REALIZATION THEORY OF INFINITE-DIMENSIONAL LINEAR SYSTEMS

By

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Abstract of Dissertation Presented to the Graduate Council of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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This work studies the problem of realization of constant linear input/output maps, which do not necessarily possess a finite-dimensional realization. A class of constant linear input/output maps is introduced. This class is then characterized as the family of continuous linear maps whose weighting patterns are measures. The natural state-space representation (realization) of such constant linear input/output maps is studied, and a new notion of observability, topological observability, is introduced. It is then seen that topological observability enables us to prove the existence and uniqueness of canonical (quasi-reachable and topologically observable) realizations. It is also shown that a certain subclass of realizations admits a functional-differential equation description. Necessary and sufficient conditions that the state space of a canonical realization be a Banach (or Hilbert) space are obtained. A new notion, topological observability in bounded time, plays a central role in deriving such conditions. A thorough study of a concrete example of such systems is given.

CHAPTER I. INTRODUCTION

In the present work we study the problem of realization of linear constant (shift-invariant) continuous-time input/output maps, which do not necessarily possess a finite-dimensional realization. In this introduction we confine ourselves to scalar (single-input/single-output) input/output maps. Multi-input/multi-output cases will be discussed in the main text.

Throughout this work we fix a field k, either $\mathbb{R}$ or $\mathbb{C}$, and every space is a locally convex Hausdorff topological vector space over k.

Let us start our discussion with some intuitive idea as to what realization theory is. The problem of realization may be viewed as an idealized way of formulating the problem of scientific model building. Frequently we encounter the situation in which a system (or a plant) is primarily defined only in terms of its external behavior. The objective of realization theory is to construct an "internal model" of the given external behavior so that the behavior of the system can be better studied on the basis of the model rather than the given external behavior. Here by an internal model (or a system) we mean some mathematical object Σ which is a dynamical system in the following modern sense:

(i) Σ accepts inputs and produces outputs. The present output is produced as a function of the (present) "state" of Σ, which represents the past history of inputs. (Classical dynamical systems, based on Newton's laws, do not have this property which is essential to treat the scientific problems of the 20th century. Note that in the mathematical literature, "dynamical system" unfortunately usually means just a "classical dynamical system.")

(ii) Each application of an input may alter the state of Σ; this state-transition is governed by causal (i.e., the present state does not depend on the future values of inputs) and deterministic rules. (We do not consider probabilistic dynamical systems in this work.)

(iii) The notion of "state" is the basic concept which needs to be axiomatized for a deeper mathematical study. (In physics one talks about "phase," as a very narrow special case of what in system theory is known as the "state.")

Of course, this is not a definition by any means. But before proceeding to a technical definition, we must clarify the meaning of "external behavior."

Intuitively external behavior of a system refers to the correspondence between inputs and outputs, i.e., whenever an input is applied to the system we can tell what the corresponding future output will be. Here we have implicitly made the assumption that the system is causal, i.e., the present value of an output does not depend on the future values of inputs. It is this property that implies the existence of an (internal) state (see KALMAN, FALB, and ARBIB [1969, Comment (10.2.4)]). Let us now make another basic assumption that the system we are concerned with is constant (this terminology is due to KALMAN; see, for instance, KALMAN [1968]), i.e., defining relations (structure) of the system do not depend on time. This property guarantees that if we execute the same experiment on the system (i.e., apply an input) under the same experimental conditions, but at a different time, we will obtain the same response. Hence the property of "constancy" enables us to carry out multiple experiments without the need to worry about structural changes of the system due to time.

It is reasonable to represent each input (output) as a k-valued function on some time set T. If $T = \mathbb{Z}$, the integers, the system is said to be discrete-time, and if $T = \mathbb{R}$, the reals, the system is said to be continuous-time. We also assume that every input has a bounded support, i.e., is identically zero except on a bounded interval. This requirement is made because, in principle, we can apply inputs only for bounded time.

Since our system is assumed to be constant, we may, without loss of generality, call the present time 0. Then what we call external behavior (or an input/output map) is the following correspondence between inputs and outputs: There is a rule to produce a future output whenever we are given an input having support in the past (i.e., $(-\infty, 0]$). Let Ω be a given space of past inputs, and Γ a given space of future outputs. Suppose that Ω and Γ are vector spaces over k. Then an input/output map $f: \Omega \to \Gamma$ is said to be linear iff f is a linear map. (We use "iff" as the common abbreviation for "if and only if.") One can easily see that causality is already built into the above setting.

REMARK. We do not, however, claim that this is the most general formulation of external behavior. Indeed, inputs may be applied <u>while</u> outputs are being observed in practical situations (see Section 5). Or an output may vary depending not only on known inputs but also on unknown inputs generated by the environment in which the system is situated. But these questions are irrelevant (as far as theoretical studies are concerned) if the system under consideration is constant and linear. For a more thorough discussion on this matter, see KALMAN, FALB, and ARBIB [1969, Chapters 1 and 10].

Thus our problem is now reduced to finding "good" input and output spaces. When the system is discrete-time, we have a very natural choice: $\Omega :=$ {sequences with bounded support contained in $(-\infty, 0] \cap \mathbb{Z}$}, $\Gamma :=$ {sequences which vanish on $(-\infty, 0] \cap \mathbb{Z}$}. By using the correspondence: $t \leftrightarrow z^{-t}$ $(t \in \mathbb{Z})$, one can easily verify the isomorphisms (as vector spaces over k): $\Omega \cong k[z]$, $\Gamma \cong z^{-1}k[[z^{-1}]]$ where z and z^{-1} denote indeterminates. In both spaces the <u>left shift operator</u> is represented by the left multiplication by z. Then the requirement that an input/output map $f\colon \Omega \to \Gamma$ be constant is very simply expressed as $zf(\omega) = f(z\omega)$. This idea is due to KALMAN [1965] and is now the standard in realization theory of discrete-time systems (one can also see its recent evolution in systems over rings and nonlinear systems: ROUCHALEAU [1972]; SONTAG [1976]).

But despite these successful predecessors, it seems to be a major problem to give a good framework for continuous-time input/output maps. Furthermore, topological considerations become essential in this case. Let us briefly examine some historical background (we do not intend to make this review very complete).

KALMAN and HAUTUS [1972] used the setting: $\Omega := \mathcal{E}'_{(-\infty,0]}$, the space of distributions with bounded support contained in $(-\infty, 0]$, $\Gamma := \mathcal{E}_{[0,\infty)}$, the space of C^∞-functions on $[0, \infty)$, and f (input/output map) is a continuous linear map from Ω to Γ. With this setting they successfully derived a differential equation description of a realization. In this case, the input/output map f is described by a C^∞ impulse response function.

KAMEN [1975, 1976] defined an input/output map as a continuous linear map $f: \underset{\sim}{D}'_+ \to \underset{\sim}{D}'_+$ that satisfies $f(\omega_1)|_{(-\infty,\tau)} = f(\omega_2)|_{(-\infty,\tau)}$ whenever $\omega_1|_{(-\infty,\tau)} = \omega_2|_{(-\infty,\tau)}$ ($\underset{\sim}{D}'_+$ denotes the space of distributions with support bounded on the left). He extended the module theoretic treatment of systems, initiated by KALMAN and HAUTUS [1972] for continuous-time systems, to a very large class of input/output maps. In this case, the input/output map f is described by a distribution weighting pattern whose support is in $[0, \infty)$.

MATSUO [1978] proposed the choice: $\Omega := M_c(-\infty, 0]$, the space of Radon measures with compact support contained in $(-\infty, 0]$, and $\Gamma := C[0, \infty)$.

Note that the above mentioned settings are given in such a way that the input space contains Dirac's "delta function" δ_0. This is a definite advantage for clarifying certain system-theoretic concepts such as impulse response functions. But at the same time the following question arises: <u>Can these ideal inputs (such as δ_0) be applied to the system while the system is in the working mode</u>? As is pointed out in KALMAN and HAUTUS [1972], this question is related to the very delicate problem of whether the truncation of inputs at an arbitrary time t is well-defined. This is the kind of difficulty that one never encounters in discrete-time systems.

On the other hand, a great many authors took the viewpoint that the input/output map is defined by a weighting pattern (or an impulse response function) so that the output is expressed as a convolution of the weighting pattern with the input; see, for example, BARAS, BROCKETT, and FUHRMANN [1974]; BARAS and BROCKETT [1975]; BROCKETT and FUHRMANN [1976], etc. Unfortunately the question, "What type of function (or distribution) should be considered as an input (or output)?" is then left out; without specifying the input and output spaces no satisfactory theory is possible.

We now propose a choice of the input space and the output space. Our requirements are (i) Ω consists of <u>functions</u> such that (a) the very difficult problem of truncation does not occur, (b) these functions can actually be applied to the system in the working mode, and (ii) Γ also consists of functions. Of course one drawback is that we cannot have the delta function as an input due to (i) (because δ_0 is merely a distribution or a measure). But under a certain regularity hypothesis on the input/output map, the input space can be indeed extended to a larger space so that this "ideal" input space contains δ_0 (Proposition (3.14)).

Fix any $T > 0$. We take the input space and the output space <u>on the interval</u> $[0, T]$ as $L^2_{[0,T]}$. (If we consider inputs in the past we take $L^2_{[-T,0]}$ instead.) Since we can apply inputs <u>only in bounded time</u> it is reasonable to take $\Omega := \bigcup_{T>0} L^2_{[-T,0]}$. On the other hand, since outputs may well continue for an indefinitely long time, we simply take Γ to be the set of all <u>locally</u> L^2-functions.

We must topologize Ω and Γ. We make an intuitive additional requirement: the topologies of Ω and Γ are so determined that the knowledge of the behavior of the system on each bounded interval $[0, T]$ is enough to know the behavior of the system in an infinitely long time period. For example, let f be a linear map from Ω to Γ. One may ask the question: <u>Is it enough to know that f is continuous on each $L^2_{[-T,0]}$ so as to conclude that f is continuous on Ω?</u> This is a very reasonable question since we can apply inputs only in bounded time. Another question could be the following: <u>Is it enough to know that $f(\omega)|_{[0,T]}$ is small in each $L^2_{[0,T]}$ so as to conclude that $f(\omega)$ itself is small in Γ?</u> This is also a reasonable question because one can observe outputs only in a finite time period (no matter how long it may be).

Fortunately there are well-known mathematical tools to handle these requirements, namely <u>inductive</u> and <u>projective limits</u> (cf. Appendix). We write $\Omega = \varinjlim L^2_{[-n,0]}$ (inductive limit) and $\Gamma = \varprojlim L^2_{[0,n]}$ (projective limit). Then a linear map $f: \Omega \to \Gamma$ is continuous iff for each m, n the map f is continuous from $L^2_{[-m,0]}$ to $L^2_{[0,n]}$. Then our questions are affirmatively answered.

Each space Ω (and Γ) is equipped with a family of shift operators $\{\sigma_t\}_{t\geq 0}$ ($\{\tilde{\sigma}_t\}_{t\geq 0}$) defined by $(\sigma_t\omega)(\tau) := \omega(\tau + t)$ if $\tau \leq - t$ and $(\sigma_t\omega)(\tau) := 0$ if $- t < \tau \leq 0$ $((\tilde{\sigma}_t\gamma)(\tau) := \gamma(\tau + t))$.

It may be a reasonable idea to define an input/output map as a continuous linear map $f: \Omega \to \Gamma$ which commutes with shifts, i.e., $\tilde{\sigma}_t f = f\sigma_t$ for all $t \geq 0$. But unfortunately this causes a tremendous problem in defining a system equation. Hence we impose one additional condition: f sends $C_o(- \infty, 0)$, the space of continuous functions with compact support in $(- \infty, 0)$, to $C[0, \infty)$, the space of continuous functions on $[0, \infty)$, and this correspondence must be continuous with respect to the topologies of $C_o(- \infty, 0)$ and $C[0, \infty)$ (cf. Section 3). Then it turns out that f is precisely the map given by a convolution of $\omega \in \Omega$ with a measure μ, i.e., $f(\omega) = \mu*\omega$ (Theorem (3.12)). The measure μ is called the weighting pattern of f.

The class of input/output maps as defined above is indeed very large. It is desirable that any input/output map can be realized by a system. Thus our question is the following: What is a good definition of systems so that (i) any input/output map can be realized (ii) systems still have a "nice" structure? Let us consider what properties we wish to require of our state spaces. For example, let f be the input/output map given by $f(\omega) = \exp\,(\exp\, t)*\omega$. Then f does not have a realization in the category of Banach spaces in any usual sense, i.e., described by a functional-differential equation. Intuitively the reason is that the weighting pattern $\exp\,(\exp\, t)$ grows too rapidly whereas a semigroup in a Banach space can grow only with exponential order. Hence the requirement that the state space be Banach may be too restrictive. (Or $\exp\,(\exp\, t)$ should not be a weighting pattern.)

We now define a linear constant system Σ as follows: Σ consists of three objects X, φ, H along with the conditions

(a) The state space X is a complete locally convex Hausdorff space;

(b) $\varphi(t, \cdot, \cdot): X \times L^2_{[0,t)} \to X$ is a continuous linear map such that $\varphi(0, x, u) = x$, $\varphi(t + s, x, u) = \varphi(t, \varphi(s, x, u), \sigma^{\ell}_t u)$ (σ^{ℓ}_t = left shift operator, see (4.1)) for all $t, s \geq 0$, $x \in X$, and $u \in L^2_{[0,t+s)}$;

(c) $H: D_o(H) \to k$ is a densely defined linear (not necessarily continuous) map such that $\varphi(t, x, 0) \in D_o(H)$ for all $x \in D_o(H)$.

(There are other technical requirements on φ and H, but we shall postpone the discussion until the main text.)

The major difference of this definition from the classical one (KALMAN, FALB, and ARBIB [1969, Chapter 1]) is (c), i.e., the <u>readout (output) map</u> H is not necessarily defined on the whole state space X. This modification is motivated by the following example.

Consider an insulated uniform rod with unknown temperature distribution. With proper normalization the temperature $v(t, \xi)$ satisfies

$$(\partial/\partial t)v(t, \xi) = (\partial^2/\partial \xi^2)v(t, \xi), \quad t > 0, \quad 0 < \xi < 1,$$

$$(\partial v/\partial \xi)(t, 0) = (\partial v/\partial \xi)(t, 1) = 0.$$

We want to observe $y(t) := v(t, 0)$. It is a standard technique in the theory of the integration of the equation of evolution to take $X = L^2_{(0,1)}$ and integrate the equation in this function space. In this case the equation is transformed into the form (by letting $x(t)(\xi) := v(t, \xi)$)

$$\frac{d}{dt}x(t) = (\partial^2/\partial \xi^2)x(t), \quad x(t)(\cdot) \in L^2_{(0,1)}.$$

See, for example, YOSHIDA [1971, Chapter XIV]. But the output equation $y(t) = v(t, 0) = x(t)(0)$ is not well-defined for every $x(t) \in L^2_{(0,1)}$! The fact is that $y(t) = Hx(t)$ makes sense only on a dense subspace $D_o(H)$ (for example, take $D_o(H) := C_{[0,1]}$). HELTON [1976] points out the same type of phenomenon with an example of a transmission line.

Note that H induces a correspondence $x \mapsto h_o(x)(t) := H\varphi(t, x, 0)$ because of the requirement $\varphi(t, x, 0) \in D_o(H)$ for all $t \geq 0$. Even though H itself is not continuous, it is quite possible that h_o gives a continuous correspondence from $D_o(H)$ to Γ. (One can make sure that this is indeed the case with the previous example.) In addition to the conditions (a), (b), (c) on systems, we require the continuity of h_o as a part of the definition. Since h_o is continuous, it has a continuous extension $h^\Sigma: X \to \Gamma$ because $D_o(H)$ is dense in X. We call h^Σ the observability map of Σ; Σ is said to be observable iff h^Σ is one-to-one.

Now take any $T > 0$, and let $X_T := \{\varphi(T, 0, u): u \in L^2_{[0,T)}\}$. X_T is the set of all elements reachable from 0 at time T with application of a suitable input. Let $X_R := \bigcup_{T>0} X_T$, and call X_R the reachable set of Σ. We say that the system Σ is (exactly) reachable iff $X_R = X$, quasi-reachable iff X_R is dense in X. Also, the system Σ is said to be weakly canonical iff it is quasi-reachable and observable.

One of the basic problems in realization theory is to associate a unique realization to the external behavior (the input/output map) without assuming a priori information not implied by the external behavior. In order that a realization be uniquely associated to the input/output map, it must not contain any redundant part in the state space. This requirement easily implies that this uniquely associated system must be at least quasi-reachable and observable, i.e., weakly canonical. One can easily construct a weakly canonical realization for any input/output map (Proposition (7.18)). But is such a realization unique? If the system is finite-dimensional, this is true, as a consequence of the classical result by KALMAN (see KALMAN, FALB, and ARBIB [1969, Chapter 10]), since quasi-reachability coincides with reachability in this case, thereby yielding the implication "weakly canonical" $\Rightarrow$ "canonical" in the classical sense. For infinite-dimensional systems, however, BARAS, BROCKETT, and FUHRMANN [1974] gave a counterexample, by proving the existence of two nonisomorphic weakly canonical systems having the same external behavior.

Many attempts have been made toward proving the uniqueness of "canonical" realizations: BENSOUSSAN, DELFOUR, and MITTER [1975, 1976]; BROCKETT and FUHRMANN [1976]; HELTON [1974]; MATSUO [1978], etc. We shall propose yet another approach to the problem.

The intuitive idea of observability is that any two different initial states are distinguishable by application of a certain "suitable procedure" to future outputs; this is pointed out in KALMAN [1968, Chapter 10]; SONTAG [1976]; SONTAG and ROUCHALEAU [1976]. Since we are interested in topological aspects of the problem, it is reasonable to demand that this "suitable procedure" mean that initial states can be determined continuously from observation data. In other words, the initial state determination procedure must be well-posed. Indeed, if the initial state determination procedure is not well-posed, it may occur that we identify two quite different initial states as the same, possibly due to observation errors. We say that a system is topologically observable iff its initial state determination procedure is well-posed.

We shall say that a system Σ is canonical iff it is quasi-reachable and topologically observable. One of the main results of this work is the claim: Every input/output map f admits a canonical realization Σ_f, and any other canonical realization of f is isomorphic to Σ_f (Theorems (9.2), (9.4)).

There are very many questions on canonical realizations that one might want to ask. Is a canonical realization nice enough so that it admits a differential equation description (at least for rather smooth inputs)? Is the character of the state space "nice," such as Hilbert or Banach or metrizable at least?

Let us discuss the first question. In order that the system be described by a differential equation, the weighting pattern of f must be smooth enough. We assume that the weighting pattern is actually a locally absolutely continuous function whose derivative belongs to L^2 on each bounded interval. With this assumption, we have the following result: If f is an input/output map as described, then the canonical realization admits a differential equation description, i.e., there exists a densely defined closed linear operator $F: D(F) \to X$ and $G \in D(F)$ such that

(a) F generates a strongly continuous semigroup $\{\Phi(t)\}_{t \geq 0}$ in X;

(b) the differential equation

$$\frac{d}{dt}x(t) = Fx(t) + Gu(t)$$

with the initial condition $x(0) = x \in D(F)$ admits a unique solution: $x(t) = \Phi(t)x + \int_0^t \Phi(t - \tau)Gu(\tau)d\tau$ at least for uniformly continuous u;

(c) $\varphi(t, x, u) = x(t) = \Phi(t)x + \int_0^t \Phi(t - \tau)Gu(\tau)d\tau$ for all $x \in D(F)$ and uniformly continuous u.

Let us consider the character of the topology of canonical realizations. We can easily prove that every canonical realization has a Fréchet (metrizable and complete) space as a state space. But in general the state space of a canonical realization cannot be a Banach space. In Chapter IV we give necessary and sufficient conditions that the state space of a canonical realization be a Banach space. A new notion, topological observability in bounded time, plays a crucial role. A concrete example will be discussed also in this chapter.

CHAPTER II. INPUT/OUTPUT MAPS

In this chapter we give the definitions of an input space, output space and linear input/output maps. We shall prove that there is a one-to-one correspondence between a Radon measure on $(0, \infty)$ and a linear input/output map. This uniquely associated Radon measure is called the weighting pattern of a linear input/output map and plays a crucial role in later chapters. We shall also give definitions of the space of Laurent functions and extended linear input/output maps. The space of Laurent functions is an analogue of the Laurent series in the continuous-time context; for the Laurent series see HAUTUS and HEYMANN [1978].

1. Input Space.

Let $L^2_{[-n,0]}$ be the set of all k-valued Lebesgue square integrable functions on the interval $[-n, 0]$ (n = a positive integer). As is well-known, $L^2_{[-n,0]}$ is a Hilbert space with the following norm:

$$(1.1)\quad \|\varphi\|_{[-n,0]} := \{ \int_{-n}^{0} |\varphi(t)|^2 dt\}^{1/2}, \quad \varphi \in L^2_{[-n,0]}.$$

Clearly we may identify the space $L^2_{[-n,0]}$ with the space of all functions defined on $(-\infty, 0]$ which vanish outside of $[-n, 0]$ and belong to $L^2_{[-n,0]}$ when restricted on $[-n, 0]$. In the sequel we shall also denote this space by $L^2_{[-n,0]}$; if a precise distinction is necessary, we shall denote it by $L^2_{o,[-n,0]}$ (the subscript 0 denotes that each member of the space has compact support).

If $a \leq b$ (a, b = positive integers), then there exists a natural inclusion $j_{ab}: L^2_{[-a,0]} \to L^2_{[-b,0]}$. Clearly j_{ab} is an isomorphism of $L^2_{[-a,0]}$ into $L^2_{[-b,0]}$. In other words, the relative topology of $L^2_{[-a,0]}$, induced from $L^2_{[-b,0]}$ as a subspace, is identical to the original topology of $L^2_{[-a,0]}$. Thus $L^2_{[-a,0]}$ can be identified with a (closed) subspace of $L^2_{[-b,0]}$ whenever $a \leq b$.

Let Ω^1 be the union of all $L^2_{[-n,0]}$ where n runs over all positive integers. Clearly Ω^1 consists of all L^2-functions with support bounded on the left, i.e., ω belongs to Ω^1 iff there exists n such that ω belongs to $L^2_{o,[-n,0]}$.

Using the inclusions $\{j_{ab}\}$ as given above, we can induce the topology of the (strict) inductive limit of the sequence $\{L^2_{[-n,0]}\}$; see the Appendix. We denote the space Ω^1, with the inductive limit topology, as $\Omega^1 = \varinjlim L^2_{[-n,0]}$.

(1.2) REMARK. It is, in principle, possible that we may induce a different topology if we take a different sequence $\{L^2_{[-a_n,0]}\}$ where $\{a_n\}$ is a sequence such that $a_n \to \infty$ as $n \to \infty$. But one can easily verify that the inductive limit topology thus induced is independent of the choice of $\{a_n\}$.

(1.3) DEFINITION. <u>The input space (with</u> m <u>input channels) is the space</u> $\Omega := (\Omega^1)^m = (\varinjlim L^2_{[-n,0]})^m$. <u>When it is necessary to refer to the number of input channels explicitly, we write</u> Ω^m.

We also define the <u>shift (operator)</u> $\sigma_t: \Omega \to \Omega$ for each $t \geq 0$ by

$$(1.4)\qquad \sigma_t(\omega)(\tau) := \begin{cases} \omega(\tau + t) & \text{if } \tau \leq -t, \\ 0 & \text{if } -t < \tau \leq 0. \end{cases}$$

We need to prove the following

(1.5) PROPOSITION. <u>The family of operators</u> $\{\sigma_t\}_{t\geq 0}$ <u>is a strongly continuous (or</u> C_o -) <u>semigroup, i.e.</u>,

(a) $\sigma_o = I$, $\sigma_t\sigma_s = \sigma_{t+s}$;

(b) <u>each</u> σ_t <u>is a continuous linear map</u>;

(c) $\lim_{t\to t_o} \sigma_t\omega = \sigma_{t_o}\omega$ <u>for every</u> $t_o \geq 0$ <u>and</u> ω <u>in</u> Ω, (<u>strong continuity</u>).

PROOF. Without any loss of generality we assume m (the number of input channels) = 1. The property (a) follows easily via direct calculation.

We prove that σ_t is continuous. By Proposition (A.2) it suffices to prove that each restriction of σ_t on $L^2_{[-a,0]}$ is continuous for every a. For sufficiently large b, we have $\sigma_t(L^2_{[-a,0]}) \subset L^2_{[-b,0]}$. Then by Proposition (A.4) we need only to prove that σ_t is continuous as a map from $L^2_{[-a,0]}$ to $L^2_{[-b,0]}$. For every $\omega \in L^2_{o,[-a,0]}$, we have the equalities:

$$\|\sigma_t\omega\|^2_{[-b,0]} = \int_{-b}^{0} |\sigma_t(\omega)(\tau)|^2 d\tau,$$

$$= \int_{-b}^{-t} |\omega(\tau + t)|^2 d\tau \quad \text{(definition of } \sigma_t\text{)},$$

$$= \int_{-a-t}^{-t} |\omega(\tau + t)|^2 d\tau,$$

$$= \int_{-a}^{0} |\omega(\tau)|^2 d\tau,$$

$$= \|\omega\|^2_{[-a,0]}.$$

Hence σ_t is continuous.

Let us prove the strong continuity (c). We may assume that $|t - t_o| < 1$. Let ω belong to $L^2_{o,[-a,0]}$. For sufficiently large b, $(\sigma_t - \sigma_{t_o})\omega$ belongs to $L^2_{o,[-b,0]}$ for all t such that $|t - t_o| < 1$. It suffices to show that $(\sigma_t - \sigma_{t_o})\omega \to 0$ in $L^2_{o,[-b,0]}$ as $t \to t_o$ since the relative topology induced from Ω^1 on $L^2_{o,[-b,0]}$ is precisely the L^2-topology of $L^2_{[-b,0]}$. Now the convergence $(\sigma_t - \sigma_{t_o})\omega \to 0$ is a well-known fact of measure theory; see, for instance, HEWITT and STROMBERG [1975, IV.13.24]. □

2. Output Space.

Let $L^2_{[0,n]}$ be the space of L^2-functions on the interval $[0, n]$ with the norm:

$$(2.1) \quad \|\varphi\|_{[0,n]} := \{\int_0^n |\varphi(t)|^2 dt\}^{1/2}.$$

If $a \geq b$, there exists the natural projection $\pi_{ab}: L^2_{[0,a]} \to L^2_{[0,b]}$ defined by $\pi_{ab}(\varphi) := \varphi|_{[0,b]}$. This map is obviously continuous and linear. Clearly we may identify the space $L^2_{[0,n]}$ with the space of all functions defined on $[0, \infty)$ which vanish outside of $[0, n]$ and belong to $L^2_{[0,n]}$ when restricted to $[0, n]$. In the sequel we shall also denote this space by $L^2_{[0,n]}$; if a precise distinction is necessary, we shall denote it by $L^2_{o,[0,n]}$.

Now let $L^2_{loc}[0, \infty) := \{\varphi: [0, \infty) \to k: \varphi$ is locally L^2, i.e., on every compact interval $[0, a]$, $\|\varphi\|_{[0,a]}$ is finite$\}$. The space $L^2_{loc}[0, \infty)$ is equipped with the countable family of seminorms $\{\|\cdot\|_{[0,n]}\}_{n=1}^{\infty}$ defined as in (2.1). This family of seminorms defines a locally convex Hausdorff topology on $L^2_{loc}[0, \infty)$.

(2.2) REMARK. In fact $\|\cdot\|_{[0,a]}$ is a seminorm on $L^2_{loc}[0, \infty)$ for every positive real number a. But the topology given by the family $\{\|\cdot\|_{[0,a]}: a > 0, \ a \in \underline{\underline{R}}\}$ is easily seen to be the same as the topology defined by the family $\{\|\cdot\|_{[0,n]}\}_{n=1}^{\infty}$.

As is proven in the Appendix (Proposition (A.7)), the space $L^2_{loc}[0, \infty)$ is the projective limit of the sequence of spaces $\{L^2_{[0,n]}\}$. We write $\Gamma^1 := L^2_{loc}[0, \infty) = \varprojlim L^2_{[0,n]}$ (projective limit).

(2.3) DEFINITION. <u>The output space (with</u> p <u>output channels) is the space</u> $\Gamma := (\Gamma^1)^p = (L^2_{loc}[0, \infty))^p$. <u>When it is necessary to refer to the number of output channels explicitly, we write</u> Γ^p.

(2.4) PROPOSITION. <u>The space</u> Γ <u>is complete</u>.

PROOF. We assume $p = 1$ without loss of generality. Clearly each $L^2_{[0,n]}$ is complete because it is a Hilbert space. Since a projective limit of complete spaces is complete (SCHAEFER [1971, II.5.3]), Γ must be complete. □

(2.5) PROPOSITION. <u>The space</u> Γ <u>is a Fréchet space, i.e., metrizable and complete</u>.

PROOF. We have only to show metrizability. We again assume $p = 1$

without loss of generality. By KÖTHE [1969, 18.2, (2)], a locally convex space is metrizable iff its topology is generated by a countable family of seminorms. But this is indeed the case for Γ^1. □

We define the shift (operator) $\tilde{\sigma}_t\colon \Gamma \to \Gamma$ for each $t \geq 0$ by

(2.6) $\tilde{\sigma}_t(\gamma)(\tau) := \gamma(\tau + t).$

(2.7) PROPOSITION. The family of operators $\{\tilde{\sigma}_t\}_{t\geq 0}$ is a strongly continuous (or C_o-) semigroup, i.e.,

(a) $\tilde{\sigma}_o = I, \quad \tilde{\sigma}_t\tilde{\sigma}_s = \tilde{\sigma}_{t+s};$

(b) each $\tilde{\sigma}_t$ is a continuous linear map;

(c) $\lim_{t\to t_o}\tilde{\sigma}_t\gamma = \tilde{\sigma}_{t_o}\gamma$ for every $t_o \geq 0$ and γ in Γ.

PROOF. (a) This is obvious via direct calculation.

(b) If $a + t \leq b$, we have the following estimate:

$$\begin{aligned}\|\tilde{\sigma}_t\gamma\|^2_{[0,a]} &= \int_0^a |\gamma(\tau + t)|^2 d\tau,\\ &= \int_t^{a+t} |\gamma(\eta)|^2 d\eta,\\ &\leq \int_0^b |\gamma(\eta)|^2 d\eta,\\ &= \|\gamma\|^2_{[0,b]}.\end{aligned}$$

Thus $\tilde{\sigma}_t$ is continuous.

(c) We assume $|t - t_o| < 1$ without loss of generality.

(2.8) $$\|(\tilde{\sigma}_t - \tilde{\sigma}_{t_o})\gamma\|^2_{[0,n]} = \int_0^n |\gamma(\tau + t) - \gamma(\tau + t_o)|^2 d\tau.$$

It is well known that the right side converges to zero as $t \to t_o$ (HEWITT and STROMBERG [1975, IV.13.24]). □

3. Input/Output Maps.

We shall give the definition of linear input/output maps and prove that to every linear input/output map there is associated a unique matrix Radon measure on $(0, \infty)$, which we shall call the weighting pattern of a linear input/output map. We need some technical preliminaries as follows.

Let $C_o(-\infty, 0)$ be the space of continuous functions on $(-\infty, 0)$ which vanish outside some compact subset of $(-\infty, 0)$. Introduce the usual inductive limit topology $\varinjlim C_o[-n, -1/n]$ to $C_o(-\infty, 0)$; see TREVES [1967, 21]. Here $C_o[-n, -1/n]$ is the space of continuous functions on $(-\infty, 0)$ which vanish outside of $[-n, -1/n]$; its topology is defined by the supremum norm:

(3.1) $$|||\varphi|||_{-n} := \sup \{|\varphi(t)|: -n \leq t \leq -1/n\}.$$

Clearly there is the natural continuous inclusion:

$$j_1: (C_o(-\infty, 0))^m \to \Omega;$$

it is easy to verify that j_1 has a dense image.

Now let $C[0, \infty)$ be the space of continuous functions on $[0, \infty)$. Its topology is given by a countable family of seminorms:

(3.2) $$|||\varphi|||_n := \sup \{|\varphi(t)|: 0 \leq t \leq n\}, \quad n = 1, 2, \ldots .$$

There is the natural continuous inclusion:

$$j_2: (C[0, \infty)^p \to \Gamma;$$

it is also easy to verify that j_2 has a dense image.

We are now ready to give

(3.3) DEFINITION. A constant (shift-invariant) linear input/output map is a continuous linear map $f: \Omega \to \Gamma$ such that

(i) *the diagram*

$$\begin{array}{ccc} \Omega & \xrightarrow{f} & \Gamma \\ \sigma_t \big\downarrow & & \big\downarrow \tilde{\sigma}_t \\ \Omega & \xrightarrow{f} & \Gamma \end{array}$$

commutes for every $t \geq 0$;

(ii) $f((C_o(-\infty, 0))^m) \subset (C[0, \infty))^p$;

(iii) f *is also continuous as a map from* $(C_o(-\infty, 0))^m$ *to* $(C[0, \infty))^p$.

(3.4) REMARK. The spaces $(C_o(-\infty, 0))^m$ and $(C[0, \infty))^p$ are shift-invariant, i.e., $\sigma_t((C_o(-\infty, 0))^m) \subset (C_o(-\infty, 0))^m$ and $\tilde{\sigma}_t((C[0, \infty))^p) \subset (C[0, \infty))^p$ for all $t \geq 0$.

(3.5) EXAMPLE. Let $\mu = (\mu_{ij})_{ij}$ $(i = 1, \ldots, p,\ j = 1, \ldots, m)$ be a matrix whose ij-entry is a Radon measure (we shall abbreviate this as a *matrix Radon measure*) on $(0, \infty)$. Consider a linear map $f\colon \Omega \to \Gamma$ given by

$$(3.6)\quad f(\omega)|_i(t) := \sum_{j=1}^{m} \int_t^{\infty} \omega_j(t - \tau)d\mu_{ij}(\tau),$$
$$= \sum_{j=1}^{m} \int_{-\infty}^{0} \omega_j(\tau)d\mu_{ij}(t - \tau),$$

where $f(\omega)|_i$ and ω_j denote i-th and j-th entry of $f(\omega)$ and ω, respectively. We also write (3.6) as

$$(3.7)\quad f(\omega) = \int_t^{\infty} \omega(t - \tau)d\mu(\tau) = \int_{-\infty}^{0} \omega(\tau)d\mu(t - \tau),$$

for simplicity of notation.

(3.8) PROPOSITION. *The linear map* f *given by* (3.6) *is a constant linear input/output map*.

PROOF. Clearly it suffices to prove the statement for the case $m = p = 1$. Observe that we need to integrate only on a bounded interval for each t in (3.6) since each ω in Ω has bounded support.

We note from L. SCHWARTZ [1966, 6.1] that $f(\omega)(t)$ exists for almost every $t \geq 0$ and $f(\omega)$ belongs to $L^2_{loc}[0, \infty)$ $(= \Gamma^1)$ for each ω in Ω^1. Furthermore, if ω belongs to $L^2_{o,[-a,0]}$, then we have the estimate:

$$(3.9) \qquad \|f(\omega)\|_{[0,b]} \leq \|\mu\|_{[0,a+b]} \cdot \|\omega\|_{[-a,0]},$$

where $\|\mu\|_{[0,\alpha]}$ denotes $\int_0^\alpha |d\mu|$; for a proof, see DIEUDONNÉ [1970, 14.9.2]. Hence f is a continuous (and obviously linear) map from $L^2_{[-a,0]}$ to Γ^1 for every a. By Proposition (A.2) it follows that f is a continuous linear map from Ω^1 to Γ^1. It is easy to verify, via direct calculation, that $\tilde{\sigma}_t f = f\sigma_t$ for all $t \geq 0$.

We again note from L. SCHWARTZ [1966, 6.1] (see also DIEUDONNÉ [1970, 14.9.2]) that $f(\omega)$ is a continuous function of t whenever ω belongs to $C_o(-\infty, 0)$. Thus $f(C_o(-\infty, 0)) \subset C[0, \infty)$.

Let ω be an element of $C_o[-a, -1/a]$ $(a > 0)$. We have the following estimate:

$$(3.10) \qquad \sup_{0 \leq t \leq b} |f(\omega)(t)| = \sup_{0 \leq t \leq b} \left| \int_{-a}^{-1/a} \omega(\tau) d\mu(t - \tau) \right|,$$

$$\leq \sup_{0 \leq t \leq b} \int_{-a}^{-1/a} |\omega(\tau)| \, |d\mu(t - \tau)|,$$

$$\leq \{ \sup_{-a \leq t \leq 0} |\omega(\tau)| \} \int_0^{a+b} |d\mu(\tau)| \quad \text{for all } b > 0.$$

Therefore f is also continuous as a map from $C_o[-a, -1/a]$ to $C[0, \infty)$ for each $a > 0$. In view of the inductive limit topology of $C_o(-\infty, 0)$, this proves the continuity of f as a map from $C_o(-\infty, 0)$ to $C[0, \infty)$ (see Proposition (A.2)). □

When a linear input/output map is given by a matrix Radon measure μ as in (3.6) (or (3.7)), we call μ the <u>weighting pattern</u> of a constant linear input/output map f.

(3.11) REMARK. If a measure μ is given by $d\mu = A(\tau)d\tau$ ($d\tau$ = the Lebesgue measure) by some function A, then (3.7) coincides with the usual convolution $\int_{-\infty}^{0} A(t - \tau)\omega(\tau)d\tau$; see L. SCHWARTZ [1966, 6.1]; we shall also call $A(\tau)$ the weighting pattern of f with slight abuse of language.

We ask the converse question: Can every constant linear input/output map f be written as (3.7) for some matrix Radon measure μ? The next theorem states that this is indeed the case.

(3.12) THEOREM. A linear map $f: \Omega \to \Gamma$ is a constant linear input/output map iff there exists μ, a unique matrix Radon measure on $(0, \infty)$, such that (3.7) ((3.6)) holds.

PROOF. We already proved the sufficiency part in Proposition (3.8). We prove the necessity for the case $m = p = 1$; the general case follows from this by considering each factor $f(\omega_j)|_i$.

Let $C_o(0, \infty)$ be the space of continuous functions on $(0, \infty)$ which vanish outside of some compact subset of $(0, \infty)$. We introduce the same topology to $C_o(0, \infty)$ as is done for $C_o(-\infty, 0)$. Consider the linear map $\check{}: \varphi \mapsto \check{\varphi}$ given by $\check{\varphi}(t) := \varphi(-t)$. Clearly $\check{}$ gives an isomorphism between $C_o(-\infty, 0)$ and $C_o(0, \infty)$, and, furthermore, $\check{\check{\varphi}} = \varphi$ for every φ.

Consider the following linear functional μ on $C_o(0, \infty)$:

(3.13) $\mu(\varphi) := f(\check{\varphi})(0)$.

Since f is a continuous linear map from $C_o(-\infty, 0)$ to $C[0, \infty)$ by hypothesis, μ is a continuous linear form on $C_o(0, \infty)$, i.e., μ is a Radon measure on $(0, \infty)$. Write $\int_0^\infty \varphi(\tau)d\mu(\tau)$ instead of $\mu(\varphi)$. Then it follows that

$$f(\omega)(t) = (\tilde{\sigma}_t f(\omega))(0) = (f(\sigma_t\omega))(0) \quad (f \text{ commutes with shifts}),$$

$$= \int_0^\infty (\sigma_t\omega)^\vee(\tau)d\mu(\tau),$$

$$= \int_0^\infty (\sigma_t\omega)(-\tau)d\mu(\tau),$$

$$= \int_0^t 0 d\mu(\tau) + \int_t^\infty \omega(t-\tau)d\mu(\tau) \quad (\text{by } (1.4)),$$

$$= \int_t^\infty \omega(t-\tau)d\mu(\tau) \quad \text{for all } \omega \text{ in } C_o(-\infty, 0).$$

Hence μ satisfies (3.6) for ω in $C_o(-\infty, 0)$. But since $C_o(-\infty, 0)$ is dense in Ω^1, this equality must hold for every ω in Ω^1.

If $\overline{\mu}$ is another Radon measure on $(0, \infty)$ that satisfies (3.6), then we must have

$$\int_0^\infty \varphi(\tau)d\mu(\tau) = \int_0^\infty \check{\varphi}(-\tau)d\mu(\tau) = f(\check{\varphi})(0) \quad (\text{by } (3.6)),$$

$$= \int_0^\infty \check{\varphi}(-\tau)d\overline{\mu}(\tau) = \int_0^\infty \varphi(\tau)d\overline{\mu}(\tau),$$

for all $\varphi \in C_o(0, \infty)$. Thus $\overline{\mu} = \mu$, i.e., μ is unique. □

There are certain occasions that we want to have the "delta function" δ_o in the input space. Unfortunately, our input space Ω does not contain δ_o. Now we show one way of introducing δ_o into our framework.

Let f be a linear input/output map with the weighting pattern μ. We assume that $d\mu = A(\tau)d\tau$ (i.e., $d\mu_{i_j} = A_{i_j}(\tau)d\tau$) and A is of class C^r $(r \geq 0)$. What we try to show is that if A is regular enough, say C^r, then there exists a (unique) continuous extension $\tilde{f}$ of f to an "ideal input space" that contains δ_o. To be more precise, let $\underset{\sim}{E}^r_{(-\infty,0]}$ be the space of C^r-functions on $(-\infty, 0]$ with the topology of uniform convergence on each compact interval for each derivative of order less than or equal to r. Now let $\underset{\sim}{E}^{r'}_{(-\infty,0]}$ be the dual space of $\underset{\sim}{E}^r_{(-\infty,0]}$, namely the set of all continuous linear forms on $\underset{\sim}{E}^r_{(-\infty,0]}$, with the topology of compact convergence, that is, the topology of uniform convergence on each compact set of $\underset{\sim}{E}^{r}_{(-\infty,0]}$. It is well known that $\underset{\sim}{E}^{r'}_{(-\infty,0]}$ is the space of distributions of order $\leq r$ with compact support contained in $(-\infty, 0]$. Thus δ_o belongs to $\underset{\sim}{E}^{r'}_{(-\infty,0]}$. It is also easy to see that Ω is contained in $\underset{\sim}{E}^{r'}_{(-\infty,0]}$ as a dense linear subspace. Further, each $\underset{\sim}{E}^{r'}_{(-\infty,0]}$ is equipped with shifts $\{\sigma_t\}_{t\geq 0}$, which are natural extensions of $\{\sigma_t\}_{t\geq 0}$ in Ω^1. We denote the space $(\underset{\sim}{E}^{r'}_{(-\infty,0]})^m$ by $\tilde{\Omega}$ (or $\tilde{\Omega}^{(r)}$, if we must specify the order). When $r = \infty$, the space $\underset{\sim}{E}'_{(-\infty,0]}$ $(= E^{\infty'}_{(-\infty,0]})$ is the input space considered in KALMAN and HAUTUS [1972]. We now claim

(3.14) PROPOSITION. Let f be a constant linear input/output map given by the weighting pattern $d\mu = A(\tau)d\tau$ for some C^r-function A $(r \geq 0)$. Then there exists a continuous extension $\tilde{f}\colon \tilde{\Omega}^{(r)} \to \Gamma$ of f such that $\tilde{\sigma}_t\tilde{f} = \tilde{f}\sigma_t$ for all $t \geq 0$.

SKETCH OF PROOF. We may assume $m = p = 1$ without loss of generality. Now in view of Remark (3.11), we must define $\tilde{f}(\tilde{\omega})$ as the convolution of $\tilde{\omega}$ in $\underset{\sim}{E}^{r'}_{(-\infty,0]}$ with a fixed element A, i.e., $\tilde{f}(\tilde{\omega})(t) := \tilde{\omega}(A(t - (\cdot)))$, where the right side denotes the value of the distribution $\tilde{\omega}$ evaluated at the C^r-function $\tau \mapsto A(t - \tau)$. This linear map $\tilde{f}$ is a continuous linear map from $\underset{\sim}{E}^{r'}_{(-\infty,0]}$ to $C[0, \infty)$; see, for instance, L. SCHWARTZ [1966, 6.4, Théorème 12 and the succeeding remark]; it is easy to modify the proof. Since the topology of $C[0, \infty)$ is finer than that of Γ, this map $\tilde{f}$ gives a continuous correspondence from $\underset{\sim}{E}^{r'}_{(-\infty,0]}$ to Γ. It is also known that $\tilde{f}$ commutes with shifts; see L. SCHWARTZ [1966, 6.3.9]. □

(3.15) PROPOSITION. Let f be a constant linear input/output map with the weighting pattern $d\mu = A(\tau)d\tau$, where $A = (A_{ij})$ is of C^r-class. Then $A_{ij} = \tilde{f}(\delta_o|_j)|_i$, where $\delta_o|_j$ denote the element having the only nonzero term δ_o in the j-th position.

PROOF. From the proof of Proposition (3.14), we have $\tilde{f}(\tilde{\omega}|_j)|_i(t) = (A_{ij}*\tilde{\omega}|_j)(t)$ for all $\tilde{\omega}|_j$ in $\underset{\sim}{E}^{r'}_{(-\infty,0]}$. Then $\tilde{f}(\delta_o|_j)|_i(t) = (A_{ij}*\delta_o|_j)(t) = A_{ij}(t)$ because δ_o is the unit element with respect to convolution. □

4. The Space of Laurent Functions.

Let $\Lambda := \Omega^1 \oplus \Gamma^1 \cong \Omega^1 \times \Gamma^1$ with the direct sum (or product) topology. We call Λ the space of Laurent functions, and its element λ a Laurent function. Since every element λ of Λ can be uniquely represented as $\lambda = \omega + \gamma$, $\omega \in \Omega^1$, $\gamma \in \Gamma^1$, λ can be regarded as a locally L^2-function with its support bounded on the left. Note that $\omega(0)$ and $\gamma(0)$ may well be different, but this does not cause any difficulty since the point 0 is of measure 0.

Let $\pi_\Omega: \Lambda^m \to \Omega$, $\pi_\Gamma: \Lambda^p \to \Gamma$ be the projections, and let $j_\Omega: \Omega \to \Lambda^p$ and $j_\Gamma: \Gamma \to \Lambda^p$ be the inclusion maps. Consider the following families of linear maps:

(4.1) $(\sigma_t^\ell \lambda)(\tau) := \lambda(\tau + t);$

(4.2) $(\sigma_t^r \lambda)(\tau) := \lambda(\tau - t), \quad t \geq 0, \quad \lambda \in \Lambda^q, \quad (q = \text{integer}).$

(4.3) PROPOSITION. The families $\{\sigma_t^\ell\}_{t\geq 0}$ and $\{\sigma_t^r\}_{t\geq 0}$ are strongly continuous semigroups in Λ^q. Furthermore, $\pi_\Omega \sigma_t^\ell j_\Omega = \sigma_t$ and $\pi_\Gamma \sigma_t^\ell j_\Gamma = \tilde{\sigma}_t$ for all $t \geq 0$.

SKETCH OF PROOF. It is easy to see that $\sigma_0^\ell = I$ and $\sigma_{t+s}^\ell = \sigma_t^\ell \sigma_s^\ell$, and $\sigma_0^r = I$, $\sigma_{t+s}^r = \sigma_t^r \sigma_s^r$.

To prove the continuity of σ_t^ℓ (and σ_t^r) it suffices to prove that σ_t^ℓ (σ_t^r) is continuous on $(L^2_{[-n-0]} \oplus \Gamma^1)^q$ for all n. But this can be done in exactly the same way as in Propositions (1.5) and (2.7).

The strong continuity, $\lim_{t\to t_0} \sigma_t^\ell(\lambda) = \sigma_{t_0}^\ell(\lambda)$ (or $\lim_{t\to t_0} \sigma_t^r(\lambda) = \sigma_{t_0}^r(\lambda)$), follows directly from the fact that this is true on each bounded interval $[-a, b]$; see HEWITT and STROMBERG [1975, IV.13.24].

The equalities $\pi_\Omega \sigma_t^\ell j_\Omega = \sigma_t$ and $\pi_\Gamma \sigma_t^\ell j_\Gamma = \tilde{\sigma}_t$ follow from direct calculation. □

5. Extended Linear Input/output Maps.

In Section 3 we defined linear input/output maps. It is assumed that every input "terminates" at 0, i.e., $\operatorname{supp} \omega \subset (-\infty, 0]$ for every ω in Ω and every output "starts" at 0, i.e., supp $f(\omega) \subset [0, \infty)$. In other words, we observe outputs only after the application of inputs. This idea has been highly successful in realization theory in the sense that "causality" is already built into the framework (KALMAN, FALB, and ARBIB [1969, Chapter 10]). Of course in the actual dynamic mode of systems, usually outputs must be observed while inputs are applied. Input/output behavior in this sense is usually described by the Laurent series in the discrete-time case; see HAUTUS and HEYMANN [1978]. We show a counterpart in the continuous-time case in this section.

Let $C[a, \infty)$ be the space of continuous functions on $(-\infty, \infty)$ which vanish outside of $[a, \infty)$. This space is a Fréchet space with countable seminorms:

$$\sup \{|\varphi(t)|: a \leq t \leq n\}, \quad n = 1, 2, \ldots .$$

Let $C_+(-\infty, \infty) := \bigcup_{-\infty<a<\infty} C[a, \infty)$, and introduce the inductive limit topology. Since each inclusion: $C[a, \infty) \to \Lambda$ is clearly continuous, the induced inclusion: $C_+(-\infty, \infty) \to \Lambda$ is continuous; it is easy to see that this inclusion has a dense image.

(5.1) DEFINITION. <u>A continuous linear map</u> $\bar{f}: \Lambda^m \to \Lambda^p$ <u>is a (strictly causal) extended constant linear input/output map iff</u>

(i) $\sigma_t^\ell \bar{f} = \bar{f}\sigma_t^\ell$, $\sigma_t^r \bar{f} = \bar{f}\sigma_t^r$, <u>for all</u> $t \geq 0$;

(ii) <u>if</u> $\lambda_1|_{(-\infty,t)} = \lambda_2|_{(-\infty,t)}$, <u>then</u> $\bar{f}(\lambda_1)|_{(-\infty,t]} = \bar{f}(\lambda_2)|_{(-\infty,t]}$ <u>(strict causality) for continuous</u> λ_1 <u>and</u> λ_2;

(iii) $\bar{f}((C_+(-\infty, \infty))^m) \subset (C_+(-\infty, \infty))^p$;

(iv) $\bar{f}$ <u>is continuous as a map from</u> $(C_+(-\infty, \infty))^m$ <u>to</u> $(C_+(-\infty, \infty))^p$.

REMARK. Property (ii) clearly implies that if $\lambda_1|_{(-\infty,t)} = \lambda_2|_{(-\infty,t)}$ then $f(\lambda_1)|_{(-\infty,t)} = f(\lambda_2)|_{(-\infty,t)}$ <u>for all</u> λ_i in Λ^m.

Our objective here is to show that <u>every strictly causal extended constant linear input/output map is necessarily derived from a constant input/output map</u>

and vice versa. Indeed, we now prove

(5.2) THEOREM. *For any constant linear input/output map* f *there is a unique strictly causal extended constant linear input/output map* $\overline{f}$ *such that the diagram*

$$\begin{array}{ccc} \Lambda^m & \xrightarrow{\ \overline{f}\ } & \Lambda^p \\ {\scriptstyle j_\Omega}\uparrow & & \downarrow{\scriptstyle \pi_\Gamma} \\ \Omega & \xrightarrow{\ f\ } & \Gamma \end{array}$$

commutes. Conversely, if $\overline{f}$ *is a strictly causal extended constant linear input/output map, then* $f := \pi_\Gamma \overline{f} j_\Omega$ *is a constant linear input/output map.*

PROOF. Assume $m = p = 1$ for simplicity of notation. Let f be a linear input/output map with the weighting pattern μ. Define a linear map $\overline{f}\colon \Lambda^m \to \Lambda^p$ by

$$(5.3) \quad \overline{f}(\lambda)(t) := \int_{-\infty}^{t} \lambda(\tau)d\mu(t - \tau) = \int_{0}^{\infty} \lambda(t - \tau)d\mu(\tau).$$

Note that if $\lambda(\tau) = 0$ for $\tau < a$, then $\overline{f}(\lambda)(t) = 0$ for $t \leq a$. This implies that $\overline{f}$ is strictly causal by linearity of $\overline{f}$. Note also that the integral (5.3) is evaluated only on a bounded interval, hence is well-defined for almost all t by L. SCHWARTZ [1966, 6.1]. Further, if t runs over a compact set, then only the values of λ on a compact set contribute to the integral (5.3). Hence $\overline{f}(\lambda)$ is locally L^2 by DIEDONNÉ [1970, 14.9.2]; $\overline{f}(\lambda)$ is a continuous function if λ is continuous, again by the same reference. Thus $\overline{f}(C_+(-\infty, \infty)) \subset C_+(-\infty, \infty)$.

We denote by $L^2_{loc}[a, \infty)$ the space of all locally L^2-functions on $(-\infty, \infty)$ which vanish outside of $[a, \infty)$. The space $L^2_{loc}[a, \infty)$ is topologized by the countable family of seminorms:

$$\|\varphi\|_{[a,n]} := \{\int_a^n |\varphi(t)|^2 dt\}^{1/2}, \quad n = 1, 2, \ldots .$$

Clearly $\Lambda = \bigcup_{-\infty<a<\infty} L^2_{loc}[a, \infty)$.

Now let λ belong to $L^2_{loc}[a, \infty)$. Then $\overline{f}(\lambda)(t) = 0$ for $t < a$. Furthermore,

$$(5.4) \quad \|\overline{f}(\lambda)\|_{[a,b]} = \{\int_a^b |\overline{f}(\lambda)(t)|^2 dt\}^{1/2} \leq \|\mu\|_{[0,b-a]} \cdot \|\lambda\|_{[a,b]}$$

($\|\mu\|_{[0,b-a]} := \int_0^{b-a} |d\mu|$) as can be shown in the same way as (3.9). Thus $\overline{f}$ is continuous as a map from $L^2_{loc}[a, \infty)$ to $L^2_{loc}[a, \infty)$. Since Λ can be regarded as the inductive limit of $L^2_{loc}[a, \infty)$ (Proposition (A.10)), this fact establishes the continuity of $\overline{f}: \Lambda \to \Lambda$ by Proposition (A.2).

Now let λ belong to $C[a, \infty)$. Then

$$(5.5) \quad \sup_{a \leq t \leq b} |\overline{f}(\lambda)(t)| \leq \{\sup_{a \leq t \leq b} |\lambda(t)|\} \cdot \|\mu\|_{[0,b-a]}.$$

This can be proved in exactly the same way as in (3.10). Hence $\overline{f}$ is continuous as a map from $C[a, \infty)$ to $C[a, \infty)$. In view of the inductive limit topology of $C_+(-\infty, \infty)$, this implies that $\overline{f}$ is continuous as a map from $C_+(-\infty, \infty)$ to $C_+(-\infty, \infty)$ (see Proposition (A.2)). Clearly $\overline{f}$ commutes with shifts. Therefore $\overline{f}$ is a strictly causal extended linear input/output map. The property $\pi_\Gamma \overline{f} j_\Omega = f$ is obvious via direct calculation.

We must prove uniqueness. Let f_1 and f_2 be two strictly causal extended linear input/output maps such that $f = \pi_\Gamma \overline{f}_1 j_\Omega = \pi_\Gamma \overline{f}_2 j_\Omega$. Take any continuous λ in Λ. Then

$$(5.6) \quad \begin{aligned} \overline{f}_1(\lambda)(t) &= (\sigma^\ell_t \overline{f}_1(\lambda))(0), \\ &= \overline{f}_1(\sigma^\ell_t \lambda)(0), \\ &= \overline{f}_1(j_\Omega \pi_\Omega \sigma^\ell_t \lambda)(0) \quad \text{(strict causality)}, \\ &= \pi_\Gamma \overline{f}_1(j_\Omega \pi_\Omega \sigma^\ell_t \lambda)(0), \\ &= f(\pi_\Omega \sigma^\ell_t \lambda)(0) \quad (\pi_\Gamma \overline{f}_1 j_\Omega = f), \\ &= \pi_\Gamma \overline{f}_2(j_\Omega \pi_\Omega \sigma^\ell_t \lambda)(0), \end{aligned}$$

$$= \bar{f}_2(\sigma_t^{\ell}\lambda)(0),$$

$$= \bar{f}_2(\lambda)(t) \quad \text{for all} \quad t \geq 0.$$

Similarly, $\bar{f}_1(\lambda)(t) = \bar{f}_2(\lambda)(t)$ for $t < 0$. Hence $\bar{f}_1 = \bar{f}_2$ on $C_+(-\infty, \infty)$. Since $C_+(-\infty, \infty)$ is dense in Λ, $\bar{f}_1$ must be equal to $\bar{f}_2$.

Conversely, let $\bar{f}$ be a strictly causal extended linear input/output map. Clearly $f := \pi_\Gamma \bar{f} j_\Omega$ is a continuous linear map from Ω to Γ, and it commutes with shifts because $f\sigma_t = \pi_\Gamma \bar{f} j_\Omega \sigma_t = \pi_\Gamma \sigma_t^{\ell} \bar{f} j_\Omega = \pi_\Gamma \sigma_t^{\ell} j_\Gamma \pi_\Gamma \bar{f} j_\Omega = \tilde{\sigma}_t \pi_\Gamma \bar{f} j_\Omega = \tilde{\sigma}_t f$. It is also clear that f maps $C_0(-\infty, 0)$ into $C[0, \infty)$ and is continuous with respect to the corresponding topologies. □

(5.7) REMARK. The condition that linear input/output maps send continuous functions to continuous functions requires, roughly speaking, that linear input/output maps do not "differentiate." In other words, if μ is the weighting pattern of a linear input/output map f, then μ must not contain terms such as δ_a', δ_a'', ... etc.

CHAPTER III. REALIZATION THEORY

We shall start by defining linear systems, objects which are of our loving concern. In Section 6, we also study some basic notions such as quasi-reachability, observability, morphisms between systems, etc. In Section 7, we define a realization and a factorization of a linear input/output map. The notion of factorization of a linear input/output map is a convenient tool to handle the problem of existence and uniqueness of canonical realizations.

It is well known (BARAS, BROCKETT, and FUHRMANN [1974]) that a weak notion of canonicity, namely quasi-reachability plus observability, does not, in general, lead us to the uniqueness of canonical realizations. A new notion of observability, which we call topological observability, is introduced in Section 8. We shall then prove the desired existence and uniqueness theorem in Section 9 as a direct consequence of topological observability; a counterexample by BARAS, BROCKETT, and FUHRMANN [1974] is discussed in order to illustrate the theorem.

In Section 10, we prove that a realization indeed produces outputs even while L^2-inputs are applied. This is not necessarily guaranteed by our definition of linear systems. In Section 11, we turn our attention to differential equation descriptions of linear systems, and prove that a canonical realization is described by a functional differential equation if the weighting pattern of the input/output map is sufficiently smooth. The notions of compatible and smooth systems are introduced.

6. Systems.

(6.1) DEFINITION. A linear (constant, continuous-time) system (with m-input, p-output channels) is a triple $\Sigma = (X, \Phi, H)$ which satisfies the following conditions:

(a) X is a complete locally convex Hausdorff space;

(b) for each fixed $t \geq 0$

$$X \times (L^2_{[0,t)})^m \to X: (x, u) \mapsto \varphi(t, x, u)$$

is a continuous linear map (when $s < t$, we denote $\varphi(s, x, u|_{[0,s)})$ by $\varphi(s, x, u)$);

(c) for every $t, s \geq 0$, φ satisfies

$$\varphi(t + s, x, u) = \varphi(t, \varphi(s, x, u|_{[0,s)}), \sigma_s^{\ell} u|_{[0,t)})$$

for all x in X, u in $L^2_{[0,t+s)}$ (for σ_s^{ℓ}, see Section 4), and

$$\varphi(0, x, u) = x \text{ for all } x \text{ in } X \text{ and } u \text{ in } L^2_{[0,t)};$$

(d) $\lim_{t \to t_0} \varphi(t, x, 0) = \varphi(t_0, x, 0)$ for all x in X;

(e) H is a densely defined (not necessarily continuous) linear operator: $D(H) \to k^p$;

(f) there exists a dense subspace $D_0(H) \subset D(H)$ such that $\varphi(t, x, 0)$ belongs to $D_0(H)$ for all $t \geq 0$ and x in $D_0(H)$;

(g) for every $t \geq 0$, $\varphi(t, 0, u)$ belongs to $D_0(H)$ for every continuous function u such that $u(0) = u(t) = 0$;

(h) under the same hypothesis on u as in (g), $\varphi(s, 0, u)$ belongs to $D(H)$ for every $0 \leq s < t$ (but not necessarily to $D_0(H)$);

(i) the correspondence: $C_0[0, t] \to k^p$ given by $u \mapsto H\varphi(t, 0, u)$ is continuous with respect to the topology of uniform convergence on $C_0[0, t]$ for every $t > 0$;

(j) there exists a continuous linear map $h_0: D(H) \to \Gamma$ such that (i) $h_0(x)$ is continuous on $[0, \epsilon)$ for some $\epsilon > 0$, (ii) $h_0(x)(0) = Hx$, (iii) $h_0(x)$ is continuous on $[0, \infty)$ if x belongs to $D_0(H)$, and (iv) $h_0(x)(t) = H\varphi(t, x, 0)$ if x belongs to $D_0(H)$.

We call X the state space, φ the state-transition map, H the readout (output) map of Σ. We also call $\varphi(t, x, u)$ the state resulting at time t from the initial state x under the action of input u.

(6.2) REMARK. It is easy to see that the first and the second conditions of (j) imply the right-continuity of $h_o(x)$ on $[0, \infty)$ for all x in $D_o(H)$. But the left-continuity is not necessarily guaranteed.

Even though Definition (6.1) may appear overly involved, the only difference of this definition from the classical one given in KALMAN, FALB, and ARBIB [1969, 1.1] is the fact that H maybe neither everywhere defined nor continuous. In fact, if H were continuous it would have a continuous extension to the whole space X, and Conditions (f) to (j) would become redundant. But as pointed out in Chapter I, requiring that H be continuous would result in excluding many interesting examples (see also Section 14).

At any rate, we must ask for some type of continuity on initial state/output correspondence; otherwise no study could be made on topological aspects of systems. Thus we require that $h_o: D(H) \to \Gamma$ exist and be continuous. Conditions (g) to (i) require that H behave "nicely" with respect to continuous inputs.

(6.3) REMARK. Causality, i.e., $\varphi(t, x, u_1) = \varphi(t, x, u_2)$ whenever $u_1|_{[0,t)} = u_2|_{[0,t)}$, is built into the definition. Indeed $\varphi(t, x, u_1)$ $\varphi(t, x, u_1|_{[0,t)}) = \varphi(t, x, u_2|_{[0,t)}) = \varphi(t, x, u_2)$ by definition (see (b) of Definition (6.1)).

We now investigate some direct consequences of Definition (6.1).

(6.4) PROPOSITION. _Let_ $\Sigma = (X, \varphi, H)$ _be a linear system. Then there exists a continuous linear map_ $h^\Sigma: X \to \Gamma$ _such that_ $h^\Sigma(x) = h_o(x) = H\varphi(\cdot, x, 0)$ _for all_ $x \in D(H)$.

PROOF. We already know that $h_o: D(H) \to \Gamma$ exists and is continuous by definition. Since $D(H)$ is dense in X and Γ is complete (Proposition (2.4)), h_o must have a unique continuous extension $h^\Sigma: X \to \Gamma$ such that $h^\Sigma|_{D(H)} = h_o$. □

(6.5) PROPOSITION. _Let_ $\Sigma = (X, \varphi, H)$ _be a linear system. Then there exist a strongly continuous semigroup_ $\{\Phi(t)\}_{t \geq 0}$ _and a continuous linear map_ $g^\Sigma: \Omega \to X$ _such that_

$$(6.6)\qquad \varphi(t,\ x,\ u) = \Phi(t)x + g^{\Sigma}(\pi_{\Omega}\sigma_t^{\ell}u)$$

for all $t \geq 0$, $x \in X$, $u \in L^2_{[0,t)}$, where u is regarded as an element of Λ^m.

PROOF. For each fixed $t \geq 0$, $\varphi(t,\ \cdot,\ 0): X \to X$ is a continuous linear map by Condition (b) of Definition (6.1). Write $\Phi(t)x$ for $\varphi(t,\ x,\ 0)$. Then

$$\begin{aligned}(6.7)\qquad \Phi(t+s)x &= \varphi(t+s,\ x,\ 0),\\ &= \varphi(t,\ \varphi(s,\ x,\ 0),\ 0),\\ &= \varphi(t,\ \Phi(s)x,\ 0),\\ &= \Phi(t)\Phi(s)x,\end{aligned}$$

and

$$(6.8)\qquad \Phi(0)x = \varphi(0,\ x,\ 0) = x \quad \text{for all } x \text{ in } X,$$

by Condition (c) of Definition (6.1). Furthermore,

$$(6.9)\qquad \lim_{t\to t_o} \Phi(t)x = \lim_{t\to t_o} \varphi(t,\ x,\ 0) = \varphi(t_o,\ x,\ 0) = \Phi(t_o)x$$

for all x in X by Condition (d) of Definition (6.1). Thus $\{\Phi(t)\}_{t\geq 0}$ is a strongly continuous semigroup.

Now let ω be an element of Ω with its support contained in $[-\alpha,\ 0]$. Define g^{Σ} by

$$(6.10)\qquad g^{\Sigma}(\omega) := \varphi(\alpha,\ 0,\ \sigma_{\alpha}^{r} j_{\Omega}\omega),$$

where $j_{\Omega}: \Omega \to \Lambda^m$ is the inclusion and σ_{α}^{r} is a right shift operator in Λ^m (see Section 4). We show $g^{\Sigma}(\omega)$ is well-defined. Indeed, if $\beta > \alpha$, we obtain

$$\begin{aligned}\varphi(\beta,\ 0,\ \sigma_{\beta}^{r} j_{\Omega}\omega) &= \varphi(\alpha,\ \varphi(\beta-\alpha,\ 0,\ \sigma_{\beta}^{r} j_{\Omega}\omega|_{[0,\beta-\alpha)}),\ \sigma_{\beta-\alpha}^{\ell}\sigma_{\beta}^{r} j_{\Omega}\omega|_{[0,\alpha)}),\\ &= \varphi(\alpha,\ \varphi(\beta-\alpha,\ 0,\ 0),\ \sigma_{\alpha}^{r} j_{\Omega}\omega|_{[0,\alpha)}),\end{aligned}$$

$$= \varphi(\alpha,\ 0,\ \sigma^r_\alpha j_\Omega \omega|_{[0,\alpha)}),$$
$$= g^\Sigma(\omega),$$

because $\varphi(s, 0, 0) = 0$ by linearity of φ. Hence $g(\omega)$ is independent of the choice of α as long as $\operatorname{supp} \omega \subset [-\alpha, 0]$. Thus g^Σ is well-defined on the whole input space Ω. Furthermore, since $\varphi(\alpha, 0, \cdot)$ is continuous for each fixed $\alpha \geq 0$, g^Σ must be continuous on each $(L^2_{[-\alpha,0]})^m$. Since Ω is the inductive limit of $\{(L^2_{[-\alpha,0]})^m\}$, g^Σ must be continuous on Ω by Proposition (A.2).

Now since $\varphi(t, \cdot, \cdot)$ is linear, one obtains

$$\varphi(t, x, u) = \varphi(t, x, 0) + \varphi(t, 0, u),$$
$$= \Phi(t)x + g^\Sigma(\pi_\Omega \sigma^\ell_t u),$$

because $\sigma^r_t j_\Omega \pi_\Omega \sigma^\ell_t u = u$ if u belongs to $(L^2_{[0,t)})^m$. □

$\{\Phi(t)\}_{t\geq 0}$ is called the <u>semigroup associated with</u> Σ.

(6.11) DEFINITION. <u>Let</u> $\Sigma = (X, \varphi, H)$ <u>be a linear system. The reachability map (of</u> Σ<u>) is</u> g^Σ <u>given by</u> (6.10), <u>and the observability map (of</u> Σ<u>) is</u> h^Σ <u>given in Proposition</u> (6.4). <u>The reachable subspace of</u> Σ <u>is</u> $X_R := g^\Sigma(\Omega)$. <u>The system</u> Σ <u>is quasi-reachable iff</u> X_R <u>is dense in</u> X, <u>(exactly) reachable iff</u> $X_R = X$, <u>and observable iff</u> h^Σ <u>is one-to-one;</u> Σ <u>is weakly canonical iff it is both quasi-reachable and observable</u>.

(6.12) PROPOSITION. <u>For every</u> $t \geq 0$, $g^\Sigma \sigma_t = \Phi(t) g^\Sigma$ <u>and</u> $\tilde{\sigma}_t h^\Sigma = h^\Sigma \Phi(t)$.

PROOF. Let ω be an element of Ω with its support contained in $[-\alpha, 0]$. By definition (see (1.4)) $\operatorname{supp} \sigma_t \omega \subset [-\alpha - t, -t]$. Then we obtain

$$g^\Sigma \sigma_t \omega = \varphi(\alpha + t,\ 0,\ \sigma^r_{\alpha+t} j_\Omega \sigma_t \omega),$$
$$= \varphi(\alpha + t,\ 0,\ \sigma^r_{\alpha+t} \sigma^\ell_t j_\Omega \omega) \quad (\sigma^\ell_t j_\Omega = j_\Omega \sigma_t),$$
$$= \varphi(\alpha + t,\ 0,\ \sigma^r_\alpha j_\Omega \omega) \quad (\sigma^r_{\alpha+t} \sigma^\ell_t = \sigma^r_\alpha),$$

$$= \varphi(t,\ \varphi(\alpha,\ 0,\ \sigma_\alpha^r j_\Omega \omega),\ 0),$$

$$= \varphi(t,\ g^\Sigma(\omega),\ 0) \quad (\text{definition of } g^\Sigma),$$

$$= \Phi(t) g^\Sigma(\omega) \quad (\text{definition of } \Phi(t)).$$

Take any element x in $D_o(H)$. Then we obtain

$$(\tilde{\sigma}_t h^\Sigma(x))(s) = h^\Sigma(x)(s + t),$$

$$= H\Phi(s + t)x \quad (\text{Proposition (6.4), definition of } \Phi(t)),$$

$$= H\Phi(s)(\Phi(t)x),$$

$$= h^\Sigma(\Phi(t)x)(s).$$

Hence $\tilde{\sigma}_t h^\Sigma = h^\Sigma \Phi(t)$ on $D_o(H)$ for all $t \geq 0$. Since $D_o(H)$ is dense in X, the conclusion follows from the continuity of h^Σ. □

The following Proposition gives a dual characterization of quasi-reachability and observability.

(6.13) PROPOSITION. Suppose that $\Sigma = (X, \varphi, H)$ is a linear system with a reflexive state space X. Consider the following statements:

(a) Σ is quasi-reachable;

(b) the adjoint $(g^\Sigma)': X' \to \Omega'$ is one-to-one;

(c) Σ is observable;

(d) the adjoint $(h^\Sigma)': \Gamma' \to X'$ has a dense image.

Then we have the equivalence (a)$\Leftrightarrow$(b) and (c)$\Leftrightarrow$(d).

PROOF. [(a)$\Leftrightarrow$(b)] We quote from TREVES [1967, Corollary 5 to Theorem 18.1] that g^Σ has a dense image iff $(g^\Sigma)'$ is one-to-one. Thus (a) is equivalent to (b).

[(c)$\Leftrightarrow$(d)] Since X and Γ are reflexive (Γ is easily seen to be reflexive as a projective limit of reflexive spaces $\{(L^2_{[0,n]})^m\}$; see SCHAEFER [1971, IV.4.4 and IV.4.5]), h^Σ can be identified with $(h^\Sigma)'': X'' \to \Gamma''$. Then, as before, $(h^\Sigma)'$ has a dense image iff $(h^\Sigma)''$ is one-to-one. Since $(h^\Sigma)''$ is identified with h^Σ, the conclusion follows. □

The following lemma will be useful later.

(6.14) LEMMA. Let Ω_o be a dense subspace of Ω. A linear system $\Sigma = (X, \varphi, H)$ is quasi-reachable iff $g^{\Sigma}(\Omega_o)$ is dense in X.

PROOF. Trivial. □

Let us now give the definition of a morphism between two systems. The following definition is a modification to the present context of the standard one.

(6.15) DEFINITION. Let $\Sigma_1 = (X_1, \varphi_1, H_1)$ and $\Sigma_2 = (X_2, \varphi_2, H_2)$ be linear systems. A morphism from Σ_1 to Σ_2 is a continuous linear map $T: X_1 \to X_2$ such that

(i) $\varphi_2(t, Tx_1, u) = T\varphi_1(t, x_1, u)$ for all $t \geq 0$, $x_1 \in X_1$, and $u \in L^2_{[0,t)}$;

(ii) there exists a dense subspace M_1 of $D_o(H_1)$ such that $TM_1 \subset D_o(H_2)$ and $H_2T = H_1$ on M_1.

We say that Σ_1 is isomorphic to Σ_2 iff T is a homeomorphism (isomorphism).

(6.16) REMARK. It is easy to see that the identity and the composition of two morphisms are morphisms. Note also that if T is a homeomorphism, then the inverse T^{-1} is automatically a morphism. For, if $\varphi_2(t, Tx_1, u) = T\varphi_1(t, x_1, u)$, then $\varphi_1(t, T^{-1}x_2, u) = T^{-1}\varphi_2(t, x_2, u)$ where $x_2 = Tx_1$. And if M_1 is a dense subspace of $D_o(H_1)$ such that $TM_1 \subset D_o(H_2)$, then $M_2 := TM_1$ is a dense subspace of $D_o(H_2)$ because T is a homeomorphism. Moreover, $T^{-1}M_2 = M_1 \subset D_o(H_1)$, and if $x_2 = Tx_1$ belongs to M_2, then $H_1T^{-1}x_2 = H_1T^{-1}Tx_1 = H_1x_1 = H_2Tx_1 = H_2x_2$ because x_1 belongs to M_1. Thus T^{-1} is a morphism.

We shall now investigate how the notion of morphisms is described in terms of reachability and observability maps. The first condition of Definition (6.15) clearly implies that $Tg^{\Sigma_1}(\omega) = T\varphi_1(\alpha, 0, \sigma^r_\alpha j_\Omega\omega) = \varphi_2(\alpha, 0, \sigma^r_\alpha j_\Omega\omega) = g^{\Sigma_2}(\omega)$ for every ω with $\operatorname{supp} \omega \subset [-\alpha, 0]$. If the system Σ_1 is quasi-reachable, we have the converse.

(6.17) PROPOSITION. *Let* $\Sigma_1 = (X_1, \varphi_1, H_1)$ *and* $\Sigma_2 = (X_2, \varphi_2, H_2)$ *be linear systems. Suppose that* Σ_1 *is quasi-reachable. If* $T: X_1 \to X_2$ *is a continuous linear map such that* $Tg^{\Sigma_1} = g^{\Sigma_2}$, *then* T *satisfies* $\varphi_2(t, Tx_1, u) = T\varphi_1(t, x_1, u)$ *for all* $t \geq 0$, $x_1 \in X$, *and* $u \in L^2_{[0,t)}$.

PROOF. Let x_1 be an element of $X_{1,R} = g^{\Sigma_1}(\Omega)$, i.e., $x_1 = g^{\Sigma_1}(\omega)$ for some ω in Ω. Then

$$\begin{aligned}
\varphi_2(t, Tx_1, u) &= \Phi_2(t)Tg^{\Sigma_1}(\omega) + g^{\Sigma_2}(\pi_\Omega \sigma_t^\ell u) \quad ((6.6)),\\
&= \Phi_2(t)g^{\Sigma_2}(\omega) + g^{\Sigma_2}(\pi_\Omega \sigma_t^\ell u) \quad \text{(by hypothesis)},\\
&= g^{\Sigma_2}(\sigma_t \omega) + g^{\Sigma_2}(\pi_\Omega \sigma_t^\ell u) \quad \text{(Proposition (6.12))},\\
&= g^{\Sigma_2}(\sigma_t \omega + \pi_\Omega \sigma_t^\ell \omega) \quad (g^{\Sigma_2} \text{ is linear}),\\
&= Tg^{\Sigma_1}(\sigma_t \omega + \pi_\Omega \sigma_t^\ell u) \quad \text{(by hypothesis)},\\
&= T\{g^{\Sigma_1}(\sigma_t \omega) + g^{\Sigma_1}(\pi_\Omega \sigma_t^\ell u)\},\\
&= T\{\Phi_1(t)g^{\Sigma_1}(\omega) + g^{\Sigma_1}(\pi_\Omega \sigma_t^\ell u)\} \quad \text{(Proposition (6.12))},\\
&= T\varphi_1(t, x_1, u) \quad ((6.6)).
\end{aligned}$$

Thus $\varphi_2(t, Tx_1, u) = T\varphi_1(t, x_1, u)$ for every x_1 in $X_{1,R}$. Since $X_{1,R}$ is dense in X_1, the conclusion follows by the continuity of $\varphi_2(t, T(\cdot), u)$ and $T\varphi_1(t, \cdot, u)$. □

Now if the second condition of Definition (6.15) is satisfied, then clearly $h^{\Sigma_2}T = h^{\Sigma_1}$ follows. Conversely, we prove

(6.18) PROPOSITION. *Let* $\Sigma_1 = (X_1, \varphi_1, H_1)$ *and* $\Sigma_2 = (X_2, \varphi_2, H_2)$ *be linear systems. Suppose that* Σ_1 *is quasi-reachable. If* $T: X_1 \to X_2$ *is a continuous linear map such that* $Tg^{\Sigma_1} = g^{\Sigma_2}$ *and* $h^{\Sigma_2}T = h^{\Sigma_1}$, *then there exists a dense subspace* $M_1 \subset D_o(H_1)$ *such that* $TM_1 \subset D_o(H_2)$ *and* $H_2T = H_1$ *on* M_1.

PROOF. Take $M_1 := g^{\Sigma_1}((C_o(-\infty, 0))^m)$. Since $(C_o(-\infty, 0))^m$ is dense in Ω, M_1 must be dense in X_1 by Lemma (6.14) because Σ_1 is quasi-reachable. By Definition (6.1) (g), M_1 must be contained in $D_o(H_1)$. Since $Tg^{\Sigma_1}(\omega) = g^{\Sigma_2}(\omega)$ for every ω in Ω, clearly $TM_1 \subset g^{\Sigma_2}((C_o(-\infty, 0))^m)$ follows. This yields $TM_1 \subset D_o(H_2)$ because $g^{\Sigma_2}((C_o(-\infty, 0))^m) \subset D_o(H_2)$ again by Definition (6.1) (g).

For every x in M_1, $H_2Tx = H_1x$ follows immediately by evaluating $h^{\Sigma_2}Tx = h^{\Sigma_1}x$ at $t = 0$. □

Combining Proposition (6.17) with Proposition (6.18), we obtain the following

(6.19) THEOREM. Let $\Sigma_1 = (X_1, \varphi_1, H_1)$ and $\Sigma_2 = (X_2, \varphi_2, H_2)$ be linear systems. Suppose that Σ_1 is quasi-reachable. A continuous linear map $T: X_1 \to X_2$ is a morphism iff $Tg^{\Sigma_1} = g^{\Sigma_2}$ and $h^{\Sigma_2}T = h^{\Sigma_1}$.

PROOF. Obvious from Propositions (6.17) and (6.18) and the remarks preceding them. □

In order to illustrate Definition (6.1) we here give two examples of systems.

(6.20) EXAMPLE (BARAS, BROCKETT, and FUHRMANN [1974]). Let $X := \ell^2$ and let $\{g_n\}$ and $\{h_n\}$ be ℓ^2-sequences. Also let $\{\lambda_n\}$ be a bounded sequence. Define φ and H by

$$\varphi(t, \{x_n\}, u)|_n := e^{\lambda_n t}x_n + \int_0^t e^{\lambda_n(t-\tau)}g_n u(\tau)d\tau;$$

$$H(\{x_n\}) := \sum_{n=1}^{\infty} h_n x_n \quad \text{for all } \{x_n\} \text{ in } \ell^2,$$

where $y|_n$ denotes the n-th coordinate of y. It is easy to see that (ℓ^2, φ, H) satisfies the axiom of systems (note that H is continuous, hence conditions (f) - (j) are automatically fulfilled).

(6.21) EXAMPLE. Let $X := \mathbb{R} \times L^2_{[-1,0]}$. We denote an element of X by $(x, z(\cdot)) \in \mathbb{R} \times L^2_{[-1,0]}$. Take

$$D(H) := \{(x, z) \in X: z \text{ is continuous on } (-\epsilon, 0] \text{ for some } \epsilon > 0\};$$

$$D_o(H) := \{(x, z) \in X: z \text{ is continuous on } [-1, 0] \text{ and } z(-1) = x\}.$$

The inclusion $D_o(H) \subset D(H)$ is obvious. We prove that $D_o(H)$ is dense in X.

For each fixed x in $\underline{\underline{R}}$, take $M_x := \{z \in L^2_{[-1,0]}: z$ is continuous and $z(-1) = x\}$. We prove M_x is dense in $L^2_{[-1,0]}$. Indeed, take an element w of $L^2_{[-1-\delta,0]}$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that $\int_{-1}^{-1+\delta} |w(t)|^2 dt < \epsilon$. Let z be a continuous function which satisfies

(i) $\int_{-1}^{-1+\delta} |z(t)|^2 dt < \epsilon$,

(ii) $\int_{-1+\delta}^{0} |w(t) - z(t)|^2 dt < \epsilon$ and $z(-1 + \delta) = 0$.

Such a function z clearly exists because $C_o[-1 + \delta, 0]$ is known to be dense in $L^2_{[-1+\delta,0]}$. It is also clear that z satisfies $\int_{-1}^{0} |w(t) - z(t)|^2 dt < 3\epsilon$, and hence M_x is dense in $L^2_{[-1,0]}$. Since $D_o(H) = \bigcup \{(x, z): z \in M_x$ and $x \in \underline{\underline{R}}\}$, $D_o(H)$ must be dense in X.

Define φ and H as follows:

(i) If $0 \leq t \leq 1$

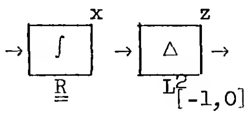
$$\varphi(t, (x, z), u) := \left(x + \int_0^t u(\tau)d\tau, \begin{cases} z(\theta - t), & \text{for } t-1<\theta\leq 0 \\ x + \int_0^{t-1-\theta} u(\tau)d\tau, & \text{for } -1\leq\theta\leq t-1 \end{cases}\right).$$

(ii) If $t \geq 1$

$$\varphi(t, (x, z), u) := \left(x + \int_0^t u(\tau)d\tau, \ x + \int_0^{t-1-\theta} u(\theta)d\theta\right);$$

$H(x, z) := z(0)$ ((x, z) belongs to $D(H)$).

Also consult the following figure:



It is now easy to check that the above defined (X, φ, H) is indeed a linear system. Note that $D(H)$ is not Φ-invariant, whereas $D_o(H)$ is (Φ = the semigroup associated with Σ).

7. Realizations and Factorizations of Input/Output Maps.

We give the definitions of realizations and factorizations of input/output maps. A factorization is not necessarily a system, hence not a realization, in general. Our main objective in this section is to prove that every quasi-reachable factorization is indeed a system, hence a realization. This enables us to study the problem of existence and uniqueness of canonical realizations in terms of factorizations.

(7.1) DEFINITION. Let $f: \Omega \to \Gamma$ be a linear input/output map. A linear system $\Sigma = (X, \varphi, H)$ is a realization of f iff $f = h^{\Sigma}g^{\Sigma}$.

Since the composition $h^{\Sigma}g^{\Sigma}$ clearly gives the correspondence: "past inputs" $\mapsto$ "future outputs" of the system Σ, this definition is the natural one. We have the following easy

(7.2) PROPOSITION. For every linear system $\Sigma = (X, \varphi, H)$, $h^{\Sigma}g^{\Sigma}$ is an input/output map.

PROOF. Clearly $h^{\Sigma}g^{\Sigma}: \Omega \to \Gamma$ is a continuous linear map.

Take any ω in $(C_o(-\infty, 0))^m$. By Condition (g) of Definition (6.1), $g^{\Sigma}(\omega)$ must belong to $D_o(H)$. Then by Condition (i) of Definition (6.1), the correspondence: $\omega \mapsto Hg^{\Sigma}(\omega)$ is continuous in view of the inductive limit topology of $(C_o(-\infty, 0))^m$ (see Proposition (A.2)). Thus Hg^{Σ} must belong to the dual space of $(C_o(-\infty, 0))^m$. Hence it must be represented by a matrix Radon measure μ as $Hg^{\Sigma}(\omega) = \int_{-\infty}^{0} \omega(\tau)d\mu(-\tau) = \int_{0}^{\infty} \omega(-\tau)d\mu(\tau)$ (consider the definition of Radon measures; see also the proof of Theorem (3.12)). Since $Hg^{\Sigma}(\omega) = h^{\Sigma}g^{\Sigma}(\omega)(0)$ by Condition (j) of Definition (6.1), the proof is complete. □

Proposition (7.2) claims that a linear system realizes some input/output map. At the end of this section, we shall answer the converse question, "Is every (linear) input/output map realized by a linear system?"

Definition (7.1) motivates the following

(7.3) DEFINITION. Let $f: \Omega \to \Gamma$ be an input/output map. A triple (X, g, h) is a factorization of f iff

(a) X is a complete locally convex Hausdorff space equipped with a strongly continuous semigroup $\{\Phi(t)\}_{t\geq 0}$;

(b) $g: \Omega \to X$ and $h: X \to \Gamma$ are continuous linear maps such that $\Phi(t)g = g\sigma_t$ and $\tilde{\sigma}_t h = h\Phi(t)$ for all $t \geq 0$;

(c) $f = hg$.

In view of Proposition (6.5) and (6.12), a realization of an input/output map is always a factorization. But the converse is false; see Counterexample (7.9).

It is convenient to say that a factorization (X, g, h) is quasi-reachable iff $g(\Omega)$ is dense in X, and observable iff h is one-to-one, and weakly canonical iff it is both quasi-reachable and observable. Since quasi-reachability and observability of systems coincide with their counterparts for factorizations (see Definition (6.11)), there is no possible danger of confusion.

We now prove the following

(7.4) PROPOSITION. Every quasi-reachable factorization (X, g, h) of a linear input/output map f is a linear system, i.e., realization. To be precise, there exists a linear system $\Sigma = (X, \varphi, H)$ such that $g^{\Sigma} = g$ and $h^{\Sigma} = h$.

PROOF. In view of (6.6) and the requirement $g^{\Sigma} = g$, we must define φ as

(7.5) $$\varphi(t, x, u) := \Phi(t)x + g(\pi_\Omega \sigma_t^{\ell} u),$$

where $\{\Phi(t)\}_{t\geq 0}$ is the semigroup associated with (X, g, h). Clearly $g^{\Sigma} = g$ follows from this definition. It is merely a routine to check that φ satisfies Conditions (b) - (d) of Definition (6.1).

Let $C_o(-\infty, 0]$ be the space of all continuous functions with compact support in $(-\infty, 0]$ which need not vanish at 0. Define $D(H)$ and $D_o(H)$ by

(7.6) $D(H) := g((C_o(-\infty, 0])^m)$;

(7.7) $D_o(H) := g((C_o(-\infty, 0))^m)$.

Clearly $D(H)$ contains $D_o(H)$; $D_o(H)$ is Φ-invariant because $(C_o(-\infty, 0))^m$ is invariant under $\{\sigma_t\}_{t \geq 0}$ (Remark (3.4)) and g commutes with shifts. By quasi-reachability of (X, g, h), $D_o(H)$ (hence, a fortiori, $D(H)$ too) is dense in X because $(C_o(-\infty, 0))^m$ is dense in Ω. Thus Conditions (f) and (g) of Definition (6.1) are satisfied.

Let u be a continuous function with its support contained in $[0, t]$. Then for each $s \in [0, t]$, $\pi_\Omega \sigma^\ell_s u$ belongs to $(C_o(-\infty, 0])^m$ Hence $\varphi(s, 0, u) = g(\pi_\Omega \sigma^\ell_s u)$ belongs to $D(H)$. Thus Condition (h) is also satisfied.

Now define $H: D(H) \to k^p$ by

(7.8) $Hx := h(x)(0)$.

Since $x = g(\omega)$ for some ω in $(C_o(-\infty, 0])^m$, $h(x) = hg(\omega) = f(\omega)$ is continuous in a neighborhood of 0 because $f(\omega)$ is given by the convolution of a matrix Radon measure μ with ω. Furthermore, if $x = g(\omega_1) = g(\omega_2)$, then $h(x) = f(\omega_1) = f(\omega_2)$, so (7.8) is well-defined. If, further, x belongs to $D_o(H)$, then $h(x) = f(\omega)$ $(\omega \in (C_o(-\infty, 0))^m)$ belongs to $(C_{[0,\infty)})^m$ by Definition (3.3); $H\Phi(t)x = h(\Phi(t)x)(0) = \tilde{\sigma}_t h(x)(0) = h(x)(t)$ for every x in $D_o(H)$. Hence Condition (j) of Definition (6.1) is satisfied.

Finally, since the correspondence: $(C_o(-\infty, 0))^m \to k^p$ given by $\omega \mapsto f(\omega)(0)$ is continuous by Definition (3.3), Condition (i) is satisfied. The equality $h^\Sigma = h$ follows from the previous identity $h(x)(t) = H\Phi(t)x$ on $D_o(H)$. □

The hypothesis that (X, g, h) is quasi-reachable is crucial in the above proof. Indeed, if (X, g, h) is not quasi-reachable, we have the following counterexample.

(7.9) COUNTEREXAMPLE. Let f be a scalar input/output map given by

$$f(\omega) := \int_{-\infty}^{0} \omega(\tau)e^{t-\tau}d\tau.$$

Clearly f is a linear input/output map by Proposition (3.8). Define

$$X := \underline{\underline{R}}^2;$$

$$g(\omega) := \left(\int_{-\infty}^{0} \omega(\tau)e^{-\tau}d\tau,\ 0\right);$$

$$h(x_1, x_2)(t) := e^t x_1 + t^{-1/3}x_2;$$

$$\Phi(t)(x_1, x_2) := (e^t x_1, x_2).$$

Clearly $(\underline{\underline{R}}^2, g, h)$ is a factorization of f; this is not quasi-reachable. And for any state $(0, x_2)$ $(x_2 \neq 0)$, $h(0, x_2)(t) = t^{-1/3}x_2$ is not continuous at 0. Hence the factorization $(\underline{\underline{R}}^2, g, h)$ cannot be a system.

We give some examples of realizations.

(7.10) EXAMPLE (BARAS, BROCKETT, and FURHMANN [1974]). Let $\{\alpha_n\}$ be an ℓ^1-sequence given by $\alpha_n = h_n g_n$, where $\{g_n\}$, $\{h_n\}$ are ℓ^2-sequences given in Example (6.20). Let $\{\lambda_n\}$ be a bounded sequence as in Example (6.20). Take

$$A(t) := \sum_{n=1}^{\infty} \alpha_n e^{\lambda_n t}.$$

This series converges uniformly on every compact interval. Hence $A(t)$ is analytic. Now define

(7.11) $$f(\omega)(t) := \int_{-\infty}^{0} A(t - \tau)\omega(\tau)d\tau,$$

$$= \sum_{n=1}^{\infty} \int_{-\infty}^{0} \alpha_n e^{\lambda_n(t-\tau)}\omega(\tau)d\tau.$$

By Proposition (3.8), f is a linear input/output map.

Let the system Σ be as given in Example (6.20). It is easy to see that the reachability map g^Σ and the observability map h^Σ are given by

$$(7.12)\quad g^\Sigma(\omega)|_n := \int_{-\infty}^{0} e^{-\lambda_n\tau} g_n \omega(\tau)d\tau;$$

$$(7.13)\quad h^\Sigma(\{x_n\})(t) := \sum_{n=1}^{\infty} h_n x_n e^{\lambda_n t}.$$

Then we obtain

$$\begin{aligned} h^\Sigma g^\Sigma(\omega)(t) &= \sum_{n=1}^{\infty} h_n g^\Sigma(\omega)|_n e^{\lambda_n t}, \\ &= \sum_{n=1}^{\infty} h_n e^{\lambda_n t} \int_{-\infty}^{0} e^{-\lambda_n\tau} g_n \omega(\tau)d\tau, \\ &= \sum_{n=1}^{\infty} h_n g_n \int_{-\infty}^{0} e^{\lambda_n(t-\tau)}\omega(\tau)d\tau, \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{0} \alpha_n e^{\lambda_n(t-\tau)}\omega(\tau)d\tau, \\ &= f(\omega)(t). \end{aligned}$$

Thus the system Σ is a realization of f.

(7.14) EXAMPLE. Define $A(t)$ as follows:

$$A(t) := 0 \text{ if } 0 \le t \le 1, \quad A(t) := 1 \text{ if } t > 1.$$

Let f be the input/output map having A as the weighting pattern, i.e.,

$$(7.15)\quad f(\omega)(t) := \int_{-\infty}^{0} A(t-\tau)\omega(\tau)d\tau,$$

$$= \begin{cases} \int_{-\infty}^{t-1} \omega(\tau)d\tau & \text{if } 0 \le t \le 1, \\ \int_{-\infty}^{0} \omega(\tau)d\tau & \text{if } t > 1. \end{cases}$$

Now let the system Σ be the one given in Example (6.21). We can easily calculate g^Σ and h^Σ as follows:

$$(7.16)\quad g^{\Sigma}(\omega) = \left(\int_{-\infty}^{0} \omega(\tau)d\tau,\ \int_{-\infty}^{-1-\theta} \omega(\tau)d\tau\right),\quad -1 \le \theta \le 0;$$

$$(7.17)\quad h^{\Sigma}(x, z)(t) = \begin{cases} z(-t) & \text{if } 0 \le t \le 1, \\ x & \text{if } t > 1. \end{cases}$$

It follows that

$$h^{\Sigma}g^{\Sigma}(\omega)(t) = \begin{cases} \int_{-\infty}^{t-1} \omega(\tau)d\tau & \text{if } 0 \le t \le 1, \\ \int_{-\infty}^{0} \omega(\tau)d\tau & \text{if } t > 1, \end{cases}$$

$$= f(\omega)(t).$$

Thus Σ is a realization of f.

Let us now prove the converse of Proposition (7.2): Given an input/output map, there exists a realization; moreover, it can be taken to be weakly canonical.

(7.18) PROPOSITION. For every linear input/output map $f\colon \Omega \to \Gamma$, there exists at least one weakly canonical realization.

PROOF. Define

(7.19) (i) $X := \overline{\text{im } f}$ (the closure is taken in Γ);

(ii) $g := f$;

(iii) $h := j\colon \overline{\text{im } f} \to \Gamma$ (the inclusion).

Then clearly $(\overline{\text{im } f}, f, j)$ is a weakly canonical factorization. Hence by Proposition (7.4), $(\overline{\text{im } f}, f, j)$ must be a realization. According to (7.5) and (7.8), φ and H in the present context are given by

$$(7.20)\quad \varphi(t, x, u) := \tilde{\sigma}_t x + f(\pi_{\Omega}\sigma_t^{\ell});$$

$$(7.21)\quad Hx := x(0),$$

where $D(H) = f((C_o(-\infty, 0])^m)$.

8. Topological Observability.

The two basic questions in realization theory are

(i) Given an input/output map, does there exist a "canonical" realization?

(ii) Is a "canonical" realization unique?

Of course, one cannot answer these questions without specifying the meaning of "canonical." Let us suppose, at the moment, that "canonical" means "weakly canonical." By Proposition (7.18) the existence question (i) is answered affirmatively. However, the uniqueness question (ii) is much more delicate if the state space is not finite-dimensional, and uniqueness does indeed fail to hold. The following counterexample is due to BARAS, BROCKETT, and FUHRMANN [1974] (see also BROCKETT and FUHRMANN [1976]).

(8.1) COUNTEREXAMPLE (BARAS, BROCKETT, and FUHRMANN [1974]). Let $\Sigma = (\ell^2, \varphi, H)$ be the system given in Example (6.20). By Example (7.10) it is known that Σ is a realization of the input/output map f given by (7.11).

Now assume the following conditions:

(a) $\{g_n/n\}$ and $\{nh_n\}$ again belong to ℓ^2;

(b) $g_n \neq 0$, $h_n \neq 0$, for all n, and $\lambda_n \neq \lambda_m$ for $n \neq m$.

Let $\hat{\Sigma} = (\ell^2, \hat{\varphi}, \hat{H})$ be the system defined by

$$\hat{\varphi}(t, \{x_n\}, u)|_n := e^{\lambda_n t} x_n + \int_0^t e^{\lambda_n (t-\tau)} (g_n/n) u(\tau) d\tau;$$

$$\hat{H}(\{x_n\}) := \sum_{n=1}^{\infty} nh_n x_n, \quad D_o(H) = D(H) = \ell^2.$$

By Condition (a), $\hat{\Sigma}$ is a system; further, $\hat{\Sigma}$ realizes the same input/output map f because of $(g_n/n)(nh_n) = g_n h_n$ (see the calculation given in Example (7.10)). By the following Lemma (8.2), Σ and $\hat{\Sigma}$ are both weakly canonical. But they are not isomorphic. In order to see this, consider the continuous linear map $T: \ell^2 \to \ell^2$ given by

$$T(\{x_n\}) := \{x_n/n\} \text{ for all } \{x_n\} \text{ in } \ell^2.$$

It is easy to check that T is indeed a morphism: $\Sigma \to \hat{\Sigma}$. However, T is not invertible, i.e., not an isomorphism. Since T is the only possible choice as a morphism (see Lemma (8.3) below), Σ and $\hat{\Sigma}$ are not isomorphic even though they are both weakly canonical realizations of the same input/output map.

(8.2) LEMMA. <u>Under the hypothesis</u> (b) <u>in Example</u> (9.1), <u>the systems</u> Σ <u>and</u> $\hat{\Sigma}$ <u>are weakly canonical.</u>

PROOF. Since $\hat{\Sigma}$ is defined by replacing $\{g_n\}$ and $\{h_n\}$ of Σ by $\{g_n/n\}$ and $\{nh_n\}$, it obviously suffices to prove the statement for Σ.

We show observability first. Suppose that $h^{\Sigma}(\{x_n\})(t) = \sum_{n=1}^{\infty} h_n x_n e^{\lambda_n t} = 0$ for all $t \geq 0$. Since $\sum_{n=1}^{\infty} h_n x_n e^{\lambda_n t}$ is clearly an analytic function of t (the series converges uniformly on every compact interval $[-n, n]$), the assumption implies that $\sum_{n=1}^{\infty} h_n x_n e^{-\lambda_n s} = 0$ for all $s \in \underline{\underline{C}}$. Nete that the series $\sum_{n=1}^{\infty} h_n x_n e^{-\lambda_n s}$ is the Laplace transform of a Radon measure $\mu = \sum_{n=1}^{\infty} h_n x_n \delta_{\lambda_n}$ (δ_{λ_n} = Dirac's point measure at λ_n). By uniqueness of Laplace inverse images (L. SCHWARTZ [1961, VI.2.4]) the measure μ itself must be zero. Let $\varphi_j(t)$ be the characteristic function of the point $\{\lambda_j\}$, i.e., $\varphi_j(t) = 1$ if $t = \lambda_j$ and $\varphi_j(t) = 0$ if $t \neq \lambda_j$. By a standard technique of measure theory, we see that μ acts on φ_j and

$$0 = \mu(\varphi_j) = \sum_{n=1}^{\infty} h_n x_n \delta_{\lambda_n}(\varphi_j) = h_j x_j$$

because $\mu = 0$ and $\lambda_i \neq \lambda_j$ $(i \neq j)$. Since $h_j \neq 0$, x_j must be zero. Thus $\{x_n\} = 0$, that is, Σ is observable.

By easy calculation, one sees that the adjoint of g^{Σ} is given by the correspondence $\{x_n\} \mapsto \sum_{n=1}^{\infty} g_n x_n e^{\lambda_n t}$ $(t \leq 0)$. Since this correspondence is of the same form as h^{Σ}, it must be one-to-one. Then by Proposition (6.13), Σ must be quasi-reachable. □

(8.3) LEMMA. <u>Let</u> $\Sigma = (X, \varphi, H)$ <u>be a quasi-reachable linear system,</u> <u>and</u> $\hat{\Sigma} = (\hat{X}, \hat{\varphi}, \hat{H})$ <u>a linear system. Suppose that</u> $T_1, T_2 \colon \Sigma \to \hat{\Sigma}$ <u>are morphisms. Then</u> $T_1 = T_2$.

PROOF. Let $x = g^{\Sigma}(\omega)$ for some ω in Ω. Then $T_1 x = T_1 g^{\Sigma}(\omega) = \hat{g}^{\Sigma}(\omega) = T_2 g^{\Sigma}(\omega) = T_2 x$ since both of T_1 and T_2 are morphisms. Thus $T_1 = T_2$ on $g^{\Sigma}(\Omega)$. Since $g^{\Sigma}(\Omega)$ is dense in X, T_1 must be equal to T_2 on X. □

Example (8.1) shows that the notion of "weakly canonical" is too weak to obtain the uniqueness theorem of "canonical" realizations. Therefore we must impose a stronger condition on "canonical" realizations in order to obtain the desired uniqueness.

The intuitive idea of observability is to determine the initial state based on suitable input/output experiments; see KALMAN [1968, Chapter 10]. Since our systems are always linear, we can reduce the problem to identifying the initial state by a suitable procedure based on observation data. Observability guarantees the abstract possibility of uniquely determining initial states from observed data.

It turns out that observability implies that initial states can also be continuously determined from observed data for finite-dimensional linear systems; see Example (8.8). In other words, the initial state determination procedure is well-posed for finite-dimensional linear systems.

However, for infinite-dimensional systems (i.e., the state space is not finite-dimensional), observability does not, in general, imply the above mentioned well-posedness; see Example (8.10).

As explained in Chapter I, we regard the well-posedness of initial state determination procedure as one of the basic properties of systems. Thus we give

(8.4) DEFINITION. *Let* $\Sigma = (X, \varphi, H)$ *be a system, and* h^{Σ} *its observability map. The system* Σ *is topologically observable iff the following statement is true:*

(8.5) *For every neighborhood* U *of* 0 *in* X, *there exists a neighborhood* V *of* 0 *in* Γ, *such that* $(h^{\Sigma})^{-1}(V) \subset U$.

(8.6) REMARK. In this definition we require that the initial state determination be a *continuous* procedure, i.e., a procedure which belongs to the scope of the categorical aspects (continuity) of the problem.

An analogous definition was used successfully in the (different) category of polynomial systems; see SONTAG [1976]; SONTAG and ROUCHALEAU [1976].

We remark that topological observability implies observability.

(8.7) PROPOSITION. If $\Sigma = (X, \varphi, H)$ is topologically observable, then it is observable.

PROOF. Suppose $h^{\Sigma}(x) = 0$. Take a neighborhood U of 0 in X. Then, by topological observability, there exists V, a neighborhood of 0 in Γ, such that $(h^{\Sigma})^{-1}(V) \subset U$. Since $h^{\Sigma}(x) = 0$, x must belong to $(h^{\Sigma})^{-1}(V) \subset U$. Thus x must belong to every neighborhood of 0 in X. Since X is a Hausdorff space, x must be 0. □

(8.8) EXAMPLE. Let $\Sigma = (X, \varphi, H)$ be a finite-dimensional linear system, i.e., $\dim X = n < \infty$. Suppose that Σ is observable, i.e., $h^{\Sigma}: X \to \Gamma$ is one-to-one. Then $h^{\Sigma}: X \to h^{\Sigma}(X)$ is a bijection and hence $\dim h^{\Sigma}(X) = n$. By SCHAEFER [1971, I.3.4], h^{Σ} must be an isomorphism. Hence if U is a neighborhood of 0 in X, then $h^{\Sigma}(U)$ is a neighborhood of 0 in $h^{\Sigma}(X)$. By definition of the subspace topology, there exists a neighborhood V of 0 in Γ such that $h^{\Sigma}(U) = V \cap h^{\Sigma}(X)$. Then $(h^{\Sigma})^{-1}(V) = (h^{\Sigma})^{-1}(V \cap h^{\Sigma}(X)) = (h^{\Sigma})^{-1}(h^{\Sigma}(U)) \subset U$. Thus Σ is topologically observable.

We give an example of a system which is observable but not topologically observable.

(8.9) EXAMPLE. Let Σ be the system given in Example (6.20). Under the assumption (b) of Counterexample (8.1), Σ is observable.

Let e^N be the element of ℓ^2 whose only nonzero entry is 1 at the N-th position. Clearly $h^{\Sigma}(e^N)(t) = h_N e^{\lambda_N t}$. Now take any T, $\epsilon > 0$. By choosing sufficiently large N, we have $\|h^{\Sigma}(e^N)\|_{[0,T]} = \|h_N e^{\lambda_N t}\|_{[0,T]} < \epsilon$ since $h_n \to 0$ as $n \to \infty$. But $\|e^N\|_{\ell^2} = 1$ for all N. This clearly contradicts topological observability.

We now give several equivalent conditions for topological observability.

(8.10) PROPOSITION. Let $\Sigma = (X, \varphi, H)$ be a linear system, and let $\{p_\alpha\}_{\alpha \in A}$ be a fundamental family of seminorms of X. The following statements are equivalent:

(a) Σ is topologically observable.

(b) Let E be a dense subspace of X. For every U_o, a neighborhood of 0 in E, there exists V, a neighborhood of 0 in Γ, such that $(h^\Sigma)^{-1}(V) \cap E \subset U_o$.

(c) For each $\alpha \in A$, there exist $T > 0$ and a constant $C_\alpha > 0$ such that

(8.11) $p_\alpha(x) \leq C_\alpha \|H\Phi(\cdot)x\|_{[0,T]}$ for all x in $D_o(H)$.

(d) For each $\alpha \in A$, there exist $T > 0$ and a constant $C_\alpha > 0$ such that

(8.12) $p_\alpha(x) \leq C_\alpha \|h^\Sigma(x)\|_{[0,T]}$ for all x in X.

(e) $h^\Sigma: X \to h^\Sigma(X)$ is an isomorphism.

(f) Let $X = X_1 \oplus X_2$, where $\dim X_1 < \infty$. The observability map h^Σ is one-to-one, and $h^\Sigma|_{X_2}: X_2 \to h^\Sigma(X_2)$ is an isomorphism.

PROOF. (a) $\Rightarrow$ (b) By definition of the subspace topology, there exists U, a neighborhood of 0 in X, such that $U_o = U \cap E$. Also, there exists V, a neighborhood of 0 in Γ, such that $(h^\Sigma)^{-1}(V) \subset U$. This implies $(h^\Sigma)^{-1}(V) \cap E \subset U \cap E = U_o$.

(b) $\Rightarrow$ (c) Take $E := D_o(H)$, and $U_o := \{x \in D_o(H): p_\alpha(x) \leq 1\}$. Then there exists V, a neighborhood of 0 in Γ, such that $(h^\Sigma)^{-1}(V) \cap D(H) \subset U_o$, i.e., if x belongs to $D_o(H)$ and $h^\Sigma(x)$ belongs to V then $p_\alpha(x) \leq 1$. Since $\{\|\cdot\|_{[0,T]}\}_{T>0}$ is a fundamental family of seminorms of Γ, there exist $\epsilon > 0$ and $T > 0$ such that $V_o :=$ $\{\gamma \in \Gamma: \|\gamma\|_{[0,T]} \leq \epsilon\}$ is contained in V. Then $(h^\Sigma)^{-1}(V_o) \cap D_o(H) \subset$ $(h^\Sigma)^{-1}(V) \cap D_o(H) \subset U_o$, i.e., if x belongs to $D_o(H)$ and $\|h^\Sigma(x)\|_{[0,T]} =$ $\|H\Phi(\cdot)x\|_{[0,T]} \leq \epsilon$, then $p_\alpha(x) \leq 1$. It follows that if $\|H\Phi(\cdot)x\|_{[0,T]} \leq \epsilon/n$ then $p_\alpha(x) \leq 1/n$. Therefore, if $\|H\Phi(\cdot)x\|_{[0,T]} = 0$, then $p_\alpha(x) = 0$.

Now take $C_\alpha := 1/\epsilon$. It is clear that if $\|h^\Sigma(x)\|_{[0,T]} = 0$, then $p_\alpha(x) \leq C_\alpha \|H\Phi(\cdot)x\|_{[0,T]}$ follows. Suppose $\|h^\Sigma(x)\|_{[0,T]} \neq 0$, and take

$y := (\epsilon/\|h^\Sigma(x)\|_{[0,T]})x$. Clearly $\|h^\Sigma(y)\| = \epsilon$. Hence $p_\alpha(y) = p_\alpha((\epsilon/\|h^\Sigma(x)\|_{[0,T]})x) = (\epsilon/\|h^\Sigma(x)\|_{[0,T]})p_\alpha(x) \leq 1$. Therefore, $p_\alpha(x) \leq C_\alpha\|h^\Sigma(x)\|_{[0,T]}$ for all x in $D_o(H)$.

(c) $\Rightarrow$ (d) Take any x in X, and let $\{x_\nu\}$ be a net on $D_o(H)$ which converges to x. Then we have

$$\begin{aligned} p_\alpha(x) &= p_\alpha(\lim_\nu x_\nu) = \lim_\nu p_\alpha(x_\nu) \quad (\text{continuity of } p_\alpha), \\ &\leq \lim_\nu C_\alpha\|h^\Sigma(x_\nu)\|_{[0,T]} \quad (x_\nu \text{ belongs to } D_o(H)), \\ &= C_\alpha\|\lim_\nu h^\Sigma(x_\nu)\|_{[0,T]} \quad (\text{continuity of } \|\cdot\|_{[0,T]}), \\ &= C_\alpha\|h^\Sigma(\lim_\nu x_\nu)\|_{[0,T]} \quad (\text{continuity of } h^\Sigma), \\ &= C_\alpha\|h^\Sigma(x)\|_{[0,T]}. \end{aligned}$$

(d) $\Rightarrow$ (e) First we prove that h^Σ is one-to-one. Let $h^\Sigma(x) = 0$ and take any $\alpha \in A$. By (d), we have $p_\alpha(x) \leq C_\alpha\|h^\Sigma(x)\|_{[0,T]} = 0$. Since X is a Hausdorff space, x must be 0. Thus $h^\Sigma: X \to h^\Sigma(X)$ is a continuous bijection. But (8.12) clearly implies that $(h^\Sigma)^{-1}: h^\Sigma(X) \to X$ is continuous.

(e) $\Rightarrow$ (f) Trivial.

(f) $\Rightarrow$ (e) We first claim that $h^\Sigma(X_1 \oplus X_2) \cong h^\Sigma(X_1) \oplus h^\Sigma(X_2)$. Since $h^\Sigma: X_1 \oplus X_2 \to h^\Sigma(X_1 \oplus X_2)$ is a bijection, $h^\Sigma(X_1 \oplus X_2) \simeq h^\Sigma(X_1) \oplus h^\Sigma(X_2)$ as vector spaces over k. Note that X_2 is a closed subspace of $X_1 \oplus X_2$ because it is a direct summand. Hence X_2 is complete because $X_1 \oplus X_2$ is complete. So $h^\Sigma(X_2)$ is complete by the isomorphism: $X_2 \cong h^\Sigma(X_2)$. Thus $h^\Sigma(X_2)$ is a closed subspace of Γ, and hence a closed subspace of $h^\Sigma(X_1 \oplus X_2)$. Furthermore the codimension of $h^\Sigma(X_2)$ in $h^\Sigma(X_1 \oplus X_2)$ is finite since $h^\Sigma(X_1 \oplus X_2) \simeq h^\Sigma(X_1) \oplus h^\Sigma(X_2)$. Therefore, the direct sum $h^\Sigma(X_1 \oplus X_2) \cong h^\Sigma(X_1) \oplus h^\Sigma(X_2)$ must also be topological, by SCHAEFER [1971, I.3.5].

Since $\dim X_1$ $(= \dim h^\Sigma(X_1)) < \infty$, $h^\Sigma: X_1 \to h^\Sigma(X_1)$ is an isomorphism by SCHAEFER [1971, I.3.2]. Therefore, $h^\Sigma: X_1 \oplus X_2 \to h^\Sigma(X_1) \oplus h^\Sigma(X_2) \cong h^\Sigma(X_1 \oplus X_2)$ is an isomorphism (because $h^\Sigma: X_2 \to h^\Sigma(X_2)$ is an isomorphism).

(e) $\Rightarrow$ (a) Since $h^{\Sigma}: X \to h^{\Sigma}(X)$ is an isomorphism, $h^{\Sigma}(U)$ is a neighborhood of 0 in $h^{\Sigma}(X)$ for every U, a neighborhood of 0 in X. By definition of the subspace topology, there exists V, a neighborhood of 0 in Γ, such that $h^{\Sigma}(U) = V \cap h^{\Sigma}(X)$. Then $(h^{\Sigma})^{-1}(V) = (h^{\Sigma})^{-1}(V \cap h^{\Sigma}(X)) = (h^{\Sigma})^{-1}(h^{\Sigma}(U)) \subset U$. □

The following Proposition (8.13) gives the dual characterization of topological observability. (The proof of the sufficiency part is very technical.)

(8.13) PROPOSITION. <u>A system</u> $\Sigma = (X, \varphi, H)$ <u>is topologically observable iff the following conditions</u> (i) <u>and</u> (ii) <u>are satisfied</u>:

(i) X <u>is reflexive</u>.

(ii) $(h^{\Sigma})': \Gamma' \to X'$ <u>is onto</u>.

PROOF. <u>Necessity</u>. By Proposition (8.10) we can identify X with $h^{\Sigma}(X)$. Since Γ is a Fréchet space (Proposition (2.5)), X is also a Fréchet space. As pointed out in the proof of Proposition (6.13), Γ is reflexive. Hence X, regarded as a closed subspace of Γ, is semi-reflexive (see SCHAEFER [1971, IV.5.7]). Since X is also barreled (because it is a Fréchet space), X must be reflexive; see TREVES [1967, Proposition (36.5)].

Take any $x^* \in X'$. Define a linear functional $\hat{x}^*$ on $h^{\Sigma}(X)$ by $\langle \hat{x}^*, h^{\Sigma}(x)\rangle := \langle x^*, x\rangle$. By the isomorphism $X \cong h^{\Sigma}(X)$, $\hat{x}^*$ is clearly a continuous linear form on $h^{\Sigma}(X)$. Then by the Hahn-Banach theorem, there exists $\gamma^* \in \Gamma'$ such that $\gamma^*|_{h^{\Sigma}(X)} = \hat{x}^*$. It follows that $\langle (h^{\Sigma})'(\gamma^*), x\rangle = \langle \gamma^*, h^{\Sigma}(x)\rangle = \langle \hat{x}^*, h^{\Sigma}(x)\rangle = \langle x^*, x\rangle$. Hence $x^* = (h^{\Sigma})'(\gamma^*)$, i.e., $(h^{\Sigma})'$ is onto.

<u>Sufficiency</u>. Since Γ' is a strong dual of a reflexive Fréchet space Γ, it is a Ptak (B-complete) space; see HUSAIN [1965, Chapter 4, Proposition 7]. The strong dual X' is a reflexive space because X is reflexive; hence it is a barreled space (SCHAEFER [1971, IV.5.7]). Then we can apply Ptak's open mapping theorem (SCHAEFER [1971, IV.8.3, Corollary 1]) to $(h^{\Sigma})': \Gamma' \to X'$, and conclude that $(h^{\Sigma})'$ is an open mapping because $(h^{\Sigma})'$ is onto. In other words, X' is isomorphic to the quotient space $\Gamma'/\ker (h^{\Sigma})'$.

Since X and Γ are reflexive, it is enough to prove that $(h^{\Sigma})'': X'' \to \Gamma''$ is an isomorphism into Γ''. By identifying X' with $\Gamma'/\ker (h^{\Sigma})'$, it suffices to prove that $\pi': (\Gamma'/\ker (h^{\Sigma})')' \to \Gamma''$ is an isomorphism into Γ'', where $\pi: \Gamma' \to \Gamma'/\ker (h^{\Sigma})'$ is the canonical projection.

We note that Γ' is a DF-space as a strong dual of a metrizable space Γ (KÖTHE [1969, 29.3]). Then by KÖTHE [1969, 29.5, (1)] we conclude that π' gives the isomorphism:
$(\Gamma'/\ker (h^{\Sigma})')' \to \pi'((\Gamma'/\ker (h^{\Sigma})')') \subset \Gamma''$. □

9. Existence and Uniqueness of Canonical Realizations

(9.1) DEFINITION. A linear system Σ is canonical iff it is quasi-reachable and topologically observable.

We prove the existence of a canonical realization first.

(9.2) THEOREM. Every input/output map f has at least one canonical realization.

PROOF. Let $\Sigma_f = (\overline{\text{im } f}, \varphi_f, H_f)$ be the system given in Proposition (7.18). We have already shown that Σ_f is weakly canonical. So we only need to show topological observability.

Recall that h^{Σ_f}, the observability map of Σ_f, is given by the inclusion $j: \overline{\text{im } f} \to \Gamma$. Let U be any neighborhood of 0 in $\overline{\text{im } f}$. By definition of the subspace topology, there exists V, a neighborhood of 0 in Γ, such that $V \cap \overline{\text{im } f} = U$. It follows that $j^{-1}(V) = V \cap \overline{\text{im } f} = U$. □

We now prove the uniqueness. We first prove the following statement, which is a counterpart of "Zeiger's lemma" (KALMAN, FALB, and ARBIB [1969, Chapter 10]).

(9.3) LEMMA. Suppose that $\Sigma_1 = (X_1, \varphi_1, H_1)$ and $\Sigma_2 = (X_2, \varphi_2, H_2)$ are both realizations of the same input/output map $f: \Omega \to \Gamma$. Suppose also that Σ_1 is quasi-reachable and Σ_2 is topologically observable. Then there exists precisely one morphism $T: X_1 \to X_2$.

PROOF. Write $g_1 := g^{\Sigma_1}$, $g_2 := g^{\Sigma_2}$, $h_1 := h^{\Sigma_1}$, $h_2 := h^{\Sigma_2}$ and $X_{1,R} := g_1(\Omega)$. Since Σ_1 is quasi-reachable, we need only to prove the existence of a continuous linear map $T: X_1 \to X_2$ such that $Tg_1 = g_2$ and $h_2 T = h_1$ by Theorem (6.19). The uniqueness of such T is already known by Lemma (8.3).

Define a linear map $\hat{T}: X_{1,R} \to X_2$ by

$$\hat{T}(x) := h_2^{-1} h_1(x) \quad \text{for } x \text{ in } X_{1,R}.$$

Since $x = g_1(\omega)$ for some ω (if $x \in X_{1,R}$), $h_1(x) = h_1 g_1(\omega) = f(\omega) = h_2 g_2(\omega) \in \operatorname{im} h_2$. Moreover, h_2 is one-to-one, so $\hat{T}(x)$ is well-defined. Clearly $h_2 \hat{T} = h_1$ on $X_{1,R}$. And for every ω in Ω, $\hat{T}g_1(\omega) = h_2^{-1} h_1(g_1(\omega)) = h_2^{-1}(f(\omega)) = h_2^{-1}(h_2 g_2(\omega)) = g_2(\omega)$ (h_2 is one-to-one). Thus $\hat{T}g_1 = g_2$.

By Proposition (8.10), $h_2^{-1}: \operatorname{im} h_2 \to X_2$ is continuous. Hence $\hat{T}$ is continuous as a composition of continuous linear maps. Then, since $X_{1,R}$ is dense in X_1 and X_2 is complete, there exists a unique continuous extension $T: X_1 \to X_2$ such that $T|_{X_{1,R}} = \hat{T}$. Clearly T satisfies $Tg_1 = g_2$ and $h_2 T = h_1$ because these equalities hold on $X_{1,R}$, which is dense in X_1. □

We are now ready to prove

(9.4) THEOREM. <u>Suppose that $\Sigma_1 = (X_1, \varphi_1, H_1)$ and $\Sigma_2 = (X_2, \varphi_2, H_2)$ are two canonical realizations of the same input/output map</u> $f: \Omega \to \Gamma$. <u>Then Σ_1 and Σ_2 are isomorphic</u>.

PROOF. Let $T_1: \Sigma_1 \to \Sigma_2$ and $T_2: \Sigma_2 \to \Sigma_1$ be morphisms as given in Lemma (9.3). Observe that $T_2 T_1: \Sigma_1 \to \Sigma_1$ and $1_{X_1}: \Sigma_1 \to \Sigma_1$ are again morphisms (Remark (6.16)). By the uniqueness of morphisms, $T_2 T_1 = 1_{X_1}$. Similarly, $T_1 T_2 = 1_{X_2}$. Hence T_1 (T_2) is an isomorphism. □

It is now clear why nonuniqueness occurs in Counterexample (8.1). As is shown in Example (8.9), the systems discussed in Counterexample (8.1) are <u>not</u> topologically observable even though they are observable.

(9.5) REMARK. One can easily check that the system given in Example (6.21) is indeed canonical.

Recall that when the weighting pattern of an input/output map f is of class C^r $(r \geq 0)$, f has a unique continuous extension $\tilde{f}: (\underset{\sim}{E}^{r'}_{(-\infty,0]})^m \to \Gamma$ by Proposition (3.14). We prove that <u>this extension of the input space does not affect the canonical realization</u>. To be more precise, we state

<u>(9.6)</u> <u>PROPOSITION</u>. <u>Let f and $\tilde{f}$ be as described above. Then $\overline{\text{im } f} = \overline{\text{im } \tilde{f}}$.</u>

PROOF. Since Ω is contained in $(\underset{\sim}{E}^{r'}_{(-\infty,0]})^m$, $\text{im } f$ is contained in $\text{im } \tilde{f}$. Hence $\overline{\text{im } f}$ must be contained in $\overline{\text{im } \tilde{f}}$. On the other hand, every element $\tilde{f}(\tilde{\omega})$ can be approximated by elements in $\text{im } f$ because Ω is dense in $(\underset{\sim}{E}^{r'}_{(-\infty,0]})^m$. Hence $\text{im } \tilde{f} \subset \overline{\text{im } f}$. Since $\overline{\text{im } f}$ is closed, $\overline{\text{im } \tilde{f}} \subset \overline{\text{im } f}$ follows. □

This Proposition (9.6) (and its proof) shows that the canonical realization $\Sigma_f = (\overline{\text{im } f}, \varphi_f, H_f)$ is determined uniquely (up to isomorphism) by its weighting pattern μ (and the output space Γ, of course) <u>irrespective of the input space</u> $\tilde{\Omega}$ as long as Ω is dense in $\tilde{\Omega}$. When $\tilde{\Omega} = (\underset{\sim}{E}^{r'}_{(-\infty,0]})^m$, we obtain the uniqueness theorem for the case treated by KALMAN and HAUTUS [1972] (modulo the difference of the output space).

(9.7) REMARK. As is clear from the proof of Lemma (9.3), topological observability leads to the uniqueness of canonical realizations for each fixed choice of output spaces (note that the equivalence (a) $\Leftrightarrow$ (e) in Proposition (8.10) is always true).

(9.8) REMARK. Note that $\overline{\text{im } f}$ is always a reflexive Fréchet space (see the proof of Proposition (8.13)).

10. <u>Realizations in the Working Mode</u>.

In Section 7 we defined a realization of a linear input/output map f as a linear system $\Sigma = (X, \varphi, H)$ which factors f, i.e., $f = h^{\Sigma} g^{\Sigma}$. This means that the outputs of the system Σ are equal to those of the external behavior <u>only after inputs are terminated</u>. Then a very natural question arises: Are those outputs, one induced by f the other induced by Σ, <u>equal even while an input is being applied</u>? One may rephrase the question as follows: For every u in $(L^2_{[0,T)})^m$, is it ture that $H\varphi(t, 0, u) = \overline{f}(u)(t)$ for almost all t in $[0, T)$? (The map $\overline{f}$ is the extended input/output map: $\Lambda^m \to \Lambda^p$ associated with f.) (We may as well assume that the initial state x is 0 since the system is linear.)

We remark that $\varphi(t, 0, u)$ does not necessarily belong to $D(H)$, and so $H\varphi(t, 0, u)$ may not be well-defined. It is guaranteed that $\varphi(t, 0, u)$ belongs to $D(H)$ only when u belongs to $(C_o[0, T])^m$ (continuous functions vanishing at 0 and T). Thus for general inputs belonging to $(L^2_{[0,T)})^m$, we must find a way of justifying the corresponding outputs.

All these technical problems do not arise (or can be solved rather easily) in discrete-time systems. But because of varied topological questions these problems become much more delicate than those in the discrete-time case. Fortunately, we can answer our questions affirmatively as follows. We start by proving the following

(10.1) PROPOSITION. Let $\Sigma = (X, \varphi, H)$ be a realization of an input/output map f. For every u in $(C_o[0, T])^m$,

(10.2) $\overline{f}(u)(t) = H\varphi(t, 0, u)$ for all $t \in [0, T)$.

PROOF. Since u belongs to $(C_o[0, T])^m$, $H\varphi(t, 0, u)$ is well-defined for every $t \in [0, T)$ by Condition (h) of Definition (6.1). Since u is continuous and $u(0) = u(T) = 0$, $\overline{f}(u)(t)$ is a continuous function of t by Theorem (5.2). Thus we obtain

$$\begin{aligned}
\overline{f}(u)(t) &= (\sigma^\ell_t \overline{f}(u))(0) = \overline{f}(\sigma^\ell_t u)(0), \\
&= (\pi_\Gamma \overline{f}(\sigma^\ell_t u))(0), \\
&= (\pi_\Gamma \overline{f} j_\Omega(\pi_\Omega \sigma^\ell_t u))(0) \quad \text{(causality of } \overline{f}), \\
&= f(\pi_\Omega \sigma^\ell_t u)(0) \quad \text{(Theorem (5.2))}, \\
&= h^\Sigma g^\Sigma(\pi_\Omega \sigma^\ell_t u)(0) \quad (\Sigma \text{ is a realization of } f), \\
&= h^\Sigma \varphi(t, 0, u)(0) \quad \text{(by (6.6))}, \\
&= H\varphi(t, 0, u) \quad \text{(by Definition (6.1) (j))}. \quad \square
\end{aligned}$$

Thus we have assured the desired property of Σ at least for continuous inputs with compact support. In order to extend our result to L^2-inputs, we need the following lemma.

(10.3) LEMMA. Let $T > 0$ be fixed. For every u in $(L^2_{[0,T)})^m$, $\varphi(t, 0, u)$ is a continuous function of $t \in [0, T]$ (with its value in X), and, furthermore, the correspondence $\bar{g}: (L^2_{[0,T)})^m \to C(0, T; X): u \mapsto \varphi(\cdot, 0, u)$ is continuous, where $C(0, T: X)$ is the set of all X-valued continuous functions on $[0, T]$ with the topology of uniform convergence on $[0, T]$.

PROOF. Take any u in $(L^2_{[0,t)})^m$. For every t in $[0, T)$, $\varphi(t, 0, u) = g^\Sigma(\pi_\Omega\sigma^\ell_t u)$ by (6.6). Then $g^\Sigma(\pi_\Omega\sigma^\ell_t u)$ converges to $g^\Sigma(\pi_\Omega\sigma^\ell_{t_0} u)$ as $t \to t_0$, since g^Σ is continuous and $\pi_\Omega\sigma^\ell_t u \to \pi_\Omega\sigma^\ell_{t_0} u$ by Proposition (4.3). Hence $\bar{g}(u)$ is a continuous function.

Now let $u_n \to 0$ in $(L^2_{[0,T)})^m$ and p_α be a continuous seminorm on X. Since g^Σ is continuous on each $(L^2_{[-T,0]})^m$, there exists $C_\alpha > 0$ such that $p_\alpha(g^\Sigma(\omega)) \leq C_\alpha\|\omega\|_{[-T,0]}$ for every ω in $(L^2_{[-T,0]})^m$. Then we obtain

$$
\begin{aligned}
p_\alpha(\varphi(t, 0, u_n)) &= p_\alpha(g^\Sigma(\pi_\Omega\sigma^\ell_t u_n)), \\
&\leq C_\alpha\|\pi_\Omega\sigma^\ell_t u_n\|_{[-T,0]}, \\
&\leq C_\alpha\|\pi_\Omega\sigma^\ell_T u_n\|_{[-T,0]}, \\
&= C_\alpha\|u_n\|_{[0,T]} \quad \text{for all } t \in [0, T].
\end{aligned}
$$

Hence $p_\alpha(\varphi(t, 0, u_n)) \to 0$ uniformly on $[0, T]$ as $n \to \infty$. Thus $\bar{g}$ is continuous. □

Now let $\underset{\sim}{X}_T := \{\varphi(\cdot, 0, u) \in C(0, T; X): u \in (L^2_{[0,T)})^m\}$, i.e., $\underset{\sim}{X}_T := \bar{g}((L^2_{[0,T)})^m)$. We now prove the following

(10.4) PROPOSITION. Let $\Sigma = (X, \varphi, H)$ be a realization of a linear input/output map f. For each $T > 0$ there exists a continuous linear $\bar{\bar{h}}: \underset{\sim}{X}_T \to (L^2_{[0,T]})^p$ such that

(10.5) $\bar{\bar{h}}(\varphi(\cdot, 0, u)) = \bar{f}(u)$ for all u in $(L^2_{[0,T)})^m$.

PROOF. Define $\bar{h}: \bar{g}((C_o[0,T]))^m \to (L^2_{[0,T]})^p$ by $\bar{h}(\varphi(\cdot, 0, u))(t) := H\varphi(t, 0, u)$. By Proposition (10.1), $\bar{h}(\varphi(\cdot, 0, u)) = \bar{f}(u)$ for all u in $(C_o[0, T])^m$. Since $\bar{g}((C_o[0, T])^m)$ is clearly dense in $\tilde{X}_T$ and $(L^2_{[0,T]})^p$ is complete, there exists a unique continuous linear extension $\bar{\bar{h}}: \tilde{X}_T \to (L^2_{[0,T]})^p$. Since $(C_o[0, T])^m$ is dense in $(L^2_{[0,T)})^m$ and $\bar{\bar{h}}\bar{g} = \bar{h}\bar{g} = \bar{f}$ on $(C_o[0, T])^m$, it follows that $\bar{\bar{h}}\bar{g} = \bar{f}$ on $(L^2_{[0,T)})^m$, i.e., $\bar{\bar{h}}(\varphi(\cdot, 0, u)) = \bar{f}(u)$. □

In other words, even while an L^2-input is applied, the system Σ keeps producing an output and this output is equal to that given by the extended input/output map $\bar{f}$.

(10.6) REMARK. Clearly $\bar{\bar{h}}$ satisfies $\bar{\bar{h}}(\varphi(\cdot, 0, u))|_{[0,t)} = \bar{\bar{h}}(\varphi(\cdot, 0, u)|_{[0,T)})|_{[0,t)}$.

11. Compatible Systems and Differential Equations.

In this section we shall prove one of our main results: If a linear input/output map f is sufficiently smooth, then its canonical realization Σ_f admits a differential equation description. In order to avoid cumbersome notation we assume that systems under consideration are single-input/single-output systems. The general case can be treated in a similar way.

Following T. KŌMURA [1968], we say that a strongly continuous semigroup (which we shall abbreviate as simply "semigroup" in the sequel) $\{\Phi(t)\}_{t\geq 0}$ in a locally convex space X is locally equicontinuous iff for every $T > 0$ the family $\{\Phi(t)\}_{0\leq t\leq T}$ is an equicontinuous family of continuous linear maps, i.e., for every continuous seminorm p there exists a continuous seminorm q such that

(11.1) $p(\Phi(t)x) \leq q(x)$ for all t in $[0, T]$.

We quote from T. KŌMURA [1968] the following facts:

(11.2) LEMMA. Let $\{\Phi(t)\}_{t\geq 0}$ be a semigroup in a complete locally convex space X with the infinitesimal generator F. Then the following statements are true:

(a) The domain of F, denoted by $D(F)$, is dense in X.

(b) *$D(F)$ is Φ-invariant; $\Phi(t)F = F\Phi(t)$ on $D(F)$; and for every x in $D(F)$, $\Phi(t)x$ is differentiable with respect to the topology of X, and*

$$\frac{d}{dt}\Phi(t)x = F\Phi(t)x = \Phi(t)Fx \quad \textit{for all} \quad t \geq 0.$$

(c) *If X is a barreled space (in particular, if X is a Fréchet space), then $\{\Phi(t)\}_{t\geq 0}$ is locally equicontinuous.*

(d) *If $\{\Phi(t)\}_{t\geq 0}$ is locally equicontinuous, then F is a closed operator.*

PROOF. See T. KŌMURA [1968]. □

Now let $\{\Phi(t)\}_{t\geq 0}$ be a locally equicontinuous semigroup in X. For every continuous seminorm p of X we define

(11.3) $p_F(x) := p(x) + p(Fx)$ for x in $D(F)$.

Clearly p_F is a seminorm on $D(F)$. *We define a locally convex Hausdorff topology τ_F on $D(F)$ by the collection of seminorms* $\{p_F\colon p$ is a continuous seminorm on $X\}$. Clearly τ_F is finer than that induced on $D(F)$ from X. Furthermore, when $\{\Phi(t)\}_{t\geq 0}$ is locally equicontinuous, we have

(11.4) PROPOSITION. *Suppose that $\{\Phi(t)\}_{t\geq 0}$ is a locally equicontinuous semigroup. Then $D(F)$ is complete with respect to τ_F. Moreover, $\{\Phi(t)\}_{t\geq 0}$ is again a (strongly continuous) semigroup in $D(F)$ with respect to τ_F.*

SKETCH OF PROOF. The completeness of $D(F)$ is an immediate consequence of the completeness of X and the fact that F is closed (Lemma (11.2)).

For strong continuity, observe that

$$p_F(\Phi(t)x - \Phi(t_o)x) = p(\Phi(t)x - \Phi(t_o)x) + p(\Phi(t)Fx - \Phi(t_o)Fx)$$

by virtue of $\Phi(t)F = F\Phi(t)$. □

Now we can give the following

(11.5) DEFINITION. A linear system $\Sigma = (X, \varphi, H)$ is compatible iff

(a) the semigroup $\{\Phi(t)\}_{t\geq 0}$ associated with Σ (given by (6.6)) is locally equicontinuous;

(b) H is well-defined on $D(F)$, and H is continuous with respect to τ_F, that is, there exists a continuous seminorm p of X such that $|Hx| \leq p_F(x) = p(x) + p(Fx)$.

(11.6) REMARK. The notion of compatible systems is clearly invariant under isomorphisms of systems.

First we show that every topologically observable system is compatible.

(11.7) PROPOSITION. Every topologically observable linear system $\Sigma = (X, \varphi, H)$ is compatible.

PROOF. By Proposition (8.10), we may identify X with $h^{\Sigma}(X)$. Take $E_o := h^{\Sigma}(X) \cap C[0, \infty)$. Clearly H is well-defined on E_o by $H(h^{\Sigma}(x)) = h^{\Sigma}(x)(0)$. The semigroup of this system is given by $\{\tilde{\sigma}_t\}_{t\geq 0}$ and its infinitesimal generator is $(\frac{d}{dt})$. Since $h^{\Sigma}(X)$ is a Fréchet space as a closed subspace of Γ, $\{\tilde{\sigma}_t\}_{t\geq 0}$ is locally equicontinuous by Lemma (11.2).

Now one can easily prove that

$$(11.8)\quad \begin{aligned} D(\tfrac{d}{dt}) &:= \{\gamma \in h^{\Sigma}(X):\ \dot{\gamma} \in h(X)\},\\ &= \{\gamma \in h^{\Sigma}(X):\ \dot{\gamma} \in L^2_{loc}[0, \infty)\},\\ &= h^{\Sigma}(X) \cap H^1_{loc}[0, \infty),\end{aligned}$$

where $H^1_{loc}[0, \infty) := \{\gamma \in L^2_{loc}[0, \infty):\ \dot{\gamma} \in L^2_{loc}[0, \infty)\}$ with generating seminorms

$$(11.9)\quad \|\gamma\|_{1,[0,T]} := \|\gamma\|_{[0,T]} + \|\dot{\gamma}\|_{[0,T]}.$$

This topology clearly induces τ_F on $D(\frac{d}{dt})$ as defined by (11.3).

Since E_o contains $D(\frac{d}{dt})$, H is well-defined on $D(\frac{d}{dt})$. We must prove that H is continuous with respect to τ_F. For every γ in $H^1_{loc}[0, \infty)$, we have

$$(11.10)\quad \gamma(0) = \gamma(t) - \int_0^t \dot{\gamma}(\tau)d\tau.$$

It follows that

$$(11.11)\quad |H\gamma| = |\gamma(0)| = \int_0^1 |\gamma(0)|dt,$$
$$\leq \int_0^1 |\gamma(t)|dt + \int_0^1\int_0^t |\dot{\gamma}(\tau)|d\tau dt,$$
$$\leq \|\gamma\|_{[0,1]} + \|\dot{\gamma}\|_{[0,1]} = \|\gamma\|_{1,[0,1]},$$

by Schwarz's inequality. Thus Σ is compatible. □

Before starting the discussion on differential equation descriptions of systems, we need to prove the following lemma.

(11.12) LEMMA. Let $\{\Phi(t)\}_{t\geq 0}$ be a locally equicontinuous semigroup in a complete locally convex space X and F its infinitesimal generator. Let G be an element of $D(F)$. Then the functional differential equation

$$(11.13)\quad \frac{d}{dt}x(t) = Fx(t) + Gu(t)$$

with the initial condition

$$(11.14)\quad x(0) = x_0 \in D(F)$$

admits a unique solution given by

$$(11.15)\quad x(t) = \Phi(t)x_0 + \int_0^t \Phi(t - \tau)Gu(\tau)d\tau,$$

at least for uniformly continuous u.

This lemma is a standard well-known fact when X is a Banach space (see, for example, YOSHIDA [1971]). The proof for the present case is essentially no different from that given for Banach spaces, in view of

Lemma (11.2) (especially (b), (d)). So we only give

SKETCH OF PROOF. The Riemann integral $\int_0^t \Phi(t - \tau)Gu(\tau)d\tau$ exists by the uniform continuity of u and local equicontinuity of $\{\Phi(t)\}_{t \geq 0}$. In order to see that (11.14) gives a solution, we observe that the equality

$$(11.15)\quad \frac{d}{dt}\int_0^t \Phi(t - \tau)Gu(\tau)d\tau = \int_0^t \frac{\partial}{\partial t}\Phi(t - \tau)Gu(\tau)d\tau + Gu(\tau),$$
$$= F\int_0^t \Phi(t - \tau)Gu(\tau)d\tau + Gu(t)$$

can be easily justified using Lemma (11.2) (b), and the facts that F is closed, and the functions $t \mapsto Gu(t)$, $t \mapsto FGu(t)$ are uniformly continuous with respect to t.

In order to see the uniqueness of solutions, observe that

$$(11.16)\quad \frac{\partial}{\partial \tau}(\Phi(t - \tau)\hat{x}(\tau)) = \Phi(t - \tau)Gu(\tau)$$

is valid for every solution $\hat{x}$ of (11.13). Integrating both sides from 0 to t, we obtain the desired uniqueness. □

We are now ready to give the definition of <u>smooth systems</u>. Roughly speaking, a smooth system is a compatible system whose state-transition is governed by a differential equation. To be more precise, we give the following

(11.17) DEFINITION. <u>A linear system</u> $\Sigma = (X, \varphi, H)$ <u>is smooth iff</u>

(i) Σ <u>is compatible</u>;

(ii) <u>there exists</u> $G \in D(F)$ <u>such that the solution</u> $x(t)$ <u>of</u> (11.13) <u>and</u> (11.14) <u>is equal to</u> $\varphi(t, x_0, u)$ <u>whenever</u> x_0 <u>belongs to</u> $D(F)$ <u>and</u> u <u>is uniformly continuous</u>.

(11.18) REMARK. Note that if the system Σ is smooth then we may regard H as defined only on $D(F)$ and drop Conditions (f) - (i) of Definition (6.1). Indeed, $D(F)$ is Φ-invariant, and for every $u \in C_0[0, T]$, $\int_0^s \Phi(s - \tau)Gu(\tau)d\tau$ belongs to $D(F)$ for every $s \in [0, T]$. Further, the correspondence: $u \mapsto \int_0^t \Phi(t - \tau)Gu(\tau)d\tau$ $(\in D(F))$ is continuous and linear as a map from $C_o[0, t]$ to $D(F)$ (endowed with τ_F). Since H is continuous with respect to τ_F, the requirement (i) can be waived.

REMARK. BARAS and BROCKETT [1975] called smooth systems "balanced" when X is a Hilbert space.

(11.19) REMARK. Observe that if a system Σ is smooth, then for every uniformly continuous input u

(11.20) $\overline{f}(u)(t) = \int_o^t H\Phi(t - \tau)Gu(\tau)d\tau$ (f = the extended input/output map of Σ).

In view of (5.3), this means that the weighting pattern of f is indeed a (locally L^2) function $A(t) = H\Phi(t)G = h^\Sigma(G)(t)$.

Moreover, we have

(11.21) PROPOSITION. If a system Σ is smooth, then its weighting pattern $A(t)$ belongs to $H^1_{loc}[0, \infty)$.

PROOF. We have already seen that $A(t) = H\Phi(t)G$ belongs to $L^2_{loc}[0, \infty)$. In view of (11.8), it suffices to prove that $h^\Sigma(G)$ $(= A)$ belongs to $D(\frac{d}{dt})$. We have

$$\lim_{t\to o} \frac{\tilde{\sigma}_t - I}{t} h^\Sigma(G) = \lim_{t\to o} \frac{\tilde{\sigma}_t h^\Sigma(G) - h^\Sigma(G)}{t},$$

$$= \lim_{t\to o} \frac{h^\Sigma(\Phi(t)G - G)}{t} \quad (h^\Sigma \text{ commutes with shifts}),$$

$$= h^\Sigma(\lim_{t\to o} \frac{\Phi(t) - I}{t} G) \quad (\text{continuity of } h^\Sigma),$$

$$= h^\Sigma(FG).$$

This means that $h^\Sigma(G)$ $(= A)$ is locally absolutely continuous and its derivative is given by $h^\Sigma(FG) \in L^2_{loc}[0, \infty)$. □

We now consider the converse: Given a weighting pattern $A(t) \in H^1_{loc}[0, \infty)$, does there exist a smooth realization? We give an affirmative answer in the following form.

(11.22) THEOREM. Suppose that f is a linear input/output map with the weighting pattern $A \in H^1_{loc}[0, \infty)$. Then its canonical realization $\Sigma_f = (\overline{\text{im } f}, \varphi_f, H_f)$ is smooth.

PROOF. We have already seen, in Proposition (11.7), that Σ_f is compatible.

By the proof of Proposition (11.7), we have $D(F) = \overline{\text{im } f} \cap H^1_{loc}[0, \infty)$, $\Phi(t) = \tilde{\sigma}_t$, and $F = (\frac{d}{dt})$. Now let $G := \tilde{f}(\delta_o) = A(t) \in D(F)$, where $\tilde{f}$ is the extension of f given in Proposition (3.14). As can be clearly seen, $H\Phi(t)G = A(t)$. By Lemma (11.12), the functional differential equation (11.13) together with the initial condition (11.14) admits a unique solution

$$x(t) = \Phi(t)x_o + \int_0^t \Phi(t - \tau)Gu(\tau)d\tau.$$

In view of the linearity $\varphi_f(t, x_o, u) = \Phi(t)x_o + \varphi_f(t, 0, u)$, we need only to prove $\int_0^t \Phi(t - \tau)Gu(\tau)d\tau = \varphi_f(t, 0, u)$.

Recall that each state of Σ_f is a function of $\eta \geq 0$. By definition (see (7.20))

$$\varphi_f(t, 0, u)(\eta) = f(\pi_\Omega \sigma^r_t u)(\eta) = \int_{-t}^{0} A(\eta - t)u(\tau + t)dt$$
$$= \int_0^t A(\eta - \tau + t)u(\tau)d\tau.$$

On the other hand,

$$(\int_0^t \Phi(t - \tau)Gu(\tau)d\tau)(\eta) = H\Phi(\eta) \int_0^t \Phi(t - \tau)Gu(\tau)d\tau,$$
$$= H \int_0^t \Phi(\eta)\Phi(t - \tau)Gu(\tau)d\tau,$$
$$= \int_0^t H\Phi(\eta - \tau + t)Gu(\tau)d\tau.$$

Since $H\Phi(\eta - \tau + t)G = A(\eta - \tau + t)$, our assertion is proved. □

CHAPTER IV. CANONICAL REALIZATIONS WITH HILBERT STATE SPACES

We have already seen that for every input/output map f the state space of its canonical realizations (which we shall abbreviate as the canonical state space in the sequel) is at least a reflexive Fréchet space (Remark (9.8)). Our main concern in this chapter is to investigate conditions under which a canonical realization has a Banach (or Hilbert) space as a state space.

We start by introducing a particular notion of topological observability, which we call topological observability in bounded time. This notion then leads to a necessary and sufficient condition that an input/output map admit a canonical realization with a Banach (Hilbert) state space.

Section 14 is devoted to the study of an example. The example is described by the wave equation of one space-dimension. We convert this description to our framework and prove that the system is indeed topologically observable in bounded time.

12. Topological Observability in Bounded Time.

Topological observability guarantees the property that initial states of the system can be continuously determined from observation data. A new notion, topological observability in bounded time, requires further that the initial state determination be done continuously based on (uniformly) bounded time observation data. To be more precise, we give

(12.1) DEFINITION. *A linear system* $\Sigma = (X, \varphi, H)$ *is topologically observable in bounded time iff there exists* $T > 0$ *such that the following statement is true*:

(12.2) *For every continuous seminorm* p_α *of* X, *there exists* $C_\alpha > 0$ *such that* $p_\alpha(x) \leq C_\alpha \|h^\Sigma(x)\|_{[0,T]}$ *for all* x *in* X.

In view of Proposition (8.10), topological observability in bounded time clearly implies topological observability. It is also obvious that

(12.2) is equivalent to the following statement:

(12.3) <u>For every continuous seminorm</u> p_α <u>of</u> X, <u>there exists</u> C_α <u>such that</u> $p_\alpha(x) \leq C_\alpha \|H\Phi(\cdot)x\|_{[0,T]}$ <u>for all</u> x <u>in</u> $D_o(H)$.

(cf. the proof of Proposition (8.10))

(12.4) REMARK. We may define <u>observability in bounded time</u> as the property that there exists $T > 0$ such that $h^\Sigma(x)|_{[0,T]} = 0$ implies $x = 0$. Clearly topological observability in bounded time implies observability in bounded time, but not vice versa. An easy counterexample is supplied by Example (8.9) (and Lemma (8.2)).

Our main objective in this section is to prove the following

(12.5) THEOREM. <u>Let</u> $\Sigma = (X, \varphi, H)$ <u>be a topologically observable system. Then</u> X <u>is a Banach space iff</u> Σ <u>is topologically observable in bounded time</u>.

PROOF. <u>Sufficiency</u>. The continuous linear map $h^\Sigma\colon X \to (L^2_{[0,T]})^p\colon x \mapsto h^\Sigma(x)|_{[0,T]}$ is one-to-one if Σ is topologically observable in bounded time T. Furthermore, $(h^\Sigma)^{-1}\colon h^\Sigma(X)|_{[0,T]} \to X$ is continuous with respect to the topology induced from $(L^2_{[0,T]})^p$ by (12.2). Hence $X \cong h^\Sigma(X)|_{[0,T]}$. Since $h^\Sigma(X)|_{[0,T]}$ is a normed linear space, so is X. But X is complete, so X must be a Banach space.

<u>Necessity</u>. Let $\|\cdot\|$ denote the norm of X. Since Σ is topologically observable, there exist $T > 0$ and $C > 0$ such that $\|x\| \leq C\|h^\Sigma(x)\|_{[0,T]}$ for all x in X by Proposition (8.10). Since any other continuous seminorm p_α of X satisfies $p_\alpha(x) \leq M_\alpha\|x\|$ for all x in X, Σ is topologically observable in bounded time. □

(12.6) COROLLARY. <u>Let</u> $\Sigma = (X, \varphi, H)$ <u>be a topologically observable system. If</u> X <u>is a Banach space, then it is isomorphic to a Hilbert space.</u>

PROOF. From the proof of the previous theorem, we see that if X is a Banach space then it is isomorphic to a closed subspace of $(L^2_{[0,T]})^p$ for some $T > 0$. Since $(L^2_{[0,T]})^p$ is a Hilbert space, X is isomorphic to a Hilbert space. □

13. Necessary and Sufficient Conditions for a Hilbert Space Canonical Realization.

Let $f: \Omega \to \Gamma$ be a linear input/output map. We ask the question: What is a condition for f to have a canonical realization with a Hilbert state space? In view of Corollary (12.6), this question is equivalent to asking whether the canonical realization possesses a Banach state space. Since we already know that a canonical state space is isomorphic to $\overline{\text{im } f}$, we only have to find conditions under which $\overline{\text{im } f}$ is a Banach space. Let us start our discussion with the following

(13.1) THEOREM. The canonical realization of an input/output map f has a Hilbert state space iff there exist $T, M, \beta > 0$ such that

$$(13.2) \quad \|f(\sigma_t\omega)\|_{[0,T]} \leq Me^{\beta t}\|f(\omega)\|_{[0,T]}$$

for all $\omega \in \Omega$ and $t \geq 0$.

PROOF. Necessity. Let $\Sigma = (X, \varphi, H)$ be the canonical realization of f with X being a Hilbert space. Let $\{\Phi(t)\}_{t \geq 0}$ be the semigroup given by (6.6). By Theorem (12.5) there exist $T, C_1 > 0$ such that $\|x\| \leq C_1\|h^\Sigma(x)\|_{[0,T]}$. It follows that $\|h^\Sigma(x)\|_{[0,T]} \leq C_2\|x\| \leq C_1C_2\|h^\Sigma(x)\|_{[0,T]}$. We now note from YOSHIDA [1971, IX.1, Proposition 1] that there exist $K, \beta > 0$ such that

$$(13.3) \quad \|\Phi(t)x\| \leq Ke^{\beta t}\|x\|,$$

since X is a Banach space. Then we obtain

$$\begin{aligned} \|h^\Sigma(\Phi(t)x)\|_{[0,T]} &\leq C_2\|\Phi(t)x\|, \\ &\leq C_2Ke^{\beta t}\|x\| \quad ((13.3)), \\ &\leq C_1C_2Ke^{\beta t}\|h^\Sigma(x)\|_{[0,T]}. \end{aligned}$$

For every ω in Ω, this yields

$$\|f(\sigma_t\omega)\|_{[0,T]} = \|h^\Sigma g^\Sigma(\sigma_t\omega)\|_{[0,T]},$$

$$= \|h^{\Sigma}(\phi(t)g^{\Sigma}(\omega))\|_{[0,T]} \quad \text{(Proposition (6.12))},$$

$$\leq C_1C_2Ke^{\beta t}\|h^{\Sigma}g^{\Sigma}(\omega)\|_{[0,T]},$$

$$= Me^{\beta t}\|f(\omega)\|_{[0,T]},$$

where $M := C_1C_2K$.

Sufficiency. Take any γ in $\overline{\text{im } f}$. Clearly,

$$\|\tilde{\sigma}_t\gamma\|_{[0,T]} \leq Me^{\beta t}\|\gamma\|_{[0,T]}.$$

We want to prove that the canonical realization $\Sigma_f = (\overline{\text{im } f}, \varphi_f, H_f)$ is topologically observable in bounded time T.

Take any $\alpha > 0$. If $\alpha \leq T$, then $\|\gamma\|_{[0,\alpha]} \leq \|\gamma\|_{[0,T]}$ for all γ in $\overline{\text{im } f}$. Suppose $\alpha \geq T$. Let ℓ be the integer that satisfies $\ell T < \alpha \leq (\ell + 1)T$. Then we obtain inductively

$$\|\gamma\|_{[0,\alpha]} \leq \|\gamma\|_{[0,T]} + \|\gamma\|_{[T,\alpha]},$$

$$= \|\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[0,\alpha-T]},$$

$$\leq \|\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[T,\alpha-T]},$$

$$\leq (1 + Me^{\beta T})\|\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[T,\alpha-T]},$$

$$\cdots$$

$$\leq (1 + Me^{\beta T} + \ldots + M^{\ell}e^{\beta\ell T})\|\gamma\|_{[0,T]}$$

for all γ in $\overline{\text{im } f}$. This shows that the canonical realization Σ_f is topologically observable in bounded time T. By Corollary (12.6), $\overline{\text{im } f}$ is indeed isomorphic to a Hilbert space. □

We can yet relax the condition (13.2) as follows:

(13.4) THEOREM. The canonical realization of an input/output map f has a Hilbert state space iff there exists $T > 0$ such that the following statement is true:

(13.5) <u>For each</u> $t \geq 0$ <u>there exists</u> $C_t > 0$ <u>such that</u> $\|\tilde{\sigma}_t\gamma\|_{[0,T]} \leq C_t\|\gamma\|_{[0,T]}$ <u>for all</u> γ <u>in</u> M, <u>where</u> M <u>is a dense linear subspace of</u> $\overline{\text{im } f}$.

PROOF. Necessity is obvious from Theorem (13.1) (take $C_t := Me^{\beta t}$).

<u>Sufficiency</u>. Note that (13.5) holds for every γ in $\overline{\text{im } f}$ by continuity. Take any $\alpha > 0$. If $\alpha \leq T$, then $\|\gamma\|_{[0,\alpha]} \leq \|\gamma\|_{[0,T]}$. Let $\alpha > T$, and ℓ the integer that satisfies $\ell T < \alpha \leq (\ell + 1)T$. It follows that

$$
\begin{aligned}
\|\gamma\|_{[0,\alpha]} &\leq \|\gamma\|_{[0,T]} + \|\gamma\|_{[T,\alpha]}, \\
&= \|\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[0,\alpha-T]}, \\
&\leq \|\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[T,\alpha-T]}, \\
&\leq (1 + C_T)\|\gamma\|_{[0,T]} + \|\tilde{\sigma}_T\gamma\|_{[T,\alpha-T]}, \\
&\quad \cdots \\
&\leq (1 + C_T + \ldots + C_T^{\ell})\|\gamma\|_{[0,T]} \quad \text{for all } \gamma \text{ in } \overline{\text{im } f}.
\end{aligned}
$$

This shows topological observability in bounded time T. Hence $\overline{\text{im } f}$ is isomorphic to a Hilbert space. □

Let $M_c(-\infty, 0] := \underset{\sim}{E}^{o'}_{(-\infty,0]}$, i.e., the space of Radon measures with compact support contained in $(-\infty, 0]$. If a weighting pattern $A(t)$ is continuous, then its associated input/output map f has an extension $\tilde{f}: (M_c(-\infty, 0])^m \to \Gamma$, as was proved in Proposition (3.14). By Proposition (9.6), it follows that $A = f(\delta_o) \in \overline{\text{im } f}$. Then <u>in order that</u> $\overline{\text{im } f}$ <u>be isomorphic to a Hilbert space it is necessary that</u> A <u>satisfy</u>

$$
(13.6) \quad \|\tilde{\sigma}_t A\|_{[0,T]} \leq Me^{\beta t}\|A\|_{[0,T]}
$$

<u>for some</u> $T, M, \beta > 0$ by Theorem (13.1). Thus we obtain an example of a weighting pattern whose canonical realization cannot have a Hilbert (Banach) state space as follows:

(13.7) COUNTEREXAMPLE. Let the weighting pattern be given by $A(t) := \exp(\exp(t))$. Clearly A does not satisfy (13.6) for any T, M, $\beta > 0$. Hence its canonical realization cannot have a Hilbert (Banach) state space.

We now give easy examples of weighting patterns whose canonical realization has a Hilbert state space.

(13.8) EXAMPLE. Let μ be a weighting pattern with support contained in $[0, T]$. Then its canonical realization has a Hilbert state space. Indeed, for every ω in Ω,

$$f(\omega)(t) = \int_t^\infty \omega(t - \tau)d\mu(\tau) = \int_t^T \omega(t - \tau)d\mu(\tau).$$

Hence $f(\omega)(t) = 0$ if $t > T$. Thus im f can be identified with a subspace of $(L^2_{[0,T]})^p$. Hence $\overline{\text{im } f}$ is isomorphic to a Hilbert space.

(13.9) EXAMPLE. Let $A(t)$ be a locally L^2-weighting pattern given by

$$A(t) = \sum_{n=-\infty}^{\infty} a_n \exp(in\pi t/L),$$

with $\sum |a_n|^2 < \infty$. Then $A(t)$ is a function of period $2L$. One can easily check that (13.2) is satisfied in this case.

14. An Example of a Topologically Observable System.

Consider the wave equation of one space-dimension:

(14.1) $(\partial^2/\partial t^2)v(t, \xi) = (\partial^2/\partial \xi^2)v(t, \xi), \quad t \geq 0, \quad 0 \leq \xi \leq 1,$

with the initial condition

(14.2) $v(0, \xi) = v_o(\xi), \quad (\partial v/\partial t)(0, \xi) = v_1(\xi),$

subject to the boundary condition

(14.3) $(\partial v/\partial \xi)(t, 0) \equiv (\partial v/\partial \xi)(t, 1) \equiv 0.$

This initial-boundary value problem occurs when we describe the vibrating string of length 1 with both ends at $\xi = 0$ and $\xi = 1$ free to slide along fixed parallel rails.

Now assume that

$$(14.4) \quad y_o(t) := v(t, 0), \quad y_1(t) := (\partial v/\partial t)(t, 0)$$

can be observed as outputs. We pose the question: Is it possible to determine unknown initial states $v_o(\xi)$, $v_1(\xi)$ by observing $y_o(t)$, $y_1(t)$ for a sufficiently long time? If it is possible, can it be done continuously? In order to answer these questions we must formulate the problem more precisely.

We convert the equation (14.1) to a first-order differential equation given in the space $H^1_{[0,1]} \times L^2_{[0,1]}$ as follows:

$$(14.5) \quad \frac{d}{dt}\begin{pmatrix} x_o(t) \\ x_1(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (\partial^2/\partial\xi^2) & 0 \end{pmatrix}\begin{pmatrix} x_o(t) \\ x_1(t) \end{pmatrix} =: F\begin{pmatrix} x_o(t) \\ x_1(t) \end{pmatrix},$$

where $x_o(t)(\xi)$ and $x_1(t)(\xi)$ are functions of ξ $(0 \leq \xi \leq 1)$ for each $t \geq 0$, and $x_o(t)(\cdot) \in H^1_{[0,1]}$, $x_1(t)(\cdot) \in L^2_{[0,1]}$.

REMARK. We drop the input term, since it is not relevant to the observability question. The space $H^1_{[0,1]}$ is the first order Sobolev space, i.e.,

$$H^1_{[0,1]} = \{z(\xi) \in L^2_{[0,1]} : (d/d\xi) z \in L^2_{[0,1]}\}$$

with the norm

$$\|z\|_1 := \{\|z\|^2 + \|dz/d\xi\|^2\}^{1/2},$$

where $\|\cdot\|$ denotes the L^2-norm.

Define the domain of F by

$$(14.6) \quad D(F) := \{\begin{pmatrix} x_o \\ x_1 \end{pmatrix} \in H^1_{[0,1]} \times L^2_{[0,1]} : x_1 \in H^1_{[0,1]}, \ (d^2/d\xi^2)x_o \in L^2_{[0,1]}, \ (dx_o/d\xi)|_{\xi=o} = (dx_o/d\xi)|_{\xi=1} = 0\}.$$

REMARK. If $(d^2/d\xi^2)x_o$ belongs to $L^2_{[0,1]}$, then $(dx_o/d\xi)$ is a continuous function of ξ. Hence $(dx_o/d\xi)|_{\xi=o}$ is well-defined.

Clearly $D(F)$ is dense in $H^1_{[0,1]} \times L^2_{[0,1]}$. However, it is a non-trivial problem to check whether F generates a strongly continuous semi-group using the Hille-Yoshida theorem; see YOSHIDA [1971, XIV.3] for this type of treatment in case of the Dirichlet boundary condition.

Fortunately we can bypass this problem by explicitly giving the formula for the strongly continuous semigroup $\{\Phi(t)\}_{t\geq 0}$ generated by F, with the aid of the Fourier series expansion of the solution of the initial-boundary value problem (14.1) - (14.3). But before doing this, we remark that the initial condition (14.2) must be rewritten as

$$(14.7) \quad x_o(0)(\xi) = v_o(\xi), \quad x_1(0)(\xi) = v_1(\xi).$$

Now for each $t \geq 0$, define a linear operator $\Phi(t)$ from $X := H^1_{[0,1]} \times L^2_{[0,1]}$ into itself by

$$(14.8) \quad \Phi(t)\begin{pmatrix} z_o \\ z_1 \end{pmatrix} := \begin{pmatrix} \sum_{n=1}^{\infty} \cos n\pi\xi[A_n \cos n\pi t + B_n \sin n\pi t] + A_o + B_o t \\ \sum_{n=1}^{\infty} \cos n\pi\xi[- n\pi A_n \sin n\pi t + n\pi B_n \cos n\pi t] + B_o \end{pmatrix},$$

where

$$(14.9) \quad A_n := 2 \int_0^1 z_o(\eta) \cos n\pi\eta d\eta, \quad B_n := (2/n\pi) \int_0^1 z_1(\eta) \cos n\pi\eta d\eta, \quad n \geq 1;$$

$$(14.10) \quad A_o := \int_0^1 z_o(\eta)d\eta, \quad B_o := \int_0^1 z_1(\eta)d\eta.$$

Note that z_o and z_1 are indeed expanded in Fourier series as

$$(14.11) \quad z_o(\xi) = \sum_{n=1}^{\infty} A_n \cos n\pi\xi + A_o,$$

$$(14.12) \quad z_1(\xi) = \sum_{n=1}^{\infty} n\pi B_n \cos n\pi\xi + B_o, \quad 0 \leq \xi \leq 1,$$

since we can always extend z_o and z_1 as <u>even</u> functions on $(-\infty, \infty)$ of period 2. Furthermore, since z_o belongs to $H^1_{[0,1]}$ and z_1 belongs to $L^2_{[0,1]}$,

$$(14.13)\quad \sum_{n=1}^{\infty} |n\pi A_n|^2 < \infty, \quad \sum_{n=1}^{\infty} |n\pi B_n|^2 < \infty.$$

(14.14) REMARK. If $z_o = \sum_{n=1}^{\infty} A_n \cos n\pi\xi + A_o$ belongs to $H^1_{[0,1]}$, then $(d/d\xi)z_o(\xi) = \sum_{n=1}^{\infty} - n\pi A_n \sin n\pi\xi$ in the sense of L^2. Indeed, $(d/d\xi)z_o$ equals the right-hand series in the sense of distributions. But since $(d/d\xi)z_o$ belongs to L^2, the series $\sum_{n=1}^{\infty} - n\pi A_n \sin n\pi\xi$ is indeed a function and belongs to $L^2_{[0,1]}$; see MIZOHATA [1973, Theorem 2.7]. But $\sum_{n=1}^{\infty} - n\pi A_n \sin n\pi\xi$ belongs to $L^2_{[0,1]}$ iff $\sum_{n=1}^{\infty} |n\pi A_n|^2 < \infty$ by Parseval's identity.

Again by Parseval's identity one obtains

$$(14.15)\quad \|z_o\|_1 = \{\|z_o\|^2 + \|dz_o/d\xi\|^2\}^{1/2},$$
$$= \{|A_o|^2 + (1/2)\sum_{n=1}^{\infty} |A_n|^2 + (1/2)\sum_{n=1}^{\infty} |n\pi A_n|^2\}^{1/2};$$

$$(14.16)\quad \|z_1\| = \{|B_o|^2 + (1/2)\sum_{n=1}^{\infty} |n\pi B_n|^2\}^{1/2}.$$

We are now ready to prove

(14.17) PROPOSITION. <u>The family of linear operators</u> $\{\Phi(t)\}_{t\geq 0}$ <u>given by</u> (14.8) - (14.10) <u>forms a strongly continuous semigroup in</u> $H^1_{[0,1]} \times L^2_{[0,1]}$, <u>and its infinitesimal generator</u> $\hat{F}$ <u>is precisely</u> F <u>given by</u> (14.5) <u>and</u> (14.6).

PROOF. Clearly $\Phi(0) = I$ by (14.11) and (14.12). The semigroup property $\Phi(t + s) = \Phi(t)\Phi(s)$ follows easily from direct calculation.

We prove strong continuity. It suffices to prove strong continuity at $t = 0$ since $H^1_{[0,1]} \times L^2_{[0,1]}$ is a Banach space (see YOSHIDA [1971, IX.1]). For the first coordinate of $\Phi(t)(z_o, z_1)$ (we shall use the row vector notation as well as the column vector notation), we have

$$(14.18)\quad \Phi(t)(z_o, z_1)|_o - z_o = \sum_{n=1}^{\infty} \cos n\pi\xi[A_n(\cos n\pi t - 1) + B_n \sin n\pi t] + B_o t.$$

By Parseval's identity

$$(14.19)\quad \|\Phi(t)(z_o, z_1)|_o - z_o\|_1^2 =$$

$$|B_o|^2 t^2 + (1/2)\sum_{n=1}^{\infty} |A_n(\cos n\pi t - 1) + B_n \sin n\pi t|^2$$

$$+ (1/2)\sum_{n=1}^{\infty} n^2\pi^2 |A_n(\cos n\pi t - 1) + B_n \sin n\pi t|^2.$$

The right hand side of (14.19) is uniformly convergent because we have the following estimate for each n:

$$(14.20)\quad |A_n(\cos n\pi t - 1) + B_n \sin n\pi t|^2 \le 2(4|A_n|^2 + |B_n|^2),$$

and because $\Sigma\, 2(4|A_n|^2 + |B_n|^2)$, $\Sigma\, 2n^2\pi^2(4|A_n|^2 + |B_n|^2)$ are finite. Hence we can take the termwise limit as $t \to 0$ in (14.19), yielding $\|\Phi(t)(z_o, z_1)|_o - z_o\|_1 \to 0$ as $t \to 0$.

For the second coordinate of $\Phi(t)(z_o, z_1)$, we have

$$(14.21)\quad \Phi(t)(z_o, z_1)|_1 - z_1 =$$

$$\sum_{n=1}^{\infty} \cos n\pi\xi[- n\pi A_n \sin n\pi t + n\pi B_n(\cos n\pi t - 1)].$$

The L^2-norm $\|\Phi(t)(z_o, z_1)|_1 - z_1\|$ can be estimated similarly as is done for $\|\Phi(t)(z_o, z_1)|_o - z_o\|_1$, and it approaches 0 as $t \to 0$. Hence $\{\Phi(t)\}_{t\ge 0}$ is strongly continuous.

We now calculate the infinitesimal generator of $\{\Phi(t)\}_{t\ge 0}$. First suppose that (z_o, z_1) belongs to $D(F)$, that is, $(d^2/d\xi^2)z_o$ belongs to $L^2_{[0,1]}$ and z_1 belongs to $H^1_{[0,1]}$ and $(dz_o/d\xi)|_{\xi=o} = (dz_o/d\xi)|_{\xi=1} = 0$. These conditions imply that

$$(14.22)\quad \sum_{n=1}^{\infty} n^4|A_n|^2 < \infty, \quad \sum_{n=1}^{\infty} n^4|B_n|^2 < \infty.$$

(Note that (z_o, z_1) belongs to $D(F)$ iff (14.22) holds.) It follows that

$$(14.23)\quad (1/t)\{\Phi(t)(z_o, z_1)|_o - z_o\} = \sum_{n=1}^{\infty} \cos n\pi\xi[A_n(\cos n\pi t - 1)/t + B_n(\sin n\pi t)/t] + B_o.$$

By the same reasoning as was given for strong continuity of $\{\Phi(t)\}_{t\geq 0}$, the right hand side of (14.23) converges to $z_1 = \sum_{n=1}^{\infty} n\pi B_n \cos n\pi\xi + B_o$ in $H^1_{[0,1]}$ as $t \to 0$ (we can take the termwise limit by virtue of the fact that $\sum n^4|A_n|^2 < \infty$ and $\sum n^4|B_n|^2 < \infty$).

For the second coordinate, we obtain

$$(14.24)\quad (1/t)\{\Phi(t)(z_o, z_1)|_1 - z_1\} = \sum_{n=1}^{\infty} \cos n\pi\xi[- n\pi A_n(\sin n\pi t)/t + n\pi B_n(\cos n\pi t - 1)/t].$$

As before, it is easily seen that the right side converges to $\sum_{n=1}^{\infty} - n^2\pi^2 A_n \cos n\pi\xi$ in $L^2_{[0,1]}$, which is equal to $(d^2/d\xi^2)z_o$.

Thus we see that the infinitesimal generator $\hat{F}$ of $\{\Phi(t)\}_{t\geq o}$ coincides with F on $D(F)$, i.e., $\hat{F}$ is an extension of F. We postpone the proof of the fact $D(\hat{F}) = D(F)$ until the end of this section. □

(14.25) REMARK. It is known that the first coordinate of $\Phi(t)(v_o, v_1)$ indeed gives the genuine solution of the initial-boundary value problem (14.1) - (14.3) for sufficiently smooth initial data v_o, v_1; see, for example, L. SCHWARTZ [1961, VII.1].

Now define the readout map $H: D(F) \to k^2$ by

$$(14.26)\quad H\begin{pmatrix} x_o \\ x_1 \end{pmatrix} := \begin{pmatrix} x_o(0) \\ x_1(0) \end{pmatrix}.$$

Also define a linear map $h_o: D(F) \to \Gamma$ by

$$(14.27)\quad h_o\begin{pmatrix} x_o \\ x_1 \end{pmatrix}(t) := H\Phi(t)\begin{pmatrix} x_o \\ x_1 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} [A_n \cos n\pi t + B_n \sin n\pi t] + A_o + B_o t \\ \sum_{n=1}^{\infty} [- n\pi A_n \sin n\pi t + n\pi B_n \cos n\pi t] + B_o \end{pmatrix},$$

where $x_o(\xi) = \sum_{n=1}^{\infty} A_n \cos n\pi\xi + A_o$, $x_1 = \sum_{n=1}^{\infty} n\pi B_n \cos n\pi\xi + B_o$.

We claim

(14.28) LEMMA. $\|h_o(x_o, x_1)\|^2_{[0,2q]} = q\{\sum_{n=1}^{\infty} (n^2\pi^2 + 1)(|A_n|^2 + |B_n|^2) +$

$$2|A_o|^2 + 2(A_o\overline{B}_o + \overline{A}_oB_o) + (14/3)|B_o|^2\},$$

where q _is a positive integer and_ $\overline{A}$ _is the complex conjugate of_ A.

PROOF. Immediate from Parseval's identity and direct calculation. □

In view of (14.15) and (14.16), Lemma (14.28) clearly implies the continuity of h_o (with respect to the topology of $X = H^1_{[0,1]} \times L^2_{[0,1]}$). Hence there exists a unique continuous extension $h^\Sigma: X \to \Gamma$, the observability map. Therefore our system is well defined. Then we claim

(14.29) THEOREM. _The map_ $h^\Sigma: X \to \Gamma$ _is an isomorphism into_ Γ, _i.e., the system is topologically observable (in bounded time)_.

PROOF. By Lemma (14.28) we obtain

$$(14.30)\quad \|h^\Sigma(x_o, x_1)\|^2_{[0,2]} = \sum_{n=1}^{\infty} (n^2\pi^2 + 1)(|A_n|^2 + |B_n|^2) +$$

$$2|A_o|^2 + 2(A_o\overline{B}_o + \overline{A}_oB_o) + (14/3)|B_o|^2,$$

$$\geq \sum_{n=1}^{\infty} (n^2\pi^2 + 1)(|A_n|^2 + |B_n|^2) + 2|A_o|^2 + 2|B_o|^2 - 2|A_o||B_o|,$$

$$\geq \sum_{n=1}^{\infty} (n^2\pi^2 + 1)(|A_n|^2 + |B_n|^2) + |A_o|^2 + |B_o|^2.$$

It now follows that

$$(14.31)\quad \|x_o\|_1^2 + \|x_1\|^2 \leq \|h^\Sigma(x_o, x_1)\|^2_{[0,2]}$$

by (14.15) and (14.16). Hence h^Σ is an isomorphism of X into Γ, and the system is topologically observable in bounded time 2. □

Let us now finish the proof of Proposition (14.17). We must prove

(14.32) LEMMA. $D(\hat{F}) = D(F)$.

PROOF. The inclusion $D(F) \subset D(\hat{F})$ has already been proved. Assume that $z = (z_0, z_1)$ belongs to $D(\hat{F})$. Consider the function

$$(14.33)\quad \alpha\colon [0, 1] \to \underline{\underline{R}}\colon t \mapsto \begin{cases} \|((\Phi(t) - I)/t)z\|_X & \text{if } t > 0, \\ \|\hat{F}z\|_X & \text{if } t = 0, \end{cases}$$

where

$$(14.34)\quad \|((\Phi(t) - I)/t)z\|_X = \sum_{n=1}^{\infty} (n^2\pi^2 + 1)|A_n(\cos n\pi t - 1)/t + B_n(\sin n\pi t)/t|^2 + \sum_{n=1}^{\infty} n^2\pi^2|A_n(\sin n\pi t)/t + B_n(\cos n\pi t - 1)/t|^2 + |B_0|^2.$$

Since z belongs to $D(\hat{F})$, α is a continuous function of t on $[0, 1]$. Furthermore, the q-partial sum of the right side of (14.34) converges monotonically to $\alpha(t)$ for each t as $q \to \infty$. Hence by Dini's theorem (RUDIN [1964, Theorem 7.13]), the q-partial sum of the right side of (14.34) converges uniformly on $[0, 1]$ as $q \to \infty$. Hence we can take the termwise limit of the right side of (14.34) as $t \to \infty$, and obtain

$$(14.35)\quad \sum_{n=1}^{\infty} n^4|A_n|^2 < \infty, \quad \sum_{n=1}^{\infty} n^4|B_n|^2 < \infty.$$

These inequalities readily imply that $z \in D(F)$. □

APPENDIX

In this appendix we review some basic facts on inductive limits and projective limits of sequences of locally convex Hausdorff spaces. We remark that an inductive limit (projective limit) is also known as a colimit (limit) in categorical terms. In most cases we shall omit proofs since they are available in the following standard references: BOURBAKI [1966]; KÖTHE [1969]; SCHAEFER [1971]; TREVES [1967].

Suppose that we are given a sequence of locally convex Hausdorff spaces $\{E_n\}_{n=1}^{\infty}$ such that for every n there exists an inclusion $j_{n,n+1}: E_n \to E_{n+1}$ and each $j_{n,n+1}$ is an isomorphism of E_n into E_{n+1}, i.e., the topology of E_n is identical to that induced on $j_{n,n+1}(E_n)$ from E_{n+1}. Clearly for every pair n, m such that $n \leq m$, there exists an isomorphic inclusion $j_{nm}: E_n \to E_m$, thereby enabling us to identify an element of E_n with an element of E_m whenever $n \leq m$. Let $E := \bigcup_{n=1}^{\infty} E_n$; E is clearly a vector space (over k) by the preceding identification. We try to introduce a natural topology on E.

Let $j_n: E_n \to E$ be the inclusion. The <u>inductive limit topology</u> on E is the finest locally convex topology on E such that all j_n are continuous; E is called the <u>(strict) inductive limit of a sequence</u> $\{E_n\}$ and is denoted as $E = \varinjlim E_n$. A strict inductive limit of a sequence has the following remarkable property.

(A.1) PROPOSITION. <u>Let</u> $E = \varinjlim E_n$ <u>be the (strict) inductive limit of a sequence</u> $\{E_n\}$. <u>Then the topology of</u> E <u>is locally convex Hausdorff and</u> $j_n: E_n \to E$ <u>is an isomorphism for every</u> n.

PROOF. See SCHAEFER [1971, II.6.4]. □

In order to check the continuity of a linear mapping $f: \varinjlim E_n \to F$, the following Proposition (A.2) is extremely useful.

(A.2) PROPOSITION. <u>Let</u> $E = \varinjlim E_n$ <u>be the (strict) inductive limit of</u> $\{E_n\}_{n=1}^{\infty}$ <u>and</u> F <u>a locally convex space. A linear map</u> $f: E \to F$ <u>is continuous iff each</u> $f \circ j_n: E_n \to F$ <u>is continuous for every</u> n, <u>where</u> $j_n: E_n \to E$ <u>is the inclusion.</u>

PROOF. See SCHAEFER [1971, II.6.1]. □

If F is also a (strict) inductive limit of a sequence $\{F_n\}$ with inclusions $i_n: F_n \to F$, and if a linear map $f: E \to F$ satisfies

(A.3) <u>for every</u> n <u>there exists</u> m <u>such that</u> $f(E_n) \subset F_m$,

we have the following

(A.4) COROLLARY. <u>Let</u> $f: E \to F$ <u>be a linear map that satisfies</u> (A.3), <u>where</u> $E := \varinjlim E_n$ <u>and</u> $F := \varinjlim F_n$. <u>Then</u> f <u>is continuous iff each</u> $f \circ j_n$ <u>is continuous as a map from</u> E_n <u>to</u> F_m.

PROOF. Necessity is obvious. Conversely, if $f \circ j_n$ is continuous as a map from E_n to F_m, it is also continuous as a map from E_n to F because $i_m: F_m \to F$ is an isomorphism by Proposition (A.1). Thus by Proposition (A.2) f is continuous. □

We now turn our attention to the dual notion of inductive limits, namely projective limits of locally convex spaces. Let E be a vector space and $\{E_n\}_{n=1}^{\infty}$ a sequence of locally convex Hausdorff spaces. Suppose that for each n there is given a linear map $\pi_n: E \to E_n$, and to each pair m, n with $m \geq n$ there is associated a continuous linear map $\pi_{mn}: E_m \to E_n$ such that

(i) $\pi_{nn} = 1_{E_n}$ for every n;

(ii) $\pi_{n\ell}\pi_{mn} = \pi_{m\ell}$ whenever $m \geq n \geq \ell$;

(iii) $\pi_{mn}\pi_m = \pi_n$ whenever $m \geq n$.

The <u>projective limit topology</u> on E is the coarsest (locally convex) topology on E such that all π_n are continuous; E is called the <u>projective limit</u> of the sequence $\{E_n\}_{n=1}^{\infty}$ and denoted as $E = \varprojlim \pi_{mn} E_n$, or simply $E = \varprojlim E_n$. The projective limit $E = \varprojlim E_n$ is <u>reduced</u> when $\pi_n(E)$ is dense in E_n for every n.

A projective limit is not a priori a Hausdorff space. Hence we quote

(A.5) PROPOSITION. <u>The projective limit</u> $E = \varprojlim E_n$ <u>is a Hausdorff</u>

space iff for each nonzero $x \in E$, there exist n and a 0-neighborhood U in E_n such that $\pi_n(x) \notin U$.

PROOF. See SCHAEFER [1971, II.5.1]. □

The following Proposition (A.6) is very useful in checking the continuity of a linear map: $F \to \varprojlim E_n$.

(A.6) PROPOSITION. Let $E = \varprojlim E_n$ be the projective limit and F a locally convex space. A linear map $f: F \to E$ is continuous iff each $\pi_n \circ f$ is continuous for every n.

PROOF. See SCHAEFER [1971, II.5.2]. □

We now prove that the space $L^2_{loc}[0, \infty)$ defined in Chapter II is indeed the projective limit of $\{L^2_{[0,n]}\}_{n=1}^{\infty}$.

(A.7) PROPOSITION. The space $L^2_{loc}[0, \infty)$, with the seminorms $\{\|\cdot\|_{[0,n]}\}$ given in Section 2, is the projective limit of the sequence $\{L^2_{[0,n]}\}$. Moreover, this topology is Hausdorff.

PROOF. Define π_n and π_{mn} as follows:

(A.8) $\pi_n(\varphi) := \varphi|_{[0,n]}, \quad \varphi \in L^2_{loc}[0, \infty);$

(A.9) $\pi_{mn}(\varphi) := \varphi|_{[0,n]}, \quad \varphi \in L^2_{[0,m]}, \quad m \geq n.$

Clearly $\pi_{nn} = I$, $\pi_{n\ell}\pi_{mn} = \pi_{m\ell}$ and $\pi_{mn}\pi_m = \pi_n$ $(m \geq n \geq \ell)$. For every $n \geq 1$, we have

$$\|\pi_n(\varphi)\| = \{\int_0^n |\pi_n(\varphi)(t)|^2 dt\}^{1/2} = \|\varphi\|_{[0,n]}.$$

Hence π_n is continuous for every n. Furthermore, it is necessary that $\|\cdot\|_{[0,n]}$ be a continuous seminorm for every n in order that each π_n be continuous. But since the topology of $L^2_{loc}[0, \infty)$ is generated by the family of seminorms $\{\|\cdot\|_{[0,n]}\}$, this must be the coarsest (clearly locally convex) topology making each π_n continuous. Hence

$L^2_{loc}[0, \infty) = \varprojlim L^2_{[0,n]}$. The topology of $L^2_{loc}[0, \infty)$ is clearly Hausdorff. This completes the proof. □

We now want to prove that the space of Laurent functions Λ is indeed the inductive limit of spaces $\{L^2_{loc}[-n, \infty)\}$, where $L^2_{loc}[-n, \infty)$ is topologized in the same way as $L^2_{loc}[0, \infty)$.

(A.10) PROPOSITION. $\Lambda = \varinjlim L^2_{loc}[-n, \infty)$.

PROOF. By definition, $\Lambda = \bigcup_{n=1}^{\infty} L^2_{loc}[-n, \infty)$. Take any n, and define $p_n: L^2_{loc}[-n, \infty) \to \varinjlim L^2_{[-n,0]}$ and $q_n: L^2_{loc}[-n, \infty) \to \varprojlim L^2_{[0,n]}$ by

$$p_n(\varphi) := \varphi|_{[-n,0]}, \quad \varphi \in L^2_{loc}[-n, \infty);$$

$$q_n(\varphi) := \varphi|_{[0,\infty)}, \quad \varphi \in L^2_{loc}[-n, \infty).$$

Clearly p_n and q_n are continuous for all n. Since $\Lambda = \varinjlim L^2_{[-n,0]} \oplus \varprojlim L^2_{[0,n]} \cong \varinjlim L^2_{[-n,0]} \times \varprojlim L^2_{[0,n]}$ by definition, there exists a unique continuous linear map $\alpha_n: L^2_{loc}[-n, \infty) \to \Lambda$ for each n such that the diagram

$$\begin{array}{ccc} & \xrightarrow{\;p_n\;} & \varinjlim L^2_{[-m,0]} \\ & & \uparrow \\ L^2_{loc}[-n, \infty) & \xrightarrow{\;\alpha_n\;} & \Lambda \\ & & \downarrow \\ & \xrightarrow{\;q_n\;} & \varprojlim L^2_{[0,m]} \end{array}$$

commutes. It then follows that α_n is the inclusion of $L^2_{loc}[-n, \infty)$ into Λ. Hence the topology of Λ is coarser than that of $\varinjlim L^2_{loc}[-n, \infty)$ in view of the definition of the inductive limit topology.

Conversely, define $\beta: \varinjlim L^2_{[-n,0]} \to \varinjlim L^2_{loc}[-n, \infty)$ and $\gamma: \varprojlim L^2_{[0,n]} \to \varinjlim L^2_{loc}[-n, \infty)$ by

$$\beta(\varphi) := \varphi, \quad \varphi \in \varinjlim L^2_{[-n,0]};$$

$$\gamma(\varphi) := \varphi, \quad \varphi \in \varprojlim L^2_{[0,n]}.$$

For any $\varphi \in L^2_{[-n,0]}$ we have $\beta(\varphi) \in L^2_{[-n,0]}$ and $\|\beta(\varphi)\|_{[-n,0]} = \|\varphi\|_{[-n,0]}$. Hence β is continuous by Corollary (A.4). Similarly, γ is continuous. Therefore there exists a unique continuous map $\delta\colon \Lambda \to \varinjlim L^2_{loc}[-n, \infty)$ such that the diagram

$$\begin{array}{ccc}
\varinjlim L^2_{[-n,0]} & \overset{\beta}{\searrow} & \\
\downarrow & & \\
\Lambda & \xrightarrow{\ \ \delta\ \ } & \lim L^2_{loc}[-n, \infty) \\
\uparrow & & \\
\varprojlim L^2_{[0,n]} & \underset{\gamma}{\nearrow} &
\end{array}$$

commutes. But δ is clearly equal to the identity. Hence the topology of Λ is finer than that of $\varprojlim L^2_{loc}[-n, \infty)$. $\square$

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