NON-REVISITING PATHS AND CYCLES IN POLYHEDRAL MAPS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1995
ACKNOWLEDGEMENTS

First and foremost, I would like to thank Professor Vince for his invaluable help and advice during the preparation of this dissertation. He has spent countless hours helping me understand the subtleties of my work. Sifting through some of the first drafts must have been quite painful, and for this, I remain deeply appreciative of him. Also, I would like to thank Professors Alladi, Davis, Mair, and White for taking the time to serve on my supervisory committee.

Finally, I would like to dedicate this work to my parents and Cynthia. Without their moral support, this endeavor would be impossible and more importantly, meaningless.
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NOTATION

\( G \) – A finite graph.
\( V(G) \) – The vertex set of \( G \).
\( E(G) \) – The edge set of \( G \).
\( S \) – A surface.
\( \chi \) – The Euler Characteristic of a surface.
\( \partial S \) – The boundary of the surface \( S \).
\( S_g \) – A surface homeomorphic to the connected sum of \( g \) tori.
\( N_k \) – A surface homeomorphic to the connected sum of \( k \) projective planes.
\( M \) – A polyhedral map.
\( M^* \) – The dual polyhedral map.
\( P \) – A polygonal representation of a polyhedral map.
\( T_P \) – The type of a polygonal representation \( P \).
In this dissertation, three problems are considered. The first problem is related to one of the most famous unsolved problems in the combinatorial theory of polytopes called the \textit{Hirsch conjecture}, which proposes a bound on the diameter of the graph of a polytope. An equivalent conjecture due to Klee and Wolfe, called the \textit{non-revisiting path conjecture}, asserts that any two vertices of a polytope can be joined by a path that does not revisit a facet. The \textit{non-revisiting path conjecture} can be extended to cell complexes that are more general than those that are the boundary complexes of polytopes. In this regard, the \textit{non-revisiting path conjecture} is known to be true for polyhedral maps on the 2—sphere, projective plane, torus, and Klein bottle. In this research, an elementary, unified proof of the validity of the \textit{non-revisiting path conjecture} for polyhedral maps on the projective plane, torus, and Klein bottle is given. In addition, it is shown that for polyhedral maps, the \textit{non-revisiting path conjecture} is false for all other surfaces except possibly surfaces homeomorphic to the connected sum of three projective planes.
The second problem concerns the existence of non-planar, non-revisiting cycles in a polyhedral map on a surface. By results due to Barnette, it is known that every polyhedral map on the projective plane, torus, and Klein bottle contains a non-planar, non-revisiting cycle. In this regard, the notion of a *polygonal representation* of a polyhedral map is introduced. This is analogous to the notion of representing a surface as a polygon in the plane with the directed sides of the polygon matched in pairs, except in this case, the representation preserves the combinatorial structure of the underlying graph of the polyhedral map. Some properties of polygonal representations are proved. As an application, an elementary, unified proof of results due to Barnette involving the existence of non-planar, non-revisiting cycles on the projective plane, torus, and Klein bottle is given.

The third problem is a graph-coloring conjecture that is shown to be true for all planar graphs and \( K_{3,3} \). As an application, it is shown that any polyhedral map on a surface homeomorphic to the connected sum of three projective planes, or the connected sum of two tori, that has a *non-separating polygonal representation*, contains a non-planar, non-revisiting cycle. This extends Barnette’s result stated in the second problem.
A *polyhedron* is the intersection of a finite collection of closed half-spaces in \( n \)-dimensional Euclidean space, and a *polytope* is a bounded polyhedron. Equivalently, a *polytope* is the convex hull of a finite set of points in Euclidean space. If a polytope is \( d \)-dimensional, then we say that it is a *d-polytope*. A *face* of a polytope \( P \) is \( \emptyset \), \( P \) itself, or the intersection of \( P \) with a supporting hyperplane. With prefixes denoting dimension, the 0-, 1-, and \((d - 1)\)-dimensional faces of a \( d \)--polytope are called *vertices*, *edges*, and *facets*. A \( d \)--polytope \( P \) is *simple* if each of its vertices are incident to precisely \( d \) edges, or equivalently, to \( d \) facets. A *d-simplex* is the convex hull of \( d + 1 \) affinely independent points, and a polytope is called *simplicial* if each of its facets is a simplex. There is a duality between the notions of *simple* and *simplicial* polytopes. That is to say, there is a bijection between the set of simple \( d \)--polytopes with \( n \) vertices and the set of simplicial \( d \)--polytopes with \( n \) facets that preserves incidences and complements dimensions.

A graph \( G \) is a finite non-empty set of objects called *vertices* together with a (possibly empty) set of unordered pairs of distinct vertices called *edges*. The *vertex set* of \( G \) is denoted by \( V(G) \), while the *edge set* of \( G \) is denoted by \( E(G) \). A graph is \( d \)--connected if the removal of fewer than \( d \) vertices yields neither a disconnected graph nor the trivial graph. A *cut-vertex* of a graph is a vertex whose removal disconnects the graph. Thus a graph is 2-connected if and only if it has no cut-vertices. The
graph of a polytope $P$ is the one-dimensional skeleton of $P$. In particular, a theorem of Steinitz and Rademacher [20] states that a graph is (isomorphic to) the graph of a 3-polytope if and only if it is planar and 3-connected. By a generalization due to Balinski [2], the graph of a $d$-polytope is $d$-connected. A directed graph consists of a set of vertices and a set of ordered pairs of distinct vertices called directed edges. A directed edge $(u, v)$ is represented by an edge with endpoints $u$ and $v$ and an arrowhead pointing towards $v$ denoting the “direction” of the directed edge $(u, v)$.

A surface $S$ is a connected 2-dimensional manifold, possibly with boundary $\partial S$. There are two kinds of closed surfaces, orientable and non-orientable. The 2-sphere, torus, double torus, and so on are orientable while the projective plane, Klein bottle and so on are non-orientable. It is well known that any orientable surface may be obtained by attaching a suitable number of handles to the sphere, while any non-orientable surface may be obtained by attaching a suitable number of Möbius bands to the sphere. An orientable surface denoted by $S_g$ is said have genus $g$, if one must add $g$ handles to the sphere to obtain its homeomorphism type. On the other hand, a non-orientable surface denoted by $N_k$ is said to have crosscap number $k$, if one must attach $k$ Möbius bands to the sphere to obtain its homeomorphism type. If $S_1$ and $S_2$ are surfaces without boundary, then their connected sum is the surface obtained by removing the interior of a disk from $S_1$ and $S_2$ and then identifying the resulting boundary components. Thus, the surface $S_g$ is homeomorphic to the connected sum of $g$ tori, while the surface $N_k$ is homeomorphic to the connected sum of $k$ projective planes. Let $G$ be a connected graph embedded on a surface $S$ such that $G \cap \partial S = \partial S$. Then the pair $(G, S)$ is called a map on $S$ and is denoted by $M$. The vertices and edges of $M$ are those of $G$, and the faces of $M$ are the closures of the connected regions in the complement of $G$ on $S$. If $G$ is embedded on the plane, then the map is called a planar map. A planar map has exactly one unbounded face. If $M = (G, S)$ is a map
on a surface without boundary, then the dual map of $M$, denoted by $M^*$ is defined as follows: for each face $f$ of $M$, place a vertex $f^*$ in its interior. Then, for each edge $e$ in $G$, draw an edge $e^*$ between the vertices just placed in the interiors of the faces containing the edge $e$. The resulting graph with vertices $f^*$ and edges $e^*$ is called the dual graph of $G$, denoted by $G^*$, and the resulting map $(G^*, S)$ is the dual map of $M$. On the other hand, if $S$ has boundary $\partial S$, then define the dual map as follows: for each face $f$ of $M$, place a vertex $f^*$ in its interior. Then, for each edge $e$ in $G$ not belonging to $\partial S$, draw an edge $e^*$ between the vertices just placed in the interiors of the faces containing the edge $e$. The resulting graph $G^*$ is called the dual graph of $G$ and the resulting map is called the dual map of $M$. An important property satisfied by 3-connected graphs embedded on the 2-sphere is that any two faces intersect on a single edge, a single vertex or not at all. Faces that meet in this way are said to meet properly. If all the faces are simply-connected and all faces meet properly, then the map $M$ is called a polyhedral map on the surface. A consequence of all faces meeting properly is that every vertex of a polyhedral map has degree at least three. By a result of Barnette [4], every polyhedral map is 3-connected, generalizing Steinitz’s Theorem.

1.2 Some History and Motivation

The feasible region of any non-empty linear programming problem is a polyhedron, and conversely, given a polyhedron $P$, it is always possible to construct a linear program with $P$ as its feasible region. Edge-following algorithms, like the Simplex algorithm, start with a vertex of the feasible region and traverse along successive edges of the region according to some prescribed rule, until an optimum vertex is
reached. The $d$-step conjecture and its relatives (including the Hirsch conjecture) play a crucial role in the study of the computational complexity of such edge-following algorithms. The $d$-step conjecture, formulated by W. M. Hirsch in 1957 and reported in 1963 by Dantzig in his book *Linear Programming And Extensions* [9], has several equivalent forms. One version, dealing with the maximum diameter $\Delta(d, n)$ of (the graphs of) $d$-dimensional polytopes with $n$ facets, asserts that $\Delta(d, 2d) = d$ for all $d$ while the Hirsch conjecture asserts that $\Delta(d, n) \leq n - d$ for all $n > d \geq 2$. It was proved by Klee and Walkup [16] that the $d$-step and the Hirsch conjectures are equivalent, though not necessarily on a dimension to dimension basis. The distance, $\delta_P(u, v)$ between two vertices $u$ and $v$ of a polytope $P$ is a lower bound on the complexity of applying an edge-following algorithm to $P$ with initial vertex $u$ and target vertex $v$. Thus $\Delta(d, n)$ is a lower bound for the worst-case behaviour of edge-following LP algorithms over all $d$-polytopes with $n$ facets. Since this applies to all edge-following algorithms, $\Delta(d, n)$ estimates the worst possible behaviour of the best possible edge-following algorithm. The $d$-step and Hirsch conjectures remain unsettled, though they have been proved in many special cases, and counterexamples have been found for slightly stronger conjectures. Specifically, the $d$-step conjecture has been proved for $d \leq 5$. Although sharper results are known for small values of $d$ and $n - d$, the best known general bounds for $\Delta(d, n)$ are due to Adler [1] and Kalai and Kleitman [12], respectively. They are as follows:

$$|n - d - \left(\frac{n - d}{\frac{5d}{4}}\right)| + 1 \leq \Delta(d, n) \leq n^{\log d + 2}.$$ 

It is generally believed that the $d$-step and Hirsch conjectures are false. However, finding counterexamples to that effect would be merely a small first step in the line of investigation related to the two conjectures. For the recent status of the conjectures
1.3 Non-Revisiting Paths and Cycles

If $\Gamma$ is a path in a polyhedral map $M$, a revisit of $\Gamma$ to a face $F$ is a pair of vertices $(x, y)$ such that $\Gamma[x, y] \cap F = \{x, y\}$ where $\Gamma[x, y]$ is the path along $\Gamma$ from $x$ to $y$. Let $(x, y)$ be a revisit of a path $\Gamma$ to a face $F$. If the two paths along $F$ from $x$ to $y$ are denoted as $F[x, y]$ and $\tilde{F}[x, y]$, then the revisit $(x, y)$ is said to be planar if either $F[x, y] \cup \Gamma[x, y]$ or $\tilde{F}[x, y] \cup \Gamma[x, y]$ bounds a cell on the surface. (Note that if one does then so does the other.). A path is non-revisiting if it has no revisits.

In research on the $d$-step and Hirsch conjectures, it has been found that the conjectures can be stated in several equivalent forms (even though no solution to any one of them seems to be in sight!). One equivalent formulation is in terms of the existence of non-revisiting paths in the graphs of convex polytopes. Part of this research is related to the following non-revisiting path conjecture of Klee and Wolfe (also called the $W_v$ conjecture): Any two vertices of a polytope $P$ can be joined by a path that does not revisit any facet of $P$. Despite an apparent greater strength of this conjecture (which prompted its original formulation), it is known [15] that the non-revisiting path conjecture is equivalent to the Hirsch conjecture.

If $P$ is a 3-polytope, then the faces of $P$ form a polyhedral map on the 2-sphere and the validity of the non-revisiting path conjecture along with some strengthened forms of the non-revisiting path conjecture are proved [3,14,15]. Klee [13] conjectured that the non-revisiting path conjecture might be true for cell complexes that are more general than the boundary complexes of convex polytopes. In this regard, Larman
[18] has shown that the conjecture is false for a very general type of 2-dimensional complex. Mani and Walkup [19] have shown that the conjecture is false for 3-spheres. Barnette [5,7] has recently shown that the non-revisiting path conjecture is indeed true for polyhedral maps on cell complexes that are homeomorphic to the projective plane and the torus. Engelhardt [10] has shown in her Ph.D. dissertation that the non-revisiting conjecture is also true for polyhedral maps on the Klein bottle. In a recent paper [8], Barnette gives counterexamples to the non-revisiting path conjecture that are polyhedral maps on the surfaces \( S_8 \) and \( N_{16} \).

Similar to the notion of a path in a polyhedral map having a disconnected intersection with a face of the polyhedral map, one may consider a cycle in the underlying graph of a polyhedral map that has a disconnected intersection with a face of the polyhedral map. A cycle of a polyhedral map refers to a cycle in the underlying graph of the polyhedral map. Let \( M = (G, S) \) be a polyhedral map and \( C \) be a cycle in \( M \). Then \( C \) is said to be non-planar if it does not bound a cell on \( S \). Suppose \( R[s, t] \) is a path along \( C \) from \( s \) to \( t \) such that for some face \( F \), \( s \) and \( t \) are on \( F \), and \( R[s, t] \) along with either path along \( F \) from \( s \) to \( t \) bounds a cell on \( S \). Then the path \( R[s, t] \) is called a planar revisit of \( C \) to the face \( F \). A cycle is non-revisiting if it does not have any revisits; in other words, for each face \( F \) of \( M \), \( C \cap F \) is either empty, or connected. According to a theorem due to Barnette [6], if \( M \) has a non-planar cycle all of whose revisits are planar, then \( M \) has a non-planar, non-revisiting cycle. It is also known that every polyhedral map on a projective plane, torus, or Klein bottle has a non-planar, non-revisiting cycle [6]. However, the problem of the existence of such cycles on other surfaces is still open.

Using the result for the three surfaces mentioned above, Barnette [6] proves that every polyhedral map on the torus is the union of two face-disjoint subcomplexes
that are annuli. Similar decomposition theorems are proved for the projective plane and Klein bottle.

1.4 A Summary of the Research

Primarily, three problems are considered in this dissertation. The first problem is related to the non-revisiting path conjecture due to Klee and Wolfe. In its original formulation, the non-revisiting path conjecture was in the context of convex polytopes. A generalization due to Klee [13] of the conjecture led to the problem of the existence of non-revisiting paths between any two vertices of a polyhedral map on a surface. Chapter 2 deals with this question. Specifically, in Section 2.1, an elementary, unified proof of the non-revisiting path conjecture for polyhedral maps on the projective plane, torus, and Klein bottle is given. Although these results are already known, the earlier proofs for the three surfaces are quite different. The proof given here uses a result due to Barnette, which states that any polyhedral map on the projective plane, torus, and Klein bottle has a non-planar, non-revisiting cycle. Furthermore, in the case of the torus and Klein bottle, cutting along this non-revisiting cycle yields an annulus. Hence in the case of the projective plane, the polyhedral map is cut along a non-revisiting cycle yielding a cell whose boundary corresponds to the non-revisiting cycle and the arguments presented pertain to the cell thus obtained. In the case of the torus and Klein bottle, the arguments pertain to an annulus whose bounding cycles correspond to the non-revisiting cycle in the polyhedral map. The unification of the proofs for the three surfaces is obtained by considering the same basic cases for all three surfaces, namely either both vertices lie on the non-revisiting cycle, one lies on the non-revisiting cycle and one does not, or neither of the two vertices lie on
the non-revisiting cycle. The proof also utilizes an important lemma due to Barnette which states that a path with only planar revisits can be modified to a non-revisiting path. This result plays a key role even in the earlier proofs. In Section 2.1, a simpler proof of this lemma is given. Section 2.2 deals with the non-revisiting path conjecture for the other surfaces. In Engelhardt’s dissertation, a proof of the validity of the non-revisiting path conjecture for the surface $S_2$ is given. In Section 2.2, it is shown that this is impossible! In fact, it is shown that the non-revisiting path conjecture is false for polyhedral maps on the surfaces $S_g, g \geq 2$, and $N_k, k \geq 4$. Thus, the non-revisiting path conjecture for polyhedral maps is settled for all surfaces except $N_3$.

The second problem concerns the existence of a non-planar, non-revisiting cycle in a polyhedral map. As stated earlier, the only surfaces that are known to contain such cycles are the projective plane, torus, and Klein bottle [6]. In this context, in Section 3.1, the notion of a polygonal representation of a polyhedral map is defined. This is analogous to that of representing a surface as a polygon whose sides are directed and matched in pairs. Except, in the case of a polygonal representation of a polyhedral map, the representation preserves the combinatorial structure of the underlying graph of the polyhedral map. In other words, a polygonal representation is a polyhedral map on a closed disc with certain matching conditions on the edges of the polyhedral map that lie on the boundary of the disc. It is shown that every polyhedral map has a polygonal representation. Next, the notion of a non-separating polygonal representation of a polyhedral map is defined. As will be evident from its definition, the existence of such a representation is a rather desirable property of a polyhedral map. An interesting question is: Which polyhedral maps have a non-separating polygonal representation? In this regard, it is shown that there exist an infinite family of polyhedral maps that do not possess a non-separating polygonal representation. Elementary Euler Characteristic arguments allow the enumeration of
all the types of polygonal representations of polyhedral maps on the projective plane, torus, and Klein bottle. In Section 3.2, an elementary, unified proof of Barnette’s result [6] on non-revisiting cycles on the projective plane, torus, and Klein bottle for polyhedral maps that possess a non-separating polygonal representation is given.

Motivated by the second problem, a graph-coloring conjecture is proposed in Section 3.3. This is the third problem considered. It is shown that if the graph-coloring conjecture is true in a special case, then every polyhedral map that has a non-separating polygonal representation, in fact, has a non-planar, non-revisiting cycle. It is shown that the graph-coloring conjecture is true for all graphs that contain a triangle, all planar graphs, and $K_{3,3}$. As a consequence, it follows that every polyhedral map on a surface homeomorphic to $N_k, k = 1, 2, 3$, and $S_g, g = 1, 2$ that has a non-separating polygonal representation, contains a non-planar, non-revisiting cycle. This extends Barnette’s result [6] to this class of polyhedral maps on the surfaces $N_3$ and $S_2$. 
This chapter deals with the non-revisiting path conjecture for polyhedral maps. In Section 2.1, the non-revisiting path conjecture is shown to be true for polyhedral maps on the projective plane, torus, and Klein bottle. Although these results are already known, the earlier proofs due to Barnette [5,7] and Engelhardt [10], are different. In Section 2.1, a simpler, unified proof for all three surfaces is provided. In the case of the projective plane, by a result due to Barnette [6], the surface is cut along a non-planar, non-revisiting cycle to yield a cell and the proof consists of the considering the following three cases:

1. Both vertices involved lie on the boundary of the cell.
2. One vertex lies on the boundary of the cell while the other lies in the interior of the cell.
3. Both vertices lie in the interior of the cell.

In the case of the torus and Klein bottle, the surface is cut along a non-revisiting cycle in the polyhedral map yielding an annulus. The unification of the proof for all three surfaces is achieved by considering the same three cases stated above.

In Section 2.2, the non-revisiting path conjecture is settled for polyhedral maps on all the remaining surfaces except $N_3$, the connected sum of three copies of the projective plane (or equivalently, the connected sum of the torus and the projective plane, or the Klein bottle and the projective plane). Specifically, it is shown that the non-revisiting path conjecture is false for all the remaining surfaces, except possibly $N_3$ and counter-examples are provided to this effect.
2.1 Non-Revisiting Paths on the Projective Plane, Torus, and Klein Bottle

Although the proof of the validity of the *non-revisiting path* conjecture for polyhedral maps on the projective plane, torus, and Klein bottle given here has many details, the ideas involved are quite elementary.

First, a simpler proof of an important lemma originally due to Barnette [5] is presented.

**Lemma 2.1.1** Let $M$ be a polyhedral map with vertices $u$ and $v$. If there is a path in $M$ joining $u$ and $v$ all of whose revisits are planar, then there is a non-revisiting path between $u$ and $v$.

**Proof.** Let $\Gamma[u, v]$ be a path in $M$ all of whose revisits are planar. If $\Gamma[u, v]$ is not a non-revisiting path, then there is a vertex $x$ on $\Gamma[u, v]$ with the following properties:

(1) There is a non-revisiting path $\Gamma_0[u, x]$ between $u$ and $x$.

(2) The path $\Gamma_0[u, x] \cup \Gamma[x, v]$ has only planar revisits.

A path satisfying (1) and (2) exists; simply take $x = u$.

(3) Among all choices for $x$ satisfying (1) and (2), choose the one which is furthest along the path $\Gamma[u, v]$.

![Figure 1](image-url). A simpler proof of a result due to Barnette.
If \( x = v \), we are done, otherwise we will obtain a contradiction. Let \((z, y)\) be a revisit of the path \( \Gamma_0[u, x] \cup \Gamma[x, v] \) to a face \( F \) of \( M \). By statement \((3)\), \( z \in \Gamma_0[u, x] \) and \( y \in \Gamma[x, v] \). Among all revisits by this path we choose \( F \) so that \( z \) is nearest to \( u \) along \( \Gamma_0[u, x] \). Now consider the path \( \Gamma_1 = \Gamma_0[u, z] \cup F[z, y] \cup \Gamma[y, v] \) from \( u \) to \( v \) (indicated by the dotted path in Figure 1) and observe the following:

(i) \( \Gamma_1 \) is a path from \( u \) to \( v \) all of whose revisits are planar. To see this note that \( \Gamma_0 \) itself has no revisits. A revisit involving vertices of \( \Gamma[y, v] \) alone has to be planar since \( \Gamma \) has only planar revisits. A non-planar revisit by \( \Gamma_1 \) cannot involve vertices of \( F[z, y] \) since the closed path \( F[z, y] \cup \Gamma[y, x] \cup \Gamma_0[x, z] \) bounds a cell. Finally, if a revisit by \( \Gamma_1 \) involves a vertex of \( \Gamma \) and a vertex of \( \Gamma_0 \), then it must be planar since \( \Gamma_0[u, x] \cup \Gamma[x, v] \) admits only planar revisits.

(ii) \( \Gamma_1[u, y] \) does not revisit any face of \( M \). A revisit by \( \Gamma_1[u, y] \) to a face \( F_1 \) must involve \( y \) and a vertex \( \bar{z} \) of \( \Gamma_0[u, z] \). Note that \( \bar{z} \neq z \); otherwise \( F \) and \( F_1 \) meet improperly at \( y \) and \( z \). Now \((\bar{z}, y)\) is a revisit of the path \( \Gamma_0[u, x] \cup \Gamma[x, v] \). This contradicts the choice of \( F \) with \( z \) nearest to \( u \) on \( \Gamma_0[u, x] \).

The existence of \( y \) contradicts the choice of \( x \) as the vertex that was furthest along \( \Gamma[u, v] \) satisfying conditions \((1)\) and \((2)\).

\[\text{Lemma 2.1.2.} \text{ Let } S \text{ be a surface with boundary } \partial S \text{ and } M = (G, S) \text{ a polyhedral map on } S \text{ such that the intersection of any face of } M \text{ with } \partial S \text{ is either empty, or connected. Then any two vertices of } M \text{ that lie in the interior of } S \text{ can be joined by path in } M \text{ that is contained in the interior of } S.\]

\textbf{Proof :} Since \( G \) is connected, there is a path \( \Gamma \) from \( u \) to \( v \) in \( M \). If \( \Gamma \) lies in the interior of \( S \), we are done; so assume that \( \Gamma \cap \partial S \neq \emptyset \) and let \( H = \Gamma[x, y] \) be a connected component of \( \Gamma \cap \partial S \) with the order of vertices along \( \Gamma \) being \( u, x, y, v \). Let \( x' \) and \( y' \) be the vertices of \( H \) that are incident to \( x \) and \( y \), respectively. It is possible
that some of \(x, x', y', \) and \(y\) are the same vertex. Denote by \(F_n, ..., F_m, m \geq n\) the faces of \(M\) that meet \(H[x', y']\) but not \(x\) or \(y\). With \(x''\) as the vertex of \(\Gamma\) not on \(H\) that is incident to \(x\), let \(F_1, ..., F_{n-1}\) be the faces of \(M\) that meet \(H\) and lie in the region determined by the sector with central angle \(x''xx'\). Likewise, let \(F_{m+1}, ..., F_N\) be the faces that meet \(H\) and lie in the region determined by the sector with central angle \(y'yy'', \) where \(y''\) is the vertex of \(\Gamma\) not on \(H\) that is incident to \(y\). For \(i = 1, ..., N-1\), let \(x_i\) be the vertex on the edge \(F_i \cap F_{i+1}\) that doesn’t belong to \(H\). Choose \(x_0\) on \(\Gamma[u, x]\) with the property that for some \(s = 1, ..., N\), the face \(F_s\) contains an edge not in \(\Gamma\) that is incident to \(x_0\). Such a vertex exists for otherwise one of the \(F_i\)'s has a disconnected intersection with \(\partial S\), which is a contradiction. Similarly, choose \(y_0\) on \(\Gamma(y, v)\) such that the face \(F_t, t \geq s,\) contains an edge not in \(\Gamma\) that is incident to \(y_0\). Note that \(x_0\) and \(y_0\) must lie in the interior of \(S\) for otherwise \(F_s\) or \(F_t\) has a disconnected intersection with \(\partial S\) (meeting \(\partial S\) at both \(H\) and \(x_0\) or \(y_0\), respectively), which is a contradiction. Construct a path from \(x_0\) to \(y_0\) that doesn’t meet \(\partial S\) as follows:

Since \(x_0\) and \(x_s\) belong to \(F_s\) and lie in the interior of \(S\), there must be a path \(F_s[x_0, x_s]\) from \(x_0\) to \(x_s\) along the face \(F_s\) that avoids \(\partial S\), for otherwise \(F_s \cap \partial S\) is disconnected. For \(k = s, ..., t - 1\), the vertices \(x_k\) and \(x_{k+1}\) lie on the face \(F_{k+1}\) and are in the interior of \(S\). Hence by the argument above, for each \(k\), there is a path \(F_{k+1}[x_k, x_{k+1}]\) along \(F_{k+1}\) that lies in the interior of \(S\). And let \(F_t[x_{t-1}, y_0]\) be the interior path from \(x_{t-1}\) to \(y_0\) along \(F_t\). By construction, 

\[
I = \left[ F_s[x_0, x_s] \cup \bigcup_{k=s}^{t-2} F_{k+1}[x_k, x_{k+1}] \cup F_t[x_{t-1}, y_0] \right] \cap \partial S = \emptyset.
\]

In \(\Gamma\), replace \(\Gamma[x_0, y_0]\) by \(I\). Now there may be repeated vertices on \(\Gamma\). In this case, remove the vertices of \(\Gamma\) that appear between successive occurrences of each repeated vertex, eventually yielding a path \(\Gamma_1\) from \(u\) to \(v\) where \(\Gamma_1 \cap H = \emptyset\). If \(\Gamma_1\) lies in
the interior of $S$, we are done; otherwise perform the same modification as above to a connected component of $\Gamma_1 \cap \partial S$. When it is no longer possible to perform any modifications, the result must be a path from $u$ to $v$ that is contained in the interior of $S$.

The following two lemmas are due to Barnette [6].

**Lemma 2.1.3.** Every polyhedral map on the projective plane has a non-planar, non-revisiting cycle.

**Lemma 2.1.4.** Every polyhedral map $M$ on the torus or Klein bottle contains a non-revisiting cycle $C$ such that cutting $M$ along $C$ yields an annulus.

We are now in a position to state and prove the main result of this section.

**Theorem 2.1.1.** Any two vertices of a polyhedral map $M$ on the projective plane, torus or Klein bottle can be joined by a non-revisiting path.

**Proof:** For each surface, we will show that any two vertices $u$ and $v$ can be joined by a path in $M$ all of whose revisits are planar. Consequently, by Lemma 2.1.1, there is a non-revisiting path joining $u$ and $v$.

First consider the case where $M$ is a polyhedral map on the projective plane. By Lemma 2.1.3, $M$ has a non-revisiting cycle $C$ such that cutting $M$ along $C$ yields a cell $H$ whose boundary corresponds to the cycle $C$. Without loss of generality, consider the following cases:

1. $u$ and $v$ lie on $C$. In this case, either of the two paths along $C$ from $u$ to $v$ must be non-revisiting (since $C$ is non-revisiting).
2. \( u \) lies on \( C \) and \( v \) does not lie on \( C \). Since every vertex of \( M \) has degree at least three, there must be a vertex \( u_1 \) of \( M \) in the interior of the cell \( H \) such that \( uu_1 \) is an edge of \( M \). Since the cycle \( C \) is non-revisiting, the intersection of any face of \( M \) with \( \partial H \) is either empty, or connected. Hence by Lemma 2.1.2, there is a path \( \Gamma_0 \) joining \( u_1 \) and \( v \) in \( M \) that is contained in the interior of \( H \). Define \( \Gamma = \Gamma_0 \cup uu_1 \). Thus \( \Gamma \) is a path joining \( u \) and \( v \) that meets the boundary of \( H \) in only \( u \). If \( \Gamma \) has only planar revisits, we are done by Lemma 2.1.1. It is clear that a non-planar revisit of \( \Gamma \) to a face \( F \) must involve a vertex \( s \) lying on \( \Gamma(u,v) \) and \( u \). Among all non-planar revisits of \( \Gamma \), choose \( F \) so that \( s \) is nearest to \( v \) along \( \Gamma \). Replace \( \Gamma \) by the path \( \Gamma_1 = F[u,s] \cup \Gamma[s,v] \) indicated by the dotted path in Figure 2.

![Figure 2. A non-planar revisit of \( \Gamma \) to \( F \) on the projective plane.](image)
Again, if $\Gamma_1$ has only planar revisits, we are done. On the other hand, if $\Gamma_1$ has a non-planar revisit to a face $F_1$, then it must involve a vertex $s_1$ of $\Gamma_1(s,v)$ and a vertex of $\Gamma_1[u,t]$ as shown in Figure 3.

![Figure 3](image)

**Figure 3.** A non-planar revisit of $\Gamma_1$ to $F_1$ on the projective plane.

Among all choices for $F_1$, choose the one for which $s_1$ is nearest to $v$ along $\Gamma_1$ and let $t_2$ be as shown. Replace $\Gamma_1$ by the path $\Gamma_2 = \Gamma_1[u,t_2] \cup F_1[t_2,s_1] \cup \Gamma_1[s_1,v]$. It can be easily checked that $\Gamma_2$ can have only planar revisits.

3. **Neither $u$ nor $v$ lies on $C$.** In this case, both $u$ and $v$ lie in the interior of the cell $H$. Hence by Lemma 2.1.2, there is a path $\Gamma$ joining $u$ and $v$ that is contained in the
interior of $H$. Such a path can have only planar revisits and by Lemma 2.1.1, we are done.

Next consider the case where $M$ is a polyhedral map on the torus. By Lemma 2.1.4, $M$ has a non-revisiting cycle $C$ such that cutting $M$ along $C$ yields an annulus $A$. Let $C_1$ and $C_2$ be the bounding cycles of $A$ corresponding to $C$ and without loss of generality, consider the following three cases:

1. $u$ and $v$ lie on $C$. In this case, either of the two paths along $C$ from $u$ to $v$ is non-revisiting (since $C$ is non-revisiting).

2. $u$ lies on $C$ and $v$ does not lie on $C$. As in the case of the projective plane, there is a path $\Gamma$ joining $u$ and $v$ that meets $\partial A$ at only $u$. Without loss of generality, assume that $\Gamma$ meets $C_1$ at only $u$ and avoids $C_2$. If all of $\Gamma$'s revisits are planar, we are done. So assume that $\Gamma$ has a non-planar revisit $(s, t)$ to a face $F$ with the vertex $s$ closer to $v$ along $\Gamma$ than the vertex $t$ is to $v$. Among all non-planar revisits of $\Gamma$ choose $F$ so that $s$ is nearest to $v$ along $\Gamma$. Note that $F$ cannot meet both $C_1$ and $C_2$ in $A$ for this would mean that the cycle $C$ revisits $F$ which is a contradiction to the assumption that $C$ is a non-revisiting cycle. Consider the following two cases:

i. $F$ contains $u$. Up to symmetry, there are two possibilities for $F$ (depending on whether $F$ meets $C_1$ or $C_2$) as shown in Figure 4.
Figure 4. The two possibilities for a non-planar revisit of $\Gamma$ to $F$ where $F$ meets the boundary of the annulus in the case of the torus.

First consider the case in Figure 4a. above. In this case, $t = u$. Let $t_0$ be as shown. Replace $\Gamma$ by the path $\Gamma_1 = F[u, s] \cup \Gamma[s, v]$ indicated by the dotted path in Figure 4a. If $\Gamma_1$ has only planar revisits, then we are done; so assume that $\Gamma_1$ has a non-planar revisit $(s_1, t_1)$ to a face $F_1$. Note that $s_1$ and $t_1$ cannot both lie on $\Gamma_1[u, s]$ since this would mean that $F$ and $F_1$ meet improperly. Hence, without loss of generality, assume that $s_1$ lies on $\Gamma_1(s, v]$ and $t_1$ lies on $\Gamma_1[u, s]$. Among all choices for $F_1$, choose the one for which $s_1$ is nearest to $v$ along $\Gamma_1$. It is easy to see that it suffices to consider the six possibilities for $F_1$ shown in Figure 5.
Figure 5. The six possibilities for a non-planar revisit of $\Gamma_1$ to $F_1$ in the case in Figure 4a.
In the cases in Figures 5a, 5b, and 5c, $\Gamma$ has a non-planar revisit to the face $F_1$ contradicting the choice of $F$ with $s$ nearest to $v$ along $\Gamma$. Consider the case in Figure 5d and let $t_1$ be as shown. Replace $\Gamma_1$ by the path $\Gamma_2 = \Gamma_1[u, t_1] \cup F_1[t_1, s_1] \cup \Gamma_1[s_1, v]$ as shown in Figure 5d. Now $\Gamma_2$ has only planar revisits, Next consider the case in Figure 5e and replace $\Gamma_1$ by the path $\Gamma_2 = F_1[u, s_1] \cup \Gamma_1[s_1, v]$, where $F_1[u, s_1]$ is a path from $u$ to $s_1$ along $F_1$ that meets $\partial A$ at only $u$. Again, it can be checked that $\Gamma_2$ can have only planar revisits. And in the case in Figure 5f, replace $\Gamma_1$ by the path $\Gamma_2 = \Gamma_1[u, t_1] \cup F_1[t_1, s_1] \cup \Gamma_1[s_1, v]$ where $F_1[t_1, s_1]$ is a path along $F_1$ that lies in the interior of $A$. Now the only possibility for a non-planar revisit of $\Gamma_2$ to a face $F_2$ is for it to involve a vertex $s_2$ of $\Gamma_2[t_1, s_1]$ and a vertex $t_2$ of $\Gamma_2[u, t_0]$. Among all such choices for $F_2$, choose the one for which $s_2$ is nearest $v$ along $\Gamma_2$. Now $\Gamma_3 = \Gamma_2[u, t_2] \cup F_2[t_2, s_2] \cup \Gamma_2[s_2, v]$ can have only planar revisits.

Next consider the case in Figure 4b. Replace $\Gamma$ by the path $\Gamma_1 = F[u, s] \cup \Gamma[s, v]$ indicated by the dotted path in Figure 4b. Again, if $\Gamma_1$ has only planar revisits, we are done. So assume that $\Gamma_1$ has a non-planar revisit $(s_1, t_1)$ to a face $F_1$. Topologically, there are three possibilities for the face $F_1$. Without loss of generality, assume that the three possibilities for $F_1$ are as shown in Figure 6.

In the cases in Figures 6a. and 6b, $\Gamma$ has a non-planar revisit to $F_1$ which contradicts the choice of $F$ with $s$ nearest to $v$ along $\Gamma$. Hence $F_1$ must be as shown in Figure 6c. Among all choices for $F_1$, choose the one for which $s_1$ is nearest to $v$ along $\Gamma_1$. Replace $\Gamma_1$ by the path $\Gamma_2 = \Gamma_1[u, t_1] \cup F_1[t_1, s_1] \cup \Gamma_1[s_1, v]$ where $F_1[t_1, s_1]$ is a path along $F_1$ from $s_1$ to $t_1$ that avoids $C_1$ and $C_2$ except possibly meeting $C_1$ at $u$ in the case where $t_1 = u$. Such a path exists for otherwise, $C$ revisits $F_1$ which is a contradiction. It can now be checked that $\Gamma_2$ can have only planar revisits.
Figure 6. The three possibilities for a non-planar revisit of $\Gamma_1$ to $F_1$ in the case in Figure 4b.
(ii) $F$ does not contain $u$. Recall that $F$ can meet at most one of $C_1$ or $C_2$. First consider the case where $F$ does not meet $C_1$. Since $F$ does not contain $u$, there must be a path along $F$ from $s$ to $t$ that is contained in the interior of $A$. Without loss of generality, assume that $F$ is as shown Figure 7.

In this case, replace $\Gamma$ by the path $\Gamma_1 = \Gamma[u, t] \cup F[t, s] \cup \Gamma[s, v]$ where $F[t, s]$ is a path along $F$ from $s$ to $t$ that is contained in the interior of $A$. If $\Gamma_1$ has only planar revisits, we are done; so assume that $\Gamma_1$ has a non-planar revisit to a face $F_1$. If this non-planar revisit involves a vertex of $\Gamma_1(s, v)$ and a vertex of $\Gamma_1[u, t]$, then the proof is identical to the one given for Figure 6c. It can be checked that the only possibility
is for the non-planar revisit to involve a vertex $s_1$ of $\Gamma_1(s,t)$ and $u$ as shown in Figure 8.

Figure 8. A non-planar revisit of $\Gamma_1$ to $F_1$ in the case in Figure 7.

Among all choices for $F_1$, choose the one for which $s_1$ is nearest to $v$ along $\Gamma_1$ and replace $\Gamma_1$ by the path $\Gamma_2 = F_1[u, s_1] \cup \Gamma_1[s_1, v]$ as shown in Figure 8. Now there are two possibilities for a non-planar revisit of $\Gamma_2$ to a face $F_2$. These are shown in Figures 9a and 9b. In both cases, among all such choices for $F_2$, choose the one for
which $s_2$ (as shown in Figures 9a and 9b) is nearest to $v$ along $\Gamma_2$ and let $t_2$ be as shown. Next replace $\Gamma_2$ by the path $\Gamma_3 = \Gamma_2[u, t_2] \cup F_2[t_2, s_2] \cup \Gamma_2[s_2, v]$. It can be checked that $\Gamma_3$ can have only planar revisits.

The case where $F$ does not meet $C_2$ only is similar to the case above. If $F$ neither meets $C_1$ nor $C_2$, the proof is again similar to the one given above.

3. Neither $u$ nor $v$ lies on $C$ In this case, $u$ and $v$ lie in the interior of the annulus $A$. By Lemma 2.1.2, $u$ and $v$ can be joined by a path $\Gamma$ that is contained in the interior of $A$. The proof that $\Gamma$ can be modified to a path joining $u$ and $v$ that has only planar revisits is identical to the one for Figure 7. Thus in all cases, $u$ and $v$ can be joined by a path in $M$ that has only planar revisits and by Lemma 2.1.1, we are done.
Figure 9. The two possibilities for a non-planar revisit of $\Gamma_2$ to $F_2$ in the case in Figure 8.

Figure 10. A non-planar revisit of $\Gamma$ to $F$ in the case where both vertices are in the interior of the annulus.
Next consider the case where $M$ is a polyhedral map on the Klein bottle and let $u$ and $v$ be vertices of $M$. The proof in this case is similar to that in the case of the torus with a few subtle differences. As before, use Lemma 2.1.4 to cut $M$ along a non-planar, non-revisiting cycle $C$ yielding an annulus $A$ with bounding cycles $C_1$ and $C_2$ and consider the following three cases:

1. $u$ and $v$ lie on $C$. In this case, the argument is identical to the one given above for the torus.

2. $u$ lies on $C$ and $v$ does not lie on $C$. Without loss of generality, assume that $u$ lies on $C_1$. As in the case of the torus, consider the path $\Gamma$ joining $u$ and $v$ shown in Figure 4. If $\Gamma$ has only planar revisits, we are done, so assume that $\Gamma$ has a non-planar revisit $(s, t)$ to a face $F$ and among all non-planar revisits of $F$, choose the one for which $s$ is nearest to $v$ along $\Gamma$. Consider the following cases:
   i. $F$ contains $u$. It suffices to consider the cases shown in Figures 4a and 4b. Replace $\Gamma$ by the path $\Gamma_1 = F[u, s] \cup \Gamma[s, v]$. If $\Gamma_1$ has only planar revisits, we are done; so assume that $\Gamma_1$ has a non-planar revisit to a face $F_1$. As in the case of the torus, such a revisit must involve a vertex $s_1$ of $\Gamma_1(s, v)$ and a vertex of $\Gamma_1[u, s]$. Without loss of generality, there are five possibilities for $F_1$ as shown in Figure 11. In the cases in Figures 11a, 11b, and 11c, $\Gamma$ has a non-planar revisit to $F_1$ contradicting the choice of $F$ with $s$ nearest to $v$. So consider the case in Figure 11d and let $t_2$ be as shown. Replace $\Gamma_1$ by the path $\Gamma_2 = \Gamma_1[u, t_2] \cup F_1[t_2, s_1] \cup \Gamma_1[s_1, v]$. It can be checked that $\Gamma_2$ can have only planar revisits. The case in Figure 11e is similar to the analogous case for the torus.
   ii. $F$ does not contain $u$. This case is similar to the analogous case for the torus.

3. Neither $u$ nor $v$ lies on $C$. Again, this case is similar to the analogous case for the torus.

This concludes the proof of the theorem.
2.2 Counter-Examples to the Non-Revisiting Path Conjecture

Barnette has recently shown that there exist polyhedral maps on the surfaces $S_8$ and $N_{16}$ for which the non-revisiting path conjecture is false. In Engelhardt's dissertation, it is claimed that the non-revisiting path conjecture is also true for polyhedral maps on the surface $S_2$. In this section, we settle the non-revisiting path conjecture for polyhedral maps on all surfaces except $N_3$. Specifically, it is shown that for each $g \geq 2$, the non-revisiting path conjecture is false for the surface $S_g$, and for each $k \geq 4$, the non-revisiting path conjecture is false for the surface $N_k$. This
of course contradicts Engelhardt’s result for the surface $S_2$. Since the non-revisiting path conjecture is already known to be true for the 2–sphere, projective plane, torus, and Klein bottle, the only surface for which the non-revisiting path conjecture is still open is $N_3$.

**The Counterexamples.** The polyhedral maps that constitute the counterexamples for the surfaces mentioned above will be described in terms of the polygons that form the faces of the polyhedral map. Thus, the vertices and edges of the polyhedral map are those of the polygons and the surface is obtained by glueing the polygons together along the edges with the same labels.

First consider the orientable case and let $F_1, \ldots, F_{16}$ be the polygons with the vertex-labelling shown in Figure 12.

![Figure 12. The faces that constitute a counter-example to the non-revisiting path conjecture for polyhedral maps on $S_2$.](image)
Paste the polygons together by identifying the edges with the same labels. It can be checked that the result is a surface $S$ without boundary with the map $M_1$ given by the union of the 16 polygons. Next, it is shown that $S$ is orientable. This is done as follows:

First note that each face has two possible directions for its boundary walk. Assign an “orientation” to each face by choosing one of these two directions. If every face can be assigned an orientation in such a way that adjacent regions induce opposite directions on every common edge, the surface $S$ is orientable. Such an orientation for the faces $F_1, ..., F_{16}$ is shown in Figure 13.

![Figure 13](image)

Figure 13. An orientation on the faces in Figure 12 that shows that the surface is orientable.

Observe that $M_1$ has 10 vertices, 28 edges, and 16 faces. Hence the Euler Characteristic of $S$ is $-2$. And since $S$ is orientable, it must be homeomorphic to $S_2$. It is
easy to check that the faces $F_1, ..., F_{16}$ meet properly. Hence $M_1$ is a polyhedral map on $S_2$. It remains to be shown that $M_1$ does not have the non-revisiting property. We will show that the vertices labelled $x$ and $y$ cannot be joined by a non-revisiting path in $M_1$. The proof is by contradiction; so assume that $\Gamma$ is a non-revisiting path joining $x$ and $y$ in $M_1$. Without loss of generality, assume that the vertex incident to $x$ along $\Gamma$ is the vertex labelled $A$ (the proof is symmetric in the other cases). Note that in this case, the path $\Gamma$ has left the faces labelled $F_3$ and $F_4$. Furthermore, the label $A$ also appears on the face $F_8$. Since $\Gamma$ was assumed to be non-revisiting and the vertex $y$ lies on $F_8$, the remainder of $\Gamma$ must lie on the face $F_8$. There are two ways of getting from $A$ to $y$ along $F_8$, namely, through the vertices labelled 2 or 3. If $\Gamma$ passes through the vertex 2, then the face $F_4$ is revisited by $\Gamma$, which contradicting the assumption that $\Gamma$ is a non-revisiting path. On the other hand, if $\Gamma$ passes through the vertex labelled 3, then $\Gamma$ revisits the face $F_3$; also a contradiction. Thus, there can be no non-revisiting path from $x$ to $y$ in $M$.

In order to prove the result for the surface $S_g, g \geq 3$, we form the connected sum of the surfaces $S_2$ and $S_{g-2}$ as follows:

Let $M_1'$ be a polyhedral map on the surface $S_{g-2}$ such that $M_1'$ has a triangular face $T$. Assign the same labelling on the vertices of $T$ as the face $F_9$ of $M_1$. Glue the polyhedral maps $M_1$ and $M_1'$ by identifying the faces $F_9$ and $T$. Then remove this face from the cell complex. The result is a map $M_1''$ on the surface $S_g$. In fact, $M_1''$ is a polyhedral map on $S_g$. In order to prove this, it suffices to show that the faces of $M_1''$ meet properly. Let $F$ and $G$ be faces of $M_1''$. If $F$ and $G$ are also faces of $M_1$, or $M_1'$, then they clearly meet properly. Without loss of generality, assume that $F$ is a face of $M_1$ and $G$ is a face of $M_1'$. The only way that they can meet in $M_1''$ is if $F$ meets $F_9$ in $M_1$ and $G$ meets $T$ in $M_1'$. In this case, $F$ and $G$ have to meet properly
for otherwise, either $F$ and $F_9$ meet improperly which is a contradiction since $M_1$ is a polyhedral map, or $G$ and $T$ meet improperly which is a contradiction since $M_1'$ was chosen to be a polyhedral map. The proof that $x$ and $y$ cannot be joined by a non-revisiting path in $M_1''$ is identical to the proof given earlier. Thus, $M_1''$ is a polyhedral map on $S_g$ without the non-revisiting property.

Next, we show that the non-revisiting path conjecture is false for the surface $N_4$. In this case, consider the 17 polygons $F_1, \ldots, F_{17}$ with the vertex-labelling shown in Figure 14.

![Figure 14](image)

Figure 14. The faces that constitute a counter-example to the non-revisiting path conjecture for polyhedral maps on $N_4$.

As in the case of the surface $S_2$, paste the polygons together by identifying edges with the same labels. Again, the result is a surface $S$ without boundary with the map $M_2$ given by the union of the faces $F_1, \ldots, F_{17}$. It can be checked that the faces
of $S$ cannot be assigned an orientation as described earlier in the proof. To check if such an assignment is possible, first assign an arbitrary orientation to a particular face. This forces an orientation of each face that shares a common edge with the original face. Since the surface is connected, the process can be continued until the orientation of each face has been forced. Either the result is an orientation for the embedding, or else the given embedding has no orientation in which case the surface is non-orientable. Hence $S$ is non-orientable. $M_2$ has 11 vertices, 30 edges, and 17 faces. Thus $S$ has Euler Characteristic $-2$ and must be homeomorphic to $N_4$. Once again, it can be checked that the faces $F_1, ..., F_{17}$ meet properly. Hence $M_2$ is a polyhedral map on $N_4$. The proof that $M$ does not have the non-revisiting property is identical to the one given for the surface $S_2$.

A counterexample for the surface $N_k, k \geq 5$ is obtained by gluing a polyhedral map on $N_{k-4}$ to $M_2$ as described in the orientable case. Again, this method yields a counterexample for each surface $N_k, k \geq 5$. 

This chapter consists of three sections. In Section 3.1, the notion of a \textit{polygonal representation} of a polyhedral map is introduced. As will be seen, this is a convenient way to represent a polyhedral map as a polygon in the plane. It is shown that every polyhedral map on a surface has such a representation and some useful properties of \textit{polygonal representations} are proved. The notion of a \textit{non-separating polygonal representation} is defined. An interesting question is: \textit{Which polyhedral maps have a non-separating polygonal representation?} It is shown that not all polyhedral maps have a \textit{non-separating polygonal representation}. In Section 3.2, \textit{polygonal representations} are used to provide a simple, unified proof of the existence of a non-planar, non-revisiting cycle in a polyhedral map on the projective plane, torus, and Klein bottle. This is done for polyhedral maps that have a \textit{non-separating polygonal representation}. And in Section 3.3, a graph-colouring problem that is motivated by the question of non-planar, non-revisiting cycles in a polyhedral map, is considered. The conjecture is shown to be true for all planar graphs and $K_{3,3}$. Consequently, Barnett's result [6] on the existence of the above mentioned cycles on the projective plane, torus, and Klein bottle is extended to the surfaces $N_3$, and $S_2$.

\section{Polygonal Representation of Polyhedral Maps}

It is well known that any compact, connected surface may be represented as a polygon in the plane with labeled and directed sides. The directed sides are matched
in pairs and the surface may be obtained by identifying the matched directed sides of the polygon. Analogously, if $M = (G, S)$ is a polyhedral map, then a *polygonal representation* of $M$ is a representation of $M$ as a polygon in the plane that preserves the combinatorial structure of $G$. Thus the sides of the polygon are in fact, edges in $G$. This notion is made more precise below.

A *polygonal map* $P$ is defined as a polyhedral map on a closed disc such that:
1. The vertices of $\partial P$ are labeled, and every label appears at least twice on $\partial P$.
2. The edges of $\partial P$ are directed and there is a matching on this set of directed edges of $\partial P$ that matches each directed edge labeled $(A, B)$ with another directed edge labeled $(A, B)$ with the same labels.

If $M$ is a polyhedral map on a surface, then a polygonal map $P$ is called a *polygonal representation* of $M$ if
1. $M$ is obtained from $P$ by identifying matched edges on $\partial P$ and,
2. after the identifications, each vertex label appears exactly once in $M$.

Note that, in general, a polyhedral map may have several polygonal representations. Figure 17 shows two polygonal representations of a polyhedral map (whose underlying graph is $K_7$) on the torus. Also if $P$ is a polygonal representation of $M$, then there can be no matched edges on $\partial P$ as in Figure 15 below. Otherwise label $A$ either appears only once on $\partial P$, contradicting statement (1) in the definition of a polygonal map, or label $A$ appears more than once in $M$, contradicting statement (2) in the definition of a polygonal representation. A polyhedral map and a polygonal representation of the polyhedral map are shown in Figure 16.
Theorem 3.1.1. Every polyhedral map $M$, not on the sphere, has a polygonal representation.

Proof. Label the vertices of $M$. Since the underlying graph of the dual map $M^*$ is connected, it has a spanning tree $T^*$. There is a bijection between the edges $e$ in $E(M)$ and the edges $e^*$ in $E(M^*)$. Here $e^*$ is the unique edge that crosses $e$. Let $E^*$ denote the complement of $T^*$ in $M^*$ and define $E = \{ e \in E(M) | e^* \in E^* \}$. Cut $M$ along the edges in $E$. Since $T^*$ is planar, the result is a planar map $P$ that satisfies all the conditions for it to be a polygonal map except condition (1). If a pair of edges are matched as in Figure 15, then glue them back together. Now every vertex label on $\partial P$ appears at least twice on $\partial P$ and the map still remains planar. Furthermore, $P$ is a map that satisfies all the conditions for it to be a polygonal representation of $M$.

Let $P$ be a polygonal map and assume that a pair of directed edges $(A, B)$ and $(B, C)$ on $\partial P$ are incident at $B$. Further assume that the respective matching edges $(A, B)'$ and $(B, C)'$ are also incident at $B$. Replace $(A, B)$ and $(B, C)$ by a single directed edge $(A, C)$; similarly replace $(A, B)'$ and $(B, C)'$ by a single directed edge $(A, C)'$. Call such a replacement a concatenation. Perform concatenations along $\partial P$ until it is no longer possible to do so. Call $\partial P$ with the resulting vertex labeling the type of $P$, denoted by $T_{\partial P}$. Figure 16 shows a polyhedral map $M$ on the torus, a polygonal representation of $M$, and the type of $P$. 

![Figure 15. An improper matching of edges on $\partial P$.](image)
Figure 16

Figure 16. An example of a polyhedral map $M$, a polygonal representation $P$ of $M$, and the type of $M$.

Figure 17.

Figure 17. Two polygonal representations of a polyhedral map on the torus.
Lemma 3.1.1. Let $M$ be a polyhedral map on a surface of Euler characteristic $\chi \neq 2$, $T_P$ the type of any polygonal representation of $M$, and $v$ the number of distinct vertex labels on $T_P$. If the vertices on $T_P$ are labeled $1, \ldots, v$ and $n_i$ is the number of occurrences of the label $i$ on $T_P$, then $n_1 + \ldots + n_v = 2v + 2 - 2\chi$. Furthermore, if $\chi \neq 1$, then $n_i \geq 3$ for $i = 1, \ldots, v$.

Proof. First note that there cannot exist a vertex label that appears exactly twice on $T_P$ except in the case where $M$ is a polyhedral map on the projective plane. To see this, suppose $B$ is a vertex label that appears exactly twice on $T_P$ and let $A$ be another vertex label such that $(A, B)$ is a directed edge on $T_P$ and $(A, B)'$ its matching edge on $T_P$. If $B = A$, then there are no more vertex labels on $T_P$. Hence the directed edge can be matched in exactly one way on $T_P$, and in this case the surface is a projective plane. Next assume $B \neq A$. If there are no more vertex labels on $T_P$, then either the directed edges $(A, B)$ and $(B, A)$ can be concatenated contradicting the fact that $T_P$ is the type of a polygonal representation of $M$, or the directed edge $(B, B)$ cannot be matched on $T_P$, which is again a contradiction. Hence, there must be another vertex with label $C$ (possibly $A$) such that $(B, C)$ is a directed edge on $T_P$. Since the vertex label $B$ appears exactly twice on $T_P$, there is only one possibility for the matching edge $(B, C)'$. But the directed edges $(A, B)$ and $(B, C)$ can be concatenated, which is a contradiction. Hence $n_i \geq 3$ for $i = 1, \ldots, v$.

Next, consider the map $M'$ with one face (the polygon $T_P$ itself) obtained by identifying matched directed edges on $T_P$ and let $e$ be the number of edges in $M'$. Since the directed edges are matched in pairs on $T_P$, $e = \frac{n_1 + \ldots + n_v}{2}$. It follows from the Euler formula $v - e + f = \chi$ that

$$n_1 + \ldots + n_v = 2v + 2 - 2\chi.$$  

(3.1)
Figure 18. The types of polygonal representations of polyhedral maps on the projective plane, torus, and Klein bottle.
Theorem 3.1.2. Let $M = (G, S)$ be a polyhedral map.

1. If $S$ is a projective plane, then $M$ has a polygonal representation of type I in Figure 18.

2. If $S$ is a torus, then $M$ has a polygonal representation of type II or type III in Figure 18.

3. If $S$ is a Klein bottle, then $M$ has a polygonal representation of type IV, type V, type VI, or type VII in Figure 18.

Proof. Consider the map $M'$ with one face (the polygon $P$ itself) obtained by identifying matched directed edges on $\partial P$. Let $v$ be the number of vertices and $e$ the number of edges on $M'$. Denote the vertex labels on $\partial P$ by $1, 2, ..., v$. Further, let $n_i$ denote the number of occurrences of the label $i$ on $\partial P$. First consider the case where $M$ is a polyhedral map on the projective plane. By Lemma 3.1.1, $v = 1, n_i = 2$; and $P$ is of type I. Next consider the case where $M$ is a polyhedral map on the torus or Klein bottle. Since $\chi = 0$ in this case, by Lemma 3.1.1, $n_i \geq 3$ for $i = 1, ..., v$ and

$$n_1 + ... + n_v = 2v + 2. \quad (3.2)$$

Since equation (3.2) has no solutions for $v > 2$, $v = 1$, or $v = 2$. Consider the following cases:

1. $v = 1$: In this case, there is exactly one vertex label on $T_P$ and $P$ must be of type III in the case of the torus and of type V or type VI in the case of the Klein bottle.

2. $v = 2$: In this case two vertex labels $A$ and $B$ appear exactly three times on $T_P$. Furthermore, $P$ must be of type II in the case of the torus and of type IV or type VII in the case of the Klein bottle. □
A face $F$ of a polygonal representation $P$ is called \textit{separating} if $F \cap \partial P$ is disconnected. That is to say, the cycle $\partial P$ revisits $F$. A polygonal representation without separating faces is called \textit{non-separating}, otherwise it is called \textit{separating}. In the example in Figure 16, the polygonal representation is non-separating, however both polygonal representations shown in Figure 17 are separating. Specifically in Figure 17b, $\partial P$ revisits the face labeled $F$. The existence of a non-separating polygonal representation is a useful property of a polyhedral map. In the context of non-revisiting paths, if a polyhedral map $M$ has a non-separating polygonal representation $P$, then any two vertices of $M$ that lie in the interior of the polygon $\partial P$ can be joined by a non-revisiting path in $M$ (see Proposition 3.1.2). Of course, given two vertices of $M$, it is not always possible to find a polygonal representation $P$ of $M$ with the property that the two vertices lie in the interior of the polygon $\partial P$ for otherwise, the \textit{non-revisiting path} conjecture would be true for all polyhedral maps. And in the context of non-revisiting cycles of a polyhedral map (this is discussed in Section 3.2.), the existence of non-separating polygonal representations enables us to give simple proofs of results on non-revisiting cycles due to Barnette [6]. In addition, it motivates the formulation of an interesting graph-colouring conjecture (discussed in Section 3.3). However, not all polyhedral maps have a non-separating polygonal representation. In fact, there is an infinite family of polyhedral maps that cannot have any non-separating polygonal representations.

\textbf{Proposition 3.1.1.} For $n \geq 7$, if $n \equiv 0, 3, 4, 7 \pmod{12}$ and $\gamma = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor$, then there exists a triangulation of the orientable surface $S_\gamma$ that is a polyhedral map $M_n$ with the property that every polygonal representation of $M_n$ is separating.

\textbf{Proof.} It is well known [11] that with $\gamma$ as above, the complete graph on $n$ vertices $K_n$ embeds on $S_\gamma$. Let $M_n = (K_n, S_\gamma)$ be a resulting map on $S_\gamma$. If $1 \leq n \leq 4$, then
\[ \gamma = 0 \] and the surface is the 2–sphere. If \( n = 5 \) or 6, then \( K_n \) embeds on the torus. However, the embedding is not a triangulation of the torus. So assume that \( n \geq 7 \).

If \( n \equiv 0, 3, 4, 7 \,(mod\,12) \), then \( \frac{(n-3)(n-4)}{12} \) is an integer and any embedding of \( K_n \) on \( S_\gamma \) is, in fact, a triangulation of \( S_\gamma \). To see this, let \( v, e, \) and \( f \) be the number of vertices, edges, and faces, respectively, of \( M_n \). Then \( v = n \) and \( e = \frac{n(n-1)}{2} \). Hence by the Euler equation for \( S_\gamma \), \( f = \frac{3}{2}e \). Hence the embedding is a triangulation of \( S_\gamma \). Since there are no multiple edges between vertices, the faces of the embedding meet properly and the map \( M_n = (K_n, S_\gamma) \) is a polyhedral map.

**Claim.** For each \( n \) as above, if \( M_n \) has a non-separating polygonal representation \( P_n \), then \( P_n^* \) is contained in \( M_n^* \) and has the following properties:

1. The faces of \( P_n^* \) are \( (n-1) \)-gons.
2. \( P_n^* \) has either two, three, or four faces.
3. The graph of \( P_n^* \) is a planar, spanning, 2–connected subgraph of the graph of \( M_n^* \).

**Proof of Claim.** Statement (1) is obvious. The vertices of \( P_n \) that lie in the interior of the polygon \( \partial P_n \) span a complete subgraph of \( K_n \) that is also contained in the interior of the polygon \( \partial P_n \). If the number of vertices of \( P_n \) that lie in the interior of the polygon \( \partial P_n \) is greater than four, then by the previous statement, the graph of \( P_n \) would be non-planar, which is a contradiction since \( P_n \) is a planar map. Consequently, \( P_n^* \) can have at most four faces. If \( P_n^* \) has no faces, then \( P_n \) must be separating, which is a contradiction. If \( P_n^* \) has exactly one face, then \( P_n \) must have faces that meet improperly, which is also a contradiction since \( P_n \) is a polyhedral map. To see statement (3), note that the graph of \( P_n^* \) is planar, has all the vertices of \( M_n^* \), and is 2–connected because \( P_n \) was assumed to be non-separating.

Hence, for each \( n \geq 7 \), if \( P_n^* \) has exactly two faces, then the graph of \( P_n^* \) has \( 2n - 6 \) vertices. On the other hand, if \( P_n^* \) has exactly three faces, then the graph of
$P_n^*$ has $3n - 7$ or $3n - 8$ vertices and if $P_n^*$ has exactly four faces, then the graph of $P_n$ has $4n - 10, 4n - 11$, or $4n - 12$ vertices. Now, by the Euler formula, $M_n^*$ has $v^* = \frac{n(n-1)}{2} - n + 2 - \frac{(n-3)(n-4)}{6}$ vertices. Note that as $n$ increases, the number of vertices of the graph of $P_n^*$ grows linearly while $v^*$ grows quadratically. In fact, for $n \geq 11$, $v^*$ is greater than each of the numbers $2n - 6, 3n - 7, 3n - 8, 4n - 10, 4n - 11$ and $4n - 12$. Thus if $n \geq 11$, the graph of $P_n^*$ cannot possibly span the graph of $M_n^*$ and we need only consider the case where $n = 7$. Let $M_7$ be the polyhedral map corresponding to the polygonal representation shown in Figure 17a. By statements (1), (2) and (3) above, $P^*$ must be a map on a closed disc with 14 vertices and 3 hexagonal faces. Hence the only possibilities for $P_7^*$ are as shown in Figure 19.

![Figure 19. The two possibilities for a 2-connected, planar map with 14 vertices and 3 hexagonal faces.](image)

However, it is easily checked that these planar maps are not contained in $M_7^*$. ■

In Section 2.2, the polyhedral maps $M_1$ and $M_2$ were counterexamples to the non-revisiting path conjecture for the surfaces $S_2$ and $N_4$, respectively. Figure 20a shows a polygonal representation of $M_1$ while Figure 20b gives a polygonal representation for $M_2$. Observe that both polygonal representations are separating, however, it is easy to construct similar counterexamples that have non-separating polygonal representations.
Figure 20. Polygonal representations for the polyhedral maps in Figures 12 and 14.

**Proposition 3.1.2.** Let $M$ be a polyhedral map that has a non-separating polygonal representation $P$. Then any two vertices of $M$ that lie in the interior of the polygon $P$ can be joined by a non-revisiting path in $M$.

**Proof:** Let $u$ and $v$ be vertices of $M$ that lie in the interior of the polygon $P$. Since $P$ is non-separating, by Lemma 2.1.2, there is a path $\Gamma$ joining $u$ and $v$ in $M$ that is also contained in the interior of the polygon $P$. Clearly, $\Gamma$ can have only
planar revisits. Hence by Lemma 2.1.1, there is a non-revisiting path joining $u$ and $v$ in $M$. ■

3.2 Polygonal Representation and Non-Revisiting Cycles

It is known that any polyhedral map on the projective plane, torus or Klein bottle has a non-planar, non-revisiting cycle. Barnette's proofs [6] of these results are not trivial and involve some details. In this section, we give a unified, elementary proof of these results in the case where the polyhedral map has a non-separating polygonal representation. This is done by considering the cycles of $M$ that lie on the boundary of a non-separating polygonal representation of $M$. Such cycles are non-planar by Lemma 3.2.1 below. And by techniques that are similar to those used in Chapter 2, it is shown that if a cycle of $M$ contained in $\partial P$ revisits a face, then it can be modified to a cycle that is non-revisiting and is also contained in $\partial P$. In the next section, a graph-colouring conjecture is proposed and it is shown that the conjecture is true for all planar graphs. Consequently, an alternate proof of the above-stated result on non-revisiting cycles is given.

**Lemma 3.2.1.** Let $M = (G, S)$ be a polyhedral map on a surface and $P$ be a polygonal representation of $M$. If $C$ is a cycle of $M$ that is contained in $\partial P$, then it must be a non-planar cycle in $M$.

**Proof.** The proof is by contradiction; so assume that $C$ bounds a cell $A$ in $M$. Let $f$ be a face of $M$ that is not contained in $A$ and that has an edge $e$ in common with $C$. Such a face exists; otherwise $A$ would contain all the faces in $M$ which is impossible. Let $e'$ be the matching edge for $e$ on $\partial P$ and let $f'$ be the face of $P$ that contains
$e'$. Since $f$ does not belong to $A$, $f'$ must belong to $A$. Also, since $P$ is connected, so is its dual $P^*$. Also, by definition, $P^* \cap \partial P = \emptyset$. Hence, with $v_f$ and $v_{f'}$ as the vertices of $P^*$ corresponding to the faces $f$ and $f'$ of $P$, respectively, there is a path $v_f, v_{f_1}, \ldots, v_{f_k}, v_{f'}$ from $v_f$ to $v_{f'}$ in $P^*$ that is contained in the interior of the polygon $\partial P$. Hence, there is a sequence of faces $f_i, i = 1, \ldots, k$ of $P$ corresponding to the vertices $v_{f_i}$ of $P^*$ such that $f \cap f_1, f_1 \cap f_2, \ldots, f_{k-1} \cap f_k, f_k \cap f'$, are all edges of $P$ that are contained in the interior of the polygon $\partial P$. But the edges of $C$ are all on $\partial P$. Hence the interior edges $f \cap f_1, f_1 \cap f_2, \ldots, f_{k-1} \cap f_k, f_k \cap f'$, are all in the cell $A$. This implies that the face $f$ is also in $A$, which is a contradiction to the choice of $f$ as a face of $M$ not in $A$. Hence $C$ does not bound a cell on the surface and must be non-planar.

**Corollary 3.2.1.** Every polyhedral map on a surface (except on the sphere) has a non-planar cycle.

**Proof.** Let $M$ be a polyhedral map and let $P$ be a polygonal representation of $M$. By the definition of a polygonal representation, every vertex on $\partial P$ appears at least twice on $\partial P$. Hence there is at least one cycle that is contained in $\partial P$ that is obtained by traveling along $\partial P$ between two consecutive vertices both labeled $A$ that have the property that there is no other pair of matched vertices that appear between the $A$'s. By Lemma 3.2.1, such a cycle must be non-planar.

**Theorem 3.2.1** Let $M$ be a polyhedral map on the projective plane, torus, or Klein bottle. If $M$ has a non-separating polygonal representation, then $M$ has a non-planar, non-revisiting cycle.
Proof. First consider the case where \( M \) is a polyhedral map on the projective plane. By Theorem 3.1.1, there is a polygonal representation \( P \) of \( M \) that is of type I. Let \( C \) be the cycle \((A, A)\) along \( \partial P \) as shown in Figure 21.

![Figure 21. Non-planar, non-revisiting cycles on the projective plane.](image)

The only possibility for a revisit of \( C \) to a face \( F \) is if \( F \cap \partial P \) is disconnected. But this contradicts the assumption that \( P \) is non-separating. Hence \( C \) must be non-revisiting. And by Lemma 3.2.1, \( C \) is also non-planar.

Next, consider the case where \( M \) is a polyhedral map on the torus. By Theorem 3.1.1, there is a polygonal representation \( P \) of \( M \) that is of type II or type III. First consider the case where \( P \) is of type II. Let \( C = (A, A) \) be the bold-faced cycle shown in Figure 22.
If $C$ has a revisit to a face $F$, then it can be easily checked that $F$ must contain both $A$ and $B$ for otherwise $\partial P$ would have to revisit $F$ which contradicts the assumption that $P$ is non-separating. Up to symmetry, $F$ must be as shown in Figure 22. Replace $C$ by the cycle $C_1 = (B, B)$ as shown. If $C_1$ revisits a face $F_1$, then it can be checked that $F_1$ must also contain the vertices labelled $A$ and $B$. But this means that $F$ and $F_1$ meet improperly at $A$ and $B$ which is a contradiction. Hence $C_1$ must be non-revisiting. Next, consider the case where $P$ is of type III and let $C$ be the cycle $(A, A)$ along $\partial P$ as shown in Figure 23. By the same argument as in the case of the projective plane, it is easy to see that $C$ is non-planar and non-revisiting.

Finally, consider the case where $M$ is a polyhedral map on the Klein bottle and let $P$ be a polygonal representation of $M$ that is of type IV. Let $C = (A, A)$ be the cycle shown in Figure 24. If $C$ is non-revisiting, we are done, so assume that $C$ revisits a face $F$. As in the case of the torus, it can be checked that $F$ must contain both $A$ and $B$. There are two possibilities for $F$ as shown in Figure 24.
In both cases, replace $C$ by the cycle $C_1 = (B, B)$ as shown. Now $C_1$ must be non-planar and non-revisiting.

If $P$ is of type V, then by the same argument as for the projective plane the cycle $C$ shown in Figure 25 is non-planar and non-revisiting. If $P$ is of type VI or type VII, then the cycles shown in Figures 26a and 26b, respectively can be easily checked to be non-planar and non-revisiting.
Figure 25. Non-planar, non-revisiting cycles on the Klein bottle.

a.

Figure 26. Non-planar, non-revisiting cycles on the Klein bottle.

b.
Thus in all cases $M$ has a non-planar, non-revisiting cycle that is contained in $\partial P$.

**3.3 A Graph-Coloring Problem and Non-Revisiting Cycles**

A path $P$ in a graph $G$ is said to be a *chord* of a cycle $C$ in $G$ if $P$ is a path joining vertices $x$ and $y$ of $C$ such that $P \cap C = \{x, y\}$. For our purposes, an *edge-coloring* of $G$ is a coloring of the edges of $G$ in which every edge can be colored with many different colors. Given such an edge-coloring of $G$, a subgraph $H$ of $G$ is said to be *monochromatic* if there is a color $C_1$ such that every edge of $H$ is colored with $C_1$. Similarly, $H$ is said to be *dichromatic* if there are colors $C_1$ and $C_2$ such that every edge of $H$ is colored with $C_1$ or $C_2$.

**Conjecture.** *Every edge-colored finite graph $G$ with no mono or dichromatic cycles contains a cycle with no monochromatic chord.*

The above conjecture is motivated by the problem of the existence of non-planar, non-revisiting cycles in a polyhedral map. If every edge is coloured using exactly two colors, then the validity of the coloring conjecture implies that every polyhedral map that has a non-separating polygonal representation, in fact, has a non-planar, non-revisiting cycle. The proof of this result follows.
Theorem 3.3.1. If the conjecture is true in the case where each edge is colored with exactly two colors, then every polyhedral map with a non-separating polygonal representation contains a non-planar, non-revisiting cycle.

Proof. Let $P$ be a non-separating polygonal representation of a polyhedral map $M$ on a surface. Let $\{F_i\}, i = 1, \ldots, k$, be the collection of faces of $M$ that have at least one edge in common with $\partial P$. Since $P$ has no separating faces, for $i = 1, \ldots, k$, $P_i = F_i \cap \partial P$ is a path in $\partial P$. For $i = 1, \ldots, k$, color the edges of the path $P_i$ using a distinct color $C_i$. Since the edges on $\partial P$ are matched in pairs, every edge in $M$ that lies on $\partial P$ is colored using exactly two colors. Now consider $\partial P$ and identify the matched edges on $\partial P$. The result is a graph $G$, where each edge is colored using exactly two colors. Note that the cycles in $G$ are the cycles of $M$ contained in $\partial P$. Recall, by Lemma 3.2.1, that the cycles that are contained in $\partial P$ are non-planar. Furthermore, there is a monochromatic cycle in $G$ if and only if, for some $i$, two vertices of $P_i$ are identified. This in turn implies the face $F_i$ of $M$ is not simply-connected which is impossible. There is a dichromatic cycle in $G$ if and only if there are faces $F_i$ and $F_j$ of $M$ that meet improperly which is also not allowed. Finally, a cycle in $G$ has a monochromatic chord using a color $C_i$ if and only if the corresponding cycle on $\partial P$ revisits the face $F_i$ of $M$. Hence, if the conjecture is true, then there must be a cycle of $M$ contained in $\partial P$ that has no monochromatic chord and hence must be non-revisiting.

Example 3.3.1. The conjecture is true for all graphs that contain a triangle.

Proof. Let $G$ be a graph and $T$ a triangle of $G$. The proof is by contradiction; so assume that $G$ has an edge-coloring with no mono or dichromatic cycles such that every cycle of $G$ has a monochromatic chord. In particular, $T$ has a monochromatic chord $P$. Since there are no multiple edges between vertices, $P$ has length at least
two. Let \( v_1 \) and \( v_2 \) be such that \( P \cap T = \{v_1, v_2\} \) and \( e = v_1v_2 \) the edge in \( T \). Then the cycle \( P[v_1, v_2] \cup \{e\} \) is a mono or dichromatic cycle, which is a contradiction. 

**Example 3.3.2.** The conjecture is true for \( K_{3,3} \).

**Proof.** The proof is by contradiction; so assume that there is an edge-coloring of \( K_{3,3} \) with no mono or dichromatic cycles such that every cycle has a monochromatic chord. Let the vertices of \( K_{3,3} \) be labeled as shown in Figure 27.

![Figure 27](image)

**Figure 27.** The graph-coloring conjecture is true for \( K_{3,3} \).

It will be shown that there must be a 4-cycle with no monochromatic chord. It is easy to see that a monochromatic chord of any 4-cycle must have length at least two. If a monochromatic chord of a 4-cycle has length greater than two, then the endpoints of the chord are adjacent to each other in \( K_{3,3} \). This is a contradiction since in this case, the monochromatic chord together with the edge joining the endpoints of the chord form a mono or dichromatic cycle. Hence a monochromatic chord of a 4-cycle must have length exactly two. First, consider the 4-cycle \( C_1 = 1A3C1 \) and let \( P_1 \) be a monochromatic chord of \( C_1 \). Up to symmetry, \( P_1 = C2A \). Next, the cycle \( C_2 = 1A2C1 \) must have a monochromatic chord \( P_2 \). The vertices labeled \( A \) and \( C \) cannot be the endpoints of \( P_2 \) for then, the paths \( P_1 \) and \( P_2 \) form a mono or dichromatic cycle. Hence \( P_2 = 1B2 \). Let \( C_3 = 1A2B1 \) and \( P_3 \) a monochromatic chord of \( C_3 \). By a similar argument as the one given for \( P_2 \), \( P_3 = A3B \). If \( P_4 \)
is a monochromatic chord for the cycle $C_4 = 2B3A2$, then $P_4 = 2C3$. Likewise, if $C_5 = 2B3C2$, then the monochromatic chord $P_5$ of $C_5$ is $B1C$; and with $C_6 = 1B3C1$, the monochromatic chord $P_6$ of $C_6$ must be $3A1$. Finally, consider the 4-cycle $C_7 = A3C2A$ and let $P_7$ be a monochromatic chord of $C_7$. There are two possibilities for $P_7$. If $P_7 = 3B2$, then $P_4$ and $P_7$ form a mono or dichromatic cycle. On the other hand, if $P_7 = C1A$, then $P_1$ and $P_7$ form a mono or dichromatic cycle. In either case, its a contradiction. Hence $C_7$ cannot have a monochromatic chord. ■

Theorem 3.3.2. The conjecture is true for all planar graphs.

Proof. The proof is by contradiction; so assume that there is a planar graph $G$ for which the conjecture is false. In other words, for some edge-coloring of $G$ with no mono or dichromatic cycles, every cycle of $G$ has a monochromatic chord. Consider an embedding of $G$ in the plane in which edges cross each other only at vertices of $G$. Let $\partial C_0$ be the boundary of the unbounded face of $G$ and $C_0$ the closed region interior to $\partial C_0$. Let $P_1$ be any monochromatic chord of $\partial C_0$. Then $P_1$ must be contained in $C_0$ and it separates $C_0$ into components $C_1$ and $B_1$ such that $C_1 \cap B_1 = P_1$ and $C_1 \cup B_1 = C_0$. Now consider the cycle $\partial C_1$ that bounds the component $C_1$ and let $P_2$ be any monochromatic chord for $\partial C_1$. If $P_2$ leaves the component $C_1$, then it must enter the component $B_1$ by crossing the monochromatic chord $P_1$ and the only way for $P_2$ to re-enter $C_1$ is for it to cross $P_1$ again. But this yields a dichromatic cycle, which is not allowed. Hence the only way $\partial C_1$ can have a monochromatic chord $P_2$ is for $P_2$ to be completely contained in $C_1$. Now $P_2$ separates $C_1$ into components $C_2$ and $B_2$ such that $C_2 \cap B_2 = P_2$ and $C_2 \cup B_2 = P_1$. For every integer $k \geq 1$, we claim that at the $k^{th}$ step, if $P_{k+1}$ is any monochromatic chord for the cycle $\partial C_k$, then the component $C_k$ is divided into components $C_{k+1}$ and $B_{k+1}$ with the following properties:
(1) \( C_{k+1} \cup B_{k+1} = C_k \).

(2) \( C_{k+1} \cap B_{k+1} = P_{k+1} \).

Conditions (1) and (2) above are equivalent to the statement that \( P_{k+1} \) is contained in \( C_k \). The proof of the above claim is by induction on \( k \). It was shown above that the claim is true for \( k = 1 \). Assume that the claim is true for each \( i < k \). If \( P_{k+1} \), a monochromatic chord for \( \partial C_k \), is contained in \( C_k \), we are done, so assume that \( P_{k+1} \) leaves \( C_k \). However, if \( P_{k+1} \) does not leave \( C_{k-1} \), then the only way it can return to \( C_k \) is by crossing the monochromatic chord \( P_k \) twice giving a dichromatic cycle in \( G \), which is not allowed. On the other hand, if \( P_{k+1} \) leaves both \( C_k \) and \( C_{k-1} \), then it eventually has to return to \( C_k \). But by condition (1) for the component \( C_{k-1} \), this means that it also returns to \( C_{k-1} \). Thus, there is a subpath \( Q_{k+1} \) of \( P_{k+1} \) that is a monochromatic chord for the cycle \( \partial C_{k-1} \) and that returns to \( C_{k-1} \) after leaving. This is a contradiction to the induction hypothesis. Hence \( P_{k+1} \) cannot leave \( C_k \), proving the claim.

It follows that \( G \) would have to be an infinite graph in order that every cycle in \( G \) have a monochromatic chord, which is a contradiction. Hence there can be no planar graph that can be a counter-example to the conjecture and the theorem is true for all planar graphs.

As an application of Theorem 3.3.1, we give another simple proof of Theorem 3.2.1

**An alternate proof of Theorem 3.2.1** In the case of each surface, let \( P \) be a non-separating polygonal representation of \( M \) and let \( G_T \) and \( G \) be the graphs obtained by identifying the edges on \( T_P \) and \( \partial P \), respectively. Since \( T_P \) is obtained from \( \partial P \) by performing concatenations along \( \partial P \), \( G \) can be obtained from \( G_T \) by
inserting vertices of $G$ that are not in $G_T$ along the interior of each edge of $G_T$. Thus $G_T$ and $G$ are homeomorphic. If $M$ is a polyhedral map on the projective plane, then by Theorem 3.1.2, $P$ is of type I and consequently, $G_T$ is isomorphic to the graph in Figure 28a. By a similar argument, if $M$ is a polyhedral map on the torus, then $G$ is homeomorphic to the graph in Figure 28b, or Figure 28c. and if $M$ is a polyhedral map on the Klein bottle, then $G$ is homeomorphic to the graph in Figure 28b, Figure 28c, or Figure 28d.

![Figure 28](image)

**Figure 28.** The boundary graphs of polyhedral maps on the projective plane, torus, and Klein bottle.

In all cases, $G$ is planar and by Theorem 3.3.2, the conjecture is true. Hence by Theorem 3.3.1, in each case, there is a non-planar, non-revisiting cycle in $M$. ■

So far, two elementary proofs of Theorem 3.2.1 have been provided. However, the scope of Theorems 3.3.1 and 3.3.2 are greater than merely giving proofs for already known results on non-planar, non-revisiting cycles on the projective plane, torus, and Klein bottle. In this regard, the following result extends Barnette’s result on non-revisiting cycles on the three surfaces mentioned above to a class of polyhedral maps on the surfaces $N_3$ or $S_2$. 
Theorem 3.3.3. Every polyhedral map on $N_3$ or $S_2$ that has a non-separating polygonal representation contains a non-planar, non-revisiting cycle.

Proof. Let $M$ be a polyhedral map on $N_3$ and assume that $M$ has a non-separating polygonal representation $P$. Let the vertices of $T_P$ be labeled $1,\ldots,v$ and for $i = 1,\ldots,v$, let $n_i$ be the number of occurrences of the label $i$ on $T_P$. Since the Euler characteristic in this case is $-1$, by Lemma 3.1.1,

$$n_1 + \ldots + n_v = 2v + 4.$$  \hspace{1cm} (3.3)

where $n_i \geq 3$ for $i = 1,\ldots,v$. It is easily checked that the above equation has no solutions for $v \geq 5$. That is to say, there are at most four different vertex labels on $T_P$. Hence the graph $G_T$, obtained by identifying matched directed edges on $T_P$ can have at most four vertices and consequently, must be planar. By the same argument given in the alternate proof to Theorem 3.2.1, the graph $G$ obtained by identifying matched edges on $\partial P$ must be homeomorphic to $G_T$ and consequently, must also be planar. By Theorem 3.3.2, the conjecture is true for the graph $G$. Hence by Theorem 3.3.1, $M$ has a non-planar, non-revisiting cycle that is contained in $\partial P$.

Next, consider the case where $M$ is a polyhedral map on $S_2$ and let $P$ be a non-separating polygonal representation of $M$. Since the Euler characteristic in this case is $-2$, by Lemma 3.1.1,

$$n_1 + \ldots + n_v = 2v + 6.$$  \hspace{1cm} (3.4)

where $n_i \geq 3$ for $i = 1,\ldots,v$. It is easily checked that the above equation can have solutions only if $v \leq 6$. If $e$ is the number of distinct directed edges on $T_P$, then $e \leq 9$. Let $G$ be the graph obtained by identifying matched edges on $T_P$. Then $G$ has at most 6 vertices and at most 9 edges. Since $K_5$ has 10 edges, $G$ is either planar, or is isomorphic to $K_{3,3}$. In either case, the coloring conjecture for two colors is true, and
by the same argument given for the surface \( N_3 \), there is a non-planar, non-revisiting cycle in \( M \) that is contained in \( \partial P \).
The non-revisiting path conjecture is now settled for all polyhedral maps except those that are homeomorphic to $N_3$. It is conceivable that the conjecture is true for polyhedral maps on surfaces homeomorphic to $N_3$, however, at this juncture, no proof is known. Considering the complexity of the proofs of the validity of the non-revisiting path conjecture for the torus and Klein bottle, a brute force method might prove to be tedious in the case of the surface $N_3$.

The problem of the existence of non-planar, non-revisiting cycles in a polyhedral map is wide open. The only surfaces for which such cycles are known to exist are the projective plane, torus, Klein bottle, and for a class of polyhedral maps on the surfaces homeomorphic to $N_3$ and $S_2$ (see Theorem 3.3.3). If every polyhedral map contains a non-planar, non-revisiting cycle, one could potentially obtain decomposition theorems for surfaces other than the torus and Klein bottle.

Even though the graph coloring conjecture proposed in Section 3.3 was motivated by the existence of non-planar, non-separating cycles in a polyhedral map, the conjecture is certainly interesting on its own merit. Apart from planar graphs, graphs with a triangle, $K_{3,3}$, and some very specific graphs (not included in this dissertation), the coloring conjecture remains unsettled.

In conclusion, this research has raised at least three different questions that remain unsolved and would make for some interesting work in the future.
REFERENCES


REFERENCES


BIOGRAPHICAL SKETCH

Hari Pulapaka was born in Bombay, India, on March 19, 1966. Upon receiving a bachelor’s degree in mathematics from St. Xavier’s College, University of Bombay, he arrived in the U.S. in 1987 to attend graduate school. In 1989, he received an M.S. in mathematics from George Mason University, Fairfax, VA, under the supervision of Dr. James Lawrence. And in 1995, he received a Ph.D. in mathematics from the University of Florida, Gainesville, FL, under the supervision of Dr. Andrew Vince.
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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August 1995

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