ON THE RATE OF CONVERGENCE OF SERIES OF RANDOM VARIABLES

By

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>TAIL SERIES STRONG LAWS OF LARGE NUMBERS I</td>
<td>9</td>
</tr>
<tr>
<td>2.1</td>
<td>Introduction and Preliminaries</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>Tail series SLLNs for Arbitrary Random Variables</td>
<td>14</td>
</tr>
<tr>
<td>2.3</td>
<td>Tail series SLLNs for Independent Random Variables</td>
<td>23</td>
</tr>
<tr>
<td>2.4</td>
<td>Examples</td>
<td>31</td>
</tr>
<tr>
<td>3</td>
<td>TAIL SERIES WEAK LAWS OF LARGE NUMBERS</td>
<td>42</td>
</tr>
<tr>
<td>3.1</td>
<td>Introductory Comments, Tail Series Inequality, and a New Proof of Klesov’s Tail Series SLLN</td>
<td>42</td>
</tr>
<tr>
<td>3.2</td>
<td>Tail Series WLLNs</td>
<td>48</td>
</tr>
<tr>
<td>4</td>
<td>TAIL SERIES STRONG LAWS OF LARGE NUMBERS II</td>
<td>55</td>
</tr>
<tr>
<td>4.1</td>
<td>Introduction and Preliminaries</td>
<td>55</td>
</tr>
<tr>
<td>4.2</td>
<td>Tail series SLLNs</td>
<td>57</td>
</tr>
<tr>
<td>4.3</td>
<td>The Weighted I.I.D. Case</td>
<td>73</td>
</tr>
<tr>
<td>5</td>
<td>SOME FUTURE RESEARCH PROBLEMS</td>
<td>81</td>
</tr>
<tr>
<td>REFERENCES</td>
<td></td>
<td>84</td>
</tr>
<tr>
<td>BIOGRAPHICAL SKETCH</td>
<td></td>
<td>89</td>
</tr>
</tbody>
</table>
Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

ON THE RATE OF CONVERGENCE OF SERIES OF RANDOM VARIABLES

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For an almost surely (a.s.) convergent series of random variables \( S_n = \sum_{j=1}^{n} X_j \), the tail series \( T_n = \sum_{j=n}^{\infty} X_j \) is a well-defined sequence of random variables which converges to 0 a.s. The rate of a.s. convergence of \( S_n \) to a random variable \( S \) is investigated through the current study of the rate in which \( T_n \) converges to 0 a.s.

Tail series strong laws of large numbers (SLLN) of the form \( b_n^{-1}T_n \rightarrow 0 \) a.s. (where \( \{b_n, n \geq 1\} \) is a sequence of positive constants with \( b_n \downarrow 0 \)) are obtained under various sets of conditions. Both the cases of (i) \( \{X_n, n \geq 1\} \) having no conditions on their joint distributions and (ii) \( \{X_n, n \geq 1\} \) being independent are investigated.

Some earlier work by Klesov on the tail series SLLN problem, which had provided tail series analogues of Petrov’s SLLNs for partial sums, is generalized to a larger class of random variables. In the case of independent summands, some tail series
analogues of Teicher’s SLLNs for partial sums are obtained as well.

Moreover, by employing the von Bahr and Esseen inequality, tail series weak laws of large numbers (WLLN) for independent random variables are obtained. The tail series WLLNs provide a bound on the rate in which \( \sup_{j \geq n} |T_j| \) converges to 0 in probability. These tail series WLLNs are compared with the tail series SLLNs and with tail series laws of the iterated logarithm of Rosalsky.

Examples are provided throughout to illustrate the current results and to compare them with other results in the literature.
CHAPTER 1
INTRODUCTION

The theory of partial sums of random variables has been at the forefront of research in statistical science for most of this century. The case of independent summands has been of especial interest. One of the most interesting problems in classical probability theory has been to determine, for a given series of random variables, the probability that the series converges. (Here, and throughout the entire sequel, the term "converges" means that the limit under consideration exists and is finite. The term "diverges" means "does not converge.") According to the famous Kolmogorov 0-1 law (see, e.g., Chow and Teicher [14], p. 64, or Chung [17], p. 254), a series of independent random variables either converges almost surely (a.s.) or diverges a.s. The primary objective of the current work is to determine the almost sure rate of convergence for a convergent series. This objective will be discussed in more detail below.

Let

$$S_n = \sum_{j=1}^{n} X_j, \ n \geq 1,$$

where \( \{X_n, \ n \geq 1\} \) are random variables. This dissertation will concentrate on a series of independent random variables, but some results are obtained without assuming independence. To establish almost sure convergence of the series \( S_n \) assuming \( \{X_n, \ n \geq 1\} \) are independent random variables, the Khintchine-Kolmogorov
convergence theorem (see, e.g., Chow and Teicher [14], p. 110) and the celebrated Kolmogorov three-series criterion (see, e.g., Chow and Teicher [14], p. 114, or Chung [17], p. 118) are very useful devices. In fact, the Kolmogorov three-series criterion provides a triumvirate of conditions which are both necessary and sufficient for the convergence of the series $S_n$ when the summands $\{X_n, n \geq 1\}$ are independent random variables.

The Khintchine-Kolmogorov convergence theorem asserts that if $\{X_n, n \geq 1\}$ are independent random variables with

$$E(X_n) = 0, \; n \geq 1 \text{ and } \sum_{n=1}^{\infty} E(X_n^2) < \infty,$$

then the series $S_n$ converges a.s. and in quadratic mean to a random variable $S$ with

$$E(S) = 0 \text{ and } E(S^2) = \sum_{n=1}^{\infty} E(X_n^2).$$

The Kolmogorov three-series criterion asserts that if $\{X_n, n \geq 1\}$ are independent random variables, then the series $S_n$ converges a.s. iff

1. $\sum_{n=1}^{\infty} P\{|X_n| > 1\} < \infty$,
2. $\sum_{n=1}^{\infty} E\left(X_n^{(1)}\right)$ converges,
3. $\sum_{n=1}^{\infty} \text{Var}\left(X_n^{(1)}\right) < \infty$,

where

$$X_n^{(1)} = X_n I_{|X_n| \leq 1}, \; n \geq 1.$$
If the series $S_n$ converges a.s. to a random variable $S$, then (set $S_0 = X_0 = 0$) the tail series

$$T_n = S - S_{n-1} = \sum_{j=n}^{\infty} X_j, \; n \geq 1$$

is a well-defined sequence of random variables and converges to $0$ a.s. In the theory of partial sums, the fact that the sum $S_n$ is well defined for every $n$ is, of course, automatic. On the other hand, in the theory of tail series, the problem as to whether $\{T_n, \; n \geq 1\}$ is well defined is a genuine one. The two classical theorems (Khintchine-Kolmogorov convergence theorem and Kolmogorov three-series criterion) play a key role in guaranteeing that the tail series $T_n$ is well defined in the case of independent summands. In this dissertation, we will focus on the rate of convergence of the series $S_n$ to a random variable $S$ or, equivalently, on the rate of convergence of the tail series $T_n$ to $0$.

We say that the sequence of random variables $\{X_n, \; n \geq 1\}$ (such that the series $S_n$ diverges a.s.) obeys the strong law of large numbers (SLLN) with norming constants $\{a_n, \; n \geq 1\}$ if

$$\frac{S_n}{a_n} \rightarrow 0 \; \text{a.s.}$$

where $\{a_n, \; n \geq 1\}$ is a sequence of positive constants with $a_n \uparrow \infty$.

In the same way, we will say that the sequence $\{X_n, \; n \geq 1\}$ obeys the tail series SLLN with norming constants $\{b_n, \; n \geq 1\}$ if the tail series $T_n$ is well defined and

$$\frac{T_n}{b_n} \rightarrow 0 \; \text{a.s.}$$

where $\{b_n, \; n \geq 1\}$ is a sequence of positive constants with $b_n \downarrow 0$. 
Of course, a SLLN has a sharper result for the slower $0 < a_n \uparrow \infty$. Similarly, a tail series SLLN has a sharper result for the faster $0 < b_n \downarrow 0$.

SLLNs for partial sums lie at the very foundation of statistical science and have been and still are the subject of vigorous research activity. In the case of partial sums, the SLLN problem was investigated prior to the LIL problem. On the other hand, the situation is reversed for tail series; the tail series LIL problem was investigated prior to the tail series SLLN problem. As will be seen, many results for partial sums $S_n$ can be paired with analogous results for tail series $T_n$. (Of course, the actual random variables in the series $S_n$ and tail series $T_n$ are necessarily different.) This duality was first discovered and investigated by Chow and Teicher [13].

Chow and Teicher [13] constructed a milestone for research about the rate of almost sure convergence of the tail series of independent random variables. They developed a tail series counterpart to the renowned Kolmogorov law of the iterated logarithm (LIL) (see, e.g., Chow and Teicher [14], p. 343, or Petrov [40], p. 292) for series of independent and bounded random variables. Studies to eliminate Chow and Teicher's boundedness assumption were conducted by Barbour [9], Heyde [23], Budianu [12], Rosalsky [41], and Klesov [29]. Barbour [9] suggested a methodology which yields a tail series analogue of the Linderberg-Feller version of the central limit theorem (CLT) (see, e.g., Chow and Teicher [14], p. 291, or Loève [34], p. 292). Using this methodology, Heyde [23] obtained tail series analogues of the CLT and LIL for a martingale difference sequence. Budianu [12] proved a tail series LIL for
series of independent unbounded random variables, which is a tail series analogue of Petrov [40, Section 10.2, Theorems 2] LIL, and extended it for two-dimensional random variables. Rosalsky [41] developed a more general tail series LIL than that of Budianu [12] for series of independent unbounded random variables, which is a tail series counterpart to Teicher’s [45] version of the LIL, and as special cases he proved tail series LILs for weighted sums of independent and identically distributed (i.i.d.) random variables (see below for the definition of the weighted i.i.d. case).

In the same year as Rosalsky’s article appeared, Klesov [29] developed a version of the tail series LIL for weighted i.i.d. unbounded random variables, but Klesov’s [29, Proposition 4] result is nothing but the special case $\beta = 0$ of Rosalsky [41, Theorem 2]. Klesov [29] also proved two tail series SLLNs for independent random variables, which are tail series analogues of Petrov [40, Section 9.3, Theorems 12 and 13] and Petrov [38, Theorem 5], respectively.

In his follow-up article, Klesov [30] extended his previous tail series SLLNs to wider classes of independent random variables, and he also obtained tail series SLLNs for several other dependence structures, viz. arbitrary sequences with no assumptions on their joint distributions, orthogonal sequences, quasistationary sequences, and martingale difference sequences.

Solntsev [43] proved a tail series SLLN, but his result is not satisfactory since his condition involves blocks

$$\sum_{j=n_{k-1}+1}^{n_k} X_j, \quad k \geq 1$$

of summands rather than individual summands. It is widely discussed in the
literature (see, e.g., Chung [16] or Loève [34], p. 270) that conditions for the classical SLLN for partial sums which involve blocks of random variables (as opposed to only the individual summands) are unsatisfactory or at best undesirable. Such criticism carries over directly to the tail series situation.

Throughout the entire sequel, all random variables are defined on a fixed but otherwise arbitrary probability space \((\Omega, \mathcal{F}, P)\), and the logarithm and iterated logarithm are conveniently defined for \(x > 0\) and a positive integer \(r\), by

\[
\log_1 x = \begin{cases} 
\log x, & \text{if } x \geq e \\
\frac{1}{e} x, & \text{if } x < e
\end{cases}
\]

and

\[
\log_r x = \log_1 \log_{r-1} x, \quad r \geq 2
\]

where \(\log x\) (when \(x \geq e\)) denotes the natural logarithm.

Some of the results herein as well as some examples illustrating them concern the weighted i.i.d. case consisting of sequences \(\{X_n, n \geq 1\}\) of the form \(X_n = a_n Y_n, n \geq 1\), where \(\{Y_n, n \geq 1\}\) are i.i.d. random variables with \(\mathbb{E}(Y_1) = 0, \mathbb{E}(Y_1^2) = 1\), and \(\{a_n, n \geq 1\}\) are nonzero constants.

This dissertation will be divided into five chapters which will now be briefly described. In Chapter 2, we will generalize some of Klesov's [30] tail series SLLNs. (Throughout this chapter and the subsequent ones, our assumptions on the random variables \(\{X_n, n \geq 1\}\) involve the individual summands rather than blocks of summands as were considered by Solntsev [43].) Furthermore, we will develop truncated versions of our new tail series SLLNs. Both of the cases
(i) \( \{X_n, n \geq 1\} \) are independent random variables, and

(ii) \( \{X_n, n \geq 1\} \) are random variables with no assumptions being imposed on their joint distributions are investigated. Also we will provide examples which demonstrate that the new results are indeed better than previous ones.

Chapter 3 is quite independent of the others. In Chapter 3, we will study the rate of convergence in probability of a series \( S_n \) of independent random variables to a random variable \( S \) or, more specifically, the rate in which \( \sup_{j \geq n} |T_j| \) converges to 0 in probability by establishing tail series weak laws of large numbers (WLLN). These tail series WLLNs take the form

\[
\frac{\sup_{j \geq n} |T_j|}{b_n} \xrightarrow{P} 0
\]

where \( \{b_n, n \geq 1\} \) is a sequence of norming constants with \( 0 < b_n \downarrow 0 \). As special cases of our tail series WLLNs, we will obtain the tail series WLLNs for weighted sums of i.i.d. random variables. Also, via the example of the harmonic series with a random choice of signs, we will find a sequence of norming constants which yields tail series WLLNs, but does not yield tail series SLLNs.

In Chapter 4, we will prove advanced tail series SLLNs for series of independent random variables, which are counterparts to Teicher's [47] SLLNs for partial sums. As special cases of these tail series SLLN's, we will investigate the tail series SLLN problem for weighted sums of i.i.d. random variables. Also we will provide an example which illustrates the new results.
Finally, in the last chapter (Chapter 5), some problems for future research work are presented.
CHAPTER 2
TAIL SERIES STRONG LAWS OF LARGE NUMBERS I

2.1 Introduction and Preliminaries

The tail series LIL has comparatively rich references; on the other hand, the
tail series SLLN has limited references. Two papers of Klesov [29 and 30] are
good references of the tail series SLLN. Although Klesov [29] proved two tail series
SLLNs for independent random variables by first establishing a tail series version
of the Kolmogorov's inequality (see, e.g., Chow and Teicher [14], p. 127 or Petrov
[40], p. 52), the proof of his tail series version of the Kolmogorov's inequality
is very complicated, and we could not quite follow his argument. His argument
rested on a tail series inequality which he did not substantiate. But, in his follow¬
up article, using a tail series analogue of the Kronecker lemma (rather than his tail
series version of the Kolmogorov's inequality), Klesov [30] extended his previous tail
series SLLNs to wider classes of independent random variables. He also developed
tail series SLLNs for arbitrary random variables (i.e., not necessarily independent)
as well.

Some of Klesov's [29, 30] work will now be described. Let $\Psi$ be the class of
functions $\psi(x)$ satisfying the following three conditions:

(i) $\psi(x)$ is positive and nondecreasing.
(ii) $x \psi\left(\frac{1}{x}\right)$ tends monotonically to 0 as $x \downarrow 0$.

(iii) $\sum_{n=1}^{\infty} \frac{1}{n \psi(n)} < \infty$.

Examples of such functions $\psi(x)$ are

$$\psi(x) = |x|^\alpha, \quad 0 < \alpha < 1$$

$$\psi(x) = (\log_1 |x|)^{1+\varepsilon}, \quad \varepsilon > 0$$

$$\psi(x) = (\log_1 |x|) (\log_2 |x|)^{1+\varepsilon}, \quad \varepsilon > 0$$

and so on.

For arbitrary random variables $\{X_n, n \geq 1\}$, without the assumption of independence, Klesov [30] developed two tail series SLLNs (Propositions 1 and 2 below). But Proposition 2 included a technical error in its formulation, and so it needs to be restated. As in Chapter 1, $\{T_n, n \geq 1\}$ denotes throughout the sequence of tail series $T_n = \sum_{j=n}^{\infty} X_j, \ n \geq 1$, corresponding to random variables $\{X_n, n \geq 1\}$. Note that the hypotheses of Proposition 1 and 2 ensure that $\{T_n, n \geq 1\}$ is well defined.

Proposition 1 (Klesov [30]). Let $0 < p \leq 1$ and let $\{X_n, n \geq 1\}$ be random variables. Furthermore, let $\{b_n, n \geq 1\}$ be a sequence of positive constants with $b_n \downarrow 0$. If the series

$$\sum_{n=1}^{\infty} \frac{\text{E}(|X_n|^p)}{b_n^p} < \infty,$$

then the tail series SLLN

$$\frac{T_n}{b_n} \to 0 \text{ a.s.}$$

obtains.
Proposition 2 (Klesov [30]). Let $0 < p \leq 1$ and let $\{X_n, n \geq 1\}$ be random variables. If the series
\[
\sum_{n=1}^{\infty} E(|X_n|^p) < \infty,
\]
then setting
\[
A_n = \sum_{j=n}^{\infty} E(|X_j|^p), \quad n \geq 1,
\]
and assuming that $A_n > 0$, $n \geq 1$, the tail series SLLN
\[
\frac{T_n}{(A_n \psi(A_n^{-1}))^\frac{1}{p}} \to 0 \ a.s.
\]
obtains for each function $\psi(x) \in \Psi$.

Proposition 2 is a tail series analogue of a SLLN of Petrov [39].

Under the assumption that $\{X_n, n \geq 1\}$ are independent random variables, Propositions 1 and 2 have been extended by Klesov [30] to Propositions 3 and 4 below, respectively, by employing a class of functions instead of a specific function $g(x) = |x|^p$, $0 < p \leq 1$. But both of them included a technical error in their formulation, and so they need to be reformulated as follows. In addition, Klesov [30] did not verify that his conditions ensure that the tail series $\{T_n, n \geq 1\}$ is indeed well defined.

Let the function $g(x)$ be positive for $x > 0$ with $g(x) \uparrow \infty$ as $x \uparrow \infty$. Assume that either of the following two conditions holds

(i) $\frac{x}{g(x)}$ is nondecreasing for $x > 0$.

(ii) $\frac{g(x)}{x}$ is nondecreasing for $x > 0$, $\frac{x^2}{g(x)}$ is nondecreasing for $x > 0$,

and $E(X_n) = 0$, $n \geq 1$. 
Proposition 3 (Klesov [30]). Let \( \{X_n, n \geq 1\} \) be independent random variables and let \( \{b_n, n \geq 1\} \) be a sequence of positive constants with \( b_n \downarrow 0 \). If the series

\[
\sum_{n=1}^{\infty} \frac{E(g(|X_n|))}{g(b_n)} < \infty,
\]

then the tail series SLLN

\[
\frac{T_n}{b_n} \rightarrow 0 \quad \text{a.s.}
\]

obtains.

Proposition 4 (Klesov [30]). Let \( \{X_n, n \geq 1\} \) be independent random variables. If the series

\[
\sum_{n=1}^{\infty} E\left(g(|X_n|)\right) < \infty,
\] (2.1.1)

then setting

\[
A_n = \sum_{j=n}^{\infty} E\left(g(|X_j|)\right), \quad n \geq 1
\]

and assuming that \( A_n > 0, n \geq 1 \), the tail series SLLN

\[
\frac{T_n}{g^{-1}(A_n \psi(A_n^{-1}))} \rightarrow 0 \quad \text{a.s.}
\]

obtains for each function \( \psi(x) \in \Psi \).

Not only do Propositions 3 and 4 reduce to two tail series SLLNs of Klesov [29], respectively, by taking \( g(x) = |x|^p \), \( 0 < p \leq 2 \), but they also are tail series analogues of Petrov [40, Section 9. 3, Theorem 11 with \( g_n \equiv g, n \geq 1 \)] and Petrov [38, Theorem 5], respectively.

Most of our results in this chapter are based on the following two lemmas.

Lemma 1 (Heyde [23], Rosalsky [41], Klesov [30]). Let \( \{x_n, n \geq 1\} \) be a sequence of constants and let \( \{b_n, n \geq 1\} \) be a sequence of positive constants with \( b_n \downarrow 0 \). If
the series

\[ \sum_{n=1}^{\infty} \frac{x_n}{b_n} \]

converges,

then

\[ \frac{1}{b_n} \sum_{j=n}^{\infty} x_j \to 0. \]

Lemma 1 is a tail series analogue of the Kronecker lemma. This lemma is initially due to Heyde [23], but Rosalsky [41] re-proved it in an alternative way because Heyde's original proof was not clear. One year after Rosalsky's [41] paper appeared, but independently from Rosalsky's paper, Klesov [30] proved the lemma in a manner similar to that of Rosalsky. As we mentioned earlier, in his previous paper, Klesov [29] proved his tail series SLLNs via a tail series version of the Kolmogorov inequality instead of the above tail series analogue of the Kronecker lemma. The approach using this analogue of the Kronecker lemma is simpler and indeed more natural.

Lemma 2 (Klesov [29]). Let \( \{c_n, n \geq 1\} \) be a sequence of nonnegative constants such that \( \sum_{n=1}^{\infty} c_n < \infty \). If

\[ C_n \equiv \sum_{j=n}^{\infty} c_j > 0, \quad n \geq 1, \]

then

\[ \sum_{n=1}^{\infty} \frac{c_n}{C_n \psi(C_n^{-1})} < \infty \]

obtains for each function \( \psi(x) \in \Psi \).

This lemma is a tail series analogue of the Abel-Dini theorem (see, e.g., Knopp [32], p. 290).
2.2 Tail series SLLNs for Arbitrary Random Variables

For arbitrary random variables \( \{X_n, n \geq 1\} \), without the assumption of independence, we obtain the following tail series SLLNs. To avoid trivial considerations, assume that \( \{X_n, n \geq 1\} \) are not eventually degenerate at 0. This assumption is in effect throughout the entire chapter and will not be repeated. The main result of this section, Theorem 1, may now be stated. It will be shown in the proof of Theorem 1 that the hypotheses ensure that \( \{T_n, n \geq 1\} \) is a well-defined sequence of random variables. The proof of Theorem 1 will be deferred until after the proof of the ensuing Lemma 4.

**Theorem 1.** Let \( \{X_n, n \geq 1\} \) be random variables and let \( \{g_n(x), n \geq 1\} \) be strictly increasing functions defined on \([0, \infty)\) such that

\[
g_n(0) = 0 \text{ and } \lim_{x \to \infty} g_n(x) = \infty, \quad n \geq 1. \tag{2.2.1}
\]

Assume that

\[
\frac{x}{g_n(x)} \text{ is nondecreasing as } 0 < x \uparrow \text{ for each } n \geq 1 \tag{2.2.2}
\]

and

\[
g_n(x) \text{ is nondecreasing in } n \text{ for each fixed } x > 0. \tag{2.2.3}
\]

If the series

\[
\sum_{n=1}^{\infty} E\left(g_n(|X_n|)\right) < \infty, \tag{2.2.4}
\]

then setting

\[
A_n = \sum_{j=n}^{\infty} E\left(g_j(|X_j|)\right), \quad n \geq 1
\]
and assuming that for some function \( \psi(x) \in \Psi \)

\[
P \left\{ \left| X_n \right| \leq g_n^{-1} \left( A_n \psi(A_n^{-1}) \right) \right\} \text{ eventually } = 1, \tag{2.2.5}
\]

the tail series SLLN

\[
\frac{T_n}{g_n^{-1} \left( A_n \psi(A_n^{-1}) \right)} \to 0 \text{ a.s.} \tag{2.2.6}
\]

obtains, where \( g_n^{-1} \) denotes the inverse function of \( g_n \) for each \( n \geq 1 \).

Remarks. (i) By the Borel-Cantelli lemma, a sufficient condition for (2.2.5) is

\[
\sum_{n=1}^{\infty} P \left\{ \left| X_n \right| > g_n^{-1} \left( A_n \psi(A_n^{-1}) \right) \right\} < \infty.
\]

(ii) From the definitions of \( A_n \) and the class \( \Psi \), we note that \( A_n \psi(A_n^{-1}) \downarrow \) and so \( g_n^{-1} \left( A_n \psi(A_n^{-1}) \right) \downarrow \). Moreover, the condition (2.2.5) is necessary for (2.2.6) to hold. This follows from the remark after the ensuing Lemma 4 by setting \( b_n = g_n^{-1} \left( A_n \psi(A_n^{-1}) \right) \), \( n \geq 1 \).

(iii) For each \( n \geq 1 \), note that (2.2.2) together with the fact that each \( g_n(x) \) is a nondecreasing function, implies that each \( g_n(x) \) is necessarily a continuous function. In order to prove this, we will show that

\[
g_n(x_0^{-}) = g_n(x_0^{+}) \text{ for arbitrary } x_0 \in (0, \infty) \text{ and for each } n \geq 1. \tag{2.2.7}
\]

Let \( 0 < s < x_0 < t \). Then (2.2.2) ensures that

\[
\frac{s}{g_n(s)} \leq \frac{x_0}{g_n(x_0)} \leq \frac{t}{g_n(t)}, \ n \geq 1.
\]

Take \( s \uparrow x_0 \) and \( t \downarrow x_0 \). Then

\[
\frac{x_0}{g_n(x_0^{-})} = \lim_{s \uparrow x_0} \frac{s}{g_n(s)} \leq \lim_{t \downarrow x_0} \frac{t}{g_n(t)} = \frac{x_0}{g_n(x_0^{+})}, \ n \geq 1.
\]
Therefore, $g_n(x_0^-) \geq g_n(x_0^+)$ for arbitrary $x_0 \in (0, \infty)$ and for each $n \geq 1$. Hence, via the monotonicity of each $g_n$, (2.2.7) follows. □

Assuming (2.2.5) (which is necessary for (2.2.6)), then not only does Theorem 1 reduce to Proposition 2 by setting

$$g_n(x) \equiv |x|^p, \ 0 < p \leq 1, \ n \geq 1,$$

but, Theorem 1 also yields Theorem 2 under the condition (i), without the independence assumption.

The proof of Theorem 1 utilizes the following two lemmas.

**Lemma 3.** Let $\{X_n, \ n \geq 1\}$ be random variables and let $\{g_n(x), \ n \geq 1\}$ be non-decreasing functions defined on $[0, \infty)$ satisfying (2.2.1) and (2.2.2). Furthermore, let $\{b_n, \ n \geq 1\}$ be a sequence of positive constants such that

$$P\{|X_n| < b_n \text{ eventually }\} = 1. \quad (2.2.8)$$

If the series

$$\sum_{n=1}^{\infty} \frac{E(g_n(|X_n|))}{g_n(b_n)} < \infty, \quad (2.2.9)$$

then the series

$$\sum_{n=1}^{\infty} \frac{X_n}{b_n} \text{ converges a.s.} \quad (2.2.10)$$

**Remark.** Since (2.2.10) ensures that $b_n^{-1}X_n \to 0$ a.s., (2.2.8) follows. Thus the condition (2.2.8) is necessary for (2.2.10) to hold.

**Proof of Lemma 3.** By (2.2.8), for almost all $\omega \in \Omega$ there exists an integer $N(\omega)$
such that \(|X_n(\omega)| < b_n\) for all \(n > N(\omega)\), and hence by (2.2.2)

\[
\frac{|X_n(\omega)|}{g_n(|X_n(\omega)|)} \leq \frac{b_n}{g_n(b_n)}, \quad n > N(\omega).
\] (2.2.11)

Next, via the Lebesgue monotone convergence theorem, (2.2.9) ensures that

\[
E\left(\sum_{n=1}^{\infty} \frac{g_n(|X_n|)}{g_n(b_n)}\right) < \infty
\]

and so

\[
\sum_{n=1}^{\infty} \frac{g_n(|X_n|)}{g_n(b_n)} < \infty \text{ a.s.}
\] (2.2.12)

Thus, for almost all \(\omega \in \Omega\)

\[
\sum_{n=1}^{\infty} \frac{|X_n(\omega)|}{b_n} = \sum_{n=1}^{N(\omega)} \frac{|X_n(\omega)|}{b_n} + \sum_{n=N(\omega)+1}^{\infty} \frac{|X_n(\omega)|}{b_n} \leq \sum_{n=1}^{N(\omega)} \frac{|X_n(\omega)|}{b_n} + \sum_{n=N(\omega)+1}^{\infty} \frac{g_n(|X_n(\omega)|)}{g_n(b_n)} \quad \text{(by (2.2.11))}
\]

\[
< \infty \quad \text{(by (2.2.12))}
\]

and therefore (2.2.10) obtains. \(\square\)

Using Lemmas 1 and 3, we obtain the following lemma.

**Lemma 4.** Let \(\{X_n, n \geq 1\}\) be random variables and let \(\{g_n(x), n \geq 1\}\) be non-decreasing functions defined on \([0, \infty)\) satisfying (2.2.1) and (2.2.2). Let \(\{b_n, n \geq 1\}\) be a sequence of positive constants satisfying (2.2.8) with \(b_n \downarrow 0\). If \(g_n(b_n) = O(1)\) and (2.2.9) holds, then the tail series SLLN

\[
\frac{T_n}{b_n} \rightarrow 0 \ a.s.
\] (2.2.13)

obtains.
Remark. The triangle inequality and the fact that $b_n \downarrow$ imply
\[
\frac{|X_n|}{b_n} \leq \frac{|T_n|}{b_n} + \frac{|T_{n+1}|}{b_{n+1}}, \quad n \geq 1
\]
and so (2.2.13) ensures $b_n^{-1}X_n \to 0$ a.s. Thus the condition (2.2.8) is necessary for (2.2.13) to hold.

Proof of Lemma 4. Note that (2.2.9) and $g_n(b_n) = O(1)$ ensure (2.2.4) which, as will be demonstrated in the proof of Theorem 1, ensures that $\{T_n, n \geq 1\}$ is well defined. Employing Lemma 3 yields (2.2.10). Since $b_n \downarrow 0$, the lemma follows from Lemma 1. □

By assuming (2.2.8) which is a necessary condition for (2.2.10) and (2.2.13), Lemmas 3 and 4 yield the ensuing Lemmas 5 and 6, respectively, but without assuming independence in the case when the condition (i) of Theorem 2 is assumed. Also Lemma 4 reduces to Proposition 1 by setting
\[
g_n(x) \equiv |x|^p, \quad 0 < p \leq 1, \quad n \geq 1.
\]

The proof of Theorem 1 may now be given.

Proof of Theorem 1. Firstly, we want to show that the tail series $\{T_n, n \geq 1\}$ is well defined. Since, via the Lebesgue monotone convergence theorem, (2.2.4) ensures
\[
E\left(\sum_{n=1}^{\infty} g_n(|X_n|)\right) < \infty,
\]
(2.2.4) \quad $\Rightarrow$ \quad $\sum_{n=1}^{\infty} g_n(|X_n|) < \infty$ a.s. \quad (2.2.14)

\[
\Rightarrow \quad g_n(|X_n|) \rightarrow 0 \text{ a.s.}
\]

\[
\Rightarrow \quad |X_n| \rightarrow 0 \text{ a.s. (by (2.2.3)).} \quad (2.2.15)
\]
Now let $N(\omega)$ be the random integer defined by

$$N(\omega) = \min \{N \geq 1 : |X_n(\omega)| \leq 1 \text{ for all } n > N\} \quad (= \infty, \text{ otherwise}).$$

Then (2.2.15) ensures that $N(\omega) < \infty$ a.s. whence by (2.2.2), for almost all $\omega \in \Omega$

$$\frac{|X_n|}{g_n(|X_n|)} \leq \frac{1}{g_n(1)}, \quad n > N(\omega). \quad (2.2.16)$$

Thus, for almost all $\omega \in \Omega$

$$\sum_{n=1}^{\infty} |X_n| = \sum_{n=1}^{N(\omega)} |X_n| + \sum_{n=N(\omega)+1}^{\infty} |X_n| \leq \sum_{n=1}^{N(\omega)} |X_n| + \sum_{n=N(\omega)+1}^{\infty} \frac{g_n(|X_n|)}{g_n(1)} \quad \text{(by (2.2.16))}$$

$$\leq \sum_{n=1}^{N(\omega)} |X_n| + M \sum_{n=N(\omega)+1}^{\infty} g_n(|X_n|) \quad \left( M = \frac{1}{g_1(1)} \right)$$

$$< \infty \quad \text{(by (2.2.14))}$$

and so

$$\sum_{n=1}^{\infty} X_n \text{ converges a.s.}$$

Therefore $\{T_n, n \geq 1\}$ is a well-defined sequence of random variables.

Next, let

$$c_n = E\left(g_n(|X_n|)\right), \quad n \geq 1$$

and observe that

$$c_n \geq 0, \quad n \geq 1 \text{ and } \sum_{n=1}^{\infty} c_n < \infty \text{ a.s.} \quad \text{(by (2.2.4)).}$$

Since $A_n = \sum_{j=n}^{\infty} c_j > 0, \quad n \geq 1$, Lemma 2 ensures that for each function $\psi(x) \in \Psi$,

$$\sum_{n=1}^{\infty} \frac{E\left(g_n(|X_n|)\right)}{g_n^{-1}(A_n \psi(A_n^{-1}))} = \sum_{n=1}^{\infty} \frac{E\left(g_n(|X_n|)\right)}{A_n \psi(A_n^{-1})} < \infty. \quad (2.2.17)$$
For each function $\psi(x) \in \Psi$, since

$$0 < g_n^{-1}(A_n \psi(A_n^{-1})) \downarrow 0,$$

then setting

$$b_n = g_n^{-1}(A_n \psi(A_n^{-1})), \ n \geq 1,$$

(2.2.8) and (2.2.9) follows directly from (2.2.5) and (2.2.17), respectively. Thus the theorem follows from Lemma 4 since $g_n(b_n) = A_n \psi(A_n^{-1}) = O(1)$ (see Remark (ii) after the statement of Theorem 1). $\Box$

We obtain the following two truncated versions of Theorem 1 as corollaries.

**Corollary 1.** Let $\{X_n, \ n \geq 1\}$ be random variables and let $\{g_n(x), \ n \geq 1\}$ be strictly increasing functions defined on $[0, \infty)$ satisfying (2.2.1), (2.2.2) and (2.2.3). If

$$\sum_{n=1}^{\infty} P\{|X_n| > C_n\} < \infty \quad (2.2.18)$$

and

$$\sum_{n=1}^{\infty} E\left(g_n(|X_n I_{\{|X_n| \leq C_n\}}|)\right) < \infty \quad (2.2.19)$$

are satisfied for some sequence of positive constants $\{C_n, \ n \geq 1\}$, then setting

$$\tilde{A}_n = \sum_{j=n}^{\infty} E\left(g_j(|X_j I_{\{|X_j| \leq C_j\}}|)\right), \ n \geq 1$$

and assuming that for some function $\psi(x) \in \Psi$

$$P\{|X_n I_{\{|X_n| \leq C_n\}}| \leq g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \ \text{eventually}\} = 1, \quad (2.2.20)$$

the tail series SLLN

$$\frac{T_n}{g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1}))} \rightarrow 0 \ a.s. \quad (2.2.21)$$
obtains.

Remarks. (i) A sufficient condition for (2.2.20) to hold is

$$\sum_{n=1}^{\infty} P \left \{ \left| X_n I_{[|X_n| \leq C_n]} \right| > g_n^{-1} \left( A_n \psi (A_n^{-1}) \right) \right \} < \infty ,$$

by the Borel-Cantelli lemma.

(ii) Since (2.2.18) asserts that \( \{ X_n, n \geq 1 \} \) and \( \{ X_n I_{[|X_n| \leq C_n]}, n \geq 1 \} \) are equivalent in the sense of Khintchine, (2.2.21) is equivalent to

$$\frac{T_n^*}{g_n^{-1} (A_n \psi (A_n^{-1}))} \to 0 \text{ a.s.} \quad (2.2.22)$$

where \( T_n^* = \sum_{j=n}^{\infty} X_j I_{[|X_j| \leq C_j]}, n \geq 1 \). Note that \( g_n^{-1} (A_n \psi (A_n^{-1})) \downarrow \). By the argument in the remark after the statement of Lemma 4, \textit{mutatis mutandis}, (2.2.22) ensures the condition (2.2.20). Thus the condition (2.2.20) is necessary for (2.2.21) to hold.

(iii) The condition (2.2.18) ensures that (2.2.20) is equivalent to the apparently stronger but structurally simpler condition

$$P \left \{ \left| X_n \right| \leq g_n^{-1} \left( A_n \psi (A_n^{-1}) \right) \text{ eventually} \right \} = 1.$$

Proof of Corollary 1. Set

$$Z_n = X_n I_{[|X_n| \leq C_n]}, n \geq 1.$$

Then, by applying Theorem 1 to the random variables \( \{ Z_n, n \geq 1 \} \), (2.2.19) ensures that the tail series \( T_n^* = \sum_{j=n}^{\infty} Z_j \) is well defined and then (2.2.22) obtains for each function \( \psi (x) \in \Psi \). Since \( \{ X_n, n \geq 1 \} \) and \( \{ X_n I_{[|X_n| \leq C_n]}, n \geq 1 \} \) are equivalent in
the sense of Khintchine, \( \{ T_n, n \geq 1 \} \) is also well defined and the corollary follows.

□

**Corollary 2.** Let \( \{ X_n, n \geq 1 \} \) be random variables and let \( \{ g_n(x), n \geq 1 \} \) be strictly increasing functions defined on \([0, \infty)\) satisfying (2.2.1), (2.2.2) and (2.2.3).

If (2.2.18) and

\[
\sum_{n=1}^{\infty} E \left( g_n(\lfloor X_n \lfloor_{[x_n] \leq C_n} - E(X_n \lfloor_{[x_n] \leq C_n})) \right) < \infty \tag{2.2.23}
\]

are satisfied for some sequence of positive constants \( \{ C_n, n \geq 1 \} \), then setting

\[
\tilde{A}_n = \sum_{j=n}^{\infty} E \left( g_j(\lfloor X_j \lfloor_{[x_j] \leq C_j} - E(X_j \lfloor_{[x_j] \leq C_j})) \right), \ n \geq 1
\]

and

\[
\tilde{T}_n = \sum_{j=n}^{\infty} \left\{ X_j - E(X_j \lfloor_{[x_j] \leq C_j}) \right\}, \ n \geq 1, \tag{2.2.24}
\]

and assuming for some function \( \psi(x) \in \Psi \) that

\[
P \left\{ \left| X_n \lfloor_{[x_n] \leq C_n} - E(X_n \lfloor_{[x_n] \leq C_n}) \right| \leq g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \text{ eventually} \right\} = 1, \tag{2.2.25}
\]

the tail series SLLN

\[
\frac{\tilde{T}_n}{g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1}))} \to 0 \tag{2.2.26}
\]

obtains.

**Remarks.** By the argument in Remarks (i), (ii), and (iii) after the statement of Corollary 1, *mutatis mutandis*, we observe, respectively, that

(i) A sufficient condition for (2.2.25) is

\[
\sum_{n=1}^{\infty} P \left\{ \left| X_n \lfloor_{[x_n] \leq C_n} - E(X_n \lfloor_{[x_n] \leq C_n}) \right| > g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \right\} < \infty.
\]
(ii) The condition (2.2.25) is necessary for (2.2.26) to hold.

(iii) The condition (2.2.18) ensures that (2.2.25) is equivalent to the condition

\[ P \{ |X_n - E(X_n I_{|X_n| \leq C_n})| \leq g_n^{-1}(\bar{A}_n \psi(\bar{A}_n^{-1})) \text{ eventually} \} = 1. \]

Proof of Corollary 2. Set

\[ Z_n = X_n I_{|X_n| \leq C_n} - E(X_n I_{|X_n| \leq C_n}), \quad n \geq 1. \]

Then the result follows from (2.2.18) and (2.2.23) by employing the argument in Corollary 1, \textit{mutatis mutandis}. □

2.3 Tail Series SLLNs for Independent Random Variables

For independent random variables \( \{X_n, n \geq 1\} \) we obtain the following tail series SLLNs. In part \( i \) of the ensuing theorem, the condition (2.2.5) of Theorem 1 is dispensed with at the expense of assuming that \( \{X_n, n \geq 1\} \) are independent. The main result of this section, Theorem 2, may now be stated. As in Theorem 1, it will be shown in the proof of Theorem 2 that the hypotheses ensure that \( \{T_n, n \geq 1\} \) is a well-defined sequence of random variables. The proof of Theorem 2 will be deferred until after the proof of the ensuing Lemma 6.

\textbf{Theorem 2.} Let \( \{X_n, n \geq 1\} \) be independent random variables and let \( \{g_n(x), n \geq 1\} \) be strictly increasing functions defined on \([0, \infty)\) such that

\[ g_n(0) = 0 \text{ and } \lim_{x \to \infty} g_n(x) = \infty, \quad n \geq 1 \tag{2.3.1} \]

and assume that

\[ g_n(x) \text{ is nondecreasing in } n \text{ for each fixed } x > 0. \tag{2.3.2} \]
Suppose that one of the following two conditions prevails

(i) \( \frac{x}{g_n(x)} \) is nondecreasing as \( 0 < x \uparrow \) for each \( n \geq 1 \).

(ii) \( \frac{g_n(x)}{x} \) is nondecreasing as \( 0 < x \uparrow \), \( \frac{x^2}{g_n(x)} \) is nondecreasing as \( 0 < x \uparrow \),

and \( \mathbb{E}(X_n) = 0 \), for each \( n \geq 1 \).

If the series

\[
\sum_{n=1}^{\infty} \mathbb{E}(g_n(|X_n|)) < \infty,
\]

(2.3.3)

then setting

\[ A_n = \sum_{j=n}^{\infty} \mathbb{E}(g_j(|X_j|)), \quad n \geq 1, \]

the tail series SLLN

\[
\frac{T_n}{g_n^{-1}(A_n \psi(A_n^{-1}))} \to 0 \text{ a.s.}
\]

obtains for each function \( \psi(x) \in \Psi \), where \( g_n^{-1} \) denotes the inverse function of \( g_n \) for each \( n \geq 1 \).

Remark. Note that for each \( n \geq 1 \), the hypotheses to (i) or (ii), together with the fact that each \( g_n(x) \) is a nondecreasing function, imply that each \( g_n(x) \) is necessarily a continuous function. Under the hypotheses to (i) the continuity of each \( g_n(x) \) follows directly from Remark (iii) after the statement of Theorem 1. So it is enough to show that each \( g_n(x) \) is a continuous function under the hypotheses to (ii). To this end, we will prove that

\( g_n(x_0^-) = g_n(x_0^+) \) for arbitrary \( x_0 \in (0, \infty) \) and for each \( n \geq 1 \).

(2.3.4)

Let \( 0 < s < x_0 < t \). Then (ii) of the theorem ensures that

\[
\frac{s^2}{g_n(s)} \leq \frac{x_0^2}{g_n(x_0)} \leq \frac{t^2}{g_n(t)}, \quad n \geq 1.
\]
Take \( s \uparrow x_0 \) and \( t \downarrow x_0 \). Then

\[
\frac{x_0^2}{g_n(x_0)} = \lim_{s \uparrow x_0} \frac{s^2}{g_n(s)} \leq \lim_{t \downarrow x_0} \frac{t^2}{g_n(t)} = \frac{x_0^2}{g_n(x_0^+)}, \quad n \geq 1.
\]

Therefore for each \( n \geq 1 \),

\[
g_n(x_0^-) \geq g_n(x_0^+) \text{ for arbitrary } x_0 \in (0, \infty) \text{ and for each } n \geq 1.
\]

Hence, via the monotonicity of each \( g_n \), (2.3.4) follows. □

Theorem 2 reduces to Proposition 4 by setting \( g_n \equiv g \), \( n \geq 1 \). And also Theorem 2, under the hypotheses to (i), follows directly from Theorem 1 by assuming the condition (2.2.5) which is a necessary condition for the result to hold. Moreover, as will become apparent, Theorem 2, under the hypotheses to (ii), owes much to the work of Klesov [30].

The proof of Theorem 2 utilizes the following two lemmas.

**Lemma 5** (Petrov [38]). Let \( \{X_n, n \geq 1\} \) be independent random variables and let \( \{g_n(x), n \geq 1\} \) be nondecreasing functions defined on \([0, \infty)\) satisfying (2.3.1). Assume that condition (i) or (ii) of Theorem 2 holds. Further, let \( \{b_n, n \geq 1\} \) be a sequence of positive constants. If the series

\[
\sum_{n=1}^{\infty} \frac{E(g_n(|X_n|))}{g_n(b_n)} < \infty, \quad (2.3.5)
\]

then the series

\[
\sum_{n=1}^{\infty} \frac{X_n}{b_n} \text{ converges a.s.} \quad (2.3.6)
\]

Lemma 5, under the condition (ii) of Theorem 2, was proved for the case \( g_n \equiv g, n \geq 1 \), by Chung [17, p. 124].
Using Lemmas 1 and 5, we obtain the following lemma.

**Lemma 6.** Let \( \{X_n, n \geq 1\} \) be independent random variables and let \( \{g_n(x), n \geq 1\} \) be nondecreasing functions defined on \([0, \infty)\) satisfying (2.3.1). Assume that condition (i) or (ii) of Theorem 2 holds. Let \( \{b_n, n \geq 1\} \) be a sequence of positive constants with \( b_n \downarrow 0 \). If \( g_n(b_n) = O(1) \) and (2.3.5) holds, then the tail series SLLN

\[
\frac{T_n}{b_n} \to 0 \text{ a.s.}
\]

obtains.

**Proof.** Note that (2.3.5) and \( g_n(b_n) = O(1) \) ensure (2.3.3) which, as will be demonstrated in the proof of Theorem 2, ensures that \( \{T_n, n \geq 1\} \) is well defined. Employing Lemma 5 yields (2.3.6). Since \( b_n \downarrow 0 \), the lemma follows from Lemma 1.

\( \square \)

Not only does Lemma 6 reduce to Proposition 3 by taking \( g_n \equiv g, n \geq 1 \), but it also is a tail series analogue of Petrov [40, Section 9. 3, Theorem 11]. Moreover, if (2.2.8) holds, then under the hypotheses to (i) of Theorem 2, Lemmas 5 and 6 follow directly from Lemmas 3 and 4, respectively.

The proof of Theorem 2 may now be given.

**Proof of Theorem 2.** Note at the outset that in the proof of Theorem 1 the condition (2.2.5) was not employed to establish that the tail series \( \{T_n, n \geq 1\} \) is well defined. Consequently, under the hypotheses to (i), \( \{T_n, n \geq 1\} \) is a well-defined sequence of random variables.

Next, it will be verified by employing the Kolmogorov three-series criterion that under the hypotheses to (ii), \( \sum_{n=1}^{\infty} X_n \) converges a.s. and hence \( \{T_n, n \geq 1\} \) is well
defined. For each \( n \geq 1 \)

\[
P\{|X_n| > 1\} = P\{g_n(|X_n|) > g_n(1)\} \\
\leq \frac{E(g_n(|X_n|))}{g_n(1)} \quad \text{(by the Markov inequality)} \\
\leq M E\left(g_n(|X_n|)\right) \left(M = \frac{1}{g_1(1)}\right)
\]

and so

\[
\sum_{n=1}^{\infty} P \{|X_n| > 1\} \leq M \sum_{n=1}^{\infty} E\left(g_n(|X_n|)\right) < \infty \quad \text{(by (2.3.3)).}
\]

Now for each \( n \geq 1 \)

\[
\left|E\left(X_n I_{\{|X_n| \leq 1\}}\right)\right| = \left|E\left(X_n I_{\{|X_n| > 1\}}\right)\right| \quad \text{(since } E(X_n) = 0) \\
\leq E\left(|X_n| I_{\{|X_n| > 1\}}\right) \\
\leq \frac{E\left(g_n(|X_n|) I_{\{|X_n| > 1\}}\right)}{g_n(1)} \quad \text{(since } \frac{x}{g_n(x)} \leq \frac{1}{g_n(1)}, \ x > 1) \\
\leq \frac{E\left(g_n(|X_n|)\right)}{g_n(1)} \\
\leq M E\left(g_n(|X_n|)\right) \left(M = \frac{1}{g_1(1)}\right).
\]

Thus,

\[
\sum_{n=1}^{\infty} \left|E\left(X_n I_{\{|X_n| \leq 1\}}\right)\right| \leq M \sum_{n=1}^{\infty} E\left(g_n(|X_n|)\right) < \infty \quad \text{(by (2.3.3))}
\]

implying that

\[
\sum_{n=1}^{\infty} E\left(X_n I_{\{|X_n| \leq 1\}}\right) \text{ converges.}
\]

Again, for each \( n \geq 1 \)

\[
\text{Var}\left(X_n I_{\{|X_n| \leq 1\}}\right) \\
\leq E\left(X_n^2 I_{\{|X_n| \leq 1\}}\right) \\
\leq \frac{E\left(g_n(|X_n|) I_{\{|X_n| \leq 1\}}\right)}{g_n(1)} \quad \text{(since } \frac{x^2}{g_n(x)} \leq \frac{1}{g_n(1)}, \ x \leq 1)\]
\[
\begin{align*}
\frac{E(g_n(|X_n|))}{g_n(1)} & \leq M E\left(g_n(|X_n|)\right) \left( M = \frac{1}{g_1(1)} \right) \\
\end{align*}
\]

implying

\[
\sum_{n=1}^{\infty} \text{Var}(X_n I_{[|X_n| \leq 1]}) \leq M \sum_{n=1}^{\infty} E\left(g_n(|X_n|)\right) < \infty \quad \text{(by (2.3.3)).}
\]

Hence the conditions of the Kolmogorov three-series criterion are satisfied thereby ensuring that

\[
\sum_{n=1}^{\infty} X_n \text{ converges a.s.}
\]

Therefore, in both cases, \{T_n, n \geq 1\} is a well-defined sequence of random variables.

Next, let

\[
c_n = E\left(g_n(|X_n|)\right), \quad n \geq 1
\]

and observe that

\[
c_n \geq 0, \quad n \geq 1, \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty \quad \text{(by (2.3.3)).}
\]

Since \( A_n = \sum_{j=n}^{\infty} c_j > 0, \quad n \geq 1, \) Lemma 2 ensures that for each function \( \psi(x) \in \Psi, \)

\[
\sum_{n=1}^{\infty} \frac{E\left(g_n(|X_n|)\right)}{g_n^{-1}\left(\mathcal{A}_n \psi(\mathcal{A}_n^{-1})\right)} = \sum_{n=1}^{\infty} \frac{E\left(g_n(|X_n|)\right)}{\mathcal{A}_n \psi(\mathcal{A}_n^{-1})} < \infty.
\]

Now for each function \( \psi(x) \in \Psi, \) since

\[
0 < g_n^{-1}\left(\mathcal{A}_n \psi(\mathcal{A}_n^{-1})\right) \downarrow 0,
\]

setting

\[
b_n = g_n^{-1}\left(\mathcal{A}_n \psi(\mathcal{A}_n^{-1})\right), \quad n \geq 1,
\]
the condition (2.3.5) holds. The theorem then follows directly from Lemma 6 since
\[ g_n(b_n) = A_n \psi(A_n^{-1}) = O(1). \]

We obtain the following two truncated versions of Theorem 2 as corollaries by employing an argument similar to that used to establish Corollaries 1 and 2 of Section 2.1.

**Corollary 3.** Let \( \{X_n, n \geq 1\} \) be independent random variables and let \( \{g_n(x), n \geq 1\} \) be strictly increasing functions defined on \([0, \infty)\) satisfying (2.3.1) and (2.3.2).

Assume that the condition (i) or (ii) of Theorem 2 holds. If

\[
\sum_{n=1}^{\infty} \mathbb{P}\{|X_n| > C_n\} < \infty \quad (2.3.7)
\]

and

\[
\sum_{n=1}^{\infty} \mathbb{E}(g_n(|X_n| I_{\{|X_n| \leq C_n\}})) < \infty \quad (2.3.8)
\]

are satisfied for some sequence of positive constants \( \{C_n, n \geq 1\} \), then setting

\[
\tilde{A}_n = \sum_{j=n}^{\infty} \mathbb{E}(g_j(|X_j| I_{\{|X_j| \leq C_j\}})), \quad n \geq 1,
\]

the tail series SLLN

\[
\frac{T_n}{g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1}))} \to 0 \text{ a.s.} \quad (2.3.9)
\]

obtains for each function \( \psi(x) \in \Psi \).

**Remark.** A necessary condition for (2.3.9) to hold is that (2.3.7) obtains with

\[
C_n = g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})), \quad n \geq 1. \quad (2.3.10)
\]

**Proof of Remark.** The triangle inequality and the fact that \( g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \downarrow \)
imply

\[
\left| X_n \right| \leq \frac{|T_n|}{g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1}))} + \frac{|T_{n+1}|}{g_{n+1}^{-1}(\tilde{A}_{n+1} \psi(\tilde{A}_{n+1}^{-1}))}, \ n \geq 1.
\]

Thus (2.3.9) ensures

\[
\frac{X_n}{g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1}))} \to 0 \text{ a.s.}
\]

Using the Borel-Cantelli lemma and the independence of \( \{X_n, \ n \geq 1\} \), we obtain

\[
\sum_{n=1}^{\infty} P\left\{ \left| X_n \right| > g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \right\} < \infty.
\]

Hence (2.3.7) holds with \( \{C_n, \ n \geq 1\} \) as in (2.3.10). □

Proof of Corollary 3. Set

\[
Z_n = X_n I_{\{|X_n| \leq C_n\}}, \ n \geq 1.
\]

Then, by applying Theorem 2 to the random variables \( \{Z_n, \ n \geq 1\} \), (2.3.8) implies that the tail series \( T_n^* \equiv \sum_{j=n}^{\infty} Z_j \) is well defined and the tail series SLLN

\[
\frac{T_n^*}{g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1}))} \to 0 \text{ a.s.}
\]

obtains for each function \( \psi(x) \in \Psi \). Since (2.3.7) implies that \( \{X_n, \ n \geq 1\} \) and \( \{X_n I_{\{|X_n| \leq C_n\}}, \ n \geq 1\} \) are equivalent in the sense of Khintchine, \( \{T_n, n \geq 1\} \) is also well defined and Corollary 3 follows. □

Corollary 4. Let \( \{X_n, \ n \geq 1\} \) be independent random variables and let \( \{g_n(x), \ n \geq 1\} \) be strictly increasing functions defined on \([0, \infty)\) satisfying (2.3.1) and (2.3.2). Assume that condition (i) or (ii) of Theorem 2 holds. If (2.3.7) and

\[
\sum_{n=1}^{\infty} E\left(g_n(|X_n I_{\{|X_n| \leq C_n\}} - E(X_n I_{\{|X_n| \leq C_n\}})|) \right) < \infty \tag{2.3.11}
\]
are satisfied for some sequence of positive constants \( \{C_n, \ n \geq 1\} \), then setting

\[
\tilde{A}_n = \sum_{j=n}^{\infty} E\left( g_j(|X_j| I_{|X_j| \leq C_j} - E(X_j I_{|X_j| \leq C_j})) \right) , \ n \geq 1
\]

and

\[
\tilde{T}_n = \sum_{j=n}^{\infty} \left\{ X_j - E\left( X_j I_{|X_j| \leq C_j} \right) \right\} , \ n \geq 1,
\]

the tail series SLLN

\[
\frac{\tilde{T}_n}{g_n^{-1}\left( \tilde{A}_n \psi(\tilde{A}_n^{-1}) \right)} \to 0 \text{ a.s.}
\]

obtains for each function \( \psi(x) \in \Psi \).

**Proof.** Set

\[
Z_n = X_n I_{|X_n| \leq C_n} - E\left( X_n I_{|X_n| \leq C_n} \right) , \ n \geq 1.
\]

Then the corollary follows from (2.3.7) and (2.3.11) by employing the argument in Corollary 3, *mutatis mutandis*. □

### 2.4 Examples

Three examples are provided to illustrate some of the current results as well as to compare them with related results in the literature.

**Example 1.** Let \( \{X_n, \ n \geq 1\} \) be random variables (not necessarily independent) such that

\[
P\left\{ X_n = \frac{1}{n^2} \right\} = 1 - \frac{1}{n^2} \text{ and } P\{X_n = e^n\} = \frac{1}{n^2}, \ n \geq 1.
\]

Then for any \( p \in (0, 1] \),

\[
E(|X_n|^p) = \frac{1}{n^{2p}} \left( 1 - \frac{1}{n^2} \right) + \frac{e^{np}}{n^2}, \ n \geq 1
\]
and so

\[ \sum_{n=1}^{\infty} \mathbb{E}(|X_n|^p) = \infty \text{ for all } p \in (0, 1). \]

Therefore the hypotheses of Proposition 2 are not met.

Let \( \frac{1}{2} < \alpha < 1 \) and let

\[ g_n(x) \equiv (\log x)^{\alpha}, \ n \geq 1. \quad (2.4.1) \]

Then, recalling the definition of \( \log x \), the conditions (2.2.1), (2.2.2), and (2.2.3) are satisfied. For each \( n \geq 1 \)

\[ \mathbb{E}(g_n(|X_n|)) = \mathbb{E}((\log |X_n|)^{\alpha}) = \frac{1}{e^\alpha} \frac{1}{n^{2\alpha}} \left(1 - \frac{1}{n^2}\right) + \frac{n^\alpha}{n^2} \]

and so

\[ \sum_{n=1}^{\infty} \mathbb{E}(g_n(|X_n|)) \leq \sum_{n=1}^{\infty} \left\{ \frac{1}{n^{2\alpha}} + n^{\alpha-2} \right\} < \infty. \]

Hence (2.2.4) holds.

Now for \( n \geq 1 \),

\[ A_n = \sum_{j=n}^{\infty} \mathbb{E}((\log |X_j|)^{\alpha}) \]

\[ = \sum_{j=n}^{\infty} \left\{ \frac{1}{e^\alpha} \frac{1}{j^{2\alpha}} \left(1 - \frac{1}{j^2}\right) + j^{\alpha-2} \right\} \]

\[ \sim M_1 n^{1-2\alpha} + M_2 n^{\alpha-1} \left(M_1 = \frac{1}{e^\alpha} \frac{1}{2\alpha - 1} \text{ and } M_2 = \frac{1}{1 - \alpha}\right). \quad (2.4.2) \]

Suppose that \( \frac{1}{2} < \alpha \leq \frac{2}{3} \). Then

\[ A_n \sim M_1 n^{1-2\alpha}. \]

If \( \psi(x) \) is taken to be the function

\[ \psi(x) = (\log x)^{1+\varepsilon} \text{ where } \varepsilon > 0, \quad (2.4.3) \]
then for all large $n$

$$\mathcal{A}_n \psi(\mathcal{A}_n^{-1}) \sim M_3 n^{1-2\alpha} (\log_1 n)^{1+\varepsilon} \left( M_3 = \frac{1}{e^\alpha (2\alpha - 1)^\varepsilon} \right)$$

$$= o(1).$$

Therefore, for all large $n$

$$\mathcal{A}_n \psi(\mathcal{A}_n^{-1}) \leq 1$$

implying

$$g_n^{-1}(\mathcal{A}_n \psi(\mathcal{A}_n^{-1})) = e \left( \mathcal{A}_n \psi(\mathcal{A}_n^{-1}) \right)^{1/\alpha} \sim M_4 n^{1/\alpha - 2} (\log_1 n)^{1+\varepsilon} \left( M_4 = e M_3^{1/\alpha} \right).$$

Thus,

$$P \left\{ |X_n| > g_n^{-1}(\mathcal{A}_n \psi(\mathcal{A}_n^{-1})) \right\} = \frac{1}{n^2} \text{ for all large } n \quad (2.4.4)$$

and so, via Remark (i) after the statement of Theorem 1, the condition (2.2.5) also holds. Hence for each $\alpha \in \left( \frac{1}{2}, \frac{2}{3} \right]$, the tail series SLLN

$$\frac{T_n}{n^{1-2} (\log_1 n)} \to 0 \text{ a.s.} \quad (2.4.5)$$

obtains by Theorem 1, i.e.,

$$\frac{n^{2-1/\alpha}}{(\log_1 n)^{1+\varepsilon/\alpha}} T_n \to 0 \text{ a.s.} \quad (2.4.5)$$

On the other hand, suppose that $\frac{2}{3} < \alpha < 1$. Then recalling (2.4.2),

$$\mathcal{A}_n \sim M_2 n^{\alpha-1}.$$

If $\psi(x)$ be taken to be the function as in (2.4.3), then for all large $n$

$$\mathcal{A}_n \psi(\mathcal{A}_n^{-1}) \sim M_5 n^{\alpha-1} (\log_1 n)^{1+\varepsilon} \left( M_5 = (1 - \alpha)^\varepsilon \right)$$

$$= o(1)$$
implying

\[ \mathcal{A}_n \psi(\mathcal{A}_n^{-1}) \leq 1 \]

for all large \( n \) and so

\[
g_n^{-1}(\mathcal{A}_n \psi(\mathcal{A}_n^{-1})) = e(\mathcal{A}_n \psi(\mathcal{A}_n^{-1}))^{\frac{1}{\alpha}} \sim M_6 n^{1 - \frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}} \left( M_6 = e M_5^{\frac{1}{\alpha}} \right). \]

Thus, (2.4.4) holds and so, via Remark (i) after the statement of Theorem 1, the condition (2.2.5) also holds. Hence for each \( \alpha \in \left(\frac{2}{3}, 1\right) \), the tail series SLLN

\[
\frac{T_n}{n^{1 - \frac{1}{\alpha}} (\log n)^{\frac{1}{\alpha}}} \rightarrow 0 \text{ a.s.}
\]

obtains by Theorem 1, i.e.,

\[
\frac{n^{\frac{1}{\alpha} - 1}}{(\log n)^{\frac{1}{\alpha}}} T_n \rightarrow 0 \text{ a.s.} \quad (2.4.6)
\]

Next, it will now be demonstrated that Corollary 1 can be also applied. Let

\( \frac{1}{2} < p \leq 1 \) and let

\[ g_n(x) \equiv |x|^p, \ n \geq 1. \quad (2.4.7) \]

Then the conditions (2.2.1), (2.2.2), and (2.2.3) are satisfied. Set

\[ C_n \equiv 2, \ n \geq 1. \quad (2.4.8) \]

Then

\[ P\{|X_n| > C_n\} = \frac{1}{n^2}, \ n \geq 1 \]

implying (2.2.18). Also

\[
E\left(g_n(|X_n I_{[|X_n| \leq C_n]}|)\right) = E\left(|X_n I_{[|X_n| \leq C_n]}|^p\right) = \frac{1}{n^{2p}} \left(1 - \frac{1}{n^2}\right), \ n \geq 1
\]
and so
\[ \sum_{n=1}^{\infty} \mathbb{E} \left( g_n \left( |X_n I_{|X_n| \leq C_n}| \right) \right) \leq \sum_{n=1}^{\infty} \frac{1}{n^{2p}} < \infty. \]

Thus the condition (2.2.19) holds. Now for \( n \geq 1 \),
\[ \tilde{A}_n = \sum_{j=n}^{\infty} \mathbb{E} \left( |X_j I_{|X_j| \leq C_j}|^p \right) = \sum_{j=n}^{\infty} \frac{1}{j^{2p}} \left( 1 - \frac{1}{j^2} \right) \sim M_7 n^{1-2p} \left( M_7 = \frac{1}{2p-1} \right). \]

If \( \psi(x) \) be taken to be the function as in (2.4.3), then for all large \( n \),
\[ \tilde{A}_n \psi(\tilde{A}_n^{-1}) \sim M_8 n^{1-2p} (\log_1 n)^{1+\epsilon} \left( M_8 = (2p-1)^r \right) = o(1). \]

Hence for all large \( n \),
\[ \tilde{A}_n \psi(\tilde{A}_n^{-1}) \leq 1 \]

implying
\[ g_n^{-1} \left( \tilde{A}_n \psi(\tilde{A}_n^{-1}) \right) = \left( \tilde{A}_n \psi(\tilde{A}_n^{-1}) \right)^{\frac{1}{p}} \sim M_9 n^{\frac{1}{p} - 2} (\log_1 n)^{\frac{1+\epsilon}{p}} \left( M_9 = M_8^{\frac{1}{p}} \right). \]

Thus, (2.4.4) holds and so, via Remark (i) after the statement of Corollary 1, the condition (2.2.20) also holds. Therefore, by Corollary 1 the tail series SLLN
\[ \frac{T_n}{n^{\frac{1}{p} - 2} (\log_1 n)^{\frac{1+\epsilon}{p}}} \rightarrow 0 \ \text{a.s.} \]

obtains, i.e.,
\[ \frac{n^{2-\frac{1}{p}}}{(\log_1 n)^{\frac{1+\epsilon}{p}}} T_n \rightarrow 0 \ \text{a.s.} \]
Not only is this result sharper than (2.4.5) for \( p > \alpha, \alpha \in \left(\frac{1}{2}, \frac{2}{3}\right) \), but, also, this result is sharper than (2.4.6) for \( p > \frac{\alpha}{3\alpha - 1}, \alpha \in \left(\frac{2}{3}, 1\right) \). Hence, for \( p \in \left(\frac{2}{3}, 1\right) \), Corollary 1 gives us a better result than that which can be obtained by Theorem 1.

In conclusion, it may be noted that Theorem 2 and Corollary 3 can also be applied to this example by setting \( \{g_n, n \geq 1\} \) as in (2.4.1) or as in (2.4.7), respectively, if \( \{X_n, n \geq 1\} \) are assumed to be independent.

**Example 2.** Let \( \{X_n, n \geq 1\} \) be random variables (not necessarily independent) such that

\[
P\{X_n = 1\} = 1 - \frac{1}{n^2} \quad \text{and} \quad P\{X_n = e^n\} = \frac{1}{n^2}, \quad n \geq 1.
\]

Then for any \( p \in (0, 1] \),

\[
E(|X_n|^p) = \left(1 - \frac{1}{n^2}\right) + \frac{e^{np}}{n^2}, \quad n \geq 1
\]

and so

\[
\sum_{n=1}^{\infty} E(|X_n|^p) = \infty, \quad \text{for all} \quad p \in (0, 1].
\]

Therefore the hypotheses of Proposition 2 are not met.

It will now be demonstrated that Corollary 2 can be applied. Let \( \frac{1}{2} < p \leq 1 \).

Then by the *argument* in Example 1, setting \( \{g_n, n \geq 1\} \) as in (2.4.7) and \( \{C_n, n \geq 1\} \) as in (2.4.8), the conditions (2.2.1), (2.2.2), (2.2.3), and (2.2.18) are satisfied. Moreover, for each \( n \geq 1 \)

\[
E\left(g_n(|X_n I_{[|X_n| \leq C_n]} - E(X_n I_{[|X_n| \leq C_n]}))\right) = E\left(|X_n I_{[|X_n| \leq C_n]} - E(X_n I_{[|X_n| \leq C_n]}))|^p\right)
\]

\[
= E\left(|X_n I_{[|X_n| \leq C_n]} - \left(1 - \frac{1}{n^2}\right)|^p\right)
\]

\[
= \frac{1}{n^{2p}} \left(1 - \frac{1}{n^2}\right) + \left(1 - \frac{1}{n^2}\right)^p \frac{1}{n^2}
\]
implying
\[ \sum_{n=1}^{\infty} E\left( g_n\left( |X_n I_{|X_n| \leq C_n} - E(X_n I_{|X_n| \leq C_n})|\right) \right) \leq \sum_{n=1}^{\infty} \left\{ \frac{1}{n^{2p}} + \frac{1}{n^{2}} \right\} < \infty. \]

Hence (2.2.23) holds.

Next for \( n \geq 1 \),
\[ \tilde{A}_n = \sum_{j=n}^{\infty} E\left( |X_j I_{|X_j| \leq C_j} - E(X_j I_{|X_j| \leq C_j})|^p \right) \]
\[ = \sum_{j=n}^{\infty} \left\{ \frac{1}{j^{2p}} \left( 1 - \frac{1}{j^2} \right) + \left( 1 - \frac{1}{j^2} \right)^{p-1} \frac{1}{j^2} \right\} \]
\[ \sim M_1 n^{1-2p} \left( M_1 = \frac{1}{2p-1} \right). \]

If \( \psi(x) \) be taken to be the function as in (2.4.3), then for all large \( n \),
\[ \tilde{A}_n \psi(\tilde{A}_n^{-1}) \sim M_2 n^{1-2p} (\log_1 n)^{1+\epsilon} \left( M_2 = (2p-1)^{1/2} \right) \]
\[ = o(1). \]

Hence, for all large \( n \)
\[ \tilde{A}_n \psi(\tilde{A}_n^{-1}) \leq 1 \]

and so
\[ g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) = (\tilde{A}_n \psi(\tilde{A}_n^{-1}))^{1/2} \sim M_3 n^{1/2-2(\log_1 n)^{1+\epsilon}} \left( M_3 = M_2^{1/2} \right). \]

Thus, for all large \( n \)
\[ P \left\{ |X_n I_{|X_n| \leq C_n} - E(X_n I_{|X_n| \leq C_n})| > g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \right\} \]
\[ = P \left\{ |I_{|X_n| \leq 2} - P\{|X_n| \leq 2\}| > g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \right\} \]
\[ \leq P \left\{ |X_n| \leq 2 \right\} \cap \left\{ |I_{|X_n| \leq 2} - P\{|X_n| \leq 2\}| > g_n^{-1}(\tilde{A}_n \psi(\tilde{A}_n^{-1})) \right\} + P\{|X_n| > 2\} \]
\[ P \left( \left| X_n \right| \leq 2 \right) \cap \left[ \left( 1 - 1 + \frac{1}{n^2} > g_n^{-1} \left( \tilde{A}_n \rho(\tilde{A}_n^{-1}) \right) \right] \right) + P \left( \left| X_n \right| > 2 \right) = P \{ \left| X_n \right| > 2 \} = \frac{1}{n^2} \]

and so by Remark (i) after the statement of Corollary 2 the condition (2.2.25) also holds. Therefore, by Corollary 2, the tail series SLLN

\[ \frac{\tilde{T}_n}{n^{p-2} (\log n)^{1+\epsilon}} \to 0 \text{ a.s.} \]

obtains for the tail series \( \{\tilde{T}_n, n \geq 1\} \) defined as in (2.2.24), i.e.,

\[ \frac{n^{2-1/p}}{(\log n)^{1+\epsilon}} \tilde{T}_n \to 0 \text{ a.s.} \]

In conclusion, it may be noted that Corollary 4 can also be applied to this example by setting \( \{g_n, n \geq 1\} \) as in (2.4.7) in the case when \( \{X_n, n \geq 1\} \) are assumed to be independent.

In the following example, we will consider the rate of almost sure convergence of the harmonic series with a random choice of signs.

**Example 3.** Let \( \{X_n, n \geq 1\} \) be independent random variables such that

\[ P \left\{ X_n = \frac{1}{n} \right\} = P \left\{ X_n = -\frac{1}{n} \right\} = \frac{1}{2}, \ n \geq 1. \]

The series of partial sums \( S_n = \sum_{j=1}^{n} X_j, \ n \geq 1, \) can be interpreted as the harmonic series with a random choice of signs. We will employ Theorem 2 to determine its rate of convergence to a random variable.

Let \( 0 < \alpha < 1 \) and let

\[ g_n(x) = n^{1-\alpha} x^2, \ n \geq 1, \text{ and } g(x) = g_1(x) = x^2, \ x \geq 0. \]
Then the conditions (2.3.1), (2.3.2), and (ii ) of Theorem 2 are satisfied. For each $n \geq 1$,

$$E(g_n(|X_n|)) = n^{-(1+\alpha)} \text{ and } E(g(|X_n|)) = n^{-2}$$

implying (2.3.3) and (2.1.1), respectively. Therefore, all the hypotheses of Theorem 2 as well as all the hypotheses of Proposition 4 (with $g(x) = g_1(x) = x^2$) are satisfied.

Now for $n \geq 1$,

$$A_n = \sum_{j=n}^{\infty} E(g_j(|X_j|)) = \sum_{j=n}^{\infty} j^{-(1+\alpha)} \sim M_1 n^{-\alpha} \quad (M_1 = \alpha^{-1}) \quad (2.4.9)$$

and

$$A_n = \sum_{j=n}^{\infty} E(g(|X_j|)) = \sum_{j=n}^{\infty} j^{-2} \sim n^{-1} \quad (2.4.10)$$

If $\psi(x)$ be taken to be the function

$$\psi(x) = \sqrt{x},$$

then

$$A_n \psi(A_n^{-1}) \sim M_2 n^{-\frac{\alpha}{2}} \quad (M_2 = \alpha^{-\frac{1}{2}})$$

and

$$A_n \psi(A_n^{-1}) \sim n^{-\frac{1}{2}}$$

implying, respectively,

$$g_n^{-1}(A_n \psi(A_n^{-1})) \sim M_3 n^{-\frac{1}{2} + \frac{\alpha}{4}} \quad (M_3 = \alpha^{-\frac{1}{4}})$$

and

$$g^{-1}(A_n \psi(A_n^{-1})) \sim n^{-\frac{1}{4}}.$$
Thus, by applying Theorem 2 and Proposition 4, the tail series SLLNs

\[ n^{1-a}_T n \to 0 \text{ a.s.} \] (2.4.11)

and

\[ n^1 T_n \to 0 \text{ a.s.} \] (2.4.12)

obtain, respectively. Hence, recalling \( \alpha < 1 \), (2.4.11) dominates (2.4.12). Therefore Theorem 2 gives us a sharper result than that which can be obtained by Proposition 4.

Next, by taking \( \psi(x) \) to be the function as in (2.4.3) of Example 1, two relations (2.4.9) and (2.4.10) yield the asymptotic relations

\[ A_n \psi(A_n^{-1}) \sim M_4 n^{-\alpha} (\log_1 n)^{1+\varepsilon} \quad (M_4 = \alpha^\varepsilon) \]

and

\[ A_n \psi(A_n^{-1}) \sim n^{-1} (\log_1 n)^{1+\varepsilon}, \]

respectively. Thus,

\[ g^{-1}_n(A_n \psi(A_n^{-1})) \sim M_5 n^{-\frac{1}{2}} (\log_1 n)^{\frac{1+\varepsilon}{2}} \quad (M_5 = \alpha^\frac{1}{2}) \]

and

\[ g^{-1}(A_n \psi(A_n^{-1})) \sim n^{-\frac{1}{2}} (\log_1 n)^{\frac{1+\varepsilon}{2}}. \]

Hence, by either Theorem 2 or Proposition 4, the tail series SLLN

\[ \frac{n^{\frac{1}{2}}}{(\log_1 n)^{\frac{1+\varepsilon}{2}}} T_n \to 0 \text{ a.s.} \] (2.4.13)

obtains for arbitrary \( \varepsilon > 0 \). Therefore, there is no advantage of Theorem 2 over Proposition 4 in this case.
Furthermore, let \( \{Y_n, \ n \geq 1\} \) be a sequence of i.i.d. random variables such that
\[
P\{Y_n = 1\} = P\{Y_n = -1\} = \frac{1}{2}, \ n \geq 1.
\]
Consider the weighted i.i.d. random variables
\[
X_n = a_n Y_n, \ \text{where} \ a_n = n^{-1}, \ n \geq 1.
\]
Then
\[
t_n^2 = \sum_{j=n}^{\infty} a_j^2 = \sum_{j=n}^{\infty} \frac{1}{j^2} \sim n^{-1}
\]
and so
\[
t_n^2 = O(t_{n+1}^2) \ \text{and} \ \frac{n a_n^2}{t_n^2} = O(1).
\]
Therefore, by a tail series LIL of Rosalsky [41, Theorem 2] where \( \beta \) therein is chosen to be 0, the tail series LIL
\[
\limsup_{n \to \infty} \frac{\sum_{j=n}^{\infty} a_j Y_j}{(2 t_n^2 \log_2 t_n^{-2})^{\frac{1}{2}}} = 1 \ \text{a.s.}
\]
obtains, i.e.,
\[
\limsup_{n \to \infty} \frac{n^{\frac{1}{2}}}{(\log_2 n)^{\frac{1}{2}}} T_n = \sqrt{2} \ \text{a.s.} \tag{2.4.14}
\]
Hence, for arbitrary \( \varepsilon > 0 \), the tail series SLLN
\[
\frac{n^{\frac{1}{2}}}{(\log_2 n)^{\frac{1}{2}+\varepsilon}} T_n \to 0 \ \text{a.s.}
\]
obtains. Thus this result of the tail series LIL of Rosalsky [41, Theorem 2] is sharper than (2.4.13) as well as (2.4.11) and (2.4.12).

This example illustrates the gap between the conclusion of the tail series SLLN (Theorem 2) and that of the tail series LIL of Rosalsky [41, Theorem 2]. Further discussion about this will be given in Chapter 4.
CHAPTER 3
TAIL SERIES WEAK LAWS OF LARGE NUMBERS

3.1 Introductory Comments, Tail Series Inequality, and a New Proof of Klesov's Tail Series SLLN

As was mentioned at the beginning of Chapter 2, in Klesov's [29, Lemma 1] proof of a tail series version of Kolmogorov's inequality for independent random variables, not only was his argument obscure, but, also, he employed a tail series inequality without proving it. After formulating and proving this tail series inequality (Proposition 5 below), we will provide an alternative proof of the tail series SLLN of Klesov [29, Proposition 1] (which is not based on the tail series version of the Kolmogorov's inequality as was used by Klesov to prove his tail series SLLN). As a direct application of this tail series inequality, we will establish tail series WLLNs for the case of independent summands. Furthermore, as special cases of these tail series WLLNs, we will also obtain tail series WLLNs for weighted sums of i.i.d. random variables. As in Chapters 1 and 2, \( \{T_n, n \geq 1\} \) denotes throughout the tail series \( T_n = \sum_{j=n}^{\infty} X_j, n \geq 1 \), corresponding to random variables \( \{X_n, n \geq 1\} \). As will be seen, the hypotheses to each of the tail series results presented below ensure that \( \{T_n, n \geq 1\} \) is a well-defined sequence of random variables. Hence \( T_n \rightarrow 0 \) a.s. or, equivalently,

\[
\sup_{j \geq n} |T_j| \overset{P}{\rightarrow} 0.
\]
As was mentioned in Chapter 1, these tail series WLLNs are of the form

\[ \sup_{j \geq n} |T_j| \frac{1}{b_n} \xrightarrow{P} 0 \]  

(3.1.1)

where \( \{b_n, n \geq 1\} \) is a suitable sequence of norming constants with \( 0 < b_n \downarrow 0 \).

Of course, if the tail series SLLN

\[ \frac{T_n}{b_n} \rightarrow 0 \text{ a.s.} \]

holds, then

\[ \sup_{j \geq n} \frac{|T_j|}{b_j} \xrightarrow{P} 0 \]

whence via \( 0 < b_n \downarrow 0 \) the tail series WLLN (3.1.1) also obtains and it involves the same sequence of norming constants.

This tail series inequality under discussion may now be formulated.

**Proposition 5** (Klesov [29]). Let \( \{X_n, n \geq 1\} \) be independent random variables with \( \mathbb{E}(|X_n|^p) < \infty, \ n \geq 1 \), for some \( p > 0 \). Assume that one of the following two conditions holds

\[(i) \ 0 < p \leq 1.\]

\[(ii) \ 1 < p \leq 2 \text{ and } \mathbb{E}(X_n) = 0.\]

If

\[ \sum_{n=1}^{\infty} \mathbb{E}(|X_n|^p) < \infty, \]  

(3.1.2)

then for every \( \varepsilon > 0 \), the inequalities

\[ \mathbb{P}\left\{ \sup_{j \geq n} |T_j| > \varepsilon \right\} \leq \frac{C_n(p)}{\varepsilon^p} \sum_{j=n}^{\infty} \mathbb{E}(|X_j|^p), \ n \geq 1 \]

obtain where \( C_n(p) \in (0, 2] \) is a sequence of constants depending only on \( p \).
The proof of Proposition 5, which will be given below, utilizes the following Lemmas 7 and 8 and the proposition, under the assumption (ii), is indeed a tail series analogue of Lemma 7 which concerns partial sums of independent random variables.

**Lemma 7.** Let \( S_n = \sum_{j=1}^{n} X_j, \ n \geq 1 \) where \( \{X_n, \ n \geq 1\} \) are independent random variables satisfying for some \( p \in (1, 2] \)

\[
E(|X_n|^p) < \infty \quad \text{and} \quad E(X_n) = 0, \ n \geq 1.
\]

Then for all \( \varepsilon > 0 \), the inequalities

\[
P\left( \max_{1 \leq j \leq n} |S_j| > \varepsilon \right) \leq \frac{2}{\varepsilon^p} \sum_{j=1}^{n} E(|X_j|^p), \ n \geq 1
\]

obtain.

**Proof.** Note at the outset that the hypotheses ensure that \( \{S_n, \mathcal{F}_n, \ n \geq 1\} \) is a martingale where \( \mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), \ n \geq 1 \), and so \( \{|S_n|^p, \mathcal{F}_n, \ n \geq 1\} \) is a submartingale (see, e.g., Chow and Teicher [14], p. 232) since the function \( \varphi(t) = |t|^p \) is convex. Then

\[
P\left( \max_{1 \leq j \leq n} |S_j| > \varepsilon \right) = P\left( \max_{1 \leq j \leq n} |S_j|^p > \varepsilon^p \right)
\]

\[
\leq \frac{E(|S_n|^p)}{\varepsilon^p}
\]

(by Doob’s submartingale maximal inequality [18, p. 314])

\[
\leq \frac{2}{\varepsilon^p} \sum_{j=1}^{n} E(|X_j|^p)
\]

by employing the von Bahr-Esseen [8] inequality. Thus the lemma follows. □

**Lemma 8.** Let \( \{X_n, \ n \geq 1\} \) be independent random variables satisfying for some
For each \( n \geq 1 \) and \( 1 \leq k \leq n \), let

\[
S_{n,j} = \sum_{i=j}^{n} X_i, \quad k \leq j \leq n.
\]

Then for all choices of \( n \) and \( k \) with \( 1 \leq k \leq n \) and for all \( \varepsilon > 0 \), the inequality

\[
P\left\{ \max_{k \leq j \leq n} |S_{n,j}| > \varepsilon \right\} \leq \frac{2}{\varepsilon^p} \sum_{j=k}^{n} E(|X_j|^p)
\]

obtains.

Proof. Fix \( n \geq 1 \) and \( 1 \leq k \leq n \). Set

\[
S_j^* = \sum_{i=1}^{j} X_{n+1-i}, \quad 1 \leq j \leq n+1-k
\]

and note that

\[
\{ S_{n,j} : j = k, ..., n \} = \{ S_j^* : j = n+1-k, ..., 1 \}.
\]

Then, applying Lemma 7 to the random variables \( \{X_n, X_{n-1}, ..., X_k\} \), it follows that for \( \varepsilon > 0 \),

\[
P\left\{ \max_{k \leq j \leq n} |S_{n,j}| > \varepsilon \right\} = P\left\{ \max_{1 \leq j \leq n+1-k} |S_j^*| > \varepsilon \right\}
\leq \frac{2}{\varepsilon^p} \sum_{j=1}^{n+1-k} E(|X_{n+1-j}|^p)
= \frac{2}{\varepsilon^p} \sum_{j=k}^{n} E(|X_j|^p)
\]

disproving the lemma. \( \square \)

Proof of Proposition 5. Let \( g_n(x) \equiv |x|^p \), \( 0 < p \leq 2 \), \( n \geq 1 \). Then, by the argument in the proof of Theorem 2 of Chapter 2, (3.1.2) ensures that \( \{T_n, n \geq 1\} \) is a well-defined sequence of random variables.
Firstly, suppose that the assumption (i) holds. Then

\[
P\left\{ \sup_{j \geq n} |T_j| > \varepsilon \right\} \leq P\left\{ \sum_{j=n}^{\infty} |X_j| > \varepsilon \right\}
\]

\[
= P\left\{ (\sum_{j=n}^{\infty} |X_j|)^p > \varepsilon^p \right\}
\]

\[
\leq \frac{1}{\varepsilon^p} E\left( (\sum_{j=n}^{\infty} |X_j|)^p \right) \text{ (by the Markov inequality)}
\]

\[
= \frac{1}{\varepsilon^p} \lim_{N \to \infty} E\left( (\sum_{j=n}^{N} |X_j|)^p \right)
\]

(by the Lebesgue monotone convergence theorem)

\[
\leq \frac{1}{\varepsilon^p} \lim_{N \to \infty} E\left( \sum_{j=n}^{N} |X_j|^p \right) \text{ (since } |a + b|^p \leq |a|^p + |b|^p, \ 0 < p \leq 1)\]

\[
= \frac{1}{\varepsilon^p} \sum_{j=n}^{\infty} E(|X_j|^p)
\]

again by the Lebesgue monotone convergence theorem. Thus, the proposition follows under the assumption (i) with \( C_n(p) \equiv 1 \).

Next, Lemma 8 will be employed to prove Proposition 5 under the assumption (ii). Note that for \( N \geq n \geq 1 \),

\[
P\left\{ \max_{n \leq j \leq N} |T_j| > \varepsilon \right\} = P\left\{ \max_{n \leq j \leq N} \lim_{M \to \infty} \sum_{i=j}^{M} X_i \right\}
\]

\[
= P\left\{ \max_{n \leq j \leq N} \lim_{M \to \infty} \sum_{i=j}^{M} X_i \right\}
\]

\[
= P\left\{ \lim_{M \to \infty} \max_{n \leq j \leq N} \sum_{i=j}^{M} X_i \right\}
\]

\[
= E\left( I_{\left\{ \lim_{M \to \infty} \max_{n \leq j \leq N} \sum_{i=j}^{M} X_i \right\} > \varepsilon} \right)
\]

\[
\leq E\left( \lim \inf_{M \to \infty} I_{\left\{ \max_{n \leq j \leq N} \sum_{i=j}^{M} X_i \right\} > \varepsilon} \right)
\]
\[
\leq \liminf_{M \to \infty} E \left( I_{\max_{n \leq j \leq N} |\sum_{i=j}^{M} X_i| > \varepsilon} \right) \quad \text{(by Fatou's lemma)}
\]
\[
= \liminf_{M \to \infty} P \left\{ \max_{n \leq j \leq N} |\sum_{i=j}^{M} X_i| > \varepsilon \right\}
\]
\[
\leq \liminf_{M \to \infty} P \left\{ \max_{n \leq j \leq M} |\sum_{i=j}^{M} X_i| > \varepsilon \right\}
\]
\[
\leq \liminf_{M \to \infty} \frac{2}{\varepsilon^p} \sum_{j=n}^{M} E(|X_j|^p) \quad \text{(by Lemma 8)}
\]
\[
= \frac{2}{\varepsilon^p} \sum_{j=n}^{\infty} E(|X_j|^p).
\]

Letting \( N \to \infty \) yields

\[
P \left\{ \sup_{j \geq n} |T_j| > \varepsilon \right\} = \lim_{N \to \infty} P \left\{ \max_{n \leq j \leq N} |T_j| > \varepsilon \right\}
\]
\[
\leq \frac{2}{\varepsilon^p} \sum_{j=n}^{\infty} E(|X_j|^p)
\]

thereby proving the proposition under the assumption (ii) with \( C_n(p) \equiv 2. \Box \)

Now, using Proposition 5, we will re-prove (in Proposition 6 below) the tail series SLLN of Klesov [29, Proposition 1] which we had questioned earlier. Of course, Proposition 6 is merely the special case \( g(x) = |x|^p, \ 0 < p \leq 2, \) of Proposition 3 of Chapter 2 as well as the special case \( g_n(x) \equiv |x|^p, \ 0 < p \leq 2, \ n \geq 1, \) of Lemma 6 of Chapter 2 but an alternative proof may be of interest.

**Proposition 6 (Klesov [29]).** Let \( \{X_n, \ n \geq 1\} \) be independent random variables with \( E(|X_n|^p) < \infty, \ n \geq 1, \) for some \( p > 0. \) Assume that either of the condition (i) or (ii) of Proposition 5 holds. Let \( \{b_n, \ n \geq 1\} \) be a sequence of positive constants with \( b_n \downarrow 0. \) If the series

\[
\sum_{n=1}^{\infty} \frac{E(|X_n|^p)}{b_n^p} < \infty,
\]

(3.1.3)
then the tail series SLLN

\[ \frac{T_n}{b_n} \rightarrow 0 \text{ a.s.} \]

obtains.

**Proof.** By the proof of Proposition 5 with \( X_n \) replaced by \( b_n^{-1}X_n, \ n \geq 1 \), \( \{\sum_{j=n}^{\infty} b_j^{-1}X_j, \ n \geq 1\} \) is a well-defined sequence of random variables. Since \( b_n \downarrow 0 \), the proposition follows from Lemma 1 of Chapter 2. □

### 3.2 Tail Series WLLNs

Using Proposition 5, we will prove tail series WLLNs of the form

\[
\sup_{j \geq n} \frac{|T_j|}{b_n} \xrightarrow{P} 0
\]

where \( \{b_n, \ n \geq 1\} \) is a sequence of norming constants with \( 0 < b_n \downarrow 0 \). This, of course, ensures that

\[
\frac{T_n}{b_n} \xrightarrow{P} 0.
\]

The following theorem is comparable with Proposition 6.

**Theorem 3.** Let \( \{X_n, \ n \geq 1\} \) be independent random variables with \( \text{E}(|X_n|^p) < \infty, \ n \geq 1, \) for some \( p > 0 \). Assume that either of the conditions (i) or (ii) of Proposition 5 holds. Let \( \{b_n, \ n \geq 1\} \) be a sequence of positive constants with \( b_n \downarrow 0 \).

If

\[
\frac{1}{b_n^p} \sum_{j=n}^{\infty} \text{E}(|X_j|^p) \rightarrow 0, \tag{3.2.1}
\]

then the tail series WLLN

\[
\sup_{j \geq n} \frac{|T_j|}{b_n} \xrightarrow{P} 0 \tag{3.2.2}
\]
obtains.

Remark. Note at the outset that Lemma 1 of Chapter 2 ensures that (3.1.3) implies (3.2.1). Thus, while in Theorem 3 we obtain a weaker conclusion than that of Proposition 6, we use a weaker assumption.

Proof of Theorem 3. Since (3.2.1) implies (3.1.2), taking $g_n(x) \equiv |x|^p$, $0 < p \leq 2$, $n \geq 1$, we see that $\{T_n, n \geq 1\}$ is a well-defined sequence of random variables by the argument in the proof of Theorem 2 of Chapter 2. Alternatively, it may be noted that (3.2.1) implies (3.1.2) whence $\{T_n, n \geq 1\}$ is well defined as was shown in Proposition 5.

In Proposition 5, replace $\varepsilon$ by $\varepsilon b_n$ for each $n \geq 1$. Then, for arbitrary $\varepsilon > 0$

$$
P \left\{ \sup_{j \geq n} \frac{|T_j|}{b_n} > \varepsilon \right\} \leq \frac{C_n(p)}{\varepsilon^p} \frac{1}{b_n^p} \sum_{j=n}^{\infty} \mathbb{E}(|X_j|^p) \quad (0 < C_n(p) \leq 2)
$$

$\rightarrow 0$ (by (3.2.1))

thereby proving (3.2.2). □

Corollary 5. Under the hypotheses to Theorem 3, the tail series WLLN

$$
\frac{T_n}{b_n} \xrightarrow{P} 0
$$

obtains.

Proof. The corollary follows immediately from (3.2.2). □

As additional corollaries of this theorem, we obtain the following two tail series WLLNs (Corollaries 6 and 7) for the weighted i.i.d. case.

Corollary 6. Let $\{Y_n, n \geq 1\}$ be i.i.d. random variables with $\mathbb{E}(Y_1) = 0$, $\mathbb{E}(Y_1^2) = \ldots$
and let \( \{a_n, \ n \geq 1\} \) be a sequence of nonzero constants. If the series

\[
\sum_{n=1}^{\infty} a_n^2 < \infty,
\] (3.2.3)

then setting

\[
t_n^2 = \sum_{j=n}^{\infty} a_j^2, \quad n \geq 1,
\]

for every \( \alpha > 0 \) and positive integer \( r \), the tail series WLLN

\[
\sup_{j \geq n} \left| \sum_{i=j}^{\infty} a_i Y_i \right| \xrightarrow{P} 0
\]

obtains.

Remark. The condition (3.2.3) is necessary for \( \{T_n, \ n \geq 1\} \) to be a well-defined sequence of random variables where \( T_n = \sum_{j=n}^{\infty} a_j Y_j, \ n \geq 1 \) (for clarification see the discussion in Section 4.3 of Chapter 4).

Proof of Corollary 6. Let \( \alpha > 0 \) and let \( r \) be positive integer. Set

\[
b_n = t_n \left( \log_r t_n^{-2} \right)^{\frac{\alpha}{2}}, \quad n \geq 1.
\]

Since (3.2.3) ensures that \( t_n^2 \downarrow 0 \),

\[
\frac{1}{b_n^2} \sum_{j=n}^{\infty} E(|a_j Y_j|^2) = \frac{t_n^2}{t_n^2 \left( \log_r t_n^{-2} \right)^{\alpha}} = \frac{1}{(\log_r t_n^{-2})^{\alpha}} \to 0
\]

and so (3.2.1) holds with \( p = 2 \) where \( X_n = a_n Y_n, \ n \geq 1 \). The corollary then follows directly from Theorem 3. \( \square \)

As a special case (\( r = 2 \) and \( \alpha = 1 \)) of Corollary 6, we obtain the following corollary which will then be compared with a tail series LIL.
Corollary 7. Under the hypotheses to Corollary 6, the tail series WLLN
\[
\sup_{j \geq n} |\sum_{i=j}^{\infty} a_i Y_i| \overline{t_n (\log_2 t_n^{-2})^{1/2}} \xrightarrow{P} 0
\] (3.2.4)
obtains.

The hypotheses of Corollary 7 (or Corollary 6) are weaker than those of some results of Rosalsky [41, Theorems 2 and 3] which provided conditions for the tail series LIL
\[
\limsup_{n \to \infty} \frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1/2}} = \sqrt{2} \text{ a.s.}
\] (3.2.5)
to obtain. Observe that the norming constants in (3.2.4) and (3.2.5) are the same.

The following two examples exhibit a sequence of norming constants \(\{b_n, n \geq 1\}\) for which a tail series WLLN holds, but a tail series SLLN does not. In the first example, the harmonic series with a random choice of signs, which was considered in Example 3 of Chapter 2, will be reconsidered.

Example 4. Let \(\{X_n, n \geq 1\}\) be independent random variables such that
\[
P\{X_n = \frac{1}{n}\} = P\{X_n = -\frac{1}{n}\} = \frac{1}{2}, n \geq 1.
\]
Let \(0 < \alpha \leq 1\). Then for arbitrary \(p \in (0, 2]\)
\[
E(|X_n|^p) = n^{-\alpha}, n \geq 1.
\]
Let \(r\) be a positive integer, and set
\[
b_n = n^{-\frac{1}{2}} (\log_r n)^{\frac{\alpha}{2}}, n \geq 1.
\]
Then
\[
\frac{E(|X_n|^p)}{b_n} = \frac{n^{-\alpha}}{n^{-\frac{p}{2}} (\log_r n)^{\frac{\alpha}{2}}} = n^{-\frac{p}{2}} (\log_r n)^{-\frac{\alpha}{2}}
\]
implying
\[\sum_{n=1}^{\infty} \frac{E(|X_n|^p)}{b_n^p} = \sum_{n=1}^{\infty} n^{-\frac{p}{2}} (\log, n)^{-\frac{ap}{2}} = \infty.\]

Hence, since \(p \in (0, 2]\) is arbitrary, the hypotheses of Proposition 6 are not met.

Indeed, it will be seen below that for \(p = 2, r = 2, \alpha = 1, \{X_n, n \geq 1\}\) obeys the tail series WLLN with norming constants \(\{b_n, n \geq 1\}\) but does not obey the tail series SLLN with those norming constants (since it obeys the tail series LIL with those constants).

Next, choose \(p = 2\). Then
\[\sum_{j=n}^{\infty} E(X_j^2) = \sum_{j=n}^{\infty} \frac{1}{j^2} \sim \frac{1}{n}.\]
Thus for \(r \geq 1\) and \(\alpha \in (0, 1]\),
\[\frac{1}{b_n^2} \sum_{j=n}^{\infty} E(X_j^2) \sim \frac{n^{-1}}{n^{-1}(\log, n)^{\alpha}} = \frac{1}{(\log, n)^{\alpha}} = o(1)\]
ensuring (3.2.1). By applying Theorem 3, the tail series WLLN
\[\sup_{j \geq n} \frac{|T_j|}{n^{-\frac{1}{2}} (\log, n)^{\frac{\alpha}{2}}} \xrightarrow{P} 0, \quad (3.2.6)\]
obeys. Choosing \(r = 2\) and \(\alpha = 1\), it follows in particular that
\[\sup_{j \geq n} \frac{|T_j|}{n^{-\frac{1}{2}} (\log_2, n)^{\frac{1}{2}}} \xrightarrow{P} 0. \quad (3.2.7)\]
(Note that \(\{X_n, n \geq 1\}\) are weighted i.i.d. random variables, i.e., \(X_n = a_n Y_n, n \geq 1\), where \(\{Y_n, n \geq 1\}\) are i.i.d. random variables with
\[P\{Y_n = 1\} = P\{Y_n = -1\} = \frac{1}{2}, n \geq 1\]
and $a_n = n^{-1}$, $n \geq 1$. Thus, by applying Corollary 6 and Corollary 7, we can also arrive at the same conclusions (3.2.6) and (3.2.7), respectively.) But, recalling the tail series LIL (2.4.14) of Example 3 of Chapter 2, it is clear that the tail series SLLN

$$\frac{T_n}{n^{-\frac{1}{2}} (\log_2 n)^{\frac{1}{2}}} \to 0 \text{ a.s.}$$

fails.

Next, an example constructed by Rosalsky [41, Example 1] of weighted i.i.d. random variables will be discussed in the context of the tail series WLLN and SLLN.

**Example 5.** Let $\{Y_n, n \geq 1\}$ be i.i.d. random variables with $E(Y_1) = 0$, $E(Y_1^2) = 1$. For each $n \geq 1$, let

$$a_n^2 = \frac{(\log_2 n)^u (\log_3 n)^v}{n \exp \{(\log_1 n)(\log_2 n)^u (\log_3 n)^v\}}, \quad -\infty < u, v < \infty$$

and assume that if $u \geq 1$, then

$$E\left(Y_1^2 (\log \left|Y_1\right|)^q\right) < \infty \text{ for some } q > u - 1.$$ 

Then the condition (3.2.3) obtains (see Rosalsky [41, Example 1]). Then, for $r \geq 1$ and $\alpha > 0$, the tail series WLLN

$$\sup_{j \geq n} \frac{|\sum_{i=j}^{\infty} a_i Y_i|}{t_n (\log_2 t_n)^{\frac{\alpha}{2}}} \to 0,$$

(3.2.8)

obtains by Corollary 6. By choosing $r = 2$ and $\alpha = 1$ (or by Corollary 7), it follows in particular that

$$\sup_{j \geq n} \frac{|\sum_{i=j}^{\infty} a_i Y_i|}{t_n (\log_2 t_n)^{\frac{1}{2}}} \to 0.$$
But by Rosalsky [41, Example 1], the tail series LIL

\[ \limsup_{n \to \infty} \frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1/2}} = \sqrt{2} \text{ a.s.} \]

obtains. Thus the tail series SLLN

\[ \frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1/2}} \to 0 \text{ a.s.} \]

fails.

**Remark.** It may be noted that the hypotheses to the tail series LILs of Rosalsky [41, Theorem 2 and 3] ensure immediately via the Chebyshev inequality that

\[ \frac{T_n}{t_n (\log_2 t_n^{-2})^{1/2}} \to P \to 0. \]

Indeed, the hypotheses to these theorems of Rosalsky [41] always entail the stronger conclusion

\[ \frac{\sup_{j \geq n} |T_j|}{t_n (\log_2 t_n^{-2})^{1/2}} \to P \to 0. \]

This observation was already made after Corollary 7 concerning the tail series LIL of Rosalsky [41, Theorems 2 and 3]. Apropos of Rosalsky [41, Theorem 1], the observation follows immediately from our Theorem 3 by taking \( p = 2 \) and \( b_n = t_n (\log_2 t_n^{-2})^{1/2}, n \geq 1. \)
CHAPTER 4
TAIL SERIES STRONG LAWS OF LARGE NUMBERS II

4.1 Introduction and Preliminaries

As was discussed at the end of Chapter 2, there is a gap between the conclusion of our tail series SLLN (Theorem 2) and that of the tail series LIL of Rosalsky [41, Theorem 2]. So, it is natural to seek a tail series SLLN whose conclusion is more akin to that of the tail series LIL of Rosalsky [41, Theorem 2]. To this end, we will establish in Theorem 4 below a tail series counterpart to the following SLLN for partial sums by Teicher [47].

Proposition 7 (Teicher [47]). Let \(1 < p < 2\) and let \(S_n = \sum_{j=1}^{n} X_j\), \(n \geq 1\), where \(\{X_n, n \geq 1\}\) are independent random variables with

\[
E(X_n) = 0, \quad E(|X_n|^p) \leq e_n < \infty, \quad B_n^p = \sum_{j=1}^{n} e_j \to \infty, \quad n \geq 1
\]

where \(\{e_n, n \geq 1\}\) are positive constants. Assume that

\[B_{n+1} = O(B_n).\]

If for some \(\alpha \in [0, \frac{1}{p})\) and some positive constants \(\delta\) and \(\varepsilon\)

\[
\sum_{n=1}^{\infty} P\{|X_n| > \delta B_n (\log_2 B_n)^{1-\alpha}\} < \infty
\]

(4.1.1)

and

\[
\sum_{n=1}^{\infty} \frac{E\left(X_n^2 I_{[\varepsilon B_n (\log_2 B_n)^{-\alpha} < |X_n| < \delta B_n (\log_2 B_n)^{1-\alpha}]}ight)}{(B_n (\log_2 B_n)^{1-\alpha})^2} < \infty,
\]

(4.1.2)
then the SLLN

\[
\frac{S_n}{B_n (\log_2 B_n)^{1-a}} \to 0 \text{ a.s.}
\]  

(4.1.3)

obtains.

**Remark.** A standard Borel-Cantelli argument reveals that (4.1.1) is a necessary condition for the conclusion (4.1.3) to obtain. Moreover, while the condition (4.1.2) is technical in nature, it is not at all *ad hoc* in that it is of the spirit of conditions employed by Chow, Teicher, Wei, and Yu [15], Egorov [19, 20], Heyde [22], Klesov [31], Petrov [37] (see Heyde [22] and the inequality (1) of Loève [34, p. 209] for clarification), Petrov [40, p. 303], Sakhanenko [42], Teicher [45, 46], Tomkins [48], and Wittmann [50] to prove LILs (or SLLNs) for partial sums of independent random variables. Moreover, the above authors also employed a condition in the same spirit as (4.1.1).

It will be seen after the statement of Theorem 4 that this theorem will yield a sharper result than that of the tail series SLLN of Theorem 2 of Chapter 2 when the hypotheses of Theorem 4 are satisfied. Furthermore, as special cases of the tail series SLLNs of this chapter, we will investigate the tail series SLLN problem for weighted sums of i.i.d. random variables. In the weighted i.i.d. case, it will also be seen after the statement of Theorem 5 that this tail series SLLN will narrow the gap between the conclusion of the tail series SLLN and that of the tail series LIL of Rosalsky [41, Theorem 2].
4.2 Tail Series SLLNs

For independent random variables \( \{X_n, n \geq 1\} \) we obtain tail series SLLNs below, which are counterparts to the SLLNs for partial sums of Teicher [47]. The main result of this chapter, Theorem 4, may now be stated. As in previous chapters, \( \{T_n, n \geq 1\} \) denotes throughout the tail series \( T_n = \sum_{j=n}^{\infty} X_j, n \geq 1 \), corresponding to random variables \( \{X_n, n \geq 1\} \). It will be shown in the proof of Theorem 4 that the hypotheses guarantee that \( \{T_n, n \geq 1\} \) is a well-defined sequence of random variables. But, the proof of Theorem 4 will be deferred until after the proof of the ensuing Lemma 10.

**Theorem 4.** Let \( 1 < p \leq 2 \) and let \( \{X_n, n \geq 1\} \) be independent random variables with

\[
E(X_n) = 0, \quad E(|X_n|^p) \leq e_n, \quad n \geq 1
\]

where \( \{e_n, n \geq 1\} \) are positive constants with \( \sum_{n=1}^{\infty} e_n < \infty \). Assume that

\[
A_n^p = O(A_{n+1}^p) \quad (4.2.1)
\]

where \( A_n^p = \sum_{j=n}^{\infty} e_j, n \geq 1 \). If for some \( \alpha \in (-\infty, \frac{1}{p}) \)

\[
\sum_{n=1}^{\infty} P\{|X_n| > \delta A_n (\log_2 A_n^{-p})^{1-\alpha}\} < \infty \text{ for some } \delta > 0 \quad (4.2.2)
\]

and for all \( \varepsilon > 0 \)

\[
\sum_{n=1}^{\infty} \frac{E\left(X_n^2 I_{[\varepsilon A_n (\log_2 A_n^{-p})^{-\alpha} < |X_n| < \delta A_n (\log_2 A_n^{-p})^{1-\alpha}]}\right)}{(A_n (\log_2 A_n^{-p})^{1-\alpha})^2} < \infty, \quad (4.2.3)
\]

then the tail series SLLN

\[
\frac{T_n}{A_n (\log_2 A_n^{-p})^{1-\alpha}} \to 0 \text{ a.s.} \quad (4.2.4)
\]
obtains.

Remarks. (i) Note that (4.2.1) ensures that $A_{n}^{-1}A_{n+1} \geq \gamma$ for some $\gamma \in (0, 1)$.

(ii) Note that if \{X_n, n \geq 1\} satisfies the hypotheses to Theorem 4, then the hypotheses to Theorem 2 of Chapter 2 are also satisfied with

$$g_n(x) \equiv |x|^p, \quad 1 \leq p \leq 2, \quad n \geq 1$$

and so for any $\psi(x) \in \Psi$

$$\frac{T_n}{(A_n \psi(A_n^{-1}))^{\frac{1}{p}}} \to 0 \text{ a.s.}$$

where

$$A_n = \sum_{j=n}^{\infty} E(|X_j|^p), \quad n \geq 1.$$ 

As for as notation is concerned, note that if \{X_n, n \geq 1\} obeys the hypotheses to Theorem 4 with $e_n = E(|X_n|^p), \quad n \geq 1$, then the sequence \{A_n, n \geq 1\} of Theorem 4 is in fact the sequence \{A_n^{\frac{1}{p}}, n \geq 1\}. However, in this case,

$$(\log_2 A_n^{-p})^{1-\alpha} = o((\psi(A_n^{-1}))^{\frac{1}{p}})$$

whence Theorem 4 yields a sharper conclusion than does Theorem 2. Of course,

(a) in general, the hypotheses of Theorem 2 may be satisfied, but not those of Theorem 4,

and

(b) Theorem 2 involves a class of norming sequences which is structurally different from that of Theorem 4.

As will become apparent, Theorem 4 owes much to the work of Teicher [47]. The proof of Theorem 4 utilizes the following two lemmas. In Lemma 9, there are no as-
Assumptions concerning the integrability of the random variables \( \exp\{t S\} \), \( \exp\{t S_n\} \), \( n \geq 1 \), in (4.2.5). Moreover, Lemma 9 cannot be proved by invoking the continuity theorem for moment generating functions unless \( S_n \), \( n \geq 1 \) and \( S \) are all defined on a common interval of the \( t \)-axis containing 0 as an interior point.

**Lemma 9.** Let \( S_n = \sum_{j=1}^{n} X_j \), \( n \geq 1 \), where \( \{X_n, n \geq 1\} \) are independent random variables with

\[
E(X_n) = 0, \; n \geq 1, \; \text{and} \; \sum_{n=1}^{\infty} E(X_n^2) < \infty.
\]

Then there exists a random variable \( S \) with \( E(S) = 0 \), \( \text{Var}(S) = \sum_{n=1}^{\infty} E(X_n^2) \) and \( S_n \to S \text{ a.s. and such that} \)

\[
\lim_{n \to \infty} E\left(\exp\{t S_n\}\right) = E\left(\exp\{t S\}\right), \; -\infty < t < \infty. \tag{4.2.5}
\]

**Proof.** The existence of a random variable \( S \) with \( E(S) = 0 \), \( \text{Var}(S) = \sum_{n=1}^{\infty} E(X_n^2) \) and \( S_n \to S \text{ a.s. follows directly from the Khintchine-Kolmogorov convergence theorem.} \)

Next, for all \( n \geq 1 \), Jensen's inequality ensures that

\[
E\left(\exp\{t T_{n+1}\}\right) \geq \exp\{t E(T_{n+1})\} = e^0 = 1
\]

and so

\[
E\left(\exp\{t S\}\right) = E\left(\exp\{t T_{n+1}\} \exp\{t S_n\}\right) \\
= E\left(\exp\{t T_{n+1}\}\right) E\left(\exp\{t S_n\}\right) \text{ (by independence) } \\
\geq E\left(\exp\{t S_n\}\right).
\]
Thus,

$$\limsup_{n \to \infty} E(\exp\{t S_n\}) \leq E(\exp\{t S\}). \quad (4.2.6)$$

Moreover,

$$E(\exp\{t S\}) = E\left( \lim_{n \to \infty} \exp\{t S_n\} \right) \leq \liminf_{n \to \infty} E(\exp\{t S_n\}) \quad \text{(by Fatou's lemma)}$$

which when combined with (4.2.6) yields the conclusion (4.2.5). □

**Lemma 10.** Let \( \{X_n, n \geq 1\} \) be independent random variables with \(|X_n| \leq M_n, n \geq 1\), where \( \{M_n, n \geq 1\} \) is a bounded sequence of positive constants and suppose that

$$E(X_n) = 0, \ n \geq 1.$$  

\( (i) \) If the series

$$\sum_{n=1}^{\infty} \sigma_n^2 < \infty, \quad (4.2.7)$$

where \( \sigma_n^2 = E(X_n^2), \ n \geq 1\), then setting

$$t_n^2 = \sum_{j=n}^{\infty} \sigma_j^2, \ n \geq 1,$$

the inequalities

$$E\left( \exp\left\{ \frac{t T_n}{t_n} \right\} \right) < \exp\left\{ \frac{t^2}{2} \left( 1 + \frac{t C_n}{2} \right) \right\}, \ n \geq 1$$

obtain for all \( t \in (0, C_n^{-1}] \) where

$$C_n = \frac{1}{t_n} \sup_{j \geq n} M_j, \ n \geq 1.$$
(ii) In addition to the assumptions in part (i), let \( \{x_n, n \geq 1\} \) be a numerical sequence satisfying
\[
0 < C_n x_n \leq u, \ n \geq 1
\]  
(4.2.8)
for some constant \( u < \infty \). Then the inequalities
\[
P\left\{ \sup_{j \geq n} T_j > \lambda x_n t_n \right\} \leq \exp\left\{ -x_n^2 \left( v \lambda - \frac{v^2}{2} \left( 1 + \frac{u v}{2} \right) \right) \right\}, \ n \geq 1
\]
obtain for all \( \lambda > 0 \) and all \( v \in (0, u^{-1}] \).

**Remark.** Observe that our \( \lambda \) in part (ii) may vary with \( n \), i.e., the above \( \lambda \) can be replaced by \( \lambda_n \).

This lemma is a tail series analogue of the exponential bounds lemma of Teicher [47, Lemma 1]. The proof of Lemma 10 employs the function \( 2^{-1} (1 + 2^{-1} x) \) playing a similar role as the function \( g(x) = x^{-2} (e^x - 1 - x) \) of Lemma 1 of Teicher [47].

**Proof of Lemma 10.** (i) The argument is contained in the proof of Theorem 2 of Chow and Teicher [13].

(ii) In order to prove part (ii) of the lemma, we will employ the argument in the proof of Proposition 5 of Chapter 3. As in the proof of Lemma 7 of Chapter 3, note that the hypotheses ensure that, for a given \( n \geq 1 \), \( \{S_{n,M}, \mathcal{F}_{n,M}, M \geq n\} \) is a martingale where
\[
S_{n,M} = \sum_{j=n}^{M} X_j, \ \mathcal{F}_{n,M} = \sigma(X_n, ..., X_M), \ M \geq n \geq 1
\]
and so for \( t > 0 \), \( \exp\left\{ \frac{t}{t_n} S_{n,M} \right\}, \ \mathcal{F}_{n,M}, \ M \geq n \} \) is a submartingale (see, e.g., Chow and Teicher [14], p. 232) since the function \( \varphi(s) = \exp\left\{ \frac{t}{t_n} s \right\} \) is convex. Then for
$N \geq n \geq 1$, $v \in (0, u^{-1}]$, and $t = v x_n$, we have

\[
P\{\max_{n \leq j \leq N} T_j > \lambda x_n t_n\} = P\{\max_{n \leq j \leq N} \lim_{M \to \infty} \sum_{i=j}^{M} X_i > \lambda x_n t_n\}
\]

\[
= P\{\lim_{M \to \infty} \max_{n \leq j \leq N} \sum_{i=j}^{M} X_i > \lambda x_n t_n\}
\]

\[
= E\left(\lim_{M \to \infty} \max_{n \leq j \leq N} \sum_{i=j}^{M} X_i > \lambda x_n t_n\right)
\]

\[
= E\left(\lim_{M \to \infty} \inf\left\{\max_{n \leq j \leq N} \sum_{i=j}^{M} X_i > \lambda x_n t_n\right\}\right)
\]

\[
\leq \lim_{M \to \infty} \inf E\left(\max_{n \leq j \leq N} \sum_{i=j}^{M} X_i > \lambda x_n t_n\right) (by \ Fatou's \ lemma)
\]

\[
= \lim_{M \to \infty} \inf P\left\{\max_{n \leq j \leq N} \sum_{i=j}^{M} X_i > \lambda x_n t_n\right\}
\]

\[
\leq \lim_{M \to \infty} \inf \exp\left\{\frac{t}{t_n} S_{j,M}\right\} > \exp\left\{\frac{t}{t_n} \lambda x_n t_n\right\}\right\} \right) (t > 0)
\]

\[
\leq \lim_{M \to \infty} \inf \frac{\exp\left\{\frac{t}{t_n} S_{n,M}\right\}}{\exp\left\{\lambda x_n\right\}}
\]

(by Doob's submartingale maximal inequality [18, p. 314])

\[
= \frac{\exp\left\{\frac{t}{t_n} T_n\right\}}{\exp\left\{\lambda x_n\right\}} \right) (by \ Lemma \ 9)
\]

\[
\leq \exp\left\{-t \lambda x_n + \frac{t^2}{2} \left(1 + \frac{C_n}{2}\right)\right\}
\]

(by part (i): note $t \in (0, C_n^{-1}]$)

\[
= \exp\left\{-v \lambda x_n^2 + \frac{1}{2} v^2 x_n^2 \left(1 + \frac{1}{2} v x_n C_n\right)\right\}
\]

\[
\leq \exp\left\{-x_n^2 \left(\lambda - \frac{v^2}{2} \left(1 + \frac{u v}{2}\right)\right)\right\} \right) (by\,(4.2.8)).
\]
Letting $N \to \infty$ yields
\[
P\left\{ \sup_{j \geq n} T_j > \lambda x_n t_n \right\} = \lim_{N \to \infty} P\left\{ \max_{1 \leq j \leq N} T_j > \lambda x_n t_n \right\} \leq \exp\left\{ -x_n^2 \left( v \lambda - \frac{v^2}{2} \left( 1 + \frac{u v}{2} \right) \right) \right\}
\]
thereby proving the lemma. □

Proof of Theorem 4. Observe at the outset that the tail series \( \{T_n, n \geq 1\} \) is well defined by taking \( g_n(x) \equiv |x|^p, \, n \geq 1 \), in Theorem 2 of Chapter 2. (Alternatively, with the above choice of \( \{g_n(x), n \geq 1\} \), Loève's [34, p. 252] generalization of the Khintchine-Kolmogorov convergence theorem ensures that \( \{T_n, n \geq 1\} \) is a well-defined sequence of random variables.)

Let \( \varepsilon > 0 \) be arbitrary and let \( 0 < \varepsilon < \frac{1}{p} \). For each \( n \geq 1 \) set
\[
U_n = X_n \ 1_{|X_n| \leq \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha}}
\]
\[
V_n = X_n \ 1_{|X_n| > \varepsilon A_n (\log_2 A_n^{-p})^{1-\alpha}}
\]
\[
W_n = X_n \ 1_{\varepsilon A_n (\log_2 A_n^{-p})^{-\alpha} < |X_n| \leq \varepsilon A_n (\log_2 A_n^{-p})^{1-\alpha}}.
\]
Then \( X_n = U_n + V_n + W_n, \, n \geq 1 \). Now, for each \( j \geq n \geq 1 \),
\[
E(|V_j|) \leq E\left( |X_j| \ 1_{\varepsilon A_j (\log_2 A_j^{-p})^{1-\alpha} < |X_j| \leq A_n (\log_2 A_n^{-p})^{-\alpha}} \right) + E\left( |X_j| \ 1_{|X_j| > A_n (\log_2 A_n^{-p})^{-\alpha}} \right) \leq A_n (\log_2 A_n^{-p})^{-\alpha} P\left\{ |X_j| > \delta A_j (\log_2 A_j^{-p})^{1-\alpha} \right\} + A_n^{1-p} (\log_2 A_n^{-p})^{\alpha(p-1)} E\left( |X_j|^p \ 1_{|X_j| > A_n (\log_2 A_n^{-p})^{-\alpha}} \right)
\]
and so
\[
E\left( \sum_{j=n}^{\infty} |V_j| \right) \leq A_n (\log_2 A_n^{-p})^{-\alpha} \sum_{j=n}^{\infty} P\left\{ |X_j| > \delta A_j (\log_2 A_j^{-p})^{1-\alpha} \right\}
\]
\begin{align*}
&+ A_n^{1-p} \left( \log_2 A_n^{-p} \right)^{\alpha(p-1)} \sum_{j=n}^{\infty} E \left( |X_j|^p I_{|X_j| > A_n (\log_2 A_n^{-p})^{-\alpha}} \right) \\
&= o\left( A_n (\log_2 A_n^{-p})^{1-\alpha} \right) \text{ (since } \alpha p < 1), \quad (4.2.9)
\end{align*}

using (4.2.2) and the fact that
\begin{equation*}
\sum_{j=n}^{\infty} E \left( |X_j|^p I_{|X_j| > A_n (\log_2 A_n^{-p})^{-\alpha}} \right) \leq A_n^p.
\end{equation*}

Note that (4.2.2) ensures via the Borel-Cantelli lemma that a.s. $V_n$ is eventually 0 and consequently so is $\sum_{j=n}^{\infty} V_j$. Thus
\begin{equation*}
\frac{\sum_{j=n}^{\infty} V_j}{A_n (\log_2 A_n^{-p})^{1-\alpha}} \to 0 \text{ a.s.}
\end{equation*}

implying via (4.2.9) that
\begin{equation*}
\frac{\sum_{j=n}^{\infty} \{V_j - E(V_j)\}}{A_n (\log_2 A_n^{-p})^{1-\alpha}} \to 0 \text{ a.s.} \quad (4.2.10)
\end{equation*}

In view of (4.2.3) and Khintchine-Kolmogorov convergence theorem
\begin{equation*}
\sum_{j=1}^{n} \frac{W_j - E(W_j)}{A_j (\log_2 A_j^{-p})^{1-\alpha}} \text{ converges a.s.}
\end{equation*}

Then by applying Lemma 1 of Chapter 2 to this we obtain
\begin{equation*}
\frac{\sum_{j=n}^{\infty} \{W_j - E(W_j)\}}{A_n (\log_2 A_n^{-p})^{1-\alpha}} \to 0 \text{ a.s.} \quad (4.2.11)
\end{equation*}

Now, observe that $E(X_n) = E(U_n) + E(V_n) + E(W_n) = 0$. Then, in view of (4.2.10) and (4.2.11), in order to show that (4.2.4) holds, it suffices to show (since $\epsilon$ is arbitrary) that $R_n = \sum_{j=n}^{\infty} \{U_j - E(U_j)\}, \; n \geq 1$, is a well-defined sequence of random variables satisfying
\begin{equation*}
\limsup_{n \to \infty} \frac{|R_n|}{A_n (\log_2 A_n^{-p})^{1-\alpha}} \leq \frac{6 \epsilon}{\gamma^2} \text{ a.s.} \quad (4.2.12)
\end{equation*}
where $0 < \gamma < 1$ is as in Remark (i) after the statement of the theorem.

To this end, firstly observe for each $n \geq 1$ that

$$
E(|U_n - E(U_n)|^p) \leq E((|U_n| + |E(U_n)|)^p)
$$

$$
\leq 2^p \{E(|U_n|^p) + |E(U_n)|^p\}
$$

$$
= 2^p \{E(|U_n|^p) + |E(U_n)|^p\}
$$

$$
\leq 2^p \{E(|U_n|^p) + E(|U_n|^p)\} \quad \text{(by Jensen's inequality)}
$$

$$
= 2^{p+1} E(|U_n|^p)
$$

and so

$$
\sum_{n=1}^{\infty} E(|U_n - E(U_n)|^p) \leq \sum_{n=1}^{\infty} 2^{p+1} E(|U_n|^p) \leq 2^{p+1} \sum_{n=1}^{\infty} e_n < \infty.
$$

Thus, by taking $g_n(x) \equiv |x|^p$, $n \geq 1$, via Theorem 2 of Chapter 2, \{R_n, n \geq 1\} is a well-defined sequence of random variables.

Next, recalling that $\frac{A_{n+1}}{A_n} \geq \gamma$ where $0 < \gamma < 1$, let

$$
n_k = \inf \{n \geq 1 : A_n \leq \gamma^k\}, \quad k \geq 1.
$$

Then, for all $k$ such that $n_k \geq 2$, since $A_{n_k-1} > \gamma^k$,

$$
A_{n_k} \geq \gamma A_{n_k-1} > \gamma^{k+1} \geq A_{n_{k+1}}.
$$

Hence \{n_k, k \geq 1\} is a strictly increasing sequence of integers. Moreover, for all $k \geq 2$ such that $n_k \geq 2$, since

$$
A_{n_k} > \gamma^{k+1} \quad \text{and} \quad A_{n_{k-1}} \leq \gamma^{k-1},
$$

it follows that

$$
\frac{A_{n_k}}{A_{n_{k-1}}} > \frac{\gamma^{k+1}}{\gamma^{k-1}} = \gamma^2. \quad (4.2.13)
$$
For each $n \geq 1$,
\[
P \left\{ R_n > \frac{6 \varepsilon}{\gamma^2} A_n (\log_2 A_n^{-p})^{1-\alpha} \text{ i.o.} \ (n) \right\}
\leq P \left\{ \max_{n_{k-1} \leq n < n_k} R_n > \frac{6 \varepsilon}{\gamma^2} A_{n_k} (\log_2 A_{n_k}^{-p})^{1-\alpha} \text{ i.o.} \ (k) \right\}
\leq P \left\{ \sup_{n \geq n_{k-1}} R_n > \frac{6 \varepsilon}{\gamma^2} A_{n_k} (\log_2 A_{n_k}^{-p})^{1-\alpha} \text{ i.o.} \ (k) \right\}
\leq P \left\{ \sup_{n \geq n_{k-1}} R_n > 6 \varepsilon A_{n_k-1} (\log_2 A_{n_k-1}^{-p})^{1-\alpha} \text{ i.o.} \ (k) \right\}
\]
(by (4.2.13) and the fact that $A_n^{-p} \uparrow$ as $k \uparrow$)
\[
= P \left\{ \sup_{n \geq n_k} R_n > 6 \varepsilon A_{n_k} (\log_2 A_{n_k}^{-p})^{1-\alpha} \text{ i.o.} \ (k) \right\}. \tag{4.2.14}
\]

Now, for each $n \geq 1$, let $r_n^2 = E(R_n^2)$. Then
\[
r_n^2 \leq \sum_{j=n}^{\infty} E \left( X_j^2 I_{[|X_j| \leq \varepsilon A_j (\log_2 A_j^{-p})^{-\alpha}]} \right)
= \sum_{j=n}^{\infty} E \left( |X_j|^p |X_j|^{2-p} I_{[|X_j| \leq \varepsilon A_j (\log_2 A_j^{-p})^{-\alpha}]} \right)
\leq \frac{\varepsilon^{2-p} A_n 2^{-p}}{(\log_2 A_n^{-p})^{\alpha(2-p)}} \sum_{j=n}^{\infty} E(|X_j|^p)
= \frac{\varepsilon^{2-p} A_n^2}{(\log_2 A_n^{-p})^{\alpha(2-p)}}. \tag{4.2.15}
\]

For each $n \geq 1$, note that, since $|U_n| \leq \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha}$,
\[
|U_n - E(U_n)| \leq 2 \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha} \equiv M_n, \ n \geq 1.
\]

Then, for each $n \geq 1$, setting
\[
C_n = \frac{1}{r_n} \sup_{j \geq n} M_j = \frac{M_n}{r_n} = \frac{2 \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha}}{r_n}
\]
\[
\lambda_n = \frac{6 \varepsilon A_n^2 (\log_2 A_n^{-p})^{1-2\alpha}}{r_n^2}
\]
\[
x_n = \frac{r_n (\log_2 A_n^{-p})^{\alpha}}{A_n}
\]
it follows that

\[
C_{n_k}x_{n_k} = 2\varepsilon \quad (4.2.16)
\]

\[
\lambda_{n_k}x_{n_k}r_{n_k} = 6\varepsilon A_{n_k} \left( \log_2 A_{n_k}^{-p} \right)^{1-\alpha} \quad (4.2.17)
\]

\[
\lambda_{n_k}x_{n_k}^2 = 6\varepsilon \left( \log_2 A_{n_k}^{-p} \right) \rightarrow \infty \quad (4.2.18)
\]

\[
x_{n_k}^2 = \frac{r_{n_k}^2 \left( \log_2 A_{n_k}^{-p} \right)^{2\alpha}}{A_{n_k}^2} \leq \varepsilon^2 (\log_2 A_n^{-p})^{\alpha p} = o(\log_2 A_n^{-p}) \quad (4.2.19)
\]

by (4.2.15) and the fact that \( \alpha p < 1 \).

Now by (4.2.14) and (4.2.17) we obtain

\[
P\left\{ R_n > \frac{6\varepsilon}{\gamma^2} \frac{A_n (\log_2 A_n^{-p})^{1-\alpha}}{\i.o. \; (n)} \right\} \leq P\left\{ \sup_{n \geq n_k} R_n > \lambda_{n_k}x_{n_k}r_{n_k} \text{ i.o. (k)} \right\}.
\]

(4.2.20)

But, for all \( k \geq 2 \) such that \( n_k \geq 2 \), letting \( u = 2\varepsilon \) and \( v \in (0, u^{-1}] \), we have

\[
P\left\{ \sup_{n \geq n_k} R_n > \lambda_{n_k}x_{n_k}r_{n_k} \right\} \leq \exp\left\{ -x_{n_k}^2 \left( v \lambda_{n_k} - \frac{v^2}{2} (1 + \varepsilon v) \right) \right\}
\]

(by part (ii) of Lemma 10 recalling (4.2.16))

\[
= \exp\left\{ -3 \left( \log_2 A_{n_k}^{-p} \right) + Kx_{n_k}^2 \right\} \quad (K = \frac{3}{16\varepsilon^2})
\]

(by taking \( v = \frac{1}{2\varepsilon} \) and by (4.2.18))

\[
\leq \exp\left\{ -2 \left( \log_2 A_{n_k}^{-p} \right) \right\} \quad (\text{by (4.2.19)})
\]

\[
= \left( \log_1 A_{n_k}^{-p} \right)^{-2}
\]

\[
\leq (-pk \log_1 \gamma)^{-2} \quad (\text{since } A_{n_k}^{-p} \geq \gamma^{-pk})
\]

and so for some constant \( C \in (0, \infty) \)

\[
\sum_{k=1}^{\infty} P\left\{ \sup_{n \geq n_k} R_n > \lambda_{n_k}x_{n_k}r_{n_k} \right\} \leq C + \frac{1}{p^2 (\log_1 \gamma)^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.
\]
Hence, by Borel-Cantelli lemma,

\[ P\left\{ \sup_{n \geq n_k} R_n > \lambda_n x_n r_n \text{ i.o.}(k) \right\} = 0 \]

implying via (4.2.20) that

\[ P\left\{ R_n > \frac{6\varepsilon}{\gamma^2} A_n (\log_2 A_n^{-p})^{1-\alpha} \text{ i.o.}(n) \right\} = 0. \]

Hence

\[ \lim_{n \to \infty} \sup R_n \leq \frac{6\varepsilon}{\gamma^2} \quad \text{a.s.} \quad (4.2.21) \]

Since \((-U_n - E(U_n)), n \geq 1\) have the same bounds and variances as those of \((U_n - E(U_n)), n \geq 1\), (4.2.21) likewise obtains with \(-R_n\) replacing \(R_n\) thereby proving (4.2.12) and the theorem. □

By taking \(p = 2\) in Theorem 4, we obtain the following two corollaries which are partial analogues of Teicher's [47] corollaries of Proposition 7 above.

**Corollary 8.** Let \(\{X_n, n \geq 1\}\) be independent random variables with

\[ E(X_n) = 0, \quad E(X_n^2) = \sigma_n^2, \quad n \geq 1, \quad \text{and} \quad t_n^2 = \sum_{j=n}^{\infty} \sigma_j^2 = o(1). \]

Assume that

\[ t_n^2 = O(t_{n+1}^2). \quad (4.2.22) \]

If for some \(\alpha \in (-\infty, \frac{1}{2})\)

\[ \sum_{n=1}^{\infty} P\{ |X_n| > \delta t_n (\log_2 t_n^{-2})^{1-\alpha} \} < \infty \quad \text{for some} \quad \delta > 0 \quad (4.2.23) \]

and for all \(\varepsilon > 0\)

\[ \sum_{n=1}^{\infty} \frac{E(X_n^2 1_{[\epsilon t_n (\log_2 t_n^{-2})^{1-\alpha} < |X_n| < \delta t_n (\log_2 t_n^{-2})^{1-\alpha}]})}{(t_n (\log_2 t_n^{-2})^{1-\alpha})^2} < \infty, \quad (4.2.24) \]
then the tail series SLLN

\[ \frac{T_n}{t_n (\log_2 t_n^{-2})^{1-\alpha}} \to 0 \text{ a.s.} \]  \hspace{1cm} (4.2.25)

obtains.

**Remark.** Observe that this corollary precludes \( \alpha = \frac{1}{2} \). In fact, the conditions (4.2.23) and (4.2.24) when \( \alpha = \frac{1}{2} \) comprise two of the three conditions for the tail series LIL of Rosalsky [41, Theorem 2].

**Corollary 9.** Let \( \{X_n, n \geq 1\} \) be independent random variables with

\[ E(X_n) = 0, \ E(X_n^2) = \sigma_n^2, \ n \geq 1, \text{ and } t_n^2 = \sum_{j=n}^{\infty} \sigma_j^2 = o(1) \]

If for some \( \alpha^* \in (0, \frac{1}{2}] \) and \( M \in (0, \infty) \),

\[ |X_n| \leq M t_n (\log_2 t_n^{-2})^{-\alpha^*} \text{ a.s., } n \geq 1, \]  \hspace{1cm} (4.2.26)

then the tail series SLLN (4.2.25) prevails for all \( \alpha < \alpha^* \).

**Proof.** For \( \alpha^* \in (0, \frac{1}{2}] \), observe at the outset that

\[ \frac{t_{n+1}^2}{t_n^2} = 1 - \frac{\sigma_n^2}{t_n^2} \geq 1 - \frac{M^2 \sigma_n^2 (\log_2 t_n^{-2})^{-2\alpha^*}}{t_n^2} = 1 - \frac{M^2}{(\log_2 t_n^{-2})^{2\alpha^*}} \to 1 \text{ (since } t_n^2 = o(1) \} \]

and hence (4.2.22) holds. Note that (4.2.26) ensures that the conditions (4.2.23) and (4.2.24) hold for any \( \alpha < \alpha^* \). The corollary then follows from Corollary 8. \( \Box \)

The two conditions (4.2.2) and (4.2.3) of Theorem 4 will now be combined into a single one in the next two Corollaries 10 and 11 which are comparable with the
Corollary 10. Let $1 \leq p \leq 2$ and let \{X_n, n \geq 1\} be independent random variables with
\[
E(X_n) = 0, \ E(|X_n|^p) \leq \varepsilon_n, \ n \geq 1
\]
where \{\varepsilon_n, n \geq 1\} are positive constants with $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Assume that (4.2.1) holds where $A^p_n = \sum_{j=m}^{\infty} \varepsilon_j, \ n \geq 1$. Let $-\infty < \alpha < \frac{1}{p}$ and $0 \leq \beta \leq 1$. If for all $\varepsilon > 0$
\[
\sum_{n=1}^{\infty} \frac{E\left(|X_n|^{2\beta} I_{|X_n| > \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha}}\right)}{\left(A_n (\log_2 A_n^{-p})^{1-\alpha}\right)^{2\beta}} < \infty, \tag{4.2.27}
\]
then the tail series SLLN (4.2.4) obtains.

Remarks. (i) Observe that a smaller $\alpha$ gives us a weaker assumption (4.2.27) as well as a weaker conclusion (4.2.4).

(ii) Also observe that for $\beta = 0$, the condition (4.2.27) reduces to
\[
\sum_{n=1}^{\infty} P\{|X_n| > \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha}\} < \infty \text{ for all } \varepsilon > 0
\]
and for $\beta = 1$, it becomes
\[
\sum_{n=1}^{\infty} \frac{E\left(X_n^2 I_{|X_n| > \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha}}\right)}{\left(A_n (\log_2 A_n^{-p})^{1-\alpha}\right)^2} < \infty.
\]

Proof of Corollary 10. Note that for all large $n$
\[
\frac{E\left(|X_n|^{2\beta} I_{|X_n| > \varepsilon A_n (\log_2 A_n^{-p})^{-\alpha}}\right)}{\left(A_n (\log_2 A_n^{-p})^{1-\alpha}\right)^{2\beta}} \geq \frac{E\left(|X_n|^{2\beta} I_{|X_n| > A_n (\log_2 A_n^{-p})^{1-\alpha}}\right)}{\left(A_n (\log_2 A_n^{-p})^{1-\alpha}\right)^{2\beta}} \geq E\left(I_{|X_n| > A_n (\log_2 A_n^{-p})^{1-\alpha}}\right) = P\{|X_n| > A_n (\log_2 A_n^{-p})^{1-\alpha}\}.
\]
Then for some constant $C \in (0, \infty)$,

$$
\sum_{n=1}^{\infty} P\{ |X_n| > A_n (\log_2 A_n^{-\beta})^{1-\alpha} \} \\
\leq C + \sum_{n=1}^{\infty} \frac{E\left[ |X_n|^{2\beta} I_{[|X_n| > A_n (\log_2 A_n^{-\beta})^{-\alpha}]} \right]}{(A_n (\log_2 A_n^{-\beta})^{1-\alpha})^{2\beta}} \\
< \infty \text{ (by (4.2.27))}
$$

implying the condition (4.2.2) with $\delta = 1$.

Next, note that for arbitrary $\varepsilon > 0$ and all $n \geq 1$

$$
E\left( |X_n|^{2\beta} I_{[\varepsilon A_n (\log_2 A_n^{-\beta})^{-\alpha} < |X_n| < A_n (\log_2 A_n^{-\beta})^{1-\alpha}]} \right) \\
\geq \frac{E\left( |X_n|^{2\beta} I_{[\varepsilon A_n (\log_2 A_n^{-\beta})^{-\alpha} < |X_n| < A_n (\log_2 A_n^{-\beta})^{1-\alpha}]} \right)}{(A_n (\log_2 A_n^{-\beta})^{1-\alpha})^{2\beta}} \\
= \frac{E\left( X_n^2 I_{[\varepsilon A_n (\log_2 A_n^{-\beta})^{-\alpha} < |X_n| < A_n (\log_2 A_n^{-\beta})^{1-\alpha}]} \right)}{(A_n (\log_2 A_n^{-\beta})^{1-\alpha})^2}.
$$

Then

$$
\sum_{n=1}^{\infty} E\left( X_n^2 I_{[\varepsilon A_n (\log_2 A_n^{-\beta})^{-\alpha} < |X_n| < A_n (\log_2 A_n^{-\beta})^{1-\alpha}]} \right) \\
\leq \sum_{n=1}^{\infty} \frac{E\left( |X_n|^{2\beta} I_{[|X_n| > \varepsilon A_n (\log_2 A_n^{-\beta})^{-\alpha}]} \right)}{(A_n (\log_2 A_n^{-\beta})^{1-\alpha})^{2\beta}} \\
< \infty \text{ (by (4.2.27))}
$$

and so the condition (4.2.3) also holds with $\delta = 1$. The corollary then follows
directly from Theorem 4. \(\square\)
Corollary 11. Let $1 \leq p \leq 2$ and let $\{X_n, n \geq 1\}$ be independent random variables with

$$E(X_n) = 0, \ E(|X_n|^p) \leq e_n, \ n \geq 1$$

where $\{e_n, n \geq 1\}$ are positive constants with $\sum_{n=1}^{\infty} e_n < \infty$. Assume that (4.2.1) holds where $A_n^p = \sum_{j=1}^{\infty} e_j, \ n \geq 1$. Let $-\infty < \alpha < \frac{1}{p}$, and $\gamma > 1$. If for all $\varepsilon > 0$

$$E\left(\left|X_n\right|^p I_{\left\{\left|X_n\right| > e_n (\log A_n)^{-\alpha}\right\}}\right) = O\left(\frac{\left(\log A_n^{-p}\right)^{-1+\alpha(1-\alpha)}}{\left(\log_1 A_n^{-p}\right)\left(\log_3 A_n^{-p}\right)^\gamma e_n}\right),$$

(4.2.28)

then the tail series SLLN (4.2.4) obtains.

Proof. Note that for arbitrary $\varepsilon > 0$ and all large $n$, (4.2.28) ensures that for some constant $C_1 \in (0, \infty)$

$$\frac{E\left(\left|X_n\right|^p I_{\left\{\left|X_n\right| > e_n (\log A_n)^{-\alpha}\right\}}\right)}{\left(\log A_n^{-p}\right)^{1-\alpha}^p} \leq C_1 \frac{e_n}{A_n^p (\log_1 A_n^{-p}) (\log_2 A_n^{-p}) (\log_3 A_n^{-p})^\gamma}.$$ 

Then for some constant $C_2 \in (0, \infty)$

$$\sum_{n=1}^{\infty} \frac{E\left(\left|X_n\right|^p I_{\left\{\left|X_n\right| > e_n (\log A_n)^{-\alpha}\right\}}\right)}{\left(\log A_n^{-p}\right)^{1-\alpha}^p} \leq C_2 + C_1 \sum_{n=1}^{\infty} A_n^p (\log_1 A_n^{-p}) (\log_2 A_n^{-p}) (\log_3 A_n^{-p})^\gamma \leq \infty$$

(by Rosalsky [41, Lemma 5])

and so the condition (4.2.27) holds with $\beta = \frac{p}{2}$. The corollary then follows from Corollary 10.

4.3 The Weighted I.I.D. Case

For i.i.d. random variables $\{Y_n, n \geq 1\}$ with $E(Y_1) = 0, \ E(Y_1^2) = 1$, and for nonzero constants $\{a_n, n \geq 1\}, \ \{a_n Y_n, n \geq 1\}$ is a sequence of weighted i.i.d.
random variables. Then there exists a random variable $S$ with $\sum_{j=1}^{n} a_j Y_j \to S$ a.s. iff $\sum_{n=1}^{\infty} a_n^2 < \infty$. (Sufficiency follows directly from the Khintchine-Kolmogorov convergence theorem whereas necessity results from the work of Kac and Steinhaus [28] or Marcinkiewicz and Zygmund [35] or Abbott and Chow [1].) In such a case, $\operatorname{E}(S) = 0$, $\operatorname{E}(S^2) = \sum_{n=1}^{\infty} a_n^2$.

Corollaries 8 and 10 reduce to Corollaries 12 and 13 below, respectively, in the weighted i.i.d. case.

Corollary 12. Let $\{Y_n, \ n \geq 1\}$ be i.i.d. random variables with $\operatorname{E}(Y_1) = 0$, $\operatorname{E}(Y_1^2) = 1$, and let $\{a_n, \ n \geq 1\}$ be nonzero constants satisfying $t_n^2 = \sum_{j=n}^{\infty} a_j^2 = o(1)$ and (4.2.22). If for some $\alpha \in (-\infty, \frac{1}{2})$

$$\sum_{n=1}^{\infty} \operatorname{P}\{|Y_1| > \delta |a_n|^{-1} t_n (\log_2 t_n^{-2})^{1-\alpha}\} < \infty \text{ for some } \delta > 0 \tag{4.3.1}$$

and for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} a_n^2 \frac{\operatorname{E}\left(Y_1^2 1_{|Y_1| < |a_n|^{-1} t_n (\log_2 t_n^{-2})^{-\alpha}} \right)}{\left(t_n (\log_2 t_n^{-2})^{1-\alpha}\right)^2} < \infty, \tag{4.3.2}$$

then the tail series SLLN

$$\frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1-\alpha}} \to 0 \text{ a.s.} \tag{4.3.3}$$

obtains.

Proof. Since the conditions (4.3.1), (4.3.2), and (4.3.3) are simply transcriptions of (4.2.23), (4.2.24), and (4.2.25), respectively the corollary follows immediately from Corollary 8. □

Corollary 13. Let $\{Y_n, \ n \geq 1\}$ be i.i.d. random variables with $\operatorname{E}(Y_1) = 0$, $\operatorname{E}(Y_1^2) = 1$, and let $\{a_n, \ n \geq 1\}$ be nonzero constants satisfying $t_n^2 = \sum_{j=n}^{\infty} a_j^2 = o(1)$ and
(4.2.22). Let \(-\infty < \alpha < \frac{1}{2}\) and \(0 \leq \beta \leq 1\). If for all \(\varepsilon > 0\)
\[
\sum_{n=1}^{\infty} \frac{|a_n|^{2\beta} E\left(|Y_1|^{2\beta} I_{|Y_1| > \varepsilon |a_n|^{-1} t_n (\log_2 t_n^{-2})^{-\alpha}}\right)}{(t_n (\log_2 t_n^{-2})^{1-\alpha})^{2\beta}} < \infty,
\]
then the tail series SLLN (4.3.3) obtains.

Proof. Since the condition (4.3.4) is a simple transcription of (4.2.27) with \(p = 2\),
the corollary follows directly from Corollary 10. □

The main result of this section, Theorem 5, which is an analogue of the tail
series LIL of Rosalsky [41, Theorem 2], may now be stated.

**Theorem 5.** Let \(\{Y_n, n \geq 1\}\) be i.i.d. random variables with \(E(Y_1) = 0, E(Y_1^2) = 1\), and let \(\{a_n, n \geq 1\}\) be nonzero constants satisfying \(t_n = \sum_{j=n}^{\infty} a_j^2 = o(1)\) and
(4.2.22). If
\[
\frac{n a_n^2}{t_n^2} = O\left((\log_2 t_n^{-2})^{-\tau}\right) \text{ for some } -\infty < \tau < \infty,
\]
then the tail series SLLN
\[
\frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1-\alpha}} \to 0 \text{ a.s.}
\]

obtains for every \(\alpha \in (-\infty, \frac{1}{2})\) provided in the case \(\tau > 2(1 - \alpha)\) that
\[
E\left(Y_1^2 (\log_2 |Y_1|)^{-2(1-\alpha)}\right) < \infty.
\]

Remark. Actually, under the assumption (4.3.5) where \(\tau < 1\), the result follows
directly from the tail series LIL of Rosalsky [41, Theorem 2]. In the case \(1 \leq \tau \leq 2(1 - \alpha)\), the additional assumption is not needed in Theorem 5, although an
alternative additional assumption in the same spirit as (4.3.6) is required for the
tail series LIL of Rosalsky when (4.3.5) holds with $\tau \geq 1$. And for $\tau > 2(1 - \alpha)$, we assumed the moment condition (4.3.6) which is weaker than the additional moment condition in the tail series LIL of Rosalsky since $\tau - 2(1 - \alpha) < \tau - 1$.

**Proof of Theorem 5.** Without loss of generality, it may be assumed that $\tau \geq 0$ and $\alpha \geq 0$. Note that (4.2.22) is tantamount to $t_n^{-2} t_{n-1}^2 \leq M_1$ for some constant $M_1 \in (0, \infty)$ and all $n \geq 2$. Then for all $n \geq 1$,
\[
t_n^{-2} = t_1^{-2} \prod_{j=2}^{n} \frac{t_j^{-2}}{t_j^2} \leq t_1^{-2} M_1^{n-1} = O(M_1^n)
\]
implying
\[
\log_2 t_n^{-2} \leq (1 + o(1)) \log_1 n. \tag{4.3.7}
\]
Moreover, for all large $j$ and for some constant $M_2 \in (0, \infty)$, observe that
\[
\frac{t_{j-1}^2}{t_j^2} = 1 + \frac{t_{j-1}^2 a_{j-1}^2}{t_{j-1}^2 t_j^2} \leq 1 + M_2 \frac{\left(\log_2 t_{j-1}^{-2}\right)^{\tau}}{j-1} \quad \text{(by (4.3.5))}
\]
\[
\leq 1 + 2 M_2 \frac{\left(\log_1 (j-1)\right)^{\tau}}{j-1} \quad \text{(by (4.3.7))}
\]
and so for all large $n_0$ and $n > n_0$
\[
\frac{t_{n_0}^2}{t_n^2} = \prod_{j=n_0+1}^{n} \frac{t_{j-1}^2}{t_j^2} \leq \prod_{j=n_0+1}^{n} \left(1 + 2 M_2 (j-1)^{-1} \left(\log_1 (j-1)\right)^{\tau}\right)
\]
\[
\leq \prod_{j=n_0}^{n-1} \exp \left\{2 M_2 j^{-1} \left(\log_1 j\right)^{\tau}\right\}
\]
\[
= \exp \left\{\sum_{j=n_0}^{n-1} 2 M_2 j^{-1} \left(\log_1 j\right)^{\tau}\right\}
\]
Thus

\[ \log_2 t_n^{-2} = O(\log_2 n). \]  

(4.3.8)

In view of Corollary 13, it is suffices to show that the condition (4.3.4) is satisfied for some \( \beta \in [0, 1] \) and all \( \varepsilon > 0 \). Recalling (4.3.5), let \( K \in (0, \infty) \) satisfy

\[ \frac{n a_n^2}{t_n^2} \leq K(\log_2 t_n^{-2})^{\gamma} \]  

(4.3.9)

for all large \( n \). For each \( n \geq 1 \), let

\[ q_n^2 = \frac{\varepsilon^2 n}{K(\log_2 t_n^{-2})^{\gamma+2\alpha}} \]  

(4.3.10)

where \( \varepsilon > 0 \) is fixed but arbitrary. Then, by (4.3.9)

\[ q_n^2 \leq \frac{\varepsilon^2 t_n^2}{a_n^2 (\log_2 t_n^{-2})^{2\alpha}}, \]  

(4.3.11)

and also by (4.3.8) and (4.3.10)

\[ \log_2 t_n^{-2} = O(\log_2 q_n). \]  

(4.3.12)

Next, it will be demonstrated that \( \{q_n, n \geq 1\} \) is eventually nondecreasing with \( q_n \to \infty \). To this end, note that

\[ \frac{n (t_n^{-2} - t_{n-1}^{-2})}{t_{n-1}^{-2} (\log_1 t_{n-1}^{-2}) (\log_2 t_{n-1}^{-2})} \overset{\text{by (4.2.22)}}{=} \frac{n a_{n-1}^2}{t_n^2 (\log_1 t_{n-1}^{-2}) (\log_2 t_{n-1}^{-2})} = \frac{O(1) (n-1) a_{n-1}^2}{t_{n-1}^2 (\log_1 t_{n-1}^{-2}) (\log_2 t_{n-1}^{-2})} \overset{\text{(by (4.2.22))}}{=} o(1) \]  

(4.3.13)
by the assumption (4.3.5). Let \( \phi(x) \) be the extension of \( \{t_n^{-2}, n \geq 1\} \) defined by linear interpolation between integers, i.e., for all \( n \geq 2 \)

\[
\phi(x) = t_{n-1}^{-2} + (t_n^{-2} - t_{n-1}^{-2})(x - n + 1), \quad x \in [n - 1, n).
\]

Then, via the mean value theorem, for all large \( n \) there exists a number \( x_n \) in \((n - 1, n)\) such that

\[
q_n^2 - q_{n-1}^2 = \frac{\varepsilon^2 n}{K(\log_2 t_n^{-2})^{\tau + 2\alpha}} - \frac{\varepsilon^2 (n - 1)}{K(\log_2 t_{n-1}^{-2})^{\tau + 2\alpha}}
\]

\[
= \frac{\varepsilon^2}{K} \left\{ \frac{1}{(\log_2 \phi(x_n))^{\tau + 2\alpha}} - \frac{(\tau + 2\alpha) x_n (t_n^{-2} - t_{n-1}^{-2})}{\phi(x_n)(\log_2 \phi(x_n))(\log_2 \phi(x_n))^{\tau + 2\alpha + 1}} \right\}
\]

\[
= \frac{\varepsilon^2}{K} (1 + o(1)) \frac{1}{(\log_2 \phi(x_n))^{\tau + 2\alpha}} \quad \text{(by (4.3.13))}
\]

\[
\geq \frac{\varepsilon^2}{2K} (\log_2 \phi(x_n))^{-(\tau + 2\alpha)}
\]

\[
\geq 0.
\]

Thus, since (4.3.10) and (4.3.8) guarantee \( q_n^2 \to \infty \), we have verified that \( \{q_n, n \geq 1\} \) is eventually nondecreasing with \( q_n \to \infty \).

Let \( 0 \leq \beta \leq 1 \) and \( \tau^* \geq 0 \), the exact choices of which will be made later. Then

\[
\sum_{n=1}^{\infty} \frac{|a_n|^{2\beta} E\left(|Y_1|^{2\beta} I_{|Y_1|>|a_n|^{-1} t_n (\log_2 t_n^{-2})^{-\alpha}}\right)}{(t_n (\log_2 t_n^{-2})^{1-\alpha})^{2\beta}}
\]

\[
\leq O(1) \sum_{n=1}^{\infty} \frac{E\left(|Y_1|^{2\beta} I_{|Y_1|>|a_n|^{-1} t_n (\log_2 t_n^{-2})^{-\alpha}}\right)}{n^{\beta (\log_2 t_n^{-2})^{2(1-\alpha)-\beta \tau}}} \quad \text{(by (4.3.9))}
\]

\[
\leq O(1) \sum_{n=1}^{\infty} \frac{E\left(|Y_1|^{2\beta} I_{|Y_1|>|q_n|}\right)}{n^{\beta (\log_2 t_n^{-2})^{2(1-\alpha)-\beta \tau}}} \quad \text{(by (4.3.11))}
\]

\[
\leq O(1) \sum_{n=1}^{\infty} \frac{\sum_{j=n}^{\infty} E\left(|Y_1|^{2\beta} I_{|Y_1|>|q_j|}ight)}{n^{\beta (\log_2 t_n^{-2})^{2(1-\alpha)-\beta \tau}}}
\]

\[
= O(1) \sum_{j=1}^{\infty} E\left(|Y_1|^{2\beta} I_{|Y_1|>|q_j|}\right) \sum_{n=1}^{j} n^{-\beta (\log_2 t_n^{-2})^{\beta (2(1-\alpha)-\tau)}}
\]
\[
\leq O(1) \sum_{j=1}^{\infty} j^{1-\beta} \left( \log_2 t_j^{-2} \right) \beta(\tau - 2(1-\alpha)) E\left( |Y_1|^{2\beta} I_{[q_j,|Y_1| \leq q_j+1]} \right)
\]
(by Rosalsky [41, Lemma 6])

\[
\leq O(1) \sum_{j=1}^{\infty} j^{1-\beta} \left( \log_2 t_j^{-2} \right) \beta(\tau - 2(1-\alpha)) \frac{E\left( Y_1^2 (\log_2 |Y_1|)^\tau I_{[q_j,|Y_1| \leq q_j+1]} \right)}{q_j^{2(1-\beta)} (\log_2 q_j)^\tau^*}
\]

\[
= O(1) \sum_{j=1}^{\infty} \left( \log_2 t_j^{-2} \right) \beta(\tau - 2(1-\alpha)) + (1-\beta)(\tau + 2\alpha) \frac{E\left( Y_1^2 (\log_2 |Y_1|)^\tau I_{[q_j,|Y_1| \leq q_j+1]} \right)}{(\log_2 q_j)^\tau^*}
\]
(by (4.3.10))

\[
\leq O(1) \sum_{j=1}^{\infty} \left( \log_2 t_j^{-2} \right)^{\tau + 2\alpha - \tau^*} E\left( Y_1^2 (\log_2 |Y_1|)^\tau I_{[q_j,|Y_1| \leq q_j+1]} \right) \tag{4.3.14}
\]
(by (4.3.12)).

If \( \tau \leq 2(1-\alpha) \), let \( \tau^* = 0 \) and \( \beta = \frac{1}{2}(\tau + 2\alpha) \) and then via (4.3.14) the series of (4.3.4) is dominated by \( O(1) E(Y_1^2) < \infty \).

Alternatively, if \( \tau > 2(1-\alpha) \), let \( \tau^* = \tau - 2(1-\alpha) \) and \( \beta = 1 \). Then again via (4.3.14), the series of (4.3.4) is dominated by \( O(1) E\left( Y_1^2 (\log_2 |Y_1|)^\tau \right) < \infty \) recalling (4.3.6). The theorem follows then directly from Corollary 13. \( \Box \)

To illustrate Theorem 5, we will revoke previous examples from Chapters 2 and 3.

**Example 6.** As was observed in Example 3 of Chapter 2, the harmonic series with a random choice of signs yields the tail series LIL (2.4.14) thereby ensuring the tail series SLLN

\[
\frac{n^{\frac{1}{2}}}{(\log_2 n)^{\frac{1}{2}+\epsilon}} T_n \to 0 \ a.s. \ (\epsilon > 0)
\]

obtains. Alternatively, the same conclusion follows directly from our tail series SLLN (Theorem 5 with \( \tau = 0 \) and \( \alpha = \frac{1}{2} - \epsilon \)).

**Example 7.** The same argument as in Example 6 can be applied to Example 5.
of Chapter 3, i.e., since

\[ \frac{n \alpha_n^2}{t_n^2} \sim (\log_2 n)^u (\log_3 n)^v \quad \text{and} \quad \log_2 t_n^{-2} \sim \log_2 n \]

(see Rosalsky [41, Example 1] for verification), the condition (4.3.5) holds with \( \tau > u \) (or \( \tau = u \) if \( v \leq 0 \)) and so from Theorem 5 the tail series SLLN

\[
\frac{\sum_{j=n}^{\infty} a_j Y_j}{t_n (\log_2 t_n^{-2})^{1-\alpha}} \rightarrow 0 \text{ a.s.}
\]

obtains whenever \( -\infty < \alpha < \frac{1}{2} \) provided (4.3.6) holds if \( \tau > 2(1-\alpha) \).

In the following example, we will see an application of the tail series SLLN to the field of time series analysis.

**Example 8.** Let \( \{S_t, t = 0, \pm 1, \pm 2, \ldots\} \) be the moving average process of infinite order given by

\[ S_t = \sum_{j=0}^{\infty} a_j X_{t-j} \quad (4.3.15) \]

where \( \{X_t, t = 0, \pm 1, \pm 2, \ldots\} \) are i.i.d. normal random variables with mean 0 and variance 1 and \( \{a_j, j \geq 0\} \) is a square summable sequence of constants. As a specific example, consider a long memory process, which is represented by (4.3.15) with \( a_0 = 1 \) and

\[
a_j = \frac{\Gamma(j+d)}{\Gamma(j+1) \Gamma(d)} = \prod_{0<k\leq j} \frac{k-1+d}{k}, \quad j \geq 1 \quad (4.3.16)
\]

where

\[
|d| < \frac{1}{2} \quad \text{and} \quad \Gamma(x) = \begin{cases} 
\int_0^{\infty} t^{x-1} e^{-t} dt, & \text{if } x > 0 \\
\infty, & \text{if } x = 0 \\
x^{-1} \Gamma(1+x), & \text{if } x < 0.
\end{cases}
\]
By Stirling’s formula

$$
\Gamma(x) \sim \sqrt{2\pi} e^{-x+1} (x - 1)^{x-\frac{1}{2}} \text{ as } x \to \infty
$$

applied to (4.3.16), we obtain (see e.g., Brockwell and Davis [11], p. 466) for \( d \neq 0 \)

$$
a_j \sim \frac{j^{d-1}}{\Gamma(d)} \text{ as } j \to \infty.
$$

Then

$$
t_n^2 = \sum_{j=n}^{\infty} a_j^2 \sim \frac{1}{(\Gamma(d))^2} \sum_{j=n}^{\infty} j^{2(d-1)} \sim \frac{n^{2d-1}}{(1 - 2d) (\Gamma(d))^2}
$$

implying

$$
\log_2 t_n^{-2} \sim \log_2 n, \ t_n^2 = O(t_{n+1}^2), \text{ and } \frac{n a_n^2}{t_n^2} = O(1).
$$

Thus the conditions (4.2.22) and (4.3.5) (with \( \tau = 0 \)) hold and so for every integer \( t \) and all \( \alpha \in (-\infty, \frac{1}{2}) \), the tail series SLLN

$$
\frac{n^{\frac{1}{2}-d}}{(\log_2 n)^{1-\alpha}} \sum_{j=n}^{\infty} a_j X_{t-j} \to 0 \text{ a.s.}
$$

follows from Theorem 5. We have thus determined an order bound on the almost sure rate in which \( \sum_{j=0}^{n} a_j X_{t-j} \) converges to \( S_t \) for every \( t \). Observe that this rate is independent of the time \( t \). Of course, \( \sum_{j=0}^{n} a_j X_{t-j} \) is structurally far simpler than \( S_t \).
CHAPTER 5
SOME FUTURE RESEARCH PROBLEMS

The current research work suggests a number of open problems or areas for future research activity. These problems or areas are discussed in this chapter.

In Chapter 2, a function $\psi(x)$ in a specific class of functions $\Psi$ defined by Klesov [29 and 30] was employed for determining the norming constants for tail series SLLNs for random variables. Actually, this class $\Psi$ is a tail series partial analogue of the class $\Psi_c$ defined by Petrov [38] as follows; a function $f$ belongs to $\Psi_c$ if it is a positive and nondecreasing function such that the series $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ converges. In the case of the SLLNs for partial sums, Egorov [21] defined a wider class of functions $F_c$ as follows; a function $f$ belongs to $F_c$ if $(f(x))^\varepsilon \in \Psi_c$ for some $\varepsilon > 0$. In the same spirit as in Chapter 2, tail series SLLNs which might exist and correspond to Egorov’s [21] SLLNs for partial sums will possibly employ a function in a class which is a tail series analogue of $F_c$. It would be particularly interesting to see whether such tail series SLLNs subsumes the results of Chapter 2.

In the Theorem 4 of Chapter 4, we established the counterpart to the SLLN for partial sums of Teicher [47]. But Theorem 4 is indeed an incomplete analogue of the SLLN of Teicher [47] because we assumed

$$\sum_{n=1}^{\infty} \frac{E(X_n^2 1_{\{\frac{A_n^{(\log_2 A_n^{-p})-\alpha}}{\delta A_n^{(\log_2 A_n^{-p})-\alpha}} < |X_n| < \delta A_n^{(\log_2 A_n^{-p})^{-\alpha}}\}}}{(A_n^{(\log_2 A_n^{-p})^{-\alpha}})^2} < \infty, \quad (5.0.1)$$
for all $\epsilon > 0$ rather than for merely some $\epsilon > 0$, as was the case in a partial sum version of condition (5.0.3) which was used by Teicher [47] to prove a SLLN. The reason for this is that our tail series exponential bound (part (ii) of Lemma 10 of Chapter 4), which is employed to prove Theorem 4 of Chapter 4, was proved only for all $v \in (0, u^{-1}]$ rather than for all $v \in (0, \infty)$. Thus, by establishing an extension of this exponential bound lemma without the restriction on $v$ (as is the case in an exponential bound for partial sums), the assumption (5.0.3) for all $\epsilon > 0$ might be able to be weakened to (5.0.3) for some $\epsilon > 0$. Conceivably, under no additional conditions or under mild conditions, the convergence of the series in (5.0.3) for some $\epsilon > 0$ guarantees convergence for all $\epsilon > 0$ but this would require some investigation.

Next, it will be a very interesting problem to establish tail series analogues of Adler and Rosalsky’s [3, 4] general SLLNs for weighted sums of (stochastically dominated or i.i.d) random variables. Adler and Rosalsky [3] established general SLLNs of the form

$$\frac{\sum_{j=1}^{n} a_j (Y_j - \gamma_j)}{b_n} \rightarrow 0 \text{ a.s.}$$

where $\{Y_n, n \geq 1\}$ are stochastically dominated by a random variable $|Y|$ and $\{\gamma_n, n \geq 1\}$ are suitable conditional expectations or are all 0. In their follow-up paper, Adler and Rosalsky [4] provided sets of necessary and (or) sufficient conditions for $\{a_n Y_n, n \geq 1\}$ to obey the general SLLN of the form

$$\frac{\sum_{j=1}^{n} a_j Y_j}{b_n} \rightarrow 0 \text{ a.s.}$$

where $\{Y_n, n \geq 1\}$ is a sequence of i.i.d. mean 0 random variables and $\{a_n, n \geq 1\}$ are nonzero constants.
Finally, tail series problems for almost surely convergent series of independent random elements taking values in normed linear spaces is an area ripe for extensive research activity. Beginning with the pioneering work of Mourier [36] (wherein an analogue of the classical Kolmogorov SLLN was proved for sums of i.i.d. random elements taking values in a real separable Banach space), an extensive literature of investigation has appeared on the SLLN and WLLN problems for partial sums of Banach space valued random elements. For some recent developments in this general direction, see the articles Adler, Rosalsky, and Taylor [5, 6, 7] and some of the references contained therein (specifically, see Beck [10], Itô and Nisio [26], Hoffmann-Jorgensen and Pisier [25], Woyczyński [51, 52], Kuelbs and Zinn [33], de Acosta [2], and Wang and Bhaskara Rao [49]). The necessary background material for reading the above papers on Banach space valued random elements may be found in Taylor [44]. Tail series versions of some of the results in the literature cited above would certainly be a worthwhile research accomplishment. Indeed, the very question as to when a series of independent Banach space valued random elements converges almost surely is one which requires more investigation. For some results in this direction, see Hoffmann-Jørgensen [24], Jain [27], and Woyczyński [51, p. 386-390 and p. 430-431].
REFERENCES


BIOGRAPHICAL SKETCH

The author was born on June 24, 1956, in Kimcheon, Republic of Korea. In 1979, he graduated from the Air Force Academy, Seoul, Republic of Korea. He was awarded a Bachelor of Science degree in mathematics in 1982 and a Master of Statistics degree in 1985, both from Seoul National University, Seoul, Republic of Korea. He then served as a full-time instructor in the Department of Mathematics of the Korean Air Force Academy until 1988. He has held the rank of Major in the Korean Air Force since 1987. He has published a paper (joint with Jong Woo Jeon and Suk Ki Han), "Some Distribution Free Tests for Exponential Distributions," *Journal of the Korean Society for Quality Control* 14 (1986), 39-46.

Since 1988, Mr. Nam has been working towards the Ph.D. in statistics from the University of Florida. He has been a member of the American Statistical Association since 1989. He is married and has two children.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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Associate Professor of Statistics

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Professor of Statistics

Richard Scheaffer
Professor of Statistics
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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