FEYNMAN-KAC SEMIGROUPS WITH DISCONTINUOUS ADDITIVE FUNCTIONALS, GAUGE THEOREMS AND APPLICATIONS TO DIRICHLET PROBLEMS

By

RENMING SONG

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RENMING SONG

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Chairman: Dr. Joseph Glover
Major Department: Mathematics

Let $X_t$ be a symmetric stable process of index $\alpha$, $0 < \alpha < 2$, in $\mathbb{R}^d$ ($d \geq 2$), let $\mu$ and $\nu$ be Radon measures on $\mathbb{R}^d$ belonging to the Kato class $K_{d,\alpha}$ and let $F$ be a Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying certain conditions. Suppose that $A_t^\mu$ and $B_t$ are the continuous additive functionals with Revuz measures $\mu$ and $\nu$, respectively, and that

$$A_t = A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s).$$

In this research we first study the following Feynman–Kac semigroup

$$T_t f(x) = E^x \{ e^{A_t} f(X_t) \}$$

and identify the bilinear form corresponding to $(T_t)_{t \geq 0}$. Then we prove a criterion, called the gauge theorem, about the boundedness of the following gauge function

$$g(x) = E^x \{ e^{A(t)} \}.$$
Finally we prove that for a suitable exterior function $f$,

$$u(x) = E^x \left( e^{A(\tau_D)} f(X(\tau_D)) \right) + E^x \int_0^{\tau_D} e^{A_t} dB_t$$

is the unique continuous solution to the Dirichlet problem of

$$\left( -(-\Delta)^{\frac{\mu}{2}} + \mu \right) u(x) + \int_{\mathbb{R}^d} \frac{G(x,y)u(y)}{|x-y|^{d+\alpha}} dy + \nu = 0$$

on $D$ with the exterior function $f$, where $G = e^F - 1$. 
CHAPTER 1
INTRODUCTION

The study of perturbations of Markov processes created by multiplicative functionals is a classical topic dating back to the early researches of Mark Kac. See [17]. This subject has been revived in the past ten years by both probabilists and mathematical physicists interested in the perturbations of the generators of Markov processes.

If \( X = (X_t, P^x) \) is a Feller process on \( \mathbb{R}^d \) and if \( A_t \) is a continuous additive functional of \( X \), then the semigroup defined by

\[
T_t f(x) = E^x \{ e^{A_t} f(X_t) \}
\]

is called a Feynman–Kac semigroup. Mathematical physicists are mainly interested in, among other things, various properties of the semigroup \((T_t)_{t \geq 0}\). For literature on this subject, one can consult [1], [2], [4], [5], [7], [31] and the references therein.

Probabilists are interested in, among other things, using the Feynman–Kac semigroup to solve various boundary value problems of second order elliptic partial differential equations. When \( X \) is a standard Brownian motion, we know, from probabilistic potential theory, that the function

\[
u(x) = E^x \{ e^{A_{\tau_D}} f(X(\tau_D)) \}
\]

(where \( D \) is an open set in \( \mathbb{R}^d \), \( \tau_D \) is the first exit time from \( D \) and \( f \) is a bounded function on \( \partial D \)) should be a candidate for a solution to the Dirichlet problem of the equation

\[
\left( \frac{\Delta}{2} + \mu \right) u = 0
\]
(where $\mu$ is the Revuz measure of $A_t$) on $D$ with boundary function $f$. One of the most important steps in making the sentence above precise is a criterion, generally known as the gauge theorem, about the boundedness of the gauge function

$$g(x) = E^x\{e^{A(t_D)}\}.$$  

This was first studied by Chung and Rao in [9] and later generalized by many others. For literature on this and some related problems, see [4], [10], [16], [22], [28], [30] and the references therein.

So far, all the literature on Feynman–Kac semigroups and their applications to various boundary value problems have dealt exclusively with the case when $A_t$ is a continuous additive functional. It is well known that, when the underlying Markov process $X$ is discontinuous, there are a lot of important discontinuous additive functionals. The most typical example of such an additive functional is as follows:

$$\sum_{0 < s \leq t} F(X_{s-}, X_s), \quad (1.1)$$

where $F$ is a Borel function on $R^d \times R^d$ vanishing on the diagonal. Naturally one would also like to study the Feynman–Kac semigroup $(T_t)_{t \geq 0}$ with the discontinuous additive functional given in (1.1). As one would expect, there are some difficulties in accomplishing this, since some of the basic tools used in the continuous case are no longer available in the discontinuous case. For example, it is no easy task to determine when

$$E^x\{e^{A_t}\}$$

is finite or bounded as a function of $x$, while the corresponding question in the continuous case can be solved by a simple use of Fubini’s theorem.

As for the application to the Dirichlet problem, although the case where $X$ is a diffusion has been thoroughly studied, no one has touched the case when $X$ is discontinuous. Even in the case when no perturbation is involved, discussions about
the Dirichlet problem for discontinuous Markov processes are scarce and unsystematic. There is a brief discussion in [20] about solving the Dirichlet problem for the symmetric stable process by using a balayage method. In [24] and [25] the Dirichlet problem for a class of infinitely divisible processes is discussed, but the formulation there is purely probabilistic and is not interpreted in analytic language. Another reference is [21] in which the Dirichlet problem for general discontinuous Markov processes is formulated and solved by using characteristic operators; and when the underlying process is a symmetric stable process an equivalent analytical formulation is also given. References [12] and [13] also contain some related discussions. When compared with the case of a second order differential operator $L$, the Dirichlet problem in an open set $D$ for the generator $A$ of a discontinuous Markov process has some new features. First, instead of a boundary function as in the diffusion case, we have to use a function which is defined on all of the complement of $D$ (which we call an exterior function) and we have to require that the solution to the Dirichlet problem coincide with the exterior function on the complement of $D$ (because when $X_t$ leaves $D$ it can jump to any point in the complement of $D$). Secondly, we have to be careful about the phrase "$u$ is a solution of $Au = 0$ in $D$", especially when we encounter it for the first time. In the historical case where $L$ is a differential operator, the phrase "$u$ is a solution to the equation $Lu = 0$ in $D$" means that $u$ is defined in $D$ and satisfies the equation $Lu(x) = 0$ for every $x \in D$. But in our case, the function $u$ has to be defined everywhere on $R^d$ because the operator $A$ is not a differential operator but is an integral operator and the integration extends over the whole space, not just over $D$.

Let $X = (X_t, P^x)$ be the symmetric stable process of index $\alpha$, $0 < \alpha < 2$ on $R^d$ ($d \geq 2$), let $\mu$ and $\nu$ be Radon measures on $R^d$ belonging to the Kato class $K_{d,\alpha}$ (see Definition 3.1.1) and let $F$ be a Borel function on $R^d \times R^d$ belonging to $A_{d,\alpha}$ (see
Definition 3.1.3). Suppose that $A_t^\mu$ and $B_t$ are the continuous additive functionals with Revuz measures $\mu$ and $\nu$, respectively, and that

$$A_t = A_t^\mu + \sum_{0<s\leq t} F(X_{s-}, X_s).$$

The purposes of this research are (a) to study the Feynman–Kac semigroup

$$T_t f(x) = E^x \{ e^{A_t f(X_t)} \}$$

and to identify the bilinear form corresponding to $(T_t)_{t\geq 0}$; (b) to use Dirichlet form theory to formulate and solve the Dirichlet problem for the equation

$$(-(-\Delta)^{\frac{\alpha}{2}} + \mu) u(x) + \int_{\mathbb{R}^d} \frac{G(x,y) u(y)}{|x-y|^{d+\alpha}} dy + \nu = 0 \quad (1.2)$$

on some bounded open set $D$, where $G = e^F - 1$.

As the title suggests, Chapter 2 serves as preparation for later chapters. In this chapter, we recall some basic properties about symmetric stable processes and killed symmetric stable processes. A section on bilinear forms and sesquilinear forms is also included. Section 2.1 and Section 2.3 contain no new results. Some of the results of Section 2.2 may have never been explicitly stated in the literature, but they are hardly surprising.

In chapter 3, we first introduce the class of additive functionals which we are going to deal with, then we study various properties of the Feynman–Kac semigroup and finally we identify the bilinear form corresponding to the Feynman–Kac semigroup.

In chapter 4, we first prove a criterion, called the gauge theorem, about the boundedness of the gauge function

$$g(x) = E^x \{ e^{A(\tau_D)} \}.$$ 

Then we formulate the Dirichlet problem of (1.2) on $D$ and prove that for a suitable exterior function $f$,

$$u(x) = E^x \left( e^{A(\tau_D)} f(X(\tau_D)) \right) + E^x \int_0^{\tau_D} e^{A_t} dB_t$$
is the unique continuous solution to the Dirichlet problem of (1.2) on $D$ with exterior function $f$. As an application of the gauge theorem and the results in Section 4.2, we prove, in Section 4.3, the existence of a solution to the Dirichlet problem for the semilinear equation

$$\left(-(-\Delta)^{\frac{\alpha}{2}} + \mu\right)u(x) + \int_{R^d} \frac{G(x, y)u(y)}{|x - y|^{d+\alpha}} dy + \xi(u) + \nu = 0$$  \hspace{1cm} (1.3)

on $D$, where $\xi$ is a continuous differentiable function on $R^1$. The results of Section 4.3 generalize those of [14].
CHAPTER 2
PRELIMINARIES

2.1 Symmetric Stable Processes

The starting point of our research is a particular symmetric Hunt process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$: the symmetric stable process of index $\alpha$, $0 < \alpha < 2$, on $\mathbb{R}^d$ ($d \geq 2$) with the characteristic function

$$e^{-t|x|^\alpha}$$

For convenience, we shall assume that $\Omega$ is the space of all right continuous maps $\omega$ from $[0, \infty)$ to $\mathbb{R}^d$. Set $X_t(\omega) = \omega(t)$, and let $\mathcal{F}_t$ and $\mathcal{F}$ be the appropriate completions of $\mathcal{F}_t^0 = \sigma\{X_s : s \leq t\}$ and $\mathcal{F}^0 = \sigma\{X_s : s \geq 0\}$. For each $t \geq 0$, $\theta_t : \Omega \rightarrow \Omega$ is the shift operator characterized by $X_s \circ \theta_t = X_{s+t}$.

As usual, we use $(P_t)$ to denote the transition semigroup of $X_t$ and $U_\gamma$, $\gamma \geq 0$, to denote the $\gamma$-potential of $(P_t)$,

$$U_\gamma f(x) = \int_0^\infty e^{-\gamma t}P_t f(x) dt.$$

The following result is well known. See, for instance, [7].

**Theorem 2.1.1** There is a function $p(t, x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that

(i) for any $t > 0$, any $x \in \mathbb{R}^d$ and any nonnegative Borel function $f$ on $\mathbb{R}^d$,

$$P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy;$$

(ii) $p(t, x, y)$ is jointly continuous on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$;
(iii) for any \( t > 0 \) and any \( x, y \in \mathbb{R}^d \),
\[
p(t, x, y) = p(t, y, x);
\]

(iv) for any \( t, s > 0 \) and any \( x, y \in \mathbb{R}^d \)
\[
p(t + s, x, y) = \int_{\mathbb{R}^d} p(t, x, z)p(s, z, y)dz;
\]

(v) for any \( t > 0 \) and any \( x, y \in \mathbb{R}^d \),
\[
p(t, x, y) = p(t, 0, y - x);
\]

(vi) for any \( t > 0 \) and any \( x, y \in \mathbb{R}^d \),
\[
0 < p(t, x, y) \leq p(t, 0, 0);
\]

(vii) for any bounded continuous function \( f \) on \( \mathbb{R}^d \),
\[
\lim_{t \to 0} \int_{\mathbb{R}^d} p(t, x, y)f(y)dy = f(x).
\]

Theorem 2.1.1 only guarantees the existence of the density \( p(t, x, y) \); it does not provide an explicit formula for \( p(t, x, y) \). In fact, except for the cases of \( \alpha = 1 \) and \( \alpha = 1/2 \), no explicit formula seems to be known for \( p(t, x, y) \). But fortunately the following two results (see, for instance, [7]) will provide all the estimates that we are going to use later on.

**Theorem 2.1.2** For any \( \eta > 0 \), there exist positive constants \( c_1 \) and \( c_2 \) which depend only on \( d, \alpha \) and \( \eta \) such that for any \( x, y \in \mathbb{R}^d \) with \( |x - y| \geq \eta \),
\[
\frac{c_1}{|x - y|^{d+\alpha}} \leq p(1, x, y) \leq \frac{c_2}{|x - y|^{d+\alpha}}.
\]
Theorem 2.1.3 For any $t > 0$ and any $x, y \in \mathbb{R}^d$,

$$p(t, x, y) = t^{-\frac{d}{\alpha}} p(1, t^{-\frac{1}{\alpha}} x, t^{-\frac{1}{\alpha}} y).$$

From the two results above we can immediately get the following lemma which will be used in the proof of the next theorem.

Lemma 2.1.1 For any $\eta > 0$,

$$\lim_{t \to 0} \sup_{|x-y| \geq \eta} p(t, x, y) = 0.$$

Proof. Fix $\eta > 0$ and $x, y \in \mathbb{R}^d$ such that $|x - y| \geq \eta$. Then by Theorem 2.1.2 and Theorem 2.1.3 we know that there exists a constant $c > 0$ such that

$$p(t, x, y) \leq t^{-\frac{d}{\alpha}} \frac{c}{|t^{-\frac{1}{\alpha}}(x - y)|^{d+\alpha}} \leq \frac{ct}{|x - y|^{d+\alpha}} \leq \frac{ct}{\eta^{d+\alpha}}.$$

Therefore the conclusion of the lemma is true.

Q.E.D.

Theorem 2.1.4 $(P_t)_{t \geq 0}$ is a strongly continuous semigroup on $C_0(\mathbb{R}^d)$. Here $C_0(\mathbb{R}^d)$ is the Banach space of all continuous real-valued functions $f$ on $\mathbb{R}^d$ such that $\lim_{|x| \to \infty} f(x) = 0$.

Proof. First, we are going to prove that, for any $t > 0$, $P_t$ maps $C_0(\mathbb{R}^d)$ into itself. Fix a $t > 0$ and an $f \in C_0(\mathbb{R}^d)$. Then for any $n > 0$, we have

$$P_tf(x) = \int_{|y| \leq n} p(t, x, y)f(y)dy + \int_{|y| > n} p(t, x, y)f(y)dy.$$
The fact that, for any \( n > 0 \), the function
\[
x \mapsto \int_{|y| \leq n} p(t, x, y)f(y)dy
\]
is in \( C_0(R^d) \) follows from the continuity of \( p \) and the lemma above. Since
\[
\lim_{n \to \infty} \sup_{x \in R^d} \int_{|y| > n} p(t, x, y)f(y)dy \leq \lim_{n \to \infty} \sup_{|y| > n} |f|(y) = 0,
\]
we know that \( P_t f \) is in \( C_0(R^d) \).

From Theorem 2.1.1 we know that for any \( f \in C_0(R^d) \) and any \( x \in R^d \),
\[
\lim_{t \to 0} P_t f(x) = f(x),
\]
thus it follows from [8] that \( (P_t)_{t \geq 0} \) is a strongly continuous semigroup on \( C_0(R^d) \).

\( Q.E.D. \)

For any \( p \in [1, \infty) \), we use \( L^p(R^d) \) to denote the real Banach space of all the real-valued functions \( f \) such that
\[
\int_{R^d} |f|^p(x)dx < \infty.
\]
The norm on \( L^p(R^d) \) will be denoted by \( \| \cdot \|_p \). \( L^\infty(R^d) \) will stand for the real Banach space of all the essentially bounded real-valued functions on \( R^d \), and the norm on this space will be denoted by \( \| \cdot \|_\infty \).

**Theorem 2.1.5** If \( 1 \leq p \leq p' \leq \infty \), then for any \( t > 0 \), \( P_t \) is a bounded operator from \( L^p(R^d) \) into \( L^{p'}(R^d) \).

**Proof.** From Theorem 2.1.1, Theorem 2.1.2 and Theorem 2.1.3 we know that for any \( t > 0 \) the function \( p(t, 0, \cdot) \) is in \( L^q(R^d) \) for any \( q \in [1, \infty] \). Applying Hölder's inequality we get that for any \( f \in L^p(R^d) \) and \( t > 0 \), \( \|P_t f\|_p \leq \|f\|_p \). Thus for any \( t > 0 \), \( P_t \) is bounded operator from \( L^p(R^d) \) into \( L^p(R^d) \).
Applying Hölder's inequality again we get that for $t > 0$, $x \in \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d)$,

$$|P_t f(x)| \leq \left( \int_{\mathbb{R}^d} p(t, x, y) |f|^p(x) dy \right)^{\frac{1}{p}} \leq p(t, 0, 0) \|f\|_p.$$ 

Thus for any $t > 0$, $P_t$ is bounded operator from $L^p(\mathbb{R}^d)$ into $L^\infty(\mathbb{R}^d)$.

Now the theorem follows easily from the two paragraphs above.

Q.E.D.

**Theorem 2.1.6** For any $p \in [1, \infty)$, $(P_t)_{t \geq 0}$ is a strongly continuous semigroup on $L^p(\mathbb{R}^d)$.

**Proof.** For any $f \in L^p(\mathbb{R}^d)$, any $t > 0$ and almost every $x \in \mathbb{R}^d$,

$$P_t f(x) - f(x) = \int_{\mathbb{R}^d} p(t, x, y)(f(y) - f(x)) dy = \int_{\mathbb{R}^d} p(t, 0, y)(f(y + x) - f(x)) dy.$$

Using Hölder's inequality we get

$$\|P_t f - f\|_p^p = \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p(t, 0, y)(f(y + x) - f(x)) dy \right|^p dx \leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} p(t, 0, y)|f(y + x) - f(x)|^p dy \right) dx = \int_{\mathbb{R}^d} \|f(y + \cdot) - f(\cdot)\|^p p(t, 0, y) dy.$$

Since the function $y \mapsto \|f(y + \cdot) - f(\cdot)\|^p$ is bounded and continuous, we get that

$$\lim_{t \to 0} \|P_t f - f\|_p^p = 0.$$

Q.E.D.

**Theorem 2.1.7** For any $t > 0$, $P_t$ maps $L^\infty(\mathbb{R}^d)$ into $bC(\mathbb{R}^d)$. Here $bC(\mathbb{R}^d)$ stands for the real Banach space of all the bounded real-valued continuous functions on $\mathbb{R}^d$.

**Proof.** It follows from the semigroup property that we only need to consider the case when $t \in (0, 1]$. Fix a $t \in (0, 1]$. For any $f \in L^\infty(\mathbb{R}^d)$, the boundedness of $P_t f$
is trivial. Let \( f \in L^\infty(R^d) \) and let \( r \) be an arbitrary positive number. Then for any \( n > 0 \),

\[
P_t f(x) = \int_{|y| \leq r + n + 1} p(t, x, y)f(y)dy + \int_{|y| > r + n + 1} p(t, x, y)f(y)dy.
\]

The continuity of the function

\[
x \mapsto \int_{|y| \leq r + n + 1} p(t, x, y)f(y)dy
\]

follows from the dominated convergence theorem. From Theorem 2.1.2 and Theorem 2.1.3 we can get

\[
\lim_{n \to \infty} \sup_{|x| \leq r} \left| \int_{|y| > r + n + 1} p(t, x, y)f(y)dy \right| \leq \|f\|_{L^\infty(R^d)} \lim_{n \to \infty} \sup_{|x| \leq r} \int_{|y| > r + n + 1} p(t, x, y)dy
\]

\[
\leq C \|f\|_{L^\infty(R^d)} \sup_{|x| \leq r} \int_{|y| > r + n + 1} \frac{1}{|x - y|^{d+\alpha}}dy
\]

\[
\leq C \|f\|_{L^\infty(R^d)} \lim_{n \to \infty} \int_0^\infty s^{-(1+\alpha)}ds
\]

\[
= 0,
\]

where \( C \) is the constant \( c_2 \) in Theorem 2.1.2. Therefore \( P_t f \) is continuous on \( \{x : |x| \leq r\} \). Since \( r \) is arbitrary, \( P_t f \) is continuous on \( R^d \). The proof is now complete.

Q.E.D.

Theorem 2.1.8 For any \( t > 0 \) and any \( p \in [1, \infty) \), \( P_t \) maps \( L^p(R^d) \) into \( C_0(R^d) \).

*Proof.* Fix an \( f \in L^p(R^d) \) and a \( t > 0 \). For any \( n > 0 \), put

\[
f_n(x) = 1_{B(0,n)}(x)f(x),
\]

with \( B(0,n) \) being the ball of radius \( n \) around the origin. Using the dominated convergence theorem we can easily see that the function

\[
x \mapsto P_t f_n(x) = \int_{B(0,n)} p(t, x, y)f(y)dy
\]
belongs to $C_0(R^d)$. Now by using Hölder's inequality we can get that, for any $x \in R^d$, 
\[
|P_t f(x) - P_t f_n(x)| = \left| \int_{R^d} p(t,x,y)(f-f_n)(y)dy \right| 
\leq \|p(t,x,y)\|_q\|f-f_n\|_p 
= \|p(t,0,y)\|_q\|f-f_n\|_p,
\]
where $q$ is such that $\frac{1}{p} + \frac{1}{q} = 1$. Since 
\[
\lim_{n \to \infty} \|f-f_n\|_p = 0,
\]
we know that $P_t f$ must be in $C_0(R^d)$. 

Q.E.D.

The following simple result will be used in the next section.

Lemma 2.1.2 If $A, B$ are disjoint closed subsets of $R^d$ with $B$ being compact, then $p(t,x,y)$ is uniformly continuous on $(0,T] \times A \times B$ for any $T > 0$.

Proof. Follows directly from Lemma 2.1.1 and the joint continuity of $p(t,x,y)$. Q.E.D.

For any $x, y \in R^d$, define 
\[
u(x,y) = \int_0^\infty p(t,x,y)dt.
\]
The function $\nu(\cdot, \cdot)$ is called the Green function of $X$. Although generally we do not have an explicit formula for $p(t,x,y)$, the following nice formula for the Green function is well known. See, for instance, [31].

Theorem 2.1.9 For any $x, y \in R^d$, 
\[
u(x,y) = C|x-y|^{-d+\alpha},
\]
where $C > 0$ is a constant depending on $d$ and $\alpha$ only.
2.2 Killed Symmetric Stable Processes

Let \( D \) be a bounded open subset of \( \mathbb{R}^d \). We are going to use \( \tau_D \) to denote the first exit time from \( D \), i.e.,

\[
\tau_D = \inf \{ t > 0 : X_t \notin D \}.
\]

Then it is well known that the process

\[
X_t^D(\omega) = \begin{cases} 
X_t(\omega), & \text{if } t < \tau_D; \\
\delta, & \text{if } t \geq \tau_D,
\end{cases}
\]

is a symmetric Hunt process on \( D_\delta = D \cup \{ \delta \} \), the one-point compactification of \( D \). This process is called the symmetric stable process of index \( \alpha \) killed upon leaving \( D \), or simply the killed symmetric stable process on \( D \). The purpose of this section is to study some of the properties of \( X^D \).

We are going to use \( (P_t^D)_{t \geq 0} \) to denote the transition semigroup of \( X^D \) and \( (U^D_\gamma)_{\gamma \geq 0} \) to denote the \( \gamma \)-potential of \( X^D \):

\[
U_\gamma^D f(x) = \int_0^\infty e^{-\gamma t} P_t^D f(x) dt.
\]

**Lemma 2.2.1** For any \( t > 0 \) and \( x \in \mathbb{R}^d \),

\[
P^x(\tau_D = t) = 0.
\]

**Proof.** Let \( n \) be a positive integer. Then

\[
\int_{n \leq |y| < n + 1} P_y^y(\tau_D \leq t) dy < \infty,
\]

for every \( t > 0 \) and hence

\[
\int_{n \leq |y| < n + 1} P_y^y(\tau_D = s) dy > 0
\]

for only a finite or countably infinite number of values of \( s \). Thus \( \int_{\mathbb{R}^d} P_y^y(\tau_D = s) dy \) for only a finite or countably infinite number of values of \( s \).
Let \( t > 0 \). By the previous paragraph there is an \( s \in (0, t) \) such that

\[
\int_{\mathbb{R}^d} P^y(\tau_D = s) dy = 0.
\]

Since

\[
\{\tau_D = t\} = \{\tau_D > t - s, t - s + \tau_D \circ \theta_{t-s} = t\} \subset \{\tau_D \circ \theta_{t-s} = s\},
\]

by the Markov property we get

\[
P_x(\tau_D = t) \leq E^x\{ P^{X(t-s)}(\tau_D = s) \}
= \int_{\mathbb{R}^d} p(t - s, x, y) P^y(\tau_D = s) dy
= 0.
\]

This completes the proof.

Q.E.D.

**Lemma 2.2.2** For any \( t > 0 \), the function \( x \mapsto P_x(\tau_D \leq t) \) is lower semicontinuous.

**Proof.** Let \( 0 < s < t \). Then

\[
P_x(X_u \notin D \text{ for some } u \in (s, t)) = \int_{\mathbb{R}^d} p(s, x, y) P^y(\tau_D \leq t - s) dy,
\]

which is continuous in \( x \) and increases to

\[
P_x(\tau_D < t) = P_x(\tau_D \leq t)
\]
as \( s \downarrow 0 \). Thus the conclusion of this lemma is true.

Q.E.D.

**Definition 2.2.1** A bounded open set \( D \) in \( \mathbb{R}^d \) is said to be regular if every point \( z \) on \( \partial D \) is regular for \( D^c \), i.e., \( P^z(\tau_D = 0) = 1 \).
In the following we are going to assume that $D$ is a fixed regular bounded open subset of $\mathbb{R}^d$.

**Theorem 2.2.1** $(P^D_t)_{t>0}$ admits an integral kernel $p^D(t, x, y)$ defined on $(0, \infty) \times D \times D$ which satisfies the following properties:

(i) for any $x \in D$ and any nonnegative Borel function $f$ on $D$,

$$P^D_t f(x) = \int_D p^D(t, x, y) f(y) dy;$$

(ii) $p^D(t, x, y)$ is jointly continuous on $(0, \infty) \times D \times D$;

(iii) for any $t, s > 0$ and $x, y \in D$,

$$p^D(t + s, x, y) = \int_D p^D(t, x, z) p^D(s, z, y) dz;$$

(iv) for any $t > 0$ and any $x, y \in D$,

$$p^D(t, x, y) = p^D(t, y, x);$$

(v) for any $t > 0$ and any $x, y \in D$,

$$0 \leq p^D(t, x, y) \leq p(t, x, y);$$

(vi) if $t_0 > 0$ and $x_0$ (or $y_0$) is a point on $\partial D$ then

$$\lim_{n \to \infty} p^D(t_n, x_n, y_n) = 0$$

whenever $(t_n, x_n, y_n) \in (0, \infty) \times D \times D$ converges to $(t_0, x_0, y_0)$.

**Proof.** For any $t > 0$ and $x, y \in D$, set

$$p^D(t, x, y) = p(t, x, y) - E^x[p(t - \tau_D, X_{\tau_D}, y); \tau_D < t].$$

(2.1)
We are going to show that the \( p^D(t, x, y) \) defined above satisfies all the requirements of the theorem.

It is clear that for any \( t > 0 \) and \( x, y \in D \),

\[
p^D(t, x, y) \leq p(t, x, y)\]

By using the strong Markov property we can prove that \( p^D(t, x, y) \) satisfies property (i) of the theorem. From this we can easily derive the following:

(a) for any \( t > 0 \) and any \( x \in D \), \( p^D(t, x, \cdot) \) is nonnegative almost everywhere on \( D \);

(b) for any \( t, s > 0 \) and any \( x \in D \), the following identity is true for almost all \( y \in D \),

\[
p^D(t + s, x, y) = \int_D p^D(t, x, z)p^D(s, z, y)dz;
\]

(c) for any \( t > 0 \) and any nonnegative Borel functions \( f, g \) on \( D \),

\[
\int_D \int_D p^D(t, x, y)f(x)g(y)dxdy = \int_D \int_D p^D(t, y, x)f(x)g(y)dxdy;
\]

(d) for any \( x \in D \) and any bounded continuous function \( f \) on \( D \),

\[
l_{t} \int_D p^D(t, x, y)f(y)dy = f(x).
\]

Let \( (t_0, y_0) \in (0, \infty) \times D \). Take a closed neighborhood \([\epsilon, t] \times A\) of \((t_0, y_0)\) such that \( \epsilon > 0 \) and \( A \subset D \). For an arbitrary sequence \( \{(t_n, y_n)\}_{n \geq 1} \subset [\epsilon, t] \times A \) such that \( (t_n, y_n) \) converges to \((t_0, y_0)\), we have, by Lemma 2.2.1 and Lemma 2.1.2,

\[
\lim_{n \to \infty} E^x[p(t_n - \tau_D, X_{\tau_D}; y_n); \tau_D < t_n] = E^x[p(t_0 - \tau_D, X_{\tau_D}; y_0); \tau_D < t_0],
\]

therefore \( p^D(t, x, y) \) is jointly continuous in \( t \) and \( y \) for any fixed \( x \in D \). Now applying the continuity of \( p^D \) in \( y \), (iii) follows from (b) above and the first inequality of (v) follows from (a) above.
Since for any \((t, y) \in (0, \infty) \times D\) the function \(p^D(t, y, \cdot)\) is bounded continuous, we get from (d) above

\[
\lim_{\epsilon \downarrow 0} \int_D p^D(\epsilon, x, z)p^D(t, y, z)dz = p^D(t, y, x).
\]  

(2.2)

Since for any \(x, y \in D\), \(p^D\) is continuous in \(t\), it follows from (iii) that

\[
\lim_{\epsilon \downarrow 0} \int_D p^D(\epsilon, x, z)p^D(t, z, y)dz = \lim_{\epsilon \downarrow 0} p^D(t + \epsilon, x, y) = p^D(t, x, y).
\]  

(2.3)

Let \(\epsilon, \eta > 0\). By (c) above we have

\[
\int_D \int_D p^D(t, u, v)p^D(\epsilon, x, u)p^D(\eta, y, v)dudv = \int_D \int_D p^D(t, v, u)p^D(\epsilon, x, u)p^D(\eta, y, v)dudv.
\]

First letting \(\epsilon \downarrow 0\) and then letting \(\eta \downarrow 0\), we obtain (iv) by applying (2.2) and (2.3) to the identity above. A consequence of (iv) is that \(p^D(t, x, y)\) is also jointly continuous in \((t, x)\) for any fixed \(y \in D\).

Let \(\{(t_n, x_n, y_n)\}_{n \geq 1} \subset (0, \infty) \times D \times D\) and \((t_n, x_n, y_n) \to (t, x, y) \in (0, \infty) \times D \times D\). Without loss of generality we may assume that for any \(n > 1\), \(S < t_n < T\) for some constants \(T > S > 0\). From (iii) and (v) we can get that

\[
\lim_{n \to \infty} p^D(t_n, x_n, y_n) = \lim_{n \to \infty} \int_D p^D(t_n - \epsilon, x_n, z)p^D(\epsilon, z, y_n)dz = p^D(t, x, y).
\]

Thus (ii) is verified.

It remains to prove (vi). Let \(x_0 \in D\) and let \(\{(t_n, x_n, y_n)\}_{n \geq 1} \subset (2\epsilon, T) \times D \times D\) be such that \((t_n, x_n, y_n) \to (t, x_0, y_0) \in (0, \infty) \times \partial D \times \overline{D}\). Then we have by (iii) and (v),

\[
p^D(t_n, x_n, y_n) = \int_D p^D(\epsilon, x_n, z)p^D(t_n - \epsilon, z, y_n)dz \leq MP^{x_n}(\epsilon < \tau_D),
\]

where \(M\) is a constant. Since the function \(x \mapsto P^x(\epsilon < \tau_D)\) is upper semicontinuous,
we can derive from the above
\[ \limsup_{n \to \infty} p^D(t_n, x_n, y_n) \leq MP^{x_0}(\epsilon < \tau_D) = 0, \]
which completes the proof of the theorem.

\[ Q.E.D. \]

From the theorem above we immediately get the following corollary.

**Corollary 2.2.1** The function
\[ u^D(x,y) = \int_0^\infty p^D(t,x,y)dt \]
is finite off the diagonal of \( D \times D \). Furthermore, it is jointly continuous off the diagonal of \( D \times D \).

**Proof.** The first assertion follows easily from Theorem 2.1.9 and conclusion (v) of Theorem 2.2.1. We are now going to the second assertion. From (2.1) we have, for any \( x,y \in D \),
\[ u(x,y) = u_D(x,y) + ES\{u(X(\tau_D),y)\}, \]
so \( ES\{u(X(\tau_D),y)\} \) is symmetric in \( x \) and \( y \) on \( D \times D \). To prove the desired result it suffices to show that \( ES\{u(X(\tau_D),y)\} \) is jointly continuous off the diagonal of \( D \times D \). Choose \( x_0, y_0 \in D \) with \( x_0 \neq y_0 \). For any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that
\[ |u(z,y) - u(z,y_0)| \leq \frac{\epsilon}{2}, \]
\[ |u(z,x) - u(z,x_0)| \leq \frac{\epsilon}{2}, \]
for \( |x - x_0| < \delta, |y - y_0| < \delta \), and \( z \in D^c \). Choose \( x, y \in D \) such that \( |x - x_0| < \delta \) and \( |y - y_0| < \delta \). Then
\[ |E_x^z\{u(X(\tau_D), y)\} - E^{x_0}_x\{u(X(\tau_D), y_0)\}| \]
\[ \leq E_x^z\{|u(X(\tau_D), y) - u(X(\tau_D), y_0)|\} + E^{y_0}_x\{|u(X(\tau_D), x) - u(X(\tau_D), x_0)|\} \]
\[ \leq \epsilon. \]

Therefore \( E_x^z\{u(X(\tau_D), y)\} \) is jointly continuous off the diagonal of \( D \times D \) as desired.

\[ Q.E.D. \]

By using the theorem above we can also get the following result which will be used in Chapter 4.

**Theorem 2.2.2** For any \( t > 0 \), \( P_t^D \) is a compact operator from \( L^\infty(D) \) to \( L^\infty(D) \).

**Proof.** Let \( \{f_n\} \) be a bounded sequence in \( L^\infty(D) \), i.e., there exists a \( C_1 > 0 \) such that for any \( n > 0 \), \( \|f_n\|_\infty \leq C_1 \). Then \( \{f_n\} \) is also bounded in \( L^k(D) \) for any \( k > 1 \). Since for any \( k > 1 \), the unit ball in \( L^k(D) \) is weakly compact, by applying diagonal argument we see that there is a subsequence of \( \{f_n\} \), say, \( \{f_n\} \) itself, which is weakly convergent to an \( f \in L^\infty(D) \). Hence for any \( x \in D \) and any \( t > 0 \),

\[ \lim_{n \to \infty} P_t^D f_n(x) = P_t^D f(x) \]

Consequently for any \( t > 0 \), \( \{P_t^D f_n\}_{n \geq 1} \) converges in measure with respect to the Lebesgue measure \( m \) on \( D \). Therefore for any \( \epsilon > 0 \), there exists an integer \( N \) such that

\[ m(\{y \in D : |P_{t/2}^D f_n(y) - P_{t/2}^D f(y)| > \epsilon\}) < \epsilon, \]

whenever \( n \geq N \). Thus for any \( x \in D \) and any \( t > 0 \),
\[ |P^D_t f_n(x) - P^D_t f(x)| = \left| \int_D p^D(t, x, y) \left( P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y) \right) dy \right| \]
\[ \leq \int_D p^D(t, x, y) |P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y)| dy \]
\[ = \int_{\{|p^D_{\frac{t}{2}} f_n - p^D_{\frac{t}{2}} f| > \epsilon\}} p^D(t, x, y) |P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y)| dy \]
\[ + \int_{\{|p^D_{\frac{t}{2}} f_n - p^D_{\frac{t}{2}} f| \leq \epsilon\}} p^D(t, x, y) |P^D_{\frac{t}{2}} f_n(y) - P^D_{\frac{t}{2}} f(y)| dy \]
\[ \leq C_2 \epsilon + \epsilon, \]
whenever \( n \geq N \), where
\[ C_2 = 2 \max(C_1, \|f\|_\infty, \|p^D(t, \cdot, \cdot)\|_\infty). \]

The proof is now complete.

Q.E.D.

2.3 Bilinear Forms and Sesquilinear Forms

Let \( E \) be a locally compact Hausdorff space with a countable base. We are going to fix a positive Radon measure \( m \) on \( E \) such that for any nonempty open subset \( O \) of \( E \), \( m(O) > 0 \). \( L^2(E, m) \) will always denote the real \( L^2 \)-space with inner product
\[ (u, v) = \int_E u(x)v(x)m(dx) \]
and \( L^2_c(E, m) \) will always denote the complex \( L^2 \)-space with inner product
\[ (u, v) = \int_E u(x) \overline{v(x)} m(dx). \]

Definition 2.3.1 Let \( D \) be a linear subspace of \( L^2(E, m) \). A map \( T \) from \( D \times D \) into \( R^1 \) is said to be a bilinear form if

(i) for any \( u_1, u_2, v \in D \) and any \( a, b \in R^1 \),
\[ T(au_1 + bu_2, v) = aT(u_1, v) + bT(u_2, v); \]
(ii) For any \( u, v_1, v_2 \in \mathcal{D} \) and any \( a, b \in \mathbb{R} \),

\[
T(u, av_1 + bv_2) = aT(u, v_1) + bT(u, v_2).
\]

\( \mathcal{D} \) is said to be the domain of \( T \). A bilinear form \((T, \mathcal{D})\) is said to be symmetric if for any \( u, v \in \mathcal{D} \)

\[
T(u, v) = T(v, u).
\]

A bilinear form \((T, \mathcal{D})\) is said to be densely defined if \( \mathcal{D} \) is dense in \( L^2(E, m) \).

In the sequel we shall always deal with densely defined bilinear forms; so from now on, whenever we talk about a bilinear form, we mean a densely defined bilinear form.

To any bilinear form \((T, \mathcal{D})\) we can associate the following bilinear form \((\hat{T}, \mathcal{D})\):

\[
\hat{T}(u, v) = \frac{1}{2} \{T(u, v) + T(v, u)\}.
\]

It is clear that \( \hat{T} \) is a symmetric bilinear form.

For any bilinear form \((T, \mathcal{D})\) and any \( \beta > 0 \), \((T_\beta, \mathcal{D})\) will always denote the following bilinear form

\[
T_\beta(u, v) = T(u, v) + \beta(u, v).
\]

**Definition 2.3.2** A bilinear form \((T, \mathcal{D})\) is said to satisfy the sector condition if there exists a \( \beta_0 > 0 \) and a constant \( K > 0 \) such that

(i) for any \( u \in \mathcal{D} \),

\[
T_{\beta_0}(u, u) \geq 0;
\]

(ii) for any \( u, v \in \mathcal{D} \),

\[
|T_{\beta_0}(u, v)| \leq KT_{\beta_0}(u, u)^{\frac{1}{2}}T_{\beta_0}(v, v)^{\frac{1}{2}}.
\]
**Definition 2.3.3** A bilinear form \((T, D)\) is said to be closed if there exist a \(\beta_0 > 0\) such that

(i) for any \(u \in D\),
\[ T_{\beta_0}(u, u) \geq 0; \]

(ii) for any \(\beta > \beta_0\), \(D\) is a real Hilbert space with respect to the inner product \(T_{\beta}\).

From the definition it is easy to see that any closed symmetric bilinear form satisfies the sector condition.

The following result is proved by Kunita. See [19].

**Theorem 2.3.1** If \((T, D)\) is a closed bilinear form satisfying the sector condition, then there exists a uniquely determined strongly continuous semigroup \((T_t)_{t \geq 0}\) on \(L^2(E, m)\) with resolvent
\[ V_{\beta}f(x) = \int_0^\infty e^{-\beta t}T_tf(x)dt \]

satisfying
\[ T_{\beta}(V_{\beta}f, u) = (f, u) \]
for any \(f \in L^2(E, m)\), \(u \in D\) and \(\beta > \beta_0\). Furthermore, when \((T, D)\) is symmetric, \((T_t)_{t \geq 0}\) is a strongly continuous semigroup of self-adjoint operators on \(L^2(E, m)\).

**Definition 2.3.4** A bilinear form \((T, D)\) is said to be a Dirichlet form if

(i) for any \(u \in D\),
\[ T(u, u) \geq 0; \]

(ii) \((T, D)\) is a closed bilinear form;

(iii) \((T, D)\) satisfies the sector condition;
(iv) the strongly continuous semigroup $(T_t)_{t \geq 0}$ determined by $(T,D)$ is Markovian, that is, for any $t > 0$ we have $0 \leq T_t f \leq 1$ whenever $f \in L^2(E,m)$, $0 \leq f \leq 1$ m-a.e.

**Definition 2.3.5** Let $(T,D)$ be a Dirichlet form and let $C_{0,0}(E)$ be the family of all continuous functions with compact supports. We shall say that $(T,D)$ is regular if $D \cap C_{0,0}(E)$ is dense in $D$ with the $\hat{T}_1$-norm and dense in $C_{0,0}(E)$ with the uniform norm.

**Definition 2.3.6** A Dirichlet form $(T,D)$ is said to be transient if there exists an m-a.e. strictly positive bounded function $f \in L^2(E,m)$ such that for any $u \in D$,

$$ \int_E |u|(x)f(x)m(dx) \leq \sqrt{T(u,u)}. $$

We know that for a Dirichlet form $(T,D)$, $D$ is a Hilbert space with inner product $\hat{T}_\beta$ for any $\beta > 0$, but $D$ is not even a pre–Hilbert space with respect to $\hat{T}$ in general. When $(T,D)$ is transient, $D$ is a pre–Hilbert space with inner product $\hat{T}$.

**Definition 2.3.7** Suppose that $(T,D)$ is a transient symmetric Dirichlet form. If we use $D_e$ to denote the completion of the pre–Hilbert space $D$ with respect to $T$, then $(T,D_e)$ is called an extended Dirichlet form associated with $(T,D)$.

**Example 2.3.1** (See [13].) Assume that $0 < \alpha < 2$, $E = R^d(d \geq 2)$ and $m$ is the Lebesgue measure on $R^d$. Put

$$ T(u,v) = \int_{R^d} \hat{u}(x)\overline{\hat{v}(x)}|x|^\alpha dx, $$

$$ D = \left\{ u \in L^2(R^d) : \int_{R^d} |\hat{u}(x)|^2|x|^\alpha dx < \infty \right\}, $$

where

$$ \hat{u}(x) = (2\pi)^{-\frac{d}{2}} \int_{R^d} e^{i(x,y)}u(y)dy. $$
Then \((T, D)\) is a transient, regular symmetric Dirichlet form and the strongly continuous semigroup determined by \((T, D)\) is nothing but the transition semigroup of the symmetric stable process of index \(\alpha\) on \(\mathbb{R}^d\). Another way of writing \((T, D)\) is as follows.

\[
T(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \, dx \, dy,
\]

\[
D = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \, dx \, dy < \infty \right\}.
\]

The extended Dirichlet form associated with \((T, D)\) can be characterized as follows.

\[
D_e = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^d) : u \text{ is a tempered distribution and } \int_{\mathbb{R}^d} |\hat{u}(x)|^2 |x|^\alpha \, dx < \infty \right\}.
\]

Another characterization of \(D_e\) is as follows.

\[
D_e = \left\{ u = R_{f}^{\alpha} f : f \in L^2(\mathbb{R}^d) \right\}
\]

where \(R_{f}^{\alpha}\) denotes the Riesz kernel of index \(\alpha\):

\[
R_{f}^{\alpha}(x) = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{2^\alpha \pi^{\frac{d}{2}} \Gamma\left(\frac{\alpha}{2}\right)} |x|^{\alpha-d}.
\]

**Example 2.3.2** (See [13].) Assume that \(D\) is a bounded open subset of \(\mathbb{R}^d\) and \((T, D)\) is the Dirichlet form given in Example 2.3.1. Put

\[
D_D = \{ u \in D : \tilde{u} = 0 \text{ q.e. on } D^c \}
\]

where \(\tilde{u}\) denotes a quasi-continuous version of the function \(u\) and "q.e." stands for quasi-everywhere. Then \((T, D_D)\) is again a regular symmetric Dirichlet form. The strongly continuous semigroup determined by \((T, D_D)\) is nothing but the transition semigroup of the killed symmetric stable process on \(D\).

**Definition 2.3.8** Let \(D\) be a linear subspace of \(L^2_\mathsf{C}(E, m)\). A map \(T\) from \(D \times D\) into \(\mathbb{C}\) is said to be a sesquilinear form if
(i) for any \( u_1, u_2, v \in D \) and any \( a, b \in C \),

\[
T(au_1 + bu_2, v) = aT(u_1, v) + bT(u_2, v);
\]

(ii) for any \( u, v_1, v_2 \in D \) and any \( a, b \in C \),

\[
T(u, av_1 + bv_2) = aT(u, v_1) + bT(u, v_2).
\]

\( D \) is said to be the domain of \( T \). A sesquilinear form \((T, D)\) is said to be symmetric if for any \( u, v \in D \),

\[
T(u, v) = \overline{T(v, u)}.
\]

A sesquilinear form \((T, D)\) is said to be densely defined if \( D \) is dense in \( L_2^C(E, m) \).

In the sequel we shall always deal with densely defined sesquilinear forms; so from now on, whenever we talk about a sesquilinear form we mean a densely defined sesquilinear form.

With each sesquilinear form \((T, D)\) is associated another sesquilinear form \((T^*, D)\) defined by

\[
T^*(u, v) = \overline{T(v, u)}.
\]

\((T^*, D)\) is called the adjoint form of \((T, D)\). For any sesquilinear form \((T, D)\), the two sesquilinear forms \((\mathcal{R}, D)\) and \((\mathcal{I}, D)\) defined by

\[
\mathcal{R} = \frac{1}{2}(T + T^*)
\]

\[
\mathcal{I} = \frac{1}{2i}(T - T^*)
\]

are symmetric and

\[
T = \mathcal{R} + i\mathcal{I}.
\]

For a sesquilinear form \((T, D)\), the set of values of \( T(u, u) \) for \( u \in D \) with \((u, u) = 1\) is called the numerical range of \( T \) and is denoted by \( \Theta(T) \).
**Definition 2.3.9** A sesquilinear form \((T, D)\) is said to be sectorial if there exist real numbers \(\gamma > 0\) and \(0 \leq \theta < \frac{\pi}{2}\) such that for any \(\zeta \in \Theta(T)\),

\[
|\arg(\zeta + \gamma)| \leq \theta.
\]

A symmetric sesquilinear form \((T, D)\) is said to be nonnegative if for any \(u \in D\),

\[
T(u, u) \geq 0.
\]

From the definition above it is easy to see that a nonnegative symmetric sesquilinear form is always sectorial.

Let \((T, D)\) be a sectorial sesquilinear form. A sequence \(\{u_n\} \subset D\) is said to be \(T\)-convergent to \(u \in L^2_C(E, m)\) if \(u_n \rightarrow u\) in \(L^2_C(E, m)\) and \(T(u_n - u_m, u_n - u_m) \rightarrow 0\) as \(n, m \rightarrow \infty\).

**Definition 2.3.10** A sectorial form is said to be closed if the \(T\)-convergence of \(\{u_n\} \subset D\) to \(u \in L^2_C(E, m)\) implies that \(u \in D\) and \(T(u_n - u, u_n - u) \rightarrow 0\) as \(n \rightarrow 0\).

With any bilinear form \((T, D)\) is associated a sesquilinear form \((T_C, D_C)\) defined as follows.

\[
D_C = D \oplus iD,
\]

\[
T_C(u_1 + iu_2, v_1 + iv_2) = T(u_1, v_1) + T(u_2, v_2)
+ iT(u_2, v_1) - iT(u_1, v_2), \quad \forall u_1, u_2, v_1, v_2 \in D.
\]

The following result is proved in [18] and will be used in the next chapter.

**Theorem 2.3.2** Let \((T, D)\) be a bilinear form. If \((T_C, D_C)\) is a closed sectorial form, then \((T, D)\) is a closed bilinear form satisfying the sector condition.
CHAPTER 3
THE FEYNMAN–KAC SEMIGROUP

3.1 The Kato Class

From now on we shall always assume that \( X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x) \) is the symmetric stable process of index \( \alpha \), \( 0 < \alpha < 2 \), on \( \mathbb{R}^d \) \( (d \geq 2) \) with the following characteristic function

\[
e^{-t|x|^\alpha}
\]

and that \((\mathcal{T}, D)\) is the regular symmetric Dirichlet form defined below

\[
\mathcal{T}(u,v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} \, dx dy,
\]

\[
D = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \, dx dy < \infty \right\}.
\]

As usual, we use \((P_t)\) to denote the transition semigroup of \( X_t \) and \( U_{\gamma}, \gamma \geq 0 \), to denote the \( \gamma \)-potential of \((P_t)\),

\[
U_{\gamma}f(x) = \int_0^\infty e^{-\gamma t} P_tf(x) \, dt.
\]

**Definition 3.1.1** A signed Radon measure \( \mu \) on \( \mathbb{R}^d \) is said to be in the Kato class \( K_{d,\alpha} \) if

\[
\limsup_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x - y| < r} \frac{\mu^*(dy)}{|x - y|^{-d-\alpha}} = 0
\]

where \( \mu^* = \mu^+ + \mu^- \) with \( \mu^+ \) and \( \mu^- \) being the positive part and the negative part of \( \mu \) respectively.
A Borel function \( q \) is said to be in the Kato class \( K_{d,\alpha} \) if the measure \( q(x)dx \) is in \( K_{d,\alpha} \).

**Lemma 3.1.1** Let \( \mu \) be a Radon measure on \( \mathbb{R}^d \). If for some \( r > 0 \),

\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \mu(dy) = C < \infty,
\]

then

\[
\sup_{x \in \mathbb{R}^d} \int_{|x-y| > r} \frac{\mu(dy)}{|x-y|^{d+\alpha}} < \infty.
\]

**Proof.** For any \( x \in \mathbb{R}^d \), we can cover \( \mathbb{R}^d \) with nonoverlapping cubes of side length \( \frac{2}{\sqrt{d}}r \) with one of the cubes centered at \( x \). Then counting from the cube at \( x \), on the \( n \)-th layer,

\[
|x-y| \geq \frac{2n-1}{2} \frac{2}{\sqrt{d}}r = \frac{2n-1}{\sqrt{d}}r.
\]

There are \((2n+1)^d - (2n-1)^d\) cubes on the \( n \)-th layer. Each cube can be covered by a ball of radius \( r \). Thus

\[
\int_{|x-y| > r} \frac{\mu(dy)}{|x-y|^{d+\alpha}} \leq \sum_{0}^{\infty} \left[ (2n+1)^d - (2n-1)^d \right] \left( \frac{(2n-1)r}{\sqrt{d}} \right)^{(d+\alpha)} C
\]

\[
\leq C \left( \frac{r}{\sqrt{d}} \right)^{(d+\alpha)} \sum_{0}^{\infty} 2d(2n+1)^{d-1}(2n-1)^{-(d+\alpha)}
\]

\[
= 2dC \left( \frac{r}{\sqrt{d}} \right)^{(d+\alpha)} \sum_{0}^{\infty} \frac{(2n+1)^{d-1}}{(2n-1)^{d+\alpha}}
\]

\[
< \infty.
\]

The proof is now complete.

\(Q.E.D.\)

**Lemma 3.1.2** Let \( \mu \in K_{d,\alpha} \). Then there exists an \( r_0 > 0 \) such that for each fixed \( 0 < r < r_0 \) we have

\[
\sup_{t \leq 1} \sup_{x \in \mathbb{R}^d} \int_{|y-y| > r} p(t, x, y)\mu^*(dy) = C(r) < \infty.
\]
Proof. From the definition of $K_{d,\alpha}$ we know that there exists an $r_0 > 0$ such that
\[
\sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq r_0} \frac{\mu^*(dy)}{|y-x|^{d-\alpha}} < 1.
\]
Therefore for any $0 < r < r_0$,
\[
\sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq r} \mu^*(dy) \leq \sup_{x \in \mathbb{R}^d} \int_{|y-x| \leq r} \frac{r^{d-\alpha} \mu^*(dy)}{|y-x|^{d-\alpha}} \leq r^{d-\alpha}.
\]
Applying Theorem 2.1.2 and Theorem 2.1.3 we know that for any $t \leq 1$,
\[
\sup_{x \in \mathbb{R}^d} \int_{|y-x| > r} p(t, x, y) \mu^*(dy) = \sup_{x \in \mathbb{R}^d} \int_{|y-x| > r} t^{-\frac{d}{\alpha}} p(1, t^{-\frac{1}{\alpha}} x, t^{-\frac{1}{\alpha}} y) \mu^*(dy)
\leq t c_2 \sup_{x \in \mathbb{R}^d} \int_{|y-x| > r} \frac{\mu^*(dy)}{|y-x|^{d+\alpha}}.
\]
Now applying Lemma 3.1.1 we immediately get the conclusion of this lemma.
\[Q.E.D.\]

Theorem 3.1.1 Let $\mu$ be a signed Radon measure on $\mathbb{R}^d$. Then $\mu \in K_{d,\alpha}$ if and only if
\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_{0}^{t} \int_{\mathbb{R}^d} p(s, x, y) \mu^*(dy) ds = 0.
\]
Proof. Suppose first that $\mu \in K_{d,\alpha}$. Lemma 3.1.2 guarantees that we can choose an $r > 0$ such that for any $t \leq 1$,
\[
\sup_{x \in \mathbb{R}^d} \int_{0}^{t} \int_{|y-x| > r} p(s, x, y) \mu^*(dy) ds \leq t C(r).
\]
On the other hand we have, by Theorem 2.1.9, for all $t \geq 0$,
\[
\int_{0}^{t} \int_{|y-x| \leq r} p(s, x, y) \mu^*(dy) ds \leq \int_{0}^{\infty} \int_{|y-x| \leq r} p(s, x, y) \mu^*(dy) ds
= C \int_{|y-x| \leq r} \frac{\mu^*(dy)}{|y-x|^{d-\alpha}}.
\]
The two inequalities above imply that
\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} \int_{0}^{t} \int_{\mathbb{R}^d} p(s, x, y) \mu^*(dy) ds = 0.
\]
Now suppose that

$$\limsup_{t \to 0} \int_{x \in \mathbb{R}^d} \int_0^t p(s, x, y) \mu^*(dy) ds = 0.$$  

We know that

$$\int_0^t \int_{\mathbb{R}^d} p(s, x, y) \mu^*(dy) ds = \int_{\mathbb{R}^d} \mu^*(dy) \int_0^t p(s, x, y) ds = \int_{\mathbb{R}^d} \mu^*(dy) \int_0^t s^{-\frac{d}{\alpha}} p(1, s^{-\frac{1}{\alpha}} x, s^{-\frac{1}{\alpha}} y) ds$$

$$= \int_{\mathbb{R}^d} \mu^*(dy) \int_{|x - y|^{d-\alpha}} u^{-\frac{d}{\alpha}} p(1, u^{-\frac{1}{\alpha}} |x - y|^{-1} x, u^{-\frac{1}{\alpha}} |x - y|^{-1} y) du.$$  

Applying Theorem 2.1.2 we get

$$\int_0^t \int_{\mathbb{R}^d} p(s, x, y) \mu^*(dy) ds \geq \int_{|y - x| < c_1 t^{\frac{1}{\alpha}}} \frac{\mu^*(dy)}{|x - y|^{d-\alpha}} \int_0^1 u^{-\frac{d}{\alpha}} \frac{c_1}{u^{-\frac{d}{\alpha}} - 1} du$$

$$= \left( c_1 \int_0^1 u du \right) \int_{|y - x| < c_1 t^{\frac{1}{\alpha}}} \frac{\mu^*(dy)}{|x - y|^{d-\alpha}};$$

thus, $\mu \in K_{d, \alpha}$.

Q.E.D.

**Definition 3.1.2** A Radon measure $\mu$ on $\mathbb{R}^d$ is said to be of finite energy integral if there exists a constant $C > 0$ such that for $v \in D \cap C_{0,0}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |v(x)| \mu(dx) \leq C \sqrt{T_1(v, v)}.$$

We shall use $S_0$ to denote the collection of measures of finite energy integrals.

By Riesz's representation theorem, a Radon measure $\mu \in S_0$ if and only if there exists an element $V_1 \mu \in D$ such that for any $u \in D \cap C_{0,0}(\mathbb{R}^d)$,

$$T_1(V_1 \mu, u) = \int_{\mathbb{R}^d} u(x) \mu(dx).$$
If for any $\gamma > 0$ and any nonnegative measure $\nu$, we use $U_\gamma \nu$ to denote the following function

$$U_\gamma \nu(x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\gamma t} p(t, x, y) \nu(\,dy\,)dt = \int_{\mathbb{R}^d} u_\gamma(x, y) \nu(\,dy\,) .$$

Then we can give the following characterization of $S_0$.

**Lemma 3.1.3** If $\nu$ is a Radon measure on $\mathbb{R}^d$, then $\nu \in S_0$ if and only if for some $\gamma > 0$,

$$< U_\gamma \nu, \nu > := \int_{\mathbb{R}^d} U_\gamma \nu(x) \nu(dx) < \infty .$$

**Proof.** If $\nu \in S_0$, then by Lemma 5.1.3 and Theorem 3.2.2 of [13] we know that

$$< U_\gamma \nu, \nu > < \infty .$$

Now let us assume that $< U_\gamma \nu, \nu >$ is finite. Then $\nu$ charges no polar sets, thus by [13] there exists an increasing sequence $\{F_n\}$ of closed sets such that

(a) $\nu(\mathbb{R}^d - \bigcup_{n=1}^\infty F_n) = 0$;

(b) for any $n \geq 1$, $1_{F_n} \cdot \nu \in S_0$.

From (a) above we get that $U_\gamma(1_{F_n} \cdot \nu)$ converges to $U_\gamma \nu$ everywhere on $\mathbb{R}^d$. From (b) above we get that for any $\nu \in \mathbf{D} \cap C_{0,0}(\mathbb{R}^d)$,

$$T_\gamma(U_\gamma(1_{F_n} \cdot \nu), \nu) = \int_{F_n} \nu(x) \nu(dx) .$$

Since for each $n$, $U_\gamma(1_{F_n} \cdot \nu)$ is quasi-continuous, we know by Theorem 3.3.2 of [13] that for each $n$,

$$T_\gamma(U_\gamma(1_{F_n} \cdot \nu), U_\gamma(1_{F_n} \cdot \nu)) = \int_{F_n} U_\gamma(1_{F_n} \cdot \nu)(x) \nu(dx) \leq < U_\gamma \nu, \nu > .$$
Therefore there exists a subsequence \( \{U_\gamma(1_{F_{n_k}} \cdot \nu)\} \) of \( \{U_\gamma(1_{F_n} \cdot \nu)\} \) which converges weakly with respect to the \( T_\gamma \)-norm to some function \( w \in D \). Combining this with the fact that \( U_\gamma(1_{F_n} \cdot \nu) \) converges everywhere to \( U_\gamma \nu \), we get that \( w = U_\gamma \nu \), hence \( U_\gamma(1_{F_{n_k}} \cdot \nu) \) converges weakly with respect to the \( T_\gamma \)-norm to \( U_\gamma \nu \). Now letting \( k \) tend to infinity in

\[
T_\gamma(U_\gamma(1_{F_{n_k}} \cdot \nu), v) = \int_{F_{n_k}} v(x) \nu(dx), \quad v \in D \cap C_{0,0}(R^d),
\]

we get that for any \( v \in D \cap C_{0,0}(R^d) \),

\[
T_\gamma(U_\gamma \nu, v) = \int_{R^d} v(x) \nu(dx),
\]

which implies that \( \nu \in S_0 \).

Q.E.D.

An immediate consequence of the result above is that for any Radon measure \( \nu \), if \( <U_\gamma \nu, \nu> \) is finite, then \( U_\gamma \nu \) is quasi-continuous. From this we can easily get that for any \( f \in L^2(R^d) \) and any \( \gamma > 0 \), \( U_\gamma f \) is quasi-continuous.

**Theorem 3.1.2** Let \( \mu \) be a Radon measure which is in \( K_{d,\alpha} \). Then for any compact subset \( F \) of \( R^d \), the restriction of \( \mu \) to \( F \), \( 1_F \cdot \mu \), is of finite energy integral.

**Proof.** From Theorem 3.1.1 we can see that

\[
\sup_{x \in R^d} \int_0^\infty e^{-s} \int_{R^d} p(s, x, y) \mu(dy)ds < \infty.
\]

In particular, the following function

\[
U_1(1_F \cdot \mu)(x) = \int_0^\infty e^{-s} \int_F p(s, x, y) \mu(dy)ds
\]

is bounded, say, by a constant \( C \). Now

\[
\int_{R^d} (U_1(1_F \cdot \mu)(x))^2 dx \leq C \int_{R^d} \int_0^\infty e^{-s} \int_F p(s, x, y) \mu(dy)ds dx = C \mu(F),
\]
so $U_1(1_F \cdot \mu) \in L^2(R^d)$. And since
\[
\lim_{t \to 0} \frac{1}{t} \left( U_1(1_F \cdot \mu) - e^{-t}P_t(U_1(1_F \cdot \mu)), U_1(1_F \cdot \mu) \right) \\
\leq \lim_{t \to 0} \frac{C}{t} \int_{R^d} \int_0^t e^{-s} \int_F p(s, x, y) \mu(dy) ds dx \\
\leq C \mu(F) < \infty,
\]
we know by Lemma 1.3.4 of [13] that $U_1(1_F \cdot \mu) \in D$. Therefore, for any $u \in D \cap C_{0,0}(R^d)$,
\[
T_1(U_1(1_F \cdot \mu), u) = \lim_{t \to 0} \frac{1}{t} \left( U_1(1_F \cdot \mu) - e^{-t}P_t(U_1(1_F \cdot \mu)), u \right) \\
= \lim_{t \to 0} \left( \frac{1}{t} \int_0^t \int_F e^{-\tau}p(s, x, y) \mu(dy) d\tau, u \right) \\
= \int_F u(x) \mu(dx).
\]
Thus $1_F \cdot \mu \in S_0$.

**Q.E.D.**

**Corollary 3.1.1** If a Radon measure $\mu$ is in $K_{d,\alpha}$, then $\mu$ does not charge polar sets.

Therefore it follows from [27] that for any Radon measure $\mu$ in $K_{d,\alpha}$, there exists a unique positive continuous additive functional $A = (A_t)_{t \geq 0}$ (in the sense of [6]) of $X$ such that for all $\gamma$-excessive functions ($\gamma \geq 0$) $h$ and all nonnegative Borel functions $f$ on $R^d$,\[
\lim_{t \to 0} \frac{1}{t} \int_{R^d} h(x) E^x \int_0^t f(X_s) dA_s dx = \int_{R^d} h(x) f(x) \mu(dx). \tag{3.1}
\]
We shall always deal with additive functionals in the sense of [6], so, from now on, whenever we talk about an additive functional we mean an additive functional in the sense of [6].

Suppose now that $\mu \in K_{d,\alpha}$. Let $(A^+)$ and $(A^-)$ be the positive continuous additive functionals associated with $\mu^+$ and $\mu^-$ respectively in the manner of (3.1).
If we set \( A_t = A_t^+ - A_t^- \), then \( (A_t) \) is a continuous additive functional of bounded variation. We shall call \( (A_t) \) the continuous additive functional associated with \( \mu \).

Now here is an easy corollary to Theorem 3.1.1.

**Theorem 3.1.3** If \( \mu \in K_{d,\alpha} \) and \( (A_t) \) is the continuous additive functional associated with \( \mu \), then

\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} E^x A_t = 0.
\]

**Proof.** Without loss of generality we can assume that \( \mu \) is nonnegative. From Lemma 5.1.4 of [13] and Theorem 3.1.2 above we know that if we denote by \( B_n \) the ball of radius \( n \) around the origin then for any \( n > 0, t > 0 \), and any nonnegative Borel function \( f \),

\[
\left( f, E^t \int_0^t 1_{B_n}(X_s) dA_s \right) = \left( f, \int_0^t \int_{B_n} p(s, x, y) \mu(dy) ds \right),
\]

consequently for any \( t > 0 \) and \( n > 0 \),

\[
E^x \int_0^t 1_{B_n}(X_s) dA_s = \int_0^t \int_{B_n} p(s, x, y) \mu(dy) ds
\]

is true for almost all \( x \in \mathbb{R}^d \). Multiplying both sides of the equality above by \( p(\epsilon, x_0, x) \) and integrating over \( x \), we get from the additivity of \( (A_t) \),

\[
E^{x_0} \int_\epsilon^t 1_{B_n}(X_s) dA_s = \int_\epsilon^t \int_{B_n} p(s, x_0, y) \mu(dy) ds.
\]

Letting \( \epsilon \downarrow 0 \) and \( n \uparrow \infty \), by the monotone convergence theorem we get that

\[
E^{x_0} A_t = \int_0^t \int_{\mathbb{R}^d} p(s, x_0, y) \mu(dy) ds.
\]

Since \( x_0 \) is arbitrary, we get that for any \( x \in \mathbb{R}^d \) and any \( t > 0 \),

\[
E^x A_t = \int_0^t \int_{\mathbb{R}^d} p(s, x, y) \mu(dy) ds.
\]

Now the conclusion of the theorem follows immediately from Theorem 3.1.1.

\[ Q.E.D. \]
Theorem 3.1.4 If \( \mu \in K_{d,\alpha} \) and \( (A_t) \) is the continuous additive functional associated with \( \mu \), then
\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} E_x(A_t)^2 = 0.
\]

**Proof.** Without loss of generality we can assume that \( \mu \) is nonnegative. Since for any \( t > 0 \),
\[
(A_t)^2 = 2 \int_0^t (A_t - A_s) dA_s,
\]
we know that for any \( x \in \mathbb{R}^d \),
\[
E_x(A_t)^2 = 2 E_x \int_0^t E^x A_{t-s} dA_s \\
\leq 2 \sup_{x \in \mathbb{R}^d} (E_x A_t)^2.
\]
Now the conclusion of the theorem follows immediately from Theorem 3.1.3.

\[Q.E.D.\]

**Definition 3.1.3** A bounded Borel function \( F(\cdot, \cdot) \) on \( \mathbb{R}^d \times \mathbb{R}^d \) is said to be admissible with respect to \( X \) if

(i) \( F \) vanishes on the diagonal;

(ii) the function
\[
z \mapsto \int_{\mathbb{R}^d} \frac{|F(z,y)|}{|z-y|^{d+\alpha}} dy
\]
is in \( K_{d,\alpha} \).

We are going to use \( \mathcal{A}_{d,\alpha} \) to denote the collection of all the admissible functions with respect to \( X \).

It is easy to see from the definition that if \( F_1, F_2 \in \mathcal{A}_{d,\alpha} \) and \( c \in \mathbb{R} \), then \( cF_1, F_1 + F_2 \) and \( F_1 F_2 \) all belong to \( \mathcal{A}_{d,\alpha} \). Furthermore we have the following
**Lemma 3.1.4** If \( F \in \mathcal{A}_{d,\alpha} \), then \( e^F - 1 \in \mathcal{A}_{d,\alpha} \).

**Proof.** It is easy to see that the bounded function \( G = e^F - 1 \) satisfies the first condition in the definition above, so we need only to check the second condition. Let \( M > 0 \) be such that for any \( x, y \in \mathbb{R}^d \),

\[
|F(x, y)| < M.
\]

Since on the interval \([-M, M]\), the function

\[
a \mapsto \frac{e^a - 1}{a}
\]

is bounded, there is a \( C > 0 \) such that for any \( a \in [-M, M] \),

\[
\left| \frac{e^a - 1}{a} \right| \leq C
\]

Consequently we have

\[
|e^{F(x, y)} - 1| < C|F(x, y)|.
\]

for any \( x, y \in \mathbb{R}^d \) Therefore for any \( z \in \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} \frac{|G(z, y)|}{|z - y|^{d+\alpha}} \, dy 
\leq \|G\|_{\infty} \cdot \int_{\{|y-z| \geq \eta\}} \frac{1}{|z - y|^{d+\alpha}} \, dy + C \int_{\{|y-z| < \eta\}} \frac{|F(z, y)|}{|z - y|^{d+\alpha}} \, dy,
\]

hence the second condition in the definition above is also satisfied.

**Q.E.D.**

**Theorem 3.1.5** If \( F \in \mathcal{A}_{d,\alpha} \), then

\[
A_t = \sum_{0 < s \leq t} F(X_{s-}, X_s),
\]

\[
A^*_t = \sum_{0 < s \leq t} |F|(X_{s-}, X_s)
\]
are finite almost surely. Furthermore, \((A_t)\) and \((A_t^*)\) are additive functionals such that
\[
\limsup_{t \to 0} E^x |A_t| = \limsup_{t \to 0} E^x A_t^* = 0.
\]

**Proof.** Without loss of generality we can assume that \(F\) is nonnegative. From [3] we know that for any \(t > 0\) and any \(x \in \mathbb{R}^d\),
\[
E^x A_t = E^x \int_0^t ds \int_{\mathbb{R}^d} \frac{F(X_s, y)}{|X_s - y|^{d+\alpha}} dy.
\]
From the third condition in the definition above and Theorem 3.1.1 we immediately see that
\[
\limsup_{t \to 0} E^x A_t = 0.
\]
The rest of the conclusions follow easily.

Q.E.D.

**Theorem 3.1.6** Suppose that \(F \in A_{d,\alpha}\). If we put
\[
A_t = \sum_{0 < s \leq t} F(X_s, X_s),
\]
then
\[
\limsup_{t \to 0} E^x (A_t)^2 = 0.
\]

**Proof.** Without loss of generality we can assume that \(F\) is nonnegative. From the fact
\[
(A_t)^2 = 2 \int_0^t (A_t - A_s) dA_s + \sum_{0 < s \leq t} F^2(X_s, X_s)
\]
we get that for any \(x \in \mathbb{R}^d\),
\[
E^x (A_t)^2 = 2 E^x \int_0^t E^{X_s} (A_{t-s}) dA_s + E^x \int_0^t ds \int_{\mathbb{R}^d} \frac{F^2(X_s, y)}{|X_s - y|^{d+\alpha}} dy
\leq 2 \sup_{x \in \mathbb{R}^d} (E^x A_t)^2 + E^x \int_0^t ds \int_{\mathbb{R}^d} \frac{F^2(X_s, y)}{|X_s - y|^{d+\alpha}} dy.
\]
Since \( F \in \mathcal{A}_{d,\alpha} \), \( F^2 \) is also in \( \mathcal{A}_{d,\alpha} \), thus by Theorem 3.1.1 we have

\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} E^x \int_0^t ds \int_{\mathbb{R}^d} \frac{F^2(X_s, y)}{|X_s - y|^{d+\alpha}} = 0.
\]

The proof is now complete.

Q.E.D.

Combining Theorem 3.1.4 and Theorem 3.1.6 we immediately get the following result.

**Theorem 3.1.7** Suppose that \( \mu \in K_{d,\alpha} \), \( F \in \mathcal{A}_{d,\alpha} \) and that \( A_t^\mu \) is the continuous additive functional of \( X \) associated with \( \mu \). If we put

\[
A_t = A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s),
\]

then

\[
\lim_{t \to 0} \sup_{x \in \mathbb{R}^d} E^x (A_t)^2 = 0.
\]

### 3.2 The Feynman–Kac Semigroup

Before we get to the Feynman–Kac semigroup, some preliminary results are in order. The following result is well known, see, for instance, [10]. A proof is given here for the sake of completeness.

**Theorem 3.2.1** Suppose that \( \mu \in K_{d,\alpha} \) and \( (A_t) \) is the continuous additive functional of \( X \) associated with \( \mu \). Then there exist constants \( C > 1 \) and \( \beta > 0 \) such that for any \( t > 0 \),

\[
\sup_{x \in \mathbb{R}^d} E^x \{ \sup_{s \leq t} e^{A_s} \} \leq C e^{\beta t}.
\]

**Proof.** Without loss of generality we can assume that \( \mu \) is nonnegative. From the assumption \( \mu \in K_{d,\alpha} \) we know, by Theorem 3.1.1, that there exists a \( t_0 > 0 \) such that

\[
\sup_{x \in \mathbb{R}^d} E^x A_{t_0} < \theta < 1.
\]
Hence for all $x \in \mathbb{R}^d$, $0 \leq t \leq t_0$ and $n \geq 2$,

\[
\frac{1}{n} E^x \{ (A_t)^n \} = E^x \left\{ \int_0^t (A_t - A_s) d(A_s)^{n-1} \right\} = E^x \left\{ \int_0^t A_{t-s} (\theta_s) d(A_s)^{n-1} \right\} = E^x \left\{ \int_0^t E^x_s [A_{t-s}] d(A_s)^{n-1} \right\} \leq \theta E^x \{ (A_t)^{n-1} \}.
\]

Therefore by induction we have for all $x \in \mathbb{R}^d$,

\[
E^x \{ (A_t)^n \} \leq n! \theta^n,
\]

hence,

\[
E^x \{ e^{A_t} \} \leq \sum_{n=0}^{\infty} \theta^n = \frac{1}{1 - \theta}.
\]

Using the additivity of $(A_t)$ we get that for any positive integer $n$ and any $(n-1)t_0 < t \leq nt_0$,

\[
E^x \{ e^{A_t} \} \leq \frac{1}{(1 - \theta)^n}.
\]

This is equivalent to the conclusion of the theorem.

\textit{Q.E.D.}

\textbf{Theorem 3.2.2} Suppose that $F \in \mathcal{A}_{d,\alpha}$ and

\[
A_t = \sum_{0<s \leq t} F(X_{s-}, X_s).
\]

Then there exist constants $C > 1$ and $\beta > 0$ such that for any $t > 0$,

\[
\sup_{x \in \mathbb{R}^d} E^x \{ \sup_{s \leq t} e^{A_s} \} \leq C e^{\beta t}.
\]

\textit{Proof}. We need only to prove the theorem for a nonnegative $F \in \mathcal{A}_{d,\alpha}$. Put $F_1 = e^{2F} - 1$ and

\[
B_t = \sum_{0<s \leq t} F_1(X_{s-}, X_s).
\]
Then it follows from [3] that

\[ B_t^p := \int_0^t \int_{\mathbb{R}^d} \frac{F_1(X_s, y)}{|X_s - y|^{d+\alpha}} dy ds \]

is the dual predictable projection of \( B_t \). Therefore \( B_t - B_t^p \) is a \( P^x \)-martingale for every \( x \in \mathbb{R}^d \). Now it follows from the exponential formula (see, for instance, [11]) that

\[
N_t = e^{B_t - B_t^p} \prod_{s \leq t} (1 + F_1(X_{s-}, X_s)) e^{-F_1(X_{s-}, X_s)}
\]

\[
= e^{-B_t^p} \prod_{s \leq t} (1 + F_1(X_{s-}, X_s))
\]

\[
= e^{2A_t - B_t^p}
\]

is a local martingale under \( P^x \) for any \( x \in \mathbb{R}^d \), where the convergence of the infinite product in the first line above is also guaranteed by the exponential formula. \( (N_t) \) is clearly a multiplicative functional of \( X \), so it is a supermartingale multiplicative functional of \( X \). Thus

\[
E^x \{ e^{2A_t - B_t^p} \} \leq 1.
\]

Applying the Schwarz inequality we get

\[
E^x \{ e^{A_t} \} = E^x \{ e^{A_t - \frac{1}{2}B_t^p + \frac{1}{2}B_t^p} \} \leq \left( E^x \{ e^{2A_t - B_t^p} \} \right)^{\frac{1}{2}} \left( E^x \{ e^{B_t^p} \} \right)^{\frac{1}{2}} \leq \left( E^x \{ e^{B_t^p} \} \right)^{\frac{1}{2}}.
\]

Now the conclusion of the theorem follows from Theorem 3.2.1 and the fact that

\( F_1 \in \mathcal{A}_{d,\alpha} \).

Q.E.D.

**Theorem 3.2.3** Suppose that \( \mu \in K_{d,\alpha} \), \( F \in \mathcal{A}_{d,\alpha} \) and that \( A_t^\mu \) is the continuous additive functional of \( X \) associated with \( \mu \). If we put

\[
A_t = A_t^\mu + \sum_{0<s \leq t} F(X_{s-}, X_s),
\]
then there exist constants $C > 1$ and $\beta > 0$ such that for any $t > 0$,

$$\sup_{x \in \mathbb{R}^d} E^x \{ \sup_{s \leq t} e^{A_t} \} \leq C e^{\beta t}.$$  

**Proof.** Direct consequence of the two results above and the Schwarz inequality. Q.E.D.

**Lemma 3.2.1** Suppose that $\mu \in K_{d,\alpha}, F \in \mathcal{A}_{d,\alpha}$ and that $A_t^\mu$ is the continuous additive functional of $X$ associated with $\mu$. If we put

$$A_t = A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s),$$

then for any $p \geq 1$,

$$\limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} E^x \left\{ |e^{A_t} - 1|^p \right\} = 0.$$  

**Proof.** Noticing that for any $a > 0$, $e^a - 1 \geq 1 - e^{-a}$, we can assume, without loss of generality, that both $\mu$ and $F$ are nonnegative. Observe that for any $a \geq 0$, $e^a - 1 \leq ae^a$. Then

$$E^x \left\{ |e^{A_t} - 1|^p \right\} \leq E^x \left( e^{pA_t} - 1 \right) \leq E^x \left( (pA_t)e^{pA_t} \right) \leq p \left( E^x (A_t)^2 \right)^{\frac{1}{2}} \left( E^x e^{2pA_t} \right)^{\frac{1}{2}}.$$  

From Theorem 3.2.3 we know that there exist constants $C > 1$ and $\beta > 0$ such that for any $t > 0$,

$$\sup_{x \in \mathbb{R}^d} E^x e^{2pA_t} \leq C e^{\beta t},$$

and from Theorem 3.1.7 we get that
\[ \limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} E^x(\mathcal{A}_t)^2 = 0, \]

therefore the conclusion of the lemma is true.

Q.E.D.

From now on we are going to fix a \( \mu \in K_{d,a} \) and an \( F \in A_{d,a} \). Let \( \mathcal{A}_t^\mu \) be the continuous additive functional associated with \( \mu \) and put

\[ A_t = \mathcal{A}_t^\mu + \sum_{0 < s < t} F(X_{s-}, X_s). \]

For any \( t > 0 \), \( x \in \mathbb{R}^d \) and any nonnegative Borel function \( f \) on \( \mathbb{R}^d \), define

\[ T_t f(x) = E^x \{ e^{\mathcal{A}_t} f(X_t) \}. \]

Then \( (T_t)_{t>0} \) is called a Feynman–Kac semigroup.

**Theorem 3.2.4** If \( 1 \leq p \leq p' \leq \infty \), then for any \( t > 0 \), \( T_t \) is a bounded operator from \( L^p(\mathbb{R}^d) \) into \( L^{p'}(\mathbb{R}^d) \).

**Proof.** For any \( t > 0 \), \( x \in \mathbb{R}^d \) and \( f \in L^p(\mathbb{R}^d) \), a simple use of Hölder’s inequality gives us

\[ |T_t f(x)| \leq (E^x |f(X_t)|^p)^{\frac{1}{p}} \left( E^x e^{\mathcal{A}_t} \right)^{\frac{1}{q}}, \quad (3.2) \]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).

From Theorem 3.2.3 we know that there exist \( C > 1 \) and \( \beta > 0 \) such that

\[ \sup_{x \in \mathbb{R}^d} E^x \{ e^{\mathcal{A}_t} \} \leq C e^{\beta t}, \quad (3.3) \]

thus

\[ \|T_t f\|_p \leq (C e^{\beta t})^{\frac{1}{p}} \left( \int_{\mathbb{R}^d} E^x |f(X_t)|^p \, dx \right)^{\frac{1}{q}} = (C e^{\beta t})^{\frac{1}{q}} \|f\|_p. \]

Hence \( T_t \) is a bounded operator from \( L^p(\mathbb{R}^d) \) into \( L^{p'}(\mathbb{R}^d) \). From (3.2), (3.3) and Theorem 2.1.5 we can easily see that \( T_t \) is bounded operator from \( L^p(\mathbb{R}^d) \) into \( L^\infty(\mathbb{R}^d) \).

By using interpolation the theorem follows easily.

Q.E.D.
**Theorem 3.2.5** For any $p \in [1, \infty)$, $(T_t)_{t \geq 0}$ is a strongly continuous semigroup on $L^p(\mathbb{R}^d)$.

**Proof.** For any $t > 0$ and $f \in L^p(\mathbb{R}^d)$,

$$\|T_t f - f\|_p \leq \|T_t f - P_t f\|_p + \|P_t f - f\|_p.$$ 

From Theorem 3.1.6 we know that

$$\lim_{t \downarrow 0} \|P_t f - f\|_p = 0$$

Thus to complete the proof, we need only to show that

$$\lim_{t \downarrow 0} \|T_t f - P_t f\|_p = 0. \quad (3.4)$$

Now by Hölder’s inequality we can get

$$\|T_t f - P_t f\|_p = \left( \int_{\mathbb{R}^d} |E^x \{e^{A_t} - 1\} f(X_t)\}^p dx \right)^{\frac{1}{p}}$$

$$\leq \left( \int_{\mathbb{R}^d} \left( E^x |e^{A_t} - 1|^q \right)^{\frac{p}{q}} \left( E^x |f|^p(X_t) \right) dx \right)^{\frac{1}{p}}$$

$$\leq \sup_{x \in \mathbb{R}^d} \left( E^x |e^{A_t} - 1|^q \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^d} E^x |f|^p(X_t) dx \right)^{\frac{1}{p}}$$

$$\leq \|f\|_p \sup_{x \in \mathbb{R}^d} \left( E^x |e^{A_t} - 1|^q \right)^{\frac{1}{q}}.$$

Since by Lemma 3.2.1,

$$\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left( E^x |e^{A_t} - 1|^q \right)^{\frac{1}{q}} = 0,$$

we know that (3.4) is true. The proof is now complete.

Q.E.D.

**Theorem 3.2.6** For any $t > 0$, $T_t$ maps $L^\infty(\mathbb{R}^d)$ into $bC(\mathbb{R}^d)$.

**Proof.** For any $t > 0$ and $f \in L^\infty(\mathbb{R}^d)$, we have

$$T_t f(x) = E^x \{e^{A_t - A_x} f(X_t)\} + E^x \{e^{A_t - A_x} (e^{A_x} - 1) f(X_t)\}$$
The first term on the right hand side of this equation is equal to $P_\epsilon[T_{t-\epsilon} f](x)$, therefore it is continuous in $x$ because of Theorem 2.1.7 and Theorem 3.2.6. The second term on the right hand side of the last identity is bounded by

$$
\|f\|_\infty \left( E^x |e^{A_\epsilon} - 1|^2 \right) \sup_{x \in \mathbb{R}^d} \left( E^x (\sup_{s \leq t} e^{2A_s}) \right)^{\frac{1}{2}}
$$

which tends to 0 uniformly as $\epsilon \downarrow 0$ because of Theorem 3.2.3 and Lemma 3.2.1. Therefore $T_t f$ is a continuous function on $\mathbb{R}^d$.

Q.E.D.

**Theorem 3.2.7** For any $t > 0$ and any $p \in [1, \infty)$, $T_t$ maps $L^p(\mathbb{R}^d)$ into $C_0(\mathbb{R}^d)$.

**Proof.** For any $t > 0$ and $f \in L^p(\mathbb{R}^d)$, we have

$$
T_t f(x) = E^x \{e^{A_{t-\epsilon} - A_\epsilon} f(X_t)\} + E^x \{e^{A_{t-\epsilon} - A_\epsilon} (e^{A_\epsilon} - 1) f(X_t)\}
$$

The first term on the right hand side of the last identity is equal to $P_\epsilon[T_{t-\epsilon} f](x)$ which belongs to $C_0(\mathbb{R}^d)$ because of Theorem 2.1.4 and Theorem 3.2.6. The second term on the right hand side of the last identity is bounded by

$$
(E^x |f|^p(X_t))^\frac{1}{p} \sup_{x \in \mathbb{R}^d} \left( E^x (\sup_{s \leq t} e^{2qA_s}) \right)^{\frac{1}{2q}} \left( E^x |e^{A_\epsilon} - 1|^2 \right)^{\frac{1}{2q}}
$$

(\text{where } \frac{1}{p} + \frac{1}{q} = 1) which tends to 0 uniformly in $x$ as $\epsilon \downarrow 0$ because of Theorem 2.1.8, Theorem 3.2.3 and Lemma 3.2.1. Therefore $T_t f \in C_0(\mathbb{R}^d)$.

Q.E.D.

Similar to the proof of Theorem 3.2.5 we can get the following result.

**Theorem 3.2.8** $(T_t)_{t \geq 0}$ is a strongly continuous semigroup on $C_0(\mathbb{R}^d)$. 
3.3 The Bilinear Form

It is well known that any function in $D$ admits a quasi-continuous version. We are going to use $\tilde{D}$ to denote the family of all the quasi-continuous functions belonging to $D$. It is clear that the equivalence classes of $\tilde{D}$ (in the sense of q.e.) are identical with the equivalence classes of $D$ (in the sense of a.e.).

In this section we are going to deal frequently with integrals of the following type:

$$\int_{R^d} u^2(x)\nu(dx)$$

where $u$ is in $D$ and $\nu$ is a nonnegative measure. The following theorem provides an estimate on the above integral.

**Theorem 3.3.1** There exists an $M_{d,\alpha} > 0$ depending only on $\alpha$ and the dimension $d$ such that for any $v \in \tilde{D}$ and any nonnegative Radon measure $\nu$ charging no polar sets we have

$$\int_{R^d} v^2(x)\nu(dx) \leq M_{d,\alpha} \left( \sup_{x \in R^d} \int_{R^d} \frac{\nu(dy)}{|x - y|^{d-\alpha}} \right) (\|\nu\|_2^2 + T(\nu, \nu)).$$

**Proof.** Put

$$u_1(x, y) = \int_0^\infty e^{-t} p(t, x, y)dt;$$

then $u_1$ is the integral kernel of $U_1$. Since $\nu$ does not charge polar sets, we know that for every Borel set $A$,

$$1_A(x) \leq P^x(T_A = 0), \quad \nu \text{-a.e.,}$$

where $T_A$ is the first hitting time of $A$ by the symmetric stable process killed with exponential rate 1.

For any set $A$, let $\kappa_A$ be the equilibrium measure of $A$. Using properties of the capacity we can get
\[
\int_{\mathbb{R}^d} v^2(x)\nu(dx) = 2\int_0^\infty t\nu(|v| \geq t)dt \\
= 2\int_0^\infty t \int_{\{|v| \geq t\}} \nu(dx)dt \\
\leq 2\int_0^\infty t \int_{\mathbb{R}^d} P^x(T_{|v| \geq t} = 0)\nu(dx)dt \\
\leq 2\int_0^\infty t \int_{\mathbb{R}^d} P^x(T_{|v| > t} < \xi)\nu(dx)dt \\
= 2\int_0^\infty t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} u_1(x, y)\kappa(|v| \geq t)(dy)\nu(dx)dt \\
\leq 2\left(\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} u_1(x, y)\nu(dy)\right) \cdot \int_0^\infty tC_1(|v| \geq t)dt \\
\leq 2M'_{d,\alpha}\left(\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\nu(dy)}{|x - y|^{d-\alpha}}\right) \cdot \int_0^\infty tC_1(|v| \geq t)dt,
\]

where \(\xi\) is the life time of the symmetric stable process killed with exponential rate 1, \(C_1\) is the capacity of this killed symmetric stable process and \(M'_{d,\alpha}\) is a constant depending only on \(d\) and \(\alpha\).

From Theorem 1.6 of [15] we get that

\[
\int_0^\infty tC_1(|v| \geq t)dt \leq M''_{d,\alpha}\inf\{<U_1\lambda, \lambda>\}
\]

where \(M''_{d,\alpha}\) depends only on \(d\) and \(\alpha\), and the infimum is taken over all nonnegative measures \(\lambda\) such that \(U_1\lambda \geq |v|\) up to a set of zero \(C_1\) capacity. The infimum in the second line above can be replaced by infimum over all measures \(\nu\) such that \(U_1\nu \geq |v|\) a.e., since \(U_1\lambda\) and \(\nu\) are both quasi-continuous. Therefore, to finish the proof it suffices to show that there exists a \(p \in \mathcal{D}\) of the form \(U_1\lambda\) for some positive measure \(\lambda\) such that

\[p \geq |v|\text{ a.e., and } \mathcal{T}_1(p, p) \leq \mathcal{T}_1(v, v).\]

Consider now the set \(\mathcal{P}\) of potentials of finite energy:

\[\mathcal{P} = \{U_1\lambda : \lambda \text{ is a nonnegative measure and } <U_1\lambda, \lambda> < \infty\}.\]
It follows from Lemma 3.1.3 that every $U_1 \lambda \in \mathcal{P}$ is in $\mathbf{D}$ and satisfies

$$T(U_1 \lambda, U_1 \lambda) = \langle U_1 \lambda, \lambda \rangle.$$ 

Hence $\mathcal{P}$ is a closed, convex subset of $\mathbf{D}$, since the set of positive measures of finite energy is complete in the energy norm (see, for instance, [20]). It follows that there exists a projection $p$ of $|v|$ on the set $\mathcal{P}$. Using the standard manipulation by quadratic equations (see, for example, Chapter 7 of [26]) we can prove that

$$T_1(p - |v|, q) \geq 0, \quad \forall q \in \mathcal{P},$$

and

$$T_1(p - |v|, p) \leq 0.$$ 

It follows that $p \geq |v|$ a.e. and that $T_1(|v| - p, p) = 0$. Hence

$$T_1(p, p) = T_1(|v|, p) \leq T_1(p, p)^{\frac{1}{2}} T_1(v, v)^{\frac{1}{2}},$$

which finishes the proof.

Q.E.D.

The following result is elementary. A proof is given here for the sake of completeness.

**Lemma 3.3.1** For every $r > 0$, there exists a sequence $\{c_j : j \geq 1\}$ of points in $\mathbb{R}^d$ such that

(i) $\{B(c_j, 3r) : j \geq 1\}$ covers $\mathbb{R}^d$;

(ii) for every $x \in \mathbb{R}^d$, $B(x, r)$ intersects at most $5^d$ balls from $\{B(c_j, 3r) : j \geq 1\}$, at most $8^d$ balls from $\{B(c_j, 6r) : j \geq 1\}$ and at most $9^d$ balls from $\{B(c_j, 7r) : j \geq 1\}$. 

where for any $c \in \mathbb{R}^d$ and any $\eta > 0$, $B(c, \eta)$ stands for the open ball of radius $\eta$ around $c$.

**Proof.** Let us start with a countable cover of $\mathbb{R}^d$ by balls of radius $r$ and let us denote this cover by $\{B_j = B(a_j, r) : j \geq 1\}$. We can take a maximal subset of $\{B_j\}$ such that any two balls in the subset are disjoint. First, we take $j_1 = 1$. Then we take $j_2 = \min\{j > j_1 : B_j \cap B_{j_1} = \phi\}$ and we continue inductively in the same way. Therefore we get a sequence $\{B_{j_k} : k \in \mathbb{N}\}$ with the following two properties:

1. For any $k \neq l$,
   $\quad B_{j_k} \cap B_{j_l} = \phi$;

2. For any $j \geq 1$ there exists a $j_k \geq 1$ such that $B_j$ and $B_{j_k}$ have a nonempty intersection.

Obviously $\{j_k : k \geq 1\}$ must be infinite. We claim that $c_k = a_{j_k}$ satisfies all the requirements of this lemma.

Let $x \in \mathbb{R}^d$. Since $\{B_j\}$ is a cover, there exists $B_j$ such that $x \in B_j$. By (2) above there exists $B_{j_k}$ such that $B_j \cap B_{j_k} \neq \phi$. Hence,

$$|x - c_k| = |x - a_{j_k}|$$
$$\leq |x - a_j| + |a_j - a_{j_k}|$$
$$< r + 2r = 3r.$$ 

It follows that $\{B(c_k, 3r) : k \geq 1\}$ is a cover of $\mathbb{R}^d$.

Let $x \in \mathbb{R}^d$, and let $B(x, r)$ intersect $B(c_k, 3r)$. Then $|x - c_k| < 4r$, i.e., $B(c_k, r)$ is contained in $B(x, 5r)$. Since the balls in $\{B(c_l, r) : l \geq 1\}$ are pairwise disjoint, there can be no more than

$$\frac{m(B(x, 5r))}{m(B(c_l, r))}$$
balls of the form $B(c_i, r)$ inside $B(x, 5r)$, where $m$ denotes the Lebesgue measure in $\mathbb{R}^d$. Since

$$m(B(x, 5r)) = \frac{(5r)^d \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)}$$

and

$$m(B(c_i, r)) = \frac{r^d \pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)},$$

there are no more than $5^d$ balls of the form $B(c_i, r)$ intersecting $B(x, r)$.

The assertion about $\{B(c_i, 6r) : i \geq 1\}$ can be proved by starting with the radius $6r$ in the paragraph above and going over the same argument. And the assertion about $\{B(c_i, 7r) : i \geq 1\}$ can be proved similarly.

Q.E.D.

Using the two results above we can get the following theorem.

**Theorem 3.3.2** If $\nu \in K_{d, \alpha}$ is nonnegative, then for any $u \in \tilde{D}$,

$$\int_{\mathbb{R}^d} u^2(x) \nu(dx) < \infty.$$  

Furthermore, for every $\epsilon > 0$, there exists a $C(\epsilon) > 0$ such that for every $u \in \tilde{D}$,

$$\int_{\mathbb{R}^d} u^2(x) \nu(dx) \leq cT(u, u) + C(\epsilon)\|u\|^2_2.$$  

**Proof.** Fix an arbitrary $\epsilon > 0$. Since $\nu \in K_{d, \alpha}$ there exists an $r_0 > 0$ such that

$$\sup_{x \in \mathbb{R}^d} \int_{|x - y| < r_0} \frac{\nu(dy)}{|x - y|^{d-\alpha}} < \epsilon.$$  

Let $r = r_0/6$ and let $\{c_j : j \geq 1\}$ be the corresponding sequence obtained in Lemma 3.3.1. Let $\varphi : \mathbb{R}^d \rightarrow [0, 1]$ be an infinitely differentiable function such that $\varphi(x) \equiv 1$ on $B(0, 3r)$ and $\varphi(x) \equiv 0$ outside of $B(0, 6r)$. For every $j \geq 1$, we denote by $\varphi_j$ the following translate of $\varphi$: $\varphi_j(x) = \varphi(x - c_j)$. Notice that there exists a constant $M = M(\epsilon) > 0$ such that $|\nabla \varphi_j|^2 \leq M$ for every $j$. 

Since \( \{B(c_j, 3r) : j \geq 1\} \) is a cover of \( \mathbb{R}^d \), and \( 1_{B(c_j, 3r)} = \chi_{B(c_j, 3r)} \varphi_j^2(x) \), we know that

\[
\int_{\mathbb{R}^d} u^2(x) \nu(dx) \leq \sum_{j=1}^{\infty} \int_{\mathbb{R}^d} 1_{B(c_j, 3r)}(x) u^2(x) \nu(dx) = \sum_{j=1}^{\infty} \int_{B(c_j, 3r)} u^2(x) \varphi_j^2(x) \nu(dx).
\]

Notice that \( u \varphi_j \in \mathcal{D} \), so we can apply Lemma 3.3.1 to every integral under the above sum:

\[
\int_{B(c_j, 3r)} u^2(x) \varphi_j^2(x) \nu(dx) \leq M_{d, \alpha} \left( \sup_{x \in \mathbb{R}^d} \int_{B(c_j, 3r)} \frac{\nu(dy)}{x - y |^{d-\alpha}} \right) \left( \mathcal{T}(u \varphi_j, u \varphi_j) + \|u \varphi_j\|^2 \right).
\]

Recall that for \( 0 < \alpha < 2 \), the Riesz potential satisfies the maximum principle; therefore

\[
\sup_{x \in \mathbb{R}^d} \int_{B(c_j, 3r)} \frac{\nu(dy)}{|x - y|^{d-\alpha}} = \sup_{x \in B(c_j, 3r)} \int_{B(c_j, 3r)} \frac{\nu(dy)}{|x - y|^{d-\alpha}} \leq \epsilon.
\]

It follows that

\[
\int_{B(c_j, 3r)} u^2(x) \varphi_j^2(x) \nu(dx) \leq M_{d, \alpha} \epsilon \left( \mathcal{T}(u \varphi_j, u \varphi_j) + \|u \varphi_j\|^2 \right).
\]

It is easy to see that

\[
\mathcal{T}(u \varphi_j, u \varphi_j) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u^2(x)(\varphi_j(x) - \varphi_j(y))^2}{|x - y|^{d+\alpha}} dx dy + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2 \varphi_j^2(y)}{|x - y|^{d+\alpha}} dx dy.
\]
It is easy to see that there exists a constant $C > 0$ depending only on $r$ such that for any $j \geq 1$ and any $x, y \in \mathbb{R}^d$, 

$$(\varphi_j(x) - \varphi_j(y))^2 \leq C|x - y|^2.$$ 

Thus for any $x \in B(c_j, 7r)$, 

$$
\int_{B(c_j, 7r)} \frac{(\varphi_j(x) - \varphi_j(y))^2}{|x - y|^{d+\alpha}} dy 
\leq C \int_{B(c_j, 8r)} \frac{|x - y|^2}{|x - y|^{d+\alpha}} dy 
\leq C \int_{|x-y|<15r} \frac{1}{|x - y|^{d+\alpha-2}} dy 
= C \int_{B(0,15r)} \frac{1}{|y|^{d+\alpha-2}} dy 
:= M_1(r) < \infty
$$
For any $x \in B(c_j, 6r)$, 
\[
\int_{B(c_j, 6r)} \frac{(\varphi_j(x) - \varphi_j(y))^2}{|x - y|^{d+\alpha}} dy \leq \int_{|x - y| > r} \frac{(\varphi_j(x) - \varphi_j(y))^2}{|x - y|^{d+\alpha}} dy \\
\leq \int_{|x - y| > r} \frac{1}{|x - y|^{d+\alpha}} dy \\
= \int_{|y| > r} \frac{1}{|y|^{d+\alpha}} dy \\
=: M_2(r) < \infty.
\] (3.9)

For any $x \notin B(c_j, 7r)$, 
\[
\int_{B(c_j, 6r)} \frac{(\varphi_j(x) - \varphi_j(y))^2}{|x - y|^{d+\alpha}} dy \\
\leq \int_{B(c_j, 6r)} \left( \frac{1}{|x - y|^{d+\alpha}} \wedge \frac{1}{|r|^{d+\alpha}} \right) dy \\
= \int_{R^d} \left( \frac{1}{|x - y|^{d+\alpha}} \wedge \frac{1}{|r|^{d+\alpha}} \right) 1_{B(c_j, 6r)}(y) dy. 
\] (3.10)

Combining (3.6) — (3.10), we get
\[
T(u \varphi_j, u \varphi_j) \\
\leq \int_{R^d} \int_{R^d} \frac{(u(x) - u(y))^2 \varphi_j^2(y)}{|x - y|^{d+\alpha}} dx dy \\
+ 2M_1(r) \int_{R^d} u^2(x) 1_{B(c_j, 7r)}(x) dx \\
+ M_2(r) \int_{R^d} u^2(x) 1_{B(c_j, 6r)}(x) dx \\
+ \int_{R^d} u^2(x) \left( \int_{R^d} \left( \frac{1}{|x - y|^{d+\alpha}} \wedge \frac{1}{|r|^{d+\alpha}} \right) 1_{B(c_j, 6r)}(y) dy \right) dx \\
\leq \int_{R^d} \int_{R^d} \frac{(u(x) - u(y))^2 \varphi_j^2(y)}{|x - y|^{d+\alpha}} dx dy \\
+ (2M_1(r) + M_2(r)) \int_{R^d} u^2(x) 1_{B(c_j, 7r)}(x) dx \\
+ \int_{R^d} u^2(x) \left( \int_{R^d} \left( \frac{1}{|x - y|^{d+\alpha}} \wedge \frac{1}{|r|^{d+\alpha}} \right) 1_{B(c_j, 6r)}(y) dy \right) dx.
\] (3.11)
Combining (3.5) and (3.11) we get

\[
\int_{B(c_j,3r)} u^2(x)\varphi_j^2(x)\nu(dx)
\leq M_{d,\alpha} \varepsilon \left\{ \int_{R^d} \int_{R^d} \frac{(u(x) - u(y))^2 \varphi_j^2(y)}{|x - y|^{d+\alpha}} dx dy \right. \\
+ \int_{R^d} u^2(x)[(1 + 2M_1(r) + M_2(r))1_{B(c_j,\tau r)}(x)]dx \\
+ \int_{R^d} u^2(x) \int_{R^d} \left( \frac{1}{|x - y|^{d+\alpha}} \wedge \frac{1}{|r|^{d+\alpha}} \right) 1_{B(c_j,6r)}(y)dy dx \left\} 
\]

which implies

\[
\int_{R^d} u^2(x)\nu(dx)
\leq M_{d,\alpha} \varepsilon \left\{ \int_{R^d} \int_{R^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \sum_{0}^\infty \varphi_j^2(y) dx dy \\
+ \int_{R^d} u^2(x)[(1 + 2M_1(r) + M_2(r))\sum_{0}^\infty 1_{B(c_j,\tau r)}(x)]dx \\
+ \int_{R^d} u^2(x) \int_{R^d} \left( \frac{1}{|x - y|^{d+\alpha}} \wedge \frac{1}{|r|^{d+\alpha}} \right) \sum_{0}^\infty 1_{B(c_j,6r)}(y)dy dx \left\}. 
\]

For every \( x \in R^d \), \( B(x, r) \) intersects at most \( 8^d \) balls from \{ \( B(c_j, 6r) : j \geq 1 \} \) and at most \( 9^d \) balls from \{ \( B(c_j, 7r) : j \geq 1 \}. \) Thus for any \( x, y \in R^d \),

\[
\sum_{0}^\infty \varphi_j^2(y) \leq 8^d, \\
\sum_{0}^\infty 1_{B(c_j,6r)}(y) \leq 8^d, \\
\sum_{0}^\infty 1_{B(c_j,7r)}(x) \leq 9^d. 
\]

Hence we obtain that
\[
\int_{\mathbb{R}^d} u^2(x)\nu(dx) \\
\leq M_{d,\alpha} \cdot \epsilon \cdot 2 \cdot 8^d T(u, u) + M_{d,\alpha} \cdot \epsilon [(1 + 2M_1(r) + M_2(r))9^d + 8^d M_3(r)]\|u\|_2^2,
\]

where

\[
M_3(r) = \int_{\mathbb{R}^d} \left( \frac{1}{|y|^{d+\alpha}} \wedge \frac{1}{|r|^{d+\alpha}} \right) dy < \infty.
\]

The proof is now complete.

\textit{Q.E.D.}

\textbf{Theorem 3.3.3} If \(G \in \mathcal{A}_{d,\alpha}\) is a nonnegative function such that the function \(G\) define by

\[
\overline{G}(x, y) = G(y, x)
\]

is also in \(\mathcal{A}_{d,\alpha}\), then for any \(u \in \tilde{D}\),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x)G(x, y)u(y)}{|x - y|^{d+\alpha}} dxdy < \infty.
\]

Furthermore, for any \(\epsilon > 0\), there exists \(C(\epsilon) > 0\) such that for any \(u \in \tilde{D}\),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x)G(x, y)u(y)}{|x - y|^{d+\alpha}} dxdy \leq \epsilon T(u, u) + C(\epsilon)\|u\|_2^2.
\]

\textbf{Proof.} Observe that

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x)G(x, y)u(y)}{|x - y|^{d+\alpha}} dxdy \\
\leq \int_{\mathbb{R}^d} u^2(x) \left( \int_{\mathbb{R}^d} \frac{G(x, y)}{|x - y|^{d+\alpha}} dy \right) dx + \int_{\mathbb{R}^d} u^2(y) \left( \int_{\mathbb{R}^d} \frac{G(x, y)}{|x - y|^{d+\alpha}} dx \right) dy.
\]

From the assumptions on \(G\) we know that the functions
\[
x \mapsto \int_{R^d} \frac{G(x,y)}{|x-y|^{d+\alpha}} \, dy
\]
\[
y \mapsto \int_{R^d} \frac{G(x,y)}{|x-y|^{d+\alpha}} \, dx
\]
are in \(K_{d,\alpha}\), therefore the conclusion of this theorem follows from Theorem 3.3.2.

\[Q.E.D.\]

If, in addition, we assume that the function \(F\) defined by \(F(x,y) = F(y,x)\) is also in \(A_{d,\alpha}\), then from Lemma 3.1.4, Theorem 3.3.2 and Theorem 3.3.3 we know that \(\mathcal{E}\) given by

\[
\mathcal{E}(u,v) = \mathcal{T}(u,v) - \int_{R^d} u(x)v(x)\mu(dx) - \int_{R^d} \int_{R^d} \frac{u(x)G(x,y)v(y)}{|x-y|^{d+\alpha}} \, dx \, dy
\]

with \(G = e^F - 1\). \(\mathcal{E}(u,v)\) is finite for all \(u,v \in \tilde{D}\) and \((\mathcal{E}, \tilde{D})\) is a bilinear form. The following theorem is the main result of this section.

**Theorem 3.3.4** If, in addition, we assume that the function \(F\) defined by \(F(x,y) = F(y,x)\) is also in \(A_{d,\alpha}\), then \((\mathcal{E}, \tilde{D})\) is a closed bilinear form satisfying the sector condition.

**Proof.** From Theorem 3.3.2 and Theorem 3.3.3 we can easily show that for any \(\epsilon > 0\) there exists a \(C(\epsilon) > 0\) such that for any \(u \in \tilde{D} \oplus i\tilde{D}\),

\[
|\mathcal{E} - \mathcal{T}|_C(u,u) \leq \epsilon \mathcal{T}(u,u) + C(\epsilon)\|u\|_2^2,
\]
i.e., the relative form bound of \((\mathcal{E} - \mathcal{T})_C\) with respect to \(\mathcal{T}_C\) is zero. Therefore by Theorem 6.1.33 of [18] we know that \((\mathcal{E} - \mathcal{T})_C, \tilde{D} \oplus i\tilde{D})\) is a closed sectorial form. Thus it follows from Theorem 2.3.2 that \((\mathcal{E}, \tilde{D})\) is a closed bilinear form satisfying the sector condition.

\[Q.E.D.\]
From Theorem 2.3.1 and the theorem above we know that \((E, D)\) uniquely determines a strongly continuous semigroup on \(L^2(R^d)\). In the next section we are going to show that this semigroup is nothing but the semigroup \((T_t)_{t>0}\) of the Section 3.2.

### 3.4 The Connection between the Semigroup \((T_t)\) and the Bilinear Form \((E, D)\)

In this section, in addition to assuming that \(F \in \mathcal{A}_{d, \alpha}\), we are going to assume that the function \(\overline{F}\) defined by \(\overline{F}(x, y) = F(y, x)\) is also in \(\mathcal{A}_{d, \alpha}\).

From Theorem 3.2.3 we know that there exist constants \(C_1, C_2 > 0\) and \(\beta_1, \beta_2 > 1\) such that

\[
\sup_{x \in R^d} E^x \{e^{A't}\} \leq C_1 e^{\beta_1 t} \tag{3.12}
\]

\[
\sup_{x \in R^d} E^x \{e^{2A't}\} \leq C_2 e^{\beta_2 t} \tag{3.13}
\]

Put \(\beta_0 = \beta_1 \vee \beta_2\). Then for any \(\beta > \beta_0\), the mapping

\[
f \mapsto E^x \int_0^\infty e^{-\beta t} e^{A't} f(x_t)dt
\]

maps \(L^\infty(R^d)\) into \(L^\infty(R^d)\) and maps \(L^2(R^d)\) into \(L^2(R^d)\).

**Theorem 3.4.1** If \(f \in L^\infty(R^d)\), then for any \(\beta > \beta_0\), the function

\[
u(x) = E^x \int_0^\infty e^{-\beta t} e^{A't} f(x_t)dt
\]

is a bounded continuous function in \(D\) such that for any \(v \in \dot{D}\),

\[
T(u, v) + \beta(u, v) = \int_{R^d} f(x) v(x)dx - \int_{R^d} u(x) v(x)\mu(dx)
\]

\[
- \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)u(y)}{|x - y|^{d+\alpha}}dxdy,
\]

where \(G = e^F - 1\).

**Proof.** The fact that \(u\) is bounded and continuous follows from the properties of the Feynman–Kac semigroup. Set

\[
M_t = e^{-\beta t} e^{A't} u(X_t) + \int_0^t e^{-\beta s} e^{A's} f(X_s)ds.
\]
Then we have for each $x \in \mathbb{R}^d$,

$$M_t = E^x \left[ \int_0^\infty e^{-\beta s} e^{A_s} f(X_s) ds | \mathcal{F}_t \right].$$

Therefore $(M_t)_{t \geq 0}$ is a continuous $P^x$-martingale for each $x \in \mathbb{R}^d$. Applying the integration by parts formula for semimartingales, we get

$$e^{-\beta t} u(X_t) = e^{-A_t} e^{-\beta t} e^{A_t} u(X_t) = e^{-A_t} \left( M_t - \int_0^t e^{-\beta s} e^{A_s} f(X_s) ds \right) = u(X_0) + \int_0^t e^{-A_s} dM_s - \int_0^t e^{-\beta s} f(X_s) ds + \sum_{0 < s \leq t} e^{-\beta s} u(X_s) (1 - e^{F(X_{s-}, X_s)}) - \int_0^t e^{-\beta s} u(X_s) dA_s^u.

We know that $\int_0^t e^{-A_s} dM_s$ is a local $P^x$-martingale for each $x \in \mathbb{R}^d$, thus

$$N_t = e^{-\beta t} u(X_t) - u(X_0) + \int_0^t e^{-\beta s} f(X_s) ds + \int_0^t e^{-\beta s} u(X_s) dA_s^u - \sum_{0 < s \leq t} e^{-\beta s} u(X_s) (1 - e^{F(X_{s-}, X_s)})$$

is a local $P^x$-martingale for each $x \in \mathbb{R}^d$. Since $u$ and $f$ are bounded, it is easy to check that $(N_t)_{t \geq 0}$ is in fact a uniformly integrable $P^x$-martingale for each $x \in \mathbb{R}^d$.

Taking expectations we get for every $x \in \mathbb{R}^d$,

$$u(x) = E^x e^{-\beta t} u(X_t) + E^x \int_0^t e^{-\beta s} f(X_s) ds + E^x \int_0^t e^{-\beta s} u(X_s) dA_s^u + E^x \int_0^t e^{-\beta s} \int_{\mathbb{R}^d} G(X_s, y) u(y) \frac{1}{|X_s - y|^{d+\alpha}} dy ds.$$

Letting $t \uparrow \infty$, we get

$$u(x) = E^x \int_0^\infty e^{-\beta s} f(X_s) ds + E^x \int_0^\infty e^{-\beta s} u(X_s) dA_s^u + E^x \int_0^\infty e^{-\beta s} \int_{\mathbb{R}^d} G(X_s, y) u(y) \frac{1}{|X_s - y|^{d+\alpha}} dy ds.$$

Therefore by Lemma 5.1.3 of [13] we know that $u \in D$. 
Applying the integration by parts formula for semimartingales again we get

\[
\begin{align*}
u(X_t) &= e^{\beta t} e^{-A_t} e^{-\beta t} e^{A_t} u(X_t) \\
&= e^{\beta t} e^{-A_t} \left( M_t - \int_0^t e^{\beta s} e^{A_s} f(X_s) ds \right) \\
&= u(X_0) + \int_0^t e^{\beta s} e^{-A_s} dM_s - \int_0^t f(X_s) ds \\
&\quad + \beta \int_0^t u(X_s) ds - \int_0^t u(X_s) dA_s + \sum_{0 < s \leq t} u(X_s)(1 - e^{F(X_{s^-}, X_s)}).
\end{align*}
\]

Thus

\[
Z_t = u(X_t) - u(X_0) - \beta \int_0^t u(X_s) ds + \int_0^t u(X_s) dA_s \\
- \sum_{0 < s \leq t} u(X_s)(1 - e^{F(X_{s^-}, X_s)}) + \int_0^t f(X_s) ds
\]

is a local $P^x$-martingale for each $x \in \mathbb{R}^d$. Again from the boundedness of $u$ and $f$ we can show that $(Z_t)_{t \geq 0}$ is uniformly $P^x$-integrable for each $x \in \mathbb{R}^d$, therefore it is a $P^x$-martingale for each $x \in \mathbb{R}^d$. Taking expectation, we get that for each $x \in \mathbb{R}^d$,

\[
u(x) = \mathbb{E}^x u(X_t) + \mathbb{E}^x \int_0^t f(X_s) ds - \beta \mathbb{E}^x \int_0^t u(X_s) ds \\
+ \mathbb{E}^x \int_0^t u(X_s) dA_s + \mathbb{E}^x \int_0^t \int_{\mathbb{R}^d} \frac{G(X_s, y) u(y)}{|X_s - y|^{d+\alpha}} dy ds.
\]

Thus from Lemma 5.1.4 of [13] we get that for any $\gamma > 0$ and any $g \in L^2(\mathbb{R}^d),$

\[
T(u, U_\gamma g) = \lim_{t \downarrow 0} \frac{1}{t} (u - P_t u, U_\gamma g) \\
= \int_{\mathbb{R}^d} f(x) U_\gamma g(x) dx - \beta(u, U_\gamma g) + \int_{\mathbb{R}^d} u(x) U_\gamma g(x) dx \\
+ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{U_\gamma g(x) G(x, y) u(y)}{|x - y|^{d+\alpha}} dxdy.
\]

From Theorem 3.3.2 we can easily see that

\[
u \mapsto \int_{\mathbb{R}^d} u(x)v(x)\mu(dx) \\
v \mapsto \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{v(x) G(x, y) u(y)}{|x - y|^{d+\alpha}} dxdy
\]
are bounded linear functionals on \( \dot{D} \) with respect to the \( T_1 \)-norm. Since \( U_\gamma(L^2(R^d)) \) is dense in \( D \), we can deduce immediately from the identity above that for any \( v \in \dot{D} \),

\[
T(u, v) + \beta(u, v) = \int_{R^d} f(x)v(x)dx + \int_{R^d} u(x)v(x)\mu(dx) + \int_{R^d} \int_{R^d} v(x)G(x, y)u(y)\mu(dx)dx.
\]

**Q.E.D.**

**Lemma 3.4.1** If \( f \in L^2(R^d) \) and \( \beta > \beta_0 \), then the function

\[
v(x) = E^x \int_0^{\infty} e^{-\beta t} e^{At} |f|(X_t)dt
\]

is finite quasi everywhere on \( R^d \).

**Proof.** From (3.2) and (3.13) we can easily get that for any \( x \in R^d \),

\[
v(x) \leq C \int_0^{\infty} e^{-(\beta - \beta_0)t} \left( E^x f^2(X_t) \right)^{\frac{1}{2}} dt \\
\leq \frac{C}{\sqrt{\beta - \beta_0}} \left( \int_0^{\infty} e^{-(\beta - \beta_0)t} E^x f^2(X_t)dt \right)^{\frac{1}{2}}.
\]

It is easy to show that for any \( \nu \in S_0 \) \( \|U_1 \nu\|_\infty < \infty \) implies \( \|U_\gamma \nu\|_\infty < \infty \) for any \( \gamma > 0 \), for any \( \nu \in S_0 \). Therefore for any \( \nu \in S_0 \) with \( \|U_1 \nu\|_\infty < \infty \), we have

\[
\int_{R^d} v^2(x)\nu(dx) \leq \frac{C}{\sqrt{\beta - \beta_0}} \int_{R^d} \int_0^{\infty} e^{-(\beta - \beta_0)t} E^x f^2(X_t)dt\nu(dx) \\
= \frac{C}{\sqrt{\beta - \beta_0}} \int_{R^d} \int_{R^d} u_{\beta - \beta_0}(x, y)f^2(y)dy\nu(dx) \\
\leq \frac{C}{\sqrt{\beta - \beta_0}} \|U_{\beta - \beta_0} \nu\|_\infty f_2^2.
\]

Thus from Theorem 3.3.2 of [13] we know that \( v \) is finite quasi everywhere.

**Q.E.D.**

**Lemma 3.4.2** Suppose that \( \{u_n\} \subset D \) and that \( u_n \rightarrow u \) in \( L^2(R^d) \). If

\[
\sup_n T(u_n, u_n) = \sup_n \int_{R^d} |x|^\alpha |\dot{u}_n|^2 dx < \infty,
\]
then \( u \in D \) and there is a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) which converges weakly to \( u \) with respect to the \( T_1 \)-norm.

**Proof.** Using Fatou's lemma we get that

\[
\int_{R^d} |x|^\alpha |\hat{u}|^2(x)dx \leq \liminf_{n \to \infty} \int_{R^d} |x|^\alpha |\hat{u}_{n_k}|^2dx < \infty,
\]

thus \( u \in D \).

From (3.14) we know that there is a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) such that \( |x|^\frac{3}{2} \hat{u}_{n_k} \) converges weakly in \( L^2(R^d) \) to some function \( w \in L^2(R^d) \). Taking convex combinations of \( \{|x|^\frac{3}{2} \hat{u}_{n_k}\} \) if necessary, we can assume that \( |x|^\frac{3}{2} \hat{u}_{n_k} \) converges strongly in \( L^2(R^d) \) to \( w \). Combining this with the fact that \( u_n \) converges to \( u \) in \( L^2(R^d) \), we can see that \( w = |x|^\frac{3}{2} \hat{u} \). Therefore \( |x|^\frac{3}{2} \hat{u}_{n_k} \) converges weakly in \( L^2(R^d) \) to \( |x|^\frac{3}{2} \hat{u} \), that is to say, for any \( f \in L^2(R^d) \),

\[
\lim_{k \to \infty} \int_{R^d} |x|^\frac{3}{2} \hat{u}_{n_k} f(x)dx = \int_{R^d} |x|^\frac{3}{2} \hat{u}(x)f(x)dx.
\]

In particular, we have, for any \( v \in D \),

\[
\lim_{k \to \infty} \int_{R^d} |x|^\alpha \hat{u}_{n_k}(x)v(x)dx = \int_{R^d} |x|^\alpha \hat{u}(x)v(x)dx,
\]

i.e.,

\[
\lim_{k \to \infty} T(u_{n_k}, v) = T(u, v).
\]

The proof is now complete.

*Q.E.D.*

**Theorem 3.4.2** If \( f \in L^2(R^d) \) and \( \beta > \beta_0 \), then the function

\[
u(x) = E^x \int_0^\infty e^{-\beta t} e^{A_t} f(X_t)dt
\]


is in $\tilde{D}$ and satisfies for any $v \in \tilde{D}$,

$$T(u, v) + \beta(u, v) = \int_{\mathbb{R}^d} f(x)v(x)dx - \int_{\mathbb{R}^d} u(x)v(x)\mu(dx)$$

$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{v(x)G(x, y)u(y)}{|x - y|^{d+\alpha}}dxdy,$$  \hspace{1cm} (3.15)

where $G = e^{F} - 1$.

**Proof.** For any $n > 0$, let

$$f_n(x) = \begin{cases} f(x), & \text{if } |f(x)| < n; \\ n, & \text{if } f(x) > n; \\ -n, & \text{if } f(x) < -n, \end{cases}$$

and define

$$u_n(x) = \mathbb{E}^x \int_0^\infty e^{-\beta t}e^{A_t} f_n(x)dt.$$

By Theorem 3.4.1 we know that for each $n$, $u_n$ is a continuous function in $D$ such that for any $v \in \tilde{D}$,

$$T(u_n, v) + \beta(u_n, v) = \int_{\mathbb{R}^d} f(x)v(x)dx - \int_{\mathbb{R}^d} u_n(x)v(x)\mu(dx)$$

$$- \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{v(x)G(x, y)u_n(y)}{|x - y|^{d+\alpha}}dxdy,$$  \hspace{1cm} (3.16)

We can easily show that $u_n$ tends to $u$ strongly in $L^2(\mathbb{R}^d)$, thus

$$\sup_n \{\|u_n\|_2\} < \infty.$$  

Now from (3.16), Theorem 3.3.2 and Theorem 3.3.3 we obtain

$$T(u_n, u_n) = -\beta(u_n, u_n) + \int_{\mathbb{R}^d} f(x)u_n(x)dx$$

$$- \int_{\mathbb{R}^d} u_n^2(x)\mu(dx) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u_n(x)G(x, y)u_n(y)}{|x - y|^{d+\alpha}}dxdy$$

$$\leq \beta\|u_n\|^2_2 + \|f\|_2\|u_n\|_2 + \frac{1}{2}T(u_n, u_n) + C\|u_n\|^2_2,$$

where $C$ is a positive constant. Therefore

$$T(u_n, u_n) \leq 2(1 + C)\|u_n\|^2 + 2\|f\|\|u_n\|,$$
consequently

\[ \sup_n T(u_n, u_n) < \infty. \]

Hence by Lemma 3.4.2 we know that \( u \in D \) and there is a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) converging to \( u \) weakly with respect to the \( T_1 \)-norm. Thus there is a sequence \( \{v_k\} \) with \( v_k \) being finite convex combinations of \( \{u_{n_k}\} \) such that \( v_k \) converges to \( u \) strongly with respect to the \( T_1 \)-norm. Now from Lemma 3.4.1 and the dominated convergence theorem we can see that \( u_n \) converges to \( u \) quasi everywhere on \( R^d \), thus \( v_k \) converges to \( u \) quasi everywhere on \( R^d \). For each \( k \), \( v_k \) is continuous since \( u_n \) is continuous for each \( n \). Therefore by Lemma 3.1.4 of [13] we know that \( u \) is quasi-continuous.

Now we are going to prove (3.15). From the facts that \( u_n \) converges to \( u \) strongly in \( L^2(R^d) \) and that \( u_{n_k} \) converges to \( u \) weakly with respect to the \( T_1 \)-norm, we can easily see that for any \( v \in \tilde{D} \),

\[
\lim_{k \to \infty} T(u_{n_k}, u_{n_k}) = T(u, u),
\]

\[
\lim_{k \to \infty} (u_{n_k}, u_{n_k}) = (u, u),
\]

\[
\lim_{k \to \infty} \int_{R^d} f(x)u_{n_k}(x)dx = \int_{R^d} f(x)u(x)dx.
\]

Therefore by (3.16) we know that in order to prove (3.15), we need only to prove that for any \( v \in \tilde{D} \),

\[
\lim_{k \to \infty} \int_{R^d} u_{n_k}(x)v(x)\mu(dx) = \int_{R^d} u(x)v(x)\mu(dx), \quad (3.17)
\]

\[
\lim_{k \to \infty} \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)u_{n_k}(y)}{|x-y|^{d+\alpha}}dxdy = \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)u(y)}{|x-y|^{d+\alpha}}dxdy. \quad (3.18)
\]

We are only going to show (3.17). (3.18) can be shown similarly.

Fix a \( v \in \tilde{D} \). By Schwarz’s inequality and Theorem 3.3.2 we know that for any \( \epsilon > 0 \) there exists a \( C(\epsilon) > 0 \) such that
\[ |\int_{\mathbb{R}^d} u_{n_k}(x)v(x)\mu(dx) - \int_{\mathbb{R}^d} u(x)v(x)\mu(dx)| \]
\[ \leq \left( \int_{\mathbb{R}^d} v^2(x)\mu^*(dx) \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} (u_{n_k} - u)^2(x)\mu^*(dx) \right)^{\frac{1}{2}} \]
\[ \leq \left( \int_{\mathbb{R}^d} v^2(x)\mu^*(dx) \right)^{\frac{1}{2}} \left( \epsilon^2 T(u_{n_k} - u, u_{n_k} - u) + C^2(\epsilon)\|u_{n_k} - u\|_2^2 \right)^{\frac{1}{2}} \]
\[ \leq M\epsilon + C(\epsilon)\|u_{n_k} - u\|_2 \left( \int_{\mathbb{R}^d} v^2(x)\mu^*(dx) \right)^{\frac{1}{2}}, \]

where
\[
M = \left( \int_{\mathbb{R}^d} v^2(x)\mu^*(dx) \right)^{\frac{1}{2}} \cdot \sup_k \left\{ \sqrt{T(u_{n_k} - u, u_{n_k} - u)} \right\}.
\]

Since \( u_n \) tends to \( u \) strongly in \( L^2(\mathbb{R}^d) \), we can take a \( K > 0 \) such that
\[
\|u_{n_k} - u\|_2 \leq \frac{\epsilon}{C(\epsilon)} \left( \int_{\mathbb{R}^d} v^2(x)\mu^*(dx) + 1 \right)^{\frac{1}{2}},
\]
whenever \( k > K \). Therefore
\[
|\int_{\mathbb{R}^d} u_{n_k}(x)v(x)\mu(dx) - \int_{\mathbb{R}^d} u(x)v(x)\mu(dx)| \leq (M + 1)\epsilon,
\]
whenever \( k > K \), and consequently
\[
\lim_{k \to \infty} \int_{\mathbb{R}^d} u_{n_k}(x)v(x)\mu(dx) = \int_{\mathbb{R}^d} u(x)v(x)\mu(dx),
\]
which is (3.17).

\[ Q.E.D. \]

From Theorem 3.4.2 we immediately get the main result of this section.

**Theorem 3.4.2** \((T_t)_{t \geq 0}\) is the unique strongly continuous semigroup on \( L^2(\mathbb{R}^d) \) determined by the bilinear form \((\mathcal{E}, \mathcal{D})\).

From Theorem 3.3.4 and the result above we obtain the following result which generalizes the main result of [29].

**Corollary 3.4.1** If \( F \leq 0 \) and \( \mu^+ = 0 \), then the Markov process \((X_t, e^{A_t})\) satisfies the sector condition.
CHAPTER 4
THE GAUGE THEOREM AND ITS APPLICATIONS

4.1 The Gauge Theorem

Recall that $\mu \in K_{d,\alpha}$ and $F \in A_{d,\alpha}$ are fixed and

$$A_t = A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s).$$

Let $A_t^{\mu+}$ and $A_t^{\mu-}$ be the continuous additive functionals associated with $\mu^+$ and $\mu^-$, respectively, and put

$$A_t^+ = A_t^{\mu+} + \sum_{0 < s \leq t} F^+(X_{s-}, X_s);$$
$$A_t^- = A_t^{\mu-} + \sum_{0 < s \leq t} F^-(X_{s-}, X_s);$$
$$A_t^* = A_t^+ + A_t^-;$$
$$L_t = e^{A_t^+};$$
$$K_t = e^{-A_t^-}.$$

Then $\{L_t, t \geq 0\}$ and $\{K_t, t \geq 0\}$ are multiplicative functionals of $X$ and furthermore for all $t \geq 0$, $e^{A_t} = K_t \cdot L_t$, $K_t \leq 1$, $L_t \geq 1$.

From now on we are going to fix a regular bounded open domain $D$ in $\mathbb{R}^d$ and define

$$g(x) = E^x(e^{A(\tau_D)}).$$

The function $g$ is called the gauge function of $(A, D)$ with respect to the process $X$, or simply the gauge function of $(A, D)$ if there is no confusion. The purpose of this section is to study the boundedness property of $g$. 

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Lemma 4.1.1 Suppose that $\nu \in K_{d,\alpha}$ is nonnegative and $A^\nu_{_i}$ is the continuous additive functional associated with $\nu$. Then the function

$$x \mapsto E^x A^\nu(\tau_D)$$

is bounded on $R^d$.

Proof. From the definition of $K_{d,\alpha}$ we see that the function

$$x \mapsto \int_D u(x, y) \nu(dy)$$

is bounded on $R^d$. From [23] we know that $A^\nu_{_{_i}}\cap \tau_D$ is the continuous additive functional of $X^D$ associated with $1_D \cdot \nu$. Therefore by repeating the argument in the proof of Theorem 3.1.3, we can get that for any $x \in D$,

$$E^x A^\nu(\tau_D) = \int_D u^D(x, y) \nu(dy). \quad (4.1)$$

Hence for any $x \in D$,

$$E^x A^\nu(\tau_D) \leq \int_D u(x, y) \nu(dy),$$

and thus the conclusion of this lemma is true.

Q.E.D.

Theorem 4.1.1 $\inf_{x \in R^d} g(x) > 0$.

Proof. From the definition of $A_{d,\alpha}$ and Lemma 4.1.1 we get that the function

$$x \mapsto E^x \left( A^{\mu^+}_{\tau_D} + A^{\mu^-}_{\tau_D} + \int_0^{\tau_D} \int_{R^d} \frac{|F(X_t, y)|}{|X_t - y|^{d+\alpha}} dy dt \right)$$

is bounded by a constant $C$ on $R^d$. Hence by Jensen's inequality we get
\[ g(x) = E^x \{ e^{A_t^D} \} \geq E^x \{ e^{-A_t^D} \} \]
\[ \geq \exp(-E^x \{ A_t^D \}) \]
\[ = \exp \left\{ -E^x \left( A_t^{\mu^+} + A_t^{\mu^-} \right) + \int_0^T \int_{R^d} \frac{|F(X_t, y)|}{|X_t - y|^{d+\alpha}} dy dt \right\} \]
\[ \geq e^{-C} > 0. \]

The proof is now complete.

Q.E.D.

**Lemma 4.1.2** There exists a \( \gamma > 0 \) such that
\[ \sup_{x \in R^d} E^x \int_0^T e^{-\beta t} dL_t < \infty. \]
whenever \( \beta > \gamma. \)

**Proof.** By using the integration by parts formula for semimartingales we can get that
\[ \int_0^1 dL_t = \int_0^1 L_t dA_t^{\mu^+} + \sum_{0 < s < t} e^{A_t^{\mu^+}} \exp \left( \sum_{0 < s < t} F^+(X_{s-}, X_s) \right) \left( e^{F^+(X_{1-}, X_t)} - 1 \right). \]

Therefore for any \( x \in R^d, \)
\[ E^x \int_0^1 dL_t \]
\[ = E^x \int_0^1 L_t dA_t^{\mu^+} + E^x \int_0^1 e^{A_t^{\mu^+}} \sum_{0 < s < t} F^+(X_{s-}, X_s) \int_{R^d} \frac{e^{F^+(X_{t-}, y)} - 1}{|X_t - y|^{d+\alpha}} dy dt \]
\[ = E^x \int_0^1 L_t dA_t^{\mu^+} + E^x \int_0^1 L_t \int_{R^d} \frac{e^{F^+(X_{t-}, y)} - 1}{|X_t - y|^{d+\alpha}} dy dt \]
\[ = E^x \int_0^1 L_t dB_t \]
\[ \leq \left( E^x(B_t)^2 \right)^{\frac{1}{2}} \left( E^x e^{2A_t^+} \right)^{\frac{1}{2}}, \]

where
\[ B_t = A_t^{\mu^+} + \int_0^t \int_{R^d} \frac{e^{F^+(X_{t-}, y)} - 1}{|X_t - y|^{d+\alpha}} dy dt. \]
Since $\mu^+$ and the function

$$x \mapsto \int_{R^d} \frac{e^{F^+(x,y)} - 1}{|x - y|^{d+\alpha}} dy$$

are both in $K_{d,\alpha}$, it follows from Theorems 3.1.4 and 3.2.3 that

$$\sup_{x \in R^d} \left( E^x (B_1^2) \right)^{\frac{1}{2}} \left( E^x e^{2A_t^+} \right)^{\frac{1}{2}} < C_1$$

for some constant $C_1$.

Now for any $x \in R^d$, we have

$$E^x \int_0^{\tau_D} e^{-\beta t} dL_t$$

$$= E^x \left\{ \sum_0^{\infty} \int_{\tau_D \wedge n}^{\tau_D \wedge (n+1)} e^{-\beta t} dL_t \right\}$$

$$= \sum_0^{\infty} E^x \left\{ n < \tau_D; \int_{n}^{\tau_D \wedge (n+1)} e^{-\beta t} dL_t \right\}$$

$$\leq \sum_0^{\infty} E^x \left\{ n < \tau_D; e^{-\beta t} L_n E^x (n) \left( \int_0^{\tau_D \wedge 1} dL_t \right) \right\}$$

$$\leq C_1 \sum_0^{\infty} e^{-\beta n} E^x L_n.$$

From Theorem 3.2.3 we know that there exist constants $C_2$ and $\gamma > 0$ such that for any $n > 0$,

$$\sup_{R^d} E^x L_n \leq C_2 e^{\gamma n}.$$

Therefore if $\beta > \gamma$, then

$$\sup_{x \in R^d} E^x \int_0^{\tau_D} e^{-\beta t} dL_t$$

$$\leq C_1 C_2 \sum_0^{\infty} e^{(\gamma - \beta) n}$$

$$< \infty.$$

Q.E.D.
Theorem 4.1.2 (The Gauge Theorem) The gauge function $g$ is either identically infinite on $D$ or bounded on $D$.

**Proof.** Lemma 4.1.2 implies that $e^{A_t}$ is compatible in the following sense:

$$\sup_{x \in D} \int_0^{T_D} e^{-\beta t} K_t dL_t < \infty,$$

for some $\beta > 0$. It follows from Theorem 2.2.2 that $U^{D}_\gamma$ is compact as a map from $L^\infty(D)$ to $L^\infty(D)$ for any $\gamma \geq 0$. The operator $U^{D,K}_\gamma$ defined by

$$f \mapsto E \int_0^{T_D} e^{-\gamma t} f(X_t) K_t dL_t$$

is dominated by $U^{D}_\gamma$, consequently $U^{D,K}_\gamma$ is compact from $L^\infty(D)$ to $L^\infty(D)$ for any $\gamma \geq 0$ by Remark (3.5.b) of [28]. It follows again from Remark (3.5) of [28] that for any $\gamma \geq 0$, $U^{D,K}_\gamma$ is non-degenerate and irreducible in the following sense: $U^{D,K}_\gamma 1$ is not identically zero and for any nonnegative $f$, $U^{D,K}_\gamma f$ is either identically zero or strictly positive on $D$. Theorem 4.1.1 above guarantees that $g$ is strictly positive. Thus all the conditions of the general Gauge Theorem (Theorem (3.4)) of [28] are satisfied, and so the gauge function is either identically infinite or bounded on $D$.

Q.E.D.

Now we are going to derive some important consequences of the boundedness of the gauge function which will be very useful in the next two sections. But first we state two lemmas which we will use to derive the results mentioned above.

**Lemma 4.1.3** There exists an $\eta > 0$ such that for any Borel subset $B$ of $D$ whose Lebesgue measure is less than $\eta$,

$$\sup_{x \in \mathbb{R}^d} E^x \{ e^{A(\tau_B)} \} < \infty.$$
Proof. The proof is similar to that of Lemma 4 of [10], we give it here for the sake of completeness.

If $P^x\{\tau_B = 0\} = 1$, then $E^x\{e^{A(\tau_B)}\} = 1$. Otherwise $P^x\{\tau_B = 0\} = 0$ and we have

$$E^x\{e^{A(\tau_B)}\} \leq E^x\{L(\tau_B)\}$$

$$= \sum_{n=0}^{\infty} E^x\{n < \tau_B; L_n = \min\{0 < \tau_B \leq 1; L(\tau_B)\}\}$$

$$\leq \sum_{n=0}^{\infty} E^x\{n < \tau_B; L_1 = \min\{L\}\}$$

$$\leq C_1 \sum_{n=0}^{\infty} E^x\{n < \tau_B; \exp(A_n^+)\}$$

where

$$C_1 = \sup_{x \in \mathbb{R}^d} E^x\{L_1\} < \infty$$

by Theorem 3.2.3. The sum above is bounded by

$$\sum_{n=0}^{\infty} (P^x\{n < \tau_B\})^{\frac{1}{2}} (E^x\{\exp(2A_n^+)\})^{\frac{1}{2}}.$$ 

Applying Theorem 3.2.3 again we know that there exist constants $C_2$ and $b$ such that

$$\sup_{x \in \mathbb{R}^d} E^x\{\exp(2A_n^+)\} \leq C_2 e^{nb},$$

Hence the last sum is bounded by

$$\sqrt{C_2} \sum_{n=0}^{\infty} (P^x\{n < \tau_B\})^{\frac{1}{2}} e^{\frac{nb}{2}}.$$ 

(4.2)

It is easy to see that there exists an $\eta > 0$ such that for any Borel subset $B$ of $D$ whose Lebesgue measure is less than $\eta$ we have

$$\sup_{x \in \mathbb{R}^d} P^x\{1 < \tau_B\} \leq e^{-2b}.$$ 

It follows from the Markov property that
Using the above inequality in (4.2) we see that the infinite series in (4.2) is convergent.

Q.E.D.

Lemma 4.1.4 If the gauge function $g$ is bounded, then for any $\eta > 0$, there exists a constant $C(\eta)$ such that for any $x \in D$,

$$\|g\|_\infty^{-1} g(x) \leq \sum_{k=0}^{\infty} E^x \{ k\eta < \tau_D; e^{A(k\eta)} \} \leq C(\eta).$$

Proof. Since the proof is the same for any $\eta > 0$ we take $\eta = 1$. For each $x \in D$ the quantity $\{ n < \tau_D; e^{A(\tau_D)} \}$ decreases to zero as $k \to \infty$. Hence for any $\epsilon > 0$, we can choose an integer $N$ and a set $H$ such that the Lebesgue measure of $D \setminus H$ is less than $\epsilon$ and

$$\sup_{x \in H} E\{ N < \tau_D; e^{A(\tau_D)} \} < \epsilon.$$

By choosing a compact subset of $H$ if necessary we may assume that $H$ itself is compact. Here $\epsilon$ is such that

$$\sup_{x \in \mathbb{R}^d} E\{ e^{2A(\tau_D \setminus H)} \} := b < \infty.$$

This is possible by Lemma 4.1.2. Now

$$E^x \{ 2N < \tau_D; e^{A(\tau_D)} \}$$

$$\leq E^x \{ N < \tau_D \setminus H; e^{A(\tau_D)} \} + E^x \{ N \geq \tau_D \setminus H, 2N < \tau_D; e^{A(\tau_D)} \}$$

$$\leq \|g\|_\infty E^x \{ N < \tau_D \setminus H; e^{A(\tau_D \setminus H)} \}$$

$$+ E^x \{ \tau_D \setminus H < \tau_D; e^{A(\tau_D \setminus H)} E^x(\tau_D \setminus H) \{ N < \tau_D; e^{A(\tau_D)} \} \}.$$

The expectation in the third line above can be estimated by

$$E^x \{ e^{2A(\tau_D \setminus H)} \}^{\frac{1}{2}} P^x \{ \tau_D \setminus H > N \}^{\frac{1}{2}}$$
which is uniformly small if \( N \) is large, by the transience of \( X \). On the set \( \{ \tau_D \cap H < \tau_D \} \), \( X_{\tau_D \cap H} \in H \), so the expectation in the fourth line above is bounded by \( \sqrt{b \varepsilon} \). Thus for large \( N \),

\[
\sup_{x \in \mathbb{R}^d} E^x \{ N < \tau_D; e^{A(\tau_D)} \}
\]
is small. Now from Theorem 4.1.1 \( g \) is bounded from below, say \( g \geq a > 0 \). Then

\[
E^x \{ N < \tau_D; e^{A(N)} \} \\
\leq \frac{1}{a} E^x \{ N < \tau_D; e^{A(N)} g(X_N) \} \\
\leq \frac{1}{a} E^x \{ N < \tau_D; e^{A(\tau_D)} \}.
\]

Therefore we can see that for \( N \) large enough

\[
\sup_{x \in \mathbb{R}^d} E^x \{ N < \tau_D; e^{A(N)} \} \leq \beta < 1.
\]

By the Markov property we have for any \( k \),

\[
\sup_{x \in \mathbb{R}^d} E^x \{ kN < \tau_D; e^{A(kN)} \} \leq \beta^k.
\]

Further if \( j < N \),

\[
E^x \{ kN + j < \tau_D; e^{A(kN+j)} \} \\
= E^x \{ j < \tau_D; e^{A(j)} \{ kN < \tau_D; e^{A(kN)} \} \\
\leq \beta^j \sup_{j \leq N} \sup_{x \in \mathbb{R}^d} E^x \{ e^{A(j)} \} := \beta^j \gamma.
\]

All these inequalities give

\[
\sum_{n=0}^{\infty} E^x \{ n < \tau_D; e^{A(n)} \} \\
= \sum_{k=0}^{\infty} \sum_{j=1}^{N} E^x \{ kN + j < \tau_D; e^{A(kN+j)} \} \\
\leq N \gamma \sum_{k=1}^{\infty} \beta^k = \frac{N \gamma}{1 - \beta} := C(1).
\]

Finally we have
\[ g(x) = \sum_{n=0}^{\infty} E^x \{ n < \tau_D \leq n + 1; e^{A(\tau_D)} \} \]
\[ \leq \sum_{n=0}^{\infty} E^x \{ n < \tau_D; e^{A(n)}g(X_1) \} \]
\[ \leq \|g\|_\infty \sum_{n=0}^{\infty} E^x \{ n < \tau_D; e^{A(n)} \} \]

which proves the result.

\[ Q.E.D. \]

Now here is one consequence of the boundedness of the gauge function.

**Theorem 4.1.3** Let \( \nu \in K_{d, \alpha} \) be nonnegative and \( B_t \) be the continuous additive functional associated with \( \nu \). If the gauge function \( g \) is bounded on \( D \), then

\[ \sup_{x \in D} E^x \int_0^{\tau_D} e^{A(t)} dB_t < \infty. \]

**Proof.** Obviously we have

\[ E^x \int_0^1 e^{A_t} dB_t \leq E^x \int_0^1 e^{A^*_t} dB_t \]
\[ \leq E^x \{ e^{A^*_t} B_1 \} \]
\[ \leq E^x \{ e^{A^*_t + B_1} \}. \]

By Theorem 3.2.3 we know that there exists a constant \( C \) such that

\[ \sup_{x \in \mathbb{R}^d} E^x \{ e^{A^*_t + B_1} \} \leq C. \]

Thus we have

\[ \sup_{x \in D} E^x \int_0^{\tau_D} e^{A_t} dB_t \]
\[ = \sup_{x \in D} E^x \left\{ \sum_{n=0}^{\infty} \int_{\tau_D \land n}^{\tau_D \land (n+1)} e^{A_t} dB_t \right\} \]
\[ = \sup_{x \in D} \sum_{n=0}^{\infty} E^x \left\{ n < \tau_D; \int_n^{\tau_D \land (n+1)} e^{A_t} dB_t \right\} \]
\[
= \sup_{x \in D} \sum_{n=0}^{\infty} E^x \left\{ n < \tau_D; e^{A_n} E^{X(n)} \left( \int_0^{\tau_D} e^{A_t} dB_t \right) \right\} \\
\leq \sup_{x \in D} \sum_{n=0}^{\infty} E^x \left\{ n < \tau_D; e^{A_n} E^{X(n)} \left( \int_0^1 e^{A_t} dB_t \right) \right\} \\
\leq C \sup_{x \in D} \sum_{n=0}^{\infty} E^x \{ n < \tau_D; e^{A_n} \} \\
< \infty,
\]

where the last inequality follows from Lemma 4.1.4.

\[Q.E.D.\]

Now here is another consequence of the boundedness of the gauge function. It looks much stronger than the assumption that the gauge function is bounded.

**Theorem 4.1.4** If the gauge function \( g \) is bounded on \( D \), then

\[ \sup_{x \in D} E^x \left\{ \sup_{0 < t \leq \tau_D} e^{A_t} \right\} < \infty. \]

**Proof.** We have

\[
e^{A_t} = 1 + \int_0^t e^{A_s} dA_s^\mu \\
+ \sum_{0 < s \leq t} e^{A_s^\mu} \exp \left( \sum_{0 < r < s} F(X_{r-}, X_r) \right) \left( e^{F(X_{s-}, X_s)} - 1 \right)
\]

\[
\leq 1 + \int_0^t e^{A_s^\mu} dA_s^\mu \\
+ \sum_{0 < s \leq t} e^{A_s^\mu} \exp \left( \sum_{0 < r < s} F(X_{r-}, X_r) \right) \left( e^{F(X_{s-}, X_s)} - 1 \right).
\]

Therefore,

\[
\sup_{0 < t \leq \tau_D} e^{A_t} \leq 1 + \int_0^{\tau_D} e^{A_t^\mu} dA_t^\mu \\
+ \sum_{0 < t \leq \tau_D} e^{A_t^\mu} \exp \left( \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) \left( e^{F(X_{s-}, X_s)} - 1 \right).
\]

Hence for any \( x \in D \),
\[
\begin{align*}
E^x \left\{ \sup_{0 < t \leq \tau_D} e^{A_t} \right\} \\
\leq 1 + E^x \int_0^{\tau_D} e^{A_t} dA_t^\alpha \\
&+ E^x \int_0^{\tau_D} e^{A_t} \exp(\sum_{0 < s < t} F(X_s, X_t)) \int_{\mathbb{R}^d} \frac{e^{\|F(X_t, y)\|} - 1}{|X_t - y|^{d+\alpha}} dy \, dt \\
\leq 1 + E^x \int_0^{\tau_D} e^{A_t} dA_t^\alpha \\
&+ E^x \int_0^{\tau_D} e^{A_t} \int_{\mathbb{R}^d} \frac{e^{\|F(X_t, y)\|} - 1}{|X_t - y|^{d+\alpha}} dy \, dt.
\end{align*}
\]

Now the conclusion of this theorem follows Theorem 4.1.3.

Q.E.D.

4.2 The Dirichlet Problem for Linear Equations

In addition to \( \mu, F \) and \( D \), we are going to fix another Radon measure \( \nu \) on \( \mathbb{R}^d \). We are going to assume that \( \nu \) is in \( K_{d,\alpha} \) and \( B_t \) is the continuous additive functional associated with \( \nu \). The purpose of this section is to formulate and solve the Dirichlet problem of the following linear equation

\[
\left(-(-\Delta)^{\frac{\alpha}{2}} + \mu\right) u(x) + \int_{\mathbb{R}^d} \frac{G(x, y)u(y)}{|x - y|^{d+\alpha}} dy + \nu = 0 \quad (4.3)
\]
on \( D \), where \( G = e^F - 1 \) as in the previous chapter.

Before we discuss the Dirichlet problem of (2.1), we are going to recall some useful facts first.

**Definition 4.2.1** A function \( u \in D_e \) is said to be harmonic with respect to \( X \) on \( D \) if \( T(u, v) = 0 \) for any \( v \in D_e \cap C_{0,0}(D) = D \cap C_{0,0}(D) \). Here \( C_{0,0}(D) \) stands for the family of continuous functions with compact support in \( D \).

In the sequel a function which is harmonic with respect to \( X \) on \( D \) will simply be said to be harmonic on \( D \). Here we need to keep in mind that a function that is
harmonic on \( D \) is required to be defined on all of \( R^d \) and even the part of the function on the complement of \( D \) is involved in the above definition.

**Lemma 4.2.1** If \( f \in D_e \) is bounded, then the function

\[
u(x) := E^x \tilde{f}(X(\tau_D))
\]

is harmonic and continuous on \( D \). Here \( \tilde{f} \) denotes a quasi-continuous version of \( f \). Furthermore, if \( f \) is continuous on \( D^c \), then \( u \) is continuous everywhere and coincides with \( f \) on \( D^c \).

**Proof.** That \( u \) is harmonic follows from the transient version of Theorem 4.4.1 of [13]. The rest are proven in [24] and [25].

\[Q.E.D.\]

**Lemma 4.2.2** If a bounded function \( u \in D_e \) is harmonic on \( D \), then

\[
u_1(x) := E^x \tilde{u}(X(\tau_D))
\]

is a version of \( u \) which is continuous in \( D \).

**Proof.** Follows immediately from the transient version of Theorem 4.4.1 of [13] and Lemma 4.2.1 above.

\[Q.E.D.\]

**Lemma 4.2.3** If \( f \) is a bounded function on \( R^d \), then

\[
u(x) := E^x \int_0^{\tau_D} f(X_t) dB_t
\]

is a bounded continuous function which belongs to \( D_D \), vanishes on \( D^c \) and

\[
T(u, v) = \int_{R^d} v(x)f(x)\nu(dx)
\]

for any \( v \in D_e \cap C_{0,0}(D) = D \cap C_{0,0}(D) \).
Proof. That \( u \) vanishes on \( D^c \) is trivial from the regularity of \( D \). The boundedness of \( u \) follows from Lemma 4.1.1. Now we are going to prove the continuity of \( u \). It follows from (4.1) that for any \( x \in D \),

\[
u(x) = \int_D u^D(x,y) f(y) \nu(dy).\]

From Corollary 2.2.1 and Theorem 2.2.1 we know that for any \( r > 0 \),

\[
u_r(x) = \int_{D \cap \{|x-y| \leq r\}} u^D(x,y) \nu(dy)
\]

is continuous on \( D \) and that, as \( x \) tends to any point \( z \) on the boundary of \( D \), \( u_r(x) \) tends to zero. By the definition of \( K_{d,\alpha} \) and the fact that for any \( x, y \in D \),

\[
u^D(x,y) \leq |x - y|^{-d+\alpha},
\]

we can easily see that \( u_r \) tends to \( u \) uniformly on \( D \), therefore \( u \) is continuous on \( D \) and \( u(x) \) tends to zero as \( x \) tends to any point \( z \) on the boundary of \( D \). Hence \( u \) is continuous on \( \mathbb{R}^d \).

The rest of the conclusions follow from the transient version of Lemma 5.1.3 of [13].

Q.E.D.

Definition 4.2.2 A bounded function \( u \in \mathbf{D}_c \) is called a solution to the equation (4.3) in \( D \) if

\[
u(x) = \int_{\mathbb{R}^d} v(x) \nu(dx) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{v(x) G(x,y) u(y)}{|x-y|^{d+\alpha}} dx dy
\]

\[
u(x) = \int_{\mathbb{R}^d} v(x) \nu(dx), \quad \forall v \in \mathbf{D} \cap C_{0,0}(D) .
\]

Again we need to bear in mind that a solution to the equation (4.3) in \( D \) is required to be defined on all of \( \mathbb{R}^d \) and even the part of \( u \) on the complement of \( D \) plays a role in the definition above.
The definition above is a generalization of a solution in distributional sense to the following Schrödinger equation

$$\left( \frac{\Delta}{2} + \mu \right) u(x) = \nu. \quad (4.5)$$

In fact, when the underlying process $X$ is the standard Brownian motion, then for any $F$ vanishing on the diagonal of $R^d \times R^d$, the additive functional

$$\sum_{0 < s < t} F(X_s, X_s)$$

is identically zero, or equivalently, $F$ can be replaced by the function which is zero everywhere, thus $G$ can be replaced by the function which is zero everywhere and so the third term in the first line of (4.4) disappears and (4.4) becomes

$$T(u, v) - \int_{R^d} u(x)v(x)\mu(dx)$$

$$= \int_{R^d} v(x)\nu(dx), \quad \forall v \in D \cap C_{0,0}(D),$$

which is exactly the definition of a solution of (4.5) in the distributional sense.

**Theorem 4.2.1** Any solution $u$ of (4.3) in $D$ has a version which is continuous in $D$.

**Proof.** Put

$$u_1(x) = E^x B(\tau_D) + \int_0^{\tau_D} u(X_t) dA^\mu_t$$

$$+ E^x \int_0^{\tau_D} \left( \int_{R^d} \frac{G(X_t, y)u(y)}{|X_t - y|^{d+\alpha}} dy \right) dt.$$

Then it follows from Lemma 4.2.3 that $u_1 \in D_D$ is a bounded continuous function on $R^d$ such that for any $v \in D \cap C_{0,0}(D)$,

$$T(u_1, v) = \int_{R^d} v(x)\nu(dx) + \int_{R^d} u(x)v(x)\mu(dx)$$

$$+ \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)u(y)}{|x - y|^{d+\alpha}} dxdy.$$
Consequently the function $u_2 := u - u_1 \in D_e$ satisfies the following condition

$$T(u_2, v) = 0,$$

for any $v \in D \cap C_{0,0}(D)$, i.e., $u_2$ is harmonic on $D$. Therefore, by Lemma (4.2.2) above, we know that

$$u_3(x) := E^x \tilde{u}_2(X_{\tau_D})$$

is a version of $u_2$ which is continuous in $D$. Thus the function $u_3 + u_1$ is a version of $u$ which is continuous in $D$.

Q.E.D.

**Definition 4.2.3** Suppose $f$ is a bounded function in $D_e$. We say that a bounded function $u$ in $D_e$ is a solution to the Dirichlet problem of (4.3) with exterior function $f$ if the following three conditions are satisfied:

(i) $u$ is a solution of (4.3) in $D$;

(ii) $u|_{D^c} = f|_{D^c}$;

(iii) for every $z \in \partial D$,

$$\lim_{D^c \ni x \to z} u(x) = f(z).$$

**Theorem 4.2.2** If the gauge function $g$ is bounded, then for any $f \in D_e$ which is bounded and continuous on $D^c$, 

$$u(x) := E^x \left( e^{A(\tau_D)} f(X(\tau_D)) \right) + E^x \int_{\tau_D}^{\tau_D} e^{A_t} dB_t$$

is the unique continuous solution to the Dirichlet problem of (4.3) in $D$ with exterior function $f$. 
Proof. It follows from the regularity of $D$, the boundedness of $f$, the boundedness of the gauge function and Theorem 4.1.3 that the function $u$ defined above is bounded on $R^d$. It follows easily from the definition of $u$ that

$$M_t := e^{A(t \wedge \tau_D)}u(X_{t \wedge \tau_D}) + \int_0^{t \wedge \tau_D} e^{A_t} dB_t$$

is a $P^x$-martingale for each $x \in D$. Using the integration by parts formula we get

$$u(X_{t \wedge \tau_D}) = e^{-A(t \wedge \tau_D)} \left( M_t - \int_0^{t \wedge \tau_D} e^{A_t} dB_t \right)$$

$$= u(X_0) + \int_0^{t \wedge \tau_D} e^{-A(s-)} dM_s - B(t \wedge \tau_D)$$

$$- \int_0^{t \wedge \tau_D} u(X_s) dA_s + \sum_{0 < s \leq t \wedge \tau_D} u(X_s) \left( 1 - e^{F(X_s, X_s)} \right).$$

Thus

$$N_t := u(X_{t \wedge \tau_D}) + B(t \wedge \tau_D)$$

$$+ \int_0^{t \wedge \tau_D} u(X_s) dA_s - \sum_{0 < s \leq t \wedge \tau_D} u(X_s) \left( 1 - e^{F(X_s, X_s)} \right)$$

is a local martingale with respect to $P^x$ for each $x \in D$. Now, for any $t > 0$,

$$N_t \leq B(\tau_D)$$

$$+ \|u\|_\infty \left( 1 + A^*(\tau_D) + \sum_{0 < s \leq \tau_D} (e^{F(X_s, X_s)} - 1) \right),$$

and for each $x \in D$,

$$E^x \sum_{0 < s \leq \tau_D} (e^{F(X_s, X_s)} - 1) = E^x \int_0^{\tau_D} \int_{R^d} \frac{e^{F(x, y)} - 1}{|x - y|^{d+\alpha}} dy ds.$$ 

Thus by Lemma 4.1.1, we know that $N_t$ is uniformly $P^x$-integrable for any $x \in D$, and consequently $N_t$ is a uniformly integrable $P^x$-martingale for any $x \in D$. Hence

$$u(x) = E^x u(X_{\tau_D}) + E^x B(\tau_D)$$

$$+ \int_0^{\tau_D} u(X_s) dA_s + E^x \sum_{0 < s \leq \tau_D} u(X_s) G(X_s, X_s)$$

$$= E^x f(X_{\tau_D}) + E^x B(\tau_D)$$

$$+ \int_0^{\tau_D} u(X_s) dA_s + E^x \int_0^{\tau_D} \int_{R^d} \frac{G(X_s, y)u(y)}{|X_s - y|^{d+\alpha}} dy ds.$$
Now it follows from Lemma 4.2.3 that
\[ u_1(x) := E^x B(\tau_D) + E^x \int_0^{\tau_D} u(X_s) dA_s + E^x \int_0^{\tau_D} \int_{R^d} \frac{G(X_s, y)u(y)}{|X_s - y|^{d+\alpha}} dy ds \]
satisfies the following:

(a) \( u_1 \in D_D \) is a bounded continuous function which vanishes outside \( D \);

(b) for any \( v \in D \cap C_{0,0}(D) \),
\[
T(u_1, v) = \int_{R^d} v(x)u(dx) + \int_{R^d} u(x)v(x)\mu(dx)
+ \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)u(y)}{|x - y|^{d+\alpha}} dxdy.
\]

And by Lemma 4.2.1 we know that
\[ u_2(x) = E^x f(X(\tau_D)) \]
satisfies the following:

(c) \( u_2 \in D_e \) is a bounded function which is continuous in \( D \);

(d) \( u_2|_{D^c} = f|_{D^c} \);

(e) for any \( z \in \partial D \),
\[
\lim_{x \to z} u_1(x) = f(z);
\]

(f) for any \( v \in D \cap C_{0,0}(D) \),
\[ T(u_2, v) = 0. \]

Therefore \( u = u_1 + u_2 \) is a continuous solution to the Dirichlet problem of (4.3) with exterior function \( f \).
Now let us assume that \( \overline{u} \) is a continuous solution to the Dirichlet problem of (4.3) with exterior function \( f \). Then by Lemma 4.2.3 we know that the function

\[
\overline{u}_1(x) := E^x B(\tau_D) + E^x \int_0^{\tau_D} \overline{u}(X_s) dA_s
\]

\[
+ E^x \int_0^{\tau_D} \int_{R^d} \frac{G(X_s, y)\overline{u}(y)}{|X_s - y|^{d+\alpha}} dy ds
\]

satisfies the following

(g) \( \overline{u}_1 \in D_D \) is a bounded continuous function which vanishes on \( D_e \);

(h) for any \( v \in D \cap C_{0,0}(D) \),

\[
T(\overline{u}_1, v) = \int_{R^d} v(x)\nu(dx) + \int_{R^d} \overline{u}(x)v(x)\mu(dx)
\]

\[
+ \int_{R^d} \int_{R^d} \frac{v(x)G(x, y)\overline{u}(y)}{|x - y|^{d+\alpha}} dx dy.
\]

Therefore \( \overline{u} - \overline{u}_1 \) is a bounded continuous function in \( D_e \) such that

\[
T(\overline{u} - \overline{u}_1, v) = 0,
\]

for any \( v \in D \cap C_{0,0}(D) \), i.e., \( \overline{u} - \overline{u}_1 \) is harmonic on \( D \). Hence by Lemma (4.2.2) we know that for any \( x \in D \),

\[
(\overline{u} - \overline{u}_1)(x) = E^x(\overline{u} - \overline{u}_1)(X_{\tau_D}) = E^x f(X_{\tau_D}),
\]

consequently

\[
\overline{u}(x) = E^x f(X_{\tau_D}) + E^x B(\tau_D)
\]

\[
+ \int_0^{\tau_D} \overline{u}(X_s) dA_s + E^x \int_0^{\tau_D} \int_{R^d} \frac{G(X_s, y)\overline{u}(y)}{|X_s - y|^{d+\alpha}} dy ds.
\]

From this we can easily get that

\[
Y_t := \overline{u}(X_{t \wedge \tau_D}) + B(t \wedge \tau_D)
\]

\[
+ \int_0^{t \wedge \tau_D} \overline{u}(X_s) dA_s + \sum_{0<s\leq t \wedge \tau_D} \overline{u}(X_s)G(X_s-, X_s)
\]
is a $P^x$-martingale for each $x \in D$. Using the integration by parts formula we get

$$e^{A(t\wedge \tau_D)}\bar{u}(X_{t\wedge \tau_D}) = \bar{u}(X_0) + \int_0^{t\wedge \tau_D} e^{A(s^-)}dY_s - \int_0^{t\wedge \tau_D} e^{At}dB_t.$$ 

Therefore

$$Z_t := e^{A(t\wedge \tau_D)}\bar{u}(X_{t\wedge \tau_D}) + \int_0^{t\wedge \tau_D} e^{At}dB_t$$

is a $P^x$-local martingale for each $x \in D$. It follows from Theorem 4.1.3 and Theorem 4.1.4 that $\{Z_t\}$ is uniformly $P^x$-integrable for each $x \in D$, so $\{Z_t\}$ is a uniformly integrable $P^x$-martingale for each $x \in D$, and consequently

$$\bar{u}(x) = E^x\left\{ e^{A(\tau_D)}\bar{u}(X(\tau_D)) \right\} + E^x\int_0^{\tau_D} e^{A_t}dB_t = \int_0^{\tau_D} e^{A_t}dB_t = u(x),$$

for each $x \in D$. The proof is now complete.

$Q.E.D.$

4.3 The Dirichlet Problem for Semilinear Equations

In this section, $D$, $\mu$, $F$, $G$ and $\nu$ are fixed as before. For convenience, we are going to use $B^+_t$, $B^-_t$ and $B^*_t$ to denote the continuous additive functionals associated with $\nu^+$, $\nu^-$ and $\nu^* = \nu^+ + \nu^-$, respectively.

The purpose of this section is to prove the existence of solutions to the Dirichlet problem for the following semilinear equation

$$(-(-\Delta)^{\frac{d}{2}} + \mu)u(x) + \int_{\mathbb{R}^d} \frac{G(x,y)u(y)}{|x-y|^{d+\alpha}}dy + \xi(u) + \nu = 0 \quad (4.6)$$

on $D$, where $\xi$ is a continuous differentiable function on $\mathbb{R}^1$.

In order to prove the existence of solutions to the Dirichlet problem for (4.6), we need the following result.
Lemma 4.3.1 Suppose that $\bar{\mu} \in K_{d,\alpha}$ is negative and that $\bar{A}_t$ is the continuous additive functional associated with $\bar{\mu}$. If the gauge function $g$ is bounded, then for any $f \in D_e$ which is bounded continuous on $D_e$, the function defined below

$$u(x) = E^x \left( e^{A(\tau_D)} f(X(\tau_D)) \right) + E^x \int_0^{\tau_D} e^{A_t} dB_t$$  \hspace{1cm} (4.7)$$
satisfies the following relation

$$u(x) = E^x \left( e^{A(\tau_D)+\bar{A}(\tau_D)} f(X(\tau_D)) \right) + E^x \int_0^{\tau_D} e^{A_t+\bar{A}_t} dB_t - E^x \int_0^{\tau_D} e^{A_t+\bar{A}_t} u(X_t) d\bar{A}_t.$$  \hspace{1cm} (4.8)$$

Proof. It follows from Theorem 4.2.2 that the function $u$ defined in (4.7) is the unique continuous solution to the Dirichlet problem for (4.3) on $D$ with exterior function $f$. Therefore $u$ is a continuous solution to the Dirichlet problem for the following equation

$$\left( -(-\Delta)_{\alpha}^f + \mu + \bar{\mu} \right) u(x) + \int_{R^d} \frac{G(x,y)u(y)}{|x-y|^{d+\alpha}} dy + (\nu - u(x)\bar{\mu}) = 0$$  \hspace{1cm} (4.9)$$
on $D$ with exterior function $f$. Now applying Theorem 4.2.2 to equation (4.9) we know that $u$ satisfies (4.8).

Q.E.D.

Definition 4.3.1 Suppose $f$ is a bounded function in $D_e$. We say that a bounded function $u$ in $D_e$ is a solution to the Dirichlet problem of (4.6) with exterior function $f$ if the following three conditions are satisfied:

(i) for any $v \in D \cap C_{0,0}(D)$,

$$T(u,v) = \int_{R^d} u(x)v(x)\mu(dx) - \int_{R^d} \xi(u(x))v(x) dx$$

$$= \int_{R^d} \int_{R^d} \frac{v(x)G(x,y)u(y)}{|x-y|^{d+\alpha}} dx dy + \int_{R^d} v(x)\nu(dx);$$
\((\text{ii}) \text{ } u|_{D^c} = f|_{D^c};\)

\((\text{iii}) \text{ for every } z \in \partial D,\)

\[
\lim_{D^c \to z} u(x) = f(z).
\]

**Theorem 4.3.1** If the gauge function \(g\) is bounded and if \(\xi\) satisfies

\[
s\xi(s) \geq 0, \quad \forall s \in R^1,
\]

then for any \(f \in D_e\) which is continuous on \(D^c\), the Dirichlet problem for (4.6) on \(D\) with exterior function \(f\) has at least one continuous solution.

**Proof.** For any \(r \geq 0\), define

\[
\rho(r) = 1 + \sup\{|s'|(s)|: -r < s < r\}.
\]

Then \(\rho\) is a continuous function on \([0, \infty)\) such that for any \(r > 0\), \(\rho(r)t - \xi(t)\) is increasing as a function of \(t\) on \([-r, r]\).

For any \(x \in R^d\), define

\[
v(x) = E^x \{e^{A(\tau_D)}|f|(X(\tau_D))\} + E^x \int_0^{\tau_D} e^{A_i} dB_i^x;
\]

\[
\psi(x) = \rho(v(x)).
\]

Then \(v\) is bounded on \(R^d\), hence \(\psi\) is bounded on \(R^d\). Furthermore, for any \(x\), \(\psi(x)t - \xi(t)\) is increasing on \([-v(x), v(x)]\) as a function of \(t\).

Define a measure on \(R^d\) as follows:

\[
\bar{\mu}(dx) = -\psi(x)dx.
\]

Then \(\bar{\mu} \in K_{d,\alpha}\) is a negative Radon measure and
\[ \overline{A}_t = - \int_0^t \psi(X_s) ds \]

is the continuous additive functional associated with \( \overline{\mu} \).

For any \( x \in \mathbb{R}^d \), define

\[
    u_0(x) = E^x \left( e^{A(\tau_D)} f^+(X(\tau_D)) + \int_0^{\tau_D} e^{A_t} dB_t^+ \right),
\]
\[
    w_0(x) = -E^x \left( e^{A(\tau_D)} f^- (X(\tau_D)) + \int_0^{\tau_D} e^{A_t} dB_t^- \right).
\]

Then \(-v \leq w_0 \leq u_0 \leq v\). From Lemma 4.3.1 we can get that

\[
    u_0(x) = E^x \left( e^{A(\tau_D)} f^+(X(\tau_D)) + \int_0^{\tau_D} e^{A_t} (dB_t^+ + (\psi u_0)(X_t) dt) \right),
\]
\[
    w_0(x) = -E^x \left( e^{A(\tau_D)} f^- (X(\tau_D)) + \int_0^{\tau_D} e^{A_t} (dB_t^- - (\psi w_0)(X_t) dt) \right),
\]

where \( \Lambda_t = A_t + \overline{A}_t \).

For any function \( h \) on \( \mathbb{R}^d \), we can define a new function \( H(h) \) on \( \mathbb{R}^d \) as follows:

\[
    H(h)(x) = \psi(x)h(x) - \xi(h(x)), \quad \forall x \in \mathbb{R}^d.
\]

Now if we define

\[
    u_1(x) = E^x \left( e^{A(\tau_D)} f(X(\tau_D)) + \int_0^{\tau_D} e^{A_t} (dB_t + H(u_0)(X_t) dt) \right),
\]
\[
    w_1(x) = E^x \left( e^{A(\tau_D)} f(X(\tau_D)) + \int_0^{\tau_D} e^{A_t} (dB_t + H(w_0)(X_t) dt) \right),
\]

then

\[
    u_0(x) - u_1(x) = E^x \left( e^{A(\tau_D)} f^-(X(\tau_D)) + \int_0^{\tau_D} e^{A_t} (dB_t^- + \xi(u_0)(X_t) dt) \right) \geq 0,
\]
\[
    w_0(x) - w_1(x) = -E^x \left( e^{A(\tau_D)} f^+(X(\tau_D)) + \int_0^{\tau_D} e^{A_t} (dB_t^+ + \xi(w_0)(X_t) dt) \right) \leq 0,
\]
\[
    u_1(x) - w_1(x) = E^x \int_0^{\tau_D} e^{A_t} (H(u_0) - H(w_0))(X_t) dt \geq 0.
\]
Therefore $u_1$ and $w_1$ satisfy the following relations:

$$v \geq u_0 \geq u_1 \geq w_1 \geq w_0 \geq -v.$$ 

For any $x \in \mathbb{R}^d$, define

$$u_2(x) = E^x \left( e^{A(T_D)} f(X(T_D)) + \int_0^{T_D} e^{A_t}(dB_t + H(u_1)(X_t)) dt \right),$$

$$w_2(x) = E^x \left( e^{A(T_D)} f(X(T_D)) + \int_0^{T_D} e^{A_t}(dB_t + H(w_1)(X_t)) dt \right),$$

then

$$u_2(x) - u_1(x) = E^x \int_0^{T_D} e^{A_t}(H(u_1) - H(u_0))(X_t) dt \leq 0,$$

$$w_2(x) - w_1(x) = E^x \int_0^{T_D} e^{A_t}(H(w_1) - H(w_0))(X_t) dt \geq 0,$$

$$u_2(x) - w_2(x) = E^x \int_0^{T_D} e^{A_t}(H(u_1) - H(w_1))(X_t) dt \geq 0.$$

Thus $u_2$ and $w_2$ satisfy the following relations:

$$v \geq u_0 \geq u_1 \geq u_2 \geq w_2 \geq w_1 \geq w_0 \geq -v.$$

By induction, we can define two sequences $\{u_n\}$ and $\{w_n\}$ of functions on $\mathbb{R}^d$ such that

(a) for any $n \geq 0$,

$$v \geq u_n \geq u_{n+1} \geq w_{n+1} \geq w_n \geq -v.$$

(b) for any $x \in \mathbb{R}^d$,

$$u_{n+1}(x) = E^x \left( e^{A(T_D)} f(X(T_D)) + \int_0^{T_D} e^{A_t}(dB_t + H(u_n)(X_t)) dt \right),$$

$$w_{n+1}(x) = E^x \left( e^{A(T_D)} f(X(T_D)) + \int_0^{T_D} e^{A_t}(dB_t + H(w_n)(X_t)) dt \right).$$
Using the monotone convergence theorem we can get two functions $u$ and $w$ such that

$$
\begin{align*}
  u(x) &= E^x \left( e^{\lambda(x_D)} f(X(x_D)) + \int_0^{x_D} e^{\lambda(t)} (dB_t + H(u)(X_t) dt) \right), \\
  w(x) &= E^x \left( e^{\lambda(x_D)} f(X(x_D)) + \int_0^{x_D} e^{\lambda(t)} (dB_t + H(w)(X_t) dt) \right).
\end{align*}
$$

Therefore by Theorem 4.2.2 we know that $u$ and $w$ are continuous solutions to the Dirichlet problem of (4.6) on $D$ with the exterior function $f$.

Q.E.D.

Similar to Theorem 4.3.1, we can get the following two results.

**Theorem 4.3.2** If the gauge function $g$ is bounded and if $\xi$ satisfies

(i) $\xi(0) = 0$;

(ii) for any $s \in (0, \infty)$, $\xi(s) \geq 0$,

then for any $f \in D_c$ which is continuous on $D^c$, the Dirichlet problem for (4.6) on $D$ with exterior function $f$ has at least one continuous solution.

**Theorem 4.3.3** If the gauge function $g$ is bounded and if $\xi$ satisfies

(i) $\xi(0) = 0$;

(ii) for any $s \in (-\infty, 0)$, $\xi(s) \leq 0$,

then for any $f \in D_c$ which is continuous on $D^c$, the Dirichlet problem for (4.6) on $D$ with exterior function $f$ has at least one continuous solution.
CHAPTER 5
CONCLUSIONS

Let \( X = (X_t, P^x) \) be the symmetric stable process of index \( \alpha \), \( 0 < \alpha < 2 \) on \( \mathbb{R}^d \) \((d \geq 2)\), let \( \mu \) and \( \nu \) be Radon measures on \( \mathbb{R}^d \) belonging to the Kato class \( K_{d,\alpha} \) (see Definition 3.1.1) and let \( F \) be a Borel function on \( \mathbb{R}^d \times \mathbb{R}^d \) belonging to \( A_{d,\alpha} \) (see Definition 3.1.3). Suppose that \( A_t^\mu \) and \( B_t \) are the continuous additive functionals with Revuz measures \( \mu \) and \( \nu \), respectively, and that

\[
A_t = A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s).
\]

The topic of this thesis is to study the perturbation of the symmetric stable process \( X_t \) by the multiplicative functional:

\[
e^{A_t}
\]

and the results can be divided into two groups. The first group consists of the results given in Chapter 3, while the second group consists of the results in Chapter 4.

The results of the first group are related to the semigroup defined by

\[
T_t f(x) = E^x \{ e^{A_t} f(X_t) \}.
\]

First we studied the properties of the semigroup \((T_t)_{t \geq 0}\) in great detail. More precisely, we proved the following:

1. If \( 1 \leq p \leq p' \leq \infty \), then for any \( t > 0 \), \( T_t \) is a bounded operator from \( L^p(\mathbb{R}^d) \) into \( L^{p'}(\mathbb{R}^d) \);

2. For any \( p \in [1, \infty) \), \((T_t)_{t \geq 0}\) is a strongly continuous semigroup on \( L^p(\mathbb{R}^d) \);
(3) For any \( t > 0 \), \( T_t \) maps \( L^\infty(\mathbb{R}^d) \) into \( bC(\mathbb{R}^d) \);

(4) For any \( t > 0 \) and any \( p \in [1, \infty) \), \( T_t \) maps \( L^p(\mathbb{R}^d) \) into \( C_0(\mathbb{R}^d) \);

(5) \( (T_t)_{t \geq 0} \) is a strongly continuous semigroup on \( C_0(\mathbb{R}^d) \).

Then we identified the bilinear form corresponding to \( (T_t)_{t \geq 0} \).

Let \( D \) be a regular domain in \( \mathbb{R}^d \). The results of the second group are related to the gauge function defined by

\[
g(x) = E^x \{ e^{A(\tau_D)} \}.
\]

First, we proved a criterion, called the gauge theorem, about the boundedness of \( g \). Then we applied the gauge theorem to prove that for a suitable exterior function \( f \),

\[
u(x) = E^x \left( e^{A(\tau_D)} f(X(\tau_D)) \right) + E^x \int_0^{\tau_D} e^{At} dB_t
\]

is the unique continuous solution to the Dirichlet problem of

\[
(-(-\Delta)^{\alpha/2} + \mu) \nu(x) + \int_{\mathbb{R}^d} \frac{G(x, y)u(y)u(y)}{|x - y|^{d+\alpha}} dy + \nu = 0,
\]
on \( D \), where \( G = e^{F} - 1 \). Finally, we proved the existence of a solution to the Dirichlet problem for the semilinear equation

\[
(-(-\Delta)^{\alpha/2} + \mu) \nu(x) + \int_{\mathbb{R}^d} \frac{G(x, y)u(y)u(y)}{|x - y|^{d+\alpha}} dy + \xi(u) + \nu = 0
\]
on \( D \), where \( \xi \) is a continuous differentiable function on \( \mathbb{R}^1 \).
REFERENCES


BIOGRAPHICAL SKETCH

Renming Song was born in Linzhang, Hebei Province, China, on April 6, 1963. He entered the Mathematics Department of Hebei University in September 1979, received his B. S. degree in July 1983 and his M. S. degree in July 1986. Upon receiving his master's degree, he joined the faculty of the Mathematics Department of Hebei University and he was promoted to lecturer one year later.

In August 1988, he came to the United States and enrolled at the Mathematics Department of the University of Florida as a graduate student to pursue his doctor's degree. Since then he has been working as a teaching assistant at the Mathematics Department of the University of Florida.

He is married to Junge Guo and they have a daughter, Linda Song.
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Joseph Glover, Chairman
Professor of Mathematics

Murali Rao
Professor of Mathematics

Zoran R. Pop-Stojanović
Professor of Mathematics
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Nicolae Dinculeanu
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Malay Ghosh
Professor of Statistics

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Dean, Graduate School