THE SYNTHESIS OF MINIMUM PHASE TRANSFER FUNCTIONS BY ZERO SHARING

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A new philosophy is presented for synthesizing ladder networks from a prescribed minimum phase transfer function, \( T(s) \). Synthesis procedures are developed in which a sharing of transmission zeros is effected between the network parameters, \( z_{12} \) and \( z_{22} \) (or \(-y_{21}\) and \(y_{22}\)), with the result that the synthesis problem is divided into two parts, a two-terminal realization and a four-terminal realization. This division usually reduces the labor involved in the synthesis and often requires a smaller number of elements in the final network than are required by other methods.

A wide range of problems exists in which the zero sharing technique may be advantageously applied. Methods are described and illustrated for the synthesis of RC, RL,
LC, and RLC networks. For the synthesis of two-element-kind networks, from transfer functions whose transmission zeros lie on the negative portion of the $\sigma$ axis or on the $j\omega$ axis, the zero sharing approach is straightforward, regardless of the complexity of the transfer function, i.e., regardless of the degree of the numerator or denominator polynomial. In the synthesis of transfer functions which have complex as well as other types of transmission zeros, the zero sharing approach is applicable to those transfer functions whose denominator polynomial is of rather low degree in the complex frequency variable $s$.

A central portion of the zero sharing approach is concerned with the choice of the parameters $z_{21}$ and $z_{22}$ (or $-y_{21}$ and $y_{22}$) from a prescribed transfer function, $T(s) = \frac{P(s)}{Q(s)}$, of the response/excitation type. It is shown that this selection is equivalent to removing certain of the poles of $1/T(s) = \frac{Q(s)}{P(s)}$ and replacing them with new poles in such a way that the modified function, $\frac{Q(s)}{P'(s)}$, has the following three properties:

1) $\frac{Q(s)}{P'(s)}$ is positive real.

2) $\frac{Q(s)}{P'(s)}$ may be broken into a sum of two parts, each of which is positive real. One part must have for its poles, all of the poles which were added in forming $\frac{Q(s)}{P'(s)}$. This part must not have any of the poles of the original function, $\frac{Q(s)}{P(s)}$. The second part must have as its poles only the poles of the original function, $\frac{Q(s)}{P(s)}$. 

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3) The poles which are selected for the modified function must not coincide with the zeros of Q(s).

The construction of the modified function Q(s)/P'(s) is easily accomplished by inspection for those transfer functions, whose transmission zeros lie on the negative portion of the σ axis or on the jω axis, and whose realization is to result in a two-element-kind network. For those transfer functions which have complex as well as other types of transmission zeros, the selection of poles for the modified function, Q(s)/P'(s), is restricted to certain regions of the s plane. The location and approximate shape of these regions are determined by a procedure similar to that used in the design of control systems by the root locus method.
CHAPTER I

INTRODUCTION

Many practical problems encountered in the design of electrical networks are concerned with the transfer of a signal from one pair of network terminals to a second pair. In the four-terminal network shown in Fig. 1.1, terminals 1-1' are considered as the input, and transmission occurs from this terminal pair to the output terminals 2-2'.

![Diagram of a four-terminal network](image)

For a given load the transfer properties of the network may be expressed in terms of a ratio of input voltage or current, $E_1$ or $I_1$, to output voltage or current, $E_2$ or $I_2$. Expressed in the frequency domain, this description has the form of a ratio of two polynomials in the complex frequency variable, $s = \sigma + j\omega$, and is termed a transfer function. The synthesis problem considered here is one of finding a ladder network which possesses the properties that are specified in terms of such a transfer function. The resulting
network is to consist of lumped, linear, bilateral, and passive elements, and is to have no mutual inductances. Realization procedures will be developed for networks consisting of RC, RL, LC, and RLC elements.

The network specifications will be given in terms of one of six transfer functions of the response/excitation type. For a current input at terminals 1-1' of Fig. 1.1, three of these transfer functions are defined as

$$Z_T(s) = \frac{E_2}{I_1} \mid_{R_2 \neq \infty}, \quad A(s) = \frac{I_2}{I_1} \mid_{R_2 \neq 0}, \quad \text{and} \quad A^*(s) = \frac{I_2}{I_1} \mid_{R_2 = 0}. \quad (1-1)$$

$T_1(s)$ will be used when referring to the set, $Z_T(s)$, $-A(s)$, or $-A^*(s)$. For a voltage input at terminals 1-1' of Fig. 1.1, three transfer functions are defined as

$$Y_T(s) = \frac{I_2}{E_1} \mid_{R_2 \neq 0}, \quad G(s) = \frac{E_2}{E_1} \mid_{R_2 \neq \infty}, \quad \text{and} \quad G^*(s) = \frac{E_2}{E_1} \mid_{R_2 = \infty}. \quad (1-2)$$

$T_2(s)$ will be used to denote $-Y_T(s)$, $G(s)$, or $G^*(s)$. For convenience $T(s)$ will be used when referring to the entire group of transfer functions, members of either $T_1(s)$ or $T_2(s)$. From (1-1) it is seen that

$$A(s) = \frac{I_2}{I_1} = -\frac{(E_2/R_2)}{I_1} = -Z_T(s)/R_2. \quad (1-3)$$
If \( A(s) \) is specified for a network which is terminated in a finite non-zero load, an equivalent \( Z_T(s) \) may be easily found from (1-3). It is sufficient, therefore, to consider only the synthesis of \( Z_T(s) \). It will be found advantageous, however, to consider the synthesis of the limiting case, \( A*(s) \), individually. Similarly, it is necessary to consider only \( Y_T(s) \) and the special case, \( G*(s) \).

The transfer specifications in terms of \( T(s) \) will be a ratio of two polynomials in \( s \) having the form

\[
T(s) = H \frac{P(s)}{Q(s)} = H \frac{s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0}{s^m + b_{m-1}s^{m-1} + \ldots + b_1s + b_0}
\]

Assuming that \( P(s) \) and \( Q(s) \) have no common factors, the finite zeros of \( T(s) \) occur at the zeros of the numerator polynomial, \( P(s) \), and additional zeros may occur at infinity for \( m > n \). The zeros of \( T(s) \) are termed "transmission zeros," since at these frequencies no transmission occurs through the network.

Due to the assumed configuration of the resulting network, only those transfer functions, \( T(s) \), whose finite transmission zeros do not occur in the right-hand half of the \( s \) plane will be considered. Such transfer functions are commonly referred to as minimum phase functions. \( T(s) \) must possess a number of additional properties in order to be realizable in a ladder network made up of lumped, linear elements. Discussion of these properties, however, will be
postponed until Chapter II.

The problem of realizing a ladder network, whose transfer function is specified, is commonly handled by first finding a pair of network parameters, and from these parameters, realizing a network. Use is made of the open circuit parameters $z_{11}$, $z_{12}$, $z_{21}$, and $z_{22}$ in the synthesis of the transfer functions, $T_1(s)$. It may easily be shown that

$$z_T(s) = -R_2A(s) = \frac{z_{21}}{1 + z_{22}/R_2} \quad (1-5)$$

Use is made of the short circuit parameters $y_{11}$, $y_{12}$, $y_{21}$, and $y_{22}$ in the synthesis of the set of transfer functions, $T_2(s)$. It may be shown that

$$-Y_T(s) = G(s)/R_2 = \frac{-y_{21}}{1 + y_{22}R_2} \quad (1-6)$$

In selecting the network parameters, the ratio specified in (1-4) is inserted in (1-5) or (1-6) for the appropriate transfer function. From the expression which results, $z_{21}$ and $z_{22}$ (or $-y_{21}$ and $y_{22}$) are each selected as a ratio of two polynomials in $s$. In general a great many possible choices exist, and to ascertain which of these choices result in a realizable pair of parameters requires a complete description of the conditions of physical realizability for the parameters of a ladder network. Such a
description is postponed until Chapter II.

To form a basis for choosing the network parameters, attention is turned to the transmission zeros of $T(s)$ and the behavior of the parameters at these zeros. From (1-5) (or (1-6)) it is evident that at a zero of $T_1(s)$ one of the following two conditions must hold:

1) $z_{21}$ (or $y_{21}$) must have a zero which is not a zero of the denominator, $1 + z_{22}/R_2$ (or $1 + y_{22} R_2$).

2) $z_{22}$ (or $y_{22}$) must have a pole which is not a pole of $z_{21}$ (or $y_{21}$).

Two standard methods for choosing the parameters represent the widely differing approaches which one may take in making this choice [1].* In the first approach $z_{21}$ and $z_{22}$ are chosen to have the same denominators, which rules out possibility 2. All transmission zeros are then the zeros of $z_{21}$. The second approach assigns a constant value to $z_{21}$ with the result that all transmission zeros are produced as private poles of $z_{22}$. This method is restricted in its use to only those transfer functions which are positive real. However, for this special case it yields a very simple solution which requires only two-terminal techniques in the synthesis.

The zero sharing method is primarily concerned with a new philosophy which may be used in selecting network

*Brackets denote entries in the List of References.
parameters from the given transfer function. An orderly procedure is developed for utilizing the simplifying properties of the second approach above while extending its application to all physically realizable minimum phase transfer functions.

1.1 The Basic Zero Sharing Procedure

Consider the network arrangement of Fig. 1.1 terminated in a normalized load, \( R_2 = 1 \). For such a termination (1-5) and (1-6), when solved for \( z_{22} \) and \( y_{22} \), have the form

\[
z_{22} = z_{21} \frac{Q(s)}{HP(s)} - 1 \tag{1-7}
\]

and

\[
y_{22} = -y_{21} \frac{Q(s)}{HP(s)} - 1 , \tag{1-8}
\]

respectively. For the special open circuit and short circuit cases, \( A^*(s) \) and \( G^*(s) \), \( z_{22} \) and \( y_{22} \) have the form

\[
z_{22} = z_{21} \frac{Q(s)}{HP(s)} \tag{1-9}
\]

and

\[
y_{22} = -y_{21} \frac{Q(s)}{HP(s)} , \tag{1-10}
\]

respectively.
The basic zero sharing procedure may be summarized in the following steps:

1) Determine whether or not $Q(s)/P(s)$ is positive real. In the event that the positive real property is satisfied, $z_{21}$ (or $-y_{21}$) may be set equal to a constant, $K$, and step 2 may be omitted.

2) In the event that $Q(s)/P(s)$ is not positive real, choose $z_{21} = KN(s)/D(s)$ (or $-y_{21} = KN(s)/D(s)$) in such a way that the product, $z_{21}Q(s)/P(s)$ (or $-y_{21}Q(s)/P(s)$) is positive real and has no zeros on the $j\omega$ axis. (In the synthesis of $A*(s)$ or $G*(s)$, however, zeros on the $j\omega$ axis are permitted.) The degree of the polynomial $N(s)$ and that of $D(s)$ are to be as low as possible and still meet the desired conditions on the product $z_{21}Q(s)/P(s)$ (or $-y_{21}Q(s)/P(s)$). $N(s)$ is chosen to have certain of the zeros of $P(s)$, as will be described later in detail. If $N(s)$ has some but not all of the zeros of $P(s)$ then the product $z_{21}Q(s)/P(s)$ (or $-y_{21}Q(s)/P(s)$) must in general satisfy certain restrictions on the residues at each of its poles in addition to being positive real. It is this case which represents a sharing of transmission zeros and is of primary interest.

3) $z_{22}$ as given in (1-7) or (1-9), or $y_{22}$ as given in (1-8) or (1-10), is next placed in the form

$$z_{22} = z'_{22} + z_s$$

or
or

\[ y_{22} = y'_2 + Y_p, \quad (1-12) \]

respectively.

Here \( z'_{22} \) (or \( y'_{22} \)) is to have the same poles as \( z_{21} \)
(or \( -y_{21} \)) and both \( z''_{22} \) (or \( y''_{22} \)) and \( Z_s \) (or \( Y_p \)) are positive
real. The decomposition indicated in (1-11) or (1-12) per-
mits a network realization of the form shown in Fig. 1.2 or
Fig. 1.3, respectively.

Fig. 1.2 Network arrangement for a prescribed transfer
function of the set \( T_1(s) \)

\[
\begin{align*}
N' & \\
\text{With} & \\
\text{parameters} & \\
z_{21}' = z_{21} \\
z_{22}' = z_{22} - Z_s
\end{align*}
\]

Fig. 1.3 Network configuration for a prescribed transfer
function of the set \( T_2(s) \)

\[
\begin{align*}
N'' & \\
\text{With} & \\
\text{parameters} & \\
y_{21}' = y_{21} \\
y_{22}' = y_{22} - Y_p
\end{align*}
\]
4) The synthesis is completed by realizing the network $N'$ of Fig. 1.2 (or the network $N''$ of Fig. 1.3) from its associated parameters $z'_{21}$ and $z'_{22}$ (or $y'_{21}$ and $y'_{22}$) in a ladder configuration. The synthesis method which will be discussed was recently introduced by Chen [1]. This method provides a well organized and unified approach to synthesizing transfer functions of RC, RL, LC, and RLC ladder networks.
CHAPTER II

CONDITIONS OF PHYSICAL REALIZABILITY

The six transfer functions, $T(s)$, defined in (1-1) and (1-2) must satisfy a number of conditions in order that they represent the transfer properties of a physical network which is made up of lumped, linear, bilateral, and passive elements arranged in a ladder configuration. Similarly, the open circuit and short circuit network parameters must meet certain conditions in order that they be the associated parameters of a physical ladder network. From the previous discussion of the basic zero sharing process and from (1-5) and (1-6), it is seen that the synthesis of a given transfer function reduces to the problem of synthesizing a pair of network parameters, $z_{21}^*$ and $z_{22}^*$, (or $-y_{21}^*$ and $y_{22}^*$.) For this reason, those conditions on the parameters will be considered which are particularly applicable to the pair of parameters $z_{21}$ and $z_{22}$ (or $-y_{21}$ and $y_{22}$.) The combined realizability conditions on both $T(s)$ and the parameters will play an important role in the development of the zero sharing method. First, these conditions aid in establishing a criterion for selecting the network parameters from $T(s)$, and second, they are instrumental in determining permissible
decompositions of the parameters $z_{22}$ or $y_{22}$ to allow the sharing of transmission zeros.

Only a summary of the realizability conditions will be given here. For proofs of these conditions, see the references [1, 2].

2.1 Conditions of Physical Realizability of the Transfer Functions, $T(s)$, With Ladder Networks [1]

The six transfer functions, $T(s)$, must satisfy the following conditions:

1) $T(s)$ is representable as a ratio of two polynomials in $s$ with real coefficients.

2) The poles of $T(s)$ if not on the $j\omega$ axis are in the left-hand half of the $s$ plane, and those on the $j\omega$ axis are simple. (Of the six transfer functions, $T(s)$, only $A^*(s)$ and $G^*(s)$ may have poles on the $j\omega$ axis.)

3) For the expressions $Z_T(s)/R_2$, $-A(s)$, $-R_2Y_T(s)$, $G(s)$, $-A^*(s)$, and $G^*(s)$ the following conditions must be satisfied: (a) All coefficients in the numerators and denominators of these expressions must be nonnegative. (b) The numerator coefficients for each of these expressions must be no greater than the corresponding denominator coefficients.

4) The residues of the transfer functions $A^*(s)$ or $G^*(s)$ at any of their imaginary poles, $s = \pm j\omega$, are imaginary.

5) The transfer functions $A^*(s)$ and $G^*(s)$ do not have a pole at either $s = 0$ or $s = \infty$. 
These conditions must hold for a transfer function if it is to be realizable in a ladder network having no mutual inductances. Also, those cases of node bridging which produce transmission zeros in the right-hand half of the s plane will not be considered. Therefore, \( T(s) \) will be a minimum phase function, and, as a sixth condition, the zeros of \( T(s) \) must be either in the left-hand half of the s plane or on the \( j\omega \) axis.

Additional restrictions are placed on the transfer functions when the network is to consist only of two kinds of elements. RC, RL, and LC network realizations will be considered separately, and the additional conditions which apply will be considered along with each individual synthesis procedure.

2.2 Conditions of Physical Realizability For the Parameters \( z_{21} \) and \( z_{22} \) or \(-y_{21} \) and \( y_{22} \) With Ladder Networks [1]

The following conditions which the network parameters must satisfy, are particularly applicable to the pairs of parameters \( z_{21} \) and \( z_{22} \) or \( y_{21} \) and \( y_{22} \):

1) \( z_{22} \) (or \( y_{22} \)) and \( z_{11} \) (or \( y_{11} \)) must be physically realizable driving point functions.

2) The poles of \( z_{21} \) (or \( y_{21} \)), \( z_{22} \) (or \( y_{22} \)), and \( z_{11} \) (or \( y_{11} \)), if not on the \( j\omega \) axis, are in the left-hand half of the s plane, and those on the \( j\omega \) axis are simple.

3) In general, the poles of \( z_{21} \) (or \( y_{21} \)) are also the
poles of $z_{22}$ (or $y_{22}$) and $z_{11}$ (or $y_{11}$), but $z_{22}$ or $z_{11}$ ($y_{22}$ or $y_{11}$) may have poles in addition to those of $z_{21}$ (or $y_{21}$). The poles of $z_{21}$ (or $y_{21}$) on the $j\omega$ axis must always be poles of $z_{22}$ (or $y_{22}$) and $z_{11}$ (or $y_{11}$).

4) All coefficients in the numerators and denominators of $z_{21}$ (or $-y_{21}$), $z_{22}$ (or $y_{22}$), and $z_{11}$ (or $y_{11}$) are nonnegative. The numerator coefficients of $z_{21}$ (or $-y_{21}$) are no greater than the corresponding numerator coefficients of $z_{22}$ (or $y_{22}$) or $z_{11}$ (or $y_{11}$) where $z_{21}$, $z_{22}$, and $z_{11}$ (or $-y_{21}$, $y_{22}$ and $y_{11}$) are placed in a form having the same denominators.

5) If $z_{22}$ (or $y_{22}$) or $z_{11}$ (or $y_{11}$) is a LC driving point function, and, therefore, a ratio of odd and even polynomials in $s$, then $z_{21}$ (or $-y_{21}$) must also be a ratio of odd and even polynomials.
CHAPTER III

TRANSFER FUNCTIONS WHICH ARE POSITIVE REAL

In the realization of a transfer function, $T(s)$, in ladder form, it is desirable to first determine whether or not $T(s)$ is positive real in addition to being physically realizable as a transfer function. In the event that $T(s)$ does satisfy the positive real condition, a simple realization may be effected using only two-terminal techniques. In the interest of completeness, this case will now be discussed.

For any of the six transfer functions under consideration, if $T(s)$ is positive real then $Q(s)/P(s)$, considered as an impedance, is a non-minimum-resistive function. This follows for $Z_T(s)$, $A(s)$, $Y_T(s)$, and $G(s)$ since these functions are not permitted to have poles on the $j\omega$ axis as discussed in Chapter II. Although $A^*(s)$ and $G^*(s)$ may have poles on the $j\omega$ axis, the residues at such poles are imaginary as the conditions of physical realizability dictate. Imaginary residues at these poles, of course, rule out the possibility of the function being positive real. One may therefore conclude that if $T(s)$ is positive real then $Q(s)/P(s)$ considered as an impedance is not a minimum resistive function.
In the following discussion a synthesis procedure is given for realizing $Z_T(s)$, $A(s)$, and $A^*(s)$. A completely dual procedure may be followed for $Y_T(s)$, $G(s)$, and $G^*(s)$. Synthesis may be carried out by first placing $Q(s)/P(s)$ in the form

$$Q(s)/P(s) = R + Z'(s)$$

(3-1)

where $Z'(s)$ is a minimum resistance driving point impedance function. When $Q(s)/P(s)$ is an RC or RL driving point function, or when all transmission zeros lie on the $j\omega$ axis, the decomposition of $Q(s)/P(s)$ is easily effected by partial fractioning. If $Q(s)/P(s)$ is a general RLC driving point function, standard methods for the series removal of a purely resistive element may be used. With $Q(s)/P(s)$ reduced as in (3-1), $z_{22}$ has the form

$$z_{22} = \frac{z_{21}}{H} (R + Z'(s)) - 1$$

(3-2)

when $Z_T(s)$ or $A(s)$ is specified, and the network is to be terminated in a normalized resistive load, $R_2 = 1$. $z_{22}$ has the form

$$z_{22} = \frac{z_{21}}{H} (R + Z'(s))$$

(3-3)

for $A^*(s)$. 
$z_{21}$ may now be chosen as a constant, $K$, and with the proper choice of values for the constants, $K$ and $H$, $z_{22}$ will be a physically realizable driving point function. In the synthesis of $Z_T(s)$ or $A(s)$ it is evident from a standpoint of physical realizability that $KR/H$ must be greater than one. When it is required that the network be synthesized within a constant multiplier it is convenient in many cases to choose a value of $K \approx 1$ and assure that the necessary condition

$$H \leq \frac{KR}{K + 1} \quad (3-4)$$

is satisfied. When $H$ is specified, an appropriate value of $K$ is easily determined from (3-4). It is seen in (3-4) that the greatest possible upper bound on $H$ is $R$. In the realization of $A^*(s)$ it is only necessary to assure that

$$H \leq R \quad (3-5)$$

in (3-3).

For $Z_T(s)$, $A(s)$, or $A^*(s)$ the resulting network has the form of Fig. 1.2 in which the network $N'$ consists of a single shunt resistance of $K$ ohms and possibly a series resistance. When equality occurs in (3-4) or (3-5) the realization will not require the series resistive element. For $Y_T(s)$, $G(s)$, or $G^*(s)$ the resulting network has the form of Fig. 1.3. Here the network $N''$ consists of a single
series resistance of 1/K ohms and possibly a shunt resistance. Equality in (3-4) or (3-5) eliminates the shunt resistive element.
CHAPTER IV

TRANSFER FUNCTIONS WITH REAL NEGATIVE TRANSMISSION ZEROS

The cases which will now be considered are those in which the resulting network is to consist of only resistive and capacitive elements, or it is to consist only of resistive elements, and inductive elements with no mutual coupling. The six transfer functions, T(s), of networks which are composed of these kinds of elements, arranged in a ladder configuration with no-node bridging, will possess the following two properties in addition to the properties of physical realizability stated in Chapter II:

1) The transmission zeros must lie on the negative portion of the σ axis including the points σ = 0 and σ = −∞.

2) The poles of T(s) must be simple and lie on the negative portion of the σ axis excluding the points σ = 0 and σ = −∞.

4.1 RC or RL Transfer Functions Which Are Positive Real

As indicated in the discussion of the general procedure to be followed, it is desirable to first determine whether or not Q(s)/P(s) is positive real. Any one of the transfer functions Z_T(s), -A(s), and -A*(s) for RC networks or -Y_T(s), G(s), and G*(s) for RL networks is positive real if and only if its reciprocal, Q(s)/P(s), is of the form
\[
\frac{Q(s)}{P(s)} = R(s) \frac{(s - p_1)(s - p_2)\cdots(s - p_n)}{(s - z_1)(s - z_2)\cdots(s - z_n)}
\]

where the \( p_i \) and \( z_j \) are real with

\[
0 > p_0 > z_1 > p_1 > z_2 > p_2 \cdots > z_n > p_n
\]

and

\[
R(s) = 1
\]

or

\[
R(s) = \frac{s - p_0}{s}.
\]

Any one of the transfer functions \(-Y_T(s), G(s), \) and \( G^*(s)\) for RC networks or \(Z_T(s), -A(s), \) and \( A^*(s)\) for RL networks is positive real if and only if its reciprocal, \( Q(s)/P(s) \), is of the form

\[
\frac{Q(s)}{P(s)} = R(s) \frac{(s - p_1)(s - p_2)\cdots(s - p_n)}{(s - z_1)(s - z_2)\cdots(s - z_n)}
\]

where the \( p_i \) and \( z_j \) are real with

\[
0 > p_1 > z_1 > p_2 > z_2 \cdots > p_n > z_n
\]

or

\[
0 > p_0 > z_1 > p_1 > z_2 \cdots > z_n > p_n.
\]
for
\[ R(s) = 1 \]

or
\[ R(s) = (s - p_0), \]

respectively. If \( T(s) \) satisfies the conditions of (4-1) or (4-2) a realization may be easily effected by following the procedure given in Chapter III. If, however, the resulting network configuration of Fig. 1.2 with a rather complex \( Z_s \) and with \( N' \) consisting of a single shunt resistor, or the network of Fig. 1.3 in which \( N'' \) consists of a single series resistor is undesirable, various degrees of zero sharing may be achieved by use of the methods which will be discussed in Section 4.3.

4.2 A Special Class of RC and RL Transfer Functions [3]

In addition to those transfer functions which are positive real, there exists another class which may be synthesized using only two-terminal techniques. For RC or RL networks, this class consists of those transfer functions, \( T(s) \), whose reciprocal, \( Q(s)/P(s) \), has the following properties:

1) \( Q(s)/P(s) \) is of the form

\[
\frac{Q(s)}{P(s)} = \frac{(s - p_1)(s - p_2)\cdots(s - p_m)}{(s - z_1)(s - z_2)\cdots(s - z_n)}, \tag{4-3}
\]

where \( n \leq m \).
2) The zeros, $p_i$, of $Q(s)$ must be simple and must lie on the negative portion of the $\sigma$ axis excluding the points $\sigma = 0$ and $\sigma = -\infty$.

3) The zeros, $z_j$, of $P(s)$ must have no greater multiplicity than two and must lie on the negative portion of the $\sigma$ axis including the points $\sigma = 0$ and $\sigma = -\infty$.

4) If one begins with the critical frequency of $Q(s)/P(s)$ nearest the origin and divides the entire group of critical frequencies into adjacent pairs (counting a double pole as two adjacent poles), then each pair of critical frequencies must consist of a pole and a zero.

The realization of transfer functions which satisfy these conditions may be carried out by using the method which is developed in Section 4.3. An example of the synthesis of such a transfer function is given in Section 4.5.

4.3 Synthesis of $T_1(s)$ With RC Networks

Attention is now turned to the problem of synthesizing an RC network for which $Z_T(s)$, $-A(s)$, or $-A^*(s)$ is specified. As mentioned previously, $T_1(s)$ is used to represent this set. The only restriction is that $T_1(s)$ be physically realizable with resistive and capacitive elements as described at the beginning of the present chapter. The transfer impedance or current-ratio function, $T_1(s)$, is given as a ratio of two polynomials in the form
\[ T_1(s) = H \frac{P(s)}{Q(s)} \]  

\[ = H \frac{(s - z_1)(s - z_2) \ldots (s - z_n)}{(s - p_1)(s - p_2) \ldots (s - p_m)} \]

with poles and zeros on the \(-s\) axis and \(m \geq n\). The expression for \(T_1(s)\) in terms of the open circuit parameters when solved for \(z_{22}\) is

\[ z_{22} = z_{21} \frac{Q(s)}{HP(s)} - 1 \]

\[ z_{22} = K/H \frac{N(s) Q(s)}{D(s) P(s)} - 1 \]

\[ = K/H \frac{Q(s)}{P'(s)} - 1 \]  

for \(T_1(s) = Z_T(s)\) or \(T_1(s) = -A(s)\) and a normalized termination, \(R_2 = 1\), or

\[ z_{22} = z_{21} \frac{Q(s)}{HP(s)} \]

\[ = K/H \frac{Q(s)}{P'(s)} \]  

for \(T_1(s) = A^*(s)\). As suggested in the outline of the general procedure to be followed, choosing the open circuit parameters \(z_{21}\) and \(z_{22}\) is the initial step in the network
realization. $z_{21}$ should be chosen as a rational function of the form

$$z_{21} = K \frac{N(s)}{D(s)} = \frac{K(s-z_1)(s-z_2)\ldots(s-z_{n-q+k})}{(s-\gamma_1)(s-\gamma_2)\ldots(s-\gamma_k)} \quad (4-7)$$

in which both poles and zeros are real. The transmission zeros $z_1, z_2, \ldots, z_{n-q+k}$ are chosen according to a set of rules which will be given and are not necessarily the first $n-q+k$ zeros of $T_1(s)$ given in (4-4). For $Z_T(s)$ or $A(s)$, $q = m$, and for $A^*(s)$, $q$ may take on either of two values, $q = m$ or $q = m + 1$, as will become evident in the following discussion. The choice of poles and zeros of $z_{21}$ is dictated by the following considerations:

1) The zeros of $z_{21}$ are selected from the set of transmission zeros of the given transfer function as indicated in (4-7).

2) The poles and zeros of $z_{21}$ must lie on the $-\sigma$ axis including the point $\sigma = 0$. Poles of $z_{21}$, however, are not permitted at $s = \infty$.

3) The poles of $z_{21}$ must not coincide with the zeros of $Q(s)$.

4) The poles and zeros of $z_{21}$ must be chosen in such a way that the product $z_{21}Q(s)/P(s)$ is an RC driving point function of the form
\[
\frac{Q(s)}{P(s)} = K \frac{N(s) Q(s)}{D(s) P(s)} = K \frac{Q(s)}{P'(s)}
\]

\[
= K \frac{(s-p_1)(s-p_2)\ldots(s-p_m)}{(s-\gamma_1)(s-\gamma_2)\ldots(s-\gamma_k)(s-z_{k+1})\ldots(s-z_q)}
\]

in which the denominator polynomial may or may not have a simple zero at \( s = 0 \). The set of transmission zeros, \( z_{k+1}, z_{k+2}, \ldots, z_q \) consists of those finite transmission zeros which do not belong to the set, \( z_1, z_2, \ldots, z_{n-q+k} \). The poles and zeros of \( z_{21} Q(s)/P(s) \) must alternate on the \(-\sigma\) axis with a pole as the leading singularity. The possible values of \( q \), \( q = m \) for \( Z_T(s) \) or \( \Lambda(s) \) and \( q = m \) or \( q = m + 1 \) for \( \Lambda^*(s) \), become evident when (4-8) is substituted in (4-5) and (4-6).

5) In order to take full advantage of the simplification which is possible using this technique one should assign as few poles and zeros to \( z_{21} \) as possible and still satisfy condition 4 above. Although such a choice of \( z_{21} \) is not essential, it does assure that two-terminal techniques will be used as much as possible throughout the remainder of the synthesis.

With \( z_{21} \) chosen according to the considerations above, one next places the product \( z_{21} Q(s)/P(s) \) in partial-fractioned form. \( z_{22} \) may then be represented as
\[ z_{22} = \frac{K}{H} \left[ \left( 1 + \frac{\alpha_1}{s-\gamma_1} + \cdots + \frac{\alpha_k}{s-\gamma_k} \right) + \left( \frac{\alpha_{k+1}}{s-z_{k+1}} + \cdots + \frac{\alpha_m}{s-z_m} \right) \right]^{-1} \]

(4-9)

for \( z_T(s) \) or \( A(s) \), and

\[ z_{22} = \frac{K}{H} \left[ (\alpha_0 + \frac{\alpha_1}{s-\gamma_1} + \cdots + \frac{\alpha_k}{s-\gamma_k}) + \left( \frac{\alpha_{k+1}}{s-z_{k+1}} + \cdots + \frac{\alpha_q}{s-z_q} \right) \right] \]

(4-10)

for \( A^*(s) \). In (4-10), \( \alpha_0 = 1 \) for \( q = m \), or \( \alpha_0 = 0 \) for \( q = m + 1 \). At this point it is necessary to determine a value for \( K/H \) in (4-9) or (4-10), and it is well to consider briefly the significance of this choice. The constant \( H \) determines the overall magnitude of the voltage or current response for a given voltage or current excitation. The constant \( K \) was introduced as a scale factor of \( z_{21} \) in order to provide flexibility in synthesizing the networks \( N' \) and \( N'' \) in Fig. 1.2 and Fig. 1.3 respectively. This flexibility is desirable since the network \( N' \) or \( N'' \) is to be synthesized by one of the well known synthesis techniques for ladder networks which is directed toward synthesizing a given transfer function within a multiplicative constant.

In a particular problem, the constant \( H \) may be specified or it may simply be required that the synthesis be carried out to within a constant multiplier. When \( H \) is specified or when a high value of \( H \) is desired, the permissible range of values for \( H \) is of interest. An upper limit
is placed on $H$ by the conditions of physical realizability for transfer functions as stated in property number 4 of Chapter II. Additional limits are placed on $H$ by the topological form of the network realization and considerable attention has been given to determining these limits for specific network configurations [4].

Two sets of limits on $K$ and $H$ exist after the zero sharing step and will now be considered. The first set is for the special case in which the synthesis is to be carried out entirely on a two-terminal basis. Here, $O(s)/P(s)$ must satisfy the conditions of section 4.2. After the application of the zero sharing techniques just discussed, a pair of open circuit parameters $z_{21}$ and $z_{22}$ will result in which $z_{21}$ is positive real. $z_{21}$ may be represented in partial fractioned form as

$$z_{21} = K \left[ \beta_0 + \frac{\beta_1}{s - \gamma_1} + \frac{\beta_2}{s - \gamma_2} + \ldots + \frac{\beta_k}{s - \gamma_k} \right], \quad (4-11)$$

where $\beta_0 = 1$ if $q = n$ or $\beta_0 = 0$ if $q \neq n$. For the realization of the network $N'$ in an L network as shown in Fig. 4.1 in which $Z_1$ and $Z_2$ are RC driving point impedances, it is both necessary and sufficient that

$$K/H - 1 \geq K\beta_0', \quad (4-12)$$
and that

\[ \frac{a_i}{\beta_i} \geq \gamma \quad \text{for } i = 1, 2, \ldots, k, \quad (4-13) \]

for \( Z_T(s) \) or \( A(s) \) and a resistive termination, \( R_2 = 1 \).

This realization for a specified \( A^*(s) \) requires that

\[ \frac{a_i}{\beta_i} \geq \gamma \quad \text{for } i = 0, 1, 2, \ldots, k. \quad (4-14) \]

![Network Diagram](image)

**Fig. 4.1** L network configuration

When synthesizing \( Z_T(s) \) or \( A(s) \), one may easily select a suitable value of \( \gamma \) from (4-13). This value of \( \gamma \) will determine an acceptable range of \( K \) as indicated in (4-12). \( K \) may then be chosen within this range on the basis of producing desirable element values in the normalized network.

If the value of \( \gamma \) is of secondary importance, one may first make a desirable choice of \( K \) and accept further restrictions on \( \gamma \). When synthesizing \( A^*(s) \), \( \gamma \) may be selected within the limits set by (4-14). The value of \( K \) is not restricted and may simply be chosen in such a way as to
produce desirable element values in the normalized network.

The second set of limits on $K$ and $H$ is for the case in which the network $N'$ is to be synthesized in a general ladder configuration of resistive and capacitive elements. For $Z_T(s)$ or $A(s)$, the open circuit parameters for the network $N'$ are

$$z'_{21} = z_{21}$$  \hfill (4-15)

and

$$z'_{22} = \frac{K}{H} \left[ 1 + \frac{a_1}{s-\gamma_1} + \frac{a_2}{s-\gamma_2} + \cdots + \frac{a_k}{s-\gamma_k} \right]^{-1} \right]. \hfill (4-16)$$

If each parameter is placed in the form of a ratio of two polynomials with the numerator in expanded form, property 4 for the network parameters may then be applied to determine limits on $H$ and $K$. It is generally desirable when synthesizing $N'$ in a general ladder configuration to select a value for the ratio $K/H$ but to leave the particular values of $K$ and $H$ unspecified. The value of $K/H$ may be selected in such a way as to control the spread of element values in the resulting network. For realizability it is necessary, but not sufficient, that the condition

$$K/H \geq 1$$  \hfill (4-17)
be satisfied. Equality may occur in (4-17) only when \( m > n \). The manner in which \( K \) and \( H \) may be determined may be best explained with the illustrations which will be given in section 4.5. In synthesizing \( A*(s) \) the role of \( K/H \) is not critical and any convenient value of \( K/H \) may be selected.

With the ratio \( K/H \) determined one continues the synthesis by realizing \( Z_s \) as an RC driving point impedance. From (4-9) or (4-10) \( Z_s \) is given by

\[
Z_s = \frac{\frac{\alpha_{k+1}}{s - z_{k+1}} + \cdots + \frac{\alpha_m}{s - z_m}}{K/H}.
\]

Finally, the network \( N' \) is synthesized either by standard two-terminal techniques for RC networks or by the methods to be discussed in section 4.4.

4.4 Synthesis of the Network \( N' \) as an RC Ladder Network

In this section a procedure will be described for synthesizing the network \( N' \), whose parameters are \( z'_{21} \) and \( z'_{22} \). As explained earlier, the driving point impedance \( z'_{22} \) is obtained by removing the series impedance \( Z_s \) from \( z_{22} \), and \( z'_{21} = z'_{21} \). The method which will be described was introduced by Chen [1]. Although it produces the same results as zero-shifting methods, it has two distinct advantages. First, it is more straightforward in its application than the zero-shifting methods, and, second, it may be extended
to the realization of transfer functions which have complex transmission zeros \([1]\). This extension will be described in Chapter VI. The underlying principles which will be considered now are applicable to LC, RC, RL, and certain RLC synthesis problems.

The basic approach to synthesizing \(N'\) consists of the following two steps. First, a network configuration is predicted which consists of a number of basic transmission-zero-producing sections. Second, \(z'_{22}\) is synthesized as a driving point function having the predicted configuration. The synthesis of \(z'_{22}\) is carried out section by section in such a way as to assure that each section produces the desired transmission zero.

Associated with the network \(N'\) is the pair of open-circuit parameters \(z'_{21}\) and \(z'_{22}\). Due to the application of the zero sharing technique of section 4.3, \(z'_{21}\) and \(z'_{22}\) have identical denominators. Therefore, all transmission zeros of the network \(N'\) are the zeros of \(z'_{21}\), and may be easily determined by inspection. These transmission zeros will be produced by either shunt or series branches of the standard ladder configuration of Fig. 4.2. This configuration which is the assumed form of \(N'\) will have transmission zeros which are produced either by a pole of a series impedance \(Z_k\) \((k \neq 0)\) or by a pole of a shunt admittance \(Y_i\). In general, each finite nonzero pole of \(Z_k\) \((k \neq 0)\) or \(Y_i\) produces a transmission zero. However, several poles of
Fig. 4.2 Standard ladder configuration for prescribed $z_{21}$ and $z_{22}$

$Z_k$ or $Y_1$ at $s = 0$ or $s = \infty$ may contribute to a simple transmission zero at $s = 0$ or $s = \infty$, respectively.

The ladder structure of Fig. 4.2 is considered here to be made up of a predicted configuration of certain building blocks. These building blocks or generalized basic sections are shown in Fig. 4.3. Each of the two types of basic sections consists of a "principal branch" which is shown as a shaded box and an "auxiliary branch" which appears as a clear box. In general, the principal branch is used to produce either one, or a complex conjugate pair of finite, nonzero value transmission zeros. The auxiliary branch contributes to a transmission zero at $s = 0$ or $s = \infty$, or it may make no contribution to a transmission zero.

Fig. 4.3 Generalized basic sections (a) A type 1 section (b) A type 2 section
Attention is now turned to the problem of synthesizing \textit{N}' as a ladder network which is made up of only resistive and capacitive elements. The first step is to predict a possible ladder configuration which has the form shown in Fig. 4.2 and consists of basic RC sections and possibly some additional RC branches. There are two basic RC sections, type 1 and type 2, as depicted in Fig. 4.4. The subscript or superscript, \( k \), indicates that this is the \( k \)th section in the ladder configuration with the sections numbered consecutively from right to left. In each basic section of Fig. 4.4 a transmission zero is produced at \( s = z_k \). In Fig. 4.4 (a), if \( R_a^{(k)} \) is set equal to zero, the principal branch of this type 1 RC section contributes to a transmission zero at \( s = \infty \). In Fig. 4.4 (b), if \( R_b^{(k)} \) is set equal to infinity, the principal branch of this type 2 RC section contributes to a transmission zero at \( s = 0 \). Transmission zeros may therefore be produced at any point on the negative \( \sigma \) axis, including \( s = 0 \) and \( s = \infty \), by the use of type 1 and type 2
RC sections. The prediction step is accomplished as follows:

1) Examine \( z_{21} \) to determine the transmission zeros of \( N' \).

2) Arrange type 1 and/or type 2 RC sections (and possibly some additional RC branches) in the ladder configuration of Figure 4.2. (Note that a shunt element appears at the extreme left, and that \( Z_0 \) may or may not be present as indicated by its presence in dotted lines.)

3) In the arrangement described in 2, the principal branch of each RC section is assigned one transmission zero which it is to produce.

4) The auxiliary branch of each RC section makes no contribution to a transmission zero.

A number of predicted configurations are possible for a given set of transmission zeros. In this particular situation two of the possible predictions are (a) a configuration consisting exclusively of type 2 RC sections with a shunt resistive branch at the far left, and (b) a configuration consisting exclusively of type 1 RC sections. It should be understood that some predictions will be realizable, while others may be nonrealizable. The appearance of negative elements will indicate a nonrealizable prediction. A simple rearrangement of the order of the transmission zeros will frequently bring a nonrealizable situation into a realizable form.

The synthesis of the predicted network follows an orderly procedure in which one basic section is realized at
a time. One begins by synthesizing the first section at the extreme right \((k = 1)\). For this section the impedance \(Z_1 = Z'_{22}\) or admittance \(Y_1 = \frac{1}{Z'_{22}}\) is known, and, as a result of prediction, a particular transmission zero has been assigned to this section. This is sufficient information to determine the elements of the first section. Upon removing the elements of this first section from the impedance \(Z_1\) or admittance \(Y_1\), one obtains the impedance or admittance looking into the next section, \(Z_2\) or \(Y_2\), respectively. This process is repeated for the entire predicted configuration.

The technique for synthesizing a Type 1 RC section is as follows:

1) Evaluate \(R_b^{(k)}\) where

\[
R_b^{(k)} = Z_k(s) \bigg|_{s = z_k}.
\] (4-19)

2) The admittance \(Y^*_k\) looking into the remaining portion of the ladder after \(R_b^{(k)}\) is removed is

\[
Y^*_k = \frac{1}{k} \left( Z_k - R_b^{(k)} \right).
\] (4-20)

3) \(Y^*_k\) is of the form

\[
Y^*_k = \frac{M_k s}{s - z_k} + Y_k + 1,
\] (4-21)
where

\[ M_k = \left[ \frac{(s - z_k)}{s} Y_k^* \right] \]

The elements of \( Y_a \) are then

\[ R_a^{(k)} = \frac{1}{M_k} \text{ and } C_a^{(k)} = \frac{M_k}{-z_k^*} \]  \hspace{1cm} (4-22)

4) The impedance looking into the next basic section is

\[ z_{k+1} = \frac{1}{Y_k^* - \frac{M_k s}{s - z_k^*}} \]  \hspace{1cm} (4-23)

The technique for synthesizing a Type 2 RC section is

1) Evaluate \( G_a^{(k)} \), where

\[ G_a^{(k)} = Y_k(s) \bigg|_{s = z_k^*} \]  \hspace{1cm} (4-24)

2) The impedance \( Z_k^* \) looking into the remaining portion of the ladder after \( G_a^{(k)} \) is removed is

\[ Z_k^* = \frac{1}{Y_k - G_a^{(k)}} \]  \hspace{1cm} (4-25)

3) \( Z_k^* \) is of the form
\[ Z^*_k = \frac{N_k}{s - z_k} + Z_k + 1, \]

where

\[ N_k = \left[ (s - z_k) Z^*_k \right]_{s= z_k} \]

The elements of \( Z_b \) are then

\[ C_b = \frac{1}{N_k} \quad \text{and} \quad R^{(k)}_b = \frac{N_k}{-z_k} \]  

(4-27)

4) The admittance looking into the next basic section is

\[ Y_k + 1 = \frac{1}{Z^*_k - \frac{N_k}{s - z_k}} \]  

(4-28)

4.5 Illustrations of the Synthesis of \( Z_T(s) \), \(-A(s)\), or \(-A^*(s)\) With RC Networks

For the first illustration a transfer function which belongs to the class described in section 4.2 will be considered. Let it be required to synthesize

\[ Z_T(s) = H \frac{(s + 2)(s + 3)}{(s + 1)(s + 4)} \]  

(4-29)

in a network with a normalized load \( R_2 = 1 \).
In accordance with the procedure given in section 4.3, \( z_{21} \) is chosen as

\[
z_{21} = K \frac{s + 2}{s}. \tag{4-30}
\]

Then

\[
z_{22} = K/H \left[ \frac{(s + 1)(s + 4)}{s(s + 3)} \right]^{-1}. \tag{4-31}
\]

Upon partial fractioning \( z_{21}Q(s)/P(s) \), \( z_{22} \) becomes

\[
z_{22} = K/H \left[ 1 + \frac{4/3}{s} + \frac{2/3}{s+3} \right]^{-1}. \tag{4-32}
\]

The limits on \( K \) and \( H \) are set by (4-11) and (4-12) as

\[
K/H - 1 \geq K \tag{4-33}
\]

and

\[
2/3 \geq H \tag{4-34}
\]

If \( H \) is chosen equal to its limiting value, that is \( H = 2/3 \), then

\[
K \geq \frac{H}{1 - H} = 2. \tag{4-35}
\]

\( K \) may be chosen equal to its limiting value, 2, in which case \( z_{22} \) and \( z_{21} \) become
\[ z_{22} = 2 + \frac{4}{s} + \frac{2}{s + 3} \]  \hspace{1cm} (4-36)

and

\[ z_{21} = 2 + \frac{4}{s}. \]

When the private pole of \( z_{22} \) is removed, the parameters of the network \( N' \) are

\[ z'_{22} = 2 + \frac{4}{s} \]

and

\[ z'_{21} = 2 + \frac{4}{s} \]  \hspace{1cm} (4-37)

The resulting network for \( Z_T(s) \) as given in (4-29) is shown in Fig. 4.5.

Fig. 4.5 Ladder network for \( Z_T(s) \) as given in (4-29)

For the second illustration a transfer function will be considered whose realization requires that the network \( N' \) have the form of a ladder with more than one section.
Let it be required to synthesize

\[
-A(s) = H \frac{P(s)}{Q(s)} = H \frac{(s+2)(s+6)(s+7)(s+9)(s+10)}{(s+1)(s+3)(s+5)(s+8)(s+12)}
\]  

(4-38)

in a network which is terminated in a normalized load, 
\[R_2 = 1.\]

As explained in section 4.3, one possible choice of 
\[z_{21}\] is

\[
z_{21} = K \frac{(s + 7)(s + 9)}{s(s + 4)}
\]  

(4-39)

For this choice of \(z_{21}\), \(z_{22}\) becomes

\[
z_{22} = K/H \left[ \frac{(s+1)(s+3)(s+5)(s+8)(s+12)}{s(s+2)(s+4)(s+6)(s+10)} \right]^{-1}.
\]  

(4-40)

The product, \(z_{21} \cap(s)/P(s)\), is next partial fractioned, which places \(z_{22}\) in the form

\[
z_{22} = K/H \left[ \frac{3}{s} + \frac{45/32}{s+2} + \frac{1}{s+4} + \frac{15/16}{s+6} + \frac{21/32}{s+10} \right]^{-1}.
\]  

(4-41)

At this point it is necessary to select a value of \(K/H\). Property 4 for the network parameters, given in Chapter II, states that the numerator coefficients of \(z'_{22}\) must be greater than or equal to the corresponding numerator coefficients of \(z'_{21}\). Application of this property places the following
condition on $H$.

$$H \leq \frac{12}{63} \quad .$$  \hfill (4-42)

For this value of $H$, $K/H$ must satisfy the condition

$$K/H \geq \frac{63}{51} \quad .$$  \hfill (4-43)

In selecting a desirable $K/H$ within this wide range of permissible values, the following three facts are helpful:

1) $z'_{22}$ is given by

$$z'_{22} = \frac{K}{H} \left[ 1 + \frac{3}{s} + \frac{1}{s+4} \right] -1 \quad .$$  \hfill (4-44)

As $K/H$ decreases from infinity to its lower limit, the zeros of $z'_{22}$ move to the left along the $-\sigma$ axis from their initial positions at the zeros of $f(s)$, where

$$f(s) = 1 + \frac{3}{s} + \frac{1}{s+4} \quad .$$  \hfill (4-45)

The zero movement is covered in detail in reference [5].

2) To prevent the occurrence of a large spread in element values in the normalized network one should avoid selecting a value of $K/H$ which places a zero of $z'_{22}$ very near to, but not touching, the transmission zero which is to be produced by the first section at the right of $N'$. 
3) This shift in the zero locations of $z'_{22}$ is identical to the shift which occurs when synthesizing a type 1 RC section.

With these facts in mind the synthesis is continued by simply selecting $K/H = 2$, the next larger integer value above the limiting value of $63/51$, for $H = 12/63$. As the first step in synthesizing the network $N'$, a network configuration is predicted as shown in Fig. 4.6. The realization of the first basic section at the right of $N'$ is begun by evaluating $Z_1(s) \bigg|_{s = -7} = z'_{22}(s) \bigg|_{s = -7}$ or $Z_1(s) \bigg|_{s = -9} = z'_{22}(s) \bigg|_{s = -9}$. Negative values are found for both $s = -7$ and $s = -9$ which indicates that a negative resistance must appear in the position of $R_b^{(1)}$ in Fig. 4.6. Therefore, for $K/H = 2$, the network prediction of Fig. 4.6 is nonrealizable. According to remarks 1 and 2 above, a different choice of $K/H$ may make the prediction of Fig. 4.6 realizable. (See reference [5]).

A second choice, $K/H = 3$, eliminates the problem just encountered. For this choice, the open circuit parameters of $N'$ are

$$z'_{22} = 2 + \frac{9}{s} + \frac{3}{s + 4} = \frac{2s^2 + 20s + 36}{s(s + 4)} \quad (4-46)$$

and

$$z'_{21} = z_{21} = K \frac{(s + 7)(s + 9)}{s(s + 4)} \quad (4-47)$$
Fig. 4.6 Network prediction for N'

For the network prediction of Fig. 4.6, the synthesis of N' is carried out as follows:

\[ Z_1 = z'_{22} = \frac{2s^2 + 20s + 36}{s(s + 4)} \]  \hspace{1cm} (4-48)

\[ R_b^{(1)} = 2/5 \quad R_a^{(1)} = \frac{4 \times 13}{25} \quad C_a^{(1)} = \frac{25}{4 \times 13 \times 9} \]

\[ Z_2 = \frac{8 \times 13}{15} \left[ \frac{s + 5/2}{s} \right] \]

\[ R_b^{(2)} = \frac{12 \times 13}{5 \times 7} \quad R_a^{(2)} = \frac{4 \times 13}{3 \times 7} \quad C_a^{(2)} = \frac{3}{4 \times 13} \]

The impedance \( Z_s \), whose position in the network is shown in Fig. 1.2, is given by

\[ Z_s = \frac{9}{32} \left[ \frac{15}{s + 2} + \frac{10}{s + 6} + \frac{7}{s + 10} \right] \]  \hspace{1cm} (4-49)
The realization of \( Z_s \) may be carried out by any of the standard synthesis procedures for RC driving point functions. The resulting network is shown in Fig. 4.7

![Network N'](image)

**Fig. 4.7** Ladder network for \( A(s) \) as given in (4-38) with \( K/H = 3 \)

The situation frequently occurs in which \( K/H \) can be chosen so that \( z_{22}' \) will possess a zero at one of the transmission zeros which are to be produced by the network \( N' \). This choice of \( K/H \) eliminates the need for the auxiliary branch in the first basic section at the right of network \( N' \). The possibility of choosing \( K/H \) on this basis depends on the value of the expression within the first parentheses of (4-9). To use this criterion in choosing \( K/H \), it is necessary that the inequality
be satisfied at one or more of the transmission zeros assigned to $z_{21}$. Some flexibility in satisfying this condition does exist, since a great many choices of $N(s)/D(s)$ are possible, each of which meet the restrictions set by conditions 1 through 5 of section 4.3. However, it is difficult to select $N(s)/D(s)$ on this basis. If (4-50) is satisfied, $K/H$ may be chosen as

$$K/H = \left[ 1 + \frac{a_1}{s - \gamma_1} + \frac{a_2}{s - \gamma_2} + \cdots + \frac{a_k}{s - \gamma_k} \right]^{-1} \bigg|_{s = z_1}$$

(4-51)

where $z_1$ is the transmission zero which is to be produced by the right most section of $N'$. The remainder of the synthesis is then carried out as described in section 4.4.

The application of the above criterion in choosing $K/H$ will now be illustrated for the transfer function given in (4-38). The expression indicated in (4-50) is first evaluated at each of the transmission zeros to be produced by $z_{21}$. From (4-39) and (4-41)

$$\left[ 1 + \frac{3}{s} + \frac{1}{s + 4} \right] \bigg|_{s = -7} = \frac{5}{21}$$

(4-52)

and
are evaluated. For \( K/H \) chosen as

\[ K/H = 15/7 \]  \hspace{1cm} (4-54)

\( z_{22} \) becomes

\[ z_{22} = \frac{8}{7} + \frac{45}{s} + \frac{15}{s+2} + \frac{15}{s+4} + \frac{15}{s+6} + \frac{15}{s+10} \]  \hspace{1cm} (4-55)

With the series removal of the impedance \( z_s \) from \( z_{22} \), \( z'_{22} \) becomes

\[ z'_{22} = \frac{8}{7} + \frac{45}{s} + \frac{15}{s+4} \]  \hspace{1cm} (4-56)

A network is then predicted having the form given in Fig. 4.6. The realization, however, will not require the resistive element \( R_b^{(1)} \). The synthesis procedure given in equations (4-19) through (4-23) is next carried out. The resulting network is shown in Fig. 4.8.

4.6 Synthesis of \( T_2(s) \) With RC Networks

A synthesis procedure will now be developed for the realization of a network for which \( Y_T(s) \), \( G(s) \), or \( G^*(s) \) is specified. The only restriction on \( T_2(s) \) is that it must
be a physically realizable transfer function of an RC network, as described at the beginning of this chapter. The procedure for synthesizing $T_2(s)$ is the same as that which was developed for $T_1(s)$ in section 4.3 except for variations in the permissible pole and zero locations in choosing the network parameters. The transfer admittance or voltage-ratio function is given in the form

$$T_2(s) = H \frac{P(s)}{Q(s)}$$

(4-57)

$$= H \frac{(s - z_1)(s - z_2)\ldots(s - z_n)}{(s - p_1)(s - p_2)\ldots(s - p_m)}$$
with poles and zeros on the negative $\sigma$ axis and $m \geq n$.

From the expression for $T_2(s)$ in terms of the short circuit parameters, $y_{22}$ is

$$y_{22} = -y_{21} \frac{Q(s)}{HP(s)} -1$$

$$= K/H \frac{N(s) \cdot Q(s)}{D(s) \cdot P(s)} -1$$

$$= K/H \frac{Q(s)}{P'(s)} -1$$

for $T_2(s) = Y_T(s)$ or $T_2(s) = G(s)$ and a normalized termination, $R_2 = 1$, or

$$y_{22} = -y_{21} \frac{Q(s)}{HP(s)}$$

$$= K/H \frac{Q(s)}{P'(s)}$$

for $T_2(s) = G*(s)$. $-y_{21}$ is chosen as

$$-y_{21} = K \frac{N(s)}{D(s)} = K \frac{(s-z_1)(s-z_2)\ldots(s-z_{n-q+k})}{(s-\gamma_1)(s-\gamma_2)\ldots(s-\gamma_k)},$$

where $q = m$ or $q = m - 1$. Equation (4-60) indicates that the number of transmission zeros assigned to $-y_{21}$ is $n-q+k$. They are not necessarily the first $n-q+k$ zeros of $T_2(s)$ as
given in (4-57). The poles and zeros of $y_{21}$ are chosen on the basis of the following considerations:

1) The zeros of $y_{21}$ must be selected from the set of transmission zeros of the given transfer function. Although they will not be discussed here, certain cases do arise in which the choice of other zeros is advantageous.

2) The zeros of $y_{21}$ may lie anywhere on the negative $\sigma$ axis, including $s = 0$ and $s = \infty$. The poles of $y_{21}$ must lie also on the negative $\sigma$ axis including the point $s = \infty$ but excluding the point $s = 0$.

3) The poles of $y_{21}$ must not coincide with the zeros of $Q(s)$.

4) The poles and zeros of $y_{21}$ must be chosen in such a way that the product $-y_{21}Q(s)/P(s)$ is an RC driving point admittance function of the form

$$
-y_{21} \frac{Q(s)}{P(s)} = K \frac{(s-p_1)(s-p_2)\ldots(s-p_m)}{(s-\gamma_1)(s-\gamma_2)\ldots(s-\gamma_k)(s-z_{k+1})\ldots(s-z_q)}
$$

(4-61)

in which $q = m$ or $q = m - 1$. The set of transmission zeros, $z_{k+1}, z_{k+2}, \ldots, z_q$ consists of those finite transmission zeros which do not belong to the set $z_1, z_2, \ldots, z_{n-q+k}$. The poles and zeros of $-y_{21}Q(s)/P(s)$ must alternate on the negative $\sigma$ axis with a zero as the leading singularity. This of course excludes the possibility of $y_{21}Q(s)/P(s)$ having a pole at $s = 0$.

5) One should assign as few poles and zeros as possible
to $y_{21}$ and still satisfy conditions 1 through 4 above. As before, various degrees of zero sharing between $y_{21}$ and $y_{22}$ may be produced by assigning more transmission zeros to $y_{21}$ than the minimum required number.

After choosing $y_{21}$ in accordance with the above considerations one next places the product $-y_{21} Q(s)/P(s)$ in partial fractioned form. $y_{22}$ may then be represented as

$$y_{22} = \frac{K}{H} \left[ \left( \frac{\alpha_{-1} s + \alpha_0}{s - \gamma_1} + \frac{\alpha_1 s}{s - \gamma_2} + \cdots + \frac{\alpha_k s}{s - \gamma_k} \right) + \left( \frac{\alpha_{k+1} s}{s - z_{k+1}} + \cdots + \frac{\alpha_q s}{s - z_q} \right) \right]^{-1}$$

(4-62)

for $Y_T(s)$ or $G(s)$ and

$$y_{22} = \frac{K}{H} \left[ \left( \frac{\alpha_{-1} s + \alpha_0}{s - \gamma_1} + \frac{\alpha_1 s}{s - \gamma_2} + \cdots + \frac{\alpha_k s}{s - \gamma_k} \right) + \left( \frac{\alpha_{k+1} s}{s - z_{k+1}} + \cdots + \frac{\alpha_q s}{s - z_q} \right) \right]^{-1}$$

(4-63)

for $G*(s)$. In (4-62) and (4-63), $\alpha_{-1} = 1$ for $q = m - 1$, or $\alpha_{-1} = 0$ for $q = m$.

At this point it is necessary to assign a value to $K/H$. As in section 4.3, two different sets of limits on $K$ and $H$ will be considered. The first is for the more restrictive
case, in which the synthesis of $N'$ is to be carried out entirely on a two-terminal basis. Here, $Q(s)/P(s)$ must satisfy the conditions of section 4.2. After the application of the zero sharing techniques above, a pair of open circuit parameters, $y_{21}$ and $y_{22}$, will result in which $y_{21}$ is an RC driving point admittance. $y_{21}$ may be represented as

$$-y_{21} = K \left[ \beta_{-1} s + \beta_0 + \frac{\beta_1 s}{s - \gamma_1} + \cdots + \frac{\beta_k s}{s - \gamma_k} \right], \quad (4-64)$$

where $\beta_{-1} = 1$ for $n = q + 1$, or $\beta_{-1} = 0$ for $n \neq q + 1$. For the realization of the network $N''$ in an L network as shown in Fig. 4.9, in which $Z_1$ and $Z_2$ are RC driving point impedances, it is both necessary and sufficient that

$$(K/H)\alpha_0 - 1 \geq K\beta_0 \quad (4-65)$$

and that

$$\frac{\alpha_i}{\beta_i} \geq H \quad \text{for } i = -1, 1, 2, \ldots, k \quad (4-66)$$

for $Y_T(s)$ or $G(s)$ with a resistive termination, $R_2 = 1$. Such a realization for a specified $G^*(s)$ requires that

$$\frac{\alpha_i}{\beta_i} \geq H \quad \text{for } i = -1, 0, 1, 2, \ldots, k. \quad (4-67)$$

When synthesizing $Y_T(s)$ or $G(s)$, $H$ may be selected in
accordance with (4-66). A rearrangement of (4-65),

\[
\frac{K/H}{\alpha_0 - \omega_0^2} \geq \frac{1}{\omega_0^2 - \omega_0^2}.
\]  

(4-68)

shows the dependence of K/H on this choice of H. If the form of the network N' is restricted to that shown in Fig. 4.9, the maximum value of H for this configuration may be realized by equating \( \alpha_0 \) to its upper limit and then choosing K/H according to (4-68). A second alternative is to choose K/H in such a way as to reduce the spread in element values in the resulting network. Examination of (4-62) reveals very quickly the proper choice of K/H to reduce this spread.

When \( G^*(s) \) is to be synthesized, H may be selected within the limits set by (4-67). The value of K is not restricted and may be chosen so as to produce convenient element values in the normalized network.

The second set of limits on K and H is applicable when
\( N'' \) is to be synthesized in a general ladder configuration composed of resistive and capacitive elements. Property 4 for the network parameters may be applied to \( y''_{21} \) and \( y''_{22} \) to determine the limits on \( H \) and \( K \). Here \( y''_{21} \) and \( y''_{22} \) are given by

\[
y''_{21} = y_{21}
\]

and

\[
y''_{22} = \frac{K}{H} \left[ a_{-1}s + a_0 + \frac{a_1s}{s - \gamma_1} + \frac{a_2s}{s - \gamma_2} + \ldots + \frac{a_k s}{s - \gamma_k} \right]. \tag{4-70}
\]

When synthesizing \( Y_T(s) \) or \( G(s) \), it is usually desirable to choose a value for \( K/H \) but to leave the individual values of \( K \) and \( H \) unspecified. The value of \( K/H \) may be selected in such a way as to control the spread of element values in the resulting network. In the synthesis of \( G^*(s) \), the value of \( K/H \) is not critical and any convenient value may be chosen.

After a suitable value of \( K/H \) has been selected, the synthesis is continued by realizing \( Y_p \) as an RC driving point admittance, where

\[
Y_p = \frac{K}{H} \left[ \frac{a_{k+1}s}{s - z_{k+1}} + \ldots + \frac{a_q s}{s - z_q} \right]. \tag{4-71}
\]

The remainder of the procedure consists of synthesizing the network \( N'' \) either by standard RC two-terminal techniques or by the methods to be discussed in section 4.7.
4.7 Synthesis of the Network \( N'' \)

The method of synthesizing the network \( N'' \) is essentially the same as that described in section 4.4 for the network \( N' \). A modification of the predicted ladder configuration is necessary due to the fact that the short circuit parameters \( y_{21} \) and \( y_{22} \) are prescribed for \( N'' \) whereas the open circuit parameters were prescribed for the network \( N' \). This modification is shown in Fig. 4.10. The predicted configuration again consists of type 1 and/or type 2 RC sections (and possibly some additional RC branches), however, in this case a series element appears at the extreme left. Except for this modification, the synthesis procedure for the realization of \( N'' \) is identical to that given in section 4.4.
CHAPTER V

TRANSFER FUNCTIONS WITH PURELY IMAGINARY TRANSMISSION ZEROS

The general technique of sharing transmission zeros between \( z_{21} \) and \( z_{22} \) (or \(-y_{21} \) and \( y_{22} \)) is based on the fact that one may replace certain poles of \( Q(s)/P(s) \) with poles which make the modified function, \( Q(s)/P'(s) = [N(s)/D(s)] \times [Q(s)/P(s)] \), positive real. The success of this technique depends on the freedom which exists in making this replacement. It was seen in Chapter IV that for the RC and RL cases, poles were replaced in such a way that alternation on the negative \( \sigma \) axis would exist between zeros and poles of the modified function. Also there were certain requirements at \( s = 0 \) and \( s = \infty \). Since these were the only restrictions on the replacement of poles, considerable freedom was permitted. In contrast to this freedom which exists for the RC and RL cases and that which will be found for RLC networks, the restrictions are so great in the LC case that the zero sharing technique is of no value in synthesizing \( Z_T(s) \), \( A(s) \), \( Y_T(s) \), or \( G(s) \).

5.1 Limitations of the Zero Sharing Technique in the Synthesis of LC Networks

It will now be shown that the zero sharing approach
cannot be applied advantageously to the synthesis of \( Z_T(s) \), \( \Lambda(s) \), \( Y_T(s) \), or \( G(s) \) with LC networks. Consider \( Q(s)/P(s) \) whose reciprocal is \( Z_T(s) \), or \(-A(s)\), or \(-Y_T(s)\), or \( G(s) \) as defined in (1-1) and (1-2). From the conditions of physical realizability for these transfer functions, it is known that the zeros of \( Q(s)/P(s) \) are restricted to the left-hand half of the \( s \) plane excluding the \( j\omega \) axis. The poles of \( Q(s)/P(s) \) which, of course, represent the zeros of transmission, are to lie on the \( j\omega \) axis. The permissible pole locations on the \( j\omega \) axis for the modified function, \( Q(s)/P'(s) \), are fixed by the restriction that the residues of the function at each pole must be real.

In the event that an odd number of finite poles are to be placed on the \( j\omega \) axis, the denominator will be of the form

\[
P'(s) = s(s - j\lambda_1)(s + j\lambda_1)(s - j\lambda_2)(s + j\lambda_2) \quad \cdots \quad (s - j\lambda_m)(s + j\lambda_m). \quad (5-1)
\]

The angle associated with the residue at any one of the poles, \( j\lambda_i \), is

\[
\arg k_i = \arg Q(j\lambda_i) - \arg \left[ \frac{P'(s)}{(s-j\lambda_i)} \right]_{s=j\lambda_i}. \quad (5-2)
\]

The contribution to \( \arg k_i \) from the poles is
\[
\arg \left( \frac{p'(s)}{(s-j\lambda_1)} \right) = \arg \frac{j\lambda_1}{(s-j\lambda_1)} - \arg \frac{j\lambda_1}{(s-j\lambda_1)} - \arg \frac{(j2\lambda_1)}{(-\lambda_i^2 + \lambda_m^2)} 
\]

\[\text{(5-3)}\]

= \text{0° or 180°}.

For \( k_i \) to be real, it is then necessary that

\[\text{arg } Q(j\lambda) = \text{0° or 180°}.\]

\[\text{(5-4)}\]

\( Q(s) \) is now represented as

\[Q(s) = \sum_{k=0}^{n} a_k s^k,\]

\[\text{(5-5)}\]

where \( a_n = 1 \). With \( s = \text{Re}^{j\theta} \), \( Q(s) \) becomes

\[Q(s) = \sum_{k=0}^{n} a_k R^k e^{jk\theta}.\]

\[\text{(5-6)}\]

In order to meet the condition given in (5-4), the imaginary part of \( Q(s) \) is set equal to zero

\[0 = \sum_{k=0}^{n} a_k R^k \sin k\theta.\]

\[\text{(5-7)}\]

The only points of interest are those on the \( j\omega \) axis, for which (5-7) is satisfied. Therefore, let \( \theta = \pi/2 \) in
(5-7) which gives
\[ 0 = \sum_{k=1}^{n} a_k R^k \sin k \pi/2. \]  
(5-8)

When \( n \), the number of finite zeros, is odd
\[ 0 = R^n (-1)^{(n-1)/2} + a_{n-2} R^{n-2} (-1)^{(n-3)/2} + \cdots + a_1 R \]  
(5-9)

which has exactly \( n \) solutions. When \( n \) is even
\[ 0 = a_{n-1} R^{n-1} (-1)^{(n-2)/2} + a_{n-3} R^{n-3} (-1)^{(n-4)/2} + \cdots + a_1 R \]  
(5-10)

which has \( n-1 \) solutions. From (5-9), (5-10) and the preceding development it is seen that if an odd number of poles are to be placed on the \( j \omega \) axis in such a way as to make \( Q(s)/P'(s) \) positive real, the permissible pole locations are restricted to a set of \( n \) points when \( n \) is odd, or to a set of \( n-1 \) points when \( n \) is even.

A similar development, when \( Q(s)/P'(s) \) is to have an even number of finite poles, shows that
\[ \arg P'(j \gamma_1) = \pm 90^\circ. \]  
(5-11)

In this case the real part of \( Q(s) \) is set equal to zero, which gives
\[ 0 = \sum_{k=0}^{n} a_k R^k \sin k \pi/2 \]  
(5-12)
for $\theta = \pi/2$. When $n$, the number of finite zeros is odd

$$0 = a_{n-1} R^{n-1} (-1)^{(n-1)/2} + a_{n-3} R^{n-3} (-1)^{(n-3)/2} + \cdots$$

(5-13)

$$+ a_2 R^2 (-1) + a_0$$

which has $n-1$ solutions. When $n$ is even

$$0 = R^n (-1)^n/2 + a_{n-2} R^{n-2} (-1)^{(n-2)/2} + \cdots a_2 R^2 (-1) + a_0$$

(5-14)

which has $n$ solutions. If an even number of poles are to be placed on the $j\omega$ axis so that $\zeta(s)/P'(s)$ is positive real, the permissible pole locations are restricted to $n-1$ points for $n$ odd, or $n$ points for $n$ even.

These severe restrictions are further strengthened by the fact that if $z_{22}$ is a ratio of odd to even or even to odd polynomials, then $z_{21}$ must also be a ratio of odd to even or even to odd polynomials. This condition stipulates that when $P(s)$ is even, $P'(s)$ must be odd, or that when $P(s)$ is odd $P'(s)$ must be even. In effect one is forced to choose precisely the same set of parameters as those which are chosen in standard LC synthesis methods [1, 5]. That is, $z_{21}$ and $z_{22}$ will have the same set of poles unless cancellation occurs. If such a cancellation is possible, it will be obvious in the application of the standard method for choosing $z_{21}$ and $z_{22}$ in the LC case.
5.2 Synthesis of $A^*(s)$ or $G^*(s)$ With LC Networks

Although the zero sharing technique cannot be advantageously applied to the synthesis of $Z_T(s)$, $A(s)$, $Y_T(s)$, or $G(s)$ with LC networks, the cases involving $A^*(s)$ and $G^*(s)$ can be handled very effectively using this method. The only restrictions placed on $A^*(s)$ and $G^*(s)$ are that they be physically realizable with inductive and capacitive elements. In addition to the conditions of physical realizability given in Chapter II, this requires that $A^*(s)$ or $G^*(s)$ have the following two properties:

1) The transmission zeros must lie on the $j\omega$ axis including the points $s = 0$ and $s = \infty$.

2) The poles of $A^*(s)$ or $G^*(s)$ must be simple and lie on the $j\omega$ axis excluding the points $s = 0$ and $s = \infty$.

Before the zero sharing technique as it applies to the synthesis of LC networks is considered, it should be pointed out that a special class of LC transfer functions, similar to that for the RC and RL cases, exists in which the entire synthesis may be carried out using only two-terminal techniques. This class consists of those transfer functions $A^*(s)$ or $G^*(s)$, which are physically realizable with LC networks, and whose reciprocal, $Q(s)/P(s)$, has the additional two properties:

1) The zeros of $P(s)$ must have no greater multiplicity than two.

2) If one begins with the critical frequency nearest
the origin and divides the group of critical frequencies on
the positive jω axis into adjacent pairs (counting a double
pole of Q(s)/P(s) as two adjacent poles), then each pair of
critical frequencies must consist of a pole and a zero. The
synthesis of this class of transfer functions differs from
the more general case only in the realization of the networks
N' or N'' shown in Fig. 2.1 and Fig. 2.2.

A method will now be developed for synthesizing A*(s)
or G*(s). Here, the only restrictions are those of physical
realizability for LC networks as stated at the beginning of
this section. The transfer current-ratio function, A*(s),
or transfer voltage-ratio function, G*(s), is given in the
form

\[
\begin{bmatrix} -A*(s) \\ G*(s) \end{bmatrix} = -\frac{H P(s)}{Q(s)}
\]

\[
= -H \frac{(s^2 + \lambda_1^2)(s^2 + \lambda_2^2) \cdots (s^2 + \lambda_n^2)}{(s^2 + \eta_1^2)(s^2 + \eta_2^2) \cdots (s^2 + \eta_m^2)}
\]

where poles and zeros lie on the jω axis and m ≥ n. The
expression for A*(s) in terms of the open circuit parameters,
or for G*(s) in terms of the short circuit parameters, may
be solved for z_{22} or y_{22} respectively. This gives

\[
z_{22} = z_{21} \frac{Q(s)}{H P(s)}
\]

or
\[ y_{22} = -y_{21} Q(s)/H_P(s) \] \hspace{1cm} (5-17)

It is convenient to choose \( z_{21} \) or \(-y_{21}\) as

\[
\begin{bmatrix} z_{21} \text{ or } -y_{21} \end{bmatrix} = H \frac{N(s)}{D(s)}
\]

\[
= \frac{H}{s} \left[ \frac{(s^2 + \lambda_1^2)(s^2 + \lambda_2^2)\cdots(s^2 + \lambda_{n-q+k}^2)}{(s + \nu_1^2)(s + \nu_2^2)\cdots(s + \nu_k^2)} \right]
\] \hspace{1cm} (5-18)

which forms the initial step in the synthesis procedure.

Here, the poles and zeros of \( z_{21} \) (or \(-y_{21}\)) lie on the \( j\omega \) axis. The transmission zeros, \( \pm j\lambda_1, \pm j\lambda_2, \ldots, \pm j\lambda_{n-q+k} \) are not necessarily the first \( n-q+k \) zeros of \(-A^*(s)\) or \(G^*(s)\) as given in (5-15). The following considerations govern the choice of poles and zeros for \( z_{21} \) (or \(-y_{21}\)):

1) The zeros of \( z_{21} \) (or \(-y_{21}\)) should be selected from the set of transmission zeros of the given transfer function \( A^*(s) \) (or \(G^*(s)\)) as indicated in (5-18).

2) The poles and zeros of \( z_{21} \) (or \(-y_{21}\)) must lie on the \( j\omega \) axis including the points \( s = 0 \) and \( s = \infty \).

3) The poles of \( z_{21} \) (or \(-y_{21}\)) must not coincide with the zeros of \( Q(s) \).

4) The poles and zeros of \( z_{21} \) (or \(-y_{21}\)) must be chosen in such a way that the product \( z_{21} Q(s)/H_P(s) \) (or \(-y_{21} Q(s)/H_P(s)\)) is an \( LC \) driving point function of the form
\[
[z_{21} \frac{Q(s)}{HP(s)} \text{ or } -y_{21} \frac{Q(s)}{HP(s)}]
\]

\[
= \frac{1}{s} \frac{(s^2 + \eta_1^2)(s^2 + \eta_2^2) - (s^2 + \eta_m^2)}{(s + \nu_1^2) - (s + \nu_k^2)(s + \lambda_k^2) - (s + \lambda_q^2)}
\]

in which the set of transmission zeros \(\pm j\lambda_{k+1}, \pm j\lambda_{k+2}, \ldots \pm j\lambda_q\) consists of those finite transmission zeros which do not belong to the set \(\pm j\lambda_1, \pm j\lambda_2, \ldots \pm j\lambda_{n-q+k}\). Acceptable values for \(q\) are \(q = m\) and \(q = m - 1\). The poles and zeros of \(z_{21} Q(s)/HP(s)\) (or \(-y_{21} Q(s)/HP(s)\)) must alternate on the \(j\omega\) axis.

5) One should assign as few poles and zeros to \(z_{21}\) (or \(-y_{21}\)) as possible and still satisfy condition 4 above.

With \(z_{21}\) (or \(-y_{21}\)) chosen according to the above conditions one next places the product \(z_{21} Q(s)/HP(s)\) (or \(-y_{21} Q(s)/HP(s)\)) in partial fractioned form. \(z_{22}\) (or \(y_{22}\)) may then be represented as

\[
[z_{22} \text{ or } y_{22}] = (a_{-1} s + \frac{a_0}{s} + \frac{a_1 s}{s^2 + \nu_1^2} + \ldots + \frac{a_k s}{s^2 + \nu_k^2})
\]

\[
+ (\frac{a_{k+1} s}{s^2 + \nu_{k+1}^2} + \ldots + \frac{a_q s}{s^2 + \nu_q^2})
\]

where \(a_{-1} = 1\) for \(q = m - 1\) or \(a_{-1} = 0\) for \(q = m\). At this point, \(Z_s\) as shown in Fig. 2.1 or \(Y_p\) as shown in Fig. 2.2
may be removed from \( z_{22} \) or \( y_{22} \), respectively. The standard LC realization for \( Z_s \) or \( Y_p \) may then be performed.

The remaining portion of the realization, the synthesis of the network \( N' \) or \( N'' \), follows one of two courses. First, if \( A^*(s) \) or \( G^*(s) \) belongs to the special class of LC transfer functions mentioned earlier, then \( N' \) or \( N'' \) may be synthesized in an L network of the form shown in Fig. 4.1 or Fig. 4.9 respectively. Only two-terminal LC synthesis techniques are required for this realization. Let \( z_{21} \) (or \(-y_{21}\)) as given in (5-18) be represented in partial fractioned form as

\[
[z_{21} \text{ or } -y_{21}] = \beta_{-1}s + \frac{\beta_0}{s} + \frac{\beta_1s}{s^2 + v_1^2} + \ldots + \frac{\beta_ks}{s^2 + v_k^2}. \tag{5-21}
\]

Then the restrictions on the gain, \( H \), for the L network configuration may be stated as

\[
\alpha_i/\beta_1 \geq H \quad \text{for } i = -1, 0, 1, 2, \ldots - k. \tag{5-22}
\]

The second course which may be taken is that of synthesizing \( N' \) or \( N'' \) in a general LC ladder configuration.

The general approach for synthesizing the networks \( N' \) and \( N'' \) was described in section 4.4, and details were given there for the synthesis of RC ladders. In synthesizing LC ladders, the same general approach is used with respect to predicting a possible network configuration and synthesizing
$z_{22}$ (or $y_{22}$) as a driving point impedance, section by section [1, 6]. Therefore, it is only necessary to introduce the basic LC sections which will make up the ladder, to indicate the procedure for synthesizing these sections, and to mention certain factors to be considered in prediction.

Two basic LC sections are shown in Fig. 5.1. The subscript or superscript, $k$, again indicates that this is the $k$th section in the predicted configuration, in which the sections are numbered from right to left consecutively. In

Fig. 5.1 Basic LC sections (a) type 1 LC section (b) type 2 LC section
each basic section of Fig. 5.1, a transmission zero is produced by the principal branch at \( s = \pm j\lambda_k \). If, in Fig. 5.1 (a), \( L_a^{(k)} \) is set equal to zero, or if \( C_a^{(k)} \) is set equal to infinity, the kth section will contribute to a transmission zero at \( s = -\infty \) or \( s = 0 \), respectively. Similarly, if, in Fig. 5.1 (b), \( L_b^{(k)} \) is set equal to infinity, or \( C_b^{(k)} \) is set equal to zero, this kth type 2 section will contribute to a transmission zero at \( s = 0 \) or \( s = \infty \), respectively.

Note that the auxiliary branches of each section may contribute either to a transmission zero at \( s = 0 \) or to a transmission zero at \( s = \infty \).

In order to synthesize a type 1 LC section, one must know the impedance looking into that section, \( Z_k \), indicated in Fig. 5.1 (a). One must also know the transmission zeros, \( s = \pm j\lambda_k \), or their special cases, \( s = 0 \) and \( s = \infty \), which are to be produced by this kth section. A knowledge of these two facts permits the synthesis of a type 1 LC section as follows:

1) Let

\[
JX_k = Z_k(s) \bigg|_{s = j\lambda_k},
\]  

then

\[
L_b^{(k)} = X_k / \lambda_k \quad \text{(if } X_k > 0), \tag{5-24}
\]

\[
C_b^{(k)} = 1 / \lambda_k |X_k| \quad \text{(if } X < 0). \tag{5-25}
\]
2) The admittance $Y_k^*$ looking into the remaining portion of the ladder, after $L_b^{(k)}$ or $C_b^{(k)}$ is removed, is

$$Y_k^* = \frac{1}{Z_k - sL_b^{(k)}}$$  \hspace{1cm} (5-26)

or

$$Y_k^* = \frac{1}{Z_k - 1/sC_b^{(k)}}$$  \hspace{1cm} (5-27)

respectively.

3) $Y_k^*$ has the form

$$Y_k^* = \frac{M_k s}{s + \lambda_k} + Y_k + 1$$  \hspace{1cm} (5-28)

where

$$M_k = \left[ \frac{(s^2 + \lambda_k^2)}{s} \right] Y_k^*$$

$$s = j\lambda_k$$  \hspace{1cm} (5-29)

The elements of $Y_a^{(k)}$ are then

$$L_a^{(k)} = 1/M_k$$  \hspace{1cm} (5-30)

and

$$C_a^{(k)} = M_k / \lambda_k^2$$  \hspace{1cm} (5-31)
4) The impedance looking into the next basic section is then

\[ Z_k + 1 = \frac{1}{M_k s} \frac{Y^*}{s^2 + \lambda_k^2}. \] (5-32)

The synthesis of a type 2 LC section requires a knowledge of \( Y_k(s) \) and the transmission zeros which are to be produced by this section. The procedure for synthesizing this section is as follows:

1) Let

\[ jB_k = Y_k(s) \bigg|_{s = \lambda_k} \] (5-33)

then

\[ C_{a(k)} = \frac{B_k}{\lambda_k} \quad \text{(if } B_k \geq 0 \text{)}, \] (5-34)

or

\[ L_{a(k)} = \frac{1}{(\lambda_k |B_k|)} \quad \text{(if } B_k < 0 \text{)} \]. (5-35)

2) The impedance \( Z^*_k \) looking into the remaining portion of the ladder after \( C_{a(k)} \) or \( L_{a(k)} \) is removed, is

\[ Z^*_k = \frac{1}{Y_k - sC_a}. \] (5-36)
or
\[ Z^*_k = \frac{1}{Y_k - 1/sL_a}, \quad (5-37) \]
respectively.

3) \( Z^*_k \) has the form
\[ Z^*_k = \frac{N_k s}{s^2 + \lambda^2_k} + Z^*_{k+1}, \quad (5-38) \]
where
\[ N_k = \left[ \frac{(s^2 + \lambda^2_k)}{s} \right] \quad (5-39) \]
The elements of \( Z^{(k)}_b \) are then
\[ C^{(k)}_b = 1/N_k \quad (5-40) \]
and
\[ L^{(k)}_b = N_k/\lambda^2_k \quad (5-41) \]

4) The admittance looking into the next basic section is then
\[ Y_{k+1} = \frac{1}{Z^*_k - \frac{N_k s}{s^2 + \lambda^2_k}}. \quad (5-42) \]
Attention is now turned to those factors which are important in predicting possible LC network configurations. When the network parameters \( z_{21} \) and \( z_{22} \) (or \( y_{21} \) and \( y_{22} \)) are chosen by the zero sharing methods above, it is certain that the parameters for the network \( N' \) (or \( N'' \)) will have the same denominator, and that \( z'_{21} \) and \( z'_{22} \) (or \( y'_{21} \) and \( y'_{22} \)) will have no common numerator factors. It is therefore assured that the zeros of \( z'_{21} \) (or \( y'_{21} \)) are the transmission zeros of the network \( N' \) (or \( N'' \)). The procedure for predicting an LC network may then be stated as follows:

1) Examine \( z_{21} \) (or \( y_{21} \)) to determine the transmission zeros of \( N' \) (or \( N'' \)).

2) Arrange type 1 and/or type 2 LC sections (and possibly some additional LC branches) in the ladder configuration of Fig. 4.2 for prescribed \( z'_{21} \) and \( z'_{22} \), or in the configuration of Fig. 4.10 for prescribed \( y'_{21} \) and \( y'_{22} \).

In the arrangement described in 2, the principal branch of each LC section is assigned a complex conjugate pair of transmission zeros which it is to produce. In the event that all transmission zeros are at \( s = 0 \), \( s = \infty \), or a combination of these two, it is necessary that certain elements in the principal branches take on the limiting values discussed earlier. The principal branch will then produce a transmission zero at \( s = 0 \) or \( s = \infty \). The auxiliary branch of each LC section will contribute to a transmission zero at either \( s = 0 \) or \( s = \infty \).

As in the case of RC and RL networks, a number of
predicted configurations is possible for a given set of transmission zeros. Some of these predictions will be realizable, while others may contain negative circuit elements and are therefore nonrealizable with passive networks. If a first prediction is nonrealizable, a rearrangement of the order of transmission zero assignments may cause the prediction to become realizable, or it may be necessary to predict a different configuration of basic sections.
CHAPTER VI

TRANSFER FUNCTIONS WITH COMPLEX CONJUGATE AND NEGATIVE REAL TRANSMISSION ZEROS

A method will be developed in this chapter for synthesizing minimum phase transfer functions with networks which, in general, will consist of resistive, capacitive, and inductive elements. The realizability conditions, which a transfer function, \( T(s) \), must satisfy, are given in Chapter II. Although these are the only restrictions on the transfer functions, it will be found that the usefulness of the method under consideration is limited to transfer functions of low complexity. In fact, only those transfer functions of third degree or less in numerator and denominator will be considered in detail. The extension of the zero sharing method to higher degree transfer functions will be discussed, but has not been placed on a firm basis. That is, in applying the technique to higher degree transfer functions, one cannot be sure that full advantage has been taken of the simplifying effects inherent in the zero sharing process.

The trivial case, in which the denominator of \( T(s) \) is a first degree polynomial with the numerator either a constant or a first degree polynomial, is always positive real and
therefore realizable with an RC or RL network in the L configuration of either Fig. 4.1 or Fig. 4.9.

6.1 Transfer Functions Whose Denominator Polynomial Is of Second Degree

It will be shown in this section that every minimum phase transfer function, \( T(s) \), whose denominator polynomial is of second degree in \( s \), can be synthesized using only two-terminal techniques. The resulting network will have the form indicated in Fig. 1.2 or Fig. 1.3, in which the networks \( N' \) or \( N'' \) may be realized in the L configuration of Fig. 4.1 or Fig. 4.9 respectively. It is convenient to consider separately five different forms of \( T(s) \) which differ in their pole-zero configurations.

1) Let \( T(s) \) have the form

\[
T(s) = \frac{H}{(s + c - jd)(s + c + jd)} ,
\]

where \( c > 0 \) and \( d \geq 0 \). \( z_{21} \) (or \(-y_{21}\)) may be chosen as

\[
z_{21} \text{ (or } -y_{21} \text{)} = K \frac{1}{s + e} ,
\]

where \( 0 \leq e \leq 2c \). The product \( z_{21} \frac{Q(s)}{P(s)} \) (or \(-y_{21} \frac{Q(s)}{P(s)} \)) then has the form

\[
\left[ z_{21} \frac{Q(s)}{P(s)} \text{ (or } -y_{21} \frac{Q(s)}{P(s)} \right] = K \frac{1}{s + e} \frac{(s+c-jd)(s+c+jd)}{1} .
\]
For $Z_T(s)$, $A(s)$, $Y_T(s)$, or $G(s)$, $z_{22}$ (or $y_{22}$) has the form

$$\left[ z_{22} \text{ (or } y_{22} \right] = K/H \left[ s + (2c-e) + \frac{c^2+d^2-e(2c-e)}{s+e} \right]^{-1} \tag{6-4}$$

For $A^*(s)$ or $G^*(s)$, $z_{22}$ (or $y_{22}$) has the form

$$\left[ z_{22} \text{ (or } y_{22} \right] = K/H \left[ s + (2c-e) + \frac{c^2+d^2-e(2c-e)}{s+e} \right] \tag{6-5}$$

Since $z_{21}$ (or $-y_{21}$) is a positive real driving point function, a two-terminal realization may always be effected. A very simple sharing of transmission zeros is represented by the fact that one transmission zero at infinity is produced by $z_{21}$ (or $-y_{21}$), while the other transmission zero at infinity is produced as a private pole of $z_{22}$ (or $y_{22}$).

2) Let $T(s)$ be of the form

$$T(s) = H \frac{s+a}{(s+c-jd)(s+c+jd)} \tag{6-6}$$

where $a \geq 0$, $c > 0$, and $d \geq 0$. It is assumed that $T(s)$ is not positive real, and therefore, that $a > 2c$. One possible choice of $z_{21}$ (or $-y_{21}$) is

$$\left[ z_{21} \text{ (or } -y_{21} \right] = K \frac{s+a}{s+e} \tag{6-7}$$

where $0 \leq e \leq 2c$. The product $z_{21} Q(s)/P(s)$ (or $-y_{21}$ $Q(s)/P(s)$) then has the form
\[
\left[ z_{21} \frac{Q(s)}{p(s)} \text{ (or } -y_{21} \frac{Q(s)}{p(s)} \right] = K \frac{(s+c-jd)(s+c+jd)}{s+e}.
\]

(6-8)

If \( z_T(s), A(s), Y_T(s), \) or \( G(s) \) is specified, then \( z_{22} \) (or \( y_{22} \)) will have the form given in (6-4). For \( A^*(s) \) or \( G^*(s) \) specified, \( z_{22} \) (or \( y_{22} \)) will have the form given in (6-5). Since \( z_{21} \) (or \( -y_{21} \)) is a positive real driving point function, a two-terminal realization may always be effected.

A sharing of transmission zeros is represented in that the finite transmission zero at \( s = -a \) is produced by \( z_{21} \) (or \( -y_{21} \)), while a transmission zero at infinity is produced as a private pole of \( z_{22} \) (or \( y_{22} \)).

3) Let \( T(s) \) have the form

\[
T(s) = \frac{(s + a - jb)(s + a + jb)}{(s + c - jd)(s + c + jd)},
\]

(6-9)

where \( a \geq 0, b \geq 0, c > 0, \) and \( d \geq 0. \) The single necessary and sufficient condition for a function of this form to be a positive real driving point function is given by

\[
\left[ \sqrt{a^2 + b^2} - \sqrt{c^2 + d^2} \right]^2 \leq 4 ac.
\]

(6-10)

It will be assumed that \( T(s) \) is not positive real and therefore that the condition given in (6-10) is not satisfied.

If \( z_{21} \) (or \( -y_{21} \)) is chosen as
\[
[z_{21} \text{ (or } -y_{21})] = K \frac{(s + a - jb)(s + a + jb)}{s + e}, 
\tag{6-11}
\]

where \(0 \leq e \leq 2a\) and \(0 \leq e \leq 2c\), a realization may be easily effected using only two-terminal techniques. The product \(z_{21}Q(s)/P(s)\) (or \(-y_{21}Q(s)/P(s)\)) then has the form given in (6-8). If \(Z_T(s), A(s), Y_T(s)\), or \(G(s)\) is specified, \(z_{22}\) (or \(y_{22}\)) will have the form given in (6-4). If \(A^*(s)\) or \(G^*(s)\) is specified, \(z_{22}\) (or \(y_{22}\)) will have the form given in (6-5). Here, \(z_{21}\) (or \(-y_{21}\)) is positive real, and, therefore, a realization may always be effected using only two-terminal techniques. No sharing of transmission zeros is possible since both members of the pair of complex conjugate transmission zeros must be assigned to \(z_{21}\) (or \(-y_{21}\)).

4) Let \(T(s)\) have the form

\[
T(s) = \frac{(s + a)(s + b)}{(s + c - jd)(s + c + jd)},
\tag{6-12}
\]

where \(a \geq 0\), \(b \geq 0\), \(c > 0\), and \(d \geq 0\), with \(a\) and \(b\) distinct. It will be assumed that \(T(s)\) is not a positive real driving point function and therefore that the condition

\[
\left[ \sqrt{ab} - \sqrt{c^2 + d^2} \right]^2 \leq 2c (a + b) \tag{6-13}
\]

is not satisfied. A realization may always be effected by choosing \(z_{21}\) (or \(-y_{21}\)) as
\[ [z_{21} \text{ (or } -y_{21})] = K \frac{(s + a)(s + b)}{s + e}, \quad (6-14) \]

in which \(0 \leq e \leq 2c\) and \(0 \leq e \leq (a + b)\). The product \(z_{21}\)
\(Q(s)/P(s)\) (or \(-y_{21}Q(s)/P(s)\)) then has the form given in
(6-8). In the synthesis of \(Z_T(s)\), \(A(s)\), \(Y_T(s)\), or \(G(s)\), \(z_{22}\)
(or \(y_{22}\)) will have the form given in (6-4). In the synthesis
of \(A^*(s)\) or \(G^*(s)\), \(z_{22}\) (or \(y_{22}\)) will have the form given
in (6-5). Since \(z_{21}\) (or \(-y_{21}\)) is positive real, the complete
realization may always be carried out using two-terminal

A realization may frequently be effected in which the
transmission zeros are shared. In order to produce this
sharing, the numerator polynomial, \(N(s)\), of \(z_{21}\) (or \(-y_{21}\))
should be chosen as either \(s + a\) or \(s + b\). Without loss of
generality \(s = -a\) is chosen as the zero of \(N(s)\). \(z_{21}\) (or
\(-y_{21}\)) is assigned a simple pole at \(s = -e\), which gives

\[ [z_{21} \text{ (or } -y_{21})] = K \frac{s + a}{s + e}. \quad (6-15) \]

The selection of \(e\) is made in such a way that the expression

\[ (\sqrt{eb} - \sqrt{c^2 + d^2})^2 \leq 2c \ (e + b) \quad (6-16) \]

is satisfied. It will be found convenient to assure that
\(e\) satisfies the restriction,
\[ [e (e - b)]^2 > (c - e)^2 + d \]
for \( e > b \), or \( (6-17) \)

\[ [b (b - e)]^2 > (c - b)^2 + d^2 \]
for \( b > e \). The product \( z_{21} Q(s)/P(s) \) (or \(-y_{21} Q(s)/P(s)\)) then has the form

\[ \left[ z_{21} \frac{Q(s)}{P(s)} \right. \text{ (or)} \left. -y_{21} \frac{Q(s)}{P(s)} \right] = (s+c-\bar{j}d)(s+c+\bar{j}d) \over (s+e)(s+b). \quad (6-18) \]

In the synthesis of \( Z_T(s), A(s), Y_T(s) \) or \( G(s) \), \( z_{22} \) (or \( y_{22} \)) will have the form

\[ [z_{22} \text{ (or } y_{22} \text{)}] = K/H \left[ (s+c-\bar{j}d)(s+c+\bar{j}d) \over (s+e)(s+b) \right] -1. \quad (6-19) \]

In the synthesis of \( A^*(s) \) or \( G^*(s) \), \( z_{22} \) (or \( y_{22} \)) will have the form

\[ [z_{22} \text{ (or } y_{22} \text{)}] = K/H \left[ (s-c-\bar{j}d)(s+c+\bar{j}d) \over (s+e)(s+b) \right] . \quad (6-20) \]

5) Let \( T(s) \) have the form

\[ T(s) = \frac{(s+a-\bar{j}b)(s+a+\bar{j}b)}{(s+c)(s+d)}, \quad (6-21) \]
where \( a \geq 0, b \geq 0, c \geq 0, \) and \( d \geq 0 \) with \( c \) and \( d \) distinct. It will be assumed that \( T(s) \) is not a positive real driving point function, and therefore, that the condition

\[
(\sqrt{cd} - \sqrt{a^2 + b^2})^2 \leq 2a(c + d) \tag{6-22}
\]

is not satisfied. A realization may always be effected by choosing \( z_{21} \) (or \(-y_{21}\)) as

\[
[z_{21} \text{ (or } -y_{21})] = K \frac{(s+a-jb)(s+a+jb)}{s+e}, \tag{6-23}
\]

in which \( 0 < e < 2a \) and \( 0 < e < (c+d) \). The product \( z_{21} Q(s)/P(s) \) (or \(-y_{21}Q(s)/P(s)\)) then has the form

\[
\left[ z_{21} \frac{Q(s)}{P(s)} \text{ (or } -y_{21} \frac{Q(s)}{P(s)} \right] = K \frac{(s+c)(s+d)}{s+e}. \tag{6-24}
\]

In the synthesis of \( Z_T(s), \Lambda(s), Y_T(s), \) or \( G(s), z_{22} \) (or \( y_{22} \)) will have the form

\[
[z_{22} \text{ (or } y_{22})] = K/H \left[ s + (c+d-e) + \frac{cd-e(c+d-e)}{s+e} \right]^{-1}, \tag{6-25}
\]

while in the synthesis of \( A^*(s) \) or \( G^*(s) \), \( z_{22} \) (or \( y_{22} \)) may be written as

\[
[z_{22} \text{ (or } y_{22})] = K/H \left[ s + (c+d-e) + \frac{cd-e(c+d-e)}{s+e} \right]. \tag{6-26}
\]
Since \( z_{21} \) (or \(-y_{21}\)) is positive real, a realization using only two-terminal techniques is possible. No sharing of transmission zeros is possible in this case since both members of the complex conjugate pair of transmission zeros must be assigned to \( z_{21} \) (or \(-y_{21}\)).

The five cases considered above include all physically-realizable transfer functions, whose denominator is of second degree, which were not included in the synthesis methods given in Chapter IV and Chapter V. The transfer functions considered in Chapters IV and V, which have denominators of second degree are found to belong to the special classes, which were defined in section 4.2 and section 5.1, respectively. It may, therefore, be concluded that all transfer functions which have denominators of second degree may be synthesized using only two-terminal techniques.

Methods have been given for the synthesis of every physically-realizable minimum phase transfer function whose denominator is of second degree. However, only those choices of \( z_{21} \) (or \(-y_{21}\)) have been considered in which the zeros of \( D(s) \) are assigned to points on the negative \( \sigma \) axis or on the \( j\omega \) axis. The possibility of choosing a complex conjugate pair of poles for \( z_{21} \) (or \(-y_{21}\)) has not been considered. As previously described, one should assign as few poles and zeros to \( z_{21} \) (or \(-y_{21}\)) as possible and still assure that the product \( z_{21} Q(s)/P(s) \) (or \(-y_{21} Q(s)/P(s)\)) is positive real. In general, when more poles and zeros than
the minimum required number are assigned to \( z_{21} \) (or \( -y_{21} \)), an excessive number of circuit elements is required in the realization. Since the synthesis of all transfer functions, \( T(s) \), whose denominator is of second degree can be carried out with \( z_{21} \) (or \( -y_{21} \)) having a single pole on the negative \( \sigma \) axis, it would be expected that the assignment of a pair of complex poles to \( z_{21} \) (or \( -y_{21} \)) will cause an excessive number of circuit elements to be necessary in the realization. Although this is true, consideration of the complex conjugate pole assignment is important from a theoretical standpoint, and will be found useful in the synthesis of transfer functions whose denominator polynomial is of third degree or higher in \( s \).

A procedure will now be given for synthesizing any of the five forms of \( T(s) \) given above, in which one has the freedom to choose the poles of \( z_{21} \) (or \( -y_{21} \)) and \( z_{22} \) (or \( y_{22} \)) as a complex conjugate pair.

Attention is first turned to a preliminary subject, the problem of constructing a positive real function, \( F(s) \), whose poles are to appear in complex conjugate pairs. With the form of \( F(s) \) given as

\[
F(s) = \frac{(s + c - jd)(s + c + jd)}{(s + e - jf)(s + e + jf)},
\]

the problem may be stated as follows.

If the zero locations, \( s = -c \pm jd \), are known, what
values may be chosen for e and f so that \( F(s) \) will be positive real? Permissible values for e and f are determined by a simple geometric construction which is explained in the following four steps: (A proof of the validity of this procedure is given in section 6.2.)

1) On a piece of graph paper construct a rectangular coordinate system to represent the \( s \) plane. With a compass, mark the point where a circle, whose center is at the origin and which passes through the point \(-c + jd\), intersects the positive \( j\omega \) axis. (This intersection is, of course, at \( s = j \sqrt{c^2 + d^2} \).) Call this point \( j\omega_1 \).

2) From the specified value of \( c \), and \( \omega_1 \) as just determined, calculate \( r \), where \( r \) is given by

\[
r = 2c + \omega_1.
\] (6-28)

(It is convenient to perform the addition in (6-28) by summing lengths on the \( j\omega \) axis.) With a compass mark the points at which a circle, whose radius is \( r \) and whose center is at \( s = j\omega_1 \), intersects the real axis. Call the point at the intersection of this circle with the negative \( \sigma \) axis \(-x_0\), and the point at the intersection with the positive \( \sigma \) axis \( x_0 \).

3) Construct a circle, \( C_1 \), whose center is at \( s = -x_0 \), and whose radius is \( r \), as given in (6-28). Construct a second circle, \( C_2 \), whose center is at \( s = x_0 \), and whose
radius is also $r$. This construction is illustrated in Fig. 6.1 for $c = 1$ and $d = 2$.

![Diagram](image)

Fig. 6.1 The construction of a region in which poles of $F(s)$ may be selected.

4) A pair of complex conjugate poles may be placed anywhere within or on the closed region whose interior is to the left and outside $C_2$, and inside $C_1$. The shaded area in Fig. 6.1 indicates such a region. For this assignment of poles, $F(s)$, as given in (6-27), is a positive real driving point function.

The case will now be considered in which the zeros of
\( F(s) \) lie on the negative portion of the \( \sigma \) axis. \( F(s) \) is to have the form

\[
F(s) = \frac{(s + c)(s + d)}{(s + e - jf)(s + e + jf)}.
\] (6-29)

The zeros at \( s = -c \) and \( s = -d \) are considered to be previously specified, and \( e \) and \( f \) are to be chosen so that \( F(s) \) will be positive real. A region in which poles of \( F(s) \) may be selected can be determined by following steps 1 through 5 with two modifications. First, the point, \( j\omega_1 \), is now given by

\[
 j\omega_1 = j \sqrt{cd} ,
\] (6-30)

and, second, the radius \( r \) is now given by

\[
r = \omega_1 + c + d.
\] (6-31)

An illustration of this construction is shown in Fig. 6.2, for \( F(s) \) as given in (6-29), with \( c = 1 \) and \( d = 2 \).

If a pair of complex conjugate poles is chosen for \( F(s) \) within one of the regions just defined, it is certain that \( F(s) \) will be positive real. Although such a choice is sufficient to assure that \( F(s) \) is positive real, it is not necessary that the poles be chosen within these regions.
Fig. 6.2 A region in which poles of \( F(s) \) may be selected, for \( F(s) \) as given in (6-29)

Larger regions have been shown to exist. However, these regions are much more difficult to construct [7].

The above procedures for choosing the poles of a positive real function may be used in synthesizing any one of the five forms of \( T(s) \) given in equations (6-1), (6-6), (6-9), (6-12), or (6-21). After choosing an acceptable pair of poles for \( z_{21} \) (or \( -y_{21} \)), the synthesis is carried out by first, selecting a value of \( K/H \), and second, synthesizing the network \( N' \) or \( N'' \). If \( z_{21} \) (or \( -y_{21} \)) is chosen as a
positive real driving point function, the synthesis may be carried out on a two-terminal basis, with the resulting network having an L configuration. For such a realization, \( \frac{K}{H} \) should be chosen in such a way that 
\[
\frac{z_{22}'}{z_{21}'} - \frac{z_{21}'}{z_{21}'} \quad \text{(or } \frac{y_{22}'}{y_{21}'} - \frac{y_{21}'}{y_{21}'})
\]
is a positive real driving point function. If, however, a general four-terminal realization of \( N' \) or \( N'' \) is to be effected, then in the selection of \( \frac{K}{H} \), it is only necessary that property 4 for the network parameters, as given in Chapter II, be satisfied.

Illustrations will be given in section 6.5 for those transfer functions, \( T(s) \), whose denominator polynomial is of second degree in \( s \).

6.2 The Derivation of Regions In Which A Positive Real Function, \( F(s) \), May Have Poles

The validity of the construction procedure, given in steps 1, 2, 3, and 4 of section 6.1, will now be established. \( F(s) \) is to be a biquadratic driving point function of the form given in (6-27) or (6-29). For specified zero locations, if the poles of \( F(s) \) are chosen to lie within a certain region of the \( s \) plane, it will be assured that \( F(s) \) is positive real. A derivation of these regions will be given in the following eight steps:

1) Definition of a C path [7]: Consider a circle which lies partly in the left-hand half of the \( s \) plane and partly in the right-hand half of the \( s \) plane, and whose center is on the real axis. A C path is, by definition, that portion
of the circle which lies in the closed left-hand half of the s plane. Also by definition, the negative direction on a C path always points to the left. Fig. 6.3 shows a typical C path with its negative direction indicated.

![Diagram](image)

Fig. 6.3 A typical C path with arrows indicating its negative direction

2) The phase function of a quadratic: Let \( Q(s) \) be a real quadratic given by

\[
Q(s) = s^2 + 2as + a^2 + B^2,
\]

with \( a \geq 0 \) and \( B \geq 0 \). The phase function of \( Q(j\omega) \) is given by

\[
\Phi_Q(\omega) = \tan^{-1} \frac{2\alpha \omega}{2 + \frac{a^2 + B^2}{\alpha + B} - \omega}
\]

3) A property of the phase function as the zeros of \( Q(s) \) are moved along a C path [7]: Let the zeros of \( Q(s) \)
move in the negative direction on a C path that intersects the \( j\omega \) axis at \( s = \pm j\omega_1 \) \((\omega_1 > 0)\). Then the phase function \( \phi_Q(\omega) \) of \( Q(j\omega) \) decreases monotonically for every fixed \( \omega \) in the interval \( \omega_1 < \omega < \infty \) and increases monotonically for every fixed \( \omega \) in the interval \( 0 < \omega < \omega_1 \). This has been shown by Steiglitz and Zemanian [7].

4) A property of a biquadratic driving point function, \( F(s) \), whose poles are restricted to lie on the \( j\omega \) axis: Let \( F(s) \) have the form

\[
F(s) = \frac{Q_1(s)}{Q_2(s)} = \frac{s^2 + 2as + a^2 + \beta^2}{s^2 + \omega_1^2} .
\]  

(6-34)

If values of \( \alpha \) and \( \beta \) \((\alpha > 0, \beta \geq 0)\) are specified, only one permissible value of \( \omega_1 \) exists for which \( F(s) \) is positive real. A proof of this fact using the general procedure given in Chapter V is as follows: \( Q_1(s) \) has the form

\[
Q_1(s) = s^2 + 2\alpha s + \alpha^2 + \beta^2 .
\]  

(6-35)

From (5-14) the permissible values of \( \omega_1 \) are the roots of the equation

\[
0 = -R^2 + \alpha^2 + \beta^2 ,
\]  

(6-36)

which gives
\[ \omega_1 = R = \sqrt{\alpha^2 + \beta^2} \quad (6-37) \]

Similarly, if \( F(s) \) has two distinct zeros on the negative \( \sigma \) axis at \( s = -c \) and \( s = -d \), then \( F(s) \) has the form

\[
F(s) = \frac{Q_1(s)}{Q_2(s)} \frac{s^2 + (c + d) s + cd}{s^2 + \omega_1'^2} . \quad (6-38)
\]

Again, only one permissible value of \( \omega_1' \) exists. This value of \( \omega_1' \) may be found by applying equation (5-14). For \( Q_1(s) \) as given in (6-38), this gives

\[
0 = -R^2 + cd \quad (6-39)
\]

and

\[
\omega_1' = R = \sqrt{cd} . \quad (6-40)
\]

5) A property of a biquadratic driving point function, \( F(s) \), which has a double pole on the negative portion of the \( \sigma \) axis: First, let \( F(s) \) have the form

\[
F(s) = \frac{Q_1(s)}{Q_2(s)} = \frac{s^2 + 2as + \alpha^2 + \beta^2}{(s + \gamma)^2} , \quad (6-41)
\]

where values of \( \alpha \) and \( \beta \) (\( \alpha \geq 0, \beta \geq 0 \)) are specified. From
the single condition of physical realizability for a biquadratic driving point function, it is necessary that the condition

$$\left[ \sqrt{\alpha^2 + \beta^2} - r \right]^2 \leq 4ar \quad (6-42)$$

be satisfied [8]. This inequality determines a segment of the negative $\alpha$ axis on which a double pole of $F(s)$ may be placed, with the result that $F(s)$ is a positive real driving point function. The end points of this segment are determined by assuming equality in (6-42), and then solving for the roots, $r_1$ and $r_2$, where $r_1 \geq r_2$. The roots, $r_1$ and $r_2$, are given by

$$r_1 = (\sqrt{\alpha^2 + \beta^2} + 2\alpha) + \sqrt{[\sqrt{\alpha^2 + \beta^2} + 2\alpha]^2 - (\alpha^2 + \beta^2)} \quad (6-43)$$

and

$$r_2 = (\sqrt{\alpha^2 + \beta^2} + 2\alpha) - \sqrt{[\sqrt{\alpha^2 + \beta^2} + 2\alpha]^2 - (\alpha^2 + \beta^2)} \quad (6-44)$$

respectively. From (6-37), $r_1$ and $r_2$ become

$$r_1 = (\omega_1 + 2\alpha) + \sqrt{(\omega_1 + 2\alpha)^2 - \omega_1^2} \quad (6-45)$$

and
respectively. $F(s)$ is a positive real driving point function if the condition

$$r_2 \leq \gamma \leq r_1 \quad (6-47)$$

is satisfied. Similarly, if $F(s)$ has two distinct zeros on the negative $\sigma$ axis at $s = -c$ and $s = -d$, then $F(s)$ has the form

$$F(s) = \frac{Q_1(s)}{Q_2(s)} = \frac{s + (c + d)s + cd}{(s + \gamma)^2}. \quad (6-48)$$

A double pole of $F(s)$ may be placed on the negative $\sigma$ axis at the point $s = -\gamma$, if $\gamma$ satisfies the condition

$$r' \leq \gamma \leq r' \quad (6-49)$$

where

$$r'_1 = (\omega'_1 + c + d) + \sqrt{(\omega'_1 + c + d)^2 - \omega'_1^2} \quad (6-50)$$

and

$$r'_2 = (\omega'_1 + c + d) - \sqrt{(\omega'_1 + c + d)^2 - \omega'_1^2} \quad (6-51)$$
For this assignment of \( r \), \( F(s) \) is a positive real driving point function.

6) A necessary and sufficient condition for a rational function, \( W(s) \), to be positive real [7]: Let \( W(s) \) be a rational function which has all its poles and zeros in the closed left-hand half of the \( s \) plane. Let the complex poles and zeros of \( W(s) \) appear in complex conjugate pairs. If the phase function \( \Phi_W(\omega) \) of \( W(j\omega) \) satisfies the condition

\[
|\Phi_W(\omega)| \leq \pi/2
\]

(6-52)

for \( \omega > 0 \), then \( W(s) \) is positive real.

7) Regions generated by \( C \) paths: Consider the rational function, \( W_1(s) \), where

\[
W_1(s) = F_1(s) \times F_2(s) = \frac{Q_1(s)}{Q_2(s)} \times \frac{Q_2(s)}{Q_3(s)}
\]

(6-53)

\[
= \frac{(s+\alpha_1-j\beta_1)(s+\alpha_1+j\beta_1)}{(s-j\omega_1)(s+j\omega_1)} \times \frac{(s-j\omega_1)(s+j\omega_1)}{(s+\alpha_2-j\beta_2)(s+\alpha_2+j\beta_2)}.
\]

Here, it is assumed that

\[
\omega_1 = \sqrt{\frac{\alpha_1^2 + \beta_1^2}{1}}.
\]

(6-54)
and, therefore, according to (6-37), $F_1(s)$ is positive real. The phase function of $F_2(j\omega)$, is given by

$$\Phi_{F_2}(\omega) = \Phi_{Q_2}(\omega) - \Phi_{Q_3}(\omega).$$  \hspace{1cm} (6-55)

If the poles of $Q_3(s)$ are moved in the negative direction along a C path which passes through the point $s = j\omega_1$, $\Phi_{F_2}(\omega)$ will increase monotonically for every fixed $\omega$ in the interval $\omega_1 < \omega < \infty$ and will decrease monotonically for every fixed $\omega$ in the interval $0 < \omega < \omega_1$. This fact was stated in 3 above.

The phase function of $W_1(j\omega)$, $\Phi_{W_1}(\omega)$, satisfies the condition given in (6-52) for $\beta_2 = 0$ and $r_2 < a_2 < r_1$ as proven in 5 above. Since the variation of $\Phi_{W_1}(\omega)$ is monotonic along every C path which passes through the point $s = j\omega_1$, $\Phi_{W_1}(\omega)$ satisfies (6-52) for every pole pair, $s = -a_2 \pm j\beta_2$, on a C path which intersects the negative $\sigma$ axis at a point, $\gamma$, provided that $\gamma$ satisfies condition (6-47). A group of these C paths is shown in Fig. 6.4. These C paths generate the shaded region shown in Fig. 6.4. This region is bounded by the C path, $C_1$, which intersects the negative $\sigma$ axis at $s = -\gamma_1$, and the C path $C_2$ which intersects the negative axis at $s = -\gamma_2$. The region has the property that a complex conjugate pair of poles lying within its interior may be assigned to $F(s)$, with the result that $F(s)$ is a positive real driving point function.
Fig. 6.4 A group of C paths on which poles of $F(s)$ may be placed

A similar argument for $W_2(s)$, where $W_2(s)$ has the form

$$W_2(s) = F_1(s) \times F_2(s)$$

$$= \frac{Q_1(s)}{Q_2(s)} \times \frac{Q_2(s)}{Q_3(s)}$$

$$= \frac{(s+c)(s+d)}{(s-j\omega_1')(s+j\omega_1')} \times \frac{(s-j\omega_1)(s+j\omega_1')}{(s+\alpha_2-j\beta_2')(s+\alpha_2+j\beta_2')}$$
94 shows that similar regions are generated for this rational function. These regions are bounded on the real axis by the C paths, C₁ and C₂, which intersect the negative γ axis at −γ₁ and −γ₂.

8) Construction of C₁ and C₂: In order to construct the paths C₁ and C₂ it is necessary to determine their radii. For the case in which the zeros of F(s) appear as a complex conjugate pair, the radii are determined as follows:

The equation of a circle which passes through the points s = −γ₁ and s = jω₁ is given by

\[(σ - σ₁)^2 + ω^2 = r^2₁, \tag{6-57}\]

where σ₁ is the center of this circle and r₁ is its radius. For σ = 0, (7-57) becomes

\[σ^2₁ + ω^2 = r^2₁, \tag{6-58}\]

and for ω = 0, (6-57) becomes

\[(-r₁ - σ₁)^2 = r^2₁. \tag{6-59}\]

If the value of r₁ given in (6-45) is substituted into (6-58) and (6-59), and these equations solved simultaneously, it is found that r₁ is given by

\[r₁ = ω₁ + 2a. \tag{6-60}\]
A similar solution may be carried out for the radius \( r_2 \) of the path \( C_2 \). Such an investigation shows that

\[
r_2 = r_1 = \omega_1 + 2\alpha.
\]

(6-61)

The case in which the zeros of \( F(s) \) are distinct and appear on the negative portion of the \( \sigma \) axis at \( s = -c \) and \( s = -d \) may be treated in a similar manner, with the result that

\[
r_1 = r_2 = \omega_1 + c + d.
\]

(6-62)

The above considerations have established the validity of the procedure given in steps 1 through 4 of section 6.1.

6.3 Transfer Functions Whose Denominator Polynomial Is of Third Degree

The synthesis of a minimum phase transfer function, \( T(s) \), whose denominator is of third degree can be handled very effectively using the zero sharing approach. The zeros of \( T(s) \) may lie anywhere in the left-hand half of the \( s \) plane including the \( j\omega \) axis. The poles of \( T(s) \) must lie in the left-hand half of the \( s \) plane and are not permitted at \( s = 0 \) or \( s = \infty \). In general, the synthesis of these transfer functions will result in networks which consist of resistive, inductive, and capacitive elements.

The synthesis is greatly facilitated by the use of a
specially built protractor called a Spirule which is available from The Spirule Company, 9728 El Venado, Whittier, California. The Spirule, which is commonly used in the design of control systems by the root locus method, consists of a transparent protractor at whose center is attached an arm which is free to rotate. Along the arm are a linear scale for measuring vector lengths, and a logarithmic scale which may be used in the multiplication of vector magnitudes. Although the Spirule is very useful in carrying out the graphical construction which will be described, all steps in the procedure can be performed using only a ruler and protractor.

Let it be required to synthesize a given transfer function, \( T(s) \). It will be assumed that \( T(s) \) is not positive real. Two forms of \( T(s) \) will be discussed. First, those transfer functions, \( T(s) \), will be considered which have a pair of complex conjugate transmission zeros, say at \( s = -a \pm jb \). As stated in section 1.1, the first step in the synthesis is to choose the parameter \( z_{21} \) (or \( -y_{21} \)) in such a way that the product \( z_{21}Q(s)/P(s) \) (or \( -y_{21}Q(s)/P(s) \)) is positive real and can be broken into two parts. The first part is to have only the poles of \( z_{21} \) (or \( -y_{21} \)) as its poles, while the second part is to have only poles which are not poles of \( z_{21} \) (or \( -y_{21} \)) as its poles. A procedure will now be given for making this selection.
A. The possibility of \( z_{21} \) (or \(-y_{21}\)) having a single pole on the negative \( \sigma \) axis

Since \( T(s) \) is to have a pair of transmission zeros at \( s = -a \pm jb \), and the denominator polynomial is of third degree, one additional transmission zero will occur on the negative \( \sigma \) axis at \( s = -c \), where \( 0 \leq c \leq \infty \). First, the assignment of this transmission zero as a zero of \( z_{21} \) (or \(-y_{21}\)) will be considered. For this assignment, the product \( z_{21}Q(s)/P(s) \) (or \(-y_{21}Q(s)/P(s)\)) has the form

\[
\left[ z_{21} \frac{Q(s)}{P(s)} \right. \left( \text{or } -y_{21} \frac{Q(s)}{P(s)} \right) = K \left( \frac{N(s)}{D(s)} \frac{Q(s)}{P(s)} = K \frac{(s+c)Q(s)}{D(s)P(s)} \right)
\]

D(s) may be determined as follows: A rectangular coordinate system should be constructed to represent the \( s \) plane.

1) Measure the angle, \( \phi_1 \), between the positive \( j\omega \) axis and the pole of \( [(s+c)Q(s)]/P(s) \) in the upper left-hand half of the \( s \) plane. \( \phi_1 \) is given by

\[
\phi_1 = \left(-a + jb - \frac{\pi}{2} \right). \tag{6-64}
\]

2) Add the angles associated with vectors from each of the zeros of \( Q(s) \) to the pole at \( s = -a + jb \), and add the angle \(-\pi/2\) to this result. Let this sum be denoted by \( \Psi_1 \). Then

\[
\Psi_1 = \sum \angle \text{vectors from the zeros to the pole} - \frac{\pi}{2}. \tag{6-65}
\]
3) Construct the unit vectors, $A_1$ and $A_2$, from the pole at $s = -a + jb$, at the angles $\theta_1$ and $\theta_2$ respectively. $\theta_1$ and $\theta_2$ are given by

$$\theta_1 = \pi + \phi + \gamma_1$$

and

$$\theta_2 = \pi - \phi + \gamma_1.$$  \hfill (6-66) \hfill (6-67)

If these vectors are extended, their intersections (if such intersections exist) with the negative real axis determine a tentatively acceptable segment of this axis for the placement of a pole of $z_{21}$ (or $-y_{21}$) and $z_{22}$ (or $y_{22}$). The extension of the vector, $A_1$, determines the right most limit of the segment while the vector, $A_2$, determines the left limit. It is possible that one or both of the vectors, $A_1$ and $A_2$, will not intersect the negative $\sigma$ axis when extended. If only the extension of $A_1$ intersects the negative $\sigma$ axis, then the entire portion of this axis to the left of its intersection with $A_1$ is a tentatively acceptable segment. If only the extension of $A_2$ intersects the negative $\sigma$ axis, then that portion of the negative $\sigma$ axis between this intersection and the origin is a tentatively acceptable segment.

A pole of $z_{21}$ and $z_{22}$ (or $-y_{21}$ and $y_{22}$) may be placed anywhere on the segment of the negative $\sigma$ axis, determined above, provided that it does not coincide with a zero of
Q(s) and that one of two additional conditions is met.

1) Either an even number of zeros of Q(s), or none of the zeros of Q(s) lie on the negative \( \sigma \) axis to the right of this pole.

2) If the pole is located to the left of an odd number of zeros of Q(s), the following is true. The product of the magnitudes of the vectors from each zero to this pole is less than the product of the magnitudes of vectors from each pole of \( N(s)Q(s)/P(s) \) times the magnitude of a vector from the origin to this pole.

In the event that there is no segment of the negative \( \sigma \) axis which is bounded by the vectors \( A_1 \) and \( A_2 \) and which satisfies either condition 1 or 2 the complex conjugate pair of transmission zeros will be assigned as zeros of \( z_{21} \) (or \( -y_{21} \)). After making this assignment, one should choose for \( z_{21} \) (or \( -y_{21} \)) the least possible number of poles which is required to place \( z_{21}Q(s)/P(s) \) (or \( -y_{21}Q(s)/P(s) \)) in the desired form.

A single pole which lies on the negative \( \sigma \) axis may be assigned to \( z_{21} \) (or \( -y_{21} \)) provided that it does not coincide with a zero of Q(s), and that each pole of \( z_{21}Q(s)/P(s) \) satisfies condition 1 or condition 2, as given above. For this pole assignment, \( z_{21} \) (or \( -y_{21} \)) has the form

\[
[z_{21} \ (\text{or} \ -y_{21})] = \frac{(s + a - jb)(s + a + jb)}{s + c} \quad (6-68)
\]
It is sometimes possible to select this pole so that \( z_{21} \) (or \( -y_{21} \)) is positive real, with the result that only two-terminal techniques are required in the synthesis. For \( z_{21} \) (or \( -y_{21} \)) to be a positive real driving point function, it is only necessary that \( c \leq 2a \).

B. The possibility of assigning to \( z_{21} \) (or \( -y_{21} \)) a complex conjugate pole pair

When the procedure described in A cannot be applied in the selection of \( z_{21} \) (or \( -y_{21} \)), one may proceed as follows: A complex conjugate pair of poles will be assigned to \( z_{21} \) (or \( -y_{21} \)) and \( z_{22} \) (or \( y_{22} \)) by following a simple organized procedure. Several of the concepts and techniques used in the design of control systems by the root locus method are found to be helpful in selecting this pole pair. Consider those \( T(s) \) which have the form

\[
T(s) = H \frac{P(s)}{Q(s)} = \frac{(s+a-jb)(s+a+jb)(s+c)}{(s+d-je)(s+d+je)(s+f)} . \tag{6-69}
\]

First, the transmission zeros, at \( s = -a \pm jb \), are assigned as zeros of \( z_{21} \) (or \( -y_{21} \)). Then, a pair of complex conjugate poles at \( s = -g \pm jh \) must be chosen. The product \( z_{21} Q(s)/P(s) \) (or \( -y_{21} Q(s)/P(s) \)) will have the form

\[
\left[ z_{21} \frac{Q(s)}{P(s)} \text{ (or } -y_{21} \frac{Q(s)}{P(s)} \right] = \frac{(s+d-je)(s+d+je)(s+f)}{(s+g-jh)(s+g+jh)(s+c)} . \tag{6-70}
\]

This product must be broken into two parts, one having only
a single pole at \( s = -c \), the other having only a complex conjugate pole pair at \( s = -g \pm jh \). To assure that such a decomposition is possible, the pole pair at \( s = -g \pm jh \) will be chosen so that the residue \( k_{gh} \) at the pole \( s = -g + jh \) satisfies the condition

\[
\left| \frac{k_{gh}}{(-g + jh - \pi/2)} \right| \leq (6-71)
\]

Regions exist in the left-hand half of the \( s \) plane for which this condition is satisfied. Often, only a single, simply connected region exists for a given set of values of \( c, d, e, \) and \( f \); however, cases do arise in which two separate, simply connected regions satisfy the condition given in (6-71). Examples of these regions are given in Fig. 6.5 and Fig. 6.6. Here, only the upper left-hand half of the \( s \) plane is shown. A corresponding region occurs in the lower left-hand half of the \( s \) plane to accommodate the other member of the complex conjugate pole pair, \( s = -g \pm jh \). Fig. 6.5 shows the continuous deformation of the region as the pair of zeros at \( s = -d \pm je \) are repositioned. It is true, in general, that the region becomes smaller as the ratio, \( e/d \), increases. Fig. 6.6 (a) and Fig. 6.6 (b) show the deformation of the region, when the pole at \( s = -c \) and the zero at \( s = -f \) undergo a translation. Fig. 6.6 (c) shows the form of the regions when \( c > f \). For \( c > f \), an additional condition must be met in the selection
Fig. 6.5 Regions of the s plane in which condition (6-71) is satisfied
Fig. 6.6 Regions of the $s$ plane in which condition (6-71) is satisfied.
of g and h. It is necessary that

\[(6-72)\]

\[|(-c+d-je)(-c+d+je)(-c+f)| < |(-c+g-jh)(-c+g+jh)|\]

be satisfied. This condition requires that the poles be chosen outside of a circle whose center is at \(s = -c\) and whose radius is given by the left-hand side of (6-72). Fig. 6.5 (c) shows these circles as dotted lines in each illustration.

It is usually difficult and time consuming to determine the exact size and shape of the region which exists for a particular pole-zero configuration. However, in the synthesis problem under consideration it is not really necessary to know the region in great detail. The required information for choosing g and h can be easily determined by a graphical procedure similar to the root locus method.

Consider the locus of points in the upper half of the s plane, for which

\[\frac{\text{Im}}{gh} = 0\]  

\[(6-73)\]

is satisfied. This locus consists of those points, s, for which

\[\angle \left[ \frac{Q(s)}{P(s)} \right] = \frac{\pi}{2} .\]  

\[(6-74)\]
An approximate locus may be sketched by following a simple set of rules, which are similar to those used in constructing root locus plots. Standard methods of constructing root locus plots are covered in detail in [9]. The rules for constructing the locus of points at which (6-73) is satisfied are as follows:

1) The loci asymptotically approach straight lines which originate at the centroid of the poles and zeros of \([\frac{Q(s)}{P(s)}] x (s+a-jb)(s+a+jb)\). In determining the centroid, poles are considered as positive masses and zeros as negative masses. The centroid is located on the real axis at a distance \(D\) from the origin, where \(D\) is given by

\[
D = \frac{\sum(\text{real parts of the poles})}{P - Z} - \frac{\sum(\text{real parts of the zeros})}{P - Z}
\]

(6-75)

Here,

(6-76)

\[P = \text{number of finite poles of } \left[\frac{Q(s)}{P(s)} x (s+a-jb)(s+a+jb)\right]\]

and

(6-77)

\[Z = \text{number of finite zeros of } \left[\frac{Q(s)}{P(s)} x (s+a-jb)(s+a+jb)\right].\]

The angle at which the asymptotes leave the centroid is given by
\[ \phi_2 = \frac{\pi/2 + n2\pi}{Z - p} \quad n = 1, 2, \ldots \]  
\hspace{10cm} (6-78) 

where \( 0 \leq \phi_2 \leq \pi \).

2) The locus starts at a zero of \([Q(s)/P(s)] \times (s+a-jb)(s+a+jb)\). Although the locus cannot lie on the \( \sigma \) axis, it can start from a zero on the negative \( \sigma \) axis.

3) The angle at which the locus leaves a particular zero of \([Q(s)/P(s)] \times (s+a-jb)(s+a+jb)\) is given by

\[ \psi_2 = \frac{\pi}{2} - \sum \text{vectors from each other zero to this zero} \\
+ \sum \text{vectors from each pole to this zero} \]  
\hspace{10cm} (6-79) 

4) A locus may originate at a zero on the negative \( \sigma \) axis if

\[ 0 = \sum \text{vectors from each other zero to this zero} \\
- \sum \text{vectors from each pole to this zero} \]  
\hspace{10cm} (6-80) 

5) A locus may terminate at a pole on the negative \( \sigma \) axis if

\[ \pi = \sum \text{vectors from each zero to this pole} \\
- \sum \text{vectors from each other pole to this pole} \]  
\hspace{10cm} (6-81)
The conditions in rules 4 and 5 may be checked very easily since only those poles and zeros on the negative portion of the $\sigma$ axis contribute to the angles in question.

The above rules are very helpful in making an approximate sketch of the paths for which (6-73) is satisfied. In general, this path lies toward the middle of the region throughout which the condition given in (6-71) is met.

Points on either side of these paths usually satisfy condition (6-71) as indicated in Fig. 6.5 and Fig. 6.6. As pointed out previously, the size of the regions generally decrease as the complex conjugate pair of zeros (if such a pair exists) moves toward the $jw$ axis. Also, it should be noted that each region reduces to a point at the $jw$ axis. From the approximate sketch, and observation of Fig. 6.5 and Fig. 6.6, one may select a possible location for the complex conjugate pair of poles, and quickly check this selection against condition (6-71) using a Spirule.

It is often possible to choose $g$ and $h$ so that $z_{21}$ (or $-y_{21}$) is positive real. If the $C$ paths, $C_1$ and $C_2$, for $z_{21}$ (or $-y_{21}$) are superimposed on the s plane plot just described, those choices of $g$ and $h$ which assure that $z_{21}$ (or $-y_{21}$) is positive real become evident. The paths, $C_1$ and $C_2$, are easily constructed as described in section 6.1, since the zeros of $z_{21}$ (or $-y_{21}$) are known to be at $s = -a \pm jb$. If $z_{21}$ (or $-y_{21}$) is positive real, the network $N'$ or $N''$ may be synthesized in the L configuration of Fig. 4.1.
The synthesis of the network $N'$ or $N''$, when $z_{21}$ (or $-y_{21}$) is not positive real, will be described in section 6.4.

The above procedure provides a basis for intelligently choosing $z_{21}$ (or $-y_{21}$) for those transfer functions, $T(s)$, which have a complex conjugate pair of transmission zeros. Next, attention is turned to the case in which all of the finite transmission zeros of $T(s)$ are real. It is assumed, of course, that $Q(s)$ does possess a complex conjugate pair of zeros. Otherwise, the techniques in Chapter IV could be applied. It is not necessary to consider this form of $T(s)$ separately since all of the methods given above apply equally well to this case with the exception of steps 1, 2, and 3 of part A of this section.

With $z_{21}$ (or $-y_{21}$) and $z_{22}$ (or $y_{22}$) determined, the remainder of the synthesis may be carried out by first, partial fractioning $z_{22}$ (or $y_{22}$); second, selecting a value of $K/H$; third, synthesizing the network $N'$ or $N''$; and fourth, synthesizing the driving point impedance $Z_g$ or admittance $Y_p$. $K/H$ should be chosen so that $z_{22}' - z_{21}$ (or $y_{22}' + y_{21}$) is a positive real driving point function if $N'$ (or $N''$) is to be synthesized in a network having an L configuration. If, however, a general four-terminal realization of $N'$ or $N''$ is to be effected, then, in selecting $K/H$ it is only necessary that property 4 for the network parameters, as given in Chapter II, be satisfied.
6.4 Synthesis of the Networks N' Or N'' As RC, RL, or RLC Ladders [1, 10]

From the driving point impedance, $z_{22}$ (or $y_{22}$), a series impedance, $Z_s$ (or shunt admittance, $Y_p$), is removed. As a result of this removal, a driving point impedance, $z'_{22}$ (or admittance, $y'_{22}$), is obtained which has the same poles as $z_{21}$ (or $-y_{21}$). A method will now be considered for synthesizing the four-terminal network N' (or N''), whose parameters are $z'_{22}$ (or $y'_{22}$) and $z'_{21} = z_{21}$ (or $-y'_{21} = -y_{21}$). The general approach which will be used is the same as that given in section 4.4. First, a network configuration is predicted which consists of basic transmission zero producing sections; and second, $z_{22}$ (or $y_{22}$) is synthesized as a driving point function which has this predicted network configuration.

The network prediction will have the form given in Fig. 4.2 for the prescribed pair of parameters $z'_{21}$ and $z'_{22}$, or it will have the form given in Fig. 4.10 when $-y'_{21}$ and $y'_{22}$ are known. The predicted configuration will be made up of complex transmission-zero producing basic sections, which will be introduced, along with the basic sections given previously. In this way, it will be possible to synthesize transfer functions with different combinations of transmission zeros.

The type 1 and type 2 generalized basic sections were described in section 4.4 and illustrated in Fig. 4.3. Two
additional generalized basic sections will now be defined. Fig. 6.7 introduces the type 3 and type 4 generalized basic sections. Here, the "principal branch", shown as a shaded box, produces a pair of complex conjugate transmission zeros at $s = -\alpha \pm j\beta$, while the "auxiliary branches", shown as clear boxes, produce no transmission zeros, provided that

$$B = \frac{A + C}{AC} \frac{\gamma^2 - 2\alpha \gamma^2 + \alpha^2 + \beta^2}{\gamma} \quad [1, 10] \quad (6-82)$$

Satisfaction of this condition will be assured when one carries out the synthesis according to the given procedure.

![Diagram](attachment:image.png)

Fig. 6.7 Generalized basic sections (a) A type 3 section (b) A type 4 section
for realizing the type 3 and type 4 sections.

The type 1, type 2, type 3, and type 4, complex transmission-zero producing sections will now be introduced along with methods for their synthesis.

1) Type 1 RLC section

The type 1 RLC section is shown in Fig. 6.8. The principal branch, \( \gamma_a^{(k)} \), produces a pair of complex conjugate transmission zeros, \( s = -\alpha_k \pm j\beta_k \), while the auxiliary branch \( z_b^{(k)} \) contributes to a transmission zero at \( s = \infty \) or \( s = 0 \). The subscript, or superscript, \( k \), indicates that this is the \( k \)th section in the ladder configuration, numbering the sections consecutively from right to left.

Fig. 6.8 Type 1 RLC section
The synthesis of the type 1 RLC section may be carried out as follows: Evaluate \( Z_k(s) \) for \( s = -\alpha_k + j\beta_k \) as

\[
Z_k(-\alpha_k + j\beta_k) = R_k + jX_k.
\]  

(6-83)

For \( X_k \geq 0 \) the elements of \( Z_b^{(k)} \) are

\[
L_b^{(k)} = \frac{X_k}{\beta_k}
\]  

(6-84)

and

\[
R_b^{(k)} = \frac{\alpha_k}{\beta_k} X_k + R_k.
\]  

(6-85)

For \( X_k < 0 \) the elements of \( Z_b^{(k)} \) are

\[
C_b^{(k)} = \frac{\beta_k}{(\alpha_k^2 + \beta_k^2)|X_k|}
\]  

(6-86)

and

\[
R_b^{(k)} = \frac{\alpha_k}{\beta_k} |X_k| + R_k.
\]  

(6-87)

\( Y^* \) is given by

\[
Y_k^* = \frac{1}{Z_k - Z_b^{(k)}}.
\]  

(6-88)

Evaluate the following.
\[ N_k = \left[ (s + \alpha_k - j\beta_k) Y_k^* \right]_{s = (-\alpha_k + j\beta_k)} \quad \tag{6-89} \]

\[ M_k = 2 \times \text{real part of } N_k = 2 \text{ Re } [N_k] \quad \tag{6-90} \]

\[ r_k = \frac{2}{M_k} \text{ Re } \left[ N_k(\alpha_k + j\beta_k) \right] \quad \tag{6-91} \]

and

\[ \phi = r_k^2 - 2\alpha_k r_k + \alpha_k^2 + \beta_k^2 \quad \tag{6-92} \]

The elements of \( Y_a^{(k)} \) are given by

\[ L_a^{(k)} = \frac{1}{M_k}, \quad R_a^{(k)} = \frac{2\alpha_k - r_k}{M_k}, \quad C_a^{(k)} = \frac{M_k}{\phi} \quad \tag{6-93} \]

and

\[ R_c^{(k)} = \frac{\phi}{M_k r_k} \quad \tag{6-94} \]

2) Type 2 RLC section

The type 2 RLC section is shown in Fig. 6.9. The principal branch, \( Z_b^{(k)} \), produces a complex conjugate pair of transmission zeros at \( s = -\alpha_k \pm j\beta_k \), while the auxiliary branch, \( Y_a^{(k)} \), contributes to a transmission zero at \( s = \infty \) or \( s = 0 \).
Fig. 6.9 Type 2 RLC section

The procedure for synthesizing the type 2 RLC section is as follows: Evaluate $Y_k(s)$ for $s = -\alpha_k + j\beta_k$ as

$$Y_k(-\alpha_k + j\beta_k) = G_k + jB_k \quad (6-95)$$

For $B_k \geq 0$ the elements of $Y_a(k)$ are

$$C_a(k) = \frac{B_k}{\beta_k} \quad (6-96)$$

and

$$G_a(k) = \frac{\alpha_k}{\beta_k} B_k + G_k \quad (6-97)$$
For $B_k < 0$ the elements of $y^{(k)}_a$ are

$$L_a^{(k)} = \frac{\beta_k}{\left(\frac{2}{\alpha_k + \beta_k}\right)^2 |B_k|}$$  \hspace{1cm} (6-98)

and

$$G_a^{(k)} = \frac{\alpha_k}{\beta_k} |B_k| + G_k$$  \hspace{1cm} (6-99)

$Z^{*}_k$ is given by

$$Z^{*}_k = \frac{1}{Y_k - Y_a^{(k)}}$$  \hspace{1cm} (6-100)

Evaluate the following:

$$N_k = \left[ (s + \alpha_k - j\beta_k) Z^{*}_k \right] \hspace{1cm} s = -\alpha_k + j\beta_k$$  \hspace{1cm} (6-101)

$$M_k = 2 \text{Re} \left[ N_k \right],$$  \hspace{1cm} (6-102)

$$r_k = \frac{2}{M_k} \text{Re} \left[ N_k (\alpha_k + j\beta_k) \right]$$  \hspace{1cm} (6-103)

and

$$\phi = \frac{r_k^2}{\alpha_k} - 2 \alpha_k r_k + \alpha_k^2 + \beta_k^2$$  \hspace{1cm} (6-104)

The elements of $Z^{(k)}_b$ are given by
\[
C_b(k) = \frac{1}{M_k}, \quad R_b(k) = \frac{M_k}{2\alpha_k - r_k}, \quad L_b(k) = \frac{M_k}{\phi} \tag{6-105}
\]

and

\[
R^L(k) = \frac{M_k r_k}{\phi}. \tag{6-106}
\]

3) Type 3 sections

The type 3 section is shown in Fig. 6.10 along with a parallel resistance, \( R^{(h)}_a \). It is often necessary to remove a shunt resistance from the driving point admittance function, \( Y_h \), before attempting the synthesis of a type 3 section. \( R^{(h)}_a \) represents this resistance. The actual element values in the type 3 section are determined by replacing this section with an equivalent RC or RLC section. The element values of the RLC section may be determined simply by realizing \( Y_A \), \( Z_B \), and \( Y_C \) as driving point impedance or admittance functions. For such a realization to exist it is necessary that the condition,

\[
2\alpha_k \geq r_k, \tag{6-107}
\]

be satisfied.

Equivalence relationships may be derived which give the element values for a type 3 RC section in the form of either a bridged T or a twin T, depending on the relationship between \( 2\alpha_k \) and \( r_k \). These relationships are treated
in detail in reference [1] and will not be given here.

\[ Z_b(k) = \frac{B_k(s + r_k)}{s + 2\alpha_k s + \alpha_k^2 + \beta_k^2} \text{ for } s = -\alpha_k \pm j\beta_k \]

Fig. 6.10 Type 3 basic section with a shunt resistance \( R_a^{(h)} \)

The procedure for synthesizing a type 3 section is as follows: For \( \alpha_h = \alpha_k \) and \( \beta_h = \beta_k \), the following symbols are defined.

\[ g_h + jb_h = \begin{bmatrix} Y_h(s) \end{bmatrix} \quad s = -\alpha_h + j\beta_h \]  

(6-108)

\[ g'_h + jb'_h = \begin{bmatrix} \frac{dY_h(s)}{ds} \end{bmatrix} \quad s = -\gamma_k \]  

(6-109)

and
\[ g_n^* = b_h \frac{b_h - \beta_h g_h' + \alpha_h b_h'}{(\alpha_h/\beta_h) b_h - \alpha_h g_h' - \beta_h b_h'} \]  \quad (6-110)

If \( g_h < g_h^* \), \( Y_h \) cannot be synthesized in a type 3 section.

For \( g_h \geq g_h^* \), the synthesis is carried out as follows:

Remove the shunt impedance \( R_a^{(h)} \), where

\[ R_a^{(h)} = \frac{1}{g_h - g_h^*} \]  \quad (6-111)

\( Y_k \) is given by

\[ Y_k = Y_h - \frac{1}{R_a^{(h)}} \]  \quad (6-112)

\( g_k \) and \( b_k \) are defined as

\[ g_k + jb_k = \left[ Y_k(s) \right] \quad s = -\alpha_k + j\beta_k \]  \quad (6-113)

\( T_k \) and \( \Lambda_k \) are then given by

\[ T_k = \frac{1}{\beta_k} \frac{\alpha_k^2 + \beta_k^2}{(g_k/b_k) + (\alpha_k/\beta_k)} \]  \quad (6-114)

and

\[ A_k = \frac{1}{b_k} \frac{g_k^2 + b_k^2}{(g_k/b_k) + (\alpha_k/\beta_k)} \]  \quad (6-115)
respectively. \( Z_k^* \) is given by

\[
Z_k^* = \frac{1}{Y_k - Y_A^{(k)}}. \tag{6-116}
\]

From \( Z_k^* \), \( B_k \) may be evaluated as follows:

\[
B_k = \left[ \frac{s^2 + 2\alpha_k s + \alpha_k^2 + \beta_k^2 \times Z_k^*}{s + \gamma_k} \right] \quad \tag{6-117}
\]

\( Y_{**} \) is given by

\[
Y_{**} = \frac{1}{Z_k^* - Z_k^B}. \tag{6-118}
\]

from which one may evaluate

\[
C_k = \left[ \frac{s + \gamma_k}{s} Y_{**} \right] \quad \tag{6-119}
\]

Using the expressions given above, one may easily determine an equivalent RC or RLC network [1, 10].

4) Type 4 sections

The type 4 section is shown in Fig. 6.11, along with a series resistance, \( R_b^{(h)} \). It is often necessary to remove a series resistance from the driving point impedance function, \( Z_h \), before attempting the synthesis of a type 4
section. $R_b^{(h)}$ represents this resistance. The actual element values in the type 4 section are determined by replacing this section with an equivalent RL or RLC section. The element values of the RLC section may be determined simply by realizing $Z_A$, $Y_B$, and $Z_C$ as driving point impedance or admittance functions. For such a realization to exist it is necessary that the condition

$$2\alpha_k \geq r_k$$

be satisfied. Equivalence relationships may be derived which give the element values for a type 4 RL section. These relationships are treated in detail in reference [1].

The procedure for synthesizing a type 4 section is as follows: For $\alpha_h = \alpha_k$ and $\beta_h = \beta_k$, the following symbols are defined.

$$r_h + jx_h = \left[ \frac{Z_h(s)}{s} \right], \quad \text{and} \quad \quad s = -\alpha_h + j\beta_h$$

$$r'_h + jx'_h = \left[ \frac{dZ_h(s)}{ds} \right], \quad \text{and} \quad \quad s = -\alpha_h + j\beta_h$$

and

$$r^*_h = \frac{x_h - \beta_h r'_h + \alpha_h x'_h}{x_h - \alpha_h r'_h + \beta_h x'_h} \quad (6-123)$$
If \( r_h < r_h^* \), \( Z_h \) cannot be synthesized in a type 4 section. For \( r_h \geq r_h^* \), the synthesis is carried out as follows:

Remove the series impedance \( R_b^{(h)} \), where

\[
R_b^{(h)} = r_h - r_h^* \quad (6-124)
\]

\( Z_k \) is given by

\[
Z_k = Z_h - R_b^{(h)} \quad (6-125)
\]

\( r_k \) and \( x_k \) are defined as
\[ r_k + jx_k = \left[ Z_k(s) \right] \quad s = -\alpha_k + j\beta_k \]  

(6-126)

\( \gamma_k \) and \( A_k \) are then given by

\[ \gamma_k = \frac{1}{\beta_k} \frac{\alpha_k^2 + \beta_k^2}{(r_k/x_k) + (\alpha_k/\beta_k)} \]  

(6-127)

and

\[ A_k = \frac{1}{x_k} \frac{r_k^2 + x_k^2}{(r_k/x_k) + (\alpha_k/\beta_k)} \]  

(6-128)

respectively. \( Y^* \) is given by

\[ Y^*_k = \frac{1}{Z_k - Z_{(k)}} \]  

(6-129)

From \( Y^*_k \), \( B_k \) may be evaluated as follows:

\[ B_k = \left[ \frac{s^2 + 2\alpha_k s + \alpha_k^2 + \beta_k^2}{s + \gamma_k} \right] \times y^*_k \quad s = -\alpha_k + j\beta_k \]  

(6-130)

\( Z^{**} \) is given by

\[ Z^{**}_k = \frac{1}{\frac{y^*_k}{Y_k} - \frac{y^*_B}{Y_B}} \]  

(6-131)

from which one may evaluate
From these expressions, one may easily determine the equivalent RL or RLC network.

6.5 Illustrations of the Synthesis of RLC Networks

For the first illustration, a transfer function will be considered whose denominator polynomial is of second degree in s. Let it be required to synthesize

\[ Y_T(s) = -H \frac{s^2 + 2s + 2}{s^2 + 12s + 45} \]  

(6-133)

in an RLC network which is terminated in a normalized load, \( R_2 = 1 \). \( Y_T(s) \) is not positive real as may be verified by applying the condition given in (6-23). As described in section 6.1, a very simple realization may be effected by choosing \(-y_{21}\) as

\[ -y_{21} = K \frac{s^2 + 2s + 2}{s} \]  

(6-134)

For this choice of \(-y_{21}\), \( y_{22} \) has the form

\[ y_{22} = \frac{K}{H} \left[ \frac{s^2 + 12s + 45}{s} \right]^{-1} \]  

(6-135)

\[ = \frac{K}{H} \left[ s + 12 + \frac{45}{s} \right]^{-1} \]  

(6-136)
From condition 3 of section 2.1, it is necessary that \( H \leq 1 \).
For \( K = 1/4 \) and \( H = 1 \) the resulting normalized network is shown in Fig. 6.12.

Fig. 6.12 First network realization for \( Y_T(s) \) as given in (6-133)

The possibility of choosing a pair of complex conjugate poles for \(-y_{21}\) will now be considered. Fig. 6.13 shows a region, \( R_1 \), in which a complex conjugate pair of poles may be chosen, with the result that \( y_{22} \) is positive real. If poles are placed in region \( R_2 \), both \( y_{22} \) and \(-y_{21}\) are positive real and the synthesis of \( N'' \) can be carried out using only two-terminal techniques. One possible choice of poles, which assures that both \( y_{21} \) and \( y_{22} \) are positive real, gives

\[-y_{21} = K \frac{s^2 + 2s + 2}{s^2 + 6s + 10} \]  \hspace{1cm} (6-137)

For this choice of \(-y_{21}\), \( y_{22} \) is given by
\[ y_{22} = \frac{K}{H} \left[ \frac{s^2 + 12s + 45}{s^2 + 6s + 10} \right]^{-1} \]  
(6-138)

Fig. 6.13 Regions in which the poles of \(-y_{21}\) may be chosen, for \(Y_T(s)\) as given in (6-133)

K/H should be chosen so that \([y_{22} - (-y_{21})]\) is a positive real driving point function. A choice of \(H = 1/2\) and \(K = 1\) permits the realization of \([y_{22} - (-y_{21})]\) with four elements. The resulting network is shown in Fig. 6.14.

This illustration bears out the fact that, in general, an excessive number of circuit elements are required in the realization when more than the minimum required number of poles are selected for \(-y_{21}\) (or \(z_{21}\)).
In the second illustration a transfer function will be considered whose denominator polynomial is of third degree in $s$. Let it be required to synthesize

$$Z_T(s) = H \frac{s^3 + 2s^2 + 2s}{s^3 + 7s^2 + 17s + 15} \quad (6-139)$$

in an RLC network which is terminated in a normalized load, $R_2 = 1$. It may easily be verified that $Z_T(s)$ is not positive real, and therefore, cannot be synthesized by the methods of Chapter III. First, the poles and zeros of $Q(s)/P(s)$ are plotted in the complex plane as shown in Fig. 6.15. In following the procedure given in section 6.3, one finds that

$$\phi_1 = 45^\circ \quad (6-140)$$
The unit vectors $A_1$ and $A_2$ are then constructed at the angles

$$\theta_1 = 225^\circ$$

and

$$\theta_2 = 135^\circ ,$$

respectively. The extension of $A_1$ intersects the negative $\sigma$ axis at $s = -2$, while the extension of $A_2$ does not intersect the negative $\sigma$ axis. Therefore, a single zero at $s = 0$ may be chosen for $z_{21}$, and a single pole may be chosen at any point on the negative $\sigma$ axis which lies to the left of...
s = -2, including s = \infty.

First, consider the choice of \( z_{21} \),

\[
z_{21} = K \frac{s}{s + 2}.
\]  (6-144)

Then,

\[
z_{22} = \frac{K}{H} \left[ \frac{s^3 + 7s^2 + 17s + 15}{(s + 2)(s + 1 - j)(s + 1 + j)} \right] - 1. \]  (6-145)

After \( Q(s)/P(s) \) is partial fractioned, \( z_{22} \) has the form

\[
z_{22} = \frac{K}{H} \left[ 1 + \frac{1/2}{s + 2} + \frac{5/2s + 5}{s^2 + 2s + 2} \right] - 1. \]  (6-146)

For \( H = 1/2 \) and \( K = 1 \), the remainder of the synthesis may be carried out using only two-terminal techniques. The resulting network is shown in Fig. 6.16.

![Network Diagram](image-url)

Fig. 6.16 First network realization for \( Z_T(s) \) as given in (6-139)
Second, consider the choice of \( z_{21} \),

\[
z_{21} = Ks. \tag{6-147}
\]

For this choice of \( z_{21} \), \( z_{22} \) has the form

\[
z_{22} = \frac{K}{H} \left[ \frac{s^2 + 7s^2 + 17s + 15}{(s + 1 - j)(s + 1 + j)} \right]^{-1}. \tag{6-148}
\]

If \( H \) and \( K \) are chosen as \( H = 1 \) and \( K = 1 \), respectively, the resulting network has the form shown in Fig. 6.17.

Fig. 6.17 Second network realization for \( Z_T(s) \) as given in (6-139)

The third illustration will demonstrate the assignment of a pair of complex conjugate transmission zeros as zeros of \( Z_{21} \). Let it be required to synthesize

\[
A(s) = -H \frac{(s + 3)(s + 2 - j3)(s + 2 + j3)}{(s + 1)(s + 2 - j2)(s + 2 + j2)} \tag{6-149}
\]

in an RLC network which is terminated in a normalized
load, \( R_2 = 1 \). First, the poles and zeros of \( Q(s)/P(s) \) are plotted in the complex plane as shown in Fig. 6.18. When the procedure given in section 6.3 is followed one finds that

\[
\phi_1 \approx 34^\circ \tag{6-150}
\]

and

\[
\psi_1 \approx 198^\circ \tag{6-151}
\]

Fig. 6.18 Permissible pole locations of \( z_{21} \) for \( A(s) \) as given in (6-149)

The unit vectors \( A_1 \) and \( A_2 \) are then constructed at the angles

\[
\theta_1 \approx 52^\circ \tag{6-152}
\]

and
\[ \theta_2 \approx -16^\circ , \]  

respectively. It is seen that neither the extension of \( A_1 \) nor the extension of \( A_2 \) intersects the negative \( \sigma \) axis. Therefore, it is necessary to assign the complex conjugate pair of transmission zeros at \( s = -2 \pm j2 \) as zeros of \( z_{21} \). Next, the pole location (or locations) for \( z_{21} \) must be chosen. Consider the choice of a simple pole at \( s = 0 \), then \( z_{21} \) and \( z_{22} \) are given by

\[
z_{21} = k \frac{(s + 2 - j3)(s + 2 + j3)}{s} \tag{6-154}
\]

and

\[
z_{22} = \frac{k}{H} \left[ \frac{(s + 1)(s + 2 - j2)(s + 2 + j2)}{s(s + 3)} \right] -1 \tag{6-155}
\]

After \( z_{21} Q(s)/P(s) \) is partial fractioned, \( z_{22} \) has the form

\[
z_{22} = \frac{k}{H} \left[ s + 2 + \frac{8/3}{s} + \frac{10/3}{s + 3} \right] -1 \tag{6-156}
\]

For \( H = \frac{8}{3 \times 13} \) and \( K = \frac{8}{3 \times 13} \), the remainder of the synthesis may be carried out using only two-terminal techniques. (An inspection of the prescribed transfer function as given in (6-149) indicates that the maximum possible value of \( H \) is \( 8/39 \).) The resulting network is shown in Fig. 6.19.
An illustration will now be given which demonstrates the synthesis of $N'$ as a type 3 RC network. The method which is used is intended to illustrate this synthesis procedure and is not necessarily the most advantageous approach for this particular problem. Coefficients in the transfer function are chosen so that one may easily follow through the procedure. Let it be required to synthesize

$$A(s) = -H \frac{(s + 2)(s^2 + 2s + 5)}{(2s + 3)(2s + 5)(s + 5)}$$

(6-157)

in an RC network which is terminated in a load, $R_2 = 1/3$. Consider the assignment of the complex conjugate pair of transmission zeros at $s = -1 \pm j2$ as zeros of $z_{21}$. In
order to assure that the product, \( z_{21}Q(s)/P(s) \), is positive
real and that the required decomposition is possible, two
poles are chosen for \( z_{21} \) on the negative portion of the \( \sigma \)
axis at \( s = -1 \) and \( s = -3 \). For this selection, \( z_{21} \) and
\( z_{22} \) are given by

\[
z_{21} = K \frac{s^2 + 2s + 5}{s^2 + 4s + 3}
\]

and

\[
z_{22} = \frac{K}{H} \left[ \frac{(2s + 3)(2s + 5)(s + 5)}{(s + 1)(s + 2)(s + 3)} \right]^{-1/3}
\]

respectively. After \( z_{21}Q(s)/P(s) \) is partial fractioned,
\( z_{22} \) has the form

\[
z_{22} = \frac{K}{H} \left[ \frac{4 + \frac{6}{s + 1} + \frac{3}{s + 2} + \frac{3}{s + 3}}{} \right]^{-1/3}
\]

A choice of \( K/H = 1/3 \) gives

\[
z_{22} = 1 + \frac{2}{s + 1} + \frac{1}{s + 2} + \frac{1}{s + 3}
\]

The synthesis of the network \( N' \), whose open circuit param-
eters are

\[
z'_{21} = K \frac{s^2 + 2s + 5}{s^2 + 4s + 3}
\]

and
\[
\frac{z'}{z} = \frac{s^2 + 7s + 10}{s^2 + 4s + 3}
\]

may be carried out according to the procedure given in section 6.4. A type 3 section, as shown in Fig. 6.10, is first predicted. This section is to produce the complex pair of transmission zeros at \( s = -1 \pm j2 \). When the relationships given in (6-108) through (6-119) are evaluated, it is found that

\[
\begin{align*}
Y_0 &= \frac{s^2 + 4s + 3}{s^2 + 7s + 10}, & g_0 + jb_0 &= 2/5 + j2/5, \\
g'_0 + jb'_0 &= 1/25 - j4/25, & g^*_0 &= 2/15, \\
R_a^{(0)} &= 15/4, & Y_1 &= 1/15 \left[ \frac{11s^2 + 32s + 5}{s^2 + 7s + 10} \right], & (6-164) \\
g_1 + jb_1 &= 2/15 + j6/15, & \gamma_1 &= 3, \\
A_1 &= 8/15, & B_1 &= 25, & C_1 &= 2/15, \\
\end{align*}
\]

and

\[
Y_2 = 1/15.
\]

From (6-164) the equivalent RC network shown in Fig. 6.20 is obtained.
Fig. 6.20 Network realization for $A(s)$ as given in (6-157)
CHAPTER VII

CONCLUSIONS

A new philosophy for the design of ladder networks has been described and illustrated in which a sharing of transmission zeros is effected between the network parameters $z_{21}$ and $z_{22}$ (or $-y_{21}$ and $y_{22}$). This approach can be applied in the synthesis of RC, RL, and RLC networks. Its value in the synthesis of LC networks depends on the network termination. For a resistive termination, it was shown that the method could not be applied advantageously, while the special cases of open-circuit and short-circuit terminations can be handled very effectively. The advantages of this method lie in the fact that the synthesis is, in general, divided into two parts, a two-terminal synthesis problem, and a four-terminal synthesis problem. This division usually reduces the labor involved in the realization and often requires less elements in the final network than are required by other methods. Throughout the presentation, methods have been stressed which allow as much of the synthesis as possible to be carried out by the use of two-terminal techniques. This approach takes full advantage of the simplifying properties of the zero sharing method. There are, however, in a great many
synthesis problems, several possible degrees of zero sharing which may be effected, and in a particular application it may be desirable to assign more than the minimum required number of transmission zeros to $z_{21}$ (or $-y_{21}$).

In Chapter IV a procedure was given for synthesizing RC and RL networks from transfer functions having real negative transmission zeros. In Chapter V a procedure was presented for synthesizing LC networks from transfer functions having purely imaginary transmission zeros. In each of these cases the zero sharing method is quite general, in the sense that it may be systematically carried out regardless of the complexity of the transfer function, i.e., regardless of the degree of the numerator or denominator polynomial. It should be pointed out that all zero sharing methods require that both $P(s)$ and $Q(s)$ be placed in factored form, a job which is usually very tedious for high degree polynomials.

In contrast to the general nature of the procedures of Chapter IV and Chapter V, the zero sharing method is limited to transfer functions of rather low complexity when networks are to be synthesized whose transfer functions have combinations of different types of transmission zeros. These different types are negative real, complex conjugate, and purely imaginary transmission zeros. The treatment of such transfer functions was limited to those whose denominator polynomial $Q(s)$ was of third degree or less. The
extension of the zero sharing method to transfer functions of higher complexity was briefly discussed. The major difficulty in applying the zero sharing approach to transfer functions of high complexity lies in the selection of poles and zeros for $z_{21}$ (or $-y_{21}$). The choice of $z_{21}$ (or $-y_{21}$) was shown to be equivalent to the following problem. For a prescribed function, $Q(s)/P(s)$, replace certain of its poles in such a way that the new function, $Q(s)/P'(s)$, has the following properties:

1) $Q(s)/P'(s)$ is positive real.

2) $Q(s)/P'(s)$ may be broken into a sum of two parts, each of which is positive real. One part must have for its poles all of the poles which were added in forming $Q(s)/P'(s)$. This part must not have any of the poles of the original function, $Q(s)/P(s)$. The second part must have only the poles of the original function, $Q(s)/P(s)$.

A method was described in Chapter VI for selecting $P'(s)$ which relied heavily on root locus techniques. This method assures that the residues of $Q(s)/P'(s)$ at each of its poles are such that the decomposition in property 2 is possible. Where $P'(s)$ is to have more than two complex conjugate pairs of zeros, the choice of these zeros becomes difficult. This is due to the fact that the residues of $Q(s)/P'(s)$ depend in a rather complicated way on the zero locations of $P'(s)$. Although the possibility of encountering the above difficulty does exist, a great many
transfer functions of high complexity may be synthesized very easily by the zero sharing method.
LIST OF REFERENCES


BIOGRAPHICAL SKETCH

William Austin Walter was born October 28, 1937, at Pittsburg, Pennsylvania. In May, 1955, he was graduated from Shades Valley High School, Birmingham, Alabama. In March, 1960, he received his Bachelor of Electrical Engineering degree from Auburn University. He worked as Graduate Assistant in the Electrical Engineering Department there from March, 1960, to December, 1961, at which time he received his Master of Electrical Engineering degree. He entered the University of Florida in February, 1962 and was employed as a Graduate Assistant in the Electrical Engineering Department until September, 1963, when he received a College of Engineering Fellowship to continue his studies toward his Doctor of Philosophy degree at the University of Florida.

William Austin Walter is married to the former Elizabeth Patton Meacham and is the father of one child. He is a member of Eta Kappa Nu, Tau Beta Pi, and the Institute of Electrical and Electronics Engineers.
This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Engineering and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 19, 1964

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