APPLICATION OF THE WIGNER DISTRIBUTION TO PROBLEMS IN TIME-VARYING SIGNAL ANALYSIS

By

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APPLICATION OF THE WIGNER DISTRIBUTION TO PROBLEMS IN TIME-VARYING SIGNAL ANALYSIS

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The Wigner distribution (WD) is a bilinear time–frequency signal representation that possesses a number of characteristics desirable in time–varying signal analysis. This dissertation investigates and extends the application of the WD and its derivatives to new areas.

The WD is placed in the general framework of nonstationary signal analysis and time–frequency distributions. The WD has been previously applied to the estimation of instantaneous frequency of phase–modulated signals. Analytical expressions for the performance of the discrete–time WD (DWD) in estimating the instantaneous frequency of linear f.m. signals in additive white Gaussian noise are derived and are verified using simulation. It is shown that the peak of the DWD provides an optimal estimate of the instantaneous frequency for such signals at sufficiently high signal–
to-noise ratios. The applicability of these results to the general case of nonlinear f.m. signals is discussed. Based on a methodology suggested by the WD, a new signal kernel is derived whose spectrum is concentrated at the frequency-rate of the arbitrary phase-modulated signal. A computationally simple implementation of the frequency-rate estimator is proposed and computer simulations are used to investigate the noise performance of this method.

The WD is applied to the detection of narrowband transient signals of unknown waveform and arrival time in additive noise. A specific class of received signals is considered, motivated by examples from underwater acoustics and vibration signal processing. A detector based on the WD is proposed that provides good time localization of the transient waveform and is not strongly sensitive to changes in the pulse duration or frequency of the received transient signal. The performance of the detector in additive Gaussian noise is analyzed.

The field of vibration signal processing is rich with potential applications for time-frequency methods. The advantage provided by the WD over the traditional short-time spectral magnitude is demonstrated by examples in the vibration monitoring of machinery, where nonstationary signals arise due to speed variations and other time-varying phenomena. Specifically, we consider the problems of envelope detecting time-varying narrowband signals, and the detection of periodic transients of low energy from the sidebands of a high frequency, time-varying carrier component.
CHAPTER 1
INTRODUCTION

Signal analysis tools for nonstationary, or time-varying signals, need to incorporate the time dependence of signal characteristics in their formulation. Most often the signal feature chosen to be represented is frequency, giving rise to time-frequency representations. Alternative signal features such as the time-extent of elementary signal components that comprise the input signal may also be of interest, giving rise to time-scale methods.

The short-time Fourier transform (STFT) and the wavelet transform (WT) are linear signal transforms that analyze the signal in terms of elementary waveforms indexed by time and another parameter, which is frequency in the case of the STFT, and time-scale in the case of the WT. When the scale parameter in the WT is chosen inversely proportional to frequency, we get a time-frequency representation similar to the STFT, but characterized by time-frequency resolution that is frequency dependent, instead of fixed. When one is interested in the energy distribution of a signal in time and frequency, it is natural to consider transforms that are characterized by a bilinear dependence on the signal. If the further constraint of invariance with respect to shifts in both time and frequency is applied, the admissible transforms reduce to those which belong to the "Cohen's class." Similarly if the transform is constrained to be invariant with respect to shifts and dilations in time, we get the affine distributions [Fla89a].

Although the question of a joint signal representation in time and frequency was first addressed as early as 1946 by Gabor and soon after by Ville, it is only in the past decade that there has been a surge of activity in the application of time-frequency
distributions to problems in signal processing. A number of properties that are desirable in a time–frequency signal representation have been identified. These are generally based on properties recognized in the spectral density functions of stationary signal analysis with the addition of an explicit time dependence. It turns out that there is no single distribution that satisfies all the requirements. Much research has been addressed toward obtaining distributions that satisfy specific properties and their interpretation. Various theoretical aspects of these distributions have been studied and a number of practical applications proposed.

The Wigner distribution (WD), a member of Cohen's class of bilinear time–frequency representations, possesses a large number of desirable properties. Of the representations that satisfy the finite support property and preserve inner products, the WD provides the most concentration in the time–frequency plane for a given signal. The WD also satisfies the marginal and local properties in time and frequency. These characteristics have made it an attractive tool in time-varying signal analysis.

A primary objective of this work is to extend the existing applications of time–frequency methods, based on the WD and its derivatives, to new areas involving time–varying signals and systems. Some of the applications considered here have traditionally benefitted from short–time spectral methods. The performance of WD–based methods is investigated and compared with available techniques.

A review of time–frequency distributions is presented in Chapter 2 and an attempt is made to place the WD in the perspective of time–frequency signal representations and spectral analysis methods. The properties of the WD that form the bases for the applications considered in this work are discussed in depth. The WD of multicomponent signals is characterized by interference terms that obscure its interpretation as a signal energy distribution. The theory of interference in the WD is given and the methods available for its suppression are described.
In Chapter 3 we address the problem of the joint estimation of frequency and frequency-rate of arbitrary phase-modulated signals, one that arises in situations where the velocity of an object and the curvature of its path must be measured simultaneously. The ability of the WD to provide a concentrated distribution at the instantaneous frequency of a narrowband signal has been exploited widely in practice. We examine analytically the performance of the peak of the discrete-time signal WD in the estimation of the instantaneous frequency of f.m. signals in additive noise. An exact expression for the variance of the estimate for the case of a linear f.m. signal in additive white Gaussian noise is derived and verified using computer simulations. The extension of this result to the more general case of nonlinear f.m. signals is discussed. Based on a methodology suggested by the WD, a new signal transform is obtained by which the frequency-rate of an arbitrary f.m. signal can be estimated. A computationally simple implementation of the frequency-rate estimator is proposed and computer simulations are used to investigate its performance in noise.

Chapter 4 investigates the application of the WD to the detection of unknown transient signals in additive noise. Since the received signal is inherently nonstationary, it is expected to benefit from the application of time-frequency methods. Motivated by examples from underwater acoustics and vibration signal processing, we consider transient signals that are monochromatic and of narrow time durations relative to the observation interval. Further the transient waveform is assumed to be embedded in a noisy background comprised of deterministic and random components. We seek to detect the transient signal and localize it in time. The viability of the spectrogram (which is closely related to existing methods for such problems) and the WD in achieving this are explored and compared. The smoothed-pseudo-WD allows the independent control of time- and frequency-resolution in the signal representation, and hence provides a more flexible tool in applications to nonstationary situations than the fixed-window spectrogram. A detector based on the WD is pres-
ented that provides good time–frequency resolution and is relatively insensitive to the
time duration of the transient signal. The noise properties of the WD estimate are
discussed and the performance of the detector for transient signals in additive Gaussian noise is derived. The detection algorithm is applied to a sample of experimental-
ly obtained milling machine data containing transient vibration due to tool chatter.

The spectral analysis of vibration signals is an integral component of machinery
monitoring and fault detection. It is typically carried out using the magnitude of the
short–time Fourier transform. While this method provides satisfactory results when
the signal is stationary over the duration of the analysis window, there are situations
when the vibration signal is nonstationary due to the time–varying nature of the
source of vibration. In Chapter 5 we demonstrate, by examples, the advantage pro-
vided by WD–based methods in the monitoring of nonstationary vibration signals.
Specifically, we investigate the application of the WD to the problems of envelope
detection of narrowband, time–varying signals, and the detection of periodic tran-
sients in time–varying signals.
CHAPTER 2
A REVIEW OF TIME-FREQUENCY DISTRIBUTIONS

2.1 The Need For A Time-Frequency Representation

The Fourier representation of a signal enables the decomposition of the signal into its individual frequency components and yields the energy density of each of these components. Standard spectral estimation methods are based on the implicit assumption that the signals being analyzed are characterized by a stationary spectrum or a spectrum that does not vary with time. In a great number of signal processing applications such an assumption is not justified. The spectral characteristics of signals arising in nature often evolve in time due to some changing characteristic of the underlying process. Examples of such time-varying signals are speech, seismic signals and biological signals. The phase-modulated signals of radar and sonar, as well as the acoustic signals from rotating machinery, also belong in this class. The analysis of such time-varying signals cannot be carried out entirely using standard Fourier techniques or other spectral estimators based on the stationarity assumption. For while these methods give the spectral components of the signal, the time localization of the frequency components is often concealed in the values of the different phases, making it difficult to access this important element in the investigation of such signals.

For the past several decades the common tool for time-varying spectral analyses has been the short-time Fourier transform, which is implemented using either a bank of time-windows or a bank of bandpass filters. The former is more common and is based on decomposing the signal into short, and possibly overlapping, time
segments in each of which the signal is assumed to be approximately stationary, and computing the Fourier spectrum of each segment to estimate the frequency content of the signal at that time. There are many situations in which such an analysis is neither adequate nor appropriate. This can occur when the signal spectrum is changing so rapidly that it is difficult to find a suitable short-time window. Any short-time spectral energy measurement requires a minimum effective bandwidth and observation time, and frequency shifts within this bandwidth as well as power variations within this observation time remain obscured. In the case of the short-time spectrum, however, the amount of time–frequency trade-off is fixed by the arbitrary, often predetermined, selection of windows/filters and the reciprocal relation between spectral and temporal resolutions. These considerations have given rise to the need to turn to the practical application of joint functions of time and frequency, historically termed “time–frequency distributions” in order to describe the energy density of a signal in time and frequency simultaneously.

In this chapter we give a historical review of time–frequency representations and summarize briefly the research to date. We discuss in greater depth the Wigner Distribution (WD), whose properties and applications are the subject of this thesis. We also discuss the spectrogram and its relation to the WD.

2.2 A Brief Historical Review

The question of a joint time–frequency distribution was first addressed by Gabor [Gab46] and Ville [Vil48], inspired by similar developments in quantum mechanics, where there is a partial mathematical resemblance to time–frequency analysis. Gabor proposed an expansion of a given waveform into the sum of elementary signals of “minimum” spread in time and frequency each with different center frequency and epoch. Since the elementary signals are chosen to be concentrated in time and frequency, a local measure of time–frequency signal energy is obtained.
Helstrom [Hel66] made Gabor's representation exact by using an integral expansion. The elementary signals of Gabor's representation overlap, and hence the kernel does not represent the energy distribution in time and frequency. Ville [Vil48] attempted to derive an exact signal energy distribution, based on one presented by Wigner in 1932 to calculate the two-dimensional probability distribution in time and frequency of a quantum mechanical particle, defining what is now known as the Wigner–Ville distribution, or simply, the Wigner distribution. Its characteristic function, introduced in modified form to radar theory by Woodward, is well known as the ambiguity function. Page [Pag52] defined an "instantaneous power spectrum" as the rate of change of the energy density spectrum of the signal segment from $-\infty$ to $T$, as $T$ is increased. Levin [Lev64] used the same definition for the segment $(T, \infty)$ and defined a new function as the average of both types of instantaneous power spectra. In an insightful paper, Rihaczek [Rih68] derived a complex energy density function based on the concepts of the one-dimensional energy density functions in time and frequency. Each of the distributions enumerated here has been derived independently from plausible time–frequency concepts and each has its own set of distinct properties that make it a good candidate for time–frequency analyses. It was realized by Cohen [Coh66] that these time–frequency representations have more in common than is readily apparent, and he proposed what has now become known as the Cohen's class of bilinear time–frequency representations, a unifying framework for several of the time–frequency distributions that have been proposed in the literature. Some recent research has been directed toward deriving representations starting from the definition of Cohen's class to satisfy specific constraints required by a particular application. Examples of this approach are the Choi–Williams distribution [Cho89] and the cone–shaped kernel distribution proposed by Zhao, Atlas and Marks [Zha90].
2.3 A Summary of Work to Date

Over the past decade there has been a renewed interest in the application of time–frequency distributions to problems in signal processing. Starting with the work of Claasen and Mecklenbrauker [Cla80a,b,c] who presented a set of fine papers on the WD, there has been much research on the theoretical aspects and on the practical applications of time–frequency distributions. Much research has been directed toward establishing distributions that satisfy specific properties deemed desirable in time–frequency analyses and in their interpretation. The quality of the estimates of time–frequency energy distribution and methods to improve these have also been studied widely. Other theoretical aspects that have received attention include the inversion of the representation and the inter–relationships between the different distributions [Coh89]. Efficient digital implementations have been developed [Wil87]. Time–frequency distributions have been applied in various fields where nonstationary signals arise. The applications have generally been based upon one of the following i) use of the distribution to present graphically the time–varying characteristics of the signal or ii) the use of a particular property of the time–frequency distribution to describe the evolution with time or frequency of an important signal parameter. We give here some examples of the wide variety of applications of time–frequency distributions that have appeared in the literature over the past few years.

Imberger and Boashash exploited the instantaneous frequency property of the WD in the analysis of geophysical and oceanographic signals [Imb86], where the time–evolution of a significant parameter is represented by the variation of signal frequency with time. Janse and Kaizer [Jan83] used the WD to graphically represent the nonstationary signals encountered in loudspeaker design. In a similar manner, the WD has been used as an aid in the design and analysis of ultrasonic transducers [Mar86]. In the area of biological signal processing, time–frequency representations have been applied to the classification of muscle sounds [Bar90] and to event–related
potentials [Cho87]. These signals are inherently nonstationary and time–frequency distributions provide consistent patterns rich in detail, making the classification easier than with standard time–domain or frequency–domain techniques alone. Starting with the work of Chester [Che84], several researchers have applied the WD in the analysis and recognition of speech. The difficulties presented by artifacts in the analysis of speech have been sought to be overcome by the use of smoothed variants of the WD [Ril87]. A WD based detector has been developed for the detection of known signals in noise [Kum84]. A general approach to the detection problem in the time–frequency plane has been formulated by Flandrin [Fla88]. Breed and Posch [Bre84] utilized the spatial WD to estimate the range of a target based on the quadratic phase variation due to waveform curvature across the array. The WD has also been applied in image analysis. The WD, defined for a two–dimensional image, is used to achieve the invariant recognition of objects [Jac84] and applied to the segmentation of textured images [Ree90]. In addition to its superiority in joint space/spatial–frequency resolution, the WD also has the advantage that it implicitly encodes phase in a single real–valued function. This is a valuable characteristic in applications to vision where the phase is a critical component.

2.4 Cohen’s Class of Bilinear Representations

A generalized class of bilinear time–frequency representations has been established by Cohen [Coh66] which provides a unifying framework for a number of distributions that have been proposed for time–frequency signal analysis. Cohen’s class is defined by,

$$C_{t}(t, \omega; \phi) = \frac{1}{2\pi} \int_{\xi} \int_{\tau} \int_{u} f(u + \tau/2) f^*(u - \tau/2) \phi(\xi, \tau) e^{j(\xi - \tau \omega - \xi u)} du d\tau d\xi$$  \hspace{1cm} (2–1)

where $$\phi$$ is an arbitrary kernel that defines the particular distribution, and all integrals go from $$-\infty$$ to $$\infty$$. The kernel is chosen to be independent of the signal and of
time and frequency, which results in the bilinearity and time–frequency shift invariance that characterizes the Cohen's class of distributions. That (2-1) provides a reasonable definition of a general time–frequency representation can be understood by rewriting it as the Fourier transform of a generalized autocorrelation function $K(t,\tau)$ of the signal $f(t)$ [Cho89]

$$C_t(t, \omega; \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(t, \tau)e^{-j\omega \tau} d\tau$$  \hspace{1cm} (2-2)

where

$$K(t, \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u + \tau/2)f^*(t - \tau/2)\phi(\xi, \tau)e^{j(\xi - \xi_0)u}dud\xi$$  \hspace{1cm} (2-3)

We see that the generalized autocorrelation function is a time–averaged and lag–weighted value of the time–dependent autocorrelation function of the signal. In general, the two weighting functions represented by the kernel $\phi(\xi, \tau)$ are not independent. Depending on the choice of the kernel different bilinear time–frequency representations are obtained. Based on the requirements of time–varying signal analysis several properties desirable of a time–frequency distribution have been identified [Cla80c]. These include shift invariance (which holds for all members of the Cohen's class), marginal properties by which the instantaneous power and spectral density of the signal are obtained, realness and positivity, obtention of instantaneous frequency and group delay from the local moments of the distribution, preservation of support in time and frequency, inner product conservation, and compatibility with linear filtering and modulation. Furthermore, a valuable characteristic of any time–frequency distribution would be for it to be concentrated mainly in those regions of the time–frequency plane where, according to intuition or experience, signal energy is expected to be located. Each of the above properties imposes different constraints on the kernel function, thus limiting the number of suitable representations obtainable.
2.5 The Wigner Distribution

The WD belongs to the Cohen's class of bilinear time-frequency distributions. It is defined for a signal $f(t)$ by,

$$W_f(t, \omega) = \int_{-\infty}^{\infty} f(t+\tau/2) f^*(t-\tau/2) \exp(-j\omega \tau) d\tau$$

and is obtained from (2-1) with the kernel $\phi(\xi, \tau) = 1$. The WD satisfies a large number of desirable properties. Since we refer to one or another of these throughout this thesis, these properties are summarized here. Proofs can be found in [Cla80a] and [DeB73].

2.5.1 Desirable Properties

Since the WD is a member of Cohen's class, time- and frequency-shifts of the signal are reflected as corresponding shifts in the WD time–frequency plane, so that for $g(t) = f(t - t_0)$, we get

$$W_g(t, \omega) = W_f(t - t_0, \omega)$$

For $g(t) = f(t) \exp(j\omega_0 t)$, we get

$$W_g(t, \omega) = W_f(t, \omega - \omega_0)$$

The WD is real-valued, being the Fourier transform of the conjugate symmetric inner product of the signal. However it is not always positive making it difficult to interpret as a true signal energy distribution.

The instantaneous signal power and the energy spectral density of the signal are obtained from the corresponding marginal quantities of the WD as,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} W_f(t, \omega) d\omega = |f(t)|^2$$
\[
\int_{-\infty}^{\infty} W_f(t, \omega) dt = |F(\omega)|^2 \quad (2-8)
\]

The WD preserves the time- and frequency-support of the signal, so that

\[
f(t) = 0 \text{ for } |t| > T \Rightarrow W_f(t, \omega) = 0 \text{ for } |t| > T \quad (2-9)
\]

\[
F(\omega) = 0 \text{ for } |\omega| > \Omega \Rightarrow W_f(t, \omega) = 0 \text{ for } |\omega| > \Omega \quad (2-10)
\]

However the WD is not necessarily zero at all time-instants and frequencies that the signal goes to zero [Coh87].

The first-order frequency moment of the WD is given by,

\[
\frac{1}{2\pi |f(t)|^2} \int_{-\infty}^{\infty} \omega W_f(t, \omega) d\omega = \Omega_f(t) \quad (2-11)
\]

where

\[
\Omega_f(t) = \text{Im} \frac{f'(t)}{f(t)} = \text{Im} \frac{d}{dt} \ln f(t) \quad (2-12)
\]

For complex signals written in the form \( f(t) = v(t) \exp(j\phi(t)) \) where \( v(t) \), the envelope, and \( \phi(t) \), the phase, are real functions, we get from eq. (2-12),

\[
\Omega_f(t) = \phi'(t) \quad (2-13)
\]

Hence the average frequency of the WD at any time is the derivative of the phase. This quantity can be interpreted as the instantaneous frequency of the signal for a certain class of signals. In an analogous manner, the first moment in time of the WD of the impulse response of a linear system turns out to be equal to the group delay [Cla80a].

The WD is invertible in that the signal \( f(t) \) can be recovered up to the constant factor \( f^*(0) \), as given by
\[ f(t) = \frac{1}{2\pi} f^*(0) \int_{-\infty}^{\infty} W_f(t/2, \omega) \exp(j\omega t) \, d\omega \quad (2-14) \]

This ambiguity is represented in the fact that the signals \( f(t) \) and \( f(t) \exp(j\alpha) \), where \( \alpha \) is a real constant, have the same WD.

An interesting relation exists between the inner product of two signals and that of the corresponding WDs, known as Moyal’s formula and is given by,

\[ | \int f(t)g^*(t) \, dt |^2 = 2\pi \int_{-\infty}^{\infty} W_f(t, \omega)W_g(t, \omega) \, dt \, d\omega \quad (2-15) \]

This property has been used to formulate the optimum detection of signals in the time–frequency plane [Fla88].

The WD preserves convolution and modulation. If \( g(t) = \int f(\tau)h(t-\tau) \, d\tau \) then

\[ W_g(t, \omega) = \int_{-\infty}^{\infty} W_f(\tau, \omega)W_h(t - \tau, \omega) \, d\tau \quad (2-16) \]

For \( g(t)=f(t)h(t) \),

\[ W_g(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_f(t, \eta)W_h(t, \omega - \eta) \, d\eta \quad (2-17) \]

In practice the WD is implemented after applying a finite–length, moving window to the data, centered at the time–instant of interest, giving rise to the pseudo–WD (pWD). For a signal \( f(t) \) and real, symmetric window \( h(t) \) the pWD is computed as

\[ pW_f(t, \omega) = \int_{-\infty}^{\infty} f(t + \tau/2)f^*(t - \tau/2)h(\tau/2)h^*(-\tau/2) \exp(-j\omega \tau) \, d\tau \quad (2-18) \]

The pWD is related to the WD by

\[ pW_f(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_f(t, \eta)W_h(0, \omega - \eta) \, d\eta \quad (2-19) \]

Since
\[ W_h(0, \omega) = \int_{-\infty}^{\infty} h^2(\tau/2) \exp(-j\omega\tau) d\tau \] (2-20)

is always a lowpass type of function, the pWD is a frequency smoothed version of the original WD. This implies that the marginal and local properties of the WD in time are maintained in the pWD, while the corresponding frequency properties are replaced by weighted frequency averages.

Further, it has been shown [Jan82] that of members of the Cohen's class that satisfy the finite support property and the Moyal's formula, the WD has the least amount of global spread about its center of gravity for any given signal. When signals of the type \( v(t) \exp(j\phi(t)) \) are considered, where \( v(t) \) is a slowly-varying, real function, and \( \phi(t) \) is a smooth, real function, so that the signal can be considered to be phase-modulated with a slowly-varying envelope, the WD is concentrated about the curve \( \phi'(t) \) and has significantly lower spread compared to any other time-frequency distribution. In the case of linear frequency-modulated signals, the WD at any time-instant is a delta function at the instantaneous frequency. An enlightening interpretation of this general behavior of the WD for phase-modulated signals, presented in [Fla84b], is given here. The WD of a signal \( f(t) \) computed over a finite data window to obtain the pWD, can be regarded as an STFT performed on a modified version of the signal given by the inner product \( f(t+\tau/2)f^*(t-\tau/2) \). The bandwidth of the short-time spectrum is related to the frequency deviation of the signal within the data window. Let us assume, without loss of generality, that the data window is centered at \( t=0 \). Then for a signal given by \( f(t) \) within the window, the instantaneous frequency of the signal is represented by \( \nu_1(t) \). However the "instantaneous frequency" of the inner product \( f(\tau/2)f^*(-\tau/2) \) is given by the quantity
\[ \frac{1}{2} \left[ \nu_1(\frac{\tau}{2}) + \nu_1(-\frac{\tau}{2}) \right] \]. This is illustrated by Figure 2-1.
We see that the total frequency deviation within the window of the inner product is less than or equal to that of the signal itself, leading to a reduction in bandwidth of the resulting spectrum. From Figure 2–1 it is evident that for a given window, the ideal situation is now only a linear frequency modulation approximation (or more generally, a skew-symmetric instantaneous frequency about the window center which gives rise to a constant frequency inner product function). This is a weaker restriction than the constant frequency law of the signal within the window required by the STFT. The WD therefore acts as a kind of autoadaptive frequency demodulator and provides an order of magnitude improvement in the tracking of time-varying frequencies. In any case, the instantaneous frequency can always be recovered exactly from the first moment of the WD as given by (2–13).

2.5.2 Interference Terms

Despite the many desirable properties of the WD, enumerated in the previous section, its application in practice has been limited by the occurrence of interference
terms, or artifacts, in the representation of multicomponent signals. This phenomenon is attributable to the bilinear structure of time-frequency distributions belonging to the Cohen's class. The WD of the sum of two signals \( f(t) \) and \( g(t) \) is given by

\[
W_{f+g}(t, \omega) = W_f(t, \omega) + W_g(t, \omega) + 2 \text{Re}[W_{f,g}(t, \omega)]
\]  

(2-21)

where

\[
W_{f,g}(t, \omega) = \int_{-\infty}^{\infty} f(t + \tau/2)g^*(t - \tau/2) \exp(-j\omega \tau) \, d\tau
\]  

(2-22)

is the cross-WD of \( f(t) \) and \( g(t) \). Thus the WD contains cross-components arising from the interaction of the distinct time-frequency components of the signal. Some general conclusions about the nature of the interference terms can be drawn from the following example [Hla84]. Consider a signal \( h(t) \) concentrated at \((t=0, \omega=0)\) and shifted in time and frequency to get the signals \( f(t) = a \cdot h(t-\tau_f) \exp(j\omega_f t) \) and \( g(t) = b \cdot h(t-\tau_g) \exp(j\omega_g t) \). The WD of the signal \( f(t)+g(t) \) is given by (2-21) where

\[
W_f(t, \omega) = |a|^2 W_{h}(t-\tau_f, \omega - \omega_f)
\]

\[
W_g(t, \omega) = |b|^2 W_{h}(t-\tau_g, \omega - \omega_g)
\]  

(2-23)

and the cross-term denoted by \( I_{f,g}(t, \omega) \) is

\[
2 \text{Re}[W_{f,g}(t, \omega)] = 2|ab| \cos[\omega_d(t-\tau_m) - \tau_d(\omega - \omega_m) + \phi_0] x W_{h}(t-\tau_m, \omega - \omega_m)
\]

with \((t_d, \omega_d) = (t_g - \tau_f, \omega_g - \omega_f)\)

\((t_m, \omega_m) = \left( \frac{\tau_f + \tau_g}{2}, \frac{\omega_f + \omega_g}{2} \right)\)  

(2-24)

Thus in addition to the auto-components given by the corresponding shifted versions of \( W_{h}(t, \omega) \), the resultant WD contains a cross-component which is a modulated
version of $W_h(t, \omega)$ located at the midpoint of the interacting components. The modulation is represented as an oscillation in the time–frequency plane in a direction that is orthogonal to the line connecting the two signal components. The frequency of this oscillation is proportional to the distance of separation between the two signal components in the time–frequency plane. Figure 2–2 illustrates these results.

![Figure 2-2. WD of a bicomponent signal [Hla84].](image)

It has been found that the above simple geometrical laws on the nature of the cross–term hold quite generally even when the interacting signal components are not of the same form. It should be noted that an interference term always arises in the WD from the interaction of every two signal components disjoint in time–frequency, and hence is present even in monocomponent signals where the various time–shifted components of the signal interact.

An interesting interpretation of the interference phenomenon is provided in [Jon89] where the WD is viewed as the matched window short–time Fourier spectrum of the multicomponent signal. While highly concentrated signal terms are produced in the WD due to the matched windowing of the input signal components, cross–terms arise from the interaction of each signal component with the remaining window components not corresponding with it.
The cross-components are an integral part of the representation. In them the relative phase relations of the different frequency components are preserved, and they enable the marginal properties to hold. However, the presence of interference terms makes the WD difficult to interpret since these have no real physical significance in terms of the energy distribution of the signal in the time–frequency plane. Cross-terms have been demonstrated to result in a loss of resolution in the WD of multicomponent signals [Jon89]. In the case of signals contaminated by additive noise, the effect of noise on the resulting distribution is severe, limiting the application of the WD to situations in which the signal–to–noise ratios are high.

Much research has been directed toward the suppression of interference terms in the WD. The most important approach is the use of two-dimensional smoothing in the time–frequency plane, the basis for this being the oscillatory nature of the cross-components [Fla84a]. This operation is given by

$$\tilde{W}_f(t, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi(t-u, \omega-v) W_f(u, v) du dv$$  \hspace{1cm} (2-25)

where \(\Pi(t, \omega)\) is the chosen smoothing function that is localized in the time–frequency plane and is sufficiently regular, but otherwise arbitrary. Since the convolution in \(2-25\) effects a lowpass filtering in the time–frequency plane, the highly oscillatory cross-components are attenuated while the auto-components are generally smeared. The optimal smoothing of the WD in terms of eliminating cross-terms while retaining a highly concentrated distribution, would require adapting the smoothing function in \(2-25\) to match the local time–frequency characteristics of the signal. An example of such an approach applied to unimodular signals has been presented [And87]. A special case of the time–frequency smoothing represented by \(2-25\) is when the function \(\Pi(t, \omega)\) is chosen to be separable in time and frequency, of the form \(\Pi(t, \omega) = g(t) x H(\omega)\), to get the smoothed–pseudo WD (spWD) [Fla84a],
\[
\text{spW}_f(t, \omega) = \int_{-\infty}^{\infty} h(\tau) \int_{-\infty}^{\infty} g(t - u)f(u + \frac{\tau}{2})f^*(u - \frac{\tau}{2})du \ e^{-j\omega\tau} \ d\tau \quad . (2-26)
\]

where \( H(\omega) \) is the spectrum of \( h(t) \). Now the frequency resolution is determined by \( h(t) \) while the time resolution is set by \( g(t) \), making available a flexible tool with two degrees of freedom for time–frequency analyses. The time–frequency separable smoothing function, however, restricts the form of the smoothing region to be parallel to the \( t- \) or \( f- \) axis.

In general, time–frequency smoothing of the WD leads to the sacrifice of the marginal and local properties. The exponential kernel of the Choi–Williams’ distribution is an example of time–frequency smoothing where the two smoothing functions are related in a specific manner that allows the marginal and local properties to hold, while the cross–terms are suppressed by their spreading out in the time–frequency plane. The resulting distribution is no longer as concentrated as the WD but provides improved interpretability in the case of multicomponent signals.

In the case of real signals, the WD contains artifacts at d.c. due to the interaction of the positive and negative frequency components of the signal. The analytic signal, derived from the real signal by suppressing the negative frequency spectrum, is usually utilized to overcome this particular problem [Boa87]. Another approach that has been studied for the elimination of artifacts in the WD of a noisy signal is via the truncation of an outer product expansion based on the SVD of the WD [Mar85]. Noise components that share the same space as the signal components are not eliminated, and some signal components may be truncated leading to signal distortion. An approach to the elimination of cross–terms at the “display” level, by using the correlation that exists between the WD and the spectrogram of the signal, has been proposed [Rao89]. The spectrogram of a signal comprised of non–overlapping frequency–time components is free of cross–terms, while providing a smoothed representation of the signal components. A comparison of the WD with the spectrogram
of the signal could provide an indication of the approximate regions in the time–frequency plane where signal components exist and hence used to derive a “mask”, which when applied to the WD would yield a representation free of cross–components but with concentrated auto–components. Simulation results show that in the case of narrowband signals in noise and multicomponent signals with individual components that are not spaced very closely, excellent results can be obtained without any a priori knowledge of the signal structure. However this method needs further refinement before it can be successfully applied to signals of more complex nature in which case it is possible that cross–terms may overlap with signal components in the WD plane, making them difficult to identify.

2.6 The Spectrogram and its Relation to the WD

The spectrogram of a signal is obtained by taking the magnitude squared of the short–time Fourier transform given for a signal $f(t)$ and window $h(t)$ by

$$F_t(\omega) = \int_{-\infty}^{\infty} f(\tau) h(\tau - t) \exp(-j\omega \tau) d\tau$$  \hspace{1cm} (2-27)

The spectrogram is then

$$S_f(t, \omega) = |F_t(\omega)|^2$$  \hspace{1cm} (2-28)

The spectrogram is a member of the Cohen’s class of bilinear time–frequency representations. The Cohen’s class can also be expressed in the following alternative form, in terms of the WD, according to,

$$C_f(t, \omega; \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(t - \tau, \omega - \xi) W_f(\tau, \xi) d\tau d\xi$$  \hspace{1cm} (2-29)

where

$$\Phi(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\xi, \tau) \exp(j[\xi t - \tau \omega]) d\tau d\xi$$  \hspace{1cm} (2-30)
Hence any member of the Cohen's class can be obtained by a linear transformation of the WD through the kernel \( \Phi(t, \omega) \) which is related to the original kernel of (2–1) by the two–dimensional Fourier transform of (2–30). It is possible therefore to examine the properties of other members of the Cohen's class based on the known properties of the WD.

The following expression relates the spectrogram of a signal to its WD [Cla80c],

\[
S_f(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_f(\tau, \xi) W_h(\tau - t, \omega - \xi) d\tau d\xi
\]

We see that the spectrogram is a time–frequency smoothed version of the WD, with the smoothing function given by the WD of the window \( h(t) \). A direct consequence of this is the fact that for a given signal waveform, the effective area in the time–frequency plane of the spectrogram is always slightly greater than that of the WD [Nut88]. The time–frequency extent of the spectrogram is minimized when the window function is matched to the signal. Further, the time–frequency properties that hold exactly in the case of the WD, hold only in the sense of averages over the duration of the window in the spectrogram. That is, the marginals of the spectrogram in time and frequency are given by

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(t, \omega) d\omega = \int_{-\infty}^{\infty} |f(\tau)|^2 |h(\tau - t)|^2 d\tau
\]

\[
\int_{-\infty}^{\infty} S_f(t, \omega) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\eta)|^2 |H(\omega - \eta)|^2 d\eta
\]

where \( H(\omega) \) is the window spectrum. Similar properties hold for the average frequency and the average time, where the corresponding desired local quantities are averaged over the duration of the window with the square of the window, or its spectrum, as the weighting function. Hence the spectrogram can provide useful information on
these quantities only for signals for which they do not vary appreciably over the extent of the window.

In contrast to the spWD of (2-26), the time-smoothing and frequency-smoothing functions applied to the WD in (2-31) to get the spectrogram, are not independent but rather, are related through the predetermined window $h(t)$. This results in the time-frequency trade-off that is characteristic of the spectrogram. However, the amount of time-frequency smoothing effected by $W_h(t, \omega)$, the WD of the window function, is sufficient to eliminate negative values and cross-terms between disjoint components in the spectrogram. The reason for this can be appreciated through the following example. Assume a signal composed of two components separated in time. If these components must be non-overlapping in the spectrogram, we need to choose a time window that is equal to or shorter than the time interval separating them. Hence the frequency spread of the WD of the window, being inversely proportional to its time-width, will be equal to or greater than a quantity inversely proportional to the time interval between the components. Since the period of oscillation in the frequency direction of the cross-term in the WD of such a signal is inversely proportional to the time interval between the two components, it will effectively be suppressed by the frequency-averaging wrought by the WD of the window function.

The nature of the spectrogram, by which it is relatively free of interference terms, has prompted research on determining windows that would maximize concentration of signals in the time-frequency plane. Since the WD provides a maximally concentrated distribution and the spectrogram is related to it via the time-frequency averaging of (2-31) it seems obvious that a window function whose WD is most concentrated in the time-frequency plane would achieve the minimal spread. The Gaussian window is such a function with its WD being a double Gaussian function in the time-frequency plane. In order to achieve optimal concentration, the parameters of the window (namely its time-/frequency-width and orientation in the time-frequency plane).
quency plane) must be matched to those of the signal components. Such an approach is embodied in the data-adaptive transform of [Jon87] where the Gaussian window parameters are adapted to minimize a concentration measure at each location in time-frequency.
CHAPTER 3
THE ESTIMATION OF FREQUENCY AND FREQUENCY-RATE USING THE WIGNER DISTRIBUTION

3.1 The Concept of Instantaneous Frequency

The need for the analysis of frequency-modulated (f.m.) signals gave rise to the mathematical description of the concept of instantaneous frequency (referred henceforth to as IF). An intuitively satisfying and useful description was first provided by Ville [Vil48]. Ville, based on earlier work by Gabor, related the IF of a real signal to the derivative of the phase of the corresponding analytic signal. For a real signal $s(t)$, the analytic signal is given by

$$s_a(t) = s(t) + j\hat{s}(t)$$

where $\hat{s}(t)$ is the Hilbert transform of $s(t)$:

$$\hat{s}(t) = (Hs)(t) = \frac{1}{\pi v.p.} \int_{-\infty}^{\infty} \frac{s(\tau)}{(t - \tau)} d\tau$$

with v.p. representing the Cauchy principle value. The analytic signal has a spectrum given by,

$$S_a(f) = \begin{cases} 2S(f) & f > 0 \\ S(0) & f = 0 \\ 0 & f < 0 \end{cases}$$

If the analytic signal is expressed in the form $s_a(t) = a(t) \exp(j\phi(t))$, then the IF of $s(t)$ is defined as

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\[ f_i(t) = \frac{1}{2\pi} \frac{d\phi(t)}{dt} \quad (3-4) \]

In order for the above definition to be meaningful, it is necessary that the analytic form of a real signal \( s(t) \), given by (3-1), correspond to the complex form given by the signal plus its imaginary quadrature component. For a real signal of the form \( a(t)\cos(\phi(t)) \), the following holds [Bed63]:

\[ a(t)\cos(\phi(t)) + jH[a(t)\cos(\phi(t))] = a(t)\exp(j\phi(t)) \quad (3-5) \]

if and only if \( A(f) \), the spectrum of \( a(t) \), lies entirely in the region \( |f| < f_0 \) and \( \sigma(\cos \phi(t)) \) exists only outside this region. If the spectra of \( a(t) \) and \( \cos(\phi(t)) \) are not separated in frequency the analytic form of their product will be a distorted version of \( a(t)\exp(j\phi(t)) \) due to the spectral foldover from overlapping spectra. Only when the above condition is met, or equivalently when the signal is narrowband, does the derivative of the phase in (3-4) have a meaningful interpretation as the IF of the signal.

### 3.2 Estimation of Instantaneous Frequency from the WD

The use of a time–frequency distribution to estimate the IF of a signal has been considered widely, and applied in practice [Boa89]. As stated in Chapter 2 a property desirable in any time–frequency distribution is for its average frequency at any time to be equal to the IF of the complex signal. The WD satisfies this property as given by (2-13). The discrete–time WD (DWD) is defined by

\[ W_s(n, f) = 2 \sum_{k=-\infty}^{k=\infty} s(n + k)s^*(n - k) \exp(-j4\pi kf) \quad (3-6) \]

The average frequency at any time–instant \( n \) is computed as,

\[ f_i(n) = \frac{1}{4\pi} \text{arg} \left[ \int_{-1/4}^{1/4} \exp(j4\pi f)W_s(n, f)df \right] \quad (3-7) \]
For a discrete-time complex signal given by,

$$s(n) = a(n) \exp(j\phi(n))$$  \hspace{1cm} (3-8)

where \(a(n)\) and \(\phi(n)\) are real functions, (3-7) evaluates to [Cla80b]

$$f_i(n) = \frac{1}{2\pi} \left[ \phi(n + 1) - \phi(n - 1) \right] \mod \pi$$  \hspace{1cm} (3-9)

Hence the average frequency of the DWD is the arithmetic mean of the forward and backward differences of the phase sequence, taken modulo \(\pi\) (since the DWD is periodic in frequency with a period equal to \(\pi\)). Although in the case of a general complex-valued signal, this reflects an inability to discriminate between frequencies in the interval \((0, \pi)\) and \((\pi, 2\pi)\), it provides an acceptable definition of the IF of an analytic signal, whose IF is constrained to an interval of length \(\pi\). (3-9) holds exactly for the pseudo-DWD computed using any (real-valued) window function. Thus the DWD can be used to estimate the IF since its first moment with respect to frequency provides an unbiased estimate of the IF for a signal of the general form of (3-8), independent of the actual amplitude modulation \(a(n)\), or phase modulation \(\phi(n)\). The presence of noise, however, leads to a serious degradation of the IF estimated from the first moment of the DWD. The reason for this is made evident by the following example. For a sequence given by,

$$s(n) = A \exp(j[2\pi fn + \phi]) + z(n)$$  \hspace{1cm} (3-10)

of a complex sinusoid in noise, where \(z(n)\) is complex white Gaussian noise (w.g.n.) with zero-mean and variance \(\sigma^2\), it has been shown [Tre85] that for a high enough input signal-to-noise ratio (SNR) \(A^2/\sigma^2\), (3-10) may be approximated by,

$$s(n) = A \exp(j[2\pi fn + \phi + w(n)])$$  \hspace{1cm} (3-11)
where \( \{w(n)\} \), the phase noise sequence, is real w.g.n. Hence for such an input signal the estimate of (3-9) which consists of simply the difference in successive samples of the noisy phase sequence with no averaging, would exhibit a high statistical variance even at high input SNR. Since the WD provides a highly concentrated distribution of signal energy in the time–frequency plane for narrowband signals, a natural alternative is the use of peak detection on the WD at any time to estimate the IF. Taking the peak of the DWD or maximizing it with respect to frequency is, as seen from (3-6), equivalent to the best least-squares fit of a complex sinusoid to the kernel sequence \( \{s(n+k)s^*(n-k)\} \) as a function of \( k \). When the kernel is indeed perfectly sinusoidal, as is the case when the input signal is of constant amplitude and quadratic phase (linear f.m.) the DWD peak estimate of IF is unbiased. For signals with nonlinear frequency modulations and also for any arbitrary amplitude modulation, the DWD peak estimate is biased to the extent that the kernel diverges from a perfect sinusoid.

The DWD peak has been applied to the estimation of the IF of phase–modulated signals in practice [Imb86, Boa89]. Experimental results comparing the performance of the DWD estimate to other conventional methods for the estimation of the IF and amplitude of f.m. signals in additive, white noise have been reported [Har88]. It is of interest, therefore, to derive an analytical expression for the performance in noise of the DWD peak estimator of IF. We begin with an analysis of the DWD peak in the estimation of IF of linear f.m. signals in additive w.g.n. The theoretical results are then verified using computer simulations. Finally we discuss, qualitatively, the behavior of the DWD peak estimate for nonlinear f.m. signals.

### 3.2.1 Performance of the DWD Peak for Linear F.M. Signals

The input signal is assumed to be given in the complex form by

\[
s(n) = A \exp(j\phi(n)) + z(n) \tag{3-12}
\]
where \( z(n) \) is complex w.g.n. with zero-mean and variance \( \sigma_z^2 \). The DWD of \( s(n) \) is given by (3-6). In practice the DWD is evaluated over a finite data record of \( N \) samples and is given by

\[
W_s(n, f) = 2 \sum_{k=L-1}^{k=L+1} s(n + k)s^*(n - k) \exp(-j4\pi kf)
\]  

(3-13)

with \( N=2L+1 \). The DWD can be looked upon as the frequency-scaled discrete-time Fourier transform (DFT) of the kernel sequence with a scaling factor of 2 and hence computed using a standard DFT routine. Taking the peak of the DWD is equivalent to finding the complex sinusoid that fits best, in the least-square's sense, the kernel sequence computed over the finite interval \((-L+1, L-1)\), at the time-instant \( n \). For the signal of (3-12) the kernel is given by

\[
s(n + k)s^*(n - k) = A^2 \exp(j[\phi(n + k) - \phi(n - k)])
\]

\[
+ A \cdot \exp(j[\phi(n + k)])z^*(n - k)
\]

\[
+ A \cdot \exp(- j[\phi(n - k)])z(n + k) + z(n + k)z^*(n - k)
\]  

(3-14)

The first term is a SxS term, the next two are SxN terms and the last a NxN term, where \( S \) refers to signal, and \( N \) to noise. For signals with a quadratic phase function, given by \( A \exp(j2\pi[\alpha_1n^2 + \alpha_0]) \) the first term in (3-14) yields the constant frequency sinusoid \( A^2\exp(j8\pi\alpha_1nk) \). The remaining three terms in (3-14) represent the noise in the kernel sequence. Since the input noise sequence \( z(n) \) is assumed to be zero-mean, white and Gaussian, the distinct noise terms in the kernel are uncorrelated with each other and with the signal term. The resultant noise power is the sum of the variances of the three uncorrelated noise components, and is white since the noise components at different lags are uncorrelated. The noise power in the kernel is thus given by
\[ N_{\text{kernel}} = 2A^2\sigma_z^2 + \sigma_t^2 \]  

(3-15)

It is not Gaussian however due to the introduction of the product term. However, at high input SNR, the product term is negligible and hence the noise is close to being Gaussian. Due to the fact that there are only \( N/2 \) independent samples in the (conjugate symmetric) kernel, the variance of the noise in the DWD spectrum is scaled by the factor \( N/2 \) rather than the expected factor \( N \) over the input SNR. Hence the SNR in the DWD spectrum is

\[ \text{SNR}_{\text{DWD}} = \frac{A^4x(N/2)}{2A^2\sigma_z^2 + \sigma_t^2} \]  

(3-16)

The problem is now reduced to that of estimating a constant frequency sinusoid in white noise with SNR given by (3-16) using the peak of the DFT magnitude. It is well known that the peak of the DFT is a maximum-likelihood estimate (MLE) of the frequency of a sinusoid in w.g.n. At high enough input SNR this MLE meets the Cramer–Rao lower bound (CRLB) on the variance of the frequency estimate [Rif74]. For an input sequence given by

\[ x(n) = A\exp(j[2\pi fn + \phi]) + z(n) \quad \text{for } n = 0, 1, \ldots, N - 1 \]  

(3-17)

where \( z(n) \) is complex w.g.n. with zero-mean and variance \( \sigma_z^2 \), the variance of the MLE of frequency at high input SNR is given by

\[ \text{var}_{\text{DFT}}(f) = \frac{6}{(2\pi)^2(SNR)(N^2 - 1)} \]  

(3-18)

where SNR in (3-18) stands for the SNR in the DFT of the noisy sequence of (3-17) and is given by \( NA^2/\sigma_z^2 \). We use this result to derive the variance of the frequency estimated from the peak of the DWD for a linear f.m. signal in w.g.n.

Due to the frequency scaling by a factor of 2 inherent in the DWD the frequency estimate obtained from its peak must be corrected by the same factor. This leads to a
reduction in variance by a factor of 4 for the DWD. Substituting the SNR of the DWD spectrum, as given by (3–16), into (3–18) and scaling by 4, we obtain the variance of the DWD peak estimate of IF for a linear f.m. signal in white Gaussian noise, which is

\[
\text{var}_{\text{DWD}}(f) = \frac{6(2A^2\sigma_z^2 + \sigma_0^2)}{(2\pi)^22A^4N(N^2 - 1)}
\] (3–19)

At high input SNR \((A^2 \gg \sigma_z^2)\) the above reduces to

\[
\text{var}_{\text{DWD}}(f) = \frac{6\sigma_z^2}{(2\pi)^2A^2N(N^2 - 1)}
\] (3–20)

which equals the variance of the MLE of the frequency of a stationary sinusoid in w.g.n., as given by (3–18).

We see therefore that the DWD peak is an optimal estimator of IF for linear f.m. signals in w.g.n. at high input SNRs. As the SNR decreases a threshold effect is expected to occur as does in the case of the DFT estimate of a stationary sinusoid in noise. The threshold effect is due the phenomenon (known as outliers) of frequency estimates falling outside the main lobe in the DWD spectrum and on one of the minor maxima leading to a sharp increase in the mean squared error (MSE). The probability of occurrence of an outlier is related to the SNR in the spectrum, and it is known from the DFT case that the threshold SNR, below which the MSE starts to increase abruptly, is about 15 dB [Rif74]. The SNR in the DWD is related to the input SNR through (3–16). Hence the threshold SNR for the DWD peak estimator is given by

\[
\text{SNR}_{\text{DWD}} = \frac{A^4x(N/2)}{2A^2\sigma_z^2 + \sigma_0^2} = 15\text{dB}
\] (3–21)

which at high input SNRs \((A^2 \gg \sigma_z^2)\) reduces to the condition,

\[
\frac{A^2N}{\sigma_z^2} = 21\text{dB}
\] (3–22)
Since the DWD is always real-valued, we can detect the peak simply from the function itself, instead of computing its absolute value. This should lead to a further reduction in the probability of an outlier since the minor maxima in the DWD spectrum oscillate in sign. However, computer simulations indicate that this reduction is slight (between 0.5 to 0.75) resulting in a nearly insignificant effect on the MSE.

3.2.2 Computer Simulations

Computer simulations were used to verify the performance of IF estimation using the DWD peak. The MSE at various input SNRs for a linear f.m. signal in w.g.n. was estimated by taking the average of the squared errors between the actual and estimated frequencies, obtained over between 300 to 3000 trials. The expression for the MSE is thus,

$$\text{MSE} = \frac{1}{M} \sum_{i=1}^{M} (f_i - \hat{f}_i)^2 \quad (3-23)$$

where $M$ is the number of trials, $f_i$ is the actual IF and $\hat{f}_i$ the estimated value. A 64-sample data window was used and a large, zero-padded FFT (of length $2^{15}$) computed in order to minimize errors in the IF estimate due to frequency quantization (in practice an efficient interpolation scheme can be substituted). Figure 3-1 compares the simulation results with the theoretical MSE given by (3–19). It is seen that at SNRs above the threshold, the experimentally obtained values closely match the theoretical values of MSE. The threshold SNR is correctly predicted at 3 dB for the given value of data window size N=64.

3.2.3 Estimation of Instantaneous Frequency of Nonlinear F.M. Signals

In the case of nonlinear f.m. signals, the DWD peak estimator is generally biased. The magnitude of the bias depends on the extent of the nonlinearity of the IF curve (or more generally, on the deviation of the IF law from skew-symmetry within
Figure 3-1. Performance of the DWD peak estimator of IF for a linear f.m. signal in additive w.g.n. (N=64); $10\log_{10}(1/\text{MSE})$ versus input SNR.
the data window, as discussed in Section 2.5.1). The bias arises in the attempt to
apply the least-square's fit of a constant frequency sinusoid to the kernel sequence
which is no longer narrowband, within the data window. The dependence of the bias
on the nature of the IF curve is illustrated well by the sinusoidal f.m. signal which is
characterized by a time-varying frequency-rate that varies continuously over a range
of values from zero to a maximum. In Figure 3-2 we plot the IF and the squared
error (SE), between the actual IF and the DWD peak estimate, versus time for a
sinusoidal f.m. signal with a data window of length approximately 7 percent of the
modulation period. It is seen that the bias of the DWD peak estimate, represented by
the squared error, reaches its maximum at the peaks of the IF versus time curve.
These time-instants represent the maximum deviation of frequency of the kernel, as
a function of lag, from a constant, within the data window. The squared error is zero,
indicating an unbiased estimate of frequency, at the mid-frequency, a point about
which the IF curve is skew-symmetric so that the kernel is reduced to a constant
frequency sinusoid within the data window.

For an arbitrary nonlinear f.m. signal, the bias can be controlled by suitably
windowing the input signal before generating the DWD. For a slowly varying f.m.
signal, the window length can be chosen small enough so that the IF law is practically
linear within the window. In this case the performance results of the DWD peak
estimator derived in the previous section can be applied directly. Due to the in-
creased main lobe width in the DWD created by the use of a shorter data window, this
approach reduces the bias at the expense of an increase in the variance of the esti-
mate in the presence of noise. Alternative window types such as, for example the
Gaussian window which weights the data nearer the window center more, can be
employed in order to reduce the trade-off between bias and variance of the estima-
tor.
Figure 3-2. Bias in the DWD peak estimate of IF of a sinusoidal f.m. signal as seen by the squared error (SE).
3.3 Joint Estimation of Frequency and Frequency–Rate

The joint estimation of instantaneous frequency and frequency–rate of phase-modulated signals is of importance in problems such as the orbit determination of satellites, motion compensation in synthetic-aperture radar processing, and the high resolution signature analysis of targets. In all of these situations, the received signal is the return from a doppler–type radar system, and the frequency and its time–derivative are important observation parameters. The joint estimation of doppler and doppler–rate allows the simultaneous measurement of object velocity and the curvature of the object path.

Several different approaches to the joint estimation of frequency and frequency–rate have been proposed. These include using a linear least–squares fit estimation of these parameters on a finite data set of frequency samples [Kno85], the maximum–likelihood estimation of the instantaneous frequency and frequency–rate of the parameters of a linear chirp signal [Aba86], and the estimation of frequency–rate of a sequence of linear f.m. samples after carrying out phase data differencing twice on the input sequence so that the problem is reduced to that of estimating a constant phase signal in colored noise [Dju89]. In all the above approaches the form of the phase function of the input sequence is assumed to be a quadratic polynomial (linear f.m.) with unknown parameters which are then estimated from the data.

We present here a new approach to the estimation of frequency–rate based on a methodology suggested by the Wigner distribution. We derive a new kernel from the signal and show that its spectrum is concentrated at the frequency–rate of the (arbitrary f.m.) input signal.

3.3.1 Derivation of a New Kernel

The first moment property of the WD by which the average frequency of the WD at any time equals the IF of the signal, suggests that it is possible to develop a similar
formulation for the frequency-rate based on the difference of phase of WD kernels. We define a new kernel given by,

\[ r_s(t, \tau_1, \tau_2) = q(t + \frac{\tau_1}{2}, \tau_2) q^*(t - \frac{\tau_1}{2}, \tau_2) \]  

where \( q(t, \tau) \) is the WD kernel of the signal \( s(t) \) and is given by,

\[ q(t, \tau) = s(t + \frac{\tau}{2}) s^*(t - \frac{\tau}{2}) \]  

We show here that for the input signal given by \( s(t) = \exp(j\phi(t)) \) the two-dimensional Fourier spectrum of \( r_s(t, \tau_1, \tau_2) \) given by \( R_s(t, \omega_1, \omega_2) \) satisfies the following property,

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega_1 \omega_2 R_s(t, \omega_1, \omega_2) d\omega_1 d\omega_2 = C \times \phi''(t) \]  

where \( C = -j(2\pi)^2 \). For \( s(t) = \exp(j\phi(t)) \), we get from (3-24),

\[ r_s(t, \tau_1, \tau_2) = \exp(j\phi_1(t, \tau_1, \tau_2)) \]  

where,

\[ \phi_1(t, \tau_1, \tau_2) = \phi(t + \frac{\tau_1}{2} + \frac{\tau_2}{2}) - \phi(t + \frac{\tau_1}{2} - \frac{\tau_2}{2}) \]  

\[ - \phi(t - \frac{\tau_1}{2} + \frac{\tau_2}{2}) + \phi(t - \frac{\tau_1}{2} - \frac{\tau_2}{2}) \]  

Since,

\[ r_s(t, \tau_1, \tau_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_s(t, \omega_1, \omega_2) \exp(j[\omega_1 \tau_1 + \omega_2 \tau_2]) d\omega_1 d\omega_2 \]  

we have,

\[ \frac{\partial^2 r_s(t, \tau_1, \tau_2)}{\partial \tau_1 \partial \tau_2} \bigg|_{\tau_1=0, \tau_2=0} = -\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \omega_1 \omega_2 R_s(t, \omega_1, \omega_2) d\omega_1 d\omega_2 \]  

But from (3-27) we get,
The result (3–26) then follows from (3–29) and (3–30).

The implementation of (3–26) involves obtaining the two-dimensional kernel function and its DFT prior to taking the joint moment, a computationally expensive procedure. It is possible to simplify the computation by an approximation at the cost of a slight loss in the accuracy of the estimate of frequency-rate. We give the simplification here.

If one of the parameters \( \tau_1 \) or \( \tau_2 \), say \( \tau_1 \), is fixed at a value \( T \), then the one-dimensional DFT of \( r_s(t, T, \tau_2) \) with respect to \( \tau_2 \), denoted by \( R_{s,T}(t, \omega_2) \) can be shown, by retracing the steps (3–27) to (3–30), to satisfy,

\[
\lim_{T \to 0} \frac{1}{T} \int_{-\infty}^{\infty} \omega_2 R_{s,T}(t, \omega_2) d\omega_2 = \frac{1}{T} \left[ \phi'(t + \frac{T}{2}) - \phi'(t - \frac{T}{2}) \right] = \phi''(t)
\]  

(3–31)

We can rewrite the above expression to obtain an estimate of the frequency-rate as,

\[
\frac{1}{T} \int_{-\infty}^{\infty} \omega_2 R_{s,T}(t, \omega_2) d\omega_2 \approx \phi''(t)
\]

(3–32)

with the approximation becoming increasingly accurate as the interval \( T \) is made smaller. For signals characterized by a purely quadratic frequency modulation, (3–31) holds without the limit, and hence the expression (3–32) holds exactly, irrespective of the choice of the time-interval \( T \).

We can simplify computation further, by basing the estimate of frequency-rate on simply the peak of \( R_{s,T}(t, \omega_2) \) with respect to \( \omega_2 \), instead of the first moment. Another advantage of this modification is the anticipated reduction in noise sensitivity of the estimator since, as in the case of the instantaneous frequency estimator discussed in Section 3.2, it is expected that the discrete-time estimator based on the
frequency moment of $R_s,T(t, \omega_2)$ as in (3-32) would be more sensitive to noise than one based on the peak of the same quantity. For f.m. signals for which the frequency-rate versus time curve is linear, $R_s,T(t, \omega_2)$ is concentrated on this curve. For example, for the signal given by $s(t) = \exp(j\pi at^2)$ (whose frequency-rate equals $6at$) the spectrum $R_s,T(t, \omega_2)$ is a delta-function at $\phi''(t) \cdot T = 6\pi at T$ and hence the peak of $R_s,T(t, \omega_2)$ at any time $t$ provides an unbiased estimate of the frequency-rate. In analogy with the behavior of the WD peak in the estimation of IF, the estimate of frequency-rate from the peak of $R_s,T(t, \omega_2)$ is expected to be biased in the case of arbitrary f.m. signals with the amount of bias depending on the deviation from skew-symmetry of the frequency-rate versus time curve. Nevertheless, we have now an order-of-magnitude improvement in the tracking of time-varying frequency-rate over methods that are based on the assumption of constant frequency-rate within the data window.

3.3.2 Computer Simulations

In this section we investigate the performance of the frequency-rate estimator based on the peak of the new transform for a discrete-time signal in additive w.g.n. Since an analytical formulation for the variance of the estimate is far too complex, we content ourselves with computer simulations to compare the MSE of the estimator (using an input signal for which it is unbiased) with the corresponding CRLB. The input signal is given by,

$$s(n) = A \exp(j\pi a_0 n^2) + z(n) \quad (3-33)$$

which is a complex, linear f.m. signal in additive, complex w.g.n. of zero-mean and variance $\sigma_z^2$, so that the SNR is equal to $A^2/\sigma_z^2$. The actual frequency-rate is
equal to $\alpha_0$ at all time. The MSE at various input SNRs is determined by averaging the squared estimation error over a large number of trials.

The estimator is implemented in discrete-time by,

$$R_{s, T=2}(n, f) = \sum_{k=-L+1}^{k=L-1} q(n + 1, k)q^*(n - 1, k) \exp(-j4\pi kf)$$

(3-34)

where,

$$q(n, k) = s(n + k)s^*(n - k) \quad k \in (-L + 1, L - 1)$$

(3-35)

It is seen that $R_{s, T}(n, f)$ can be computed easily as the (frequency-scaled) DFT of the product of DWD kernels separated in time by $T$. For the signal of (3-33) the kernel of (3-34) is equal to $\exp(j4\pi \alpha_0 k)$ and hence the peak of (3-34) with respect to $f$ provides an unbiased estimate of the frequency-rate.

Figure 3-3 shows the simulation results using a data window of length 64 samples, and compares the MSE obtained, with the CRLB for the variance of any frequency-rate estimator of the signal (3-33). This lower bound is given by [Dju89]

$$\text{var}(\hat{\alpha}) = \frac{90\sigma^2}{\pi^2 A^2 N(N^2 - 1)(N^2 - 4)}$$

(3-36)

We see that the MSE of the frequency-rate estimate almost meets the CRLB at high input SNR. As the SNR decreases a threshold is reached at which the MSE rises sharply due to the occurrence of outliers in the spectrum $R_{s, T}(n, f)$. The relatively high value of threshold SNR is attributable to the noisy nature of the kernel of (3-34) which is a quadratic function of the signal, and hence contains a number of signal x noise terms of various powers. Above the threshold SNR, the MSE of the frequency-rate estimate is low. For a given value of SNR, it is lower than that of the IF estimate from the DWD peak. The reason for this is the narrower main lobe width in the spectrum $R_{s, T}(n, f)$ which is the convolution of DWD's as seen from (3-34).
Figure 3–3. Performance of the frequency-rate estimator for a linear f.m. signal in additive w.g.n. (N=64); $10\log_{10}(1/\text{MSE})$ versus input SNR.
4.1 Detection of Transient Signals

A transient signal can be defined as a waveform of arbitrary shape with a time duration that is short compared with the observation interval. The detection and analysis of transient signals is a problem of importance in fields such as biomedical signal processing, seismic signal processing and acoustics. The detection of a signal requires processing that is determined by what is known of the signal characteristics, and when noise is present, what is known about the noise. Depending on the available knowledge, various methods have been developed for the particular signal detection problem. A commonly occurring case is that of a signal of known form in additive Gaussian noise. The appropriate processing is then based on a matched filter. A classic problem in seismic signal processing, underwater surveillance and radar is the detection of transient signals with unknown waveforms. When the signal can be described in terms of unknown parameters it is sometimes appropriate to consider the signal as a sample function of a random process. When the signal statistics are known, this knowledge can be used to design a suitable detector [Hel68]. When the signal is deterministic but of unknown form, a multitude of different detection methods is available ranging from simple energy detection [Urk67] to adaptive signal processing techniques that are based on assumed signal models. The unknown signal parameters are estimated from the observed data in order to derive the detection statistic [Por86, Liv88]. Another type of unknown signal
situation is that of a composite signal of possibly overlapping, unknown, multiple wavelets which is effectively treated by cepstrum techniques [Kem72].

Here we address the problem of the detection and localization of narrowband transient signals of unknown waveform and short duration that are embedded in a noisy background comprised of quasi-harmonic and random components. Such signal conditions are found in passive sonar and in vibration signal monitoring, where the transients are typically of low energy and only an approximate knowledge of their frequency location is available. We begin with a review of some recently proposed methods for the detection of such signals. We then give some practical examples of the problem we seek to solve and describe desirable detector characteristics. We discuss and compare systematically, the application of the Wigner distribution and the spectrogram (which is closely related to some of the existing approaches to the similar problem). We propose a detection scheme based on the Wigner distribution and derive analytical results for the performance of the detector for transient signals in additive Gaussian noise.

### 4.2 Existing Approaches To The Detection And Localization Of Narrowband Transient Signals

A recently proposed approach to the detection and localization of transient signals is based on the Gabor representation, a time–frequency representation which is closely related to the short–time Fourier transform [Fri89]. The Gabor representation is inherently localized and therefore well suited for the modeling of the transients that are assumed to be concentrated in time and frequency. The Gabor coefficients \( \{ C_{m,n} \} \) of the given signal, that contains possibly the transient waveform and additive noise, are computed from the following expression,

\[
y(t) = \sum_{m,n=\ldots}^{\infty} C_{m,n} \ g(t - na) \exp(j2\pi m\beta t) .
\]  (4-1)
where \( y(t) \) is the received signal, \( g(t) \) is a chosen window function and \( \alpha \) and \( \beta \) are detector parameters, with \( \alpha \beta = 1 \). The detector, based on a generalized likelihood ratio test, examines sets of coefficients \( \{ C_{m,n} \} \) spanning relatively short time intervals in order to detect the transient waveform and localize it in time. The performance of the detector is dependent on how closely the window function, \( g(t) \), matches the transient waveform. Thus the Gabor detector is not robust to parameter variations in the transient waveform. The authors remark, in particular, the need to increase the robustness of the detector to time–scale variations of the signal. This problem of the variability in detector performance with changes in time–scale of the transient signal has been addressed through the use of time–scale analysis [Gar90].

The wavelet transform is proposed for the detection of transient signals with unknown arrival times and waveforms that are known to within a scale factor. When the analyzing wavelet is chosen to be of the form of the transient signal to be detected, the wavelet transform provides good localization of the transient in time–time–scale space. Since the parameters that are represented by the wavelet transform are the time of arrival and scale, the frequency content of the transient cannot be arbitrary but rather must be a strict function of the time–scale. The author restricts himself to the class of transient signals that are monochromatic and have a frequency that is inversely proportional to the time duration, an example of which is the class of vibration transients with a damping factor proportional to the frequency.

The short–time power spectrum is widely applied in passive sonar signal processing to detect transient signals of short duration by the temporary increase in energy in portions of the received signal spectrum, caused by the arrival of a transient [Kni81, Wol83]. This procedure is equivalent to performing narrowband filtering within energy detection. The detector parameters are the filter bandwidths and averaging times which determine its performance for given signal conditions.
All of the approaches to transient signal detection and localization discussed above are based on the analysis of the received signal using fixed window functions. Since the signal waveform is not known a priori the window function cannot be optimally chosen, and hence these methods suffer from performance constraints imposed by the particular window and the consequent lack of robustness to parameter variations of the transient waveform. Further the time and frequency resolutions obtainable in the representation of the received signal are not independent but are characterized by a fixed trade-off determined by the window function.

4.3 Desirable Detector Characteristics

To motivate our approach to transient detection, we give some practical examples of the type of problem we have attempted here. One such is the detection and classification of mechanical parameters of ships based on underwater acoustic sounds [Lou87]. Ship propeller noise can be modeled as a burst of pressure pulses caused by collapsing cavitation bubbles once every rotation for each blade. The received signal is further contaminated by noise from other sources like gearboxes, turbines and electric motors. Detection of the cavitation and the power in the cavitation frequency is of interest and needs to be achieved without a precise knowledge of the cavitation frequency or bandwidth. A similar problem arises in the detection of transient vibration signals in the field of machine monitoring. The transient vibration signal is typically of low energy and is buried in a background comprised of a number of harmonic components of relatively high energy, contributed by other sources interacting with the system in which the transient vibration originates. Additionally the vibration signal may contain random noise.

We concern ourselves here with the class of transient signals that are monochromatic and of short, but otherwise arbitrary, time durations. The transient
signal is possibly in background noise comprised of deterministic and random components. Since the transient signals of interest are localized in time and in frequency, it is natural to consider the application of a time–frequency representation to the detection problem. Our goal is to develop a detector which:

(i) requires no a priori information on the actual time waveform or the frequency of the transient signal;

(ii) provides high enough frequency resolution to discriminate the transient signal from the background signal components, so that the detector output is not influenced by energy changes in the background, that are non–overlapping in frequency with the transient signal. Further, we would like the detector to provide high time resolution, so that the transient signal is well localized in time;

(iii) is robust to time-scale changes of the transient waveform, so that the detector maintains a reasonable level of performance over a wide range of transient time durations.

Since we do not intend to make any assumption on the time waveform of the transient signal but intend to detect the transient by recognizing its spectral signature, or the temporary increase in energy in a region of the spectrum, we are restricted to the choice of a nonparametric method for processing the received signal. The short–time power spectrum and the Wigner distribution are among the possible time–frequency representations that suggest themselves for the detection problem. As discussed in Section 2.6, for a given waveform, the WD provides a more concentrated representation in the time–frequency plane than does the spectrogram. The extent of time–frequency spread of the spectrogram is minimized when it is computed using a window matched to the signal waveform, which is not possible when the transient is of unknown form.

The application of the WD to the detection of signals in noise has been considered before [Kum84, Fla88]. Flandrin [Fla88] has shown how the optimum
detection of a signal of known structure (known deterministic signal, or random signal of known statistics) in Gaussian noise can be achieved using the WD. The inner product invariance property of the WD permits the implementation of the optimum detector in the time–frequency plane, which for nonstationary input signals can lead to a considerable simplification in detector structure. In this chapter, we address the application of time–frequency methods to the detection of signals of unknown form in additive noise.

In the following subsections, we discuss the application of the WD and the spectrogram to the detection of transient signals in noise, where the received signal satisfies the assumptions outlined in this section, and compare their performance.

4.3.1 High Time–Frequency Resolution

Since the received signal contains the transient waveform embedded in a noisy background comprised of spectral components originating in other interacting phenomena and which could be varying slowly in amplitude, we need high frequency resolution in our signal representation, in order to isolate the energy contribution in the narrow spectral band where the transient is expected to be concentrated. For relatively closely spaced components, this would require us to choose a greater transform length or data window size. In the case of the spectrogram, an increase in data window length has a direct bearing on the time resolution of the transient signal. Since a longer data window would lead to the spreading of the transient signal energy over a greater time duration, it is expected that the time localization of the transient waveform would degrade as the window length is increased. Further, increasing the data window length also contributes to a decrease in the SNR and hence reduces the detectability of the signal in noise. The pseudo–WD, however, allows the increase of data window size to achieve high enough frequency resolution without a sacrifice in time localization. The difference in combined time–frequency resolution between
the spectrogram and the pWD is illustrated by Figures 4-1 to 4-3. Figure 4-1 shows a time signal, comprised of the sum of a sinusoid of constant amplitude at frequency 0.09 and a transient waveform with a Gaussian envelope at frequency 0.12. In this example and all other examples in this chapter, we use the complex form of the signal when computing both the DWD and short-time power spectrum. The simulated complex signals have spectra confined to the frequency region $[0, \pi]$ to avoid aliasing in the DWD [Cla80]. (The DWD shown here, is computed using a standard FFT routine and hence is a frequency-scaled (by a factor of two) version of the actual signal DWD.) Spectrograms obtained with two different data window lengths and spanning the time-instant of arrival of the transient are compared in Figure 4-2. (In this and all other figures in this chapter, $N$ refers to the data window length in samples and $M$ to the length in samples of the time-averaging window used to compute the spDWD. We use rectangular data and time-averaging windows throughout.) It is seen that the longer data window provides the superior frequency resolution of the distinct signal components, but the energy of the transient is spread in time, so that the actual time of arrival of the transient signal is not obvious. In contrast, Figure 4-3 which shows the spDWD of the signal using the same two data windows, does not display any perceptible trade-off in the time localization of the transient with the improved frequency resolution of the longer data window.

One may be concerned about the effect of cross-terms in the WD on the detection of the transient waveform when the received signal contains multiple, closely spaced frequency components. The cross-terms contributed by these components degrade the resolution of the WD and may lie in the spectral region of the transient signal and hence influence the detector output. In such cases it is necessary to employ some time-smoothing of the WD in order to improve the frequency resolution by attenuating the cross-terms. This operation leads to a corresponding loss of time resolution. However, since the time-smoothing window is
Figure 4–1. Gaussian transient in sinusoid.
Figure 4–2. Effect of data window length on spectrogram of Gaussian transient in sinusoid. (a) N=128 (b) N=512.
Figure 4–3. Effect of data window length on spDWD of Gaussian transient in sinusoid. (a) N=128, M=40 (b) N=512, M=40.
chosen independently of the data window, there is no immediate trade-off between time and frequency resolutions as in the short-time power spectrum. For instance, we can set the data window duration in time, long enough to achieve the frequency resolution we need, and then depending on the extent of cross-term interference, choose the appropriate length for the time-averaging window. In most cases a time-smoothing window that is much shorter in duration than the data window will be adequate to attenuate the cross-terms sufficiently to prevent them from influencing the output of the detector. Hence using the spWD allows the independent control of time and frequency resolutions. Figures 4-4 to 4-6 illustrate this point well. Figure 4-4 shows the time waveform of a Gaussian transient of frequency 0.12, embedded in a signal comprised of two harmonics closely spaced in frequency with the transient signal. Figures 4-5 and 4-6 show respectively the spectrogram and the spDWD of the waveform in Figure 4-4, spanning the time of arrival of the transient. The spectrogram was computed using a data window length of 84 samples, to match the duration of the transient waveform, so as to achieve good time localization. We see that the distinct frequency components are just about well resolved in the spectrogram. Figure 4-6 shows the 512-sample spDWD of the signal with a time-smoothing window length of 40 samples, in order to attenuate the cross-term generated by the sinusoids. It is seen that we achieve better time and frequency resolutions in the spDWD over the spectrogram.

4.3.2 Robustness to Transient Duration

In order to maintain a good detection performance even with relatively short signal durations, it is necessary for the detector not to be strongly sensitive to changes in the time duration of the received transient waveform. For this condition to hold, we must base our detection statistic on a parameter or feature of the signal that is invariant to such changes, but represents only the transient signal amplitude. A
Figure 4-4. Gaussian transient in sum of two closely spaced sinusoids.
Figure 4–5. Spectrogram of Gaussian transient in sum of two closely spaced sinusoids (N=84).
Figure 4-6. spDWD of Gaussian transient in sum of two closely spaced sinusoids (N=512, M=40).
change in the time duration of the transient signal corresponds to a change in the total signal energy, and therefore will affect the magnitude of the peak of the transient signal spectrum. In other words, a decrease in pulse-width will lead to a corresponding decrease in magnitude of the peaks of both, the spectrogram and the WD, at the frequency and time of arrival of the transient signal and hence a degradation in performance of any detector based on these quantities. The instantaneous signal power, on the other hand, is a quantity that reflects the amplitude of the transient signal and does not depend on its time duration. The WD of the received signal can be used to obtain an estimate of the instantaneous signal power.

The WD itself is simply the Fourier spectrum of the signal kernel and hence the magnitude of the peak of the WD is a function of the total signal energy. However, from the time-marginal property of the WD we have for a signal \( u(t) \),

\[
\int_{-\infty}^{\infty} W_u(t,f) df = |u(t)|^2 .
\] (4-2)

We can thus recover the instantaneous signal power at any time by integrating the WD over all frequency, at that time. Equation (4-2) holds also for the pWD irrespective of the time window used to compute it. The spectrogram, on the other hand, gives the average signal energy over the duration of the window \( h(t) \) with the square of the window as a weighting function,

\[
\int_{-\infty}^{\infty} S_u(t,f) df = \int_{-\infty}^{\infty} |u(\tau)|^2 |h(\tau-t)|^2 d\tau .
\] (4-3)

Figure 4-8 compares the robustness of the time-marginals of the spectrogram and the WD, computed using fixed data windows, to variations in the time duration of the received transient signal. Figure 4-7 shows a time waveform containing three
Figure 4-7. Sum of Gaussian transients of same peak amplitude, but differing pulse durations.
Figure 4–8. Time–marginals of the spectrogram and the WD – a comparison; — spectrogram; — WD.
transient signals of the same center frequency and peak amplitude but different time durations. Figure 4–8 compares the quantities (4–2) and (4–3) as a function of time for this waveform. The magnitude of the integral of the spectrogram is a strong function of the time duration of the transient, since it reflects the total energy of the signal over the duration of the window. The integral of the WD on the other hand is proportional to the instantaneous signal power and hence reaches the same peak value for all three transient signals.

Since the transient signals of interest are assumed to be concentrated in frequency, the above property of the WD by which the instantaneous signal power can be recovered, suggests a detector based on integrating the WD of the received signal over a narrow bandwidth corresponding to the region of significant spectral support of the transient signal. If the WD of the signal \( u(t) \) is contained entirely in the region \((f-\Delta f, f+\Delta f)\) at any time and is zero outside, then the instantaneous signal power may be recovered by integrating the WD only over this frequency band thus,

\[
\int_{f-\Delta f}^{f+\Delta f} W_u(t,f) df = |u(t)|^2
\]  

(4–4)

Integrating the WD in frequency provides the further advantage of smoothing out cross-components between multiple transient signals that are closely spaced in time. Signal components that are separated in time give rise to cross-terms in the WD midway between the two interacting components. These cross-terms are oscillatory in the frequency direction and, therefore, are suppressed by averaging the WD in frequency, an effect achieved by (4–4). Figure 4–9 shows the time waveform of two almost overlapping transient signals. Figure 4–10 shows the 200-sample spectrogram and the 512-sample DWD, for this signal. Cross-terms are clearly seen in the DWD and in this case, due to the overlap between signal components, in the spectrogram as well. Figure 4–11 compares the quantities of (4–2) and (4–3) for this signal, where the DWD is computed using a 512-sample data window and the
Figure 4-9. Sum of two closely spaced in time Gaussian transients.
Figure 4-10. Cross-terms in the representation of overlapping Gaussian transients. (a) Spectrogram (N=200) (b) pDWD (N=512).
Figure 4-11. Time-marginals of the spectrogram and the WD of overlapping transients – a comparison; — spectrogram; — WD.
spectrogram using a 128-sample window. The transient signals are well resolved in time in the DWD, but clearly are not as well resolved in the spectrogram inspite of having matched the spectrogram window duration to the duration of each of the transient signals.

4.4 A Detector Based on Instantaneous Power

Based on the considerations of the earlier sections we propose here a detector for transient signals in additive noise based on monitoring the received signal power in localized regions of the spectrum. The received signal is assumed to meet the assumptions stated in Section 4.3. Figure 4-12 shows the configuration of the detector. The received signal is filtered so as to bandlimit the noise to the

approximately known broad spectral region that the transient signal is expected to lie within, while the signal of interest is passed unattenuated. This operation is necessitated by the fact that the WD estimate for a signal in additive, white noise is characterized by infinite variance. (Detailed analytical results on the noise properties of the WD are presented in the next section.) The input signal is next windowed by the finite-length window \( v(t) \) and then the WD is computed, incorporating time-smoothing if necessary. The WD is integrated over a narrow

Figure 4–12. Narrowband transient signal detector
frequency band surrounding the center frequency of the transient waveform to obtain an estimate of the instantaneous signal power as,

$$r(t, f) = \int_{t-\Delta f}^{t+\Delta f} W_y(t, \nu) d\nu$$

This quantity is then compared with a threshold to determine whether a transient signal is present. When the exact frequency of the transient signal is not known, we obtain a maximum likelihood estimate of the frequency by computing (4-5) for several (overlapping) frequency bands throughout the approximately known spectral region of the transient signal, and choosing the maximum output to detect the presence of the transient signal. The quantity $\Delta f$ is a parameter of the detector and is fixed at a predetermined value. It represents the largest bandwidth or equivalently, the narrowest duration transient pulse, that can be effectively detected.

We illustrate the high combined time–frequency resolution characteristic of the WD–based detector via the example of the time–series of Figure 4-13. The signal is comprised of two short-duration Gaussian pulses in an additive background created by a closely-spaced (in frequency) narrowband component with slowly–varying amplitude. Figure 4–14 shows the detection statistic of (4–5) with the region of integration being the narrow frequency band surrounding the center frequency of the transient pulse. It is seen that the detection statistic is influenced mostly by the arrival of the transient signals. Figure 4–15 shows the similar detection statistic computed using the spectrogram with two different data windows. The spectrogram succeeds in resolving the transient signal energy from the background spectral energy changes only at the expense of a loss in the time localization of the transient.

In the following sections, we analyze the performance of the WD–based transient signal detector in the presence of additive random noise. We begin by studying the SNR in the WD estimated from a noisy signal and then derive the performance of the detector.
Figure 4-13. Transient pulses in a narrowband background with slowly varying amplitude.
Figure 4-14. The WD detection statistic (N=512, M=128) at the frequency of the transient signal.
Figure 4-15. The detection statistic based on the spectrogram with different data window lengths: —— N=128; —— N=256, at the frequency of the transient signal.
4.5 Noise Performance of the WD

In the previous sections we have considered the effect of deterministic, background signal components on the detection of transient signals using the WD. We now examine the properties of the WD estimated from a signal contaminated with additive random noise, in order to investigate the noise performance of the detector. Nuttall [Nut88] has derived the mean and the variance of the WD estimated from a such a noisy waveform, and we begin by summarizing his results. We then discuss the bearing of these results on the choice of detector parameters, and compare the performance in noise of the WD estimate with the performance of the magnitude of the short-time Fourier transform under similar signal conditions. (Unless otherwise indicated, all integrals go from $-\infty$ to $\infty$.)

The received signal is given by,

$$x(t) = u(t) + n(t)$$  \hspace{1cm} (4-6)

where $u(t)$ is the transient signal waveform, and $n(t)$ is zero–mean, stationary, complex Gaussian noise that satisfies,

$$\overline{n(t)} = 0 \hspace{0.5cm} \text{and} \hspace{0.5cm} \overline{n(t_1)n(t_2)} = 0.$$  \hspace{1cm} (4-7)

The noise power spectral density is given by

$$G_n(f) = \int \overline{\left[ n(t + \frac{\tau}{2}) n^*(t - \frac{\tau}{2}) \right]} \exp(-j2\pi f \tau) \, d\tau.$$  \hspace{1cm} (4-8)

To compute the WD, we must window the received signal. The windowed waveform is given by,

$$y(t) = v(t)[u(t) + n(t)]$$  \hspace{1cm} (4-9)

where $v(t)$ is the chosen, finite-duration window function. The WD is then calculated as,
$$W_y(t, f) = \int y(t + \frac{\tau}{2}) y^*(t - \frac{\tau}{2}) \exp(-j2\pi fr) \, d\tau$$

$$= a + b + c + d \quad (4-10)$$

where

$$a = \int R_{vv}(t, \tau) \ u(t + \frac{\tau}{2}) \ u^*(t - \frac{\tau}{2}) \ \exp(-j2\pi fr) \, d\tau \quad (4-11)$$

$$b = \int R_{vv}(t, \tau) \ n(t + \frac{\tau}{2}) \ n^*(t - \frac{\tau}{2}) \ \exp(-j2\pi fr) \, d\tau \quad (4-12)$$

$$c = \int R_{vv}(t, \tau) \ u(t + \frac{\tau}{2}) \ n^*(t - \frac{\tau}{2}) \ \exp(-j2\pi fr) \, d\tau \quad (4-13)$$

$$d = \int R_{vv}(t, \tau) \ n(t + \frac{\tau}{2}) \ u^*(t - \frac{\tau}{2}) \ \exp(-j2\pi fr) \, d\tau \quad (4-14)$$

where

$$R_{vv}(t, \tau) = v(t + \frac{\tau}{2}) \ v^*(t - \frac{\tau}{2}) \quad (4-15)$$

The quantities $a$ and $b$, are respectively $S \times S$ and $N \times N$ terms, while $c$ and $d$ are $S \times N$ terms, where $S$ refers to signal and $N$ to noise. Based on the assumptions stated earlier in this section, Nuttall [Nut88] has derived the following expressions for the mean and the variance of the WD estimate of the noisy signal. The mean is given by,

$$\overline{W_y(t, f)} = W_v(t, f) \ast_f [W_u(t, f) + G_n(f)] \quad (4-16)$$

where $W_v(t, f)$ is the WD of the window $v(t)$ and $W_u(t, f)$ that of $u(t)$, and $\ast_f$ refers to convolution in the frequency variable. The variance of the WD estimate is given by

$$\text{var}[W_y(t, f)] = 2W_v^2(t, f) \ast_f G_{nn}(f) + 2 \int G_n(\nu) \ |B(t, f - \frac{\nu}{2})|^2 \, d\nu \quad (4-17)$$

$$\quad (N \times N) \quad (S \times N)$$
where

\[ G_{nn}(f) = G_n(f)^* G_n(f) \]  \hspace{1cm} (4-18)

\[ B(t, f) = \int W_v(t, f - \frac{\nu}{2}) U(\nu) \exp(j2\pi\nu t) \, d\nu \]  \hspace{1cm} (4-19)

and \( U(f) \) is the Fourier transform of \( u(t) \). The variance of the WD estimate is comprised of two terms, one each contributed by the \( N\times N \) portion and the \( S\times N \) portion of the total noise. It can be seen from the \( N\times N \) term in (4-17) that the variance of the WD estimate depends on both the window function \( v(t) \) and on the noise power spectral density \( G_n(f) \). If we set \( v(t) = 1 \) for all \( t \), or do not use any time-windowing of the data, then \( W_v(t, f) = \delta(f) \), and the \( N\times N \) term in (4-17) becomes infinite. Furthermore, irrespective of the time window, if the input noise is white, \( G_n(f) = N_0 \) for all \( f \), making \( G_{nn}(f) \) and hence the resultant variance infinite. Hence, in order for the variance to be finite it is necessary for the time window \( v(t) \) to be of finite length and further, also necessary for the noise to be bandlimited.

Figure 4-17 compares the 512-sample DWD with the 256-sample spectrogram of the noisy signal of Figure 4-16, consisting of a Gaussian transient in w.g.n. of power spectral density \( N_0 \) and total transient signal energy \( E_0 \) so that \( E_0/N_0 = 7.6 \). The DWD is very noisy compared to the spectrogram. This is due in part to the longer data window used in computing the DWD but, more significantly, due to the noisy nature of the DWD attributable to the existence of \( N\times N \) as well as \( S\times N \) terms, and the fact that the noise is white.

We next study, quantitatively, the SNR of the WD estimated from a noisy waveform for an example case. Our aim is to compare the SNR of the WD with that of the magnitude of the short-time Fourier transform for transients with time durations that are short with respect to the observation window. For the sake of
Figure 4–16. Gaussian transient in white, Gaussian noise ($E_0/N_0 = 7.6$).
Figure 4-17. Effect of additive noise on (a) Spectrogram (N=256) (b) pDWD (N=512).
analytical tractability we choose the signal, window and filter to be Gaussian in form. The received signal is given by (4-6) where,

\[ u(t) = a \exp(-t^2/2\sigma_0^2) \quad (4-20) \]

and \( n(t) \) is white Gaussian noise of power spectral density \( N_0 \). The received signal is windowed in time by a window \( v(t) \) of effective length \( p \), and filtered by a transfer function \( H(f) \) of effective bandwidth \( B \) so that the WD estimate has finite variance. The window length \( p \) and the filter bandwidth \( B \) are assumed to be large enough so that the transient signal \( u(t) \) is passed essentially unaltered, while the noise is attenuated. The configuration is as shown in Figure 4-12, where

\[ v(t) = \exp(-t^2/2p^2) \quad (4-21) \]

\[ H(f) = \exp(-f^2/2B^2) \quad (4-22) \]

The filtered noise spectrum is therefore,

\[ G_n(f) = N_0 |H(f)|^2 = N_0 \exp(-f^2/B^2) \quad (4-23) \]

The quantities required to calculate the mean and the variance in (4-16) and (4-17) for this example are given below.

\[ U(f) = a\sigma_0 \sqrt{2\pi} \exp(-\sigma_0^2 2\pi^2 f^2) \quad (4-24) \]

\[ W_u(t, f) = 2a\sigma_0 \sqrt{\pi} \exp(-t^2/\sigma_0^2 - \sigma_0^2 4\pi^2 f^2) \quad (4-25) \]

\[ W_v(t, f) = 2p \sqrt{\pi} \exp(-t^2/p^2 - p^2 4\pi^2 f^2) \quad (4-26) \]

\[ G_{nn}(f) = \frac{\sqrt{\pi}}{2} N_0^2 B \exp(-2f^2/B^2) \quad (4-27) \]
We are interested in the SNR of the WD estimate given by

\[
\text{SNR} = \frac{\text{difference of mean outputs}}{\text{standard deviation of output}}
\]  

(4-28)

where the numerator refers to the difference in mean output with signal present from that with signal absent. We compute the SNR of the WD estimate at the time of arrival \( t=0 \), and the center frequency \( f=0 \), of the transient signal,

\[
\text{SNR}_{WD} = \frac{W_v(t, f) \ast W_u(t, f)}{[\text{var}(W_y(t, f))]^{1/2}} \quad \text{at} \quad t = 0, f = 0
\]  

(4-29)

with the variance computed as in (4-17). The expression (4-29) was evaluated numerically over a range of values of pulse width \( \alpha_0 \), keeping the parameters \( a, p, B \) and \( N_0 \) fixed. The values chosen are \( p = 40.0, B = 4.0, N_0 = 0.2, \) and \( a = 2.0 \). Figure 4-18 shows a plot of the SNR of the WD estimate versus pulse width. In order to get a better understanding of what the numbers represent, we plot on the same graph the SNR of the, more familiar, magnitude of the STFT under identical conditions, i.e. signal, noise, window and filter. The analytical expression for the SNR of the magnitude of the STFT estimated at the time of arrival and center frequency of the transient is derived below.

The STFT is computed at \( t=0 \), by applying the window \( v(t) \) to the received signal \( x(t) \) to get,

\[
S_y(0, f) = \int x(\tau) v(-\tau) \exp(-j2\pi f \tau) \, d\tau
\]  

(4-30)

The mean signal output at \( t=0, f=0 \) is then,

\[
\overline{S_y(0, 0)} = \int U(f) V(f) \, df
\]  

(4-31)

The variance of the output noise is,

\[
\text{var}\{S_y(0, 0)\} = \int G_n(f) |V(f)|^2 \, df
\]  

(4-32)
Figure 4-18. Output SNR of the WD and |STFT| estimates of a Gaussian pulse in w.g.n. versus pulse duration.
where $U(f)$ and $G_n(f)$ are given by (4-24) and (4-23) respectively, and $V(f)$ is the spectrum of $v(t)$,

$$V(f) = p \sqrt{2\pi} \exp(-p^2 2\pi^2 f^2)$$ \hspace{1cm} (4-33)

Substituting the above expressions in (4-28) for the output SNR and simplifying, we get the SNR in the magnitude of the STFT estimate as,

$$\text{SNR}_{\text{STFT}} = \frac{a \sigma_0}{[N_0(\sigma_0^2 + p^2)]^{1/2}} \times \frac{1}{B^2 2\pi} + 4\pi^2 p^2 \frac{1}{4} \hspace{1cm} (4-34)$$

From Figure 4-18 we see that for the parameters of this example, the SNR in the WD estimate is comparable to, and in fact slightly better than, that of the magnitude of the STFT. The WD estimate however, degrades rapidly with an increase in filter bandwidth $B$. The STFT magnitude is not as sensitive to changes in $B$. In summary, with the proper choice of filter and window parameters it is possible to achieve in the estimation of the WD a noise performance on the same order as that of the magnitude of the STFT, while obtaining the superior time-frequency resolution of the WD.

### 4.6 Noise Performance of the WD Transient Detector

A transient signal detector based on estimating the instantaneous signal power using the WD, was presented in Section 4.4. We use the results presented in the previous section on the noise performance of the WD to analyze the performance of the detector of Figure 4-12 for a transient signal in additive noise. The detection is based on the integral of the WD estimated from the noisy signal, computed over a fixed frequency region given by $(f-\Delta f, f+\Delta f)$. At time $t$ and frequency $f$, this quantity may be written in the following equivalent form,

$$r(t, f) = \int_{f-\Delta f}^{f+\Delta f} W_y(t, \nu) \, d\nu = W_y(t, f) \ast f D(f) \hspace{1cm} (4-35)$$

where,
\[ D(f) = \begin{cases} 1 & \text{for } f \in (-\Delta f, +\Delta f) \\ 0 & \text{else} \end{cases} \]  
(4-36)

But \( y(t) = v(t)x(t) \) and hence we have,
\[ W_y(t, f) = W_v(t, f) \ast f W_x(t, f) \]  
(4-37)

Substituting in (4-35) we get,
\[ r(t, f) = W_v(t, f) \ast f W_x(t, f) \ast f D(f) \]  
(4-38)

or,
\[ r(t, f) = W_{eq}(t, f) \ast f W_x(t, f) \]  
(4-39)

where,
\[ W_{eq}(t, f) = W_v(t, f) \ast f D(f) = \int_{f-\Delta f}^{f+\Delta f} W_v(t, \nu) \, d\nu \]  
(4-40)

From (4-39) we see that the effect of integrating the WD of the received signal over the frequency interval \((f-\Delta f, f+\Delta f)\) is equivalent to a modification of the window \(v(t)\) so that its WD is now given by (4-40). For the window given by (4-21), this expression evaluates to
\[ W_{eq}(t, f) = \frac{1}{2} \exp\left(\frac{-t^2}{p^2}\right) \left[ \text{erf}(2\pi p f + \Delta f) - \text{erf}(2\pi p f - \Delta f) \right] \]  
(4-41)

where \( \text{erf} \) refers to the error function and is given by
\[ \text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y \exp(-t^2) \, dt \]  
(4-42)
The mean detector output when the signal is present is then obtained from (4-16) by replacing \( W_v(t, f) \) with \( W_{eq}(t, f) \) to get

\[
E_1[r(t, f)] = W_{eq}(t, f) \ast f [W_u(t, f) + G_n(f)] \quad . \quad (4-43)
\]

The mean detector output when the signal is absent is

\[
E_0[r(t, f)] = W_{eq}(t, f) \ast f G_n(f) \quad . \quad (4-44)
\]

The variance of the detection statistic when the signal is present is obtained from (4-17) and is given by,

\[
\text{var}_1[r(t, f)] = 2W_{eq}^2(t, f) \ast f G_{mn}(f) + 2 \int G_n(v) \left| B(t, f - \frac{v}{2}) \right|^2 dv \quad \text{[SxN]} \quad (4-45)
\]

where

\[
B(t, f) = \int W_{eq}(t, f - \frac{v}{2}) U(v) \exp(j2\pi vt) \ dv \quad . \quad (4-46)
\]

The variance of the detection statistic with only noise present is simply the first term \([\text{NxN}]\) of (4-45). Since an exact derivation of the probability distribution of the detection statistic is not mathematically tractable due to the presence of nonlinear noise terms, we do not attempt to determine the probability of detection. We characterize the performance of the detector instead by a SNR measure given by,

\[
\text{SNR}_r = \frac{E_1[r(t, f)] - E_0[r(t, f)]}{\sqrt{\frac{1}{2} (\text{var}_1[r(t, f)] + \text{var}_0[r(t, f)])}} \quad . \quad (4-47)
\]

Using the window and filter of (4-21) and (4-22), the above expression was evaluated at the time of arrival, \( t=0 \), and center frequency, \( f=0 \), of the transient signal \( u(t) \) of (4-20), using error function tables and numerical integration over a range of pulse widths. The constant parameters were: \( p=40.0, \quad B=4.0, \quad a=2.0, \) and \( N_0 = 0.2 \). The results are plotted in Figure 4-19 which shows the output SNR as a function of
the transient pulse width $\sigma_0$ relative to the window length $p$, for different values of $\Delta f$. The energy contained in the transient signal is proportional to the pulse width $\sigma_0$ and hence Figure 4–19 also represents the variation of output SNR with input SNR (given by signal energy /noise power spectral density), with peak signal amplitude kept constant. The SNR obtained in the WD peak, which is the limiting case with $\Delta f \to 0$, is also plotted for comparison. From Figure 4–19 we see that the output SNR for $\Delta f > 0$, rises rapidly with transient pulse duration, and reaches a level that is then maintained as the transient duration increases further. The insensitivity of the output SNR to the energy of the received transient signal beyond a threshold pulse duration, is due to the detection statistic being based on an estimate of instantaneous signal power, or equivalently on the signal amplitude, rather than on the total received energy. For a given $\Delta f$, we obtain from (4–35), an estimate of the signal power in the bandwidth $(-\Delta f, +\Delta f)$ at the instant $t$. As the pulse width $\sigma_0$ is increased starting from a small value, the bandwidth of the WD spectrum of the transient pulse decreases correspondingly. This leads to an increasingly greater fraction of the transient signal’s WD spectrum falling within the region of integration $(-\Delta f, +\Delta f)$, and hence the rapid rise in output SNR. Above a threshold pulse width $\sigma_0$, the entire transient signal WD spectrum is concentrated within the frequency region $(-\Delta f, +\Delta f)$, so that now any further increase in $\sigma_0$ does not add to the mean signal output in (4–43), but leaves it unchanged. The variance of the output is also a function of the pulse width, when all other parameters are fixed as is the case we are considering here. The power in the $SxN$ term of (4–45) is dependent on both the received signal power and the noise power. Since only a signal parameter, i.e. the pulse width, is being varied here, the received noise power is expected to remain constant. Hence the power in the $SxN$ term increases with the pulse duration until the threshold pulse width and then maintains a constant level for any further increases of pulse width. The power in the $NxN$ term of (4–45) is constant since it does not depend on the pulse
Figure 4-19. SNR of the WD–based detection statistic versus pulse width, at various $\Delta f$. 
The above discussion on the dependence of the mean and variance of the detector output on the received transient pulse width, with all other parameters kept constant, explains the behavior of the plot of output SNR versus pulse width. When the detector parameter $\Delta f$ is varied we get a different curve in Figure 4–19. As $\Delta f$ is increased we are integrating over an increasing region of the noisy WD spectrum, which leads to the increase in the total noise power and hence the decrease in the output SNR of the estimate for a given received signal pulse width. On the other hand, for higher $\Delta f$ the output SNR attains a constant level more rapidly and at a lower threshold pulse width since now the wider region of integration includes the entire WD spectrum of the transient waveform at $t=0$ for even narrower pulse widths.

The WD peak is a limiting case (obtained as $\Delta f \to 0$). As $\sigma_0$ is increased, the WD spectrum at $t=0$ of the transient pulse becomes increasingly concentrated and hence the mean detector output, which is simply the value of the WD at $f=0$ increases. The power of the NxN term remains constant while that of the SxN term increases with $\sigma_0$. The net effect is a rapid rise of output SNR with increasing pulse width.

4.7 Analysis of Experimental Milling Machine Data

Detection of tool chatter during high-speed milling operations is a problem that has benefitted from spectral analysis methods [Smi89]. In stable milling conditions the vibration signal from the cutting tool is characterized by a spectrum containing primarily the tooth frequency and its harmonics. As the cut becomes unstable due to changes in operating conditions, such as the depth of cut or spindle speed, a new component, the chatter frequency, begins to dominate the vibration signal spectrum. The exact frequency of the chatter induced vibration is generally not known a priori. In [Smi90] the magnitude of the short-time Fourier spectrum of the vibration signal is used to detect the occurrence of chatter by observing the peaks in the spectrum. This method has proven adequate for the commonly occurring condition of fixed
spindle speed (characterized by a stationary signal spectrum) and a long duration of chatter. Experiments on vibration data from a cut obtained while continuously increasing the axial depth of cut, so that chatter set in and continued to increase after the threshold depth was crossed, indicate that the spectrogram and the WD perform equivalently in the detection of the onset of chatter. It might be expected however, from the discussions of the earlier sections, that if the chatter were to be intermittent, lasting for time durations that are short compared to the observation interval, then the WD would perform better in the detection and localization of chatter compared to short-time spectral techniques. Another case that would benefit greatly from the application of the WD is when the vibration signal is nonstationary, so that it is characterized by time-varying frequency components as might happen if the spindle speed were to vary with time. In this case the short-time power spectrum would be smeared due to the time-varying tooth frequency and harmonics, possibly obscuring the chatter frequency component and making it difficult to detect.

We present here the analysis of a data sample of the acoustic signal collected from a high-speed milling machine during a cut. Figure 4–20 shows a portion of the time signal in which the axial depth of cut was gradually increased and then decreased so that the tool entered the chatter mode for a short time duration, marked by the increase in signal power near the center of the time-frame. The sampling rate is 4000 Hz, and the total duration of the signal in Figure 4–20 is 0.38 sec. The spindle speed was 4000 rpm, the number of teeth equal to 6 and the feed rate was set at 3000 mm/min. Figure 4–21 shows the 256-sample spectrogram for a narrow time frame of 0.2 sec spanning the occurrence of chatter, and the spDWD computed using a 512-sample data window and 50-sample time-smoothing window for the same signal region. The development and the decay of the chatter frequency component are clearly perceptible in both the spectrogram and the spDWD. The tooth frequency and its harmonics are also visible. While the spectrogram and the spDWD show
Figure 4-20. Milling machine data.
Figure 4-21. Milling machine data (a) Spectrogram (N=256) (b) spDWD (N=512, M=50).
similar time resolution, the latter displays better frequency resolution due to the longer data window used. In Figures 4-22 and 4-23 we plot the quantities \( \sum_{k=k_i-\Delta k}^{k_i+\Delta k} S_y(n,k) \) and \( \sum_{k=k_i-\Delta k}^{k_i+\Delta k} W_y(n,k) \) respectively, versus time-instant \( n \), where \( W_y(n,k) \) is the 512-sample spDWD with a 50-sample time-smoothing window, and \( S_y(n,k) \) is the 256-sample spectrogram. The region of the frequency summation \( (k_i - \Delta k, k_i + \Delta k) \) is chosen to be a small fixed interval centered at each of three distinct closely spaced frequencies including the chatter frequency. Since we do not have a definite knowledge of the true signal spectrum, it is difficult to interpret the performance of the spectrogram and spWD as represented by these plots. We, however, make the following general observations. The power in the chatter frequency component is seen to rise rapidly with time in both, but is more concentrated in time in the spDWD. The curves at the other closely spaced frequencies arise from the “leakage” of the chatter frequency spectrum into the neighbouring frequency bands and hence could be representative of the frequency resolution. With this interpretation it is seen that the frequency resolution of the spectrogram is poorer than that of the spDWD.
Figure 4–22. Estimation of power in chatter frequency region (continuous line), and two other closely spaced frequency regions (broken lines) using the spectrogram.
Figure 4-23. Estimation of power in chatter frequency region (continuous line), and two other closely spaced frequency regions (broken lines) using the spDWD.
CHAPTER 5
APPLICATION OF THE WIGNER DISTRIBUTION TO VIBRATION SIGNAL MONITORING

5.1 Spectral Analysis in Vibration Signal Processing

The vibration monitoring of machines, structures and other systems is an important process in the effective maintenance and operation of these systems. Vibration signals emitted by machinery carry valuable information about the general condition of the various system components. Vibration signals may be periodic or random, stationary or nonstationary and may further contain transient phenomena and mechanical shocks. Each component of the system contributes its peculiar spectral signature. For example, in the case of rotating machinery the vibration signals are characterized by high frequencies originating in rolling element bearings, gear teeth, turbomachinery blading, etc., lower frequencies from shaft misalignments and the essentially random contributions associated with fluid flow, jet noise and cavitation. Spectral analysis is necessary to reveal the individual frequency components that make up the wideband vibration signal.

Vibration signal monitoring is typically carried out using the short-time spectral magnitude computed in an FFT analyzer. This method provides satisfactory results when the signal is stationary over the duration of the analysis window. There are situations however, when the vibration signal is nonstationary due to the time-varying nature of the source of vibration. Short-time spectral techniques are then not always adequate for the monitoring of such signals. The purpose of this chapter is to show, by examples, what can be gained by the application of time-frequency meth-
ods to the problem of vibration monitoring involving nonstationary signals. Specifically, we investigate the application of the WD to the problems of envelope detection of narrowband, time-varying signals, and the detection of periodic transients in time-varying signals. It must be noted that the WD has been applied previously in vibration signal studies [Fla89b] where time-frequency signatures of the vibration signal were used to detect and identify abnormal machine operating conditions. Here we propose to apply the WD in a more quantitative formulation.

5.2 Envelope Detection

The estimation of the envelope of a specific component of the vibration signal related to a phenomenon of interest is a common operation in machinery fault detection and monitoring. The observation and analysis of the vibration level at specific frequencies, such as for example the structural resonances of the system, can indicate the onset or development of a fault. Typically, envelope detection is achieved via a narrowband filter centered at the frequency of interest in order to isolate the particular component. The filter output is rectified and lowpass filtered to obtain the envelope, which may further be subjected to spectrum analysis, if necessary. Very often the component of interest is nonstationary due, for example, to changes in speed, and the above method must be modified to allow for the tracking of the time-varying frequency component. A commonly used method to achieve this is by the use of an analog tracking filter that is adaptively tuned to the frequency of interest, and determines the level of the signal in a small bandwidth around this frequency. Since the WD provides a highly concentrated representation of narrowband, time-varying signals, its application to the envelope detection of such signals is of interest. We review here the properties of the WD relevant to such an application.

For complex signals of the form \( a(t)\exp(j\phi(t)) \), with \( \phi(t) \) a real, smooth function and \( a(t) \) slowly varying, the WD spectrum is concentrated about the instanta-
neous frequency dφ/dt. The average frequency of the WD, at any given time instant, exactly equals the instantaneous frequency of the signal. Further, the instantaneous signal power can be recovered from the WD by integrating the WD spectrum over all frequency at the time instant of interest. That is:

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} W_f(t, \omega) d\omega = |f(t)|^2 \quad (5-1)
\]

The above time-domain properties holds for the pWD irrespective of the data window applied. For signals of the form \(a(t)\exp(j\phi(t))\), (5–1) yields \(|a(t)|^2\) which is the square of the time-varying amplitude of the signal. Although (5–1) indicates that the integration of the WD must be performed over all frequency, in the case of narrow-band signals whose WD spectrum is concentrated in a narrow frequency band locally defined around the instantaneous frequency at any time instant, the instantaneous signal power can be obtained by integrating over this narrow subrange. This property will be applied to the problem of the amplitude tracking of nonstationary signal components. Vibration signals are typically expected to be multicomponent and we need to contend with the cross-terms that arise from the application of the WD. By using the analytic signal derived from the real signal, the cross-terms that arise from the interaction between positive and negative spectral components can be eliminated. Since the intent is to recover the amplitudes of the individual signal components that comprise the input signal by integrating the WD over narrow bandwidths in frequency, care must be taken to ensure that the effect of the other signal components and the cross-components on the integral is minimal. This can be achieved through the spWD. The extent of smoothing necessary for the effective suppression of cross-terms depends on the separation of the interacting primary signal components, with those cross-terms that arise from the interaction of closely spaced signal components needing a relatively greater amount of smoothing. However, smoothing the WD results in the smearing of the auto-terms, leading to a corresponding loss of concen-
tration in the time–frequency plane. The integral over frequency of the spWD at any time, would yield the average signal power over the duration of the smoothing time–window which would imply a corresponding loss of accuracy in the tracking of time–varying amplitudes. Hence there are trade–offs to be considered when choosing the smoothing window.

The above considerations suggest that the spWD can provide a powerful and flexible tool in the amplitude tracking of nonstationary, narrowband signals. We discuss here a specific application, that of the monitoring of vibration levels in a high–power gas turbine. A brief overview of the application and the WD–based algorithm for vibration–level monitoring are described. Simulations are used to evaluate the performance of the method. A detailed implementation of the real–time digital vibration monitor is presented in [Rao90b].

5.2.1 Vibration Monitoring in the Gas Turbine

The gas turbine consists of two separate rotating units coupled by air pressure. These are the Gas Generator (GG) and the Power Turbine (PT) as illustrated in Figure 5–1. Since there is no mechanical coupling between the two units their rotational frequencies are not the same. The available vibration signal is the output of an accelerometer mounted on each turbine unit. Based on the analyses of these signals the levels of vibration of each of the two units must be monitored in real–time in order to facilitate decisions on speed, loading, and emergency shutdown of the turbine. The output of each accelerometer contains two dominant components one each at the rotational frequency of the GG and the PT. These are likely to vary in time due to changes in rotation speed. However, the frequency bands in which vibration analysis is performed are non–overlapping. The vibration signal also contains a relatively low level of background noise contributed by other sources coupled to the turbine.
Tachometer signals are available that provides data continually on the rotational velocities of each of the two units.

![Diagram of Gas Turbine Setup]

**Figure 5-1.** Gas turbine setup

### 5.2.2 Algorithm and Simulations

In Figure 5-2 the magnitude spectrum of the simulated (stationary) turbine vibration velocity signal from the accelerometer connected to the PT is shown. The signal is comprised of the two dominant frequency components contributed by each of the GG and PT, and a background made up of harmonics from various other sources coupled to the system. The low noise floor in the spectrum is generated by the lowpass filtering of a series of impulses. In order to demonstrate the WD-based monitoring method, we simulate the time-varying nature of the vibration signal by frequency modulating the PT and GG frequency components over frequency bands specified by the manufacturer. The vibration components are specified to lie between 15 hz and 70 hz for the PT, and 84 hz to 200 hz for the GG. The WD is used to track the simulated amplitude changes of each of the vibration components in the velocity signal in conjunction with information provided by the tachometer signal on the actual, or instantaneous, vibration frequencies at any time. The sampling frequency is set at 500 hz.
Figure 5-2. Magnitude spectrum of the turbine vibration velocity signal.
We use the analytic signal derived from the samples of the real signal in order to compute the discrete-time spWD (spDWD). This allows us to sample the input signal at the Nyquist rate and also eliminates cross-terms in the DWD that arise from the interaction of positive and negative spectral components of the real signal. The analytic signal can be obtained from the input signal by using an FIR filter implementation of the Hilbert transform. The Hilbert tranform is well approximated by a non-causal and symmetric FIR filter of order 79 [Boa87]. So in order to compute the analytic signal at each input sample we need to "look ahead" about 39 time samples.

In order to estimate the vibration amplitude in the velocity signal at a time instant n, the sampled velocity signal denoted by \( f(n) \) is processed as follows:

1. Select a data set \{ \( f(n-N/2), \ldots, f(n), \ldots, f(n+N/2-1) \) \} and compute the corresponding analytic signal \( z(n) \) using the FIR filtering implementation. Since the FIR filter is symmetric and of order 79 samples, we discard 39 samples at each end of the output data set and replace the discarded samples with zeros.

2. Generate the spDWD kernel by using the time-averaging window \( g(m) \) of duration \( M \), and data-window \( h(m) \) of length \( N \). Take the DFT of the kernel, given by,

\[
W_z(n, k) = 2 \sum_{m=-N/2}^{m=N/2-1} |h(m)|^2 e^{-j \frac{2\pi mk}{N}} \sum_{i=n-M/2}^{i=n+M/2-1} g(i) z(i+m)z^*(i-m)
\]

(5-2)

where \( \omega = \frac{k\pi}{NT} \) is the analog frequency in radians/sec, \( T \) is the sampling period and an \( N \)-point FFT is utilized. Figures 5–3 (a) and (b) show the DWD (N=512) and spDWD (N=512, M=40) and \( h(n) \) is a Gaussian window of half–power width 200 samples) computed from the analytic signal derived from the simulated velocity signal. We have taken \( g(m) \) as simply a rectangular window, but another smoothing function could have been chosen so as to weight those products nearer the time instant \( n \) more, relative to those that are far removed. The time resolution between plots is 50 samples (0.1s). The time–varying frequencies are clearly tracked by both the DWD
Figure 5–3. Effect of time–smoothing on the pDWD of the simulated turbine vibration signal. (a) pDWD (N=512) of vibration signal. (b) spDWD (N=512, M=40) of vibration signal.
and the spDWD, but the artifacts in the DWD are suppressed to a significant extent in the spDWD, while the signal components are slightly smeared.

3. Estimate the instantaneous frequency $k_{i=1,2}$ of the 2 signal components from the tachometer output. Sum the spDWD values over each frequency interval $(k_i - D_i, k_i + D_i)$ to obtain an estimate of the power in the respective component at the time-instant $n$. The frequency intervals, $D_i$, should be chosen broad enough to include the significant spectral region of the possibly smeared signal component, but narrow enough to preclude the effect of spurious components in the spDWD and the background noise. What we more nearly estimate is the average signal power over the time interval $(n - \frac{M}{2}, n + \frac{M}{2} - 1)$. For values of $M$ that are small relative to the time-variation of signal amplitudes this still represents an accurate tracking of the instantaneous signal power.

Figures 5-4 and 5-5 show the results of the tracking the amplitudes of each of the vibration components (that were varied independently) using the above algorithm and parameters $(N=512, M=40, D_1=D_2=20)$. The time interval between successive estimates is 25 samples (0.05 sec). It is seen that the time-varying amplitude is closely tracked for both components. Since we have used a 512-point transform we have “looked ahead” 256 samples or about 0.5 sec of the input signal in order to estimate the amplitude at any time.

Figure 5-6 shows the experiment of Figure 5-4 repeated with the addition of random noise to the vibration signal. White Gaussian noise at an SNR equal to 5 with respect to the PT component is added to the simulated signal. The white noise contributes randomly varying spectral components (noise x noise and signal x noise terms) that lie within the signal bandwidth and lead to a high variance in the instantaneous power estimate computed from the spDWD. This gives rise to the random fluctuations in the amplitude estimate of Figure 5-6. The fluctuations can be reduced by
Figure 5-4. Comparison of estimated and actual amplitude versus time for the FT component of the vibration signal.
Figure 5-5. Comparison of estimated and actual amplitude versus time for the GG component of the vibration signal.
Figure 5-6. Comparison of estimated and actual amplitude versus time for the PT component of the vibration signal with additive white noise at SNR=5.
increasing the extent of time–frequency smoothing of the DWD, at the cost of a corresponding degradation in tracking performance. Another solution that comes to mind is to apply a threshold to the spDWD prior to the integration, in order to eliminate low-lying noise components and their integral effect on the estimate of instantaneous power.

5.3 Detection of Periodic Transients

Incipient faults in the toothmesh of a gearbox, such as due to a cracked tooth, create a series of sharp pulses at the toothmeshing frequency (rotational frequency x number of teeth) [Ang87]. These pulses are generally of low energy, but show up in the frequency spectrum as an increase in the level of sidebands spaced with rotation speed above, as well as below, the toothmeshing frequency. Such faults are then detected by observing the sidebands around this frequency created by the periodic transients contributed by the cracked tooth. When the vibration signal is absolutely stable the sidebands created by the periodic transient signal, are expected to be clearly visible in the short–time spectrum of the signal. However, this is not always the case and often the rotational speed varies with time resulting in the smearing of the short–time spectrum obscuring the sideband structure altogether. Since the WD is capable of providing a far more concentrated spectrum of the time–varying tooth-meshing frequency, at any time, it would be expected to produce a clearer representation of the distorted vibration signal spectrum. We compare in this section, the short–time magnitude spectrum and the WD for a simulated signal representing vibration data from a gearbox containing a cracked tooth. The cracked tooth, due to its weakened condition, deflects more when it goes into mesh under load than the healthy teeth. We assume that the rotational speed varies linearly with time so that the vibration signal is well modeled by a linear chirp modulated by the envelope of sharp pulses created by the cracked tooth. Figure 5–7 shows the simulated vibration
signals, both normal and distorted due to tooth damage, superimposed on a low frequency wave contributed by other interacting phenomena. The number of teeth is 12. In Figures 5-8 and 5-9 we compare the spectrogram and the spDWD of the normal and distorted vibration signals. Both are computed using windows that span several gear rotation periods. (In all the spectra, only a small region spanning the region of interest, namely the toothmeshing frequency, is plotted.) We see that the toothmeshing frequency and sidebands are heavily smeared in the spectrogram since these frequency components vary significantly over the window length. The sideband activity created by the broken tooth is far more readily perceived in the spDWD.
Figure 5-7. Time waveforms of the simulated vibration data from healthy and damaged gearboxes.
Figure 5-8. Comparison of spectrograms. (a) Spectrogram of normal data. (b) Spectrogram of data characterizing damaged tooth.
Figure 5-9. Comparison of spDWDs. (a) spDWD of normal data. (b) spDWD of data characterizing damaged tooth.
CHAPTER 6
SUMMARY

Some new applications of the Wigner distribution, a time–frequency representation that possesses a number of properties that are valuable in time–varying signal analysis, have been proposed and investigated. The WD provides a highly concentrated representation of nonstationary signals in time–frequency. It allows, further, the easy estimation of important signal parameters such as the instantaneous power and frequency through its marginal and local properties. These characteristics form the basis for the applications that have been studied here.

The application of the WD to the estimation of frequency and frequency–rate of phase–modulated signals has been investigated. The average frequency of the WD at any time, provides an unbiased estimate of the IF of the signal. It is shown that the presence of noise, however, leads to the severe degradation of the first moment estimate. The application of the peak of the WD to the estimation of IF is examined. An exact expression for the variance of the estimate from the peak of the DWD for a linear f.m. signal in additive white Gaussian noise is derived. It is shown that the DWD peak provides an optimal estimate of the IF at high enough SNRs. The DWD peak estimate is biased in the case of nonlinear f.m. signals. Windowing of the input signal, so that its IF curve is approximately linear within the data window, is suggested to reduce the bias at the expense of an increase in the variance of the estimate.

A new approach to the estimation of frequency–rate of an arbitrary phase–modulated signal is proposed, based on taking the phase difference of WD kernels separated in time. A two–dimensional spectral transform is derived whose joint moment in frequency at any time is shown to be proportional to the frequency–rate of the
input signal. The exact implementation of the frequency–rate estimator, however, is computationally complex. A simplification based on taking the peak of the new transform is presented, which provides an unbiased estimate only when the frequency–rate versus time curve is linear within the data window. Still, we now have an order–of–magnitude improvement in the tracking of time–varying frequency–rate over the available estimators based on the assumption of constant frequency–rate within the data window. The computer simulation of a linear f.m. signal in additive w.g.n. shows that the variance of this estimator of frequency–rate meets the corresponding Cramer–Rao lower bound at high values of SNR. The threshold SNR, below which the estimator breaks down, is relatively high due to the noisy nature of the quadratic signal functional that it is based upon.

The problem of the detection and localization of transient signals in additive noise is considered. The transient signals are assumed to be monochromatic but of unknown waveform and center frequency, and of short time duration relative to the observation interval. The received signal also contains a noisy background comprised of deterministic and random components of relatively high energy. We do not assume any knowledge of the transient waveform including its duration, but base our approach on recognizing the temporary increase in received energy in a localized region of the spectrum caused by the arrival of the transient (assuming that the signal and the background noise components are non–overlapping in frequency). Examples of practical situations where such signals arise are presented, and desirable detector characteristics are stated. Existing methods applied to similar problems are generally based upon the short–time power spectrum of the received signal. Based on the important relationship between the spectrogram and the WD, by which the former is a time–frequency smoothed version of the WD characterized by a reciprocal relation between time and frequency resolutions, we have investigated the application of the WD to the problem of transient detection. The smoothed–pseudo–WD
which permits the independent setting of time and frequency resolutions, has been shown through simulated examples to provide the superior time and frequency resolutions of the received signal. A detector based on monitoring the received power in localized regions of the received signal spectrum is derived by employing as the detection statistic, the integral of the WD over a narrow bandwidth in frequency. Such an operation yields an estimate of the instantaneous signal power provided the signal is narrowband so that its WD is concentrated entirely within the spectral region in question, at any time. By computing such an estimate over a range of overlapping frequency intervals and choosing the maximum output to detect the presence of a transient, we obtain a detector whose performance is not strongly sensitive to changes in pulse duration or frequency, and also provides good time localization of the transient waveform. A detailed detector configuration is presented and its performance for a Gaussian transient signal in additive Gaussian noise is derived based on known noise properties of the WD estimate. The utility of such a detector in the detection of transient vibrations is discussed with reference to an analysis of real milling machine data. The characteristics of the detector proposed here would also be valuable in situations where the received transient frequency signature is likely to vary with time such as occurs in passive sonar. Further, the proposed detection statistic can be used to estimate the envelope of the transient waveform which information can be valuable in the classification of the transient.

The field of vibration signal processing is rich with potential applications for time-frequency methods. Two specific examples of the monitoring of nonstationary vibration signals have been considered here in order to demonstrate the advantage of WD-based methods over the traditional short-time magnitude spectrum. A detailed algorithm is presented for vibration level monitoring in a high-power gas turbine, together with simulation results based on realistic data. The proposed technique, generally speaking, is applicable to the tracking of the envelopes of any narrowband,
time-varying signal components that comprise an input signal provided these are well separated in frequency. The presence of a low level of background noise does not affect the performance of the method. Any a priori knowledge of the signal and noise components, although not essential, can be used to advantage in the choice of algorithm parameters. The choice of parameters becomes critical only when the distinct signal components are closely spaced in frequency. In such a situation, there is a trade-off between the accuracy of amplitude estimation and the ability to achieve good tracking. The WD is also applied to the detection of a gearbox fault created by the presence of a cracked tooth, when the vibration signal is time-varying due to changes in rotation speed with time. The nonstationarity of the signal leads to the smearing of the sideband structure due to the periodic transients created by the cracked tooth, in the short-time spectrum. Since the WD provides better frequency resolution of the time-varying signal, the spectral distortion is far more readily perceived in the WD.
FORTRAN Program To Compute The Smoothed-Pseudo-DWD

C PROGRAM spDWD
C This program finds the smoothed-pseudo DWD of the
C input time series at the specified time-instant.
C Expects complex input data.

REAL ITRY(2048), ETRY(2048), PITRY, GITRY
COMPLEX FITRY(1024), SITRY(2048)
REAL W(1024)
CHARACTER*8 NAMEI, NAMEO
COMPLEX RIP
INTEGER RSMV, SMV

PRINT *, 'Enter the sampled data file name'
READ(5,110) NAMEI

110 FORMAT(A8)

PRINT *, 'Enter the spDWD output file name'
READ(5,110) NAMEO

PRINT *, 'Enter the data window length in samples'
READ *, M

PRINT *, 'Enter the power-of-2 size of the spDWD'
READ *, M1

PRINT *, 'Enter the Gaussian data window parameter'
READ *, SIGMA

PRINT *, 'Enter the time instant (sample #) for
1 which spDWD must be computed'
READ(5,*) ND

PRINT *, 'Enter the length of the smoothing window
1 in samples (odd number)'
READ(5,*) RSMV

OPEN(UNIT=1, NAME=NAMEI, STATUS='OLD')
OPEN(UNIT=2, NAME=NAMEO, STATUS='NEW')
K = ND-M
MN= 2M+2
N = 2**M1
SMV = (RSMV-1)/2

DO 100 I=K-1

100 READ(1,*)PITRY,GITRY

READ(1,*,END=900)(ITRY(I),ETRY(I), I=1,2*M+1)

DO 125 I=1,2*M+1

125 SITRY(I) = CMPLX(ITRY(I),ETRY(I))

C Generate the Gaussian data window (if required)

FACT = 2.0*(SIGMA**2.0)
DO 150 J=1,M+1
KO = (M/2+1)-J

150 W(J) = EXP(-(KO**2)/FACT)

C Compute the kernel array of inner products

DO 200 J=1,M/2+1
RIP = 0.
MS = M/2 + J

C Generate the time-averaged inner product for each lag J

DO 225 L=-SMV,SMV
RIP = RIP+SITRY(MS+L)*CONJG(SITRY(MN+L-MS))
RIP = RIP/(1000.*RSMV)

225 CONTINUE

FITRY(J) = RIP

C apply the window function to kernel (if required)

FITRY(J) = RIP * (W(J)**2.)

FITRY(M+2-J)=CONJG(FITRY(J))

200 CONTINUE

FITRY(1) = CMPLX(0.0,0.0)

C Re-order to input properly to the (standard) FFT routine which expects the samples in the normal time sequence

DO 250 I=1,M/2
SITRY(I)=FITRY(I+M/2)

250 SITRY(M/2 + I) = FITRY(I)

C Zero-pad the kernel (equivalent to zero-padding the input data) if DWD-size > data window length
KO = (N-M)/2
DO 11 I=1,M/2
11 CITRY(I)=SITRY(I)
DO 12 I=M/2+1,N-M/2
12 CITRY(I)=0.0
DO 13 I=N-M/2+1,N
13 CITRY(I)=SITRY(I-N+M)

C Compute the DFT of the kernel sequence
CALL FFT(CITRY,N,M1,-1.0)

C Now CITRY contains the (real-valued) SDWD
DO 300 I=1,N
300 WRITE(2,*)REAL(CITRY(I))

900 CONTINUE
CLOSE(1)
CLOSE(2)
STOP
END

SUBROUTINE FFT(A,N,M,RI)

C Taken from Oppenheim & Schafer, "Digital Signal

COMPLEX A(1024),U,W,T,CMPLX
NV2 = 2**(M-1)
NM1 = N-1
J = 1
DO 7 I=1,NM1
IF(I .GE. J) GO TO 5
T = A(J)
A(J) = A(I)
A(I) = T
5 K = NV2
6 IF(K .GE. J) GO TO 7
J = J-K
K = K/2
GO TO 6
7 J = J+K
PI = 3.141592653589793
DO 20 L=1,M
LE = 2**L
LE1 = LE/2
U = CMPLX(1.0,0.0)
W = CMPLX(COS(PI/FLOAT(LE1)),RI*SIN(PI/FLOAT(LE1)))
DO 20 J=1,LE1
DO 10 I=J,N,LE
IP = I+LE1
20 CONTINUE
10 CONTINUE
200 U = CMPLX(1.0,0.0)
300 W = CMPLX(COS(PI/FLOAT(LE1)),RI*SIN(PI/FLOAT(LE1)))
400 DO 300 I=1,M/2
500 W = CMPLX(COS(PI/FLOAT(LE1)),RI*SIN(PI/FLOAT(LE1)))
600 DO 500 J=1,LE1
700 CONTINUE
800 CONTINUE
900 CONTINUE
CLOSE(1)
CLOSE(2)
STOP
END
T = A(IP)*U
A(IP) = A(I)-T
10 A(I) = A(I)+T
20 U = U*W
   IF(RI .LE. 0.0) GO TO 35
   DO 45 I=1,N
45 A(I) = A(I)/N
35 RETURN
END
REFERENCES


BIOGRAPHICAL SKETCH

Preeti Rao was born in India in 1961. She received the bachelor's degree in electrical engineering at the Indian Institute of Technology, Bombay, in 1984. She began her graduate studies at the University of Florida in Fall, 1985. She received the MS in electrical engineering in 1987. She joined the High-Speed Digital Architecture Lab as a research assistant in 1988, and expects to graduate from the University of Florida with the Ph.D. in electrical engineering in August, 1990.
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Professor of Electrical Engineering

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