To my mom, dad and my wife
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NUMERICAL MODELING OF WAVES AND WAVE-INDUCED CURRENTS IN THE NEARSHORE ZONE

By

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Ocean waves and wave-induced currents are two of the most important components of nearshore hydrodynamics. This study focuses on transformations of waves and wave-induced currents in the nearshore area through numerical modeling efforts. The first portion of this dissertation documents the development of a reduced wave spectral model, which describes the evolution equations of wave moments instead of individual wave spectral components as in the full spectral wave models (e.g., SWAN and WAVEWATCH III). This wave model is computationally much cheaper than full spectral models and more accurate than monochromatic-based wave models for wave spectra with finite wave frequency and directional bandwidths, but relatively compact shapes with clear peak directions and frequencies. The model can provide information on some of the most important wave properties including RMS wave height, mean frequency, mean direction, and frequency and directional spreading. Based on the wave model, formulations for the radiation stresses including the effects of the frequency and directional spreading are derived. The radiation stress components can be overestimated or underestimated by over 50% with the monochromatic approximation when waves have finite frequency and directional bandwidths. The simulation results showed that the model simulates shoaling and
refraction of irregular waves with Gaussian-shaped spectra very well for simple bathymetries, and reasonably well for complex bathymetries such as planar beach incised with periodic rip channels. Following that, a 2-D finite volume steady circulation model based on the nonlinear shallow water equations is developed using the pressure-correction solution method and validated by comparing to analytical solutions and to the available benchmark solutions, and the circulation model is then coupled with the reduced wave spectral model. Using the coupled models, wave-induced steady longshore currents are simulated over planar beach bathymetries. For the approximately longshore uniform barred beach at DUCK, both the wave field and the longshore currents are accurately simulated by the models compared against the DUCK94 data. To simulate the time domain solution of nearshore currents, an unsteady version extended from the steady circulation model is developed. Very low frequency motions, shear waves or shear instabilities of longshore currents, are modeled and compared with results of a linear stability analysis. The steady-state wave-induced rip currents and circulations are simulated and the relative importance of various wave and current parameters is evaluated. Numerical results suggest that the wave-current interaction produces a negative forcing on the wave forcing, hence decreasing the rip current intensity. Effects of the wave frequency and directional spreading on mean properties of wave-induced longshore currents and rip currents are studied, and the simulation results suggest that the wave frequency and especially directional bandwidths have significant impacts on both the mean and the instability properties of the longshore currents. For rip currents, however, the impacts are much less significant because refraction and breaking are the dominant processes in this case. Finally, numerical experiments on rip currents show that wave-current interaction has significant impacts on the unsteady flow and the instabilities are sensitive to a small variation range of the angle of wave incidence.
CHAPTER 1
INTRODUCTION

1.1 General Introduction

In the twentieth century, coastal regions have become one of the most important areas on the earth for their economic vitality and crowded population (Bookman et al., 1999). About half the U.S. population now resides within 50 miles of the coastline and the coastal population continues to grow (Dean and Dalrymple, 2002). More recently, recreation and tourism at beaches have become more important economically. During the past several decades, coastal engineering has received increasing attention with the objectives of obtaining a better understanding of natural coastal processes and developing more effective coastal protection (NRC 1999).

Ocean waves and wave-induced currents are two of the most important components of nearshore hydrodynamics. Waves can be seen in all sizes and forms almost in every body of water open to the atmosphere, depending on magnitude of the forces acting on the water (Dean and Dalrymple, 1998). Ocean waves usually consist of wind waves and swell waves. Wind waves are generated locally by wind blowing over the water surface. Swell waves are seas that have moved away from the area in which they were formed. Swell can be waves from a distant storm or combination of waves from different storm systems. As waves propagate from deep water to shallow water, wave transformations, such as shoaling, refraction, diffraction and dissipation, could take place. As water continues to become shallower, waves eventually become unstable and break at some depth, dissipating energy in the form of turbulence in the surf zone (Lippmann et al., 1996). The breaking of waves transfers momentum into the water column, thus generating currents (Longuet-Higgins and Stewart, 1964). Depending on the incident wave angle and the specific bathymetry, the wave-induced currents can be either alongshore currents (oblique incident waves on beaches without major alongshore variations) or nearshore
circulation cells (shore-normal incidence over complex bathymetries). Since turbulent fluctuations in the surf zone lift many sediment particles from the seabed and mix them with the fluid, nearshore currents will carry sediments and drive sediment transport, and eventually shape the evolution of the beach.

Historically, a tremendous effort has been invested in predicting nearshore waves and wave-induced currents. A brief literature review in simulation of nearshore waves and wave-induced nearshore currents is given in next section. The motivations and objectives of this study are described in Section 1.3.

1.2 Background

1.2.1 Simulation of Nearshore Waves

Ocean waves have significant impacts on human activities, which has inspired scientists, researchers and engineers to study and to understand them. Water wave theory has a very long history that has been over one and half centuries, and many water wave theories have been developed. Two categories can be roughly classified for all these wave theories: phase resolving wave models, and spectral wave models. The mild-slope wave models and Boussinesq wave models both belong to the phase resolving wave model. Spectral wave models are based on wave action conservation and treat the wave field as a stochastic phenomenon, belonging to the phased-averaged model category.

To reduce the complexity, monochromatic waves (waves that only have one frequency and one direction) have been traditionally assumed in developing wave theories in early years. A number of monochromatic wave theories have been developed, and these include the relatively lately developed mild-slope wave models. Berkhoff (1972) first introduced the mild-slope equation, which assumes a slowly varying bathymetry and can be used to study the effects of
combined refraction-diffraction. Since its debut, the mild-slope equation has been widely used in simulating wave transformations over complex bathymetries and received a great amount of efforts to improve its performance and to extend the range of its validity. These efforts include developing the parabolic mild-slope equations which are much easily to be solved than the original elliptic equation (Radder, 1979; Booij, 1981; Kirby, 1986), accounting for the effect of ambient currents (Kirby and Dalrymple, 1984), including the effects of the nonlinearity of waves (Kirby and Dalrymple, 1986; Hedge, 1987; Li, et al., 2003). The monochromatic wave models based on mild-slope equation have been developed and some of them are widely used now such as REF/DIF1 (Kirby and Dalrymple, 1994). Like the mild-slope equations, the Boussinesq-type equations describe evolution of surface elevation and Boussinesq wave models belong to phase resolving model too. Since the introduction of standard Boussinesq equations for variable water depth by Peregrine (1967), tremendous effort has been devoted to the development of extended equations with improved the dispersion and nonlinearity properties (e.g. Madsen, et al., 1991, Nwogu, 1993, Wei et al., 1995, Gobbi et al., 2000). Numerical models based on these equations have been developed and have been shown to give good predictions in comparison with field data when applied within their range of validity. One popular Boussinesq wave model is FUNWAVE, which was developed by Kirby et al. (1998).

Monochromatic wave theories provide insights about the natures of waves and build the theoretical foundation for studying the real irregular ocean waves. In practice, ocean engineers have often utilized the monochromatic wave models to investigate the transformation of ocean waves by using a representative monochromatic wave to approximate the irregular waves. However, many researchers have shown that such an approximation may result in significant errors due to vast dissimilarity in the refraction-diffraction patterns from monochromatic and
irregular wave fields even if the wave energy spectra are narrowly distributed (Goda, 1985; Vincent and Briggs, 1989; Panchang et al., 1990).

Ocean waves have irregular wave heights and periods and the sea surface is continuously varying, which make it infeasible to use a deterministic approach to describe sea surface. On the other hand, statistical properties of the surface like mean wave height, mean wave periods and directions appear to vary slowly in time and space compared to typical wave periods and wave lengths. The surface displacement in the ocean can be seen as the sum of many harmonic waves that are statistically independent. Since the Fourier series approach was applied to random waves, the statistical theory of ocean waves has developed rapidly in past decades. In spectral wave models, waves generally are described with direction-frequency (or other forms) variance spectrum \( F(f, \theta, \bar{x}, t) \), which characterizes the statistical properties of sea surface, or wave action \( N(f, \theta, \bar{x}, t) = F / \sigma \). Evolution of wave spectrum is described by the spectral action balance equation. Several such models are currently popular and have been applied to open oceans, shelf seas and nearshore regions, including WAM (WAMDI, 1988), WAVEWATCH III (Tolman, 1991; Tolman and Chalikov, 1996), SWAN (Holthuijsen et al., 1993; Ris et al., 1994; Booij et al., 1999; Ris et al., 1999).

### 1.2.2 Simulation of Nearshore Wave-induced Currents

Waves become unstable and begin to break as they propagate into shallow nearshore areas, dissipating energy in the form of turbulence in the surf zone. Water piles up onshore due to breaking, leading an increase of the mean water level (setup). This gradient in mean water level is balanced by the gradient of the excess flow of momentum due to the presences of waves, i.e. radiation stress. The gradients in surface wave level and momentum fluxes give rise to the generation of wave-induced nearshore circulation. Two typical wave-driven flow patterns are
alongshore currents (Longuet-Higgins, 1970, among many others) and rip currents (Shepard et al., 1941, among many others), see Figure 1-1 and 1-2. Alongshore currents are often generated by obliquely incident waves breaking on a relatively alongshore uniform beach. Cell-like circulations can occur when waves are nearly at normal incidence. Often described as narrow, strong seaward-directed currents that extend from the inner surf zone out through the breaking line, rip currents are integral parts of these circulation cells, fed by the longshore currents converging nearly at the shoreline.

The introduction of the concept of radiation stress by Longuet-Higgins and Stewart (1964) has made it possible to simulate wave-induced currents numerically. In past decades, a variety of current models have been developed and used to study the wave-induced currents in coastal areas.

Early models simultaneously developed by Longuet-Higgins (1970), Bowen (1969) and Thornton (1970) were reduced to one dimension and used to predict the wave-induced alongshore currents. Thornton and Guza (1986) developed a model that takes into account irregular waves, and successfully reproduced the measured alongshore current profile on a planar beach. For alongshore uniform barred beaches, models (Church and Thornton, 1993; Feddersen et al., 1998) predicted longshore current profiles with maxima not over the bar crests, which differ from the observations. Ruessink et al. (2001) demonstrated that including the rollers in the wave forcing greatly improved the predictions.

Waves (nearly) normally incident on alongshore non-uniform bathymetries typically leads to two-dimensional horizontal flow pattern, i.e., cellular circulation or rip current. A variety of two-dimensional models have been developed in past decades, early studies including Bowen (1969), Noda (1972), Liu and Mei (1976), and Ebersole and Dalrymple (1980), among others. These models are highly approximated for simplicity, for example, neglecting the nonlinear
convective terms and/or neglecting wave-current interaction. More recently, more sophisticated modeling effects have been reported. Haas et al. (1998) conducted numerical simulation of a laboratory rip current system and found that the offshore extension of rip currents can be significantly reduced by considering wave-current interaction. Chen et al. (1999) utilized Boussinesq equations to simulate the same laboratory set up. Yu and Slinn (2003) conducted a numerical study of rip currents on a barred beach with gentle sinusoidal alongshore variations. Studies of 3-D variations of nearshore currents has also been conducted by many researchers in past years, including Van Dongeren et al. (1994), Putrevu and Svendsen (1999), Haas et al. (2003), and Newberger and Allen (2007), among others.

1.3 Study Motivations and Objectives

Currently, transformations of ocean waves in coastal areas are simulated either using the full spectral wave models or using the monochromatic wave models. Traditional spectral wave models have full directional and spectral description of the waves from offshore to onshore and do not need a priori assumption of the spectral shape. However, in spectral wave models the wave spectrum needs to be solved in spectral space, besides in the temporal and spatial space, which makes them very time-consuming. The complexity of these models, meanwhile, makes them difficult to use.

Recently, methods using monochromatic wave models to compute directional spectral sea state, based on discretization of the offshore wave spectrum into individual direction-frequency components, have been developed. Theoretical background of this kind of methods can refer to Izumiya and Horikawa (1987). Numerical models based on this idea have been developed including Panchang et al (1990) and Grassaa (1990), among others. Chawla et al (1998) using this approach extended the monochromatic wave model REF/DIF to a spectral version. In these
models, the individual wave components are computed separately using monochromatic model, and the stochastic characteristics of the spectrum are obtained by assembling the wave components by linear superposition. This type of methods can give good results in certain circumstances, however, it is unable to predict wave-wave interaction due to the assumption that individual wave components propagate independently. Besides, intuitively, one can tell this approach is not computationally efficient because it needs to go through the whole computation process for every single wave component.

Many beaches are dominated by narrow-banded waves or swell waves. In this case, using the full spectral wave model is expensive and unnecessary if a reduced spectral wave model, which could be order of magnitudes faster and also gives quite good results, is available. So it is worthy of efforts to develop a relatively simple spectral wave model as an alternative of the traditional spectral models under certain circumstances. One of main purposes of this study is to develop a reduced wave spectral model describing the evolution of wave moments instead of wave action, which is the fundamental variable in the full spectral wave models. Extended from the concept of moments of a frequency spectrum (Holthuijsen, 2007, among others), the general wave moment \( E_n(x,t) \) could be defined by:

\[
E_n(x,t) = \int_0^{\infty} \int_{-\pi}^{\pi} w_n(\sigma, \theta) F(x,t; \sigma, \theta) d\sigma d\theta
\]

where \( w_n(\sigma, \theta) \) are designated weighting functions, and the subscript \( n \) is the index of weighting function and wave moment. With the definition of wave moment, the evolution equations of wave moments can be obtained by integrating the standard wave action balance equation multiplied by the weighting functions \( w_n(\sigma, \theta) \) over angular frequencies and directions. Compared with conventional spectral wave models, in the present model, the dependent
variables are wave moments, and any wave moment $E_n(x,t)$ is only a function of time and geographical coordinates.

As discussed above, a moments-based wave model is an approximation to the full spectral wave models, but it could be computationally much cheaper. To solve the wave action balance equation (the basic equation used in full spectral wave models), at every computational location and time the wave spectrum is usually discretized into tens of frequency bins and direction directional bins. For instance, 30 bins in frequency and 36 bins in direction are typical discretizations, which will result in 1080 components. Combined with dimensions in space and time, it is very computationally expensive to solve the wave spectrum completely. Since moments of a wave spectrum are independent from the specific energy distribution over frequencies and directions, equations of wave moments do not need to be solved in the spectral domain. Instead, several equations of wave moments will be solved only in space and time (for unsteady problems). Therefore, within its applicative range, a moments-based wave model that gives sufficiently accurate results with much less computational effort is very useful. On the other hand, the moments-based wave model solves wave moments, which can contain information on other important wave parameters than the RMS wave height, mean frequency, and mean wave angle provided by using the monochromatic-based models, for example, wave frequency bandwidth and directional bandwidth. Consequently, the moments-based model is more accurate that the monochromatic-based models. In addition, the monochromatic approximation tends to do a very poor job in calculating the wave radiation stresses (see Section 2.3.3.4, chapter 2), especially when the wave spectrum have finite frequency bandwidth and directional spreading. When incorporated with a circulation model as the wave driver, the moments-based model can give a much better estimate of wave radiation stresses.
As for modeling nearshore wave-induced currents, most of current models are time-dependent and developed by finite difference methods, and some of them are very computer intensive and require very small time steps to reach steady-state solutions. For a stationary wave field, a steady wave-induced current field usually results. In this case, a steady circulation model that is robust and computationally much less time consuming will be very useful in practical sense. Up to date, to our knowledge steady current models exclusively developed for nearshore wave-induced circulation are not widely available. Ruessink (2001) developed a 1-D steady model to study the wave-induced longshore currents. For complex bathymetries, a 2-D model is definitely necessary. In this study, a 2-D steady circulation model based on the nonlinear shallow water equation is developed using the finite volume method. The model used the pressure-correction method solving the governing equations, which is suitable for steady flow problems, and is robust and efficient.

Another main goal of this study is utilize the newly developed wave model and circulation model as research tools to investigate various aspects of nearshore waves and wave-induced current. Objectives of this research can be summarized as follows

- To develop and validate the moments-based spectral wave model
- To develop and validate a robust and efficient steady 2-D nearshore circulation model
- Using the wave model to investigate the effects of wave frequency and directional bandwidth on wave transformations and on the wave radiation stresses
- Incorporate the two models to reproduce the wave-induced alongshore current data, to investigate the effects of wave frequency and directional bandwidths on wave-induced longshore currents and rip currents
- To develop a unsteady circulation model to investigate the very low frequency motions of nearshore currents, including shear instabilities of alongshore currents and rip current instability
- To investigate effects of wave-current interaction on waves as well as on rip currents
Figure 1-1. Wave-induced longshore currents.

Figure 1-2. Rip currents.
CHAPTER 2
REDUCED SPECTRAL WAVE MODEL

2.1 Introduction

Tremendous effort has been devoted to nearshore wave modeling. Two types of wave models can be roughly classified: the phase-resolving models and phase-averaged models (spectral models). The former models include the two well known and intensely investigated categories: mild-slope wave model and Boussinesq wave model. The spectral wave models are developed based on wave action conservation.

In the early years of nearshore wave modeling, the coastal engineers typically approximated the offshore irregular sea state by a representative monochromatic (e.g., significant) wave. Great achievements in developing the monochromatic-base wave models have been earned in past years, and many coastal and ocean engineering projects have benefited greatly by these achievements. However, many researchers such as Goda (1985) (using an analytical approach), Vincent and Briggs (1989) (by experimental study), and Panchang et al. (1990) (by numerical study) have shown that such an approximation may result in significant errors even if the wave energy spectra are narrowly distributed.

In the past several decades, extensive research efforts have been invested in developing spectral wave models and several models are now available and widely used including WAM (WAMDI, 1988), WAVEWATCH III (Tolman, 1991; Tolman and Chalikov, 1996), and SWAN (Holthuijsen et al., 1993; Ris et al., 1994; Booij et al., 1999; Ris et al., 1999). These models entail full directional and spectral description of the ocean waves and do not need a priori assumption of the spectral shape. These models are doing very good jobs in predicting the sea state either in open oceans or in coastal areas. The complexity of these models, meanwhile, can make them difficult to use and generally time consuming.
Many beaches are dominated by narrow-banded wave spectra or swell waves. In this case, a full spectral wave model may be unnecessary if a simplified spectral wave model, which could be order of magnitudes faster and also gives quite good results, is available. In this study, evolution equations of several wave moments of the wave energy density spectrum for wave spectra with relative compact shape with a clear peak frequency and direction are derived. Based on these equations, a reduced spectral model is developed, which gives better results than the monochromatic-based models by accounting for the directional and frequency bandwidth of a wave spectrum. For traditional spectral wave models, the wave spectrum (or wave action) needs to be solved in spectral space, besides in the temporal and spatial space. On the other hand, the model in this study does not need solve the wave spectrum in spectral space, instead only several wave moments need to be solved. Consequently, the model is computationally more efficient in comparison with spectral wave models.

2.2 Literature Review of Wave Models

2.2.1 Phase Resolving Wave Models

Propagation of water waves over irregular bathymetries involves many processes, including shoaling, refraction, diffraction and energy dissipation. Airy wave theory includes the phenomena of shoaling, refraction and energy dissipation, but not wave diffraction. The combined refraction-diffraction equation, which includes diffraction and refraction phenomena explicitly, is introduced first by Berkhoff (1972). It is also known as mild-slope equation since in deriving the equation it was assumed that the bottom slopes are mild. Radder (1979) developed a parabolic mild-slope equation, which is easy to deal with the down wave boundary and to be solved numerically too. However, the parabolic model is restricted to waves propagating within $45^\circ$ of the assumed direction. More recently, Booij (1981) and Kirby (1986) have improved the
parabolic model and extended the range of validity of the model. The combined refraction-diffraction wave model REF/DIF is developed based on the parabolic mild-slope equation (Kirby 1986). Chawla et al. (1998) suggested that random sea state can be discretized into wave components and wave components are calculated simultaneously using the parabolic wave model, and the statistical properties can then be evaluated afterwards.

Based on the non-dispersive linear long wave theory, Boussinesq-type wave equations have been developed including the lowest order effects of nonlinearity and frequency dispersion. Boussinesq wave equations have provided a sound basis for studying wave transformation in coastal regions since their debut. Peregrine (1967) derived the Boussinesq equations for variable water depth for the first time using the depth-averaged velocities as the independent variables, which are called the standard Boussinesq equations. The standard Boussinesq equations only work well for shallow water areas and weak nonlinearity cases due to the assumption of weak dispersion and weak nonlinearity. Numerous efforts have been invested to improve the dispersion and nonlinearity properties of the standard Boussinesq equations. Madsen et al. (1991), Madsen and Sørensen (1992) and Nwogu (1993), among others, have derived extended forms of Boussinesq equations. Madsen et al improved the dispersion property by introducing additional small value terms, which are governed by the constraint of obtaining the best possible linear dispersion relation. Nwogu used a different method of derivation: the velocity at certain water depth was used as the dependent variable instead of the depth averaged velocity, and the choice of the representative depth was again determined by the goal of obtaining the best dispersion relation. Both of their equations have been proved to be valid for water depths from relatively deep to shallow water. To enable the Boussinesq equations to simulate strong nonlinearity, Wei et al. (1995) derived the fully nonlinear Boussinesq equations by adapting the approach of
Nwogu (1993). The equations retain the full representation of fluid kinematics in nonlinear surface boundary condition terms by not weak nonlinearity. Gobbi et al. (2000) further derived a new set of equations by using a new dependent variable which is defined as a weighted average of the velocity at two distinct water depths.

2.2.2 Spectral Wave Model

2.2.2.1 Wave Spectrum

The sea surface elevation at one location can be described as superposition of many wave trains:

\[ \eta(t) = \sum_i a_i \cos(\sigma_i t + \epsilon_i) \]  

(2-1)

where \( \eta \) is the sea surface elevation; \( a_i, \sigma_i \) and \( \epsilon_i \) are the amplitude, angular frequency and phase of the \( i \)th wave component respectively. Ocean waves are chaotic and in practice it is impossible to specify the sea state in complete detail. One usually uses a statistical description, in which the probability of finding a particular sea state is considered. A good approximate description is provided by the covariance function of the sea surface elevation.

\[ <\eta(t), \eta(t + \tau)> \]  

(2-2)

The Fourier transform of the covariance function is defined as the wave spectrum, or the variance density spectrum

\[ F(f) = \int_{-\infty}^{+\infty} <\eta(t), \eta(t + \tau)> e^{-2\pi j f \tau} d\tau \]  

(2-3)

The physical meaning of wave spectrum is the density function specifying the distribution of the variance of the sea-surface elevation over the frequencies of the harmonic components. Also, the integral of wave spectrum over all wave components is proportional to total wave
energy per unit area. From Equation (2-3), the variance of sea surface elevation can be given by (Holthuijsen, 2007):

\[ \langle \eta^2 \rangle = \int_0^\infty F(\omega) d\omega \] (2-4)

and the potential wave energy per unit surface area can be written as

\[ E = \frac{1}{2} \rho g \langle \eta^2 \rangle \] (2-5)

In cases of no ambient currents, the energy (variance) of a wave package is a conserved quantity. With the presence of currents, the energy or variance of a spectral component is no longer conserved since there will be some mean momentum transfer of waves done by currents (Longuet-Higgins and Stewart, 1961, 1962). Fortunately, the wave action density \( A = E / \sigma \) (\( \sigma \) is the intrinsic angular frequency) is still conserved (Whitham, 1965). Therefore, the wave action density spectrum \( N = F / \sigma \) is usually chosen as the fundamental quantity to be solved in spectral wave models.

When the random sea-elevation is treated as a stationary, Gaussian process, then all its statistical characteristics can be expressed in terms of moments of the variance density spectrum \( F(\omega) \) (Holthuijsen, 2007):

\[ m_n = \int_0^\infty \omega^n F(\omega) d\omega, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots \] (2-6)

Obviously, the zeroth-order moment is equal to the variance:

\[ m_0 = \int_0^\infty F d\omega = \langle \eta^2 \rangle \] (2-7)

Compared with Equation (2-5), it is clear that the zeroth moment of a wave spectrum is simply proportional to wave energy.
2.2.2.2 Evolution of Wave Action Balance Equation

Wave energy density spectrum and wave action density spectrum have been introduced above. Willebrand (1975) derived the evolution equation of wave action for variable water depth and ambient currents. There is an alternative way to derive the wave action balance equation. In this section, we first briefly introduce the derivation by following Willebrand, followed by using the alternative method.

It was noted by Willebrand (1975) that the conservation of wave action holds for every single wave component separately

$$\frac{\partial}{\partial t} N_n + \nabla x \cdot \left[ \nabla \left( \Omega_n \right) N_n \right] = 0$$

(2-8)

where, $N_n(x,t) = 2|a_n|^2 / \sigma_n$. $\nabla_x$ is the horizontal gradient operator. $\Omega_n$ is the absolute frequency, which has the following form

$$\Omega_n = \sigma(k_n) + k_n \cdot u$$

(2-9)

Wave number vector and wave frequency are linked by conservation of waves and by the dispersion relation too:

$$\frac{\partial}{\partial t} k_n + \nabla \Omega_n = 0$$

(2-10)

$$\sigma_n^2 = g k_n \tanh k_n h$$

(2-11)

Willebrand (1975) argued that the action density could be introduced in the form:

$$A(k,x,t) dk \cong \sum_n d^k N_n(x,t)$$

(2-12)

where the superscript $dk$ on the summation sign indicates that the sum is taken only over values of $n$ for which $k_n(x,t)$ lies in the range $dk$ around the fixed wavenumber $k$. Instead of the
index \( n \), the initial values \( P_n = k_n(x, 0) \) of the wavenumber vectors are used to identify different wave components:

\[
N_n = N(P_n, x, t) \\
k_n = k(P_n, x, t)
\]  

(2-13) 

(2-14)

Without losing generality, assume \( P_n \) is independent of \( x \) and then Equation (2-12) is equivalent to:

\[
A(k_n, x, t) d\mathbf{k} = A(k_n, x, t) D(P_n, x, t) dP = N(P_n, x, t) dP
\]  

(2-15)

where, \( dP \) is a volume element in \( P \) space corresponding to \( d\mathbf{k} \) in \( \mathbf{k} \) space, and

\[
D(P_n, x, t) = \frac{\partial k_n}{\partial P_n}
\]  

is the Jacobian of Equation (2-14). From Equation (2-15),

\[
N_n = A(k_n, x, t) D(P_n, x, t)
\]  

(2-16)

The derivative of Equation (2-10) with respect to \( P_n \) gives:

\[
\frac{\partial}{\partial t} D(P_n, x, t) + \nabla_x \left[ \frac{\partial \Omega_{n, x}}{\partial k_n} D(P_n, x, t) \right] = 0
\]  

(2-17)

Substituting Equation (2-16) into (2-8), combined with Equation (2-17), it follows that

\[
\frac{\partial}{\partial t} A(k_n, x, t) + \frac{\partial \Omega_{n, x}}{\partial k_n} \cdot \nabla_x A(k_n, x, t) = 0
\]  

(2-18)

It should be noted that the derivatives with respect to \( x \) and \( t \) are taken for fixed value of \( P_n \).

Introducing derivatives for fixed \( k_n(x, t) \), Equation (2-18) can be further written as

\[
\left( \frac{\partial A}{\partial t} \right)_{k_n} + \frac{\partial \Omega_{n, x}}{\partial k_n} \cdot \nabla_x A + \frac{\partial}{\partial k_n} \left( \frac{\partial k_n}{\partial t} + \frac{\partial \Omega_{n, x}}{\partial k_n} \cdot \nabla_x k_n \right) = 0
\]  

(2-19)

Again from Equation (2-10), it can be shown that
\[
\frac{\partial \mathbf{k}}{\partial t} + \frac{\partial \Omega}{\partial \mathbf{k}} \cdot \nabla \cdot \mathbf{k} = -\left( \nabla \cdot \Omega \right)_{k_x} \tag{2-20}
\]

Substituting Equation (2-20) into (2-19) and dropping indices, we finally obtain

\[
\frac{\partial A}{\partial t} + \nabla_k \Omega \cdot \nabla \cdot A - \nabla_k A \cdot \nabla \Omega = 0 \tag{2-21}
\]

Using the following trivial identity

\[
\nabla_x \cdot (\nabla_k \Omega) = \nabla_k \cdot (\nabla \Omega) \tag{2-22}
\]

Equation (2-21) can be rewritten in the conservative form:

\[
\frac{\partial}{\partial t} \left( \frac{F(\mathbf{k}, \mathbf{x}, t)}{\sigma} \right) + \nabla_x \cdot \left[ \left( \mathbf{c_x} + \mathbf{u} \right) \frac{F(\mathbf{k}, \mathbf{x}, t)}{\sigma} \right] - \nabla_k \cdot \left( \nabla \Omega \right) \frac{F(\mathbf{k}, \mathbf{x}, t)}{\sigma} = 0 \tag{2-23}
\]

This form conserves the total wave action and also holds in other spectral coordinates, for example in term of \( k, \theta \).

An alternative way, which may be more straight and easier to understand, to derive the wave balance equation is given as follows. For simplicity, the 1-D equation including ambient currents is presented and extension to 2-D is straightforward. Starting with the 1-D wave energy balance equation,

\[
\frac{\partial}{\partial t} N + \frac{\partial}{\partial x} \left[ \left( c_x + u \right) N \right] = 0 \tag{2-24}
\]

in which, \( N = E / \sigma \) is the wave action. \( E = \frac{1}{8} \rho g H_{rms}^2 \) is the total wave energy. For a wave spectrum, \( N \) can be regarded as

\[
N = \sum_{k=0}^{\infty} S(k) \Delta k \tag{2-25}
\]
where $S(k) = F(k)/\sigma$ is the wave action density spectrum. It should be pointed out that $S(k) = S[x, t, k(x, t)]$ and special care is needed when computing $\partial S / \partial t$ and $\partial S / \partial x$.

Substituting Equation (2-25) into (2-24), we arrive at

$$
\frac{\partial}{\partial t} \left[ \sum S(k) \Delta k \right] + \frac{\partial}{\partial x} \left[ (c_g + u) \sum S(k) \Delta k \right] = 0 \quad (2-26)
$$

Thus for any individual component

$$
\frac{\partial}{\partial t} [S(k) \Delta k] + \frac{\partial}{\partial x} [(c_g + u) S(k) \Delta k] = 0 \quad (2-27)
$$

Expanding the first term in Equation (2-27)

$$
\frac{\partial}{\partial t} [S(k) \Delta k] = S(k) \frac{\partial \Delta k}{\partial t} + \Delta k \frac{\partial}{\partial t} S(k) + \Delta k \frac{\partial S(k)}{\partial k} \frac{\partial k}{\partial t} \quad (2-28)
$$

The second term can be rewritten as

$$
\frac{\partial}{\partial x} [(c_g + u) S(k) \Delta k] = S(k) \Delta k \frac{\partial}{\partial x} (c_g + u) + (c_g + u) \frac{\partial}{\partial x} [S(k) \Delta k] \quad (2-29)
$$

In which,

$$
\frac{\partial}{\partial x} [S(k) \Delta k] = \Delta k \left[ \frac{\partial}{\partial x} S(k) + \frac{\partial S(k)}{\partial k} \frac{\partial k}{\partial x} \right] + S(k) \frac{\partial \Delta k}{\partial x} \quad (2-30)
$$

Plugging Equations (2-28)-(2-30) into (2-27), it gives

$$
\frac{\partial}{\partial t} [S(k) \Delta k] + \frac{\partial}{\partial x} [(c_g + u) S(k) \Delta k] = S(k) \Delta k \frac{\partial}{\partial t} + \Delta k \frac{\partial}{\partial t} S(k) + \Delta k \frac{\partial S(k)}{\partial k} \frac{\partial k}{\partial t} + S(k) \Delta k \frac{\partial}{\partial x} (c_g + u) + (c_g + u) \Delta k \frac{\partial S(k)}{\partial k} \frac{\partial k}{\partial x} + (c_g + u) S(k) \frac{\partial \Delta k}{\partial x} \quad (2-31)
$$

In which,

$$
S(k) \frac{\partial \Delta k}{\partial t} + (c_g + u) S(k) \frac{\partial \Delta k}{\partial x} = S(k) \left[ \frac{\partial \Delta k}{\partial t} + \frac{\partial}{\partial x} [\Delta k (c_g + u)] \right] - \Delta k \frac{\partial}{\partial x} (c_g + u) \quad (2-32)
$$
According to conservation of waves, the underlined terms in Equation (2-32) is equal to 0.

So, substituting Equation (2-32) back into (2-31) and divided by $\Delta k$, we arrive at

$$\frac{\partial}{\partial t} S(k) + \frac{\partial}{\partial x} \left[ (c_g + u) S(k) \right] - \left[ S(k) \frac{\partial}{\partial x} (c_g + u) - (c_g + u) \frac{\partial k}{\partial x} \frac{\partial S(k)}{\partial k} \frac{\partial k}{\partial t} \right] = 0 \quad (2-33)$$

Since the target equation we are aiming at is

$$\frac{\partial}{\partial t} S(k) + \frac{\partial}{\partial x} \left[ (c_g + u) S(k) \right] - \frac{\partial}{\partial k} \left[ (k \frac{\partial u}{\partial x} + \frac{\partial \sigma}{\partial h} \frac{\partial h}{\partial x}) S(k) \right] = 0 \quad (2-34)$$

Now the problem becomes to prove that the last term in Equation (2-33) is equal to the last term in Equation (2-34). It can be shown that

$$\frac{\partial}{\partial k} \left[ \left( k \frac{\partial u}{\partial x} + \frac{\partial \sigma}{\partial h} \frac{\partial h}{\partial x} \right) S(k) \right] = S(k) \frac{\partial}{\partial x} (c_g + u) - (c_g + u) \frac{\partial k}{\partial x} \frac{\partial S(k)}{\partial k} \frac{\partial k}{\partial t} (k u + \sigma) \quad (2-35)$$

Using the conservation of waves law again, Equation (2-35) can be rewritten as

$$\frac{\partial}{\partial k} \left[ \left( k \frac{\partial u}{\partial k} + \frac{\partial \sigma}{\partial h} \frac{\partial h}{\partial k} \right) S(k) \right] = S(k) \frac{\partial}{\partial x} (c_g + u) - (c_g + u) \frac{\partial k}{\partial k} \frac{\partial S(k)}{\partial k} \frac{\partial k}{\partial t} \quad (2-36)$$

Substituting Equation (2-36) into (2-34), we obtain the 1-D wave action balance equation.

Extension to 2-D equation is straightforward but lengthy, and the detailed derivation will not be reproduced here.

In practice, several ‘modes’ of wave spectrum have been used. The spectrum $F(f, \theta, x, t)$ is used in WAM model (WAMDI, 1988). The wavenumber-direction spectrum $F(k, \theta, x, t)$ is used in WAVEWATCH-III (1999). The spectrum $F(\sigma, \theta, x, t)$ distributing wave energy over radial frequencies $\sigma$ and directions $\theta$ is used in SWAN model. The different modes of wave spectrum can be switched using straightforward Jacobean transformations:

$$F(f, \theta) = \frac{\partial k}{\partial f} F(k, \theta) = \frac{2\pi}{c_g} F(k, \theta) \quad \text{and}$$
\[ F(\sigma, \theta) = \frac{\partial k}{\partial \sigma} F(k, \theta) = \frac{1}{c_g} F(k, \theta) \]  

(2-37)

The advantage of the wavenumber-direction spectrum \( F(k, \theta) \) is its invariance characteristics with respect to physics of wave growth and decay for variable water depths. The more traditional direction-frequency spectrum \( F(\sigma, \theta) \) or \( F(f, \theta) \) is physically easy to understand and may be more convenient to deal with in the numerical computation processes.

### 2.3 Development of the Reduced Spectral Wave Model

In previous sections, the definition of wave spectrum and the physical meaning of the wave spectrum have been discussed. Two methods to derive the wave action balance equation were given. Wave spectra in different spectral coordinates were briefly reviewed and relations between different wave spectrum forms were discussed. In this section, we will discuss the incentive to develop a moments-based reduced spectral wave model and the underlying idea for the model. The detailed derivation of governing equations of the model will be shown, followed by several numerical tests.

#### 2.3.1 Introduction to Wave Spectral Moment Equations

The wave action balance equation is the basic equation for all traditional spectral wave models. In this study, it will be the starting point for developing the reduced spectral model. As mentioned above, various modes of wave spectrum have been used as dependent variables in spectral models. The spectrum \( F(\sigma, \theta, x, t) \) distributing wave energy over radial frequencies \( \sigma \) and directions \( \theta \) will be used in this study. As discussed above, wave action density \( N(x, t; \sigma, \theta) = F / \sigma \) is the always conserved quantity rather than the energy density \( F \). The conservation of wave action equation can be written as (Hasselmann et al., 1973)
\[
\frac{\partial}{\partial t} \left( \frac{F}{\sigma} \right) + \nabla \cdot \left[ (c_\sigma + U) \frac{F}{\sigma} \right] + \frac{\partial}{\partial \sigma} \left( c_\sigma \frac{F}{\sigma} \right) + \frac{\partial}{\partial \theta} \left( c_\theta \frac{F}{\sigma} \right) = \frac{S_{\text{tot}}}{\sigma} \tag{2-38}
\]

\[
c_\sigma = \frac{\partial \sigma}{\partial h} (U \cdot \nabla h) - c_s k \cdot \frac{\partial U}{\partial s} \tag{2-39}
\]

\[
c_\theta = -\frac{1}{k} \left( \frac{\partial \sigma}{\partial h} \frac{\partial h}{\partial m} + k \cdot \frac{\partial U}{\partial m} \right) \tag{2-40}
\]

where \( \nabla \) is two-dimensional geographical gradient operator; \( c_\sigma \) and \( c_\theta \) are propagation velocities in spectral spaces \( \sigma \)-space and \( \theta \)-space, respectively. \( h \) is water depth, \( s \) is the coordinate in the direction of \( \theta \) and \( m \) is the coordinate perpendicular to \( s \). \( k \) is wave number vector and \( k \) the modulus of \( k \). \( U = (u,v) \) is the current velocity vector. In Equation (2-38), the second term on left hand side describes the propagation of wave energy in two-dimensional geographical space. The third term stands for the effects of shifting of the intrinsic frequency due variations in water depth and currents. The forth term represents depth-induced and current-induced wave refraction. The right hand side term \( S_{\text{tot}} \) is the source/sink term which presents all physical processes of wave generation, dissipation, nonlinear wave-wave interaction.

To solve the wave action balance equation numerically, the continuous wave spectrum \( F(\sigma, \theta) \) is always approximated at a finite number of discrete frequencies and directions. In addition, the evolution equation has to be discretized into computational spatial grids to be solved numerically. Thus, a tradition spectral model is actually a four dimensional model besides time step marching for unsteady cases. Obviously, it is computationally heavily laden, especially when accuracy is required which means high resolutions both in spatial and spectral spaces are necessary. The tremendous computational costs may be greatly reduced if full discretization of the wave spectrum into spectral space can be avoided. This actually is able to be achieved by looking at the evolution of moments of wave spectrum instead of wave spectrum itself.
With the concept of moments of a frequency spectrum discussed in Section 2.2.2.1, for any frequency-direction spectrum \( F(x,t;\sigma,\theta) \), wave moment \( E_l(x,t) \) corresponding to an arbitrary weighting function \( w_l(\sigma,\theta) \) could be expressed as

\[
E_l(x,t) = \int_0^{+\infty} \int_{-\pi}^{\pi} w_l(\sigma,\theta) F(x,t;\sigma,\theta) \, d\sigma \, d\theta
\]

(2-41)

where, the subscript \( l \) is the index of weighting function and wave moment. With the definition of wave moment, the evolution equations of wave moments can be obtained by integrating the standard wave action balance equation multiplied by the weighting functions \( w_l(\sigma,\theta) \) over angular frequencies and directions. The dependent variables in the derived equations are wave moments, and any wave moment \( E_l(x,t) \) is only a function of time and geographical coordinates. Therefore, the governing equations do not need to be solved in spectral domain and consequently the computational burden can be greatly relieved.

There may be many possible forms one can choose for the weighting function. A good weighting function should be easy to treat mathematically, and more importantly the resulting wave moments can provide information about the most important wave parameters such as the RMS wave height, mean frequency, mean wave angle, and wave frequency and directional spreading etc. In this study, the weighting function \( \sigma^n e^{im\theta} \) is used and the corresponding moment \( E_{n,m} \) is given by

\[
E_{n,m}(x,t) = \int_0^{+\infty} \int_{-\pi}^{\pi} \sigma^n e^{im\theta} F(x,t;\sigma,\theta) \, d\sigma \, d\theta
\]

(2-42)

where, \( n,m = 0,1,2... \) are the indices of moments. It should be noticed that when \( n = 1 \) and \( m = 0 \), Equation (2-42) reduces to the definition of \( E_{0,0} \) which is the variance of surface elevation \( <\eta^2> \) as discussed in Section 2.2.2. With the definition of wave moments given by Equation (2-
42), the mean angular frequency \( \sigma_m \) is given by \( \left( \frac{E_{i,0}}{E_{0,0}} \right) \), which is less dependent on high-frequency noise of the wave spectrum compared to the mean zero-crossing period (Holthuijsen, 2007). According to the definition proposed by Longuet-Higgins (1975), the frequency bandwidth parameter is given by \( S_\sigma = \left( \frac{E_{0,0}E_{2,0}}{E_{i,0}^2} - 1 \right)^{1/2} \). Mean wave direction and directional bandwidth can be computed using moments \( E_{0,m} \), and their specific formulations will be given in late sections.

To derive the \( E_{n,m} \) moment equation, we start with the wave action balance equation. Multiplying Equation (2-38) by \( \sigma^{n+1} e^{im\theta} \) and then putting it inside all the derivatives by using the product rule of derivative, but dropping the right hand side source term temporarily for convenience, we will get

\[
\frac{\partial}{\partial t} \left( \sigma^n e^{im\theta} F \right) + \nabla \left[ \left( \mathbf{e}_g + \mathbf{U} \right) \sigma^n e^{im\theta} F \right] + \frac{\partial}{\partial \sigma} \left( c_\sigma \sigma^n e^{im\theta} F \right) + \frac{\partial}{\partial \theta} \left( c_\theta \sigma^n e^{im\theta} F \right) - (n+1)c_\sigma \sigma^{n-1} e^{im\theta} F - imc_\sigma \sigma^n e^{im\theta} F = 0
\]  

(2-43)

Integrating Equation (2-43) over frequencies and directions, combined with the definition of moment \( E_{n,m} \), gives:

\[
\frac{\partial}{\partial t} E_{n,m} + \nabla \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \mathbf{e}_g + \mathbf{U} \right) \sigma^n e^{im\theta} F d\sigma d\theta + \int_{-\pi}^{\pi} \frac{\partial}{\partial \sigma} \left( c_\sigma \sigma^n e^{im\theta} F \right) d\sigma d\theta 
+ \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} \left( c_\theta \sigma^n e^{im\theta} F \right) d\sigma d\theta - (n+1) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} c_\sigma \sigma^{n-1} e^{im\theta} F d\sigma d\theta = 0
\]  

(2-44)

Physically, since \( F(\sigma = 0) = 0 \) and \( F(\sigma = \infty) = 0 \) for sufficiently small \( n \)

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \sigma} \left( c_\sigma \sigma^n e^{im\theta} F \right) d\sigma d\theta = 0
\]  

(2-45)
\[
\int_{0}^{+\infty} \int_{-\pi}^{\pi} \frac{\partial}{\partial \theta} \left( c_\theta \sigma^n e^{im\theta} F \right) d\sigma d\theta = 0
\]  

(2-46)

The wave spectrum moment equation for \( E_{n,m} \) is reduced to

\[
\frac{\partial}{\partial t} E_{n,m} + \nabla \cdot \int_{0}^{+\infty} \int_{-\pi}^{\pi} \left( c_u + U \right) \sigma^n e^{im\theta} F d\sigma d\theta - (n + 1) \int_{0}^{+\infty} \int_{-\pi}^{\pi} c_\sigma \sigma^{n-1} e^{im\theta} F d\sigma d\theta 
- im \int_{0}^{+\infty} \int_{-\pi}^{\pi} c_\sigma \sigma^n e^{im\theta} F d\sigma d\theta = 0
\]  

(2-47)

This equation, actually, is not truly a closed system since wave spectrum \( F(x,t;\sigma,\theta) \) is still in the equation. If we can find a way to approximate the three integrals in Equation (2-47) in terms of moment \( E_{n,m} \) and/or moments of other orders, a system of evolution of moments of spectrum can then be established. Details in establishment of such a system will be described in next section, including the closure assumptions that will be made and methods used to evaluate these integrals.

### 2.3.2 Governing Equations of the Reduced Wave Spectral Model

Theoretically, the moments of wave spectrum can be taken to any order as necessary. However, the values of higher-order moments are rather sensitive to noise in the high-frequency range of the spectrum (Holthuijsen, 2007) and problems with convergence may be encountered for too many moments involved. Generally, more information can be obtained from more moments. At the same time, however, more moments means a more complicated system. So in practice, the number of moments used should be determined by the specific wave spectrum and by the accuracy requirement of problem interested. In this study, a system involving five moments, \( E_{0,0}, E_{1,0}, E_{2,0}, (E_{0,1})^R, \) and \( (E_{0,1})^I \), is developed. This five-parameter system is chosen on purpose because it is simple but still contains information on: wave RMS height, mean period, mean direction, frequency bandwidth and directional bandwidth. The system itself can be used to
replace the traditional spectral wave models to some degree, especially when well-defined wave spectra involved, such as narrow-banded Gaussian-type spectra. What is even more important is that this system can serve as a foundation for a more comprehensive model that is able to simulate accurately the realistic wave spectra.

Referring to Equation (2-47), it is straightforward to give the governing equations of the five parameter system as follows

\[
\frac{\partial}{\partial t} E_{00} + \nabla \cdot \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} (c_x + U)Fd\sigma d\theta - \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{c_x}{\sigma} Fd\sigma d\theta = 0
\]

(2-48)

\[
\frac{\partial}{\partial t} E_{10} + \nabla \cdot \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sigma (c_x + U)Fd\sigma d\theta - 2\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} c_\Phi Fd\sigma d\theta = 0
\]

(2-49)

\[
\frac{\partial}{\partial t} E_{20} + \nabla \cdot \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sigma^2 (c_x + U)Fd\sigma d\theta - 3\int_{-\pi}^{\pi} \sigma \sigma c_\Phi Fd\sigma d\theta = 0
\]

(2-50)

\[
\frac{\partial}{\partial t} E_{01} + \nabla \cdot \int_{-\pi}^{\pi} e^{i\theta} (c_x + U)Fd\sigma d\theta - \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i\theta} \frac{c_x}{\sigma} Fd\sigma d\theta - 2\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{i\theta} c_\Phi Fd\sigma d\theta = 0
\]

(2-51)

It should be noted that Equation (2-51) is a complex equation and consists of two equations: the real equation and an imaginary equation. It is a five parameter system with five equations and seems to be a closed system. However, we still need to eliminate the $F(\mathbf{x}, t; \sigma, \theta)$ from the equations and to make them only in terms of wave moments.

As we all know, wave group velocity $c_g$ is a function of wave frequency and direction:

\[c_g = (\cos \theta, \sin \theta) c_g (\sigma, h)\]. Ambient current is a known quantity and only a function of geographical location and time: $U = U(x, t)$. Propagation velocities in spectral spaces are functions of coordinates of all dimensions: $c_\sigma = c_\sigma (\sigma, \theta, x, t), c_\phi = c_\phi (\sigma, \theta, x, t)$. Now the problem becomes how those integrals in the governing equations can be expressed in terms of wave moments. Essentially, once this problem is addressed, the system is closed.
2.3.2.1 Closure Assumptions and Evaluation of Integrals

To close the system introduced above, the integrals have to be able to be expressed in terms of wave moments. In this section, we will list the assumptions required and give detailed description of evaluation of the integrals appearing in the wave moment equations. The first assumption needs to be made is as follows

- **Assumption 1**: frequency-direction spectrum \( F(\sigma, \theta) \) is separable and can be written as \( F(\sigma, \theta) = E_{0,0} M(\sigma) D(\theta) \).

where \( E_{0,0} = H_{rms}^2 / 8 \) is the total wave energy. \( M(\sigma) \) and \( D(\theta) \) are the frequency spectrum and directional spectrum, respectively. They are normalized so that \( \int_0^{\infty} M(\sigma) d\sigma = 1 \) and \( \int_{-\pi}^{\pi} D(\theta) d\theta = 1 \). The real ocean wave spectra are rarely separable, but this provides a springboard to evaluate the unsolved integrals.

Now, we look at the integral over frequency \( \sigma \) first. According to the definition of moments of the wave spectrum density, it is clear that the integrands of integrals in Equation (2-47) should be able to be expressed as polynomials of the frequency \( \sigma \) such that the integrals are possible to be evaluated in terms of moments. However, neither of integrands \( c_g, c_\sigma, c_\theta \) is a polynomial of \( \sigma \). The simplest way to address this problem may be using the Taylor Series approximation, specifically, taking the Taylor expansion of each velocity about the mean frequency \( \sigma_m \). For example, the group velocity \( c_g \) can be approximated by

\[
 c_g(\sigma) = \left( c_g \right)_{\sigma_m} + \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_m} (\sigma - \sigma_m) + \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_m} (\sigma - \sigma_m)^2 + \cdots
\]

(2-52)
Substituting this approximated $c_g$ into an integral, for instance, \[ \int_0^{+\infty} c_g \sigma^a M(\sigma) d\sigma \], the integral now can be evaluated in terms of $E_{n,0}$, $E_{n+1,0}$, \ldots, etc. As known, using a higher order of Taylor series is more accurate. However, it also results in more moments involved and may make the system very difficult to solve. After a negotiation between the two conflicting factors, the Taylor expansion up to 2nd order will be used in this study. To avoid significant errors introduced by using the Taylor expansion approximation, the frequency spectra are required to be narrow-banded. As discussed above, besides the significant wave height, mean frequency and mean direction, the directional bandwidth and frequency bandwidth may be the other most important wave parameters highly interested in ocean wave simulations. The frequency bandwidth can be defined as $S_\sigma = \left(1 - E_{1,0}^2 / E_{0,0} E_{2,0}\right)^{1/2}$, a larger value of $S_\sigma$ corresponding to a broader frequency spectrum. Combined with the Taylor expansion approximation, the integral over frequency can be translated into terms of wave moments for a precisely defined spectrum such as a Gaussian-distributed spectrum, JONSWAP spectrum and the PM spectrum, among others. It is straightforward to calculate the mean frequency and frequency bandwidth if a Gaussian-shaped spectrum is assumed owing to its formulation:

\[
M(\sigma) = \frac{1}{\sqrt{2\pi} S_\sigma} \exp \left[ -\frac{1}{2} \left( \frac{\sigma - \sigma_m}{S_\sigma} \right)^2 \right]
\]

(2-53)

in which, $\sigma_m$ is the mean intrinsic frequency; $S_\sigma$ represents the standard deviation (bandwidth) of the spectrum in frequencies. Advantages of using Gaussian-shape wave spectra include: (1) they are precisely defined by spectral parameters of interest. (2) It is relatively easy to evaluate integrals. In this model, the Gaussian-shaped spectrum is assumed although other spectra can be used too. Therefore, we introduce the second assumption as follows
• **Assumption 2**: narrow-banded Gaussian-shaped frequency spectra are assumed and

\[ S_\sigma / \sigma_m \leq 0.25 \] to ensure the narrow-banded limitation.

Ocean wave spectra generally contain a combination of local wind-generated seas with swell waves, and their shapes are usually irregular and might be very complex. It is also obvious that no perfect Gaussian-shape wave spectra can be seen in real sea. While, for some coasts where local wind is not the main driver for the generation of ocean waves, narrow banded wave spectra or swell waves dominate. Approximation of this kind of spectra using Gaussian-shape spectra may be valid. Moreover, it is instructive to build a system of moments even a crude assumption of the wave spectra. In addition, the realistic spectra can be well approximately by combining suitably several (or some number) Gaussian-shape spectra. A comprehensive model based on this idea can be developed.

In the moment equation \( E_{n,m} \), higher order moments concerning the directional spectrum \( E_{n,m+1} \) and \( E_{n,m+2} \) appear (see Appendix B). To make a closed system, additional higher order moments are required to be expressed in terms of \( E_{n,m} \) and/or lower moments. Investigation of possible relations between these moments using different spectra suggests that there is no general rule that can be drawn, and the relation depends largely on the specific type of spectra. For simplicity, the current model is limited to the specific type of directional spectra as shown in the following assumption

• **Assumption 3**: Gaussian-shaped directional spectra are assumed. That is,

\[
D(\theta) = \frac{1}{\sqrt{2\pi}S_\theta} \exp \left[ -\frac{1}{2} \left( \frac{\theta - \theta_m}{S_\theta} \right)^2 \right]
\] (2-54)
where, $\theta_m$ is the mean wave direction. $S_{\theta}$ is one-sided the directional spread, with a larger value implying broader spectrum. Combination of equations (2-53) and (2-54) provides the description of the ‘target’ spectra in this model.

With the target wave spectra, calculation of the general moment $E_{n,m}$ of the spectrum density according to the definition gives

$$E_{n,m} = \frac{E_{0,0}}{2\pi} e^{-n^2S_{\theta}^2/2} e^{i\theta_m} \left\{ \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} S^2 - \frac{1}{2} \sigma_m^{2(n-k+1)/2} \Gamma\left[ \frac{n-k+1}{2} \right], \quad n-k=0,2,4,... \right\} \quad \text{(2-55)}$$

where $\Gamma\left[ \frac{n-k+1}{2} \right]$ is the gamma function. Detailed derivations refer to Appendix A.

Substituting corresponding values of $n$ and $m$ into Equation (2-55), it is straightforward to obtain the following relations

$$\sigma_m = E_{1,0} / E_{0,0}, \quad S_{\sigma}^2 = E_{2,0} / E_{0,0} - \sigma_m^2, \quad E_{0,1} = E_{0,0} e^{-S_{\theta}^2/2} e^{i\theta_m} \quad \text{(2-56)}$$

Now it is very clear that this system, describing evolution of five moments of the spectrum density, can also be described by five spectral parameters: total variance $E_{0,0}$, mean angular frequency $\sigma_m$, frequency bandwidth $S_{\sigma}$, mean direction $\theta_m$ and directional bandwidth $S_{\theta}$.

With the assumptions and approximations discussed above, integrals in Equation (2-47) can be evaluated in terms of wave moments. Before solving the integrals, we first look closely at the spectral velocities (see Equation (2-39) and (2-40)). The wave ray coordinates $s$ and $m$ involved in spectral velocities can be given by a coordinate transformation with a clockwise rotation of the geographic coordinate system $x$ and $y$. The relation between these two coordinate systems is:

$$\begin{bmatrix} s \\ m \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{(2-57)}$$
From the linear dispersion relation \( \sigma^2 = gk \tanh kh \), we can have

\[
\frac{\partial \sigma}{\partial h} = \frac{\sigma k}{\sinh 2kh} = \frac{1}{h} \left( n - \frac{1}{2} \right) \sigma
\]  

(2-58)

where \( h \) is the water depth and \( n = c_g / c \) is the ratio of group velocity and wave phase velocity.

Then propagation velocity in \( \sigma \) - space can be rewritten as

\[
c_\sigma = \frac{1}{h} (\mathbf{U} \cdot \nabla h) \left( n - \frac{1}{2} \right) \sigma - n \sigma \left[ \cos^2 \theta \frac{\partial u}{\partial x} + \sin \theta \cos \theta \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \sin^2 \theta \frac{\partial v}{\partial y} \right]
\]  

(2-59)

Similarly, the propagation velocity in \( \theta \) - space is

\[
c_\theta = -\frac{1}{h} \left( c_g - \frac{1}{2} c \right) \left( -\sin \theta \frac{\partial h}{\partial x} + \cos \theta \frac{\partial h}{\partial y} \right) - \left[ \cos^2 \theta \frac{\partial u}{\partial x} - \sin \theta \cos \theta \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) - \sin^2 \theta \frac{\partial v}{\partial x} \right]
\]  

(2-60)

In velocity \( c_\sigma \), \( n \) is a function of \( \sigma \), and \( c_g - c / 2 \) in \( c_\theta \) is also a function of \( \sigma \). These two terms need the Taylor expansion approximation as the group velocity

\[
n = n(\sigma_m) + \left( \frac{\partial n}{\partial \sigma} \right)_{\sigma_m} (\sigma - \sigma_m) + \frac{1}{2} \left( \frac{\partial^2 n}{\partial \sigma^2} \right)_{\sigma_m} (\sigma - \sigma_m)^2 + \cdots
\]  

(2-61)

For convenience, let \( cc = c_g - c / 2 \). Then,

\[
cc = cc(\sigma_m) + \left( \frac{\partial cc}{\partial \sigma} \right)_{\sigma_m} (\sigma - \sigma_m) + \frac{1}{2} \left( \frac{\partial^2 cc}{\partial \sigma^2} \right)_{\sigma_m} (\sigma - \sigma_m)^2 + \cdots
\]  

(2-62)

Several derivatives used in Taylor series approximations are listed

\[
\frac{\partial c_g}{\partial \sigma} = \frac{\partial c_g}{\partial k} \frac{\partial k}{\partial \sigma} = \frac{\partial (nc)}{\partial k} \frac{1}{c_g} = \frac{1}{k} (n-1) + \frac{1}{n} \frac{\partial n}{\partial \sigma} \\
\frac{\partial^2 c_g}{\partial \sigma^2} = \frac{\partial}{\partial k} \left[ \frac{\partial c_g}{\partial \sigma} \right]_{c_g} \frac{1}{c_g} = \left( \frac{n'k - n + 1}{k^2} + \frac{nn'' - n'^2}{n^2} \right) \frac{1}{c_g} \\
\frac{\partial cc}{\partial \sigma} = \frac{\partial c_g}{\partial \sigma} - \frac{1}{2} \frac{\partial c}{\partial \sigma} \frac{\partial^2 c_g}{\partial \sigma^2} = \frac{\partial^2 c_g}{\partial \sigma^2} - \frac{1}{2} \frac{\partial^2 c}{\partial \sigma^2} \\
\frac{\partial c}{\partial \sigma} = \frac{1}{nk} (n-1)
\]  

(2-63-2-66)
\[
\frac{\partial^2 c}{\partial \sigma^2} = \frac{n'k-n(n-1)}{n^2 k^2} \frac{1}{c_g}
\]  
(2-67)

\[
n' = \frac{\partial n}{\partial k} = \frac{h}{\sinh 2kh} - \frac{2kh^2}{\sinh 2kh \tanh 2kh}
\]  
(2-68)

\[
n'' = \frac{\partial^2 n}{\partial k^2} = \frac{-4h^2}{\sinh 2kh \tanh 2kh} + \frac{4kh^3 (\cosh 2kh \tanh 2kh + \sinh 2kh \sec h^2 2kh)}{(\sinh 2kh \tanh 2kh)^2}
\]  
(2-69)

Errors will be introduced by using Taylor expansion approximation, so it is necessary to check the significance of errors. Comparison of \(c_g\), \(c_g/c\) and \(c_g - c/2\) between approximation using Taylor expansion and exact values are conducted and results are plotted in Figure 2-1. The wave period of 5 s is used for water depths ranging from deep, intermediate to shallow water. Figure 2-1 suggests that all the three quantities approximated using Taylor expansion are reasonably accurate within a not too broad frequency range. Approximations are very good in shallow water and approximated factor \(n\) and \(c_g - c/2\) match the exact values well too in deep water. For frequencies fairly off the mean value, group velocity is not very well approximated in deep water and in intermediate water depth all approximations have maximum error about 10%-20%. For relatively narrow-banded wave spectra case which is interested in this study, wave energy belonging to where approximation is not good enough is only a very small fraction of total energy. Overall, it is reasonable to use the Taylor expansion approximation.

Substituting these Taylor series approximations (to 2nd power) into the integrals, it is lengthy but not difficult to evaluate them. Detailed derivation and the final governing equation of this reduced wave model refer to Appendix B.

2.3.2.2 An Improvement to the Original Governing Equations

Recall Equation (2-56), \(\sigma_m = \frac{E_{i,0}}{E_{0,0}}\) and \(S_\sigma = \left(\frac{E_{2,0}}{E_{0,0}} - \sigma_m^2\right)^{1/2}\). With the narrow-banded limitation \(S_\sigma / \sigma_m \leq 0.25\), the frequency bandwidth \(S_\sigma\) is usually much smaller than the
mean angular frequency $\sigma_m$. This makes $S_\sigma$ very vulnerable and numerical errors in $\sigma_m$ which are not of significance can greatly contaminate $S_\sigma$. Actually, this problem was first observed through some numerical experiments. For initial condition problems, numerical spurious oscillations were seen and the system is not stable. A more well-conditioned set of equations is required to make the system more stable and more accurate. One simple way to achieve this is to redefine $E_{2,0}$, while other moments remain unchanged:

$$E_{2,0} = \int_0^{\infty} \int_{-\pi}^{\pi} (\sigma - \sigma_m)^2 F d\sigma d\theta$$

(2-70)

Correspondingly, the $E_{2,0}$ equation becomes

$$\frac{\partial}{\partial t} E_{2,0} + \nabla \cdot \int_{-\pi}^{\pi} (\sigma - \sigma_m)^2 (c_g + U) F d\sigma d\theta + 2\nabla \sigma_m \cdot \int_{-\pi}^{\pi} (c_g + U) (\sigma - \sigma_m) F d\sigma d\theta$$

$$- \int_{-\pi}^{\pi} c_\sigma \frac{F}{\sigma} (\sigma - \sigma_m) (3\sigma - \sigma_m) d\sigma d\theta = 0$$

(2-71)

where,

$$E_{2,0} = E_{0,0} S_{\sigma}^2$$

(2-72)

From the expressions, the slightly modified system is more well-conditioned because $E_{2,0}$ is only a function of $E_{0,0}$ and $S_\sigma$, and numerical errors in $\sigma_m$ will not have significant impact on $S_\sigma$ any more. Numerical experiments also show a better performance and better stability of the modification (not shown).

### 2.3.2.3 Spectral Wave Breaking Model

For irregular waves, the spectral and directional details of depth-induced wave breaking are not well understood and little is known about how to simulate numerically. Fortunately, the total energy dissipation due to depth-induced breaking has been studied and can be well modeled based on different theories, such as the bore-based breaking model (Battjes and Jassen 1978,
Laboratory observations indicate that the shape of a wave spectrum is conserved as waves propagate across a simple beach profile, based on which Eldeberky and Battjes (1996) developed a spectral version of breaking model of Battjes and Jassen (1978). This breaking model is used in SWAN and the formulation is

\[ S_{br}(\sigma, \theta) = \frac{D_{tot}}{E_{tot}} F(\sigma, \theta) \quad (2-73) \]

where \( S_{br} \) is the source term due to wave breaking, \( D_{tot} \) is the rate of total energy dissipation due to wave breaking and \( E_{tot} \) is the total wave energy, defined by

\[ E_{tot} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F d\sigma d\theta \quad (2-74) \]

Lim et al. (2003) incorporated Rattanapitikon (1998)’s breaking model and Thornton and Guza (1983)’s breaking model into the SWAN and evaluated their performance by comparing with experimental data, and concluded that model of Thornton and Guza (1983) yielded the smallest errors. In the present model, the breaking model of Thornton and Guza will be incorporated, and the formulation will be used is a slightly modified version by Whitford (1988), based on a best fit to field data.

\[ D_{tot} = \frac{3\sqrt{\pi}}{16} f_p B^3 C_1 \frac{H_{rms}^3}{h} \left\{ 1 + \left[ 1 + \left( \frac{H_{rms}}{\gamma h} \right)^2 \right]^{-2.5} \right\} \quad (2-75) \]

where,

\[ C_1 = 1 + \tanh \left[ 8 \left( \frac{H_{rms}}{\gamma h} - 0.99 \right) \right] \quad (2-76) \]

in which \( f_p \) is the peak frequency, the coefficient \( B = 0.8 \) (measuring the intensity of wave breaking) and \( \gamma = 0.42 \).
To develop a wave breaking model fit to the current reduced wave spectral model, similar operations apply to the breaking dissipation term $S_{br}(\sigma, \theta)$. Multiplying Equation (2-73) by $\sigma^n e^{im\theta}$ and integrating over frequencies and directions, we can have

$$
\int_{0}^{+\infty} \int_{-\pi}^{\pi} \sigma^n e^{im\theta} S_{br} d\sigma d\theta = \int_{0}^{+\infty} \int_{-\pi}^{\pi} \frac{D_{tot}}{E_{tot}} \sigma^n e^{im\theta} F d\sigma d\theta = \frac{D_{tot}}{E_{tot}} E_{n,m}
$$

(2-77)

In which, $E_{tot} = E_{0,0}$. The root mean square wave height in $D_{tot}$ can be evaluated by $E_{0,0}$ with the following relation

$$
H_{rms} = \sqrt{8E_{0,0}}
$$

(2-78)

According to Equation (2-77), the breaking dissipation terms associated with $E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1}$ equation are as the following, respectively

$$
\int_{0}^{+\infty} \int_{-\pi}^{\pi} S_{br} d\sigma d\theta = D_{tot}
$$

(2-79)

$$
\int_{0}^{+\infty} \int_{-\pi}^{\pi} \sigma S_{br} d\sigma d\theta = D_{tot} \sigma_m
$$

(2-80)

$$
\int_{0}^{+\infty} \int_{-\pi}^{\pi} (\sigma - \sigma_m)^2 S_{br} d\sigma d\theta = D_{tot} S_\sigma^2
$$

(2-81)

$$
\int_{0}^{+\infty} \int_{-\pi}^{\pi} e^{i\theta} S_{br} d\sigma d\theta = D_{tot} \frac{E_{01}}{E_{00}}
$$

(2-82)

Thus, the dissipation due to breaking has been built into this wave model by including the breaking terms, given by equations (2-79) to (2-82). The energy dissipation rate $D_{tot}$ is infinitesimally small when no breaking happens but increases to finite value when waves start breaking. It may be worthy mentioning that no assumption has been made in developing this breaking model. Therefore, this breaking model essentially does not limit to narrow-banded regular spectra.
Wave radiation stress gradients define the wave forcing, which drives the currents in the surf zone and leads to shoreline setup (Longuet-Higgins and Stewart, 1964). Accurate estimates of the radiation stresses are required in nearshore circulation modeling. For simplicity, radiation stresses computed using the wave statistics ($H_{rms}$, peak frequency and mean direction) have been used to drive the nearshore circulation models (e.g., Church and Thornton, 1993; Ruessink et al., 2001). For random waves, however, the wave radiation stress is also a function of frequency-directional bandwidth and the monochromatic (or narrow-banded) approximation can result in significant errors in radiation stresses (Battjes, 1972). Feddersen (2004) compared the exact radiation stresses with approximations for an empirical wave spectrum, and showed that the component $S_{xy}$ could be overestimated up to 60% using the narrow-banded approximation, which is roughly consistent with the combined Duck94 and SandyDuck observations.

The present wave model solves the frequency and directional bandwidth in the entire computational domain besides the wave height, mean direction and mean frequency. Therefore, it can provide a much more accurate estimate of the radiation stress than the monochromatic approximation by including the effect of frequency and directional spreading. For a monochromatic (single frequency and single direction) wave of height $H$, the radiation stress can be written as (Longuet-Higgins and Stewart, 1964)

$$
S_{xx} = E \left[ n \left( \cos^2 \theta + 1 \right) - \frac{1}{2} \right], \quad S_{xy} = E \left[ n \left( \sin^2 \theta + 1 \right) - \frac{1}{2} \right], \quad S_{yy} = \frac{1}{2} En \sin 2\theta \quad (2-83)
$$

where, $E = 1/8\rho g H^2$ and $n = c_s / c$. For random waves, Battjes (1972) proposed a similar formulation but in terms of wave energy density spectrum $F(f, \theta)$. To be consistent with the notations used before, the formulation can be written as follows.
\[ S_{xx} = \rho g \int_{\pi}^{+\infty} \int_{-\pi}^{\pi} \frac{c_g}{c} \left( \cos^2 \theta + 1 \right) - \frac{1}{2} \] \( F(\sigma, \theta) \) d\sigma d\theta \] (2-84)

\[ S_{yy} = \rho g \int_{\pi}^{+\infty} \int_{-\pi}^{\pi} \frac{c_g}{c} \left( \sin^2 \theta + 1 \right) - \frac{1}{2} \] \( F(\sigma, \theta) \) d\sigma d\theta \] (2-85)

\[ S_{xy} = \rho g \int_{\pi}^{+\infty} \int_{-\pi}^{\pi} \frac{c_g}{c} \sin \theta \cos \theta F(\sigma, \theta) \) d\sigma d\theta \] (2-86)

where \( c_g \) and \( c \) are the group and phase velocity, respectively, and they are functions of the radial frequency \( \sigma \). Assuming \( F(\sigma, \theta) \) is separable:

\[ F(\sigma, \theta) = M(\sigma)D(\theta) \] (2-87)

where \( M(\sigma) \) is the function of frequency distribution, and \( \int_{0}^{\infty} M(\sigma) d\sigma = E_{0,0} = H_{rms}^2 \) \( \frac{\pi}{8} \). \( D(\theta) \) is normalized so that \( \int_{-\pi}^{\pi} D(\theta) d\theta = 1 \). With Equation (2-87), the expressions for radiation stress components become:

\[ S_{xx} = \rho g \int_{\pi}^{+\infty} \frac{c_g(\sigma)}{c(\sigma)} M(\sigma) d\sigma \left[ 1 + \int_{-\pi}^{\pi} D(\theta) \cos^2 \theta d\theta \right] - \frac{1}{2} \rho g E_{0,0} \] (2-88)

\[ S_{yy} = \rho g \int_{\pi}^{+\infty} \frac{c_g(\sigma)}{c(\sigma)} M(\sigma) d\sigma \left[ 1 + \int_{-\pi}^{\pi} D(\theta) \sin^2 \theta d\theta \right] - \frac{1}{2} \rho g E_{0,0} \] (2-89)

\[ S_{xy} = \rho g \int_{\pi}^{+\infty} \frac{c_g(\sigma)}{c(\sigma)} M(\sigma) d\sigma \int_{-\pi}^{\pi} D(\theta) \sin \theta \cos \theta d\theta \] (2-90)

**Frequency Spreading**

To examine the effect of frequency spreading, the unidirectional waves are assumed temporarily, i.e. \( D(\theta) = \delta(\theta - \bar{\theta}) \) is a delta function. Then Equation (2-88) becomes

\[ S_{xx} = \rho g (1 + \cos^2 \bar{\theta}) \int_{\pi}^{+\infty} \frac{c_g(\sigma)}{c(\sigma)} M(\sigma) d\sigma - \frac{1}{2} \rho g E_{0,0} \] (2-91)
For any given $M(\sigma)$ and water depths, the integral in the above equation can be computed numerically. The amount of errors introduced by using the monochromatic approximation, \((S_{xx})_{mono}\), is evaluated by the ratio

\[
\frac{S_{xx}}{(S_{xx})_{mono}} = \frac{(1 + \cos^2 \bar{\theta})\int_0^{\infty} c_g(\sigma)/c(\sigma)M(\sigma)d\sigma - \frac{1}{2}E_{0,0}}{E_{0,0} \left[ c_g(\bar{\sigma})/c(\bar{\sigma})(1 + \cos^2 \bar{\theta}) - \frac{1}{2} \right]}
\]  

(2-92)

Ratios for the other two components can be derived in the same way and will not be reproduced here. For analysis purpose, two types of empirical frequency spectra are examined. One is the Gaussian-type distribution around the mean frequency $\sigma_m$ with one-sided bandwidth $S_\sigma$. The other is the Pierson and Moskowta (1964) spectrum:

\[
F(f) = \frac{\alpha g^2 f^{-5}}{(2\pi)^4} \exp \left[ -1.25 \left( \frac{f}{f_p} \right)^{-4} \right]
\]  

(2-93)

where $\alpha$ defines the total wave energy, $f_p$ is the peak frequency.

For Gaussian frequency spectra, ratios of exact radiation stresses to approximated ones for intermediate water depth are computed, and they bias from 1 and decrease monotonously with increasing the frequency spreading bandwidth (see Figure 2-2), i.e. radiation stresses are overestimated. At the fairly broad spectra $S_\sigma/\sigma_m = 0.5$, ratios are all larger than 0.91. Thus, for Gaussian-shaped frequency spectra, the frequency bandwidth does not have strong influences on the radiation stresses. For the PM spectra, the ratios are calculated for varying the free parameter $f_p$, and all radiation stress components are underestimated in this case (Figure 2-3). However, the errors are not significant either. Although a realistic frequency spectrum can by no means be a Gaussian distribution or PM spectrum, this strongly suggests that the frequency spreading does
not change the wave radiation stresses significantly and the monochromatic approximation in
calculating radiation stresses could be reasonable for most cases.

**Directional Spreading**

In examining the effects of directional spreading on radiation stress, the assumption of

\[ M(\sigma) = E_{0,0} \delta(\sigma - \sigma_m) \]

is made. Substituted into Equation (2-90), it gives:

\[ S_{xy} = \rho g E_{0,0} \int_{-\pi}^{\pi} D(\theta) \sin \theta \cos \theta d\theta \]  \hspace{1cm} (2-94)

If \( D(\theta) = \delta(\theta - \bar{\theta}) \) is a delta function, it is straightforward to show that the ratio for \( S_{xy} \) is as simple as

\[ \frac{S_{xy}}{(S_{xy})_{mono}} = \exp(-2S^2_{\theta}) \]  \hspace{1cm} (2-95)

where \( S_{\theta} \) the standard deviation of spectrum in directions. The above equation is valid for any non-zero mean wave angle. If \( \bar{\theta} = 0^\circ \), both \( S_{xy} \) and \( (S_{xy})_{mono} \) are equal to zero. Expressions for \( S_{xx} \) and \( S_{yy} \) are

\[ \begin{align*}
\frac{S_{xx}}{(S_{xx})_{mono}} &= \frac{n(\bar{\sigma}) \left[ 1 + \frac{1}{2} (1 + \cos 2\bar{\theta} e^{-2S_{\theta}^2}) \right] - \frac{1}{2}}{n(\bar{\sigma})(1 + \cos^2 \bar{\theta}) - \frac{1}{2}} \hspace{1cm} (2-96) \\
\frac{S_{yy}}{(S_{yy})_{mono}} &= \frac{n(\bar{\sigma}) \left[ 1 + \frac{1}{2} (1 - \cos 2\bar{\theta} e^{-2S_{\theta}^2}) \right] - \frac{1}{2}}{n(\bar{\sigma})(1 + \sin^2 \bar{\theta}) - \frac{1}{2}} \hspace{1cm} (2-97)
\end{align*} \]

If assuming a very small wave angle (i.e. \( \cos \bar{\theta} \approx 1 \)) and shallow water, the ratios become

\[ \frac{S_{xx}}{(S_{xx})_{mono}} = \left( 1 + e^{-2S_{\theta}^2} \right) / 3 \quad \text{and} \quad \frac{S_{yy}}{(S_{yy})_{mono}} = 2 - e^{-2S_{\theta}^2} \]

respectively. Figure 2-4 shows how the ratios vary with the wave directional bandwidth \( S_{\theta} \). For shallow water and small non-zero
mean wave angles, $S_{xx}$ and $S_{xy}$ will be overestimated by using the monochromatic approximation, whereas $S_{yy}$ is underestimated. When the directional spreading $S_\theta \approx 34^\circ$, 18%, 51% overestimation of $S_{xx}$ and $S_{xy}$ respectively are observed, and $S_{yy}$ is underestimated by over 50%.

2.3.3 Numerical Implementation

The reduced wave model derived above consists of five equations that are nonlinear and intricately coupled. It may be extremely difficult to apply an implicit scheme if not impossible. So in this study an explicit propagation scheme is used.

2.3.3.1 Temporal Differencing

The simplest method for unsteady problems is the explicit Euler method, but it only has the first order truncation error in time and requires very small step sizes to keep stability. The Runge-Kutta type formulas are widely used because of their good stability characteristics. Another advantage of the Runge-Kutta methods is that they have a very loose restriction on the step size. The 2nd order formula and 4th order formula are two most widely used ones, which has been proved to be very stable. In the following, application of them to governing equations will be described in detail.

For convenience, we rewrite the governing equation in a simple way and here we only write down the $E_{0,0}$ equation:

$$\frac{\partial}{\partial t} E_{0,0} + f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1}) = 0 \quad (2-98)$$

The 2nd order Runge-Kutta formula can be treated as two steps. Step 1 calculates the estimators using explicit Euler approximation. Step 2 computes the new time level values using estimators. Step 1:
\[
(E_{0,0})^{n+1/2} = (E_{0,0})^n - \frac{\Delta t}{2} f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^n
\]  
(2-99)

The same implementation should be made simultaneously on other moment equations. For simplicity, we will not show all of them. After all moments are computed at time level \( n + 1/2 \), we arrive at

Step 2:

\[
(E_{0,0})^{n+1} = (E_{0,0})^n - \Delta t f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^{n+1/2}
\]

(2-100)

The 4th order Runge-Kutta formula is more accurate than the 2nd order. Meanwhile it requires more steps to do and is more expensive. It has four steps, and each step calculates an intermediate estimator. The estimator is then used in the following step to compute the new estimator. The intermediate estimators are interdependent and the order given in the following must be followed.

Step 1:

\[
(E_{0,0})^{n+1/2} = (E_{0,0})^n - \frac{\Delta t}{2} f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^n
\]

(2-101)

Step 2:

\[
(E_{0,0})^{n+1/2} = (E_{0,0})^n - \frac{\Delta t}{2} f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^{n+1/2}
\]

(2-102)

Step 3:

\[
(E_{0,0})^{n+1} = (E_{0,0})^n - \Delta t f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^{n+1/2}
\]

(2-103)

Step 4:

\[
(E_{0,0})^{n+1} = (E_{0,0})^n - \frac{\Delta t}{6} \left[ f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^n + 2 f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^{n+1/2} + 2 f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^{n+1/2} + f(E_{0,0}, E_{1,0}, E_{2,0}, E_{0,1})^{n+1} \right]
\]

(2-104)
2.3.3.2 Spatial Differencing

Several spatial differencing schemes have been built into the model, including the upwind differencing (UD) scheme, the central differencing (CD), the 2\textsuperscript{nd} order upwind scheme and 3\textsuperscript{rd} order upwind method.

Upwind (upwave) differencing method can be simply illustrated by

\[
\frac{\partial \phi}{\partial x} = \frac{\phi_i - \phi_{i-1}}{\Delta x}
\]  

(2-105)

The upwind scheme is simple, robust and efficient, and it has been widely applied in CFD calculation. The scheme accounts for the direction of the flow, so transportiveness is naturally built into the formulation. Solution of upwind method is bounded, hence no ‘wiggles’ occur in the solution. The upwind scheme is based on the backward differencing formula so the accuracy is only first order on the basis of the Taylor Series truncation error. The main drawback of the upwind theme is that it causes distribution of transported properties to become smeared in some types of problems, such as initial value problem (a sharp bump at the beginning of propagation in a computational domain; results and figures will be shown in next section). The resulting error has a diffusion-like appearance and is usually referred to as false diffusion, or dispersive error.

Using higher order discretization can minimize the dispersion errors. Higher order schemes involve more neighboring points and reduce the errors by bringing in a wider influence. The central differencing is 2\textsuperscript{nd} order accuracy proved to be unconditionally unstable when used in conjunction with explicit Euler method for convective problems. And it is also found that the central differencing theme does not possess the transportiveness property under certain circumstances. The 2\textsuperscript{nd} and 3\textsuperscript{rd} order upwind themes are higher order schemes which also retain the transportiveness property. For uniform grids, the 2\textsuperscript{nd} order upwind scheme is as follows
The $3^{rd}$ order upwind scheme can be written as

$$\frac{\partial \phi}{\partial x} = \frac{1}{2\Delta x} \left( 3\phi_i - 4\phi_{i-1} + \phi_{i-2} \right)$$

(2-106)

This scheme may be viewed as a parabolic correction to linear interpolation for the cell face values, hence it’s also called the well-known QUICK (Quadratic Upstream Interpolation for Convective Kinetics). Here the $2^{nd}$ and $3^{rd}$ schemes may be combined into a single expression:

$$\frac{\partial \phi}{\partial x} = \frac{1}{8\Delta x} \left( 3\phi_i + 3\phi_i - 7\phi_{i-1} + \phi_{i-2} \right)$$

(2-107)

(2-108)

where, $\theta = 0$ yields to the $2^{nd}$ upwind theme; $\theta = 0.75$ the QUICK; $\theta = 1$ the central differencing theme.

2.4 Preliminary Tests

The present model was tested by analytically solving the energy conservation equation or the wave action balance equation for several simple test cases: (1) 1-D evolution of an initial sharp bump in deep water, (2) shoaling on over a 1-D planar beach, (3) wave transformations over a 2-D beach.

2.4.1 One-dimensional Evolution of an Initial Sharp Bump in Deep Water

It is logical to test a new model first for the simple cases and then for complex ones. In this study, a one-dimensional version of the wave model was first developed. Evolution of an initial surface sharp bump in deep water is simulated to test the validity of physical description of governing equations and performance of numerical schemes. For the 1-D problem, the governing equations can be reduced as follows with the assumption of no ambient currents for simplicity,
\[
\frac{\partial}{\partial t} E_{0,0} + \frac{\partial}{\partial x} \left[ \left( c_g \right)_{\sigma_n} E_{0,0} + \frac{1}{2} \left( \frac{\partial^2}{\partial \sigma^2} c_g \right)_{\sigma_n} E_{2,0} \right] = 0 \quad (2-109)
\]

\[
\frac{\partial}{\partial t} E_{1,0} + \frac{\partial}{\partial x} \left[ \left( c_g \right)_{\sigma_n} E_{1,0} + \left( \frac{\partial}{\partial \sigma} c_g \right)_{\sigma_n} E_{2,0} + \frac{1}{2} \left( \frac{\partial^2}{\partial \sigma^2} c_g \right)_{\sigma_n} \frac{E_{1,0} E_{2,0}}{E_{0,0}} \right] = 0 \quad (2-110)
\]

\[
\frac{\partial}{\partial t} E_{2,0} + \frac{\partial}{\partial x} \left[ \left( c_g \right)_{\sigma_n} E_{2,0} + \frac{3}{2} \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_n} \left( \frac{E_{2,0}}{E_{0,0}} \right)^2 \right] + 2 \frac{\partial \sigma_m}{\partial x} \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_n} E_{2,0} = 0 \quad (2-111)
\]

(a) Monochromatic Waves

First, we look at the monochromatic wave case. For monochromatic waves, the 1-D unsteady wave energy conservation equation for constant deep water can be simply written as

\[
\frac{\partial}{\partial t} E + \frac{g T}{4 \pi} \frac{\partial}{\partial x} E = 0 \quad (2-112)
\]

where, \( E = \frac{1}{8} \rho g H^2 \) is wave energy, \( g \) is the acceleration of gravity; \( T \) is wave period.

Equation (2-112) is a one-dimensional unsteady advection equation. The exact transient solution of the equation over time \( \Delta t \) is as simple as

\[
E(x, \Delta t) = E \left(x - \frac{g T}{4 \pi} \Delta t, 0\right) \quad (2-113)
\]

This test is aiming to check the model’s capability in simulating a pure convective problem. Consider an initial sharp Gaussian profile travels over a flat bottom with water depth \( h = 30m \). Wave period is 3 seconds, and group velocity \( c_g = 2.342 \frac{m}{s} \). The computational domain was chosen long enough such that no influences of wave conditions at either boundary can reach to the bump profile, and then the convection characteristics of the problem can be clearly observed. Simulations are conducted using different numerical schemes discussed in Section 2.3.3. Figure 2-5 shows the comparison of the exact solutions, first order upwind solutions and QUICK solutions. As discussed above, the first order upwind scheme introduces ‘diffusion error’, and the
initial profile is spread out considerably and the peak amplitude has decreased significantly but nowhere has the solution exceeded the initial bounds. In contrast, QUICK method has a very good agreement with the analytical solutions. For the present reduced spectral wave model, the monochromatic wave case actually can be regarded as an extreme wave spectrum, \( S_\sigma = 0 \).

Equations (2-109)-(2-111) are essentially equivalent to one another, and all of them can be reduced to the wave energy conservation equation given by Equation (2-112).

(b) Irregular Waves

For monochromatic waves, the analytical solution of the advection equation can be easily obtained as discussed above. For irregular waves, the wave energy conservation equation can not be used any more. Instead, the wave action balance equation should be referred. For 1-D, no current and constant deep water and ignoring source terms, the wave action balance equation is reduced to

\[
\frac{\partial}{\partial t} F(\sigma,x,t) + \frac{gT}{4\pi} \frac{\partial}{\partial x} F(\sigma,x,t) = 0
\]  

(2-114)

With the target Gaussian spectra

\[
F(\sigma,x,t) = E_{0,0}(x,t) \frac{1}{\sqrt{2\pi S_\sigma}} \exp \left[ -\frac{1}{2} \left( \frac{\sigma - \sigma_m}{S_\sigma} \right)^2 \right]
\]  

(2-115)

The initial surface bump is given by \( E_{0,0}(x,0) \). At any time \( t \), the analytical solution of Equation (2-114) is

\[
F(x,t,\sigma) = E_{0,0}(x - gT / 4\pi) \frac{1}{\sqrt{2\pi S_\sigma}} \exp \left[ -\frac{1}{2} \left( \frac{\sigma - \sigma_m}{S_\sigma} \right)^2 \right]
\]  

(2-116)

And then it is straightforward to compute \( E_{0,0} \), \( \sigma_m \) and \( S_\sigma \).
The reduced model uses spectral parameters or wave moments as dependent variables. This implies that it has been assumed that an input wave spectrum at offshore boundary will remain as a Gaussian distribution at any time and at any computational grids. However, wave components with different frequencies tend to separate from each other, known as the wave dispersion effect. Consequently, the shape of a wave spectrum may change significantly as wave transformations, such as shoaling, refraction, diffraction and dissipation, take place. Two simulations are presented out of many experimental runs: shorter time simulation (the total time is 600 s) for the spectrum with bandwidth $S_\sigma = 0.1 \text{Hz}$, longer time simulation (2000 s) for the broader spectrum with bandwidth $S_\sigma = 0.2 \text{Hz}$. For the first simulation, the predicted results agree very well with the exact solutions, the wave energy at various times simulated almost perfectly as well as the mean frequency and bandwidth well predicted, as shown in Figure 2-6. The last panel gives the comparison of the real wave spectrum with the predicted spectrum at a monitored location at $t = 600 \text{s}$. Difference between the predicted spectrum and the real spectrum is minor, implying that in this case the input wave spectrum roughly remains as a Gaussian distribution. For the second test, the wave energy distribution is well modeled while spurious oscillations begin to appear in the mean frequency and especially in the bandwidth (Figure 2-7). By checking the spectrum at $x = 5200 \text{m}$ and $t = 2000 \text{s}$, obviously, the real spectrum has double peaks and can by no means be approximated by a Gaussian distribution. This significant disagreement in spectrum is believed to be the main reason for the poor simulation of frequency bandwidth, in addition to the errors introduced by using Taylor series expansion.
2.4.2 Wave Shoaling over a One-dimensional Sloping Beach

Shoaling of monochromatic waves and irregular waves was simulated on a planar beach with the slope of 1:50. The bathymetry consists of a flat bottom connected to a constant slope shown in Figure 2-8. In this test, no energy dissipation was assumed for comparison purpose.

(a) Monochromatic waves

The offshore boundary condition is given by generating steady 4 s, monochromatic waves. The 1-D steady energy conservation equation (no dissipation) can be written as \( \frac{\partial}{\partial x} \left( c_g E \right) = 0 \), and wave energy can be easily computed analytically using the wave energy and group velocity at the offshore boundary. Figure 2-9 shows the comparison between computational results with exact solutions. The model predicts wave shoaling very accurately for monochromatic waves.

(b) Irregular waves

In irregular wave case, shoaling of waves with the mean period of 4 s and varying frequency bandwidth is predicted. Recall the one-dimension, no currents wave balance equation:

\[
\frac{\partial}{\partial t} F + \frac{\partial}{\partial x} \left( c_g F \right) = 0,
\]

where \( F(\sigma,x,t) \) is the wave energy spectrum density. For stationary incoming waves at offshore boundary condition (wave input), it is essentially a steady-state problem and the time differencing term \( \frac{\partial F}{\partial t} \) can be dropped. The analytical solution of the equation \( \frac{\partial}{\partial x} \left( c_g F \right) = 0 \) can be used to test the performance of the model. Since both \( c_g \) and \( F \) are known at the offshore boundary, the energy density \( F \) at every location can be computed analytically. It is worth pointing out that the way to deriving the analytical solutions is valid for any spectral shape.

Field measurements of ocean wave spectrum have shown that the one-dimensional frequency spectrum appears to roughly have a universal shape: the JONSWAP spectrum.
(Hasselmann et al., 1973). The current wave model uses the Gaussian-shaped wave spectra, which are different from JONSWAP spectra to some degree. Therefore, it is necessary to examine the differences between using JONSWAP spectra and using Gaussian spectra as an approximation. Figure 2-10(a) shows the comparison of the JONSWAP wave frequency spectrum and its approximations using two different Gaussian-shaped spectra. The first Gaussian spectrum use the peak frequency of the JONSWAP spectrum as its mean (or peak) frequency and the frequency bandwidth \( S_\sigma / \sigma_m = 0.11 \) is chosen to fit the JONSWAP. The second Gaussian spectrum has the same mean frequency as the JONSWAP spectrum and its frequency bandwidth is given so that the moment \( E_{2,0} \) is identical to that of the JONSWAP spectrum. All the three spectra give the same RMS wave height of 1 m. Comparison of analytical results for shoaling of the JONSWAP spectrum, these two Gaussian-shaped spectra and the modeled results for the Gaussian spectra is shown in Figure 2-10(b). Non-breaking waves are assumed for comparison purpose. For the first Gaussian-shaped spectrum, the maximum difference between the modeled wave heights and analytical values is only about 1%. The maximum difference between analytical solutions for the JONSWAP spectrum and modeled results using the first Gaussian spectrum is about 3.6% at the shallowest water depth, and the RMS difference is less than 1%. For the second Gaussian, the analytical results are more different between the JONSWAP spectrum and the Gaussian spectrum due to the vast discrepancy in the spectral shapes. In addition, since the second Gaussian spectrum is too broad and the assumption of relatively narrow-banded spectrum does not hold here, the model did not match the analytical results accurately.

Figure 2-11 shows the comparison between modeling results and analytical solutions. For \( S_\sigma = 0.1 \text{Hz} \), the model predicts the spatial variation of energy and the mean frequency variation
accurately but slightly overestimates the frequency bandwidth. As wave spectra become broad, the model is less accurate but overall giving reasonable results. For $S_\sigma = 0.3\text{Hz}$, the frequency bandwidth is poorly predicted and even the wave energy is not predicted very well, suggesting that the model may lose its accuracy or even give bad results if wave spectra are broad. This in fact is what we expect because the narrow-banded wave spectrum is one of the base assumptions for developing the reduced model. Effects of the frequency bandwidth on wave shoaling are also tested (see Figure 2-12). Generally, waves with broader spectra shoal more quickly than narrow-banded waves if there is no any dissipation and no breaking.

This test roughly demonstrated the capability of the reduced model as well as some of its limitations. Overall, the model can simulate shoaling of waves with Gaussian spectra well, while, considerable errors are observed for broad wave spectra. These errors are believed to be introduced because of two reasons. One reason is that some quantities are approximated using the Taylor expansion which is not accurate when the wave spectrum is relatively broad. The other reason is that the assumption that an input Gaussian spectrum will maintain its Gaussian shape is more likely to break down when the wave spectrum is broad.

2.4.3 Shoaling and Refraction of Spectral Waves over a Planar Beach

In Section 2.4.2, shoaling of on a 1-D, sloping beach bathymetry with either monochromatic or spectral waves was simulated numerically. In this section, a simple two-dimensional test is used to check the capability of the model in simulating shoaling and refraction of spectral waves.

For slopes with straight bottom contours parallel to the shoreline, the $y$-derivative of any quantity is equal to zero. For monochromatic waves, conservation of wave energy shows that for any two points of interest, $E_1 c_{g1} \cos \theta_1 = E_2 c_{g2} \cos \theta_2$. According to Snell’s law, the refraction
coefficient is straightforward to compute. Combined with the shoaling coefficient, wave energy at any point of interest can be computed exactly using wave conditions at incident boundary. So, the analytical solutions can be easily established for monochromatic waves. For spectral waves, however, the analytical solutions are not as apparent, and need more efforts. The basic idea to derive the analytical solutions is splitting the total wave energy into many components, and each component corresponds to a direction and a frequency and satisfies the energy conservation equation. Each component is solved analytically, and then wave spectrum corresponding to each component can be computed by dividing the bin sizes in frequency and direction. Once having the wave spectra everywhere, the wave moments can be computed following their definitions. One thing should be aware of is that $\Delta \theta_j$ varies spatially due to wave refraction. Specifically,

1. $\left(E_{i,j}\right) = F(\sigma_i, \theta_j) \Delta \sigma_i \Delta \theta_j$ (wave energy contained in certain spectral bin $(i, j)$)
2. $\frac{\partial}{\partial x}\left(c_{s_i} \cos \theta_j E_{i,j}\right) = 0$ (using wave energy conservation to compute $E_{i,j}$ at every grid location)
3. $F(\sigma_i, \theta_j) = E_{i,j} / (\Delta \sigma_i \Delta \theta_j)$ (obtain the wave spectrum)
4. compute wave moments and spectral parameters.

In this test, the 2-D bathymetry used is y-direction uniform and the across-shore depth profile is the same as the 1-D bathymetry (Figure 2-8). Waves of 4 s (mean period if irregular waves), 1 m in height ($H_{rms}$ if irregular waves) at the offshore boundary, and the incident angle (mean direction if multidirectional waves) is 30°.

Figure 2-13 shows wave refraction and wave shoaling as the offshore monochromatic waves propagate towards the shoreline. The top panel compares predicted wave heights with exact values. The bottom panel gives comparison of wave direction between simulated values with values given by the Snell’s law. Clearly, the model predicts wave refraction and wave shoaling very well for monochromatic incident waves.
Simulations for waves with a single frequency (i.e. \( S_\sigma = 0 \)) but multiple directions have been conducted. Figure 2-14 gives the comparison of the simulation results with analytical solutions for waves with various directional bandwidths \( S_\theta = 10^\circ, 20^\circ, 30^\circ \). For relatively narrow directional spreading cases \( S_\theta = 10^\circ \) and \( S_\theta = 20^\circ \), very good agreement between predicted \( H_{rms} \), \( \theta_m \), \( S_\theta \) and analytical values is observed. For \( S_\theta = 30^\circ \), as waves approach to the shoreline, the predicted \( S_\theta \) gradually deviate from analytical values and differences in mean direction and wave height can also be seen. It is also found that wave heights become smaller with the increase in directional bandwidth and bandwidth \( S_\theta \) itself decreases as waves approach to the shoreline. This is what is supposed to happen because the characteristics of wave refraction suggest that waves tend to travel in the direction normal to bottom contours, thus all wave components bend to \( x \) direction and consequently both the mean direction and the directional bandwidth become smaller.

Simulations for incident waves with non-zero frequency bandwidth and non-zero directional bandwidth are also conducted. Model results of waves with \( S_\sigma = 0.1 \text{Hz} \) and \( S_\theta = 10^\circ \) are presented and compared to the analytical results as shown in Figure 2-15. Across-shore variations of \( H_{rms} \), \( \sigma_m \) and \( \theta_m \) are accurately modeled, and noticeable underestimation in \( S_\sigma \) and overestimation in \( S_\theta \) are observed. Overall, this test demonstrated the model’s capability to predict shoaling and refraction of irregular waves with certain frequency bandwidth and directional bandwidth.
2.4.4 Transformation of Irregular Waves over a Beach with Periodic Rip Channels

Simulating wave transformation over a beach with periodic rip channels is a great challenge for this model, especially for waves with a broad directional spectrum. The bathymetry used in this test is defined by

\[ h(x, y) = 0.025x \left[ 1 + 20 \exp \left( -3(x/20)^{1/3} \right) \sin^8 \left( \frac{\pi y}{80} \right) \right] \]  

(2-117)

The bathymetry is similar to that used in Noda (1974). Bottom contours of the bathymetry are shown in Figure 2-16. Periodic boundary condition is applied to the lateral boundaries. Waves of 4 s, \( H_{rms} = 0.5m \) at offshore boundary are used throughout this section. Numerical experiments are carried out for two objectives: (1) to check the capability of this model in predicting wave transformations over this relatively complex 2-D bathymetry; (2) to investigate the effects of frequency and directional bandwidths on wave properties. Both the normal incidence and oblique incidence (30° from x-direction) are studied. For each case, monochromatic waves, waves with non-zero frequency bandwidths and waves with non-zero directional bandwidths are simulated.

For normal wave incidence, waves bend from rip channels and consequently wave focusing occurs in the region between rip channels (see Figure 2-17). As the water depth decreases, waves break very strongly outside rip channels. For incident waves with directional bandwidth \( S_\theta = 10^\circ \) (Figure 2-18) and \( S_\theta = 20^\circ \) (Figure 2-19), the wave height distribution and wave direction are similar to the monochromatic case. The model predicted very large \( S_\theta \) in the area near to the shoreline outside rip channels (see Figure 2-18, 2-19). The possible explanation is that wave caustics begin to happen and the model is not able to give correct values of \( S_\theta \).

Figure 2-20 gives \( H_{rms} \) along the two transects for cases with various values of directional
bandwidth. Difference in wave heights due to the presence of directional bandwidth is not of significance but is not negligible either for the case \( S_{\theta} = 20^\circ \). Modeled \( H_{rms} \) for waves with non-zero frequency bandwidth is very similar to the monochromatic wave case and will not be presented here. Figure 2-21 gives the distribution of the mean angular frequency and the frequency bandwidth for the case \( S_{\sigma} = 0.3 \)Hz. It can be seen that both the mean angular frequency \( \sigma_m \) and bandwidth \( S_{\sigma} \) decreases from offshore to onshore with strong variation outside the rip channels where strong wave refraction happens. Wave height \( H_{rms} \) along the two transects are also compared for different frequency bandwidths (Figure 2-22). No appreciable changes are observed with the presence of frequency bandwidth because wave refraction and breaking are the dominant factors and overwhelm others in this case.

Another set of experiments is carried out for oblique wave incidence with an angle of 30° from the x-direction. Figure 2-23 shows the simulated results for monochromatic waves. Similar to the normal incidence case, contour plots of \( H_{rms} \) did not change very much with the presence of directional bandwidth \( S_{\theta} = 10^\circ \) (Figure 2-24) and \( S_{\theta} = 20^\circ \) (Figure 2-25). This is further demonstrated by comparing \( H_{rms} \) along the two transects for various directional bandwidths (Figure 2-26). The mean wave direction along transect s1 and s2 is only slightly changed by the presence of directional bandwidth (Figure 2-27). Again, the frequency bandwidth does not have considerable impacts on wave field. Figure 2-28 shows the spatial distribution of the computed mean frequency and frequency bandwidth.

2.4.5 Physical Effects of Current on Waves

In addition to the topographical variation, ambient currents also lead to wave transformations. Several general conclusions have been drawn about effects of currents on waves
including: when the currents and waves are in the same direction, waves will be lengthened and the wave height will be decreased. However, waves are shorten and steepened by the current in opposite direction, and strong enough currents can completely block the propagation of waves, resulting in wave breaking.

In this section, numerical simulations of wave transformations due to artificial alongshore currents were conducted, and for monochromatic waves the modeling results were compared to the analytical solutions. Good agreement between the analytical solutions and the numerical predictions demonstrated the wave model’s ability to accurately predict wave transformations due to ambient currents. The model then was used to investigate effects of frequency/ directional bandwidth upon evolution of waves with the presence of these artificial ambient currents. Two simple cases, whose analytical solutions are available, were used: waves obliquely incident on an alongshore current, and waves with currents in the same direction. With the purpose of studying effects of currents on waves, the constant water depth was used in most case to avoid wave transformations due to topographical variations. For a sloping beach bathymetry, modeling results for monochromatic wave cases only are presented here. As waves propagate over the sloping beach, effects of frequency and direction bandwidth are relatively small compared to the impacts of wave shoaling, refraction and breaking.

**Waves with Oblique Incidence on Alongshore Currents**

With the presence of currents, the absolute angular frequency can be written as

$$\omega = 2\pi / T = U \cdot k + \sigma \quad (2-118)$$

and the intrinsic frequency is defined by the linear dispersion relation

$$\sigma^2 = gk \tanh kh \quad (2-119)$$
For steady waves and current, assuming no energy input or energy dissipation, the wave action conservation equation is reduced to

$$\nabla \cdot \left( \left( \mathbf{U} + C_g \right) \frac{E}{\sigma} \right) = 0$$

(2-120)

If both the currents and the bathymetry are longshore uniform, Equation (2-120) can be further reduced to

$$\frac{\partial}{\partial x} \left( E C_g \cos \theta / \sigma \right) = 0$$

(2-121)

Since $\partial / \partial y = 0$ and $\nabla \times \mathbf{k} = 0$, it is easy to show that

$$k_y = k \sin \theta = (k \sin \theta)_0 = \text{const}$$

(2-122)

where, the subscript ‘0’ represents values at the reference location (often at offshore boundary).

From (2-121), the wave energy variation due to the effects of current can be found

$$\frac{E}{E_0} = \frac{\sigma C_{g0} \cos \theta_0}{\sigma_0 C_g \cos \theta}$$

(2-123)

in which, $\sigma$ can be computed according to Equation (2-119) and wave direction is calculated using Equation (2-122).

The artificial alongshore current profile $v(x) = 4 \sin^4 \left( \frac{2\pi x}{l_x} \right)$ was used for this test case. $l_x$ is the computational domain in $x$-direction. Waves of $H_0 = 0.61m$, $T = 4s$ and $\theta_0 = 22.5^0$ propagate through the alongshore currents. Figure 2-29 shows the comparison of modeled results with the analytical solutions for monochromatic waves on constant water depth of 100 m. Spatial variation of wave number due to the presence of current is shown in panel (b), and the red dashed line indicates $k_y$ which is constant (from Equation (2-122)). Very interesting conclusions have been drawn about the relation between $k_y$ and the minimum $k$: If $k_y < k_{\text{min}}$, wave rays can...
penetrate the current after deflection; if \( k_y > k_{\text{min}} \), wave caustics will happen and wave rays incident from outside must bend backward after touching the caustic edge; for the extreme situation, \( k_y < 0 \), which means waves propagate against the current, waves can be trapped near the current peaks. For more details refer to Mei’s textbook (1983). As expected, the wave model failed for wave caustics cases and we will limit ourselves to \( k_y < k_{\text{min}} \) case. Strong wave refraction around the current peaks is observed, as shown in panel (c), and variation in energy along x-direction is plotted in panel (d). The wave number, wave direction and energy variation were accurately simulated by the model.

To investigate the effects of frequency and directional bandwidths, simulations with same current profile and wave conditions but with nonzero \( S_\sigma \) and \( S_\theta \), were also conducted. Increasing \( S_\sigma \) from 0 to 0.3 Hz, the mean wave number slightly decreases and \( \theta \) also reduces a bit around the current peaks (Figure 2-30). While, the wave energy is not affected significantly, which can be explained by considering that the effect of the changes in wave number cancels out that of the changes in wave direction. The effects of directional spreading \( S_\theta \) are demonstrated in Figure 2-31, and a weaker current with the same shape is used for this run to avoid the possible instability problem. For a large directional spreading \( S_\theta \approx 17^\circ \), the mean wave angle increases significantly around the current peaks and surprisingly the energy variation pattern is completely changed. Wave energy increases instead of decreasing for unidirectional waves or waves with sufficiently small directional spreading. Figure 2-32 shows the simulation results and analytical solutions of the same wave condition but on a planar beach with a slope of 1:20. In this test, no wave breaking was assumed such that analytical solutions are available.

**Waves with Paralleling Currents**
For the case of waves with paralleling currents, we can assume that waves and currents are in the x-direction, and then the current is given by \( u = u(x), v = 0 \). It should be noted that this kind of current requires a vertical velocity component to satisfy the continuity equation. To illustrate the ability of the wave model to simulate how depth-averaged currents affect wave propagation, however, attention will only focus on horizontal components here. For 1-D (alongshore uniform) topography, the wave action conservation Equation (2-120) can be written as

\[
\frac{\partial}{\partial x} \left[ E \left( C_g + u \right)/\sigma \right] = 0
\]

(2-124)

and the energy variation can be obtained by

\[
\frac{E}{E_0} = \frac{\sigma \left( c_{g0} + u_0 \right)}{\sigma_0 \left( c_g + u \right)}
\]

(2-125)

In this set of experiments, the artificial velocity \( u(x) = 0.6245 \sin \left( \frac{2\pi x}{l} \right) \) was used. Figure 2-33 shows the comparison of computational results with analytical solutions of wave transformations due to the paralleling currents over the flat bed. Currents in the same direction of waves lengthen the waves and decrease the wave number. On the other hand, opposing currents increases the wave number and shorten the waves (Figure 2-33(b)). Wave energy increases when waves meet the opposing currents and decreases when waves come across the downstream currents, see Figure 2-33(c). Increasing frequency bandwidth \( S_\sigma \) results in the strengthening of wave number variation due to the current, as shown in Figure 2-34(b). However, the wave energy was not altered much by increasing \( S_\sigma \) from \( 0.1\sigma_m \) to \( 0.3\sigma_m \). As expected, the directional spreading does not have significant influences since the current is parallel to the propagation direction of waves, which can be seen in Figure 2-35. Figure 2-36 shows the
A comparison of modeling results with analytical solutions for monochromatic waves over the sloping beach bathymetry, and good agreement was obtained.

2.5 Discussion and Summary

A new type of wave model, which is different from the traditional spectral wave models and approximate monochromatic-based models, has been developed and several simple tests have been conducted in this chapter. Unlike the full wave spectral models, this reduced model describes the evolution of wave moments instead of individual wave spectral components. Evolution equation for the general moment $E_{n,m}$ was introduced by integrating the standard wave action balance equation multiplied by the weighting function, $\sigma^{n+1}e^{im\theta}$, over angular frequencies and directions. As a preliminary study in developing this moments-based wave model, a relatively simple system with moments $E_{0,0}, E_{1,0}, E_{2,0}, E_{1,0}$ as fundamental variables, which contain some of the most important wave properties including $H_{rms}$, mean frequency, mean direction, frequency and directional bandwidth, is developed. To uniquely define all terms in the governing equations in terms of wave moments, several approximations and assumptions were made including

1. The input wave spectrum is a separable frequency-direction spectrum.
2. In order to use the Taylor expansion approximation, the input frequency spectrum was required to be narrow-banded.
3. Only the Gaussian-shaped spectra are used in this study and thus the higher order moments existed in equation $E_{n,m}$ can be computed using the wave spectral parameters including the wave energy $E_{0,0}$, mean angular frequency $\sigma_m$, mean direction $\theta_m$, directional bandwidth $S_\theta$ and frequency bandwidth $S_\sigma$. 
4. An input spectrum was assumed to conserve its spectral shape.

All these approximations and assumptions introduce errors and/or limit the application of the model to some degree. It should be pointed out that other wave frequency spectra such as the JONSWAP spectra and PM spectra rather than the Gaussian-shaped spectra can be simulated using the evolution equations of wave moments although we limit to Gaussian-shaped spectra in this model. In addition, theoretically the directional spectrum is not necessary to be a Gaussian distribution either. Waves rarely maintain their spectral shapes as they propagate from offshore to the shoreline. Redistribution of wave energy over the spectrum can be caused by many processes including refraction, shoaling, diffraction and nonlinear wave-wave interactions. For the directional spectrum, the shape at certain locations can be very different from that of the incident spectrum at offshore, especially when strong wave refraction-diffraction happens. Reilly and Guza (1991) investigated the spectral transformation of a symmetric, single-peak directional spectrum over a circular shoal using the backward ray tracing method. They found that directional spectra at locations behind the shoal can be very complex and several peak directions were observed.

Despite all the approximations and limitations in applicability, for beaches typically having wave spectra with finite relatively compact shapes with clear peak direction and frequency, the reduced spectral wave model can give better results than monochromatic-based wave models and is much computationally cheaper than a full spectral wave model. Therefore, this moments-based model is useful and can be an alternative to the traditional spectral wave models within its applicative range. Another advantage of this model is that it can directly investigate effects of directional/frequency bandwidth upon wave fields and then the wave-induced currents. In
addition, the realistic spectra can be well approximately by combining suitably several Gaussian-shape spectra. A comprehensive model based on this idea can be developed.

It is also worth pointing out that this model does not include wind input, energy dissipation due to whitecapping and wave-wave interaction. Wave reflection and diffraction can not be simulated by this model either. So, future studies to lift limitations and improve the performance of the model include:

1. Developing a more sophisticated model which can deal with more realistic wave spectra, for example, spectra with multi-peaks in both frequency and direction
2. Seeking a sound way to approximate higher order moments of the directional spectrum appearing in equation $E_{n,m}$ such that the assumption that a directional spectrum conserve its shape can be removed. This is important because the directional spectrum may change its shape vastly under certain circumstances
3. Accounting for wave diffraction and wave-wave interaction, which may play very important roles in evolution of spectral waves in nearshore areas, especially for complex bathymetries
Figure 2-1. Comparison of quantities (existing in the propagation velocities) approximated using Taylor expansion with exact values at various water depths. (a) for group velocity $c_g$; (b) for $n = c_g/c$; (c) for $c_g - c/2$, solid lines represent approximated values, dots for exact values. The vertical dot lines mark the position of $\sigma_m$. 
Figure 2-2. Ratios of the exact radiation stress components to approximated ones for Gaussian spectra with varying frequency bandwidth. $\sigma_m = 2\pi / 6$, $\theta = 30^\circ$ and water depth $h = 6m$ are used. $S_{xx} / (S_{xx})_{\text{mono}}$ (circle), $S_{yy} / (S_{yy})_{\text{mono}}$ (x-mark), $S_{xy} / (S_{xy})_{\text{mono}}$ (square).

Figure 2-3. Ratios of the exact radiation stress components to approximated ones for PM spectra with varying peak frequency. $\theta = 30^\circ$ and water depth $h = 6m$ are used. $S_{xx} / (S_{xx})_{\text{mono}}$ (circle), $S_{yy} / (S_{yy})_{\text{mono}}$ (x-mark), $S_{xy} / (S_{xy})_{\text{mono}}$ (square).

Figure 2-4. Ratios of the exact radiation stress components to approximated ones for Gaussian directional spectra with varying directional bandwidth $S_\theta \cdot \bar{\theta} = 0.1^\circ$ and shallow water are assumed. $S_{xx} / (S_{xx})_{\text{mono}}$ (circle), $S_{yy} / (S_{yy})_{\text{mono}}$ (x-mark), $S_{xy} / (S_{xy})_{\text{mono}}$ (square).
Figure 2-5. Evolution of an initial sharp bump with incident monochromatic waves of 3 s over constant water depth $h = 30m$. 1st order upwind solutions (left), QUICK solutions (right). — Analytical solution, --- Computed results.

Figure 2-6. Evolution of an initial sharp bump with incident Gaussian wave spectrum with frequency bandwidth $S_c = 0.1Hz$. Wave energy (first panel), mean frequency (second panel), frequency bandwidth (third panel) and the last panel shows comparison between computed and the real wave spectra at $x = 2800m$, $t = 600s$. — Analytical solution, --- Computed results.
Figure 2-7. Same as Figure 2-6 but with $S_\sigma = 0.2\, (Hz)$; The last panel shows comparison between computed and the real wave spectra at $x = 5200\, m, t = 2000\, s$. — Analytical solution, --- computed results.

Figure 2-8. The 1-D bathymetry with a slope of 1:50 resting on a flat bed with 10 m water depth.
Figure 2-9. Shoaling of monochromatic waves on the 1-D bathymetry (Figure 2-8). Numerical results (solid line), analytical solutions (circles). Non-breaking waves assumed for comparison purpose.

Figure 2-10. (a) The JONSWAP wave frequency spectrum and its approximations using the Gaussian-shaped spectra: Gaussian spectrum 1 has the same peak frequency 0.25Hz as the JONSWAP spectrum and the frequency bandwidth \( \sigma / \sigma_m = 0.11 \); Gaussian spectrum 2 has the same mean frequency and moment \( E_{2,0} \) as the JONSWAP spectrum. All the three spectra give the same RMS wave height of 1m. (b) Comparison of analytical results for shoaling of the JONSWAP spectrum, the two Gaussian-shaped spectra and modeled results for the two Gaussian spectra. Non-breaking waves assumed for comparison purpose.
Figure 2-11. Shoaling of irregular waves with various frequency bandwidths on the 1-D bathymetry (Figure 2-8). (a) $S_\sigma = 0.1\text{Hz}$; (b) $S_\sigma = 0.2\text{Hz}$; (c) $S_\sigma = 0.3\text{Hz}$. Non-breaking waves assumed.

Figure 2-12. Effects of frequency bandwidth on the wave shoaling over the 1-D bathymetry (Figure 2-8). Lines from bottom to top stand for $S_\sigma = 0$, 0.1, 0.2, 0.3 (Hz), respectively. Non-breaking waves assumed.
Figure 2-13. Comparison of wave height and direction between numerical results (solid lines) and analytical solutions (circles) for monochromatic wave propagation over alongshore uniform 2-D bathymetry (water depth profile is the same as Figure 2-8). Non-breaking waves assumed.

Figure 2-14. Transformations of spectral waves over the sloping beach (Figure 2-8). Incident Gaussian waves are defined: $\sigma = 2\pi / 4$, $S_\sigma = 0$, $\theta_m = \pi / 6$ and various $S_\theta$ (Lines from top to bottom represent $S_\theta = 10^\circ, 20^\circ, 30^\circ$, respectively. Symbols indicate analytical solutions). Non-breaking waves assumed.
Figure 2-15. Comparison of computational results (lines) with analytical solutions (circles) for offshore wave spectra: $\theta_m = \pi / 6$, $S_{\sigma} = 0.2\text{Hz}$, $S_{\theta} = 10^\circ$. Non-breaking waves assumed.

Figure 2-16. Bottom contours of the beach bathymetry with periodic rip channels. Two across-shore transects are marked by the thick dashed lines.
Figure 2-17. Contour plots of computed wave height (left) and wave direction (right) for normally incident monochromatic waves. $H = 0.5m$ and $T = 4s$ at the offshore boundary.

Figure 2-18. Contour plots of computed wave height (left), mean direction (middle) and directional bandwidth (right) for irregular waves: $H_{rms} = 0.5m$, $T_m = 4s$, $\theta_m = 0^\circ$, $S_\sigma = 0$ and $S_\theta = 10^\circ$ at the offshore boundary.
Figure 2-19. Contour plots of computed wave height (left), mean direction (middle) and directional bandwidth (right) for waves: $H_{rms} = 0.5m$, $T_m = 4s$, $\theta_m = 0^\circ$, $S_\sigma = 0$ and $S_\theta = 20^\circ$ at the offshore boundary.

Figure 2-20. Comparison of $H_{rms}$ along transect s1 (black lines), s2 (red lines) for various directional bandwidths: $S_\theta = 0^\circ$ (solid lines), $S_\theta = 10^\circ$ (short-dashed lines) and $S_\theta = 20^\circ$ (long-dashed lines).
Figure 2-21. Contour plots of computed $\sigma_m$ (left) and $S_{\sigma}$ (right) for waves: $H_{rms} = 0.5m$, $T_m = 4s$, $\theta_m = 0^\circ$, $S_{\sigma} = 0.3Hz$ and $S_{\theta} = 0^\circ$ at the offshore boundary.

Figure 2-22. Comparison of $H_{rms}$ along transect s1 (black lines), s2 (red lines) for various directional bandwidths: $S_{\sigma} = 0$ (solid lines), $S_{\sigma} = 0.1Hz$ (short-dashed lines), $S_{\sigma} = 0.2Hz$ (long-dashed lines) and $S_{\sigma} = 0.3Hz$ (dash-dot lines).
Figure 2-23. Contour plots of computed wave height (left) and wave direction (right) for monochromatic waves: $H = 0.5m$, $T = 4s$ and $\theta = 30^\circ$ at the offshore boundary.

Figure 2-24. Contour plots of computed wave height (left), mean direction (middle) and directional bandwidth (right) for waves: $H_{rms} = 0.5m$, $T_m = 4s$, $\theta_m = 30^\circ$, $S_\sigma = 0$ and $S_\theta = 10^\circ$ at the offshore boundary.
Figure 2-25. Contour plots of computed wave height (left), mean direction (middle) and directional bandwidth (right) for waves: $H_{rms} = 0.5m$, $T_m = 4s$, $\theta_m = 30^\circ$, $S_\sigma = 0$ and $S_{\theta} = 20^\circ$ at the offshore boundary.

Figure 2-26. Comparison of $H_{rms}$ along transect s1 (black lines), s2 (red lines) for various directional bandwidths: $S_{\theta} = 0^\circ$ (solid lines), $S_{\theta} = 10^\circ$ (short-dashed lines) and $S_{\theta} = 20^\circ$ (long-dashed lines).
Figure 2-27. Comparison of $\theta_m$ along transect s1 (black lines), s2 (red lines) for various directional bandwidths: $S_\theta = 0^\circ$ (solid lines), $S_\theta = 10^\circ$ (short-dashed lines) and $S_\theta = 20^\circ$ (long-dashed lines).

Figure 2-28. Contour plots of computed $\sigma_m$ (left) and $S_\sigma$ (right) for waves: $H_{rms} = 0.5m$, $T_m = 4s$, $\theta_m = 30^\circ$, $S_\sigma = 0.3Hz$ and $S_\theta = 0^\circ$ at the offshore boundary.
Figure 2-29. Monochromatic waves obliquely incident on a longshore current over constant deep water $h = 100m$. The offshore wave conditions are $H_0 = 0.61m$, $T = 4s$ and $\theta_0 = 22.5^\circ$. — modeling results; o analytical solutions. (a) longshore velocity distribution; (b) wave number along x-direction, red dash line indicates $k_y$; (c) wave direction; (d) normalized wave energy distributed along x-direction.

Figure 2-30. Unidirectional waves with various frequency bandwidths obliquely incident on a longshore current over constant depth. The offshore wave conditions are the same as Figure 2-29 but with various frequency bandwidths: (solid curve), $S_\sigma = 0$; (dotted line), $S_\sigma = 0.1\sigma_m$; (dashed line), $S_\sigma = 0.2\sigma_m$; (dash-dotted line) $S_\sigma = 0.3\sigma_m$. 
Figure 2-31. Waves with various directional bandwidths obliquely incident on the alongshore current over constant depth. The offshore wave conditions are the same as Figure 2-29 but with various directional bandwidths: $S_\theta = 0.1\, rad = 5.73^\circ$, dotted line; $S_\theta = 11.46^\circ$, dashed line; $S_\theta = 17.19^\circ$, dash-dotted line.

Figure 2-32. Monochromatic waves obliquely incident on the alongshore current over a planar beach with the slope of 1:20. For comparison purpose, non-breaking waves are assumed.
Figure 2-33. Monochromatic waves with paralleling currents over constant water depth of 100m. The offshore wave conditions are $H_o = 0.61m$, $T = 4s$. — modeling results; o analytical solutions. (a) cross-shore velocity distribution; (b) wave number along x-direction; (c) normalized wave energy.

Figure 2-34. Waves with various frequency bandwidths with paralleling currents over deep water. The offshore wave conditions are the same as Figure 2-33 but with various frequency bandwidths: $S_\sigma = 0$; (dotted line), $S_\sigma = 0.1\sigma_m$; (dashed line), $S_\sigma = 0.2\sigma_m$; (dash-dotted line) $S_\sigma = 0.3\sigma_m$. 

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Figure 2-35. Waves with various directional bandwidths with paralleling currents over deep water. The offshore wave conditions are the same as Figure 2-33 but with various directional bandwidths: $S_\sigma = 0.1\, \text{rad} = 5.73^\circ$, dotted line; $S_\sigma = 11.46^\circ$, dashed line; $S_\sigma = 17.19^\circ$, dash-dotted line.

Figure 2-36. Monochromatic waves with paralleling currents over a plane beach with the slope of 1:20. The offshore wave conditions are $H_0 = 0.61\, \text{m}$, $T = 4\, \text{s}$. — modeling results; o analytical solutions. For comparison purpose, non-breaking waves are assumed.
CHAPTER 3
STEADY TWO-DIMENSIONAL NEARSHORE CIRCULATION MODEL

3.1 Introduction

Wave-induced nearshore currents, including alongshore currents, circulation cells and rip currents, carry sediments and are directly responsible for beach morphology evolution. Energetic nearshore currents, especially rip currents, are dangerous for beach swimmers. According to the United States Lifesaving Association, rip currents alone cause more than 100 deaths each year in United States alone, which on average is more deaths than hurricanes, tropical storms, lightning and tornadoes combined (Lascody, 1998). In addition, nearshore circulations make exchange of offshore and onshore water, affecting the cross-shore mixing of heat, nutrient and chemical substances in nearshore regions. The importance of the nearshore circulation has inspired extensive research efforts by numerous scientists and coastal engineers in past decades.

Numerous field observations and laboratory studies have been conducted all over the world and great achievements in improving the understanding of the mechanism, forcing, and stability of the wave-induced nearshore currents, have been accomplished. Mathematical models, which do not have some limitations of the field or laboratory measurements, can study the nearshore circulation systems in more detail and more systematically. A variety of numerical models studying the nearshore circulation system have been developed since the pioneering work on introducing the concept of wave-induced radiation stresses for small amplitude waves by Longuet-Higgins and Stewart (1964). Among them, early works include Bowen (1969), Longuet-Higgins (1970), Thornton (1970), and Noda (1974). More recently, more sophisticated modeling efforts include Van Dongeren et al. (1994), Sorensen et al. (1998), Chen et al. (1999), Haas and Svendsen (2000), and Yu and Slinn (2003), among others.
Most of nearshore circulation models are generally based on either the Nonlinear Shallow Water Equations (NSWE) (e.g. Dongeren and Svendsen, 1997) or the Boussinesq-type equations (e.g. Chen et. al, 1999), and the former is used more often. Most of these models are time-dependent and were developed using the finite difference method, and some of them are very computer intensive and require very small time steps to reach steady-state solutions. Aside from current instabilities which can happen under certain circumstances, steady wave forcing usually leads to steady current motions including longshore currents, rip currents and circulation cells, depending on the wave forcing itself and the specific bathymetry. In this case, a steady circulation model that is robust and computationally much less time consuming than time-dependent models will be very useful in a practical sense. Up to date, to our knowledge steady current models exclusively developed for nearshore circulation are rare.

In this study, a steady circulation model based on the two-dimension depth-averaged NSWE is developed by the finite volume method (FVM). In computational fluid dynamics (CFD) field, the FVM method has become more and more popular owing to its advantages over the finite difference method (FDM) and the finite element method (FEM). For example, FVM naturally satisfies many conservation laws, is computationally efficient and can accommodate in any types of grids. Unlike most of the existing models, in this model the pressure-correction method is used to solve the NSWE, which is very efficient and robust especially for steady problems. The model is developed using the boundary fitted structured non-orthogonal grids. Consequently, the model can simulate currents in relatively complex bathymetries.

In this chapter, we focus on the development of the 2-D steady nearshore circulation model. The numerical procedures, discretization of governing equations, treatment of various boundary conditions, and the linear algebraic solvers will be discussed in detail. The model validation and
convergence tests are conducted using simple computational tests. Applications of this circulation model incorporated with the wave model introduced in the previous chapter in simulating nearshore wave-induced currents are presented in chapter 4.

3.2 Formulations

In modeling nearshore two-dimensional horizontal circulations, the 2-D nonlinear shallow water equations (NSWE) are often used to describe depth averaged mean flows. The NSWE describe incompressible flow when the ratio of the vertical scale to the horizontal scale is a small value, derived based on the depth-integrated (or depth averaged) Navier-Stokes equations. Hydrostatic pressure is usually assumed in modeling the NSWE since hydrostatic balance is accurate when the characteristic horizontal length scale is much larger than vertical length scale.

Hydrostatic balance states that gravity balances the pressure gradient in the vertical equation of motion, implying that vertical acceleration are negligible:

\[ \frac{\partial P}{\partial z} = -\rho g \] (3-1)

where, the density \( \rho \) is constant. This equation implies that the horizontal pressure gradient is independent of \( z \). This is the key simplification that underlies the 2-D shallow water system.

For steady problem, dropping the time derivative terms of the traditional shallow water equations, the governing equations of this steady circulation model are written as follows

**Continuity Equation:**

\[ \frac{\partial}{\partial x} (hu) + \frac{\partial}{\partial y} (hv) = 0 \] (3-2)

**Momentum Equations:**

\[ \frac{\partial}{\partial x} (huu) + \frac{\partial}{\partial y} (huv) = -\frac{h}{\rho} \frac{\partial P}{\partial x} (\tau_x) + F_x + \tau'_x \] (3-3)
\[
\frac{\partial}{\partial x}(huv) + \frac{\partial}{\partial y}(hvv) = -\frac{h}{\rho} \frac{\partial P}{\partial y} - (\tau_{x})_{y} + F_{x} + \tau'_{x} \quad (3-4)
\]

where \((u, v)\) are the depth-averaged velocities in \((x, y)\) directions, \(h\) the local water depth, \(\rho\) the water density, \(P = \rho g (\eta - z)\) is the pressure, \(\eta\) the mean water surface displacement relative to the undisturbed free surface, \(g\) the gravity, \((\tau_{x})_{x}\) and \((\tau_{y})_{y}\) denote the bottom friction effect in \(x\) and \(y\) coordinates respectively. \(\tau'_{x}\) and \(\tau'_{y}\) represents the eddy viscosity (or lateral mixing in oceanic processes) effect. \(F_{x} = -\frac{1}{\rho} \left( \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xy}}{\partial y} \right)\) and \(F_{y} = -\frac{1}{\rho} \left( \frac{\partial S_{yx}}{\partial x} + \frac{\partial S_{yy}}{\partial y} \right)\) are the wave forcing term in \(x\) and \(y\) direction, respectively. In which \(S_{xx}, S_{xy}, S_{yy}\) are wave radiation stress components.

It is worth mentioning that the rigid-lid approximation has been used in the governing equations given above for simplicity. The rigid-lid approximation neglects the effect of surface elevation \(\eta\) on the continuity equation. Justifications of this approximation in simulating nearshore currents can be found in Allen et al. (1996), Özkan-Haller and Kirby (1999), among others.

**Bottom Friction**

Bottom friction plays a very important role in the wave-induced nearshore circulation. In the surf zone, bottom stress could be the major force to balance the wave forcing under certain conditions. It is very difficult to develop a rigorous formula for bottom friction, especially when both waves and currents are present. Jonsson et al. (1974) have proposed a semi-empirical formulation for the combined case

\[
\tau_{b} = \frac{1}{2} \rho f w |w| \quad (3-5)
\]
where, $\mathbf{w} = \mathbf{u}_c + \mathbf{u}_w$ is the vector sum of the current velocity vector and wave orbital velocity vector. The friction coefficient $f = f_w + (f_c - f_w) \sin \alpha$, and $\alpha$ is the angle between the total velocity vector $\mathbf{w}$ and the wave orbital velocity $\mathbf{u}_w$. In this general formulation, information about $f_w$ is rarely available and $\alpha$ is also difficult to obtain, which make it extremely difficult to apply directly. Instead, the simple linear damping formulations are often used in practice by assuming that the incident waves approach the shore with a small angle and the current is weak compared to the wave orbital velocity:

$$\tau_b = c_f |U_0| \mathbf{u} = \mu \mathbf{u}$$  \hspace{1cm} (3-6)$$

where, $|U_0|$ is the amplitude of wave horizontal orbital velocity at the sea bottom. According to the linear shallow water theory, $|U_0|$ can be evaluated by (Thornton and Guza, 1983)

$$|U_0| = \frac{1}{4} \sqrt{\frac{g \pi}{h} H_{rms}}$$  \hspace{1cm} (3-7)$$

Furthermore, the drag coefficient $c_f$ is also a major source of uncertainty. Constant $c_f$ with the typical value of $O(0.01)$ from offshore to shoreline has been used by many investigators (Özkan-Haller and Kirby, 1999; Slinn et al. 2000; Yu and Slinn, 2003). Ruessink et al. (2001) suggested that $c_f$ could be parameterized with the Manning-Strickler equation (Sleath, 1984)

$$c_f = 0.015 \left( \frac{k_a}{h} \right)^{1/3}$$  \hspace{1cm} (3-8)$$

where, $k_a$ is the apparent bed roughness and is assumed to be constant and time-independent. According to Ruessink et al. (2001), $k_a$ is typically in the range of 0.01~0.06m in simulating longshore currents. In the present circulation model, bottom friction uses ‘weak current, small incident angle’ with variable drag coefficient $c_f$ given by Equation (3-8).
**Lateral Momentum Mixing**

There are three sources of lateral momentum mixing: (a) mixing owing to turbulence induced by wave breaking; (b) mixing due to bottom-induced turbulence; (c) mixing due to the vertical non-uniformity of horizontal velocities. The first two sources are turbulence induced stress and can be modeled by utilizing the eddy viscosity approach (Sancho, 1997)

\[
\tau'_\alpha = \frac{\partial S'_{\alpha\beta}}{\partial x_\beta}, \quad \text{and} \quad S'_{\alpha\beta} = \nu_t \left( \frac{\partial hu_\alpha}{\partial x_\beta} + \frac{\partial hu_\beta}{\partial x_\alpha} \right)
\]  

(3-9)

where, \(\nu_t\) is the turbulence eddy viscosity and accounts for both breaking turbulence and bottom turbulence.

Putrevu and Svendsen (1999) proposed the following formulation to approximate the momentum mixing due to vertical variation in horizontal velocities

\[
\tau^d_\alpha = \frac{\partial}{\partial x_\beta} \left( D_{\alpha\beta} \frac{\partial hu_\alpha}{\partial x_\delta} + D_{\alpha\delta} \frac{\partial hu_\beta}{\partial x_\delta} \right)
\]  

(3-10)

They stated that \(D_{\alpha\beta}\) is roughly proportional to the wave-induced fluxes in the horizontal directions. Therefore, for waves that are obliquely incident at small angles, the volume fluxes due to waves will most occur in \(x\) direction. Under this assumption, we know the dominant dispersive mixing term is \(D_{xx}\) term and only this term will remain as a first approximation, which gives

\[
\tau^d_x = 2 \frac{\partial}{\partial x} \left( D_{xx} \frac{\partial hu}{\partial x} \right) + \frac{\partial}{\partial y} \left( D_{xx} \frac{\partial hv}{\partial x} \right), \quad \tau^d_y = \frac{\partial}{\partial x} \left( D_{xx} \frac{\partial hv}{\partial x} \right)
\]  

(3-11)

Combining the turbulent momentum mixing and the dominant dispersive mixing gives

\[
\tau'_x = 2 \frac{\partial}{\partial x} \left[ (\nu_t + D_{xx}) \frac{\partial hu}{\partial x} \right] + \frac{\partial}{\partial y} \left[ (\nu_t + D_{xx}) \frac{\partial hv}{\partial x} \right] + \frac{\partial}{\partial y} \left( \nu_t \frac{\partial hu}{\partial y} \right)
\]  

(3-12)
\[ \tau'_y = \frac{\partial}{\partial x} \left[ (v_t + D_{xx}) \frac{\partial hv}{\partial x} \right] + 2 \frac{\partial}{\partial y} \left( v_t \frac{\partial hv}{\partial y} \right) + \frac{\partial}{\partial x} \left( v_t \frac{\partial hu}{\partial y} \right) \] (3-13)

Svendsen and Putrevu (1994) showed that \( D_{xx} \) can be much larger than \( \nu_t \). Therefore, it is convenient and reasonable to remain only the \( \nu = (v_t + D_{xx}) \) terms, and then the terms are reduced to

\[ \tau'_x = 2 \frac{\partial}{\partial x} \left( v \frac{\partial hu}{\partial x} \right) + \frac{\partial}{\partial y} \left( v \frac{\partial hv}{\partial x} \right), \quad \tau'_y = \frac{\partial}{\partial x} \left( v \frac{\partial hv}{\partial x} \right) \] (3-14)

The combined eddy viscosity \( \nu \) is parameterized as follows

\[ \nu = c_1 U_0 h + M h \left( \frac{\epsilon_b}{\rho} \right)^{1/3} \] (3-15)

where, \( c_1 \) is a constant coefficient typically being \( O(0.01) \) and \( U_0 \) is the wave bottom orbital velocity. \( M \) is the mixing coefficient and is a constant being 0.06~0.5 suggested by Özkan-Haller and Kirby (1999). \( \epsilon_b = \rho g D_{tot} \) represents the energy dissipation due to wave breaking, and \( D_{tot} \) is defined in chapter 1.

### 3.3 Numerical Procedures

#### 3.3.1 Discretization Approach

Three discretization approaches widely used in computational fluid dynamics (CFD) are: Finite Difference Method (FDM), Finite Volume Method (FVM) and Finite Element Method (FEM). FDM is the oldest method for numerical solution of partial differential equations, believed to have been introduced by Euler back to 18th century. FDM is also the easiest one to understand and to use for simple geometries. One main disadvantage of FDM is that the conservation is not satisfied naturally unless special care is taken. In addition, the restriction to simple geometries is another significant unfavorable feature of FDM.
The distinguishing feature of FEM is that the equations are multiplied by a weight function before they are integrated over the entire domain. The most important advantage of FEM is the ability to deal with arbitrary geometries. The grids are easily generated and local refinement can be easily done too. The principle drawback is that the matrices of the algebraic equations are not as well structured as those for regular grids.

FVM uses the integral form of the conservation equations as its starting points. The solution domain is subdivided into a finite number of contiguous control volumes (CVs), and the integrated conservation equations are applied to each CV, as well as to the solution domain as a whole. If we sum equations for all CVs, the global conservation equations will be obtained since the surface integrals of inner CV faces cancel out. Thus, global conservation is built into the method, which is one of its principle advantages. In addition, the FVM can accommodate any type of grids, so it is suitable for complex geometry. All terms using FVM have physical meaning, which makes it easy to understand and implement. FVM is also usually more computationally efficient than FEM. In this study, the FVM is used to discretise the integral form of the governing equations.

3.3.2 Grid and Variable Arrangement

Computational grids can be roughly divided into two categories: structured (regular) grid and unstructured grid. Structured grids consist of families of grid lines with the property that members of a single family do not cross each other and cross each member of other families only once. Unlike the structured grids, unstructured grids do not have a restriction on the shape of the elements or control volumes nor number of neighbor elements or nodes. Unstructured grids are flexible and can fit an arbitrary solution domain boundary. However, the matrix of the algebraic
equation system is no longer well structured. Consequently, it is difficulty to find a good solver, and solvers are usually slower than those for structured grids.

In order to be able to calculate flows in relatively complex geometries at the same time to use structured grids, the boundary fitted structured non-orthogonal grids are used in this circulation model. Since the grid lines can follow the boundaries, the boundary conditions are more easily implemented. One disadvantage of such grids is that more terms introduced into the discretized equations thereby increasing both the difficulty of coding and the cost of solving the equations. Moreover, for a grid with strong non-orthogonality, unphysical solutions may result, and accuracy and efficiency of the algorithm will be affected. Nevertheless, this can be circumvented by making the grid as orthogonal as possible in practice. In practice, for a simple geometry the orthogonal grids are favorable since it is easier to generate and the algebraic system associated is well-structured too.

In this model, the principles of discretization and solution methods are developed using the general non-orthogonal grids and they are of course valid for orthogonal grids, which are essentially a special case of non-orthogonal grids. Figure 3-1 shows a typical non-orthogonal 2-D control volume (CV) and the notation used. The CV surface consists of four plane faces, denoted by lower-case letters corresponding to their direction ( \( w, n, e \) and \( s \) ) with respect to the central node \( P \). The neighboring CV centers are represented by upper-case letters corresponding to their directions too. This notation system is used throughout this chapter.

There two ways to arrange unknown dependent variables on a grid: staggered arrangement and collocated arrangement, shown as Figure 3-2. The staggered arrangement makes the stencil for the continuity equation very compact, requiring no interpolations in evaluation of mass fluxes. In addition, the pressure-gradient terms in the momentum equations require no interpolation.
These two features make the staggered grid approach responsive to grid-level fluctuations, thus uniquely capturing all the resolvable modes and preventing accumulation of spurious energy at the grid level. However, staggered-grid schemes become very awkward to generalize to complex geometries. It also has been pointed out that the use of staggered grid for complex geometries leads to either high memory requirements, or inefficient solution methods (Zang et al, 1994). Unlike staggered arrangement, the collocated (non-staggered) arrangement stores all variables at the same set of grid points and all variables share the same control volumes. Generally, using a collocated grid requires lower memory and less computational cost. Moreover, the collocated arrangement has significant advantages in complicated computational domain, especially when the boundaries have slope discontinuity. The main drawback of collocated arrangement is the occurrence of spurious oscillation in pressure field due to the weak coupling between the velocities and the pressure. Fortunately, this problem has been greatly alleviated since the improved pressure-velocity coupling algorithms were developed in 1980’s. The current circulation model is developed based on collocated variable arrangement.

3.3.3 Solution Algorithm: SIMPLE Algorithm

Solving the 2-D depth-averaged Navier-Stokes equations is very complicated by the natures of the equations themselves: the convective terms in the momentum equations are nonlinear; dependent variables are intricately coupled, and there is no apparent dominant variable for each equation; there is no independent equation for the pressure, whose gradients contributes to momentum equations. Solution of steady equations is even more difficult since many methods such as fractional step methods and some explicit time advance schemes that work well for unsteady problems may not work for steady problems.
The pressure-correction method, which belongs to the segregated algorithm, is an approach widely used for solving the Navier-stokes equations for fluid flow problems. The SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) algorithm, proposed by Patankar and Spalding (1972), was the first such algorithm widely used in the literature.

The distinguishing characteristic of SIMPLE-family algorithm is that a pressure-correction term is introduced to the calculation procedure of each segregated solution step (outer iteration level) to improve the velocity field computed from the linearized momentum equation as the intermediate value. The pressure-correction equation is built up by substituting the corrected velocities into the discretized continuity equation. Once the pressure correction obtained, the velocities solved from linearized momentum equations can be updated and the modified velocities satisfy the mass conservation for each control volume at each outer iteration level, which is critical for iteration convergence.

Now we introduce the SIMPLE algorithm in some detail. At the iteration level $m$, the discretized momentum equations can be written as

$$A^m_p u^m_{i,P} + \sum_l A^m_{i,l} u^m_l = S^m_u - SP^m_{u}$$

in which, the subscript ‘$P$’ denotes the node at which the partial differential equation is approximated and index $l$ represents neighbor nodes involved in finite volume approximation, the source term $S^m_u$ contains all of the terms other than pressure term, that might depend on the $u^m$ and/or other variables at the level $m$. SP is the pressure term, $SP = \left( \frac{h}{\rho} \frac{\partial P}{\partial x} \right)_p dV$.

Due to the non-linearity and coupling of the underlying differential equations, Equation (3-16) can only be solved using iterative methods. The iterations within one time step, in which the coefficient and source matrices are updated, are called the outer iteration to distinguish them
from the inner iterations performed on linear algebraic systems with fixed matrix coefficients.

On each outer iteration, the equations to be solved are:

\[ A^u_{i,p} u^m_{i,p} + \sum_i A^u_{i,i} u^m_{i,i} = S^{m-1}_{u_i} - S P^{m-1}_{u_i} \]  

(3-17)

The superscript \( m^* \) stands for solution to be sought. The right hand side of the equation is evaluated using the variables at the preceding outer iteration. Rewriting Equation (3-17) gives,

\[ u^m_{i,p} = \frac{S^{m-1}_{u_i} - \sum_i A^u_{i,i} u^m_{i,i}}{A^u_{i,i}} - \frac{1}{A^u_{i,i}} \left( \frac{h}{\rho} \frac{\delta P^{m-1}_i}{\delta x} \right)_i \]  

(3-18)

where \( \delta / \delta x \) represent a discretized spatial derivative. Since the pressure used in Equation (3-18) was obtained from the previous outer iteration, the velocities computed from Equation (3-18) do not necessarily satisfy the discretized continuity equation.

The discretized continuity equation may be written as following:

\[ \frac{\delta}{\delta x_i} (h u^m_{i,i}) = 0 \]  

(3-19)

Now we define the pressure correction \( P' \) as the difference between the corrected pressure field and the previous pressure field \( P^{m-1} \). Similarly velocity corrections \( u' \) and \( v' \) are introduced as small corrections to the intermediate velocities computed from the linearized momentum equations:

\[ u^m = u^{m*} + u', \quad v^m = v^{m*} + v' \quad \text{and} \quad P^m = P^{m-1} + P' \]  

(3-20)

Substitution of Equation (3-20) into the momentum equations (3-18) yields the relation between the velocity correction and pressure correction as follows

\[ u'_{i,p} = -\frac{\sum_i A^u_{i,i} u'_{i,i}}{A^u_{i,i}} - \frac{1}{A^u_{i,i}} \left( \frac{h}{\rho} \frac{\delta P'}{\delta x_i} \right)_i \]  

(3-21)
For the sake of convenience, the first term on the right hand side of the above equations is represented by $\tilde{u}^\prime_{i,p}$:

$$u^\prime_{i,p} = \tilde{u}^\prime_{i,p} - \frac{1}{A_p} \left( \frac{h}{\rho} \delta P' \right)_p$$  \hspace{1cm} (3-22)

Substituting Equation (3-22) into the discretized continuity Equation (3-19), we obtain the following pressure correction equation:

$$\frac{\delta}{\delta x_i} \left[ \frac{h}{A_p} \left( \frac{h}{\rho} \delta P' \right)_p \right] + \left[ \frac{\delta}{\delta x_i} \left( h\tilde{u}^m \right)_p \right] + \left[ \frac{\delta}{\delta x_i} (h\tilde{u}^\prime) \right] = 0$$  \hspace{1cm} (3-23)

The velocity corrections $\tilde{u}^\prime_i$ are unknown at this point, so the common practice is to drop them and neglecting this term does not influence the final steady-state solution. The above equation is then reduced to a normal Poisson equation and can be solved using regular methods. Omission of the last term is the main approximation of the SIMPLE algorithm and is probably the major reason why the resulting method does not converge very rapidly. Various modified SIMPLE methods have been developed, and they are called SIMPLE-family methods. The detailed descriptions of these methods can be found in many CFD textbooks such as Patankar (1980), Fletcher (1991) and Ferziger and Peric (2002) etc.

### 3.3.4 Discretization of Governing Equations

In this section, the finite volume schemes used to discretise the governing equations based on the SIMPLE algorithm on a structured non-orthogonal collocated grid is described in some detail. Details of discretization of terms in governing equations are given in Appendix C.

Once the approximations for all terms are completed, rearranging all coefficients and source terms, we can obtain the following algebraic equation systems for momentum equations:

$$A^m_p u^m_{i,p} + \sum_l A^{l_i}_i u^m_{l,i} = Q^m_{u_i}, \quad l = E, W, N, S$$  \hspace{1cm} (3-24)
where,

\[ A_{E}^{w} = \min(m_e, 0) - \frac{(v)_e S_e h_e}{L_{PE}} \]  \hspace{1cm} (3-25)

\[ A_{W}^{w} = \min(m_w, 0) - \frac{(v)_w S_w h_w}{L_{PW}} \]  \hspace{1cm} (3-26)

\[ A_{N}^{w} = \min(m_n, 0) - \frac{(v)_n S_n h_n}{L_{PN}} \]  \hspace{1cm} (3-27)

\[ A_{S}^{w} = \min(m_s, 0) - \frac{(v)_s S_s h_s}{L_{PS}} \]  \hspace{1cm} (3-28)

\[ A_p^w = -\left( A_{E}^{w} + A_{W}^{w} + A_{N}^{w} + A_{S}^{w} \right) + \frac{\mu}{h_p} \Delta \Omega \]  \hspace{1cm} (3-29)

\[ Q_u = m_e u_{i,e}^{m-1} - \max(m_e, 0) u_{i,P,E}^{m-1} - \min(m_e, 0) u_{i,P,E}^{m-1} + m_w u_{i,w}^{m-1} - \max(m_w, 0) u_{i,w}^{m-1} - \min(m_w, 0) u_{i,w}^{m-1} + m_n u_{i,n}^{m-1} - \max(m_n, 0) u_{i,n}^{m-1} + m_s u_{i,s}^{m-1} - \max(m_s, 0) u_{i,s}^{m-1} - \min(m_s, 0) u_{i,s}^{m-1} + (v)_e S_e \left[ \frac{\partial h_{u_i}}{\partial n} \right]_{e} - (v)_{w} S_w \left[ \frac{\partial h_{u_i}}{\partial n} \right]_{w} - (v)_{n} S_{n} \left[ \frac{\partial h_{u_i}}{\partial t} \right]_{n} \]  \hspace{1cm} (3-30)

\[ + (v)_s S_s \left[ \frac{\partial h_{u_i}}{\partial n} \right]_{s} - (v)_{s} S_{s} \left[ \frac{\partial h_{u_i}}{\partial t} \right]_{s} + R_{S} + S_{P} \]

Solving the linearized momentum equations using the iterative solver gives the intermediate values of velocities *u* and *v*, which may not satisfy the continuity equation since ‘old’ mass fluxes and pressure from previous outer iteration were used in discretizing the momentum equations. So:

\[ m_e^* + m_w^* + m_n^* + m_s^* = \Delta m_p^* \]  \hspace{1cm} (3-31)

where the mass fluxes are calculated using the intermediate velocities *u* and *v*. \( \Delta m_p^* \) stands for the residual of the continuity equation.

Substituting Equation (3-31) into Equation (3-23) leads to the pressure-correction equation:

\[ A_p^p p'_p + \sum_i A_i^p p'_i = -\Delta m_p^* - \Delta m_{p'}^* \hspace{1cm} (3-32) \]

in which,
The term $\Delta m_p'$ is analogous to $\Delta m_p^*$, with $(\tilde{u}', \tilde{v}')$ replacing $(u^*, v^*)$, see Equation (3-21) and (3-31). Since the velocity corrections at this point are not known prior to the solution of the pressure-correction equation, the $\Delta m_p'$ term is neglected if using SIMPLE algorithm as mentioned above.

### 3.3.5 Solution of Linear Algebraic Equations

In the previous section, we presented a detailed description of discretization of the governing equation and three sets of algebraic equations were obtained, two for the momentum equations and one from the pressure-correction equations. All the three systems have the same form and the algebraic equation for a control volume (see Figure 3-4) can be written as

$$A_p\phi_p + A_w\phi_w + A_e\phi_e + A_s\phi_s + A_n\phi_n = Q_p$$  \hspace{1cm} (3-37)

Over the whole computational domain, the matrix version of the algebraic system is given by:

$$[A][\phi] = [Q]$$  \hspace{1cm} (3-38)

The coefficient matrix is a five-diagonal matrix, two other grid nodes in each direction involved.

There are two families of solution techniques for linear algebraic equations: direct methods and iterative methods. Simple examples of direct methods are Cramer’s rule matrix inversion, Gauss elimination and LU decomposition. Direct methods are designed to deal with full matrices and are usually very expensive. Iterative methods are based on the repeated application of a
relatively simple algorithm leading to eventual convergence after certain number of repetitions. For sparse matrices, if each iteration is very cheap and the number of iterations is small, the iterative methods may be efficient and cost much less than a direct method. Well-known iterative solvers include Jacobi methods, Gauss-Seidel methods, Conjugate Gradient methods and others. The iterative method called strongly implicit procedure (SIP) proposed by Stone (1968) is used in this circulation model.

The SIP method is essentially a version of incomplete lower-upper decomposition method. The ILU methods are based on the idea of using an approximate LU factorization of the coefficient matrix $A$ as the iteration matrix $M$:

$$M = LU = A + N$$

(3-39)

where $L$, $U$ are the lower and upper matrix respectively and $N$ is small compared to $A$. Since the convergence is faster if the pre-conditioner matrix $M$ is a good approximation of coefficient matrix $A$, we wish to select $L$ and $U$ such that $M$ is as good approximation to $A$ as possible. Unfortunately, this method converges slowly. Stone (1968) proposed that convergence can be improved by allowing $N$ have non-zero elements on the diagonals corresponding to all non-zero diagonals of $LU$. And elements on non-zero diagonals of $N$ are chosen so that $N\phi \approx 0$.

Detailed description and the algorithm for five-diagonal matrices can be found in the CFD textbook Ferziger and Peric (2002).

The SIP method is designed for equation system arising from discretization of partial differential equations. Incomplete $LU$ factorization of matrix $A$ need be calculated only once, prior to the first iteration. Usually, convergence is obtained in a small number of iterations and it always has a smoothing error distribution, which makes it a good solver by itself and a good smoother when the multigrid method is applied.
3.4 Boundary Conditions

Careful treatment of boundary conditions is important for a numerical model to simulate a physical flow correctly and accurately. The finite volume method requires that fluxes at boundaries either be known or expressed in terms of known quantities and/or interior nodal values. Since no additional equations are given from boundary conditions, no additional unknowns should be introduced either. The boundary conditions often encountered in modeling nearshore circulation have been built in this model. Implementation of those boundary conditions is described in some detail.

3.4.1 No-Slip Wall Boundary Condition

The no-slip wall is one of most often used boundaries in computational fluid dynamics. No-slip condition means:

\[
(u_i)_{wall} = 0
\]

In words, it is saying that at a wall boundary the velocity of fluid is the same as the wall velocity. Meanwhile, the no-slip wall boundary also implies

\[
\left( \frac{\partial v_s}{\partial s} \right)_{wall} = 0 \quad \Rightarrow \quad \left( \frac{\partial v_n}{\partial n} \right)_{wall} = 0
\]

where, \( n \) is the local coordinate normal to the wall outwardly; \( s \) is the local coordinate parallel to the wall. \( v_s, v_n \) are the velocity components in \( s \) and \( n \) direction, respectively.

Since there are no fluxes across the wall, convective fluxes of all quantities are zero. The diffusive fluxes should be treated carefully. The diffusive flux in the \( u \) momentum equation at the wall can be written as

\[
F_u^d = \nu S_w \left( \frac{\partial u}{\partial n} \right)_w
\]

The velocity \( u \) can be written as
\[ u = v_n x_n + v_s x_s, \quad x_n = \vec{x} \cdot \vec{n}, \quad x_s = \vec{x} \cdot \vec{s} \]  

(3-43)

Substituting Equation (3-43) into (3-42), combined with Equation (3-41), we arrive at:

\[ F_u^d = (v)_w S_w \frac{\partial}{\partial n} (h v_x x_s)_w = (v)_w S_w (x_s)_w \left[ \frac{(h v_s)_p - (h v_s)_w}{L_{p,w}} \right] \]  

(3-44)

The quantity \((h v_s)_p\) can be evaluated according to Equation (C-13). The matrix coefficients for \(u\) at the wall boundary can be collected according to Equation (3-44). Treatment of the \(v\) momentum equation is the same and will not be reproduced here.

At the wall boundary the fluid velocity is always equal to the wall velocity, so in the process of solving the pressure-correction equation the velocity correction \(u'\) is zero. Consequently, the matrix coefficient for the pressure-correction equation should be set correspondingly.

### 3.4.2 Periodic Boundary Condition

A periodic boundary condition is another often used boundary condition in nearshore circulation models. Periodic boundary conditions are usually used when the bathymetry and flow pattern are approximately periodic with certain length scale, such as periodic rip channels. Periodic boundary condition is also used as an approximation sometimes when information at a boundary is not available.

Figure 3-5 shows the sketch of a pair of periodic boundaries. The periodic boundary condition could be regarded that the opposite ends are tied together. All quantities at one periodic boundary should be exactly the same as those at the other boundary. Velocities at boundaries are interpolated using values of the two neighboring CV centers:

\[ (u,v)_h = f (u,v)_i + (1-f)(u,v)_o \]  

(3-45)
where $f$ is the linear interpolation factor. The mass fluxes and convective fluxes at boundaries can be computed using the interpolated velocities. The diffusive fluxes at boundaries are also convenient to be evaluated using the interpolated velocities. Unlike the wall boundary condition, the pressure corrections at periodic boundaries usually are not zero. The pressure correction equations at CVs around the boundaries should be treated the same way as those at inner CVs, except the notation of neighboring CVs.

### 3.4.3 Symmetric Boundary Condition

In many flows, there are one or more symmetry planes. At a symmetry plane, the following conditions apply:

$$v_n = 0, \quad \frac{\partial v_t}{\partial n} = 0$$

And thus the convective fluxes of all quantities are zero. Also, the normal gradients of the velocity components parallel to symmetry plane and of all scalar quantities are zero too.

The diffusive flux in the $u$ momentum at a symmetry plane can be written as

$$F_u^d = (v)_w S_w \frac{\partial}{\partial n} (h v_n x_n)_w = (v)_w S_w (x_n)_w \frac{(h v_n)_p - (h v_n)_w}{L_{p_w}}$$

It is similar to the case of wall boundaries, except displacing the normal gradient of shear velocity $v_t$ with that of norm velocity $v_n$.

Treatment of a symmetry boundary condition in the pressure-correction is quite simple since the mass flux is zero there.

### 3.4.4 Inlet Boundary Condition

At an inlet boundary, all quantities are prescribed. In this model, the three fundamental dependent variables $(u, v, P)$ need to be known at inlet boundaries. All the convective fluxes and mass fluxes can be calculated, and the diffusive fluxes are approximated using known boundary
values of the variables. One-sided finite difference is used to compute the gradients. Since the velocities and the pressure are known at boundaries, both the velocity correction and the pressure correction should be set to zero at boundaries.

3.4.5 Outlet Boundary Condition

At outgoing boundaries, the flow properties are usually not known. For this reason, the outgoing boundaries should be selected as far from the downstream of the region of interest as possible. If the flow at the outgoing boundaries reaches a fully developed state and variation of all quantities \((u, v, P)\) in the flow direction is small, it may be reasonable to make the approximation of zero gradients along the boundary grid line. The zero gradient condition on a grid line can be implemented very easily. For example, the outgoing boundary is located on the east side of domain. The zero gradient condition gives:

\[
u_e = u_p, \quad v_e = v_p\quad (3-48)
\]

The condition can be implemented implicitly, and the algebraic equation for a CV next to boundary can be written as

\[
(A_p + A_E \phi_p + A_w \phi_w + A_S \phi_S + A_N \phi_N) = Q_p
\quad (3-49)
\]

The zero gradient approximation is convenient to implement. However, it does not guarantee the mass conservation of neither single CVs nor over the computational domain as a whole. If higher accuracy is required, mass conservation rule can be used to approximate the velocities at the outgoing boundaries. For a CV at the boundary, mass flux out should be equal to mass flux coming into the CV, which gives:

\[
(\bar{v}_n)_e = \frac{-(m_s + m_n + m_w)}{h_S}\quad (3-50)
\]
where, \((\tilde{v}_n)_e\) is the velocity component normal to the east cell face. The velocity component parallel to east face can be given still using zero gradient approximation. To ensure the global mass conservation, the normal velocity computed by Equation (3-50) should be further corrected by:

\[
(\nu_n)_e = (\tilde{v}_n)_e \times \frac{(\text{Mass})_{\text{in}}}{(\text{Mass})_{\text{out}}}
\]  

(3-51)

where \((\text{Mass})_{\text{in}}\), \((\text{Mass})_{\text{out}}\) are the total mass flux coming in and going out of the entire computational domain. \((\text{Mass})_{\text{out}}\) can be computed approximately using \((\tilde{v}_n)_e\).

The velocity at the outgoing boundaries is not corrected by means of pressure corrections. Hence in the discretized pressure-correction equation, the matrix coefficient \(A_E = 0\).

**3.4.6 Prescribed Pressure Boundary Condition**

In some situations, the exact details of the flow distribution are unknown but the boundary values of pressure are prescribed. A common way of dealing with this type of boundary condition is to extrapolate the velocities at the boundary. Since the pressure is prescribed, the pressure correction is set to zero at boundary. Unlike the inlet boundary condition, the velocity correction is not zero for this boundary condition.

In order to test the implementation of a prescribed pressure boundary condition, two simple flow problems, Poiseuille flow and Couette flow, are simulated using the current model. The Poiseuille flow is known as the incompressible, viscous flow in two paralleling stationary plates. If one plate is moving in its own plane, it becomes the Couette flow problem. The geometry and boundary conditions for the two flow problems are shown schematically in Figure 3-6.

The governing equation for these two flow problems is
\[
\frac{d^2 u}{dy^2} = \frac{1}{\mu} \frac{dp}{dx}
\]  \hspace{1cm} (3-52)

Prescribed pressure values at the up and down boundaries are given. For Poiseuille flow, the lateral boundary conditions are

\[u(y = 0) = 0, \quad u(y = h) = 0\]  \hspace{1cm} (3-53)

And the analytical solution of the Poiseuille flow is

\[u(y) = -\frac{1}{2\mu} \frac{dp}{dx} y(h - y)\]  \hspace{1cm} (3-54)

For Couette flow, the lateral boundary conditions are

\[u(y = 0) = 0, \quad u(y = h) = U\]  \hspace{1cm} (3-55)

where \(U\) is the speeding of the moving plate. The analytical solution is

\[u(y) = \frac{yU}{h} + \alpha \frac{yU}{h^2}(h - y)\]  \hspace{1cm} (3-56)

in which, \(\alpha = -\frac{h^2}{2\mu U} \frac{dp}{dx}\) is the non-dimensional pressure gradient. When \(\alpha > 0\), pressure decreases in flow direction and \(u\) is positive everywhere; when \(\alpha < 0\), pressure increases in flow direction and the adverse pressure gradient works against the local viscous forces. Flow will reverse when \(\alpha < -1\), which indicates the adverse pressure gradient is larger than the viscous forces.

In simulating the two flow cases, convection and bottom friction in the circulation model are turned off and the values of pressure at both ends are input. Figure 3-7 shows the comparison between the simulation results and analytical results. It can be seen that good agreement with analytical solutions has been achieved, and confidence in dealing the prescribed pressure boundary conditions of this model has been gained through the two simple tests.
3.5 Numerical Tests

3.5.1 Pure Diffusion in a Rectangular Basin

Generally, the possible best way to test a newly developed model is to simulate test examples whose analytical solutions are available, allowing a thorough investigation of models’ performance, convergence and accuracy. An artificial test, pure diffusion in a rectangular basin, is designed and analytical solutions are available with some simplifications.

Neglecting the advection terms and assuming constant water depth, the shallow water equations (3-2)-(3-3) are reduced to:

\[
\begin{align*}
    u_x + v_y &= 0 \quad (3-57) \\
    \frac{1}{\rho} \frac{\partial P}{\partial x} &= -\mu \frac{u}{h^2} + \nu \nabla^2 u + \frac{F_x}{h} \quad (3-58) \\
    \frac{1}{\rho} \frac{\partial P}{\partial y} &= -\mu \frac{v}{h^2} + \nu \nabla^2 v + \frac{F_y}{h} \quad (3-59)
\end{align*}
\]

where \( F_x \) and \( F_y \) are the input forcing in x- and y-direction, respectively.

Using the stream function method, equations (3-57)-(3-59) can be combined to just one equation. In detail, it starts with the relation between the stream function \( \psi \) and velocities.

\[
\begin{align*}
    u &= -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \quad (3-60)
\end{align*}
\]

Taking \( y \)-derivative with Equation (3-58) gives:

\[
\begin{align*}
    \frac{1}{\rho} \frac{\partial^2 P}{\partial x \partial y} &= -\mu \frac{\partial u}{\partial y} + \nu \frac{\partial}{\partial y} \left( \nabla^2 u \right) + \frac{1}{h} \frac{\partial F_x}{\partial y} \quad (3-61)
\end{align*}
\]

Taking \( x \)-derivative with Equation (3-59) gives:

\[
\begin{align*}
    \frac{1}{\rho} \frac{\partial^2 P}{\partial x \partial y} &= -\mu \frac{\partial v}{\partial x} + \nu \frac{\partial}{\partial x} \left( \nabla^2 v \right) + \frac{1}{h} \frac{\partial F_y}{\partial x} \quad (3-62)
\end{align*}
\]

Combining equations (3-61) and (3-62) and eliminating the pressure derivatives, it gives
\[ \frac{\mu}{h^2} \omega - \nu \nabla^2 \omega + \frac{1}{h} \left( \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) = 0 \quad (3-63) \]

where, \( \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) is the vorticity. Using Equation (3-60), it is straightforward to obtain:

\[ \omega = \nabla^2 \psi \quad (3-64) \]

Rewriting Equation (3-63) in stream function, it becomes:

\[ \frac{\mu}{h^2} \nabla^2 \psi - \nabla^4 \psi + \frac{1}{h} \left( \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) = 0 \quad (3-65) \]

For a random forcing, finding the analytical solutions of Equation (3-65) may be not easy at all. However, with special forcing and boundary conditions analytical solutions are not too difficult to obtain. For example, we assumed the following forcing:

\[ F_x = -\frac{Ah}{k_y} \left[ \nu \left( k_x^4 + k_y^4 + 2k_x^2 k_y^2 \right) + \frac{\mu}{h^2} \left( k_x^2 + k_y^2 \right) \right] \sin k_x x \cos k_y y, \quad F_y = 0 \quad (3-66) \]

where \( k_x = 2\pi / L_x, \ k_y = 2\pi / L_y \). \( L_x \) and \( L_y \) are the size of the rectangular basin. We also assumed that all the four sides of the rectangular basin are symmetry boundaries, shown as Figure 3-8. The analytical solutions for this designed forcing distribution given above and symmetry boundaries are not difficult to obtain, and they are

The velocity field:

\[ u = -Ak_{y} \sin k_{x} x \cos k_{y} y, \quad v = Ak_{x} \cos k_{x} x \sin k_{y} y \quad (3-67) \]

The stream function:

\[ \psi = Ak_{y} \sin k_{x} x \sin k_{y} y \quad (3-68) \]

The vorticity:

\[ w = -A \left( k_{x}^2 + k_{y}^2 \right) \sin k_{x} x \sin k_{y} y \quad (3-69) \]
Figure 3-9 gives the plots of the input forcing and the analytical solutions on the 49×49 CV uniform grids. The designed forcing is symmetric about the horizontal centerline and anti-symmetric about the vertical centerline. This spatial distribution of forcing leads to that at a boundary the normal velocity component is equal to zero. It is the main reason why the symmetry boundary is used for all the four sides of the rectangular basin. Both the streamline contour and the vorticity contour are point symmetric about the center point. The velocity field composed of four circulation cells is point symmetric too.

Calculations for the same forcing and boundary conditions using the circulation model were conducted. Figure 3-10 shows comparison of velocities between numerical solutions and analytical solutions along horizontal and vertical central lines. The computational velocities match the analytical solutions very well.

Since the analytical solutions are available for this problem, computational errors can be easily calculated. A systemic and rigorous assessment of the computation errors can be made and then credibility of the model can be established. Second order schemes are used in discretizing the governing equations. Therefore, the model is expected to be 2\textsuperscript{nd} order accurate. The root mean square values of difference between computational and analytical velocity field have been calculated. Computational errors using uniform grids closely follow the theoretical reduction path for second order schemes as grids become finer, as shown in Figure 3-11.

In order to evaluate performance of the model on non-orthogonal grids, the same test was conducted on a non-orthogonal grid (Figure 3-12). Velocity comparison along a sample grid line is shown in Figure 3-12 (the right panel). It can be seen that accurate results are achieved on the non-orthogonal grid too. Estimation of computation errors on non-orthogonal grids also shows the second order accuracy as well (Figure 3-13).
3.5.2 Lid-Driven Cavity Flow

The incompressible flow in a square cavity with the top wall moving with a uniform velocity in its own plane is usually called the lid-driven cavity flow problem. This two-dimensional flow problem has been used by many authors for testing and evaluating their models. Benchmark solutions for this problem are available in the literature (Ghia et al., 1982). The geometry and boundary conditions are shown schematically in Figure 3-14. The moving lid creates a strong vortex around the center and sequence of weaker vortices in the corners. The Reynolds number is defined by:

\[
\operatorname{Re} = \frac{U_{lid} L}{\nu} \tag{3-70}
\]

where \( \nu \) is the viscosity.

Calculations are carried out for \( \operatorname{Re} = 100, 1000, 5000 \). For \( \operatorname{Re} = 100 \), 80×80 uniform grids are used. The 120×120 uniform grids are used for the case of \( \operatorname{Re} = 1000 \). For the very high Reynolds number case \( \operatorname{Re} = 5000 \), a 150×150 non-uniform Cartesian mesh is used. Figure 3-15 shows the computational results for \( \operatorname{Re} = 100 \). The \( u \) velocity profile along the vertical line and \( v \) velocity profile along the horizontal line passing through geometric center of the cavity show good agreement with the results of Ghia et al. (1982). Computational results and comparison of velocity profiles for \( \operatorname{Re} = 1000 \) are shown in Figure 3-16. Figure 3-18 shows the simulation results of the high Reynolds number flow \( \operatorname{Re} = 5000 \) on the non-uniform Cartesian grids (see Figure 3-17).

As seen from Figure 3-15, 3-16 and 3-18, the center of the primary vortex moves toward the geometric center of the cavity as Reynolds number increases. For \( \operatorname{Re} = 1000 \) and \( \operatorname{Re} = 5000 \), the secondary vortices at bottom corners become obvious, and a secondary vortex at the top left
corner can be apparently seen for the \( \text{Re} = 5000 \) case. Velocity profiles along the geometric centerlines for all \( \text{Re} \) numbers match very well with the results of Ghia (1982).

Two important numerical techniques have been adapted in this circulation model: Under-Relaxation technique and Momentum Interpolation method. For details, see Appendix D. To test the techniques and to demonstrate the effects of the modified momentum interpolation method, numerical simulations have been conducted for this cavity flow problem with \( \text{Re} = 100 \). The \( 30 \times 30 \) uniform grids are used throughout these calculations. The under-relaxation factor for velocities varying from 0.3 to 0.9 were used in these calculations, and the optimum under-relaxation factor for pressure correction \( \alpha_p = 1.1 - \alpha_u \) is used to update the pressure correction at every outer iteration level. Convergence is declared when the maximum RMS residual of the three governing equations is less than \( 10^{-6} \).

Use of the modified momentum interpolation method is expected to give unique converged results for different under-relaxation factors. This is well illustrated by Table 3-1. The velocities at monitored location are independent of under-relaxation factors and unique results are obtained after convergence. Number of iterations required for convergence increases quickly as the velocity under-relaxation factor decreases from 0.9 to 0.3 (see Figure 3-19).

### 3.6 Wave-induced Alongshore Currents on a Planar Beach

In chapter 2, the development of the present reduced spectral wave model accounting for currents has been described in great detail. In previous sections of this chapter we introduced the development of the steady 2-D nearshore circulation model, including the formulation, numerics and treatment of various boundary conditions. The model verification was followed using several simple numerical experiments. With the concept of radiation stress, the wave model and the circulation model can be coupled easily in logical sense. Special care should be taken in
computing radiation stress when irregular waves are involved, as discussed in Chapter 1. The finite volume method (FVM) is used in both the wave model and the circulation model, making it easier to couple the two models. The main challenge for coupling the two models lies in the fact the wave model is in time domain while the circulation model is time-invariant. This, however, does not prevent from coupling the two models. For steady wave fields, the wave model, in fact, can be regarded as using a transient method to solve steady problems. The wave field is computed with a pseudo time marching and the final solutions that do not vary with time can be obtained after a sufficient run time. The wave model itself of course can be used to predicted time varying wave fields.

The detailed procedure of coupling the wave model and the circulation model can be put in the following way. The wave model is originally executed with zero ambient currents, giving an approximate wave field. The radiation stresses and wave forcing can then be calculated and input to the circulation model as a priori. After the circulation model reached to a convergent solution, we re-run the wave model with the newly computed currents with the old wave field as the initial condition. The loop should be repeated until variations of both wave properties and current field from last loop are within the tolerance value. It should be noted that this is only a loose two-way coupling.

In this section, the coupled wave/circulation model is used to simulate the wave-induced steady alongshore currents over a longshore uniform planar beach. Using this simple bathymetry it is easier to verify the new wave/circulation model, by avoiding the complexity that a more complicated bathymetry may cause. It is also convenient to examine effects of bottom friction and lateral mixing effects on currents and influences of wave-current feedback both on waves and on currents on a relatively simple bathymetry. The formulations of the bottom friction and
lateral momentum mixing used in this circulation model were introduced in Section 3.2 and the free parameters $k_a$ and $M$ are very important but usually not known as a priori in nearshore circulation modeling. To investigate the effect of bottom friction on alongshore currents, simulations with varying bottom friction factors are conducted. In addition, a set of simulations with varying lateral mixing coefficient is carried out to study the effects of lateral mixing. The effects of wave-current interaction on the alongshore currents are also studied.

The artificial planar beach with a constant slope of 1:20 is used here, and the offshore water depth is 3 m (Figure 3-20). For illustration purpose, a small computational domain of 40 m in width, 60 m in length and coarse uniform grids (60×40) are used. The no-slip wall boundary condition is applied to the shoreline boundary, and lateral boundaries are periodic. The offshore wave conditions are $H = 0.61 m$, $T = 4 s$, and $\theta = 22.5^\circ$. Since whether the wave is monochromatic or irregular is irrelevant to the investigation of effects of the bottom friction, lateral momentum mixing and wave-interaction, regular waves are used here. Effects of frequency/directional spreading on waves and currents will be studied separately in chapter 4. The typical across-shore variations of energy dissipation due to breaking, wave bottom orbital velocity, eddy viscosity and bottom friction coefficient for $M = 0.2$ and $k_a = 0.03 m$ are shown in Figure 3-21. For a fixed $M$, a decrease in $k_a$ increases the magnitude of longshore current $v$ without significantly altering the shape of $v$ velocity profile (Figure 3-22). Simulations for varying values of $M$ with the fixed $k_a = 0.03 m$ show that lateral mixing smoothes the cross-shore distribution of $v$ without shifting the location of $v_{\text{max}}$ (Figure 3-23). The momentum balance in the longshore direction resulting from a simulation for $M = 0.25$ and $k_a = 0.03 m$ is
shown in Figure 3-24. The wave forcing is mostly dissipated by the bottom friction and the mixing process is represented by the eddy viscosity term.

As discussed in chapter 2, both the wave height and wave direction can vary significantly as waves propagate through a strong shear current with a considerable angle of incidence. Figure 3-25 displays the computed wave height and wave direction with and without the influences of wave-current interaction. The wave height decreases slightly with the presence of longshore currents in the region right before wave breaking starts because of current-induced wave refraction, which can be seen more clearly in the plot of wave direction comparison (see Figure 3-25b). Wave-current interaction, in this case, postpones the outset of wave breaking. Closer to the shoreline, wave breaking dominates and wave-current interaction has less influence on wave height. Results also show that the effect of w-c interaction on currents is not very significant but is not negligible neither (Figure 3-26). Generally, taking into account the interaction shifts the velocity profile shoreward and decreases the current peak slightly.
Table 3-1. Effect of the under-relaxation factor on number of iteration required for convergence and effect of MMIM on the converged computational results.

<table>
<thead>
<tr>
<th>Under-relaxation factor $\alpha_u = \alpha_r$</th>
<th>Monitored velocities</th>
<th>Iteration required</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>U(m/s)</td>
<td>V(m/s)</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.1140</td>
<td>0.0755</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.1140</td>
<td>0.0755</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.1140</td>
<td>0.0755</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.1140</td>
<td>0.0755</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.1140</td>
<td>0.0755</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.1140</td>
<td>0.0755</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.1140</td>
<td>0.0755</td>
</tr>
</tbody>
</table>
Figure 3-1. A typical non-orthogonal 2-D control volume and notations.

Figure 3-2. Types of variable arrangement: collocated (left) and staggered (right).

Figure 3-3. A typical 2-D control volume and auxiliary nodes used.

Figure 3-4. A control volume with neighboring CVs.
Figure 3-5. An example of the periodic boundaries.

Figure 3-6. Sketch of Poiseuille flow (left) and Couette flow (right).

Figure 3-7. Comparison of the velocity profile between modeling results (solid line) and analytical solutions (circles) for Poiseuille flow (left); Comparison of normalized velocity distribution with various $\alpha$ values for Couette flow (right), modeling results (solid lines), analytical solutions (circles).
Figure 3-8. Sketch of the rectangular basin with symmetry boundaries.

Figure 3-9. Forcing distribution (top left) and analytical solutions for the pure diffusion problem. Stream function contours (top right), vorticity contours (bottom left) and the velocity field (bottom right).
Figure 3-10. Comparison of $u$ (left) and $v$ (right) at a sample section between modeling results (solid lines) and the analytical solutions (circles).

Figure 3-11. Computational error as functions of grid number (uniform Cartesian grids used); the model is expected to be 2nd order accuracy since 2nd order discretization schemes were used.

Figure 3-12. The non-orthogonal grid used (left) and velocity comparison (right) along a monitored grid line between computed (solid line) and exact values (red points).
Figure 3-13. Computational error as functions of number of grid (non-orthogonal grid used).

Figure 3-14. Lid-driven square cavity flow geometry and boundary conditions.
Figure 3-15. Velocity field (left top), vorticity contour (right top) and comparison of velocities along the two geometric centerlines for Re = 100 (80×80 uniform grids used).

Figure 3-16 Velocity field (left top), vorticity contour (right top) and comparison of velocities along the two geometric centerlines for Re = 1000 (120×120 uniform grids used).
Figure 3-17. The 150×150 non-uniform Cartesian grids used for the Re = 5000.

Figure 3-18. Velocity field (left top), vorticity contour (right top) and comparison of velocities along the two geometric centerlines for Re = 5000 (150×150 non-uniform grids used).
Figure 3-19. Number of iterations required to reach convergence using various velocity under-relaxation factors ($Re = 100$, the $30 \times 30$ uniform grids used).

Figure 3-20. A planar beach with a slope of 1:20.

Figure 3-21. (a) energy dissipation due to wave breaking, (b) viscosity coefficient $\nu$, (c) wave horizontal orbital velocity near water bottom, (d) bottom friction coefficient $\mu$. $M$ is the mixing coefficient and $k_a$ is the apparent bed roughness.
Figure 3-22. Effect of the bottom friction factor $k_a$ on the alongshore current distributed along the cross-shore distance. $M = 0.2$ is used for the experimental runs.

Figure 3-23. Effect of the lateral mixing coefficient $M$ on the alongshore current. $k_a = 0.03m$ is used for the experimental runs.

Figure 3-24. Alongshore momentum balance for $M = 0.25$ and $k_a = 0.03m$. Bottom friction term $-\mu v$ (solid line), wave forcing term (dashed line), lateral mixing term (dotted line), and residual (dash-dotted line).
Figure 3-25. Comparison of wave properties with (solid lines) and without (dashed lines) effects of wave-current interaction. (a) normalized wave height; (b) wave angle. $M = 0.25$ and $k_a = 0.03m$ are used.

Figure 3-26. Comparison of alongshore current profiles between neglecting wave-current interaction case (dashed line) and including interaction case (solid line). $M = 0.25$ and $k_a = 0.03m$ are used.
CHAPTER 4
APPLICATIONS OF THE WAVE/CIRCULATION MODEL

In previous two chapters, we introduced the reduced spectral wave model and the steady circulation model, and the coupling of the two models was briefly discussed, followed by application of the coupled wave/circulation model to wave-induced longshore currents over a planar beach. In this chapter, we focus on further applications of the wave/circulation model to nearshore currents, including longshore currents over barred beaches and rip currents. The DUCK94 field experiment is simulated and compared to measured data. Another typical current pattern in the nearshore area, rip current, is also studied using the coupled models. In addition, the effects of frequency/directional spreading on the wave field and wave-driven currents are investigated.

Oscillations in velocity field with long periods have been observed in field measurements, laboratory experiments and numerical studies for longshore currents (Oltman-Shay et al., 1989; Allen et al., 1996; Özkan-Haller and Kirby, 1999; Slinn et al., 2000), rip currents (Shepard and Inman, 1950; Sonu, 1973; Haller and Dalrymple, 2001; Yu and Slinn, 2003; Kennedy and Zhang, 2008), and the entire circulation cells (MacMahan et al., 2004). An understanding and predictive capability of the unsteady flow motions is very important and necessary for engineering practices. In order to study the instabilities of longshore currents and rip currents, an unsteady circulation model extended from the steady version is developed and coupled with the wave model in the time domain, detailed description given in Section 4.2.3.

4.1 Wave-induced Alongshore Currents over Barred Beach

4.1.1 Introduction

The important role of alongshore currents in the surf zone in coastal processes has long been recognized. Measurement and prediction of alongshore currents generated by obliquely
incident waves breaking near the shoreline have gained tremendous efforts by numerous scientists and coastal engineers. Models of alongshore currents are often based on the onedimensional alongshore momentum flux balance between wave, wind and tidal forcing, bottom stresses and lateral mixing (Longuet-Higgins, 1970; Thornton and Guza, 1986). Breaking waves are the most important forcing mechanism in the surf zone, although wind (Feddersen et al., 1998) and tidal (Houwman and Hoekstra, 1998) forcing can contribute significantly. Bottom friction also plays a very important role in the wave-induced nearshore circulation. In the surf zone, bottom stress could be the major force to balance the wave forcing under certain conditions. Unfortunately, no rigorous formula accurately describing the complicated near bottom processes is available. In practice, parameterized formulations are usually used in modeling nearshore circulation. Lateral momentum mixing from wave breaking turbulence is another factor in alongshore currents and more complicated nearshore circulations. Other than using a constant eddy viscosity in and out the surf zone, the variable parameterized expressions of lateral mixing have been proposed (e.g. Özkan-Haller and Kirby, 1999). The widely used formulations for bottom friction and lateral mixing, and their effects on the alongshore currents over a planar beach have been discussed in chapter 3.

Barred beaches are another type of topography that is often seen in the nature. For wave-induced longshore currents over barred beaches, laboratory experiments suggest that the maximum alongshore velocity is located on the bar (Reniers and Battjes, 1997). However, early numerical studies (e.g., Church and Thornton, 1993) predicted a strong, narrow jet on seaward side of the bar and near-zero flow in the bar trough, which differs from the laboratory results and field data. This failure may result from the alongshore variations in bathymetry and/or the neglect of wave surface rollers in wave forcing. Variations of bathymetry in longshore direction
may alter the longshore current field significantly, but which is not a subject of this study. The surface rollers carry the energy dissipated from wave breaking and release it later as the rollers become smaller, thus shifting the maximum velocity shoreward and increasing the bar trough velocity as well. The roller model proposed by Stive and DeVriend (1994) is discussed in Section 4.1.2, and improvement in predictions of the measured alongshore currents at Duck, North Carolina on Oct, 14, 1994 by including rollers in the wave forcing, is also discussed in Section 4.1.2.

Monochromatic wave drivers were traditionally used in simulating wave-induced longshore currents. However, the monochromatic wave approximation is very crude and may overestimate or underestimate the radiation stress components significantly, which was discussed and quantitatively analyzed in Chapter 2. Due to the difference in wave radiation stresses, the resulting longshore currents are supposed to be different between from monochromatic waves and from irregular waves. However, the quantitative change in longshore currents due to the presence of wave directional and frequency spreading is not well known. Using the current wave/circulation model, the effects of the wave frequency and directional spreading on the wave-driven alongshore currents are investigated, and the difference in alongshore currents of DUCK94 owing to the presence of frequency/directional bandwidth is demonstrated in Section 4.1.2.

### 4.1.2 Comparison to DUCK94 Field Experiment Data

Models neglecting the wave rollers predicted current profiles with a maximum velocity seaward of the bar or even in the trough (e.g., Church and Thornton, 1993). Many researchers have proved that by including wave rollers can give better prediction (e.g. Ruessink et al., 2001). The wave surface roller concept is introduced before presenting the simulation results. To
author’s knowledge, narrow-banded waves have been usually assumed in current roller models, which means that the effects of frequency/directional spreading were not being accounted for. A roller model including the effects of wave frequency/directional spreading can give a more accurate representation. In the following, a brief literature review in wave roller models is given, followed by introducing the spectral version of the roller model.

**Surface Wave Roller**

As waves break, a volume of turbulent water is generated, which rushes down the front face of the waves. This body of aerated water detached from the wave form is called the surface wave roller. The surface rollers carry the energy imparted from wave breaking shoreward. As a result, the wave momentum is imparted into the water column and transferred shoreward, and consequently the surface rollers have an important impact on surf zone dynamics, including wave setup/setdown and wave-induced currents (Svendsen, 1984; Roelvink and Stive, 1989; Fredsoe and Diegaard, 1992; Smith et al., 1994). Svendsen proposed that the turbulent water body is pushed by the wave front with horizontal velocity equal to the wave phase speed, so the roller energy travels with wave phase speed. It also has been pointed out that as soon as the surface rollers are generated by wave breaking they are dissipated locally by the shear layer between the rollers and the wave motions.

The one-dimensional roller energy balance equation has been given by Stive and De Vriend (1994), and Reniers and Battjes (1997), independently. When extended to a two-dimensional time-dependent formulation, the energy balance for roller can be written as

\[
\frac{\partial}{\partial t} E_r + \nabla \cdot \left[ (c + u) E_r \right] = -D_r + D_{br}
\]  

(4-1)

where \(E_r\) is the roller energy density, derived from the kinetic energy density of the roller volume with unit crest length (Svendsen, 1984a,b). \(c\) and \(u\) are the wave phase speed vector and
ambient current vector, respectively. The first term on the right hand side is the roller sink term, given by (Duncan, 1981; Dieggard, 1993):

\[ D_r = \frac{E_r g \sin \alpha}{c} \] (4-2)

where \( \alpha \) is the wave-front slope and usually is assumed to be \( \alpha < 10^0 \) (Lippmann et al. 1996).

\( D_{br} \) stands for the wave breaking dissipation and is the source term for the roller energy balance equation.

Usually, an approximation of Equation (4-1), assuming that a wave spectrum is narrow-banded in frequency and direction (Lippmann et al., 1996; Ruessink et al., 2001), is used in practice, which is a gross approximation for broad wave spectra. For a well-defined wave spectrum, for instance the Gaussian-shaped wave spectrum, in fact, the formulation for the roller balance equation can be rigorously derived using the same strategy as to the wave action balance equation. The detailed algebras of derivation will not be reproduced here. In the surface roller energy balance equation, wave phase velocity \( c \) is weakly relevant to frequency \( \sigma \) since surface rollers only exist after wave breaking occurs, therefore, the equation only considering the directional spreading as following is used:

\[
\frac{\partial}{\partial t}E_r + \frac{\partial}{\partial x} \left[ \left( c \left( E_{0,1} \right)_r / E_{0,0} + u \right) E_r \right] + \frac{\partial}{\partial y} \left[ \left( c \left( E_{0,1} \right)_i / E_{0,0} + v \right) E_r \right] = -D_r + D_{br} \] (4-3)

\( \left( E_{0,1} \right)_r \) and \( \left( E_{0,1} \right)_i \) are the real and imaginary part of wave moment \( E_{0,1} \) as defined previously.

\( \left( E_{0,1} \right)_r / E_{0,0} \) and \( \left( E_{0,1} \right)_i / E_{0,0} \) contains not only information on the mean wave direction but also the wave directional spreading. The Equation (4-3) is solved after solving the wave moments with the same numerical schemes. \( E_r \) is set to be zero at the offshore boundary.
The formulation of roller radiation stress is similar to the wave energy stress. Since \( c_g = c \) in shallow water, the roller radiation stress is given by

\[
S_{ij,r} = E_r \left( \frac{k_i k_j}{k^2} + \frac{\delta_{ij}}{2} \right)
\]  

(4-4)

The wave roller energy is coupled to the circulation model through the wave forcing in the same manner as the wave energy:

\[
\tau^r_y = -\frac{1}{\rho} \left( \frac{\partial S_{xy,r}}{\partial x} + \frac{\partial S_{yx,r}}{\partial y} \right)
\]  

(4-5)

Model-Data Comparison

Alongshore currents driven by wave forcing over the barred beach are simulated and comparison to DUCK94 data is made. The bathymetry and experimental data used here were downloaded from the DUCK94 Experiment Data Server. The Duck94 data were collected from August to October 1994, during the near shore field experiments at the U.S. Army Corps of Engineers Field Research Facility (FRF) located in Duck, North Carolina. Directional wave statistics (\( H_{\text{sig}} \), peak period, and mean wave direction) were measured at 8-m water depth. Pressure and velocity observations were obtained at more than 15 cross-shore locations extending from near the shoreline to 8-m depth.

The bathymetry used in this study (as shown in Figure 4-1) was the bathymetry of daily minigrid 3D survey data at around 10:00 am, October 14, 1994. All the wave and current data are 3 hour-averaged for a storm event. The 8-m depth wave statistics are: \( H_{\text{sig}} = 2.07m \), \( T_p = 8.866s \), and \( \theta = -20^\circ \). The pressure gage nearest to the offshore end of the measured bathymetry is the gage P19, located 347.4 m from shoreline. In order to take advantage of the available wave conditions at gage P19, the offshore boundary of the computational domain is
chosen to be at pressure sensor gage P19. The $H_{sig}$ at P19 is of 1.373 m, and $T_p = 6.4s$. The mean wave direction at P19 is calculated using the Snell’s law.

Alongshore uniformity in morphology, waves and currents is assumed through this numerical experiment. The influence of wave-front slope $\alpha$, which determines how fast the roller energy dissipates (large $\alpha$ for fast roller energy dissipation), is examined using the bathymetry and wave conditions of Duck94, Oct 14. The bed roughness $k_a$ is fixed to be 0.07 m and the mixing coefficient $M = 0.6$ for all numerical simulations. With no roller, the current jets are located on the seaward side of the sandbar and near the shoreline, with near-zero current in the trough area (50-100 m). The rollers stored the energy from water breaking and released later, which results in shift of the maximum along current onshore and broadens the current jets by increasing the currents in the trough. With decreasing $\alpha$, the $v_{max}$ is shifted further onshore and velocity in the trough increases continually. For $\alpha$ up to 0.2, the roller effect is not significant any more (Figure 4-2), consistent with the concept that for large dissipation rate (associated with large $\alpha$) the roller energy is dissipated quickly.

The roller energy has a significant impact on the total radiation stress (sum of wave radiation stress and roller radiation stress) distributed along the cross-shore distance from the shoreline, peak of roller energy located on where waves break strongly, i.e. where the trough of wave radiation stress located (Figure 4-3a). Consequently, the cross-shore wave forcing peak shifts shoreward and the forcing distribution become broader. The magnitude of $\tau_{y_{max}}$ is decreased considerably (Figure 4-3b).

The wave model predicted the observed $H_{rms}$ very well at all sensors. The $H_{rms}$ errors for individual sensors vary between 0.01 m and 0.12 m, with an average of 0.04 m for all sensors.
The larger errors occur near the bar crest area (Figure 4-4a). Figure 4-4b shows comparison of the predicted alongshore current (with and without roller model) with measured velocity profile. The roller model greatly improved the performance of the current model in simulating the alongshore current distribution by shifting the current jets onshore and lifting the trough velocity. Modeled alongshore velocity with $\alpha = 0.05$, $k_a = 0.07m$ and $M = 0.6$ agree most well with measured data for this run. The $\alpha = 0.05$ is consistent with values used by Walstra et al. (1996) and Ruessink et al. (2001), while, both $M = 0.6$ and $k_a = 0.07m$ are equal to and slightly over the upper range suggested by Özkan-Haller and Kirby (1999). Terms in the y-momentum equation are plotted in Figure 4-5. Obviously, the wave forcing is balanced mostly by the bottom friction in this case.

4.1.3 Effects of Frequency/Directional Spreading

For simplicity, the wave-driven nearshore circulation models (e.g., Church and Thornton, 1993; Ruessink et al., 2001) often use the wave statistics ($H_{rms}$, peak frequency and mean direction) to compute the radiation stress and then the wave forcing. For waves with finite frequency/directional bandwidths, this approximation can result in significant errors (Battjes, 1972). Feddersen (2004) compared the exact radiation stresses and showed that the component $S_{xy}$ could be overestimated up to 60% using the narrow-banded approximation, which is roughly consistent with the combined Duck94 and SandyDuck observations. As discussed by Battjes (1972), errors in radiation stresses will result in modeled current errors or, when fitting a model to observations, in biased model parameters such as bottom friction coefficient, eddy viscosity etc. The current wave model solves not only the wave statistics but also the frequency and directional bandwidths, and then the radiation stresses including the effects of frequency and directional spreading can be obtained.
To demonstrate effects of frequency spreading upon the modeled wave height, radiation stress, and finally the longshore velocity, the field data of DUCK94, 10am, October 14 is re-simulated assuming that the frequency spectrum is a Gaussian distribution with different values of frequency bandwidth $S_\sigma$ ($S_\sigma / \sigma_m = 0, 0.2, 0.4$ are examined). As expected, no appreciable differences in predicted $H_{rms}$ (Figure 4-6(a)) are observed between different frequency bandwidths because the wave breaking is so strong that other factors are not very important. As discussed above, the radiation stresses are weakly dependent on frequency spreading in intermediate water, less than 10% variation even for a fairly large frequency bandwidth. In shallow water, the changes in radiation stress $S_{xy}$ with frequency spreading are even less (Figure 4-6(b)). The small variation in $S_{xy}$ can only result in slight changes in alongshore velocity profiles (Figure 4-6(c)), the alongshore velocities decreasing slightly with the increase of $S_\sigma$. Free parameters $\alpha = 0.05$, $k_\sigma = 0.04m$ and $M = 0.5$ were used for these experimental runs.

Figure 4-7 demonstrates effects of directional spreading upon the modeled wave height and alongshore velocity profile on DUCK94 bathymetry. Due to the strong wave breaking, the $H_{rms}$ is not obviously affected by directional spreading (panel (a)). $S_{xy}$ changes considerably for $S_\theta = 23^\circ$ (panel (b)) compared to the no directional spreading case (about 40% larger before waves break strongly), which is consistent with the analysis theoretical conclusion discussed above. The predicted maximum $v_{max}$ for the no spreading case is 43% larger than with $S_\theta = 23^\circ$.

4.2 Shear Stability of Alongshore Currents

4.2.1 Introduction

In 1986, observations during the SUPERDUCK field experiment (Oltman-Shay, et al., 1989) revealed a surprising behavior of longshore currents. The longshore currents began to
oscillate with very low frequencies that typically fall into the lower end of the infragravity band ($0.001 < f < 0.01\text{Hz}$). However, the alongshore wavelengths of this motion are more than an order of magnitude less than the wavelengths predicted by edge wave or gravity wave theories, which suggested that these wave motions are not surface gravity waves (Oltman-Shay, et al., 1989). Bowen and Holman (1989), in a companion paper, interpreted the energetic, propagating perturbations in alongshore currents as shear instability of the mean currents using linear analysis, and their explanation has lead to these wave-like motions being referred as ‘shear waves’. This wave motion, occurring in the horizontal plane and moving with a speed less than the mean alongshore current, causes the alongshore current to move back and forth across the surf zone.

To test the hypothesis that the newly observed wave-like motion is dynamically derived from the shear instability of mean longshore currents, Bowen and Holman (1989) performed an analytical study of a simple, three-piece longshore current profile over a constant water depth. They showed that a shear instability leads to growing, nearly non-dispersive shear waves propagating in the direction of the alongshore current with a phase speed between $\frac{1}{4}$ and $\frac{3}{4}$ of the maximum value of the mean longshore current $V(x)$, which was qualitatively in agreement with observations (Oltman-Shay, et al., 1989). The restoring mechanism of these shear waves is potential vorticity consisting of a relative vorticity and a background vorticity. The background vorticity is defined by the shear of the mean alongshore currents. Dodd, Oltman-Shay and Thornton (1992) extended the linear instability theory to include a linearized bottom friction and applied to more realistic alongshore currents and beach profile. They obtained good agreement of wavelengths and wave speeds from observations and from theoretical predictions based on the most unstable linear mode.
As shear waves reach to finite-amplitude, the assumptions of linearity would fail. In order to investigate the finite-amplitude behavior of shear waves, Feddersen (1998) carried out an analytical study on the weakly nonlinear shear waves using the standard perturbation method. Using the same base-state longshore currents and bathymetry as in the numerical study of Allen et al. (1996), Feddersen obtained reasonably consistent results in the weakly nonlinear regime. However, this analytical model is mathematically complex and becomes invalid when the instabilities are strongly nonlinear.

Since numerical models based on the nonlinear shallow water equations can simulate shear waves from linear, weakly nonlinear to strongly nonlinear, numerical simulations of nonlinear instability of longshore currents have received a great number of studies in literature. Falquès et al. (1994) performed numerical experiments of nonlinear shear waves with the rigid-lid assumption for a planar beach, including the bottom friction effects and turbulence mixing. They obtained some preliminary results on the finite-amplitude behavior of these shear instabilities. Using the rigid-lid nonlinear shallow water equations on a planar beach with an idealized base mean longshore velocity profile and linearized bottom friction, Allen et al. (1996) conducted a set of numerical experiments and studied the effects of various bottom friction coefficients and varying longshore width of the modeling domain. They found that for flows from low to high nonlinearity (low nonlinearity defined by high friction coefficient with weakly unstable base longshore current), a wide range of behavior is observed. For weak nonlinearity, equilibrated steady finite-amplitude disturbances are observed. As the nonlinearity increases, the disturbances display modulated amplitudes and eventually irregular eddies. They also found that for modeling domain substantially larger than the wavelength of the most unstable linear mode, unstable waves subsequently evolve into larger-scale, nonlinear propagating wave-like disturbances.
Allen et al. (1996) also pointed out that the measured mean current profiles in the ocean correspond to the final mean current in the presence of the fluctuations rather than fluctuation-free initial state.

Slinn et al. (1998) extended the study of Allen (1996) over barred beach topography with a more realistic forced current computed using the Thornton and Guza (1986) alongshore current model. They found that instabilities cause substantial lateral mixing of momentum in the surf zone and alter the initial current profile significantly. Soon after, Özkan-Haller and Kirby (1999) simulated the nonlinear instabilities of alongshore currents of the SUPERDUCK field experiment, including the bottom friction and lateral mixing effects. They obtained satisfactory results in the propagation velocities of the instabilities as well as the maximum mean alongshore currents including the shear instabilities. They carried out simulations for varying friction and mixing coefficients and pointed out that a smaller friction coefficient leads to a stronger mean current and faster and more energetic vortex structures and less energetic flow structures result from an increase in mixing coefficient. They also found that shear instabilities induce significant horizontal momentum mixing, as a result, the resulting current profile including effects of shear instabilities displays a smaller peak value and increased velocities in the trough.

Numerical simulations of the finite-amplitude development of shear instabilities have gained great achievements and provided a better understanding of shear waves. The effects of friction and mixing on shear instabilities have gained intensive studies by many investigators. Most of the studies, however, used the monochromatic wave drivers. That how the instabilities of the mean alongshore currents driven by waves with frequency/directional spreading differ, has not been carefully studied to date. Since the wave forcing tends to be overestimated by using
monochromatic wave models (as discussed in previous section), both the resulting steady alongshore currents and shear instabilities will alter considerably for irregular waves.

In this section, one of our goals is to investigate the effects of directional spreading of incident waves on the resulting shear instabilities of alongshore currents. Considering the fact that wave forcing simulated by the present reduced wave model for a narrow spectrum does not differ significantly from that of monochromatic waves, the effects of frequency spreading on shear instabilities are not very noticeable and will not be presented here.

4.2.2 Linear Stability Analysis

Following Bowen and Holman (1989), the velocity field is assumed to consist of a steady longshore current $V(x)$ and perturbations in alongshore currents. Therefore, the total velocity vector is

$$u(x, y, t) = \left( u(x, y, t), \ V(x) + v(x, y, t) \right)$$  \hspace{1cm} (4-6)

where, $v(x, y, t)$ is the longshore perturbation velocity and is assumed that $v \sqsubset V$. Substituting Equation (4-6) into the nonlinear shallow water equations and retaining only the linear terms, including bottom friction and lateral mixing, following equations can be obtained

$$u_t + V u_x = -g \eta_x - \frac{\mu}{h} u + \frac{V}{h} \nabla^2 (hu)$$  \hspace{1cm} (4-7)

$$v_t + V v_y + V_x u = -g \eta_y - \frac{\mu}{h} v + \frac{V}{h} \nabla^2 (hv)$$  \hspace{1cm} (4-8)

where, $\eta$ is the free surface elevation, $\mu$ is the bottom friction coefficient given by Equation (3-6). The above perturbation equations were derived under the rigid-lid approximation and have neglected the lateral mixing term, which have been proved reasonable for linear stability analysis (Allen et al., 1996). Under the assumption of non-divergence, a stream function $\psi(x, y, t)$ can be introduced, such that
\[ u = -\psi_y / h, \ v = \psi_x / h \] (4-9)

which allows the momentum equations to be combined and yields

\[ \left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial y} + \frac{\mu}{h} \right) \left( \psi_{yy} / h + (\psi_x / h)_x \right) - \frac{V}{h} \nabla^2 (\psi_{xx} + \psi_{yy}) \]
\[ = \psi_x (V_x / h)_x - \left( \frac{\psi_x}{h} \right)_x \left( \frac{\mu}{h} \right)_x + \left( \frac{V}{h} \right)_x \nabla^2 \psi_{xx} \] (4-10)

assuming a solution of the following form:

\[ \psi(x, y, t) = \text{Re} \left\{ \psi(x) e^{i(ky - \sigma t)} \right\} \] (4-11)

where, the simple alongshore harmonic is assumed. \( k \) is the longshore wavenumber and is a real number. But \( \psi(x) \) and \( \sigma \) may be complex. Substituting Equation (4-11) into (4-10) gives the linear stability equation

\[ (V - i\mu / kh - c) \psi_{xx} - h \psi_x / h - k^2 \psi - h (V_x / h)_x \psi - i(\mu / h)_x \psi_x / k \]
\[ + \frac{iv}{k} (\psi_{xxxx} - 2k^2 \psi_{xx} + k^4 \psi) - \frac{ih}{k} (V / h)_x (\psi_{xxxx} - k^2 \psi_{xx}) = 0 \] (4-12)

Generally, the equation above need to be solved numerically. The finite difference method is utilized to solve the linear stability problem and the following second-order center difference scheme is used for spatial discretization:

\[ \psi_x = (\psi_{i+1} - \psi_{i-1}) / (2\Delta x) \] (4-13)

The following boundary conditions are imposed at both ends of the domain:

\[ \psi = \psi_{xx} = 0, \quad x = 0, L \] (4-14)

Discretizing Equation (4-12) and rearranging, we arrive at an algebraic eigenvalue problem

\[ A\psi = cB\psi \] (4-15)

where, only values of \( \psi \) at \( i = 2, 3, ..., n - 1 \) are required to be solved. \( A \) and \( B \) are matrices. The above equation may be solved much more efficiently after the following operation:
A simple linear stability model programmed in Matlab is developed and applied to two cases. In order to verify the capability of the present model, the case of Bowen and Holman (1989) is reproduced as the first test. In the second test, a predicted steady alongshore current profile on a barred beach is studied including the bottom friction term with across-shore variable $\mu$. This barred bathymetry is also used to study the nonlinear evolution of shear instability, which is presented in following Section 4.2.3.

**Case 1: Bowen and Holman (1989)**

A three-piece longshore current profile used by Bowen and Holman (1989) is shown in Figure 4-8. In this case, no bottom friction and no lateral mixing are assumed in order to compare with their analytical results.

Results of the linear stability calculations in terms of growth rate $\sigma_m$ for the most unstable mode as a function of alongshore wavenumber $k$ are shown in Figure 4-9. The propagation velocity associated with the most unstable mode $c_r$ as a function of $k$ is also shown. The alongshore wavenumber $k$ of the most unstable linear mode is approximately $0.026m^{-1}$ (and wavelength $2\pi/k_0 \approx 242m$), with a growth rate of about $0.00335\ s^{-1}$. The corresponding propagation velocity $c_r = 0.335m/s$ which gives a period of $2\pi/(k_0c_r) = 721s$. These calculation results are in a very good agreement with Bowen and Holman’s results.

Figure 4-10 shows the spatial form of $\psi(x,y)$ of the fastest growing waves. The domain in alongshore direction is chosen to be 484 m, which is two times of the wavelength of the most unstable mode. As discussed by Bowen and Holman (1989), progressive waves in the alongshore direction are observed. The pattern of total velocity, in which the fluctuation velocities are scaled.
such that its peak magnitude equals the peak of the mean alongshore current for illustration purpose, is shown in panel (c). It is obvious that the alongshore currents meander along its peak position and secondary circulation cells can also be seen.

**Case 2: Longshore Currents on a Barred Beach**

In this particular test case and in next Section 4.2.3, a barred beach bathymetry approximately fitting topography measured at Duck, North Carolina, on October 11, 1990, given by Lippmann et al. (1999) is used

\[
h(x) = A_1 \tanh \left( \frac{b_1 x}{A_1} \right) + A_2 \tanh \left( \frac{b_2 x}{A_2} \right) - A_1 \exp \left[ -5 \left( \frac{x - x_c}{x_c} \right)^2 \right] \tag{4-17}
\]

where \( b_1 = \tan(0.075) \), \( b_2 = \tan(0.0064) \), \( \gamma_1 = b_1 / b_2 \). \( x_c = 80m \) is location of the sandbar.

\( A_1 = 2.97 \) and \( A_2 = 1.5 \). This bathymetry is plotted in Figure 4-11.

Monochromatic waves of \( H = 1.373m \), \( T = 6.4s \) and with the incident angle of 15° at offshore boundary are modeled over this barred bathymetry. The mean alongshore currents driven by wave forcing are simulated for varying bottom friction coefficients. Large bottom friction results in relatively weaker currents without significantly changing the shape of the current profile (Figure 4-12(a)). The across-shore variations in bottom friction coefficients \( \mu \) for \( k_a = 0.01, 0.02, 0.03m \) are calculated and used during simulations of the mean currents (Figure 4-12(b)), and they are also utilized in linear stability calculations. The resulting wavenumber of the most unstable linear mode for different bottom friction coefficients are about the same, \( k_0 = 0.032m^{-1} \), which leads to a wavelength of \( 2\pi / k_0 \approx 200m \) (Figure 4-12(c)). However, the growth rate increases greatly as \( k_a \) decreases. A growth time scale of 4.4 minutes is resulted for \( k_a = 0.03m \). The corresponding propagation velocity \( c_r \) increases considerably by decreasing \( k_a \).
from 0.03 m to 0.01 m. As a result, the period $2\pi/(k_\omega c_r)$ of shear waves is smaller for higher bottom friction (Figure 4-12(d)). For $k_\omega = 0.03 m$, the period is about 4 minutes.

4.2.3 Numerical Simulation of Nonlinear Shear Instability

4.2.3.1 Unsteady Circulation Model

The time-independent 2-D shallow water equations with the rigid-lid approximation were used in the steady circulation model described in chapter 3. To simulate the time domain solution of nearshore currents, an unsteady version extended from the steady circulation model is developed based on the time-dependent 2-D shallow water equations. In the steady circulation model, the rigid-lid approximation was used for simplicity. While, for unsteady flow problems treatment of the free surface is not as difficult as for steady flows using the pressure-correction method. Therefore, the rigid-lid approximation is not necessary in the unsteady version of circulation model, and the effect of the free surface is included. The time-dependent depth-averaged NSWE averaged over incident wave timescales, including forcing and dissipation, can be written as

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (Hu) + \frac{\partial}{\partial y} (Hv) = 0 \quad (4-18)$$

$$ \begin{align*}
(Hu)_x + (Huu)_x + (Huv)_y &= -gH\eta_x + F_x + \tau_x + \tau'_x \\
(Hv)_y + (Huu)_y + (Huv)_y &= -gH\eta_y + F_y + \tau_y + \tau'_y
\end{align*} \quad (4-19)$$

where $\eta$ is the wave-averaged water surface elevation (setup), $H = h + \eta$ is the total water depth. $(\tau_x, \tau_y)$ are bottom friction terms, given by Equation (3-6), $(\tau'_x, \tau'_y)$ are lateral mixing terms, given by Equation (3-14).

In chapter 3, the SIMPLE algorithm used to solve steady shallow water equations have been described in great detail. Since the governing equations for unsteady flows involve transient terms and the total water depth $H$ is time varying too, the solution for unsteady shallow water
equations is required. The easiest and most direct way is to extend the SIMPLE algorithm to
transient calculations. The discretized momentum equations now include transient terms, and the
resulting discretized $u$-equation at time level $n+1$, outer iteration level $m$, becomes
\[
\begin{align*}
(A_{u}^{m})_{\text{unsteady}} &= (A_{u}^{m})_{\text{steady}} + H^{m-1} \Delta \Omega / \Delta t, \text{ and} \\
(S_{u}^{m})_{\text{unsteady}} &= (S_{u}^{m})_{\text{steady}} + (Hu)^{n} \Delta \Omega / \Delta t
\end{align*}
\]
(4-21)
where, $(A_{u}^{m})_{\text{steady}}$ and $(S_{u}^{m})_{\text{steady}}$ are the main diagonal coefficients and source terms for steady
flow problems respectively, more details referred to chapter 3. $\Delta \Omega$ is the control volume, $\Delta t$
being the time step. All other matrix coefficients remain the same. It is similar for the $v$-equation
and will not be presented here. For the continuity equation at time level $n+1$, outer iteration
level $m$, the discretization equation is
\[
(\eta^{m} - \eta^{n}) \Delta \Omega / \Delta t + \sum_{l} m_{l}^{*} = 0
\]
(4-22)
where $\sum_{l} m_{l}^{*}$ is the summation of mass fluxes computed using the intermediate velocities from
the momentum equations, for details refer to chapter 3. $\eta^{m} = \eta^{m-1} + \eta'$ and $\eta' = p' / (\rho g)$ is the
correction of the mean surface elevation from last outer iteration. Substituted into Equation (4-
22), the main diagonal coefficients and the source terms for the pressure correction equations can
be given by
\[
\begin{align*}
(A_{p}^{m})_{\text{unsteady}} &= (A_{p}^{m})_{\text{steady}} + \Delta \Omega / (\rho g \Delta t), \text{ and} \\
(S_{p}^{m})_{\text{unsteady}} &= (S_{p}^{m})_{\text{steady}} - (\eta^{m-1} - \eta^{n}) \Delta \Omega / \Delta t
\end{align*}
\]
(4-23)
At each time level, the iterative procedure is applied until convergence is achieved. Considering
that it is easy to get, only a little effort required from the solution method used in the present
steady circulation model, and that it is very robust, this transient solution method is used for
solving the transient equations, although some other solution methods which could be more efficient, can be developed.

4.2.3.2 Simulation Results

The same bathymetry described in previous section (see Equation (4-17) and Figure 4-11), is used in this section. No-slip boundary conditions are incorporated at the offshore boundary and at the shore, and the offshore boundary is far enough from the surf zone so that the vortices can not reach the offshore boundary. The periodic boundary condition is imposed in the alongshore direction. The linear stability analysis in previous section suggests the typical alongshore wavelength is about 200 m, although it may be affected slightly by using different bottom friction coefficients, mixing coefficients and with the presence of direction spreading. In this modeling effort, the computational domain is chosen to be 10 times of the predicted most unstable wavelength so that a sufficient number of waves can exist in the modeling domain and the flow is not influenced by the finite size in y-direction. The computation grids with uniform spacing (\( dx = dy = 5m \)) are used, and the implicit Euler method with \( dt = 4s \) is used. Sufficient iterations are used at every time level to ensure an accurate time evolution of the shear instabilities.

Numerical simulations are carried out to investigate the effects of bottom friction, lateral mixing, especially the wave directional spreading on the resulting mean alongshore currents and then their shear instabilities. Many different experiment simulations are carried out and nine of them are presented here. Simulations are conducted for the directional spreading 0°, 10°, 20° while keeping bottom friction coefficient \( k_a = 0.03m \) and the mixing coefficient \( M = 0.5 \). Simulations are also carried out for varying bottom friction coefficient with a fixed \( M \) and \( S_\theta \).
Simulation conditions and maximum of the resulting mean alongshore current $V_{\text{max}}$ are given in Table 4-1.

All simulations have the initial condition consisting of the mean base flow superimposed with a small fluctuating component

$$v(x, y, t = 0) = V(x) \times [1 + \varepsilon S(y)]$$

(4-24)

where, $\varepsilon = 0.001$ is the perturbation amplitude and experiment results proved that the long time behavior of flows are independent of the perturbation amplitude. The function $S(y)$ is given by

$$S(y) = \sum_{j=1}^{J} \cos \left( 2\pi \frac{y_j}{L_y} - \phi_j \right)$$

(4-25)

where, $J = 10$ is the number of times of the most unstable wavelength fits into the modeling domain. $\phi_j$ are random phases. Initial values of cross-shore velocity, surface elevation and wave forcing are not perturbed. This initial condition ensures that the mean alongshore currents are perturbed at the most unstable wavelength as well as longer wavelengths can also coexist in the modeling domain. Different ways to set initial conditions have been used in the literature of nonlinear shear instability simulation. For example, Slinn et al. (2000) started simulations with the fluid at rest but the wave forcing terms are perturbed. It is found that the flow behavior after long enough time run ($t > 4$ hour) is independent of the way to set the initial condition. The way to set initial conditions used in this study requires less computer time for the instabilities to reach finite amplitudes.

For incoming waves of $H_{\text{rms}} = 1.373 \text{m}$, $T = 6.4 \text{s}$, $\theta_0 = 15^0$, the predicted base-state steady alongshore velocity profiles for the tested experimental conditions (see Table 4-1) are shown in Figure 4-13. In general, large alongshore velocity results from smaller bottom friction coefficients and smaller mixing coefficients. Waves with directional spreading lead to relatively
weak currents without altering the shape of the velocity profile or shifting the peak location. Decreasing the mixing coefficient $M$ will increase the maximum velocity over the bar, whereas decrease the velocity in the trough.

As discussed above, the resulting steady mean currents are affected considerably by the existence of directional spreading in wave directional spectra. Consequently, characters of shear instability of alongshore currents induced by monochromatic waves are supposed to be different from those of currents induced by waves with finite wave directional bandwidth. For the purpose of investigating the effects of directional spreading on shear instabilities, simulation results with fixed $k_a$ and $M$ but varying $S_\theta$ are compared. Figure 4-14 shows the time series of velocities $(u, v)$ at $x = 420 m, y = 100 m$ for different values of $S_\theta$ while keeping $k_a = 0.03 m$ and $M = 0.5$. The time series show that fluctuations reach finite amplitude within about 90 min. For $S_\theta = 20^\circ$ the instabilities reach finite-amplitude later than that for other two cases. This is because the base-state mean flow associated with $S_\theta = 20^\circ$ is more stable. For all three cases, time series of across-shore velocity show regular fluctuations similar to wavelike pattern with nearly constant amplitudes. For $S_\theta = 0^\circ$, the amplitude of fluctuations in $u$ is about 0.21 m and the period is about 5.5 minutes. For $S_\theta = 10^\circ$, the amplitude is about 0.19 m and the period is about 6 minutes. For $S_\theta = 20^\circ$, the amplitude decreases to 0.16 m and the period increases to 7 minutes. In summary, the amplitude is decreasing and the period is increasing as direction spreading $S_\theta$ increases. This is consistent with the results of linear stability analysis. A longer period and smaller amplitude is resulted from a more stable base flow. Time series in alongshore velocity $v$ at the same location indicate that the alongshore velocity begins to oscillate as the same time as fluctuations in $u$ reaches to finite amplitude. Fluctuations in across-shore velocity $v$ are
irregular although oscillations in $u$ are regular. This may be explained by the fact that the kinetic momentum $\frac{1}{2}(u^2 + v^2)$ at a fixed location is not constant over time due to the instability of the flow. Again, relatively weak oscillations are found for $S_\theta = 20^\circ$ than those for smaller $S_\theta$. For all three cases, the time-averaged $v$ is well less than the base mean flow velocity (neglecting shear instability) as expected.

For analysis of the results of the numerical experiments, it is useful to introduce the vertical component of vorticity

$$\zeta = v_x - u_y, \quad (4-26)$$

and to define an alongshore average and a time average as follows

$$\bar{Q}(x,t) = \frac{1}{L_y} \int_0^{L_y} Q(x, y, t)dy \quad (4-27)$$

$$<Q>(x,y) = \frac{1}{t_e - t_s} \int_{t_s}^{t_e} Q(x, y, t) dt \quad (4-28)$$

where, $Q(x, y, t)$ some variable. $L_y$ is the length of the computation domain in $y$-direction.

Using these definitions, the velocities can be defined by

$$(u, v) = (<\bar{u}>, <\bar{v}>) + (u', v') \quad (4-29)$$

The spatial structure of the flow is best illustrated by snapshot of the vorticity field. Figure 4-15 shows the contours of vorticity at several times. It can be seen the disturbances are essentially of constant form propagating in the direction of steady mean alongshore current both for $S_\theta = 0^\circ$ and $S_\theta = 20^\circ$ cases. This flow character is similar to the flow described by Slinn et al. (1998) as equilibrated shear waves. The main differences between $S_\theta = 0^\circ$ and $S_\theta = 20^\circ$ are the magnitude of the vorticity and the propagation speed. For $S_\theta = 0^\circ$, the time variability of the
flow is further demonstrated by contour plots of the alongshore-averaged alongshore velocity \( \overline{v}(x,t) \) and the alongshore-averaged perturbation kinetic energy density \( \frac{1}{2}\left( \overline{u'^2} + \overline{v'^2} \right)(x,t) \) shown in Figure 4-16. The approximately equilibrated nature of the disturbances is indicated by the non-zero and nearly constant values of the \( \overline{v} \) and \( \frac{1}{2}\left( \overline{u'^2} + \overline{v'^2} \right) \).

**Convergence Test**

For numerical simulations, it is very important to verify that the computational results are independent to the grid size and the time step size. Because of limited computational resources and the tremendous time needed for each run, convergence tests were performed for a single case E01. Spatial resolutions with \( dx = dy = 5.5m \), \( dx = dy = 5m \), and \( dx = dy = 4.5m \) were used to examine the effects of grid size on computational results. To test whether the computational results vary with varying time step size, simulations with time step sizes of \( dt = 3.6s, 4s, 4.4s \) were conducted. These tests had all parameters the same as the test case E01 other than the spatial and/or time resolution. Figure 4-17(a) shows comparison of time- and alongshore-averaged alongshore velocity of simulation results using different time step sizes, and panel (b) shows the comparison of perturbation kinetic density. Similarly, Figure 4-18 shows the time- and alongshore-averaged alongshore velocity profiles and perturbation kinetic density for different grid sizes. Both the mean alongshore velocity profile and the perturbation kinetic density distribution do not change much with varying the spatial resolution and time step size. In addition, fluctuating quantities were also compared between these simulations (not shown here) and they are very close to each other. Therefore, it further confirms that the simulation results are independent from grid size and time step size.

**Effects of Directional Spreading**
The time- and alongshore-averaged alongshore velocities for cases E01, E02, and E03 are shown in Figure 4-19(a). The alongshore averages are performed over the entire width of the domain, while the time averages are obtained neglecting the first 3 hours of the computed time series. The maximum values of the three velocity profiles are 1.134 m/s, 1.0675 m/s and 0.9 m/s respectively, corresponding to $V_{\text{max}}$ values of 1.350 m/s, 1.258 m/s and 1.026 m/s (see Table 4-1). For case E01 the maximum alongshore velocity decreases about 0.216 m/s, while for E03 it only decreases 0.126 m/s. This may suggest that more energy is dissipated due to the instabilities for more unstable base-state alongshore currents. The cross-shore distributions of the perturbation kinetic density associated with these cases are shown in Figure 4-19(b). All three cases display similar shapes with two local maxima, shoreward of the location of mean current maximum and seaward of the trough of the velocity profiles. The perturbation kinetic energy decreases in the surf zone very quickly and completely disappears toward the offshore direction. The differences in kinetic energy density for E01 and E03 are pronounced in the bar trough region. The maximum of kinetic energy density for E01 is as large as over 2 times of that for case E03. The time- and alongshore-averaged across-shore momentum flux $\langle hu^c \rangle$ for the three cases is plotted as a function of $x$ coordinate in Figure 4-20. It is indicated that the across-shore momentum flux decreases as $S_\theta$ increases, with maxima located around the sandbar crest.

In simulation E01, E02 and E03, the relatively large friction is used. As a result, all the computed flows display approximately steady wavelike motion and the presence of a finite directional spreading does not alter the flow regime although quantities describing the flows are certainly affected. Next, a smaller bottom friction coefficient $k_u$ is used in the second set of experiments, i.e. E04, E05 and E06 (for details see Table 4-1).
As the friction factor is decreased, the maximum velocity $V_{\text{max}}$ of the resulting alongshore currents increases and the currents become more unstable. Time series of velocities for the three cases are shown in Figure 4-21. Compared to cases with $k_a = 0.03m$, the oscillations in across-shore velocity $u$ have become irregular (E04 and E05), and the amplitudes of oscillations in both across-shore and alongshore velocity also increase. For case E06, however, approximately steady wavelike flow pattern characters of equilibrated shear waves are observed since the base-state flow is less unstable compared to E04 and E05, and the overall flow pattern is different too. The influences of directional spreading can be further demonstrated by comparing the snapshots of vorticity field for case E04 ($S_\theta = 0^\circ$) and case ($S_\theta = 20^\circ$), see Figure 4-22. For case E04, the shear waves develop into long-wavelength propagating nonlinear waves. The nonlinear disturbances form, interact and propagate in a complicated manner such that at different times several separate individual disturbances with different alongshore scales are observed. For example, at $t = 9.00h$, three disturbances with length scales of approximately 200 m, 600 m, and 800 m can be seen (Figure 4-22(a)). The changes in the vorticity with time are also pronounced. For case E06 (Figure 4-22(b)), propagating regular disturbances are observed, which are similar to those found in cases with $k_a = 0.03m$. For case E04, the behavior of the flow is also illustrated in the $\bar{v}(x,t)$ and $\frac{1}{2}(\bar{u}^2 + \bar{v}^2)(x,t)$ contour plots in Figure 4-23. Small variations in $\bar{v}$ with time are observed in the bar trough area. Fluctuations in kinetic energy density $\frac{1}{2}(\bar{u}^2 + \bar{v}^2)$ on a very long time scale of about 2-3 hours are evident in Figure 4-23(b). The time- and alongshore-averaged velocity profiles and the perturbation kinetic density for cases E04, E05 and E06 are plotted in Figure 4-24. The time- and alongshore-averaged momentum flux $<hu^2>$ for $S_\theta = 20^\circ$
is much narrower and the magnitude is apparently smaller compared to the other two cases, see Figure 4-25.

To further examine the effects of wave directional spreading on the shear instabilities, another set of experiments (E11, E12 and E13) were performed. In these three experimental runs, a constant mixing coefficient $M = 0.3$ was used. Non-zero directional bandwidth $S_\theta = 10^\circ$ was used in E12 and $S_\theta = 20^\circ$ in E13. The bottom friction factor $k_a = 0.0235m$ for test E12 and $k_a = 0.012m$ for E13 were obtained by fitting the base-state velocity profiles to that of test case E11, in which $k_a = 0.03m$ (see Table 4-1). Comparison of snapshots of the vorticity field for the three cases is shown in Figure 4-26. For all three cases, time varying nonlinear shear waves propagating in the alongshore direction are observed. However, shear waves with larger alongshore wavelengths from 400 m to 900 m are developed in case E12 and E13 compared to the case E11. The overall flow patterns of the three cases are apparently different from each other although the different model parameter settings produce quite close velocity profiles. Strong energy dissipation due to shear instabilities is evident for all the three cases by comparing their time- and alongshore-averaged alongshore velocities to the base-state longshore velocity profile (Figure 4-27a). While, the $\langle \bar{v} \rangle$ profiles are close, which may suggest that the amount of energy dissipated is about the same although their flow patterns are different. The time- and alongshore-averaged perturbation kinetic energy $\frac{1}{2} \left( \bar{u'}^2 + \bar{v'}^2 \right)$ is shown in Figure 4-27b, and it also indicates that energy dissipation due to the instabilities is close for the three cases.

Effects of Lateral Mixing

Simulation results of experimental runs E04, E07 and E08, in which fixing $k_a = 0.01m$ and $S_\theta = 0$, are compared to investigate the effects of lateral mixing coefficient $M$ on shear
instabilities. For these three cases the steady base-state alongshore current profiles (Figure 4-13) clearly show that the velocity gradient is smaller for larger $M$. Simulations of nonlinear evolution of shear instabilities of these base-state velocity profiles are carried out with the method discussed above. Time series of velocities for E04 ($M = 0.5$), E04 ($M = 0.3$) and E04 ($M = 0.2$) are plotted in Figure 4-28. It can be seen that initial perturbations took longer to reach finite amplitudes for larger $M$, showing the damping nature of the lateral mixing terms. It is evident that the instabilities are more energetic for lower $M$. This is partly because the base-state velocity profile is more unstable with lower $M$, but also because the weaker eddy viscosity mixing effects reinforce the instabilities. For $M = 0.3$ and $M = 0.2$, the oscillations in velocities display underlying longer time scales of about 45 minutes. For $M = 0.3$, this longer time scale flow behavior is even approximately stable. No obvious long timescale oscillations are observed for $M = 0.5$. Snapshots of the vorticity field for case E04, E07 and E08 at $t = 10.0h$ are shown in Figure 4-29. As $M$ decreases from 0.5 to 0.2, the maxima of both the positive and negative vorticity increase. For $M = 0.5$, the vorticity field is relatively dispersive, near-zero vorticity existing in most region of the domain. For $M = 0.3$, areas with strong positive vorticity are obviously narrower. For $M = 0.2$, the features display a longer alongshore scale and vortex shedding occurs.

The time- and alongshore-averaged perturbation kinetic energy density (Figure 4-30) also shows that the motions are more energetic over most of the computation domain for smaller $M$. Figure 4-31 shows the across-shore momentum flux $\langle h u^2 \rangle$ for the three cases. The across-shore flux of the across-shore momentum is not altered significantly in the region between bar crest and the shoreline by varying the mixing coefficient $M$, while it increases dramatically as $M$ decreases in the region from sandbar crest to the offshore boundary. In addition, local
maxima of $\langle hu^2 \rangle$ are observed around $x = 250m$ for $M = 0.2, 0.3$. Finally, it is somewhat surprising to find that the time- and alongshore-averaged velocity $\langle \bar{v} \rangle$ is very close for varying $M$, although the their basic state velocity profiles are quite different and the instability climate is also very different (see Figure 4-32). Especially, the $\langle \bar{v} \rangle$ over sandbar and in the trough is almost the same for the three cases. The current $V'$ profile neglecting the instabilities is remarkably different, suggesting that the instabilities induce momentum mixing.

**Effects of Bottom Friction**

Fixing the mixing coefficient and the directional spreading, simulations with varying bottom friction factor $k_a$ are carried out and analyzed to study the effects of bottom friction on the instabilities. This set of experiments consists of run E01 ($k_a = 0.03m$), E04 ($k_a = 0.01m$), E09 ($k_a = 0.02m$) and E10 ($k_a = 0.005m$). Again, the steady base-state alongshore current profiles corresponding to these four cases can be found in Figure 4-13.

Time series of velocities at the monitoring location are shown in Figure 4-33. Generally, as the bottom friction coefficient $k_a$ decreases, the amplitude of oscillations in the time series of velocities increases, and the fluctuations become more irregular. For $k_a = 0.03m$, the oscillations are in regular, wavelike motions and are approximately steady. For $k_a = 0.01m$, the motions are more energetic, and oscillations with the longer time scales in time series of $v$ can be roughly seen. At the very small friction factor $k_a = 0.005m$, oscillations are initially of frequencies of about the order of 5 min. These waves, however, break down eventually after about $t = 2.5h$ and develop to motions with longer time scales of about 50 min. The alongshore velocity oscillations are very energetic and reach to about 0.5m/s at times with the undisturbed steady velocity of about 2.5m/s. The spatial flow structures are further demonstrated by snapshots of the vorticity
field at several times (see Figure 4-34). Generally, one large-scale disturbance is observed at all five times. The disturbance is strongly nonlinear and propagates in the alongshore direction, at the same time, it travels back and forth in the across-shore direction irregularly with time. In addition, the offshore extension of the vorticity disturbance is greatly increased compared with higher $k_a$, reaching the offshore boundary.

The time- and alongshore-averaged velocity $<\bar{v}>$ profiles and steady current profiles $V$ for the four cases are plotted in Figure 4-35(a). For all four cases, the mean velocities are considerably damped and the shape of velocity profiles are also altered somehow by the instabilities. The instabilities play the roles of momentum mixing, stretching out the steady velocity profiles in the across-shore directions. As $k_a$ decreases, the velocity profile is altered more significantly by the instabilities. The time and alongshore-averaged kinetic energy density (Figure 4-35(b)) suggests that strong oscillations occur at the location of about 420 m. For $k_a = 0.005 m$, the kinetic energy is much larger around the bar crest area than that for higher friction, and there is no the secondary maximum in this case as in cases with higher frictions. Figure 4-33 indicates that the across-shore momentum flux increases substantially as $k_a$ decreases. For $k_a = 0.005 m$, strong across-shore momentum flux presents outside the surf zone, indicating that eddies break away from the mean currents towards the offshore direction.

### 4.3 Simulation of Rip Currents

There have been a great number of efforts in simulating the rip currents using numerical models since the introduction of radiation stress by Longuet-Higgins and Stewart in a series of papers (1960, 1961, 1962, 1964). The depth-averaged two-dimensional shallow water equations are usually used as the governing equations in rip current models, and the wave forcing described by the gradients in radiation stress is associated as an input to drive the circulation. Numerical
studies have advantages over field or laboratory experiments in respect of allowing direct
computation and thus rip currents can be investigated in more details.

A variety of models in rip currents have been developed in past decades, early studies
including Bowen (1969), Noda (1974), Liu and Mei (1976), and Ebersole and Dalrymple (1980),
among others. These studies have generally assumed the currents are weak such that their effects
on waves are negligible. Some of them also neglected the nonlinear convective terms for
simplicity. More recently, more sophisticated modeling efforts have been reported. Haas et al.
(1998) conducted numerical simulations of a laboratory rip current system. They found that the
offshore extension of rip currents can be significantly reduced by including the wave-current
interaction. Chen et al. (1999) utilized Boussinesq equations to simulate the same laboratory set
up. In their model, the wave motion and induced currents were always solved simultaneously,
therefore the mutual interaction is always included. Therefore, it is not possible to compare the
results with and without interaction. Yu and Slinn (2003) conducted a numerical study of rip
currents on a barred beach with gentle sinusoidal alongshore variations. They investigated the
effects of wave-current interaction using the linear bottom friction model and ignoring the lateral
mixing.

Most the modeling efforts in rip currents to date have been for monochromatic (single
frequency and unidirectional) waves. The wave forcing of irregular waves, as discussed in
previous sections, can be different to some degree. Therefore, the resulting wave-induced rip
currents may be influenced somewhat by the presence of wave frequency and directional
bandwidth. It is still uncertain that how the frequency/directional spreading affects rip currents.
Using the present wave model, the effects of frequency/directional spreading on rip currents can
be studied in detail, which is very useful.
Oscillations or pulsations in rip currents over periods of minutes or tens of minutes have been noticed and recorded since very early in rip current studies. Shepard and Inman (1950) observed fluctuations with periods varying from 1.7 minutes to 7.8 minutes. Many other field observations (Mackenzie, 1958; Sonu, 1973; Smith and Largier, 1995; Huntley et al., 1998; Brander and Short, 2001; MacMahan et al., 2004) have also observed energetic rip current velocity pulsations. In addition, Haller and Dalyrmple (2001) documented oscillations of the rip current jet-like motions in a laboratory study. Many field experiments have overlooked the amount of energy associated with the rip current pulsations due to not sampling long enough to resolve them. Ogston and Sternberg (2003) re-analyzed some measurements using longer records and found that there is a great amount of energy associated with the low frequency oscillations. Fluctuations in rip current strength at a location can significantly increase the danger to swimmers. Therefore, it is very important to study the rip current pulsations and their causes. Two causes have been recognized to be responsible for rip current pulsations: time-varying wave groups and flow instabilities. In present study, we will concentrate on flow instabilities rather than wave groups.

In this section, two main topics are investigated: wave-current interaction on rip currents, and rip current instabilities. The objectives of this section include: improving the understanding of the physical processes of wave-current interaction, and its effects on rip currents; searching the relation between frequency/directional spreading and rip current features; and finally, rip current stabilities.

4.3.1 Steady Rip Currents

It has been known that wave-current interaction could change the structure of wave forcing dramatically and is very important in numerical simulation of rip currents (Haas et al., 1998;
Numerical simulations including effects of bottom friction and lateral mixing were carried out in order to obtain a better understanding of the physical processes involved in the wave-current interaction as it affects waves and rip currents. For cases where currents are strong, convergence could be difficult to achieve using the two-way coupling of the wave model and the steady version of the circulation model as discussed in chapter 3. Here, we used the wave model coupled with the unsteady version of the circulation model described in Section 4.2.3. Specifically, both the wave equations and the shallow water equations were solved in the time domain. At each time step, the wave model computed the instantaneous wave forcing which was then used to drive the circulation model, and the wave field and velocity field computed were then used as the initial values for the next time step. This time marching process was continued until a convergent solution had been reached both for the wave field and for the velocities. The velocity fields for cases without wave-interaction were computed by using the steady circulation model along with the wave model.

The bathymetry from Kennedy and Zhang (2008) was used throughout this section. The bathymetry was parameterized as a planar beach with a superimposed bar. The bar is Gaussian in cross-section and itself gives the alongshore variations in bathymetry, i.e.

\[ h(x, y) = h_0 - \alpha(x - x_0) - h_b(y) \exp\left(-\frac{(x - x_b)^2}{2\sigma_b^2}\right) \]  

(4-30)

where \( h_0 \) is the offshore water depth, and \( x_0 \) is its corresponding cross-shore coordinate, \( \alpha \) is the slope of the planar beach, \( x_b = 80m \) is the cross-shore location of the bar crest, and \( \sigma_b \) is the standard derivation of the Gaussian bar, which determines the width of the bar in across-shore direction. For all experiments, \( \sigma_b = 50/(3\sqrt{2})m \) was used. The longshore variation of the bar height is given by \( h_b \).
\[ h_b(y) = \left(1 - Cn^2\left(U_e, M_e\right)\right)A_b \] 
\[(4-31)\]

where \( Cn \) is a Jacobian elliptic function, and \( U_e \) and \( M_e \) are the elliptic argument and modulus respectively. \( M_e \) is in the range of 0 to 1, and a larger value of \( M_e \) gives a narrower rip channel. \( A_b \) is the bar amplitude and equal to 1m for all cases. For most cases, \( M_e = 0.95 \) (a narrow rip channel) was used. In this section, \( M_e = 0.95 \) unless stated otherwise. The argument

\[ U_e(y) = 2(y/L_y - 1/2)K_e, \]

where \( K_e(M_e) \) is the complete elliptic integral of the first kind.

Figure 4-37 shows the rip channel bathymetry with \( M_e = 0.95 \).

The no-slip wall boundary condition was applied to both the offshore boundary and the shoreline boundary. In the alongshore direction, periodic boundary conditions are imposed on \((u, v, \eta)\) because of the topography and the flow pattern we are interested in. For incoming waves, an equivalent deep water wave height of \( H_{rms0} = 1m \), and thus the RMS wave height at the offshore computational boundary is computed according to

\[ H_{rms} = H_{rms0} \sqrt{\frac{c_{g0} \cos \theta_0}{c_g \cos \theta}}, \]

with angle \( \theta_0 = 0 \), period \( T_m = 10s \).

For all the experiments presented below, \( \Delta x = \Delta y = 5m \) and the lateral mixing coefficient \( M = 0.8 \) were used. For cases including the wave-current interaction, \( \Delta t = 0.2s \) was used (relatively small because of the stability of the wave model). For most cases, the computational domain is \( L_x = 300m \) and \( L_y = 200m \).

To conduct a comprehensive study of the rip current system, a number of cases were simulated and one case was selected as the base case for detailed study. Departures from this base case were then examined to evaluate the importance of various parameters. These variations
included the effects of bottom friction, wave height, rip channel width and frequency-directional spreading as well.

**Base Case**

In the base case, monochromatic waves with equivalent deep water wave height is 1 m and the bottom friction factor $k_a = 0.08m$. Figure 4-38 shows the computed wave field without (a) and with (b) the wave-current interaction for the base case. Figure 4-39(a) shows velocity field without taking into account the wave-current interaction. For the same parameters, the results with the wave-current interaction are given in Figure 4-39(b). Some typical features on the influences of the wave-current interaction upon the wave field and the circulation system may be summarized as follows

1. Without the w-c interaction, wave shoaling occurs as the normally incident waves propagate toward the shore. After approaching to the bar region, waves refract from the rip channel and focus over the bar crest area, which leads to stronger breaking over the bar than in the rip channel (Figure 4-38(a)). The longshore variation in wave height then resulted into a classic current pattern with circulation cells and a narrow offshore-directed jet.

2. With the interaction, noticeable changes in wave height due to the ambient currents are observed: in the region from $x = 90m$ to $160m$ around the rip channel, the offshore-directed currents shorten the waves and consequently the wave height is increased. On the other hand, waves are lengthened by the onshore current in the area from the bar to the shore, which decreases the intensity of wave breaking and consequently wave height is slightly increased. Owing to the refraction effects by the circulation cells, dramatic changes in wave direction are observed. The changes in wave field due to the w-c interaction results in a forcing effect
opposite to that due to the topography. As a result, with the interaction the rip currents are noticeably weaker and circulations are mostly within 150 m from the shoreline in this case.

3. Without the interaction, rip currents reach to the offshore boundary even with the relative large parameter values of bottom friction factor $k_a$ and lateral mixing coefficient $M$. With the interaction, the offshore extent of rip currents is greatly reduced. Figure 4-40(a) shows the comparison of the u-velocity along the center of rip channel with and without the interaction. Rip currents are also noticeably broader with the interaction and u-velocity comparison along the longshore transect at $x = 100m$ is given in Figure 4-40(b). Figure 4-41 shows comparison of the wave set-up along the rip channel and the bar crest. It was observed that the w-c interaction did not alter the wave set-up significantly and the maximum difference is only about 1 cm.

**Bottom Friction**

In modeling nearshore currents, the bottom friction is usually a source of uncertainty because the bottom friction coefficient is usually unknown and also the formulations of bottom friction are roughly approximated. To test the effects of bottom friction, several different values of the bottom friction factor $k_a$ were used while other parameters were remained the same as in the base case. Figure 4-42 shows the u-velocity profiles along the rip channel center with various values of $k_a$. The shapes of velocity profiles are quite similar for all the bottom frictions and have maxima (offshore velocities) located at $x = 105m$ and minima (onshore velocities) located around $x = 50m$. The maximum offshore velocity is not sensitive to the bottom friction coefficient and changes only about 10% from a small friction coefficient case ($k_a = 0.05m$) to an unrealistic large friction case ($k_a = 0.2m$). As $k_a$ increases, the onshore velocities decrease apparently although the offshore velocities only decrease slightly. Figure 4-43 shows the
offshore extent and the maximum/minimum velocities versus the bottom friction factor $k_s$. The offshore extents are given by the across-shore locations where the offshore velocities decrease to 0.01 m/s. We can see that the offshore extent of the rip currents is weakly dependent on the bottom friction. It is also shown that all the velocity extreme values are smaller for a larger bottom friction, which is certainly reasonable since more dissipation associated with a larger friction.

It should be also noted that both the velocities and the offshore extent only change slightly as the bottom friction varies in a very wide range, which definitely suggest that the bottom friction becomes much less influential to the prediction of the circulation system. This is not the case when the effects of w-c interaction are not taken into account because the bottom friction is the only major dissipative effect on the flows. Results of the numerical experiments without the w-c interaction also suggest that the strong dependence of the predicted flow on bottom friction (not shown). With the interaction, decreasing the bottom friction will result in a stronger current, which in turn leads to decreased forcing to the current. This self-generated dissipation is dominant in deep waters where bottom friction is relatively small. It also explains why the bottom friction has more impact on velocities in the region from the alongshore bar to the shoreline.

**Wave Height**

As wave heights increase, the breaking induced currents are expected to become stronger. However, wave-current interaction acts as an opposing factor to focus wave breaking in the rip channel and thus reduce forcing, and breaking will saturate offshore of the bar for large wave heights, so the overall behavior may be very complex and is not well understood. To test the effects of the wave heights, another set of experiments was performed over equivalent deep
water wave heights of 0.6, 0.8, 1.0, 1.2 and 1.5 m. Again, normal incident waves are used and the bottom friction factor \( k_a \) is fixed to be 0.08 m in these experiments.

In Figure 4-44(a), the mean water level (wave setup/set-down) relative to the still water level along the line \( y = 100m \) for various wave heights is plotted versus the across-shore distance from the shoreline. The magnitude of wave setup increases monotonically with the deep water wave height, and the setup at the shoreline increases approximately linearly with \( H_0 \). It is also noted that as \( H_0 \) increases, the onset location of wave set-down moves seaward considerably.

Figure 4-44(b) shows the u-velocity profiles with different wave heights. For \( H_0 \geq 0.8m \), the maximum rip current velocity is located approximately at \( x = 105m \). For \( H_0 = 0.6m \), it moves shoreward slightly to \( x = 95m \). The maximum velocity increases significantly as \( H_0 \) varies from 0.6 m to 0.8 m but only slightly as \( H_0 \) increases from 0.8 m to 1.5 m. This is because for \( H_0 \geq 0.8m \) wave height distributions are similar, while for \( H_0 = 0.6m \) wave breaking occurs only in very shallow water and results in a different wave height distribution pattern. It is also evident that the maximum rip current velocity increases with wave height, as shown in Figure 4-45. The onshore maximum velocity shows a little variation as wave height increases and the maximum velocity of the feeder currents only increases slightly with wave height. The offshore extent of rip currents initially decreases quickly with \( H_0 \) and then shows a slow decrease when \( H_0 \geq 1.2m \).

**Rip Channel Width**

Rip currents have been observed for a wide range of topographies. Almost any geometry that induces a significant longshore variation in wave forcing will see rip currents when the incident waves are close to shore-normal. Kennedy et al. (2008) showed in a laboratory
experiment that rip currents with varying gap widths had similar peak velocities although the flows could be very different. In this set of experiments, the effects of relative rip channel width on the present system are investigated by varying the rip channel width. A relative rip channel width may be defined as the ratio of rip channel width to rip current periodic length. For the bathymetry described above, this is controlled by the elliptic modulus $M_e$ which for all previous tests was set to $M_e = 0.95$ (relative rip channel width 0.336). Figure 4-46 shows the longshore variation of bar/rip channel with varying $M_e$. $M_e = 0$ corresponds to a sinusoidal variation (relative width 0.5). The narrowest rip channel used had $M_e = 0.99$ (relative width 0.2). The relative rip channel width is taken at the longshore location where the bar height decreases to half its maximum height.

In this set of experiments, waves are normally incident with $H_o = 1m$ and the bottom friction factor $k_e$ is fixed to be 0.08. Figure 4-47 shows the across-shore $u$-velocity profiles for various $M_e = 0, 0.50, 0.75, 0.95, 0.99$. Generally, the stronger current is observed for a narrower channel (larger $M_e$), and the velocity profile does not change its across-shore shape as the rip channel width changes. The longshore $u$-velocity profile at $x = 100m$ changes significantly as $M_e$ varies from 0 to 0.99, as shown in Figure 4-48. This implies that the rip current becomes narrower as the rip channel becomes narrower, which can be seen even more clearly in Figure 4-49. As rip channel becomes broader, it is also observed that the maximum velocity of the rip current decreases noticeably, while the maximum feeder velocity increases slightly. The maximum onshore velocity does not change monotonously with rip channel width, initially decreasing then beginning to increasing with rip channel width.

**Frequency/Directional Spreading**
Monochromatic waves have been used in all the previous tests. As discussed in previous sections, with certain frequency and directional bandwidth the resulting wave radiation stresses could be considerably different from that of monochromatic waves, especially for irregular waves with a relatively broad directional spreading. Consequently, the wave forcing and then the flow pattern could be different. The effects of frequency/directional bandwidth on wave-induced longshore currents and on the nonlinear instability of longshore currents are investigated in Section 4.2. To investigate the frequency/directional spreading on this wave-induced rip current system, a series of numerical experiments was conducted with different values of frequency bandwidth $S_\sigma$ and directional bandwidth $S_\theta$. For all irregular wave cases, a Gaussian-shaped spectrum was used with mean wave period of 10 s and mean incident angle of $0^\circ$ (shore-normal). The equivalent deep water wave height $H_{\text{rms0}} = 1$ m and the friction factor $k_a = 0.08 m$ are used in these simulations.

To examine the effects of frequency bandwidth, results from cases with relative frequency bandwidth $S_\sigma/\sigma_m = 0.1, 0.2, 0.3$ are compared to those of the monochromatic wave case. Comparison of the u-velocity profiles of the resulting rip currents in Figure 4-50 does not show appreciable differences between these cases. This is not a big surprise partly because as shown in previous sections the wave field simulated by the present wave model is not significantly affected by a relative small frequency bandwidth with the Gaussian spectrum assumption. It is also because the topography induced wave transformations are dominant as irregular waves propagate over such an irregular bathymetry. In addition, the assumption that an input spectrum conserves its spectral shape used in the wave model may not hold here because strong wave focusing and scattering have occurred in these cases.
To assess the effects of the directional bandwidth on this rip current system, simulations with different directional bandwidth were carried out. A numerical instability problem was encountered for larger directional bandwidth \((S_\theta \geq 15^\circ)\), and it is believed because wave caustics may occur when the tested directional spectra are too broad. So the directional bandwidth used here is limited to be no larger than \(10^\circ\). Figure 4-51 shows the u-velocity profiles at \(x = 100m\) corresponding to the directional bandwidth \(S_\theta = 0^\circ, 4^\circ, 8^\circ, 10^\circ\). It can be seen that the u-velocity profile of the rip current only shows a little variation with directional bandwidth, with almost the same maximum velocity of the rip current but the maximum onshore velocity decreasing noticeably with \(S_\theta\). These observations are confirmed in Figure 4-52, and it also shows that the maximum velocity of the feeder currents do not change much either.

### 4.3.2 Stability of Rip Currents

Field measurements indicate that rip currents can have very long period oscillations (e.g. Shepard and Inman, 1950; Sonu 1973; Huntley et al. 1988; MacMahan et al., 2004, and many others). The low-frequency motions of rip currents have also been recognized in laboratory experiments (e.g. Haller and Dalrymple, 2001; Kennedy and Thomas, 2004). Two mechanisms have been suggested to be responsible for the pulsations in rip currents: long period modulations in incoming wave heights i.e. wave grouping, and instability of the hydrodynamic flow itself (Haller and Dalrymple, 2001). In this study, we will restrict ourselves to flow instabilities. Haller and Dalrymple (2001) performed an analytical study to investigate the linear stability characteristics of rip currents in a laboratory experiment, and they modeled the vicinity of the rip by considering it as a slowly varying, long narrow jet and obtained good agreement with laboratory data in the vicinity of the rip neck. To obtain typical pictures of instability over the whole circulation domain, Kennedy and Zhang (2008) developed a numerical model to perform
the linear stability analysis to a wave-induced rip current system by using the similar strategy as used to the longshore currents (see Section 4.2.2). The effects of wave-current interaction, lateral mixing and bottom friction were included in their model.

Numerical simulations of the low-frequency motions of rip current and circulation cell have not received sufficient studies to date. Slinn et al. (2000) simulated the flow instabilities over rip channel topography. However, all computations were performed using a very large incident angle of 45 degree, which resulted in longshore currents rather than rip currents for all cases. Yu and Slinn (2003) modeled instabilities of rip currents induced by normal (or nearly) incident waves breaking over artificial rip channel bathymetries. They found that the instabilities of the cellular circulations are sensitive to the angle of wave incidence and rip channel spacing. They also observed that the inclusion the wave-current interaction not only reduced significantly the mean forcing available to the rip current but also changed the instabilities of rip current: onset of instabilities occurs at the nearshore region, rather than offshore at rip heads with no-current interaction. These findings are very useful and can provide us useful insights into some typical features of rip current instabilities. While they used a gentle rip channel bathymetry with the sinusoidal perturbation in the y-direction and small bar heights (0.1 m and 0.2 m used), and the resulting currents were relatively weak, with the maximum rip current velocity of about 0.2m/s-0.3m/s for most cases. For strong rip currents, for example in which the rip velocity is as large as 0.5m/s or larger, the stability characteristics may be quite different from those for weak rip currents and were not sufficiently studied. In addition, the lateral mixing effects were not included in their model and they used the simple linear bottom friction model with a constant bottom friction coefficient. In this study, we specifically simulate the instabilities of rip currents
with a maximum rip velocity of about 0.5m/s, including the effects of lateral mixing and bottom friction with across-shore variable coefficients.

In the previous section, the simulation results showed that steady strong rip currents with a maximum offshore directed velocity of around 0.5m/s or larger resulted with the fairly large bottom friction \( k_a \geq 0.05m \) and lateral mixing \( M = 0.8 \) for both cases including w-c interaction and neglecting the interaction. Using the same wave parameters and bathymetry, the rip currents will become unsteady and the instabilities begin to grow if the bottom friction coefficient and the lateral mixing keep decreasing. The wave field is expected to vary temporally due to the existence of the time-dependent circulation field, and the resulting wave forcing in turn will affect the current field. This feedback mechanism, i.e. wave-current interaction, can be simulated using a time-dependent wave model along with an unsteady circulation model. While, numerical difficulties have been encountered in simulating the unsteady energetic rip current system using the present coupled wave/circulation model. Possible reasons that are believed to be responsible for the failure include: (1) possible occurrence of wave ray crossing, caustics and/or wave blocking, which is beyond the capability of the present wave model; (2) The explicit time differencing used in the wave model is not compatible very well with the implicit time differencing scheme in circulation model. Because of the difficulties, the instabilities of rip currents are simulated excluding the effects of wave-current interaction in this study. Specifically, the steady wave forcing is used to drive the unsteady rip current system instead of the time-dependent forcing.

### 4.3.2.1 Experimental Plan

Although the feedback between waves and currents can not be fully simulated by the present wave/circulation model due to some numerical difficulties when the currents are
unsteady and energetic, the impact of wave-current interaction on the resulting unsteady rip currents can be investigated to some degree through comparing the unsteady flows driven by wave forcing with and without the wave-current interaction. Two sets of numerical experiments were performed: one completely neglecting the wave-current interaction; the other using time-invariant wave forcing in which the wave-current interaction is included. Specifically, in the first set of simulations, the wave forcing was calculated based on the steady wave field simulated by neglecting the effects of the ambient current on waves, and the wave forcing was assumed invariant as the resulting rip currents become unsteady. In the second set, the wave forcing was from a steady-state solution of the wave and current system with sufficiently large bottom dissipation and lateral mixing coefficient including the wave-current interaction (as did in the previous section). The wave forcing then was used to drive the circulation.

Rip currents are known to be strongest with normally incident waves. However, both the mean properties and the stability of rip currents may change considerably with relative small changes in incident angle. In their study cases, Yu and Slinn (2003) found that instabilities of the rip currents were sensitive to the angle of wave incidence, ‘when the flow is dominated by cellular circulations a small longshore flow tends to destabilize the circulations, e.g., $\theta = 1^\circ$ and $3^\circ$. As the longshore component becomes strong, it suppresses the circulation cells, and the flow starts behaving like a longshore currents’. For relative strong rip currents, it is still uncertain whether the angle of wave incidence will affect the development of rip current instability in the same way. Another set of experiments were performed to investigate the effects of wave incident angle on rip current instability. Experiment setup and parameter settings are given in Table 4-2.

In test cases T03 and T04, the wave forcing is computed based on the wave field without including the wave-current interaction and remains invariant during the simulation of the
unsteady rip currents. In all other test cases, the wave forcing is from the simulation results of steady state wave and rip current system with the wave-current interaction being accounted for, in which the bottom friction factor $k_a = 0.06m$ and the lateral momentum mixing coefficient $M = 0.7$ were used.

4.3.2.2 Results

To decrease the influence of the lateral boundaries on the unsteady flow motions, simulations are carried out for the bathymetry composed of three rip channels. For all simulations, the computation domain is: $L_x = 300m$ and $L_y = 600m$. Rip channels are located at $y = 100, 300$ and 500 m. The rip channel bathymetry and incident wave conditions are the same as used in the base steady case in previous section except that the incident wave angle may be not normal for some cases. The no-slip wall condition is applied to both the offshore and onshore boundary, and the lateral boundaries are periodic. For all cases, the grid size of $(dx = dy = 5m)$ is used. Because the wave model does not need to be solved and the robust implicit discretization scheme is incorporated in the circulation model, a fairly large time step of $2 s$ is used for most cases. All computations cover duration of 15 hours. The time averages are obtained using the last 10 hours of the time series. The lateral mixing coefficient $M = 0.3$ was used for all the simulations.

Convergence

Convergence tests were performed for test cases T01 and T08. For each test case, simulations with several grid sizes of $dx = dy = 6m, 5m, 4m$ were conducted to investigate the effects spatial resolution upon the simulation results, and different time step sizes of $dt = 2.5s, 2.0s, 1.6s$ were used to examine the effects of the time step size on computational results. Figure 5-53 shows the comparison of the time-averaged vorticity field between
simulations with different grid sizes and time step sizes for test case T01. For the different time steps, the vorticity fields are very close. Using different grid sizes, while, evident differences in vorticity field appear. For the finer grid resolutions, the rip currents reach further offshore, especially, for the grid size of \( dx = dy = 4m \). Comparison of the across-shore velocity profiles along the rip channel also suggests that the time step sizes have been small enough, while the grid resolutions affect to some degree the mean velocity (see, Figure 5-54). For the test case T08, comparison of the time-averaged vorticity field is shown in Figure 5-55. It can be seen that simulations with these different time step sizes and grid sizes are very close in the mean vorticity field. Figure 5-56 shows the comparison of across-shore velocity profiles along the rip channel for these simulations. The time series of velocities at several monitored locations were also compared and no significant difference was observed. Since it is difficult to show the comparison of time varying quantities, only the comparison of mean quantities is presented.

Generally, these convergence tests verify that the time step of 2 s is small enough and using small time steps will not affect the computational results much. The grid resolution has more impact on the simulation results, and using finer resolutions than \( dx = dy = 5m \) could change the results to some degree. But the overall flow pattern will not alter significantly. As the result of negotiation between computational time and accuracy, the grid size of \( dx = dy = 5m \) and \( dt = 2s \) are used in all the following unsteady rip current simulations.

**Wave-Current Interaction**

The effects of wave-current interaction on the time evolution of rip currents are examined by comparing test cases T01, T02, T03 and T04. The bottom friction factor \( k_a = 0.02m \) is used in T01 and T03, and a smaller friction \( k_a = 0.01m \) is used in T02 and T04. Figure 4-57 shows the time series of velocities \((u, v)\) at the location of \((x, y) = (100m, 302.5m)\) for the four cases. The
flows are time dependent for all the four cases. For \( k_a = 0.02m \) (T01 and T03), fluctuations in velocities grow to be apparent after several hours of simulation (about 5 hours for T01 and 8 hours for T03). For the smaller friction \( k_a = 0.01m \) cases (T02 and T04), the instabilities grow much more quickly. Regular oscillations with the period of 12 minutes in velocities are observed for the simulation results of case T01 after the flow developing into an equilibrated state, and oscillations are somewhat irregular for all other three cases. For wave forcing without the effects wave-current interaction (T03 and T04), fluctuations in velocities display longer time scales (20~50 minutes) and are more irregular compared to T01 and T02.

Comparison of vorticity between case T02 and T04 is shown in Figure 4-58. The snapshots are separated by a 12-minute time interval. For test case T02, the vorticity is mainly concentrated between the shoreline and 150 m from the shoreline. For wave forcing without wave-current interaction (test case T04), however, the spatial distribution of vorticity is all over the computation domain, positive and negative vorticity existing even in the adjacent region of the offshore boundary. In inspection of temporal variations of the vorticity for T02 shows that the very low frequency (VFL) motions in rip currents are more or less trapped within the offshore extent of 150 m and they do not propagate seaward away from the surf zone. In contrast, in case T04 the VFL motions move back and forth from the surf zone to the offshore boundary. The strength of the vorticity for the two cases is surprisingly close although the shape and position are quite different from each other. It should be also noted that the flow patterns at different rip channels are quite different and the flows are complex for both test cases.

The cross-shore mass flux and momentum flux associated with a rip current system may be as a measurement of the intensity of the rip currents, and may also be of interest in the study of sediment transport phenomena and then morphological evolution in the nearshore areas. The
magnitude and spatial distribution of fluctuations of the rip current can be examined by the time-averaged turbulence kinetic energy, \( TKE = \frac{1}{2} \left[ (u - \bar{u})^2 + (v - \bar{v})^2 \right] \). Figure 4-59 shows the spatial distribution of the time-averaged across-shore mass flux \( \overline{hu} \), time-averaged across-shore momentum flux \( \overline{huu} \) and time-averaged turbulence kinetic energy \( TKE \) for test cases T01, T02, T03 and T04. For cases with wave forcing neglecting the wave-current interaction (T03 and T04), more water and more momentum are transported offshore through the rip channels compared to case T01 and T02. Although energetic, irregular oscillations in velocities are observed in T01 and T02, the mean flow pattern (not shown) is similar to the steady state rip currents. It should be also noted that \( \overline{hu} \) and \( \overline{huu} \) are generally slightly smaller for the smaller bottom friction cases, which may imply that more energy is dissipated by the instabilities.

The turbulent kinetic energy shows that the fluctuations are mainly localized in the feeder currents and rip currents, with local maxima of TKE located in the rip channels for case T01 and T02. For case T03, however, the TKE is associated with the offshore rip heads. The TKE for test case T04 suggests that oscillations occur over the entire circulations with local maxima associated with the ‘feeder’ and the rip currents. It is also evident that a smaller bottom friction results into more TKE by comparing T01 and T02, and T03 and T04.

**Wave Incident Angle**

To examine the response of the resulting unsteady flow to small deviation in angle of wave incidence from the shore-normal direction, computational experiments were performed for wave incident angles of 1°, 2°, 3°, 4°, 6° and 8°, with the same parameter settings in the wave model and circulation model (for details, see Table 4-2). In these simulations, the wave forcing used is
calculated based on the simulation results of steady wave fields with wave-current interaction being accounted for, in which \( k_a = 0.06m \) and \( M = 0.7 \).

Figure 4-60 shows the time series of velocities \((u, v)\) for simulation cases with various wave incident angles at the location of \( x = 100m \) and \( y = 302.5m \), where is seaward of the transverse sandbar around the rip channel center. At the very small angle of incidence \( \theta = 1^0 \), oscillations are initiated intermediately and develop to an approximately constant pattern after about 4 hour simulation. The time series show that the oscillations have the period of O(10 min) with magnitude of about 0.2m/s. Oscillations in the across-shore velocity also display the underlying longer time scales of about 80 minutes. When \( \theta = 2^0 \), significant differences in time series of velocities from the case \( \theta = 1^0 \) are observed: high frequency oscillations are observed at first. After about 3.5 hours, oscillations begin to display much longer time scales of about 53 minutes with smaller magnitudes; the mean value of the across-shore velocity is also reduced significantly. At \( \theta = 3^0 \), oscillations in velocities can still be roughly seen, but the magnitudes of fluctuations decrease further to very small values. As further increasing \( \theta \) to \( 4^0 \), a very different flow pattern results and energetic fluctuations with time scales of about 135 minutes are observed. It appears that within one cycle of the periodic case, the across-shore velocity \( u \) is close to 0.1m/s most of the time and increases to over 0.5m/s rapidly in a short time period, meanwhile the longshore velocity \( v \) is also at its peak value. At a still larger angle of \( \theta = 6^0 \), evident longshore character in the mean flow is observed, but the fluctuations in velocities have smaller magnitudes, especially in the cross-shore velocity, and shorter time scales of about 80 minutes. At the largest incident wave angle of \( \theta = 8^0 \) in this set of experiments, the longshore current character is more pronounced and the unsteady flow develops into a wave motion propagating in the longshore direction. The time series of velocities suggest the oscillations are
harmonic with a period of about 26 minutes. It should be also noted that both the mean value and
the oscillating magnitude of the across-shore velocity are small for this case, whereas the mean
longshore velocity is about 0.45 m/s. This also implies that the flow becomes longshore current
dominated.

From plots of time series of velocities \((u,v)\), i.e. Figure 4-60 along with Figure 4-57
\((\theta = 0^\circ\), case T01\), it seems that the resulting flow is quite sensitive to the wave incident angle.
For the case with shore-normal incidence, the flow instabilities take a few hours to develop and
very strong oscillation in velocities are observed. A small incident wave angle tends to trigger
the instabilities much quickly, however, the magnitudes of oscillations decreases, e.g. \(\theta = 1^\circ, 2^\circ\)
and \(3^\circ\). Irregular oscillations in velocities with the time scales of from 1 to 2 hours are observed
and longshore character begins to be evident for larger wave angles, i.e. \(\theta = 4^\circ, 6^\circ\). As further
increasing the angle of incidence to a certain value, i.e. \(\theta = 8^\circ\), longshore currents become
dominant and the flow acts as the typical ‘shear waves’.

Snapshots of vorticity for cases with the angles of incidence \(\theta = 2^\circ, 4^\circ \) and \(8^\circ\), each of
which represents a typical flow pattern (see Figure 4-60), are compared at several times, as
shown in Figure 4-61. For \(\theta = 2^\circ\), the sequence of vorticity fields roughly covers one cycle of
periodic oscillations in Figure 4-58. Let us focus on the patches of vorticity around the north rip
channel. During the first half of the cycle, the patch of the positive vorticity close to the shoreline
stretches clockwise with decrease in strength and the negative patch below shrinks. In the second
half of the cycle, this process reverses. The vorticity patches around other rip channels do not
appear to change their shapes and position significantly. For \(\theta = 4^\circ\), snapshots of vorticity only
cover about half cycle of the periodic case in Figure 4-60. During the time from \(t = 14.0h\) to
\(t = 14.6h\), the positive patch close to the shoreline around the ‘south’ rip channel stretches to the
north and also seaward and the strength is decreasing. In the meantime, the positive patch around the ‘north’ channel undergoes an opposite process. In addition, the vorticity pattern around the center rip channel does not alter very much during this time period. At $t = 15.0h$ when both the across-shore and longshore velocities are at their peak values (see Figure 4-60), however, strong circulation cells are observed and the offshore patch the positive vorticity turns clockwise toward the offshore direction. For $\theta = 8^\circ$, the vorticity fields suggest the circulation cells have virtually disappeared and unsteady longshore currents have been developed both close to the shoreline and over the sandbar. Fluctuations in the vorticity fields appear to have the longshore wavelength of 200 m, which is also the longshore length scale of the rip channel bathymetry. This suggests that the topography has its signature on the flow instabilities.

The time-averaged vorticity field, across-shore mass flux $\langle hu \rangle$, and turbulent kinetic energy $TKE = \frac{1}{2} \left[ (u - \bar{u})^2 + (v - \bar{v})^2 \right]$ for the set of experiments are shown in Figure 4-62. As the wave incident angle $\theta$ increases from $1^\circ$ to $8^\circ$, the rip currents gradually tilt away from the shore-normal direction and the circulation cells develop to longshore currents. At $\theta = 2^\circ$ and $3^\circ$, the similarity of time-averaged mean flow patterns at different rip channels breaks down, and the cause for this is not clear. At other angles, the similarity is observed. The time-averaged across-shore mass flux decreases considerably as $\theta$ increases from $1^\circ$ to $8^\circ$. The time-averaged turbulent kinetic energy provides a good measure of the intensity and distribution the fluctuations due to the flow instabilities. Strongest turbulent kinetic energy is observed when $\theta = 4^\circ$ and $6^\circ$, which is consistent with energetic oscillations in the time series of velocities shown in Figure 4-53. For $\theta = 1^\circ$, the TKE is mainly associated with the rip currents and shows similar pattern at different rip channels. For $\theta = 2^\circ$ and $3^\circ$, the TKE is small and can be only
seen at one rip channel. At $\theta = 8^0$, the TKE is not energetic but clearly associated with the meandering longshore currents, nearly absent from the rip channels.

4.3.3 Summary

The wave-induced rip currents over an artificial rip channel bathymetry have been simulated using the coupled wave/circulation model in this section. Both the steady state rip currents and unsteady motions (instabilities) of rip currents were examined. In the simulations of steady state rip currents, the wave-current interaction is fully represented by using the unsteady circulation model along with the wave model through the radiation stress concept. However, the wave feedback was not included in simulating the unsteady rip currents due to some numerical difficulties, and the stationary wave forcing was used. The bottom friction model with across-shore variable friction coefficient is used in these simulations. The lateral momentum mixing is also included.

From simulation results for the steady state rip currents, several conclusions can be drawn. For 10 s waves with equivalent deep water wave height of 1 m, fairly strong rip currents with maximum offshore velocity over 0.5 m/s result over this artificial rip channel bathymetry (see Equation (4-30) and Figure 4-37) even for moderately large bottom friction and lateral mixing. Both the wave field and the current field are altered considerably by including the wave-current interaction. Generally, the w-c interaction produced an opposite forcing effect to that due to topography. As a result, the rip currents are weaker, slightly broader and dramatically reduced in offshore extent with the w-c interaction being accounted for. Second, with the interaction the rip currents are much less sensitive to changes of the bottom friction. This is because that decreasing the bottom friction will result in a stronger current, which in turn leads to decreased forcing to the current. This self-generated dissipation mechanism supersedes the effects of bottom friction.
significantly. These two conclusions are in good agreement with the previous simulation results by Yu and Slinn (2003). It is also found that larger incident waves lead to a larger wave setup and wave setup at the shoreline increases approximately linearly with the wave height. For incident wave heights leading to a similar wave-breaking pattern, however, the resulting rip currents alter only slightly. The rip channel width appears to have a bigger impact on rip currents compared to the incident wave height. A narrow rip channel leads to a strong and narrow rip current without changing the shape of the across-shore velocity profile along the rip channel center. Moreover, it is suggested that the presence of limited wave frequency and direction spreading does not change the rip current apparently by the results of simulation experiments.

For rip current instabilities, we find that the very low frequency motions in rip currents are confined in the surf zone by using the wave forcing with w-c interaction rather than moving back and forth from the surf zone to the offshore boundary as predicted without including w-c interaction in the wave forcing. Finally, the instabilities are sensitive to the angle of wave incidence in terms of periods and magnitudes of the oscillations in velocities. The unsteady rip currents display a wide range of flow patterns as the wave incident angle varies from shore-normal to 8°. Under the current experiment conditions, the strongest fluctuations with very large time scales (from 1 to 2 hours) are resulted from the wave incident angle of $\theta = 4^\circ$ and 6°. As $\theta$ increases from 0 to 3°, the energy content of the instabilities decreases continually. At larger values $\theta = 6^\circ$ and 8°, circulation cells begin to disappear and the flow acts like ‘shear waves’ with the spatial length scale in longshore direction equal to that of the bathymetry.
Table 4-1. Experimental conditions and parameter settings for shear instability simulations. $k_a$ is the bottom friction coefficient, $M$ the lateral momentum mixing coefficient, $S_{θ}$ the directional bandwidth of incident waves, $V_{max}$ the maximum longshore velocity of the base mean longshore currents.

<table>
<thead>
<tr>
<th>Run name</th>
<th>$k_a$ (m)</th>
<th>$M$</th>
<th>$S_{θ}$ (0)</th>
<th>$V_{max}$ (m/s)</th>
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</tr>
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Table 4-2. Experimental conditions and parameter settings for rip current instability simulations. $WF$-interaction stands for whether wave forcing is computed with the wave-current interaction being accounted for; $θ_0$ is the angle of wave incidence, $k_a$ the bottom friction coefficient, $M$ the lateral momentum mixing coefficient.

<table>
<thead>
<tr>
<th>Run name</th>
<th>$WF$-interaction</th>
<th>$θ_0$ (0)</th>
<th>$k_a$ (m)</th>
<th>$M$</th>
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</tr>
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</tr>
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<tr>
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<td>8</td>
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<td>0.3</td>
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Figure 4-1. Sea bottom elevation relative to mean sea level versus cross-shore distance on October 14, 1994 at DUCK, NC and selected cross-shore locations of pressure sensors and/or current meters.

Figure 4-2. Predicted alongshore current versus cross-shore distance without roller model (points) and including the rollers with various wave-front slope values: $\alpha = 0.02$ (dash-dotted line), $\alpha = 0.05$ (solid line), $\alpha = 0.1$ (long-dashed line), $\alpha = 0.2$ (short-dashed line).

Figure 4-3. (a) Radiation stress $S_{xy}$, the wave radiation stress (short-dashed line), the roller radiation stress (long-dashed line), and the total (solid line); (b) Wave forcing in cross-shore direction, no roller (dashed line), including roller model with $\alpha = 0.05$ (solid line).
Figure 4-4. Model-data comparison: (a) measured (circle) and modeled wave height $H_{rms}$; (b) alongshore velocity, measured(circle), no roller (dashed line), roller model with $\alpha = 0.05$ (solid line); (c) bathymetry used.

Figure 4-5. Alongshore momentum balance: wave forcing (solid line), bottom friction (long-dashed line), lateral mixing (short-dashed line), and residual (dash-dotted line).
Figure 4-6. Effects of frequency spreading on the modeled (a) $H_{rms}$, (b) $S_{xy}$ and (c) alongshore velocity. Bathymetry, offshore $H_{rms}$, mean period and direction are the same as in Figure 4-4. Single frequency (short-dashed line), $S_\sigma / \sigma_m = 0.2$ (solid line), $S_\sigma / \sigma_m = 0.4$ (long-dashed line).

Figure 4-7. Effects of directional spreading on the modeled (a) $H_{rms}$, (b) $S_{xy}$ and (c) alongshore velocity. Bathymetry, offshore $H_{rms}$, mean period and direction are the same as in Figure 4-4. No directional spreading (short-dashed line), $S_\theta = 0.2 rad \approx 11.5^\circ$ (solid line), $S_\theta \approx 23^\circ$ (long-dashed line).
Figure 4-8. The artificial longshore current profile of the study case Bowen and Holman (1989), a constant water depth of 1 m is used.

Figure 4-9. Results of linear stability calculations for the alongshore velocity profile (Figure 4-8) in terms of growth rate (top) and propagation velocity (bottom) versus alongshore wave number. The thick dashed lines show wave number of the most unstable linear mode.
Figure 4-10. Spatial form of shear wave associated with the most unstable linear mode \( k_0 = 0.026 m^{-1} \). (a) stream function pattern; (b) fluctuation velocity field; (c) total velocity pattern, the shear wave has been scaled so that its peak magnitude equals the peak of the mean alongshore current.

Figure 4-11. Bathymetry for a barred beach using the formulation given by Slinn et al. (2000), which is an approximate fit to topography measured at Duck, North Carolina, on October 11, 1990.
Figure 4-12. Linear stability calculations for wave-induced alongshore currents over the barred beach (Figure 4-11) for varying bed apparent roughness, $k_a = 0.01m$ (short dashed lines), $k_a = 0.02m$ (long dashed lines), $k_a = 0.03m$ (solid lines). (a) Mean velocity profiles, (b) bottom friction coefficient, (c) growth rate versus alongshore wavenumber, (d) propagation velocity versus alongshore wavenumber.

Figure 4-13. The predicted steady alongshore velocity $V(x)$ for various experimental conditions (see Table 4-1).
Figure 4-14. Time series of velocities $u$, $v$ at $(x, y) = (420m,100m)$ with $k_\theta = 0.03m$ and $M = 0.5$ for different values of wave directional spreading $S_\theta$ from top to bottom $S_\theta = 0^\circ, 10^\circ, 20^\circ$ (Run name E01, E02, E03).
Figure 4-15. Snapshots of the contour plots of vorticity at times separated by 0.25 h for $k_a = 0.03 m$ and $M = 0.5$. Top panels for $S_\theta = 0^\circ$, bottom panels for $S_\theta = 20^\circ$.

Figure 4-16. The alongshore-averaged alongshore velocity $\bar{v}(x,t)$ (m/s) (top) and the alongshore-averaged perturbation kinetic energy density $\frac{1}{2}(u'^2 + v'^2)(x,t)$ (m$^2$s$^{-2}$) (bottom) as a function of the across-shore coordinate $x$ and time $t$ for $k_a = 0.03 m$, $M = 0.5$ and $S_\theta = 0^\circ$.
Figure 4-17. Comparison of time- and alongshore-averaged (a) alongshore velocity \( \langle \vec{v} \rangle \) (m/s) and (b) perturbation kinetic density \( \frac{1}{2} \langle u'^{2} + v'^{2} \rangle (x) \) (m\(^2\)s\(^{-2}\)) between simulations using different time step sizes to check whether the computations results are independent to the time step size. All other parameters are the same with the test E01.

Figure 4-18. Comparison of time- and alongshore-averaged (a) alongshore velocity \( \langle \vec{v} \rangle \) (m/s) and (b) perturbation kinetic density \( \frac{1}{2} \langle u'^{2} + v'^{2} \rangle (x) \) (m\(^2\)s\(^{-2}\)) between simulations using different grid sizes to check whether the computations results are independent to spatial resolution. All other parameters are the same with the test E01.
Figure 4-19. Time- and alongshore-averaged (a) alongshore velocity $< v >$ (m/s) and (b) perturbation kinetic density $\frac{1}{2} (u'^2 + v'^2)(x)$ (m$^2$s$^{-2}$) for $k_a = 0.03m$ and $M = 0.5$. $S_\theta = 0^0$ (solid lines), $S_\theta = 10^0$ (long dashed lines), $S_\theta = 20^0$ (short dashed lines).

Figure 4-20. Time- and alongshore-averaged across-shore momentum flux $< hu'^2 >$ for $k_a = 0.03m$ and $M = 0.5$. $S_\theta = 0^0$ (solid line), $S_\theta = 10^0$ (long dashed line), $S_\theta = 20^0$ (short dashed line).
Figure 4-21. Time series of velocities \( u, v \) at \( (x, y) = (420m, 100m) \) with \( k_a = 0.01m \) and \( M = 0.5 \) for different values of wave directional spreading \( S_{\theta} \) from top to bottom \( S_{\theta} = 0^0, 10^0, 20^0 \).

Figure 4-22. Snapshots of the contour plot of vorticity at times separated by 0.25 h for \( k_a = 0.01m \) and \( M = 0.5 \). (a) for \( S_{\theta} = 0^0 \), (b) for \( S_{\theta} = 20^0 \).
Figure 4-23. The alongshore-averaged alongshore velocity \( \overline{v}(x,t) \) (m/s) (top) and the alongshore-averaged perturbation kinetic energy density \( \frac{1}{2}(u'^2 + v'^2)(x,t) \) (m²s⁻²) (bottom) as a function of the across-shore coordinate \( x \) and time \( t \) for \( k_a = 0.01 m, M = 0.5 \) and \( S_\phi = 0^0 \).

Figure 4-24. Time- and alongshore-averaged (a) alongshore velocity \( \langle v \rangle \) (m/s) and (b) perturbation kinetic density \( \frac{1}{2}(u'^2 + v'^2)(x) \) (m²s⁻²) for \( k_a = 0.01 m \) and \( M = 0.5 \). \( S_\phi = 0^0 \) (solid lines), \( S_\phi = 10^0 \) (long dashed lines), \( S_\phi = 20^0 \) (short dashed lines).
Figure 4-25. Time- and alongshore-averaged across-shore momentum flux $< \overline{h u'^2} >$ for $k_a = 0.01m$ and $M = 0.5$. $S_\theta = 0^\circ$ (solid lines), $S_\theta = 10^\circ$ (long dashed lines), $S_\theta = 20^\circ$ (short dashed lines).

Figure 4-26. Snapshots of the contour plots of vorticity at times separated by $0.25\text{ h}$ for $M = 0.3$. Top panels for $S_\theta = 0^\circ$ and $k_a = 0.03m$; middle panels for $S_\theta = 10^\circ$ and $k_a = 0.0235m$; bottom panels for $S_\theta = 20^\circ$ and $k_a = 0.012m$. 

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Figure 4-27. Time- and alongshore-averaged (a) alongshore velocity $\langle v \rangle$ (m/s) and (b) perturbation kinetic density $\frac{1}{2} \langle u'^2 + v'^2 \rangle(x)$ (m$^2$s$^{-2}$) for $M = 0.5$. Thick dashed line in (a), the base-state velocity profile; solid line, $S_\theta = 0^\circ$; short dashed line, $S_\theta = 10^\circ$; long dashed line, $S_\theta = 20^\circ$.

Figure 4-28. Time series of velocities $u$, $v$ at $(x, y) = (420 m, 100 m)$ with $k_x = 0.01 m$ and $S_\theta = 0^\circ$ for different values of lateral mixing coefficient from top to bottom $M = 0.5, 0.3, 0.2$ (case E04, E07 and E08 respectively).
Figure 4-29. Snapshots of the contour plot of vorticity at \( t = 10h \) for \( k_a = 0.01m \) and \( S_\theta = 0 \) and varying lateral mixing coefficient \( M \).

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CHAPTER 5
CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

This study sought to develop numerical models to study water waves and wave-induced currents in nearshore area. A reduced wave spectral model was developed based on the evolution equations of wave moments of wave energy density spectrum. This new moments-based wave model is different from the traditional monochromatic wave models or the full spectral wave models because the dependent variables to be solved are the moments of wave spectrum. The present wave model can serve as an intermediate ground in the field of nearshore wave modeling between the two traditional types of wave models. Generally, the wave model is computationally cheaper than full spectral wave models and more accurate than monochromatic-based wave models for wave spectra with finite wave frequency and directional bandwidths, but relatively compact shapes with clear peak directions and frequencies. To study the wave-induced currents, a finite volume, steady two-dimensional circulation model was first developed using the pressure-correction solution method. Coupling of the wave model and the steady circulation model was utilized to simulate steady wave field and wave-induced current cases. The simulation results showed good agreement with the DUCK 94 field data of waves and longshore currents. It has long been known unsteady flow motions in nearshore currents can result from time invariant input waves. To be able to simulate the unsteady flow motions, a time-dependent circulation model extended from the steady solver was created and coupled with the wave model through the radiation stress concept. Shear instabilities of the longshore currents over a barred beach (approximating the DUCK90 bathymetry) were simulated and compared to the results of analytical linear stability analysis. Rip current instabilities were also studied using the coupled
wave/circulation model system. Overall, achievements achieved through this study can be summarized as follows

- The monochromatic wave models including the mild-slope wave models and the Boussinesq wave models, and the full spectral wave model (e.g. SWAN) were reviewed. The standard spectral wave action balance equation was re-derived using two different approaches.

- Starting with the standard wave action balance equation, the definition of the general wave moment $E_l(x,t)$ was introduced and the basic idea to build a moments-based wave model was presented. Based on the idea, the evolution equation of the general wave moment using a weighting function with the form of $\sigma^{n+1}e^{i\theta}$ was derived. A simple five-parameter wave model, which gives the RMS wave height, mean frequency, mean direction, and the frequency and directional bandwidths, was developed with some assumptions and approximations. One major assumption in this model is that the input wave spectra were assumed to be Gaussian-shaped distributions both in frequencies and directions and the input wave spectrum was assumed to reserve its spectral shape.

- Simulation of the evolution of an initial bump in wave heights for the irregular wave conditions showed good agreement with analytical solutions for both the wave heights and the wave spectra when the wave spectra are relatively narrow-banded. As wave spectra become broader, the model still predicted the wave heights reasonably well, however, the mean frequency, and especially the frequency bandwidth were not predicted well because the shape of wave spectrum at some locations is far from a Gaussian distribution. Simulations of non-breaking wave transformation on a sloping beach were conducted and compared with the analytical solutions. It was shown that the reduced model simulated not only the shoaling and refraction of irregular waves very well for this simple bathymetry, but also the limited evolution of the frequency bandwidth and the strong evolution of the directional bandwidth. Numerical tests of the wave model on the relative complex bathymetry were performed, and reasonable results were obtained.

- The wave model was used to investigate wave transformation due to the presence of ambient currents. For monochromatic waves, current-induced wave transformation was simulated very well by comparing with analytical solutions. The wave model was then used to investigate effects of wave frequency and directional bandwidths upon evolution of waves with the existence of artificial ambient currents. Generally, the wave field is more sensitive to directional spreading than frequency spreading. For waves with a considerable directional spread, the wave transformation due to ambient currents can be very different from that of monochromatic waves.

- The radiation stresses computed based on the monochromatic approximation are often used to drive the wave-induced currents in the nearshore circulation models. This approximation, however, may cause significant errors in the radiation stress components when the waves have finite frequency and directional bandwidths. Formulations of radiation stresses including the effects of frequency and directional bandwidths were
derived, and a much more accurate estimate than the monochromatic approximation was provided.

- A 2-D finite volume steady circulation model based on the nonlinear shallow water equations was developed using the pressure-correction solution method. The model was built on the boundary fitted structured non-orthogonal computational grids and can be used to simulate flows in complex geometries. The models' performance, convergence and accuracy were investigated using an artificial test with the analytical solutions available. It was proved that the model has 2nd order accuracy for both Cartesian and non-orthogonal computations grids. The model was further tested using the well known lid-driven cavity flow case, and the computed results showed a good agreement with the benchmark solutions (Ghia et al., 1982).

- The wave model and the steady circulation model were coupled and used to simulate wave transformations and wave-induced longshore currents over artificial and field bathymetries. Simulation results of the coupled models were compared to the DUCK94 field data. Both the wave field and current field were well predicted by the models. A wave roller model including the effects of wave directional spreading was derived, simulation results showed that inclusion of surface rollers improved the prediction of the across-shore velocity profile by shifting the maximum velocity shoreward and increasing the bar trough velocity over the barred beach. Effects of wave frequency and directional spreading were studied by numerical experiments on the DUCK94 bathymetry. Generally, the wave height distribution is relatively insensitive to wave frequency and directional spreading, however, the magnitude of the resulting longshore currents was considerably decreased by the presence of a finite directional bandwidth.

- The linear shear instabilities of longshore currents over a barred beach were studied by an analytical stability analysis model including the bottom friction. The analytical results suggested that the shear instabilities of alongshore currents give rise to alongshore propagating shear (vorticity) waves, which is consistent with previous studies. However, studying the fully nonlinear dynamics of finite-amplitude shear waves requires numerical simulations. A time-dependent circulation model extended from the steady version was created and coupled with the wave model. Using the coupled models, sets of simulations were performed to investigate the effects of bottom friction, lateral mixing, especially wave directional spreading on shear instabilities. Numerical results suggested that more unsteady, energetic instabilities are associated with a smaller bottom friction, a weaker lateral mixing and a stronger mean current. Shear instabilities can cause significant damping of the mean current by inducing the horizontal momentum mixing. A finite wave directional bandwidth leads to a less unstable flow, consequently, the shear instability character can be altered quantitively and qualitatively due to the presence of the directional bandwidth.

- Steady-state wave-induced rip currents on an artificial rip channel bathymetry were simulated using the wave model along with the unsteady circulation model. Simulation results showed that the wave-current interaction has significant impacts on both the wave field and the resulting circulation system. Generally, the w-c interaction resulted in a forcing effect opposite to that due to the topography. As a result, the rip currents were
noticeably weaker and slightly broader and the offshore-directed rips were mostly restricted in the surf zone when including the w-c interaction. In addition, the wave-current interaction is of such significance that the resulting rip currents are much less sensitive to a change of bottom friction coefficient. It was also observed that the strength and offshore extent of rip currents are subject to the incident wave height and the rip channel width. Limited spreading in wave frequency and direction do not have significant effects on the resulting rip currents because topography- and current-induced wave transformations are dominant factors in generating the rip currents.

• The instabilities of rip currents were investigated by numerical experiments focusing on the following two aspects: effects of wave-current interaction on the development of rip current instabilities; and dependency of unsteady rip currents on the angle of wave incidence. The simulation results suggested that low frequency fluctuations in rip currents are restricted in the surf zone when using the wave forcing with the wave-current interaction rather than moving back and forth from the surf zone to offshore boundary as predicted without including wave-current interaction in the wave forcing. With no wave-current interaction, the time-averaged across-shore mass and momentum transported offshore through the rip channels may be overestimated, and offshore extent of transport processes could be over-predicted considerably too. It was also observed that the instabilities of rip currents are sensitive to the angle of wave incidence. The unsteady rip currents displayed a wide range of flow patterns as the wave incident angle increases from shore-normal to 8°. The energy content of the instabilities initially decreased as the angle continually from 0 to 3 degrees. Most energetic fluctuations with periods of about 2 hours were observed for wave incident angle of 4°. Circulation cells began to disappear and the longshore current character became pronounced as the angle further increased to 6° and 8°.

5.2 Discussion and Recommendations for Future Work

In this study, the reduced wave spectral model and the circulation model were developed and used to simulate the nearshore waves and currents. Satisfactory simulation results have been achieved by using the newly developed models, especially in the aspect that the models allow for the direct investigation of the effects of the wave frequency and directional spreading on nearshore waves and currents. However, there are uncertainties associated with various assumptions, approximations and simplifications in the models, especially in the wave model. There are also some approximations that have been made in the applications of the models. The following recommendations are made for further studies.

• Derivation of the evolution equations of wave moments, by integrating the wave action balance equation multiplied by some weighting functions over frequencies and directions,
does not need any prior assumptions. However, closure assumptions are necessary in order to evaluate the unsolved integrals in terms of wave moments. In this study, the input Gaussian-shaped spectra have been assumed exclusively. More realistic wave frequency spectra such as JONSWAP (Hasselmann, 1973), TMA (Bonuws et al., 1985) and PM Spectrum (Pierson and Moskowitz, 1964) should be used in the future.

- One major difficulty in developing the moments-based wave model is that more dependent wave moments will be involved than the number of evolution equations. To define a close system, extra relations between different moments are required. For simplicity, the spectral shape of the input spectrum was assumed to reserve in this study. This, however, is not the case, especially for the directional spectrum when strong wave focusing and/or defocusing take place. One possible improvement of this is to use an external estimator for the extra wave moments using the available wave moments, such as the maximum entropy technique.

- The effects of wave diffraction are not accounted for in the present wave model, like the typical full spectral wave model. To accommodate the wave diffraction, one could use the phase-decoupled refraction-diffraction approximation initially proposed for the SWAN wave model by Holthuijsen et al. (2003). For a complete simulation of nearshore waves, wind input, energy dissipation due to whitecapping, and wave-wave interactions should also be included.

- The present wave model is limited to wave spectra with single peak frequency and direction. One possible improvement is to develop a more sophisticated model that can deal with wave spectra with multi-peaks in both frequency and direction.

- In both the steady and unsteady circulation models, the SIMPLE algorithm was used as the solution method. One could use enhanced versions of SIMPLE to make the model more efficient and more robust. For the unsteady circulation model, a more sophisticated time differencing scheme rather than the implicit Euler method should be used for accuracy and stability.

- Although the circulation models were developed on the boundary-fitted non-orthogonal structured grid system, tests on realistic coastal topographies using non-orthogonal grids were not conducted. More verifications of the model on computation domain with complex external and/or internal boundaries should be performed before any application.

- In the present circulation models, a stationary computation grid was used. To model the shoreline movement, capability of dealing with the dynamic boundary should be built into the models. One simple way to do that is following the drying-wetting scheme developed by Casulli and Cheng (1992).

- In simulating the nonlinear shear instabilities of longshore currents, the effects of wave-current interaction on the development of were not included. It has been found that wave-current interaction can significantly alter the finite amplitude behavior of the shear instabilities (Özkan-Haller and Li, 2003). Therefore, for a complete study, wave-current interaction should be accounted for.
In simulating the rip current instabilities, the wave-current interaction was only partially represented by using the time invariant wave forcing including the wave-current interaction. In reality, temporal changes in the wave field will be induced due to the changes of the current field, and so does the wave forcing. Therefore, the full representation of wave-current interaction is necessary for a complete study.

The present models have been mostly used in artificial or idealized systems. More simulations of field data should be conducted in future work.
APPENDIX A
DERIVATION OF MOMENTS OF THE GAUSSIAN-SHAPED SPECTRA

With the assumption of the Gaussian-shaped wave spectra

\[ F(x, y; t; \sigma, \theta) = E_{0,0} \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ -\frac{1}{2} \left( \frac{x}{\sigma} \right)^2 \right] \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ -\frac{1}{2} \left( \frac{\theta}{\sigma} \right)^2 \right] \] (A-1)

According to its definition, the general moment of the wave spectrum density (A-1) \( E_{n,m} \) can be written as

\[ E_{n,m} = E_{0,0} \frac{1}{\sqrt{2\pi \sigma}} \frac{1}{\sqrt{2\pi \sigma}} \int_0^\infty \sigma^n e^{\frac{1}{2} \left( \frac{(x - \sigma)}{\sigma} \right)^2} d\sigma \int_0^\infty \sigma^m e^{\frac{1}{2} \left( \frac{(\theta - \sigma)}{\sigma} \right)^2} d\theta \] (A-2)

First, look at the integral \( \int_0^\infty \sigma^n e^{\frac{1}{2} \left( \frac{(x - \sigma)}{\sigma} \right)^2} d\sigma \). Let \( \frac{\sigma - \sigma_m}{\sigma} = t \), which gives

\[ \int_0^\infty \sigma^n e^{\frac{1}{2} \left( \frac{(x - \sigma)}{\sigma} \right)^2} d\sigma = \int_{-\infty}^\infty (S_\sigma t + \sigma_m)^n e^{\frac{1}{2} t^2} S_\sigma dt \] (A-3)

In the case \( \left| \frac{-\sigma_m}{S_\sigma} \right| \) is not too small, the integral above can be approximated as follows

\[ \int_0^\infty \sigma^n e^{\frac{1}{2} \left( \frac{(x - \sigma)}{\sigma} \right)^2} d\sigma = \int_{-\infty}^\infty (S_\sigma t + \sigma_m)^n e^{\frac{1}{2} t^2} S_\sigma dt \approx \int_{-\infty}^\infty (S_\sigma t + \sigma_m)^n e^{\frac{1}{2} t^2} S_\sigma dt \] (A-4)

Because the exponential function \( e^{\frac{1}{2} t^2} \) decreases quickly to 0 as \( |t| \) increases to about 3. With the narrow-banded assumption, \( \left| \frac{-\sigma_m}{S_\sigma} \right| \), in fact, is always larger than 3. Therefore, approximation from Equation (A-3) to (A-4) is reasonable. According to the binomial theorem, we have

\[ (S_\sigma t + \sigma_m)^n = \sum_{k=0}^n \frac{n!}{k!(n-k)!} S_\sigma^{n-k} \sigma_m^k t^{n-k} \] (A-5)

Substituted into Equation (A-4) gives
\[
\int_{0}^{\infty} \sigma^n e^{-\frac{1}{2}(\sigma-\sigma_m)/S_\sigma} d\sigma = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} S_{\sigma}^{n-k+1} \sigma_m^k \Gamma\left[(n-k+1)/2\right] \int_{0}^{\infty} t^{n-k} e^{-\frac{1}{2}t^2} dt
\]

\[
= \left\{ \begin{array}{ll}
\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} S_{\sigma}^{n-k+1} \sigma_m^k 2^{(n-k+1)/2} \Gamma\left[(n-k+1)/2\right], & (n-k)=0,2,4,... \\
0, & \text{otherwise}
\end{array} \right.
\]  

(A-6)

Where, \(\Gamma\left[(n-k+1)/2\right]\) is the gamma function.

Similar process is applied to the integral \(\int_{-\pi}^{\pi} e^{-\frac{1}{2}(\theta-\theta_m)/S_{\theta}} d\theta\). Substituting \(\frac{\theta-\theta_m}{S_{\theta}} = t\) into the integral and assuming that both \(\left| (\pi - \theta_m) / S_{\theta} \right| \) and \(\left| (-\pi - \theta_m) / S_{\theta} \right|\) are not small so that the approximation followed can be made

\[
\int_{-\pi}^{\pi} e^{im\theta} e^{-\frac{1}{2}(\theta-\theta_m)/S_{\theta}} d\theta = \int_{(-\pi-\theta_m)/S_{\theta}}^{(\pi-\theta_m)/S_{\theta}} e^{im(S_{\theta}t+\theta_m)} e^{-\frac{1}{2}t^2} S_{\theta} dt \approx \int_{-\infty}^{+\infty} e^{im(S_{\theta}t+\theta_m)} e^{-\frac{1}{2}t^2} S_{\theta} dt
\]

(A-7)

And,

\[
\int_{-\infty}^{+\infty} e^{im(S_{\theta}t+\theta_m)} e^{-\frac{1}{2}t^2} dt = \sqrt{2\pi}e^{-m^2S_{\theta}^2/2} (\cos m\theta_m + i \sin m\theta_m)
\]

(A-8)

So,

\[
\int_{-\pi}^{\pi} e^{im\theta} e^{-\frac{1}{2}(\theta-\theta_m)/S_{\theta}} d\theta \approx \sqrt{2\pi}S_{\theta}e^{-m^2S_{\theta}^2/2} (\cos m\theta_m + i \sin m\theta_m)
\]

(A-9)

Substituting equations (A-6) and (A-9) into (A-2), the moment \(E_{n,m}\) is finally given by

\[
E_{n,m} = \frac{E_{0,0} \sqrt{2\pi} e^{-m^2S_{\theta}^2/2} (\cos m\theta_m + i \sin m\theta_m) *}
\]

\[
= \left\{ \begin{array}{ll}
\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} S_{\sigma}^{n-k+1} \sigma_m^k 2^{(n-k+1)/2} \Gamma\left[(n-k+1)/2\right], & (n-k)=0,2,4,... \\
0, & \text{otherwise}
\end{array} \right.
\]  

(A-10)
As mentioned above, \( |\sigma_m / S_\sigma|, |(\pi - \theta_m) / S_\theta| \) and \( |(-\pi - \theta_m) / S_\theta| \) were assumed to be no less than 3 in order to take advantage of the definite integral containing exponential functions. To maintain the assumption valid, following limitations are forced:

\[
S_\sigma / \sigma_m \leq 0.25, \ (\pi - \theta_m) / S_\theta \geq 5 \text{ and } (-\pi - \theta_m) / S_\theta \leq -5
\]  

(A-11)

After deriving the general form of wave moment \( E_{n,m} \), it is straightforward to give the moments appearing in the current wave model:

\[
E_{1,0} = E_{0,0} \sigma_m
\]  

(A-12)

\[
E_{2,0} = E_{0,0} \left( \sigma_m^2 + S_\sigma^2 \right)
\]  

(A-13)

\[
E_{0,1} = E_{0,0} e^{-S_\theta^2 / 2} \left( \cos \theta_m + i \sin \theta_m \right)
\]  

(A-14)

\[
E_{0,-1} = E_{0,0} e^{-S_\theta^2 / 2} \left( \cos \theta_m - i \sin \theta_m \right)
\]  

(A-15)

\[
E_{0,2} = E_{0,0} e^{-2S_\theta^2} \left( \cos 2\theta_m + i \sin 2\theta_m \right)
\]  

(A-16)

\[
E_{0,3} = E_{0,0} e^{-3S_\theta^2 / 2} \left( \cos 3\theta_m + i \sin 3\theta_m \right)
\]  

(A-17)
APPENDIX B

EVALUATION OF THE INTEGRALS IN GOVERNING EQUATIONS

This appendix gives the derivations of integrals in the equation of the general moment $E_{n,m}$. Integrals in the equations of the reduced spectral wave model are straightforward to obtain from the three general integrals.

Recall Equation (2-47)

$$
\frac{\partial}{\partial t} E_{n,m} + \nabla \cdot \int_{-\pi}^{\pi} \left( c_g + u \right) \sigma^n e^{im\theta} F d\sigma d\theta - im \int_{\pi}^{\pi} \int_{-\pi}^{\pi} c_0 \sigma^n e^{im\theta} F d\sigma d\theta 
$$

$$
-(n+1) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} c_g \sigma^{n-1} e^{im\theta} F d\sigma d\theta = 0 \tag{B-1}
$$

1. First integral

$$
I_1 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( c_g + u \right) \sigma^n e^{im\theta} F d\sigma d\theta
$$

$$
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} c_g (\cos \theta, \sin \theta) \sigma^n e^{im\theta} F d\sigma d\theta + (u, v) E_{n,m}
$$

$$
= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \frac{e^{i\theta} + e^{-i\theta}}{2}, \frac{e^{i\theta} - e^{-i\theta}}{2i} \right) c_g \sigma^n e^{im\theta} E_{0,0} M(\sigma) D(\theta) d\sigma d\theta + (u, v) E_{n,m} \tag{B-2}
$$

$$
= \frac{1}{2} \int_{-\pi}^{\pi} c_g \sigma^n M(\sigma) d\sigma \int_{-\pi}^{\pi} \left( e^{i(m+1)\theta} + e^{i(m-1)\theta} - i \left( e^{i(m+1)\theta} - e^{i(m-1)\theta} \right) \right) E_{0,0} D(\theta) d\theta + (u, v) E_{n,m}
$$

$$
= \frac{1}{2} \left( E_{0,m+1} + E_{0,m-1}, -i \left( E_{0,m+1} - E_{0,m-1} \right) \right) \int_{0}^{\infty} c_g \sigma^n M(\sigma) d\sigma + (u, v) E_{n,m}
$$

Using the Taylor expansion approximation of $c_g$, we have:

$$
\int_{0}^{\infty} c_g \sigma^n G(\sigma) d\sigma 
$$

$$
\approx \left( c_g \right)_{\sigma_m} \int_{0}^{\infty} \sigma^n G(\sigma) d\sigma + \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_m} \int_{0}^{\infty} (\sigma - \sigma_m) \sigma^n G(\sigma) d\sigma + \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_m} \int_{0}^{\infty} (\sigma - \sigma_m)^2 \sigma^n G(\sigma) d\sigma
$$

$$
= \left( c_g \right)_{\sigma_m} \int_{0}^{\infty} \sigma^n G(\sigma) d\sigma + \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_m} \left( \int_{0}^{\infty} \sigma^{n+1} G(\sigma) d\sigma - \sigma_m \int_{0}^{\infty} \sigma^n G(\sigma) d\sigma \right)
$$

$$
+ \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_m} \left( \int_{0}^{\infty} \sigma^{n+2} G(\sigma) d\sigma - 2\sigma_m \int_{0}^{\infty} \sigma^{n+1} G(\sigma) d\sigma + \sigma_m^2 \int_{0}^{\infty} \sigma^n G(\sigma) d\sigma \right)
$$

(B-3)
Inserting the above equation back into Equation (B-2), we arrive at

\[
I_1 = \frac{1}{2} \left( E_{n,m+1} + E_{n,m-1}, -i \left( E_{n,m+1} - E_{n,m-1} \right) \right) \left[ \left( c_g \right)_{\sigma_n} - \sigma_m \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_n} + \frac{1}{2} \sigma_n^2 \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_n} \right] \\
+ \frac{1}{2} \left( E_{n+1,m+1} + E_{n+1,m-1}, -i \left( E_{n+1,m+1} - E_{n+1,m-1} \right) \right) \left[ \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_n} - \sigma_m \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_n} \right] \\
+ \frac{1}{2} \left( E_{n+2,m+1} + E_{n+2,m-1}, -i \left( E_{n+2,m+1} - E_{n+2,m-1} \right) \right) \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_n} + \left( u, v \right) E_{n,m}
\]  

(B-4)

2. Second integral

Similar to the first integral, the second integral can be given by

\[
I_2 = \int_{-\pi}^{\pi} \int_{0}^{2\pi} c_g \sigma^n e^{i\sigma \theta} F d\sigma d\theta = -\frac{1}{2h} \left( c_g \frac{\partial}{\partial \sigma} \right) G(\sigma) \sigma^n d\sigma \left[ i \frac{\partial h}{\partial x} \left( E_{0,m+1} - E_{0,m-1} \right) + \frac{\partial h}{\partial y} \left( E_{0,m+1} + E_{0,m-1} \right) \right] \\
- \frac{1}{4} \left( \frac{\partial u}{\partial y} \right) \left( 2E_{n,m} + E_{n,m+2} + E_{n,m-2} \right) - \frac{i}{4} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \left( E_{n,m+2} - E_{n,m-2} \right) + \frac{1}{4} \left( \frac{\partial v}{\partial x} \right) \left( 2E_{n,m} - E_{n,m+2} - E_{n,m-2} \right) \\
- \frac{1}{2h} \left[ \left( \frac{\partial c}{\partial \sigma} \right)_{\sigma_n} - \sigma_m \left( \frac{\partial^2 c}{\partial \sigma^2} \right)_{\sigma_n} + \frac{1}{2} \sigma_n^2 \left( \frac{\partial^2 c}{\partial \sigma^2} \right)_{\sigma_n} \right] \left[ \frac{i}{\partial \sigma} \left( E_{n,m+1} - E_{n,m-1} \right) + \frac{\partial h}{\partial \sigma} \left( E_{n,m+1} + E_{n,m-1} \right) \right] \\
- \frac{1}{2h} \left[ \left( \frac{\partial^2 c}{\partial \sigma^2} \right)_{\sigma_n} - \sigma_m \left( \frac{\partial^2 c}{\partial \sigma^2} \right)_{\sigma_n} \right] \left[ \frac{i}{\partial \sigma} \left( E_{n+1,m+1} - E_{n+1,m-1} \right) + \frac{\partial h}{\partial \sigma} \left( E_{n+1,m+1} + E_{n+1,m-1} \right) \right] \\
- \frac{i}{4} \left( \frac{\partial^2 c}{\partial \sigma^2} \right)_{\sigma_n} \left[ \frac{i}{\partial \sigma} \left( E_{n+2,m+1} - E_{n+2,m-1} \right) + \frac{\partial h}{\partial \sigma} \left( E_{n+2,m+1} + E_{n+2,m-1} \right) \right] - \frac{1}{4} \left( \frac{\partial u}{\partial y} \right) \left( 2E_{n,m} + E_{n,m+2} + E_{n,m-2} \right) \\
- \frac{i}{4} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \left( E_{n+1,m+1} - E_{n+1,m-1} \right) + \frac{1}{4} \left( \frac{\partial v}{\partial x} \right) \left( 2E_{n,m} - E_{n,m+2} - E_{n,m-2} \right) \\
\]

(B-5)
3. Third integral

\[ I3 = \int_0^{\pi} \int_{-\pi}^{\pi} c_n e^{i\theta} d\theta \]

\[ = \frac{E_{0,m}}{h} (U \cdot \nabla h) \int_0^{\pi} \left( \frac{c_g}{c} - \frac{1}{2} \right) G(\sigma) d\sigma - \frac{1}{4} \int_0^{\pi} \frac{c_g}{c} G(\sigma) d\sigma \left( \frac{\partial u}{\partial x} (2E_{0,m} + E_{0,m+2} + E_{0,m-2}) \right) \]

\[ -i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) (E_{0,m+2} - E_{0,m-2}) + \frac{\partial v}{\partial y} (2E_{0,m} - E_{0,m+2} - E_{0,m-2}) \]

\[ = \frac{1}{h} (U \cdot \nabla h) \left\{ \left( \frac{c_g}{c} \right) - \sigma_m \left( \frac{\partial c_g}{\partial \sigma} \right) + \frac{1}{2} \sigma_m^2 \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right) \right\} E_n,m + \]

\[ \left\{ \left( \frac{\partial c_g}{\partial \sigma} \right) - \sigma_m \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right) \right\} E_{n+1,m} + \frac{1}{2} \sigma_m \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right) E_{n+2,m} \]

\[ - \left\{ \left( \frac{c_g}{c} \right) - \sigma_m \left( \frac{\partial c_g}{\partial \sigma} \right) + \frac{1}{2} \sigma_m^2 \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right) \right\} \]

\[ -i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) (E_{n,m+2} - E_{n,m-2}) + \frac{\partial v}{\partial y} (2E_{n,m} - E_{n,m+2} - E_{n,m-2}) - \left\{ \left( \frac{\partial c_g}{\partial \sigma} \right) - \sigma_m \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right) \right\} \]

\[ + \frac{1}{4} \left( \frac{\partial u}{\partial x} \right) \left( 2E_{n+1,m} + E_{n+1,m+2} + E_{n+1,m-2} \right) - i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left( E_{n+1,m+2} - E_{n+1,m-2} \right) \]

\[ + \frac{\partial v}{\partial y} \left( 2E_{n+1,m} - E_{n+1,m+2} - E_{n+1,m-2} \right) - \frac{1}{8} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right) \left[ \frac{\partial u}{\partial x} \left( 2E_{n+2,m} + E_{n+2,m+2} + E_{n+2,m-2} \right) \right] \]

\[ - i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) (E_{n+2,m+2} - E_{n+2,m-2}) + \frac{\partial v}{\partial y} \left( 2E_{n+2,m} - E_{n+2,m+2} - E_{n+2,m-2} \right) \]

(B-6)

For convenience, let

\[ A1 = \left( \frac{c_g}{c} \right) - \sigma_m \left( \frac{\partial c_g}{\partial \sigma} \right) + \frac{1}{2} \sigma_m^2 \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right), \quad A2 = \left( \frac{\partial c_g}{\partial \sigma} \right) - \sigma_m \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right), \quad A3 = \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right) \]

\[ B1 = -\frac{1}{h} \left( \frac{c g}{c} \right) - \sigma_m \left( \frac{\partial c g}{\partial \sigma} \right) + \frac{1}{2} \sigma_m^2 \left( \frac{\partial^2 c g}{\partial \sigma^2} \right) \]

\[ B2 = -\frac{1}{h} \left( \frac{\partial c g}{\partial \sigma} \right) - \sigma_m \left( \frac{\partial^2 c g}{\partial \sigma^2} \right) \]

\[ B3 = -\frac{1}{2h} \left( \frac{\partial^2 c g}{\partial \sigma^2} \right) \]
\[ C_1 = \left( \frac{c_g}{c} \right)_{\sigma_m} - \sigma_m \left( \frac{\partial c_g}{\partial \sigma \ c} \right)_{\sigma_m} + \frac{1}{2} \sigma_m^2 \left( \frac{\partial^2 c_g}{\partial \sigma^2 \ c} \right)_{\sigma_m}, \]
\[ C_2 = \left( \frac{\partial c_g}{\partial \sigma \ c} \right)_{\sigma_m} - \sigma_m \left( \frac{\partial^2 c_g}{\partial \sigma^2 \ c} \right)_{\sigma_m}, \quad C_3 = \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2 \ c} \right)_{\sigma_m}, \]
\[ X_{n,m} = \left[ i \frac{\partial h}{\partial x} \left( E_{n,m+1} - E_{n,m-1} \right) + \frac{\partial h}{\partial y} \left( E_{n,m+1} + E_{n,m-1} \right) \right] \]
\[ Y_{n,m} = \frac{1}{4} \left[ \frac{\partial u}{\partial x} \left( 2E_{n,m} + E_{n,m+2} + E_{n,m-2} \right) - i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \left( E_{n,m+2} - E_{n,m-2} \right) + \frac{\partial v}{\partial y} \left( 2E_{n,m} - E_{n,m+2} - E_{n,m-2} \right) \right] \]
\[ Z_{n,m} = -\frac{1}{4} \frac{\partial u}{\partial y} \left( 2E_{n,m} + E_{n,m+2} + E_{n,m-2} \right) \]
\[ \quad - i \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \left( E_{n,m+2} - E_{n,m-2} \right) + \frac{1}{4} \frac{\partial v}{\partial x} \left( 2E_{n,m} - E_{n,m+2} - E_{n,m-2} \right) \]
Then, we obtain much shorter expressions for the three integrals

\[ I_1 = \frac{A_1}{2} \left( E_{n,m+1} + E_{n,m-1} - i \left( E_{n,m+1} - E_{n,m-1} \right) \right) + \frac{A_2}{2} \left( E_{n+1,m+1} + E_{n+1,m-1} - i \left( E_{n+1,m+1} - E_{n+1,m-1} \right) \right) \]
\[ + \frac{A_3}{2} \left( E_{n+2,m+1} + E_{n+2,m-1} - i \left( E_{n+2,m+1} - E_{n+2,m-1} \right) \right) + \left( u, v \right) E_{n,m} \quad \text{(B-7)} \]
\[ I_2 = \frac{1}{2} \left( B_1 X_{n,m} + B_2 X_{n+1,m} + B_3 X_{n+2,m} \right) + Z_{n,m} \quad \text{(B-8)} \]
\[ I_3 = \frac{1}{h} \left( \mathbf{U} \cdot \nabla h \right) \left( C_1 E_{n,m} - 0.5 E_{n,m} + C_2 E_{n+1,m} + C_3 E_{n+2,m} \right) \]
\[ - C_1 Y_{n,m} - C_2 Y_{n+1,m} - C_3 Y_{n+2,m} \quad \text{(B-9)} \]
After lots of algebra, the governing equations for the reduced wave model are given by

\[ \frac{\partial}{\partial t} E_{0,0} + \nabla \cdot \left[ \left( \frac{c_g}{c} \right)_{\sigma_m} + \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2 \ c} \right)_{\sigma_m} \right] \left( \frac{\partial}{\partial \sigma} \right)_{\sigma_m} \left( E_{0,1} \right)_R \left( E_{0,1} \right)_I + E_{0,0} \left( u, v \right) \]
\[ - \left[ \left( \frac{c_g}{c} \right)_{\sigma_m} + \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2 \ c} \right)_{\sigma_m} \right] \left( \frac{1}{h} \left( \mathbf{U} \cdot \nabla h \right) E_{0,0} - Y_{0,0} \right) + \frac{E_{0,0}}{2h} \left( \mathbf{U} \cdot \nabla h \right) = 0 \quad \text{(B-10)} \]
\[ \frac{\partial}{\partial t} E_{1,0} + \nabla \cdot \left[ \left( \frac{c_g}{c} \right)_{\sigma_m} + \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_m} \right] \left( \frac{\partial}{\partial \sigma} \right)_{\sigma_m} + \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_m} \left( E_{1,0} \right)_R \left( E_{1,0} \right)_I + E_{1,0} \left( u, v \right) \]
\[ - 2 \left[ \left( \frac{c_g}{c} \right)_{\sigma_m} + \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_m} \right] S_{\sigma}^2 + \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_m} \left( \frac{1}{h} \left( \mathbf{U} \cdot \nabla h \right) E_{1,0} - \sigma_m Y_{0,0} \right) + \frac{E_{1,0}}{h} \left( \mathbf{U} \cdot \nabla h \right) = 0 \quad \text{(B-11)} \]
\[
\frac{\partial}{\partial t} E_{2,0} + \nabla \cdot \left[ \left( c_g \right)_{\sigma_n} S_2^2 + \frac{3}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_n} S_4^2 \right] \left( (E_{0,1})_R \cdot (E_{0,1})_I \right) + E_{2,0} (u, v) \right] \\
+ 2 \nabla \sigma_m \cdot \left\{ \left( \frac{\partial c_g}{\partial \sigma} \right)_{\sigma_n} S_2^2 \left( (E_{0,1})_R \cdot (E_{0,1})_I \right) \right\} + \frac{3}{2h} (U \cdot \nabla h) E_{2,0} \\
- \left[ 3 \left( \frac{c_g}{c} \right)_{\sigma_n} + 2 \sigma_m \left( \frac{\partial c_g}{\partial \sigma} c \right)_{\sigma_n} + \frac{9}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} c \right)_{\sigma_n} \right] \left[ \frac{1}{h} (U \cdot \nabla h) E_{2,0} - Y_{0,0} S_2^2 \right] = 0 \\
\frac{\partial}{\partial t} E_{0,m} + \nabla \cdot \left[ \left( c_g \right)_{\sigma_n} + \frac{1}{2} \left( \frac{\partial^2 c_g}{\partial \sigma^2} \right)_{\sigma_n} S_2^2 \right] \left[ \left( \frac{E_{0,m+1} + E_{0,m-1}}{2} \right) - i \left( \frac{E_{0,m+1} - E_{0,m-1}}{2} \right) \right] + (u, v) E_{0,m} \\
- \frac{1}{2h} \left( c c \right)_{\sigma_n} + \frac{1}{2} \left( \frac{\partial^2 c c}{\partial \sigma^2} \right)_{\sigma_n} S_2^2 \left[ \frac{1}{h} (U \cdot \nabla h) E_{0,m} - Y_{0,m} \right] = 0
\]

(B-12)

(B-13)

in which,

\[
Y_{0,0} = \left[ \frac{1}{2} \left( \frac{\partial u}{\partial x} (E_{0,0} + (E_{0,2})_R) + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) (E_{0,2})_I + \frac{\partial v}{\partial y} (E_{0,0} - (E_{0,2})_I) \right) \right] \\
X_{0,m} = \left[ \frac{i}{h} \left( E_{0,m+1} - E_{0,m-1} \right) + \frac{\partial h}{\partial y} (E_{0,m+1} + E_{0,m-1}) \right] \\
Y_{n,m} = \frac{1}{4} \left[ \frac{\partial u}{\partial x} (2E_{n,m} + E_{n,m+2} + E_{n,m-2}) - i \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) (E_{n,m+2} - E_{n,m-2}) + \frac{\partial v}{\partial y} (2E_{n,m} - E_{n,m+2} - E_{n,m-2}) \right] \\
Z_{0,m} = -\frac{1}{4} \frac{\partial u}{\partial y} (2E_{0,m} + E_{0,m+2} + E_{0,m-2}) - \frac{i}{4} \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) (E_{0,m+2} - E_{0,m-2}) + \frac{1}{4} \frac{\partial v}{\partial x} (2E_{0,m} - E_{0,m+2} - E_{0,m-2})
\]
APPENDIX C
DISCRETIZATION OF TERMS IN THE SHALLOW WATER EQUATIONS

Mass Flux

The midpoint rule of approximation is used to compute the mass fluxes. The method is demonstrated by applying to the east face of a typical 2-D control volume (see figure 3-1), and it is the same to apply to other faces. As stated previously, on the outer iteration level $m$, all nonlinear terms are approximated by a product of ‘old’ (from the preceding outer iteration) and a ‘new’ value. Thus the mass flux through each CV face is evaluated using the latest available velocity field:

$$m_e = \int_{S_e} h\vec{v} \cdot ndS \approx \left( h\vec{v}^{m-1} \cdot n \right)_e S_e$$  \hspace{1cm} (C-1)

$$n_e = \left[ \left( y_{n_e} - y_{se} \right)i - \left( x_{n_e} - x_{se} \right)j \right] / S_e$$  \hspace{1cm} (C-2)

Where $n_e$ is the outward unit normal vector at the face ‘e’ and $S_e$ is the surface area. Unless stated otherwise, all variables in this section are about the $m$th outer iteration.

Convective Flux

The convective flux of $u_i$-momentum equation through ‘e’ face of CV is:

$$F_{i,e} = \int_{S_e} hu_i \vec{v} \cdot ndS \approx \int_{S_e} hu_i \vec{v}^{m-1} \cdot ndS = m_i u_{i,e}$$  \hspace{1cm} (C-3)

The CV face value of $u_i$ used in this expression is not necessary to be the same as that used to calculate the mass flux. Linear interpolation is the easiest second order approximation, known as the central difference scheme (CDS).

Some iteration methods fail to converge when applied to the algebraic equation system derived from central difference approximations of convective fluxes since the matrices may not be diagonally dominant. For large Peclet number flow case, central difference approximation may lead to physically impossible solutions. An approach called deferred correction is very
useful to ensure the convergence of iterative solver when high order approximation is used. Deferred correction approach combines an explicit high order approximation (here is CDS) and an implicit lower order approximation (here is the Upwind Differencing Scheme, UDS) in the following way:

\[ F_{i,e}^c = m_e u_{i,e}^{UDS} + m_e \left( u_{i,e}^{CDS} - u_{i,e}^{UDS} \right)^{m-1} \]  

(C-4)

The term in parenthesis on the right hand side of the above equation is evaluated using values from previous iteration. As convergence approaching, the UDS contributions cancel out, the solution is still the second order approximation (CDS). So this procedure always makes iterative solver used more robust and efficient, meanwhile, keeps the high order accuracy.

**Diffusive Flux**

The lateral momentum mixing terms given by Equation (3-14) involve second derivatives. Applying the finite volume method to the integral form of the momentum equations, the lateral mixing terms become diffusive fluxes. For convenience, we use the general term ‘\( \nabla (hu_i) \)’ for demonstration, and it is straightforward to obtain the specific expression for each second derivatives from the general form. The midpoint rule approximation applied to the integrated diffusive flux gives:

\[ F_{i,e}^d = \int_{S_e} \nabla \left( hu_i \right) \cdot n dS \approx \left[ \nabla \left( hu_i \right) \cdot n \right]_{e} S_e \]

(C-5)

The spatial gradient of \( hu_i \) at the cell face center can be defined in terms of the derivatives with respect to either global Cartesian coordinates \((x, y)\) or local orthogonal coordinates\((n, t)\):

\[ \nabla \left( hu_i \right) = \frac{\partial hu_i}{\partial x} \mathbf{i} + \frac{\partial hu_i}{\partial y} \mathbf{j} = \frac{\partial hu_i}{\partial n} \mathbf{n} + \frac{\partial hu_i}{\partial t} \mathbf{t} \]

(C-6)
Where, \( n \) and \( t \) denote respectively the normal and tangential coordinate direction to the cell face.

A better way to approximate the diffusive flux is that the gradient evaluated with respect to the local orthogonal coordinates \((n,t)\), since only the derivative in the \( n \)-direction contributes to the diffusive flux:

\[
F_{i,e}^d = (\nu)_e \left( \frac{\partial hu_i}{\partial n} \right)_e S_e
\]

(C-7)

For the non-orthogonal grid, the line norm to the cell face ‘e’ may not pass through the CV center nodes ‘P’ and ‘E’ (see figure 3-3). As a result, the derivative of \( hu_i \) with respect to the direction normal to cell face at cell center ‘e’ can not be approximated using the values at nodes ‘P’ and ‘E’. Muzaferija (1994) suggested using the derivative with respect to the coordinate \( t \), along the line connecting nodes ‘P’ and ‘E’, as an implicit flux approximation, if the \( t \) direction is nearly orthogonal to the cell face, Equation (C-7) can be approximated as

\[
F_{i,e}^d \approx (\nu)_e \left( \frac{\partial hu_i}{\partial t} \right)_e S_e = (\nu)_e S_e \frac{(hu_i)_E - (hu_i)_P}{L_{P,E}}
\]

(C-8)

where \( L_{P,E} \) denotes the distance from ‘P’ to ‘E’.

Muzaferija (1994) suggested a deferred correction part to the diffusive flux, and then Equation (C-8) becomes

\[
F_{i,e}^d = (\nu)_e \left( \frac{\partial hu_i}{\partial t} \right)_e S_e + (\nu)_e S_e \left[ \left( \frac{\partial hu_i}{\partial n} \right)_e - \left( \frac{\partial hu_i}{\partial t} \right)_e \right]^{m-1}
\]

(C-9)

in which,

\[
\left( \frac{\partial hu_i}{\partial n} \right)_e^{m-1} = \text{grad}(hu_i)_e \cdot n
\]

(C-10)
The gradient at cell face center ‘e’ is calculated by interpolating the gradients at CV centers ‘P’ and ‘E’. It should also be noted that if \( n = t \), the last term in Equation (C-9) becomes zero.

When the non-orthogonality is not severe, the method discussed above is second order accurate. The approximations discussed above assumed that the line connecting nodes ‘P’ and ‘E’ passes through the cell face center ‘e’. In irregular grid case, however, the line connecting ‘P’ and ‘E’ may not pass through ‘e’, and the approximations for \((hu_i)_e\) and \(\left(\frac{\partial hu_i}{\partial t}\right)_e\) used above are second order accurate at a point ‘e’ rather than at the face center ‘e’. If ‘e’ is close to the corners ‘se’ or ‘ne’, the approximation reduces to first order accurate.

To give an effective cure, Ferziger and Peric (2002) suggest using values at auxiliary points ‘P”, ‘E” and ‘e’. ‘P” and ‘E” lie at the intersections of the cell face normal \( n \) and straight lines connecting cell face centers ‘n’ and ‘s’ of each CV. ‘e” lies at the intersection of line connecting ‘P” and ‘E” and line connecting CV corners, see figure 3-3. Then the modified diffusive flux is written as

\[
F_{i,e}^d = (v)_e S_e \left( \frac{\partial hu_i}{\partial t} \right)_e^c + (v)_e S_e \left[ \left( \frac{\partial hu_i}{\partial n} \right)_e - \left( \frac{\partial hu_i}{\partial t} \right)_e \right]^{m-1}
\]  

where,

\[
\left( \frac{\partial hu_i}{\partial t} \right)_e^c = \frac{(hu_i)_E - (hu_i)_P}{L_{P,E}} \]  

\[
\left( \frac{\partial hu_i}{\partial n} \right)_e = \frac{(hu_i)_E - (hu_i)_P}{L_{P,E}}
\]

\( L_{P,E} \) stands for the distance between ‘P” and ‘E”. The values of \((hu_i)_E\) and \((hu_i)_P\) can be calculated either by bilinear interpolation or by using the gradient at CV center:

\[
(hu_i)_P = (hu_i)_P + \text{grad} \left( hu_i \right)_P \cdot (r_p - r_P)
\]
\[(h_u)_E = (h_u)_E + \text{grad}(h_u)_E \cdot (r_E - r_E) \]  
\hspace{2cm} (C-15)

**Bottom Friction Term**

In this circulation model, the bottom friction term has the following general form

\[
(\tau_b)_x = \mu u, \quad (\tau_b)_y = \mu v
\]  
\hspace{2cm} (C-16)

Where, \( \mu \) is the bottom friction coefficient and the formulation is given by Equation (3-6). For the circulation model, the values of \( \mu \) at every grid points are known as a priori.

The bottom friction terms are linear expressions in terms of velocities, so they can be easily treated implicitly. In discretized momentum equations, implicit treatment of bottom friction term increases the diagonal dominance of coefficient matrix and thus usually has a positive effect on many iterative solution methods. The midpoint rule approximation applied to the volume integral of the bottom friction term gives:

\[
\int_{\Omega} \mu u_i d\Omega = \mu (u_i)_p \Delta \Omega
\]  
\hspace{2cm} (C-17)

where \( \Delta \Omega \) is the volume of the interested CV.

**Wave Forcing Term**

As for the circulation model, the wave forcing is an input quantity as a surface force term in the momentum equations. This term only contributes to the source terms of the discretized momentum equations. The second order midpoint rule approximation is applied:

In the \( x \)-momentum equation:

\[
\tau'_x = \frac{1}{\rho} \int_S \left( S_{x,i} + S_{x,j} \right) \cdot ndS
\]  
\hspace{2cm} (C-18)

In the \( y \)-momentum equation:

\[
\tau'_y = \frac{1}{\rho} \int_S \left( S_{y,i} + S_{y,j} \right) \cdot ndS
\]  
\hspace{2cm} (C-19)
Pressure Term

The pressure term in momentum equations can be treated either as surface forces (conservative approach) or as body forces (non-conservative approach).

In the first case (as surface forces):

\[
SP_p = \int_s - \left( \frac{h}{\rho} P^{m-1} \right)_i \cdot ndS \approx \sum_l \left( \frac{h}{\rho} P^{m-1} \right)_i \cdot (ndS)_i \quad , \quad l = e, w, n, s \quad (C-20)
\]

In the second case (as body forces):

\[
SP_p = \int_{d\Omega} - \frac{h}{\rho} \frac{\partial P^{m-1}}{\partial x_i} d\Omega \approx - \left( \frac{h}{\rho} \frac{\partial P^{m-1}}{\partial x_i} \right)_p \Delta\Omega \quad (C-21)
\]

Equation (C-21) is essentially equivalent to (C-20) if the derivative \( \left( \frac{h}{\rho} \frac{\partial P^{m-1}}{\partial x_i} \right)_p \) is calculated using Gauss’ theorem.
APPENDIX D
TWO NUMERICAL ISSUES FOR THE CIRCULATION MODEL: UNDER-RELAXATION
TECHNIQUE AND MOMENTUM INTERPOLATION METHOD

Under-Relaxation Technique

For the algebraic equation for the variable $\phi$ on outer iteration level $m$

$$A_p\phi_p^m + \sum_i A_i\phi_i^m = Q_p \quad (D-1)$$

in practice, only a fraction of the would-be difference is allowed to be changed for stabilization consideration:

$$\phi^m = \phi^{m-1} + \alpha_\phi (\phi^{new} - \phi^{m-1}), \quad 0 < \alpha_\phi < 1 \quad (D-2)$$

where $\alpha_\phi$ is called the ‘under-relaxation factor’. $\phi^{new}$ is the solution of Equation (D-1) and can be written as

$$\phi_p^{new} = \frac{Q_p - \sum_i A_i\phi_i^m}{A_p} \quad (D-3)$$

Substituting Equation (D-3) into (D-2), it gives:

$$\frac{A_p}{\alpha_\phi} \phi_p^m + \sum_i A_i\phi_i^m = Q_p + \frac{1 - \alpha_\phi}{\alpha_\phi} A_p\phi_p^{m-1} \quad (D-4)$$

where $A_p^*$ and $Q_p^*$ are modified main diagonal elements and source terms respectively. When convergence approaches, the terms involving $\alpha_\phi$ will cancel out and the solution of the original problem will be obtained. Compared to the explicit application of the under-relaxation technique according to Equation (D-2), this treatment has been shown to have positive effects on many iterative solution methods since the diagonal dominance of the coefficient matrix is increased (Patankar, 1980).
The optimum under-relaxation factors, which require least iterations to converge, may vary from case to case. A good strategy is to use a small value at the beginning of iterations and increase it towards unity as convergence is approached. For solving the incompressible Navier-Stokes equations based on the pressure-correction method, the two momentum equations usually use one under-relaxation factor \( \alpha_u \) and the pressure correction equation should be assigned its own factor \( \alpha_p \). Pantanker (1980) suggested using \( \alpha_u = 0.5 \) and \( \alpha_p = 0.8 \) in the SIMPLE algorithm. Ferziger (2002) proposed that \( \alpha_p = 1.1 - \alpha_u \) gives the best results for the lid-driven cavity flow problem.

**Momentum Interpolation Method**

The collocated grids have advantages over staggered grids in respect of reducing required storage memory and computational costs, and in respect of dealing with complex geometries. Nonetheless, the main drawback of the collocated grids is the occurrence of nonphysical oscillations of pressure and/or velocities, causing severe numerical instability in some cases. The spurious oscillations are mainly due to the decoupling between the pressure and the velocities. To address the problem, Rhie and Chow (1983) first proposed to employ a special interpolation practice, the momentum interpolation method (MIM), instead linear interpolation for calculating the velocities at the faces of CVs. The success of this method reported by Rhie and Chow (1983) has stimulated more research efforts on this subject. Following the same basic idea, Peric (1985) and Majumdar (1986) have worked on subsequent refinements. Later, Majumdar (1988) observed that the converged results are under-relaxation-dependent during trial runs. Such dependence on under-relaxation factor is certainly an undesirable feature of the MIM. To remove the dependency, Majumdar (1988) analyzed the reason for the dependency and proposed a modified momentum interpolation method illustrated for one-dimensional flow. In the following,
we will give a brief review of the momentum interpolation method and then present the modified version for 2-D steady flow based on the same principle of method proposed by Majumdar (1988). For simplicity, they will be demonstrated on Cartesian grids.

Taking out the pressure term from the source term, the algebraic equations for discretized $u$ momentum equation can be written as

$$A_p u_p + \sum_i A_u u_i = Su + Su_p$$

where $Su_p$ is the source term due to pressure gradient. And,

$$Su_p = -\frac{h_p}{\rho} \left( p_e S_e - p_w S_w \right)$$

From equations (D-5)-(D-6), we can get

$$u_p = \frac{\left( Su - \sum_i A_u u_i \right)_p}{\left( A_p \right)_p} - \frac{h_p}{\rho} \frac{\left( p_e S_e - p_w S_w \right)_p}{\left( A_p \right)_p}$$

The right hand side of Equation (D-7) consists of two parts, the influences of velocities at the surrounding CV centers denoted by the first term, and the influence of the pressure gradient represented by the second term. Following the expression, the $u$ at the CV center ‘$E$’ can be written as

$$u_E = \frac{\left( Su - \sum_i A_u u_i \right)_E}{\left( A_p \right)_E} - \frac{h_E}{\rho} \frac{\left( p_e S_e - p_w S_w \right)_E}{\left( A_p \right)_E}$$

Mimicking Equation (D-7) and (D-8), we can express the interface velocity at cell face ‘e’ as

$$u_e = \frac{\left( Su - \sum_i A_u u_i \right)_e}{\left( A_p \right)_e} - \frac{h_e S_e}{\rho} \frac{\left( p_e - p_r \right)}{\left( A_p \right)_e}$$

This equation gives the original momentum interpolation method (OMIM).
When the under-relaxation is incorporated for stability purpose, Equation (D-9) becomes to:

$$u_e = \frac{\alpha_u \left( Su' - \sum_i A_i u_i \right)}{(A_p)_e} - \frac{\alpha_u h S_e \left( p_e - p_p \right)}{\rho (A_p)_e} + (1 - \alpha_u) \left[ f_e^+ u_E^0 + (1 - f_e^+) u_p^0 \right]$$  \hspace{1cm} (D-10)

where $u_p^0$ and $u_E^0$ are values at the previous iteration level. $f_e^+ = L_{pe} / L_{PE}$ is the linear interpolation factor. $Su'$ is the modified $Su$ (see Equation (D-6)). Equation (D-10) is still the formulation of the OMIM, and the only difference from Equation (D-9) is the incorporation of under-relaxation. Majumdar (1988) suggested that a slight change of Equation (D-10) could eliminate the under-relaxation dependent

$$\frac{\alpha_u \left( Su' - \sum_i A_i u_i \right)}{A_p} = u_p - \frac{\alpha_u Su_p}{A_p} - (1 - \alpha_u) u_p^0$$  \hspace{1cm} (D-11)

This expression for cell face velocities can achieve a unique solution that is under-relaxation factor independent. For a better understanding of Equation (D-11), we will take a close look at it.

It is easy to show that:

$$\frac{\alpha_u \left( Su' - \sum_i A_i u_i \right)}{A_p} = f_e^+ u_E + (1 - f_e^+) u_p + (1 - \alpha_u) \left[ f_e^+ u_E^0 + (1 - f_e^+) u_p^0 \right]$$  \hspace{1cm} (D-12)

Using Equation (D-12), we can get

$$\alpha_u \left( Su' - \sum_i A_i u_i \right) = f_e^+ u_E + (1 - f_e^+) u_p + (1 - \alpha_u) \left[ f_e^+ u_E^0 + (1 - f_e^+) u_p^0 \right]$$  \hspace{1cm} (D-13)

Substituting Equation (D-13) into (D-11), we finally arrive at:
The expression of cell face velocities by Equation (D-14) is composed of two parts. The part from linear interpolation is given by the first term on the right side of the equation. The part from momentum interpolation is denoted by all terms in the curly braces. The last term in the braces can be called ‘MMIM correction term’ since it is introduced by using the modified momentum method. Without this term Equation (D-14) becomes to be equivalent to the MIM. Obviously, the part from momentum interpolation can be regarded as a correction to the linear interpolation. The first three terms in the braces have the function of smoothing the pressure field, and then the unrealistic pressure oscillation could be removed. As iterations approach to convergence, $u_p = u_p^0$, $u_E = u_E^0$ and $u_e = u_e^0$. Pulling out the MMIM correction term and reorganizing, Equation (D-14) can be rewritten as a formulation that every term is containing the multiplier $\alpha_u$. Divided by $\alpha_u$, it yields an expression of $u_e$ that is not a function of $\alpha_u$.

Therefore, the dependency of under-relaxation factor is certainly eliminated. Verification by numerical tests is shown in Section 3.5.2.
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