groups, which are comprised of the scales.

The primary instantaneous variables of interest are nondimensionalized by the following scales in Equation B-1. The underscoring of a variable with a tidal (~) will indicate that the variable is nondimensionalized.

\[
\begin{align*}
\chi_i &= \frac{x_i}{h}, \quad u_i = \frac{u_i}{U}, \quad t = \frac{t}{h/U}, \quad p = \frac{p}{\rho_f U^2}, \\
\rho &= \frac{\rho}{\rho_f}, \quad u_{id} = \frac{u_{id}}{w_s}, \quad C = \frac{C}{\rho_f},
\end{align*}
\]

(B-1)

These lead to the ensemble-averaged nondimensional variables in Equation B-2.

\[
\begin{align*}
\bar{u}_i &= \frac{\bar{u}_i}{U}, \quad \bar{p} = \frac{\bar{p}}{\rho_f U^2}, \quad \bar{u}_{id} = \frac{\bar{u}_{id}}{W_s}, \quad \bar{C} = \frac{\bar{C}}{\rho_f},
\end{align*}
\]

(B-2)

The velocity scale $U$ is assumed to be a representative magnitude appropriate to the specific case. The scaling of the differential particle velocity, $u_{id}$, is taken as a representative fall velocity of the sediment, $W_s$; for example, the Stokes fall velocity. The turbulent perturbation variables would nondimensionalize as defined in Equation B-3.

\[
\begin{align*}
u_{id}' &= \frac{u_{id}'}{W_s}, \quad u_i' = \frac{u_i'}{\sigma_{u'}}, \quad \rho' = \frac{\rho'}{\rho_f}, \quad C' = \frac{C'}{\rho_f}
\end{align*}
\]

(B-3)

These variables vanish on ensemble averaging, just as the dimensional variables vanish.

386
The turbulent velocity deviation scale $\sigma_\prime$ is assumed to be representative of the turbulent velocity fluctuations, for example, the standard deviation of the turbulent fluctuations $(\sigma_\prime = \sqrt{2k})$.

The scalar magnitude of the derivatives of a primary variable is based on the constancy of the scales as shown in Equation B-4.

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial U u_i}{\partial h x_i} = \left(\frac{U}{h}\right) \frac{\partial u_i}{\partial x_i}$$  

(B-4)

When ensemble averaging is performed on the products of fluctuating variables the time-average of the product will not necessarily scale by the product of the variable scales. For example, an appropriate scale for the term $\rho u'_i$ will not be $\sigma_\rho \sigma_\prime$, but rather an independent scale associated with the net turbulent mass flux. Call that turbulent flux $Q'_\rho$, where it is hypothesized that in general $Q'_\rho \leq \sigma_\rho \sigma_\prime$. If the terms in the product are the same variable, e.g. $u'_i u'_i$, then by definition the ensemble average will be the variance of $u'_i$. Consequently, if the two variables to be averaged are perfectly correlated, then the ensemble average is the product of the standard deviations of the two variables. If the two variables are less than perfectly correlated, then the product of standard deviations is an upper bound on the ensemble average. If they are not correlated at all, then the ensemble average will approach zero.

This is illustrated by the development of the identity for the scaling of the nondimensional turbulent correlation in Equation B-5.

$$\overline{\rho' u'} = \frac{1}{T} \int_{t_i}^{t_i+T} (\rho' u') dt = \frac{1}{T} \int_{t_i}^{t_i+T} (\sigma_\rho \sigma_\prime)(\rho' u') dt = \left(\sigma_\rho \sigma_\prime\right) \frac{1}{T} \int_{t_i}^{t_i+T} (\rho' u') dt = \left(\sigma_\rho \sigma_\prime\right) \overline{\phi' \phi}$$  

(B-5)
This would be appropriate for scaling $\overline{\rho' u'_j}$ only if $\overline{\rho' u'_i}$ is close to unity, which is the root of the problem. In a turbulent flow where the mean value of the perturbation terms is by definition zero, the mean of the product of nondimensional variables should be much less than unity.

Therefore, assuming that these scaling values for the second order ensemble averaged terms are for now unknown but unique to each term, we introduce the nondimensional scaling shown in Equation B-6.

\[
\begin{align*}
\overline{u'_{ij} C'} &= \frac{\overline{u'_{m} C'}}{Q'_{c}}, \\
\overline{\rho' u'_i} &= \frac{\overline{\rho' u'_i}}{Q'_\rho}, \\
\overline{\rho' u'_i u'_i} &= \frac{\overline{\rho' u'_i u'_i}}{Q'_m}, \\
\overline{\rho' u'_i u'_i} &= \frac{\rho' u'_i u'_i}{Q'_E}, \\
\overline{\rho' u'_i u'_i} &= \frac{\rho' u'_i u'_i}{\Lambda_0}
\end{align*}
\]  

(B-6)

Note that since $\tau_y = -\overline{\rho' u'_i u'_i} = \rho u_*^2$ it is appropriate to use $Q'_m = \rho_j u_*^2$, scaling for the turbulent momentum flux. It may be possible to extend the use of $u_*$ and select $Q'_\rho = \sigma_{\rho' u_*}$ and $Q'_E = \rho_j u_*^3$, representing the scales for turbulent mass flux and turbulent energy flux. The term $\Lambda_0$ is a representative scale value for the turbulent pressure-velocity correlation. For the context being developed here the pressure fluctuations will result from fluctuations in the density of the fluid associated primarily with the suspended sediment concentration and the turbulent velocity fluctuations.

Furthermore, when the ensemble average involves terms of mixed derivatives a key assumption must be invoked relative to the appropriate scale. That is, for example, each of the following terms are assumed to have the same appropriate nondimensional scale
\[
\frac{\partial \rho' u'_i}{\partial x_i}, \quad \rho f \frac{\partial u'_i}{\partial x_i}, \quad u'_i \frac{\partial \rho'}{\partial x_i}.
\]
These should all scale by a common scale of \( \frac{Q'_c}{h} \). This commonality of scaling for the mixed derivatives is invoked, in part, due to the ability to move the spatial derivatives outside the ensemble integral.

So the power of the above assumption is that the nondimensional scaling for the ensemble averaged terms simplifies down to the correlations among the primary variables.

**B.1 Continuity Equation**

Inserting the nondimensional equalities above into the full continuity Equation A-1 without turbulent decomposition we obtain Equation B-7.

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0
\]  
(B-7)

The scaling terms cancel each other out, leaving little additional information. However, inserting the nondimensional terms into the derived continuity equation (A-46) accounting for the impact of the presence of suspended sediment we obtain.

\[
\frac{\partial \overline{u}_i}{\partial x_i} + \left( \frac{\rho f W_s}{\rho_s U} \right) \left[ \frac{\partial \overline{u}_i}{\partial x_i} + \left( \frac{Q'_c}{W_s \rho_f} \right) \frac{\partial u'_i}{\partial x_i} \right] = 0
\]  
(B-8)

The nondimensional term \( \left( \frac{W_s}{U} \right) \) is simply the ratio between the differential velocity of the sediment relative to the fluid and the velocity of the fluid. The ratio between the fluid density and the sediment floc density appears along with the velocity ratio. The second imbedded nondimensional term \( \left( \frac{Q'_c}{W_s \rho_f} \right) \) can be interpreted as the ratio between the turbulent flux of the sediment relative to the fluid and the flux of fluid relative to the sediment.
B.2 Momentum Equation

Inserting the nondimensional variables into Equations A-56\textsubscript{i} we obtain:

\[
\left( \frac{\rho_{j}U}{h/U} \right) \frac{\partial \bar{p} \bar{u}_{i}}{\partial \bar{t}} + \left( \frac{Q_{\rho}}{h/U} \right) \frac{\partial \bar{p} \bar{u}'_{i}}{\partial \bar{t}} + \left( \frac{\rho_{j}U^{2}}{h} \right) \frac{\partial \bar{u}_{i}}{\partial \bar{x}_{j}} + \left( \frac{UQ'_{\rho}}{h} \right) \frac{\partial \bar{u}_{i} \rho' \bar{u}'_{i}}{\partial \bar{x}_{j}} + \\
+ \left( \frac{UQ'_{\rho}}{h} \right) \frac{\partial \rho' \bar{u}'_{i}}{\partial \bar{x}_{j}} + \left( \frac{\rho_{j}u_{i}^{2}}{h} \right) \frac{\partial \rho' \bar{u}'_{i}}{\partial \bar{x}_{j}} = \bar{F}_{i}
\]

where:

\[
\bar{F}_{i} = - \left( \frac{\rho_{j}U^{2}}{h} \right) \frac{\partial \bar{p}}{\partial \bar{x}_{i}} + \left( \rho_{j} \right) \bar{p} \bar{g}_{i} \\
+ \frac{\partial}{\partial \bar{x}_{j}} \left\{ v \left( \frac{\rho_{j}U}{h^{2}} \right) \bar{p} \left( \frac{\partial \bar{u}_{i}}{\partial \bar{x}_{j}} + \frac{\partial \bar{u}_{j}}{\partial \bar{x}_{i}} \right) + v \left( \frac{Q'_{\rho}}{h^{2}} \right) \left( \frac{\rho' \bar{u}'_{i}}{\partial \bar{x}_{j}} + \frac{\rho' \bar{u}'_{j}}{\partial \bar{x}_{i}} \right) \right\}
\]

B-10\textsubscript{i})

Dividing through Equations B-9\textsubscript{i} and B-10\textsubscript{i} by \( \left( \frac{\rho_{j}U^{2}}{h} \right) \) yields Equation B-11\textsubscript{i}.

\[
\frac{\partial \bar{p} \bar{u}_{i}}{\partial \bar{t}} + \frac{\partial \bar{p} \bar{u}_{j} \bar{u}_{i}}{\partial \bar{x}_{j}} + \left( \frac{Q'_{\rho}}{\rho_{j}U} \right) \left\{ \frac{\partial \rho' \bar{u}'_{i}}{\partial \bar{t}} + \frac{\partial \bar{u}_{j} \rho' \bar{u}'_{i}}{\partial \bar{x}_{j}} + \frac{\partial \rho' \bar{u}'_{j}}{\partial \bar{x}_{j}} \right\} \\
+ \left( \frac{u_{i}^{2}}{U^{2}} \right) \frac{\partial \rho' \bar{u}'_{i}}{\partial \bar{x}_{j}} = - \frac{\partial \bar{p}}{\partial \bar{x}_{i}} + \left( \frac{g_{i}h}{U^{2}} \right) \bar{p}
\]

B-11\textsubscript{i})

The nondimensional terms that results are:
\[
\frac{\nu}{hU} = \frac{1}{Re}
\]
where \(Re\) is the Reynolds number, the ratio of advective momentum transport to molecular dissipation of momentum.

\[
\frac{g h}{U^2} = \frac{1}{Fr^2}
\]
where \(Fr\) is Froude number, the ratio of inertia to gravitational acceleration.

\[
\frac{u_\ast}{U} = \Psi
\]
is the ratio of turbulent intensity to mean velocity.

\[
\frac{Q'_\rho}{\rho_f U}
\]
is the ratio of turbulent density flux to advective flux.

If the scaling for the turbulent density flux, \(Q'_\rho\), is taken to be \(\sigma_{\rho'} u_*\) then we would get

\[
\frac{Q'_\rho}{\rho_f U} = \Psi \Gamma, \text{ where } \Gamma = \frac{\sigma_{\rho'}}{\rho_f} \text{ is ratio of the density fluctuations to the fluid density.}
\]

### B.3 TKE Equation

The TKE equation introduces several additional nondimensional scales. Inserting the nondimensional terms into Equation A-105 yields Equation B-12.

\[
\left( \frac{\rho_i u_i^2 U}{h} \right) \left( \frac{\partial k}{\partial t} + \frac{\rho_i}{h} \frac{\partial k_i}{\partial x} \right) + \left( \frac{\rho_f u_f^3}{2h} \right) \frac{1}{h} \frac{\partial R_{ij}}{\partial x_j} + \left( \frac{U^2 Q'_\rho}{h} \right) \left( \frac{\partial u_i}{\partial t} + \frac{\rho_i}{h} \frac{\partial u_i}{\partial x} \right) \frac{1}{2} \frac{\partial R_{ij}}{\partial x_j} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}
\]

\[
= - \left( \frac{\rho_f u_f^2 U}{h} \right) \left( \frac{\partial R_{ij}}{\partial x} \right) \frac{\partial u_i}{\partial x_j} + \left( \frac{\Lambda_0}{h} \right) \frac{\partial u_i}{\partial x_i} \frac{\partial \rho'_i}{\partial x_j} + \left( Q'_\rho g_i \right) \frac{\partial \rho'_i}{\partial x_j} + \left( \frac{\rho_f \nu u_f^2}{h^2} \right) \frac{\partial u_i}{\partial x_i} \frac{\partial \sigma'_{ij}}{\partial x_j}
\]

Dividing through Equation B-12 by \(\frac{\rho_f u_f^2 U}{h}\) results in Equation B-13.
\[
\left\{ \frac{\partial k}{\partial t} + \frac{\partialu}{\partialx} \right\} + (\Psi) \frac{1}{2} \frac{\partial \rho}{\partialx} = \\
+ \left( \frac{1}{\Psi} \rho, \frac{Q^*}{\rho, u_*} \right) \left\{ \frac{\partial \rho}{\partial t} + \frac{1}{2} \frac{\partial \rho}{\partial \rho} + \frac{\partial \rho}{\partial \rho} \right\} = \\
= - \left\{ \frac{\rho}{\partial \rho} \right\} + \left( \frac{\Lambda_0}{U \rho, u_*} \right) \left\{ \frac{\partial \rho}{\partial \rho} \right\} + \left( \frac{1}{\Psi} \frac{Q^*}{\rho, u_*} \right)^2 \left\{ \frac{g_{h, h}}{U^2} \right\} + \left( \frac{1}{\rho, u_*} \right) \left\{ \frac{\partial \sigma^*}{\partial \rho} \right\} \\
\]

**B.4 Dissipation Equation**

The rate of dissipation of TKE, \( \varepsilon \), (Equation A-146) is scaled similarly resulting in the nondimensional Equation B-14.
\[
\begin{align*}
\frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u u) &= \nabla \cdot \left( \mu \nabla u + \frac{2}{3} \rho \nabla \Psi \right) + \frac{\partial \rho u_j}{\partial x_j} + \frac{\partial \rho u_j}{\partial x_j} \\
+ 2\nu &\left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( \frac{\partial \rho u_j}{\partial x_j} + \frac{\partial \rho u_j}{\partial x_j} \right) \\
+ 2 &\left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( \frac{1}{\Psi \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) \\
+ 2 &\left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( \frac{1}{\Psi \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) \\
+ 2 &\left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( \frac{1}{\Psi \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) \\
+ 2 &\left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( \frac{1}{\Psi \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) \\
+ 2 &\left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( 1 \frac{Q'_\rho}{\Psi^2 \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) + \left( \frac{1}{\Psi \rho U} \right) \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_j}{\partial x_j} \right) \\
\end{align*}
\]
And the RHS results for Equation B-14 are given in Equation B-15.

\[
\text{RHS} = -2(\Psi)\frac{\partial u_i'}{\partial x_m} \frac{\partial^2 p'}{\partial x_i \partial x_i} + \left(\frac{Q_{\rho}'}{\Psi \rho U F_r^{1/2}}\right) 2\nu \frac{\partial u_i'}{\partial x_m} \frac{\partial \rho'}{\partial x_i} g_i \\
+ \left(\frac{1}{R_e}\right) \left[v^2 \frac{\partial^2 e_i}{\partial x_i^2} + v \frac{\partial^2 e_i}{\partial x_j \partial x_j} + v \frac{\partial \rho}{\partial x_j} \frac{\partial e_i}{\partial x_j}\right] + 2\left(\frac{1}{\Psi R_e}\right) v \frac{\partial^2}{\partial x_m \partial x_i} \left(\frac{\partial u_i'}{\partial x_i} + \frac{\partial \Pi}{\partial x_i}\right) \rho' \frac{\partial u_i'}{\partial x_m} \\
+ \left(\frac{1}{\Psi R_e}\right) \left[2v \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_i \partial x_j} \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial \Pi}{\partial x_j}\right) + 2v \frac{\partial u_i'}{\partial x_m} \frac{\partial \rho'}{\partial x_i} \frac{\partial}{\partial x_i} \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial \Pi}{\partial x_j}\right) \right] \\
+ \left(\frac{1}{R_e}\right) \left\{ 2v \frac{\partial^2 \rho}{\partial x_m \partial x_i} \frac{\partial u_i'}{\partial x_j} \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial \Pi}{\partial x_j}\right) + 2v \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_i} \frac{\partial^2 u_j'}{\partial x_i \partial x_j} + 2v \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_i} \frac{\partial^2 u_j'}{\partial x_j \partial x_j} \right\} \\
+ 2v^2 \frac{\partial^2 u_i'}{\partial x_m \partial x_j} \frac{\partial^2 u_j'}{\partial x_i \partial x_j} - 2v \frac{\partial \rho}{\partial x_m} \frac{\partial u_i'}{\partial x_i} \frac{\partial^2 u_j'}{\partial x_i \partial x_j} - 2v \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_i \partial x_j} \frac{\partial u_j'}{\partial x_j} - 2v \frac{\partial \rho}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_i \partial x_j} \frac{\partial \Pi}{\partial x_j} \\
- \left(\frac{2}{3} - \frac{\kappa}{\mu}\right) 2\left(\frac{1}{R_e}\right) \nu \phi_j + \left(\frac{1}{\Psi \rho U}\right) \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + \left(\frac{1}{\Psi \rho U}\right) \frac{\partial^2 \rho'}{\partial x_k \partial x_k} \\
= \left(\frac{1}{\Psi \rho U}\right) \left[ \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_k \partial x_k} + \frac{\partial \rho'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \right] \frac{\partial \Pi}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \\
= \left(\frac{Q'_{\rho}}{\rho U F_r^{1/2}}\right) \left[ \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_k \partial x_k} + \frac{\partial \rho'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \right] \frac{\partial \Pi}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial \rho'}{\partial x_k} \\
= \left(\frac{Q'_{\rho}}{\rho U F_r^{1/2}}\right) \left[ \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_k \partial x_k} + \frac{\partial \rho'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \right] \frac{\partial \Pi}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial \rho'}{\partial x_k} \\
\text{(B-15)}
\]

Here, as before, if the scaling for the turbulent density flux is taken to be \(Q'_{\rho} = \sigma_{\rho u}\), then we would get \(\frac{Q'_{\rho}}{\rho U F_r^{1/2}} = \Psi \Gamma\), where \(\Gamma = \frac{\sigma_{\rho u}}{\rho_f}\) is ratio of the density fluctuations to the fluid density.

Returning to the mean momentum equation and dropping second order terms in \(\Gamma\) and \(\Psi\) leaves Equation B-16.
\[
\frac{\partial p}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} + \left( \Psi^2 \right) \frac{\partial \rho' u'_i}{\partial x_j} = - \frac{\partial \rho}{\partial x_i} + \left( \frac{1}{R_c^2} \right) \bar{\rho}_t \\
+ \frac{\partial}{\partial x_j} \left[ \left( \frac{1}{R_c} \right) \left\{ \bar{\rho}_t \left( \frac{\partial u_j}{\partial x_j} + \frac{\partial u_i}{\partial x_i} \right) - \left( \frac{2}{3} \right) \frac{\bar{\rho}}{\nu} \delta_{ij} \rho' \frac{\partial u_i}{\partial x_k} \right\} \right]
\]

(B-16)

However, for the TKE equation none of the terms can be dropped as second order.

Expressing Equation B-13 with the nondimensional bulk variables gives Equation B-17.

\[
\left\{ \bar{\rho} \frac{\partial k}{\partial t} + \bar{\rho} \frac{\partial \rho u_j}{\partial x_j} \right\} + \left( \Psi \right) \frac{1}{2} \frac{\partial R_{ij}}{\partial x_j} + \left( \frac{\Gamma}{\Psi} \right) \left\{ \bar{\rho} \frac{\partial \rho u_i}{\partial t} + \frac{1}{2} \frac{\partial \rho u_j \rho' u'_j}{\partial x_j} + \frac{1}{2} \frac{\partial \rho u_i \rho' u'_j}{\partial x_j} \right\} = 0
\]

(B-17)

If the turbulent pressure velocity correlation, \( \Lambda_0 \), is scaled by \( \rho_j u^3 \), we get Equation B-18.

\[
\left\{ \bar{\rho} \frac{\partial k}{\partial t} + \bar{\rho} \frac{\partial \rho u_j}{\partial x_j} \right\} + \left( \Psi \right) \frac{1}{2} \frac{\partial R_{ij}}{\partial x_j} + \left( \frac{\Gamma}{\Psi} \right) \left\{ \bar{\rho} \frac{\partial \rho u_i}{\partial t} + \frac{1}{2} \frac{\partial \rho u_j \rho' u'_j}{\partial x_j} + \frac{1}{2} \frac{\partial \rho u_i \rho' u'_j}{\partial x_j} \right\} = 0
\]

(B-18)

In the dissipation equation, using the scaling for the turbulent mass flux, \( Q' \), as \( \sigma, u^3 \) only a few terms drop out leaving Equations B-19 and B-20.
\[
\begin{align*}
\frac{\partial p e}{\partial t} + \frac{\partial \bar{u} \rho e}{\partial x_j} + e \left( \frac{\partial \bar{p}}{\partial t} + \frac{\partial \bar{u} \rho}{\partial x_j} \right) + 2 \nu u_{i} \frac{\partial^2 \bar{u}_{i}}{\partial x_m \partial x_n} + \frac{\partial \bar{p}}{\partial t} - \frac{\partial \bar{u}_{i}}{\partial x_j} \right) \\
+ \left\{ 2\left( \frac{\partial \bar{u}_j}{\partial t} \right) + 2 \left( \frac{\partial \bar{u}_i}{\partial x_j} \right) \right\} + 2 \left( \frac{\bar{u}_{i} \rho}{\partial t} \right) + 2 \left( \frac{\bar{u}_{i} \rho}{\partial x_j} \right) + 2 \left( \frac{\bar{u}_{i} \rho}{\partial x_j} \right) + 2 \left( \frac{\bar{u}_{i} \rho}{\partial x_j} \right) \\
+ 2 \left( \frac{\bar{u}_{i} \rho}{\partial x_j} \right) + 2 \left( \frac{\bar{u}_{i} \rho}{\partial x_j} \right) + 2 \left( \frac{\bar{u}_{i} \rho}{\partial x_j} \right) + 2 \left( \frac{\bar{u}_{i} \rho}{\partial x_j} \right)
\end{align*}
\]

\begin{align*}
\text{And on the RHS of Equation B-19 (Equation B-20) no terms can be dropped as second order.}
\end{align*}
\[ \text{RHS} = -2(\Psi)v \left( \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 p'}{\partial x_m \partial x_m} + \left( \frac{\Gamma}{\Psi F_r} \right) \right) 2v \left( \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 \rho'}{\partial x_m \partial x_m} g_i \right) \\
+ \left( \frac{1}{R_c} \right) \left[ \nu \frac{\partial^2 e_i}{\partial x_j^2} + \nu \frac{\partial^2 e_i}{\partial x_i \partial x_j} + \nu \frac{\partial \rho}{\partial x_j} \frac{\partial e}{\partial x_j} \right] + 2 \left( \frac{1}{\Psi R_c} \right) \nu \frac{\partial^2}{\partial x_m \partial x_j} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial \pi_j}{\partial x_j} \right) \rho' \frac{\partial u_i'}{\partial x_m} \\
+ \left( \frac{1}{R_c} \right) \left[ 2v \frac{\partial^2 \rho}{\partial x_m \partial x_j} \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial \pi_j}{\partial x_j} \right) + 2v \frac{\partial \rho}{\partial x_j} \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_j \partial x_j} + 2v \frac{\partial \rho}{\partial x_j} \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_j \partial x_j} \right] \\
+ \left( \frac{1}{R_c} \right) \left[ -2v \frac{\partial^2 u_i'}{\partial x_m \partial x_j} - 2v \frac{\partial \rho}{\partial x_j} \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_j \partial x_j} - 2v \frac{\partial^2 u_i'}{\partial x_m \partial x_j} \frac{\partial^2 u_i'}{\partial x_j \partial x_j} \right] \\

- \left( \frac{2}{3} - \frac{\kappa}{\mu} \right) \frac{1}{R_c} v^3 g_i + \left( \frac{\Gamma}{\Psi} \right) \left[ \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + \frac{\partial \rho}{\partial x_j} \frac{\partial u_i'}{\partial x_m} \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + \frac{\partial^2 \rho}{\partial x_m \partial x_m} \frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_k} \right] + \left( \frac{\Gamma}{\Psi} \right) \left[ \frac{\partial^2 u_i'}{\partial x_m \partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial \rho}{\partial x_j} + \frac{\partial^2 \pi_j}{\partial x_m \partial x_j} \frac{\partial u_i'}{\partial x_k} \frac{\partial \rho}{\partial x_k} \right] + \left( \frac{\Gamma}{\Psi} \right) \left[ \frac{\partial^3 u_i'}{\partial x_m \partial x_k \partial x_k} + \frac{\partial^3 \pi_j}{\partial x_m \partial x_j \partial x_j} \frac{\partial u_i'}{\partial x_k} \frac{\partial \rho}{\partial x_k} \right] + \left( \frac{\Gamma}{\Psi} \right) \left[ \frac{\partial^3 u_i'}{\partial x_m \partial x_k \partial x_k} \frac{\partial u_i'}{\partial x_k} \frac{\partial \rho}{\partial x_k} \right] \right] \right] \right) \]  

(B-20)

The standard form of the $k-\varepsilon$ model has essentially the fully derived $k$-equation intact. However, because of the complexity of the fully derived $\varepsilon$-equation, even for the homogeneous case, the standard form utilized for the conservation equation for $\varepsilon$ is essentially empirical. It takes the same general form as the $k$ equation: total derivative equals a source term plus a diffusive term and a decay term. And since the dissipation equation is empirical, the $k$ equation
is also typically developed with empirical coefficients (Speziale, 1998; Hanjalic, 2004). The general form is as shown in Equation B-21.

\[
\frac{\partial k}{\partial t} + \frac{\partial \bar{u}_j k}{\partial x_j} = \frac{\partial}{\partial x_j} \left[ v + \frac{v_T}{\sigma_k} \right] \frac{\partial k}{\partial x_j} = v_T \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} + \delta_{j3} \frac{g v_T}{\bar{p} \sigma_T} \frac{\partial \bar{p}}{\partial x_j} - \varepsilon \\
+ \delta_{j3} \left(1 - c_{3e}\right) \frac{\varepsilon}{k} \frac{g v_T}{\bar{p} \sigma_T} \frac{\partial \bar{p}}{\partial x_j} - c_{2e} \frac{\varepsilon^2}{k} 
\]

The empirical coefficients in the dissipation equation are presented in Table B-1.

<table>
<thead>
<tr>
<th>$c_\mu$</th>
<th>$c_{\varepsilon 1}$</th>
<th>$c_{\varepsilon 2}$</th>
<th>$c_{\varepsilon 3}$</th>
<th>$c_{\varepsilon 4}$</th>
<th>$\sigma_k$</th>
<th>$\sigma_\varepsilon$</th>
<th>$\sigma_\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.09</td>
<td>1.44</td>
<td>1.92</td>
<td>0.8</td>
<td>0.33</td>
<td>1.0</td>
<td>1.3</td>
<td>0.9</td>
</tr>
</tbody>
</table>

The nondimensionalized version s of the standard k-\varepsilon model are given in Equation B-22.

\[
\frac{\partial k}{\partial t} + \frac{\partial \bar{u}_j k}{\partial x_j} = \left\{ v + \frac{v_T}{\sigma_k} \right\} \frac{\partial^2 k}{\partial x_j^2} = \left\{ v_T U_h \right\} \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \frac{\partial \bar{u}_i}{\partial x_j} + \delta_{j3} \left\{ \frac{g v_T}{\sigma_T U_h^2} \right\} \frac{1}{\bar{p}} \frac{\partial \bar{p}}{\partial x_j} - \varepsilon \\
+ \delta_{j3} \left(1 - c_{3e}\right) \frac{\varepsilon}{k} \frac{g v_T}{\sigma_T U_h^2} \frac{1}{\bar{p}} \frac{\partial \bar{p}}{\partial x_j} - c_{2e} \frac{\varepsilon^2}{k} 
\]

These dimensionless numbers within the brackets in Equation B-22 are versions of a turbulent Reynolds number. The first term, $\frac{v + \frac{v_T}{\sigma_k}}{U_h}$, is the ratio of the advection of turbulent energy and turbulent diffusion of turbulent energy. The first term in the dissipation equation is
essentially the same ratio. The term $\frac{v_f U}{h u_*}$ can be expressed as $\frac{v_f U}{h u_*} \left(\frac{U}{u_*}\right)^2 = \frac{1}{R'_e} \frac{1}{\nu^2}$, where $R'_e$ is a turbulent Reynolds number, in that the turbulent eddy viscosity is used rather than the molecular viscosity. Also note that when the eddy viscosity dominates over molecular viscosity.

$$\frac{\nu + \frac{v_f}{\sigma_k}}{U h} \approx \frac{1}{R'_e}$$

The buoyancy term has led to another nondimensional number, $\frac{g v_f}{\sigma_f U u_*}$. This term can be rearranged to be $\frac{1}{R'_e} \frac{1}{\psi'}$, where $R'_e = \frac{u_h}{v_f}$. The form of Equation B-22 can be restated as Equation B-23.

$$\frac{\partial k}{\partial t} + \frac{\partial u_j k}{\partial x_j} = \left\{ \frac{1}{R'_e} \frac{\partial^2 k}{\partial x_j^2} \right\} + \left\{ \frac{1}{(R'_e)^2} \left[ \frac{\partial u_j}{\partial x_j} + \frac{\partial u_i}{\partial x_i} \right] \frac{\partial u_j}{\partial x_j} + \delta_{j3} \left[ \frac{1}{F'_e \psi'} \right] \frac{1}{\rho} \frac{\partial \rho}{\partial x_j} - \frac{\partial k}{\rho} \frac{\partial u_j}{\partial x_j} \right\}$$

(B-23)

**B.5 Sediment Transport**


$$\left(\frac{\rho_f U}{h}\right) \left\{ \frac{\partial C}{\partial t} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} u'_n \bar{C}_n \right\} + \left(\frac{Q'_e}{h}\right) \frac{\partial}{\partial x_j} \sum_{n=1}^{N} u'_n \bar{C}_n =$$

$$= \left(\frac{\rho_f U}{h}\right) \sum_{n=1}^{N} \bar{S}_n + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left\{ \left(\frac{\rho_f D_{ai}}{h^2}\right) \frac{\partial \bar{C}_n}{\partial x_j} \right\}$$

(B-24)

Dividing through Equation B-24 by $\left(\frac{U \rho_f}{h}\right)$ yields Equation B-25.
\[
\left\{ \frac{\partial \overline{C}}{\partial t} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \overline{u_{zn}c_n} \right\} + \left( \frac{Q'_{c}}{\rho_f U} \right) \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \overline{u_{zn}c_n'} = \sum_{n=1}^{N} \overline{S_n} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left\{ \left( \frac{D_m}{hU} \right) \frac{\partial \overline{c_n}}{\partial x_j} \right\} \tag{B-25} \]

If the turbulent sediment flux, \( Q'_{c} \), is scaled by \( \sigma_c u_* \), we obtain Equation B-26.

\[
\left\{ \frac{\partial \overline{C}}{\partial t} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \overline{u_{zn}c_n} \right\} + \left( \Psi \Gamma_c \right) \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \overline{u_{zn}c_n'} = \sum_{n=1}^{N} \overline{S_n} + \frac{\partial}{\partial x_j} \sum_{n=1}^{N} \left\{ \left( \frac{D_m}{hU} \right) \frac{\partial \overline{c_n}}{\partial x_j} \right\} \tag{B-26} \]

Where a new nondimensional term, \( \Gamma_c \) has been introduced which is the ratio of the turbulent sediment concentration scale and the freshwater density, \( \frac{\sigma_c}{\rho_f} \).
This appendix provides a detailed step-by-step description of the analysis steps in the development of the particle size distributions presented in Chapters 2 and 6.

A particle size distribution has been developed from the analysis of a video capture of in situ suspended cohesive flocs. The density of the flocs has been estimated from the observed settling velocity computed from Stokes' settling law. The data are tabulated below. Construct a particle size distribution plot based on the volumetric concentration (in ppm) using the distribution density based on the logarithm of the particle size.

Assume the following: the clear fluid density is $\rho_w = 1025 \text{ kg/m}^3$, the mineral density is $\rho_s = 2650 \text{ kg/m}^3$ and the spacing of the size classes in log space is uniform

$$d(\log d) = \log(d_{i+1}) - \log(d_i) = 0.075051.$$ 

What is the total concentration?

Table C-1. Input data for floc size analysis

<table>
<thead>
<tr>
<th>Equivalent spherical diameter (ESD) (microns)</th>
<th>Number of flocs per unit volume (# /m$^3$)</th>
<th>Estimated floc density (kg/m$^3$)</th>
<th>Equivalent spherical diameter (ESD) (microns)</th>
<th>Number of flocs per unit volume (# /m$^3$)</th>
<th>Estimated floc density (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_i$</td>
<td>$dN_i$</td>
<td>$\rho_f$</td>
<td>$d_i$</td>
<td>$dN_i$</td>
<td>$\rho_f$</td>
</tr>
<tr>
<td>32.830</td>
<td>0</td>
<td>219.698</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
<tr>
<td>39.023</td>
<td>0</td>
<td>261.143</td>
<td>110.058</td>
<td>18037760</td>
<td>1076.2</td>
</tr>
<tr>
<td>46.384</td>
<td>7754551</td>
<td>1092.4</td>
<td>130.820</td>
<td>13823330</td>
<td>1082.9</td>
</tr>
<tr>
<td>55.134</td>
<td>27815237</td>
<td>1179.3</td>
<td>155.498</td>
<td>10114632</td>
<td>1060.4</td>
</tr>
<tr>
<td>65.534</td>
<td>32703976</td>
<td>1184.5</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
<tr>
<td>77.897</td>
<td>32872553</td>
<td>1135.2</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
<tr>
<td>92.592</td>
<td>27478083</td>
<td>1098.0</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
<tr>
<td>110.058</td>
<td>18037760</td>
<td>1076.2</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
<tr>
<td>130.820</td>
<td>13823330</td>
<td>1082.9</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
<tr>
<td>155.498</td>
<td>10114632</td>
<td>1060.4</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
<tr>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
<td>184.831</td>
<td>5563047</td>
<td>1082.2</td>
</tr>
</tbody>
</table>
Step-by-step development: refer to Tables 1 and 2 below. This analysis was performed with Microsoft Excel.

- Columns A, B and C are the input data provided.

- Column D: The volume of the floc ESD in m$^3$. \( V = \frac{\pi d_i^3}{6} \), with \( d_i \) converted to meters

- Column E: The lower boundary diameter for this size class, taken as the diameter resulting from averaging the logarithms of the adjacent size classes as follows:

\[
d_{iL} = 10^{\frac{[\log_{10}(d_{i-1}) + \log_{10}(d_i)]}{2}}
\]

- Column F: The upper boundary diameter for this size class, taken as the diameter resulting from averaging the logarithms of the adjacent size classes. Note that the lower boundary for class \( i \) is the upper boundary for class \( i-1 \).

\[
d_{iH} = 10^{\frac{[\log_{10}(d_{i+1}) + \log_{10}(d_i)]}{2}}
\]

- Column G: The difference, in microns, between the upper and lower class boundaries.

\[
\Delta d_i = d_{iH} - d_{iL}
\]

- Column H: \( V_{low} \), the volume of the equivalent spherical diameter \( d_{iL} \), the lower bound. Use same equation as for column D above.

- Column I: \( V_{high} \), the volume of the equivalent spherical diameter \( d_{iH} \). Use same equation as for column D above.

- Column J: The differential individual particle volume spanned by this size class, \( dV_{pi} \). This is computed as \( V_{high} - V_{low} \).

- Column K: \( D_{con} \), is the differential sediment concentration within this size class. It is the product of the number of flocs (Column B) times the average floc density (Column C) times the individual floc volume (Column D).

\[
D_{con} = dN_i \rho_{fl} V_{pi} . \text{ Units} = (1/ \ m^3) \times (kg/ \ m^3) \times (m^3)
\]

- Column L: \( n_d \), the probability density for the distribution by particle diameter. The generic units of \( n_d \) are number of particles (of size \( d_i \) per unit volume of solution per unit span of the particle diameter. In this case: \#/\ m^3/micron. The value is computed from:

\[
n_d = dN_i / \Delta d_i , \text{Column B/Column G}
\]
• Column M: \( n_v \), the probability density for the distribution by particle diameter. The generic units of \( n_v \) are number of particles (of size \( d_i \)) per unit volume of solution per unit span of the particle volume. In this case: \#/m\(^3\)/m\(^3\). The value is computed from:

\[
n_v = \frac{dN_i}{dV_{pi}} \quad \text{Column B/ Column J}
\]

Note that columns L and M are not necessary to compute the volumetric distribution, but are useful in understanding the distribution.

• Column N: \( dV_i \), is the differential sediment volume within this size class. It is computed as the product of the particle number density (column B) times the volume of each floc in this class (column D). The units of \( dV_i \) are dimensionless (m\(^3\)/m\(^3\)), volume per volume.

\[
dV_i = dN_i \cdot V_{pi}
\]

• Column O: \( \frac{dV_i}{d\log d} \), This is the equivalent probability density expressed per unit span of the logarithm of the particle size rather than per span of the particle size. This value is computed as the ratio of the differential sediment volume in each class (column N) divided by the uniform differential span of each size class used to define the classes

\[
d \left( \log d \right) = \log (d_{i+1}) - \log (d_i) = 0.075051 \quad \text{as given.}
\]

\[
\frac{dV_i}{d\log (d_{pi})} = \frac{dV_i}{0.075051}
\]

The units of \( \frac{dV_i}{d\log (d_{pi})} \) are “per log micron”, or volume per volume per log micron.

• Column P: This is just converts the value in column O into units of ppm (parts per million volumetric concentration density)

Total concentration: the sum of column K provides the estimate of the total concentration in kg/m\(^3\). \( C_{tot} = 0.140 \) kg/m\(^3\)
Table C-2. Example calculations for development of the volumetric concentration.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Given</td>
<td>Given</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>d_i</td>
<td>dN_i</td>
<td>density</td>
<td>V_i</td>
<td>d_i</td>
<td>V_i</td>
<td>d_i</td>
<td>V_i</td>
<td>dV_i</td>
<td>D_con</td>
</tr>
<tr>
<td>---</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
</tr>
<tr>
<td>1</td>
<td>32.830</td>
<td>1.85E-14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>39.023</td>
<td>3.11E-14</td>
<td>35.792</td>
<td>42.544</td>
<td>6.752</td>
<td>2.40089E-14</td>
<td>4.03E-14</td>
<td>1.63116E-14</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>46.384</td>
<td>5.23E-14</td>
<td>42.544</td>
<td>50.570</td>
<td>8.026</td>
<td>4.03205E-14</td>
<td>6.77E-14</td>
<td>2.73936E-14</td>
<td>2.45844E-12</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>65.534</td>
<td>1.47E-13</td>
<td>60.110</td>
<td>71.449</td>
<td>11.339</td>
<td>1.13719E-13</td>
<td>1.91E-13</td>
<td>7.72599E-14</td>
<td>3.17077E-11</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>77.897</td>
<td>2.47E-13</td>
<td>71.449</td>
<td>84.927</td>
<td>13.478</td>
<td>1.90979E-13</td>
<td>3.21E-13</td>
<td>1.2975E-13</td>
<td>5.12977E-11</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>92.592</td>
<td>4.16E-13</td>
<td>84.927</td>
<td>100.948</td>
<td>16.021</td>
<td>3.20729E-13</td>
<td>5.39E-13</td>
<td>2.17902E-13</td>
<td>6.96544E-11</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>155.498</td>
<td>1.97E-12</td>
<td>142.626</td>
<td>169.531</td>
<td>26.905</td>
<td>1.51914E-12</td>
<td>2.55E-12</td>
<td>1.03209E-12</td>
<td>1.17285E-10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>184.831</td>
<td>3.31E-12</td>
<td>169.531</td>
<td>201.512</td>
<td>31.981</td>
<td>2.55123E-12</td>
<td>4.28E-12</td>
<td>1.73329E-12</td>
<td>1.10555E-10</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>219.698</td>
<td>5.55E-12</td>
<td>201.512</td>
<td>239.526</td>
<td>38.014</td>
<td>4.28452E-12</td>
<td>7.2E-12</td>
<td>2.91089E-12</td>
<td>6.02137E-11</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>261.143</td>
<td>9.32E-12</td>
<td>239.526</td>
<td>284.710</td>
<td>45.185</td>
<td>7.19541E-12</td>
<td>1.21E-11</td>
<td>4.88853E-12</td>
<td>7.33763E-11</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>310.405</td>
<td>1.57E-11</td>
<td>284.710</td>
<td>338.419</td>
<td>53.708</td>
<td>1.20839E-11</td>
<td>2.03E-11</td>
<td>8.20977E-12</td>
<td>4.67536E-11</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>438.562</td>
<td>4.42E-11</td>
<td>402.259</td>
<td>478.141</td>
<td>75.883</td>
<td>3.40812E-11</td>
<td>5.72E-11</td>
<td>2.31546E-11</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>521.293</td>
<td>7.42E-11</td>
<td>478.141</td>
<td>568.339</td>
<td>90.197</td>
<td>5.72357E-11</td>
<td>9.61E-11</td>
<td>3.88857E-11</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>736.519</td>
<td>2.09E-10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum</td>
<td>1.0</td>
<td></td>
<td>7.77603E-10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table C-2. Example calculations for development of the volumetric concentration (concluded)

<table>
<thead>
<tr>
<th></th>
<th>A Given</th>
<th>L Corrected</th>
<th>M Corrected</th>
<th>N</th>
<th>O</th>
<th>P</th>
<th>Q</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>d_i (microns)</td>
<td>Dcon (kg/ m^3)</td>
<td>dNi (#/m^3)</td>
<td>n_d (#/ )</td>
<td>n_v</td>
<td>dVi (m^3/m^3)</td>
<td>dV/dLOGd (1/log(μm))</td>
<td>dV/dLOGd (ppm/log(μm))</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>32.830</td>
<td>0.0000E+00</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>39.023</td>
<td>4.4264E-04</td>
<td>7754681</td>
<td>966221</td>
<td>2.8307E+20</td>
<td>4.0519E-07</td>
<td>5.3989E-06</td>
<td>5.4</td>
</tr>
<tr>
<td>3</td>
<td>55.134</td>
<td>2.8785E-03</td>
<td>27815702</td>
<td>2915758</td>
<td>6.0461E+20</td>
<td>2.4408E-06</td>
<td>3.2522E-05</td>
<td>32.5</td>
</tr>
<tr>
<td>4</td>
<td>65.534</td>
<td>5.7089E-03</td>
<td>32704522</td>
<td>2884153</td>
<td>4.2329E+20</td>
<td>4.8196E-06</td>
<td>6.4217E-05</td>
<td>64.2</td>
</tr>
<tr>
<td>5</td>
<td>77.897</td>
<td>9.2358E-03</td>
<td>32873102</td>
<td>2438935</td>
<td>2.5335E+20</td>
<td>8.1357E-06</td>
<td>0.0001084</td>
<td>108.4</td>
</tr>
<tr>
<td>6</td>
<td>92.592</td>
<td>1.2540E-02</td>
<td>27478542</td>
<td>1715150</td>
<td>1.2610E+20</td>
<td>1.1421E-05</td>
<td>0.00015218</td>
<td>152.2</td>
</tr>
<tr>
<td>7</td>
<td>110.058</td>
<td>1.3550E-02</td>
<td>18038061</td>
<td>947212</td>
<td>4.9291E+19</td>
<td>1.2591E-05</td>
<td>0.00016776</td>
<td>167.8</td>
</tr>
<tr>
<td>8</td>
<td>130.820</td>
<td>1.7548E-02</td>
<td>13823561</td>
<td>610698</td>
<td>2.2493E+19</td>
<td>1.6204E-05</td>
<td>0.00021591</td>
<td>215.9</td>
</tr>
<tr>
<td>9</td>
<td>155.498</td>
<td>2.1115E-02</td>
<td>10114801</td>
<td>375935</td>
<td>9.8001E+18</td>
<td>1.9912E-05</td>
<td>0.00026532</td>
<td>265.3</td>
</tr>
<tr>
<td>10</td>
<td>184.831</td>
<td>1.9905E-02</td>
<td>5563140</td>
<td>173950</td>
<td>3.2095E+18</td>
<td>1.8392E-05</td>
<td>0.00024507</td>
<td>245.1</td>
</tr>
<tr>
<td>11</td>
<td>219.698</td>
<td>1.0841E-02</td>
<td>1854380</td>
<td>48781</td>
<td>6.3704E+17</td>
<td>1.0296E-05</td>
<td>0.00013719</td>
<td>137.2</td>
</tr>
<tr>
<td>12</td>
<td>261.143</td>
<td>1.3211E-02</td>
<td>1348640</td>
<td>29847</td>
<td>2.7587E+17</td>
<td>1.2575E-05</td>
<td>0.00016756</td>
<td>167.6</td>
</tr>
<tr>
<td>13</td>
<td>310.405</td>
<td>8.4179E-03</td>
<td>505740</td>
<td>9416</td>
<td>6.1601E+16</td>
<td>7.9196E-06</td>
<td>0.00010552</td>
<td>105.5</td>
</tr>
<tr>
<td>14</td>
<td>368.960</td>
<td>4.6059E-03</td>
<td>168580</td>
<td>2641</td>
<td>1.2226E+16</td>
<td>4.4334E-06</td>
<td>5.9072E-05</td>
<td>59.1</td>
</tr>
<tr>
<td>15</td>
<td>438.562</td>
<td>0.0000E+00</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>521.293</td>
<td>0.0000E+00</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>619.630</td>
<td>0.0000E+00</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>736.519</td>
<td>0.0000E+00</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Sum</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.140</td>
<td></td>
</tr>
</tbody>
</table>
Figure D-1. Particle size distribution for example video analysis.

This example analysis is intended to demonstrate the steps involved in the analysis of the San Francisco Bay video imaging analysis for the determination of the particle size distribution.
APPENDIX D
OUTLINE OF KEY SEDIMENT SUBROUTINES IN COMPUTATIONAL MODEL

This appendix provides a detailed description of the computational steps and equations used in several of the key subroutines within the research computational program. The primary subroutines detailed here are those that develop the sediment properties and interaction processes. These subroutines are the implement the aggregation/disaggregation model (subroutine AGGFLUX), the development of settling velocities (subroutine FALLVEL) and the bed exchange (subroutine BEDXCHG). The general architecture of the computational program was outlined in Chapter 4. The development of the sediment transport over the simulation time involves calling each of these subroutines during each time step.

D.1 AGGFLUX

Subroutine AGGFLUX computes the mass fluxes between size classes associated with aggregation, disaggregation due to both hydrodynamic shear and floc breakage associated with particle-particle collisions. The logical flow of the subroutine is described in the following program sequence, with reference to the equations used.

The logic below is repeated for each spatial discretization over the water depth and for each cohesive sediment class.

1. The current values of the mass fluxes are stored as the values for the previous time step, and the current fluxes initialized to zero in preparation for being updated during this pas through the subroutine.

2. The following computations are performed for each cohesive size class:
   a. Collision frequencies are developed based on Equations 4-40 through 4-42, for Brownian motion, turbulent shear and differential settling. The collision frequencies are developed for all parings of size classes.
   b. If a probabilistic treatment is used for differential settling, Equation 4-42 is integrated over the two probability distributions for the two class settling velocities. This is
accomplished via a subroutine for numerical integration of the probability distributions for each possible pairing of size classes.

c. The numbers of flocs formed per second are computed based on Equation 4-38 for each pairing of size classes.

3. A check is made of the summation of all possible floc formations involving each size class, to ensure that there are sufficient numbers of flocs present to satisfy the total. Adjustments are made to all floc formation rates constrained by a criterion for time step reduction. If the adjustment fraction is less than the constraining criterion then the program stops, flagging the need to reduce the time step. The limiting factor used was 0.75.

4. Aggregation losses, \( L_{iA} \), from each of the classes involved in the floc aggregation are computed based on Equation 4-48.

5. Aggregation gains, \( G_{iA} \), to each size class are computed by Equation 4-45 and 4-46, using the distribution factors developed by Equation 4-47.

6. Floc breakage due to local shear stresses was computed based on Equation 4-53.

7. If probabilistic treatment is used, then Equation 4-53 in integrated over the two probability distributions for \( \tau \) and \( \tau_{fi} \).

8. Disaggregation breakage loss flux, \( L_{iB} \), is then computed using Equation 4-52 for each size class.

9. Disaggregation breakage gains, \( G_{iB} \), are summed from the distribution of the remnants of the breakage losses, using a two-fragment assumption with masses 3/16 and 13/16 of the original floc. This is given in Equation 4-54.

10. Collision related disaggregation is developed from the interparticle shearing computed from Equation 4-55.

11. The collision efficiency of floc breakage is computed from Equation 4-61, with either the mean value form for classical treatment or with an integrated probabilistic treatment of the two shear stress distributions.

12. Collision disaggregation losses, \( L_{iC} \), are estimated based on Equation 4-62.

13. Collision dissaggregation gains, \( G_{iC} \), are developed by summing up the redistribution of the fragments of the disaggregation losses, using the two-fragment 3/16-13/16 redistribution of the floc masses.

All of the aggregation flux terms (gains and losses) become contributions to the governing equations for each sediment size class. Those equations are solved in subroutines SEDCOH and SEDCOHG for the cohesive sediment classes and the tagged sediment classes, respectively.
D.2 FALLVEL

Subroutine FALLVEL computes the settling velocities for every cell over the water depth and for every sediment size class, both cohesive and noncohesive, for each model time step. The requirement each time step comes from the evolution of the sediment concentration field, which in turn affects the flocculation and hindered settling effects on the settling velocities.

The computational steps in the subroutine are:

1. Settling velocities for the current time level are stored as the old time level.
2. The free settling velocities for all particle classes (both cohesive and noncohesive) are solved iteratively using Equations 2-25 and 2-26.
3. The exponents, $n_f(d)$, from Equations 2-44 and 2-46 are calculated by solving Equation 2-44 to yield
   \[
   n_f(d) = \frac{\ln \left( \frac{w_g(d)}{w_{s\text{max}}} \right)}{\ln \left( \frac{C_f}{C_h} \right)}
   \]
   The variables $w_{s\text{max}}$, $C_f$ and $C_h$ are specified as input, allowing for the calculation of the exponent for each cohesive size class.
4. If $C > C_f$, then the effects of concentration and internal shear on the setting velocities for each cohesive size class are computed based on Equation 2-46.
5. If $C \leq C_f$ the concentration and shear corrected settling velocity remains $w_{sf}$.
6. The effects of hindered settling are then applied to the concentration and shear corrected cohesive settling velocities based on Equation 2-40. This application is applied without condition on concentration since at low concentrations the correction vanishes explicitly. This correction is applied for each size class for every computational cell in the water column.
7. The effects of hindered settling are then also applied to the concentration and shear corrected noncohesive free settling velocities based on Equation 2-40.
D.3 BEDXCHG

The primary treatment of the focus of the research here is developed in subroutine BEDXCHG, where the exchange fluxes for erosion (EROS and EROSG) and deposition (DEP and DEPG) are developed for each size class, for both tagged and untagged cohesive sediments. The silt bed fluxes (ENTR and SETS) are also computed in the subroutine for each silt size class. These flux terms are then used in the governing equations for each size class as source and sink terms for the bottom computational cell. The units of the erosion terms are kg m\(^{-2}\) s\(^{-1}\), which is the actual erosive flux. The units on the deposition terms are m/s, since these variables are multiplied by the concentration in the bottom computational cell to define the deposition flux.

The bed exchange is specified by two general options. The first is the decision of whether to treat the primary variables as mean values or probabilistically. The second option is whether to treat erosion and deposition as exclusive or allow for simultaneous erosion and deposition.

For sensitivity testing and analysis, the option for either no deposition and/or no erosion can be invoked.

The logical flow of subroutine BEDXCHG can be summarized as follows.

1. The current bed exchange variables are assigned to the values for the previous time step, and the current fluxes initialized to zero.

2. For each cohesive size class, \(i\), the following computations are executed for the mean value option.

   a. For the case when \(\tau_b < \tau_{el}\),

      i. If \(\tau_b < \tau_{di}\), then the probability of deposition is estimated as \(P_{Di} = 1 - \frac{\tau_b}{\tau_{di}}\).

         Otherwise, \(P_{Di} = 0\)

      ii. If the option for simultaneous erosion and deposition is used, then set \(P_{Di} = 1\)

      iii. If the option for no deposition is chosen, set \(P_{Di} = 0\)
iv. For the case when the shear stress is below the deposition threshold, set the erosion flux to zero \((\text{EROS}_i = 0; \text{EROSG}_i = 0)\).

v. Set the deposition variable, \(\text{DEP}_i = P D w_{sc}\), where \(w_{sc}\) is the aggregation and hindered settling adjusted fall velocity for the bottom cell for sediment class \(i\) estimated from Equations 2-26, 2-40 and 2-46

b. When \(\tau_b > \tau_{ei}\),

i. The probability of erosion is estimated as \(P_{ei} = \frac{\tau_b}{\tau_{ei}} - 1\)

ii. The erosion rate for this size class is then estimated as \(\varepsilon_i = C_b c u_i f_i P_{ei}\), based on the empirical coefficient, \(C_b\) (kg/m³), the shear velocity and the fraction of the total bed mass associated with this size class, \(f_i\).

iii. If the option of no erosion is invoked, then set \(\varepsilon_i = 0\).

iv. The only difference between the application of the erosion logic for tagged and untagged cohesive sediment of the same size class is use of the appropriate fraction of the bed mass.

v. \(\text{EROS} = \varepsilon_i\). \(\text{EROSG} = \varepsilon_{Gi}\), the erosion rates for the untagged and tagged sediment.

3. For each cohesive size class, \(i\), the following computations are executed for the probabilistic option.

a. The probability of deposition is calculated from the two probability curves for \(\tau_b\) and \(\tau_{di}\), based on Equation 4-76 for \(P_{Di} = P (\tau_b < \tau_{di})\).

b. If applying simultaneous erosion and deposition, then set \(P_{Di} = 1\)

c. If the option for no deposition is chosen, set \(P_{Di} = 0\)

d. Set the deposition variable, \(\text{DEP}_i = P_{Di} w_{si}\), where \(w_{si}\) is the aggregation and hindered settling adjusted fall velocity for the bottom cell for sediment class \(i\) estimated from Equations 2-26, 2-40 and 2-46

e. Calculate the erosion potential integral, \(E_i\), for this size class from Equation 4-33, which incorporates the shear stress dependent time response inside the probability integral.

f. The erosion for the untagged and tagged sediments, if being simulated, differs only by the relative fractions of the class concentrations in the bed.

g. For the option of no erosion, set the erosion flux to zero \((\text{EROS}_i = 0; \text{EROSG}_i = 0)\).
The exclusive paradigm for the probabilistic treatment involves the use of the probability of deposition in modifying the deposition flux. For the simultaneous paradigm the probability of deposition is set to 1.

4. The silt critical shear stresses for mobility, $\tau_{csi}$, are determined by size class based on Equation 4-55.

5. For the use of mean variables the silt bed exchange logic is:

   a. If the bottom shear stress is less than the threshold shear stress ($\tau_b < \tau_{csi}$) then:
      
      i. The probability of deposition is computed as: $P_{Di} = 1 - \frac{\tau_b}{\tau_{csi}}$
      
      ii. If applying simultaneous erosion and deposition, then set $P_{Di} = 1$
      
      iii. If the option for no deposition is chosen, set $P_{Di} = 0$
      
      iv. The deposition velocity is defined as $SETS = w_{si} P_{Di}$. The silt settling velocity $w_{sn}$ is corrected for hindered settling.
      
      v. The erosion flux is set to zero.

   b. If the bed shear stress is larger than the threshold shear stress ($\tau_b > \tau_{csi}$) then:
      
      i. The normalized excess shear stress is computed as: $P_{ei} = \frac{\tau_b}{\tau_{csi}} - 1$
      
      ii. If the simultaneous option for erosion and deposition is selected then the deposition velocity is set to the settling velocity for this silt class, with a correction for hindered settling. ($SETS = w_{si}$)
      
      iii. If the exclusive erosion or deposition paradigm is selected, set $SETS = 0$.
      
      iv. The erosion flux is calculated as $ENTR = C_{bs} w_{si} f_i P_{ei}$

   c. For a probabilistic treatment for silt size classes:
      
      i. The probability of deposition is calculated from a numerical integration of the two probability curves to yield $P_{Di} = P (\tau_b < \tau_{csi})$ based on Equation 4-76.
      
      ii. If applying simultaneous erosion and deposition, then set $P_{Di} = 1$
      
      iii. If the option for no deposition is chosen, set $P_{Di} = 0$
      
      iv. The deposition velocity is defined as: $SETS = w_{si} P_{Di}$.
v. The expected value of the normalized excess shear stress, $P_{e_{i}}$, is computed as the numerical integration of the double integral of the excess shear stress using the probability distribution functions for the bottom shear stress and the critical shear stress for mobility of this silt class. (Equation 4-76)

vi. The erosion flux is computed as: $\text{ENTR} = C_{bs} w_{si} f_{i} P_{e_{i}}$

6. The erosion flux terms are checked against the mass of material for each size class in the bed plus the depositional flux for the size class for the case of simultaneous erosion and deposition. If the computed flux exceeds the available sediment it is limited to that mass divided by the time step.

   a. $\text{EROS}(i) = \text{MIN}\{\text{EROS}(i), \text{mass in bed of class } i/\Delta t + \text{DEP}(i)\}$.

   b. $\text{EROSEG}(i) = \text{MIN}\{\text{EROSEG}(i), \text{mass in bed of class } i/\Delta t + \text{DEPG}(i)\}$.

   c. $\text{ENTR}(i) = \text{MIN}\{\text{ENTR}(i), \text{mass in bed of class } i/\Delta t + \text{SETS}(i)\}$

The primary results from subroutine BEDXCHG are the bed exchange fluxes for each cohesive class $i$ from 1 to MCLASS, the number of cohesive size classes, DEP($i$), DEPG($i$), EROS($i$), and EROSG($i$), and the silt exchange fluxes, SETS($j$) and ENTR($j$), with $j$ from 1 to NSILT, the number of silt size classes.
LIST OF REFERENCES


Ariathurai, R., R. C. MacArthur, and R. B. Krone, 1977: A finite element model for sediment transport in estuaries. U. S. Army Engineer Waterways Experiment Station, Vicksburg, MS.


Dent, G. L., 1999: Aspects of particle sedimentation in dilute flows at finite Reynolds numbers, Ph.D. dissertation, Brown University, Providence, RI.


Sanford, L. P. and J. P. Halka, 1993: Assessing the paradigm of mutually exclusive erosion and deposition of mud, with examples from upper Chesapeake Bay. *Marine Geology, 114,* 37-57.


——, 2000: Underflow spreading from an open-pipe line disposal. ERDC TN-DOER-N7, U. S. Army Engineer Research and Development Center, Vicksburg, MS, 9p.


U. S. Army Corps of Engineers, 1963: Typical major tidal hydraulic problems in United States and research sponsored by the Corps of Engineers Committee on Tidal Hydraulics. Technical Bulletin No. 6, Committee on Tidal Hydraulics, Vicksburg, MS, 35p.


BIOGRAPHICAL SKETCH

Joseph V. Letter, Jr. was born in Gainesville, Florida to Joseph V. Letter, Sr. and Helen Emery Letter, when his father was a student at the University of Florida. His maternal grandfather, Paul B. Emery, was an employee of the University of Florida for several decades as a technician in the Botany Department. He grew up in Jacksonville, Lake Worth, Tampa and, Miami, Florida. He graduated from Miami Norland High School in 1968. He graduated from Georgia Institute of Technology in 1972 with a Bachelor of Civil Engineering. He received a Master of Engineering degree in 1973 from the University of Florida in Coastal and Oceanographic Engineering. On March 16, 1974, he married Linda Marie Arlotta, the love of his life. He extended his graduate studies in 1979 through 1980 at the University of Miami in Ocean Engineering. He returned to the University of Florida in 2000 to continue graduate studies in Civil and Coastal Engineering.

He has been employed by the U. S. Army Engineer Research and Development Center (formerly known as the U.S. Army Corps of Engineers Waterways Experiment Station) in Vicksburg, Mississippi since 1974. He is currently Group Leader, Long Waves Group, Estuarine Engineering Branch, Coastal and Hydraulics Laboratory.