Honor’s Thesis:  
Portfolio Construction &  
Modern Portfolio Theory

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Abstract

The purpose of this project is to explain the Modern Portfolio Theory, how it can be used, and illustrate an example by analyzing a hypothetical portfolio using Microsoft Excel with macros. To do this, it is important to understand the characteristics of the Markowitz model and how quadratic programming can be used to optimize this model. It is our goal that with this thesis, the basic knowledge of investment can be expanded, as well as to outline the fact that many mathematical models used in operations research and other engineering fields can be used extensively in other fields such as finance.

Introduction

The mean-variance portfolio selection is one of the cornerstones of modern portfolio theory. This theory attempts to maximize the expected return of a portfolio given a certain level of portfolio risk, or equivalently attempts to minimize the portfolio’s risk given a certain level of expected return. Its goal is to lower the risk by diversifying a portfolio of assets rather than investing in any individual asset. Harry Markowitz, a University of Chicago graduate student introduced this theory in an 1952 article and in a 1958 book, after a stockbroker suggested him to study the stock market. He later received a share of the 1990 Nobel Prize in Economics for the introduction of this theory. Today, this theory is widely used in the fields of finance, investment, and operations research.

The goal of this paper is to explicitly explain the mean-variance model of the modern portfolio theory along with the concept of quadratic programming and how they are used together to optimize the selection of investment portfolios. It is worth noting that even though this theory, just as any other theory, has its own limitations; it is still considered to be the best way of constructing the most efficient investment portfolio that allows for the maximization of expected return and minimization of risk.
Investment Management

To understand the concept of modern portfolio theory and portfolio analysis, it is important to understand the concept of Investment Management. This concept involves five steps. These are:

1. **Setting investment objectives**
   This includes an analysis of the investment objectives of the investor whose funds are being managed. This involves institutional investors (pension funds, insurance companies, mutual funds, government agencies), and/or individual investors.

2. **Establishing an investment policy to satisfy investment objectives**
   This step begins with asset allocation decisions, which means that a decision must be made as to how the funds will be allocated among different classes of assets. Asset classes include a variety of investment products, such as:
   - **Equities or common stocks** – These include large, medium, and small capitalization stocks, and growth, value, domestic, foreign and emerging markets stocks.
   - **Fixed income securities or bonds** – These include treasury, municipal, corporate, high yield, mortgage backed securities, asset backed securities, and domestic, foreign, and emerging market bonds.
   - **Cash** – Includes T-Bills, money market, savings
   - **Alternatives** – These alternative investments include commodities, real state, currencies, private equity, and hedge funds.

   In the development of an investment policy it is important to consider client, and regulatory constraints, as well as tax and accounting issues.

3. **Selecting an investment strategy**
   Portfolio strategy can be classified as active or passive. *Active portfolio strategy* uses available information and forecasting techniques to seek a better performance than a portfolio that is simply diversified broadly. In contrast, *passive portfolio strategy* involves minimal inputs; instead it relies on diversification to match the performance of some market index. It assumes that the marketplace will reflect all available information in the price paid for securities. In the bond space, a *structured portfolio strategy* has also been used, in which a portfolio is designed to achieve the performance of some predetermined liabilities that must be paid out. The question now becomes, which investment strategy must be selected? And
the answer depends on the client’s view of how price efficient the market is, the client’s tolerance to risk, and the nature of the client’s liabilities.

4. **Selecting the specific assets**

   This step attempts to construct an efficient portfolio, which is described as a portfolio that provides the greatest expected return for a given level of risk, or the lowest risk for a given expected return. For this step, three key inputs are required, these are: future expected return, variance of asset returns, and correlation of asset returns.

5. **Measuring and evaluating investment performance**

   This involves measuring the performance of the portfolio and evaluating that performance relative to some benchmark.

**Portfolio Selection and the Markowitz Model**

The goal of the portfolio selection is the construction of portfolios that maximize expected returns given a certain level of risk. Professor Harry Markowitz came up with a model that attempts to do this by diversifying the portfolio. This model is called the *Markowitz model* or the *mean-variance model*, because it attempts to maximize the mean (or expected return) of the entire portfolio, while reducing the variance as a measure of risk. This model shows that assets should not be selected individually, but rather as a portfolio, in order to reduce risk and maximize expected return. For this, it is necessary to consider how each asset’s price change relatively with the other assets in the portfolio. Portfolios that have the highest level of expected return given a certain level of risk are called *efficient portfolios*. In order to construct these portfolios, the Markowitz model makes five key assumptions. These are:

- **Assumption 1**: The expected return and the variance are the only parameters that affect an investor’s decision.
- **Assumption 2**: Investors are risk averse, meaning investors prefer investments with lower risks, given a certain level of expected return.
- **Assumption 3**: All investors’ goal is to achieve the highest level of expected return given a certain level of risk.
• **Assumption 4:** All investors have the same expectations concerning expected return, variance, and covariance.

• **Assumption 5:** All investors have a one period investment horizon.

After these assumptions are clear, portfolios can be constructed in a two-stage process: First, the investor needs to evaluate the available securities on the basis of their future perspectives to select the best securities. Then, the investor needs to decide the allocation of current capital to the chosen securities in order to meet investor’s policy, goals, and objectives.

In order to calculate the actual return on a portfolio of assets over some period of time, the following formula is used:

\[ R_p = w_1R_1 + w_2R_2 + \cdots + w_GR_G \]

Where,
- \( R_p \) = Rate of return of the portfolio
- \( R_G \) = Rate of return on asset \( g \) over the period
- \( w_G \) = weight of asset \( g \) in the portfolio
- \( G \) = number of assets in the portfolio

Thus, the expected return from a portfolio of risky assets is calculated by:

\[ E(R_p) = w_1E(R_1) + E(R_2) + \cdots + w_GE(R_G) \]

Where,
- \( E(R_G) \) = the expected return of asset \( G \)

To quantify the concept of risk, Markowitz used the statistical measures of variance and covariance. In order to measure the risk of a portfolio comprised of more than two assets, the following formula is used:

\[ \text{var}(R_p) = \sum w_g^2 \text{var}(R_g) + \sum \sum 2w_gw_h \text{cov}(R_g, R_h) \]
The variance of a portfolio not only depends on the variance of the assets but also upon the covariance of any two assets, this is how closely the returns on every two assets in the portfolio move with respect to each other, this is mainly due to the fact that financial markets interact, meaning that the investments do not vary independently. The covariance on any two assets can be calculated as:

\[
Cov(R_i, R_j) = p_1[r_{i1} - E(R_i)][r_{j1} - E(R_j)] + p_2[r_{i2} - E(R_i)][r_{j2} - E(R_j)] + \ldots + p_N[r_{iN} - E(R_i)][r_{jN} - E(R_j)]
\]

Where,

- \( r_{iN} = \text{the nth possible rate of return for asset } i \)
- \( r_{jN} = \text{The nth possible rate of return for asset } j \)
- \( p_N = \text{The probability of attaining the rate of return } n \text{ for assets } i \text{ and } j \)
- \( N = \text{the number of possible outcomes for the rate of return} \)

A positive covariance means that the return on two assets move in the same direction, while a negative covariance means the returns tend to move in opposite directions. In order to have a diversified portfolio, a negative covariance is desired. This will allow investors to reduce their exposure to individual asset risk, while allowing a desired expected return.

In the same way, the correlation of two assets can be calculated in order to see the degree to which two assets move together. Correlations show an easier way to evaluate behaviors. Correlations range from +1.0 to -1.0. A correlation of +1.0 means that there is perfect movement of the return of two assets in the same direction, while a correlation of -1.0 denotes a perfect movement in opposite directions. In the other hand, a correlation of zero implies that the returns are uncorrelated. Correlation can be defined as:

\[
\text{corr}(R_i, R_j) = \frac{Cov(R_i, R_j)}{SD(R_i)SD(R_j)}
\]

Where,

- \( SD(R_i) = \sqrt{\text{Var}(R_i)} \), denotes standard deviation or \( R_i \).
Model Inputs

In order to use the mean-variance model for portfolio construction, one must obtain estimates of the return, variance and covariance for each investment of interest. When N stocks are evaluated, there will be N return and variance estimates, and [N(N-1)]/2 covariance estimates, making a total of {2N + [N(N-1)]/2} estimates. This tells us that even though this model is the most comprehensive and practical to use, it is of little use when solving problems with large number of securities, because of the overwhelming amount of input estimates needed.

Model Limitations

Even though the mean-variance theory is stable and has enhanced the portfolio management process, it has some limitations to it. The disadvantages of the Markowitz model include:

- **The effects of estimation error** – Since the inputs of the model are created by analyzing historical data, they are considered estimates (expected returns, expected standard deviations, and expected correlations), and thus contain some sort of error. This can cause over and under investment in certain assets, and make an efficient portfolio look inefficient and an inefficient portfolio look efficient. To limit this error, constrained optimization can be used to set the maximum or minimum allocation of assets, and prevent assets with favorable inputs from dominating a portfolio.

- **Unstable solutions** – Small changes in inputs can lead to bigger changes in portfolio outputs, thus making the result of this model unstable. Because of instability, an update to a small change in the expected return or standard deviation can lead to a different portfolio allocation. To minimize these changes, a sensitivity analysis can be used; with this analysis one can select an efficient portfolio and alter the inputs to see how close to efficient the portfolio is. The goal here is to identify a set of asset class weight that will be close to efficient under different set of inputs.

- **Reallocation costs** – If there are two portfolios one with higher expected return than the other with the same risk level, one cannot simply alter one of the portfolio’s allocation to match the other, because depending on the asset classes within the portfolios and the
magnitude of the quantities involved, it might be costly to reallocate one portfolio to match the other’s expected return. It might be better to retain the current portfolio allocation despite its lack of efficiency.

Alternatives to the Markowitz Model

Due to the Markowitz mean-variance model limitations, many authors have looked at alternatives for this model. However, other alternatives also have serious limitations of their own. Other alternatives to the mean-variance model include:

- **Non-variance risk measures** – The semi-variance or semi-standard deviation of return measures only returns below the mean because the variance of returns above it is not considered risk by investors.
- **Utility function optimization** – Maximizing expected utility of wealth is the basis of rational decision making under uncertainty for many financial economists. The mean variance theory is consistent with this only if any of the following two conditions hold true. 1) Asset returns are normally distributed, or 2) utility function is quadratic.
- **Multi-period objectives** – The mean-variance model is a single-period model for investment behavior. However, many investors have long-term horizons. To address these objectives, the mean-variance model can be based on long term units of time or consider the multi-period distribution of the geometric mean of return.
- **Monte Carlo financial planning** – In this model, a computer model simulates the random functioning of an investment and changes its liabilities over time.
- **Linear programming optimizations** – This is a special case of quadratic programming. The main difference is that linear programming excludes portfolio variance. To exclude the variance from the model, there are two things that can be done: 1) Set the objective to maximize expected portfolio return subject to a variety of linear equality and inequality constraints on the structure of the portfolio, or 2) assume that the risk function is given by the absolute deviation of the rate of return, and not the standard deviation as illustrated in the Markowitz model.
Asset Allocation vs. Equity Portfolio Optimization

The two most common uses of the mean-variance optimization theory involve asset allocation and equity portfolio optimization. In both cases, the objective is to find efficient allocations of capital to maximize expected return and minimize risk, subject to certain constraints. In asset allocation, there is a limited amount of risky assets usually less than 50 with a small number of constraints. The starting points for input estimates include sample means, variances, and correlations based on historic data.

On the other hand, an equity portfolio optimization usually includes many securities. Domestic equity optimizations usually include 100-500 stocks, while international equity optimizations typically include 4,000-5,000 stocks. These optimizations also include many constraints on portfolio characteristics, trading cost restrictions, and industry or sector membership. For the purpose of this thesis, an asset allocation example has been made, and will be fully explained in the following pages.

Methodology

The basis of the Markowitz model can be formulated by minimizing risk subject to a set of constraints, as noted below:

\[
\min \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} x_i \ x_j
\]

Subject to:

\[
\sum_{i=1}^{n} \eta_i \ x_i \geq \bar{R}_i
\]

\[
\sum_{i=1}^{n} x_i = 1
\]

\[
x_i \geq 0
\]
In this case, \( r_i \) represents the expected rate of return of asset i, \( \bar{R}_i \) is the minimum level of return for the portfolio chosen by the investor, \( C_{ij} \) is the covariance between assets i and j, and \( x_i \) is the fraction (weight) of the portfolio invested in asset i.

The first constraint specifies the minimum level of return the investor expects from the portfolio. The next constraint indicates that the sum of the weights of all assets invested must be equal to the investment in the portfolio. The final constraint, the nonnegativity constraint, represents that no short sales are allowed in the portfolio. To calculate this model and obtain the efficient frontier, an understanding of quadratic programming is required.

The objective of a quadratic program is a quadratic function of decision variables, and its constraints constitute linear functions of variables. In a Markowitz mean-variance portfolio optimization problem the objective is the portfolio’s variance with linear constraints specifying a lower bound for the portfolio’s return, as shown in the quadratic formulation above. Because of linear constraints, quadratic programs have only one feasible region with an optimal solution lying anywhere within the region. Quadratic programs can be either easy or extremely hard to solve. The way of knowing this, is by seeing if the quadratic objective function and constraints are convex or non-convex. If the objective is convex with a convex feasible region, then there will be only one optimal solution, a globally optimal solution. However, if the objective function and feasible region are non-convex, then it might have many locally optimal points, meaning there are many more iterations to find a globally optimal solution, making the quadratic program difficult to solve.

The process knowing whether a quadratic program is convex or non-convex is beyond the scope of this paper. For the purpose of this thesis, it is important to know that portfolio optimization problems are of quadratic objective functions that are convex, and this is why they are quadratic programs that are easier to solve.

With Markowitz’ optimization model and the ability to solve it using quadratic programming, one can construct an efficient frontier. This frontier is a set of efficient portfolios with the highest level of return for any given level of risk. It is an essential tool to see if the selected portfolio is
performing well, given a certain amount of risk. Portfolios that lie below the efficient frontier are not efficient, because they do not provide enough return for the level of risk, while portfolios that lie to the right are also inefficient, because they have a higher level of risk for the given rate of return. The figure below depicts an example of an efficient frontier. The construction of an efficient frontier will be explained under the example program section.

![Efficient Frontier](image)

The red point in the above graph is called the global minimum variance (GMV) portfolio, which is the portfolio with the highest expected return and the lowest risk. The green line in the above graph constitutes the capital allocation line (CAL). This line is an extension of the efficient frontier that takes into account a risk-free asset. An example of a risk-free asset is a three-month T-bill rate from the United States Treasury; this is because it is very unlikely that the US Treasury will default within the next three months.

The tangency portfolio, where the CAL and efficient frontier cross, is considered to be the most efficient portfolio out of all possible portfolios (assuming the same time horizon, risk, and return levels), since this portfolio maximizes the Sharpe ratio. The Sharpe ratio is a formula that tracks the performance of a portfolio, taking into account the return of the portfolio, the return of the risk-free asset, and the standard deviation (risk) of the portfolio. It tells the extra portfolio return, given a level of risk. The greater a portfolio’s Sharpe ratio, the better its performance. There is only one most efficient portfolio (the tangency portfolio) where the Sharpe ratio is maximized, portfolios below or above the tangency portfolio decrease the Sharpe ratio. A negative Sharpe
ratio indicates that a risk-free asset would have better performance than the portfolio being analyzed. To calculate Sharpe ratio, the following formula is used:

$$SR = \frac{(\bar{r}_p - r_f)}{\sigma_p}$$

Where,

$\bar{r}_p = Expected \ portfolio \ return$

$r_f = Risk \ free \ rate$

$\sigma_p = Portfolio \ Standard \ deviation$

The rest of this paper will use the theory already explained to construct a portfolio using Microsoft Excel.

**Example Program**

To show the capabilities of the Markowitz portfolio optimization model, an example portfolio was created using Microsoft Excel. The purpose of this portfolio is to show an application of the Markowitz theory. In addition, the following steps have been automated using macros to make the program more user friendly. The only required inputs from the user are the historic monthly data of the 15 stocks and the current 3-month Treasury bill monthly rate.

The first step on the construction of the portfolio is to select 15 US stocks that represent different sectors of the market. The companies that make up the list of chosen stocks for the purpose of this project are: AT&T, Ford Motor Co., Mondelez International, Inc., Interpublic Group of Cos., Inc., Apple, Exxon Mobil, American Eagle Energy Corp., JP Morgan Chase & Co., The Goldman Sachs Group, Laboratory Corp. of America Holdings, Wal-Mart Stores, Google, Southwest Airlines, and Atmos Energy Corp. The second step is to record the stock price per month of each of these companies for a period of three years, from December 2009 to November 2013. This data can be acquired from Yahoo Finance or any other finance website. The stock price for each of the companies that make up the chosen portfolio are shown in the figure below.
In the “Return %” tab of the Excel program, the percentage of expected return is calculated per stock per month using the natural logarithm function in Excel. These expected returns along with their corresponding standard deviations, form the basis of the portfolio construction model. The figure below depicts the Return % tab.

![Figure 2: Stock prices per month per company](image)

![Figure 3: Percentage Expected Return per stock](image)
Next, the correlation and covariance matrices are calculated. These tables are constructed using their corresponding functions in the “Matrices” tab and copied to the “Solver” tab. The covariance matrix shows the movement of stocks relative to each other to use in the analysis, while the correlation matrix shows an easier scale to visualize. The figure below shows the solver tab with covariance and correlation matrices.

Figure 4: Correlation and Covariance matrices

Figure 4 also depicts two tables showing portfolio weights, these weights which have to sum to 1 depict the percentage of assets that need to be invested in every stock of the portfolio to obtain an efficient portfolio. These cells are used in the solver part of the calculation.

To find an efficient portfolio with quadratic objective function and linear constraints, the solver option in excel had to be used. First the global minimum risk that returns the largest return needed to be defined. This is accomplished by relaxing the constraint for expected returns and adding the following data in the solver.

- **Set Target Cell**: $Q49$ (This is the portfolio’s variance of return)
- **Equal to**: Min
- **By Changing Cells**: $A45:O45$ (These are the portfolio weights)
- **Subject to the Constraints**:
  - $A45:O45\leq 1$
$A$45:$O$45 => 0 (This is for no short selling.)

$Q$45 = 1 (This makes sure our portfolio weight is always 100%, or not borrowing.)

This returns the global minimum risk. Next, it is required to find the maximum return ignoring the risk by entering the following data on the solver.

- **Set Target Cell:** $Q$65 (Portfolio’s expected return, ignoring variance)
- **Equal to:** Max
- **By Changing Cells:** $A$45:$O$45 (These are the portfolio weights.)
- **Subject to the Constraints:**
  - $A$45:$O$45 <= 1
  - $A$45:$O$45 => 0 (This is for no short selling.)
  - $Q$45 = 1 (This makes sure our portfolio weight is always 100%, or not borrowing.)

This returns the best allocation of stocks to have the maximum return. It is important to note that the solver will allocate 100% of the funds to one stock (considered to be the one with higher expected return) in general. This is a good way to quickly validate the work.

In order to create an efficient frontier, we need the extreme points of the curve (the minimum risk with the highest return and the highest return ignoring risk). Now add points in between by using predetermined expected returns based of these extremes. In this example, predetermined steps were used to select risks. The following data was used to find one expected return to plot the efficient frontier.

- **Set Target Cell:** $Q$49 (This is the portfolio’s variance of return)
- **Equal to:** Min
- **By Changing Cells:** $A$45:$O$45 (These are the portfolio weights.)
- **Subject to the Constraints:**
  - $A$45:$O$45 <= 1
  - $A$45:$O$45 => 0 (This is for no short selling.)
Finally, it is necessary to find the best Sharpe ratio for this portfolio to help create a Capital Allocation Line. This can be done by entering the following information in the solver.

- **Set Target Cell:** $D$68 [this is the Sharpe ratio: (Portfolio expected return - risk free asset return)/Portfolio standard deviation of returns]
- **Equal to:** Max
- **By Changing Cells:** $A$45:$O$45 (These are the portfolio weights.)
- **Subject to the Constraints:**
  - $A$45:$O$45 <= 1
  - $A$45:$O$45 => 0 (This is for no short selling.)
  - $Q$45 = 1 (This makes sure our portfolio weight is always 100 %.)

Once all of these points are calculated, the table should look as the one below. Every result from the solver is showed with their respective weights (% holding) and each individual return per stock. These points are used to plot the efficient frontier as seen on Figure 1.
All this information is summarized on the “Portfolio” tab with the name of the 15 companies, their industries, ticker symbol, calculated expected return, risk, % holding per stock and a weighted stock return. Additionally, there are three graphs to visualize the data. The first one is the “Efficient Frontier” line graph of the given stocks. Next is the “Stock Return” bar graph that displays the individual weighted return of a selected portfolio. Finally, the “% Holding” pie chart show how your funds are allocated per company. On this tab, the user is able to choose a desired portfolio return based on his or her risk tolerance. The data will dynamically adjust to the user’s choice.
Limitations of Program

Even though the example program is meant to be user friendly, there are still some limitations to this program. The first limitation is the fact that the user can only input 15 stocks in the desired portfolio and there must be 48 monthly data entries. The second limitation is that the “Solver” tab cannot be modified by the user because it will crash the program.

Additionally, in order for the program to work, Microsoft Excel 2010 or a newer version may be required. Another possible error may occur if the user does not have Excel solver installed on the Excel worksheet and enabled on the VB Developer console.

To install the “Solver” functionality in Excel, select the Add-Ins option on the Excel’s tools menu. If the solver option is shown on the list, check the box in front of its name. If the solver option is not found, click Browse and navigate to the add-in file. After it appears on the add-in list, select its checkbox. To set a reference to an add-in, the Solver must first be installed on
Excel. Next, on the VB Editor’s Tools menu, select References. This lists all open workbooks and installed add-ins, as well as a list of resources installed on the host computer. Find “Solver” in the list, and check the box in front of its name.

**Conclusion**

To conclude, this thesis has been focused on the Modern Portfolio theory and its cornerstones, the mean-variance model and the efficient frontier. The purpose is to explain the logic behind constructing an efficient portfolio and how this model is facilitated by quadratic programming to find the most efficient portfolio that minimizes the variance (risk) of the portfolio given a set of linear constraints that specify a lower bound for portfolio return. We hope to use the acquired knowledge on the creation of this thesis and Excel program as a foundation to expand on Markowitz work in the near future, and become more knowledgeable investors. The main learning point of this thesis is that mathematical models and engineering can be used to model and solve problems in any field with the right assumptions.
References


