DEVELOPMENTS IN THE PERTURBATION THEORY OF ALGEBRAICALLY SPECIAL SPACETIMES

By

LARRY R. PRICE

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Larry R. Price

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The detection of gravitational waves is the most exciting prospect for experimental relativity today. With ground based interferometers such as LIGO, VIRGO and GEO online and the space based LISA project in preparation, the experimental apparatus necessary for such work is steadily taking shape. Yet, however capable these experiments are of taking data, the actual detection of gravitational waves relies in a significant way on making sense of the collected signals. Some of the data analysis techniques already in place use knowledge of expected waveforms to aid the search. This is manifested in template based data analysis techniques. For these techniques to be successful, potential sources of gravitational radiation must be identified and the corresponding waveforms for those sources must be computed. It is in this context that black hole perturbation theory has its most immediate consequences.

This dissertation presents a new framework for black hole perturbation theory based on the spin coefficient formalism of Geroch, Held and Penrose. The two main components of this framework are a new form for the perturbed Einstein equations and a Maple package, GHPtools, for performing the necessary symbolic computation. This framework provides a powerful tool for performing analyses generally applicable to the entire class of Petrov type D solutions, which include the Kerr and Schwarzschild spacetimes.

Several examples of the power and flexibility of the framework are explored. They include a proof of the existence of the radiation gauges of Chrzanowski in Petrov type

II spaces as well as a derivation of the Teukolsky-Starobinsky relations that makes no reference to separation of variables. Furthermore, a method of determining the non-radiated multipoles in type D spaces is detailed.

CHAPTER 1
INTRODUCTION

Einstein's theory of general relativity, introduced in 1915, to this day remains as one of the final frontiers of fundamental physics. Since its inception progress in the field has been largely theoretical because of the tremendous difficulty inherent in making gravitational measurements. In particular, one of the most exciting and fundamental predictions of general relativity—the existence of gravitational waves—has remained elusive. Not for long. With ground based interferometers such as LIGO, VIRGO and GEO online and the space based LISA mission in preparation, the detection of gravitational waves is all but imminent. These experiments bring with them the task of analyzing the data they collect. For some of the promising sources of gravitational waves, the collision of two black holes, the method of choice for data analysis, known as matched filtering, requires knowledge of the expected waveforms. In the past two years the field of numerical relativity has undergone a revolution and promises to provide the most accurate waveforms for situations involving the collision of two black holes of comparable masses—situations that require the use of full nonlinear general relativity. There is however, one promising source of gravitational waves that is currently out of reach for numerical relativity—the situation where the larger black hole is roughly a million times more massive than the smaller one, known as an extreme mass ratio inspiral, or EMRI. This problem lies squarely in the realm of perturbation theory, the subject of the present work.

In particular, the "solution" of the EMRI problem requires moving beyond the test mass approximation of general relativity to describe the motion of the small black hole (treated as a particle in the spacetime of the larger black hole because of the huge mass difference)—one must account for the first order corrections to the motion of the small black hole, due to self-force. The appropriate equations of motion have been determined in general by Mino, Sasaki and Tanaka [1] and Quinn and Wald [2] and are referred to as the

MiSaTaQuWa equations. In practice, the more widely used prescription for computing the self force is due to Detweiler and Whiting [3]. In either case, the fundamental object of interest is the metric perturbation, h_{ab}, introduced by the particle on the large black hole's spacetime. Therefore the EMRI problem also requires us to compute the metric perturbation, before we can compute the self-force on the particle. This is the piece of the problem to which the present work aims to contribute. Determining the metric perturbation is a task that depends quite sensitively on the spacetime being perturbed. For spherically symmetric backgrounds, this problem is well understood and most of the remaining problems are computational in nature. However, for the more interesting and astrophysically relevant situation where the larger black hole is rotating, our understanding is not quite complete. It is on this more general situation that we focus. Before we continue, we note that all of the astrophysically interesting spacetimes, including the Kerr and Schwarzschild metrics, possess curvature tensors with the same basic algebraic structure. We will elaborate on this more fully in the next chapter, but for now we merely point out that these spacetimes belong to the larger class of *algebraically special* spacetimes.

The remainder of this chapter is devoted to providing a review of the literature [4]. Every attempt has been made to phrase the current discussion in generally accessible language. Many of these results will be explored in further detail in later chapters, after the appropriate formalism has been developed.

1.1 Perturbations of Spherically Symmetric Spacetimes

Historically, the subject of black hole perturbation theory got its start with the pioneering work of Regge and Wheeler [5] (henceforth RW), who provided an analysis of first order perturbations of the Schwarzschild solution (which was later completed by Zerilli [6, 7]). The fact that the background is spherically symmetric is crucial to their analysis. The basics will be presented here. A more complete discussion, in a very different language, is provided in Chapter 3.

Let us begin by considering a small perturbation, h_{ab}, of the Schwarzschild geometry. Thus our spacetime metric is

$$g_{ab} = g^S_{ab} + h_{ab}, \tag{1–1}$$

where

$$g^S_{ab}dx^a dx^b = \Big(1 - \frac{2M}{r}\Big)dt^2 - \Big(1 - \frac{2M}{r}\Big)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1–2}$$

is the Schwarzschild metric in Schwarzschild coordinates. Putting Equation 1–1 into the Einstein equations and keeping only terms linear in h_{ab} leads us to the perturbed Einstein equations:

$$\mathcal{E}_{ab} = -\frac{1}{2}\nabla^c\nabla_c h_{ab} - \frac{1}{2}\nabla_a\nabla_b h^c{}_c + \nabla^c\nabla_{(a}h_{b)c} + \frac{1}{2}g_{ab}(\nabla^c\nabla_c h^d{}_d - \nabla^c\nabla^d h_{cd}) = 0, \tag{1–3}$$

where ∇_a is the derivative operator compatible with the background geometry 1–2 and the indices are raised and lowered with the background metric. Henceforth we will refer to $\mathcal{E}_{ab}$ as the Einstein tensor, and the expression to the right of it as the Einstein equations (dropping the qualifier "perturbed" for brevity).

Essentially every perturbative analysis of the Schwarzschild spacetime makes extensive use of its spherical symmetry. The first step in this direction is to decompose the components of the metric perturbation into scalar, vector and tensor harmonics. Heuristically, we write

$$h_{ab} = \begin{pmatrix} s_1 & s_2 & v_1 & v_1 \\ s_2 & s_3 & v_2 & v_2 \\ v_1 & v_2 & t + s_4 & t \\ v_1 & v_2 & t & t - s_4, \end{pmatrix} \tag{1–4}$$

where s, v and t stand for scalar, vector and tensor, respectively and the subscripts distinguish between the various scalars and vectors.

Consider the metric of the two-sphere:

$$\gamma_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2. \tag{1–5}$$

Since the usual scalar harmonics, $Y_{\ell m}$, define a complete set of functions on the two-sphere, we can use them to construct two types of vectors. The first is the so-called even parity vector defined (up to a constant) by[1]

$$\nabla_A Y_{\ell m}, \tag{1–6}$$

where ∇_A is the derivative compatible with γ_{AB} (Equation 1–5). The other vector is the odd-parity (pseudo-) vector

$$\gamma_{BC}\epsilon^{AB}\nabla_A Y_{\ell m}, \tag{1–7}$$

where ϵ^{AB} is just the standard Levi-Civita symbol. To define tensor harmonics, we essentially just take one more derivative of Equations 1–6 and 1–7. The even parity tensors are given by

$$\nabla_A \nabla_B Y_{\ell m}, \quad \text{and} \quad \gamma_{AB} Y_{\ell m}, \tag{1–8}$$

and the odd-parity (pseudo-) tensor by

$$\gamma_{AC}\epsilon^{CD}\nabla_D \nabla_B Y_{\ell m}. \tag{1–9}$$

Even parity objects pick up minus signs under a parity transformation ($\theta \to \pi - \theta, \phi \to \pi + \phi$) according to $(-1)^\ell$, and odd parity objects pick up minus signs according to $(-1)^{\ell+1}$. For this reason the even parity parts are sometimes referred to as "electric" and the odd parity parts "magnetic" in the older literature. Because parity is an inherent symmetry of spherically symmetric backgrounds, it provides a natural way of decoupling the two degrees of freedom of the gravitational field. Note, however, that parity is not a good symmetry in even slightly less symmetric spacetimes (e.g. Kerr). We will return to

[1] The tensor harmonics defined in this chapter are not those generally used, but have been chosen for their heuristic value. See Thorne's review [8] for the standard tensor harmonics and their relation to various other representations of the sphere, or Appendix D for the spin-weighted spherical harmonics which provide another alternative for the angular decomposition.

this subject in Chapter 3. Continuing in our cartoon language (Equation 1–4), we now consider the two sectors of the metric perturbation independently, writing

$$h_{ab}^{\mathrm{odd}} = \begin{pmatrix} 0 & 0 & v_1^{\mathrm{odd}} & v_1^{\mathrm{odd}} \\ 0 & 0 & v_2^{\mathrm{odd}} & v_2^{\mathrm{odd}} \\ v_1^{\mathrm{odd}} & v_2^{\mathrm{odd}} & t^{\mathrm{odd}} & t^{\mathrm{odd}} \\ v_1^{\mathrm{odd}} & v_2^{\mathrm{odd}} & t^{\mathrm{odd}} & t^{\mathrm{odd}} \end{pmatrix} \tag{1–10}$$

and

$$h_{ab}^{\mathrm{even}} = \begin{pmatrix} s_1 & s_2 & v_1^{\mathrm{even}} & v_1^{\mathrm{even}} \\ s_2 & s_3 & v_2^{\mathrm{even}} & v_2^{\mathrm{even}} \\ v_1^{\mathrm{even}} & v_2^{\mathrm{even}} & t^{\mathrm{even}} + s_4 & t^{\mathrm{even}} \\ v_1^{\mathrm{even}} & v_2^{\mathrm{even}} & t^{\mathrm{even}} & t^{\mathrm{even}} - s_4. \end{pmatrix} \tag{1–11}$$

The final step before appealing to the Einstein equations consists of choosing a gauge. Equation 1–3 is invariant under the transformation

$$h_{ab} \to h_{ab} + \pounds_\xi g_{ab} = h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a, \tag{1–12}$$

where ξ_a is an arbitrary vector and $\pounds_\xi$ is the Lie derivative. Taking the odd-parity sector as an example, the Regge-Wheeler gauge vector takes the form

$$\xi^a = (0, 0, \Lambda \epsilon^{AB} \nabla_B Y_{lm}), \tag{1–13}$$

where Λ is a function chosen so that the odd parity part of the metric perturbation 1–10 takes the form

$$h_{ab}^{\mathrm{odd}} = \begin{pmatrix} 0 & 0 & 0 & v_1^{\mathrm{odd}} \\ 0 & 0 & 0 & v_2^{\mathrm{odd}} \\ 0 & 0 & 0 & 0 \\ v_1^{\mathrm{odd}} & v_2^{\mathrm{odd}} & 0 & 0. \end{pmatrix} \tag{1–14}$$

Similar simplifications arise in the even-parity sector.

Returning to the Einstein equations with this simplified description of the metric perturbation leads, after some manipulation, to the Regge-Wheeler-Zerilli equations, which can be compactly described in a single expression, namely

$$-\frac{\partial^2 \phi^{\rm o,e}_{\ell m}}{\partial t^2}+\frac{\partial^2 \phi^{\rm o,e}_{\ell m}}{\partial r^{*2}}-V^{\rm o,e}_{\ell}(r)\phi^{\rm o,e}_{\ell m}=0, \tag{1–15}$$

where the letters 'o' and 'e' stand for odd and even, respectively, $r^*=r+\ln(\frac{r}{2M}-1)$ just pushes the horizon out to infinity and $\phi_{\ell m}$ is the appropriate master variable. Two aspects of this result are noteworthy: (1) the two degrees of freedom have completely decoupled and (2) these equations are separable in the Schwarzschild background. These two features are desirable for any perturbative description of any background spacetime.

1.2 Perturbations of Kerr Black Hole Spacetimes

Unfortunately, the techniques used by RW to obtain a perturbative description of the Schwarzschild spacetime are of little use when the background geometry possesses only axial symmetry. Such is the case for the Kerr geometry, which describes a rotating black hole. In Boyer-Lindquist coordinates, its metric takes the form

$$\begin{aligned} ds^2 &= \Big(1-\frac{2Mr}{\tilde\rho^2}\Big)dt^2+\frac{4Mar\sin^2\theta}{\tilde\rho^2}dtd\phi \\ &\quad -\frac{\tilde\rho^2}{\Delta}dr^2-\tilde\rho^2 d\theta^2-\sin^2\theta\Big(r^2+a^2+\frac{2Ma^2r\sin^2\theta}{\tilde\rho^2}\Big)d\phi^2, \end{aligned} \tag{1–16}$$

where $\tilde\rho^2=r^2+a^2\cos^2\theta$, $\Delta=r^2-2Mr+a^2$, M is the mass and $a=J/M$ is the angular momentum per mass of the black hole. The spin coefficient formalism of Geroch, Held and Penrose [9] developed in the next chapter has proved to be fundamental in virtually every perturbative description of the Kerr spacetime.

The first successful perturbation analysis of the Kerr geometry was performed by Teukolsky in a series of papers beginning in 1973 [10–12]. Teukolsky took as his starting point the perturbed Bianchi identities in a spin coefficient formalism. Each quantity is perturbed away from its background value and only first order terms are kept. Equivalently, though with considerably more effort, Teukolsky's result can also be seen as arising from a wave equation for the perturbed Riemann tensor, using standard methods [13]. In either case, the result, written here in Boyer-Lindquist coordinates, is Teukolsky's *master equation* (written here in accord with [14])

$$\begin{aligned}\Bigg\{\frac{\partial}{\partial r}\Delta\frac{\partial}{\partial r} - \frac{1}{\Delta}\left\{(r^2+a^2)\frac{\partial}{\partial t} + a\frac{\partial}{\partial \phi} - s(r-M)\right\}^2 - 4s(r+i\,a\cos\theta)\frac{\partial}{\partial t} \\ + \frac{\partial}{\partial\cos\theta}\sin^2\theta\frac{\partial}{\partial\cos\theta} + \frac{1}{\sin^2\theta}\left\{a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\phi} + i\,s\cos\theta\right\}^2\Bigg\} \\ \times\Delta^{s/2}\psi_s = 4\pi\Delta^{s/2}\Sigma T_s,\end{aligned} \tag{1–17}$$

where $s = \pm 2$ correspond to the Weyl scalars ψ_0 and $\psi_2^{-4/3}\psi_4$, respectively. The Weyl scalars are perturbations of the extremal spin components of the curvature tensor. The significance of the Weyl scalar ψ_4 is that far away from the source of gravitational radiation

$$\psi_4 \sim \ddot{h}_+ - i\ddot{h}_\times, \tag{1–18}$$

where h_+ and $h_\times$ are the two polarizations of outgoing gravitational radiation in the transverse traceless gauge. Similar results hold for ψ_0 and incoming radiation. For other values of s, solutions correspond to fields of other spin: $s = 0$ is the massless scalar wave equation, $s = \pm 1/2$ the Weyl neutrino, $s = \pm 1$ the Maxwell field, $s = \pm 3/2$ the Rarita-Schwinger field, and so on. Note that angular separation necessarily involves time separation for $a \neq 0$.

Separated solutions to Equation 1–17 are of the form $\psi_s = e^{-i\omega t}e^{im\phi}{}_sR(r)_sS(a\omega,\theta)$ (omitting the ℓ, m and ω subscripts). The angular functions, ${}_sS(a\omega,\theta)$, are generally referred to as "spin weighted spheroidal harmonics". In the limit that $a\omega = 0$, ${}_sS_{\ell m}(\theta)$ reduce to the standard spin weighted spherical harmonics (cf. Appendix D), which are interrelated by the spin raising and lowering operators, ð and ð′ [15], developed in the following chapter. For $a\omega \neq 0$, solutions correspond to functions of different spin weight, but the ${}_sS(a\omega,\theta)$ no longer share common eigenvalues. Thus a metric reconstruction based on spin weight ± 2 functions would be incompatible with one based on spin weight 0 functions. This incompatibility does not arise for Schwarzschild, where reconstruction from solutions of the RW equation can translate into comparable metric reconstruction from the Weyl scalars, since there is a unique way of representing tensors on the sphere.

The spin weighted spherical (and spheroidal) harmonics fail to be defined for $\ell < |s|$ and thus the Teukolsky equation can give us no information about the $\ell = 0, 1$ modes. This is not a surprise since ψ_0 and ψ_4 are components of the curvature tensor, which carries information about the quadrupole (and higher multipole) generated gravitational waves. In fact, Wald has shown [16] that for vacuum perturbations each of ψ_0 and ψ_4 is sufficient to characterize the perturbation of the spacetime, up to shifts in mass and angular momentum. In Schwarzschild, these lower multipole moments can be expressed appropriately in terms of spherical harmonics using the RW formalism, but any comparable expressions for the Kerr case would be incompatible with metric coefficients constructed from spin weight ± 2 functions (i.e., they would be expressed in different bases). Yet, these low-ℓ multipole moments are urgently sought, since they convey information about the energy and both the axial and non-axial components of the angular momentum of a particle in orbit around the black hole. Moreover, in recent calculations demonstrating the precise relation of the $\ell = 0, 1$ multipoles in Schwarzschild to shifts in the mass and angular momentum, Detweiler and Poisson [17] emphatically point out that such shifts are just as important as the radiating multipoles for describing the motion of a small black hole orbiting a supermassive black hole. The non-radiated multipole moments are the subject of Chapter 6.

Solutions of the Teukolsky equation lead quite naturally to metric perturbations through the use of Hertz potentials which solve Equation 1–17. We now turn our attention to this subject.

1.3 Metric Perturbations of Black Hole Spacetimes

The first explicit solutions for metric perturbations given in terms of Hertz potentials were written down by Chrzanowski [18] and Cohen and Kegeles [19]. This work was carried out at a time when there was not a strong urge to obtain solutions related to very specific sources, and so it gave a successful way of creating metric perturbations in vacuum. Recent interest in EMRIs as a source for gravitational waves has developed a need for metric perturbations related to known sources, for which curvature perturbations may be obtained by solving Teukolsky's equation. For this class of problem, the source is highly localized, and most of the perturbed spacetime can still be treated as vacuum. We give first a description of solutions to the inversion problem in vacuum, paying special attention to limitations of each approach. Before we proceed, it will be helpful to give a brief overview of Hertz potentials in the more immediately familiar context of Maxwell's equations in flat space. Note that the methods presented here rely crucially on spin coefficient methods, though we have attempted to keep reference to such methods minimal for now.

1.3.1 Hertz Potentials in Flat space

To illustrate the essentials of Hertz potential methods we consider the source-free Maxwell equations in flat spacetime, in essentially the form Cohen and Kegeles attempted to generalize to curved spacetime [20]:

$$\nabla_a F^{ab} = 0 \qquad \text{and} \qquad \epsilon^{abcd}\nabla_a F_{cd} = 0. \tag{1–19}$$

As usual a vector potential, A_a, is introduced and the Lorentz gauge, $\nabla_a A^a = 0$, is imposed so that the Maxwell equations lead directly to $\Box A_a = 0$.

Then a Hertz potential H^{ab} is introduced via $A^a = \nabla_b H^{ab}$, where $H^{ab} = -H^{ba}$, so that the Maxwell field, F_{ab}, is obtainable by two derivatives of H_{ab}. However, H^{ab} is only defined up to a transformation of the type

$$H^{ab} \to H^{ab} + \nabla_c M^{cab} + \nabla^a C^b - \nabla^b C^a, \tag{1–20}$$

where M^{cab} is completely antisymmetric and $\Box C^a = 0$. It is easy to see that in flat spacetime, where derivatives commute, the transformation Equation 1–20 only changes

A^a by the addition of the term $\nabla^a \nabla_b C^b$ and therefore, in the Lorentz gauge, contributes nothing to the fields. In practice, Equation 1–20 is used to reduce the Hertz bivector potential to a single complex (or two real) scalar potential(s). Herein lies the power of the method. However, moving to curved-space naturally complicates things. While the wave equations are modified to include curvature pieces, the transformation in Equation 1–20 is retained (see Cohen and Kegeles [20] and Stewart [21]). As a result, the field equations are still satisfied and the six components of H^{ab} are still reduced to two, but the transformation in Equation 1–20 explicitly breaks the Lorentz gauge because derivatives no longer commute. In this way a new gauge is introduced that brings with it complications for the inclusion of sources. The necessary and sufficient conditions for the existence of this gauge are the subject of Chapter 4.

1.3.2 The Inversion Problem for Gravity

The formulation of the gravitational Hertz potential proceeds analogously to that of its (flat space) electromagnetic counterpart, with a few differences. For one, the result is a metric perturbation in one of two complimentary gauges. Additionally, the potential itself is a solution to the Teukolsky equation for $s = +2$ (or $s = -2$; the choice of the sign of s determines which gauge the metric perturbation is in), though it is not the curvature perturbation of the metric perturbation it generates. In analogy to the electromagnetic example above, the components of the metric perturbation are given by two derivatives of the potential. The natural language in which to express the metric perturbation arising from the Hertz potential is again the spin coefficient formalism of Newman and Penrose [22], or its modification due to Geroch, Held and Penrose [9]. Thus we postpone the formal development of the subject until Chapter 3, when the necessary formalism is in place, and instead offer an overview of the general process and documented research on the topic of reconstructing the metric perturbation from solutions to the Teukolsky equation (assuming the form of metric perturbation is prescribed), which we will refer to as the inversion problem.

The problem is that of finding a Hertz potential, given a solution (or both solutions) to the Teukolsky equation. To make this more precise, we look to the expression of the curvature perturbations, ψ_0 ($s = +2$) and ψ_4 ($s = -2$), in terms of the Hertz potentials. If we take the potential to satisfy the $s = -2$ Teukolsky equation then the perturbation exists in the ingoing radiation gauge (IRG) and we have that

$$2\psi_0 = DDDD\left[\bar{\Psi}_{\rm IRG}\right], \qquad \text{and} \tag{1–21}$$

$$2\rho^{-4}\psi_4 = \frac{1}{4}\left[\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\mathcal{L}}\bar{\Psi}_{\rm IRG} - 12\rho^{-3}\psi_2\partial_t\Psi_{\rm IRG}\right], \tag{1–22}$$

where $\bar{\mathcal{L}} = -[\partial_\theta + s\cot\theta - i\csc\theta\partial_\varphi] + ia\sin\theta\partial_t$ and $D = \Delta^{-1}[(r^2+a^2)\partial_t + \Delta\partial_r + a\partial_\phi]$ define derivatives in (orthogonal) null directions, $\rho = -(r - ia\cos\theta)^{-1}$ and $\Psi_{\rm IRG}$ is the potential. While for a potential satisfying the $s = +2$ Teukolsky equation, we have a perturbation in the outgoing radiation gauge (ORG), where

$$2\rho^{-4}\psi_4 = \Delta^2\widehat{\Delta}\widehat{\Delta}\widehat{\Delta}\widehat{\Delta}\left[\Delta^2\bar{\Psi}_{\rm ORG}\right], \qquad \text{and} \tag{1–23}$$

$$2\psi_0 = \frac{1}{4}\left[\mathcal{L}\mathcal{L}\mathcal{L}\mathcal{L}\bar{\Psi}_{\rm ORG} + 12\rho^{-3}\psi_2\partial_t\Psi_{\rm ORG}\right], \tag{1–24}$$

where $\widehat{\Delta} = \frac{1}{2}\rho\bar{\rho}[(r^2+a^2)\partial_t - \Delta\partial_r + a\partial_\phi]$ and $\mathcal{L}$, the complex conjugate of the operator defined above, are also derivatives in null directions (mutually orthogonal to each other and those defined by the operators in the IRG). These are the equations we would like to invert for the potentials $\Psi_{\rm IRG}$ and $\Psi_{\rm ORG}$. Once this is done, the potential may then be used to construct the metric perturbation. We now look at several different approaches to this problem.

1.3.2.1 Ori's construction for Kerr

In principle, with solutions for the Weyl curvature perturbations on all of spacetime, one could integrate along null directions to undo the derivatives in Equations 1–21 or 1–22 (or their ORG counterparts). Ori [23] has recently performed this task—integrating Equation 1–21 in order to find the potential $\Psi_{\rm IRG}$ in terms of ψ_0.

As we noted previously, angular separation is dependent on separation in time in the Kerr spacetime. Ori's analysis therefore takes place in the frequency domain. Ori's construction is effective in the vacuum situation, for which Ψ satisfies Teukolsky's Equation 1–17 with $s = -2$, so it does provide a complete solution in the frequency domain.

For incorporating sources, Ori continues to take Equation 1–21 as correct, where now ψ_0 is a source-dependent, non-vacuum solution. Equation 1–21 allows the freedom to add to $\bar{\Psi}_{\mathrm{IRG}}$ any function that is killed by the four derivatives there. Ori utilizes this freedom to choose functions that reproduce the discontinuity at the source and, by extension, ψ_0. However, Equation 1–17 no longer applies for Ψ, nor does Equation 1–22 for ψ_4 in the form it has here.[2] Furthermore, any metric reconstructed in the radiation gauges is ill-defined at the location of the source, a fact that is proven in Chapter 4. Moreover, Ori also finds a problem in the "shadow regions", which occur wherever null rays (say, incoming from infinity) have been blocked by the source. Apparently, the shadow region has to be thought of as being identified for each mode independently. For point sources, discontinuities are thought to develop across the shadow regions, although they have not been observed in simple flat spacetime model calculations. Nevertheless, no complete proposal has yet been developed to deal with these expected discontinuities. Earlier work by Barack and Ori [24] suggests that gauge freedom may play a role in resolving these issues.

1.3.2.2 Time domain treatment for Schwarzschild

In a different approach to the inversion problem, Lousto and Whiting [25] have chosen to work in the time domain. Because of this choice their result is only valid in the

[2] The term multiplying Ψ in Equation 5–2 arose by repeated use of the Teukolsky equation in quite a complicated expression, initially given correctly by Stewart [21], and also obtainable from the results of Chapters 2 and 3 here. The full form of the expression may still apply here.

Schwarzschild background, where angular separation is not dependent on separation in time. Nevertheless, with the formulation of the Hertz potentials being set in the context of radiation gauges, Lousto and Whiting were effectively unable to introduce sources into their treatment. Regardless, several results of their analysis are noteworthy. Similar to the analysis of Regge and Wheeler, Lousto and Whiting made use of angular and parity decompositions, two features that have eluded application in the Kerr background.

One unexpected feature of Lousto and Whiting's work is how algebraically special frequencies emerge in a fundamental way. Algebraically special solutions arise when one of ψ_0 or ψ_4 is zero while the other is not, and then only for specific (complex) frequencies. While this is inherently a frequency domain phenomena, it plays a crucial role in this time domain approach. The algebraically special equation here has a source term depending on the initial data for the Hertz potential—this term effectively corresponds to that which arises for a Laplace transform. For the Schwarzschild background, all the algebraically special frequencies are known and the algebraically special solutions have been found explicitly [26], so the equations for this analysis could be solved by quadrature [25]. Attempts to generalize this technique to the Kerr background have to date remained unsuccessful.

1.3.2.3 Working in the Regge-Wheeler gauge

The RW formalism has been extensively used for Schwarzschild perturbations, and its implications have been thoroughly investigated. In particular, full sets of gauge invariant quantities are known, and in chosen cases these have been directly related to the perturbed Weyl scalars ψ_0 and ψ_4, which are naturally gauge invariant [27]. Lousto [28] has recently chosen to work with such a formulation, rather than with a Hertz potential formulation. This immediately gives him freedom over gauge choice and it circumvents the problems previously encountered with the introduction of sources. Having calculated explicitly the dependence on sources, and knowing also how to represent all relevant quantities through gauge invariant entities, Lousto thus succeeded in reconstructing perturbations of Schwarzschild in a way that includes sources.

Lousto actually uses both ψ_0 and ψ_4 in his construction. For concreteness and for access to a vast body of prior experience, Lousto also chose to work in a gauge known as the RW gauge. Note that Equations 1–21 and 1–22 are only valid in the IRG. However, ψ_0 and ψ_4 are easily expressible in terms of an arbitrary metric perturbation, which allows them to be written in terms of the RW variables for any choice of gauge. In the RW gauge, ψ_0 and ψ_4 become algebraic in the even parity sector and first order operators in the odd parity sector. To provide enough conditions to solve for all the components of the metric perturbation in terms of the Weyl scalars, Lousto must turn to the Einstein equations (with sources), also in the RW gauge. It is in this way that reconstruction with sources is accomplished.

The identification of gauge invariant quantities, beyond ψ_0 and ψ_4, is virtually nonexistent in the Kerr spacetime and as pointed out several times before, the angular decomposition there is not as robust as that available in spherically symmetric backgrounds. In short, Lousto's work is quite notable for its inclusion of sources, but its reliance on RW tools and techniques make it difficult to see how to extend the method to the Kerr background.

1.4 This Work

Motivated by the success of spin coefficient formalisms in describing perturbations of type D spacetimes and the incompleteness of current approaches, this dissertation presents a new framework for perturbation theory that exploits the best features of both standard treatments of perturbation theory and those based in the methods of a spin coefficient formalism. As we will see, a natural feature of this formalism is that it applies to general algebraically special spacetimes with little extra effort. Though our framework is quite general and provides a new means of understanding perturbations of a wide variety of spacetimes, we will keep our focus more narrow than that. In particular, the applications we present here are primarily aimed at providing the missing pieces in the Hertz potential approach to metric perturbations.

In Chapter 2, we will develop the formalism necessary for building our framework. Additionally, the framework will be presented, which includes a new form for the perturbed Einstein equations as well as a Maple package that aids not only in their application, but any computation in the formalism of Geroch, Held and Penrose. Chapter 3 then provides a further discussion of both the RW and Teukolsky formalisms, phrased in our framework. In Chapter 4, the necessary and sufficient conditions for the existence of the IRG (in a larger class of spacetimes than we consider elsewhere) are determined with the aid of our form of the Einstein equations. Chapter 5 then uses the IRG metric perturbation to derive some important relationships between the curvature perturbations represented by ψ_0 and ψ_4, which are of importance for the inversion problem described in this chapter. Furthermore, this application showcases some of our Maple package's most useful features. In Chapter 6 we then present a very different application of our framework in conjunction with more standard techniques to address the issue of the non-radiated multipoles.

CHAPTER 2
NEW TOOLS FOR PERTURBATION THEORY

In this chapter we develop the basic formalism we will be working within for the remainder of this work. We begin with a description of the spin coefficient formalism of Newman and Penrose [22] and introduce the modifications of it due to Geroch, Held and Penrose [9]. Within the latter formalism, we develop the properties of the general class of spacetimes with which we will be working. Included is a discussion of gauge and the general framework of relativistic perturbation theory. The chapter ends with the introduction to the framework we will exploit in subsequent chapters.

2.1 NP

The Newman-Penrose (henceforth NP) formalism has its roots in the spinor formulation of General Relativity. Despite the great beauty and generality of the spinor approach, we will approach the subject as a special case of the tetrad formalism. In this view, the NP formalism is developed by (1) introducing a basis of null vectors for the spacetime and (2) contracting everything in sight with unique combinations of the aforementioned basis vectors.

We begin by introducing an orthogonal tetrad of null vectors, l^a, n^a, m^a and $\bar{m}^a$, with l^a and n^a being real and m^a and $\bar{m}^a$ being complex conjugates. We will impose a relative normalization

$$l_a n^a = -m_a m^a = 1, \tag{2–1}$$

with all other inner products vanishing. As an example to keep in mind, consider an orthonormal tetrad on Minkowski space, (t^a, x^a, y^a, z^a), such that $t^a t_a = -x^a x_a = -y^a y_a = -z^a z_a = 1$. Since the vectors are properly normalized, it is easy to verify that

$$\begin{aligned} l^a &= \frac{1}{\sqrt{2}}(t^a + z^a), & n^a &= \frac{1}{\sqrt{2}}(t^a - z^a) \\ m^a &= \frac{1}{\sqrt{2}}(x^a + iy^a), & \bar{m}^a &= \frac{1}{\sqrt{2}}(x^a - iy^a), \end{aligned} \tag{2–2}$$

defines a null tetrad. It is important to note that there is some ambiguity implicit in the above assignment, e.g. we can swap the roles of z^a and x^a (or y^a) in the above definitions without changing the character (real or complex) of the null vectors or modifying their inner products. We will return to this issue later in this section.

For simplicity, we introduce the following notation for our tetrad (borrowed from Chandrasekhar [29]):

$$e^a_{(i)} = (l^a, n^a, m^a, \bar{m}^a),$$

where the tetrad index $(i) = \{1, 2, 3, 4\} = \{l, n, m, \bar{m}\}$. In a further attempt to avoid confusion we'll take spacetime indices from the beginning of the alphabet $(a, b, c \ldots)$ and tetrad indices from later in the alphabet $(i, j, k \ldots)$. Just as the vector index can be raised or lowered with the spacetime metric

$$e^a_{(i)} g_{ab} = e_{(i)b} \quad \text{and} \quad e_{(i)a} g^{ab} = e^b_{(i)},$$

we may introduce a similar object for raising and lowering tetrad indices

$$\eta_{(i)(j)} = e^a_{(i)} e^b_{(j)} g_{ab}.$$

For a properly normalized (Equation 2–1) null tetrad

$$\eta_{(i)(j)} = \eta^{(i)(j)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0. \end{pmatrix}$$

It then follows that we can express our spacetime metric as

$$g_{ab} = e_{(i)a} e_{(j)b} \eta^{(i)(j)} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)} \tag{2–3}$$

where $l_{(a}n_{b)} = \frac{1}{2}(l_a n_b + l_b n_a)$.

An important notion in curved space is the connection. In standard tensor language it makes an appearance through the Christoffel symbols. The equivalents in NP language are called the spin coefficients and are defined, in general, by

$$\gamma_{(i)(j)(k)} = e^a_{(i)} e^b_{(j)} \nabla_a e_{(k)b}. \tag{2–4}$$

It follows from the definition that

$$\gamma_{(i)(j)(k)} = -\gamma_{(i)(k)(j)}. \tag{2–5}$$

There is a total of twelve spin complex coefficients, individually named as follows

$$\begin{aligned}
\kappa &= l^a m^b \nabla_a l_b, & \nu &= -n^a \bar{m}^b \nabla_a n_b, \\
\sigma &= m^a m^b \nabla_a l_b, & \lambda &= -\bar{m}^a \bar{m}^b \nabla_a n_b, \\
\rho &= \bar{m}^a m^b \nabla_a l_b, & \mu &= -m^a \bar{m}^b \nabla_a n_b, \\
\tau &= n^a m^b \nabla_a l_b, & \pi &= -l^a \bar{m}^b \nabla_a n_b,
\end{aligned} \tag{2–6}$$

and

$$\begin{aligned}
\beta &= \frac{1}{2}(m^a n^b \nabla_a l_b - m^a \bar{m}^b \nabla_a m_b), \\
\alpha &= \frac{1}{2}(\bar{m}^a m^b \nabla_a \bar{m}_b - \bar{m}^a l^b \nabla_a n_b), \\
\epsilon &= \frac{1}{2}(l^a n^b \nabla_a l_b - l^a \bar{m}^b \nabla_a m_b), \\
\gamma &= \frac{1}{2}(n^a m^b \nabla_a \bar{m}_b - n^a l^b \nabla_a n_b).
\end{aligned} \tag{2–7}$$

Our $e^a_{(i)}$ naturally define four independent, non-commuting directional derivatives

$$e_{(i)} \equiv e^a_{(i)} \frac{\partial}{\partial x^a},$$

which are also given individual names:

$$\begin{aligned}
D &= l^a \frac{\partial}{\partial x^a}, & \Delta &= n^a \frac{\partial}{\partial x^a}, \\
\delta &= m^a \frac{\partial}{\partial x^a}, & \bar{\delta} &= \bar{m}^a \frac{\partial}{\partial x^a}.
\end{aligned} \tag{2–8}$$

The field equations are obtained from the splitting of the Riemann tensor into a trace-free part and its traces according to

$$R_{abcd} = C_{abcd} + \frac{1}{2}(g_{ac}R_{bd} + g_{bd}R_{ac} - g_{bc}R_{ad} - g_{ad}R_{bc}) - \frac{1}{2}(g_{ac}g_{bd} - g_{bc}g_{ad})R. \tag{2–9}$$

where $C_{abcd}, R_{abcd}, R_{ab}$ and R denote the Weyl tensor, Riemann tensor, Ricci tensor and Ricci scalar, respectively. Since both the Ricci tensor and the Ricci scalar vanish in the absence of sources, the Weyl and Riemann tensors are identical in source-free spacetimes. In that sense the Weyl tensor represents the purely gravitational degrees of freedom. The Riemann tensor is then expressed purely in terms of the spin coefficients and their derivatives by contracting all four vector indices with $e^a_{(i)}$'s and making use of the Ricci identity,

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)v_k = R_{abc}{}^d v_d = R_{abcd}v^d, \tag{2–10}$$

where v_d is an arbitrary vector. In four dimensions the Riemann tensor has twenty independent components and the Ricci tensor has ten, leaving the Weyl tensor with ten independent components. In the NP formalism, this translates into five complex scalars:

$$\begin{aligned}
\psi_0 &= -C_{abcd}l^a m^b l^c m^d, \\
\psi_1 &= -C_{abcd}l^a n^b l^c m^d, \\
\psi_2 &= -\frac{1}{2}C_{abcd}(l^a n^b l^c n^d + l^a n^b m^c \bar{m}^d), \\
\psi_3 &= -C_{abcd}l^a n^b \bar{m}^c n^d, \\
\psi_4 &= -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d.
\end{aligned} \tag{2–11}$$

The Ricci tensor is represented by the following ten scalars:

$$
\begin{aligned}
\Phi_{00} &= -\frac{1}{2}R_{11}, & \Phi_{21} &= -\frac{1}{2}R_{24}, \\
\Phi_{11} &= -\tfrac{1}{4}(R_{12}+R_{34}), & \Phi_{02} &= -\frac{1}{2}R_{33}, \\
\Phi_{01} &= -\frac{1}{2}R_{13}, & \Phi_{22} &= -\frac{1}{2}R_{22}, \\
\Phi_{12} &= -\frac{1}{2}R_{23}, & \Phi_{20} &= -\frac{1}{2}R_{44}, \\
\Phi_{10} &= -\frac{1}{2}R_{14}, & \Pi &= \tfrac{1}{24}R.
\end{aligned}
\tag{2–12}
$$

The field equations then follow from Equations 2–9 and 2–10. A full set of equations for the NP formalism is composed of the commutators, the equations involving dependence on matter, and the Bianchi identities. This is given in Appendix A.

2.2 GHP

In 1973 Geroch, Held and Penrose (GHP) [9] introduced some convenient modifications of the NP formalism. Specifically, they identified the notions of spin and boost weight and make explicit use of an inherent discrete symmetry of the NP equations.

In the NP formalism, there is an implicit invariance under a certain interchange of the basis vectors which GHP have built on through the introduction of the prime ($'$) operation, defined by its action on the tetrad vectors:

$$
\begin{aligned}
(l^a)' &= n^a, & (n^a)' &= l^a, \\
(m^a)' &= \bar{m}^a, & (\bar{m}^a)' &= m^a.
\end{aligned}
\tag{2–13}
$$

A glance at Equations 2–6 and 2–7 suggests the adoption of a change in notation:

$$
\kappa' = -\nu, \qquad \sigma' = -\lambda, \qquad \rho' = -\mu, \qquad \tau' = -\pi, \qquad \beta' = -\alpha, \qquad \epsilon' = -\gamma, \tag{2–14}
$$

and similarly for the directional derivatives of Equation 2–8

$$
D' = \Delta \quad \text{and} \quad \delta' = \overline{\delta}. \tag{2–15}
$$

While the metric is invariant under a Lorentz transformation, the tetrad vectors are not. In the null tetrad formalism, a Lorentz transformation, which in general is described by six parameters, is broken up into three classes of tetrad rotations. We will consider only a tetrad rotation of Type III here[1] . In the language of our Minkowski space example, this amounts to a boost in the $z-t$ plane and a rotation in the $x-y$ plane. Under such a transformation

$$\begin{aligned}
\tilde{z}^a &= \frac{z^a - vt^a}{(1-v^2)^{1/2}}, \\
\tilde{t}^a &= \frac{t^a - vz^a}{(1-v^2)^{\frac{1}{2}}}, \\
\tilde{x}^a &= \cos\theta x^a - \sin\theta y^a, \\
\tilde{y}^a &= \sin\theta x^a + \cos\theta y^a,
\end{aligned}$$

which translates to

$$\begin{aligned}
\tilde{l}^a &= rl^a, \\
\tilde{n}^a &= r^{-1}n^a, \\
\tilde{m}^a &= e^{i\theta}m^a, \\
\tilde{\bar{m}}^a &= e^{-i\theta}\bar{m}^a,
\end{aligned} \tag{2–16}$$

where $r = \sqrt{(1-v)/(1+v)}$. The two transformations can be combined into one using $\zeta^2 = re^{i\theta}$. Then Equation 2–16 may be summarized by

$$\begin{aligned}
l^a &\to \zeta\bar{\zeta}l^a, & n^a &\to \zeta^{-1}\bar{\zeta}^{-1}n^a, \\
m^a &\to \zeta\bar{\zeta}^{-1}m^a, & \bar{m}^a &\to \zeta^{-1}\bar{\zeta}\bar{m}^a.
\end{aligned} \tag{2–17}$$

A quantity, χ, is then said to be of type $\{p,q\}$ if, under Equation 2–17, $\chi \to \zeta^p\bar{\zeta}^q\chi$. Alternatively [9], we may say that χ possesses spin weight $s = (p-q)/2$ and boost weight $b = (p+q)/2$. The p and q values for the tetrad vectors can be read off from Equation 2–17. They allow one to determine the spin and boost weights of the spin

[1] Descriptions of the other types of tetrad rotation can be found in [30] or [29].

coefficients in Equation 2–6, while the spin coefficients in Equation 2–7 have no well defined spin or boost weight since, under Equation 2–17, they pick up terms involving derivatives of ζ. When acting on a quantity of well defined spin and boost weight, the directional derivatives of Equation 2–8 by themselves also fail to create another quantity of well defined weight. However, it is possible to combine the spin coefficients in Equation 2–7 with the action of derivative operators in Equation 2–8 to construct derivative operators that do produce new quantities with well defined spin and boost weights. With χ taken to be of type $\{p, q\}$, we can define these operators as follows:

$$\begin{aligned} Þ\chi &= (D - p\epsilon - q\bar{\epsilon})\chi, \qquad & Þ'\chi &= (D' + p\epsilon' + q\bar{\epsilon}')\chi, \\ ð\chi &= (\delta - p\beta + q\bar{\beta}')\chi, \qquad & ð'\chi &= (\delta' + p\beta' - q\bar{\beta})\chi, \end{aligned} \tag{2–18}$$

where Þ and ð are Icelandic characters named "thorn" and "edth", respectively. Each of these derivatives has some well defined type $\{r, s\}$ in the sense that when they act on a quantity of type $\{p, q\}$, a quantity of type $\{r + p, s + q\}$ is produced. These new derivative operators inherit their type from their corresponding tetrad vectors:

$$\begin{aligned} Þ &: \{1, 1\}, \qquad & Þ' &: \{-1, -1\}, \\ ð &: \{1, -1\}, \qquad & ð' &: \{-1, 1\}. \end{aligned} \tag{2–19}$$

It is quite often useful to think of ð (Þ) and ð′ (Þ′) as spin (boost) weight raising and lowering operators, respectively. The derivatives in Equation 2–18 can be combined to form a covariant derivative operator:

$$\begin{aligned} \Theta_a &= l_a Þ' + n_a Þ - m_a ð' - \bar{m}_a ð \\ &= \nabla_a - \frac{1}{2}(p + q) n^b \nabla_a l_b + \frac{1}{2}(p - q) \bar{m}^b \nabla_a m_b. \end{aligned} \tag{2–20}$$

We note in passing that this definition defines the "GHP connection." Our primary use for Equation 2–20 will be to express things in GHP language via the replacement $\nabla_a \to \Theta_a$. With these definitions, all equations in the NP formalism can be translated into GHP equations. Note that under prime, $\{p,q\}' \rightarrow \{-p,-q\}$, and under complex conjugation, $\overline{\{p,q\}} \rightarrow \{q,p\}$. A basic set of the GHP equations is given in Appendix A.

2.3 Killing Tensors and Commuting Operators

2.3.1 Specialization to Petrov Type D

In this section we provide a brief explanation of why the NP and GHP formalisms are so specially equipped to handle problems in black hole space-times. For an arbitrary space-time there are precisely four null vectors, k^a, that satisfy

$$k^b k^c k_{[e} C_{a]bc[d} k_{f]} = 0, \tag{2–21}$$

where C_{abcd} is the Weyl tensor introduced in Equation 2–9 and the square brackets $[\,]$ denote anti-symmetrization. The vectors k^a define the so-called principal null directions of the space-time. For some space-times, one or more of the principal null vectors coincide. The general classification of space-times based on the number of unique principal null directions of the Weyl tensor was given in 1954 by Petrov [31] and bears his name. It turns out that all the black hole solutions of astrophysical interest—including Schwarzschild, Kerr and Kerr-Newman—are of Petrov type D, meaning they possess two principal null vectors, each with degeneracy two. According to the Goldberg-Sachs theorem [32] and its corollaries, for a space-time of type D with l^a and n^a aligned along the principal null directions of the Weyl tensor, the following hold (and reciprocally):

$$\kappa = \kappa' = \sigma = \sigma' = \psi_0 = \psi_1 = \psi_3 = \psi_4 = 0. \tag{2–22}$$

This is equivalent to the statement that both l^a and n^a are both geodesic and shear-free. Thus, in the NP and GHP formalisms, all black hole space-times are on equal footing. In the Kerr spacetime, the commonly used tetrad (aligned with the principal null directions) is the so-called Kinnersley tetrad [33], which takes the form

$$l^a = \left(\frac{r^2+a^2}{\Delta}, 1, 0, \frac{a}{\Delta}\right), \tag{2–23}$$

$$n^a = \frac{1}{2\left(r^2+a^2\cos^2\theta\right)}\left(r^2+a^2, -\Delta, 0, a\right), \tag{2–24}$$

$$m^a = \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left(ia\sin\theta, 0, 1, i/\sin\theta\right). \tag{2–25}$$

Clearly, Equations 2–22 help simplify the GHP equations tremendously. However, type D spacetimes are so special that their description in terms of the GHP formalism is even further simplified. Such simplification is due in large part to the existence of various objects satisfying suitable generalizations (and specializations) of Killing's equation.

2.3.2 The Killing Vectors and Tensor

Virtually all of the "magic" that happens when one considers type D spacetimes can be traced back to the existence of a two-index Killing spinor. Without delving into the world of spinors we remark that a two index Killing spinor [34–36], $\chi_{AB} = \chi_{(AB)}$, is a solution to[2]

$$\nabla_{A'(A}\chi_{BC)} = 0, \tag{2–26}$$

where A and A' are spinor indices and the parentheses denote symmetrization. The first consequence of the existence of χ_{AB} is that the quantity

$$\xi^a = \nabla^{A'B}\chi^A_B = \psi_2^{-1/3}(-\rho' l^a + \rho n^a + \tau' m^a - \tau\bar{m}^a), \tag{2–27}$$

is a Killing vector—ξ^a satisfies

$$\nabla_{(a}\xi_{b)} = 0. \tag{2–28}$$

The proof of this in spinor language can be found in [36], and the GHP expression can be verified directly by making the replacement $\nabla_a \to \Theta_a$ and utilizing the expressions in

[2] Equation 2–26 is also known as the twistor equation, which provides a different means of understanding its relevance.

Appendix A. Generally speaking, ξ^a is complex, and its real and imaginary parts satisfy Equation 2–28 independently [36], so all type D spacetimes possess two independent Killing vectors. These two Killing vectors each give rise to a constant of motion along a geodesic. In other words, if u^a is tangent to a geodesic ($u^b\nabla_b u^a = 0$), then $\xi_a u^a$ is conserved along u^a:

$$\begin{aligned} u^b\nabla_b\xi_a u^a &= u^a u^b\nabla_b\xi_a + \xi_a u^b\nabla_b u^a \\ &= 0, \end{aligned} \tag{2–29}$$

where the first term vanishes as a consequence of (Killing's) Equation 2–28 and the second because u^a is tangent to a geodesic.

In addition to the existence of two Killing vectors, the Killing spinor also gives rise to the conformal Killing tensor [35, 37]:

$$P_{ab} = \chi_{AB}\bar{\chi}_{A'B'} = \frac{1}{2}(\psi_2\bar{\psi}_2)^{-1/3}(l_{(a}n_{b)} + m_{(a}\bar{m}_{b)}), \tag{2–30}$$

which also exists in every type D background. The conformal Killing tensor is alternatively defined as a solution to

$$\nabla_{(c}P_{ab)} = \frac{1}{3}g_{(ab}\nabla^d P_{c)d}. \tag{2–31}$$

Conformal Killing tensors are useful because they give rise to conserved quantities along null geodesics. If k^a is tangent to a null geodesic ($k^b\nabla_b k^a = 0$ and $k^a k_a = 0$) then the quantity $P_{ab}k^a k^b$ is conserved along k^a:

$$\begin{aligned} k^c\nabla_c(P_{ab}k^a k^b) &= k^a k^b k^c\nabla_c P_{ab} + 2P_{ab}k^c k^{(a}\nabla_c k^{b)} \\ &= k^a k^b k^c\nabla_{(c}P_{ab)} \\ &= \frac{1}{3}(k_a k^a)k^c\nabla^b P_{bc} \\ &= 0, \end{aligned}$$

where we used the fact that k^a is tangent to a geodesic in the second line and null in the fourth line, along with Equation 2–31.

In certain instances we can extend this idea to provide a first integral of the motion for timelike and spacelike geodesics as well. Such a notion can be realized by defining a tensor, $K_{ab} = K_{(ab)}$, that satisfies

$$\nabla_{(a} K_{bc)} = 0. \tag{2–32}$$

A quantity satisfying this relation is called a Killing-Staeckel tensor. Note that by definition the metric and symmetric outer products of Killing vectors both satisfy Equation 2–32. We reserve the name Killing-Staeckel tensor for an object that does not reduce in this way. This is to be distinguished from the antisymmetric Killing-Yano tensor satisfying

$$\nabla_{(a} Y_{b)c} = 0,$$

which can be generally related to the Killing-Staeckel tensor via $K_{ab} = Y_{ac} Y^c{}_b$ [38, 39]. Because we will not make use of Killing-Yano tensors here, we will follow conventional language and refer to the Killing-Staeckel tensor as simply a Killing tensor. Returning to the main line of development, given the existence of a Killing tensor, we can recycle the argument above (now using Equation 2–32 instead of Equation 2–31) for the conformal Killing tensor to show that the quantity $K_{ab} u^a u^b$ is conserved for any u^a tangent to a geodesic, regardless of whether it be timelike, spacelike or null. The question then arises: When can we find a K_{ab} that satisfies Equation 2–32? To answer this question, we begin by decomposing the Killing tensor into its trace-free part and its trace, according to

$$K_{ab} = P_{ab} + \frac{1}{4} K g_{ab}, \tag{2–33}$$

with $P_{ab} g^{ab} = 0$ and $K = K_{ab} g^{ab}$. Using this in (Killing's) Equation 2–32 and dividing the resulting expression into trace-free and trace parts gives two equations. The trace-free part is simply Equation 2–31 and so P_{ab} is the conformal Killing tensor (as we anticipated with our notation) which exists in every type D background. The trace part becomes

$$\nabla_a P^a{}_b + \frac{3}{4}\nabla_b K = 0. \tag{2–34}$$

The existence of a K satisfying this condition is both necessary and sufficient for the existence of the Killing tensor. By making the appropriate substitution ($\nabla_a \to \Theta_a$), using Equation 2–30 and taking components with respect to the tetrad vectors, we are led to the following:

$$\begin{aligned} \text{Þ}K &= (\psi_2\bar{\psi}_2)^{-1/3}(\rho+\bar{\rho}), \qquad & \eth K &= -(\psi_2\bar{\psi}_2)^{-1/3}(\tau+\bar{\tau}'), \\ \text{Þ}'K &= (\psi_2\bar{\psi}_2)^{-1/3}(\rho'+\bar{\rho}'), \qquad & \eth K &= -(\psi_2\bar{\psi}_2)^{-1/3}(\tau'+\bar{\tau}). \end{aligned} \tag{2–35}$$

By applying all the commutators in Appendix A to K and making use of Equation 2–35, we arrive at a series of relations which we compactly write (following Chandrasekhar [29]) as

$$\frac{\rho}{\bar{\rho}} = \frac{\rho'}{\bar{\rho}'} = -\frac{\tau}{\bar{\tau}'} = -\frac{\tau'}{\bar{\tau}}. \tag{2–36}$$

These integrability conditions are both necessary and sufficient for the existence of a K satisfying Equation 2–34 and thus provide necessary and sufficient conditions for existence of the Killing tensor in a type D background. They are satisfied for every non-accelerating type D spacetime. These relations are the primary result of this section. It is straightforward to verify that $K = \frac{1}{2}(e^{-2ic}\psi_2^{-2/3} + e^{2ic}\bar{\psi}_2^{-2/3})$, where e^{2ic} is a phase factor whose origins will be described below in Equation 2–41. It follows that the Killing tensor may be expressed as

$$K_{ab} = (\psi_2\bar{\psi}_2)^{-1/3} l_{(a}n_{b)} - \frac{1}{8}(e^{-ic}\psi_2^{-1/3} + e^{ic}\bar{\psi}_2^{-1/3})^2 g_{ab}. \tag{2–37}$$

Historically, the Killing tensor was discovered by Carter [40, 41] while considering the separation of the Hamilton-Jacobi equation in the Kerr background. The constant of motion derived from the Killing tensor is thus known as the Carter constant.

In a non-accelerating spacetime, where the full Killing tensor is available, the Killing vector in Equation 2–27 is real up to a complex phase. If we specialize to the Kerr spacetime and the Kinnersley tetrad, it takes the value $M^{-1/3}t^a$, where t^a is the timelike Killing vector of the Kerr spacetime. To see this more generally we need to establish one more fact. Consider the GHP equation and Bianchi identity:

$$\text{Þ}\rho = \rho^2 \tag{2–38}$$

$$\text{Þ}\psi_2 = 3\rho\psi_2. \tag{2–39}$$

We can rewrite Equation 2–39 with the help of Equation 2–38 as

$$\begin{aligned} \text{Þ} \ln \psi_2 &= 3\rho \\ &= \text{Þ}(3 \ln \rho), \end{aligned}$$

which gives us

$$\psi_2 = C\rho^3, \tag{2–40}$$

where C is a (possibly complex) function annihilated by Þ. This is in fact not a proof, but rather the first step in one. A full proof would consist of showing that this is consistent with the rest of the GHP equations and Bianchi identities. The coordinate-free integration technique introduced in Chapter 5 is ideally suited for this. For now we take it as given that the Equation 2–40 is true in every type D background, for some complex[3] constant, C. It follows that

$$\frac{\rho}{\bar{\rho}} = \frac{C^{1/3}}{\bar{C}^{1/3}} \frac{\psi_2^{1/3}}{\bar{\psi}_2^{1/3}} \equiv e^{2ic} \frac{\psi_2^{1/3}}{\bar{\psi}_2^{1/3}}, \tag{2–41}$$

which defines the phase factor introduced in Equation 2–37. It turns out that in all type D spacetimes not possessing NUT charge, $c = 0$. More importantly, we now have the relations

$$e^{2ic} \frac{\psi_2^{1/3}}{\bar{\psi}_2^{1/3}} = \frac{\rho}{\bar{\rho}} = \frac{\rho'}{\bar{\rho}'} = -\frac{\tau}{\bar{\tau}'} = -\frac{\tau'}{\bar{\tau}}, \tag{2–42}$$

[3] In all type D spacetimes not possessing NUT charge, C is M, the mass of the spacetime.

which make it straightforward to see that for spacetimes without acceleration Equation 2–27 is real up to a complex phase (e^{2ic}). Note also that $\xi'_a = -\xi_a$. What happened to the other (linearly independent) Killing vector? It is given by

$$\begin{aligned}\eta_b = \xi^a K_{ab} \;&=\; \frac{1}{8}\psi_2^{-1/3}\Big\{[e^{-ic}\psi_2^{-1/3} - e^{ic}\bar{\psi}_2^{-1/3}]^2(\rho' l_b - \rho n_b) \\ &\qquad - [e^{-ic}\psi_2^{-1/3} + e^{ic}\bar{\psi}_2^{-1/3}]^2(\tau' m_b - \tau \bar{m}_b)\Big\}. \end{aligned} \tag{2–43}$$

Proving that this expression satisfies Killing's equation in general is a bit involved, and since we'll have no direct use for Equation 2–43 in subsequent chapters, we refer the interested reader elsewhere [36] for details. Once again, using Equations 2–42, it is straightforward to see that Equation 2–43 is real up to a phase. Using the Kinnersley tetrad in the Kerr spacetime, Equation 2–43 becomes

$$\eta^b = \frac{a^2}{M} t^b + \frac{a}{M}\phi^b, \tag{2–44}$$

where t^a is the timelike Killing vector and ϕ^a is the axial Killing vector. Because η^b is proportional to a, it clearly vanishes in the Schwarzschild spacetime. This can also been seen by noting that, in the Schwarzschild spacetime, $\tau = \tau' = 0$ and thus comparisons of Equations 2–27 and 2–43 reveal that the two Killing vectors are not linearly independent [42]. In [36] it is shown how one can infer spherical symmetry from this fact.

2.3.3 Commuting Operators

An important property of Killing vectors is the fact that they commute with all of the tetrad vectors:

$$\begin{aligned}\pounds_\xi g_{ab} = 2\nabla_{(a}\xi_{b)} &= 0 \\ &= 2\pounds_\xi (l_{(a}n_{b)} - m_{(a}\bar{m}_{b)}) \\ &= 2(l_{(a}\pounds_\xi n_{b)} + n_{(a}\pounds_\xi l_{b)} - m_{(a}\pounds_\xi \bar{m}_{b)} - \bar{m}_{(a}\pounds_\xi m_{b)}),\end{aligned}$$

where the first line follows from the definition of the Killing vector and the second and third from Equation 2–3. By contracting the last line with each of the tetrad vectors and making further use of Equation 2–32, we establish that

$$\pounds_\xi l_a = \pounds_\xi n_a = \pounds_\xi m_a = \pounds_\xi \bar{m}_a = 0.$$

Recall that for any two vectors, A and B, their commutator is given by $[A, B] = \pounds_A B$, which establishes that the Killing vectors of the spacetime commute with all of the tetrad vectors.

In this light, it is reasonable to expect that we can construct an operator, $\mathcal{V}$, related to the Killing vector that commutes with all four of the GHP derivatives. Because of the fact that spin- and boost-weights enter explicitly into the commutators (Equations A–1–A–3), we would also expect that any such operator would carry spin- and boost-weight dependence. In fact, such an operator can be constructed. By taking as our ansatz:

$$\mathcal{V} = \xi^a \Theta_a + pA + qB,$$

and computing all of the commutators, we can find explicit expressions for A and B. However, this also requires that Equations 2–36 are satisfied, which implies a Killing tensor exists. For non-accelerating spacetimes we then have

$$\mathcal{V} = \psi_2^{-1/3}(\tau' \eth - \tau \eth' - \rho' \text{Þ} + \rho \text{Þ}' + \frac{1}{2}\psi_2 p + \frac{\rho}{2\bar{\rho}}\bar{\psi}_2 q), \tag{2–45}$$

where p and q refer to the GHP type of the object being acted on. This result has been noted by Jeffryes [43], who arrived at it from spinor considerations. If we specialize to the Kerr spacetime and the Kinnersley tetrad, it is easy to see that it takes the value $M^{-1/3}\partial_t + bM^{2/3}(r^2 + a^2\cos^2\theta)^{-1}$, where b is the boost-weight of the quantity being acted on. Despite this difference between the vector ξ^a and the operator $\mathcal{V}$, we will refer to them interchangeably as a Killing vector. Similarly, we can follow the same procedure that led to Equation 2–45 to obtain a similar operator associated with η^a (Equation 2–43):

$$\begin{aligned}
\mathcal{P} = \frac{1}{8}\psi_2^{-1/3}\Big\{&[e^{-ic}\psi_2^{-1/3} - e^{ic}\bar{\psi}_2^{-1/3}]^2(\rho'\text{Þ} - \rho\text{Þ}') \\
&- [e^{-ic}\psi_2^{-1/3} + e^{ic}\bar{\psi}_2^{-1/3}]^2(\tau'\eth - \tau\eth') \\
&+ 2(p-q)\rho\rho'\psi_2^{-1/3}(e^{-2ic}\psi_2^{-1/3} - \bar{\psi}_2^{-1/3}) \\
&- 2(p+q)\tau\tau'\psi_2^{-1/3}(e^{-2ic}\psi_2^{-1/3} + \bar{\psi}_2^{-1/3}) \\
&- \frac{1}{2}p\, e^{-2ic}\psi_2^{1/3}(e^{-4ic} - 2\psi_2^{-1/3}\bar{\psi}_2^{1/3} - \psi_2^{2/3}\bar{\psi}_2^{-2/3}) \\
&- \frac{1}{2}q\psi_2^{1/3}(e^{4ic} - 2\psi_2^{1/3}\bar{\psi}_2^{-1/3} - \psi_2^{-2/3}\bar{\psi}_2^{2/3})\Big\},
\end{aligned} \tag{2–46}$$

which also commutes with all four GHP derivations.

On a final note, we remark that in recent work Beyer [44] obtained an operator related to Killing tensor that commutes with the scalar wave equation. The operator has the feature that it is first order in time. In this context it is tempting to ask if there exists an operator analogous to those defined for the Killing vectors that commutes with each of the GHP derivatives. The answer is currently unclear and so we leave it for future investigation.

2.4 The Simplified GHP Equations for Type D Backgrounds

With Equations 2–36 in hand, we are now in a position to completely simplify the GHP equations for the special case of type D backgrounds. Our starting point is the GHP equations and Bianchi identities adapted to a Type D background:

$$\text{Þ}\rho = \rho^2 \tag{2–47}$$

$$\text{Þ}\tau = \rho(\tau - \bar{\tau}') \tag{2–48}$$

$$\eth\rho = \tau(\rho - \bar{\rho}) \tag{2–49}$$

$$\eth\tau = \tau^2 \tag{2–50}$$

$$\text{Þ}'\rho - \eth'\tau = \rho\bar{\rho}' - \tau\bar{\tau} - \psi_2 \tag{2–51}$$

$$\text{Þ}\psi_2 = 3\rho\psi_2 \tag{2–52}$$

$$\eth\psi_2 = 3\tau\psi_2, \tag{2–53}$$

where we have omitted those equations that can be obtained directly by utilizing the operations of prime and complex conjugation. By applying the commutators to ψ_2 and making use of the equations above, we learn that

$$\text{Þ}\rho' = \text{Þ}'\rho \tag{2–54}$$

$$\eth\tau' = \eth'\tau \tag{2–55}$$

$$\text{Þ}\tau' = \eth'\rho. \tag{2–56}$$

Note that the preceding equations hold for all type D spacetimes. Next we specialize to non-accelerating spacetimes by making use of Equation 2–36 in the form $\tau' = -\frac{\rho\bar{\tau}}{\bar{\rho}}$ in Equation 2–56 to obtain

$$\text{Þ}\tau' = \eth'\rho = 2\rho\tau'. \tag{2–57}$$

Now we compute the commutator $[\text{Þ}, \text{Þ}']\rho$ and use the GHP equations and the appropriate version of Equation 2–57 until we arrive at an expression in which the only derivatives are $\eth'\tau$ and $\text{Þ}'\rho$. This expression can then be used with Equations 2–51 and 2–36 to find the following two relations:

$$\text{Þ}'\rho = \rho\rho' + \tau'(\tau - \bar{\tau}') - \tfrac{1}{2}\psi_2 - \tfrac{\rho}{2\bar{\rho}}\overline{\psi}_2 \tag{2–58}$$

$$\eth'\tau = \tau\tau' + \rho(\rho' - \bar{\rho}') + \tfrac{1}{2}\psi_2 - \tfrac{\rho}{2\bar{\rho}}\overline{\psi}_2, \tag{2–59}$$

(2–60)

and our task is complete. It is worth pointing out that due to Equations 2–36, these expressions are not unique. This is a sign that there is some redundancy in the GHP equations, which is to be expected when we consider such a special class of spacetimes. We also point out that having expressions for every derivative on every quantity of interest is sufficient (but not necessary) to completely integrate the background GHP equations. This is a task that was first performed for the NP equations by Kinnersley [33] and later by Held [45] for the GHP equations. In Chapter 5, we will discuss the latter of these methods in more detail.

2.5 Issues of Gauge in Perturbation Theory

One of the most important subtleties associated with perturbation theory in general relativity is the concept of gauge invariance. The principal of general covariance tells us that the interesting questions to ask are those that have answers that every observer agrees upon. In the context of full (non-perturbative, nonlinear) relativity, this is ensured by focusing on quantities that remain unchanged under coordinate transformations. In perturbation theory, however, there is a new twist to the problem—there are now two spacetimes of interest: the unperturbed background spacetime consisting of a manifold, M with metric g_{ab} (henceforth denoted by (M, g_{ab})) and the physical spacetime, (M', g'_{ab}), that includes both the background and the perturbation. The question of how to relate perturbations of quantities on (M', g'_{ab}) to quantities on (M, g_{ab}) in an unambiguous way is fundamental for a well defined perturbation theory in general relativity. A complete analysis, in the context of the GHP formalism, of this question was performed by Stewart and Walker [46], whose basic results will be developed here. Before we address the relativistic problem, we very briefly review first-order perturbation theory in a flat spacetime. In that instance, we think of the quantity of interest, $q = q(\lambda)$, as being parameterized by some λ, so that $q(0)$ corresponds to the unperturbed quantity and $q(1)$ is the fully perturbed quantity whose first-order perturbations we would like to consider. It follows from writing $q(\lambda)$ as a Taylor series in λ that the first-order perturbation, δq, is given by

$$\delta q = \frac{dq(\lambda)}{d\lambda}\Big|_{\lambda=0}.$$

To adapt this idea to our curved space problem, it helps to think of both the background and physical spacetimes as members of a one-parameter family of spacetimes, $(M_\lambda, g_{ab}(\lambda))$, with $\lambda = 1$ corresponding to the physical spacetime and $\lambda = 0$ corresponding to the background. Now suppose we've identified some geometric quantity of interest (could be scalar, vector, tensor, etc.; for simplicity we write it with no indices), $Q = Q(\lambda)$, and we are interested in its first order perturbation, δQ, towards the physical spacetime, evaluated in the background. Before we can compute anything we must confront the issue of how to relate quantities on two different curved manifolds. One can imagine introducing a (suitably well-behaved) vector field, ξ^a, that connects points in the physical spacetime to points in the background. Then, to compute δQ, we evaluate Q at some point $p+\delta p$ in the physical spacetime, pull the result back along ξ^a to the background spacetime, subtract from it the value of Q at a point p in the background, divide by δp and take the limit as $\delta p \to 0$. The mathematical apparatus for performing this task is the Lie derivative. Thus, the first order perturbation, δQ, to a quantity, Q, evaluated in the background spacetime is given by

$$\delta Q = \pounds_\xi Q(\lambda)\Big|_{\lambda=0}. \tag{2–61}$$

The important point about this prescription is the fact that ξ^a not only fails to be unique, but there is, in general, no preferred choice for it. A choice of ξ^a is more commonly known as a choice of gauge. According to Equation 2–61, the difference between δQ computed with ξ^a and η^a is given by

$$\delta Q_\xi - \delta Q_\eta = \pounds_{\xi-\eta} Q,$$

and so we define $\tilde{\delta Q}$, the gauge transformation of δQ by

$$\tilde{\delta Q} = \delta Q - \pounds_\xi Q. \tag{2–62}$$

Note that a gauge transformation in this sense represents a change in the way we identify points in the physical spacetime with points in the background. This is to be distinguished from a coordinate transformation, which changes the labeling of coordinates in both the physical and background spacetimes.

The significance of Equation 2–62 is that unless $\pounds_\xi Q = 0$ for every ξ^a, there is some ambiguity in identifying the perturbation—we can't differentiate between the contributions of the perturbation (δQ) and the background ($\pounds_\xi Q$). Quantities that satisfy $\pounds_\xi Q = 0$ for every ξ^a are therefore called gauge invariant. It is straightforward to see that the perturbation of Q is gauge invariant if and only if: (1) Q vanishes in the background, (2) Q is a constant scalar in the background or (3) Q is a constant linear combination of Kroenecker deltas. This is a result originally due to Sachs [47]. A direct consequence of this fact is that the metric perturbation, arguably the most fundamental quantity we deal with, fails to be gauge invariant. Fortunately, type D spacetimes come equipped with two gauge invariants, ψ_0 and ψ_4, which have simple expressions in terms of the components of the metric perturbation. As we will see, appropriate use of gauge freedom simplifies our computations tremendously.

2.6 GHPtools - A New Framework for Perturbation Theory

With the basic formalism in place, we are ready to present the tools that form the basis of the subsequent chapters. The motivation for our framework comes from two places: (1) the desire to take advantage of gauge freedom in standard metric perturbation theory and (2) the success of the GHP formalism in perturbation theory. As mentioned in the previous chapter, gauge freedom proved absolutely crucial for the RW analysis and that of Cohen & Kegeles [20], Chrzanowski [18], and Stewart [21], and it will certainly play a central role in any future description of metric perturbations. The second ingredient, the GHP formalism comes with several advantages. First of all, the inherent coordinate independence and notational economy makes calculations in general spacetimes tractable. Furthermore, by virtue of the Goldberg-Sachs theorem, we can deal with the entire class of type D spacetimes at once. Additionally, spin- and boost- weights provide useful bookkeeping and, as we'll see, a useful context for understanding the roles that various quantities play. Last but not least, the use of a spin coefficient formalism has proved absolutely crucial for studying perturbations of anything other than spherically symmetric spacetimes. We will put these ideas together to compute the perturbed Einstein equations in a mixed tetrad-tensor form. This is the heart of our work.

2.6.1 Einstein's New Clothes

The main idea behind our framework is to reorganize the tensors of interest into their tetrad components. The metric perturbation, for example, has the decomposition

$$\begin{aligned} h_{ab} = {} & h_{ll} n_a n_b + h_{nn} l_a l_b + 2h_{ln} l_{(a} n_{b)} + 2h_{m\bar{m}} m_{(a} \bar{m}_{b)} \\ & - 2h_{lm} n_{(a} \bar{m}_{b)} - 2h_{l\bar{m}} n_{(a} m_{b)} - 2h_{nm} l_{(a} \bar{m}_{b)} - 2h_{n\bar{m}} l_{(a} m_{b)} \\ & + h_{mm} \bar{m}_a \bar{m}_b + h_{\bar{m}\bar{m}} m_a m_b, \end{aligned} \tag{2–63}$$

so that, for example, $h_{ll} = h_{ab} l^a l^b$. In order for this to be valid within the GHP formalism, each component of Equation 2–63 must have a well-defined spin- and boost- weight. Because the background metric (Equation 2–3) is invariant under a spin-boost (Equation 2–17) it has type $\{0, 0\}$, which must also be the type of the metric perturbation, h_{ab}. Therefore the type of the individual components of the metric perturbation are determined by their tetrad indices:

$$\begin{aligned} h_{ll} &: \{2, 2\} & h_{nn} &: \{-2, -2\} \\ h_{lm} &: \{2, 0\} & h_{n\bar{m}} &: \{-2, 0\} \\ h_{l\bar{m}} &: \{0, 2\} & h_{nm} &: \{0, -2\} \\ h_{mm} &: \{2, -2\} & h_{\bar{m}\bar{m}} &: \{-2, 2\} \\ h_{ln} &: \{0, 0\} & h_{m\bar{m}} &: \{0, 0\}. \end{aligned} \tag{2–64}$$

All of the vectors and tensors we will concern ourselves with can be treated in this way.

It is worthwhile to stop here and take a look at what Equation 2–63 really means. Comparing with our treatment of Schwarzschild (Equation 1–4), we note that the scalar parts of the metric are "mixed up" in h_{ll}, h_{ln} and h_{nn}, all of which have spin weight zero but differ in boost weight. Similarly, the vector parts are given by h_{lm}, h_{nm} and their complex conjugates and likewise the tensor pieces are given here by $h_{mm}, h_{\bar{m}\bar{m}}$ and $h_{m\bar{m}}$. However, these identifications are completely independent of the background spacetime. Thus, in a certain sense, Equation 2–63 provides a generalization of the RW mode decomposition that takes into account both spin- and boost- weight. In the next chapter we will make some more precise statements in this direction.

Recall our expression for the perturbed Einstein equations:

$$\mathcal{E}_{ab} = -\frac{1}{2}\nabla^c\nabla_c h_{ab} - \frac{1}{2}\nabla_a\nabla_b h^c{}_c + \nabla^c\nabla_{(a}h_{b)c} + \frac{1}{2}g_{ab}(\nabla^c\nabla_c h^d{}_d - \nabla^c\nabla^d h_{cd}).$$

By making the replacement $\nabla_a \to \Theta_a$ and understanding h_{ab} as referring to the tetrad components of the metric perturbation given in Equation 2–63, we arrive at the perturbed Einstein equations in GHP form:

$$\mathcal{E}_{ab} = -\frac{1}{2}\Theta^c\Theta_c h_{ab} - \frac{1}{2}\Theta_a\Theta_b h^c{}_c + \Theta^c\Theta_{(a}h_{b)c} + \frac{1}{2}g_{ab}(\Theta^c\Theta_c h^d{}_d - \Theta^c\Theta^d h_{cd}), \tag{2–65}$$

which (right now, at least) don't look all that different! The tetrad components of Equation 2–65 for an *arbitrary* algebraically special background spacetime are given in Appendix B. Aside from the obvious cosmetic differences, there are several key distinctions between Equation 2–65 and the standard form of metric perturbation theory worth pointing out. First of all, our form lacks the background Einstein equations present in the standard treatment. Taking their place are the background GHP equations and Bianchi identities. Perhaps more importantly is the inherent coordinate independence. Coupled with the concepts of spin- and boost-weight, this allows for a certain structural intuition not present in coordinate based techniques. This point of view will be stressed throughout.

Writing Equation 2–65 is one thing, but actually computing it is another question entirely, which we now turn our attention to.

2.6.2 GHPtools - The Details

To perform such a computation for an arbitrary background spacetime is no small task, even (or rather especially) in the standard tensor language. For this the aid of Maple was enlisted. Unfortunately, at the time the computation was performed, there were no Maple packages available for performing all such computations at the level of generality required[4] . Naturally, one was developed. It has been dubbed GHPtools and the Maple code for it is the content of Appendix C. The remainder of this chapter is devoted to explaining its basic use and functionality through a simple Maple worksheet.

Every session begins by invoking GHPtools:

```
> restart;

> with(GHPtoolsv1);
```

$[BI1,\ BI1c,\ BI1p,\ BI1pc,\ BI2,\ BI2c,\ BI2p,\ BI2pc,\ BI3,\ BI3c,\ BI3p,\ BI3pc,\ BI4,\ BI4c,\ BI4p, BI4pc,\ COM1,\ COM1c,\ COM1p,\ COM1pc,\ COM2,\ COM2c,\ COM2p,\ COM2pc,\ COM3,\ COM3c,\ COM3p,\ COM3pc,\ DGHP,\ GHP1,\ GHP1c,\ GHP1p,\ GHP1pc,\ GHP2,\ GHP2NP,\ GHP2c,\ GHP2p,\ GHP2pc,\ GHP3,\ GHP3c,\ GHP3p,\ GHP3pc,\ GHP4,\ GHP4c,\ GHP4p,\ GHP4pc,\ GHP5,\ GHP5c,\ GHP5p,\ GHP5pc,\ GHP6,\ GHP6c,\ GHP6p,\ GHP6pc,\ GHPconj,\ GHPmult,\ GHPprime,\ NPconj,\ NPexpand,\ NPprime,\ comm,\ ezcomm,\ flatxyz,\ getpq,\ schw,\ tdsimp,\ tdspec,\ tetcon,\ tetdnK,\ tetdnS,\ tetdnSB,\ tetupK,\ tetupS,\ tetupSB,\ typed]$

To begin with, each variable is directly specified by its usual name. For example $\bar{\rho}$ would be entered in Maple as `conjugate(rho)`. The primed variables have a '1' appended to the end, so that $\bar{\rho}'$ would be entered as `conjugate(rho1)`. The Weyl scalars are recognized as capital Ψ's with the appropriate number, e.g. `Psi2`. The derivatives Þ, ð, Þ′ and ð′ are recognized in Maple as `th()`, `eth()`, `thp()` and `ethp()`, respectively. GHPtools recognizes the tetrad vectors as labels indicating the position of the index with the actual index in parentheses. For example l^a and $\bar{m}_c$ would be input as `lup(a)` and `conjugate(mdn)(c)`. Finally, GHPtools contains an arbitrary function, ϕ (in Maple: `phi`), that is quite useful for general calculations. Amongst

[4] There is however a series of papers describing rather sophisticated Maple packages that perform some of the manipulations that we want [48, 49], called GHP and GHPII. We stress that GHPtools is no way intended to compete with these or any other Maple packages.

these variables, GHPtools computes the primes and complex conjugates through the procedures `GHPprime()` and `GHPconj()`:

```
> GHPconj(GHPprime(rho+conjugate(rho)));
```

$$\rho 1 + \overline{\rho 1}$$

```
> GHPconj(Psi2);
```

$$\overline{\Psi 2}$$

```
> GHPprime(conjugate(mdn)(a));
```

$$\mathrm{mdn}(a)$$

The $\{p, q\}$ type of any quantity may be obtained by the use of the `getpq` function, which returns p and q, in that order:

```
> getpq(Psi2);
```

$$0,\ 0$$

```
> getpq(rho1);
```

$$-1,\ -1$$

```
> getpq(phi);
```

$$pp,\ pq$$

Note that ϕ is given the general type `pp,qq`. Before any computation begins it is often useful to specify the spacetime in which subsequent computations are to take place by specifying the value of the global variable `spacetime`:

```
> spacetime := typed;
```

$$\mathit{spacetime} := \{\Phi 20 = 0,\ \Phi 21 = 0,\ \kappa = 0,\ \Phi 10 = 0,\ \Phi 22 = 0,\ \Pi = 0,\ \sigma 1 = 0,\ \Phi 12 = 0,\ \varepsilon = 0,\ \Psi 3 = 0,\ \Psi 4 = 0,\ \Psi 1 = 0,\ \Phi 11 = 0,\ \Psi 0 = 0,\ \kappa 1 = 0,\ \sigma = 0,\ \Phi 00 = 0,\ \Phi 01 = 0,\ \Phi 02 = 0,\ 0 = 0,\ \overline{\Phi 12} = 0,\ \overline{\Phi 20} = 0,\ \overline{\Phi 11} = 0,\ \overline{\Phi 21} = 0,\ \overline{\Phi 22} = 0,\ \overline{\sigma 1} = 0,\ \overline{\kappa} = 0,\ \overline{\sigma} = 0,\ \overline{\Phi 02} = 0,\ \overline{\Phi 10} = 0,\ \overline{\kappa 1} = 0,\ \overline{\Psi 0} = 0,\ \overline{\Psi 1} = 0,\ \overline{\Psi 3} = 0,\ \overline{\Psi 4} = 0,\ \overline{\Phi 00} = 0,\ \overline{\Phi 01} = 0\}$$

Aside from `typed`, acceptable values for `spacetime` are `flatxyz` (Minkowski space in Cartesian coordinates, where all the GHP quantities vanish), `schw` (the Schwarzschild spacetime, a specialization of type D where $\tau = \tau' = 0$ and all other quantities are real) and `none` (completely arbitrary spacetime; this is the default if `spacetime` is unspecified). The user is free to change the value of `spacetime` in the middle of a worksheet and only subsequent evaluations will be affected. For simplicity, we will henceforth restrict our attention to examples with `typed` specified. The GHP equations and Bianchi identities (as well as their primes, complex conjugates and conjugate primes) are implemented as Maple procedures so that their specification to the declared spacetime is returned. For example, if `typed` is specified, then

```
> GHP1();
> GHP1pc();
```

$$\mathrm{th}(\rho) = \rho^2$$

$$\mathrm{thp}(\overline{\rho 1}) = \overline{\rho 1}^2$$

```
> BI2();
> BI2p();
```

$$\mathrm{th}(\Psi 2) = 3\,\rho\,\Psi 2$$

$$\mathrm{thp}(\Psi 2) = 3\,\rho 1\,\Psi 2$$

The real usefulness of GHPtools comes not from its bookkeeping abilities, but rather its ability to perform symbolic computations within the GHP formalism. These abilities begin with the `DGHP()` procedure, which expands derivatives of objects occurring in an expression in accordance with the rules of derivations. For example

```
> expr := rho*rho1 + tau*tau1 - (conjugate(rho)*rho1)^3 +
> ln(conjugate(tau)*conjugate(tau1));
```

$$expr := \rho\,\rho 1 + \tau\,\tau 1 - \rho 1^3\,\overline{\rho}^3 + \ln(\overline{\tau}\,\overline{\tau 1})$$

```
> th(expr);
```

$$\mathrm{th}(\rho\,\rho 1+\tau\,\tau 1-\rho 1^3\,\overline{\rho}^3+\ln(\overline{\tau}\,\overline{\tau 1}))$$

```
> DGHP(%);
```

$$\mathrm{th}(\rho)\,\rho 1+\rho\,\mathrm{th}(\rho 1)+\mathrm{th}(\tau)\,\tau 1+\tau\,\mathrm{th}(\tau 1)-3\,\rho 1^2\,\mathrm{th}(\rho 1)\,\overline{\rho}^3-3\,\rho 1^3\,\overline{\rho}^2\,\mathrm{th}(\overline{\rho})+\frac{\mathrm{th}(\overline{\tau})}{\overline{\tau}}+\frac{\mathrm{th}(\overline{\tau 1})}{\overline{\tau 1}}$$

To date, DGHP() can handle powers and logarithms (the only functions this author has encountered in the GHP formalism), but the procedure can be easily modified to accommodate just about any function. Building complicated expressions involving linear combinations of derivative and multiplicative operators is easily achieved with the help of the GHPmult() procedure. These expressions can then be expanded with DGHP(). As an example, consider the expression $(\text{Þ}-\rho)^4\phi$:

```
> th4phi :=
> GHPmult(th-rho,GHPmult(th-rho,GHPmult(th-rho,GHPmult(th-rho,phi))));
```

$$\mathit{th4phi} := \mathrm{th}(\mathrm{th}(\%1-\rho\,\mathrm{th}(\phi)+\rho^2\,\phi)-\rho\,\%1+\rho^2\,\mathrm{th}(\phi)-\rho^3\,\phi)-\rho\,\mathrm{th}(\%1-\rho\,\mathrm{th}(\phi)+\rho^2\,\phi)+\rho^2\,\%1-\rho^3\,\mathrm{th}(\phi)+\rho^4\,\phi$$

$$\%1 := \mathrm{th}(\mathrm{th}(\phi)-\rho\,\phi)$$

```
> DGHP(th4phi);
```

$$-6\,\rho^2\,\mathrm{th}(\rho)\,\phi+12\,\rho\,\mathrm{th}(\rho)\,\mathrm{th}(\phi)+3\,\mathrm{th}(\rho)^2\,\phi+4\,\rho\,\mathrm{th}(\mathrm{th}(\rho))\,\phi-4\,\rho^3\,\mathrm{th}(\phi)+\rho^4\,\phi-\mathrm{th}(\mathrm{th}(\mathrm{th}(\rho)))\,\phi-4\,\mathrm{th}(\mathrm{th}(\rho))\,\mathrm{th}(\phi)-6\,\mathrm{th}(\rho)\,\mathrm{th}(\mathrm{th}(\phi))+\mathrm{th}(\mathrm{th}(\mathrm{th}(\mathrm{th}(\phi))))-4\,\rho\,\mathrm{th}(\mathrm{th}(\mathrm{th}(\phi)))+6\,\rho^2\,\mathrm{th}(\mathrm{th}(\phi))$$

Simplifying such expressions is, in the context of type D spacetime without acceleration, handled by the tdsimp() procedure that substitutes the known values of the derivatives of the spin coefficients (stored in the globally available list tdspec; such a procedure can be easily generalized to encompass any spacetime, should the need arise) into its argument. Thus our previous example simplifies considerably:

```
> tdsimp(DGHP(th4phi));
```

$$\mathrm{th}(\mathrm{th}(\mathrm{th}(\mathrm{th}(\phi)))) - 4\,\rho\,\mathrm{th}(\mathrm{th}(\mathrm{th}(\phi)))$$

Perhaps even more useful is the `comm()` procedure which commutes derivatives on an expression. It takes two arguments: the first is the term whose first two derivatives will be commuted and the second is the expression into which the result will be substituted. Consider the following examples:

```
> commute_me1 := eth(th(ethp(ethp(phi)))) -
> th(eth(ethp(ethp(phi))));
```

$$\mathit{commute\_me1} := \mathrm{eth}(\mathrm{th}(\mathrm{ethp}(\mathrm{ethp}(\phi)))) - \mathrm{th}(\mathrm{eth}(\mathrm{ethp}(\mathrm{ethp}(\phi))))$$

```
> DGHP(comm(eth(th(ethp(ethp(phi)))),commute_me1));
```

$$-\overline{\rho}\,\mathrm{eth}(\mathrm{ethp}(\mathrm{ethp}(\phi))) + \overline{\tau 1}\,\mathrm{th}(\mathrm{ethp}(\mathrm{ethp}(\phi))) - 2\,\overline{\rho}\,\overline{\tau 1}\,\mathrm{ethp}(\mathrm{ethp}(\phi)) \\ - \overline{\rho}\,\overline{\tau 1}\,\mathrm{ethp}(\mathrm{ethp}(\phi))\,\mathit{pq}$$

```
> commute_me2 := th(th(thp(phi)))-th(thp(th(phi)));
```

$$\mathit{commute\_me2} := \mathrm{th}(\mathrm{th}(\mathrm{thp}(\phi))) - \mathrm{th}(\mathrm{thp}(\mathrm{th}(\phi)))$$

```
> tdsimp(comm(thp(th(phi)),commute_me2));
```

$$\mathrm{th}(\mathrm{eth}(\phi))\,\overline{\tau} + \mathrm{eth}(\phi)\,\overline{\rho}\,\overline{\tau} - \mathrm{eth}(\phi)\,\overline{\rho}\,\tau 1 - \mathrm{th}(\mathrm{eth}(\phi))\,\tau 1 - 2\,\mathrm{eth}(\phi)\,\rho\,\tau 1 + \mathrm{th}(\mathrm{ethp}(\phi))\,\tau \\ + \mathrm{ethp}(\phi)\,\rho\,\tau - \mathrm{ethp}(\phi)\,\rho\,\overline{\tau 1} - \mathrm{th}(\mathrm{ethp}(\phi))\,\overline{\tau 1} - 2\,\mathrm{ethp}(\phi)\,\overline{\rho}\,\overline{\tau 1} + \mathit{pp}\,\mathrm{th}(\phi)\,\tau\,\tau 1 \\ + 3\,\mathit{pp}\,\phi\,\rho\,\tau\,\tau 1 - \mathit{pp}\,\phi\,\rho\,\tau 1\,\overline{\tau 1} - \mathit{pp}\,\mathrm{th}(\phi)\,\Psi 2 - 3\,\mathit{pp}\,\phi\,\rho\,\Psi 2 + \mathit{pq}\,\mathrm{th}(\phi)\,\overline{\tau}\,\overline{\tau 1} \\ + 3\,\mathit{pq}\,\phi\,\overline{\rho}\,\overline{\tau}\,\overline{\tau 1} - \mathit{pq}\,\phi\,\overline{\rho}\,\tau 1\,\overline{\tau 1} - \mathit{pq}\,\mathrm{th}(\phi)\,\overline{\Psi 2} - 3\,\mathit{pq}\,\phi\,\overline{\rho}\,\overline{\Psi 2}$$

Computing the perturbed Einstein equations and Weyl scalars necessarily requires the ability to contract various combinations of the tetrad vectors. This functionality is provided by the `tetcon()` procedure, which also takes two arguments. The first is the expression that contains the uncontracted vectors and the second is a list of the indices to be contracted over. Take the example of computing the trace of the metric:

```
> gdn := ldn(a)*ndn(b) + ldn(b)*ndn(a) - mdn(a)*conjugate(mdn)(b) -
> mdn(b)*conjugate(mdn)(a);
> gup := subs({ldn=lup, ndn=nup, mdn=mup},gdn);
```

$$gdn := \mathrm{ldn}(a)\,\mathrm{ndn}(b) + \mathrm{ldn}(b)\,\mathrm{ndn}(a) - \mathrm{mdn}(a)\,\overline{mdn}(b) - \mathrm{mdn}(b)\,\overline{mdn}(a)$$

$$gup := \mathrm{lup}(a)\,\mathrm{nup}(b) + \mathrm{lup}(b)\,\mathrm{nup}(a) - \mathrm{mup}(a)\,\overline{mup}(b) - \mathrm{mup}(b)\,\overline{mup}(a)$$

```
> tetcon(gdn*gup,{a,b});
```

$$4$$

Finally, GHPtools provides some functionality for translating expressions into NP expressions that can subsequently be converted to ordinary coordinate expressions. This functionality is provide by the aptly named procedure `GHP2NP()`, which takes as its input a GHP expression. The functionality provided by the procedure is limited to expressions involving at most two derivatives. Furthermore, the derivatives must appear in a specified order according to the following rules: (1) ð and ð′ must always appear to the left of Þ and Þ′, (2) ð must appear to the left of ð′ and (3) Þ must appear to the left of Þ′. Take the following example:

```
> GHP2NP(th(thp(hln))+eth(ethp(hln)));
```

$$\mathrm{DD}(\Delta(hln)) + \varepsilon\,\Delta(hln) + \overline{\varepsilon}\,\Delta(hln) + \delta(\overline{\delta}(hln)) + \beta\,\overline{\delta}(hln) + \overline{\beta 1}\,\overline{\delta}(hln)$$

In order to aid in the conversion of such quantities into coordinate expressions, GHPtools contains, as lists of arrays, some commonly used tetrads in the Kerr spacetime. They are: the Kinnersley tetrad with indices up `tetupK` and down `tetdnK`, the symmetric tetrad (`tetupS, tetdnS`) and the symmetric tetrad boosted by a function $B(t, r, \theta, \phi)$ and spun by a function $S(t, r, \theta, \phi)$ (`tetupSB, tetdnSB`). These are called simply by invoking their names:

```
> tetdnK;
```

$$\left\{mdn = \left[\frac{\frac{1}{2}\,I\,a\sin(\theta)\,\sqrt{2}}{r + a\cos(\theta)\,I},\ 0,\ -\frac{1}{2}\,(r - a\cos(\theta)\,I)\,\sqrt{2},\ \frac{\frac{-1}{2}\,I\,(r^2 + a^2)\sin(\theta)\,\sqrt{2}}{r + a\cos(\theta)\,I}\right],\right.$$

$$ndn = \left[\frac{1}{2}\,\frac{r^2 - 2\,M\,r + a^2}{(r + a\cos(\theta)\,I)\,(r - a\cos(\theta)\,I)},\ \frac{1}{2},\ 0,\ -\frac{1}{2}\,\frac{a\,(r^2 - 2\,M\,r + a^2)\sin(\theta)^2}{(r + a\cos(\theta)\,I)\,(r - a\cos(\theta)\,I)}\right],$$

$$ldn = \left[1,\ -\frac{(r + a\cos(\theta)\,I)\,(r - a\cos(\theta)\,I)}{r^2 - 2\,M\,r + a^2},\ 0,\ -a\sin(\theta)^2\right],$$

$$\left.\overline{mdn} = \left[\frac{\frac{-1}{2}\,I\,a\sin(\theta)\,\sqrt{2}}{r - a\cos(\theta)\,I},\ 0,\ -\frac{1}{2}\,(r + a\cos(\theta)\,I)\,\sqrt{2},\ \frac{\frac{1}{2}\,I\,(r^2 + a^2)\sin(\theta)\,\sqrt{2}}{r - a\cos(\theta)\,I}\right]\right\}$$

Though we may not make explicit reference to it, GHPtools will be (and has been in earlier parts of this chapter) used extensively both to obtain and verify the result at hand.

CHAPTER 3
REGGE-WHEELER & TEUKOLSKY

As a first application of our framework, we will provide a more detailed discussion of the Regge-Wheeler and Teukolsky equations. This leads quite naturally to a discussion of the metric perturbation generated from a Hertz potential, which will play a major role in subsequent chapters. Our starting point is a general discussion of parity that does not assume either spherical symmetry or angular separation from the outset.

3.1 Parity Decomposition of Spin- and Boost-Weighted Scalars

One crucial feature of the Regge-Wheeler analysis is the identification of even and odd-parity modes. In the context of spherically symmetric backgrounds, where angular dependence can be separated off using spherical harmonics, it is sufficiently simple to achieve this decomposition by considering the behavior of the spherical harmonics under a parity transformation directly. For (scalar, vector or tensor) functions defined on more general 2-surfaces, this task can be cumbersome, if not outright impossible. Furthermore, narrowing our focus to angular functions obfuscates the fact that there is something more fundamental happening. It is the goal of this section to provide a more general description of the parity decomposition, applicable to more general 2-surfaces without appealing to separation of variables. We will also see that the GHP formalism is uniquely suited to this description. The decomposition theorems we make use of are proven by Detweiler and Whiting [50].

Our first assumption is that our spacetime manifold, M, admits a spacelike, closed 2-surface, $\mathcal{S}$, topologically a 2-sphere, with positive Gaussian curvature and a positive definite metric given by

$$\sigma_{ab} = -m_a\bar{m}_b - \bar{m}_a m_b, \tag{3–1}$$

where m_a and $\bar{m}_a$ are two members of a null tetrad. For a spherically symmetric background σ_{ab} is proportional to the metric of the (round) 2-sphere and m^a and $\bar{m}^a$ can be directly associated with the background metric. More generally, we allow for the possibility that m^a and $\bar{m}^a$ are in fact not the null vectors associated with the background spacetime, but rather just two (complex) null vectors tangent to $\mathcal{S}$. In that case l^a and n^a will then be identified with the two null directions orthogonal to $\mathcal{S}$. We will use the same notation regardless of whether the tetrad is globally defined or just in some neighborhood of $\mathcal{S}$ and the application at hand will dictate the appropriate interpretation. The metric's other role as the projector into $\mathcal{S}$ can be realized by simply raising one index

$$\sigma_a{}^b = -m_a\bar{m}^b - \bar{m}_a m^b. \tag{3–2}$$

For example consider some vector, v^a, defined in the spacetime:

$$v^a = v_l n^a + v_n l^a - v_m\bar{m}^a - v_{\bar{m}} m^a.$$

The restriction of v^a to $\mathcal{S}$ is simply given by the projection

$$\sigma_a{}^b v_b = -v_m\bar{m}_a - v_{\bar{m}} m_b,$$

which generalizes to n-indexed tensors by projecting each index individually. We can then carry out covariant differentiation within $\mathcal{S}$ by simply taking the full covariant derivative and projecting back into $\mathcal{S}$ with $\sigma_a{}^b$. The final object we introduce is the Levi-Civita symbol on $\mathcal{S}$, which, in tetrad language takes the following form:

$$\epsilon_{ab} \equiv \epsilon_{lnab} = i(m_a\bar{m}_b - \bar{m}_a m_b). \tag{3–3}$$

These are all the tools necessary for what follows.

We begin by considering the projection of vectors defined in the spacetime onto $\mathcal{S}$. To identify the odd and even parity pieces we start by decomposing a general vector on $\mathcal{S}$

$$\begin{aligned}\xi_a &= \sigma_a{}^b\Theta_b\xi_{\text{even}} + \epsilon_a{}^b\Theta_b\xi_{\text{odd}} \\ &= -m_a(\eth'\xi_{\text{even}} - i\eth'\xi_{\text{odd}}) - \bar{m}_a(\eth\xi_{\text{even}} + i\eth\xi_{\text{odd}})\end{aligned} \tag{3–4}$$

where ξ_{even} and ξ_{odd} are real spin-weight 0 scalars (type $\{b, b\}$; b indicating the boost-weight). Thus, given a quantity with boost-weight b and spin-weight 1, the even parity piece is simply $\eth\xi_{\text{even}}$ and the odd parity piece is $i\eth\xi_{\text{odd}}$[1] . Similarly the complex conjugate of such a quantity (same boost weight, but spin-weight -1) has even parity piece $\eth'\xi_{\text{even}}$ and odd parity piece $-i\eth'\xi_{\text{odd}}$. The relative minus sign between an odd-parity object and its complex conjugate is a possible source of confusion, so we must be careful when performing parity decompositions.

Symmetric, trace-free two-indexed tensors on $\mathcal{S}$ also have a simple parity decomposition. It is easy to recognize the (two) components of such tensors as spin-weight ± 2 scalars. That is, the components are of type $\{b \pm 2, b \mp 2\}$. We consider the parity decomposition on $\mathcal{S}$ by creating the tensor from a vector on $\mathcal{S}$,χ_a, with boost-weight b and spin-weight 0:

$$\chi_{ab} = \sigma_a{}^c\Theta_c\chi_b + \sigma_b{}^c\Theta_c\chi_a - \sigma_{ab}\sigma^{cd}\Theta_c\chi_d, \tag{3–5}$$

which can in turn be further decomposed into its even and odd parity pieces by applying Equation 3–4 to yield

$$\begin{aligned}\chi_{ab} &= (2\sigma_{(a}{}^c\sigma_{b)}{}^d\Theta_c\Theta_d - \sigma_{ab}\sigma^{cd}\Theta_c\Theta_d)\chi_{\text{even}} + 2\sigma_{(a}{}^c\epsilon_{b)}{}^d\Theta_c\Theta_d\chi_{\text{odd}} \\ &= 2m_{(a}m_{b)}(\eth'\eth'\chi_{\text{even}} - i\eth'\eth'\chi_{\text{odd}}) + 2\bar{m}_{(a}\bar{m}_{b)}(\eth\eth\chi_{\text{even}} + i\eth\eth\chi_{\text{odd}}),\end{aligned} \tag{3–6}$$

which provides us with a means of identifying the even and odd bits of symmetric trace-free tensors on $\mathcal{S}$. This result generalizes quite easily to n-indexed symmetric trace-free tensors (with components of spin-weight $\pm n$ and boost-weight b) on $\mathcal{S}$:

$$\tau_{a_1\ldots a_n} = (-1)^n n![m_{(a_1}\ldots m_{a_n)}(\eth'^n\tau_{\text{even}} - i\eth'^n\tau_{\text{odd}}) + \bar{m}_{(a_1}\ldots\bar{m}_{a_n)}(\eth^n\tau_{\text{even}} + i\eth^n\tau_{\text{odd}})]. \tag{3–7}$$

[1] This agrees with the correspondence between the even and odd parity vector and tensor spherical harmonics and the spin-weighted spherical harmonics (see Thorne's review [8] for details) (i.e., the "i" comes along for the ride).

Finally, we remark that scalars naturally arising from contractions of tensors in the spacetime with various combinations of l^a and n^a have no components in $\mathcal{S}$ and are thus all of even-parity. Note that such objects necessarily have zero spin-weight. This provides enough information to characterize the parity of arbitrary objects.

In practice, we are generally given some spin- and boost-weighted scalar, ψ (and/or its complex conjugate), and we merely want to identify the even- and odd-parity pieces without explicitly decomposing it according to Equation 3–7. In this case Equation 3–7 allows us to do so by simply writing

$$\psi = \psi_{\text{even}} + i\psi_{\text{odd}}. \tag{3–8}$$

In the context of a spacetime where l^a and n^a are fixed by considerations other than being orthogonal to m^a and $\bar{m}^a$ (e.g. Petrov type D, where we would like them aligned with the principal null directions), but m^a and $\bar{m}^a$ fail to form a closed 2-surface (the Kerr spacetime provides one such example; this can be seen by noting that ð and ð′ don't commute), the question arises of whether or not something like Equation 3–8 is still useful to consider. It appears so. In such a case the decomposition theorems (the first lines of Equations 3–5 and 3–4) fail to be true, but this isn't a serious issue. Because σ_{ab} and ϵ_{ab} still allow us to decompose tensors into their "proper" and "pseudo" pieces, in place of Equation 3–7 we have

$$\tau_{a_1 \ldots a_n} = (-1)^n n! [m_{(a_1} \ldots m_{a_n)} (\bar{\tau}_{\text{“even”}} - i\bar{\tau}_{\text{“odd”}}) + \bar{m}_{(a_1} \ldots \bar{m}_{a_n)} (\tau_{\text{“even”}} + i\tau_{\text{“odd”}})], \tag{3–9}$$

where "even" and "odd" are written in quotes to emphasize the fact that they really refer to real and imaginary in this context and the bar over tau indicates the proper spin- and boost-weight. Clearly, Equation 3–9, lacks the advantage present in Equation 3–7 of being able to put all of the angular dependence into ð and ð′ and regard the entire tensor as arising from the two real scalars τ_{even} and τ_{odd}. Nevertheless it provides a useful decomposition of spin- and boost-weighted scalars, without separation of variables, that allows us to use Equation 3–8 in arbitrary backgrounds. Furthermore in the limit that m^a and $\bar{m}^a$ become surface-forming (e.g. the $a \to 0$ limit of the Kerr spacetime), Equation 3–9 becomes Equation 3–7. This is one avenue for understanding why parity plays such an important role in the perturbation theory of spherically symmetric backgrounds. In the context of null tetrad formalisms we can see the seemingly unmotivated act of performing parity decomposition, which does not generalize well, as arising from the quite natural (and perhaps more fundamental) act of separating quantities into their real and imaginary parts, which is entirely general. In this light, it makes sense that our attention would be focused on parity because the first perturbative analysis took place in the spherically symmetric Schwarzschild background in which one cannot differentiate between the two decompositions but parity has significance there. Regardless, the only use we make of these results, except for some remarks in Chapter 5, is below in the case of the Schwarzschild background where the point is moot.

3.2 Regge-Wheeler

In this section we will provide a perturbative analysis equivalent to that of Regge and Wheeler for the odd-parity sector of the Schwarzschild spacetime. Though the results are well known, our methods and language are sufficiently different and original that they shed some new light on and bring an interesting perspective to the subject. The two keys to our analysis are essentially the same as those of RW: the parity decomposition and the RW gauge. Having already discussed the former, we will look now at the latter before proceeding with the analysis.

3.2.1 The Regge-Wheeler Gauge

Regge and Wheeler describe their gauge choice in terms of the ℓ-mode decomposition of a gauge vector. This description is inadequate for our purpose and so our first task is to translate the RW gauge into mode-independent form. This has been performed by Barack and Ori [24] who obtained

$$\sin^2\theta h_{\theta\theta} - h_{\phi\phi} = 0, \tag{3–10}$$

$$h_{\theta\phi} = 0, \tag{3–11}$$

$$\sin\theta\partial_\theta(\sin\theta h_{t\theta} + \partial_\phi h_{t\phi}) = 0, \tag{3–12}$$

$$\sin\theta\partial_\theta(\sin\theta h_{r\theta} + \partial_\phi h_{r\phi}) = 0, \tag{3–13}$$

as the mode-independent expression of the RW gauge. Now we can transform this description into GHP language. It is a relatively straightforward process now to write the tetrad components of the metric perturbation (h_{ll}, h_{ln}, etc.) in terms of the coordinate components of the metric perturbation (h_{tt}, h_{rr}, etc.) and invert the relations. With this knowledge in hand, it becomes evident that Equations 3–10 and 3–11 are simply combinations of

$$h_{mm} = 0 \qquad \text{and} \qquad h_{\bar{m}\bar{m}} = 0.$$

The effect of these conditions is to remove the spin-weight ± 2 pieces from the metric perturbation. After a quick look at the coordinate form of the ð and ð′ operators, we note that Equations 3–12 and 3–13 are combinations of

$$ð' h_{lm} + ð h_{l\bar{m}} = 0 \qquad \text{and} \qquad ð h_{n\bar{m}} + ð' h_{nm} = 0,$$

which restricts the form of the spin-weight ± 1 parts of the metric perturbation. Note that the essence of the RW gauge lies in the fact that all of the information about gravitational radiation gets pushed into the spin-1 components of the metric perturbation.

In this language, it is natural to generalize these conditions to more general type D spacetimes on the basis of spin-weight considerations. The spirit of the RW gauge suggests that we keep the requirement that no spin 2 components enter the metric perturbation. The requirement on the spin 1 components is easily generalizable by putting in pieces proportional to τ and τ' which both vanish in the Schwarzschild background.

The resulting proposal for a generalized RW gauge is

$$\begin{aligned}
h_{mm} &= 0, \\
h_{\bar{m}\bar{m}} &= 0, \\
(\eth + \bar{a}\tau + b\bar{\tau}')h_{l\bar{m}} + (\eth' + a\bar{\tau} + \bar{b}\tau')h_{lm} &= 0, \\
(\eth' + b\bar{\tau} + \bar{a}\tau')h_{nm} + (\eth + \bar{b}\tau + a\bar{\tau}')h_{n\bar{m}} &= 0,
\end{aligned} \tag{3–14}$$

where a and b are (generally complex) constants that must be determined by some other means. Note that the form of Equations 3–14 is restricted by requiring the gauge restrictions to be invariant under both prime and complex conjugation. The full utility of the generalized RW gauge remains to be explored, but it is clear that any simplification it brings will apply uniformly to all type D spacetimes.

3.2.2 The Regge-Wheeler Equation

With the pieces in place, we turn our attention to the odd-parity perturbations of the Schwarzschild spacetime. Starting with the description of the background, we have

$$\rho = \bar{\rho}, \quad \rho' = \bar{\rho}', \quad \text{and} \quad \psi_2 = \bar{\psi}_2, \tag{3–15}$$

with all other background quantities vanishing, so the situation is immediately simplified. Next we proceed with the parity decomposition by writing the components of the metric perturbation as, for example, $h_{lm} = h^{\text{even}}_{lm} + ih^{\text{odd}}_{lm}$, $h_{l\bar{m}} = h^{\text{even}}_{l\bar{m}} - ih^{\text{odd}}_{l\bar{m}}$, etc. Note the relative minus signs between the odd-parity bits and their complex conjugates. From here on we will specialize to odd-parity and thus drop the “odd” labels and factors of i since no confusion can arise. With this specialization, our gauge conditions now read:

$$\begin{aligned}
h_{mm} &= 0 \\
h_{\bar{m}\bar{m}} &= 0 \\
\eth' h_{lm} - \eth h_{l\bar{m}} &= 0 \\
\eth' h_{nm} - \eth h_{n\bar{m}} &= 0.
\end{aligned} \tag{3–16}$$

Putting these simplifications into the (odd-parity) Einstein equations, we see that

$$\mathcal{E}_{ll} = \mathcal{E}_{nn} = \mathcal{E}_{ln} = \mathcal{E}_{m\bar{m}} = 0,$$

is satisfied identically by virtue of the gauge conditions. Furthermore, we have that $\mathcal{E}_{lm} = -\mathcal{E}_{l\bar{m}}$ (and likewise for the nm and $n\bar{m}$ components), which is no surprise because, as a tensor, $\mathcal{E}_{ab}$ respects the parity decomposition. This leaves us with $\mathcal{E}_{lm}$, $\mathcal{E}_{nm}$ and $\mathcal{E}_{mm}$. Starting with the last piece, we can commute the derivatives to write

$$\mathcal{E}_{mm} = \eth((\text{Þ}' - \rho')h_{lm} + (\text{Þ} - \rho)h_{nm}) = 0,$$

which we can "integrate" by peeling off the $\eth$ to give us

$$\begin{aligned} (\text{Þ}' - \rho')h_{lm} + (\text{Þ} - \rho)h_{nm} &= 0 &\quad (3\text{–}17)\\ (\text{Þ}' - \rho')h_{l\bar{m}} + (\text{Þ} - \rho)h_{n\bar{m}} &= 0, &\quad (3\text{–}18)\end{aligned}$$

where the second relation follows from complex conjugation of the first (or integration of $\mathcal{E}_{\bar{m}\bar{m}}$), and we have set the "integration constant" to zero for convenience (it would cancel below). We now turn our attention to $\mathcal{E}_{lm}$. By successive applications of Equation 3–17 we can eliminate all terms involving $\text{Þ}h_{nm}$, arriving at

$$\mathcal{E}_{lm} = \frac{1}{2}\Big\{(\eth\eth' - 2\text{Þ}\text{Þ}' + 4\rho\text{Þ}' - 2\rho\rho' - 4\psi_2)h_{lm} - 2\rho^2 h_{nm}\Big\} \qquad (3\text{–}19)$$

Taking the prime of this (which introduces an overall minus sign because of the parity decomposition) leads to a similar expression for $\mathcal{E}_{n\bar{m}}$. Next we take the (sourcefree) combination

$$(\text{Þ}' - 2\rho')\eth'\mathcal{E}_{lm} + (\text{Þ} - 2\rho)\eth\mathcal{E}_{n\bar{m}} = 0. \qquad (3\text{–}20)$$

We can remove from this expression all references to $\eth h_{l\bar{m}}$ and $\eth' h_{nm}$ using the gauge conditions in Equations 3–16, which, after some serious commuting leads to the quite simple expression:

$$\{\text{Þ}'\text{Þ} - \eth'\eth - \rho'\text{Þ} - \rho\text{Þ}' \ + 4\psi_2\}\psi_2^{-2/3}(\text{Þ}\eth h_{n\bar{m}} - \text{Þ}'\eth' h_{lm}) = 0. \tag{3–21}$$

This is the Regge-Wheeler equation. We can clean it up a bit by recognizing the object being acted on as $2\dot{\psi}_2^{\text{odd}} = \text{Þ}\eth h_{n\bar{m}} - \text{Þ}'\eth' h_{lm}$, the odd-parity piece of the perturbation of ψ_2. Furthermore the operator in Equation 3–21 is the wave operator, $\Box$, in the Schwarzschild background up to a factor of 1/2. Making these identifications, we now have for the Regge-Wheeler equation:

$$(\Box + 8\psi_2)\psi_2^{-2/3}\dot{\psi}_2^{\text{odd}} = 0. \tag{3–22}$$

A similar equation for $\dot{\psi}_2^{\text{odd}} = \text{Im}(\dot{\psi}_2)$ was previously derived by Price [51] (whose only relation to the present author is Equation 3–22), who showed that (modulo angular dependence), $\text{Im}(\dot{\psi}_2)$ is the time derivative of the Regge-Wheeler variable. Moreover, without reference to $\text{Im}(\dot{\psi}_2)$ Jezierski [52] arrived at an equation for odd-parity perturbations that is essentially identical to Equation 3–22, though phrased in more standard language. Additionally, an analysis by Nolan [53] who looked at the perturbed Weyl scalars in terms of gauge invariants of the metric perturbation showed explicitly the relation between $\text{Im}\dot{\psi}_2$ and the gauge invariant quantity associated with the RW variable. Furthermore, Nolan points out that because ψ_2 is real in the background, the perturbation of its imaginary part is, when we restrict our attention to odd-parity, gauge invariant in the sense discussed in Chapter 2. Perhaps more surprisingly, Nolan further asserts that this is true of the perturbations of *all* the Weyl scalars, which emphasizes the fact that odd-parity perturbations of spherically symmetric spacetimes are obtainable by virtually any means.

One thing that sets our treatment of RW apart from others is our sparing use of spherical symmetry. The only place we make explicit use of it is in Equations 3–15, which defines the background GHP quantities. This certainly simplifies the subsequent calculations considerably, but fails to fully exploit the background symmetry. In particular, our implementation of the parity decomposition without separation of variables generalizes quite nicely to spacetimes where parity isn't a good symmetry because we didn't actually take the step of writing the components of the metric perturbation as spin-weight 0 scalars with the appropriate number of ð's or ð′'s. The fact that this process has eluded generalization to the Kerr spacetime has more to do with difficulties there than the particular techniques we utilized, which are fairly general. This stands in contrast to existing treatments that fully exploit spherical symmetry from the outset and are thus exclusively applicable in these situations.

The Zerilli equation [7] describing even-parity perturbations of the Schwarzschild spacetime has so far eluded a direct description in terms of gauge invariant perturbations of the Weyl scalars. However, the information contained within the Zerilli equation can be obtained through the metric perturbation that follows from the Teukolsky equation, which is the focus of the remainder of this chapter.

3.3 The Teukolsky Equation

In contrast to the RW equation, which has it origins in the description of metric perturbations, the Teukolsky equation [10–12] came directly from considering perturbations of the Weyl scalars. We, however, are interested in obtaining it directly from the Einstein equation. Using Teukolsky's expressions for the sources of ψ_0 and ψ_4, we can obtain this directly. The sources of the Teukolsky equation are given by

$$\begin{aligned} T_0 &= (ð - \bar{\tau}' - 4\tau)[(Þ - 2\bar{\rho})\mathcal{T}_{lm} - (ð - \bar{\tau}')\mathcal{T}_{ll}] \\ &\quad +(Þ - 4\rho - \bar{\rho})[(ð - 2\bar{\tau}')\mathcal{T}_{lm} - (Þ - \bar{\rho})\mathcal{T}_{mm}], \end{aligned} \tag{3–23}$$

$$\begin{aligned} T_4 &= (ð' - \bar{\tau} - 4\tau')[(Þ' - 2\bar{\rho}')\mathcal{T}_{n\bar{m}} - (ð' - \bar{\tau})\mathcal{T}_{nn}] \\ &\quad +(Þ' - 4\rho' - \bar{\rho}')[(ð' - 2\bar{\tau})\mathcal{T}_{n\bar{m}} - (Þ' - \bar{\rho}')\mathcal{T}_{\bar{m}\bar{m}}], \end{aligned} \tag{3–24}$$

where T_0 and T_4 are the sources for ψ_0 and ψ_4, respectively. Making the replacement $\mathcal{T}_{ab} = \frac{1}{8\pi}\mathcal{E}_{ab}$ in the equations above leads (after properly rearranging the derivatives with the help of GHPtools) to the Teukolsky equations. They are

$$[(\text{Þ} - 4\rho - \bar{\rho})(\text{Þ}' - \rho') - (\eth - 4\tau - \bar{\tau}')(\eth' - \tau') - 3\psi_2]\psi_0 = 4\pi T_0, \quad (3\text{–}25)$$

$$[(\text{Þ}' - 4\rho' - \bar{\rho}')(\text{Þ} - \rho) - (\eth' - 4\tau' - \bar{\tau})(\eth - \tau) - 3\psi_2]\psi_4 = 4\pi T_4, \quad (3\text{–}26)$$

where, in terms of the components of the metric perturbation

$$\psi_0 = \frac{1}{2}\Big\{(\eth - \bar{\tau}')(\eth - \bar{\tau}')h_{ll} + (\text{Þ} - \bar{\rho})(\text{Þ} - \bar{\rho})h_{mm} - [(\text{Þ} - \bar{\rho})(\eth - 2\bar{\tau}') + (\eth - \bar{\tau}')(\text{Þ} - 2\bar{\rho})]h_{(lm)}\Big\}, \quad (3\text{–}27)$$

$$\psi_4 = \frac{1}{2}\Big\{(\eth' - \bar{\tau})(\eth' - \bar{\tau})h_{nn} + (\text{Þ}' - \bar{\rho}')(\text{Þ}' - \bar{\rho}')h_{\bar{m}\bar{m}} - [(\text{Þ}' - \bar{\rho}')(\eth' - 2\bar{\tau}) + (\eth' - \bar{\tau})(\text{Þ}' - 2\bar{\rho}')]h_{(n\bar{m})}\Big\}, \quad (3\text{–}28)$$

and where the parentheses, (), around the tetrad indices denote symmetrization. It is both interesting and important to note that, in the Kerr spacetime, the coordinate description of Equation 3–26 does not lead to the separable equation discussed in Chapter 1 (Equation 1–17). To obtain a separable equation, an extra factor of $\psi_2^{-4/3}$ must be brought in, resulting in the following expression:

$$[(\text{Þ}' - \bar{\rho}')(\text{Þ} + 3\rho) - (\eth' - \bar{\tau})(\eth + 3\tau) - 3\psi_2]\psi_2^{-4/3}\psi_4 = 4\pi\psi_2^{-4/3}T_4. \quad (3\text{–}29)$$

Below we will see the same expression arising from very different considerations.

3.4 Metric Reconstruction from Weyl Scalars

The solutions of the Teukolsky equation lead quite naturally to a metric perturbation in several different ways. The original result, due to Cohen and Kegeles [20] used spinor methods. Shortly after that, Chrzanowski [54] obtained essentially the same result using factorized Green's functions. Some time later, Stewart [21] entered the game and provided a new derivation rooted in spinor methods. Eventually, Wald [55] introduced a comparatively simple derivation of the same result. This is the approach we will follow here.

Wald's method is centered around the notion of adjoints. Consider some linear differential operator, $\mathcal{L}$, that takes n-index tensor fields into m-index tensor fields. Its adjoint, $\mathcal{L}^\dagger$, which takes m-index tensor fields into n-index tensor fields is defined by

$$\alpha^{a_1\ldots a_m}(\mathcal{L}\beta)_{a_1\ldots a_m} - (\mathcal{L}^\dagger\alpha)^{b_1\ldots b_n}\beta_{b_1\ldots b_n} = \nabla_a s^a, \tag{3–30}$$

for some tensor fields $\alpha^{a_1\ldots a_m}$ and $\beta^{b_1\ldots b_n}$ and some vector field s^a. If $\mathcal{L}^\dagger = \mathcal{L}$, then $\mathcal{L}$ is self-adjoint. An important property of adjoints is that for two linear operators, $\mathcal{L}$ and $\mathcal{M}$, $(\mathcal{L}\mathcal{M})^\dagger = \mathcal{M}^\dagger\mathcal{L}^\dagger$. Now let $\mathcal{E} = \mathcal{E}(h_{ab})$ denoted the linear Einstein operator, $\mathcal{S}$ the operator that gives either of the Teukolsky equations from $\mathcal{E}$ (Equation 3–23 or 3–24), $\mathcal{O} = \mathcal{O}(\psi_0 \text{ or } \psi_4)$ the source-free Teukolsky operator (Equation 3–25 or 3–29) and $\mathcal{T} = \mathcal{T}(h_{ab})$ the operator that acts on the metric perturbation to give ψ_0 or ψ_4 (Equation 3–23 or 3–24). Then the Teukolsky equations can be written concisely as

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}. \tag{3–31}$$

It follows by taking the adjoint that

$$\mathcal{E}^\dagger\mathcal{S}^\dagger = \mathcal{E}\mathcal{S}^\dagger = \mathcal{T}^\dagger\mathcal{O}^\dagger, \tag{3–32}$$

where we have used the fact that the perturbed Einstein equations are self-adjoint. Thus, if Ψ satisfies $\mathcal{O}^\dagger\Psi = 0$, then $\mathcal{S}^\dagger\Psi$ is a solution to the perturbed Einstein equations! This remarkably simple and elegant result holds for any system having the form of Equation 3–31, whenever $\mathcal{E}$ is self-adjoint.

In order to apply this result to the Teukolsky equation we note that scalars are all self-adjoint and the adjoints of the GHP derivatives are given by

$$\begin{aligned} Þ^\dagger &= -(Þ - \rho - \bar{\rho}), & (Þ')^\dagger &= -(Þ' - \rho' - \bar{\rho}'), \\ ð^\dagger &= -(ð - \tau - \bar{\tau}'), & (ð')^\dagger &= -(ð' - \tau' - \bar{\tau}). \end{aligned} \tag{3–33}$$

We may express this more concisely by introducing $\mathcal{D} = \{\text{Þ}, \text{Þ}', \eth, \eth'\}$, so that

$$\mathcal{D}^\dagger = -(\psi_2\bar{\psi}_2)^{1/3}\mathcal{D}(\psi_2\bar{\psi}_2)^{-1/3}. \tag{3–34}$$

Suppose now that we have a solution to the Teukolsky equation for ψ_0, so that $\mathcal{O}$ is given by the left side of Equation 3–25 and $\mathcal{S}$ is given by the right side of Equation 3–23 (with T_{ab} replaced with $\mathcal{E}_{ab}$) . Wald's method then tells us that if $\mathcal{O}^\dagger\Psi = 0$, then $h_{ab} = \mathcal{S}^\dagger\Psi$ is a solution to the perturbed Einstein equations. Using Equations 3–33 we can compute $\mathcal{S}^\dagger\Psi$:

$$\begin{aligned} h_{ab} \;\; &= \;\; \{l_a l_b(\eth - \tau)(\eth + 3\tau) - l_{(a}m_{b)}[(\text{Þ} - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\text{Þ} + 3\rho)] \\ &\quad + m_a m_b(\text{Þ} - \rho)(\text{Þ} + 3\rho)\}\Psi + \text{c.c.}, \end{aligned} \tag{3–35}$$

where we've added the complex conjugate (c.c.) to make the metric perturbation real and Ψ remains to be specified. Using Equations 3–33, it is clear that the adjoint of Equation 3–25 is

$$[(\text{Þ}' - \bar{\rho}')(\text{Þ} + 3\rho) - (\eth' - \bar{\tau})(\eth + 3\tau) - 3\psi_2]\Psi = 0, \tag{3–36}$$

which is precisely the equation satisfied by $\psi_2^{-4/3}\psi_4$ (c.f. Equation 3–29), previously obtained through separability considerations in the Kerr spacetime. However, obtaining Equation 3–36 required no reference to separation of variables in a particular spacetime and thus applies to all type D spacetimes. It is important to note that although Ψ satisfies the same equation as $\psi_2^{-4/3}\psi_4$, it is not the perturbation of ψ_4 for the metric it generates (Equation 3–35). In Chapter 5 we will explore Ψ's connection to ψ_4 more carefully.

Though the derivation of Equation 3–35 was quite simple, it fails to yield any information about the gauge in which the metric perturbation exists. In this particular instance, it is fairly straightforward to verify that the metric perturbation we've been led to obeys

$$l^a h_{ab} = 0, \tag{3–37}$$

$$g^{ab} h_{ab} = 0, \tag{3–38}$$

which is known in the literature as the ingoing radiation gauge (IRG), an unfortunate name because ingoing radiation is carried by l^a and Equation 3–37 tells us that the metric perturbation is completely orthogonal to l^a. Thus there is only outgoing radiation in the ingoing radiation gauge! Obtaining the gauge conditions in Equations 3–37 and 3–38 is more natural in the approaches of Cohen and Kegeles [20] and Stewart [21]. One startling aspect of the gauge conditions is that there are five of them. This being the case, we must be concerned about the circumstances under which the metric perturbation in the IRG is well-defined. This is the subject of the next chapter.

Our derivation began with the Teukolsky equation for ψ_0. Had we instead started with the Teukolsky equation for $\psi_2^{-4/3}\psi_4$, we would be led to a metric perturbation in terms of a Hertz potential, Ψ', that satisfies the Teukolsky equation for ψ_0. The resulting metric perturbation and gauge conditions are then simply the GHP prime of Equations 3–35, 3–37 and 3–38, respectively. In this case, the metric perturbation exists in the so-called outgoing radiation gauge (ORG). For the remainder of this work, we will focus our attention on the IRG metric perturbation, but all the results hold for the ORG perturbation as well.

On a final note we remark that the Teukolsky equation for ψ_0 (Equation 3–25) actually exists in the more general type II spacetimes, without its companion for ψ_4. In this case, Wald's method also leads to metric perturbation (in the IRG; no ORG exists here), with a potential, Ψ, satisfying the adjoint of Equation 3–25, which, in this instance, is not the equation for the perturbation of ψ_4.

CHAPTER 4
THE EXISTENCE OF RADIATION GAUGES

In the previous chapter, it was seen that the perturbations of the Weyl scalars lead quite naturally to metric perturbations in the radiation gauges, (seemingly over-) specified by five conditions. In this chapter we will explore the precise circumstances under which one can impose all five of these conditions. This will require us to examine the perturbed Einstein tensor, which presents the need to integrate some of the components. For this, we will appeal to a coordinate-free integration technique based on the GHP formalism, due to Held [45, 56]. The generality of these methods allow us to prove the result for a much broader class of spacetimes than we have encountered so far, namely, Petrov type II, which we will see is the largest class of spacetime for which the radiation gauges are defined. We begin with a more thorough discussion of the radiation gauges and their origin. Most of this chapter is taken from published work [57].

4.1 The Radiation Gauges

The ingoing radiation gauge (IRG) is a crucial ingredient for the reconstruction of metric perturbations of Petrov type D spacetimes from curvature perturbations. They first appear, unexplained, in the work of Cohen and Kegeles [58] (for perturbations of Petrov type II spacetimes) and Chrzanowski [54] (who considered perturbations of Petrov type D spacetimes), but the work that comes closest to our contribution in describing their origin is that of Stewart [21], again for the more general case of type II spacetimes.

In type II background spacetimes, the IRG is defined by the conditions

$$l^a h_{ab} = 0, \tag{4–1}$$

$$g^{ab} h_{ab} = 0, \tag{4–2}$$

where l^a is aligned with the repeated PND of the background Weyl tensor. If n^a rather than l^a is a repeated PND, we instead define the outgoing radiation gauge (ORG) by

$$n^a h_{ab} = 0, \tag{4–3}$$

$$g^{ab} h_{ab} = 0. \tag{4–4}$$

In type II background spacetimes, only one or the other of these options exists (IRG or ORG), whereas in Petrov type D background spacetimes, there is the possibility of defining both gauges. For the remainder of this work we focus on the IRG. Results for the ORG can be obtained by making the replacement $l^a \leftrightarrow n^a$.

Equations 4–1 translate into algebraic conditions on the components of the metric perturbation. We will refer to the four conditions in (4–1) as the $l \cdot h$ gauge conditions.[1] In terms of the tetrad components of the metric perturbation, these gauge conditions read:

$$h_{ll} = 0, \qquad h_{ln} = 0, \qquad h_{lm} = 0, \qquad h_{l\bar{m}} = 0. \tag{4–5}$$

The condition in Equation 4–2 will be referred to as the trace condition and can be expressed in terms of the components of the metric perturbation as $h_{ln} - h_{m\bar{m}} = 0$, which, when Equation 4–5 is imposed, simply reads

$$h_{m\bar{m}} = 0. \tag{4–6}$$

Because the IRG constitutes a total of five conditions on the metric perturbation, instead of the four one might expect for a gauge condition, it is necessary to ensure that the extra condition does not interfere with any physical degree of freedom in the problem,

[1] Recently, when applied specifically to the Schwarzschild spacetime, these conditions were given a geometrical interpretation, and referred to as *light-cone gauge conditions* [59], though they are not the conditions originally introduced for gravitation with that name [60]. It may well be that this description is suitable more generally, although presumably without the specific geometrical interpretation of [59].

such as one coming from a source. The importance of this consideration can be seen immediately from Equation B–1 of Appendix B, in which every term would be removed by Equations 4–5 and 4–6, rendering Equation B–1 inoperable whenever it has a non-zero source. In the next section we will look to the perturbed Einstein equations to determine the circumstances under which we can safely impose all five of the conditions that constitute the IRG.

It is useful to note the similarity between the full IRG, (4–1), and the more commonly known transverse traceless (TT) gauge defined by

$$\nabla^a h_{ab} = 0, \qquad g^{ab} h_{ab} = 0, \tag{4–7}$$

which, at a glance, also appears to be over-specified. In fact, the TT gauge exists for any vacuum perturbation of an arbitrary, globally hyperbolic, vacuum solution [61], because imposing the differential part of the gauge does not exhaust all of the available gauge freedom. Interestingly enough, Stewart's analysis in terms of Hertz potentials [21] begins by considering a metric perturbation in the TT gauge. However, in order to construct the curved space analogue of a Hertz potential, he is compelled to perform a transformation that destroys Equation 4–7 and instead yields a metric perturbation in the IRG.[2] Furthermore it appears that the restriction to type II spacetimes is essential for Stewart's analysis. From these observations, we expect radiation gauges to exist under conditions less general than those required for the existence of the TT gauge. At the same time, we should not be surprised that the IRG inherits the feature of *residual gauge freedom*.

Consider a gauge transformation on the metric perturbation generated by a gauge vector, ξ_a. To create a transformed metric in the $l \cdot h$ gauge, the gauge conditions in

[2] See [21] or the electromagnetic example in Chapter 1 for a more detailed explanation.

Equations 4–5 require

$$l^a(h_{ab} - \xi_{(a;b)}) = 0, \tag{4–8}$$

where the semicolon denotes the covariant derivative. In terms of components this reads

$$\begin{aligned} 2\text{Þ}\xi_l &= h_{ll}, \\ \text{Þ}'\xi_l + \text{Þ}\xi_n + (\tau + \bar{\tau}')\xi_{\bar{m}} + (\bar{\tau} + \tau')\xi_m &= h_{ln}, \\ (\text{Þ} + \bar{\rho})\xi_m + (\eth + \bar{\tau}')\xi_l &= h_{lm}, \\ (\text{Þ} + \rho)\xi_{\bar{m}} + (\eth' + \tau')\xi_l &= h_{l\bar{m}}. \end{aligned} \tag{4–9}$$

Similarly, for the trace condition in Equation 4–6 to be satisfied by the gauge transformed metric, we require

$$\eth'\xi_m + \eth\xi_{\bar{m}} + (\rho' + \bar{\rho}')\xi_l + (\rho + \bar{\rho})\xi_n = h_{m\bar{m}}. \tag{4–10}$$

Any extra gauge transformation that satisfies $l^a\xi_{(a;b)} = 0$—solves the homogeneous form of Equation 4–9—preserves the four $l\cdot h$ gauge conditions in Equations 4–5. This is what is meant by residual gauge freedom. We will explicitly use this residual gauge freedom to impose the $l\cdot h$ and trace conditions simultaneously, thus establishing the IRG. We will find that some gauge freedom still remains, as explained in Section 4.3.

Now, we turn our attention to the general case of type II background spacetimes.

4.2 Imposing the IRG in type II

In order to show that residual gauge freedom can be used to impose the IRG, we need to solve for the residual gauge freedom as well as examine any perturbed Einstein equation that might impede the imposition of the trace condition of the IRG. For this, we turn to a coordinate-free integration method develop by Held. Rather than give a detailed explanation, we present the basics and refer the interested reader to the literature for an in-depth account [45, 46].

The first step is to introduce new derivative operators $\tilde{\text{Þ}}'$, $\tilde{\eth}$ and $\tilde{\eth}' = \bar{\tilde{\eth}}$ such that they commute with Þ when acting on quantities that Þ annihilates,[3] :

$$[\text{Þ}, \tilde{\text{Þ}}']x^\circ = 0, \qquad [\text{Þ}, \tilde{\eth}\,]x^\circ = 0, \qquad [\text{Þ}, \tilde{\eth}']x^\circ = 0, \tag{4–11}$$

where $[a, b]$ denotes the commutator between a and b. The explicit form of the operators is given in Appendix C. The next step, the heart of Held's method, is to exploit the GHP equation $\text{Þ}\rho = \rho^2$, and its complex conjugate, $\text{Þ}\bar{\rho} = \bar{\rho}^2$, to express everything as a polynomial in terms of ρ and $\bar{\rho}$, with coefficients that are annihilated by Þ. Held's method is then brought to completion by choosing four independent quantities to use as coordinates [56, 62]. In this work, we will not take this extra step. For type II spacetimes (and the accelerating C-metrics), this step has not been carried out, while for all remaining type D spacetimes, it has been carried through to completion [45, 46].

In a spacetime more general than type II, there is no possibility of having a repeated PND. When a repeated PND exists, we can appeal to the Goldberg-Sachs theorem [32] and set $\kappa = \sigma = \Psi_0 = \Psi_1 = 0$ in Equations B-1–B-7. Following Held's partial integration of Petrov type II backgrounds [56], we also perform a null rotation (keeping l^a fixed, but changing n^a) to set $\tau = 0$. As a consequence, it follows from the GHP equations that $\tau' = 0$. Now we are in a position to address the question of when the full IRG can be imposed. First we apply the $l{\cdot}h$ gauge conditions in Equations 4–5 to Equations B-1–B-7. While most of the perturbed Einstein equations depend on several components of the metric perturbation, after imposing Equations 4–5, the expression for $\mathcal{E}_{ll}$ depends only on $h_{m\bar{m}}$ and the ll-component of the perturbed Einstein tensor simply becomes

$$\{\text{Þ}(\text{Þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{m\bar{m}} \equiv \{(\text{Þ} - 2\rho)(\text{Þ} + \rho - \bar{\rho})\}h_{m\bar{m}} = 8\pi \mathcal{T}_{ll}, \tag{4–12}$$

[3] Such quantities are denoted with the degree mark, $^\circ$, as in $\text{Þ}x^\circ = 0$.

in which the first form indicates that the equation is real, while the second form and its complex conjugate (which follow from the fact that $\text{Þ}\rho = \rho^2$ and $\text{Þ}\bar{\rho} = \bar{\rho}^2$) is the one we will use to integrate the equation below. If we had not made use of the Goldberg-Sachs theorem, there would be terms such as $\sigma\rho h_{\bar{m}\bar{m}}$ appearing in Equation 4–12 and our argument would not hold. We immediately see that $\mathcal{T}_{ll} = 0$ is necessary to satisfy the trace condition in Equation 4–6. Next we turn our attention to the question of whether the condition $\mathcal{E}_{ll} = 0$, is sufficient to impose Equation 4–6 using residual gauge freedom.

In order to address this question we will integrate $\mathcal{E}_{ll} = 0$ and the residual gauge vector, given by the homogeneous form of Equations 4–9. Full integration of the homogeneous form of Equations 4–9 is carried out in Appendix C, but we will work through the integration of $\mathcal{E}_{ll} = 0$ here to illustrate the method. We begin by rewriting Equation 4–12, with the help of $\text{Þ}\rho = \rho^2$ and its complex conjugate, as:

$$\{(\text{Þ} - 2\rho)(\text{Þ} + \rho - \bar{\rho})\}h_{m\bar{m}} = \rho^2\text{Þ}\left[\frac{\bar{\rho}}{\rho^3}\text{Þ}\left(\frac{\rho}{\bar{\rho}}h_{m\bar{m}}\right)\right] = 0. \tag{4–13}$$

Integrating once gives

$$\text{Þ}\left(\frac{\rho}{\bar{\rho}}h_{m\bar{m}}\right) = b^\circ\frac{\rho^3}{\bar{\rho}}, \tag{4–14}$$

and another integration leads to

$$h_{m\bar{m}} = \bar{a}^\circ\frac{\bar{\rho}}{\rho} + \frac{1}{2}b^\circ(\rho + \bar{\rho}). \tag{4–15}$$

However, $h_{m\bar{m}}$ is, by definition, a real quantity, so we add the complex conjugate and use b° to represent a real quantity in the second term. The final result is that

$$h_{m\bar{m}} = a^\circ\frac{\rho}{\bar{\rho}} + \bar{a}^\circ\frac{\bar{\rho}}{\rho} + b^\circ(\rho + \bar{\rho}). \tag{4–16}$$

Similarly, integration of Equations 4–9, as carried out in Appendix C, leads to the following solution for the components of the residual gauge vector:

$$\begin{aligned}
\xi_l &= \xi_l{}^\circ, \\
\xi_n &= \xi_n{}^\circ + \frac{1}{2}\Big(\frac{1}{\rho} + \frac{1}{\bar\rho}\Big)\tilde{\text{Þ}}'\xi_l{}^\circ + \frac{1}{2}(\Psi_2{}^\circ\rho + \bar\Psi_2^\circ\bar\rho)\xi_l{}^\circ, \\
\xi_m &= \frac{1}{\bar\rho}\xi_m{}^\circ - \tilde{\eth}\xi_l{}^\circ, \\
\xi_{\bar m} &= \frac{1}{\rho}\xi_{\bar m}{}^\circ - \tilde{\eth}'\xi_l{}^\circ,
\end{aligned} \tag{4–17}$$

where $\Psi_2{}^\circ$ is related to the background curvature via $\Psi_2 = \Psi_2{}^\circ\rho^3$. In order to use this residual gauge freedom to impose the full IRG, we return to the gauge transformation for $h_{m\bar m}$ (Equation 4–10) which becomes, after some manipulation (using Equations C–6–C–9 and Equation C–13),

$$h_{m\bar m} = \frac{\rho}{\bar\rho}\Big[\tilde{\eth}'\xi_m{}^\circ + \tilde{\text{Þ}}'\xi_l{}^\circ\Big] + \frac{\bar\rho}{\rho}\Big[\tilde{\eth}\xi_{\bar m}{}^\circ + \tilde{\text{Þ}}'\xi_l{}^\circ\Big] + (\rho+\bar\rho)[-\frac{1}{2}(\tilde{\eth}'\tilde{\eth} + \tilde{\eth}\tilde{\eth}' - \rho'^\circ - \bar\rho'^\circ)\xi_l{}^\circ + \xi_n{}^\circ]. \tag{4–18}$$

In this form it is clear that we can impose the trace condition (Equation 4–6) of the full IRG if we choose our gauge vector so that

$$\tilde{\eth}'\xi_m{}^\circ + \tilde{\text{Þ}}'\xi_l{}^\circ = a^\circ, \qquad -\frac{1}{2}(\tilde{\eth}'\tilde{\eth} + \tilde{\eth}\tilde{\eth}' - \rho'^\circ - \bar\rho'^\circ)\xi_l{}^\circ + \xi_n{}^\circ = b^\circ. \tag{4–19}$$

We have now shown by construction that the condition $\mathcal{T}_{ll} = 0$ is both necessary and sufficient for imposing the full IRG in a type II background. We turn next to discussing the complete extent of the residual gauge freedom in more detail.

4.3 Remaining Gauge Freedom

Although Equations 4–19 involve three real degrees of freedom (a° is complex), it turns out that only two real degrees of gauge freedom are required to fully remove any solution of Equation 4–13 for the trace $h_{m\bar m}$. To see this we introduce the following identity:

$$\frac{\rho}{\bar\rho} - \frac{\bar\rho}{\rho} = (\rho+\bar\rho)\Big(\frac{1}{\bar\rho} - \frac{1}{\rho}\Big) \equiv (\rho+\bar\rho)\Omega^\circ, \tag{4–20}$$

which also defines Ω°, a quantity annihilated by Þ. Then we can rewrite Equation 4–16 as

$$h_{m\bar{m}} = \frac{1}{2}(a^\circ + \bar{a}^\circ)\Big(\frac{\rho}{\bar{\rho}} + \frac{\bar{\rho}}{\rho}\Big) + [\frac{1}{2}(a^\circ - \bar{a}^\circ)\Omega^\circ + b^\circ](\rho + \bar{\rho}). \tag{4–21}$$

In a similar fashion, we rewrite Equation 4–18 as

$$\begin{aligned} h_{m\bar{m}} = {} & \Big[\frac{1}{2}(\tilde{\eth}'\xi_m{}^\circ + \tilde{\eth}\xi_{\bar{m}}{}^\circ) + \tilde{\text{Þ}}'\xi_l{}^\circ\Big]\Big(\frac{\rho}{\bar{\rho}} + \frac{\bar{\rho}}{\rho}\Big) + \Big[\frac{1}{2}(\tilde{\eth}'\xi_m{}^\circ - \tilde{\eth}\xi_{\bar{m}}{}^\circ)\Omega^\circ \\ & - \frac{1}{2}(\tilde{\eth}'\tilde{\eth} + \tilde{\eth}\tilde{\eth}' - \rho'^\circ - \bar{\rho}'^\circ)\xi_l{}^\circ + \xi_n{}^\circ\Big](\rho + \bar{\rho}), \end{aligned} \tag{4–22}$$

in which each coefficient in big square brackets is purely real. Now, suppose we have a particular solution for $\mathcal{E}_{ll} = 0$ (i.e., a°, $\bar{a}^\circ$ and b° are fixed) and our task is to solve for the components of the gauge vector which removes this solution. By comparing Equations 4–21 and 4–22 we see that, for any given $\xi_m{}^\circ$ and $\xi_{\bar{m}}{}^\circ$, we can fix $\xi_l{}^\circ$ (up to a solution of $\tilde{\text{Þ}}'\xi_l{}^\circ = 0$) via

$$\tilde{\text{Þ}}'\xi_l{}^\circ = \frac{1}{2}(a^\circ + \bar{a}^\circ) - \frac{1}{2}(\tilde{\eth}'\xi_m{}^\circ + \tilde{\eth}\xi_{\bar{m}}{}^\circ), \tag{4–23}$$

and we can fix $\xi_n{}^\circ$ by setting

$$\xi_n{}^\circ = \frac{1}{2}(a^\circ - \bar{a}^\circ)\Omega^\circ + b^\circ + \frac{1}{2}(\tilde{\eth}'\tilde{\eth} + \tilde{\eth}\tilde{\eth}' - \rho'^\circ - \bar{\rho}'^\circ)\xi_l{}^\circ - \frac{1}{2}(\tilde{\eth}'\xi_m{}^\circ - \tilde{\eth}\xi_{\bar{m}}{}^\circ)\Omega^\circ, \tag{4–24}$$

to completely eliminate the nonzero $h_{m\bar{m}}$, thus imposing the full IRG while still leaving two completely unconstrained degrees of gauge freedom, $\xi_m{}^\circ$ and $\xi_{\bar{m}}{}^\circ$. Once in the IRG, Equations 4–23 and 4–24, with a°, $\bar{a}^\circ$ and b° set to zero and $\xi_m{}^\circ$ and $\xi_{\bar{m}}{}^\circ$ arbitrary, give the remaining components of a gauge vector preserving the IRG. It is currently unclear how to take advantage of this remaining gauge freedom to simplify the analysis of perturbations in the full IRG.

4.4 Imposing the IRG in type D

Type D background metrics are of considerable theoretical and observational interest since they include both the Schwarzschild and Kerr black hole spacetimes. Kinnersley first obtained all type D metrics by integrating the Newman-Penrose equations [33]. While the results of the previous section are general enough to encompass the special case of type D backgrounds, the tetrad choice we made (with $\tau = 0$) is incompatible with the complete integration of the background field equations which is possible in type D spacetimes [45]. The complete integration requires that each of l^a and n^a be aligned with one of the two PNDs. In that case we can exploit the full power of the Goldberg-Sachs theorem and its corollaries to set $\kappa = \kappa' = \sigma = \sigma' = \Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$, while maintaining $\tau \neq 0$ and $\tau' \neq 0$. In this section we repeat the previous calculation with this different choice of tetrad.

The result of integrating $\mathcal{E}_{ll} = 0$ is the same as in the case of a type II background, given in Equation 4–16. The residual gauge vector, however, now has the following, more complex, form (details of the integration are given in Appendix C):

$$\begin{aligned}
\xi_l &= \xi_l{}^\circ, \\
\xi_n &= \xi_n{}^\circ + \frac{1}{2}\Psi^\circ \xi_l{}^\circ \rho + \frac{1}{2}\bar{\Psi}^\circ \xi_l{}^\circ \bar{\rho} + \tau^\circ \bar{\tau}^\circ \xi_l{}^\circ \rho\bar{\rho} + \frac{1}{2}\pi^\circ \bar{\pi}^\circ \xi_l{}^\circ \Big(\frac{1}{\rho^2} + \frac{1}{\bar{\rho}^2}\Big) \\
&\quad + \Big[\frac{\pi^\circ}{\rho}(\tilde{\eth} + \bar{\alpha}^\circ) + \frac{\bar{\pi}^\circ}{\bar{\rho}}(\tilde{\eth}' + \alpha^\circ)\Big]\xi_l{}^\circ + \frac{1}{2}\Big(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\Big)\tilde{\text{Þ}}'\xi_l{}^\circ \\
&\quad - [\bar{\tau}^\circ \rho(\tilde{\eth} + \bar{\alpha}^\circ) + \tau^\circ \bar{\rho}(\tilde{\eth}' + \alpha^\circ)]\xi_l{}^\circ + \bar{\tau}^\circ \xi_m{}^\circ \frac{\rho}{\bar{\rho}} + \tau^\circ \xi_{\bar{m}}{}^\circ \frac{\bar{\rho}}{\rho} \\
&\quad - \pi^\circ \xi_m{}^\circ \frac{1}{\bar{\rho}^2} - \bar{\pi}^\circ \xi_{\bar{m}}{}^\circ \frac{1}{\rho^2} - \alpha^\circ \xi_m{}^\circ \frac{1}{\bar{\rho}} - \bar{\alpha}^\circ \xi_{\bar{m}}{}^\circ \frac{1}{\rho}, \\
\xi_m &= \xi_m{}^\circ \frac{1}{\bar{\rho}} - \bar{\pi}^\circ \xi_l{}^\circ \frac{1}{\rho} + \tau^\circ \xi_l{}^\circ \bar{\rho} - (\tilde{\eth} + \bar{\alpha}^\circ)\xi_l{}^\circ, \\
\xi_{\bar{m}} &= \xi_{\bar{m}}{}^\circ \frac{1}{\rho} - \pi^\circ \xi_l{}^\circ \frac{1}{\bar{\rho}} + \bar{\tau}^\circ \xi_l{}^\circ \rho - (\tilde{\eth}' + \alpha^\circ)\xi_l{}^\circ,
\end{aligned} \tag{4–25}$$

where the quantities Ψ°, τ°, π° and α° determine properties of the background spacetime.[4] Now the gauge transformation for $h_{m\bar{m}}$ becomes

$$
\begin{aligned}
h_{m\bar{m}} = \frac{\rho}{\bar{\rho}}\Big[\tilde{ð}'\xi_m{}^\circ + \tilde{\text{Þ}}'\xi_l{}^\circ - B^\circ\Big] + \frac{\bar{\rho}}{\rho}\Big[\tilde{ð}\xi_{\bar{m}}{}^\circ + \tilde{\text{Þ}}'\xi_l{}^\circ + B^\circ\Big] \\
+ (\rho + \bar{\rho})\Big[-\frac{1}{2}(\tilde{ð}'\tilde{ð} + \tilde{ð}\tilde{ð}' - \rho'^\circ - \bar{\rho}'^\circ)\xi_l{}^\circ + \xi_n{}^\circ - A^\circ\Big],
\end{aligned}
\tag{4–26}
$$

where we have introduced (note that B° is purely imaginary)

$$
\begin{aligned}
A^\circ &= \frac{1}{2}\{2\alpha^\circ\tilde{ð} + \tilde{ð}(\alpha^\circ) + \alpha^\circ\bar{\alpha}^\circ - \pi^\circ\tau^\circ\}\xi_l{}^\circ + \text{c.c.}, \\
B^\circ &= \tfrac{1}{4}\{4\pi^\circ\tilde{ð} + \tilde{ð}(\pi^\circ) + 5\bar{\alpha}^\circ\pi^\circ - 2\pi^\circ\bar{\pi}^\circ\Omega^\circ\}\xi_l{}^\circ - \text{c.c.},
\end{aligned}
\tag{4–27}
$$

with c.c. indicating the complex conjugate. Integration of the backgrounds where $\pi^\circ \neq 0$ and $\alpha^\circ \neq 0$ using the Held technique has not made its way into the literature and is beyond the scope of the present work. As a result, derivatives of π° and α° appear in Equations 4–27 but do no harm to our argument. Choosing any gauge vector that satisfies

$$
\tilde{ð}'\xi_m{}^\circ + \tilde{\text{Þ}}'\xi_l{}^\circ - B^\circ = a^\circ, \qquad -\frac{1}{2}(\tilde{ð}'\tilde{ð} + \tilde{ð}\tilde{ð}' - \rho'^\circ - \bar{\rho}'^\circ)\xi_l{}^\circ + \xi_n{}^\circ - A^\circ = b^\circ, \tag{4–28}
$$

will serve to impose the trace condition in the full IRG. Once again we have established that $\mathcal{T}_{ll} = 0$ is both a necessary and sufficient condition for the existence of the full IRG. Note that by setting $\pi^\circ = \alpha^\circ = 0$ (i.e., ignoring the C-metrics) in the background, $A^\circ = B^\circ = 0$, and the result is virtually identical to Equations 4–18 and 4–19. There is one simplification in that now $\bar{\rho}'^\circ = \rho'^\circ$ [46]. The full extent of the remaining residual gauge freedom in Equations 4–28 can be demonstrated along the same lines as used in Section 4.3. As for the case of a type II background, it resides chiefly in the freely specifiable $\xi_m{}^\circ$ and $\xi_{\bar{m}}{}^\circ$.

[4] For example, $\pi^\circ \neq 0$ leads to the accelerating C-metrics. The condition $\pi^\circ = 0$ implies $\alpha^\circ = 0$ and so α° is also related to parameters in the C-metric.

4.5 Discussion

With our new form of the perturbed Einstein equations, use of NP methods has allowed us to treat the quite general class of type II spacetimes without either choosing coordinates or finding a metric. In this context, the Held technique has allowed us to exploit our form of the equations by enabling partial integration in solving $\mathcal{E}_{ll} = 0$ while investigating the existence of the IRG. Additionally, the Held technique has allowed us to completely characterize the residual gauge freedom and use it in the radiation gauge construction. By explicit demonstration, this work establishes our new form of the perturbed Einstein equations as a powerful tool within perturbation theory, both in conjunction with the Held technique and otherwise.

For perturbations with $\mathcal{T}_{ll} = 0$, our characterization of the residual gauge freedom is sufficiently complete that we can explicitly demonstrate the required gauge choice to remove any non-zero solution for the trace obtained via $\mathcal{E}_{ll} = 0$. Thus, in type II spacetimes, radiation gauges can be established by a genuine gauge choice, even if only after a solution of $\mathcal{E}_{ll} = 0$ is chosen.

There are subtle differences between the general type II case and the more restricted type D case, as there are also in the construction of Hertz potentials for the two cases. Stewart [21] writes out the type II case rather fully for an IRG. In this case, the perturbation in Ψ_0 is tetrad and gauge invariant, while the potential satisfies the adjoint (in the sense detailed by Wald [55]) of the $s = +2$ Teukolsky equation. Remarkably, in the type D case, this adjoint is actually the $s = -2$ Teukolsky equation, also satisfied by the gauge and tetrad invariant perturbation in Ψ_4. In the type II case, the adjoint equation is the same as in type D, but Ψ_4 is no longer tetrad invariant. Compared to the type D result, the expression for Ψ_4 given by Stewart has many extra terms depending on κ' and σ', so presumably it does not satisfy the same equation as the potential. As a consequence, metric reconstruction would be restricted to being built around the perturbation for Ψ_0 (c.f. the comments at the end of Chapter 3).

In the context of a specific type D background, Wald [63] has argued that mass and angular momentum perturbations are not given by any solution to the Teukolsky equations, and Stewart [21] has shown that these cannot be represented in a radiation gauge in terms of a potential. What we have done is identify the gauge freedom which remains in the fully satisfied radiation gauges, neither interfering with the radiation gauge prescription nor ruling out the possibility of mass and angular momentum perturbations. By realizing the explicit construction of the radiation gauges for type II background spacetimes and by identifying the remaining gauge freedom which they allow, we have, in a sense, completed a task initially embarked upon by Stewart [21], though in the different context of Hertz potentials.

CHAPTER 5
THE TEUKOLSKY-STAROBINSKY IDENTITIES

Having established the conditions for the existence of the radiation gauges, we will use the corresponding metric perturbations to establish some useful relationships between the perturbed Weyl scalars known generally (and quite loosely) as the Teukolsky-Starobinsky identities. Because Hertz potentials are solutions to the Teukolsky equation, these identities have immediate relevance for metric reconstruction in the IRG, both in the time-domain approach of Lousto and Whiting [25] and the frequency domain approach of Ori [23].

The original analysis of Teukolsky [11, 12] was based on the asymptotic form of the solutions of the separated angular and radial functions in the Kerr spacetime as well as a theorem due to Starobinsky and Churilov [64]. Only later did Chandrasekhar provide a full analysis, which is nicely summarized in his book [29]. Our analysis, however, will be entirely symbolic, involving only GHP quantities. This approach has the advantage not only of applying to a larger class of spacetimes, but displaying the structure inherent in the identities in a much more obvious way. A similar analysis of some of the identities we will discuss was previously undertaken in the NP formalism by Torres del Castillo [65] and later translated into GHP by Ortigoza [66]. These prior analyses made use of the most general type D spacetime and translated back and forth between coordinate-based and coordinate-free expressions. In contrast, our approach will not make any reference to the choice of coordinates or a tetrad (other than requiring it to be aligned with the principal null directions). Because of this, our approach will showcase one of GHPtools' greatest strengths–the ability to commute several derivatives with relative ease.

Our starting point is the (source-free) IRG metric perturbation given by

$$\begin{aligned} h_{ab} = \{&l_a l_b(\eth - \tau)(\eth + 3\tau) - l_{(a}m_{b)}[(\text{Þ} - \rho + \bar{\rho})(\eth + 3\tau) + (\eth - \tau + \bar{\tau}')(\text{Þ} + 3\rho)] \\ &+ m_a m_b(\text{Þ} - \rho)(\text{Þ} + 3\rho)\}\Psi + \text{c.c.}. \end{aligned} \tag{3–35}$$

As a consequence of Equation 3–35, the actual perturbed Weyl scalars follow directly from Equations 3–27 and 3–28.[1] The expressions are at first sight quite complicated, but by commuting derivatives so that they appear in a standard order and using the fact that the potential satisfies the Teukolsky equation, they become:

$$\psi_0 = \frac{1}{2}Þ^4\bar{\Psi}, \quad (5\text{–}1)$$

$$\psi_4 = \frac{1}{2}\Big\{ð'^4\bar{\Psi} - 3\psi_2^{4/3}[\psi_2^{-1/3}(\tau' ð - \tau ð' - \rho' Þ + \rho Þ' - 2\psi_2)]\Psi\Big\}. \quad (5\text{–}2)$$

The term in square brackets [] in Equation 5–2 is actually just the operator form of the (generally complex) Killing vector (acting on Ψ, which has type $\{-4, 0\}$) discussed in Chapter 2. We can further combine the relations in Equations 5–1 and 5–2 to eliminate any reference to the potentials. The first step is to act on Equation 5–2 with $Þ\psi_2^{-4/3}$, which gives us

$$Þ^4\psi_2^{-4/3}\psi_4 = \frac{1}{2}\Big\{Þ^4\psi_2^{-4/3}ð'^4\bar{\Psi} - 3Þ^4\mathcal{V}\Psi\Big\}. \quad (5\text{–}3)$$

Commuting the eight derivatives on the first term (using GHPtools, of course) yields the useful identity

$$Þ^4\psi_2^{-4/3}ð'^4\bar{\Psi} = ð'^4\psi_2^{-4/3}Þ^4\bar{\Psi}, \quad (5\text{–}4)$$

which we will have occasion to exploit again. Commuting the derivatives in the second term of Equation 5–3 poses no problem because $\mathcal{V}$ commutes with everything. Now it is a simple matter to identify the resulting expression with the terms in Equations 5–1 and 5–2 to arrive at the following

$$Þ^4\psi_2^{-4/3}\psi_4 = ð'^4\psi_2^{-4/3}\psi_0 - 3\mathcal{V}\bar{\psi}_0 \quad (5\text{–}5)$$

$$Þ'^4\psi_2^{-4/3}\psi_0 = ð^4\psi_2^{-4/3}\psi_4 + 3\mathcal{V}\bar{\psi}_4, \quad (5\text{–}6)$$

[1] We thank John Friedman and Toby Keidl for noting missing factors of $\frac{1}{2}$ in several earlier papers. Stewart [21] and Chrzanowski [18] have these factors correct, the latter with different sign conventions.

where the second expression follows from taking the prime of the first. We will refer to these relations as the first form of the Teukolsky-Starobinsky identities. Note that the use of $\mathcal{V}$ as a commuting operator restricts the validity of these relations to non-accelerating type D metrics. In the analysis of Torres del Castillo [65] and Ortigoza [66], where explicit coordinate expressions were used, Equations 5–5 and 5–6 both appear to be true. This fact appears to be coincidental since it is unclear how it follows in general from the fundamental equations of perturbation theory. The remainder of the identities we will present have not appeared in the literature in this form and we can only claim they hold for non-accelerating type D spacetimes.

Before we continue, we'll take a look at the content of Equations 5–5 and 5–6 in the context of the Kerr spacetime. If we write $\psi_0 \sim R_{+2}(r)S_{+2}(\theta,\phi)$ and $\psi_2^{-4/3}\psi_4 \sim R_{-2}(r)S_{-2}(\theta,\phi)$ and understand the time dependence of each to be given by $e^{-i\omega t}$, then Equation 5–5 tells us: (1) the result of four radial derivatives on R_{+2} is proportional to R_{-2} and (2) the result of four angular derivatives on S_{-2} is proportional to S_{+2}. The same is true of Equation 5–6 with the $+$'s and $-$'s swapped. Note that Equations 5–1 and 5–2 (and their primes in the ORG) say essentially the same thing with the subtle difference that the angular and radial functions are not obviously solutions to the same perturbation. No such ambiguity arises in Equations 5–5 and 5–6.

Remarkably, we can actually take things a step further and arrive at expressions for ψ_0 and ψ_4 independently. We begin by acting $\text{Þ}'^4\bar{\psi}_2^{-4/3}$ on Equation 5–5:

$$\text{Þ}'^4\bar{\psi}_2^{-4/3}\text{Þ}^4\psi_2^{-4/3}\psi_4 = \text{Þ}'^4\bar{\psi}_2^{-4/3}\eth'^4\psi_2^{-4/3}\psi_0 - 3\text{Þ}'^4\bar{\psi}_2^{-4/3}\mathcal{V}\bar{\psi}_0. \tag{5–7}$$

By recalling that Ψ has the same type as $\psi_2^{-4/3}\psi_4$ (ψ_2 carries no weight), we can simply take the prime and conjugate of Equation 5–4, and use it to commute the derivatives on the first term on the right as follows

$$\begin{aligned}
Þ'^4\bar{\psi}_2^{-4/3}ð'^4\psi_2^{-4/3}\psi_0 &= ð'^4\bar{\psi}_2^{-4/3}Þ'^4\psi_2^{-4/3}\psi_0 \\
&= ð'^4\bar{\psi}_2^{-4/3}(ð^4\psi_2^{-4/3}\psi_4 + 3\mathcal{V}\bar{\psi}_4) \\
&= ð'^4\bar{\psi}_2^{-4/3}ð^4\psi_2^{-4/3}\psi_4 + 3\mathcal{V}ð'^4\bar{\psi}_2^{-4/3}\bar{\psi}_4, \qquad (5\text{–}8)
\end{aligned}$$

where we made use of Equation 5–6 in the second line and commuted everything through $\mathcal{V}$ in the third line. The second term in Equation 5–7 becomes

$$\begin{aligned}
3Þ'^4\bar{\psi}_2^{-4/3}\mathcal{V}\bar{\psi}_0 &= 3\mathcal{V}Þ'^4\bar{\psi}_2^{-4/3}\bar{\psi}_0 \\
&= 3\mathcal{V}(ð'^4\bar{\psi}_2^{-4/3}\bar{\psi}_4 + 3\bar{\mathcal{V}}\psi_4) \\
&= 3\mathcal{V}ð'^4\bar{\psi}_2^{-4/3}\bar{\psi}_4 + 9\mathcal{V}\bar{\mathcal{V}}\psi_4, \qquad (5\text{–}9)
\end{aligned}$$

where we made use of the complex conjugate of Equation 5–6. Combining these results gives us

$$Þ'^4\bar{\psi}_2^{-4/3}Þ^4\psi_2^{-4/3}\psi_4 = ð'^4\bar{\psi}_2^{-4/3}ð^4\psi_2^{-4/3}\psi_4 - 9\mathcal{V}\bar{\mathcal{V}}\psi_4 \qquad (5\text{–}10)$$

$$Þ^4\bar{\psi}_2^{-4/3}Þ'^4\psi_2^{-4/3}\psi_0 = ð^4\bar{\psi}_2^{-4/3}ð'^4\psi_2^{-4/3}\psi_0 - 9\mathcal{V}\bar{\mathcal{V}}\psi_0, \qquad (5\text{–}11)$$

where we took the prime of the first equation to get the second one "for free." These are the second form of the Teukolsky-Starobinsky identities. We note in passing that in the context of the separated solutions of ψ_0 and $\psi_2^{-4/3}\psi_4$, the relations above allow for the determination of the magnitude of the proportionality constant relating R_{+2} and R_{-2} [29].

Surprisingly, this isn't the end of the story. Recall that in a type D spacetime we also have at our disposal the outgoing radiation gauge where

$$\psi_0 = \frac{1}{2}\left\{ð^4\bar{\Psi}' + 3\psi_2^{4/3}\mathcal{V}\Psi'\right\}, \qquad (5\text{–}12)$$

$$\psi_4 = \frac{1}{2}Þ'^4\bar{\Psi}', \qquad (5\text{–}13)$$

which are easily obtained by taking the prime of Equations 5–1 and 5–2. Note that whereas Ψ satisfies the Teukolsky equation for $\psi_2^{-4/3}\psi_4$, Ψ' satisfies the adjoint equation—the Teukolsky equation for ψ_0. From the complex conjugate of the preceding equations and their IRG counterparts, we get the following:

$$\begin{aligned} Þ^4\Psi &= ð'^4\Psi' - 3\bar{\psi}_2^{4/3}\bar{\mathcal{V}}\bar{\Psi}', & (5–14) \\ Þ'^4\Psi' &= ð^4\Psi + 3\bar{\psi}_2^{4/3}\bar{\mathcal{V}}\bar{\Psi}, & (5–15) \end{aligned}$$

the first form of the Teukolsky-Starobinsky relationships for potentials. Note the difference between the above and Equations 5–5 and 5–6, particularly the missing factors of $\psi_2^{-4/3}$ and the fact that $\bar{\mathcal{V}}$ appears. As with Equations 5–5 and 5–6, we can obtain relations for each potential individually by acting $Þ'^4\bar{\psi}_2^{-4/3}$ on Equation 5–14 and further exploiting (the primed conjugate of) Equation 5–4. The result is that

$$\begin{aligned} Þ'^4\bar{\psi}_2^{-4/3}Þ^4\Psi &= ð'^4\bar{\psi}_2^{-4/3}ð^4\Psi - 9\bar{\mathcal{V}}\mathcal{V}\psi_2^{4/3}\Psi, & (5–16) \\ Þ^4\bar{\psi}_2^{-4/3}Þ'^4\Psi' &= ð^4\bar{\psi}_2^{-4/3}ð'^4\Psi' - 9\bar{\mathcal{V}}\mathcal{V}\psi 2^{4/3}\Psi'. & (5–17) \end{aligned}$$

We can summarize this last identity by writing

$$\begin{aligned} [Þ'^4\bar{\psi}_2^{-4/3}Þ^4 - ð'^4\bar{\psi}_2^{-4/3}ð^4 + 9\bar{\mathcal{V}}\mathcal{V}\psi_2^{4/3}]\{\psi_2^{-4/3}\psi_4, \Psi\} &= 0, & (5–18) \\ [Þ^4\bar{\psi}_2^{-4/3}Þ'^4 - ð^4\bar{\psi}_2^{-4/3}ð'^4 + 9\bar{\mathcal{V}}\mathcal{V}\psi 2^{4/3}]\{\psi_2^{-4/3}\psi_0, \Psi'\} &= 0. & (5–19) \end{aligned}$$

Bardeen has recently pointed out an issue in the standard treatment of the Teukolsky-

Starobinsky identities [67]. In particular, he finds that, in the Schwarzschild background, there is a hitherto unnoticed relative sign difference between the odd- and even-parity in the term proportional to ∂_t (alternatively ω when time separation is performed), which by continuity presumably persists in the Kerr background. Bardeen argues using standard techniques that don't make clear the difference between the ψ's and their complex conjugates on the right-hand-sides of Equations 5–5 and 5–6. However, recalling our discussion of parity in Chapter 3, a glance at these equations reveals that one should in fact expect a relative sign because of the occurrence of $\psi_{0,4}$ and its complex conjugate in the same expression. Moreover, this must occur even in the Kerr spacetime, where we have the real-imaginary separation instead of the parity separation. Such a consideration makes clear the obvious advantage of treating the Teukolsky-Starobinsky identities in terms of the fundamental GHP quantities. Beginning at this level and then performing the separation of variables allows for no ambiguity in the resulting expressions.

CHAPTER 6
THE NON-RADIATED MULTIPOLES

In this chapter we will address the issue of the non-radiated multipoles alluded to in Chapter 1. The issue is that the metric constructed from a Hertz potential is incomplete in the sense that its multipole decomposition necessarily begins at $\ell = 2$ because the angular dependence of the potential is that of a spin-weight ± 2 angular function. To see this explicitly, we focus our attention on the IRG metric perturbation (Equation 3–35) in the Schwarzschild spacetime, where the potential, Ψ, can be decomposed into some radial function, $R(r)$, with exponential time dependence, $e^{-i\omega t}$, and a spin-weight 2 spherical harmonic, ${}_{-2}Y_{\ell m}(\theta, \phi)$ (see Appendix D, for details about the spin-weighted spherical harmonics). Ignoring the radial and time dependence, we see that the components of the metric perturbation have angular dependence given by

$$h_{ll} \sim ð^2{}_{-2}Y_{\ell m} = [(\ell - 1)\ell(\ell + 1)(\ell + 2)]^{1/2}{}_0Y_{\ell m}, \tag{6–1}$$

$$h_{l\bar{m}} \sim ð{}_{-2}Y_{\ell m} = [(\ell - 1)(\ell + 2)]^{1/2}{}_{-1}Y_{\ell m}, \tag{6–2}$$

$$h_{\bar{m}\bar{m}} \sim {}_{-2}Y_{\ell m}, \tag{6–3}$$

and similarly for h_{lm} and h_{mm}. Because the spin-weighted spherical harmonics are undefined for $|s| > \ell$, the above expressions make it clear that the metric perturbation in this gauge has no $\ell = 0, 1$ pieces and therefore provides an incomplete description of the physical spacetime. By continuity, the situation persists in the Kerr spacetime. How incomplete is this description?

For the majority of this work, we have focused our attention on gravitational radiation in type D spacetimes. This information is contained in the perturbation of either ψ_0 or ψ_4, a result established by Wald [16]. In particular, Wald was able to show that well-behaved perturbations of ψ_0 and ψ_4 determine each other and furthermore that either one characterizes the entire perturbation of the spacetime up to "trivial" perturbations in mass and angular momentum. With ψ_0 and ψ_4 determined by the Hertz potential

(Equations 5–1 and 5–2) this begs the question of why we should concern ourselves with such trivialities.

The answer is, in part, that these trivial perturbations represent the largest contribution to the self-force, as shown by Detweiler and Poisson [17]. Although it is unclear if such contributions persist in all gauge invariant quantities of interest, such as certain characterizations of the orbital motion of the particle [68], there is in fact a more compelling reason to be concerned with the non-radiated multipoles. In recent work, Keidl, Friedman and Wiseman [69] have looked at the problem of computing the self-force in a radiation gauge in the context of a static particle in the Schwarzschild spacetime. In their calculation, they found the perturbations of mass and angular momentum arising in the construction of a Hertz potential. Thus, although the Hertz potential cannot be used to determine these perturbations, it must still "know" about them and they must be determined by some other means.

In this chapter we will present a general prescription for computing the non-radiated multipoles. More specifically, we will consider the problem of computing the shifts in mass and angular momentum due to a point source in a circular (geodesic), equatorial orbit around a black hole. Specifically we are after expressions for δM and δa, the shifts in mass and angular momentum, in terms of the orbiting particle's mass, μ, and orbital parameters. The idea is quite simple: match an interior spacetime, (g^-_{ab}, M^-), to an exterior spacetime, (g^+_{ab}, M^+), differing only in mass and angular momentum, on a hypersurface (of codimension 1), Σ_p, containing the perturbation. The basic conditions for a good matching are (1) that the metric is continuous across Σ_p and (2) the first derivatives of the metric are continuous except where the source is infinite. These conditions are compatible with Israel's quite general junction conditions [70].

Before we can do any matching, we must first determine the geometry of Σ_p. In spherically symmetric spacetimes, the obvious choice is the simplest—the (round) 2-sphere, as we'll see below in our calculation in Schwarzschild. For the Kerr spacetime, which possesses only axial symmetry, the situation is immensely more complicated. This issue will be discussed below.

Once we've agreed on a Σ_p, fulfilling our first matching condition requires us to simply equate the components of the metric (on Σ_p). In other words,

$$[g_{ab}] \equiv g^{+}_{ab|_{\Sigma_p}} - g^{-}_{ab|_{\Sigma_p}} = 0, \tag{6–4}$$

where $|_{\Sigma_p}$ indicates the restriction to Σ_p. The only (slight) complication that arises here is ensuring that there is enough freedom in the metric perturbation to perform the matching. This will generally require performing a gauge transformation on the interior and exterior spacetimes. Although this introduces some gauge dependence into the problem, the end result - δM or δa - is in fact gauge invariant, as we will see below.

Imposing the second condition is a bit more involved because of the presence of the source. By choosing a good matching surface, Σ_p, we can effectively "smear out" the angular dependence of the source. If, for example, Σ_p is a 2-sphere, we can use the completeness relations to write the angular delta function according to

$$\delta(\phi - \phi')\delta(\cos\theta - \cos\theta') = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \bar{Y}_{\ell m}(\theta', \phi')Y_{\ell m}(\theta, \phi). \tag{6–5}$$

Similar relations hold for complete sets of functions on different closed 2-surfaces. The source now consists solely of a radial delta function. To handle this, we impose the perturbed Einstein equations as, for example,

$$\lim_{\epsilon\to 0}\left(\int_{r_0-\epsilon}^{r_0+\epsilon} \mathcal{E}_{ab}\, dr = \int_{r_0-\epsilon}^{r_0+\epsilon} 8\pi\mathcal{T}_{ab}\, dr\right), \tag{6–6}$$

where $\mathcal{E}_{ab}$ denotes the perturbed Einstein tensor and $\mathcal{T}_{ab}$ denotes the stress-energy tensor of the source and r_0 is the location of Σ_p as seen from both sides. For a delta function source due a particle of mass μ in a circular equatorial orbit of the Kerr spacetime,

$$\mathcal{T}_{ab} = \frac{\mu u^a u^b}{u^t\sqrt{-g}}\delta(r - r_0)\delta(\cos\theta)\delta(\phi - \Omega t), \tag{6–7}$$

where $u^a = (\frac{dt}{d\tau}, 0, 0, \frac{d\phi}{d\tau})$ is the four-velocity of the particle parameterized by proper time (τ), r_0 is the radius of the orbit and $\Omega = \frac{d\phi}{dt}$. For circular equatorial geodesics

$$\begin{aligned} r &= r_0, && (6\text{–}8)\\ \theta &= \frac{\pi}{2}, && (6\text{–}9)\\ r_0^2\frac{dt}{d\tau} &= \frac{(r_0^2+a^2)T}{\Delta} + a(\tilde{L} - a\tilde{E}), && (6\text{–}10)\\ r_0^2\frac{d\phi}{d\tau} &= \tilde{L} - a\tilde{E} + \frac{aT}{\Delta}, && (6\text{–}11) \end{aligned}$$

with

$$T = (r_0^2 + a^2)\tilde{E} - a\tilde{L}, \qquad (6\text{–}12)$$

where $\tilde{E} = E/\mu$ and $\tilde{L} = L/\mu$ are the energy and angular momentum per unit mass, respectively. We can recover the corresponding result for the Schwarzschild spacetime by simply taking $a \to 0$. Because the integration in Equation 6–6 is purely radial, it is clear that the only terms that actually participate in the integral on the left side are those involving two radial derivatives. This is where our form of the perturbed Einstein equations comes in. While it is generally quite tedious and impractical to compute the perturbed Einstein tensor for a background more general than Schwarzschild and pick out the terms involving two derivatives, it is a quite trivial task for the Einstein equations in GHP form. All we need to do is pick out the pieces involving two of Þ and Þ$'$ (a mindless task with the aid of GHPtools), plug in our favorite tetrad and voíla! Note that these conditions on the second derivatives are generally invariant with respect to choice of tetrad. Because of this, we will write the jump conditions out in the symmetric tetrad, which is obtainable from the Kinnersley tetrad by a simple spin-boost (Equation 2–16) (and thus leaves the PNDs intact). The tetrad is given by:

$$\begin{aligned}
l^a &= \left[\frac{r^2+a^2}{\sqrt{2\Delta\tilde{\rho}^2}}, \sqrt{\frac{\Delta}{2\tilde{\rho}^2}}, 0, \frac{a}{\sqrt{2\Delta\tilde{\rho}^2}}\right], \\
n^a &= \left[\frac{r^2+a^2}{\sqrt{2\Delta\tilde{\rho}^2}}, -\sqrt{\frac{\Delta}{2\tilde{\rho}^2}}, 0, \frac{a}{\sqrt{2\Delta\tilde{\rho}^2}}\right], \\
m^a &= \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left[ia\sin\theta, 0, 1, \frac{i}{\sin\theta}\right].
\end{aligned} \tag{6–13}$$

With this tetrad choice, the radial jump conditions are:

$$\partial_r^2 h_{m\bar{m}} = 16\pi\frac{\tilde{\rho}^2}{\Delta}\mathcal{T}_{ll}, \tag{6–14}$$

$$\partial_r^2 h_{m\bar{m}} = 16\pi\frac{\tilde{\rho}^2}{\Delta}\mathcal{T}_{ln}, \tag{6–15}$$

$$\partial_r^2 (h_{ll}+h_{nn}+2h_{ln}-2h_{m\bar{m}}) = 16\pi\frac{\tilde{\rho}^2}{\Delta}\mathcal{T}_{m\bar{m}}, \tag{6–16}$$

$$\partial_r^2 (h_{lm}+h_{nm}) = 16\pi\frac{\tilde{\rho}^2}{\Delta}\mathcal{T}_{lm}, \tag{6–17}$$

$$\partial_r^2 h_{mm} = 16\pi\frac{\tilde{\rho}^2}{\Delta}\mathcal{T}_{mm}, \tag{6–18}$$

where the omitted equations follow by taking the prime and/or complex conjugate of those listed (the factors of Δ and $\tilde{\rho}^2$ remain unchanged; a feature of the symmetric tetrad), and it is understood that equality only holds in the sense of Equation 6–6. At a glance Equations 6–14–6–18 may appear inconsistent, with the same left-hand-side being equated to different right-hand-sides. In fact, the circular geodesic nature of u^a ensures that this is not the case.

What we have not yet addressed is the question of what, precisely, we mean by mass and angular momentum. Suitable definitions arise from the Hamiltonian treatment of General Relativity initiated by Arnowitt, Deser and Misner [71]. The general idea is that because Minkowski space provides an unambiguous notion of energy and angular momentum through time translations and rotations, respectively, we can adapt these notions to curved spaces if the metric becomes Minkowskian at spacelike infinity. Thus the ADM definitions require us to restrict our attention to asymptotically flat spacetimes,

spacetimes that become flat near infinity. The most precise definition of asymptotic flatness requires a detailed analysis of the conformal structure of spacetime [72], but for our purposes it will suffice to simply consider the asymptotic falloff of the components of the metric. More precisely, for a set of coordinates (x, y, z) in a metric, g_{ab}, and $r = \sqrt{x^2 + y^2 + z^2}$, we require[1]

$$\begin{aligned}\lim_{r\to\infty} g_{ab} &= \eta_{ab} + \mathcal{O}\Big(\frac{1}{r}\Big),\\ \lim_{r\to\infty} \partial_c g_{ab} &= \mathcal{O}\Big(\frac{1}{r^2}\Big).\end{aligned} \tag{6–19}$$

These conditions are satisfied by the Schwarzschild and Kerr spacetimes we wish to consider, but we must be careful to choose an appropriate gauge for the metric perturbation to ensure that Equations 6–19 are satisfied. Assuming an asymptotically flat spacetime, the ADM mass is defined by

$$M = \frac{1}{16\pi} \lim_{S\to\infty} \oint_S (D^b \kappa_{ab} - D_a \kappa) r^a \, dS, \tag{6–20}$$

where the symbols need a bit of explanation: we denote the hypersurface of constant t by Σ_t and its boundary by S. The three-metric on Σ_t is γ_{ab}. Then $\kappa_{ab} = \gamma_{ab} - \gamma^0_{ab}$, with γ^0_{ab} being the metric of flat spacetime (in the same coordinates as γ_{ab}) and $\kappa = \kappa_{ab}(\gamma^0)^{ab}$. Additionally, D_a is the covariant derivative compatible with γ^0_{ab}, r^a is the unit normal to S, and dS is the surface element on S. For an arbitrary metric perturbation, h_{ab}, this evaluates to

$$\delta M = \frac{1}{16\pi} \lim_{r\to\infty} \int_0^{2\pi} \int_0^{\pi} 2r \sin\theta h_{rr} \, d\theta d\phi, \tag{6–21}$$

[1] In general, having a well-defined angular momentum actually requires a faster falloff than that given below. However, because we're restricting our attention to spacetimes with axial killing vectors, the falloff required for asymptotic flatness is sufficient.

where we've omitted the terms that will vanish in the limit as a result of requiring asymptotic flatness. Similarly, we define angular momentum by

$$J = \frac{1}{8\pi} \lim_{S\to\infty} \oint_S (K_{ab} - K\gamma_{ab})\phi^a r^b \, dS, \tag{6–22}$$

where we have introduced the extrinsic curvature, K_{ab}, of Σ_t and the rotational Killing vector ϕ^a. For a generic metric perturbation of the Kerr spacetime, we have

$$\delta J = \frac{1}{8\pi} \lim_{r\to\infty} \int_0^{2\pi} \int_0^{\pi} r \sin\theta h_{t\phi} - \frac{1}{2} r^2 \sin\theta \partial_r h_{t\phi} \, d\theta d\phi. \tag{6–23}$$

Though these definitions provide the most general prescription for computing the mass and angular momentum, for stationary and axially symmetric spacetimes (those containing both timelike and axial Killing vectors), the Komar formulae [73] evaluated at infinity allow us to compute the value of the perturbations[2] of M and J. though not the entire perturbation in the interior and exterior spacetime. The formulae are given by

$$\delta M = 2\int_\Sigma (\mathcal{T}_{ab} - \frac{1}{2}\mathcal{T} g_{ab}) n^a t^b \sqrt{h} d^3x, \tag{6–24}$$

$$\delta J = -\int_\Sigma (\mathcal{T}_{ab} - \frac{1}{2}\mathcal{T} g_{ab}) n^a \phi^b \sqrt{h} d^3x, \tag{6–25}$$

where Σ is spacelike hypersurface that extends to infinity, n^a is the unit normal to it, t^a and ϕ^a are the timelike and axial Killing vectors and $\sqrt{h}d^3x$ is the volume element on Σ. Because our stress-energy tensor is confined to a spacelike hypersurface, Σ_p, at $r = r_0$, to compute the ADM mass we must take the limit as $r_0 \to \infty$. In this limit, with the source given by Equations 6–7–6–12, the Komar formulae give (for the Kerr spacetime)

$$\delta M = \mu\tilde{E}, \tag{6–26}$$

$$\delta J = \mu\tilde{L}. \tag{6–27}$$

[2] We thank John Friedman for suggesting the use of the Komar formulae.

These results are to be expected because of the axisymmetric nature of both the perturbations and the background spacetime. We now turn our attention to the mass and angular momentum perturbations in the Schwarzschild background.

6.1 Schwarzschild

The Schwarzschild spacetime provides the perfect testbed for our technique. Moreover, because of the spherical symmetry of the background, matching the spacetime is quite straightforward. In this case we can always choose the matching hypersurface, Σ_p, to be a (round) 2-sphere and exploit the orthogonality and completeness of the spin-weighted spherical harmonics to smear out the delta source on Σ_p. The only caveat is that we must choose Σ_p outside of the innermost stable circular orbit. If the location of Σ_p is r_0, then this amounts to requiring $r_0 \geq 6M$.

6.1.1 Mass perturbations

Our first task is to construct a suitable description of source-free mass perturbations of the Schwarzschild spacetime. We will then glue two such spacetimes together, as described above. We will write the Schwarzschild metric as

$$ds^2 = f dt^2 - f^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6–28}$$

where $f = 1 - 2M/r$. According to Birkhoff's theorem, the only static, spherically symmetric solution to the Einstein equations is the Schwarzschild solution. Thus, we are assured from the outset that perturbing the mass will simply lead us to another Schwarzschild spacetime with a mass $M + \delta M$. The nonzero components of the corresponding metric perturbation are given by

$$\begin{aligned} h_{tt} &= -2\frac{\delta M}{r} \\ h_{rr} &= -2\frac{\delta M}{r f^2}, \end{aligned} \tag{6–29}$$

which is easily obtained by linearizing a mass perturbation of Equation 6–28. In order to characterize mass perturbations more generally, we will introduce more freedom by performing a gauge transformation. To that end we introduce the gauge vector

$$\xi^a = (P(t,r)Y_{\ell m}(\theta,\phi), Q(t,r)Y_{\ell m}(\theta,\phi), 0, 0), \tag{6–30}$$

where we've taken a cue from Regge & Wheeler and decomposed the gauge vector into spherical harmonics. Note the absence of ξ_θ and ξ_ϕ components in our gauge vector. We have deliberately omitted these components on the grounds that they interfere with the form invariance of the metric. In order to determine the functions $P(t,r)$ and $Q(t,r)$ as well as the appropriate ℓ and m, we'll look at their contribution to the metric perturbation. Our gauge transformation, $\xi_{ab} = \pounds_\xi g_{ab}$, has the form

$$\xi_{ab} = \begin{pmatrix} -2(f\partial_t P + Mr^{-2}Q) & (-f\partial_r P + f^{-1}\partial_t Q) & -fP\partial_\theta & -fP\partial_\phi \\ \text{sym} & 2f^{-1}(\partial_r Q - Mr^{-2}f^{-1}Q) & f^{-1}Q\partial_\theta & f^{-1}Q\partial_\phi \\ \text{sym} & \text{sym} & 2rQ & 0 \\ \text{sym} & \text{sym} & \text{sym} & 2r\sin^2\theta Q \end{pmatrix} Y_{\ell m}, \tag{6–31}$$

where "sym" means symmetric and we've dropped the functional dependencies for simplicity. First, we'll further specialize the gauge transformation by insisting on preserving the form of Equation 6–28. A consequence of this is that

$$h_{tr} = (-f\partial_r P(t,r) + f^{-1}\partial_t Q(t,r)) = 0.$$

Because $Q(t,r)$ appears in other parts of the metric perturbation, allowing it to carry a time dependence would destroy the static nature of the perturbation and put us at odds with Birkhoff's theorem. Therefore we require $Q = Q(r)$, which immediately leads us to $P = P(t)$. Further consequences of our form invariance requirement are

$$\begin{aligned} h_{t\theta} &= -fP\partial_\theta Y_{\ell m} = 0, \\ h_{t\phi} &= -fP\partial_\phi Y_{\ell m} = 0, \\ h_{r\theta} &= f^{-1}Q\partial_\theta Y_{\ell m} = 0, \\ h_{t\phi} &= f^{-1}Q\partial_\phi Y_{\ell m} = 0. \end{aligned} \tag{6–32}$$

Thus we must impose $\partial_\theta Y_{\ell m} = \partial_\phi Y_{\ell m} = 0$, which translates into $\ell = m = 0$ and we have established that the angular dependence of the metric perturbation is purely $Y_{00}(\theta, \phi) =$ constant. Also, in order to keep the perturbation static, the time dependence of $P(t)$ must be, at most, linear. Without loss of generality, we set $P(t) = \alpha t$. Finally, our falloff conditions in Equations 6–19 require $Q(r) = \mathcal{O}\left(\frac{1}{r^2}\right)$. Thus we have arrived at a description of source-free mass perturbations in the Schwarzschild spacetime in a family of asymptotically flat gauges that preserve the form of the metric. The physical spacetime ($g_{ab} = g_{ab}^{\text{Schw}} + h_{ab}$) has components

$$
\begin{aligned}
g_{tt} &= f(1 - 2\alpha Y_{00}) - \frac{2MQ(r)Y_{00}}{r^2} - \frac{2\delta M}{r}, && (6\text{–}33)\\
g_{rr} &= -f^{-1}\left(1 - \frac{2mQY_{00} + r^2 f Y_{00}\partial_r Q - 2r\delta M}{r^2 f}\right), && (6\text{–}34)\\
g_{\theta\theta} &= -r^2\left(1 - \frac{2QY_{00}}{r}\right), && (6\text{–}35)\\
g_{\phi\phi} &= -r^2\sin^2\theta\left(1 - \frac{2QY_{00}}{r}\right). && (6\text{–}36)
\end{aligned}
$$

We can give an interpretation to α by considering Equation 6–33 with $\delta M = Q = 0$, in which case it is clear that α is just a rescaling of the time coordinate.

In order to perform the matching, we need to adapt our generic perturbation to the interior and exterior spacetimes and choose a particular gauge to perform the matching. We will begin with the description of the metric on the interior, g^-_{ab}. Here $\delta M = 0$, so the perturbation is pure gauge. Furthermore, on the interior there is no need to impose asymptotic flatness. Instead, we will choose $Q^-(r)$ so that the interior metric is regular on the horizon and leave the form of $P^-(t)$ untouched. A suitable choice is

$$
\begin{aligned}
P^-(t) &= \alpha^- t,\\
Q^-(r) &= \beta\left(\frac{r - 2M}{r_0 - 2M}\right)^i,
\end{aligned}
\qquad (6\text{–}37)
$$

where $r = r_0$ is the location of Σ_p and β is a constant inserted for dimensional reasons and $i > 0$. The values of α^- and β will be determined from the jump conditions. Proceeding to the description of the exterior spacetime, g^{+}_{ab}, we choose

$$\begin{aligned} P^{+}(t) &= \alpha^{+}t, \\ Q^{+}(r) &= \beta\left(\frac{r_0-2M}{r-2M}\right)^{j}, \end{aligned} \tag{6–38}$$

where, in anticipation of the matching, we've chosen the same dimensional constant, β, that we used in the description of the interior spacetime and $j \geq 2$. With both metrics specified we now turn our attention to matching the spacetimes.

Because both background metrics are the same, it will suffice to match the perturbations only. By imposing $[h_{ab}] = 0$, we arrive at three unique conditions:

$$\frac{\delta M}{r_0} + f_0[\alpha]Y_{00} + \frac{M}{r_0^2}[Q]Y_{00} = 0, \tag{6–39}$$

$$r_0^2 f_0\left[\frac{dQ}{dr}\right]Y_{00} - M[Q]Y_{00} - r_0\delta M = 0, \tag{6–40}$$

$$[Q] = 0, \tag{6–41}$$

where we used $f_0 = f(r_0)$. Our choices for Q^{+} and Q^{-} (6–38,6–37) ensure that the third condition is satisfied. We can solve Equations 6–39 and 6–40 to get equations for $[\alpha]$ and δM:

$$[\alpha] = -\left[\frac{dQ}{dr}\right] = \frac{\beta(i+j)}{r_0-2M}, \tag{6–42}$$

$$\delta M = (r_0-2M)\left[\frac{dQ}{dr}\right]Y_{00} = -\beta(i+j)Y_{00}, \tag{6–43}$$

where we've made use of Equations 6–38 and 6–37. Next we will use the jump conditions to solve for β.

Application of the jump conditions (Equations 6–14–6–18) is simplified by the fact that our metric perturbation is pure spin-0. Thus we only need consider the jump conditions for the spin-0 components of the metric perturbation (h_{ll}, h_{ln}, h_{nn} and $h_{m\bar{m}}$). For simplicity we will work with Equation 6–15, though it can be directly verified that the remaining spin-0 jump conditions all yield the same result. With the source given by the $a \to 0$ limit of Equation 6–7, we have for the tetrad components of the relevant objects:

$$h^-_{m\bar{m}} = \frac{2\beta}{r}\left(\frac{r-2M}{r_0-2M}\right)^i Y_{00} \tag{6–44}$$

$$h^+_{m\bar{m}} = \frac{2\beta}{r}\left(\frac{r_0-2M}{r-2M}\right)^j Y_{00} \tag{6–45}$$

$$16\pi\frac{\tilde{\rho}^2}{\Delta}\mathcal{T}_{ln} = \frac{8\pi\mu\tilde{E}}{r_0^2 f_0}\delta(r-r_0)\delta(\cos\theta)\delta(\phi-\Omega t), \tag{6–46}$$

with all the δM dependence replaced according to Equation 6–43. Imposing Equation 6–6 then leads to

$$\left[\frac{\partial h_{m\bar{m}}}{\partial r}\right] = \frac{4\pi\tilde{E}}{r_0^2 f_0}\delta(\cos\theta)\delta(\phi-\Omega t),$$

or

$$-\frac{2\beta(i+j)}{r_0^2 f_0}Y_{00}(\theta,\phi) = \frac{8\pi\mu\tilde{E}}{r_0^2 f_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\bar{Y}_{\ell m}(\pi/2,\Omega t)Y_{\ell m}(\theta,\phi), \tag{6–47}$$

where we've decomposed the angular delta functions according to Equation 6–5. We can eliminate the sum on the right side of Equation 6–48 by multiplying both sides by $\bar{Y}_{00}(\theta,\phi)$, integrating over the sphere and exploiting the orthogonality of the spherical harmonics. The result is that

$$\beta = -\frac{(4\pi)^{1/2}\mu\tilde{E}}{i+j}, \tag{6–48}$$

where we've used $\bar{Y}_{00}(\theta,\phi) = Y_{00}(\theta,\phi) = (4\pi)^{-1/2}$. Finally, we have

$$[\alpha] = -\frac{(4\pi)^{1/2}\mu\tilde{E}}{r_0-2M} \tag{6–49}$$

$$\delta M = \mu\tilde{E}. \tag{6–50}$$

These equations complete our construction of the matched spacetime. Note that the above only restricts the *difference* between α on the interior and exterior. If we recall Equation 6–41, we see that the same is generally true of $Q(r)$ as well if we drop the requirements of regularity in the interior and asymptotic flatness in the exterior.

6.1.2 Angular momentum perturbations

Treating angular momentum perturbations is a bit more involved. One reason for this is the fact that it inherently changes the form of the metric. From Equation 6–23, it is clear that our metric perturbation will acquire an $h_{t\phi}$ component. Realizing this as a perturbation towards the Kerr spacetime, we will write it as

$$h^{+}_{t\phi} = \frac{2\delta a M \sin^2\theta}{r}, \tag{6–51}$$

which is just the linearization about $a = J/M$ of the corresponding component of the (background) Kerr metric. Because of this, there will be nonzero contributions to h_{lm}, h_{nm} and their complex conjugates which means that we must now take parity into consideration. To that end we will introduce a gauge vector with components

$$\xi_t = P(t,r)Y_{\ell m}(\theta,\phi) \tag{6–52}$$

$$\xi_r = Q(t,r)Y_{\ell m}(\theta,\phi) \tag{6–53}$$

$$\begin{aligned}\xi_\theta &= [R(t,r)\frac{1}{2}(\eth + \eth') + S(t,r)\frac{i}{2\sin\theta}(\eth - \eth')]Y_{\ell m}(\theta,\phi) \\ &= R(t,r)Y^{+}_{\ell m}(\theta,\phi) + S(t,r)\frac{Y^{-}_{\ell m}(\theta,\phi)}{\sin\theta}\end{aligned} \tag{6–54}$$

$$\begin{aligned}\xi_\phi &= [R(t,r)\frac{i}{2}(\eth - \eth') - S(t,r)\frac{\sin\theta}{2}(\eth + \eth')]Y_{\ell m}(\theta,\phi) \\ &= R(t,r)Y^{-}_{\ell m}(\theta,\phi) - S(t,r)\sin\theta Y^{+}_{\ell m}(\theta,\phi),\end{aligned} \tag{6–55}$$

where we've defined $Y^{+}_{\ell m} = \frac{1}{2}(\eth + \eth')Y_{\ell m} = \frac{1}{2}({}_1Y_{\ell m} + {}_{-1}Y_{\ell m})$ and $Y^{-}_{\ell m} = \frac{i}{2}(\eth - \eth')Y_{\ell m} = \frac{i}{2}({}_1Y_{\ell m} - {}_{-1}Y_{\ell m})$, where ${}_{\pm 1}Y_{lm}$ are the spin-weight ± 1 spherical harmonics discussed in Appendix D. This form of the gauge vector was obtained by considering $\xi_a = \xi_l n_a + \xi_n l_a - \xi_m \bar{m}_a - \xi_{\bar{m}} m_a$ and making use of the parity decomposition discussed in Chapter 3. This makes it easy to see that P, Q and R represent the even-parity degrees of gauge freedom and S represents the only odd-parity gauge freedom available. A natural question to ask is what parity the perturbation in Equation 6–51 has. For an answer, we look to the source terms. A quick computation reveals that $\mathcal{T}_{lm} = \mathcal{T}_{nm} = -\mathcal{T}_{l\bar{m}} = -\mathcal{T}_{n\bar{m}}$, from which it follows that the even-parity (real) parts of the source vanish identically and thus the angular momentum perturbation is completely odd-parity (imaginary). However, we still have pieces in the source (such as $\mathcal{T}_{ll}$) that contribute to the even parity perturbation. We will treat each parity individually.

6.1.2.1 Odd-parity angular momentum perturbations

Because our gauge vector only has one nonzero component, our task is greatly simplified. The contribution of the odd-parity gauge vector to the metric perturbation takes the form

$$h^{-}_{ab} = \begin{pmatrix} 0 & 0 & -(\partial_t S)(\sin\theta)^{-1}Y^{-} & (\partial_t S)\sin\theta Y^{+} \\ 0 & 0 & -(\partial_r S - 2r^{-1}S)(\sin\theta)^{-1}Y^{-} & (\partial_r S - 2r^{-1}S)\sin\theta Y^{+} \\ \text{sym} & \text{sym} & 0 & -S\big[(\sin\theta)^{-1}\partial_\phi Y^{-} + \cos\theta Y^{+} - \sin\theta\partial_\theta Y^{+}\big] \\ \text{sym} & \text{sym} & \text{sym} & 0 \end{pmatrix}, \tag{6–56}$$

where the "-" on h_{ab}, referring to the interior spacetime, is to be distinguished from the "-" on $Y^{-}_{\ell m}$, which refers to a combination of spin-weight 1 spherical harmonics. In this situation, we must modify our requirement of form invariance (which is already broken by the perturbation) to the requirement that only $h^{-}_{t\phi}$ remains nonzero, which preserves the minimum freedom to match to the exterior. First we set $h^{-}_{t\theta} = 0$, which implies $Y^{-}_{\ell m} = 0$ or ${}_1Y_{\ell m} = {}_{-1}Y_{\ell m}$. This can only hold if $m = 0$, which means the perturbation is axially symmetric. Moving on, we turn our attention to eliminating $h^{-}_{\theta\phi}$. This entails

$$\cos\theta Y^{+}_{\ell 0} - \sin\theta\partial_\theta Y^{+}_{\ell 0} = 0,$$

which has the solution

$$Y^{+}_{\ell 0} = \frac{1}{2}\Big({}_1Y_{\ell 0} + {}_{-1}Y_{\ell 0}\Big) \sim \sin\theta.$$

This is just the statement that $\ell = 1$, which is to be expected from the fact that the perturbation appears in the spin-1 part of the metric. The angular dependence of the interior perturbation is then characterized by

$$ {}_{\pm 1}Y_{10} = -\sqrt{\frac{3}{4\pi}}\sin\theta. \tag{6–57} $$

Finally, it is easy to set

$$ h^-_{r\phi} \sim \partial_r S - 2r^{-1}S = 0, $$

by imposing $S(t,r) = r^2 S(t)$. Note that because of the quadratic dependence on r, we cannot perform this gauge transformation in the exterior spacetime if we wish to preserve asymptotic flatness. This is not a problem because the angular momentum perturbation provides the necessary freedom for matching. Finally, the piece in $h^-_{t\phi}$ is proportional to the time derivative of $S(t,r)$, which suggests we choose $S(t) = \gamma t$, to keep the perturbation static. In summary, we have for the interior and exterior metric perturbations

$$ h^-_{t\phi} = \gamma r^2 \sin\theta Y^+_{10}, \tag{6–58} $$

$$ h^+_{t\phi} = \frac{2\delta a M \sin^2\theta}{r}, \tag{6–59} $$

with all other components vanishing.

Continuity of the metric perturbation ($[h_{ab}] = 0$) requires

$$ \delta a = \frac{\gamma r_0^3\,{}_1Y_{10}}{2M\sin\theta}, \tag{6–60} $$

where we've used the equality of ${}_{\pm 1}Y_{10}$ to expand Y^+_{10}. As before, the radial jump conditions will determine γ. In this case we'll use the odd-parity (imaginary) part of Equation 6–17. The relevant tetrad components are given by:

$$ h^-_{lm} - h^-_{l\bar{m}} = h^-_{nm} - h^-_{n\bar{m}} = -\frac{i\gamma r^2\,{}_1Y_{10}}{f^{1/2}}, \tag{6–61} $$

$$ h^+_{lm} - h^+_{l\bar{m}} = h^+_{nm} - h^+_{n\bar{m}} = -\frac{i\gamma r_0^3\,{}_1Y_{10}}{r^2 f^{1/2}}, \tag{6–62} $$

$$ 16\pi\frac{\tilde{\rho}^2}{\Delta}\Big(T_{lm} - T_{l\bar{m}}\Big) = \frac{i16\pi\mu\tilde{L}}{r_0^3 f_0^{1/2}}\delta(r-r_0)\delta(\cos\theta)\delta(\phi-\Omega t). \tag{6–63} $$

Imposing the radial jump conditions then gives us

$$\begin{aligned}\frac{i6\gamma}{f_0^{1/2}}\,{}_1Y_{10} &= \frac{i16\pi\mu\tilde{L}}{r_0^3 f_0^{1/2}}\delta(\cos\theta)\delta(\phi-\Omega t)\\ &= \frac{i16\pi\mu\tilde{L}}{r_0^3 f_0^{1/2}}\sum_{l=0}^{\infty}\sum_{m=-\ell}^{\ell}{}_1\bar{Y}_{\ell m}(\pi/2,\Omega t)\,{}_1Y_{\ell m}(\theta,\phi).\end{aligned}$$

After exploiting the orthogonality of the spin-weight 1 spherical harmonics and using

Equation 6–57 we obtain

$$\gamma = -\frac{4\pi\mu\tilde{L}}{r_0^3\sqrt{3}}. \tag{6–64}$$

It then follows from Equation 6–60 that

$$\delta J = \delta a M = \mu\tilde{L}, \tag{6–65}$$

which is precisely what is to be expected—all of the angular momentum in the (otherwise non-rotating) spacetime comes from the angular momentum of the particle. Once again we can verify directly from Equation 6–23 that we have correctly identified the angular momentum of the spacetime.

6.1.2.2 Even-parity dipole perturbations

First let's review what we already know: (1) We only need consider the $\ell = 1$ pieces. This was established above when we found the angular momentum to have $\ell = 1$ angular dependence. (2) The even-parity perturbation cannot contribute to the mass or angular momentum of the spacetime. These perturbations have already been accounted for. Furthermore, it is clear that in the absence of a source, there would be no $\ell = 1$ perturbation in the even parity sector.

In contrast to the situation with mass and angular momentum perturbations, where it was easy to write down the general form of the perturbations, we have no general form for the metric perturbation. Without prior knowledge of the perturbation, we must resort to solving the Einstein equations to determine the perturbation. This has been carried out by both Zerilli [7]and Detweiler & Poisson [17]. The result is a metric perturbation that can be written

$$
\begin{aligned}
h_{tr} &= -\frac{\mu\tilde{E}}{M}\Omega r_0^2 \sin\theta \sin(\phi - \Omega t)\delta(r - r_0), \\
h_{rr} &= 2\frac{\mu\tilde{E}}{M}\frac{r_0^2}{r_0 - 2M}\sin\theta\cos(\phi - \Omega t)\delta(r - r_0), \\
h_{r\theta} &= \frac{\mu\tilde{E}}{M}r_0^2\cos\theta\cos(\phi - \Omega t)\delta(r - r_0), \\
h_{r\phi} &= -\frac{\mu\tilde{E}}{M}r_0^2\sin\theta\sin(\phi - \Omega t)\delta(r - r_0).
\end{aligned}
$$

Note that the singular nature of this metric perturbation inherently excludes it from our analysis, as it destroys the continuity of the metric perturbation across Σ_p. It is well known [7] that the gauge transformation leading to this description can be interpreted as a transformation from a non-inertial frame tethered to the central black hole to the center of mass reference frame.

6.2 Kerr

In contrast to the situation in the Schwarzschild background, mass and angular momentum perturbations in the Kerr background are much more complicated. There is, however, one simplifying feature of the mass and angular momentum perturbations. Namely, the fact that both perturbations are stationary. Therefore the angular dependence is not given by the spin-weighted spheroidal harmonics, ${}_sS(a\omega, \theta, \phi)$, but rather their $a\omega = 0$ limit—the spin-weighted spherical harmonics.

The primary issue with treating the non-radiated multipoles in the context of matched spacetimes is the choice of the matching surface, Σ_p. Most of our discussion will be focused on this issue.

6.2.1 Mass Perturbations

In place of Birkhoff's theorem there is Wald's theorem [16], described earlier, assuring us that infinitesimal mass perturbations of the Kerr solution lead to other Kerr solutions (with infinitesimally different masses, of course) because such perturbations do not contribute the perturbations of ψ_0 or ψ_4 (which we will verify shortly). Thus we have the nonzero components of the mass perturbation given by

$$
\begin{aligned}
h_{tt} &= -\frac{2r\delta M}{r^2 + a^2\cos^2\theta}, && \text{(6–66)}\\
h_{rr} &= -\frac{2r(r^2 + a^2\cos^2\theta)\delta M}{(r^2 - 2Mr + a^2)^2}, && \text{(6–67)}\\
h_{t\phi} &= \frac{2ar\sin^2\theta\delta M}{r^2 + a^2\cos^2\theta}, && \text{(6–68)}\\
h_{\phi\phi} &= -\frac{2a^2 r\sin^4\theta\delta M}{r^2 + a^2\cos^2\theta}. && \text{(6–69)}
\end{aligned}
$$

Because the calculations in the Kerr spacetime are significantly more complicated, we will take a shortcut to determining the angular dependence of the perturbation by looking at the tetrad components of the metric perturbation, a result which we will in any case use shortly. In the symmetric tetrad (Equations 6–13) we have

$$
\begin{aligned}
h_{ll} &= -\frac{2r\delta M}{\Delta}, && \text{(6–70)}\\
h_{nn} &= -\frac{2r\delta M}{\Delta}, && \text{(6–71)}
\end{aligned}
$$

with all other components vanishing. Because both h_{ll} and h_{nn} are spin-weight 0, they have a natural decomposition into $\ell = 0$, $m = 0$ scalar (ordinary) spherical harmonics. Furthermore, utilizing Equation 3–27 we see that

$$\psi_0 = (\eth - \bar{\tau}')(\eth - \bar{\tau}')h_{ll} = \eth^2 h_{ll} - 2\bar{\tau}'\eth h_{ll} = 0, \qquad \text{(6–72)}$$

and similarly for ψ_4. Therefore, according to Wald's theorem, we are ensured that Equations 6–70 and 6–71 are a perturbation towards another Kerr solution.

With the angular dependence determined, we are led to consider a gauge vector of the form

$$\xi^a = (P(t), Q(r), 0, S(t,\phi)), \qquad \text{(6–73)}$$

which represents the largest class of gauge transformations consistent with form invariance. This requirement also restricts

$$S(t,\phi) = \beta t + S(\phi), \tag{6–74}$$

while stationarity again necessitates

$$P(t) = \alpha t. \tag{6–75}$$

Next we turn our attention to the matching problem.

In order to clarify the issues involved in the matching problem, we'll take a look at the matching conditions themselves. Suppose we've chosen some Σ_p, but have yet to specify it explicitly. That is, we have not yet written (or imposed) $r =$ something. The full set of matching conditions now take the form

$$h_{tt} \quad : \quad [\alpha](\tilde{\rho}^2 + 2rM) + 2[\beta]amr\sin^2\theta - 2r\delta M = 0, \tag{6–76}$$

$$\begin{aligned} h_{t\phi} \quad : \quad & 2[\alpha]amr - [\beta](a^2\Delta\cos^2\theta + r^2(r^2+a^2) + 2amr) \\ & +2amr\left[\frac{dS}{d\phi}\right] - 2ar\delta M = 0, \end{aligned} \tag{6–77}$$

$$h_{rr} \quad : \quad \left[\frac{dQ}{dr}\right] - \frac{r\delta M}{\Delta} = 0, \tag{6–78}$$

$$h_{\theta\theta} \quad : \quad [Q] = 0, \tag{6–79}$$

$$h_{\phi\phi} \quad : \quad a^2 r\sin^2\theta\delta M - \left[\frac{dS}{d\phi}\right](a^2\Delta\cos^2\theta + r^2(r^2+a^2) + 2amr) = 0, \tag{6–80}$$

where $\Delta = r^2 - 2Mr + a^2$ and $\tilde{\rho}^2 = r^2 + a^2\cos^2\theta$ as before and we have imposed the condition in Equation 6–79 in the others. Note that this reduces to the Schwarzschild result in Equations 6–39–6–41 by taking $a \to 0$ and setting $r = r_0$. This set of equations has a solution given by

$$
\begin{aligned}
\delta M &= \frac{\Delta}{r}\left[\frac{dQ}{dr}\right], && (6\text{–}81)\\
[\alpha] &= \frac{(r^2+a^2)^2}{(a^2\Delta\cos^2\theta + r^2(r^2+a^2) + 2amr)}\left[\frac{dQ}{dr}\right], && (6\text{–}82)\\
[\beta] &= \frac{\Delta}{(r^2+a^2)^2}[\alpha], && (6\text{–}83)\\
\left[\frac{dS}{d\phi}\right] &= -\frac{a^2\Delta\sin^2\theta}{(r^2+a^2)^2}[\alpha], && (6\text{–}84)
\end{aligned}
$$

which is again easily seen to reduce to the Schwarzschild result in the appropriate limit. From these equations we can see clearly the issues involved in choosing a matching surface. First, because the left sides of Equations 6–81–6–84 are all constant, this must be reflected in the right sides as well, which currently exhibit dependence on both r and θ. Presumably, some choice of $r = r(\theta)$ will enforce this, though it is currently unclear what that choice might be. Note that because of this, $r =$ constant surfaces do not appear to be good for matching.

What we have encountered appears to be an instance of a longstanding problem with matching the Kerr solution to a source [74, 75]. Namely, there is no known matter solution that correctly reproduces the multipole structure of the full Kerr geometry. In our problem, we're trying to force the issue by specifying both the metric and the source. On the other hand, because we're not matching the entire source, which includes quadrupole and higher moments, but only the non-radiated multipoles that merely take us from one Kerr solution to the next, it is not clear that the matching (in this instance) should fail. Though we are unable to perform the matching here, we maintain that nothing forbids it.

Most authors faced with this issue turn to the "slow rotation" approximation and keep only terms linear in a. In this approximation the Kerr metric can be viewed as the first order perturbation of the Schwarzschild solution to the Kerr solution. That is, the background is given by Schwarzschild plus a term identical to that in Equation 6–59. It is no surprise, then, that the resulting background geometry possesses enough spherical symmetry to allow for a straightforward treatment of the problem. It can be directly verified that such a procedure would remove the θ dependence in Equations 6–81–6–84 and allow for a matching on $r =$ constant surfaces (which are round 2-spheres in this case). Because this approach fails to shed new light on the situation in the full Kerr spacetime, we will not follow it here. Instead, we will focus on Equations 6–66-6–69, which we know to be correct.

Let's review the situation. We have established that the metric perturbation in

Equations 6–66–6–69 is a perturbation towards another Kerr solution with differing mass. Furthermore, we previously established that $\delta M = \mu \tilde{E}$ (Equation 6–26). The problem is that we are currently unable to perform the matching. In practice, the relevant portion of the spacetime is the exterior where gravitational radiation and the non-radiated multipoles are observed far away from the source. Because of this, we contend that considerations from the Komar formula and Wald's theorem together provide the correct perturbation in the exterior spacetime, independently of any matching considerations. Thus our result is likely useful in the EMRI problem even though we lack the metric perturbation everywhere in the spacetime. Moreover, the perturbation is still simple to interpret and asymptotically flat, so it is amenable to some analysis.

This being the case, we remark that mass perturbations of the Kerr background remain confined to the $s = 0$ sector of the perturbation. It is likely that this is true in general (at least in type D), but a general proof of this remains elusive. Furthermore, contrary to what one might expect in the Kerr spacetime, the mass perturbation does not mix spherical harmonic ℓ-modes, but is purely $\ell = 0$. We now turn our attention to angular momentum perturbations.

6.2.2 Angular Momentum Perturbations

Our lack of success in matching mass perturbations extends to angular momentum perturbations in precisely the same way, though the expressions involved are more complicated. This being the case, we will focus our attention on the general features of the angular momentum perturbation that can be obtained independently of a good matching. We begin by noting that the nonzero components on the metric perturbation are given by

$$
\begin{aligned}
h_{tt} &= \frac{4Mar\cos^2\theta\delta a}{(r^2-2Mr+a^2)^2}, && \text{(6–85)}\\
h_{rr} &= \frac{2a(r^2\sin^2\theta+2rM\cos^2\theta)\delta a}{(r^2-2Mr+a^2)^2}, && \text{(6–86)}\\
h_{t\phi} &= \frac{2Mar\sin^2\theta(r^2-a^2\cos^2\theta)}{(r^2+a^2\cos^2\theta)^2}, && \text{(6–87)}\\
h_{\theta\theta} &= -2a\cos^2\theta\delta a, && \text{(6–88)}\\
h_{\phi\phi} &= -\frac{2a\sin^2\theta[a^2(r^2+a^2\cos^2\theta)+r^3(r+2M\sin^2\theta)]\delta a}{(r^2+a^2\cos^2\theta)^2}. && \text{(6–89)}
\end{aligned}
$$

The corresponding tetrad components (in the symmetric tetrad) are given by

$$
\begin{aligned}
h_{ll} &= h_{nn} = \frac{a\delta a[(r^2-a^2)\sin^2\theta+2Mr(\cos^2\theta+1)]}{\tilde{\rho}^2\Delta}, && \text{(6–90)}\\
h_{ln} &= -\frac{a\delta a\sin^2\theta}{\tilde{\rho}^2}, && \text{(6–91)}\\
h_{m\bar{m}} &= -\frac{a\delta a(\cos^2\theta+1)}{\tilde{\rho}^2}, && \text{(6–92)}\\
h_{lm} &= h_{nm} = \frac{-i\delta a(a^2-Mr)\sin^2\theta}{(r+ia\cos\theta)\sqrt{\tilde{\rho}^2\Delta}}, && \text{(6–93)}\\
h_{mm} &= \frac{a\delta a\sin^2\theta}{(r+ia\cos\theta)^2}, && \text{(6–94)}
\end{aligned}
$$

where we have omitted the complex conjugates. Though it is not immediately obvious, this perturbation makes no contribution to ψ_0 or ψ_4, ensuring that this is a valid angular momentum perturbation.

In light of relatively straightforward results for mass perturbations, the nontrivial form of Equations 6–90–6–94 comes as a surprise. Unlike mass perturbations, angular momentum perturbations are not confined to a single s sector, whereas one might expect them to be exclusively $s = 1$, as intuition from working in the Schwarzschild background would lead us to believe. Note that although the perturbation appears in the $s = \pm 2$ sector of the metric, the vanishing of the $s = \pm 2$ components of the Weyl curvature keep the perturbation from contributing to gravitational radiation. More importantly, this is a sign that our intuition needs adjustment for working in the Kerr spacetime. In further contrast to our prior results, the complicated θ dependence in the tetrad components of the metric perturbation leads to mixing of the (spin-weighted) spherical harmonic ℓ-modes, a complication not previously encountered.

Another surprising feature is the fact that the perturbation is complex and thus exhibits both types of "parity". Although the static nature of the perturbation guarantees spin-weighted spherical harmonic angular dependence, we must be careful not to speak of parity in the Schwarzschild sense, but rather the real and imaginary parts of the perturbation. In any case the implications of this fact are presently unclear and remain to be determined in future work.

6.2.3 Discussion

In this section we discuss in more detail the possible problems with our matching by looking more closely at the assumptions that we made. This will lead naturally to ideas about future work that is beyond our present scope.

First off, one may speculate that our requirement of form invariance is perhaps too strict to allow for a proper matching. This does not appear to be the case. A result of Carter [76] implies that, due to stationarity and axial symmetry, the Kerr metric (in Boyer-Lindquist coordinates) has precisely the minimum number of nonzero components. Having established independently that the mass and angular momentum perturbations preserve theses properties of the background, Carter's result suggests that the problem lies elsewhere.

This leads us to consider whether the introduction of an infinitesimally thin shell of matter (which is effectively what Σ_p is), necessarily introduces non-Kerr perturbations. A shell (of some currently unspecified shape) would presumably be a differentially rotating object. It is unclear whether this disrupts the stationarity or axial symmetry of the exterior spacetime by the introduction of perturbations that we have neglected to account for. Wald's theorem [16] actually specifies two other types of perturbations that ψ_0 and ψ_4 cannot account for: perturbations towards the accelerating C-metrics and perturbations toward the NUT solution. In the work of Keidl, et. al. [69], where they concerned themselves with a static particle in the Schwarzschild geometry, it was found that the spacetime on the interior differs from that on the exterior by a perturbation towards the C-metrics. This makes physical sense because a static particle is not on a geodesic of the Schwarzschild spacetime and thus requires acceleration to keep it in place. Though we have no obvious physical reason to expect these perturbations for circular, equatorial orbits of the Kerr geometry and evidence from the Schwarzschild calculation suggests they should not contribute, we have not yet proven a result either way.

Finally, one question that we have overlooked entirely is the question of the stability of a thin shell. In the Schwarzschild background, this problem has been solved by Brady, Louko and Poisson [77], who showed that a thin shell is stable and satisfies the dominant energy condition almost all the way up to the location of the circular photon orbit (located at $r = 3M$). There are no such results to report on for the Kerr spacetime. The closest thing to a step in this direction is the work of Musgrave and Lake [78], who consider the matching of two Kerr spacetimes with different values of mass and angular momentum. Unfortunately, these authors were forced to resort to the slow rotation approximation discussed earlier. Strictly speaking, without knowledge of the existence of a stable shell of matter sufficiently close to the black hole, we are left to question the validity of our procedure. This is a problem we leave for future work.

CHAPTER 7
CONCLUSION

7.1 Summary

Currently, there is much effort being devoted to computing theoretical waveforms for gravitational wave detection. As a consequence, many researchers are looking for ways around the longstanding problems inherent in metric reconstruction in the Kerr spacetime. We feel that our new framework for perturbation theory presents a novel and robust tool for investigation in this area.

First and foremost, by taking advantage of the GHP formalism, our framework emphasizes and exploits those features common to all black hole spacetimes—their null structure as manifested in their Petrov type—which, since Teukolsky's derivation of the equation that bears his name, has been the only proven road to progress in this difficult subject. Such features have made an appearance through the simplification in the background GHP equations discussed in Chapter 1. These have lead to useful simplifications throughout. Besides these features, the built-in concepts of spin- and boost-weight have allowed us some intuitive insight into the nature of the fundamental quantities, without resorting to separation of variables.

The creation of GHPtools is the only reason any of this work was feasible in the first place. Coordinate-independence comes at the price of having to perform many nontrivial symbolic computations. GHPtools has not only allowed us to perform such computations, but also to present them in a fully simplified way, bringing some clarity even to previously known results. This is perhaps most evident in our treatment of the Teukolsky-Starobinsky identities, where the use of GHPtools masked all of the horrendous computational complexity involved in their derivation, by providing simple and concise results in the end.

Furthermore, the coordinate-free nature of our framework has further allowed us to work in great generality. This was seen in our treatment of the commuting operators of type D spaces in Chapter 1. Perhaps the best example of this is our proof of the existence of radiation gauges in sourcefree regions of spacetime. Our form of the Einstein equations and Held's integration technique is a powerful combination that allowed us to prove the result in arbitrary type II backgrounds, where the background integration isn't even complete.

Finally, our treatment of the non-radiated multipoles demonstrates the power of our framework when combined with existing techniques. Our results in the Kerr spacetime represent the first attempt at treating this part of the perturbation. Though we were unable to obtain the description in terms of a matched spacetime, we nevertheless provided a perturbation suitable for use in metric reconstruction.

7.2 Future Work

For all the generality inherent in the framework we developed, the applications we presented were narrowly focused around the problem of metric reconstruction in the Kerr spacetime. This leaves many problems to be explored, both within the realm of metric perturbations of Kerr and otherwise. We detail some of these below.

Perhaps most pressing is the generalization of our result for the non-radiated multipoles in the Kerr spacetime to encompass more general orbits. In particular, orbits not lying in the equatorial plane are of particular interest. Such orbits necessarily contain off-axis angular momentum, which in turn are widely thought to be related to Carter's constant (associated with the Killing tensor). For such orbits the Komar formulae fail to completely characterize these off-axis angular momentum components, so it is clear that we must look elsewhere for a solution. One potential avenue for progress is the Einstein equations themselves. As we noted in the previous chapter, mass and angular momentum perturbations are both stationary perturbations with angular dependence characterized by the spin-weighted spherical harmonics. The simplifications this brings for working with the Einstein equations is immense and may prove to make the problem tractable, without recourse to purely numerical methods. In any case, it seems clear that our framework, either alone or in conjunction with various other techniques, will help to clarify the problem enormously.

Another avenue worth pursuing is the commuting operator associated with the Killing tensor due to Beyer [44] (cf. Chapter 1). Recall that Beyer's operator commutes with the scalar wave equation in Kerr. It is very tempting to think that such an operator would exist for the Teukolsky equation as well. The GHP formalism, and GHPtools (of course), provide the ideal environment in which to study such questions. Furthermore, in the context of work performed by Jeffryes [79] concerning the implications of the existence of the Killing spinor (which includes a discussion of the Teukolsky-Starobinksy identities), it is natural to think that such an operator may in fact shed some new light on the Teukolsky-Starobinsky identities in the form presented in Chapter 5. Additionally, the existence of a generalization of Beyer's operator carries with it the possibility of new decomposition of functions in the Kerr spacetime—just as the existence of the Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ lead to separation in t and ϕ according to $e^{-i\omega t}$ and $e^{im\phi}$ (respectively), the eigenfunctions of a generalized Beyer operator may provide a new separation of variables in the Kerr spacetime. This is certainly a possibility worth pursuing.

Finally, both GHPtools and our form of the perturbed Einstein equations are entirely general and ready for use by researchers interested in more general (or even more specialized) backgrounds than Petrov type D. In particular, the class of type II spacetimes seems a likely candidate for further analysis, especially with the aid of the integration technique of Held. We have only begun to scratch the surface of the wide variety of problems these tools can help solve.

APPENDIX A
THE GHP RELATIONS

In this appendix, we give the GHP commutators, field equations and Bianchi identities, as well as the derivatives of the tetrad vectors. The full set of equations is obtained by applying to those listed prime, complex conjugation or both. When acting on a quantity of type $\{p, q\}$, the commutators are:

$$
\begin{aligned}
[\text{Þ}, \text{Þ}'] &= (\bar{\tau} - \tau')\eth + (\tau - \bar{\tau}')\eth' - p(\kappa\kappa' - \tau\tau' + \psi_2 + \Phi_{11} - \Pi) \\
&\quad - q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\psi}_2 + \Phi_{11} - \Pi), && \text{(A–1)} \\
[\text{Þ}, \eth] &= \bar{\rho}\eth + \sigma\eth' - \bar{\tau}'\text{Þ} - \kappa\text{Þ}' - p(\rho'\kappa - \tau'\sigma + \psi_1) \\
&\quad - q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}' + \Phi_{01}), && \text{(A–2)} \\
[\eth, \eth'] &= (\bar{\rho}' - \rho')\text{Þ} + (\rho - \bar{\rho})\text{Þ}' + p(\rho\rho' - \sigma\sigma' + \psi_2 - \Phi_{11} - \Pi) \\
&\quad - q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \bar{\psi}_2 - \Phi_{11} - \Pi). && \text{(A–3)}
\end{aligned}
$$

The GHP equations are:

$$
\begin{aligned}
\eth\rho - \eth'\sigma &= (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa - \psi_1 + \Phi_{01}, && \text{(A–4)} \\
\text{Þ}\rho - \eth'\kappa &= \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \tau'\kappa + \Phi_{00}, && \text{(A–5)} \\
\text{Þ}\sigma - \eth\kappa &= (\rho + \bar{\rho})\sigma - (\tau + \bar{\tau}')\kappa + \psi_0, && \text{(A–6)} \\
\text{Þ}\tau - \text{Þ}'\kappa &= (\tau - \tau')\rho + (\bar{\tau} - \tau')\sigma + \psi_1 + \Phi_{01}, && \text{(A–7)} \\
\eth\tau - \text{Þ}'\sigma &= -\rho'\sigma - \bar{\sigma}'\rho + \tau^2 + \kappa\bar{\kappa}' + \Phi_{02}, && \text{(A–8)} \\
\text{Þ}'\rho - \eth'\tau &= \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' - \psi_2 - 2\Pi. && \text{(A–9)}
\end{aligned}
$$

The Bianchi identities are given by:

$$
\begin{aligned}
\text{Þ}\psi_1 - \eth'\psi_0 - \text{Þ}\Phi_{01} + \eth\Phi_{00} &= -\tau'\psi_0 + 4\rho\psi_1 - 3\kappa\psi_2 + \bar{\tau}'\Phi_{00} - 2\bar{\rho}\Phi_{01} \\
&\quad - 2\sigma\Phi_{10} + 2\kappa\Phi_{11} + \bar{\kappa}\Phi_{02}, && \text{(A–10)}
\end{aligned}
$$

$$
\begin{aligned}
\text{Þ}\psi_2 - \eth'\psi_1 - \eth'\Phi_{01} + \text{Þ}'\Phi_{00} + 2\text{Þ}\Pi &= \sigma'\psi_0 - 2\tau'\psi_1 + 3\rho\psi_2 - 2\kappa\psi_3 + \bar{\rho}'\Phi_{00} \\
&\quad -2\bar{\tau}\Phi_{01} - 2\tau\Phi_{10} + 2\rho\Phi_{11} + \bar{\sigma}\Phi_{02}, \qquad \text{(A–11)} \\
\text{Þ}\psi_3 - \eth'\psi_2 - \text{Þ}\Phi_{21} + \eth\Phi_{20} - 2\eth'\Pi &= 2\sigma'\psi_1 - 3\tau'\psi_2 + 2\rho\psi_3 - \kappa\psi_4 - 2\rho'\Phi_{10} \\
&\quad +2\tau'\Phi_{11} + \bar{\tau}'\Phi_{20} - 2\bar{\rho}\Phi_{21} + \bar{\kappa}\Phi_{22}, \qquad \text{(A–12)} \\
\text{Þ}\psi_4 - \eth'\psi_3 - \eth'\Phi_{21} + \text{Þ}'\Phi_{20} &= 3\sigma'\psi_2 - 4\tau'\psi_3 + \rho\psi_4 - 2\kappa'\Phi_{10} + 2\sigma'\Phi_{11} \\
&\quad +\bar{\rho}'\Phi_{20} - 2\bar{\tau}\Phi_{21} + \bar{\sigma}\Phi_{22}. \qquad \text{(A–13)}
\end{aligned}
$$

Finally, the derivatives of the tetrad vectors are given by

$$
\begin{aligned}
\Theta_a l_b &= -l_a(\bar{\tau}m_b + \tau\bar{m}_b) - n_a(\bar{\kappa}m_b + \kappa\bar{m}_b) \\
&\quad +m_a(\bar{\sigma}m_b + \rho\bar{m}_b) + \bar{m}_a(\bar{\rho}m_b + \sigma\bar{m}_b) \qquad \text{(A–14)} \\
\Theta_a m_b &= -l_a(\kappa' l_b + \tau n_b) - n_a(\bar{\tau}' l_b + \kappa n_b) \\
&\quad +m_a(\bar{\rho}' l_b + \rho n_b) + \bar{m}_a(\bar{\sigma}' l_b + \sigma n_b) \qquad \text{(A–15)}
\end{aligned}
$$

APPENDIX B
THE PERTURBED EINSTEIN EQUATIONS IN GHP FORM

In this appendix we write the components of the perturbed Einstein tensor for an arbitrary algebraically special (Petrov type II) background. We have assumed the PND is aligned with l^a and made use of the Goldberg-Sachs theorem. Note that the equations for $\mathcal{E}_{lm}$, $\mathcal{E}_{nm}$ and $\mathcal{E}_{mm}$ are complex, so $\mathcal{E}_{l\bar{m}} = \bar{\mathcal{E}}_{lm}$ and so on:

$$\begin{aligned}
\mathcal{E}_{ll} \quad = \quad & \{(\eth' - \tau')(\eth - \bar{\tau}') + \rho(\text{Þ}' + \rho' - \bar{\rho}') - (\text{Þ} - \rho)\rho' + \Psi_2\}h_{ll} \\
& + \{-(\rho + \bar{\rho})(\text{Þ} + \rho + \bar{\rho}) + 4\rho\bar{\rho}\}h_{ln} \\
& + \{-(\text{Þ} - 3\bar{\rho})(\eth' - \tau' + \bar{\tau}) + \bar{\tau}\text{Þ} - \bar{\rho}\eth'\}h_{lm} \\
& + \{-(\text{Þ} - 3\rho)(\eth + \tau - \bar{\tau}') + \tau\text{Þ} - \rho\eth\}h_{l\bar{m}} \\
& + \{\text{Þ}(\text{Þ} - \rho - \bar{\rho}) + 2\rho\bar{\rho}\}h_{m\bar{m}},
\end{aligned} \tag{B–1}$$

$$\begin{aligned}
\mathcal{E}_{nn} \quad = \quad & \{2\kappa'\bar{\kappa}'\}h_{ll} \\
& + \{(\eth' - \bar{\tau})(\eth - \tau) + \bar{\rho}'(\text{Þ} - \rho + \bar{\rho}) - (\text{Þ}' - \bar{\rho}')\bar{\rho} + \bar{\Psi}_2 + 2\bar{\rho}\bar{\rho}'\}h_{nn} \\
& + \{-(\rho' + \bar{\rho}')(\text{Þ}' + \rho' + \bar{\rho}') + 4\rho'\bar{\rho}' - (\eth' - 2\bar{\tau})\bar{\kappa}' - (\eth - 2\tau)\kappa'\}h_{ln} \\
& + \{(\text{Þ}' - \bar{\rho}')\kappa' + \kappa'(\text{Þ}' - \rho' - \bar{\rho}') - \bar{\kappa}'\sigma'\}h_{lm} \\
& + \{(\text{Þ}' - \rho')\bar{\kappa}' + \bar{\kappa}'(\text{Þ}' - \bar{\rho}' - \rho') - \kappa'\bar{\sigma}'\}h_{l\bar{m}} \\
& + \{-(\text{Þ}' - 3\rho')(\eth' + \tau' - \bar{\tau}) + \tau'\text{Þ}' - \rho'\eth' - \kappa'\text{Þ} \\
& + (\text{Þ} - 2\rho + \bar{\rho})\kappa' + (\eth - 3\tau + \bar{\tau}')\sigma' + \eth(\sigma') - \Psi_3\}h_{nm} \\
& + \{-(\text{Þ}' - 3\bar{\rho}')(\eth + \bar{\tau}' - \tau) + \bar{\tau}'\text{Þ}' - \bar{\rho}'\eth - \bar{\kappa}'\text{Þ} \\
& + (\text{Þ} - 2\bar{\rho} + \rho)\bar{\kappa}' + (\eth' - 3\bar{\tau} + \tau')\bar{\sigma}' + \eth'(\bar{\sigma}') - \bar{\Psi}_3\}h_{n\bar{m}} \\
& + \{-(\eth' - 2\bar{\tau})\kappa' - \sigma'(\text{Þ}' - \rho' + \bar{\rho}')\}h_{mm} \\
& + \{-(\eth - 2\tau)\bar{\kappa}' - \bar{\sigma}'(\text{Þ}' - \bar{\rho}' + \rho')\}h_{\bar{m}\bar{m}} \\
& + \{\text{Þ}'(\text{Þ}' - \rho' - \bar{\rho}') + \kappa'(\tau - \bar{\tau}') + \bar{\kappa}'(\bar{\tau} - \tau') + 2\sigma'\bar{\sigma}' + 2\rho'\bar{\rho}'\}h_{m\bar{m}},
\end{aligned} \tag{B–2}$$

$$
\begin{aligned}
\mathcal{E}_{ln} \quad = \quad & \frac{1}{2}\{\rho'(Þ' - \rho') + \bar{\rho}'(Þ' - \bar{\rho}') + (ð - 2\bar{\tau}')\kappa' + (ð' - 2\tau')\bar{\kappa}' + 2\sigma'\bar{\sigma}'\}h_{ll} \\
& + \frac{1}{2}\{\rho(Þ - \rho) + \bar{\rho}(Þ - \bar{\rho})\}h_{nn} \\
& + \frac{1}{2}\{-(ð' + \tau' + \bar{\tau})(ð - \tau - \bar{\tau}') - (ð'ð + 3\tau\tau' + 3\bar{\tau}\bar{\tau}') + 2(\bar{\tau} + \tau')ð \\
& + (Þ - 2\bar{\rho})\rho' + (Þ' - 2\rho')\bar{\rho} - \bar{\rho}'(Þ + \rho) - \rho(Þ' + \bar{\rho}') - \Psi_2 - \bar{\Psi}_2\}h_{ln} \\
& + \frac{1}{2}\{(Þ' - 2\bar{\rho}')(ð' - \tau') + \bar{\tau}(Þ' + \rho' + \bar{\rho}') - \tau'(Þ' - \rho') \\
& \quad - (2ð' - \bar{\tau})\bar{\rho}' - (Þ - 2\bar{\rho})\kappa' + \sigma'(\tau - \bar{\tau}')\}h_{lm} \\
& + \frac{1}{2}\{(Þ' - 2\rho')(ð - \bar{\tau}') + \tau(Þ' + \bar{\rho}' + \rho') - \bar{\tau}'(Þ' - \bar{\rho}') \\
& \quad - (2ð - \tau)\rho' - (Þ - 2\rho)\bar{\kappa}' + \bar{\sigma}'(\bar{\tau} - \tau')\}h_{l\bar{m}} \\
& + \frac{1}{2}\{(Þ - 2\rho)(ð' - \bar{\tau}) + (\tau' + \bar{\tau})(Þ + \bar{\rho}) - 2(ð' - \tau')\rho - 2\bar{\tau}Þ\}h_{nm} \\
& + \frac{1}{2}\{(Þ - 2\bar{\rho})(ð - \tau) + (\bar{\tau}' + \tau)(Þ + \rho) - 2(ð - \bar{\tau}')\bar{\rho} - 2\tau Þ\}h_{n\bar{m}} \\
& + \frac{1}{2}\{-(ð' - \bar{\tau})(ð' - \tau') + \bar{\tau}(\bar{\tau} - \tau') - \sigma'\rho\}h_{mm} \\
& + \frac{1}{2}\{-(ð - \tau)(ð - \bar{\tau}') + \tau(\tau - \bar{\tau}') - \bar{\sigma}'\bar{\rho}\}h_{\bar{m}\bar{m}} \\
& + \frac{1}{2}\{(ð' + \tau' - \bar{\tau})(ð - \tau + \bar{\tau}') + (ð'ð - \tau\tau' - \bar{\tau}\bar{\tau}' + \tau\bar{\tau}) - (\Psi_2 + \bar{\Psi}_2) \\
& \quad + (Þ' - 2\rho')\bar{\rho} + (Þ - 2\bar{\rho})\rho' + \rho(3Þ' - 2\bar{\rho}') + \bar{\rho}'(3Þ - 2\rho) \\
& - 2Þ'Þ + 2\rho\bar{\rho}' + 2ð'(\tau) - \tau\bar{\tau}\}h_{m\bar{m}}, \qquad \text{(B–3)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_{lm} \quad = \quad & \frac{1}{2}\{(Þ' - \rho')(ð - \bar{\tau}') + (ð - \tau - 2\bar{\tau}')\bar{\rho}' - (ð - \tau)\rho' + \tau(Þ' + \rho') \\
& \quad + \bar{\sigma}'(ð' - \tau' + \bar{\tau}) + \bar{\Psi}_3 + \bar{\rho}\bar{\kappa}'\}h_{ll} \\
& + \frac{1}{2}\{-(Þ - \rho + \bar{\rho})(ð + \tau - \bar{\tau}') - (ð - 3\tau + \bar{\tau}')\bar{\rho} - 2\rho\bar{\tau}'\}h_{ln} \\
& + \frac{1}{2}\{-(Þ' + \bar{\rho}')(Þ - 2\bar{\rho}) + \rho(Þ' + 2\rho' - 2\bar{\rho}') - 4\rho'\bar{\rho} + 2\Psi_2 \\
& + (ð' + \bar{\tau})(ð - 2\bar{\tau}') - \tau(ð' + \tau' - 2\bar{\tau}) - \tau'(\tau - 4\bar{\tau}')\}h_{lm} \\
& + \frac{1}{2}\{-ð(ð - 2\tau) - \bar{\sigma}'(Þ + 2\bar{\rho} - 4\rho) - 2\bar{\tau}'(\tau - \bar{\tau}')\}h_{l\bar{m}} \\
& + \frac{1}{2}\{Þ(Þ - 2\rho) + 2\bar{\rho}(\rho - \bar{\rho})\}h_{nm}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\{-(\text{Þ}-\bar\rho)(\eth'-\tau'+\bar\tau)+2\bar\tau\bar\rho\}h_{mm}\\
&+\frac{1}{2}\{(\text{Þ}+\rho-\bar\rho)(\eth+\bar\tau'-\tau)+2\bar\tau'(\text{Þ}-2\rho)\\
&-(\eth-\tau-\bar\tau')\bar\rho+2\rho\tau\}h_{m\bar m},
\end{aligned}
\tag{B–4}
$$

$$
\begin{aligned}
\mathcal{E}_{n\bar m} \quad = \quad &\frac{1}{2}\{(\text{Þ}'-\rho')\kappa'+\kappa'\text{Þ}'+\bar\kappa'\sigma'\}h_{ll}\\
&+\frac{1}{2}\{(\text{Þ}-\rho+\bar\rho)(\eth'-\bar\tau)-(\eth'-2\tau'+\bar\tau)\rho+\tau'(\text{Þ}-\bar\rho)\}h_{nn}\\
&+\frac{1}{2}\{(-(\text{Þ}'-\rho'+\bar\rho')(\eth'+\tau'-\bar\tau)-(\eth'-3\tau'+\bar\tau)\bar\rho'+(\eth-\tau+\bar\tau')\sigma'\\
&\quad -2\sigma'\eth-\Psi_3-2\rho'\bar\tau\}h_{ln}\\
&+\{\sigma'(\rho'-2\bar\rho')-\kappa'(\tau'-2\bar\tau)+\frac{1}{2}\Psi_4\}h_{lm}\\
&+\frac{1}{2}\{(\text{Þ}'(\text{Þ}'-2\rho')+-\kappa'(\eth-2\tau+2\bar\tau')+\bar\kappa'(\eth'-4\tau'+2\bar\tau)\\
&\quad +2\bar\rho'(\rho'-\bar\rho')+2\sigma'\bar\sigma'\}h_{l\bar m}\\
&+\frac{1}{2}\{-\eth'(\eth'-2\tau')+\sigma'(\text{Þ}-2\rho+2\bar\rho)-2\bar\tau(\tau'-\bar\tau)\}h_{nm}\\
&+\frac{1}{2}\{-(\text{Þ}'-\bar\rho')(\text{Þ}+2\bar\rho)+\rho(\text{Þ}'-2\bar\rho')+2\bar\rho'(\text{Þ}-\rho)-\Psi_2-2\bar\Psi_2\\
&+(\eth'-3\bar\tau)\eth+\bar\tau'(2\eth'-\bar\tau+4\tau')-\tau(\eth'-2\bar\tau)\}h_{n\bar m}\\
&+\frac{1}{2}\{-(\eth'-\tau')\sigma'-\sigma'\eth'\}h_{mm}\\
&+\frac{1}{2}\{-(\text{Þ}'-\bar\rho')(\eth-\tau+\bar\tau')+2\bar\tau'\bar\rho'-\bar\kappa'(\text{Þ}-2\rho+2\bar\rho)+\eth'(\bar\sigma')-\bar\tau\bar\sigma'\}h_{\bar m\bar m}\\
&+\frac{1}{2}\{(\text{Þ}'+\rho'-\bar\rho')(\eth'-\tau'+\bar\tau)+2\bar\tau(\text{Þ}'-2\rho')-(\eth'-\tau'-\bar\tau)\bar\rho'+2\rho'\tau'\\
&+(\eth-\tau-\bar\tau')\sigma'+\sigma'\eth-\kappa'\text{Þ}-\Psi_3\}h_{m\bar m},
\end{aligned}
\tag{B–5}
$$

$$
\begin{aligned}
\mathcal{E}_{mm} \quad = \quad &\{(\text{Þ}'-2\rho')\bar\sigma'+\bar\kappa'(\eth+\tau-\bar\tau')\}h_{ll}\\
&+\{-\eth(\eth-\tau-\bar\tau')-2\tau\bar\tau'+\bar\sigma'(\rho-\bar\rho)\}h_{ln}\\
&+\{(\text{Þ}'-\rho')(\eth-\bar\tau')-(\eth-\tau-\bar\tau')\rho'+\tau(\text{Þ}'+\rho'-\bar\rho')-(\text{Þ}-2\bar\rho)\bar\kappa'\\
&-\bar\tau'(\text{Þ}+\bar\rho')+\bar\tau\bar\sigma'-\bar\Psi_3\}h_{lm}\\
&+\{-(\eth-\tau-\bar\tau')\bar\sigma'-\bar\sigma'(\eth-\tau)\}h_{l\bar m}
\end{aligned}
$$

$$+\{(Þ-\bar\rho)(ð-\tau)-(ð-\tau-\bar\tau')\bar\rho-\tau(Þ+\rho)+\bar\tau'(Þ-\rho+\bar\rho)\}h_{nm}$$
$$+\{-(Þ'-\rho')(Þ-\bar\rho)+(ð-\tau)\tau'-\tau(ð'+\tau'-\bar\tau)+\Psi_2\}h_{mm}$$
$$+\{(Þ-2\bar\rho)\bar\sigma'+(\tau+\bar\tau')ð+(\tau-\bar\tau')^2\}h_{m\bar m}, \tag{B–6}$$

$$\begin{aligned}
\mathcal{E}_{m\bar m} \quad = \quad & \frac{1}{2}\{Þ'(Þ'-\rho'-\bar\rho')+2\rho'\bar\rho'+\kappa'(\tau-\bar\tau')-\bar\kappa'(\bar\tau-\tau')+2\sigma'\bar\sigma'\}h_{ll} \\
& +\frac{1}{2}\{Þ(Þ-\rho-\bar\rho)+2\rho\bar\rho\}h_{nn} \\
& +\frac{1}{2}\{-(Þ'+\rho'-\bar\rho')(Þ-\rho+\bar\rho)-Þ'(Þ+\rho)+\rho(Þ'+\rho'-\bar\rho')-\bar\Psi_2 \\
& +(ð'-\bar\tau)(ð-\tau-\bar\tau')+ð'ð-(ð-2\bar\tau')\tau'-\bar\tau(2ð+\bar\tau') \\
& -2\tau(ð'-\bar\tau)+2\tau'\bar\tau'+\bar\rho\bar\rho'\}h_{ln} \\
& +\frac{1}{2}\{-(Þ'-2\rho')(ð'-2\bar\tau)+\bar\tau(Þ'+2\rho'-2\bar\rho')+2(ð-\bar\tau')\sigma'-\sigma'ð \\
& \quad -2\tau'\bar\rho'-2\kappa'(\rho-\bar\rho)-\Psi_3\}h_{lm} \\
& +\frac{1}{2}\{-(Þ'-2\bar\rho')(ð-2\tau)+\tau(Þ'+2\bar\rho'-2\bar\rho)+2(ð'-\tau')\bar\sigma'-\bar\sigma'ð' \\
& \quad -2\bar\tau'\rho'-2\bar\kappa'(\bar\rho-\rho)-\bar\Psi_3\}h_{l\bar m} \\
& +\frac{1}{2}\{-(Þ-2\bar\rho)(ð'-2\tau')+\tau'(Þ-2\rho-2\bar\rho)-2\rho\bar\tau+4\tau'\bar\rho\}h_{nm} \\
& +\frac{1}{2}\{-(Þ-2\rho)(ð-2\bar\tau')+\bar\tau'(Þ-2\bar\rho-2\rho)-2\bar\rho\tau+4\bar\tau'\rho\}h_{n\bar m} \\
& +\frac{1}{2}\{-\bar\tau(ð'-\bar\tau)-\tau'(ð'-\tau')-(Þ-2\bar\rho)\sigma'\}h_{mm} \\
& +\frac{1}{2}\{-\tau(ð-\tau)-\bar\tau'(ð-\bar\tau')-(Þ-2\rho)\bar\sigma'\}h_{\bar m\bar m} \\
& +\frac{1}{2}\{2Þ'Þ-(Þ'-\bar\rho')\bar\rho-(Þ-\rho)\rho'-\rho(Þ'-\rho'+\bar\rho')-\bar\rho'(Þ+\rho-\bar\rho) \\
& -(ð'-2\tau')\bar\tau'+\tau(ð'+2\bar\tau)-\tau'(ð-\bar\tau')+\bar\tau(ð+\tau)-ð'(\tau) \\
& -\Psi_2-\bar\Psi_2\}h_{m\bar m}.
\end{aligned} \tag{B–7}$$

APPENDIX C
INTEGRATION À LA HELD

We provide details of the integration that lead to Equation 4–17 and 4–25. As it turns out, the type II calculation is actually much simpler than the the type D calculation because it uses a tetrad in which $\tau = \tau' = 0$. Therefore we will work out the type D calculation in detail and the type II result mostly follows by setting certain quantities to zero, as indicated below.

We will need some results (and their complex conjugates) from the integration of the type D background:

$$\begin{aligned}
\tilde{\eth}'\rho &= -\pi^\circ\frac{\rho}{\bar{\rho}} - \alpha^\circ\rho - \bar{\tau}^\circ\rho^2, &\text{(C–1)}\\
\rho' &= \rho'^\circ\bar{\rho} - \frac{1}{2}\Psi^\circ\rho^2 - (\tilde{\eth}\bar{\tau}^\circ + \frac{1}{2}\Psi^\circ)\rho\bar{\rho} - \tau^\circ\bar{\tau}^\circ\rho^2\bar{\rho} + \bar{\tau}^\circ\bar{\pi}^\circ\rho \\
&\quad +\bar{\tau}^\circ\bar{\alpha}^\circ\rho^2 + \frac{1}{2}\pi^\circ\bar{\pi}^\circ\bar{\rho}\Big(\frac{1}{\rho^2} + \frac{1}{\bar{\rho}^2}\Big) + \frac{1}{2}\bar{\rho}\Big(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\Big)(\tilde{\eth} + \bar{\alpha}^\circ)\pi^\circ, &\text{(C–2)}\\
\tau &= -\bar{\pi}^\circ - \bar{\alpha}^\circ\rho + \tau^\circ\rho\bar{\rho}, &\text{(C–3)}\\
\tau' &= -\pi^\circ - \bar{\tau}^\circ\rho^2, &\text{(C–4)}\\
\Psi_2 &= \Psi^\circ\rho^3. &\text{(C–5)}
\end{aligned}$$

As noted in the text, $\pi^\circ \neq 0$ leads to the accelerating C-metrics, which we include for full generality. Henceforth the corresponding quantities in type II spacetimes can be obtained by setting $\tau^\circ = \pi^\circ = \alpha^\circ \Rightarrow 0$ and $\Psi^\circ \Rightarrow {\Psi_2}^\circ$[1] in the type D result. Thus, in type II

[1] This arises from the fact that in type D spacetimes there is only one non-vanishing Weyl scalar, Ψ_2. In type II spacetimes, however, both Ψ_3 and Ψ_4 are in general also nonzero. Though we do not refer to any of the other Weyl scalars in this work, we would like maintain agreement with the standard conventions.

spacetimes we have

$$\rho' = \rho'^{\circ}\bar{\rho} - \frac{1}{2}\Psi_2{}^{\circ}(\rho^2 + \rho\bar{\rho}), \tag{C–6}$$

$$\tau = 0, \tag{C–7}$$

$$\tau' = 0, \tag{C–8}$$

$$\Psi_2 = \Psi_2{}^{\circ}\rho^3, \tag{C–9}$$

the equation for $\tilde{ð}'\rho$ not following from the limiting process mentioned above. Note that the quantity $\tilde{ð}'\rho$ is never used in any of the integrations we perform in the type II background spacetime. We will also need the definitions of the new operators:

$$\tilde{Þ}' = Þ' - \bar{\tau}\tilde{ð} - \tau\tilde{ð}' + \tau\bar{\tau}(\frac{p}{\bar{\rho}} + \frac{q}{\rho}) + \frac{1}{2}(\frac{p\Psi_2}{\rho} + \frac{q\bar{\Psi}_2}{\bar{\rho}}), \tag{C–10}$$

$$\tilde{ð} = \frac{ð}{\bar{\rho}} + \frac{q\tau}{\rho}, \tag{C–11}$$

$$\tilde{ð}' = \frac{ð'}{\rho} + \frac{p\bar{\tau}}{\bar{\rho}}, \tag{C–12}$$

where p and q label the GHP type of the quantity being acted on. Additionally, in Sections 4.2 and 4.4 we make use of the commutator

$$\begin{aligned}[\tilde{ð}, \tilde{ð}'] = \frac{\bar{\rho}' - \rho'}{\rho\bar{\rho}}Þ + \Big(\frac{1}{\bar{\rho}} - \frac{1}{\rho}\Big)\tilde{Þ}' + p\Big\{\frac{\rho'}{\bar{\rho}} + \frac{1}{2}\Psi_2\Big(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\Big) + \tilde{ð}\Big(\frac{\bar{\tau}}{\bar{\rho}}\Big)\Big\} \\ - q\Big\{\frac{\bar{\rho}'}{\rho} + \frac{1}{2}\bar{\Psi}_2\Big(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\Big) + \tilde{ð}'\Big(\frac{\tau}{\rho}\Big)\Big\},\end{aligned} \tag{C–13}$$

which is valid in type D and (with $\tau = 0$) type II spacetimes.

We now begin with

$$Þ\xi_l = 0, \tag{C–14}$$

which integrates trivially to give

$$\xi_l = \xi_l{}^{\circ}. \tag{C–15}$$

With this information in hand, we can now integrate the equation governing ξ_m:

$$(Þ + \bar{\rho})\xi_m + (ð + \bar{\tau}')\xi_l = 0. \tag{C–16}$$

Rewriting the Þ piece and using Equation C–11 with $p = 1$ leads to

$$\frac{1}{\bar{\rho}}\text{Þ}(\bar{\rho}\xi_m) + \bar{\tau}'\xi_l + \bar{\rho}\tilde{\eth}\xi_l - \frac{\bar{\rho}\tau}{\rho}\xi_l = 0, \tag{C–17}$$

which, after substituting Equation C–3, the complex conjugate of Equation C–4 and Equation C–15 along with some rearranging, yields

$$\text{Þ}(\bar{\rho}\xi_m) = -\bar{\pi}^\circ\xi_l{}^\circ\Big(\frac{\bar{\rho}^2}{\rho} - \bar{\rho}\Big) + 2\tau^\circ\xi_l{}^\circ\bar{\rho}^3 - \bar{\rho}^2(\tilde{\eth} + \bar{\alpha}^\circ)\xi_l{}^\circ. \tag{C–18}$$

Integration then gives us

$$\xi_m = \xi_m{}^\circ\frac{1}{\bar{\rho}} - \bar{\pi}^\circ\xi_l{}^\circ\frac{1}{\rho} + \tau^\circ\xi_l{}^\circ\bar{\rho} - (\tilde{\eth} + \bar{\alpha}^\circ)\xi_l{}^\circ, \tag{C–19}$$

and the solution for $\xi_{\bar{m}}$ then follows from complex conjugation

$$\xi_{\bar{m}} = \xi_{\bar{m}}{}^\circ\frac{1}{\rho} - \pi^\circ\xi_l{}^\circ\frac{1}{\bar{\rho}} + \bar{\tau}^\circ\xi_l{}^\circ\rho - (\tilde{\eth}' + \alpha^\circ)\xi_l{}^\circ. \tag{C–20}$$

Finally, we are in a position to deal with ξ_n, by writing

$$\text{Þ}'\xi_l + \text{Þ}\xi_n + (\tau + \bar{\tau}')\xi_{\bar{m}} + (\bar{\tau} + \tau')\xi_m = 0, \tag{C–21}$$

in terms of Held's operators (Equations C–1, C–3 and C–4) as

$$\begin{aligned}&\text{Þ}\xi_n + \tilde{\text{Þ}}'\xi_l + \bar{\tau}\tilde{\eth}\xi_l + \tau\tilde{\eth}'\xi_l - \tau\bar{\tau}\Big(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\Big)\xi_l \\ &\quad - \frac{1}{2}\Big(\frac{\Psi_2}{\rho} + \frac{\bar{\Psi}_2}{\bar{\rho}}\Big)\xi_l + (\tau + \bar{\tau}')\xi_{\bar{m}} + (\bar{\tau} + \tau')\xi_m = 0.\end{aligned} \tag{C–22}$$

Substituting Equations C–3, C–4, C–5, C–15, C–19 and C–20, rearranging terms and letting the dust settle leads to

$$\begin{aligned}\text{Þ}\xi_n = &-\tilde{\text{Þ}}'\xi_l{}^\circ + \frac{1}{2}\Psi^\circ\xi_l{}^\circ\rho^2 + \frac{1}{2}\bar{\Psi}^\circ\xi_l{}^\circ\bar{\rho}^2 - \pi^\circ\bar{\pi}^\circ\xi_l{}^\circ\Big(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\Big) \\ &+ \tau^\circ\bar{\tau}^\circ\xi_l{}^\circ(\rho^2\bar{\rho} + \rho\bar{\rho}^2) - [\bar{\tau}^\circ\rho^2(\tilde{\eth} + \bar{\alpha}^\circ) + \tau^\circ\bar{\rho}^2(\tilde{\eth}' + \alpha^\circ)]\xi_l{}^\circ \\ &- [\pi^\circ(\tilde{\eth} + \bar{\alpha}^\circ) + \bar{\pi}^\circ(\tilde{\eth}' + \alpha^\circ)]\xi_l{}^\circ + 2\pi^\circ\xi_m{}^\circ\frac{1}{\bar{\rho}} + 2\bar{\pi}^\circ\xi_{\bar{m}}{}^\circ\frac{1}{\rho} \\ &+ \tau^\circ\xi_{\bar{m}}{}^\circ\Big(\frac{\bar{\rho}^2}{\rho} - \bar{\rho}\Big) + \bar{\tau}^\circ\xi_m{}^\circ\Big(\frac{\rho^2}{\bar{\rho}} - \rho\Big) + \alpha^\circ\xi_m{}^\circ + \bar{\alpha}^\circ\xi_{\bar{m}}{}^\circ.\end{aligned} \tag{C–23}$$

Integration then results in

$$\begin{aligned}
\xi_n = {} & {\xi_n}^\circ + \frac{1}{2}\Psi^\circ {\xi_l}^\circ \rho + \frac{1}{2}\bar{\Psi}^\circ {\xi_l}^\circ \bar{\rho} + \tau^\circ \bar{\tau}^\circ {\xi_l}^\circ \rho\bar{\rho} + \frac{1}{2}\pi^\circ \bar{\pi}^\circ {\xi_l}^\circ \Big(\frac{1}{\rho^2} + \frac{1}{\bar{\rho}^2}\Big) \\
& + \Big[\frac{\pi^\circ}{\rho}(\tilde{\eth} + \bar{\alpha}^\circ) + \frac{\bar{\pi}^\circ}{\bar{\rho}}(\tilde{\eth}' + \alpha^\circ)\Big]{\xi_l}^\circ + \frac{1}{2}\Big(\frac{1}{\rho} + \frac{1}{\bar{\rho}}\Big)\tilde{\text{Þ}}'{\xi_l}^\circ \\
& - [\bar{\tau}^\circ \rho(\tilde{\eth} + \bar{\alpha}^\circ) + \tau^\circ \bar{\rho}(\tilde{\eth}' + \alpha^\circ)]{\xi_l}^\circ + \bar{\tau}^\circ {\xi_m}^\circ \frac{\rho}{\bar{\rho}} + \tau^\circ {\xi_{\bar{m}}}^\circ \frac{\bar{\rho}}{\rho} \\
& - \pi^\circ {\xi_m}^\circ \frac{1}{\bar{\rho}^2} - \bar{\pi}^\circ {\xi_{\bar{m}}}^\circ \frac{1}{\rho^2} - \alpha^\circ {\xi_m}^\circ \frac{1}{\bar{\rho}} - \bar{\alpha}^\circ {\xi_{\bar{m}}}^\circ \frac{1}{\rho},
\end{aligned} \tag{C–24}$$

and our task is complete.

APPENDIX D
SPIN-WEIGHTED SPHERICAL HARMONICS

In this appendix, we present the basics of the theory of spin-weighted spherical harmonics [15, 80]. These functions have a natural place in the GHP formalism and provide a simple alternative to the more complicated tensor spherical harmonics. The discussion in this section takes place on the round 2-sphere. In that case, the action of ð on some quantity, χ, of spin-weight s is given by

$$ð\chi = -(\sin\theta)^s \left[\frac{\partial}{\partial\theta} + i\csc\theta\frac{\partial}{\partial\phi}\right](\sin\theta)^{-s}\chi, \tag{D–1}$$

and the action of ð′ is

$$ð'\chi = -(\sin\theta)^{-s} \left[\frac{\partial}{\partial\theta} - i\csc\theta\frac{\partial}{\partial\phi}\right](\sin\theta)^{s}\chi. \tag{D–2}$$

The spin-weighted spherical harmonics, ${}_sY_{\ell m}(\theta,\phi)$, are then defined in terms of the ordinary spherical harmonics by

$${}_sY_{\ell m}(\theta,\phi) = \begin{cases} \sqrt{\frac{(\ell-s)!}{(\ell+s)!}}ð^s Y_{\ell m}(\theta,\phi) & 0 \le s \le \ell, \\ \sqrt{\frac{(\ell+s)!}{(\ell-s)!}}(-1)^s (ð')^{-s} Y_{\ell m}(\theta,\phi) & -\ell \le s \le 0, \end{cases} \tag{D–3}$$

but are undefined for $|s| > \ell$. The basic properties of the ${}_sY_{\ell m}$ are easily seen to be

$${}_s\bar{Y}_{\ell m} = (-1)^{m+s}{}_{-s}Y_{\ell m}, \tag{D–4}$$

$$ð{}_sY_{\ell m} = \sqrt{(\ell-s)(\ell+s+1)}\;{}_{s+1}Y_{\ell m}, \tag{D–5}$$

$$ð'{}_sY_{\ell m} = -\sqrt{(\ell+s)(\ell-s+1)}\;{}_{s-1}Y_{\ell m}, \tag{D–6}$$

$$ð'ð{}_sY_{\ell m} = -(\ell-s)(\ell+s+1)\;{}_sY_{\ell m}. \tag{D–7}$$

For each value of s, the spin-weighted spherical harmonics are complete:

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} {}_s\bar{Y}_{\ell m}(\theta,\phi){}_sY_{\ell m}(\theta',\phi') = \delta(\phi-\phi')\delta(\cos\theta-\cos\theta'), \tag{D–8}$$

and orthogonal:

$$\int d\Omega\ {}_s\bar{Y}_{\ell m}(\theta,\phi)\,{}_sY_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}. \tag{D–9}$$

APPENDIX E
MAPLE CODE FOR GHPTOOLS

```
GHPtoolsv1:= module()

        description "GHPtools: Tools for working in the GHP (and NP)

                     formalism(s).";

        export

                schw, typed, flatxyz, GHPprime, GHPconj, NPprime, NPconj,

                DGHP, NPexpand, ezcomm, comm, getpq, tetcon, GHPmult,

                GHP2NP, tetupK, tetdnK, tetupS, tetdnS, tetupSB, tetdnSB,

                tdspec, tdsimp, GHP1, GHP1p, GHP1c, GHP1pc, GHP2, GHP2p,

                GHP2c, GHP2pc, GHP3, GHP3p, GHP3c, GHP3pc, GHP4, GHP4p,

                GHP4c, GHP4pc, GHP5, GHP5p, GHP5c, GHP5pc, GHP6, GHP6p,

                GHP6c, GHP6pc, COM1, COM1p, COM1c, COM1pc, COM2, COM2p,

                COM2c, COM2pc, COM3, COM3p, COM3c, COM3pc, BI1, BI1p,

                BI1c, BI1pc, BI2, BI2p, BI2c, BI2pc, BI3, BI3p, BI3c,

                BI3pc, BI4, BI4p, BI4c, BI4pc;

        local

                times, ssubs, THORN, THORNP, ETH, ETHP, GHPcomm,

                D_delta, D_deltabar, D_DD, D_Delta, GE1, GE2, GE3, GE4,

                GE5, GE6;

        option

                package;


schw:={kappa1=0, conjugate(kappa1)=0, sigma1=0, conjugate(sigma1)=0,

kappa=0, conjugate(kappa)=0, sigma=0, conjugate(sigma)=0, epsilon=0,

conjugate(epsilon)=0, tau1=0, conjugate(tau1)=0, tau=0,

conjugate(tau)=0,conjugate(rho1)=rho1, conjugate(rho)=rho, Psi1=0,
```

```
conjugate(epsilon1)=epsilon1, conjugate(beta)=beta1,
conjugate(beta1)=beta1, Psi0=0, conjugate(Psi0)=0, Psi1=0,
conjugate(Psi1)=0, conjugate(Psi2)=Psi2, Psi3=0,conjugate(Psi3)=0,
Psi4=0, conjugate(Psi4)=0, Phi00=0, conjugate(Phi00)=0, Phi01=0,
conjugate(Phi01)=0, Phi02=0, conjugate(Phi02)=0, Phi10=0,
conjugate(Phi10)=0, Phi11=0, conjugate(Phi11)=0, Phi12=0,
conjugate(Phi12)=0, Phi20=0, conjugate(Phi20)=0, Phi21=0,
conjugate(Phi21)=0, Phi22=0, conjugate(Phi22)=0, PI=0,
conjugate(0)=0};

typed:={Psi0=0, conjugate(Psi0)=0, Psi4=0, conjugate(Psi4)=0,
kappa1=0, conjugate(kappa1)=0, sigma1=0, conjugate(sigma1)=0, kappa=0,
conjugate(kappa)=0, sigma=0, conjugate(sigma)=0, epsilon=0, Psi1=0,
conjugate(Psi1)=0, Psi3=0,conjugate(Psi3)=0, Phi00=0,
conjugate(Phi00)=0, Phi01=0, conjugate(Phi01)=0, Phi02=0,
conjugate(Phi02)=0, Phi10=0, conjugate(Phi10)=0, Phi11=0,
conjugate(Phi11)=0, Phi12=0, conjugate(Phi12)=0, Phi20=0,
conjugate(Phi20)=0, Phi21=0, conjugate(Phi21)=0, Phi22=0,
conjugate(Phi22)=0, PI=0, conjugate(0)=0};

flatxyz:={kappa1=0, conjugate(kappa1)=0, sigma1=0,
conjugate(sigma1)=0, kappa=0, conjugate(kappa)=0, sigma=0,
conjugate(sigma)=0, epsilon=0, conjugate(epsilon)=0, tau1=0,
conjugate(tau1)=0, tau=0, conjugate(tau)=0,rho1=0, conjugate(rho1)=0,
rho=0, conjugate(rho)=0, Psi1=0, epsilon1=0, conjugate(epsilon1)=0,
beta=0, conjugate(beta)=0, beta1=0, conjugate(beta1)=0, Psi0=0,
conjugate(Psi0)=0, Psi1=0, conjugate(Psi1)=0, Psi2=0,
```

```
conjugate(Psi2)=0, Psi3=0,conjugate(Psi3)=0, Psi4=0,
conjugate(Psi4)=0, Phi00=0, conjugate(Phi00)=0, Phi01=0,
conjugate(Phi01)=0, Phi02=0, conjugate(Phi02)=0, Phi10=0,
conjugate(Phi10)=0, Phi11=0, conjugate(Phi11)=0, Phi12=0,
conjugate(Phi12)=0, Phi20=0, conjugate(Phi20)=0, Phi21=0,
conjugate(Phi21)=0, Phi22=0, conjugate(Phi22)=0, PI=0,
conjugate(0)=0};

# ssubs performs the simple task of substituting the spacetime list into its
# argument

ssubs := proc(expr)
        if evalb(eval(spacetime)= 'spacetime') or
evalb(eval(spacetime)='none') then
                return(expr);
        else
                return(subs(spacetime,expr));
        end if;
end proc;

getpq := proc(expr)
        local p,q;
        if   evalb(expr=hnn) then p:=-2; q:=-2
        elif evalb(expr=hln) then p:=0; q:=0
        elif evalb(expr=hnmb) then p:=-2; q:=0
        elif evalb(expr=hnm) then p:=0; q:=-2
        elif evalb(expr=hll) then p:=2; q:=2
```

```
elif evalb(expr=hlmb) then p:=0; q:=2
elif evalb(expr=hlm) then p:=2; q:=0
elif evalb(expr=hmbmb) then p:=-2; q:=2
elif evalb(expr=hmmb) then p:=0; q:=0
elif evalb(expr=hmm) then p:=2; q:=-2
elif evalb(expr=rho) then p:=1; q:=1
elif evalb(expr=conjugate(rho)) then p:=1; q:=1
elif evalb(expr=rho1) then p:=-1; q:=-1
elif evalb(expr=conjugate(rho1)) then p:=-1; q:=-1
elif evalb(expr=kappa) then p:=3; q:=1
elif evalb(expr=conjugate(kappa)) then p:=1; q:=3
elif evalb(expr=kappa1) then p:=-3; q:=-1
elif evalb(expr=conjugate(kappa1)) then p:=-1; q:=-3
elif evalb(expr=tau) then p:=1; q:=-1
elif evalb(expr=conjugate(tau)) then p:=-1; q:=1
elif evalb(expr=tau1) then p:=-1; q:=1
elif evalb(expr=conjugate(tau1)) then p:=1; q:=-1
elif evalb(expr=sigma) then p:=3; q:=-1
elif evalb(expr=conjugate(sigma)) then p:=-1; q:=3
elif evalb(expr=sigma1) then p:=-3; q:=1
elif evalb(expr=conjugate(sigma1)) then p:=1; q:=-3
elif evalb(expr=conjugate(Psi0)) then p:=0; q:=4
elif evalb(expr=conjugate(Psi1)) then p:=0; q:=2
elif evalb(expr=conjugate(Psi2)) then p:=0; q:=0
elif evalb(expr=conjugate(Psi3)) then p:=0; q:=-2
elif evalb(expr=conjugate(Psi4)) then p:=0; q:=-4
elif evalb(expr=Psi0) then p:=4; q:=0
```

```
        elif evalb(expr=Psi1) then p:=2; q:=0
        elif evalb(expr=Psi2) then p:=0; q:=0
        elif evalb(expr=Psi3) then p:=-2; q:=0
        elif evalb(expr=Psi4) then p:=-4; q:=0
        elif evalb(expr=phi) then p:=pp; q:=pq
        elif evalb(expr=conjugate(phi)) then p:=pq; q:=pp
        elif evalb(expr=phi1) then p:=-pp; q:=-pq
        elif evalb(expr=conjugate(phi1)) then p:=-pq; q:=-pp
        else
                p:=UNKNOWN; q:=UNKNOWN
        end if;
        return(p,q);
end proc;

GHPprime := proc(expr)
    return(subs({ldn=ndn, lup=nup, ndn=ldn, nup=lup,
    mdn=conjugate(mdn), mup=conjugate(mup), conjugate(mup)=mup,
    conjugate(mdn)=mdn, hll=hnn, hnn=hll, hlm=hnmb, hnmb=hlm, hlmb=hnm,
    hnm=hlmb, hmm=hmbmb, hmbmb=hmm, th=thp, thp=th, eth=ethp, ethp=eth,
    rho=rho1, conjugate(rho)=conjugate(rho1), rho1=rho,
    conjugate(rho1)=conjugate(rho), kappa=kappa1,
    conjugate(kappa)=conjugate(kappa1), kappa1=kappa,
    conjugate(kappa1)=conjugate(kappa), tau=tau1,
    conjugate(tau)=conjugate(tau1), tau1=tau,
    conjugate(tau1)=conjugate(tau), sigma=sigma1,
    conjugate(sigma)=conjugate(sigma1), sigma1=sigma,
    conjugate(sigma1)=conjugate(sigma), epsilon=epsilon1,
```

```
    conjugate(epsilon)=conjugate(epsilon1), epsilon1=epsilon,
    conjugate(epsilon1)=conjugate(epsilon), beta=beta1,
    conjugate(beta)=conjugate(beta1), beta1=beta,
    conjugate(beta1)=conjugate(beta), Psi0=Psi4,
    conjugate(Psi0)=conjugate(Psi4), Psi4=Psi0,
    conjugate(Psi4)=conjugate(Psi0), Psi1=Psi3,
    conjugate(Psi1)=conjugate(Psi3), Psi3=Psi1,
    conjugate(Psi3)=conjugate(Psi1), Phi00=Phi22, conjugate(Phi00)=Phi22,
    Phi01=Phi21, conjugate(Phi01)=Phi12, Phi02=Phi20,
    conjugate(Phi02)=Phi02, Phi10=Phi12, conjugate(Phi10)=Phi21,
    Phi12=Phi10, conjugate(Phi12)=Phi01, Phi20=Phi02,
    conjugate(Phi20)=Phi20, Phi21=Phi01, conjugate(Phi21)=Phi10,
    Phi22=Phi00, conjugate(Phi22)=Phi00, phi=phi1, phi1=phi,
    conjugate(phi)=conjugate(phi1), conjugate(phi1)=conjugate(phi), p=-p,
    q=-q, pp=-pp, pq=-pq,hl=hn,hn=hl\},expr));
end proc;

NPprime := proc(expr)
        return(GHPprime(expr));
end proc;

GHPconj := proc(expr)
        return(subs({mdn=conjugate(mdn), mup=conjugate(mup),
    conjugate(mdn)=mdn, conjugate(mup)=mup, hlm=hlmb, hlmb=hlm, hnm=hnmb,
    hnmb=hnm, hmm=hmbmb, hmbmb=hmm, ethp=eth,eth=ethp,
    beta=conjugate(beta), conjugate(beta)=beta, beta1=conjugate(beta1),
    conjugate(beta1)=beta1,  epsilon=conjugate(epsilon),
```

```
    conjugate(epsilon)=epsilon, epsilon1=conjugate(epsilon1),
    conjugate(epsilon1)=epsilon1, kappa=conjugate(kappa),
    conjugate(kappa)=kappa, kappa1=conjugate(kappa1),
    conjugate(kappa1)=kappa1, sigma=conjugate(sigma),
    conjugate(sigma)=sigma, sigma1=conjugate(sigma1),
    conjugate(sigma1)=sigma1, rho=conjugate(rho), conjugate(rho)=rho,
    rho1=conjugate(rho1), conjugate(rho1)=rho1, tau=conjugate(tau),
    conjugate(tau)=tau, tau1=conjugate(tau1), conjugate(tau1)=tau1,
    Psi0=conjugate(Psi0), conjugate(Psi0)=Psi0, Psi1=conjugate(Psi1),
    conjugate(Psi1)=Psi1, Psi2=conjugate(Psi2), conjugate(Psi2)=Psi2,
    Psi3=conjugate(Psi3), conjugate(Psi3)=Psi3, Psi4=conjugate(Psi4),
    conjugate(Psi4)=Psi4,Phi01=Phi10,Phi10=Phi01,Phi02=Phi20,Phi20=Phi02,P
    hi12=Phi21,Phi21=Phi12, phi=conjugate(phi), conjugate(phi)=phi,
    phi1=conjugate(phi1), conjugate(phi1)=phi1,
    p=q,q=p,pp=pq,pq=pp},expr));
end proc;


NPconj := proc(expr)
        return(GHPconj(expr))
end proc;


THORN := proc(f)
        local i, rest, temp;
        if type(f, 'symbol') then return map( th, f)
        elif type(f, 'constant') then 0
        elif type( f, list ) then map( THORN, f)
        elif type( f, set  ) then map( THORN, f)
```

```
elif type( f, `=`  ) then map( THORN, f)
elif type( f, `+`  ) then map( THORN, f)
elif type( f, `*`  ) then
    rest := mul(op(i,f), i=2..nops(f));
    THORN(op(1,f))*rest + op(1,f)*THORN(rest);
elif type( f, `^` ) then
    op(2,f)*op(1,f)^(op(2,f)-1)*THORN(op(1,f))
elif type( f, function ) then
    if op(0,f) = 'th' then
        temp:=THORN(op(f));
        return map(THORN, temp);
    elif op(0,f) = 'thp' then
        return apply(th, f);
    elif op(0,f) = 'eth' then
        return apply(th, f);
    elif op(0,f) = 'ethp' then
        return apply(th, f);
    elif op(0,f) = 'conjugate' then
        return apply(th, f);
    elif op(0,f) = 'T' then
        return apply(th,f);
    elif op(0,f) = 'ln' then
        return THORN(op(1,f))/op(1,f);
    else
        error "routine not built to handle that
            function: %1", op(0,f);
    end if;
```

```
        else
            error "routine not built to handle that type: \%1",
                whattype(f);
        end if;
end proc;

THORNP := proc(f)
        local i, rest, temp;
        if type(f, 'symbol') then return map( thp, f)
        elif type(f, 'constant') then 0
        elif type( f, list ) then map( THORNP, f)
        elif type( f, set  ) then map( THORNP, f)
        elif type( f, `=`  ) then map( THORNP, f)
        elif type( f, `+`  ) then map( THORNP, f)
        elif type( f, `*`  ) then
            rest := mul(op(i,f), i=2..nops(f));
            THORNP(op(1,f))*rest + op(1,f)*THORNP(rest);
        elif type( f, `^` ) then
            op(2,f)*op(1,f)^(op(2,f)-1)*THORNP(op(1,f))
        elif type( f, function ) then
            if op(0,f) = 'th' then
                return apply(thp, f);
            elif op(0,f) = 'thp' then
                temp:=THORNP(op(f));
                return map(THORNP, temp);
            elif op(0,f) = 'eth' then
                return apply(thp, f);
```

```
            elif op(0,f) = 'ethp' then
                return apply(thp, f);
            elif op(0,f) = 'conjugate' then
                return apply(thp, f);
            elif op(0,f) = 'T' then
                return apply(thp,f);
            elif op(0,f) = 'ln' then
                return THORNP(op(1,f))/op(1,f);
            else
                error "routine not built to handle that
                    function: %1", op(0,f);
            end if;
        else
            error "routine not built to handle that type: %1",
                whattype(f);
        end if;
end proc;

ETH := proc(f)
        local i, rest, temp;
        if type(f, 'symbol') then return map( eth, f)
        elif type(f, 'constant') then 0
        elif type( f, list ) then map( ETH, f)
        elif type( f, set  ) then map( ETH, f)
        elif type( f, `=`  ) then map( ETH, f)
        elif type( f, `+`  ) then map( ETH, f)
        elif type( f, `*`  ) then
```

```
        rest := mul(op(i,f), i=2..nops(f));
        ETH(op(1,f))*rest + op(1,f)*ETH(rest);
    elif type( f, `^` ) then
        op(2,f)*op(1,f)^(op(2,f)-1)*ETH(op(1,f))
    elif type( f, function ) then
        if op(0,f) = 'th' then
            return apply(eth, f);
        elif op(0,f) = 'thp' then
            return apply(eth, f);
        elif op(0,f) = 'eth' then
            temp:=ETH(op(f));
            return map(ETH, temp);
        elif op(0,f) = 'ethp' then
            return apply(eth, f);
        elif op(0,f) = 'conjugate' then
            return apply(eth, f);
        elif op(0,f) = 'T' then
            return apply(eth,f);
        elif op(0,f) = 'ln' then
            return ETH(op(1,f))/op(1,f);
        else
            error "routine not built to handle that
                function: %1", op(0,f);
        end if;
    else
        error "routine not built to handle that type: %1",
            whattype(f);
```

```
        end if;
end proc;


ETHP := proc(f)
        local i, rest, temp;
        if type(f, 'symbol') then return map( ethp, f)
        elif type(f, 'constant') then 0
        elif type( f, list ) then map( ETHP, f)
        elif type( f, set  ) then map( ETHP, f)
        elif type( f, `=`  ) then map( ETHP, f)
        elif type( f, `+`  ) then map( ETHP, f)
        elif type( f, `*`  ) then
            rest := mul(op(i,f), i=2..nops(f));
            ETHP(op(1,f))*rest + op(1,f)*ETHP(rest);
        elif type( f, `^` ) then
            op(2,f)*op(1,f)^(op(2,f)-1)*ETHP(op(1,f))
        elif type( f, function ) then
            if op(0,f) = 'th' then
                return apply(ethp, f);
            elif op(0,f) = 'thp' then
                return apply(ethp, f);
            elif op(0,f) = 'eth' then
                return apply(ethp, f);
            elif op(0,f) = 'ethp' then
                temp:=ETHP(op(f));
                return map(ETHP, temp);
            elif op(0,f) = 'conjugate' then
```

```
                return apply(ethp, f);
            elif op(0,f) = 'T' then
                return apply(ethp,f);
            elif op(0,f) = 'ln' then
                return ETHP(op(1,f))/op(1,f);
            else
                error "routine not built to handle that
                    function: %1", op(0,f);
            end if;
        else
            error "routine not built to handle that type: %1",
                whattype(f);
        end if;
end proc;

DGHP := proc(expr)
        local result;
        result:=subs({th=THORN,thp=THORNP,eth=ETH,ethp=ETHP},ssubs(expr));
        return(expand(eval(result)));
end proc;

D_delta := proc(f)
        local i, rest, temp;
        if type(f, 'symbol') then return map( delta, f)
        elif type(f, 'constant') then 0
        elif type( f, list ) then map( D_delta, f)
        elif type( f, set  ) then map( D_delta, f)
```

```
elif type( f, `=` ) then map( D_delta, f)
elif type( f, `+` ) then map( D_delta, f)
elif type( f, `*` ) then
    rest := mul(op(i,f), i=2..nops(f));
    D_delta(op(1,f))*rest + op(1,f)*D_delta(rest);
elif type( f, `^` ) then
    op(2,f)*op(1,f)^(op(2,f)-1)*D_delta(op(1,f))
elif type( f, function ) then
    if op(0,f) = 'delta' then
        temp:=D_delta(op(f));
        return map(D_delta, temp);
    elif op(0,f) = 'conjugate(delta)' then
        return apply(delta, f);
    elif op(0,f) = 'Delta' then
        return apply(delta, f);
    elif op(0,f) = 'DD' then
        return apply(delta, f);
    elif op(0,f) = 'conjugate' then
        return apply(delta, f);
    else
        error "routine not built to handle that
            function: %1", op(0,f);
    end if;
else
    error "routine not built to handle that type: %1",
        whattype(f);
end if;
```

```
end proc;


D_deltabar := proc(f)
        local i, rest, temp;
        if type(f, 'symbol') then return map( conjugate(delta), f)
        elif type(f, 'constant') then 0
        elif type( f, list ) then map( D_deltabar, f)
        elif type( f, set  ) then map( D_deltabar, f)
        elif type( f, `=`  ) then map( D_deltabar, f)
        elif type( f, `+`  ) then map( D_deltabar, f)
        elif type( f, `*`  ) then
            rest := mul(op(i,f), i=2..nops(f));
            D_deltabar(op(1,f))*rest + op(1,f)*D_deltabar(rest);
        elif type( f, `^` ) then
            op(2,f)*op(1,f)^(op(2,f)-1)*D_deltabar(op(1,f))
        elif type( f, function ) then
            if op(0,f) = 'delta' then
                return apply(conjugate(delta), f);
            elif op(0,f) = 'conjugate(delta)' then
                temp:=D_deltabar(op(f));
                return map(D_deltabar, temp);
            elif op(0,f) = 'Delta' then
                return apply(conjugate(delta), f);
            elif op(0,f) = 'DD' then
                return apply(conjugate(delta), f);
            elif op(0,f) = 'conjugate' then
                return apply(conjugate(delta), f);
```

```
            else
                error "routine not built to handle that
                    function: %1", op(0,f);
            end if;
        else
            error "routine not built to handle that type: %1",
                whattype(f);
        end if;
end proc;

D_DD := proc(f)
        local i, rest, temp;
        if type(f, 'symbol') then return map( DD, f)
        elif type(f, 'constant') then 0
        elif type( f, list ) then map( D_DD, f)
        elif type( f, set  ) then map( D_DD, f)
        elif type( f, `=`  ) then map( D_DD, f)
        elif type( f, `+`  ) then map( D_DD, f)
        elif type( f, `*`  ) then
            rest := mul(op(i,f), i=2..nops(f));
            D_DD(op(1,f))*rest + op(1,f)*D_DD(rest);
        elif type( f, `^` ) then
            op(2,f)*op(1,f)^(op(2,f)-1)*D_DD(op(1,f))
        elif type( f, function ) then
            if op(0,f) = 'delta' then
                return apply(DD, f);
            elif op(0,f) = 'conjugate(delta)' then
```

```
            return apply(DD, f);
        elif op(0,f) = 'Delta' then
            return apply(DD, f);
        elif op(0,f) = 'DD' then
            temp:=D_DD(op(f));
            return map(D_DD, temp);
        elif op(0,f) = 'conjugate' then
            return apply(DD, f);
        else
            error "routine not built to handle that
                function: %1", f;
        end if;
    else
        error "routine not built to handle that type: %1",
            whattype(f);
    end if;
end proc;

D_Delta := proc(f)
    local i, rest, temp;
    if type(f, 'symbol') then return map( Delta, f)
    elif type(f, 'constant') then 0
    elif type( f, list ) then map( D_Delta, f)
    elif type( f, set  ) then map( D_Delta, f)
    elif type( f, `=`  ) then map( D_Delta, f)
    elif type( f, `+`  ) then map( D_Delta, f)
    elif type( f, `*`  ) then
```

```
            rest := mul(op(i,f), i=2..nops(f));
            D_Delta(op(1,f))*rest + op(1,f)*D_Delta(rest);
        elif type( f, `^` ) then
            op(2,f)*op(1,f)^(op(2,f)-1)*D_Delta(op(1,f))
        elif type( f, function ) then
            if op(0,f) = 'delta' then
                return apply(Delta, f);
            elif op(0,f) = 'conjugate(delta)' then
                return apply(Delta, f);
            elif op(0,f) = 'Delta' then
                temp:=D_Delta(op(f));
                return map(D_Delta, temp);
            elif op(0,f) = 'DD' then
                return apply(Delta, f);
            elif op(0,f) = 'conjugate' then
                return apply(Delta, f);
            else
                error "routine not built to handle that
                    function: %1", f;
            end if;
        else
            error "routine not built to handle that type: %1",
                whattype(f);
        end if;
end proc;


NPexpand := proc(expr)
```

```
        local result;
        result:=subs({delta=D_delta,conjugate(delta)=D_deltabar,
            DD=D_DD,Delta=D_Delta},expr);
        return(expand(eval(result)));
end proc;

times := proc(x,y)
        local i,z,result;
        result:=0;
        for i from 1 to nops(x) do
                if nops(x) <> 1 then z:=op(i,x) else z:=x end if;
                if (z='delta' or z='conjugate(delta)' or z='DD' or
                    z='Delta' or op(0,z)='delta' or
                    op(0,z)='conjugate(delta)' or op(0,z)='DD' or
                    op(0,z)='Delta') then result:=result+apply(z,y)
                else result:=result+z*y
                end if;
        end do;
        return expand(result);
end proc;

GHP2NP := proc(expr)
        local i,result,z,p,q,w;
        result:=expr;
        for i from 1 to nops(expr) do
                w:=op(i,expr);
                if (op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='th'
```

```
    and op(0,op(1,op(nops(w),w))) ='th') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w +
        w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w)))))*(times((DD -(p+1)*epsilon
        -(q+1)*conjugate(epsilon)),times((DD - p*epsilon -
        q*conjugate(epsilon)),op(1,op(1,op(nops(w),w))))));
elif (op(0,w)='th' and op(0,op(1,w))='th') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((DD - (p+1)*epsilon
        - (q+1)*conjugate(epsilon)),times((DD -
        p*epsilon - q*conjugate(epsilon)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='thp'
    and op(0,op(1,op(nops(w),w))) ='thp') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w +
        w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w)))))*(times((Delta +
        (p-1)*epsilon1+(q-1)*conjugate(epsilon1)),times((Delta
        +p*epsilon1+q*conjugate(epsilon1)),
        op(1,op(1,op(nops(w),w))))));
elif (op(0,w)='thp' and op(0,op(1,w))='thp') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((Delta +
        (p-1)*epsilon1+(q-1)*conjugate(epsilon1)),times((Delta
        +p*epsilon1 + q*conjugate(epsilon1)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='th'
```

```
    and op(0,op(1,op(nops(w),w))) ='thp') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w +
        w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w)))))*(times((DD -(p-1)*epsilon
        -(q-1)*conjugate(epsilon)),times((Delta + p*epsilon1 +
        q*conjugate(epsilon1)),op(1,op(1,op(nops(w),w))))));
elif (op(0,w)='th' and op(0,op(1,w))='thp') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((DD - (p-1)*epsilon
        - (q-1)*conjugate(epsilon)),times((Delta +
        p*epsilon1 + q*conjugate(epsilon1)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='eth'
    and op(0,op(1,op(nops(w),w))) ='eth') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w
        +w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w)))))
        *(times((delta - (p+1)*beta +
        (q-1)*conjugate(beta1)),times((delta - p*beta +
        q*conjugate(beta1)),op(1,op(1,op(nops(w),w))))));
elif (op(0,w)='eth' and op(0,op(1,w))='eth') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((delta - (p+1)*beta
        + (q-1)*conjugate(beta1)),times((delta - p*beta +
        q*conjugate(beta1)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='eth'
```

```
    and op(0,op(1,op(nops(w),w))) ='th') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w+
        w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w)))))*(times((delta -
        (p+1)*beta +(q+1)*conjugate(beta1)),times((DD -
        p*epsilon-q*conjugate(epsilon)),
        op(1,op(1,op(nops(w),w))))));
elif (op(0,w)='eth' and op(0,op(1,w))='th') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((delta - (p+1)*beta
        + (q+1)*conjugate(beta1)),times((DD - p*epsilon -
        q*conjugate(epsilon)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='eth'
    and op(0,op(1,op(nops(w),w))) ='thp') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w +
        w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w))))) *(times((delta -
        (p-1)*beta +(q-1)*conjugate(beta1)),times((Delta +
        p*epsilon1+q*conjugate(epsilon1)),
        op(1,op(1,op(nops(w),w))))));
elif (op(0,w)='eth' and op(0,op(1,w))='thp') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((delta - (p-1)*beta
        + (q-1)*conjugate(beta1)),times((Delta + p*epsilon1
        +q*conjugate(epsilon1)),op(1,op(1,w))));
```

```
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='eth'
    and op(0,op(1,op(nops(w),w))) ='ethp') then\\
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result-w+w/op(1,op(0,op(nops(w),w)))
    (op(0,op(1,op(nops(w),w)))(op(1,op(1,op(nops(w),w)))))
    *(times((delta - (p-1)*beta +(q+1)*conjugate(beta1)),
    times((conjugate(delta) + p*beta1 -
    q*conjugate(beta)),op(1,op(1,op(nops(w),w))) )));
elif (op(0,w)='eth' and op(0,op(1,w))='ethp') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((delta - (p-1)*beta
        + (q+1)*conjugate(beta1)),times((conjugate(delta)
        + p*beta1 -q*conjugate(beta)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w)))
    ='ethp' and op(0,op(1,op(nops(w),w))) ='ethp') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w +w/op(1,op(0,op(nops(w),w)))
        (op(0,op(1,op(nops(w),w)))(op(1,op(1,op(nops(w),w)))))
        *(times((conjugate(delta) + (p-1)*beta1
        -(q+1)*conjugate(beta)),times((conjugate(delta) +
        p*beta1- q*conjugate(beta)),
        op(1,op(1,op(nops(w),w))))));
elif (op(0,w)='ethp' and op(0,op(1,w))='ethp') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((conjugate(delta) +
        (p-1)*beta1 - (q+1)*conjugate(beta)),
        times((conjugate(delta) + p*beta1-
```

```
        q*conjugate(beta)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w)))
    ='ethp' and op(0,op(1,op(nops(w),w))) ='th') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w +
        w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w)))))*(times((conjugate(delta)
        +(p+1)*beta1 - (q+1)*conjugate(beta)),times((DD -
        p*epsilon - q*conjugate(epsilon)),
        op(1,op(1,op(nops(w),w))) )));
elif (op(0,w)='ethp' and op(0,op(1,w))='th') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((conjugate(delta) +
        (p+1)*beta1 -(q+1)*conjugate(beta)),times((DD -
        p*epsilon -q*conjugate(epsilon)),op(1,op(1,w))));
elif (op(0,w)=`*` and op(1,op(0,op(nops(w),w)))
    ='ethp' and op(0,op(1,op(nops(w),w))) ='thp') then
    (p,q):=getpq(op(1,op(1,op(nops(w),w))) );
    result:=result - w +
        w/op(1,op(0,op(nops(w),w)))(op(0,op(1,op(nops(w),w)))
        (op(1,op(1,op(nops(w),w)))))*(times((conjugate(delta)
        +(p-1)*beta1 - (q-1)*conjugate(beta)),times((Delta +
        p*epsilon1 + q*conjugate(epsilon1)),
        op(1,op(1,op(nops(w),w))) )));
elif (op(0,w)='ethp' and op(0,op(1,w))='thp') then
    (p,q):=getpq(op(1,op(1,w)));
    result:=result - w + times((conjugate(delta) +
```

```
        (p-1)*beta1 -(q-1)*conjugate(beta)),times((Delta +
        p*epsilon1 + q*conjugate(epsilon1)),op(1,op(1,w))));
elif(op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='th') then
    (p,q):=getpq(op(1,op(nops(w),w)) );
    result:=result - w +
        w/op(0,op(nops(w),w))(op(1,op(nops(w),w)))
        *(times((DD - p*epsilon -q*conjugate(epsilon)),
        op(1,op(nops(w),w)) ));
elif(op(0,w)='th') then
    (p,q):=getpq(op(1,w));
    result:=result - w + times((DD - p*epsilon -
        q*conjugate(epsilon)),op(1,w));
elif(op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='thp') then
    (p,q):=getpq(op(1,op(nops(w),w)) );
    result:=result - w +
        w/op(0,op(nops(w),w))(op(1,op(nops(w),w)))
        *(times((Delta + p*epsilon1+
        q*conjugate(epsilon1)),op(1,op(nops(w),w))));
elif(op(0,w)='thp') then
    (p,q):=getpq(op(1,w));
    result:=result - w + times((Delta + p*epsilon1
        + q*conjugate(epsilon1)),op(1,w));
elif(op(0,w)=`*` and op(1,op(0,op(nops(w),w))) ='eth')
    then
    (p,q):=getpq(op(1,op(nops(w),w)) );
    result:=result - w +
        w/op(0,op(nops(w),w))(op(1,op(nops(w),w)))
```

```
                    *(times((delta - p*beta +
                    q*conjugate(beta1)),op(1,op(nops(w),w))));
            elif(op(0,w)='eth') then
                (p,q):=getpq(op(1,w));
                result:=result - w + times((delta - p*beta +
                    q*conjugate(beta1)),op(1,w));
            elif(op(0,w)=`*` and op(1,op(0,op(nops(w),w)))
                ='ethp') then
                (p,q):=getpq(op(1,op(nops(w),w)) );
                result:=result - w +
                    w/op(0,op(nops(w),w))(op(1,op(nops(w),w)))
                    *(times((conjugate(delta) +p*beta1 -
                    q*conjugate(beta)),op(1,op(nops(w),w))));
            elif(op(0,w)='ethp') then
                (p,q):=getpq(op(1,w));
                result:=result - w + times((conjugate(delta) +
                    p*beta1 - q*conjugate(beta)),op(1,w));
            else
                result:=result;
            end if;
        end do;
        return(NPexpand(result));
end proc;

GE1:=th(rho)-ethp(kappa)=rho^2+sigma*conjugate(sigma)-conjugate(kappa)
*tau-tau1*kappa+Phi00;
GE2:=th(sigma)-eth(kappa)=sigma*(rho+conjugate(rho))-kappa*(tau+conjug
```

```
ate(tau1))+Psi0;
GE3:=th(tau)-thp(kappa)=rho*(tau-conjugate(tau1))+sigma*(conjugate(tau
)-tau1)+Psi1+Phi01;
GE4:=eth(rho)-ethp(sigma)=tau*(rho-conjugate(rho))+kappa*(conjugate(rh
o1)-rho1)-Psi1+Phi01;
GE5:=eth(tau)-thp(sigma)=-rho1*sigma-conjugate(sigma1)*rho
+tau^2+kappa*conjugate(kappa1)+Phi02;
GE6:=thp(rho)-ethp(tau)=rho*conjugate(rho1)+sigma*sigma1-tau*conjugate
(tau)-kappa*kappa1-Psi2-2*PI;

GHP1 := proc()
        return(DGHP(GE1));
end proc;

GHP1p := proc()
        return(DGHP(GHPprime(GE1)));
end proc;

GHP1c := proc()
        return(DGHP(GHPconj(GE1)));
end proc;

GHP1pc := proc()
        return(DGHP(GHPconj(GHPprime(GE1))));
end proc;

GHP2 := proc()
```

```
        return(DGHP(GE2));
end proc;

GHP2p := proc()
        return(DGHP(GHPprime(GE2)));
end proc;

GHP2c := proc()
        return(DGHP(GHPconj(GE2)));
end proc;

GHP2pc:= proc()
        return(DGHP(GHPconj(GHPprime(GE2))));
end proc;

GHP3 := proc()
        return(DGHP(GE3));
end proc;

GHP3p := proc()
        return(DGHP(GHPprime(GE3)));
end proc;

GHP3c := proc()
        return(DGHP(GHPconj(GE3)));
end proc;
```

```
GHP3pc := proc()

        return(DGHP(GHPconj(GHPprime(GE3))));

end proc;


GHP4 := proc()

        return(DGHP(GE4));

end proc;


GHP4p := proc()

        return(DGHP(GHPprime(GE4)));

end proc;


GHP4c := proc()

        return(DGHP(GHPconj(GE4)));

end proc;


GHP4pc := proc()

        return(DGHP(GHPconj(GHPprime(GE4))));

end proc;


GHP5 := proc()

        return(DGHP(GE5));

end proc;


GHP5p := proc()

        return(DGHP(GHPprime(GE5)));

end proc;
```

```
GHP5c := proc()
        return(DGHP(GHPconj(GE5)));
end proc;

GHP5pc := proc()
        return(DGHP(GHPconj(GHPprime(GE5))));
end proc;

GHP6 := proc()
        return(DGHP(GE6));
end proc;

GHP6p := proc()
        return(DGHP(GHPprime(GE6)));
end proc;

GHP6c := proc()
        return(DGHP(GHPconj(GE6)));
end proc;

GHP6pc := proc()
        return(DGHP(GHPconj(GHPprime(GE6))));
end proc;

COM1 := proc()
```

```
    return(DGHP(th(thp(z))-thp(th(z))=(conjugate(tau)-tau1)*eth(z)
        +(tau-conjugate(tau1))*ethp(z)-p*(kappa1*kappa-tau*tau1+Psi2+Phi11-PI)*z
        -q*(conjugate(kappa1)*conjugate(kappa)-conjugate(tau)*conjugate(tau1)+
        conjugate(Psi2)+Phi11-PI)*z));
end proc;

COM1p := proc()
        return(GHPprime(COM1()));
end proc;

COM1c := proc()
        return(GHPconj(COM1()));
end proc;

COM1pc := proc()
        return(GHPconj(GHPprime(COM1())));
end proc;

COM2 := proc()

    return(DGHP(th(eth(z))-eth(th(z))=conjugate(rho)*eth(z)+sigma*ethp(z)-
        conjugate(tau1)*th(z)-kappa*thp(z)-p*(rho1*kappa-tau1*sigma+Psi1)*z-
        q*(conjugate(sigma1)*conjugate(kappa)-conjugate(rho)*conjugate(tau1)
        +Phi01)*z));
end proc;

COM2p := proc()
```

```
        return(GHPprime(COM2()));
end proc;

COM2c := proc()
        return(GHPconj(COM2()));
end proc;

COM2pc := proc()
        return(GHPconj(GHPprime(COM2())));
end proc;

COM3 := proc()

    return(DGHP(eth(ethp(z))-ethp(eth(z))=(-rho1+conjugate(rho1))*th(z)
        +(rho-conjugate(rho))*thp(z)+p*(rho*rho1-sigma*sigma1+Psi2-Phi11-PI)*z
        -q*(conjugate(rho)*conjugate(rho1)-conjugate(sigma)*conjugate(sigma1)+
        conjugate(Psi2)-Phi11-PI)*z));
end proc;

COM3p := proc()
        return(GHPprime(COM3()));
end proc;

COM3c := proc()
        return(GHPconj(COM3()));
end proc;
```

```
COM3pc := proc()
        return(GHPconj(GHPprime(COM3())));
end proc;

BI1 := proc()

    return(DGHP(th(Psi1)-ethp(Psi0)-th(Phi01)+eth(Phi00)=-tau1*Psi0+4*rho*
        Psi1-3*kappa*Psi2+conjugate(tau1)*Phi00-2*conjugate(rho)*Phi01
        -2*sigma*Phi10+2*kappa*Phi11+conjugate(kappa)*Phi02));
end proc;

BI1p := proc()
        return(GHPprime(BI1()));
end proc;

BI1c := proc()
        return(GHPconj(BI1()));
end proc;

BI1pc:= proc()
        return(GHPconj(GHPprime(BI1())));
end proc;

BI2 := proc()

    return(DGHP(th(Psi2)-ethp(Psi1)-ethp(Phi01)+thp(Phi00)+2*th(PI)=sigma1
        *Psi0-2*tau1*Psi1+3*rho*Psi2-2*kappa*Psi3+conjugate(rho1)*Phi00
```

```
        -2*conjugate(tau)*Phi01-2*tau*Phi10+2*rho*Phi11+conjugate(sigma)*Phi02));
end proc;

BI2p := proc()
        return(GHPprime(BI2()));
end proc;

BI2c := proc()
        return(GHPconj(BI2()));
end proc;

BI2pc:= proc()
        return(GHPconj(GHPprime(BI2())));
end proc;

BI3 := proc()

    return(DGHP(th(Psi3)-ethp(Psi2)-th(Phi21)+eth(Phi20)-2*ethp(PI)=2*sigma1
        *Psi1-3*tau1*Psi2+2*rho*Psi3-kappa*Psi4-2*rho1*Phi10
        +2*tau1*Phi11+conjugate(tau1)*Phi20-2*conjugate(rho)*Phi21
        +conjugate(kappa)*Phi22));
end proc;

BI3p := proc()
        return(GHPprime(BI3()));
end proc;
```

```
BI3c := proc()
        return(GHPconj(BI3()));
end proc;

BI3pc:= proc()
        return(GHPconj(GHPprime(BI3())));
end proc;

BI4 := proc()

    return(DGHP(th(Psi4)-ethp(Psi3)-ethp(Phi21)+thp(Phi20)=3*sigma1*Psi2-4
        *tau1*Psi3+rho*Psi4-2*kappa1*Phi10+2*sigma1*Phi11
        +conjugate(rho1)*Phi20-2*conjugate(tau)*Phi21+conjugate(sigma)*Phi22));
end proc;

BI4p := proc()
        return(GHPprime(BI4()));
end proc;

BI4c := proc()
        return(GHPconj(BI4()));
end proc;

BI4pc:= proc()
        return(GHPconj(GHPprime(BI4())));
end proc;
```

```
#ezcomm forms the basis of comm
ezcomm := proc(expr,sexpr)
        local d1, d2, lo, comm, P, Q;
# first do some parsing of the expression - get the derivatives to
commute (d1,d2) and the leftover (lo)
        d1:=op(0,expr);
        d2:=op(0,op(1,expr));
        lo:=op(1,op(1,expr));
# return an error if we don't get the expected input
        if ((d1=`symbol`) or (d2=`symbol`)) then
                error "Not enough derivatives in %1 to commute",
                    expr;
        elif (d1=d2) then
                error "No commutin' to be done here! %1", expr;
        end if;
# now figure out which commutator we need
        if   (d1=`th`)   and (d2=`thp`)  then
            comm:=th(thp(z))=solve(COM1(),th(thp(z)));
        elif (d1=`thp`)  and (d2=`th`)   then
            comm:=thp(th(z))=solve(COM1(),thp(th(z)));
        elif (d1=`th`)   and (d2=`eth`)  then
            comm:=th(eth(z))=solve(COM2(),th(eth(z)));
        elif (d1=`eth`)  and (d2=`th`)   then
            comm:=eth(th(z))=solve(COM2(),eth(th(z)));
        elif (d1=`th`)   and (d2=`ethp`) then
            comm:=th(ethp(z))=solve(COM2c(),th(ethp(z)));
        elif (d1=`ethp`) and (d2=`th`)   then
```

```
            comm:=ethp(th(z))=solve(COM2c(),ethp(th(z)));
        elif (d1=`thp`)  and (d2=`ethp`) then
            comm:=thp(ethp(z))=solve(COM2p(),thp(ethp(z)));
        elif (d1=`ethp`) and (d2=`thp`)  then
            comm:=ethp(thp(z))=solve(COM2p(),ethp(thp(z)));
        elif (d1=`thp`)  and (d2=`eth`)  then
            comm:=thp(eth(z))=solve(COM2pc(),thp(eth(z)));
        elif (d1=`eth`)  and (d2=`thp`)  then
            comm:=eth(thp(z))=solve(COM2pc(),eth(thp(z)));
        elif (d1=`eth`)  and (d2=`ethp`) then
            comm:=eth(ethp(z))=solve(COM3(),eth(ethp(z)));
        elif (d1=`ethp`) and (d2=`eth`)  then
            comm:=ethp(eth(z))=solve(COM3(),ethp(eth(z)));
        else error "Can't commute %1 and %2", d1, d2;
        end if;
# add up p and q values from the components of the metric perturbation
        P:=0 + 2*(numboccur(lo,hll) + numboccur(lo,hlm) +
            numboccur(lo,hmm)) - 2*(numboccur(lo,hnn) + numboccur(lo,hnmb) +
            numboccur(lo,hmbmb)) + numboccur(lo,th) + numboccur(lo,eth) -
            numboccur(lo,thp) - numboccur(lo,ethp);
        Q:=0 + 2*(numboccur(lo,hll) + numboccur(lo,hlmb) +
            numboccur(lo,hmbmb)) - 2*(numboccur(lo,hnn) + numboccur(lo,hnm) +
            numboccur(lo,hmm)) + numboccur(lo,th) + numboccur(lo,ethp) -
            numboccur(lo,thp) - numboccur(lo,eth);
# now add up p and q values from all other objects
# this is where we can modify the procedure to recognize new things
        if (has(lo,rho) and not(has(lo,conjugate(rho)))) then
```

```
        P:=P+1;
        Q:=Q+1;
elif has(lo,conjugate(rho)) then
        P:=P+1;
        Q:=Q+1;
elif (has(lo,rho1) and not(has(lo,conjugate(rho1)))) then
        P:=P-1;
        Q:=Q-1;
elif has(lo,conjugate(rho1)) then
        P:=P-1;
        Q:=Q-1;
elif (has(lo,kappa) and not(has(lo,conjugate(kappa)))) then
        P:=P+3;
        Q:=Q+1;
elif has(lo,conjugate(kappa)) then
        P:=P+1;
        Q:=Q+3;
elif (has(lo,kappa1) and not(has(lo,conjugate(kappa1)))) then
        P:=P-3;
        Q:=Q-1;
elif has(lo,conjugate(kappa1)) then
        P:=P-1;
        Q:=Q-3;
elif (has(lo,tau) and not(has(lo,conjugate(tau)))) then
        P:=P+1;
        Q:=Q-1;
elif has(lo,conjugate(tau)) then
```

```
        P:=P-1;
        Q:=Q+1;
elif (has(lo,tau1) and not(has(lo,conjugate(tau1)))) then
        P:=P-1;
        Q:=Q+1;
elif has(lo,conjugate(tau1)) then
        P:=P+1;
        Q:=Q-1;
elif (has(lo,sigma) and not(has(lo,conjugate(sigma)))) then
        P:=P+3;
        Q:=Q-1;
elif has(lo,conjugate(sigma)) then
        P:=P-1;
        Q:=Q+3;
elif (has(lo,sigma1) and not(has(lo,conjugate(sigma1)))) then
        P:=P-3;
        Q:=Q+1;
elif has(lo,conjugate(sigma1)) then
        P:=P+1;
        Q:=Q-3;
elif (has(lo,Psi0) and not(has(lo,conjugate(Psi0)))) then
        P:=P+4;
elif has(lo,conjugate(Psi0)) then
        Q:=Q+4;
elif (has(lo,Psi1) and not(has(lo,conjugate(Psi1)))) then
        P:=P+2;
elif has(lo,conjugate(Psi1)) then
```

```
        Q:=Q+2;
elif (has(lo,Psi3) and not(has(lo,conjugate(Psi3)))) then
        P:=P-2;
elif has(lo,conjugate(Psi3)) then
        Q:=Q-2;
elif (has(lo,Psi4) and not(has(lo,conjugate(Psi4)))) then
        P:=P-4;
elif has(lo,conjugate(Psi4)) then
        Q:=Q-4;
elif has(lo,xi_l) then
        P:=P+1;
        Q:=Q+1;
elif has(lo,xi_n) then
        P:=P-1;
        Q:=Q-1;
elif has(lo,xi_m) then
        P:=P+1;
        Q:=Q-1;
elif has(lo,xi_mb) then
        P:=P-1;
        Q:=Q+1;
elif (has(lo,phi) and not(has(lo,conjugate(phi)))) then
        P:=P+pp;
        Q:=Q+pq;
elif has(lo,conjugate(phi)) then
        P:=P+pq;
        Q:=Q+pp;
```

```
elif (has(lo,phi1) and not(has(lo,conjugate(phi1)))) then
        P:=P-pp;
        Q:=Q-pq;
elif has(lo,conjugate(phi1)) then
        P:=P-pq;
        Q:=Q-pp;
elif (has(lo,chi1) and not(has(lo,conjugate(chi1)))) then
        P:=P+4;
elif has(lo,conjugate(chi1)) then
        Q:=Q+4;
elif (has(lo,chi2) and not(has(lo,conjugate(chi2)))) then
        P:=P-4;
elif has(lo,conjugate(chi2)) then
        Q:=Q-4;
elif (has(lo,omega1) and not(has(lo,conjugate(omega1)))) then
        P:=P+4;
elif has(lo,conjugate(omega1)) then
        Q:=Q+4;
elif (has(lo,omega2) and not(has(lo,conjugate(omega2)))) then
        P:=P-4;
elif has(lo,conjugate(omega2)) then
        Q:=Q-4;
elif (has(lo,eta1) and not(has(lo,conjugate(eta1)))) then
        P:=P+4;
elif has(lo,conjugate(eta1)) then
        Q:=Q+4;
elif (has(lo,eta2) and not(has(lo,conjugate(eta2)))) then
```

```
                P:=P-4;
        elif has(lo,conjugate(eta2)) then
                Q:=Q-4;
        elif (has(lo,xi1) and not(has(lo,conjugate(xi1)))) then
                P:=P+4;
        elif has(lo,conjugate(xi1)) then
                Q:=Q+4;
        elif (has(lo,xi2) and not(has(lo,conjugate(xi2)))) then
                P:=P-4;
        elif has(lo,conjugate(xi2)) then
                Q:=Q-4;
        elif has(lo,h) then
                P:=P+0;
                Q:=Q+0;
        elif has(lo,hl) then
                P:=P+1;
                Q:=Q+1;
        elif has(lo,hn) then
                P:=P-1;
                Q:=Q-1;
        end if;
        return(DGHP(subs(subs({p=P,q=Q,z=lo},comm),sexpr)));
end proc;

GHPcomm := proc(whichcom, solvefor, whichvar)
    local a,b;
    (a,b):=getpq(whichvar);
```

```
    return(solvefor=solve(subs({z=whichvar,p=a,q=b},
        whichcom),solvefor));
end proc;

# comm applies ezcomm until a given expression is completely commuted

comm:=proc(expr1,expr2)
        local ans;
        ans:=expr2;
        while has(ans,expr1) do
                ans:=DGHP(ezcomm(expr1,ans));
        end do;
        return(ans)
end proc;

# tetcon is an exercise in working around maple; it essentially wor

tetcon := proc(expr,indcs)
        local i, lexpr, lup_pieces, nup_pieces, mup_pieces,
            mbup_pieces, z1, z2, z3, z4, z5, z6, z7, z8, z9, z10, z11, z12,
            o1, o1a, o2, o2a, o3, o3a, o4, o4a;
        lexpr:=expand(expr);
        for i in indcs do
                lup_pieces := select(has,lexpr,lup(i))+xxxyyyzzz;
                if expand(lup_pieces-xxxyyyzzz) <> 0 then
                        z1:=select(has,lup_pieces,ldn(i));
                        z2:=select(has,lup_pieces,mdn(i));
```

```
        z3:=select(has,lup_pieces,conjugate(mdn)(i));
        o1:=select(has,lup_pieces,ndn(i));
        o1a:=expand(o1/(lup(i)*ndn(i)));
else
        z1:=0; z2:=0; z3:=0; o1:=0; o1a:=0;
end if;
nup_pieces := select(has,lexpr,nup(i))+xxxyyyzzz;
if expand(nup_pieces-xxxyyyzzz) <> 0 then
        z4:=select(has,nup_pieces,ndn(i));
        z5:=select(has,nup_pieces,mdn(i));
        z6:=select(has,nup_pieces,conjugate(mdn)(i));
        o2:=select(has,nup_pieces,ldn(i));
        o2a:=expand(o2/(nup(i)*ldn(i)));
else
        z4:=0; z5:=0; z6:=0; o2:=0; o2a:=0;
end if;
mup_pieces := select(has,lexpr,mup(i))+xxxyyyzzz;
if expand(mup_pieces-xxxyyyzzz) <> 0 then
        z7:=select(has,mup_pieces,ldn(i));
        z8:=select(has,mup_pieces,ndn(i));
        z9:=select(has,mup_pieces,mdn(i));
        o3:=select(has,mup_pieces,conjugate(mdn)(i));
        o3a:=expand(o3/(mup(i)*conjugate(mdn)(i)));
else
        z7:=0; z8:=0; z9:=0; o3:=0; o3a:=0;
end if;
mbup_pieces := select(has,lexpr,conjugate(mup)(i))+xxxyyyzzz;
```

```
            if expand(mbup_pieces-xxxyyyzzz) <> 0 then
                    z10:=select(has,mbup_pieces,ldn(i));
                    z11:=select(has,mbup_pieces,ndn(i));
                    z12:=select(has,mbup_pieces,conjugate(mdn)(i));
                    o4:=select(has,mbup_pieces,mdn(i));
                    o4a:=expand(o4/(conjugate(mup)(i)*mdn(i)));
            else
                    z10:=0; z11:=0; z12:=0; o4:=0; o4a:=0;
            end if;
            lexpr:=expand(lexpr - (z1+z2+z3+z4+z5+z6+z7+z8+z9
                +z10+z11+z12 + o1+o2+o3+o4)+ o1a+o2a-o3a-o4a)
        end do;
    return(expand(lexpr));
end proc;


GHPmult := proc(x,y)
        local i,z,result;
        result:=0;
        for i from 1 to nops(x) do
                if nops(x) <> 1 then z:=op(i,x) else z:=x end if;
                if (z='th' or z='thp' or z='eth' or z='ethp' or
                    op(0,z)='th' or op(0,z)='thp' or op(0,z)='eth' or
                    op(0,z)='ethp') then
                        result:=result+apply(z,y)
                else result:=result+z*y
                end if;
        end do;
```

```
        return expand(result);
end proc;


tdspec :=
DGHP({th(pp)=0,eth(pp)=0,thp(pp)=0,ethp(pp)=0,th(pq)=0,eth(pq)=0,thp(pq)=0
,ethp(pq)=0,eth(rho) = rho*tau-tau*conjugate(rho), th(rho) = rho^2,
thp(rho1) = rho1^2, th(conjugate(rho)) = conjugate(rho)^2,
thp(conjugate(rho1)) = conjugate(rho1)^2, th(tau) =
rho*tau-rho*conjugate(tau1), thp(tau1) =
rho1*tau1-rho1*conjugate(tau), th(conjugate(tau)) =
conjugate(rho)*conjugate(tau)-conjugate(rho)*tau1,
thp(conjugate(tau1)) =
conjugate(rho1)*conjugate(tau1)-conjugate(rho1)*tau, ethp(rho1) =
rho1*tau1-tau1*conjugate(rho1), ethp(conjugate(rho)) =
conjugate(rho)*conjugate(tau)-conjugate(tau)*rho, eth(conjugate(rho1))
= conjugate(rho1)*conjugate(tau1)-conjugate(tau1)*rho1, eth(tau) =
tau^2, ethp(tau1) = tau1^2, ethp(conjugate(tau)) = conjugate(tau)^2,
eth(conjugate(tau1)) = conjugate(tau1)^2, th(Psi2) = 3*rho*Psi2,
thp(Psi2) = 3*rho1*Psi2, th(conjugate(Psi2)) =
3*conjugate(rho)*conjugate(Psi2), thp(conjugate(Psi2)) =
3*conjugate(rho1)*conjugate(Psi2), ethp(Psi2) = 3*tau1*Psi2, eth(Psi2)
= 3*tau*Psi2, eth(conjugate(Psi2)) =
3*conjugate(tau1)*conjugate(Psi2), ethp(conjugate(Psi2)) =
3*conjugate(tau)*conjugate(Psi2), ethp(rho) = 2*rho*tau1,
eth(conjugate(rho)) = 2*conjugate(rho)*conjugate(tau1), eth(rho1) =
2*rho1*tau, ethp(conjugate(rho1)) = 2*conjugate(rho1)*conjugate(tau),
th(tau1) = 2*rho*tau1, thp(rho) =
```

```
rho*rho1-tau1*conjugate(tau1)+tau*tau1-1/2*rho*conjugate(Psi2)/conjuga
te(rho)-1/2*Psi2, th(conjugate(tau1)) =
2*conjugate(rho)*conjugate(tau1), thp(tau) = 2*rho1*tau,
thp(conjugate(tau)) = 2*conjugate(rho1)*conjugate(tau),
thp(conjugate(rho)) =
conjugate(rho)*conjugate(rho1)-tau1*conjugate(tau1)+conjugate(tau)*con
jugate(tau1)-1/2*conjugate(rho)*Psi2/rho-1/2*conjugate(Psi2),
ethp(tau) =
rho*rho1+tau*tau1-1/2*rho*conjugate(Psi2)/conjugate(rho)+1/2*Psi2-rho*
conjugate(rho1), eth(conjugate(tau)) =
conjugate(rho)*conjugate(rho1)+conjugate(tau)*conjugate(tau1)-1/2*conj
ugate(rho)*Psi2/rho+1/2*conjugate(Psi2)-rho1*conjugate(rho), th(rho1)
=
rho*rho1-tau1*conjugate(tau1)+tau*tau1-1/2*rho*conjugate(Psi2)/conjuga
te(rho)-1/2*Psi2, th(conjugate(rho1)) =
conjugate(rho)*conjugate(rho1)-tau1*conjugate(tau1)+conjugate(tau)*con
jugate(tau1)-1/2*conjugate(rho)*Psi2/rho-1/2*conjugate(Psi2),
eth(tau1) =
rho*rho1+tau*tau1-1/2*rho*conjugate(Psi2)/conjugate(rho)+1/2*Psi2-rho*
conjugate(rho1), ethp(conjugate(tau1)) =
conjugate(rho)*conjugate(rho1)+conjugate(tau)*conjugate(tau1)-1/2*conj
ugate(rho)*Psi2/rho+1/2*conjugate(Psi2)-rho1*conjugate(rho)\});

tdsimp := proc(expr)

return(DGHP(subs(tdspec,DGHP(subs(tdspec,DGHP(subs(tdspec,DGHP(subs(td
spec,DGHP(subs(tdspec,DGHP(subs(tdspec,DGHP(subs(tdspec,expr)))))))))
```

```
)))));

end proc;

tetupK := {lup = vector([(r^2+a^2)/(r^2-2*M*r+a^2), 1, 0,
a/(r^2-2*M*r+a^2)]), mup =
vector([1/2*I*a*sin(theta)*2^(1/2)/(r+I*a*cos(theta)), 0,
1/2*2^(1/2)/(r+I*a*cos(theta)),
1/2*I*2^(1/2)/(sin(theta)*(r+I*a*cos(theta)))]), nup =
vector([1/2*(r^2+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))),
-1/2*(r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))), 0,
1/2*a/((r+I*a*cos(theta))*(r-I*a*cos(theta)))]), conjugate(mup) =
vector([-1/2*I*a*sin(theta)*2^(1/2)/(r-I*a*cos(theta)), 0,
1/2*2^(1/2)/(r-I*a*cos(theta)),
-1/2*I*2^(1/2)/(sin(theta)*(r-I*a*cos(theta)))])\};

tetdnK := \{mdn =
vector([1/2*I*a*sin(theta)*2^(1/2)/(r+I*a*cos(theta)), 0,
-1/2*(r-I*a*cos(theta))*2^(1/2),
-1/2*I*(r^2+a^2)*sin(theta)*2^(1/2)/(r+I*a*cos(theta))]), ndn =
vector([1/2*(r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))),
1/2, 0,
-1/2*a*(r^2-2*M*r+a^2)*sin(theta)^2/((r+I*a*cos(theta))*(r-I*a*cos(the
ta)))]), ldn = vector([1,
-(r+I*a*cos(theta))*(r-I*a*cos(theta))/(r^2-2*M*r+a^2), 0,
-a*sin(theta)^2]), conjugate(mdn) =
vector([-1/2*I*a*sin(theta)*2^(1/2)/(r-I*a*cos(theta)), 0,
```

```
-1/2*(r+I*a*cos(theta))*2^(1/2),
1/2*I*(r^2+a^2)*sin(theta)*2^(1/2)/(r-I*a*cos(theta))])\};

tetupS := \{lup =
vector([1/2*(r^2+a^2)*2^(1/2)/((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I
*a*cos(theta)))^(1/2),
1/2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))))^
(1/2), 0,
1/2*a*2^(1/2)/((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I*a*cos(theta)))^
(1/2)]), nup =
vector([1/2*(r^2+a^2)*2^(1/2)/((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I
*a*cos(theta)))^(1/2),
-1/2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))))
^(1/2), 0,
1/2*a*2^(1/2)/((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I*a*cos(theta)))^
(1/2)]), mup = vector([1/2*I*a*sin(theta)*2^(1/2)/(r+I*a*cos(theta)),
0, 1/2*2^(1/2)/(r+I*a*cos(theta)),
1/2*I*2^(1/2)/(sin(theta)*(r+I*a*cos(theta)))]), conjugate(mup) =
vector([-1/2*I*a*sin(theta)*2^(1/2)/(r-I*a*cos(theta)), 0,
1/2*2^(1/2)/(r-I*a*cos(theta)),
-1/2*I*2^(1/2)/(sin(theta)*(r-I*a*cos(theta)))])\};

tetdnS := \{mdn =
vector([1/2*I*a*sin(theta)*2^(1/2)/(r+I*a*cos(theta)), 0,
-1/2*(r-I*a*cos(theta))*2^(1/2),
-1/2*I*(r^2+a^2)*sin(theta)*2^(1/2)/(r+I*a*cos(theta))]),
conjugate(mdn) =
```

```
vector([-1/2*I*a*sin(theta)*2^(1/2)/(r-I*a*cos(theta)), 0,
-1/2*(r+I*a*cos(theta))*2^(1/2),
1/2*I*(r^2+a^2)*sin(theta)*2^(1/2)/(r-I*a*cos(theta))]), ndn =
vector([1/2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(th
eta))))^(1/2),
1/2*2^(1/2)*((r+I*a*cos(theta))*(r-I*a*cos(theta))/(r^2-2*M*r+a^2))^(1
/2), 0,
-1/2*a*sin(theta)^2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*
a*cos(theta))))^(1/2)]), ldn =
vector([1/2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(th
eta))))^(1/2),
-1/2*2^(1/2)*((r+I*a*cos(theta))*(r-I*a*cos(theta))/(r^2-2*M*r+a^2))^(
1/2), 0,
-1/2*a*sin(theta)^2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*
a*cos(theta))))^(1/2)])\};

tetupSB := \{lup = vector([1/2*B(t, r, theta,
phi)*(r^2+a^2)*2^(1/2)/((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I*a*cos(
theta)))^(1/2), 1/2*B(t, r, theta,
phi)*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))))
^(1/2), 0, 1/2*B(t, r, theta,
phi)*a*2^(1/2)/((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I*a*cos(theta)))
^(1/2)]), nup = vector([1/2*(r^2+a^2)*2^(1/2)/(B(t, r, theta,
phi)*((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I*a*cos(theta)))^(1/2)),
-1/2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))))
^(1/2)/B(t, r, theta, phi), 0, 1/2*a*2^(1/2)/(B(t, r, theta,
phi)*((r^2-2*M*r+a^2)*(r+I*a*cos(theta))*(r-I*a*cos(theta)))^(1/2))]),
```

```
conjugate(mup) = vector([-1/2*I*exp(-I*S(t, r, theta,
phi))*a*sin(theta)*2^(1/2)/(r-I*a*cos(theta)), 0, 1/2*exp(-I*S(t, r,
theta, phi))*2^(1/2)/(r-I*a*cos(theta)), -1/2*I*exp(-I*S(t, r, theta,
phi))*2^(1/2)/(sin(theta)*(r-I*a*cos(theta)))]), mup =
vector([1/2*I*exp(I*S(t, r, theta,
phi))*a*sin(theta)*2^(1/2)/(r+I*a*cos(theta)), 0, 1/2*exp(I*S(t, r,
theta, phi))*2^(1/2)/(r+I*a*cos(theta)), 1/2*I*exp(I*S(t, r, theta,
phi))*2^(1/2)/(sin(theta)*(r+I*a*cos(theta)))])\};

tetdnSB := \{conjugate(mdn) = vector([-1/2*I*exp(-I*S(t, r, theta,
phi))*a*sin(theta)*2^(1/2)/(r-I*a*cos(theta)), 0, -1/2*exp(-I*S(t, r,
theta, phi))*(r+I*a*cos(theta))*2^(1/2), 1/2*I*exp(-I*S(t, r, theta,
phi))*(r^2+a^2)*sin(theta)*2^(1/2)/(r-I*a*cos(theta))]), mdn =
vector([1/2*I*exp(I*S(t, r, theta,
phi))*a*sin(theta)*2^(1/2)/(r+I*a*cos(theta)), 0, -1/2*exp(I*S(t, r,
theta, phi))*(r-I*a*cos(theta))*2^(1/2), -1/2*I*exp(I*S(t, r, theta,
phi))*(r^2+a^2)*sin(theta)*2^(1/2)/(r+I*a*cos(theta))]), ndn =
vector([1/2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(th
eta))))^(1/2)/B(t, r, theta, phi),
1/2*2^(1/2)*((r+I*a*cos(theta))*(r-I*a*cos(theta))/(r^2-2*M*r+a^2))^(1
/2)/B(t, r, theta, phi), 0,
-1/2*a*sin(theta)^2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*
a*cos(theta))))^(1/2)/B(t, r, theta, phi)]), ldn = vector([1/2*B(t, r,
theta,
phi)*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*a*cos(theta))))
^(1/2), -1/2*B(t, r, theta,
phi)*2^(1/2)*((r+I*a*cos(theta))*(r-I*a*cos(theta))/(r^2-2*M*r+a^2))^(
```

```
1/2), 0, -1/2*B(t, r, theta,
phi)*a*sin(theta)^2*2^(1/2)*((r^2-2*M*r+a^2)/((r+I*a*cos(theta))*(r-I*
a*cos(theta))))^(1/2)])\};

end module:
```

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BIOGRAPHICAL SKETCH

Larry was born in 1978, in El Paso, Texas. He is the eldest child of (the elder) Larry Price and Pamela Villa. At last count, he has approximately 6 siblings.

From the ages of about five to twelve, he attended a funny sort of school where the students were all forced to dress the same and gather on Fridays to listen to a man in a dress read from a big book. He was treated well there, but his entry into the Texas public school system in the fifth grade proved to be a good move. In middle school, Larry realized he understood algebra much better than his teacher (who happened to also be the school's basketball coach), a point that he made clear in class at every opportunity. It goes without saying that his initial desire to publicly humiliate jocks subsequently grew into a much deeper interest in mathematics and physics. These interests were furthered in high school, where Larry explored other areas as well. Among these is the theater. Few people are aware that Larry has performed in leading roles in several musicals, as well as an operetta.

Upon graduating high school in 1997, Larry decided that it would be best to get as far away from El Paso as he could. To this end, he attended a small liberal arts school named Reed College in Portland, Oregon, where he spent some of the best years of his life. Reed provided a valuable opportunity for Larry to further pursue the sciences and read some really great books at the same time. It also gave him the opportunity to interact with many interesting people from widely different backgrounds. It was there that Larry came in contact with Nick Wheeler, a truly unique individual who remains a trusted mentor. Alas, all good things must come to an end, and so Larry graduated from Reed with a B.A. in physics in 2001.

With his path uncertain at the time, Larry decided to stay in Portland for the following year. There Larry tried his hand as a computational chemist for Schrödinger, Inc. The people there were fantastic and the paychecks weren't bad, but he need more from his work. Graduate school seemed like a good remedy. This is how Larry came to Florida.

Larry entered the graduate program at the University of Florida in 2002. There he had the good fortune to work with Bernard Whiting, who introduced Larry to the subject of general relativity. To Larry's surprise, he finished the doctoral program in five years, graduating in the summer of 2007. What the future holds for Larry is uncertain. What is certain is that in the fall Larry will continue his tour of the country in Milwaukee, Wisconsin, where he has accepted a postdoctoral position at the University of Wisconsin.