QUANTIZATION
AND
REPRESENTATION INDEPENDENT PROPAGATORS

By
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For Marie-Jacqueline and Anne-Sophie
"The highest reward for a man's toil is not what he gets for it, but what he becomes by it."

- John Ruskin
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QUANTIZATION AND REPRESENTATION INDEPENDENT PROPAGATORS

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The quantization of physical systems moving on group and symmetric spaces has been an area of active and on-going research over the past three decades. It is shown in this work that it is possible to introduce a representation independent propagator for a real, separable, connected and simply connected Lie group with irreducible, square integrable representations. For a given set of kinematical variables this propagator is a single generalized function independent of any particular choice of fiducial vector and the irreducible representations of the Lie group generated by these kinematical variables, which nonetheless, correctly propagates each element of a continuous representation based on the coherent states associated with these kinematical variables.

Furthermore, it is shown that it is possible to construct regularized lattice phase-space path integrals for a real, separable, connected and simply connected Lie group with irreducible, square integrable representations, and although the configuration space is in general a multidimensional curved manifold, it is shown that the resulting lattice phase-space path integral has the form of a lattice phase-space path integral on a multidimensional flat manifold. Hence, a novel and extremely natural phase-space
path integral quantization is obtained for \textit{general} physical systems whose kinematical variables are the generators of a connected and simply connected Lie group. This novel phase-space path integral quantization is (a) more general than, (b) exact, and (c) free from the limitations of the previously considered path integral quantizations of \textit{free} physical systems moving on group manifolds.

To illustrate the general theory, a representation independent propagator is explicitly constructed for $SU(2)$ and the affine group.
CHAPTER 1
INTRODUCTION

In non-relativistic quantum mechanics the states of a quantum mechanical system are given by unit vectors, such as \( \psi, \phi, \) or \( \eta, \) in some complex, separable Hilbert space \( \mathbf{H}. \) For a single canonical degree of freedom problem the basic kinematical variables are represented on \( \mathbf{H} \) by two unbounded self-adjoint operators \( P, \) the momentum, and \( Q, \) the position, with a common dense invariant domain \( \mathbf{D}. \) These operators satisfy the Canonical (Heisenberg) Commutation Relation (CCR)

\[
[Q, P] = iI,
\]

where \( \hbar = 1. \) Let \( \mathcal{H}(P, Q) \) be the Hamilton operator of a quantum system, then the time evolution of this quantum system in the state \( \psi \in \mathbf{D}(\mathcal{H}) \) is given by the time-dependent Schrödinger equation

\[
i\partial_t\psi(t) = \mathcal{H}(P, Q)\psi(t).
\]

Since only self-adjoint operators may be exponentiated to give one-parameter unitary groups which give the dynamics of a quantum system, it will always be assumed that the Hamilton operator is essentially self-adjoint, i.e. its closure is self-adjoint. If the Hamilton operator is not explicitly time dependent then a solution to Schrödinger’s equation is given in terms of the strongly continuous one-parameter unitary Schrödinger group, \( U(t) = \exp(-it\mathcal{H}), \) by

\[
\psi(t'') = U(t'' - t')\psi(t').
\]

For a single particle system one traditionally chooses for the Hilbert space \( \mathbf{H} \) the spaces \( L^2(\mathbb{R}, dq), \) or \( L^2(\mathbb{R}, dp). \) On these Hilbert spaces the basic kinematical
variables $P$ and $Q$ are realized by the following two unbounded symmetric operators $-id/dq$ and $q$, or $p$ and $id/dp$, respectively. These operators are essentially self-adjoint on the dense subspace $S(\mathbb{R})$ of $L^2(\mathbb{R})$, the space of infinitely often differentiable functions that together with their derivatives fall off faster than the inverse of any polynomial. Furthermore, since these operators leave $S(\mathbb{R})$ invariant, one can choose $S(\mathbb{R}) \subset L^2(\mathbb{R})$ as the common dense invariant domain for these operators. One calls these operators together with their common dense invariant domain $S(\mathbb{R})$ the Schrödinger representation on q-space, or on p-space, which is denoted by $\psi(q) \in S(\mathbb{R})$ or $\tilde{\psi}(p) \in S(\mathbb{R})$, respectively (cf. [11, Appendix V]).

However, we would like to emphasize that there is nothing sacred about this choice of representation other than the time-honored custom of doing so. One can also choose one of the so-called continuous representations based on canonical coherent states (cf. [60]). In this representation the states are given by certain bounded, continuous, square integrable functions of two real parameters $p$ and $q$. We denote these functions by $\psi_\eta(p, q)$. The functions $\psi_\eta(p, q)$ span a subspace $L^2_\eta(\mathbb{R}^2)$ of $L^2(\mathbb{R}^2)$, where the subscript $\eta$ denotes a unit fiducial vector in the Hilbert space $H$ on which the canonical coherent states are based. Let $\eta(x) \in L^2(\mathbb{R})$ be a fixed normalized function and $\psi(x) \in L^2(\mathbb{R})$ be arbitrary, then an explicit representation of the functions $\psi_\eta(p, q)$ can be given as follows

$$\psi_\eta(p, q) = \int \frac{1}{\sqrt{2\pi}} \overline{\eta(x)} \exp(-ipx) \psi(x + q) dx. \quad (1.1)$$

From this form of the representation it may be seen that one obtains the Schrödinger representation in q-space or in p-space in appropriate limits. In particular, one obtains the Schrödinger representation on q-space by suitably scaling the $\psi_\eta(p, q)$ so that the scaled fiducial vector approaches in the limit a delta function, and the Schrödinger representation on p-space as the fiducial vector approaches the indicator function of $\mathbb{R}$ times the constant $e^{ipq}$ (see [61]).
With each of these various representations one can associate a propagator:

in q-space by

\[ \psi(q'', t'') = \int J(q'', t''; q', t')\psi(q', t')dq', \]

in p-space by

\[ \tilde{\psi}(p'', t'') = \int L(p'', t''; p', t')\tilde{\psi}(p', t')dp', \]

and finally in the continuous representation based on canonical coherent states by

\[ \psi_\eta(p'', q'', t'') = \int K_\eta(p'', q'', t''; p', q', t')\psi_\eta(p', q', t')dp'dq'. \]

Of course each one of these propagators generally depends on the representation one has chosen. Physically, these propagators represent the probability amplitude for the quantum system under discussion to undergo a transition from an initial configuration to some final configuration, and they contain all the relevant dynamical information for the quantum system. Let us ask whether it is possible to find a single propagator \( K(p'', q'', t''; p', q', t') \) such that

\[ \psi_\eta(p'', q'', t'') = \int K(p'', q'', t''; p', q', t')\psi_\eta(p', q', t')dp'dq', \]

holds for an arbitrary fiducial vector. Stated otherwise, is there a propagator that is independent of the chosen continuous representation, but which, nonetheless, propagates the elements of any representation space \( L^2(\mathbb{R}^2) \) in such a way that they stay in the representation space \( L^2_\eta(\mathbb{R}^2) \)? The answer is yes. We now outline the construction of this propagator for the Heisenberg Weyl group; for an alternative construction of this propagator see Klauder [65].

1.1 The Fiducial Vector Independent Propagator for the Heisenberg Weyl Group

Let \( P, Q, \) and \( I \) be an irreducible, self-adjoint representation of the Heisenberg algebra, \([Q, P] = iI, \hbar = 1, [Q, I] = [P, I] = 0 \) on \( H \). Then for an arbitrary normalized vector \( \eta \in H \) define the following set of states

\[ \eta(p, q) \equiv \frac{1}{\sqrt{2\pi}}V(p, q)\eta, \]
where $V(p, q) \equiv \exp(-iqP) \exp(ipQ)$. In fact these states are the familiar canonical coherent states which form a strongly continuous, overcomplete family of states for a fixed, normalized fiducial vector $\eta \in H$ and they admit the following resolution of identity:

$$I_H = \int \eta(p, q) \langle \eta(p, q), \cdot \rangle dp dq.$$  

The map $C_\eta : H \to L^2(\mathbb{R}^2, dp dq)$, defined for any $\psi \in H$ by:

$$[C_\eta \psi](p, q) = \psi_\eta(p, q) \equiv \langle \eta(p, q), \psi \rangle = \frac{1}{\sqrt{2\pi}} \eta, V^*(p, q) \psi,$$

yields a representation of the Hilbert space $H$ by bounded, continuous, square integrable functions on a proper closed subspace $L^2_\eta(\mathbb{R}^2)$ of $L^2(\mathbb{R}^2)$. Using the resolution of identity one finds

$$\psi_\eta(p, q) = \int K_\eta(p, q; p', q') \psi_\eta(p', q') dp' dq', $$

where,

$$K_\eta(p, q; p', q') \equiv \langle \eta(p, q), \eta(p', q') \rangle = \langle \eta, \frac{1}{\sqrt{2\pi}} V^*(p, q) V(p', q') \frac{1}{\sqrt{2\pi}} \eta \rangle$$

is the reproducing kernel, which is the kernel of a projection operator from $L^2(\mathbb{R}^2)$ onto the reproducing kernel Hilbert space $L^2_\eta(\mathbb{R}^2)$. Let $\tilde{D}$ be the common dense invariant domain of $P$ and $Q$ that is also invariant under $V(p, q)$, then one can easily show that the following relations hold on $\tilde{D}$:

$$-i\partial_q V^*(p, q) = V^*(p, q)P, \quad (1.2)$$
$$ (q + i\partial_p)V^*(p, q) = V^*(p, q)Q. \quad (1.3)$$

Notice, that the operator $V^*(p, q)$ intertwines the representation of the Heisenberg-algebra on the Hilbert space $H$, with the representation of the Heisenberg-algebra by right invariant differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(\mathbb{R}^2)$. Furthermore, these differential operators are essentially self-adjoint.
and an appropriate core for these operators is given by the continuous representation of $\tilde{D}$, $\tilde{D}_\eta = C_\eta(\tilde{D})$. Let $\mathcal{H}(P, Q)$ be the essentially self-adjoint Hamilton operator of a quantum system on $\mathcal{H}$, then using the intertwining relations (1.2) and (1.3) one finds for the time evolution of an arbitrary element $\psi_\eta(p, q, t)$ of $\tilde{D}_\eta \subset L^2_\eta(\mathbb{R}^2)$ the following

$$
\psi_\eta(p, q, t) = \int K_\eta(p, q, t; p', q', t') \psi_\eta(p', q', t') dp' dq'
$$

where,

$$
K_\eta(p, q, t; p', q', t') = \\
= \langle \eta(p, q), \exp[-i(t - t')\mathcal{H}(P, Q)]\eta(p', q') \rangle \\
= \langle \eta, \frac{1}{\sqrt{2\pi}} V^*(p, q) \exp[-i(t - t')\mathcal{H}(P, Q)]V(p', q') \frac{1}{\sqrt{2\pi}} \eta \rangle \\
= \exp[-i(t - t')\mathcal{H}(-i\partial_q, q + i\partial_p)]\langle \eta, \frac{1}{\sqrt{2\pi}} V^*(p, q)V(p', q') \frac{1}{\sqrt{2\pi}} \eta \rangle,
$$

where the closure of the Hamilton operator has been denoted by the same symbol. This construction holds for any $\eta \in \mathcal{H}$, therefore, one can choose any complete orthonormal system $\{\phi_j\}_{j=1}^\infty$ in $\mathcal{H}$ and write down the following propagator

$$
K(p, q, t; p', q', t') = \\
= \sum_{j=1}^\infty K_{\phi_j}(p, q, t; p', q', t') \\
= \exp[-i(t - t')\mathcal{H}(-i\partial_q, q + i\partial_p)]\sum_{j=1}^\infty \langle \phi_j, \frac{1}{\sqrt{2\pi}} V^*(p, q)V(p', q') \frac{1}{\sqrt{2\pi}} \phi_j \rangle, \\
= \exp[-i(t - t')\mathcal{H}(-i\partial_q, q + i\partial_p)]\frac{1}{2\pi} \text{tr}[V^*(p, q)V(p', q')]
$$

Let us now evaluate $\frac{1}{2\pi} \text{tr}[V^*(p, q)V(p', q')]$. Let $\{\phi_i(x)\}_{i=1}^\infty$ and $\{\psi_k(x)\}_{k=1}^\infty$ be two complete orthonormal systems in $L^2(\mathbb{R})$. Then using the representation in (1.1) we can formally write
the fourth line follows from using the completeness relations for the \( \phi_l(x) \) and \( \psi_k(x) \).

Hence, the propagator \( K(p, q, t; p', q', t') \) is given by:

\[
K(p, q, t; p', q', t') \equiv \exp[-i(t - t')H(-i\partial_q, q + i\partial_p)]\delta(p - p')\delta(q - q'). \tag{1.4}
\]

As shown in [65] this propagator propagates the elements of any reproducing kernel Hilbert space \( L^2_\eta(\mathbb{R}^2) \) correctly, i.e.

\[
\psi_\eta(p, q, t) = \int K(p, q, t; p', q', t')\psi_\eta(p', q', t')dp'dq', \tag{1.5}
\]

The propagator in (1.4) is clearly independent of the chosen fiducial vector. A sufficiently large set of test functions for this propagator is given by \( C(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \), where \( C(\mathbb{R}^2) \) is the set of all continuous functions on \( \mathbb{R}^2 \). Hence, every element of \( L^2_\eta(\mathbb{R}^2) \) is an allowed test function for this propagator. From (1.4) it is easily seen that the fiducial vector independent propagator is a weak solution to Schrödinger's equation, i.e.

\[
i\partial_t K(p, q, t; p', q', t') = \mathcal{H}(-i\partial_q, q + i\partial_p)K(p, q, t; p', q', t')
\]

Taking in (1.4) the limit \( t \to t' \) one finds the following initial value problem

\[
i\partial_t K(p, q, t; p', q', t') = \mathcal{H}(-i\partial_q, q + i\partial_p)K(p, q, t; p', q', t')
\]

\[
\lim_{t \to t'} K(p, q, t; p', q', t') = \delta(p - p')\delta(q - q'). \tag{1.6}
\]
We now interpret (1.6) as a Schrödinger equation appropriate to two separate and independent canonical degrees of freedom. Hence, \( p \) and \( q \) are viewed as "coordinates," and we are looking at the irreducible Schrödinger representation of a special class of two-variable Hamilton operators, ones where the classical Hamiltonian is restricted to have the form \( \mathcal{H}(k, q - x) \) instead of the most general form \( \mathcal{H}(k, x, q, p) \). In fact the operators given by equation (1.2) and (1.3) are elements of the right invariant enveloping algebra of a two dimensional Schrödinger representation. Based on this interpretation following standard procedures (cf. [63]) one can give the fiducial vector independent propagator for the Heisenberg Weyl group the following regularized standard phase space lattice prescription:

\[
K(p, q, t; p', q', t') = 
\lim_{N \to \infty} \int \cdots \int \exp \left\{ i \sum_{j=0}^{N} \left[ x_{j+1/2} (p_{j+1} - p_j) + k_{j+1/2} (q_{j+1} - q_j) \right. \right.
\left. - \epsilon \mathcal{H}(k_{j+1/2}, (q_{j+1} + q_j)/2 - x_{j+1/2}) \right]\}
\prod_{j=0}^{N} dp_j dq_j \prod_{j=0}^{N} dk_{j+1/2} dx_{j+1/2},
\]

where \( (p_{N+1}, q_{N+1}) = (p, q) \), \( (p_0, q_0) = (p', q') \), and \( \epsilon \equiv (t - t')/(N + 1) \). Observe that the Hamiltonian has been used in the special form dictated by the differential operators in equations (1.2) and (1.3) and that Weyl ordering has been adopted. After a change of variables (see [65]) the fiducial vector independent propagator for the Heisenberg Weyl Group becomes

\[
K(p, q, t; p', q', t') = 
\lim_{N \to \infty} \int \cdots \int \exp \left\{ i \sum_{j=0}^{N} \left[ \frac{1}{2} (q_{j+1} + q_j) (p_{j+1} - p_j) - x_{j+1/2} (p_{j+1} - p_j) \right. \right.
\left. + k_{j+1/2} (q_{j+1} - q_j) - \epsilon \mathcal{H}(k_{j+1/2}, x_{j+1/2}) \right]\}
\prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} dk_{j+1/2} dx_{j+1/2},
\]

Taking an improper limit by interchanging the limit with respect to \( N \) with the integrals one finds the following formal standard phase space path integrals

\[
K(p, q, t; p', q', t') = \mathcal{M} \int \exp \left\{ i \int [x \dot{p} + k \dot{q} - \mathcal{H}(k, q - x)] dt \right\} DpDqDkDx,
\]
and

\[ K(p, q, t; p', q', t') = \mathcal{M} \int \exp \left\{ i \int [q\dot{p} - x\dot{q} + k\dot{q} - \mathcal{H}(k, x)] dt \right\} DpDqDkDx, \]

here "k" and "x" denote "momenta" conjugate to the "coordinates" "q" and "p", respectively.

Despite the fact that the fiducial vector independent propagator has been constructed as a propagator appropriate to two (canonical) degrees of freedom, it is nonetheless true that its classical limit refers to a single (canonical) degree of freedom (cf. [65]).

1.1.1 Examples of the Fiducial Vector Independent Propagator

1.1.1.1 Vanishing Hamiltonian

We now look at two examples of the fiducial vector independent propagator. Our first example is that of the vanishing Hamiltonian which leads to

\[ K(p, q, t; p', q', t') = \mathcal{M} \int \exp \left[ i \int (q\dot{p} + k\dot{q} - x\dot{p}) dt \right] DpDqDkDx \]

\[ = \mathcal{N} \int \exp \left( i \int \dot{q} dt \right) \delta(\dot{q}) \delta(\dot{p}) DpDq \]

\[ = \delta(p - p') \delta(q - q'). \]

This is of course a trivial example; however, it shows that the fiducial vector independent propagator fulfills the correct initial condition as is expected from equation (1.6).

1.1.1.2 The Hamiltonian \( \mathcal{H} = (P^2 + \omega^2 Q^2)/2 \)

The second example we consider is that of the Hamiltonian \( \mathcal{H}(k, x) = (k^2 + \omega^2 x^2)/2 \). Here the fiducial vector independent propagator takes the following form:

\[ K(p, q, t; p', q', t') \]

\[ = \mathcal{M} \int \exp \left\{ i \int [q\dot{p} + k\dot{q} - x\dot{p} - (k^2 + \omega^2 x^2)/2] dt \right\} DpDqDkDx \]

\[ = \mathcal{N} \int \exp \left\{ i \int [q\dot{p} + \dot{q}^2 + \dot{p}^2/\omega^2] dt \right\} DpDq \]
\[
= \frac{\csc(\omega T/2)}{4\pi i} \times \exp \left( i \left\{ \frac{1}{2} (q + q')(p - p') + \frac{1}{4} \cot(\omega T/2) \left\{ \frac{1}{\omega} (p - p')^2 + \omega (q - q')^2 \right\} \right\} \right),
\]

where \( T = t - t' \). This is an unusual result for the propagator of the harmonic oscillator. This result has the appearance of a propagator for a two-dimensional free particle in a uniform magnetic field (cf. [37, p. 64]). However, when one brings up an element of any of the reproducing kernel Hilbert spaces \( L^2_k(\mathbb{R}^2) \) then this propagator acts like the conventional propagator for the harmonic oscillator. Moreover, in the appropriate limits one can recover the usual propagators in the Schrödinger representation (see [65]).

### 1.2 General Overview of the Thesis

This thesis is organized into six chapters and three appendices. The first chapter is this introduction and the last chapter is a conclusion. The results of our research are contained in chapters 3, 4, and 5. The three appendices have been added to make this thesis reasonably self-contained.

In chapter 2 we discuss the construction of path integrals on group and symmetric spaces. In section 2.1 we review the Feynman path integral on flat, group, and symmetric spaces. Section 2.2 is devoted to the study of group coherent states associated with a compact group and the construction of coherent state path integrals based on group coherent states associated with a compact group.

In chapter 3 we introduce the notations and basic definitions used throughout the thesis. The main result of this chapter is Theorem 3.2.1, in which we derive an operator version of the generalized Maurer-Cartan form.

Chapter 4 contains the construction of the representation independent propagator for a real, separable, locally compact, connected and simply connected Lie group with irreducible, square integrable representations. We refer hereafter to a real, separable,
locally compact, connected and simply connected Lie group\textsuperscript{1} with irreducible square integrable representations\textsuperscript{2} as a \textit{general Lie group}. For a given set of kinematical variables this propagator is a single generalized function independent of any particular choice of fiducial vector \textit{and} the irreducible representation of the general Lie group generated by these kinematical variables. In section 4.1 we define coherent states for a general Lie group and prove Lemma 4.1.4 and the Corollary 4.1.5 which we apply in the construction of the representation independent propagator and the construction of regularized lattice phase-space path integral representations of the representation independent propagator.

Prior to constructing the representation independent propagator for a general Lie group, we construct in section 4.2 the representation independent propagator for any real compact Lie group. It is shown in Theorem 4.2.2 that the representation independent propagator for any compact group correctly propagates the elements of any reproducing kernel Hilbert space associated with an arbitrary irreducible unitary representation of $G$. As an example the representation independent propagator for SU(2) is constructed.

In section 4.3 this construction is then suitably extended to a general Lie group and we show in Theorem 4.4.2 that the result obtained in Theorem 4.2.2 holds for a general Lie group. In Proposition 4.4.4 we establish that it is possible to construct regularized phase-space path integrals for a general Lie group. Even though generally the group space is a multidimensional \textit{curved} manifold, it is shown that the resulting phase-space path integral has the form of a lattice phase-space path integral on a multidimensional \textit{flat} manifold. Hence, we obtain a novel and very natural phase-space path integral quantization for systems whose kinematical variables are the generators of a general Lie group. To illustrate the general theory the representation independent propagator for the affine group is constructed.

\textsuperscript{1}See Appendix A.4 for a definition of these terms.

\textsuperscript{2}See Section 4.1 and Appendix A.5 for the definition of these terms.
In chapter 5 we discuss the classical limit of the representation independent propagator of a general Lie group and show that its classical limit refers indeed to the degrees of freedom associated with the general Lie group. Sections 5.1 and 5.2 contain a detailed discussion of the classical limit of the coherent state propagator for compact Lie groups and non-compact Lie groups.

In section 5.3 we prove that the equations of motion obtained from the action functional of the representation independent propagator for a general Lie group imply the equations of motion obtained from the most general action functional of the coherent state propagator for a general Lie group (cf. Proposition 5.3.1).
CHAPTER 2
A REVIEW OF SOME MEANS TO DEFINE THE FEYNMAN PATH INTEGRAL ON GROUP AND SYMMETRIC SPACES

This chapter is somewhat independent of the rest of this thesis and serves as an introduction to some of the ways of constructing path integrals on group and symmetric spaces. Our arguments will be largely heuristic, but we will confront the issue again with rigor in chapter 4. In section 2.1 the construction of the Feynman path integral on \( \mathbb{R}^d \), group, and symmetric spaces is discussed. Section 2.2 is devoted to a preliminary study of group coherent states, we take this subject up in more detail in chapter 4. The remaining part of section 2.2 is devoted to the construction of coherent state path integrals based on group coherent states.

2.1 The Feynman Path Integral on \( \mathbb{R}^d \), Group, and Symmetric Spaces

2.1.1 Introduction

The year 1925 can be seen as the beginning of modern quantum mechanics marked by the two almost simultaneously published papers of Heisenberg [52] and Schrödinger [91]. The former proposes the formalism of matrix mechanics, while the latter proposes the formalism of wave mechanics. Schrödinger [92] first showed that the two formulations are physically equivalent. Both of these approaches where combined heuristically by Dirac [24] into a more general formulation of quantum mechanics. The mathematically rigorous development of this general formulation of quantum mechanics was subsequently carried out by von Neumann [104].

This general formulation of quantum mechanics is based on an analogy with the Hamilton formalism of classical mechanics. It is well known that the Lagrangian formalism of classical mechanics has almost no place in this general formulation of
quantum mechanics, except in the suggestive derivation of Schrödinger's wave equation from the Hamilton-Jacobi equation by the substitution, $S = -i\hbar \ln(\psi)$, where $S$ denotes the Hamilton principal function.

The first hint of the possible importance of the Lagrangian in quantum mechanics was given by Dirac [23]; he remarked that the quantum transformation $\langle q_t | q_{t_0} \rangle$ corresponds to the classical quantity $\exp[(i/\hbar) \int_{t_0}^{t} L dt]$. It was this remark by Dirac that led Feynman in 1941, then a student at Princeton, to a new formulation of quantum mechanics (see the account in [47, 126–129]). This new approach did certainly not break any barriers that could not be overcome from the operator or Hamiltonian point of view. Nevertheless, one might have gained in two ways from Feynman’s work [35] and the ensuing work of other authors [21, 20, 22, 40, 58, 60, 61, 80, 97]. From a practical point of view, as pointed out by Feynman [35], this approach to quantum mechanics allows one to reduce a problem that involves the interaction of system A with system B, to a problem, let us say, involving system A alone. This is clearly useful if one wants to restrict oneself to questions concerning only one system. Another way one has benefitted from Feynman’s approach to quantum mechanics is in the conceptual understanding of quantum mechanics, specifically in the understanding of the connection of quantum mechanics and classical mechanics (cf. [22, 60, 61]).

There are several books and review articles on the subject of path integrals. The selection presented is not meant to be comprehensive but is rather reflective of the author’s taste. Feynman and Hibbs [37] give a heuristic introduction to the subject, whereas Schulman [95], gives a more rigorous introduction to the Feynman path integral on configuration space and considers a number of applications of the method in different fields of physics. For a good and thorough introduction to the subject of phase-space path integrals Klauder’s Bern Lecture Notes [58] and his Lectures [63] are an excellent choice. The review articles by Berry and Mount [9] and Marinov [75] also deserve to be mentioned. In addition, Kleinert’s [71] recently published book
contains many applications of the path integral method to problems in quantum mechanics, statistical, and polymer physics. Moreover, Inomata et al. [54] discuss various techniques of path integration not covered in the aforementioned monographs.

2.1.2 The Feynman Path Integral on $\mathbb{R}^d$

We will now describe a simple derivation of Feynman’s path integral on the basis of the canonical formalism of quantum mechanics which was first published by Tobocman [97]. The idea is to find an appropriate approximation for the time evolution operator, $U(t'' - t') = \exp[-(i/\hbar)(t'' - t')\mathcal{H}]$ (introduced in chapter 1), at small times and then to construct step by step the time evolution operator at finite times. We start from the identity

$$U(t'' - t') = [U((t'' - t')/(N + 1))]^{N+1},$$

which holds for any $N > 0$. Let us now consider the case of large $N$, then the step size $\epsilon = (t'' - t')/(N + 1)$ is small and we have the following approximate identity to first order in $\epsilon$

$$U(\epsilon) \approx 1 - \frac{i}{\hbar} \epsilon \mathcal{H}.$$  

To ensure that the quantized Hamilton operator $\mathcal{H}(P, Q)$ is unambiguous, i.e. in order to avoid operator ordering problems, we consider the following simple Hamiltonian

$$H_{ad}(P, Q) = \frac{1}{2} p^2 + V(q), \quad (2.1)$$

where $q = (q_1, \ldots, q_d)$ and $p = (p_1, \ldots, p_d)$. Furthermore, we use the mixed $(p, q)$ matrix element of the time evolution operator $U(t)$:

$$\langle q'' | U(t'' - t') | q' \rangle = \int \langle q'' | p' \rangle \langle p' | U(t'' - t') | q' \rangle dp',$$

to obtain a simple expression for the matrix elements of the Hamilton operator $\mathcal{H}$. Then for small $\epsilon$ we have

$$\langle p | U(\epsilon) | q \rangle \approx \langle p | (I - \frac{i}{\hbar} \epsilon \mathcal{H}) | q \rangle$$
\[
[1 - \frac{i}{\hbar} \epsilon \hat{H}(p, q)] \langle p | q \rangle 
\approx \exp \left[ -\frac{i}{\hbar} \epsilon \hat{H}(p, q) \right] \langle p | q \rangle, \tag{2.2}
\]
valid to first order in \( \epsilon \). Here, \( \hat{H}(p, q) \) is defined as

\[
\hat{H}(p, q) = \frac{\langle p | H(P, Q) | q \rangle}{\langle p | q \rangle}
\]

For the simple Hamilton operator \( H(P, Q) = (1/2)P^2 + V(Q) \) we are considering \( \hat{H}(p, q) \) coincides with the classical Hamiltonian \( H_{cl}(p, q) \). Note that for more complicated Hamilton operators this has no longer to be true (see below). Using (2.2) and the fact that \( \langle p | q \rangle = (2\pi)^{-d} \exp[-(i/\hbar)pq] \) we find that

\[
J(q'', t''; q', t') = \langle q'' | U(t'' - t') | q' \rangle 
\]

\[
= \lim_{N \to \infty} \langle q'' | [U(\epsilon)]^{N+1} | q' \rangle 
\]

\[
= \lim_{N \to \infty} \int \ldots \int \prod_{j=0}^{N} \langle q_{j+1} | U(\epsilon) | q_j \rangle \prod_{j=1}^{N} dq_j 
\]

\[
= \lim_{N \to \infty} \int \ldots \int \exp \left\{ \frac{i}{\hbar} \sum_{j=0}^{N} [p_{j+1/2} \cdot (q_{j+1} - q_j) - \epsilon H_{cl}(p_{j+1/2}, q_j)] \right\} 
\]

\[
\times \prod_{j=1}^{N} dq_j \prod_{j+1/2=0}^{N} \frac{dp_{j+1/2}}{(2\pi)^d}, \tag{2.3}
\]

where \( q_0 = q' \) and \( q_{N+1} = q'' \). It follows from (2.3) that the q-space propagator \( J(q'', t''; q', t') \) satisfies the following initial condition:

\[
\lim_{t'' \to t'} J(q'', t''; q', t') = \delta(q'' - q'),
\]
as it should by its very definition. Observe that in the phase-space path integral representation (2.3) there is always one more integral over the \( p \) than there is over the \( q \). The sum in the exponent has a natural interpretation: it is the finite sum approximation of the classical action functional along a path in phase space with fixed endpoints \( q'' \) and \( q' \). Now taking an improper limit by interchanging the limit
with respect to $N$ with the integrals we find the following formal standard phase-space path integral

$$J(q'', t''; q', t') = \int \exp \left[ \frac{i}{\hbar} I_{cl}(p(t), q(t)) \right] \mathcal{D}q \mathcal{D}p,$$

where,

$$I_{cl}(p(t); q(t)) = \int_{t'}^{t''} [p\dot{q} - H_{cl}(p, q)] dt.$$  \hfill (2.5)

This formal phase-space path integral for the q-space propagator $J(q'', t''; q', t')$ was first written by Feynman [36, Appendix B.], and then was subsequently rediscovered by other authors (see for instance Davies [20] and Garrod [40]). The integration ranges over all paths in 2d-dimensional phase-space which are pinned at $q = q'$ and $q = q''$, while the integration over the momenta is unrestricted.

The Lagrangian form of the path integral, as originally proposed by Feynman [35], can be obtained form (2.3) by integrating out the momenta. That this can be done follows from the fact that the momenta enter quadratically. Hence, if we carry out the $N + 1$ Fourier transformations in (2.3) which are of the form:

$$\int \exp \left\{ \frac{i}{\hbar} \left[ \frac{p_{j+1/2} \cdot (q_{j+1} - q_j)}{2} - \frac{\epsilon^2}{2} p_{j+1/2}^2 \right] \right\} \frac{dp_{j+1/2}}{(2\pi)^d} = \left( \frac{1}{2\pi i \hbar \epsilon} \right)^{d/2} \exp \left[ \frac{i(q_{j+1} - q_j)^2}{2\hbar \epsilon} \right]$$

then we find the following result:

$$J(q'', t''; q', t') = N \int \ldots \int \exp \left[ \frac{i}{\hbar} I_{cl}(q(t)) \right] \mathcal{D}q,$$

$$I_{cl}(q(t)) = \int_{t'}^{t''} \left[ \frac{1}{2} \dot{q}^2 - V(q) \right] dt,$$  \hfill (2.7)

This is the formal Feynman path integral over paths in configuration space pinned at $q'$ and $q''$. Before leaving this subsection we would like to make a number of remarks concerning the just presented derivation of the Feynman path integral.

(i) **Canonical transformations.** Since the measure in (2.3) is a product of Liouville measures $dp dq$ one may be tempted to assume that the phase-space integral is
invariant under general canonical transformations. However, this is not the case. As shown by Klauder [63, section II] the regularized lattice phase-space prescription (2.3) for the q-space propagator is only invariant, or better covariant, under the subset of point transformations among all canonical transformations.

(ii) **Operator ordering.** If the Hamiltonian is no longer of the simple form we have considered in (2.1) but has a more complicated \((p, q)\)-dependence, then one has to confront the issue of operator ordering in the Hamiltonian. For example, this is the case for a free particle moving on a Riemannian manifold, for which

\[ H_d(p, q) = \frac{1}{2} g^{ij}(q) p_i p_j. \]

The basic principles one uses for the resolution of the operator ordering problem are (a) the Hamilton operator has to be symmetric and, (b) if the classical system has a symmetry group, the corresponding quantum theory must have this symmetry. As Marinov remarks, “the first of these conditions is evident, while the second is more arbitrary and not always constructive” [75, p. 13]. In particular, condition (a) implies that we should associate with the classical Hamiltonian \( H_d(p, q) = F(q) p \) the following quantized Hamilton operator \( \mathcal{H}(P, Q) = (1/2)[PF(Q) + F(Q)P] \). Using the principles (a) and (b) in the resolution of the operator ordering problem might lead to additional correction terms proportional to \( \hbar^2 \) in the action functional (cf. [95]).

(iii) **Integral over configuration-space trajectories.** The Feynman path integral in (2.7) was obtained for the Hamiltonian (2.1). If the \( p \)-dependence of the Hamiltonian is no longer simply quadratic but more complicated, then the integral in (2.6) is no longer a simple Gaussian integral and does not result in the classical Lagrangian of the free particle. In this case the Feynman path integral (2.7) can not be used as a starting point for quantum theory.
2.1.3 The Feynman Path Integral on Group Spaces

The quantization of a free particle moving on a group manifold has been considered in a number of works [12, 13, 14, 28, 29, 48, 49, 56, 70, 75, 76, 77, 94]. Schulman [94] introduced, starting from the known semiclassical approximation, a propagator for a free particle moving on the group manifolds of $SO(3)$ and $SU(2)$. However, Schulman did not present a simple path integral solution for the problem, (cf. the remarks in Ref. 71, chapter 8). Dowker [28, 29] extended Schulman’s approach to simple Lie groups, considering explicitly the motion of a free particle on the group manifold of $SU(n)$. It is shown in Ref. 28 that the semiclassical approximation is only exact for the motion of a free particle on the group manifold of a semisimple Lie group and that it can in general not be expected that the semiclassical approximation is exact for all symmetric spaces, since it is not exact for the n-sphere,

$$S^n \cong SO(n + 1)/SO(n), \quad n > 3.$$  

The question as to what is the largest class of spaces for which the semiclassical approximation is exact seems still to be an open one. The beauty of the above result, as Dowker points out, is that in the cases for which the semiclassical approximation is exact, the propagator is obtained by summing only over classical paths. A Feynman path integral treatment of the motion of a free particle on compact simple Lie groups and spheres of arbitrary dimension has been proposed by Marinov and Terentyev [76, 77].

Before we consider their proposal we briefly outline the construction of path integrals on Riemannian manifolds. DeWitt [22] observed that for a free physical system moving on a d-dimensional unbounded Riemannian manifold with constant scalar curvature $R$ and metric tensor $[g_{ij}(q)]$ the propagator for infinitesimal time is given by the semiclassical approximation:

$$J^{cl}(q'', t''; q', t') = A(q'', t''; q', t') \exp \left[ \frac{i}{\hbar} I_{cl}(q'', t''; q', t') \right], \quad (2.8)$$
where

\[ A(q'', t''; q', t') = (2\pi\hbar)^{-d/2}[g^{-1/2}(q'')D(q'', t''; q', t')g^{-1/2}(q')]^{1/2}, \]

and

\[ D(q'', t''; q', t') = \det \left[ -\frac{\partial^2 I_{\text{cl}}}{\partial q^j \partial q^n} \right], \]

is van Vleck's determinant. Here \( g(q) \equiv |\det[g_{ij}(q)]| \) and \( I_{\text{cl}} = \frac{1}{2} \int_{t'}^{t''} g_{ij} \dot{q}^i \dot{q}^j dt \) is the classical action functional. As observed by Marinov [75], this simple form of the semiclassical approximation is only valid for unbounded Riemannian manifolds, since the proof of the theorem that two points \( q'' \) and \( q' \) at fixed small \( t'' - t' \) may be connected by only one classical path (cf. [108, pp. 58-64]) uses the unboundedness of the manifold in an essential way. On the other hand, as is pointed out in Refs. 29 and 75, if one is dealing with bounded Riemannian manifolds there might exist a number of classical paths connecting two points on the manifold, each of these paths then enters into (2.8) possibly with a phase; see also in this respect the review article by Berry and Mount [9]. As an example we mention the case when the bounded manifold is multiply connected, the classical paths connecting two points on the manifold then divide into distinct homotopy classes; see the example below of a free particle moving on a circle and Schulman [95, 197-205] for a discussion of this point. For this case the semiclassical approximation takes the following form:

\[ J^{\text{cl}}(q'', t''; q', t') = \sum_m A_m(q'', t''; q', t') \exp \left[ \frac{i}{\hbar} I_{\text{cl}}^m(q'', t''; q', t') - \frac{i}{2} \pi \gamma_m \right], \]  \( (2.9) \)

where the sum is over all classical paths connecting \( q'' \) and \( q' \), \( I_{\text{cl}}^m \) is the classical action functional along the \( m \)th path, and \( \gamma_m \) is an integer that depends in general on all the arguments. Also note, as is remarked in Ref. 75, that the semiclassical approximation is only applicable if the action functional \( I_{\text{cl}} \) is generically dominated by a term proportional to \( t^{-1} \) as \( t \) goes to zero, so that \( I_{\text{cl}} \) becomes large in this limit and puts one into the semiclassical regime. This is of course the case for the free
For the case of an unbounded Riemannian manifold the propagator at finite times is constructed by folding \( N + 1 \) propagators of the form (2.8)

\[
J(q'', t''; q', t') = \int \prod_{k=0}^{N} J^c(q_{k+1}, t_{k+1}; q_k, t_k) \prod_{k=1}^{N} dq_k,
\]

(2.10)

where \( dq_k = \sqrt{g} \prod_{j=1}^{d} dq_j \). Taking the limit \( N \to \infty \) one obtains a functional integral over all intermediate coordinates that can be interpreted as the path integral. The final result is a path integral of the form (2.7); however, the Lagrangian needs to be modified by a term proportional to \( \hbar^2 R \). DeWitt [22] found this term to be \( \hbar^2 (R/12) \); this modification of the Lagrangian was also discussed by McLaughlin and Schulmann [73]. In the context of curvilinear coordinates the reason for modifying the path integral has been discussed by Arthurs [2, 3], Edwards and Gulyaev [32], and in the context of quantization of non-linear field theories by Gervais and Jevicki [42] and Salomonson [90].

If on the other hand, one applies this approach to a bounded Riemannian manifold, as is the case for compact Lie groups, then the resulting path integral, as pointed out in Ref. 75, is far from simple. Since one then has to use the semiclassical approximation presented in (2.9) and in addition to integrating over all intermediate coordinates, one also has to sum over all the different classical paths connecting \( q'' \) and \( q' \).

Nevertheless, if the bounded manifold \( \mathcal{M} \) in question is isomorphic to a quotient space \( \mathcal{N}/\Gamma \), i.e. \( \mathcal{M} \cong \mathcal{N}/\Gamma \), where \( \Gamma \) is a transformation group acting on \( \mathcal{N} \) and \( \mathcal{N} \) is an unbounded Riemannian manifold, then one can, as proposed by Marinov and Terentyev [77], construct a propagator on \( \mathcal{M} \) by summing over the group \( \Gamma \):

\[
J_{\mathcal{M}}(q'', t''; q', t') = \sum_{\gamma \in \Gamma} J_{\mathcal{N}}(q'', t''; \gamma \cdot q', t') = \sum_{\gamma \in \Gamma} J_{\mathcal{N}}(\gamma \cdot q'', t''; q', t'),
\]

(2.11)

provided the propagator on \( \mathcal{N} \) is known. As pointed out above, since \( \mathcal{N} \) is an unbounded Riemannian manifold, by assumption a path integral representation for \( J_{\mathcal{N}} \) can be constructed and one sums over \( \Gamma \) in the last step to obtain \( J_{\mathcal{M}} \). It has been
shown by Marinov and Terentyev [77] that this approach is valid for any compact Lie group. In their work, see Ref. 77, Marinov and Terentyev take for \( \mathcal{N} \) the Lie algebra that is associated with the Lie group they wish to consider and for \( \Gamma \) the characteristic lattice of the group.

As an application of this general formalism of Marinov and Terentyev we now consider the free motion of a particle on a circle. We will revisit this problem in chapter 4 where we present an exact path integral treatment of this problem without reliance on the semiclassical approximation.

Let us consider a particle of mass \( m \) constrained to move on a circle of radius \( r \). If we choose the arclength the particle has traveled as our generalized coordinate, the Lagrangian is given by

\[
L(\phi, \dot{\phi}) = \frac{1}{2} \dot{\phi}^2,
\]

Here \( I = m r^2 \) denotes the moment of inertia of the particle and the angular variable \( \phi \) ranges from \( 0 \leq \phi \leq 2\pi \), where we identify the points \( \phi = 0 \) and \( \phi = 2\pi \). The solution of the equations of motion is found to be

\[
\phi = \phi_0 + \omega t,
\]

where \( \phi_0 \) and \( \omega \) are arbitrary integration constants. Considering a motion starting at \( \phi' \) and ending at \( \phi'' \), we find for the classical action functional:

\[
S_{\text{cl}}(\phi'', t''; \phi', t') = \frac{I}{2(t'' - t')}(\phi'' - \phi' - 2\pi n)^2,
\]  

(2.12)

where \( n = 0, \pm 1, \pm 2, \ldots \). So we find that the classical action functional does not only depend on the initial and final position but also on the so-called winding number \( n \), the number of times the particle moves counterclockwise minus the number of times it moves clockwise past the point \( \phi' \). Hence, the paths break up into distinct classes labeled by \( n \); these classes are the so called homotopy classes. Any two paths in the same class with the same beginning and end point can be continuously deformed into each other (see [95, 197–205]).
The canonical quantization for this example is straightforward, since \( p_\phi = \partial_\phi L = I\dot{\phi} \), we find for the Hamilton operator
\[
H = -\frac{\hbar^2}{2I} \partial_\phi^2,
\]
which has the following normalized eigenfunctions and eigenvalues:
\[
\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(i m \phi), \quad E_m = \frac{1}{2I} (\hbar m)^2,
\]
where \( m = 0, \pm 1, \pm 2, \ldots \) The propagator is given in terms of the eigenfunctions \( \psi_m(\phi) \) by the following spectral expansion:
\[
J(\phi'', t''; \phi', t') = \sum_{n=-\infty}^{+\infty} \psi_n(\phi'') \overline{\psi_n(\phi')} \exp \left[ -\frac{i}{\hbar} E_n(t'' - t') \right]
\]
\[
= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \exp \left[ i(\phi'' - \phi') n - i \frac{1}{2I} \hbar (t'' - t') n^2 \right].
\]
The sum over \( n \) is related to the Jacobi theta function,
\[
\theta_3(z, t) = \sum_{n=-\infty}^{+\infty} \exp(i \pi t n^2 + 2inz).
\]
Therefore, with the following identifications we can write the propagator in closed form. Let
\[
t \equiv -\frac{\hbar(t'' - t')}{2\pi I} ; \quad z \equiv \frac{\phi'' - \phi'}{2},
\]
then we find
\[
J(\phi'', t''; \phi', t') = \frac{1}{2\pi} \theta_3 \left( \frac{\Delta \phi}{2}, -\frac{\hbar T}{2\pi I} \right),
\]
where \( \Delta \phi \equiv \phi'' - \phi' \) and \( T \equiv t'' - t' \). Using the following property of \( \theta_3 \), which follows from the Poisson summation formula (see [19, pp. 63-65]),
\[
\theta_3(z, t) = (it)^{-1/2} \exp \left( -i \frac{z^2}{\pi t} \right) \theta_3 \left( \frac{z}{t}, -\frac{1}{t} \right),
\]
the propagator can also be written as a sum over classical paths, i.e. as a semiclassical series,
\[
J(\phi'', t''; \phi', t') = \sum_{n=-\infty}^{+\infty} \tilde{J}(\phi'', t''; \phi', t')
\]
\[
\tilde{J}(\phi'', t''; \phi', t') = \left( \frac{I}{2\pi i \hbar (t'' - t')} \right)^{1/2} \exp \left[ \frac{i I (\phi'' - \phi' - 2\pi n)^2}{\hbar} \right]. \quad (2.13)
\]
Observe that each of the propagators \( \bar{J} \) in the series (2.13) is of the same form as the
propagator of a free particle moving on the real line \( \mathbb{R} \) and that the series as a whole
is a function of period \( 2\pi \). The series (2.13) is a particular example of the general
principle (2.11).

Folding the propagator in (2.13) \( N + 1 \)-times leads to the following path integral
representation

\[
J(\phi'', t''; \phi', t') = \lim_{N \to \infty} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j=0}^{N} \sum_{n_j=-\infty}^{+\infty} \left( \frac{I}{2\pi i \hbar \epsilon} \right)^{1/2} \exp \left[ \frac{i}{\hbar} \frac{I(\phi_{j+1} - \phi_j - 2\pi n_j)^2}{2\epsilon} \right] \prod_{j=1}^{N} d\phi_j,
\]

where \( \phi_{N+1} = \phi'', \phi_0 = \phi', \) and \( \epsilon = T/(N+1) \). If we now shift the integration variable
at each step, we can extend the \( N \) intermediate integrals to the whole real line, i.e.

\[
\sum_{n_j=-\infty}^{+\infty} \int_0^{2(n_j+1)\pi} d\phi_j \to \int_{-\infty}^{+\infty} d\Phi_j.
\]

As a final result, we find

\[
J(\phi'', t''; \phi', t') = \lim_{N \to \infty} \left[ \frac{I}{2\pi i \hbar \epsilon} \right]^{(N+1)/2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp \left[ \frac{i}{\hbar} \sum_{j=0}^{N} \frac{I(\Phi_{j+1} - \Phi_j)^2}{2\epsilon} \right] \prod_{j=1}^{N} d\Phi_j,
\]

where \( \Phi_{N+1} = \phi'' \) and \( \Phi_0 = \phi' + 2\pi n \). Note that the circle is the group manifold of
the simplest compact Lie group \( U(1) \) whose faithful irreducible representations are
given by \( D^1(\phi) = \exp(i\phi) \) and \( D^{-1}(\phi) = \exp(-i\phi) \). It is well known that the one-
dimensional abelian translation group \( T_1 \) of the real line \( \mathbb{R} \) is the universal covering
group of \( U(1) \). Furthermore, the translations by \( 2\pi n, n = 0, \pm 1, \pm 2, \ldots \) form the
 cyclic subgroup \( \langle 2\pi \rangle \) of \( T_1 \), which is the kernel of the homomorphism \( T_1 \ni x \to f(x) = \exp(ix) \in U(1) \). Therefore, by the Fundamental Homomorphism Theorem we have
that \( U(1) \cong T_1/(2\pi) \). Hence, as stated in Ref. 77, the path integral representation
(2.14) is a particular example of the general statement (2.11). For more complicated
cases one might expect not to be able to obtain such a simple representation as (2.14)
for the path integral, but one which involves the summation over the lattice group at each infinitesimal step. Nevertheless, Marinov and Terentyev have shown that in the case of the motion of a free particle on the group manifold of a compact simple Lie group the resulting path integral representation is of the from (2.14), with the only difference that the Lagrangian has to be modified to include a 'quantum' potential proportional to \( \hbar^2 \). One might ask if the approach of Marinov and Terentyev could be extended to more general systems than the free particle? The answer is no, since, as we have mentioned above, the semiclassical approximation is only exact for the case of the free particle moving on the group manifold of a semisimple Lie group.

2.1.4 The Feynman Path Integral on Symmetric Spaces

More recently Böhm and Junker have used zonal spherical functions to construct path integral representations for a free particle moving on the group manifolds of compact and non-compact rotation groups (Böhm and Junker [12, 14]), the Euclidian group (Böhm and Junker [13]), and on symmetric spaces\(^1\) (Junker [56]). However, a careful analysis of the construction presented in [56] reveals that it applies only to the case of a compact transformation group \( G \) acting on a compact symmetric space of the form \( G/H \), where \( H \) is a massive\(^2\) subgroup of \( G \). We will extend this construction below to a general unimodular transformation group \( G \) acting on a symmetric space \( M = G/H \), where \( H \) is a massive compact subgroup of \( G \). This will complete the argument of Junker [56] and achieve his proposed unification of the work.

---

\(^1\)Let \( (S,T) \) and \( (T,U) \) be two topological spaces. A continuous one-to-one map \( f \) of \( S \) onto \( T \) is called a \textit{homeomorphism} if \( f^{-1} \) is continuous. A topological space \( (S,T) \) is called \textit{homogeneous} if for any pair \( u,v \in S \) there exists a homeomorphism \( f \) of \( (S,T) \) onto itself such that \( f(u) = v \). Let \( G \) be a connected Lie group and let \( \sigma \) be an involutive automorphism of \( G \), i.e. \( \sigma^2 = 1 \) and \( \sigma \neq 1 \). Denote by \( G_{\sigma} \) the closed subgroup of \( G \) consisting of all elements \( g \) that are fixed points of \( \sigma \), i.e. \( \sigma(g) = g \), and by \( G_{\sigma}^I \) the identity component of \( G_{\sigma} \). Let \( H \) be a closed subgroup of \( G \) such that \( G_{\sigma}^I \subset H \subset G_{\sigma} \), then one calls the quotient space \( G/H \) a \textit{symmetric (homogeneous) space} (defined by \( \sigma \)). The n-sphere \( S^n \) is an example of a symmetric space.

\(^2\)Let \( T_\alpha \) be an irreducible representation of the group \( G \) on the space \( \mathcal{R} \). An element \( \alpha \in \mathcal{R} \) is called \textit{invariant} relative to the closed subgroup \( H \) if for all \( h \in H \) one has \( T_h \alpha = \alpha \). A representation is called a \textit{representation of class 1} relative to \( H \) if its representation space contains elements that are invariant relative to \( H \) and if the restriction of \( T_\alpha \) to \( H \) is unitary, i.e. \( (T_h \phi, T_h \psi) = (\phi, \psi) \) \( \forall \ h \in H \). One calls \( H \) \textit{massive}, if there is only one normalized invariant element \( \alpha \in \mathcal{R} \) in the representation space \( \mathcal{R} \) of any representation of class 1 relative to \( H \).
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valid to first order in \( \epsilon \), and \( dx_j \) denotes the invariant measure on \( \mathcal{M} \). Note that for the cases we are considering, where \( G \) is a unimodular Lie group and \( H \) is a massive compact subgroup, an invariant measure always exists (cf. [7, Corollary 4.3.1]). In what follows we ask that the short time propagator (2.16) be invariant under the transformation group \( G \), i.e.

\[
J(gx_{j+1}, gx_j; \epsilon) = J(x_{j+1}, x_j; \epsilon) \quad \forall \ g \in G, \tag{2.17}
\]

for \( j = 0, 1, \ldots, N \). As we will see below, this is a crucial assumption since it implies that \( \Delta \) is an invariant elliptic\(^4\) operator in the enveloping algebra\(^5\) on \( G \).\(^6\) This can be seen by using the form (2.16) of the short time propagator valid to first order in \( \epsilon \) in (2.17). From which it follows that the Hamilton operator \( \mathcal{H} \) has to be an invariant operator for \( G \) if (2.17) is to hold, this in turn implies the above statement.

\(^4\)Denote by \( \alpha = (\alpha_1, \ldots, \alpha_m) \) a multiindex consisting of \( m \) non-negative integers. Define the length of \( \alpha \) by

\[
|\alpha| = \sum_{j=1}^{m} \alpha_j.
\]

For every \( x \in \mathbb{R}^m \) let

\[
x^\alpha = \prod_{j=1}^{m} x_j^{\alpha_j}.
\]

Let \( P \) be a polynomial of \( m \) variables of degree \( r \), which has the form

\[
P(x) = \sum_{|\alpha| \leq r} c_\alpha x^\alpha,
\]

where \( c_\alpha \) are arbitrary complex numbers and \( c_\alpha \neq 0 \) for at least one \( \alpha \) with \( |\alpha| = r \). Then we denote the formal differential operator generated by \( P \) by:

\[
P(-i\nabla) = \sum_{|\alpha| \leq r} c_\alpha (-i\nabla)^\alpha = \sum_{|\alpha| \leq r} c_\alpha (-i)^{|\alpha|} \prod_{j=1}^{m} \partial_{x_j}^{\alpha_j},
\]

where \( \nabla = (\partial_{x_1}, \ldots, \partial_{x_m}) \). The formal differential operator \( P(-i\nabla) \) is called elliptic, if there exists a \( C \geq 0 \) such that

\[
1 + |P(x)| \geq C(1 + |x|^2)^{r/2} \quad \forall \ x \in \mathbb{R}^m.
\]

\(^5\)For the definition of the enveloping algebra see Appendix A.4.2

\(^6\)If \( \mathcal{M} \) is a rank one symmetric space, then every invariant differential operator \( C \) is a polynomial in the second order Casimir operator of \( G \) which in a proper coordinate system on \( \mathcal{M} \) is proportional to the Laplace-Beltrami operator (cf. [7, Theorem 15.1.1]).
Let \( a \) be a fixed point of \( \mathcal{M} \) whose stability group is \( H \), i.e. one has \( ha = a \) for all \( h \in H \). Since \( G \) acts transitively on \( \mathcal{M} \) we can write each \( x_j \in \mathcal{M} \) as

\[
x_j = g_j a \quad \text{for some } g_j \in G.
\] (2.18)

Hence, using this construction one can view the short time propagator as a function on the group \( G \):

\[
J(x_{j+1}, x_j; \epsilon) = J(g_{j+1}, g_j; \epsilon).
\]

Using the translation invariance of the short time propagator it follows that the short time propagator can only be a function of \( g_j^{-1}g_{j+1} \), hence,

\[
J(g_{j+1}, g_j; \epsilon) = J(gg_{j+1}, gg_j; \epsilon) = J(g_j^{-1}g_{j+1}; \epsilon).
\] (2.19)

Since \( ha = a \) for any \( h \in H \), we see that (2.18) is invariant with respect to right multiplication with elements of the stability group \( H \). This implies that the short time propagator is invariant with respect to right multiplication with elements of \( H \). From (2.19) we see that the short time propagator is also invariant with respect to left multiplication with elements of \( H \). Hence, we conclude that the short time propagator \( J(g; \epsilon) \) is a constant function on the two sided cosets \( HgH \) with respect to the subgroup \( H \), i.e.

\[
J(h_1 g h_2; \epsilon) = J(g; \epsilon), \quad \text{for any } h_1, h_2 \in H.
\]

Let \( U^\xi \) be an unitary irreducible representation of class 1 on the Hilbert space \( \mathcal{R}^\xi \). Let us choose any complete orthonormal system \( \{ \phi_j \}_{j=0}^{\dim \mathcal{R}^\xi - 1} \) in \( \mathcal{R}^\xi \), then we can associate with \( U^\xi \) the following matrix elements

\[
D^\xi_{ij}(g) = \langle \phi_i, U^\xi_g \phi_j \rangle.
\] (2.20)

These matrix elements play a special role in the theory of representations of groups. Unfortunately they are explicitly known for only a handful of cases (see [7, chapter 7]). Nevertheless, it can be shown that for simple Lie groups the matrix elements
$D_{ij}(g)$ are the regular eigenfunctions of a maximal set of commuting operators in the enveloping algebra, if this maximal set of commuting operators contains an elliptic operator (cf. [7, Proposition 14.2.2]). This property is often the starting point for an explicit calculation of the $D_{ij}(g)$, in section 4.3 we consider such a calculation in some detail for the case of $SU(2)$. If the maximal set of commuting operators does not contain an elliptic operator then the matrix elements $D_{ij}^\zeta(g)$ are generalized functions, i.e. distributions (cf. [7, Theorem 14.2.1]). In chapter 4 we will consider the construction of path integrals for the cases in which the matrix elements $D_{ij}^\zeta$ are either not explicitly known, or are generalized functions. This construction will make no explicit use of the functions $D_{ij}^\zeta(g)$ but will only use the facts that they exist and form a complete orthonormal set. Note that for the cases considered in this chapter the set of maximal commuting operators always contains the Laplace-Beltrami operator $\Delta$, which, as we have remarked above, is an elliptic operator in the center of the enveloping algebra of $G$. Hence, the matrix elements $D_{ij}^\zeta(g)$ are regular functions on $G$. This shows that the assumption that the short time propagator should be invariant under the transformation group $G$ is crucial and can not be relaxed.

Since $H$ is a massive subgroup of $G$, there exists a unique normalized vector $\alpha \in \mathcal{R}^\zeta$ that is invariant relative to $H$. Using the Gram-Schmidt orthogonalization procedure we can choose our complete orthonormal basis in such a way that $\phi_0 = \alpha$. Our interest now focuses on the $(00)$-matrix elements

$$D_{00}^\zeta(g) = \langle \phi_0, U_g^\zeta \phi_0 \rangle. \quad (2.21)$$

One can easily convince oneself that this function is constant on the two-sided cosets $HgH$ with respect to $H$. The function defined in (2.21) is called the zonal spherical function of the irreducible representation $U^\zeta$ relative to $H$. If we take $G = SO(3)$, the group of rotations of $\mathbb{R}^3$, and $H = SO(2)$, the group of rotations of the plane, then $M = S^2$, the two sphere and the zonal spherical functions are given by the Legendre polynomials $P_\zeta(\cos \theta)$. 
Let us denote by $\hat{G}$ the set of all inequivalent irreducible unitary class 1 representations of $G$ relative to $H$. Then it is known, since $H$ is a massive subgroup of the unimodular group $G$ that any function $f(g)$ that is constant on the two-sided cosets $HgH$ with respect to $H$, can be decomposed in zonal spherical functions $D^{\zeta}_{00}(g)$, $\zeta \in \hat{G}$, of unitary irreducible representations of class 1 (see [103, pp.50-55]):

$$f(g) = \sum_{\zeta \in \hat{G}} d_{\zeta} c_{\zeta} D^{\zeta}_{00}(g), \quad (2.22)$$

$$c_{\zeta} = \int_{G} \overline{D^{\zeta}_{00}(g)} f(g) dg. \quad (2.23)$$

Here $\sum_{\zeta \in \hat{G}}$ stands for the discrete or continuous orthogonal sum of all inequivalent irreducible unitary representations of class 1 of $G$ with respect to $H$. The constant $d_{\zeta}$ appearing in (2.22) is given by

$$\int_{G} \overline{D^{\zeta'}_{00}(g)} D^{\zeta}_{00}(g) dg = d^{-1}_{\zeta} \delta(\zeta, \zeta'), \quad (2.24)$$

where in suitable coordinates

$$\delta(\zeta, \zeta') = \begin{cases} 
\delta_{\zeta'} & \text{if } \hat{G} \text{ is discrete,} \\
\delta(\zeta - \zeta') & \text{if } \hat{G} \text{ is continuous.}
\end{cases}$$

For the case of compact groups the constant $d_{\zeta}$ is the dimension of the representation space $\mathcal{R}^{\zeta}$ of the unitary irreducible representation $U^{\zeta}$, see also in this respect remark 4.1.1.

We have now collected all the tools we need to construct the path integral representation for a free particle moving on $\mathcal{M}$. We have seen above that the short time propagator is a constant function on the two-sided cosets $HgH$ with respect to $H$, hence using (2.22) we can decompose it in zonal spherical functions:

$$J(g_{j+1}^{-1}g_{j}; \epsilon) = \sum_{\zeta \in \hat{G}} d_{\zeta} c_{\zeta}(\epsilon) D^{\zeta}_{00}(g_{j}^{-1}g_{j+1}), \quad (2.25)$$

$$c_{\zeta}(\epsilon) = \int_{G} \overline{D^{\zeta}_{00}(g_{j}^{-1}g_{j+1})} J(g_{j}^{-1}g_{j+1}; \epsilon) dg_{j}. \quad (2.26)$$
Moreover, let $f \in L^1(G)$, where $L^1(G)$ is the space of all integrable functions on $G$, then one has (cf. [7, Corrolary 4.3.1])

$$\int_G f(g) dg = \int_M \int_H f(gh) dh dx,$$

which reduces for $f \in L^1(H \setminus G/H)$ to

$$\int_G f(g) dg = \int_M \int_H f(g) dx \int_H dh = \int_M f(g) dx,$$

(2.27)

since $f(h_1gh_2) = f(g)$ $\forall h_1, h_2 \in H$ and where we have chosen $\int_H dh = 1$, because $H$ is compact. Using (2.25) and (2.27) in (2.15) one finds

$$J(x'', t''; x', t') = \lim_{N \to \infty} \int_G \cdots \int_G \prod_{j=0}^{N} \left[ \sum_{\zeta_j+1 \in \hat{G}} d_{\zeta_j+1} c_{\zeta_j+1}(\epsilon) D_{00}^{\zeta_j+1}(g^{-1}g_{j+1}) \right] \prod_{j=1}^{N} dg_j. \quad (2.28)$$

Using the orthogonality relations for the functions $\sqrt{d_{\zeta_j}} D_{ij}^{\zeta}$ and the left invariance of $dg$ one can easily show that the following relation holds

$$\sqrt{d_{\zeta_j+1}} d_{\zeta_j} \int_G D_{00}^{\zeta_j+1}(g^{-1}g_{j+1})D_{00}^{\zeta_j}(g^{-1}g_j) dg_j = \delta(\zeta_j, \zeta_{j+1})D_{00}^{\zeta_j}(g^{-1}_j g_{j+1}). \quad (2.29)$$

Using (2.29) the $N$ intermediate integrations in (2.28) can easily be performed and one finds as a final result that

$$J(x'', t''; x', t') = \sum_{\zeta \in \hat{G}} \left\{ \lim_{N \to \infty} \left[ c_{\zeta}(\epsilon) \right]^{N+1} \right\} d_{\zeta} D_{00}^{\zeta}(g^{-1}g''). \quad (2.30)$$

Let us now evaluate the limit $N \to \infty$ in the above expression, for large $N$ one can write $c_{\zeta}(\epsilon)$ as

$$c_{\zeta}(\epsilon) = c_{\zeta}(0) + \hat{c}_{\zeta}(0) \epsilon$$

valid to first order in $\epsilon$. The value of $c_{\zeta}(0)$ can be found from (2.26), using the fact that the short time propagator satisfies the following initial condition

$$\lim_{\epsilon \to 0} J(g^{-1}_j g_{j+1}; \epsilon) = \delta_{\epsilon}(g^{-1}_j g_{j+1})$$

Hence, one finds $c_{\zeta}(0) = D^{\zeta}_{00}(\epsilon) = 1$. Therefore, one can write the limit in (2.30) as

$$\lim_{N \to \infty} \left[ 1 + \frac{t'' - t'}{N + 1} \hat{c}_{\zeta}(0) \right]^{N+1} = \exp[(t'' - t') \hat{c}_{\zeta}(0)].$$
Or if we set $E_\zeta = i\hbar \dot{\zeta}(0)$, we find

$$\lim_{N \to \infty} [c_\zeta(\epsilon)]^{N+1} = \exp \left( -\frac{i}{\hbar} (t'' - t') E_\zeta \right).$$

It is shown in Ref. 56 that the $\dot{\zeta}(0)$ are the eigenvalues of the Laplace-Beltrami operator $\Delta$ on $\mathcal{M}$. Since in the proof of this statement no use is made of the compactness of $\mathcal{M}$ it applies to the present situation as well. Finally using the group property $D^\zeta(g^{-1}g'') = \sum_k D^\zeta_{k0}(g') D^\zeta_{k0}(g'')$ we can write (2.30) in the more familiar form

$$J(x'', t''; x', t') = \sum_{\zeta \in \mathcal{G}} \sum_k \exp \left[ -\frac{i}{\hbar} (t'' - t') E_\zeta \right] Y^\zeta_k(x') Y^\zeta_k(x''),$$

(2.31)

where

$$Y^\zeta_k(g) = \sqrt{d_\zeta} D^\zeta_{k0}(g)$$

One calls the matrix elements $D^\zeta_{m0}(g)$ the associated spherical functions and they are the eigenfunctions of the Laplace-Beltrami operator on $\mathcal{M}$. For the case that $\mathcal{M}$ is the two sphere $S^2$ the functions $\sqrt{d_\zeta} D^\zeta_{m0}(\theta, \phi)$ are the classical spherical harmonics $Y_{lm}(\theta, \phi)$, which, as is well known, are the eigenfunctions of the Laplace-Beltrami operator on the two sphere $S^2$. Hence, (2.31) is the well known spectral expansion of the propagator in terms of normalized eigenfunctions of the Hamilton operator. For specific examples we refer the interested reader to Ref. 56.

Let us close this section with two remarks, the first is that this approach can also be applied to Lie groups if we choose $H$ as the closed subgroup consisting of the identity element, $e$, i.e. $H = \{e\}$. Then instead of using the zonal spherical functions one has to use the matrix elements $D^\zeta_{ij}(g)$. However, it should be clear from the remarks after (2.20) that one can construct path integrals this way only for a handful of groups. In chapter 4 we will overcome the reliance on the matrix elements $D^\zeta_{ij}(g)$, this will allow us to consider more general Hamilton operators than just the Laplace-Beltrami operator. The second remark is of a more general nature and concerns the question: Why should one study quantum dynamics on group manifolds? Clearly
as we have seen in this chapter and will see in chapter 4 the study of quantum dynamics on group manifolds uses interesting and deep mathematics. It is therefore, of considerable mathematical interest. Nevertheless, there are also physical reasons why the study of quantum dynamics on group manifolds is of interest, for instance the dynamics on a group manifold is of interest in some modern quantum field theories such as $\sigma$-models and non-abelian lattice gauge field theories.

2.2 Coherent States and Coherent State Path Integrals

2.2.1 Introduction

The origin of coherent states can be traced back to the beginning of modern quantum mechanics. Schrödinger [93] introduced a set of non-orthogonal wave functions to describe non-spreading wave packets for quantum oscillators. In 1932 von Neumann [104] used a subset of these wave functions to study the position and momentum measurement process in quantum theory. It was not until thirty four years later that the detailed study of coherent states began ([6, 8, 58]). Klauder [58] introduced boson and fermion coherent states and used them both in the construction of path integrals for boson and spinor fields, respectively, whose action functional in each case is given by the familiar classical c-number expression. In 1963 Glauber [44, 45, 46] named the set of wave functions introduced by Schrödinger "coherent states" and used them in the field of quantum optics [67, 83] for the quantum theoretical description of a coherent laser beam. At about the same time Klauder published two papers [59, 60] dealing with the formulation of continuous representation theory, that contain the seminal ideas for the construction of coherent states on general Lie groups. Coherent states for the non-compact affine group or $ax+b$ group and the continuous representation theory using the affine group where introduced by Aslaksen and Klauder [4, 5] in 1968. Radcliff [86] constructed coherent states for the compact group $SU(2)$ in 1970. In 1971 Perelomov [84] gave a general construction for coherent states for both compact and suitable non-compact Lie groups.
Several books and review articles consider the definition and properties of coherent states ([17, 34, 68, 85, 109]). Klauder and Skagerstam [68] provide an introduction to the subject of coherent states in the form of a primer and offer a comprehensive overview of the literature until 1985 in the form of reprinted relevant articles dealing with the subject of coherent states. Perelomov [85] considers the usefulness of coherent states in the study of unitary representations of Lie groups and considers a number of applications. The review article by Zhang et al. [109] and the recently published proceedings of the International Symposium on Coherent States [34] also deserve to be mentioned.

2.2.2 Coherent States: Minimum Requirements

Let us denote by \( H \) a complex separable\(^7\) Hilbert space, and by \( \mathcal{L} \) a topological space, whose finite dimensional subspaces are locally euclidian. For a family of vectors \( \{|l\rangle\}_{l \in \mathcal{L}} \) on \( H \) to be a set of coherent states it must fulfill the following two conditions. The first condition is:

**Continuity:** The vector \( |l\rangle \) is a strongly continuous function of the label \( l \).

That is for all \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|||l'\rangle - |l\rangle|| < \epsilon \quad \text{for all } l' \in \mathcal{L} \text{ with } |l' - l| < \delta.
\]

Here, \( || \cdot || \) denotes the norm on \( H \) induced by the inner product on \( H \), i.e. \( || \cdot || = \langle \cdot, \cdot \rangle^{1/2} \). Or stated differently, the family of vectors \( \{|l\rangle\}_{l \in \mathcal{L}} \) on \( H \) form a continuous (usually connected) submanifold of \( H \). We assume that \( \langle l | l \rangle > 0 \) for all \( l \in \mathcal{L} \). In the applications we are considering the continuity property is always fulfilled.

The second condition a set of coherent states has to fulfill is:

**Completeness (Resolution of the Identity):** There exists a sigma-finite positive measure \( d\mu(l) \) on \( \mathcal{L} \) such that the identity operator \( I_H \) admits the following

\(^7\)A Topological space is called *separable* if it contains a countable dense subset.
In general, as pointed out in Ref. 68, p. 5, "one has to interpret this formal resolution of identity in the sense of weak convergence, namely, that arbitrary matrix elements of the indicated expression converge as desired."

2.2.3 Group Coherent States

To avoid unnecessary mathematical complication at this point we restrict our discussion to compact Lie groups. However, we would like to point out to the reader that the discussion applies to a general Lie group, as defined in chapter 1. Let us denote by $G$ a compact $d$-dimensional Lie group. It is well known that for compact groups all representations of the group are bounded and that all irreducible representations are finite dimensional. Moreover, one can always choose a scalar product on the representation space in such a way that every representation of $G$ is unitary, (cf. [7, Theorem 7.1.1]). Therefore, without loss in generality we assume that we are dealing with a finite dimensional strongly continuous irreducible unitary representation $U$ of $G$ on a $d_\xi$-dimensional representation space $H_\xi$. Let us denote by $\{X_k\}_{k=1}^d$ the set of finite dimensional self-adjoint generators of the representation $U$. The $X_k$, $k = 1, \ldots, d$, form an irreducible representation of the Lie algebra $L$ associated with $G$, whose commutation relations are given by

$$[X_i, X_j] = i \sum_{k=1}^d c_{ij}^k X_k,$$

where $c_{ij}^k$ denote the structure constants. The physical operators are defined by $\hat{X}_k \equiv \hbar X_k$. For definiteness it is assumed that there exists a parameterization for $G$ such that

$$U_{g(l)} = \exp(-il^1 X_1) \ldots \exp(-il^k X_k),$$

up to some ordering and where $l \in \mathcal{L}$. Here $\mathcal{L}$ denotes the compact parameter space for $G$. For all $l \in \mathcal{L}$ and a fixed normalized fiducial vector $\eta \in \mathcal{E}_\xi$ we define the

$$I_{H} = \int_{\mathcal{L}} |l\rangle \langle l| \, d\mu(l)$$

(2.32)
following set of vectors on $H_\xi$

$$\eta(l) = \sqrt{d_\xi} U_g(l) \eta. \quad (2.33)$$

It follows from the strong continuity of $U_g(l)$ that the set of vectors defined in (2.33) forms a family of strongly continuous vectors on $H_\xi$. Furthermore, let us consider the operator

$$O = \int_L \eta(l') \langle \eta(l'), \cdot \rangle dg(l'), \quad (2.34)$$

where $dg(l)$ denotes the normalized, invariant measure on $G$. It is not hard to show, using the invariance of $dg$, that the operator $O$ commutes with all $U_g(l), l \in L$. Since $U_g(l)$ is a unitary irreducible representation one has by Schur’s Lemma that $O = \lambda I_{H_\xi}$. Taking the trace of both sides of (2.34) we learn that

$$\text{tr}(\lambda I_{H_\xi}) = \lambda d_\xi = \int_L \text{tr}[\eta(l') \langle \eta(l'), \cdot \rangle] dg(l')$$

$$\lambda d_\xi = d_\xi \|\eta\|^2 \int_L dg(l')$$

$$\lambda = 1.$$

Hence, the family of vectors defined in (2.33) gives rise to the following resolution of identity:

$$I_{H_\xi} = \int_L \eta(l) \langle \eta(l), \cdot \rangle dg(l). \quad (2.35)$$

Therefore, we find that the family of vectors defined in (2.33) satisfies the requirements set forth in subsection 2.2.2 for a set of vectors to be a set of coherent states. So we conclude that the vectors defined in (2.33) form a set of coherent states for the compact Lie group $G$, corresponding to the irreducible unitary representation $U_g(l)$.

2.2.4 Continuous Representation

Analogously to standard quantum mechanics one can use the set of coherent states defined in (2.33) to give a functional representation of the space $H_\xi$. Let us define the map
\[ C_\eta : \mathbf{H}_\zeta \rightarrow L^2(G, dg) \]
\[ \psi \mapsto [C_\eta \psi](l) = \psi_\eta(l) \equiv \langle \eta(l), \psi \rangle. \]

This yields a representation of the space \( \mathbf{H}_\zeta \) by bounded, continuous, square integrable functions\(^8\) on some closed subspace \( L^2_\eta(G) \) of \( L^2(G) \). Let us denote by \( B \) any bounded operator on \( \mathbf{H}_\zeta \), then using the map \( C_\eta \) and the resolution of identity we find that

\[ (2.35) \]
\[ \langle \eta(l), B \psi \rangle = \int \langle \eta(l), B \eta(l') \rangle \langle \eta(l'), \psi \rangle dg(l') \]

holds. Choosing \( B = I_{\mathbf{H}_\zeta} \) we find

\[ \psi_\eta(l) = \int \mathcal{K}_\eta(l; l') \psi_\eta(l')dg(l'), \]

where

\[ \mathcal{K}_\eta(l; l') = \langle \eta(l), \eta(l') \rangle. \]

One calls (2.37) the reproducing property. Furthermore, as shown in Appendix B.2, the kernel \( \mathcal{K}_\eta(l'); l \) is an element of \( L^2_\eta(G) \) for fixed \( l \in \mathcal{L} \). Therefore, the kernel \( \mathcal{K}_\eta(l'; l) \) is a reproducing kernel and \( L^2_\eta(G) \) is a reproducing kernel Hilbert space (cf. Appendix B.2). Note that a reproducing kernel Hilbert space can never have more than one reproducing kernel (cf. Claim B.2.1). Therefore, since \( L^2_\eta(G) \) is a space of continuous functions, \( \mathcal{K}_\eta(l'; l) \) is unique. Moreover, since the coherent states are strongly continuous the reproducing kernel \( \mathcal{K}_\eta(l'; l) \) is a jointly continuous function, nonzero for \( l = l' \), and therefore, nonzero in a neighborhood of \( l = l' \). This means that (2.37) is a real restriction on the admissible functions in the continuous representation of \( \mathbf{H}_\zeta \). Of course a similar equation holds for the Schrödinger representation, however there one has \( \langle q | q' \rangle = \delta(q - q') \) which poses no restriction on the allowed functions. In fact, the reproducing kernel \( \mathcal{K}_\eta(l'; l) \) is the integral kernel of a projection operator

\(^8\)See Appendix B.1.
from $L^2(G)$ onto the reproducing kernel Hilbert space $L^2_\eta(G)$ (cf. Claim B.2.2). This ends our discussion of the kinematics (framework) and brings us to the subject of dynamics.

2.2.5 The Coherent State Propagator for Group Coherent States

Let $\psi \in H_\zeta$ and denote by $\mathcal{H}(\tilde{X}_1, \ldots, \tilde{X}_d)$ the bounded Hamilton operator of the quantum system under discussion, then the Schrödinger equation on $H_\zeta$ is given by

$$i\hbar \partial_t \psi = \mathcal{H}(\tilde{X}_1, \ldots, \tilde{X}_d)\psi$$

since $\mathcal{H}$ is assumed to be self-adjoint and does not explicitly depend on time, a solution to Schrödinger’s equation is given by:

$$\psi(t'') = \exp \left[ -\frac{i}{\hbar} (t'' - t') \mathcal{H} \right] \psi(t').$$

Now making use of (2.36) we find

$$\psi_\eta(l'', t'') = \int K_\eta(l'', t''; l', t') \psi_\eta(l', t') dg(l'),$$

where

$$K_\eta(l'', t''; l', t') = \langle \eta(l''), \exp \left[ -\frac{i}{\hbar} (t'' - t') \mathcal{H} \right] \eta(l') \rangle.$$  

Note that the coherent state propagator $K_\eta(l'', t''; l', t')$ satisfies the following initial condition

$$\lim_{t'' \to t'} K_\eta(l'', t''; l', t') = K_\eta(l''; l').$$

Hence, as $t'' \to t'$ we obtain the reproducing kernel $K_\eta(l''; l')$, which, as we have remarked above, is the integral kernel of a projection operator from $L^2(G)$ onto $L^2_\eta(G)$. Moreover, since $K_\eta(l''; l')$ is unique, we see that if we change the fiducial vector from $\eta$ to $\eta'$, save for a change of phase, then the resulting coherent state propagator is no longer a propagator for the elements of the reproducing kernel Hilbert space $L^2_\eta(G)$, but is a propagator for the elements of the reproducing kernel Hilbert space $L^2_\eta(G)$. Hence, we see that the coherent state propagator $K_\eta(l'', t''; l', t')$ depends strongly on the fiducial vector $\eta$. 
Following standard methods in Refs. 63 and 68 we now derive a coherent state path integral representation for the coherent state propagator. We start as in section 2.1.2 from the basic idea

\[ \exp \left[ -\frac{i}{\hbar} (t'' - t') \mathcal{H} \right] = \left[ \exp \left( -\frac{i}{\hbar} \epsilon \mathcal{H} \right) \right]^{N+1}, \]

where \( \epsilon = (t'' - t')/(N + 1) \), therefore, we find

\[
K_\eta(l'', t''; l', t') = \langle \eta(l''), \exp \left[ -\frac{i}{\hbar} (t'' - t') \mathcal{H} \right] \eta(l') \rangle = \langle \eta(l''), \left[ \exp \left( -\frac{i}{\hbar} \epsilon \mathcal{H} \right) \right]^{N+1} \eta(l') \rangle.
\]

Inserting the resolution of identity (2.35) \( N \)-times this becomes

\[
K_\eta(l'', t''; l', t') = \int \ldots \int \prod_{j=0}^{N} \langle \eta(l_{j+1}), \exp \left( -\frac{i}{\hbar} \epsilon \mathcal{H} \right) \eta(l_j) \rangle \prod_{j=1}^{N} dl_j,
\]

where \( l_{N+1} = l'' \) and \( l_0 = l' \). This expression holds for any \( N \), and therefore, it holds as well in the limit \( N \to \infty \) or \( \epsilon \to 0 \), i.e.

\[
K_\eta(l'', t''; l', t') = \lim_{\epsilon \to 0} \int \ldots \int \prod_{j=0}^{N} \langle \eta(l_{j+1}), \exp \left( -\frac{i}{\hbar} \epsilon \mathcal{H} \right) \eta(l_j) \rangle \prod_{j=1}^{N} dl_j. \tag{2.38}
\]

Hence, one has to evaluate \( \langle \eta(l_{j+1}), \exp \left( -\frac{i}{\hbar} \epsilon \mathcal{H} \right) \eta(l_j) \rangle \) for small \( \epsilon \). For small \( \epsilon \) one can make the approximation

\[
\langle \eta(l_{j+1}), \exp \left( -\frac{i}{\hbar} \epsilon \mathcal{H} \right) \eta(l_j) \rangle \approx \langle \eta(l_{j+1}), \left( 1 - \frac{i}{\hbar} \epsilon \mathcal{H} \right) \eta(l_j) \rangle
\]

\[
= \langle \eta(l_{j+1}), \eta(l_j) \rangle \left[ 1 - \frac{i}{\hbar} \epsilon \langle \eta(l_{j+1}), \mathcal{H} \eta(l_j) \rangle \right]
\]

\[
= \mathcal{K}_\eta(l_{j+1}; l_j) \left[ 1 - \frac{i}{\hbar} \epsilon \mathcal{H}_\eta(l_{j+1}; l_j) \right]
\]

\[
\approx \mathcal{K}_\eta(l_{j+1}; l_j) \exp \left[ -\frac{i}{\hbar} \epsilon \mathcal{H}_\eta(l_{j+1}; l_j) \right], \tag{2.39}
\]

where

\[
H_\eta(l_{j+1}; l_j) \equiv \frac{\langle \eta(l_{j+1}), \mathcal{H} \eta(l_j) \rangle}{\langle \eta(l_{j+1}), \eta(l_j) \rangle}.
\]
Inserting (2.39) into (2.38) yields

\[ K_\eta(l'', t''; l', t') = \lim_{\epsilon \to 0} \int \ldots \int \prod_{j=0}^{N} K_\eta(l_{j+1}; l_j) \exp \left[ \frac{-i}{\hbar} \epsilon H_\eta(l_{j+1}; l_j) \right] \prod_{j=1}^{N} dg(l_j). \] (2.40)

This is the form of the coherent state path integral one typically encounters in the literature. It is worth reemphasizing that the coherent state path integral representation of the coherent state propagator (2.40) depends strongly on the fiducial vector.

2.2.5.1 Formal Coherent State Path Integral

Even though there exists no mathematical justification whatsoever we now take in analogy to what we have done in sections 2.1 and 2.2 an improper limit of (2.40) by interchanging the operation of integration with the limit \( \epsilon \to 0 \). As pointed out in Ref. 68, p. 63, one can imagine as \( \epsilon \to 0 \) that the set of points \( l_j, j = 1, \ldots, N \), defines in the limit a (possibly generalized) function \( l(t), t' \leq t \leq t'' \). Following Ref. 68, pp. 63–64, we now derive an expression for the integrand in (2.40) valid for continuous and differentiable paths \( l(t) \). Note that the set of coherent states \( \eta(l) \) we have defined in (2.33) is not normalized, but is of constant norm given by \( d_\zeta^{1/2} \). We now rewrite the reproducing kernel \( K_\eta(l_{j+1}; l_j) = \langle \eta(l_{j+1}), \eta(l_j) \rangle \) in the following way

\[ \langle \eta(l_{j+1}), \eta(l_j) \rangle = \langle \eta(l_{j+1}), \eta(l_{j+1}) \rangle - \langle \eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j) \rangle \]

\[ = d_\zeta [1 - d_\zeta^{-1} \langle \eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j) \rangle] \]

\[ \approx d_\zeta \exp[-d_\zeta^{-1} \langle \eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j) \rangle], \]

this approximation is valid whenever \( ||\eta(l_{j+1}) - \eta(l_j)|| \ll 1, j = 0, \ldots, N \). Hence, as \( \epsilon \to 0 \) the approximation becomes increasingly better since the \( \eta(l) \) form a continuous family of vectors. Therefore, one finds

\[ K_\eta(l_{j+1}; l_j) \approx d_\zeta \exp[-d_\zeta^{-1} \langle \eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j) \rangle], \] (2.41)

for \( ||\eta(l_{j+1}) - \eta(l_j)|| \ll 1, j = 0, \ldots, N \). Using (2.41) in (2.40) and taking the limit \( \epsilon \to 0 \) the integrand in (2.40) takes for continuous and differentiable paths the
following form:

$$d\zeta \exp \left[ -\frac{1}{\hbar \zeta} \int_{t'}^{t''} \langle \eta(l), d\eta(l) \rangle - \frac{i}{\hbar d\zeta} \int_{t'}^{t''} H_\eta(l(t)) dt \right],$$

where

$$H_\eta(l(t)) = \langle \eta(l), \mathcal{H}(\hat{X}_1, \ldots, \hat{X}_k) \eta(l) \rangle$$

and where we have introduced the coherent state differential

$$d\eta(l) \equiv \eta(l + dl) - \eta(l).$$

Hence, we find the following formal coherent state path integral expression for the coherent state propagator:

$$K_\eta(l'', t''; l', t') = \int \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ i\hbar \langle \eta(l), \frac{d}{dt} \eta(l) \rangle - H_\eta(l) \right] dt \right\} \mathcal{D}g(l),$$

(2.42)

where

$$\eta(l) = U_{g(l)} \eta \quad \text{and} \quad \mathcal{D}g(l) \equiv \lim_{N \to \infty} (d\zeta)^{N+1} \prod_{j=1}^{N} dg(l_j).$$

A discussion of what is right and what is wrong with (2.42) can be found in Ref 68, pp. 64–66, we only remark here that (2.42) depends strongly on the choice of the fiducial vector and on the choice of the irreducible unitary representation of $G$. Hence, one has to reformulate the path integral representation for the coherent state propagator every time one changes the fiducial vector and keeps the irreducible representation the same, or if one changes the irreducible unitary representation of $G$. Now in many applications it is often convenient to choose the fiducial vector as the ground state of the Hamilton operator $\mathcal{H}$ of the quantum system one considers; see for instance Troung [100, 101]. Hence, one has to face the problem of various fiducial vectors. In chapter 4 we develop a representation independent propagator, which nevertheless, propagates the elements of any reproducing kernel Hilbert space $L^2_\alpha(G)$ associated with any irreducible, square integrable unitary representation of $G$. Hence, we can overcome the above limitation.
Also note that coherent state path integrals afford an alternative way of constructing path integrals for quantum systems moving on group manifolds and on homogeneous spaces. For instance Klauder has used in Ref. 62 the coherent state path integral to describe the motion of a quantum system with spin $s$ moving on the two sphere $S^2$ and in Ref. 64 to describe the motion of a quantum system on the Lobachevsky plane. Klauder has also discussed a quantization procedure for physical systems moving on group manifolds and homogeneous spaces using the action functional in (2.42), see Refs. 60 and 61, and has therefore, provided an alternative method of quantization to the quantization methods discussed in subsections 2.1.3 and 2.1.4.
3.1 Notations

In this chapter, $G$ is a real, separable, connected and simply connected, locally compact Lie group\(^1\) with fixed left invariant Haar measure $dg$, i.e. $d(hg) = dg$. Let $\Delta(g)$ be the modular function for the group $G$; i.e. $d(gh) = \Delta(h)dg$. If $\Delta(g) \equiv 1$ then the group $G$ is called unimodular. It is known that the following Lie groups are unimodular (cf. [39, p. 250] and [53, chapter X, §1]):

(i) Every compact Lie group.

(ii) Every semisimple Lie group.

(iii) Every connected nilpotent Lie group.

The affine group, which we will consider in chapter 4, is an example of a non-unimodular Lie group. Let $D(G)$ be the space of regular Bruhat functions with compact support on $G$ (cf. [15] and [78, pp. 68-69]). Let $T$ be a closeable operator on some Hilbert space $H$, then we denote its closure by $\overline{T}$. Let $L$ be the Lie algebra corresponding to $G$ with basis $x_1, \ldots, x_d$. Then we denote by $X_1 = U(x_1), \ldots, X_d = U(x_d)$ a representation of the basis of the Lie algebra $L$ by symmetric operators on some Hilbert space $H$ with common dense invariant domain $D$. The commutation relations take the form $[X_i, X_j] = i \sum_{k=1}^{d} c_{ij}^k X_k$. A vector $\psi \in H$ is called an analytic vector for a symmetric operator $X$ acting on some dense domain in $H$ if for some $s > 0$, the series $\sum_{n=0}^{\infty} \frac{(-is)^n}{n!} X^n \psi$ is defined and $\sum_{n=0}^{\infty} \frac{[s^n/n!]||X^n\psi||}{n!} < \infty$. We say that

\(^1\)For a summary of the basic facts of the theory of linear operators, Lie algebras, Lie groups, and the representation theory of Lie groups see Appendix A.
the representation $U$ of the Lie algebra $L$ satisfies Hypothesis (A) if and only if $U$ is a representation of the Lie algebra $L$ on a dense invariant domain $D$ of vectors that are analytic for all symmetric representatives $X_k = U(x_k)$ of a basis $x_1, \ldots, x_d$. If Hypothesis (A) is satisfied then by Theorem 3 of Flato et al. [38] the representation $X_1, \ldots, X_d$ of the Lie algebra $L$ on $H$ is integrable to a unique unitary representation of the corresponding connected and simply connected Lie group $G$ on $H$. We will always assume that a representation of $L$ by symmetric operators satisfies Hypothesis (A). Therefore, the representation of $L$ by symmetric operators is integrable to a unique global unitary representation of the associated connected and simply connected Lie group $G$ on $H$. Let there exist a parameterization of $G$ such that the unitary representation $U$ of $G$ can be written in terms of the $X_k$ as

$$U_q(t) = \prod_{j=1}^{d} \exp(-ilt_j X_j) \equiv \exp(-ilt_1 X_1) \cdots \exp(-ilt_d X_d), \quad (3.1)$$

$$U^*_q(t) = \prod_{j=1}^{d} \exp(ilt_j X_j) \equiv \exp(ilt_d X_d) \cdots \exp(ilt_1 X_1), \quad (3.2)$$

for some ordering, where $l$ is an element of a $d$-dimensional parameter space $G$. The parameter space $G$ is all of $\mathbb{R}^d$ if the group is non-compact and a subset of $\mathbb{R}^d$ if the group is compact or has a compact subgroup.

Remark 3.1.1: Note that one obtains in this way a representation of all elements of $G$ that are connected to the identity element. Since we are considering a connected and simply connected Lie group we have that $U_q(t)$ is a representation of $G$. Since $G$ is a manifold one needs in general a collection of proper coordinate charts that cover $G$ (see Appendix A.4), we will relabel the coordinates in each of these charts by the $d$-tuple $l$. Nevertheless, in practice it is often possible to work with a single proper coordinate chart (parameterization) as the following example shows. However, it is important to note that one does not obtain all the group elements when one is working with a single proper coordinate chart. Let us consider the two-dimensional unimodular unitary group $SU(2)$. The group generators of $SU(2)$ are given by one
half times the Pauli matrices

\[
  X_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
  X_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},
  X_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and satisfy the following well known commutation relations

\[
  [X_i, X_j] = i\epsilon_{ijk}X_k,
\]

where,

\[
  \epsilon_{ijk} = \begin{cases} 
    +1 & \text{for } (ijk) \text{ an even permutation of (123)}, \\
    -1 & \text{for } (ijk) \text{ an odd permutation of (123)}, \\
    0 & \text{otherwise}.
  \end{cases}
\]

One possible parameterization of $SU(2)$ is given in terms of the Euler angles by

\[
  u(\theta, \phi, \xi) = \exp(-i\phi X_3) \exp(-i\theta X_2) \exp(-i\xi X_3) = 
  \begin{pmatrix}
    e^{-i(\phi+\xi)/2} \cos(\theta/2) & -e^{-i(\phi-\xi)/2} \sin(\theta/2) \\
    e^{i(\phi-\xi)/2} \sin(\theta/2) & e^{i(\phi+\xi)/2} \cos(\theta/2)
  \end{pmatrix},
\]

where,

\[
  0 \leq \phi < 2\pi, \quad 0 < \theta < \pi, \quad -2\pi \leq \xi < 2\pi.
\]

Note that the points $\theta = 0$ and $\theta = \pi$ have to be excluded since at these points only $\phi + \xi$ and $\phi - \xi$ are determined, respectively. The Euler angles are analogous to geographical coordinates on the sphere $S^2$ in $\mathbb{R}^3$. Just as, on $S^2$, geographical coordinates are not uniquely determined at the north and south pole, the parameters $\phi, \xi$ are not uniquely determined at the singular points $\theta = 0$ and $\theta = \pi$. Hence, at these singular points of the parameter space, $\phi$ and $\xi$ no longer define a unimodular unitary matrix uniquely. Therefore, the set of matrices for which the parameterization introduced above is unique is a proper subset of $SU(2)$. However, as far as integration over the group $SU(2)$ is concerned the above parameterization is adequate since the set of singular points forms a set of measure zero.

As pointed out by Wybourne [106, pp. 38–39], a suitable choice of parameterization for the Lie group $G$ one considers can generally be made as follows: One first obtains a matrix representation of the Lie algebra (Ado's Theorem [7, Theorem
1.2.1) associated with the Lie group one considers; these matrices are then taken as the group generators. Just as in the case of SU(2), considered above, one then determines a suitable parameterization and generates the group elements connected to the identity element in parameterized form by exponentiating the group generators. Note that the choice of parameterization of $G$ can be made in many ways and should ideally be made such that singularities in $G$ about the identity element are avoided. Finally, a representation of the form (3.1) is obtained by exponentiating the self-adjoint representatives $\{X_k\}$ of the basis $\{x_k\}$ of the Lie algebra associated with $G$ using the parameters one has determined in the representation of the group elements connected to the identity element.

Let $(U, H)$ and $(U', H')$ be unitary representations of $G$. A densely defined closed operator $S$ from $H$ to $H'$ is called semi-invariant with weight $\sigma$ if

$$U'_{g}SU_{g}^{*} = \sigma(g)S, \quad \forall g \in G.$$ 

In what follows we shall need a common dense invariant domain for $X_1, \ldots, X_d$ that is also invariant under the one-parameter groups $\exp(itX_k)$, $k = 1, \ldots, d$. Define $\tilde{D}$ as the intersection of the domains of all monomials $X_{i_1} \ldots X_{i_k}$ for all $1 \leq i_1, \ldots, i_k \leq d$. By definition $\tilde{D}$ contains $D$, hence is dense in $H$. Then by Lemma 3 of Ref. 38 the restriction of $X_1, \ldots, X_d$ to $\tilde{D}$ is a representation of $L$ and by Lemma 4 of Ref. 38 $\tilde{D}$ is invariant under all one-parameter groups $\exp(itX_k)$, $k = 1, \ldots, d$.

Let $\lambda_m^k(g(l))$ and $\rho_m^k(g(l))$ be functions such that on $\tilde{D}$ the following relations hold:

$$\prod_{a=m+1}^{d} \exp(il^aX_a) \prod_{b=m+1}^{d} \exp(-il^bX_b) = \sum_{k=1}^{d} \lambda_m^k(g(l))X_k, \quad (3.3)$$

$$\prod_{a=1}^{m-1} \exp(-il^aX_a) \prod_{b=1}^{m-1} \exp(il^bX_b) = \sum_{k=1}^{d} \rho_m^k(g(l))X_k. \quad (3.4)$$

Note that, the parameterization of the Lie group $G$ is chosen in such a way that $\det[\lambda_m^k(g(l))] \neq 0$ and $\det[\rho_m^k(g(l))] \neq 0$, respectively. Therefore, the inverse matri-
ces $[\lambda^{-1} m^k(g(l))]$ and $[\rho^{-1} m^k(g(l))]$ exist. Furthermore, let $U(l)$ be the $d \times d$ matrix whose $mk$-element is $U_m^k(l)$ such that on $\tilde{D}$

$$U_{g(l)^*}^* \overline{X}_m U_{g(l)} = \sum_{k=1}^d U_m^k(l) \overline{X}_k,$$  

(3.5)

$$U_{g(l)} \overline{X}_m U_{g(l)^*} = \sum_{k=1}^d U_{m^{-1}}^k(l) \overline{X}_k,$$  

(3.6)

holds. One can easily check that $U(l)$ is given by exponentiating the adjoint representation of $L$, 

$$U(l) = \prod_{k=1}^d \exp(t^k c_k),$$

here $c_k$ denotes the matrix formed from the structure constants such that $c_k = -c_k(t^j)$.

3.2 Preliminaries

**Theorem 3.2.1** On the common dense invariant domain $\tilde{D}$ of $\overline{X}_1, \ldots, \overline{X}_d$, the following relations hold,

(i) For all $l \in G$, $U_{g(l)}^* dU_{g(l)} = -i \sum_{k,m=1}^d \lambda_m^k(g(l)) dl_m \overline{X}_k,$

and $L_{g(l)\lambda}^m(g(l)) = \lambda_m^k(g(l))$.

(ii) For all $l \in G$, $dU_{g(l)} U_{g(l)}^* = -i \sum_{k,m=1}^d \rho_m^k(g(l)) dl_m \overline{X}_k,$

and $R_{g(l)\rho}^m(g(l)) = \rho_m^k(g(l))$.

**Proof.** (i) Let $\psi \in \tilde{D}$ be arbitrary, then since $U_{g(l)}$ leaves $\tilde{D}$ invariant, i.e. $U_{g(l)}\tilde{D} \subset \tilde{D}$, we define the differential of $U_{g(l)}$ as follows:

$$dU_{g(l)} \psi \equiv \sum_{m=1}^d \lim_{\Delta l_m \to 0} \left[ \frac{U_{g(l_1, \ldots, l_m + \Delta l_m, \ldots, l_d)} \psi - U_{g(l_1, \ldots, l_d)} \psi}{\Delta l_m} \right] dl_m. \quad (3.7)$$

Now since $U_{g(l)}$ is the product of one-parameter unitary groups one finds for the differential of $U_{g(l)}$

$$dU_{g(l)} \psi = \sum_{m=1}^m \prod_{a=1}^m \exp(-il^a \overline{X}_a)(-i\overline{X}_m) \prod_{b=m+1}^d \exp(-il^b \overline{X}_b) \psi dl_m.$$
Therefore,
\[
U_{g(l)}^* dU_{g(l)} \psi = -i \sum_{m=1}^{d} \prod_{a=m+1}^{d} \exp(it^a X_a) \ X_m \prod_{b=m+1}^{d} \exp(-it^b X_b) \psi dl^m
\]
\[
= -i \sum_{m,k=1}^{d} \lambda_m^k (g(l)) dl^m X_k \psi.
\]

Since \( \psi \in \tilde{D} \) was arbitrary, one finds that on \( \tilde{D} \subset H \), the following relation holds
\[
U_{g(l)}^* dU_{g(l)} = -i \sum_{k,m=1}^{d} \lambda_m^k (g(l)) dl^m X_k,
\] (3.8)

To establish the second part of (i) let \( \psi \in \tilde{D} \) be arbitrary then
\[
U_{g(l)}^* dU_{g(l)} \psi = U_{g(l)}^* U_{g(l_0)} U_{g(l_0)}^* dU_{g(l)} \psi = U_{g^{-1}\{l_0\}g(l)}^* dU_{g^{-1}\{l_0\}g(l)} \psi.
\]

Therefore, using (3.8) and the fact that both \( \{X_k\}_{k=1}^{d} \) and \( \{dl^m\}_{m=1}^{d} \) are linearly independent families one finds
\[
L_{g(l_0)} \lambda_m^k (g(l)) \equiv \lambda_m^k (g^{-1}\{l_0\}g(l)) = \lambda_m^k (g(l)).
\]

(ii) The first part of (ii) is similar to the first part of (i). To prove the second part of (ii) one can proceed as follows, let \( \psi \in \tilde{D} \) be arbitrary, then
\[
dU_{g(l)} U_{g(l)}^* \psi = dU_{g(l)} U_{g(l_0)} U_{g(l_0)}^* U_{g(l)}^* \psi = dU_{g(l)} U_{g(l_0)} U_{g(l)}^* \psi.
\]

Therefore, by the same reasoning as above
\[
R_{g(l_0)} \rho_m^k (g(l)) \equiv \rho_m^k (g(l)g(l_0)) = \rho_m^k (g(l)). \quad \square
\]

Since the \( \lambda_m^k (g(l)) \) are left invariant functions on the Lie group \( G \), the relation (i) can be viewed as an operator version of the generalized Maurer-Cartan form on \( G \), (cf. [18, p. 92]).

**Corollary 3.2.2** The functions \( \lambda_m^k (g(l)) \) and \( \rho_m^k (g(l)) \) are related as follows:
\[
\lambda_m^k (g(l)) = \sum_{c=1}^{d} \rho_m^c (g(l)) U_c^k (l)
\]
Proof. Let \( \psi \in \tilde{D} \) be arbitrary then by Theorem 3.2.1 (ii),

\[
dU_{g(l)} U_{g(l)}^* \psi = -i \sum_{c,m=1}^{d} \rho_m \epsilon(g(l)) d\lambda^m \overline{X}_c U_{g(l)} U_{g(l)}^* \psi.
\]

Since \( U_{g(l)} \) leaves \( \tilde{D} \) invariant, set \( \phi \equiv U_{g(l)}^* \psi \in \tilde{D} \), then multiplying the resulting relation from the left by \( U_{g(l)}^* \) yields,

\[
U_{g(l)}^* dU_{g(l)} \phi = -i \sum_{c,m=1}^{d} \rho_m \epsilon(g(l)) d\lambda^m \overline{X}_c U_{g(l)} \phi.
\]

Using Theorem 3.2.1 (i) and the definition of \( U_m \, \text{k}(l) \) the Corollary easily follows. \( \square \)

**Corollary 3.2.3** The functions \( \rho_m \, \text{k}(g(l)) \) and \( \lambda_m \, \text{k}(g(l)) \) satisfy the following equations

\[
(i) \sum_{n=1}^{d} \{ \partial_{n} [\rho^{-1}_k a_j (g(l))] \rho^{-1}_k n(g(l)) - \partial_{m} [\rho^{-1}_m a_j (g(l))] \rho^{-1}_j n(g(l)) \} = - \sum_{f=1}^{d} c_{j k}^f \rho^{-1}_f a_j (g(l)),
\]

\[
(ii) \sum_{n=1}^{d} \{ \partial_{n} [\lambda^{-1}_k a_j (g(l))] \lambda^{-1}_k n(g(l)) - \partial_{m} [\lambda^{-1}_m a_j (g(l))] \lambda^{-1}_j n(g(l)) \} = \sum_{f=1}^{d} c_{j k}^f \lambda^{-1}_f a_j (g(l)),
\]

\[
(iii) \sum_{s=1}^{d} \lambda_s (g(l)) \partial_{l m} [\lambda^{-1}_s b_j (g(l))] = \sum_{s=1}^{d} \rho_m \epsilon(g(l)) \partial_{l m} [\rho^{-1}_m b_j (g(l))].
\]

where \( c_{j k}^f \) are the structure constants for \( G \).

**Proof.** (i) Let \( \psi \in \tilde{D} \) be arbitrary then one easily finds using (3.7) that

\[
\partial_{l m} \partial_{l n} U_{g(l)} \psi = \partial_{l m} \partial_{l n} U_{g(l)} \psi,
\]

holds. Now picking out the terms \( \partial_{l m} U_{g(l)} \) and \( \partial_{l n} U_{g(l)} \) in Theorem 3.2.1 (ii) one finds

\[
\sum_{a=1}^{d} \partial_{l m} \left[ -i \rho_n a(g(l)) \overline{X}_a U_{g(l)} \right] \psi = \sum_{a=1}^{d} \partial_{l n} \left[ -i \rho_m a(g(l)) \overline{X}_a U_{g(l)} \right] \psi
\]

\[
\left\{ \sum_{a=1}^{d} -i \partial_{l m} [\rho_n a(g(l))] \overline{X}_a - \sum_{a,b=1}^{d} \rho_n a(g(l)) \rho_n b(g(l)) \overline{X}_a \overline{X}_b \right\} U_{g(l)} \psi = \left\{ \sum_{a=1}^{d} -i \partial_{l n} [\rho_m a(g(l))] \overline{X}_a - \sum_{a,b=1}^{d} \rho_m a(g(l)) \rho_n b(g(l)) \overline{X}_a \overline{X}_b \right\} U_{g(l)} \psi.
\]
Since \( U_{g(l)} \) leaves \( \bar{D} \) invariant, one can set \( \phi = U_{g(l)} \psi \) and rearranging the terms yields
\[
\sum_{f=1}^{d} \left\{ -i \partial_m [\rho_n^f (g(l))] + i \partial_n [\rho_m^f (g(l))] \right\} \bar{X}_f \phi = \sum_{a,b=1}^{d} \rho_n^a (g(l)) \rho_m^b (g(l)) [X_a, \bar{X}_b] \phi.
\]
Now making use of the commutation relations \([X_a, \bar{X}_b] = i \sum_{f=1}^{d} c_{ab}^f \bar{X}_f\) this equation becomes
\[
\sum_{f=1}^{d} \left\{ \partial_m [\rho_n^f (g(l))] - \partial_n [\rho_m^f (g(l))] + \sum_{a,b=1}^{d} \rho_n^a (g(l)) \rho_m^b (g(l)) c_{ab}^f \right\} \bar{X}_f \phi = 0.
\]
Finally using the fact that the operators \( \{\bar{X}_k\}_{k=1}^{d} \) form a basis for \( g \) and that \( \phi \in \bar{D} \) is arbitrary one concludes
\[
\{ \partial_m [\rho_n^f (g(l))] - \partial_n [\rho_m^f (g(l))] \} = - \sum_{s,t=1}^{d} \rho_n^s (g(l)) \rho_m^t (g(l)) c_{st}^f. \tag{3.9}
\]
Now contracting both sides of (3.9) with \( \rho^{-1}_{f} a^s (g(l)) \) yields
\[
\sum_{f=1}^{d} \{ \partial_m [\rho^{-1}_{f} a^a (g(l))] \rho_n^f (g(l)) - \partial_n [\rho^{-1}_{f} a^a (g(l))] \rho_m^f (g(l)) \} =
\sum_{f=1}^{d} \sum_{s,t=1}^{d} \rho_n^s (g(l)) \rho_m^t (g(l)) c_{st}^f \rho^{-1}_{f} a^a (g(l)),
\]
where,
\[
\sum_{k=1}^{d} \partial_v [\rho_m^k (g(l))] \rho^{-1}_{k} n^m (g(l)) = - \sum_{k=1}^{d} \partial_v [\rho^{-1}_{k} n^m (g(l))] \rho_m^k (g(l)),
\]
has been used. Finally contract both sides with \( \rho^{-1}_{k} m^m (g(l)) \rho^{-1}_{f} j^n (g(l)) \) to obtain the desired relation,
\[
\sum_{n=1}^{d} \{ \partial_m [\rho^{-1}_{j} a^a (g(l))] \rho^{-1}_{k} n^m (g(l)) - \partial_n [\rho^{-1}_{k} a^a (g(l))] \rho^{-1}_{j} n^m (g(l)) \} =
\sum_{f=1}^{d} c_{jk}^f (g(l)) \rho^{-1}_{f} a^a (g(l)).
\]
(ii) The proof of (ii) is similar to the proof of (i).
(iii) Using Corollary 3.2.2 (iii) can be rewritten as
\[
\sum_{t=1}^{d} \partial_t [U^{-1}_{h} (l) \rho^{-1}_{l} (g(l))] = - \sum_{f,s,t=1}^{d} U^{-1}_{h} (l) \rho^{-1}_{t} a^s (g(l)) \partial_t [\rho_m^s (g(l))] \rho^{-1}_{s} b (g(l))
\]
This equation can be simplified as follows
\[
\sum_{h,j,t=1}^{d} \rho_n^j(g(l))U_j^h(l)\partial_l\left[U^{-1}_h(t)\rho^{-1}_t(b)(g(l))\right] = -\sum_{s=1}^{d} \partial_m[\rho_m^s(g(l))]\rho^{-1}_s(b)(g(l))
\]
\[
\sum_{h,j=1}^{s} U^{-1}_h(f(l))\partial_l\left[\rho_n^j(g(l))U_j^h(l)\right] = \partial_m[\rho_m^f(g(l))].
\]
Differentiating the product and rearranging the terms yields:
\[
\partial_l[\rho_n^f(g(l))] - \partial_m[\rho_m^f(g(l))] = -\sum_{j,h=1}^{d} \rho_n^j(g(l))\partial_l[U_j^h(l)]U^{-1}_h(f(l)).
\]
Next using \(\partial_lU_j^h(l) = \rho_m^s(g(l))c_{js}^nU_n^h(l)\), which is proved along the same lines as Theorem 3.2.1 (ii), we find
\[
\partial_l[\rho_n^f(g(l))] - \partial_m[\rho_m^f(g(l))] = -\sum_{j,s=1}^{d} \rho_n^j(g(l))\rho_m^s(g(l))c_{jd}^f,
\]
which is equation (3.9), and therefore, establishes (iii). □

One could ask if it is really necessary to use unbounded symmetric operators in the representation theory of Lie algebras or stated differently, can one develop a representation theory of Lie algebras using only bounded symmetric operators. We could then discard almost all the technical difficulties we have encountered in this chapter. This interesting problem has been considered by Doebner and Melsheimer [27] who have shown that

**Theorem 3.2.4** (Doebner and Melsheimer [27]) A nontrivial representation of a non-compact Lie algebra by symmetric operators contains at least one unbounded operator.

Since we are interested in quantum physics, we have to represent our basic kinematical variables by self-adjoint operators. Hence, we have to choose a representation of the basis of a given Lie algebra by symmetric operators satisfying Hypothesis (A), so that the closures of these operators are self-adjoint and the representation is integrable to a unique unitary representation of the associated Lie group. Therefore, in
light of the above result, we can not avoid the use of unbounded symmetric operators when we are dealing with non-compact Lie algebras and Lie groups.
4.1 Coherent States for General Lie Groups

Let $U$ be a fixed continuous, unitary, irreducible representation of a $d$-dimensional real, separable, locally compact, connected and simply connected Lie group $G$ on the Hilbert space $H$. Let $\phi, \xi \in H$ then the function $G \ni g \rightarrow \langle U_g \xi, \phi \rangle$ is called a coefficient of the representation $U$.

**Definition 4.1.1** A continuous, unitary irreducible representation $U$ is called square integrable if it has a nonzero square integrable coefficient, i.e. if there exist vectors $\phi, \xi \in H$ such that $\langle U_g \xi, \phi \rangle \neq 0$ and

$$\int_G |\langle U_g \xi, \phi \rangle|^2 dg < \infty.$$ 

By a general Lie group $G$ we mean in the following a real, separable, locally compact, connected and simply connected Lie group $G$ with continuous, irreducible, square integrable, unitary representations.

For continuous, irreducible, square integrable, unitary representations one can prove the following Theorem:

**Theorem 4.1.2** (Duflo and Moore [30]) Let $U$ be a continuous, irreducible, square integrable, unitary representation of $G$, then there exists a unique operator $K$ in $H$, self-adjoint, positive, semi-invariant with weight $\Delta^{-1}(g)$, and satisfying the following conditions:

1. Let $\phi, \xi \in H$, $\phi \neq 0$. Then $\langle U_g \xi, \phi \rangle$ is square integrable if and only if $\xi \in D(K^{-1/2})$. 

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(ii) Let $\chi, \chi' \in \mathbf{H}$, and $\xi, \xi' \in D(K^{-1/2})$. Then one has
\[
\int \langle \chi, U_g \xi \rangle \langle U_g \xi', \chi' \rangle dg = \langle \chi, \chi' \rangle \langle K^{-1/2} \xi', K^{-1/2} \xi \rangle. \tag{4.1}
\]

(For the proof see Ref. 30, Theorem 3.)

Remark 4.1.1 Condition (i) shows that if a representation is square integrable then there exists a dense set $\mathbf{S}$ of vectors in $\mathbf{H}$ such that for $\xi \in \mathbf{S}$ the factor $\langle U_g \xi, \phi \rangle$ is square integrable for all $\phi \in \mathbf{H}$. One refers to condition (ii) as the orthogonality relations for $U$. A result similar to (ii) has been obtained by Carey [16, Theorem 4.3] by realizing the square integrable representation $U$ in a reproducing kernel Hilbert space. For the Heisenberg-Weyl group these orthogonality relations have first been proved by Moyal [81]. The operator $K$ is called the formal degree of the representation $U$. When $G$ is unimodular, $K$ is a scalar multiple of the identity operator which is the usual formal degree. ☐

Let $X_1, \ldots, X_d$ be an irreducible representation of the basis of the Lie algebra $L$ corresponding to $G$, by symmetric operators on $\mathbf{H}$ satisfying Hypothesis (A), then $L$ is integrable to a unique unitary representation of $G$ on $\mathbf{H}$. Let there exist a parameterization of $G$ such that,
\[
U_{g(l)} = \prod_{k=1}^{d} \exp(-i l^k X_k) = \exp(-i l^1 X_1) \ldots \exp(-i l^d X_d);
\]
where $l \in \mathcal{G}$.

Now let $\eta \in D(K^{1/2})$; then we define the set of coherent states for $G$, corresponding to the fixed continuous, irreducible, square integrable, unitary representation $U_{g(l)}$ as:
\[
\eta(l) \equiv U_{g(l)} K^{1/2} \eta; \quad \eta \in D(K^{1/2}) \quad \text{and} \quad ||\eta|| = 1. \tag{4.2}
\]
It follows directly from (4.1) that these states give rise to a resolution of identity of the form
\[
I_{\mathbf{H}} = \int_{\mathcal{G}} \eta(l) \langle \eta(l), \cdot \rangle dg(l), \tag{4.3}
\]
where $dg(l)$ is the left invariant Haar measure of $G$ given in the chosen parameterization by

$$dg(l) = \gamma(l) \prod_{k=1}^{d} dl^k,$$

where $\gamma(l) \equiv |\det[\lambda_{m}^{k}(g(l))]|$.

**Remark 4.1.2** It follows from the strong continuity of $U_{g(l)}$ that the family of states defined in (4.2) is strongly continuous. Moreover, these states give rise to the resolution of identity (4.3). Hence, the family of states defined in (4.2) satisfies the requirements set forth in subsection 2.2.2 for a family of states to be a family of coherent states. ♦

The map $C_{\eta} : H \rightarrow L^2(G)$, defined for any $\psi \in H$ by:

$$[C_{\eta} \psi](l) = \psi_{\eta}(l) \equiv \langle \eta(l), \psi \rangle = \langle U_{g(l)}K^{1/2}\eta, \psi \rangle,$$

yields a representation of the Hilbert space $H$ by bounded, continuous, square integrable functions on a proper closed subspace $L_{\eta}^2(G)$ of $L^2(G)$; see Appendix B.1. Using the resolution of identity one finds

$$\psi_{\eta}(l) = \int K_{\eta}(l; l')\psi_{\eta}(l')dg(l'),$$

where

$$K_{\eta}(l; l') \equiv \langle \eta(l), \eta(l') \rangle = \langle \eta, K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}\eta \rangle$$

and $K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}$ denotes the closure of the operator $K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}$. One calls (4.6) the reproducing property. Furthermore, as shown in Appendix B.2, the kernel $K_{\eta}(l'; l)$ is an element of $L_{\eta}^2(G)$ for fixed $l \in \mathcal{G}$. Therefore, the kernel $K_{\eta}(l'; l)$ is a reproducing kernel and $L_{\eta}^2(G)$ is a reproducing kernel Hilbert space; see Appendix B.2. One easily verifies (see Appendix B.1) that the map $C_{\eta}$ is an isometric isomorphism from $H$ to $L_{\eta}^2(G)$. Now let the map $\Lambda : G \ni g \mapsto \Lambda_{g}$ be defined by left translation, i.e.

$$(\Lambda_{g(l)}\phi)(g(l')) = \phi(g^{-1}(l)g(l')), \ \forall \phi \in L^2(G), \ \forall g(l), g(l') \in G.$$
It is straightforward to show (see Appendix A.5) that the map defined in (4.7) is a continuous, unitary representation of $G$ on $L^2(G)$. This representation is called the left regular representation of $G$.

**Lemma 4.1.3** The isometric isomorphism $C_\eta$ intertwines the representation $U_{g(l)}$ on $H$ with a subrepresentation of the left regular representation $\Lambda_{g(l)}$ on $L^2_\eta(G)$.

**Proof:** Let $\psi \in H$ be arbitrary, then we have

$$
[C_\eta U_{g(l')}(l)](l) = \langle \eta(l), U_{g(l')}(l) \psi \rangle = \langle U_{g^{-1}(l')} U_{g(l)} K^{1/2} \eta, \psi \rangle = \psi_\eta (g^{-1}(l')g(l)) = \Lambda_{g(l')} \psi_\eta(g(l)) = \Lambda_{g(l')}[C_\eta \psi](l).
$$

Since, $\psi \in H$ was arbitrary and $C_\eta$ is bounded we conclude that

$$
C_\eta U_{g(l')} = \Lambda_{g(l')} C_\eta;
$$

hence, $C_\eta$ intertwines the representation $U$ with a subrepresentation of the left regular representation $\Lambda$ on $L^2_\eta(G)$. □

Therefore, $(U, H)$ is unitarily equivalent to a subrepresentation of the left regular representation $(\Lambda, L^2(G))$.

**Lemma 4.1.4** The unitary representation $U_{g(l)}$ intertwines the operator representation $\{\overline{X}_m\}_{m=1}^d$ of $L$ on $H$, with the representation of $L$ by right and left invariant differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G) \subset L^2(G)$.

In fact setting $\nabla_l = (\partial_{l^1}, \ldots, \partial_{l^d})$ the following relations hold:

(i) Let $\tilde{x}_k(-i \nabla_l, l) \equiv \sum_{m=1}^d \rho_k^{-1} m(g(l)) (-i \partial_{l^m}), k = 1, \ldots, d$, then:

$$
\tilde{x}_k(-i \nabla_l, l) U_{g(l)}^* \psi = U_{g(l)}^* \overline{X}_k \psi, \quad \forall \psi \in \tilde{D}.
$$
(ii) Let \( \tilde{x}_k(i\nabla_l, l) \equiv \sum_{m=1}^{d} \lambda^{-1} k^m (g(l))(i\partial_l^m), \ k = 1, \ldots, d, \) then:

\[
\tilde{x}_k(i\nabla_l, l)U_{g(l)}\psi = U_{g(l)}\tilde{X}_k\psi, \ \forall \psi \in \tilde{D}.
\]

A common dense invariant domain for these differential operators on any one of the \( L^2_\eta(G) \subset L^2(G) \) is given by the continuous representation of \( \tilde{D}, \) i.e. \( \tilde{D}_\eta \equiv C_\eta(\tilde{D}). \)

**Proof.** (i) Let \( \psi \in \tilde{D} \) be arbitrary, then using the fact that \( \partial_l U_{g(l)}U^*_{g(l)} = -U_{g(l)}\partial_l U^*_{g(l)} \psi \) it follows from Theorem 3.2.1 (ii) that

\[
-i\partial_l U^*_{g(l)} \psi = \sum_{c=1}^{d} \rho^c(g(l))U^*_{g(l)}\tilde{X}_c\psi, \ m = 1, \ldots, d.
\]

After contracting both sides with \( \rho^{-1} k^m (g(l)) \) one finds

\[
\sum_{m=1}^{d} \rho^{-1} k^m (g(l))(-i\partial_l)U^*_{g(l)} \psi = U^*_{g(l)}\tilde{X}_k\psi, \ k = 1, \ldots, d,
\]

hence,

\[
\tilde{x}_k(-i\nabla_l, l)U^*_{g(l)} \psi = U^*_{g(l)}\tilde{X}_k\psi \quad k = 1, \ldots, d.
\]

Using Corollary 3.2.3 (i) one obtains,

\[
[\tilde{x}_i(-i\nabla_l, l), \tilde{x}_j(-i\nabla_l, l)] = \sum_{m,n=1}^{d} \left[ \rho^{-1} i^m (g(l))(-i\partial_l), \rho^{-1} j^n (g(l))(-i\partial_l) \right]
\]

\[
= i \sum_{m=1}^{d} \sum_{n=1}^{d} \left( \partial_l [\rho^{-1} i^m (g(l))] \rho^{-1} j^n (g(l)) - \partial_l \rho^{-1} j^n (g(l)) \rho^{-1} i^m (g(l)) \right) (-i\partial_l) \]

\[
= i \sum_{k=1}^{d} \sum_{m=1}^{d} \rho^{-1} i^m (g(l))(-i\partial_l) \]

\[
= i \sum_{k=1}^{d} \sum_{m=1}^{d} \rho^{-1} i^m (g(l))(-i\partial_l).
\]

Therefore, the differential operators \( \{\tilde{x}_k(-i\nabla_l, l)\}_{k=1}^{d} \) with common dense invariant domain \( \tilde{D}_\eta \) form a representation of \( L \) on any one of the reproducing kernel Hilbert spaces \( L^2_\eta(G). \)

(ii) The proof of (ii) is similar to proof of (i). \( \square \)
Corollary 4.1.5  The differential operators \( \tilde{x}_k(-i\nabla_l, l) \) \( k = 1, \ldots, d \) are essentially self-adjoint on any one of the reproducing kernel Hilbert spaces \( L^2_\eta(G) \) and can be identified with the generators \( \{ \Lambda(X_k) \} \) \( k = 1, \ldots, d \) of a subrepresentation of the left (right) regular representation of \( G \) on \( L^2_\eta(G) \).

**Proof.** Let \( \psi \in \tilde{D} \) then it follows from Lemma 4.1.4 that

\[
\tilde{x}_k(-i\nabla_l, l)\psi_\eta(l) = [C_\eta \bar{X}_k \psi](l), \quad k = 1, \ldots, d.
\]

On the other hand, let \( U_{g_k(t)} = \exp(-itX_k), \ k = 1, \ldots, d, \) be one-parameter subgroups of \( G \). Then \([C_\eta \bar{X}_k \psi](l)\) can also be written as

\[
[C_\eta \bar{X}_k \psi](l) = \left[ C_\eta \lim_{t \to 0} \frac{U_{g_k(t)} - I}{it} \psi \right](l) \\
= \lim_{t \to 0} \frac{1}{it} \left( \langle U_{g_k^{-1}(t)g(l)} \eta, \psi \rangle - \langle U_{g(l)} \eta, \psi \rangle \right) \\
= \lim_{t \to 0} \frac{\psi_\eta(g_k^{-1}(t)g(l)) - \psi_\eta(g(l))}{it} \\
= \lim_{t \to 0} \frac{\Lambda_{g_k(t)} - I}{it} \psi_\eta(l) \\
= \Lambda(X_k) \psi_\eta(l), \quad k = 1, \ldots, d
\]

where the \( \Lambda(X_k) \equiv \lim_{t \to 0} \frac{\Lambda_{g_k(t)} - I}{it}, \ k = 1, \ldots, d, \) are the generators of a subrepresentation of the left regular representation of \( G \) on \( L^2_\eta(G) \). Hence, one can identify \( \tilde{x}_k(-i\nabla_l, l) \) with \( \Lambda(X_k) \) on \( \tilde{D}_\eta \), i.e.

\[
\tilde{x}_k(-i\nabla_l, l) = \Lambda(X_k), \quad k = 1, \ldots, d.
\]

Clearly, the operators \( \tilde{x}_k(-i\nabla_l, l), \ k = 1, \ldots, d, \) are symmetric, since the operators \( \bar{X}_k, \ k = 1, \ldots, d, \) are symmetric on \( H \) and since \( C_\eta \) is an isometric isomorphism from \( H \) onto \( L^2_\eta(G) \).

The essential self-adjointness of the operators \( \tilde{x}_k(-i\nabla_l, l), \ k = 1, \ldots, d, \) on \( L^2_\eta(G) \) is established as follows. Since each of the operators \( \bar{X}_k, \ k = 1, \ldots, d, \) has a dense set \( D \subset H \) of analytic vectors, see section 3.1, we have by Lemma 5.1 in Ref. 82 that
each \( X_k \), \( k = 1, \ldots, d \), is self-adjoint. Hence, the restriction of each \( X_k \), \( k = 1, \ldots, d \), to \( \tilde{D} \) is essentially self-adjoint. Since, \( C_\eta \) is an isometric isomorphism from \( H \) onto \( L^2_\eta(G) \) we have that the closure of each \( \tilde{x}_k(-i\nabla, l) \), \( k = 1, \ldots, d \), contains a dense set of analytic vectors, namely, \( C_\eta(D) \), hence, is by Lemma 5.1 in Ref. 82 self-adjoint. In particular, each \( \tilde{x}_k(-i\nabla, l) \), \( k = 1, \ldots, d \), is essentially self-adjoint on \( \tilde{D}_\eta \).

Similarly one can prove that the operators \( \{\tilde{x}_k(i\nabla, l)\}_{k=1}^d \) are essentially self-adjoint and that they can be identified with the generators \( \{P(X_k)\}_{k=1}^d \) of a subrepresentation of the right regular representation of \( G \) on \( L^2_\eta(G) \). \( \square \)

**Corollary 4.1.6** The family of right invariant differential operators \( \{\tilde{x}_k(-i\nabla, l)\}_{k=1}^d \) commutes with the family of left invariant differential operators \( \{\tilde{x}_k(i\nabla, l)\}_{k=1}^d \).

**Proof:** Let \( \tilde{x}_i(-i\nabla, l) \) and \( \tilde{x}_j(i\nabla, l) \) be arbitrary, then

\[
[\tilde{x}_i(-i\nabla, l), \tilde{x}_j(i\nabla, l)] = \\
= \sum_{m,n=1}^d \left[ \rho^{-1}_{i} m(g(l))(-i\partial_{im}), \lambda^{-1}_{j} n(g(l))(i\partial_{in}) \right] \\
= \sum_{n=1}^d \left( \sum_{m=1}^d \left\{ \rho^{-1}_{i} m(g(l))\partial_{im}[\lambda^{-1}_{j} n(g(l))] - \lambda^{-1}_{j} n(g(l))\partial_{im}[\rho^{-1}_{i} m(g(l))] \right\} \partial_{in} \right) \\
= \sum_{n=1}^d \left( \sum_{m=1}^d \rho^{-1}_{i} m(g(l)) \sum_{f,s=1}^d \left\{ \lambda^{-1}_{j} f(g(l))\partial_{if}[\rho^{-1}_{s} n(g(l))] \rho_{m,s}(g(l)) \right. \\
- \left. \sum_{m=1}^d \lambda^{-1}_{j} m(g(l))\partial_{im}[\rho^{-1}_{i} s(g(l))] \right\} \partial_{in} \right) \\
= \sum_{n=1}^d \left\{ \sum_{f=1}^d \lambda^{-1}_{j} f(g(l))\partial_{if}[\rho^{-1}_{i} n(g(l))] - \sum_{m=1}^d \lambda^{-1}_{j} m(g(l))\partial_{im}[\rho^{-1}_{i} n(g(l))] \right\} \partial_{in} \\
= 0,
\]

where we have used Corollary 3.2.3 (iii) in the fourth line. Therefore,

\[
[\tilde{x}_i(-i\nabla, l), \tilde{x}_j(i\nabla, l)] = 0,
\]

and since, \( \tilde{x}_i(-i\nabla, l) \) and \( \tilde{x}_j(i\nabla, l) \) were arbitrary, this establishes the Corollary. \( \square \)
4.2 The Representation Independent Propagator for Compact Lie Groups

In this section we follow our presentation in Ref. 98. Let $G$ be a $d$-dimensional, connected and simply connected, real compact Lie group $G$. For compact Lie groups all irreducible representations are finitely dimensional (cf. [7, Theorem 7.1.3]). Hence, let us denote the finite dimensional irreducible representations of $G$ by $U^\xi$ and their finite dimensional representation spaces by $H_\xi$. We denote the dimension of the representation space $H_\xi$ by $d_\xi$. One calls $d_\xi$ the degree of the representation $U^\xi$. Let $X_1, \ldots, X_d$ be an irreducible representation of the basis of $L$ by bounded symmetric operators on $H_\xi$. Then Hypothesis (A) is trivially fulfilled for this family of operators since all vectors in $H_\xi$ are analytic vectors for these operators, hence this representation of the Lie algebra $L$ is integrable to a unique unitary representation of $G$ on $H_\xi$. Let there exist a parameterization of $G$ such that,

$$U^\xi_{g(l)} = \prod_{k=1}^{d} \exp(-il^k X_k) \equiv \exp(-il^1 X_1) \ldots \exp(-ild X_d),$$

where $l \in G$. Since $G$ is compact, the parameter space $G$ is a bounded set, therefore, all irreducible representations are trivially square integrable. The positive self-adjoint operator $K$ is given by $K = d_\xi I$, hence, we can choose any normalized vector $\eta \in H_\xi$ and the coherent states for a compact Lie group $G$ corresponding to a fixed irreducible unitary representation become:

$$\eta(l) = \sqrt{d_\xi} U^\xi_{g(l)} \eta,$$

see equation (4.2). As we have seen in chapter 2, the resolution of identity has the form

$$I_{H_\xi} = \int_{G} \eta(l) \langle \eta(l), \cdot \rangle dg(l),$$

here $dg(l)$ is given by

$$dg(l) = \frac{\gamma(l)}{|G|} \prod_{k=1}^{d} dl^k,$$
Since all operators $X_k$, $k = 1, \ldots, d$ are bounded we have by Lemma 4.1.4 for any
\( \psi \in H_\zeta \), using the continuous representation $C_\eta : H_\zeta \to L^2(G)$, that
\[
\tilde{x}_k(-i\nabla I, l)[C_\eta \psi](l) = [C_\eta X_k \psi](l), \quad k = 1, \ldots, d,
\]
Note that this relation holds independently of $\eta$.

Since $G$ is compact the center of the von Neumann algebra $A(\Lambda)$ generated by
the left regular representation $\Lambda$ of $G$ contains a compact self-adjoint operator whose
eigenspaces are $\Lambda$-invariant (cf. [78, Lemma IV.3.1]). Hence, $\Lambda$ can be decomposed
into a direct sum of irreducible representations. In fact $\Lambda$ is completely reducible into
a direct sum of all irreducible unitary representations of $G$, where each $U^\zeta$ occurs
with multiplicity $d_\zeta$ (see [7, Theorem 7.1.4]), i.e.
\[
\Lambda = \bigoplus_{\zeta \in \hat{\mathcal{G}}} d_\zeta U^\zeta,
\]
where $\hat{\mathcal{G}}$ denotes the dual space of $G$; $\hat{\mathcal{G}}$ is the set of equivalence classes of all con-
tinuous, irreducible unitary representations of $G$.

Denote by $\mathcal{H}(X_k)$ the self-adjoint Hamilton operator of a quantum mechani-
cal system on $H_\zeta$. Then for $U^\zeta_{g(l)}$ the continuous representation of the solution to
Schrödinger's equation, $\psi(t) = \exp[-i(t - t')H(X_k)]\psi(t')$, where $\hbar = 1$, is given on
$L^2_\eta(G)$ by
\[
\psi_\eta(l, t) = \int K_\eta(l, t; l', t')\psi_\eta(l', t')dg(l'),
\]
where,
\[
K_\eta(l, t; l', t') = \langle \eta(l), \exp[-i(t - t')\mathcal{H}(X_k)]\eta(l') \rangle
= [C_\eta \exp[-i(t - t')\mathcal{H}(X_k)]\eta(l')](l)
= U(t - t')|C_\eta(\eta(l'))(l)
= U(t - t')d_\zeta \langle \eta, U^\zeta_{g(l)} U^\zeta_{g(l')} \eta \rangle,
\]
where,
\[
U(t - t') = \exp[-i(t - t')\mathcal{H}(\tilde{x}_k(-i\nabla I, l))].
\]
In this construction \( \eta \) was arbitrary, hence it holds for any \( \eta \in \mathbf{H}_\zeta \). Therefore, one can choose any orthonormal basis (ONB) \( \{ \phi_j \}_{j=1}^{d_\zeta} \) in \( \mathbf{H}_\zeta \) and write down the following generalized propagator

\[
K_{\mathbf{H}_\zeta}(l, t; l', t') = \sum_{j=1}^{d_\zeta} K_{\phi_j}(l, t; l', t') = \mathcal{U}(t - t') d_\zeta \text{tr}[U_{g(l)}^{\zeta*} U_{g(l')}^{\zeta}]
\]

\[
= \mathcal{U}(t - t') d_\zeta \chi_\zeta(g^{-1}(l)g(l')),
\]

(4.8)

where \( \chi_\zeta(g^{-1}(l)g(l')) \equiv \text{tr}[U_{g(l)}^{\zeta*} U_{g(l')}^{\zeta}] \).

**Lemma 4.2.1** The propagator \( K_{\mathbf{H}_\zeta}(l, t; l', t') \) given in (4.8) correctly propagates all elements of any reproducing kernel Hilbert space \( L^2_\eta(G) \), associated with the irreducible unitary representation \( U_{g(l)}^{\zeta*} \) of the compact Lie group \( G \).

**Proof.** Let \( \eta \in \mathbf{H}_\zeta \) be arbitrary, then for \( \psi_\eta(l', t') \in L^2_\eta(G) \) one has

\[
\int K_{\mathbf{H}_\zeta}(l, t; l', t') \psi_\eta(l', t') dg(l') =
\]

\[
= \int \mathcal{U}(t - t') d_\zeta \chi_\zeta(g^{-1}(l)g(l')) \psi_\eta(l', t') dg(l')
\]

\[
= \sum_{j=1}^{d_\zeta} d_\zeta \mathcal{U}(t - t') \int \langle \phi_j, U_{g(l)}^{\zeta*} U_{g(l')}^{\zeta} \phi_j \rangle \langle \eta(l'), \psi(t') \rangle dg(l')
\]

\[
= \sum_{j, n=1}^{d_\zeta} d_\zeta \langle \eta, \phi_n \rangle \mathcal{U}(t - t') \int \langle \phi_j, U_{g(l)}^{\zeta*} U_{g(l')}^{\zeta} \phi_j \rangle \sqrt{d_\zeta} U_{g(l')}^{\zeta} \phi_n, \psi(t') \rangle dg(l')
\]

\[
= \mathcal{U}(t - t') \sqrt{d_\zeta} U_{g(l)}^{\zeta*} \sum_{n=1}^{d_\zeta} \langle \phi_n, \eta \rangle \phi_n, \psi(t') \rangle
\]

\[
= [C_\eta \exp(-i(t - t') \mathcal{H}(X_k)] \psi(t')](l)
\]

\[
= \psi_\eta(l, t).
\]

Therefore,

\[
\psi_\eta(l, t) = \int K_{\mathbf{H}_\zeta}(l, t; l', t') \psi_\eta(l', t') dg(l') \ \forall \eta \in \mathbf{H}_\zeta,
\]

i.e., the propagator \( K_{\mathbf{H}_\zeta}(l, t; l', t') \) propagates the elements of any reproducing kernel Hilbert space \( L^2_\eta(G) \) correctly. \( \square \)
Hence, we have succeeded in constructing for the irreducible representation $U^\zeta_{g(l)}$ a propagator $K_{H^\zeta}$ that correctly propagates each element of an arbitrary reproducing kernel Hilbert space $L^2_\eta(G)$, i.e., we have succeeded in constructing a *fiducial vector independent propagator* for a fixed irreducible unitary representation of $G$. Using the fact that the set $\{\phi_j\}_{j=1}^{d^\zeta}$ is an ONB one can rewrite the group character $\chi^\zeta(g^{-1}(l)g(l'))$ in terms of the matrix elements $D^\zeta_{ij}(l) = \langle \phi_i, U^\zeta_{g(l)}\phi_j \rangle$ of $U^\zeta$ as follows,

$$\chi^\zeta(g^{-1}(l)g(l')) = \sum_{i,j=1}^{d^\zeta} D^\zeta_{ij}(l) D^\zeta_{ij}(l').$$

(4.9)

Therefore, $K_{H^\zeta}$ can be written alternatively as

$$K_{H^\zeta}(l, t; l', t') = U(t - t') \sum_{i,j=1}^{d^\zeta} d^\zeta D^\zeta_{ij}(l) D^\zeta_{ij}(l').$$

(4.10)

In this construction the unitary irreducible representation $U^\zeta_{g(l)}$ was arbitrary, hence one can introduce such a propagator for each inequivalent unitary representation of $G$, i.e. one can write down the following propagator for the left regular representation $\Lambda_{g(l)}$ of $G$ on $L^2(G)$

$$K(l, t; l', t') = \sum_{\zeta \in \hat{G}} K_{H^\zeta}(l, t; l', t') = U(t - t') \sum_{\zeta \in \hat{G}, i,j=1}^{d^\zeta} d^\zeta D^\zeta_{ij}(l) D^\zeta_{ij}(l').$$

Now it is well known from the Peter-Weyl Theorem that the functions

$$\sqrt{d^\zeta} D^\zeta_{ij}(l), \quad \zeta \in \hat{G}, \quad 1 \leq i,j \leq d^\zeta,$$

form a complete orthonormal system (ONS) in $L^2(G)$ (cf. [7, Theorem 7.2.1]). The completeness relation of this ONS is given by

$$\sum_{\zeta \in \hat{G}, i,j=1}^{d^\zeta} d^\zeta D^\zeta_{ij}(l) D^\zeta_{ij}(l') = \delta_\varepsilon(g^{-1}(l)g(l')),$$

where the sum holds as a weak sum and $\delta_\varepsilon(g^{-1}(l)g(l'))$ is defined as

$$\delta_\varepsilon(g^{-1}(l)g(l')) = \frac{|G|}{\gamma(l)} \prod_{k=1}^{d} \delta(l^k - l'^k),$$

(4.11)
Therefore, we find as our final result

\[ K(l, t; l', t') = \exp \left[ -i(t - t') \mathcal{H}(-i \nabla_1, l) \right] \delta_e(g^{-1}(l)g(l')) \]  

(4.12)

This propagator, which is a tempered distribution, is clearly independent of the fiducial vector and the representation chosen for the basic kinematical variables \( \{X_k\}_{k=1}^d \). A sufficiently large set of test functions for this propagator is given by \( C(G) \), the set of all continuous functions on \( G \). Hence, we have shown the first part of the following Theorem:

**Theorem 4.2.2** The propagator \( K(l, t; l', t') \) in (4.12) is a propagator for the left regular representation of the compact Lie group \( G \) on \( L^2(G) \), which correctly propagates all elements of any reproducing kernel Hilbert space \( L^2_\eta(G) \), associated with an arbitrary irreducible unitary representation \( U^\zeta_{g(l)} \) of the compact Lie group \( G \), \( \zeta \in \hat{G} \).

**Proof.** To prove the second part of Theorem 4.2.2, let \( U^\zeta_{g(l)} \) and \( \eta \in \mathbf{H}_\zeta \) be arbitrary, then for any \( \psi_\eta(l) \) in some \( L^2_\eta(G) \), associated with \( U^\zeta_{g(l)} \) one clearly has that \( \psi_\eta(l) \in C(G) \). Hence, one can write

\[
\int K(l, t; l', t')\psi_\eta(l', t')dg(l') = \sum_{\zeta \in \hat{G}} \int K_{H_\zeta}(l, t; l', t')(\sqrt{d_\zeta} U^\zeta_{g(l')}\eta, \psi(t'))dg(l') \\
= \int K_{H_\zeta}(l, t; l', t')\psi_\eta(l', t')dg(l') \\
= \psi_{\eta_\zeta}(l, t).
\]

The second equality holds since the elements of different representation spaces are mutually orthogonal, hence, only the \( \zeta \)-term remains. In the last step Lemma 4.2.1 has been used. □

Hence, for any compact Lie group \( G \) we have constructed a propagator that is independent of the chosen irreducible unitary representation of \( G \). We call this propagator a representation independent propagator for \( G \).

We will see in section 4.4 that the representation independent propagator for compact Lie groups can be given the following d-dimensional lattice phase-space path
integral representation (see Proposition 4.4.4 or [98]):

\[
K(l'', t''; l', t') = \frac{|G|}{\sqrt{\gamma(l'') \gamma(l')}} \lim_{N \to \infty} \int \ldots \int \sum_{\{p\}} \\
\times \exp \left\{ i \sum_{j=0}^{N} \left[ p_{j+1/2} \cdot (l_{j+1} - l_j) - \epsilon \mathcal{H}(\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j)) \right] \right\} \\
\times \prod_{j=1}^{N} dl_{j+1} \ldots dl_j,
\]

(4.13)

where \( l_{N+1} = l'' \), \( l_0 = l' \) and \( \epsilon = (t'' - t')/(N + 1) \). The sum \( \sum_{\{p\}} \) appearing in the above expression is defined as

\[
\sum_{\{p\}} = \frac{1}{K} \sum_{p_{N+1/2}} \frac{1}{K} \sum_{p_{N-1/2}} \ldots \frac{1}{K} \sum_{p_0/2} \frac{1}{K} \sum_{p_{1/2}}
\]

the sums are over the spectrum of the operator \(-i\hat{\mathcal{V}}_1\), defined in section 4.4.2, and where \( K \) is the appropriate normalization constant such that

\[
\frac{|G|}{K \sqrt{\gamma(l'') \gamma(l')}} \sum_{p_1, \ldots, p_k} \exp \left[ i \sum_{k=1}^{d} p_k (l''^k - l^k) \right] = \delta_{\epsilon(g^{-1}(l'')(g(l')))},
\]

The arguments of the Hamiltonian in (4.13) are given by the following functions:

\[
\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j) = \sum_{m=1}^{d} \rho^{-1}_k m \left( g(l_{j+1}) \right) + \rho^{-1}_k m \left( g(l_j) \right) p_m j_{j+1/2}, \quad k = 1, \ldots, d.
\]

Remark 4.2.1 Observe, that the lattice expression for the representation independent propagator exhibits the correct time reversal symmetry, which means that

\[
\overline{K}(l'', t''; l', t') = K(l', t'; l'', t'').
\]

Also note that in the construction of the representation independent propagator for compact Lie groups and its path integral representation no explicit use is made of the ONS \( \sqrt{d_\zeta} D_j \zeta(l) \), \( \zeta \in \hat{G} \) and \( i, j = 1, \ldots, d_\zeta \), in \( L^2(G) \) whose existence is guaranteed by the Peter-Weyl Theorem, but merely the facts that it exists and is complete are used. Moreover, we have made no assumptions about the nature of the physical systems we are considering, other than that its Hamilton operator be self-adjoint. Therefore, the
path integral representation (4.13) can be used to describe the motion of a general physical system, not just that of a free particle, on the group manifold of any compact Lie group and it does not matter if the $D^\zeta_{ij} (l)$ are explicitly known or not. Hence, (4.13) represents a clear improvement over the path integral formulations describing the motion of a free particle on a group manifold presented in chapter 2.

4.3 Example: The Representation Independent Propagator for SU(2)

While the Peter-Weyl Theorem assures that the ONS $\sqrt{d^\zeta} D^\zeta_{ij} (l)$, $\zeta \in \check{G}$ and $i, j = 1, \ldots, d^\zeta$ exists and is complete, the construction of such a set is frequently a difficult task. The functions $\sqrt{d^\zeta} D^\zeta_{ij} (l)$ are known only for a limited class of groups and will now be constructed for SU(2). It turns out that this is an exercise in harmonic analysis. We will now explicitly describe the maximal set of commuting operators in $L^2(SU(2))$. We will take the set of infinitely often differentiable functions, $C^\infty(SU(2))$, as their common dense invariant domain. Since SU(2) is a rank one group, there exists one two-sided invariant operator $C_1$ in the center of the enveloping algebra $\mathcal{E}$ of SU(2). Moreover, since SU(2) is compact the maximal set of commuting right (left) invariant differential operators in the right (left) invariant enveloping algebra $\mathcal{E}^R \ (\mathcal{E}^L)$, can be associated with the Casimir operator of the maximal subgroup $U(1)$ of SU(2).

Let $S_1$, $S_2$, and $S_3$ be an arbitrary irreducible representation of the Lie algebra $su(2)$ by self-adjoint operators satisfying the commutation relations

$$[S_i, S_j] = i \sum_{k=1}^{3} \epsilon_{ijk} S_k.$$ 

Since the Casimir operator of SU(2), $C_1 \equiv \sum_{j=1}^{3} S_j^2$, commutes with all the generators of the Lie algebra $su(2)$, its eigenspaces are invariant under the Lie algebra, and all vectors of an irreducible representation must be eigenvectors of $C_1$ with a fixed eigenvalue. We denote by $H^\zeta$ the $(2\zeta + 1)$-dimensional complex eigenspace corresponding to the eigenvalue $\zeta(\zeta + 1)$, $\zeta = 0, 1/2, 1, \ldots$. We denote the pairwise non-equivalent
irreducible representations of $SU(2)$ on any of the $H_\zeta$ by $U_\zeta$. One can show that every irreducible representation $U$ of $SU(2)$ is equivalent to one of the representations $U_\zeta$, $\zeta = 0, 1/2, 1, \ldots$, (cf. [103, Theorem III.5.1]).

For $SU(2)$ in the Euler angle parameterization an arbitrary unitary irreducible representation of $SU(2)$ is given by

$$U_{g(\theta, \phi, \xi)}^\zeta = \exp(-i\phi S_3) \exp(-i\theta S_2) \exp(-i\zeta S_3),$$

where the domain of the parameters $\theta$, $\phi$, and $\xi$ is given by

$$0 < \theta < \pi, \quad 0 \leq \phi < 2\pi, \quad -2\pi \leq \xi < 2\pi.$$

With this choice of parameterization of $SU(2)$ the operators $\{\tilde{s}_k\}_{k=1}^3$ defined in Lemma 4.1.4 (i) are given by:

\begin{align*}
\tilde{s}_1(-i\partial_\theta, -i\partial_\phi, -i\partial_\xi, \theta, \phi, \xi) &= i \sin \phi \partial_\theta + i \cot \theta \cos \phi \partial_\phi - i \cos \phi \csc \theta \partial_\xi, \\
\tilde{s}_2(-i\partial_\theta, -i\partial_\phi, -i\partial_\xi, \theta, \phi, \xi) &= -i \cos \phi \partial_\theta + i \cot \theta \sin \phi \partial_\phi - i \sin \phi \csc \theta \partial_\xi, \\
\tilde{s}_3(-i\partial_\theta, -i\partial_\phi, -i\partial_\xi, \theta, \phi, \xi) &= -i \partial_\phi.
\end{align*}

(4.14)

By Corollary 4.1.5 these operators can be identified with the generators of a subrepresentation of the left regular representation of $SU(2)$, (i.e. belong to the right invariant Lie algebra of $SU(2)$). Similarly the operators $\{\hat{s}_k\}_{k=1}^3$ defined in Lemma 4.1.4 (ii) are given by:

\begin{align*}
\hat{s}_1(i\partial_\theta, i\partial_\phi, i\partial_\xi, \theta, \phi, \xi) &= i \sin \xi \partial_\theta - i \csc \theta \cos \xi \partial_\phi + \cot \theta \cos \xi \partial_\xi, \\
\hat{s}_2(i\partial_\theta, i\partial_\phi, i\partial_\xi, \theta, \phi, \xi) &= i \cos \xi \partial_\theta + i \csc \theta \sin \xi \partial_\phi - i \cot \theta \sin \xi \partial_\xi, \\
\hat{s}_3(i\partial_\theta, i\partial_\phi, i\partial_\xi, \theta, \phi, \xi) &= i \partial_\xi.
\end{align*}

(4.15)

and can be identified with the generators of a subrepresentation of the right regular representation, (i.e. belong to the left invariant Lie algebra of $SU(2)$). From (4.14) and (4.15) we easily identify the Casimir operators of $U(1)$ as

$$A_1 = i \partial_\phi \quad B_1 = i \partial_\xi.$$ 

(4.16)
For the Casimir operator of SU(2) one finds

\[ C_1 = -(1 - z^2)\partial_z^2 + 2z\partial_z - \frac{1}{1 - z^2}(\partial_\phi^2 - 2z\partial_\phi + \partial_\xi^2), \quad (4.17) \]

where \( z = \cos \theta \) and the identity \(-\sin \theta \partial_{\cos \theta} = \partial_\theta \) has been used. Since \( C_1 \) commutes with all elements of the enveloping algebra \( \mathcal{E} \) and \( U^\zeta_{g(\theta,\phi,\xi)} \) is irreducible, \( C_1 \) is a multiple of the identity on any one of the reproducing kernel Hilbert spaces \( L^2_\eta(SU(2)) \) associated with the irreducible representation \( U^\zeta_{g(\theta,\phi,\xi)} \), i.e.

\[ C_1 = \zeta(\zeta + 1)I_{L^2_\eta(SU(2))} \quad (4.18) \]

Let \( \{\psi_n\}_{n=0}^\infty \) be an orthonormal basis in \( H_\zeta \), then we can associate with each irreducible representation \( U^\zeta_{g(\theta,\phi,\xi)} \), where \( \zeta = 0, 1/2, 1, \ldots \), the following matrix elements

\[ D^\zeta_{mn}(\theta, \phi, \xi) = \langle \psi_m, U^\zeta_{g(\theta,\phi,\xi)} \psi_n \rangle, \quad -\zeta \leq m, n \leq \zeta. \]

We shall now determine the matrix elements \( D^\zeta_{mn}(\theta, \phi, \xi) \) as the common eigenfunctions of the operators \( A_1, B_1, C_1 \). Equations (4.16) and (4.17) suggest that the common eigenfunctions of the operators \( A_1, B_1, \) and \( C_1 \) are of the form

\[ D^\zeta_{mn}(\theta, \phi, \xi) = e^{-i(m\phi + n\xi)} P^\zeta_{mn}(\cos \theta). \]

Using this form of \( D^\zeta_{mn}(\theta, \phi, \xi) \) in (4.18) one finds:

\[-(1 - z^2) \frac{d^2}{dz^2} P^\zeta_{mn}(z) + 2z \frac{d}{dz} P^\zeta_{mn}(z) + \frac{1}{1 - z^2}(m^2 + n^2 - 2mnz) = \zeta(\zeta + 1) P^\zeta_{mn}(z). \]

The functions \( P^\zeta_{mn}(z) \), which are known as the Wigner functions, are given by

\[ P^\zeta_{mn}(z) = \frac{1}{2^m} \sqrt{\frac{(\zeta - m)!(\zeta + m)!}{(\zeta - n)!(\zeta + n)!}} (1 - z)^{\frac{m-n}{2}} (1 + z)^{\frac{m+n}{2}} P^\zeta_{-m,m+n}(z), \]

where \( P^\zeta_{-m,m+n}(z) \) are Jacobi polynomials, (see [103, p. 125]). Also observe that \( P^\zeta_{mn}(z) = (-1)^{n-m} P^\zeta_{mn}(z) \). Therefore, one finds for the matrix elements of the irreducible representation \( U^\zeta_{g(\theta,\phi,\xi)} \) the following:

\[ D^\zeta_{mn}(\theta, \phi, \xi) = e^{-i(m\phi + n\xi)} P^\zeta_{mn}(\cos \theta), \]
as pointed out above these functions form a complete ONS on \( L^2(SU(2)) \). The completeness relation for this ONS takes the form, see (4.11),

\[
\sum_{\zeta \in \mathcal{G}} \sum_{m, n = -\zeta}^{\zeta} (2\zeta + 1) \frac{D_{mn}^{\zeta}(\theta'', \phi'', \xi'') D_{mn}^{\zeta}(\theta', \phi', \xi')}{\sin \theta''} \delta(\theta'' - \theta') \delta(\phi'' - \phi') \delta(\xi'' - \xi').
\]

By equation (4.12) the representation independent propagator for SU(2) is then found to be:

\[
K(t'', t'; l', t') = 16\pi^2 \exp[-i(t'' - t') \mathcal{H}(\tilde{s}_2(-i\nabla_l, l), \tilde{s}_3(-i\nabla_l, l))] \\
\times \frac{1}{\sin \theta''} \delta(\theta'' - \theta') \delta(\phi'' - \phi') \delta(\xi'' - \xi'),
\]

where \( l = (\theta, \phi, \xi) \) and \( \nabla_l = (\partial_\theta, \partial_\phi, \partial_\xi) \). By Equation 4.13 the regularized lattice phase-space path integral representation for the representation independent propagator for SU(2) is given by

\[
K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') = \\
\frac{16\pi^2}{\sqrt{\sin \theta'' \sin \theta'}} \lim_{N \to \infty} \int \cdots \int \sum_{\{\alpha, \beta, \gamma\}} \exp\left\{i \sum_{j=0}^{N} \left[ \alpha_{j+1/2}(\theta_{j+1} - \theta_j) + \beta_{j+1/2}(\phi_{j+1} - \phi_j) + \gamma_{j+1/2}(\xi_{j+1} - \xi_j) - c\mathcal{H}(\tilde{s}_k(p_{j+1/2}; l_{j+1}, l_j)) \right] \right\} d\theta_j d\phi_j d\xi_j,
\]

where,

\[
\tilde{s}_1(p_{j+1/2}; l_{j+1}, l_j) = -\frac{1}{2}(\sin \phi_{j+1} + \sin \phi_j) \alpha_{j+1/2} \\
\quad - \frac{1}{2}(\cot \theta_{j+1} \cos \phi_{j+1} + \cot \theta_j \cos \phi_j) \beta_{j+1/2} \\
\quad + \frac{1}{2}(\cos \phi_{j+1} \csc \theta_{j+1} + \cos \phi_j \csc \theta_j) \gamma_{j+1/2},
\]

\[
\tilde{s}_2(p_{j+1/2}; l_{j+1}, l_j) = \frac{1}{2}(\cos \phi_{j+1/2} + \cos \phi_j) \alpha_{j+1/2} \\
\quad - \frac{1}{2}(\cot \theta_{j+1} \sin \phi_{j+1} + \cot \theta_j \sin \phi_j) \beta_{j+1/2} \\
\quad + \frac{1}{2}(\csc \theta_{j+1} \sin \phi_{j+1} + \csc \theta_j \sin \phi_j) \gamma_{j+1/2},
\]

\[
\tilde{s}_3(p_{j+1/2}; l_{j+1}, l_j) = \beta_{j+1/2}.
\]

As an example let us calculate this propagator for the following two Hamilton operators
(i) $\mathcal{H}(\bar{s}_1(-i\nabla, l), \bar{s}_2(-i\nabla, l), \bar{s}_3(-i\nabla, l)) = \frac{1}{2I} \bar{s}_3^2$.

(ii) $\mathcal{H}(\bar{s}_1(-i\nabla, l), \bar{s}_2(-i\nabla, l), \bar{s}_3(-i\nabla, l)) = \frac{1}{2I} (\bar{s}_1^2 + \bar{s}_2^2 + \bar{s}_3^2)$.

4.3.1 The Hamilton Operator $\mathcal{H}(\bar{s}_1, \bar{s}_2, \bar{s}_3) = \frac{1}{2I} \bar{s}_3^2$

As announced in chapter 2 we now revisit the free particle moving on a circle and present its exact path integral treatment. The Hamiltonian $\mathcal{H}(\bar{s}_1, \bar{s}_2, \bar{s}_3) = \beta^2/2I$ describes a free particle moving on a circle with fixed axis, like a bead on a hoop. We analyze this problem in two steps. First we proceed naively, assuming that the Hamilton operator is self-adjoint. In particular we assume that $\bar{s}_3 = -i\partial_\phi$ is self-adjoint on $L^2([0, 2\pi])$ and has a spectrum of the form $\beta = n$, $n = 0, \pm 1, \ldots$. Then in a second step we reexamine this assumption and show that $\bar{s}_3 = -i\partial_\phi$ self-adjoint with spectrum $\beta = n$ is only one particular choice of uncountably many. With this choice of Hamilton operator the representation independent propagator takes the form

$$K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') =$$

$$= \frac{16\pi^2}{\sin \theta'' \sin \theta'} \lim_{N \to \infty} \int \cdots \int \sum_{\{\alpha, \beta, \gamma\}} \exp\left\{ i \sum_{j=0}^{N} [\alpha_{j+1/2}(\theta_{j+1} - \theta_j) + \beta_{j+1/2}(\phi_{j+1} - \phi_j)$$

$$+ \gamma_{j+1/2}(\xi_{j+1} - \xi_j) - \frac{\beta^2_{j+1/2}}{2I}] \right\} \prod_{j=1}^{N} d\theta_j d\phi_j d\xi_j$$

$$= \frac{16\pi^2}{\sin \theta'' \sin \theta'} \delta(\theta'' - \theta') \delta(\xi'' - \xi') \lim_{N \to \infty} \int \cdots \int \left( \frac{1}{2\pi} \right)^{N+1}$$

$$\times \sum_{\beta_{N+\frac{1}{2}} = -\infty}^{\infty} \cdots \sum_{\beta_{\frac{1}{2}} = -\infty}^{\infty} \exp\left\{ i \sum_{j=0}^{N} [\beta_{j+1/2}(\phi_{j+1} - \phi_j) - \frac{\epsilon}{2I} \beta^2_{j+1/2}] \right\} \prod_{j=1}^{N} d\phi_j.$$

This last integral can be evaluated as follows

$$\lim_{N \to \infty} \int \cdots \int \left( \frac{1}{2\pi} \right)^{N+1} \sum_{\beta_{N+1/2} = -\infty}^{\infty} \cdots \sum_{\beta_{1/2} = -\infty}^{\infty} \exp\left\{ i \sum_{j=0}^{N} [\beta_{j+1/2}(\phi_{j+1} - \phi_j)$$

$$- \frac{\epsilon}{2I} \beta^2_{j+1/2}] \right\} \prod_{j=1}^{N} d\phi_j$$

$$= \lim_{N \to \infty} \left( \frac{1}{2\pi} \right)^{N+1} \sum_{\beta_{N+1/2}} \cdots \sum_{\beta_{1/2}} \int \cdots \int \exp\{i[\beta_{N+1/2}\phi'' - \beta_{1/2}\phi']$

$$- \frac{\epsilon}{2I} \beta^2_{j+1/2}] \right\} \prod_{j=1}^{N} d\phi_j$$
\[- \sum_{j=1}^{N} \phi_j (\beta_{j+1/2} - \beta_{j-1/2}) - \frac{\epsilon}{2I} \sum_{j=0}^{N} \beta_{j+1/2}^2 \] \prod_{j=1}^{N} d\phi_j

= \lim_{N \to \infty} \frac{1}{2\pi} \sum_{\beta_{1/2}} \cdots \sum_{\beta_{N+1/2}} \exp \left\{ i \left[ \beta_{N+1/2} \phi'' - \beta_{1/2} \phi' - \frac{\epsilon}{2I} \sum_{j=0}^{N} \beta_{j+1/2}^2 \right] \right\} \prod_{j=1}^{N} \delta_{\beta_{j+1/2} \beta_{j-1/2}}

= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \exp \left\{ i \left[ \beta (\phi'' - \phi') - \frac{T}{2I} \beta^2 \right] \right\}.

Hence we find for the representation independent propagator

\[ K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') = \frac{8\pi}{\sin \theta''} \delta(\theta'' - \theta') \delta(\xi'' - \xi') \times \sum_{n=-\infty}^{\infty} \exp \left\{ i \left[ n(\phi'' - \phi') - \frac{T}{2I} n^2 \right] \right\}. \]

The sum over n is related to the Jacobi theta function,

\[ \theta_3(z, t) = \sum_{n=-\infty}^{\infty} \exp(i\pi tn^2 + 2inz). \]

Therefore, with the following identifications we can write the representation independent propagator in closed form. Let

\[ t \equiv -\frac{T}{2\pi I}; \quad z \equiv \frac{(\phi'' - \phi')}{2}, \]

then our final result for the representation independent propagator becomes

\[ K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') = \frac{8\pi}{\sin \theta''} \delta(\theta'' - \theta') \delta(\xi'' - \xi') \theta_3 \left[ \frac{(\phi'' - \phi')}{2}, \frac{-T}{2\pi I} \right]. \]

This result agrees with the one found by Schulman [94] expect for an arbitrary phase factor.

We now follow our analysis in Ref. 99, section III.c. It is well known that the symmetric operator \(-i\partial_\phi\) on \(L^2([0, 2\pi])\) with domain

\[ \mathcal{D}(-i\partial_\phi) = \{ \psi : \int_0^{2\pi} (|\psi|^2 + |\psi'|^2) d\phi < \infty, \psi(0) = 0 = \psi(2\pi) \} \]

has deficiency indices (1,1), hence, it can be extended to a self-adjoint operator but not uniquely. In fact there exists a one-parameter family of self-adjoint extensions
which we denote by $-i\partial^\delta_\phi = -i\partial_\phi$ with the domain

$$
\mathcal{D}(-i\partial^\delta_\phi) = \{ \psi : \int_0^{2\pi} (|\psi|^2 + |\psi'|^2) d\phi < \infty, \psi(2\pi) = e^{i\delta}\psi(0) \},
$$

where $\delta \in [-\pi, \pi)$ (see [87, pp. 257-259]). Note that the choice $\delta = 0$ corresponds to the case of periodic boundary conditions, which we have assumed above. The spectrum of each $-i\partial^\delta_\phi$ is straightforwardly found as follows, let $\lambda \in \mathbb{R}$ then

$$
-i\partial^\delta_\phi \psi(\phi) = \lambda \psi(\phi)
$$

this implies that the eigenfunctions are given by

$$
\psi(\phi) = e^{i\lambda \phi}.
$$

Fitting the boundary conditions $\psi(2\pi) = e^{i\delta}\psi(0)$, yields the following set of eigenvalues,

$$
\lambda_n = \left( n + \frac{\delta}{2\pi} \right) \quad n \in \mathbb{Z} = \{ \ldots, -2, -1, 0, +1, +2, \ldots \}
$$

Therefore, the spectrum of $-i\partial^\delta_\phi$ is given by

$$
\text{spec}(-i\partial^\delta_\phi) = \left\{ \lambda : \lambda = \left( n + \frac{\delta}{2\pi} \right), \ \delta \in [-\pi, \pi) \text{ and } n \in \mathbb{Z} \right\}.
$$

Hence, the choice of periodic boundary conditions is only one of uncountable many possibilities. If we choose instead the boundary conditions $\psi(2\pi) = e^{i\delta}\psi(0)$, where $\delta \in [-\pi, \pi)$ is arbitrary, our expression for the representation independent propagator becomes

$$
K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t')
\begin{align*}
&= \frac{8\pi}{\sin \theta''} \delta(\theta'' - \theta') \delta(\xi'' - \xi') \\
&\times \sum_{n=-\infty}^{\infty} \exp \left\{ i \left[ \left( n + \frac{\delta}{2\pi} \right) (\phi'' - \phi') - \frac{T}{2I} \left( n + \frac{\delta}{2\pi} \right)^2 \right] \right\} \\
&= \frac{8\pi}{\sin \theta''} \delta(\theta'' - \theta') \delta(\xi'' - \xi') \exp \left[ \frac{i\delta(\phi'' - \phi')}{2\pi} - \frac{iT\delta^2}{8\pi^2 I} \right] \\
&\times \sum_{n=-\infty}^{\infty} \exp \left\{ i \left[ n \left( \frac{\phi'' - \phi'}{2\pi I} - \frac{T\delta}{2I} \right) - \frac{Tn^2}{2I} \right] \right\}
\end{align*}
$$
Therefore, with the following identifications we can write the representation independent propagator again in closed form. Let

\[ t \equiv \frac{-T}{2\pi I} ; \quad z \equiv \frac{(\phi'' - \phi')}{2} - \frac{T\delta}{4\pi I}, \]

then our final result for the representation independent propagator with arbitrary \( \delta \) becomes

\[
K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') = \frac{8\pi}{\sin \theta''} \delta(\theta'' - \theta') \delta(\xi'' - \xi') \times \exp \left[ \frac{i\delta(\phi'' - \phi')}{2\pi} - \frac{iT\delta^2}{8\pi^2 I} \right] \theta_3 \left[ \frac{(\phi'' - \phi')}{2} - \frac{T\delta}{4\pi I}, \frac{-T}{2\pi I} \right].
\]

This result exhibits the same \( \phi \)-dependence as does the one found Schulman [94], which also encompasses all spins.

4.3.2 The Hamilton Operator \( \mathcal{H}(\vec{s}_1, \vec{s}_2, \vec{s}_3) = \frac{1}{2I}(\vec{s}_1^2 + \vec{s}_2^2 + \vec{s}_3^2) \)

Our second example is that of the Hamilton operator \( \mathcal{H}(\vec{s}_1, \vec{s}_2, \vec{s}_3) = C_1(\theta, \phi, \xi) \), where \( C_1(\theta, \phi, \xi) \) is the Casimir operator of \( \text{SU}(2) \) given in (4.17). Note that the Hamilton operator \( \mathcal{H}(\vec{s}_1, \vec{s}_2, \vec{s}_3) = C_1(\theta, \phi, \xi) \) is essentially self-adjoint since \( C_1(\theta, \phi, \xi) \) is a symmetric and elliptic central element of the enveloping algebra \( \mathfrak{e} \) of \( \text{SU}(2) \), (cf. [78, Corollary VI.3.1]). This Hamiltonian describes the motion of a free particle on the group manifold of \( \text{SU}(2) \). With this choice of the Hamilton operator the representation independent propagator becomes:

\[
K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') = 16\pi^2 \exp \left[ -i \frac{T}{2I} C_1(\theta'', \phi'', \xi'') \right] \frac{1}{\sin \theta''} \delta(\theta'' - \theta') \delta(\phi'' - \phi') \delta(\xi'' - \xi') \times \sum_{\zeta \in \mathcal{G}} \sum_{m,n=-\zeta} (2\zeta + 1) D_{mn}(\theta'', \phi'', \xi'') D_{mn}(\theta', \phi', \xi') \chi_{\zeta}(g^{-1}(\theta'', \phi'', \xi'')g(\theta', \phi', \xi')).
\]

where \( T \equiv t'' - t' \) and equations (4.8) and (4.18) have been used. Here \( \chi_{\zeta}(g) \) denotes the character for the representation \( U_{g(\theta, \phi, \xi)}^\zeta \), i.e. if one denotes the Euler angles of
the element $g^{-1}(\theta'', \phi'', \xi'')g(\theta', \phi', \xi')$ by $(\bar{\theta}, \bar{\phi}, \bar{\xi})$ one finds:

$$
\chi_{\zeta}(g(\bar{\theta}, \bar{\phi}, \bar{\xi})) = \sum_{m=-\zeta}^{\zeta} \exp[-im(\bar{\phi} + \bar{\xi})]P_{m\zeta}(\cos \bar{\theta})
$$

Observe that, the character of the group can be expressed as a function of a single variable as follows. It is well known that the character as a function of the group is constant on conjugacy classes, i.e. for any two elements $g$ and $g_1$ one has

$$
\chi_{\zeta}(g_1gg_1^{-1}) = \chi_{\zeta}(g)
$$

Therefore, to show that $\chi_{\zeta}(g)$ is a function of one variable, it is sufficient to show that the conjugacy classes of SU(2) can be labeled by a single parameter. As is well known from linear algebra any unitary unimodular $2 \times 2$ matrix $g$ can be written as $g = g_1 \gamma g_1^{-1}$, where $g_1 \in SU(2)$ and $\gamma$ is of the following diagonal matrix

$$
\gamma = \begin{pmatrix}
e^{i(\Gamma/2)} & 0 \\
0 & e^{-i(\Gamma/2)}
\end{pmatrix}
$$

Furthermore, among all matrices equivalent to $g$ there exists only one other diagonal matrix $\gamma'$ obtained from $\gamma$ by complex conjugation. Therefore, each conjugacy class of elements of SU(2) is labeled by one parameter $\Gamma$, ranging from $-2\pi \leq \Gamma \leq 2\pi$ and where $\Gamma$ and $-\Gamma$ give the same class. Hence, the characters $\chi_{\zeta}(g)$ can be regarded as functions of one variable $\Gamma$ that varies between 0 and $2\pi$. The geometrical meaning of the parameter $\Gamma$ is that it is equal to the angle of rotation corresponding to the matrix $g$. In terms of the Euler angles $(\theta'', \phi'', \xi'')$ and $(\theta', \phi', \xi')$ $\Gamma$ is given by

$$
\Gamma = \arccos[\cos(\theta'' - \theta') \cos(\phi'' - \phi') \cos(\xi'' - \xi') - \cos(\theta'' + \theta') \sin(\phi'' - \phi') \sin(\xi'' - \xi')]
$$

(4.19)

One can derive an explicit formula for $\chi_{\zeta}(g)$ as a function of $\Gamma$. Note that the matrix $U^\zeta_{\gamma(0,\Gamma,0)}$ that corresponds to $\gamma \in SU(2)$ is given by the diagonal matrix of rank $2\zeta + 1$ having diagonal elements $e^{-ia\Gamma}$, $-\zeta \leq a \leq \zeta$. Now let $g = g_1 \gamma g_1^{-1}$, then

$$
\chi_{\zeta}(g) = \text{tr}[U^\zeta_{\gamma(0,\Gamma,0)}] = \sum_{a=-\zeta}^{\zeta} e^{-ia\Gamma} = \frac{\sin(\zeta + 1/2)\Gamma}{\sin \Gamma/2},
$$
Hence, the group character can be written as

$$
\chi_\zeta(g^{-1}(\theta'', \phi'', \xi'')g(\theta', \phi', \xi')) = \frac{\sin(\zeta + 1/2)\Gamma}{\sin \Gamma / 2},
$$

where $\Gamma$ is given in (4.19). Therefore, one finds for the representation independent propagator

$$
K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') =
\begin{align*}
&= \frac{16\pi^2}{\sqrt{\sin \theta'' \sin \theta'}} \lim_{\epsilon \to 0} \int \ldots \sum_{(\alpha, \beta, \gamma)} \exp\{i \sum_{j=0}^{N} [\alpha_{j+1/2}(\theta_{j+1} - \theta_j) + \beta_{j+1/2}(\phi_{j+1} - \phi_j) \\
&\quad + \gamma_{j+1/2}(\xi_{j+1} - \xi_j) - \epsilon C_1(\theta_{j+1}, \phi_{j+1}, \xi_{j+1}; \theta_j, \phi_j, \xi_j)]\} \prod_{j=1}^{N} d\theta_j d\phi_j d\xi_j \\
&= 16\pi^2 \sum_{\zeta \in C} (2\zeta + 1) \exp \left[ -i \frac{(t'' - t')}{2\Gamma} \zeta (\zeta + 1) \right] \frac{\sin(\zeta + 1/2)\Gamma}{\sin \Gamma / 2}.
\end{align*}
$$

This result agrees with the one found by Schulman [94] which was obtained by the methods mentioned in chapter 2.

4.4 The Representation Independent Propagator for General Lie Groups

4.4.1 Construction of the Representation Independent Propagator

Now let $G$ be a general Lie group. Let us again denote by $U$ an arbitrary, fixed, square integrable unitary representation of $G$. Then it is a direct consequence of Lemma 4.1.4 (i) that for any $\psi \in \tilde{D}$

$$
\tilde{x}_k(-i\nabla_l, l)[C_\eta \psi](l) = [C_\eta \overline{X}_k \psi](l), \quad k = 1, \ldots, d.
$$

holds independently of $\eta$. Therefore, the isometric isomorphism $C_\eta$ intertwines the representation of the Lie algebra $L$ on $H$, with a subrepresentation of $L$ by right-invariant, essentially self-adjoint differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G)$. To summarize, we found in section 4.1 that any square integrable representation $U$ of $G$ is unitarily equivalent to a subrepresentation of the left regular representation $\Lambda$ on $L^2_\eta(G)$. Furthermore, the generators of $G$ are represented by right invariant, essentially self-adjoint differential operators on $L^2_\eta(G)$. 
Let \((\pi, H)\) be a representation of \(G\), then we denote by \(A(\pi)\) the von Neumann algebra generated by the operators \(\pi_g, g \in G\) (cf. Appendix A.2). By Proposition 5.6.4 in Ref. 25 there exists a projection operator \(P_I\) in the center of the von Neumann algebra \(A(\Lambda)\) such that the restriction \(\Lambda_I\) of \(\Lambda\) to the closed subspace \(P_I[L^2(G)]\) of \(L^2(G)\) is of type I, and such that the restriction of \(\Lambda\) to the orthogonal complement of \(P_I[L^2(G)]\) has no type I part. Since \(G\) is separable and locally compact there exists by Theorem 5.1 in Ref. 30 a standard Borel measure \(\nu\) on \(\hat{G}\), the set of all inequivalent irreducible unitary representations of \(G\), and a \(\nu\)-measurable field \((U^\xi, H^\xi)_{\xi \in \hat{G}}\) of unitary representations of \(G\), such that the type I part of \(\Lambda, \Lambda_I\), can be decomposed into a direct integral,

\[
\Lambda_I = \int_{\hat{G}} U^\xi \otimes I_\xi \, d\nu(\xi),
\]

where \(U^\xi \otimes I_\xi\) is a representation of \(G \times G\) on \(H^\xi \otimes H^\xi\).

Denote by \(\mathcal{H}(\bar{X}_k)\) the essentially self-adjoint Hamilton operator of a quantum system on \(H^\xi\). Then the continuous representation of the solution to Schrödinger's equation, \(\psi(t) = \exp[-i(t - t')\mathcal{H}(\bar{X}_k)]\psi(t')\), takes, on \(L^2_\eta(G)\), the following form

\[
\psi_\eta(l, t) = \int K_\eta(l, t; l', t') \psi_\eta(l', t') \, dg(l')
\]

where,

\[
K_\eta(l, t; l', t') = \langle \eta(l), \exp[-i(t - t')\mathcal{H}(\bar{X}_k)]\eta(l') \rangle \\
= \langle \mathcal{C}_\eta \exp[-i(t - t')\mathcal{H}(\bar{X}_k)]\eta(l'), \eta(l) \rangle \\
= \mathcal{U}(t - t')[\mathcal{C}_\eta \eta(l')](l) \\
= \mathcal{U}(t - t')\langle \eta, K^{1/2}U_g^{-1}(l)_{g(l')}K^{1/2}\eta \rangle \\
= \mathcal{U}(t - t')K_\eta(l; l'),
\]

(4.20)

where,

\[
\mathcal{U}(t - t') = \exp[-i(t - t')\mathcal{H}(\bar{x}_k(-i\nabla_l, l))].
\]

and \(K^{1/2}U_g^{-1}(l)_{g(l')}K^{1/2}\) denotes the closure of the operator \(K^{1/2}U_g^{-1}(l)_{g(l')}K^{1/2}\).
Note that for non-compact Lie groups it is not true that every symmetric Hamilton operator is also essentially self-adjoint, as was the case for compact Lie groups. To illustrate this important fact, we consider the following two examples:

Example 4.4.1: Let $G$ be the non-compact two parameter group of transformations $x' = p^{-1}x - q$, $0 < p < \infty$, $-\infty < q < \infty$ of the real line $\mathbb{R}$ and let $\mathbf{H} = L^2(\mathbb{R}^+)$, where $\mathbb{R}^+ = (0, \infty)$. An irreducible unitary representation of $G$ on $\mathbf{H}$ is given by the formula:

$$(U_{g(p,q)}\phi)(k) = p^{-1/2}e^{-ikq}e^{ipk}, \quad \phi \in \mathbf{H}.$$ 

The generators of the one-parameter unitary subgroups are given by

$$U(X_1) = k, \quad U(X_2) = \frac{i}{2} \left( k \frac{d}{dk} + \frac{d}{dk}k \right).$$

We choose the set $\mathcal{S}^{+}(\mathbb{R}^+)$ as the common dense invariant domain for these operators.

As our first example we consider the operator

$$T_1 = U(X_1^2 + X_2) = k^2 + \frac{i}{2} \left( k \frac{d}{dk} + \frac{d}{dk}k \right),$$

$$\mathcal{D}(T_1) = \mathcal{S}^{+}(\mathbb{R}^+).$$

Clearly the operator $T_1$ is symmetric. To show that $T_1$ is essentially self-adjoint it is necessary and sufficient to show that the kernel of the operator $T_1^* + iI$, $\ker(T_1^* + iI)$, consists only of the zero vector, i.e. $\ker(T_1^* + iI) = \{0\}$. In other words we have to show that the equation:

$$T_1^* \phi_{\pm}(k) = \pm i\phi_{\pm}(k),$$

has no solutions in $\mathbf{H}$ other than $\phi_{\pm}(k) = 0$. One finds the following solution for the above equation:

$$\phi_+(k) = k^{1/2} \exp \left( \frac{i}{2} k^2 \right),$$

$$\phi_-(k) = k^{-3/2} \exp \left( \frac{i}{2} k^2 \right),$$

$$\phi_{\pm}(k) = k^{1/2} \exp \left( \frac{i}{2} k^2 \right),$$

$$\phi_{\pm}(k) = k^{-3/2} \exp \left( \frac{i}{2} k^2 \right),$$
Both of these functions are not in $\mathbf{H}$, since they are not square integrable. The function $|\phi_+(k)|^2$ diverges at infinity and the function $|\phi_-(k)|^2$ diverges at the origin. Therefore, we conclude that $T_1$ is essentially self-adjoint.

As our second example we consider the operator

$$T_2 = U(X_1^2X_2 + X_2X_1^2) = k^2 \left( \frac{i}{2} \left( k \frac{d}{dk} + \frac{d}{dk} k \right) + \frac{i}{2} \left( k \frac{d}{dk} + \frac{d}{dk} k \right) \right)$$

$$= k^2 \left[ k \left( \frac{i}{2} \frac{d}{dk} + \frac{i}{2} \right) + \left( \frac{i}{2} \frac{d}{dk} \right) k - \frac{i}{2} \right] k^2$$

$$= k^3 \left( \frac{i}{2} \frac{d}{dk} + \left( \frac{i}{2} \frac{d}{dk} \right) \right) k^3$$

$$\mathbf{D}(T_2) = \mathbf{S}(\mathbb{R}^*_+)$$.

This operator is clearly symmetric, and one determines the following solutions for the equation $T_2^* \phi_\pm(k) = \pm i \phi_\pm(k)$:

$$\phi_+(k) = k^{-3/2} \exp\left(- \frac{1}{4k^2}\right),$$

$$\phi_-(k) = k^{-3/2} \exp\left( \frac{1}{4k^2}\right).$$

Clearly $\phi_-(k)$ is not integrable since it has a non-removable singularity at the origin, however $\phi_+(k)$ is square integrable, and hence, belongs to $\mathbf{H}$. Therefore, even though the operator $T_2$ is symmetric it is not essentially self-adjoint and can also not be extended to a self-adjoint operator since it has deficiency indices (1, 0).

We now proceed with our construction of the representation independent propagator. Let $\alpha, \beta \in \mathcal{D}(G)$, then put

$$U(\alpha) = \int \alpha(g(l)) \ U_{g(l)} \ dg(l),$$

$$\alpha^*(g(l)) \equiv \Delta(g^{-1}(l)) \alpha(g^{-1}(l)),$$

and define the map $\mathcal{D}(G) \times \mathcal{D}(G) \ni (\alpha, \beta) \rightarrow \alpha \ast \beta \in \mathcal{D}(G)$ as follows:

$$(\alpha \ast \beta)(g(l)) = \int \alpha(g(l')) \beta(g^{-1}(l')g(l)) \ dg(l').$$
With these definitions we find that:

\[ \mathcal{K}_\eta(\alpha, \beta) = \]

\[ = \int \int \mathcal{K}_\eta(l; l') \alpha(g(l)) \beta(g(l')) dg(l) dg(l') \]

\[ = \int \int \langle \eta, K^{1/2} U_g^{-1}(l') g(l') K^{1/2} \eta \rangle \alpha(g(l)) \beta(g(l')) dg(l) dg(l') \]

\[ = \int \langle \eta, K^{1/2} U_g(l') K^{1/2} \eta \rangle \left[ \int \alpha(g(l)) \beta(g(l')) dg(l') \right] dg(l') \]

\[ = \int \langle \eta, K^{1/2} U_g(l') K^{1/2} \eta \rangle (\alpha^* \beta)(g(l')) dg(l') \]

\[ = \langle \eta, K^{1/2} U(\alpha^* \beta) K^{1/2} \eta \rangle. \]

Note that \( \mathcal{K}_\eta(\alpha, \beta) \) is a bilinear, separately continuous form on \( \mathcal{D}(G) \times \mathcal{D}(G) \). We call the bilinear separately continuous forms on \( \mathcal{D}(G) \times \mathcal{D}(G) \) kernels on \( G \). Also observe that \( \mathcal{K}_\eta(\alpha, \beta) \) is a left invariant kernel, that is

\[ \mathcal{K}_\eta(L_g \alpha, L_g \beta) = \mathcal{K}_\eta(\alpha, \beta), \text{ for every } g \in G, \alpha, \beta \in \mathcal{D}(G). \]

Therefore, we can write (4.20) as

\[ K_\eta(\alpha, t; \beta, t') = U(t - t') \mathcal{K}_\eta(\alpha, \beta). \]

In the above construction \( \eta \in \mathcal{D}(K^{1/2}) \) was arbitrary, furthermore as shown elsewhere [30, Corollary 2] for \( \alpha \in \mathcal{D}(G) \) the operator \( K^{1/2} U(\alpha) K^{1/2} \) is trace class. Therefore, we can choose any ONS \( \{\phi_j\}_{j \in \mathbb{N}} \) in \( \mathcal{D}(K^{1/2}) \) and write

\[ \mathcal{K}_H(\alpha, \beta) = \sum_{j=1}^{\infty} \mathcal{K}_{\phi_j}(\alpha, \beta) = \text{tr}[K^{1/2} U(\alpha^* \beta) K^{1/2}]. \]

Note that \( \mathcal{K}_H(\alpha, \beta) \) is a left invariant kernel on \( G \), since each \( \mathcal{K}_{\phi_j}(\alpha; \beta) \) is a left invariant kernel on \( G \). Therefore, by Proposition VI.6.5 in Ref. 78 there exists a unique distribution \( S \) in \( \mathcal{D}'(G) \) such that \( \mathcal{K}_H(\alpha, \beta) = S(\alpha^* \beta) \). In fact we see that \( S(g^{-1}(l) g(l')) = \text{tr}[K^{1/2} U_{g^{-1}(l)} g(l') K^{1/2}] \). Therefore, we find the following propagator which is an element of \( \mathcal{D}'(G) \):

\[ K_H(l, t; l', t') = U(t - t') \text{tr}[K^{1/2} U_{g^{-1}(l)} g(l') K^{1/2}]. \quad (4.21) \]
Remark 4.4.1 This propagator is clearly independent of \( \eta \) the fiducial vector that fixes a coherent state representation. However, this propagator is in general no longer a continuous function but a linear functional acting on \( D(G) \). We will see below that the elements of any reproducing kernel Hilbert space lie in the set of test functions for this propagator. \( \diamond \)

Lemma 4.4.1 The propagator \( K_H(l, t; l', t') \) given in (4.21) correctly propagates all elements of any reproducing kernel Hilbert space \( L^2_\eta(G) \), associated with the irreducible, square integrable unitary representation \( U_{g(l)} \) of the general Lie group \( G \).

Proof. Let \( \eta \in D(K^{1/2}) \) be arbitrary, then for \( \psi_\eta(l', t') \in L^2_\eta(G) \) one can write

\[
\int K_H(l, t; l', t')\psi_\eta(l', t')dg(l') = \]

\[
= \int U(t-t')\text{tr}[K^{1/2}U_{g^{-1}(l)}K^{1/2}]\psi_\eta(l', t')dg(l') \]

\[
= \sum_{j=1}^{\infty} U(t-t')\int (\phi_j, K^{1/2}U_{g^{-1}(l)}K^{1/2}\phi_j) (U_{g(l')}K^{1/2}\eta, \psi(t'))dg(l') \]

\[
= U(t-t')(K^{1/2}U_{g(l)}\sum_{j=1}^{\infty} (\phi_j, \eta)\phi_j, \psi(t')) \]

\[
= [C_\eta\exp[-i(t-t')]\mathcal{H}(X_k)]\psi(t')(l) \]

\[
= \psi_\eta(l, t), \]

where the fourth equality holds by Theorem 4.1.2. Therefore,

\[
\psi_\eta(l, t) = \int K_H(l, t; l', t')\psi_\eta(l', t')dg(l'), \quad \forall \eta \in D(K^{1/2}),
\]

i.e. the propagator propagates the elements of any \( L^2_\eta(G) \) correctly. \( \Box \)

In the above construction the unitary irreducible representation \( U_{g(l)} \) was arbitrary, hence we can introduce such a propagator for each inequivalent unitary representation of \( G \). By Corollary 5.1 in Ref. 30 there exists a positive, \( \sigma \)-finite standard Borel measure \( \nu \) on \( \hat{G} \), a \( \nu \)-measurable decomposition \( (U_{g(l)}^\xi, H_\xi)_{\xi \in \hat{G}} \) of \( \Lambda_I \), and a measurable field \( (K_\xi)_{\xi \in \hat{G}} \) of nonzero, positive, self-adjoint operators such that \( K_\xi \) is
a semi-invariant operator of weight $\Delta(g^{-1})$ in $H_\zeta$ for $\nu$-almost all $\zeta \in \hat{G}$ such that for $\alpha, \beta \in P_\lfloor[D(G)]$

$$
\delta_e(\alpha^* \ast \beta) = \int_G \text{tr}[K^{1/2}_\zeta U^\zeta(\alpha^* \ast \beta)K^{1/2}_\zeta]d\nu(\zeta),
$$

is well defined; see Appendix B.3. Here,

$$
\delta_e(\alpha^* \ast \beta) = \int \int \delta_e(g^{-1}(l)g(l'))\alpha(g(l))\beta(g(l'))dg(l)dg(l'),
$$

and $\delta_e(g^{-1}(l)g(l'))$ is given in the chosen parameterization by

$$
\delta_e(g^{-1}(l)g(l')) = \frac{1}{\gamma(l)} \prod_{k=1}^d \delta(l^k - l'^k).
$$

Hence, we can write down the following propagator for $\Lambda_I$ of $G$ on $L^2(G)$,

$$
K(\alpha, t; \beta, t') = \int_k K^{H_\zeta}(\alpha, t; \beta, t')d\nu(\zeta)
$$

$$
= U(t-t') \int \text{tr}[K^{1/2}_\zeta U^\zeta(\alpha^* \ast \beta)K^{1/2}_\zeta]d\nu(\zeta)
$$

$$
= U(t-t') \delta_e(\alpha^* \ast \beta).
$$

Therefore, we find the following propagator for the type I part of the left regular representation $\Lambda_I$:

$$
K(l, t; l', t') = \exp[-i(t-t')\mathcal{H}(\bar{x}_k(-i\nabla_i, l))] \delta_e(g^{-1}(l)g(l')).
$$

**Remark 4.4.2.** Observe, that this propagator is clearly independent of the fiducial vector and the irreducible, square integrable unitary representation one has chosen for $G$. A sufficiently large set of test functions for this propagator is given by $C(G) \cap L^2(G)$, where $C(G)$ is the set of all continuous functions on $G$. Hence, the elements of any reproducing kernel Hilbert space $L^2_\eta(G)$ are allowed test functions for the propagator given by (4.23), and therefore, for the propagator given by (4.21). Therefore, we have shown the first part of the following Theorem:
Theorem 4.4.2 The propagator $K(l, t; l', t')$ in (4.23) is a propagator for the type I part of the left regular representation of the general Lie group $G$ which correctly propagates all elements of any reproducing kernel Hilbert space $L^2_{\eta'}(G)$ associated with an arbitrary irreducible, square integrable unitary representation $U^\zeta_{g(l)}$ of $G$, $\zeta \in \hat{G}$.

Proof. To prove the second part of Theorem 4.4.2, let $U^\zeta_{g(l)}$ and $\eta' \in D(K_{\zeta'}^{1/2})$ be arbitrary. For any $\psi_{\eta'}(l) \in L^2_{\eta'}(G)$, associated with $U^\zeta_{g(l)}$, we can write

$$
\int_G K(l, t; l', t') \psi_{\eta'}(l', t') dg(l') = \int_G U(t - t') \delta_\zeta(g^{-1}(l)g(l'))(U^\zeta_{g(l')} K_{\zeta'}^{1/2} \eta', \psi_{\zeta'}(t')) dg(l')
$$

$$
= U(t - t')[C_{\eta'}, \psi_{\zeta'}(t')](l)
$$

$$
= [C_{\eta'}, \exp[-i(t - t')\mathcal{H}(X_k)] \psi_{\zeta'}(t')](l)
$$

$$
= \psi_{\eta'}(l, t).
$$

Therefore,

$$
\psi_{\eta'}(l, t) = \int K(l, t; l', t') \psi_{\eta'}(l', t') dg(l'),
$$

for all $\eta' \in D(K_{\zeta'}^{1/2})$ and any $\zeta' \in \hat{G}$, i.e. this propagator propagates all elements of any reproducing kernel Hilbert space $L^2_{\eta'}(G)$ associated with an arbitrary irreducible representation $U^\zeta_{g(l)}$ correctly. □

Hence, we have succeeded in constructing a representation independent propagator for a general Lie group.

4.4.2 Path Integral Formulation of the Representation Independent Propagator

From (4.23) it is easily seen that the representation independent propagator is a weak solution to Schrödinger's equation, i.e.

$$
\begin{align*}
\frac{id}{dt} K(l, t; l', t') &= \mathcal{H}(\tilde{x}_1(-i\nabla_1, l), ..., \tilde{x}_d(-i\nabla_1, l)) K(l, t; l', t'),
\end{align*}
$$

(4.24)

Taking in (4.23) the limit $t \to t'$ yields the following initial value problem

$$
\begin{align*}
\frac{id}{dt} K(l, t; l', t') &= \mathcal{H}(\tilde{x}_1(-i\nabla_1, l), ..., \tilde{x}_d(-i\nabla_1, l)) K(l, t; l', t'), \\
\lim_{t \to t'} K(l, t; l', t') &= \delta_\zeta(g^{-1}(l)g(l')).
\end{align*}
$$

(4.25)
Remark 4.4.3 Observe that the coherent state propagator given in (4.20) is also a weak solution to the Schrödinger equation (4.24). However, it satisfies the initial value problem

\[ i\partial_t K_\eta(l, t; l', t') = \mathcal{H}(\hat{x}_1(-i\nabla, l), \ldots, \hat{x}_d(-i\nabla, l))K_\eta(l, t; l', t'), \]

\[ \lim_{t \to t'} K_\eta(l, t; l', t') = \mathcal{K}_\eta(l; l'). \] (4.26)

Therefore, we can write

\[ i\partial_t K_\#(l, t; l', t') = \mathcal{H}(\hat{x}_1(-i\nabla, l), \ldots, \hat{x}_d(-i\nabla, l))K_\#(l, t; l', t'), \] (4.27)

where \( K_\# \) denotes either \( K_\eta \) or \( K \). Note that the initial conditions, i.e. either (4.25) or (4.26) determine which function is under consideration. 

We now interpret the Schrödinger equation (4.27) with the initial condition (4.25) as a Schrödinger equation appropriate to \( d \) separate and independent canonical degrees of freedom. Hence, \( l^1, \ldots, l^d \) are viewed as \( d \) "coordinates", and we are looking at the irreducible Schrödinger representation of a special class of \( d \)-variable Hamilton operators, ones where the classical Hamiltonian is restricted to have the form \( \mathcal{H}(\hat{x}_1(p, l), \ldots, \hat{x}_d(p, l)) \), instead of the most general form \( \mathcal{H}(p_1, \ldots, p_d, l^1, \ldots, l^d) \). In fact the differential operators given in Lemma 4.1.4(i) are elements of the right invariant enveloping algebra of the \( d \)-dimensional Schrödinger representation on \( L^2(G) \). Based on this interpretation one can give the representation independent propagator the following standard formal phase-space path integral formulation in which the integrand assumes the form appropriate to continuous and differentiable paths

\[ K(l'', t''; l', t') = \mathcal{M} \int \exp \left\{ i \int_{t \in [t', t'']} \left[ \sum_{m=1}^{d} p_m(i^m - \mathcal{H}(\hat{x}_1(p, l), \ldots, \hat{x}_d(p, l))) \right] dt \right\} \prod_{t \in [t', t'']} dl(t) dp(t), \] (4.28)

where "\( p_1 \),...,"\( p_d \)" denote "momenta" conjugate to the "coordinates" "\( l^1 \),...,"\( l^d \).

Note that we have used the special form of Hamiltonian and that its arguments are
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normalized such that

\[
\langle l^\mu_1, \ldots, l^\mu_d | l'^\mu_1, \ldots, l'^d \rangle = \delta_e (g^{-1}(l'') g(l')) ,
\]

\[
\langle p^\mu_1, \ldots, p^\mu_d | p'^\mu_1, \ldots, p'^d \rangle = \prod_{k=1}^d \delta(p_k'' - p_k'),
\]

where

\[
\delta(p_k'' - p_k') = \begin{cases} \delta(p_k'' - p_k') & \text{if the spectrum of } \overline{P_{\xi}} \text{ is discrete} \\ \delta(p_k'' - p_k') & \text{if the spectrum of } \overline{P_{\xi}} \text{ is continuous} \end{cases}
\]

and giving rise to the resolutions of identity

\[
\int_{\text{spec}(G^1) \times \cdots \times \text{spec}(G^d)} |l\rangle \langle l| \, dg(l) = I,
\]

\[
\int_{\text{spec}(\overline{P_{\xi}}) \times \cdots \times \text{spec}(\overline{P_{\xi}})} |p\rangle \langle p| \, dp = I,
\]

where \(|l\rangle \equiv |l^1, \ldots, l^d\rangle\) and \(|p\rangle \equiv |p_1, \ldots, p_d\rangle\).

**Remark 4.4.4** If the spectrum of \(\overline{P_{\xi}}\) is discrete then \(dp_k\) denotes a pure point measure such that the integration over \(p_k\) reduces to summation over \(\text{spec}(\overline{P_{\xi}})\).

On \(L^2(G)\) these operators can be represented as

\[ L^a = l^a \quad \text{and} \quad P_{\xi^a} = -i \partial_{l^a} - i [\partial_{l^a} + \frac{1}{2} \Gamma^a (l)], \]

where \(D_S = S(G)\), the set of functions of rapid decrease on \(G\), is chosen as the common dense invariant domain of these operators. Here \(\Gamma^a (l)\) is defined as \(\Gamma^a (l) \equiv \partial_{l^a} \ln \gamma(l)\) and where \(\gamma(l)\) is given in (4.2). It is easily seen that these operators satisfy the CCR, are symmetric on \(L^2(G)\), and that \(-i \tilde{\nabla}\) has the following generalized eigenfunctions

\[
\langle l|p \rangle = \gamma^{-1/2}(l) \exp \left( i \sum_{k=1}^d p_k l^k \right),
\]

where \(\tilde{\nabla} = (\partial_{l^1}, \ldots, \partial_{l^d})\) We normalize these functions so that

\[
\frac{1}{K \sqrt{\gamma(l') \gamma(l)}} \int \exp \left[ i \sum_{k=1}^d p_k (l''^k - l'^k) \right] dp_1 \ldots dp_d = \delta_e (g^{-1}(l'') g(l')) ,
\]

where \(K\) denotes the normalization constant. Therefore, we find for the normalized generalized eigenfunctions of \(-i \tilde{\nabla}\):

\[
\langle l|p \rangle = \frac{1}{\sqrt{K \gamma(l)}} \exp \left( i \sum_{k=1}^d p_k l^k \right). \quad (4.30)
\]
We call (4.29) a d-dimensional Schrödinger representation on $L^2(G)$. Moreover, the differential operators $\{\tilde{x}_k(-i\nabla, l)\}_{k=1}^d$ can be written as follows:

**Lemma 4.4.3** Using the differential operators $\{-i\hat{\partial}_a\}_{a=1}^d$ given in (4.29) the right invariant differential operators $\{\tilde{x}_k(-i\nabla, l)\}_{k=1}^d$ defined in Lemma 4.1.4 (i) can be written as:

$$\tilde{x}_k(-i\nabla, l) = \sum_{m=1}^d \frac{1}{2} [\rho^{-1}_{-k} m(g(l))(-i\partial_l m) + (-i\partial_l m)\rho^{-1}_{-k} m(g(l))], \quad k = 1, \ldots, d, \quad (4.31)$$

where $\nabla_l = (\partial_1, \ldots, \partial_d)$.

**Proof.** Since $-i\partial_a = -i\hat{\partial}_a + (i/2)\Gamma^a(l), \ a = 1, \ldots, d$, the differential operators $\{\tilde{x}_k(-i\nabla, l)\}_{k=1}^d$ become after substitution of this expression:

$$\tilde{x}_k(-i\hat{\partial}_l + (i/2)\Gamma^l(l), \ldots, -i\hat{\partial}_d + (i/2)\Gamma^d(l), l^1, \ldots, l^d) =$$

$$\sum_{m=1}^d \rho^{-1}_{k} m(g(l))[-i\partial_l m + \frac{i}{2} \Gamma^m(l)]$$

$$= \sum_{m=1}^d \frac{1}{2} [\rho^{-1}_{-k} m(g(l))(-i\partial_l m) + \rho^{-1}_{-k} m(g(l))(-i\partial_l m)] + \frac{i}{2} \rho^{-1}_{-k} m(g(l))\Gamma^m(l).$$

Using $[\rho^{-1}_{-k} m(g(l)), -i\partial_l m] = i\partial_l m \rho^{-1}_{-k} m(g(l))$ and the definition of $\Gamma^m(l)$ yields

$$\tilde{x}_k(-i\hat{\partial}_l + (i/2)\Gamma^l(l), \ldots, -i\hat{\partial}_d + (i/2)\Gamma^d(l), l^1, \ldots, l^d) =$$

$$\sum_{m=1}^d \frac{1}{2} [\rho^{-1}_{-k} m(g(l))(-i\partial_l m) + (-i\partial_l m)\rho^{-1}_{-k} m(g(l))]$$

$$+ \frac{i}{2\gamma(l)} \sum_{m=1}^d \partial_l m [\rho^{-1}_{-k} m(g(l))\gamma(l)].$$

Since the operators $\tilde{x}_k(-i\nabla, l)$ are essentially self-adjoint on any reproducing kernel Hilbert-space $L^2_\eta(G)$ (cf. Corollary 4.1.5) and since $\gamma(l) \neq 0$ one concludes that

$$\sum_{m=1}^d \partial_l m [\rho^{-1}_{-k} m(g(l))\gamma(l)] = 0, \quad k = 1, \ldots, d,$$

and therefore,

$$\tilde{x}_k(-i\nabla, l) = \sum_{m=1}^d \frac{1}{2} [\rho^{-1}_{-k} m(g(l))(-i\partial_l m) + (-i\partial_l m)\rho^{-1}_{-k} m(g(l))], \quad k = 1, \ldots, d.$$
Remark 4.4.5 This Lemma shows that the differential operators \( \{ \tilde{x}_k(-i\tilde{\nabla}_l,l) \}_{k=1}^d \) are elements of the right invariant enveloping algebra of the d-dimensional Schrödinger representation on \( L^2(G) \). 

Adapting methods used in Refs. 63 and 68 we can give the representation independent propagator the following regularized lattice prescription.

**Proposition 4.4.4** Let \( \mathcal{H}_c = \mathcal{H}/(I + c\mathcal{H}^2) \) be a sequence of regularized Hamilton operators on \( \mathbf{H} \), where \( c = (t'' - t')/(N + 1) \). Then provided the indicated integrals exist (see below) the representation independent propagator in (4.23) can be given the following d-dimensional lattice phase-space path integral representation:

\[
K(l'', t''; l', t') = \frac{1}{\sqrt{\gamma(l'')\gamma(l')}} \lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ p_{j+1/2} \cdot (l_{j+1} - l_j) - c \mathcal{H}_c(\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j)) \right] \right\} 
\prod_{j=1}^{N} dl^1_j \ldots dl^d_j \prod_{j+1/2=0}^{N} dp_{j+1/2} \ldots dp_{d+1/2}, \tag{4.32}
\]

where \( l_{N+1} = l'' \), \( l_0 = l' \) and the arguments of the Hamiltonian are given by the following functions:

\[
\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j) = \sum_{m=1}^{d} \frac{\rho_{-1}^{-1} m(g(l_{j+1})) + \rho_{-1}^{-1} m(g(l_j))}{2} p_{m_{j+1/2}}, \quad k = 1, \ldots, d.
\]

Remark 4.4.6 If part of the parameter space \( \mathcal{G} \) is compact then we denote by \( \mathcal{R} \) the class of momenta conjugate to the restricted range or periodic "coordinates". If \( p_k \in \mathcal{R} \) then \( dp_k \) denotes a pure point measure such that the integration over \( p_{k+1/2} \) reduces to summation over the discrete spectrum of \( \mathcal{P}_{L^k} \). For the case of a compact parameter space \( \mathcal{G} \) (4.32) reduces to (4.13).

**Proof.** Since the Hamilton operator \( \mathcal{H} \) is in general an unbounded operator, we introduce the following sequence of regularized Hamilton operators on \( \mathbf{H} \)

\[
\mathcal{H}_\delta = \frac{\mathcal{H}}{I + \delta \mathcal{H}^2}, \quad \delta > 0.
\]
Then it is straightforward to show, by using the Spectral Theorem and the Monotone Convergence Theorem, that for all $\psi \in \mathcal{D}(\mathcal{H}) \subset \mathcal{H}$ one has

$$s-lim_{\delta \to 0} \mathcal{H}_\delta = \mathcal{H},$$

and that on all of $\mathcal{H}$ one has

$$s-lim_{N \to \infty} [I - i\epsilon \mathcal{H}_\epsilon]^{N+1} = \exp[-i(t'' - t')\mathcal{H}],$$

where $\epsilon \equiv (t'' - t')/(N + 1)$. Now in order to obtain the lattice phase-space path integral in (4.32) one can proceed as follows. Let $\{\phi_j\}_{j=1}^\infty$ be an arbitrary ONS in $\Phi \subset \mathcal{H}$, then

$$K(l'', t''; l', t') = \exp[-i(t'' - t')\mathcal{H}(\tilde{x}_k(-i\delta_{lo}, l^a))] \langle l'' | l' \rangle$$

$$= \langle l'' | \exp[-i(t'' - t')\mathcal{H}(\tilde{x}_k(P_{L^a}, L^a))] | l' \rangle$$

$$= \sum_{j,k=1}^{\infty} \langle l'' | \phi_k \rangle \langle \phi_k, \exp[-i(t'' - t')\mathcal{H}(\tilde{x}_k(P_{L^a}, L^a))] \phi_j \rangle \langle \phi_j | l' \rangle$$

$$= \lim_{N \to \infty} \sum_{j,k=1}^{\infty} \langle l'' | \phi_k \rangle \langle \phi_k, [1 - i\epsilon \mathcal{H}_\epsilon(\tilde{x}_k(P_{L^a}, L^a))]^{N+1} \phi_j \rangle \langle \phi_j | l' \rangle$$

$$= \lim_{N \to \infty} \langle l'' | [1 - i\epsilon \mathcal{H}_\epsilon(\tilde{x}_k(P_{L^a}, L^a))]^{N+1} | l' \rangle$$

$$= \lim_{N \to \infty} \int \cdots \int \prod_{j=0}^{N} \langle l_{j+1} | [1 - i\epsilon \mathcal{H}_\epsilon(\tilde{x}_k(P_{L^a}, l_j, l_{j+1}))] | l_j \rangle \prod_{j=1}^{N} \gamma(l_j) dl_j^1 \cdots dl_j^d,$$

where $l'' = l_{N+1}$, $l' = l_0$, and $\langle \cdot | \cdot \rangle$ denotes the generalized inner product. Note that the third line holds true since each $\phi \in \Phi$ gives rise to a linear functional acting on $\Phi$ in the following manner $L_\phi(\psi) = \langle \phi | \psi \rangle \equiv \langle \phi, \psi \rangle$ for all $\psi \in \Phi$. Hence, one has that $\langle \phi_k | \exp[-i(t'' - t')\mathcal{H}] | \phi_j \rangle = L_{\phi_k}(\exp[-i(t'' - t')\mathcal{H}]\phi_j) = \langle \phi_k, \exp[-i(t'' - t')\mathcal{H}] | \phi_j \rangle$. The fourth line follows from the fact that $\Phi \subset \mathcal{H}$ and that the approximation we are using holds for all elements of $\mathcal{H}$ (see above). That we can interchange the limit with the infinite sum at the third step above follows from Moore's Interchange of
Limits Theorem (see [31, Lemma 1.7.6]). Hence, we find the following expression for $K(l'', t''; l', t')$:

$$K(l'', t'', l', t') = \lim_{N \to \infty} \int \cdots \int \prod_{j=0}^{N} \langle l_{j+1} | [1 - \imath \epsilon \mathcal{H}_e(\tilde{x}_k(P_{L^d}; l_{j+1}, l_j))] | l_j \rangle \prod_{j=1}^{N} \gamma(l_j) dl_j^1 \cdots dl_j^d.$$  

(4.33)

Therefore, we have to evaluate $\langle l_{j+1} | [1 - \imath \epsilon \mathcal{H}_e] | l_j \rangle$. This can be done as follows:

$$\langle l_{j+1} | [1 - \imath \epsilon \mathcal{H}_e(\tilde{x}_k(P_{L^d}; l_{j+1}, l_j))] | l_j \rangle = \int \langle l_{j+1} | p_{j+1/2} \rangle \langle p_{j+1/2} | [1 - \imath \epsilon \mathcal{H}_e(\tilde{x}_k(P_{L^d}; l_{j+1}, l_j))] | l_j \rangle dp_{j+1/2}$$

$$= \int \langle l_{j+1} | p_{j+1/2} \rangle \langle l_j | p_{j+1/2} \rangle \left[ 1 - \imath \epsilon \mathcal{H}_e(\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j)) \right] dp_{j+1/2},$$

where

$$\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j) = \sum_{m=1}^{d} \rho^{-1} \rho_k^{m}(g(l_{j+1})) + \rho^{-1} \rho_k^{m}(g(l_j)) \rho_{m_j+1/2}, \quad k = 1, \ldots, d.$$  

Substituting the right hand side of (4.30) into the above expression yields

$$\langle l_{j+1} | [1 - \imath \epsilon \mathcal{H}_e] | l_j \rangle = \frac{1}{\sqrt{\gamma(l_{j+1}) \gamma(l_j)}} \int e^{p_{j+1/2} \cdot (l_{j+1} - l_j)} [1 - \imath \epsilon \mathcal{H}_e(\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j))] \frac{dp_{j+1/2}}{K}$$  

(4.34)

Now inserting (4.34) into (4.33) yields

$$K(l'', t''; l', t') = \frac{1}{\sqrt{\gamma(l'') \gamma(l')}} \lim_{N \to \infty} \int \cdots \int \times \exp \left\{ \imath \sum_{j=0}^{N} [p_{j+1/2} \cdot (l_{j+1} - l_j)] \right\} \prod_{j=0}^{N} [1 - \imath \epsilon \mathcal{H}_e(\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j))] \prod_{j=1}^{N} dl_j^1 \cdots dl_j^d \prod_{j=1/2=0}^{N} dp_{j+1/2} \cdots dp_{j+1/2} \frac{K^d}{K^d}.$$  

(4.35)

Equation (4.35) represents a valid lattice phase-space path integral representation of the propagator $K(l'', t''; l', t')$. One can now interpret the term $1 - \imath \epsilon \mathcal{H}_e(\tilde{x}_k)$ as the first order approximation of $\exp[-\imath \epsilon \mathcal{H}_e(\tilde{x}_k)]$ for small $\epsilon$. Hence, provided the indicated integrals (or sums as necessary) exist one may replace (4.35) by the more suggestive
expression:

\[ K(l'', t''; l', t') = \frac{1}{\sqrt{\gamma(l'')}\gamma(l')} \lim_{N \to \infty} \int \ldots \int \]

\[ \times \exp \left\{ i \sum_{j=0}^{N} [p_{j+1/2} \cdot (l_{j+1} - l_j) - \epsilon \mathcal{H}_c(\tilde{x}_k(p_{j+1/2}; l_{j+1}, l_j))] \right\} \]

\[ \times \prod_{j=1}^{N} dl_1^j \ldots dl_d^j \prod_{j=1/2}^{N} \frac{dp_{1j+1/2} \ldots dp_{dj+1/2}}{K^d}, \]

which is the desired expression. □

**Remark 4.4.7** Observe that even though the group manifold is a *curved* manifold the regularized lattice expression for the representation independent propagator - save for the prefactor \(1/\sqrt{\gamma(l'')}\gamma(l')\) - has the conventional form of a lattice phase-space path integral on a d-dimensional *flat* manifold. Also note that the lattice expression for the representation independent propagator exhibits the correct time reversal symmetry.

Furthermore, we have made no assumptions about the nature of the physical systems we are considering, other than that their Hamilton operators be essentially self-adjoint. Hence, one can use (4.32) in principle to describe the motion of a general physical system, not just that of a free particle, on the group manifold of a general Lie group \(G\). In addition, there are no \(\hbar^2\) corrections present in the Lagrangian. Therefore, we have arrived at an extremely natural path integral formulation for the motion of a *general physical system* on the group manifold of a general Lie group that is (a) more general than, (b) exact, and (c) free from the limitations present in the path integral formulations for the motion of a *free physical system* on the group manifold of a *unimodular* general group discussed in chapter 2. ♦

**4.5 Example: A Representation Independent Propagator for the Affine Group**

We now introduce a representation independent propagator for the affine group. The affine group is the group of linear transformations without reflections on the real line, \(\mathcal{R} \ni x \to p^{-1}x - q, \) where \(0 < p < \infty\) and \(-\infty < q < \infty\). This group has been used by Klauder [64] for the coherent state path integral quantization of
one-dimensional systems for which the canonical momentum $p$ is restricted to be positive for all times. For further applications of the affine group in quantum physics the reader is referred to Ref. 64 and references there in. The affine group is also an example of a locally compact, non-unimodular Lie group, its modular function in the adopted parameterization is given by $\Delta(g(p, q)) = p^{-1}$ and its left invariant Haar measure is given by $dg(p, q) = dpdq$.

4.5.1 Affine Coherent States

Let us denote by $X_1$ and $X_2$ a representation of the basis of the Lie algebra associated with the affine group by self-adjoint operators with common dense invariant domain $\tilde{D}$ on some Hilbert space $H$. Since $X_1$ and $X_2$ are a representation of the basis of the Lie algebra associated with the affine group, it follows that these operators satisfy the commutation relations

$$[X_1, X_1] = 0, \quad [X_2, X_2] = 0, \quad \text{and} \quad [X_1, X_2] = -iX_1.$$  

From these commutation relations it is easily seen that the Lie algebra associated with the affine group is solvable, therefore, the affine group is a solvable Lie group. Since $X_1$ and $X_2$ are chosen to be self-adjoint they can be exponentiated to one-parameter unitary subgroups of the affine group, cf. example 4.4.1. Since the affine group is a connected solvable Lie group every group element can be written as the product of these one-parameter unitary subgroups (cf. [7, Theorem 3.5.1]). With the above parameterization the map:

$$g(p, q) \rightarrow U_{g(p,q)} = \exp(-iqX_1) \exp(i \ln pX_2)$$  

provides a unitary representation of the affine group on $H$, for all $(p, q) \in P^+$, where $P^+ = \{(p, q) : 0 < p < \infty, -\infty < q < \infty\}$. The unitary representations of the affine group have been studied by Aslaksen and Klauder [4] and Gel'fand and Neumark [41] and it is known that there exist only two (faithful) inequivalent irreducible unitary representations for this group, one for which $X_1$ is a positive self-adjoint operator.
and one for which \( X_1 \) is a negative self-adjoint operator. We denote the irreducible unitary representation of the affine group corresponding to \( X_1 \) positive by \( U_{g(p,q)}^1 \) and to \( X_1 \) negative by \( U_{g(p,q)}^2 \), respectively.

The continuous representation theory using the affine group has been investigated by Aslaksen and Klauder [5] where it was shown that for \( \xi, \phi \in \mathbf{H} \), \( \phi \neq 0 \) the factor \( \langle U_{g(p,q)}^\zeta \xi, \phi \rangle \), \( \zeta = 1, 2 \), is square integrable if and only if \( \xi \in \mathbf{D}(C^{-1/2}) \), where the operator \( C \) is given by \( C = \frac{1}{2\pi} X_1 \) and \( X_1 \) is restricted to be positive. Hence, the irreducible unitary representations of the affine group are square integrable for a dense set of vectors in \( \mathbf{H} \). Moreover, in Ref. 5 the following orthogonality relations have been established for the irreducible unitary representations of the affine group:

\[
\int \langle \chi, U_{g(p,q)}^\zeta \xi \rangle \langle U_{g(p,q)}^\zeta \xi', \chi' \rangle dpdq = \langle \chi', \chi \rangle \langle C^{-1/2} \xi', C^{-1/2} \xi \rangle, \quad \zeta = 1, 2
\]

where \( \chi, \chi' \in \mathbf{H} \), and \( \xi, \xi' \in \mathbf{D}(C^{-1/2}) \). Hence, each of the irreducible unitary representations can be used to define a set of coherent states:

\[
\eta(p, q) = U_{g(p,q)}^\zeta C^{1/2} \eta, \quad \zeta = 1, 2,
\]

where \( \eta \in \mathbf{D}(C^{1/2}) \) and \( \| \eta \| = 1 \). These states give rise to a resolution of identity and a continuous representation of the Hilbert space \( \mathbf{H} \) on any one of the reproducing kernel Hilbert spaces \( L_\eta^2(\mathbf{P}^+) \subset L^2(\mathbf{P}^+) \).

### 4.5.2 The Representation Independent Propagator

Using Theorem 3.2.1(ii) we find:

\[
id U_{g(p,q)}^\zeta U_{g(p,q)}^{\zeta^*} = X_1 dq + \left( \frac{q}{p} X_1 - \frac{1}{p} X_2 \right) dp, \quad \zeta = 1, 2
\]

from which we identify the following \( 2 \times 2 \) coefficient matrix \( [\rho_m^k(g(p,q))] \):

\[
[\rho_m^k(g(p,q))] = \begin{pmatrix}
\frac{1}{q} & 0 \\
\frac{q}{p} & -\frac{1}{p}
\end{pmatrix}.
\]

Inverting this \( 2 \times 2 \) matrix we find:

\[
[\rho_m^{-1} g(p,q)] = \begin{pmatrix}
1 & 0 \\
q & -p
\end{pmatrix}.
\]
With these coefficients we find by Lemma 4.1.4 for the differential operators that describe the action of the affine operators $X_1$ and $X_2$ on any reproducing kernel Hilbert space $L^2_\eta(\mathbb{P}^+)$ the following:

$$
\tilde{x}_1 = -i\partial_q,
\tilde{x}_2 = ip\partial_p - iq\partial_q.
$$

Thus, if we denote by $\mathcal{H}(X_1, X_2)$ the essentially self-adjoint Hamilton operator of a quantum mechanical system on $\mathcal{H}$, cf. example 4.4.1, then by Theorem 4.4.2 the representation independent propagator for the affine group is given by:

$$
K(p'', q'', t''; p', q', t') = \exp[-i(t'' - t')\mathcal{H}(\tilde{x}_1, \tilde{x}_2)] \delta(p'' - p')\delta(q'' - q').
$$

By Proposition 4.4.4 we can give the representation independent propagator for the affine group the following regularized lattice phase-space path integral representation:

$$
K(p'', q'', t''; p', q', t') = \lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^N \left[ x_{j+1/2}(p_{j+1} - p_j) + k_{j+1/2}(q_{j+1} - q_j) 
- \epsilon \mathcal{H} \left( k_{j+1/2}, \frac{1}{2}[k_{j+1/2}(q_{j+1} + q_j) - x_{j+1/2}(p_{j+1} + p_j)] \right) \right]\right\}
\times \prod_{j=1}^N dp_j dq_j \prod_{j=0}^N \frac{dk_{j+1/2} dx_{j+1/2}}{(2\pi)^2},
$$

where $(p_{N+1}, q_{N+1}) = (p'', q'')$, $(p_0, q_0) = (p', q')$, and $\epsilon = (t'' - t')/(N + 1)$. In this expression one can preform the following three consecutive variable changes. For all $j$, one first lets $x_{j+1/2} \to x_{j+1/2} + (q_{j+1} + q_j)k_{j+1/2}/(p_{j+1} + p_j)$, followed by the substitution $x_{j+1/2} \to -2x_{j+1/2}/(p_{j+1} + p_j)$, and finally one lets $k_{j+1/2} \to \frac{1}{2}(p_{j+1} + p_j)k_{j+1/2}$. Then the resulting regularized phase-space path integral is given by

$$
K(p'', q'', t''; p', q', t') = \lim_{N \to \infty} \int \ldots \int \exp \left( i \sum_{j=0}^N \left\{ \frac{1}{2}k_{j+1/2}[(q_{j+1} + q_j)(p_{j+1} - p_j) + (p_{j+1} + p_j)(q_{j+1} - q_j)] \right\} \right)
$$
\[-x_{j+1/2} \left( \frac{2(p_{j+1} - p_j)}{(p_{j+1} + p_j)} \right) - \epsilon \mathcal{H} \left( \frac{1}{2}(p_{j+1} + p_j)k_{j+1/2}, x_{j+1/2} \right) \}
\times \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} dk_{j+1/2} dx_{j+1/2} \left( 2\pi \right)^2.

Therefore, taking an improper limit by interchanging the operation of integration with
the limit with respect to \( N \) we find the following formal phase-space path integral
representation for the representation independent propagator for the affine group:

\[
K(p'', q'', t''; p', q', t') = \mathcal{M} \int \exp \left\{ i \int \left[ k(\tilde{q} \tilde{p}) - x(\ln p) - \mathcal{H}(pk, x) \right] dt \right\} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x,
\]

where \( \left( \right) \) denotes \( d/dt(\cdot) \). This expression agrees with the one found in Ref. 69 up to
a numerical factor \( M \), given by \( M = \langle \eta, X_1^{-1} \eta \rangle \), which is used in the normalization of
the resolution of identity in the definition of coherent states for the affine group due to
Aslaksen and Klauder [5]. We now formally evaluate the representation independent
propagator for two soluble examples. For the exact lattice calculation of these two
examples see Appendix C.

4.5.2.1 The Free Particle

Our first example is that of the free particle where \( \mathcal{H}(X_1, X_2) = \frac{X_1^2}{2m} \). Since
\( X_1 \) is self-adjoint and \( g(x) = x^2 \) is a real-valued Borel function on \( \mathbb{R} \), \( g(X_1) = X_1^2 \) is
self-adjoint on \( D_g = \{ \phi : \int_{-\infty}^{\infty} \lambda^2(x)\phi(x)dx < \infty \} \) (cf. [87, Theorem VIII.6]). In
this case the representation independent propagator becomes

\[
K(p'', q'', t''; p', q', t') = \mathcal{M} \int \exp \left\{ i \int \left[ k(\tilde{q} \tilde{p}) - x(\ln p) - \frac{(pk)^2}{2m} \right] dt \right\} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x
\]
\[
= \mathcal{N} \int \exp \left\{ i \int \left[ \frac{m}{2} (\tilde{q} \tilde{p})^2 - x \left( \frac{\tilde{p}}{p} \right) \right] dt \right\} \mathcal{D}q \prod_{t \in [t', t'']} \frac{dp(t)}{p(t)} \mathcal{D}x
\]
\[
= \mathcal{N} \int \exp \left\{ i \int \left[ \frac{m}{2} \left( \dot{q} + \frac{\dot{p}}{p} \right)^2 \right] dt \right\} \delta(p) \mathcal{D}q \mathcal{D}p
\]

Carrying out the remaining two integrations we obtain as our final result,

\[
K(p'', q'', t''; p', q', t) = \sqrt{\frac{m}{2\pi i(t'' - t')}} \delta(p'' - p') \exp \left[ \frac{im}{2(t'' - t')} (q'' - q')^2 \right].
\]
Observe that, up to the presence of the delta function $\delta(p'' - p')$, this result is in perfect agreement with the usual result for the free particle, even though we only consider the positive or negative half of phase-space, i.e. $p$ is constrained to be either positive or negative.

4.5.2.2 The Hamilton Operator $\mathcal{H}(X_1, X_2) = \frac{1}{2m} X_1^2 + \omega X_2$

The second example we consider is that of the Hamilton operator $\mathcal{H}(X_1, X_2) = X_1^2/2m + \omega X_2$. We have seen in example 4.4.1 that this Hamilton operator is essentially self-adjoint. The representation independent propagator takes the following form

$$K(p'', q'', t''; p', q', t') = \mathcal{M} \int \exp \left\{ i \int_0^T \left[ k(pq) - x(\ln p) - \frac{(pk)^2}{2m} - \omega x \right] dt \right\} \mathcal{D}p \mathcal{D}q \mathcal{D}k \mathcal{D}x$$

$$= \mathcal{N} \int \exp \left\{ i \int_0^T \left[ \frac{m}{2} \left( \frac{\dot{p} q}{p} \right)^2 - x \left( \frac{\dot{p}}{p} + \omega \right) \right] dt \right\} \mathcal{D}q \prod_{t \in [t', t'']} \frac{dp(t)}{p(t)} \mathcal{D}x$$

$$= \mathcal{N} \int \exp \left\{ i \int_0^T \left[ \frac{m}{2} \left( \dot{q} + q \frac{\dot{p}}{p} \right)^2 \right] dt \right\} \delta(\dot{p} + \omega p) \mathcal{D}q \mathcal{D}p$$

$$= \delta (e^{\omega T/2} p'' - e^{-\omega T/2} p') \mathcal{N} \int \exp \left\{ i \int_0^T \left[ \frac{m}{2} (\dot{q} - \omega q)^2 \right] dt \right\} \mathcal{D}q,$$

where $T \equiv t'' - t'$. The final path integral we have to solve is a Lagrangian path integral for a quadratic Lagrangian which can be done using extremal methods; see Ref. 95. The action for this Lagrangian path integral is given by

$$I_{cl} = \frac{m}{2} \int_0^T (\dot{q} - \omega q)^2 dt$$

variation of which yields the equation of motion

$$\ddot{q} = \omega^2 q,$$

which has the general solution

$$q(t) = A \sinh(\omega t) + B \cosh(\omega t).$$
After imposing the proper boundary conditions, one finds the evaluated classical action to be

\[ S_d = \frac{m\omega}{2\sinh(\omega T)} \left\{ [(q'')^2 + (q')^2] \cosh(\omega T) - 2q''q' \right\} - \frac{m\omega}{2} [(q'')^2 - (q')^2]. \]

So that our final result for the representation independent propagator with this Lagrangian becomes:

\[ K(p'', q'', t''; p', q', t') = \sqrt{\frac{m\omega}{2\pi i \sinh(\omega T)}} \delta \left( e^{\omega T/2p''} - e^{-\omega T/2p'} \right) \times \exp \left( \frac{im\omega}{2\sinh(\omega T)} \left\{ [(q'')^2 + (q')^2] \cosh(\omega T) - 2q''q' \right\} - \frac{im\omega}{2} [(q'')^2 - (q')^2] \right). \]

Observe that the evaluated action functional in the exponent of this propagator, save for the term \(- (m\omega/2)[(q'')^2 - (q')^2]\), agrees with the evaluated action functional one obtains for the propagator of the harmonic oscillator in imaginary time formulation although it is not of the same “physical origin.”
CHAPTER 5
CLASSICAL LIMIT OF THE REPRESENTATION INDEPENDENT
PROPAGATOR

Even though the regularized lattice phase-space path integral representation for the representation independent propagator has been constructed by interpreting the appropriate Schrödinger equation (4.27) as a Schrödinger equation for \textit{d separate and independent canonical} degrees of freedom, it should, nevertheless, be true that the classical limit for the representation independent propagator refers to the degree(s) of freedom associated with the Lie group \(G\). In particular we will show that this is true for a general Lie group since the classical equations of motion obtained from the action functional for the representation independent propagator imply the classical equations of motion obtained from the most general classical action functional of the coherent state propagator for \(G\). We first discuss the classical limit for compact real Lie groups and then turn our attention to the classical limit for general non-compact real Lie groups.

5.1 Classical Limit for Compact Lie Groups

It is known that any compact Lie group is the direct product of its connected center\(^1\) and a finite number of simple subgroups (cf. [7, Theorem 3.8.2]) and that all irreducible unitary representations of compact Lie groups are finite dimensional (cf. [78, Lemma IV.3.2]). Subsequently we consider the classical limit of semisimple compact Lie groups, i.e. compact Lie groups which have a discrete center. This includes the physically important examples of \(SU(2)\) and \(SU(3)\), whose centers are given by \(C = \{-e, e\}\) and \(C = \{-e, \exp\left(\frac{2\pi i}{3}\right)e, e\}\), respectively. The problem of taking

\(^1\)By the center of a group \(G\) we mean the set of elements of \(G\) which commute with every element of \(G\), that is, \(C = \{a \in G : ax = xa \forall x \in G\}\).
the classical limit of semisimple compact Lie groups has been previously considered by Gilmore [43] and Simon [96]. In Ref. 96 the classical limit of quantum partition functions is discussed, extending previous work by Lieb [72] to any semisimple compact Lie group. The classical limit is taken by using coherent states built up from a maximal weight vector in an irreducible representation. While in Refs. 43 and 96 the general problem of taking the classical limit of operators belonging to a semisimple compact Lie algebra is considered, we, on the other hand, consider the problem of taking the classical limit of the most general action functional appropriate to the coherent state propagator for a semisimple compact Lie group. Let us denote by $\hat{X}_j = \hbar X_j$, $j = 1, \ldots, d$, the physical operators. Then the most general action functional appropriate to the d-dimensional semisimple compact Lie group $G$ is given by (see Chapter 2, Eq. 2.42):

$$I = \int \left[ i\hbar \langle \eta(l) \rangle \frac{d}{dt} \langle \eta(l) \rangle - \langle \eta(l) \rangle \mathcal{H}(\hat{X}_1, \ldots, \hat{X}_d) \eta(l) \right] dt. \quad (5.1)$$

Let us assume that the semisimple compact Lie group $G$ we are considering has rank $n < d$, i.e. there exist $n$ self-commuting operators $H_r$, $r = 1, \ldots, n$, that form the Cartan subalgebra $H$ of the Lie algebra $L$ associated with the Lie group $G$. Moreover, let us denote by $m = (m_1, \ldots, m_n)$ the highest weight of the finite dimensional irreducible unitary representation $(U, H)$ of $G$. Using the non-degenerate Cartan metric tensor $g_{ls} \equiv \sum_{j,k=1}^{d} c_{l,k}^j c_{s,j}^k$ we construct the Casimir operator

$$C_2 = - \sum_{l,s=1}^{d} g^{ls} \hat{X}_l \hat{X}_s$$

which satisfies

$$[C_2, \hat{X}_k] = - \sum_{l,s=1}^{d} g^{ls} (\hat{X}_l \hat{X}_k + \hat{X}_l [\hat{X}_s, \hat{X}_k])$$

$$\quad = - \sum_{l,s=1}^{d} g^{ls} \left( \sum_{t=1}^{d} c_{l,k}^t \hat{X}_t \hat{X}_s + \sum_{j=1}^{d} c_{s,j}^k \hat{X}_l \hat{X}_j \right)$$
\[ c_{rst} = \sum_{l=1}^{d} c_{rl} g_{lt} \]

since \( c_{rst} = \sum_{l=1}^{d} c_{rl} g_{lt} \) is totally antisymmetric under any interchange of its indices.

Since the Cartan metric tensor is symmetric and is non-degenerate for semisimple compact Lie algebras it can be diagonalized, i.e. may therefore be taken in the form \( g_{ls} = -\delta_{ls} \). Hence, without loss of generality we can assume that the Casimir operator is given by:

\[ C_2 = \sum_{l=1}^{d} \hat{X}_l^2. \]

The operator \( C_2 \) can be written in the standard Cartan-Weyl basis of the Lie algebra \( L \) as follows:

\[ C_2 = \sum_{r=1}^{n} \hbar^2 H_r^2 + \sum_{\alpha} \hbar^2 E_\alpha E_{-\alpha}, \tag{5.2} \]

where \( \sum_{\alpha} \) denotes the sum over the nonzero roots of the Lie algebra \( L \). When this operator acts on the highest weight vector \( \omega_m \) of the irreducible unitary representation \( U \), one obtains, because of the condition \( E_\alpha \omega_m = 0 \) for positive roots,

\[ C_2 \omega_m = \left\{ \sum_{r=1}^{n} \hbar^2 m_r^2 + \sum_{\alpha > 0} \hbar^2 [E_\alpha, E_{-\alpha}] \right\} \omega_m = \hbar^2 \left[ \sum_{r=1}^{n} m_r^2 + \sum_{\alpha > 0} m_r \sum_{j=1}^{n} \alpha^j H_j \right] \omega_m = \hbar^2 \left[ \sum_{r=1}^{n} \left( m_r^2 + \sum_{\alpha > 0} \alpha^r m_r \right) \right] \omega_m. \]

It is well known that every irreducible unitary representation is characterized by the components of the highest weight \( m \). By Schur’s lemma every invariant operator in the carrier space of the irreducible unitary representation \( U \) is proportional to the identity operator, i.e. \( C_2 = \lambda I \), where \( \lambda \) is given in terms of the components of the highest weight \( m \), in particular we have that
\[ C_2 = \hbar^2 \sum_{r=1}^{n} (m_r^2 + 2g^r m_r)I \]

where
\[ g^r = \frac{1}{2} \sum_{\alpha > 0} \alpha^r. \]

We now consider the classical limit of the action functional given in (5.1). Since we want to deal with general fiducial vectors we have to consider
\[ \langle \eta, \hat{X}_k \eta \rangle = \hbar \chi_{k\eta}, \quad k = 1, \ldots, d, \]
where the \( \chi_{k\eta}, k = 1, \ldots, d, \) are real numbers given by \( \chi_{k\eta} = \langle \eta, X_k \eta \rangle. \) We insist on vanishing dispersion as \( \hbar \to 0 \) and \( m \to \infty \) (i.e., \( m^r \to \infty, \) for each \( r \)), namely, that
\[ \lim_{\hbar \to 0, m \to \infty} \sum_{k=1}^{d} \langle \eta, (\hat{X}_k - \langle \eta, \hat{X}_k \eta \rangle) \rangle^2 \eta = 0, \quad (5.3) \]
where the limit \( \hbar \to 0 \) and \( m \to \infty \) is taken in such a way that the product \( \mu \equiv \hbar m \) stays finite. We denote the set of fiducial vectors that satisfy (5.3) by \( \mathcal{F}. \) If we choose for the fiducial vector the highest weight vector of the irreducible representation \( U \) then we find
\[ \lim_{\hbar \to 0, m \to \infty} \sum_{k=1}^{d} \langle \omega_m, (\hat{X}_k - \langle \omega_m, \hat{X}_k \omega_m \rangle) \rangle^2 \omega_m \rangle = \lim_{\hbar \to 0, m \to \infty} (\langle \omega_m, C_2 \omega_m \rangle - \sum_{k=1}^{d} \langle \omega_m, \hat{X}_k \omega_m \rangle^2) \]
\[ = \lim_{\hbar \to 0, m \to \infty} [\hbar^2 \sum_{r=1}^{n} (m_r^2 + 2g^r m_r) - \hbar^2 \sum_{r=1}^{n} m_r^2] \]
\[ = \lim_{\hbar \to 0, m \to \infty} \hbar^2 \sum_{r=1}^{n} 2g^r m_r = 0 \]

Hence, the highest weight vector satisfies (5.3), and therefore, the set \( \mathcal{F} \) contains at least one vector. Since for fixed \( l \in \mathcal{L}, U_{g(l)} \) is a unitary operator on \( \mathcal{H} \) there exists a \( l_\eta \) such that
\[ \eta = U_{g(l_\eta)} \omega_m. \]
Therefore, we find
\[ \chi_{k\eta} = \langle \eta, X_k \eta \rangle = \langle \omega_m, U^*_{g(l)} X_k U_{g(l)} \omega_m \rangle = \sum_{t=1}^{d} U_k^t(l\eta) \langle \omega_m, X_{t\omega_m} \rangle. \]

Only the terms for which \( \langle \omega_m, X_{t\omega_m} \rangle \neq 0 \) contribute to this sum, hence we find
\[ \chi_{k\eta} = \sum_{r \in I} U_k^r(l\eta) \mu_i(r), \]
where \( I = \{ r \in \{1, \ldots, d\} : X_r \in H \} \) and \( \mu_i(r) \) denotes the component of the highest weight \( m \) for which \( X_r = H_i \). For finite \( \hbar \) the term that represents the classical Hamiltonian in the coherent state propagator for the Lie group \( G \) is given by
\[ H_\eta(l) = \langle \eta(l), H(\hat{X}_1, \ldots, \hat{X}_d) \eta(l) \rangle = \langle \eta, H \left( \sum_{m=1}^{d} U_k^m(l) \hat{X}_m \right) \eta \rangle, \]
where we assume that \( H(\hat{X}_1, \ldots, \hat{X}_d) \) is an arbitrary polynomial of the physical operators \( \{ \hat{X}_k \}_{k=1}^{d} \). Therefore, if we now take the limit \( \hbar \to 0 \) and \( m \to \infty \) in the above mentioned sense, then the classical Hamiltonian is given by
\[ \lim_{\hbar \to 0, m \to \infty} H_\eta(l) = H \left( \sum_{b=1}^{d} U_k^b(l) v_b \right), \]
where
\[ v_k = \lim_{m \to \infty} \lim_{\hbar \to 0} \langle \eta, X_k \eta \rangle = \lim_{m \to \infty} \lim_{\hbar \to 0} \hbar \chi_{k\eta} = \sum_{r \in I} U_k^r(l\eta) \mu_i(r), \quad k = 1, \ldots, d. \]
Hence, the classical limit of the action functional given in (5.1) becomes
\[ I_{cl} = \lim_{m \to \infty} \lim_{\hbar \to 0} \int \left[ i\hbar \langle \eta(l), \frac{d}{dt} \eta(l) \rangle - \langle \eta(l), H(\hat{X}_1, \ldots, \hat{X}_d) \eta(l) \rangle \right] dt \]
\[ = \int \left[ \sum_{k,m=1}^{d} \lambda_m^k(g(l)) i^m v_k - H \left( \sum_{b=1}^{d} U_1^b(l) v_b, \ldots, \sum_{b=1}^{d} U_d^b(l) v_b \right) \right] dt, \quad (5.4) \]

Extremal variation of this action functional, with respect to the independent labels \( l^b \), holding the end points fixed, yields the equations of motion
\[ \sum_{b,s=1}^{d} v_s \left\{ \partial_c \lambda_b^s(g(l)) - \partial_{cb} \lambda_c^s(g(l)) \right\} l^b = \sum_{a,f=1}^{d} \mathcal{H}_a \partial_c [U_{a f}(l)] v_f, \quad (5.5) \]
where \( \mathcal{H}_a \) denotes the partial derivative of \( \mathcal{H} \) with respect to the \( a \)-th argument \( a = 1, \ldots, d \).
5.2 Classical Limit of Non-Compact Lie Groups

Our discussion of the classical limit of compact semisimple Lie groups cannot be generalized to non-compact semisimple Lie groups, since they do not admit faithful unitary finite-dimensional representations (cf. [7, Corollary 8.1.4]). Hence, we must follow a different route to achieve a well defined classical limit of the most general action functional appropriate to a general non-compact Lie group $G$, given by

$$I = \| K^{1/2} \xi \|^{-2} \int \left[ i \hbar \langle \xi(l), \frac{d}{dt} \xi(l) \rangle - \langle \xi(l), \mathcal{H}(\overline{X}_1, \ldots, \overline{X}_d) \xi(l) \rangle \right] dt,$$

where $\xi(l) = U_{g(l)} K^{1/2} \xi$ and it is assumed that $K^{1/2} \xi \in \mathcal{D}$. Without loss in generality we can set $\eta = K^{1/2} \xi / \| K^{1/2} \xi \|$, then our most general action functional becomes

$$I = \int \left[ i \hbar \langle \eta(l), \frac{d}{dt} \eta(l) \rangle - \langle \eta(l), \mathcal{H}(\overline{X}_1, \ldots, \overline{X}_d) \eta(l) \rangle \right] dt,$$

where $\eta(l) = U_{g(l)} \eta$ and where it is assumed that $\eta \in \mathcal{D} \cap \mathcal{D}(K^{-1/2})$.

In our discussion of the classical limit of the action functional given in (5.6) we use an abstract formalism for taking the $\hbar \to 0$ limit developed by Yaffe [107]. Yaffe [107] considers a family of quantum theories characterized by some parameter $\chi$, such as $\hbar$, and studies the limit of these theories as $\chi$ approaches zero. It is assumed that each theory is defined on some Hilbert space $H_\chi$ with some Hamilton operator $\mathcal{H}_\chi$. Furthermore, it is assumed that there exists a Lie group $G$, with associated Lie algebra $L$, that has on each Hilbert space $H_\chi$ a unitary representation $U^\chi$. We assume for definiteness that $U^\chi$ is parameterized as

$$U^\chi_{g(l)} = \prod_{k=1}^{d} \exp \left( \frac{-i}{\chi} l^k \overline{X}_k \right),$$

up to some ordering. Then the first assumption, which restricts the choice of the group, is

**Assumption 1.** Each unitary representation of $G$ on $H_\chi$ is irreducible.

Hence, on each Hilbert space $H_\chi$ one can define a set of coherent states $\eta^\chi(l) = U^\chi_{g(l)} \eta^\chi$. 

For any operator $O$ acting on $H_x$, we define the upper symbol $O_{\eta_x}(l)$ by

$$O_{\eta_x}(l) = \langle \eta_x(l), O\eta_x(l) \rangle \text{ for all } l \in \mathcal{L},$$

i.e. the upper symbol is a set of coherent state expectation values. The second assumption restricts the possible fiducial vectors $\eta_x$ one can choose. For each value of $\chi$ we require

**Assumption 2.** *Zero is the only observable whose upper symbol identically vanishes.*

By an observable we mean a family of self-adjoint operators consisting of one self-adjoint operator acting in each Hilbert space $H_x$. An example in which assumption 2 is not valid is given by SU(2) coherent states based on a fiducial vector that is not the highest (or lowest) weight vector, in this case a unique specification of any observable by its upper symbol may not be possible (cf. [68, p. 34]). Note that assumption 2 implies that two different operators cannot have the same upper symbol. Hence, one can uniquely recover any operator from its symbol. As pointed out in Ref. 107, p. 411, “this means that it is sufficient to study the behavior of the symbols of various operators in order to characterize the theory completely.”

Observe that the $\chi \to 0$ limit of an arbitrary observable does not have to exist. In order to have some control over the $\chi \to 0$ limit one introduces the concept of a *classical observable*. According to Yaffe [107] an observable $O$ is called a *classical observable* if the limits of its coherent state matrix elements exist,

$$\lim_{\chi \to 0} \frac{\langle \eta_x(l), O\eta_x(l') \rangle}{\langle \eta_x(l), \eta_x(l') \rangle},$$

and are finite for all $l, l' \in \mathcal{L}$. The set of all classical observables is denoted by $O_c$. Clearly the set $O_c$ is a subset of all possible observables, hence it is possible that measurements using only observables of $O_c$ may fail to distinguish between different coherent states. Therefore, two different coherent states, $\eta_x(l)$ and $\eta_x(l')$ are called classically equivalent if for all $O \in O_c$ one has

$$\lim_{\chi \to 0} \langle \eta_x(l), O\eta_x(l) \rangle = \lim_{\chi \to 0} \langle \eta_x(l'), O\eta_x(l') \rangle.$$
The third assumption states that classically inequivalent coherent states become orthogonal in the χ → 0 limit. In particular,

**Assumption 3.** The limit \( \Phi[\eta_\chi(l), \eta_\chi(l')] = -\lim_{\chi \to 0} \chi \ln(\eta_\chi(l), \eta_\chi(l')) \) exists for all \( l, l' \in \mathcal{L} \) and \( \Phi[\eta_\chi(l), \eta_\chi(l')] \) satisfies the conditions

(i) \( \Re\{\Phi[\eta_\chi(l), \eta_\chi(l')]\} > 0 \) if \( \eta_\chi(l) \) and \( \eta_\chi(l') \) are classically inequivalent.

(ii) \( \Re\{\Phi[\eta_\chi(l), \eta_\chi(l')]\} = 0 \) if \( \eta_\chi(l) \) and \( \eta_\chi(l') \) are classically equivalent, and

\[
i \partial_t \{\Phi[\eta_\chi(l), \exp\left(\frac{-i}{\chi} tX\right) \eta_\chi(l)] - \Phi[\eta_\chi(l), \exp\left(\frac{-i}{\chi} tX\right) \eta_\chi(l')]}\} = 0 \ orall X \in \mathcal{L}.
\]

As shown in Ref. 107 assumption 3 implies that classical observables cannot "move" the coherent states. Hence, any fixed \( U^\chi_{g(l)} \) cannot be a classical observable except \( U^\chi_e = I_{\mathcal{H}_\chi} \). However, as shown in Ref. 107 assumption 3 implies that any \( X \in \mathcal{L} \) is an acceptable classical observable. Moreover, as pointed out in Ref. 107, assumption 3 implies that if \( \eta_\chi(l) \) and \( \eta_\chi(l') \) are classically equivalent then

\[
\lim_{\chi \to 0} \frac{\langle \eta_\chi(l), O \eta_\chi(l') \rangle}{\langle \eta_\chi(l), \eta_\chi(l') \rangle} = \lim_{\chi \to 0} O_{\eta_\chi(\chi)} \quad \text{for all } O \in \mathcal{O}_c.
\]

As shown in Ref. 107, this fact together with assumption 3 allows one to establish the following factorization for any pair of classical observables \( O \) and \( O' \):

\[
\lim_{\chi \to 0} \left[ (O O')_{\eta_\chi(l)} - O_{\eta_\chi(l)} O'_{\eta_\chi(l)} \right] = 0 \quad (5.7)
\]

With these three assumptions one gains some control over the \( \chi \to 0 \) limit, however the quantum dynamics is left completely unrestricted. In order to gain complete control over the \( \chi \to 0 \) limit one has to require

**Assumption 4.** \( \mathcal{H}_\chi \) is a classical observable.

As shown in Ref. 107 this set of assumptions is sufficient to show that a quantum theory reduces to a classical theory as \( \chi \to 0 \).

We now discuss the classical limit of the action functional in (5.6). Since we are working with Lie groups that have irreducible square integrable representations
assumption 1 is automatically satisfied. We assume that we have selected the fiducial vector $\eta$ such that assumption 2 is satisfied and we also assume that assumption 3 is satisfied. To satisfy assumption 4 we restrict ourselves to Hamilton operators that are arbitrary polynomials of the generators $\{X_k\}_{k=1}^d$. Then the most general classical action functional appropriate to the coherent state propagator for $G$ is given by

$$I_{cl} = \lim_{\hbar \to 0} \int \left[ i\hbar \langle \eta(l), \frac{d}{dt} \eta(l) \rangle - \langle \eta(l), \mathcal{H}(\overline{X_1}, \ldots, \overline{X_d}) \eta(l) \rangle \right] dt$$

$$= \int \left[ \sum_{k,m=1}^d \lambda_m^k(g(l)) \dot{v}_k - \mathcal{H}(\overline{X_k} \eta(l)) \right] dt$$

$$= \int \left[ \sum_{k,m=1}^d \lambda_m^k(g(l)) \dot{v}_k - \mathcal{H} \left( \sum_{b=1}^d U_{1b}^b(l) v_b, \ldots, \sum_{b=1}^d U_{db}^b(l) v_b \right) \right] dt, \quad (5.8)$$

where we have used (5.7) and (3.5). The $v_k \equiv \lim_{\hbar \to 0} \langle \eta, \overline{X_k} \eta \rangle$, $k = 1, \ldots, d$, are real constants.

Extremal variation of this action functional, with respect to the independent labels $l^b$, holding the end points fixed, yields the equations of motion

$$\sum_{b, s=1}^d v_s \{ \partial_{l^b} \lambda_s^b(g(l)) - \partial_{l^b} \lambda_c^s(g(l)) \} i^b = \sum_{a, f=1}^d \mathcal{H}^a \partial_{l^f} [U_{af}^f(l)] v_f, \quad (5.9)$$

where $\mathcal{H}^a$ denotes the partial derivative of $\mathcal{H}$ with respect to the $a$-th argument; $a = 1, \ldots, d$.

**Remark 5.2.1** Generally the constants $v_1, \ldots, v_d$ are nonzero and are the vestiges of the coherent state representation induced by $\eta$ that remain even after the limit $\hbar \to 0$ has been taken. A similar statement also applies to the case when $G$ is compact. ♦

### 5.3 Classical Limit of the Representation Independent Propagator

In the case of the representation independent propagator one identifies the classical action functional as (see Proposition 4.4.4)

$$I_{cl} = \int \left[ \sum_{j=1}^d p_j \dot{i}^j - \mathcal{H}(\overline{x_1}(p, l), \ldots, \overline{x_d}(p, l)) \right] dt$$

$$= \int \left[ \sum_{j=1}^d p_j \dot{i}^j - \mathcal{H} \left( \sum_{j=1}^d \rho^{-1}_{1j}(g(l)) p_j, \ldots, \sum_{j=1}^d \rho^{-1}_{dj}(g(l)) p_j \right) \right] dt. \quad (5.10)$$
Therefore, we can choose a set of integration constants, \(c_i,...,c_d\), such that

\[
p_j = \sum_{m=1}^{d} \lambda_j^m(g(l))c_m. \tag{5.17}
\]

Substitution of this form of \(p_j\) into (5.11) and (5.12), yields the following set of 2d equations

\[
\dot{i}^b = \sum_{a=1}^{d} \mathcal{H}^a \left( \sum_{s=1}^{d} U_k^s(l)c_s \right) \rho^{-1}_{a b}(g(l)),
\]

\[
\sum_{s=1}^{d} \partial_t [\lambda_c^s(g(l))]c_s = -\sum_{a=1}^{d} \sum_{j,m=1}^{d} \mathcal{H}^a \left( \sum_{s=1}^{d} U_k^s(l)c_s \right) \partial_t [\rho^{-1}_{a j}(g(l))]\lambda_j^m(g(l))c_m.
\]

After differentiation with respect to time these equations take the form

\[
\dot{i}^b = \sum_{a=1}^{d} \mathcal{H}^a \rho^{-1}_{a b}(g(l)),
\]

\[
\sum_{b,s=1}^{d} \partial_t [\lambda_c^s(g(l))]i^b c_s = -\sum_{a=1}^{d} \sum_{j,m=1}^{d} \mathcal{H}^a \partial_t [\rho^{-1}_{a j}(g(l))]\lambda_j^m(g(l))c_m. \tag{5.19}
\]

Next contract (5.18) with \(\sum_{s=1}^{d} \partial_t [\lambda_b^s(g(l))]c_s\) and find

\[
\sum_{b,s=1}^{d} \partial_t [\lambda_b^s(g(l))]i^b c_s = \sum_{a=1}^{d} \sum_{b,s=1}^{d} \mathcal{H}^a \rho^{-1}_{a b}(g(l))\partial_t [\lambda_b^s(g(l))]c_s, \tag{5.20}
\]

\[
\sum_{b,s=1}^{d} \partial_t [\lambda_c^s(g(l))]i^b c_s = -\sum_{a=1}^{d} \sum_{j,m=1}^{d} \mathcal{H}^a \partial_t [\rho^{-1}_{a j}(g(l))]\lambda_j^m(g(l))c_m. \tag{5.21}
\]

Subtracting (5.21) from (5.20) yields

\[
\sum_{b,s=1}^{d} c_s \{ \partial_t \lambda_b^s(g(l)) - \partial_t \lambda_c^s(g(l)) \} i^b = \sum_{a,f=1}^{d} \mathcal{H}^a \partial_t [U_a^f(l)]c_f, \tag{5.22}
\]

where Corollary 3.2.2 has been used. Among all possible allowed values of \(c_1,...,c_d\) are those that coincide with \(v_1,...,v_d\) for an arbitrary fiducial vector. Hence, for this choice of \(c_1,...,c_d\) the above equations coincide with the equations of motion obtained from the most general classical action functional for the coherent propagator for \(G\) [see Eq.(5.5) and Eq.(5.9)]. Therefore, the set of classical equations of motion obtained from the classical action functional of the representation independent propagator
implies the set of classical equations of motion obtained from the most general classical action functional of the coherent state propagator for $G$. Thus, we find that the set of solutions of the representation independent classical equations of motion appropriate to the representation independent propagator for a general Lie group $G$ with square integrable, irreducible representations includes every possible solution of the classical equations of motion appropriate to the most general coherent state propagator for $G$. We summarize all this in the following Proposition:

**Proposition 5.3.1** Let $G$ be a real, separable, locally compact, connected and simply connected Lie group whose unitary irreducible representations are square integrable. If the fiducial vector satisfies (5.3) when $G$ is compact and Assumption 2 when $G$ is non-compact, then the equations of motion obtained from the action functional of the representation independent propagator imply the equations of motion obtained from the most general classical action functional for the coherent state propagator for $G$. 
CHAPTER 6
SUMMARY AND CONCLUSION

In this chapter, as before, we mean by a general Lie group a real, separable, locally compact, connected and simply connected Lie group with irreducible, square integrable unitary representations, unless we explicitly state otherwise.

We have seen in chapter 2 that the quantization of physical systems moving on group and symmetric spaces has been an area of active and on-going research over the past three decades; see for instance Refs. 12, 13, 14, 28, 29, 48, 49, 56, 70, 75, 76, 77 and 94.

In particular we have reviewed in subsection 2.1.3 in detail the approach of Marinov and Terentyev [76, 77] to the construction of path integral representations for a free particle moving on group manifolds of compact simple Lie groups and spheres of arbitrary dimension. As we have pointed out in subsection 2.1.3, in this approach the Lagrangian needs to be modified to include a 'quantum' potential proportional to $\hbar^2$. However, the need to include a correction term of order $\hbar^2$ into the Lagrangian has also been found to be necessary by other investigators who, have starting form the semiclassical approximation, constructed path integral representations of the free particle moving on unbounded Riemannian manifolds; see for instance Refs. 22 and 73.

More importantly, as we have pointed out in subsection 2.1.3, this approach cannot be extended to more general physical systems than the free particle since the semiclassical approximation, which has been used in an essential way by Marinov and Terentyev, is exact only for the case of a free particle moving on the group manifold of a semisimple Lie group.

We also extended in subsection 2.1.4 a method used by Junker [56] to construct
path integral representations for a free particle moving on a compact symmetric space to symmetric spaces of the form $\mathcal{M} = G/H$, where $G$ is a general, not necessarily compact, unimodular transformation group acting on $\mathcal{M}$ and $H$ is a massive compact subgroup of $G$. Again we found that this method could not be extended to more general physical systems moving on $\mathcal{M}$ than the free particle, since we were asking that the short time propagator be invariant under the transformation group $G$. We found that this assumption implied that the Laplace Beltrami operator, was an element of the enveloping algebra of the transformation group $G$. This in turn implied that the matrix elements $D_{ij}^G(g)$ were regular functions on $G$. This showed that the assumption that the short time propagator should be invariant under the transformation group $G$ was crucial and could not be relaxed. Moreover, as we have remarked at the end of subsection 2.1.4, one can construct path integrals this way only for a handful of groups, since one needs to know the explicit form of the spherical zonal functions $D_{00}^G(g)$ in order to carry out the construction; see also in this respect the remarks after equation 2.20.

In subsection 2.1.5 we have discussed the construction of coherent state path integrals. We found that the final path integral representation for the coherent state propagator exhibited a strong dependence on the choice of the fiducial vector and on the choice of the square integrable, irreducible, unitary representation of the general Lie group $G$ under consideration. Hence, one has to reformulate the path integral representation for the coherent state propagator every time one changes the fiducial vector and keeps the irreducible representation the same, or if one changes the irreducible representation of $G$. As we have pointed out in subsection 2.1.5, in many applications it is often convenient to choose the fiducial vector as the ground state of the Hamilton operator $\mathcal{H}$ of the quantum system one considers; see for instance Refs. 100, and 101. Hence, one has to face the problem of various fiducial vectors.
In chapter 3 we have introduced the notations and basic definitions used throughout the thesis. The main result of this chapter was Theorem 3.2.1, in which we derived an operator version of the generalized Maurer-Cartan form.

In chapter 4 we have constructed the representation independent propagator. In section 4.1 we have defined coherent states for a general Lie group $G$ and have proved Lemma 4.1.4 and the Corollary 4.1.5 which we have applied in the construction of the representation independent propagator and the construction of regularized lattice phase-space path integral representations of the representation independent propagator.

Prior to the construction of the representation independent propagator for a general Lie group we have constructed in section 4.2 the representation independent propagator for any compact Lie group $G$. It has been shown in Theorem 4.2.2 that the representation independent propagator for any compact group correctly propagates the elements of any reproducing kernel Hilbert space associated with an arbitrary irreducible unitary representation of $G$. Hence, it solves the problem of various fiducial vectors. We observed that in the construction of the representation independent propagator for compact Lie groups and its path integral representation no explicit use has been made of the ONS $\sqrt{d_\zeta}D^\zeta_{ij}(l)$, $\zeta \in \hat{G}$ and $i, j = 1, \ldots, d_\zeta$, in $L^2(G)$ whose existence is guaranteed by the Peter-Weyl Theorem, but merely the facts that it exists and is complete have been used. Moreover, we have made no assumptions about the nature of the physical systems we were considering, other than that its Hamilton operator be self-adjoint. Therefore, the path integral representation (4.13) can be used in principle to describe the motion of a general physical system, not just that of a free particle, on the group manifold of any compact Lie group and it does not matter if the matrix elements $D^\zeta_{ij}(l)$ are explicitly known or not. Hence, we found that the path integral quantization (4.13) represented a clear improvement over the path integral quantizations describing the motion of a free particle on a compact group manifold.
presented in chapter 2. As an example we have then constructed the representation independent propagator for $SU(2)$ and presented the exact path integral treatment of a free particle moving on a circle and on the group manifold of $SU(2)$.

In section 4.3 this construction has then been suitably extended to a general Lie group and we have shown in Theorem 4.4.2 that the result obtained in Theorem 4.2.2 holds also for a general Lie group. In Proposition 4.4.4 we have established that it is possible to construct regularized phase-space path integrals for a general Lie group $G$. Even though the group space generally is a multidimensional curved manifold, we have shown that the resulting phase-space path integral has the form of a lattice phase-space path integral on a multidimensional flat manifold. Furthermore, since we have made no assumptions about the nature of the physical systems we were considering, other than that their Hamilton operators be essentially self-adjoint, we found that the path integral representation presented in Proposition 4.4.4 can be used in principle to describe the motion of a general physical system, not just that of a free particle, on the group manifold of $G$. In addition, we found that there were no $\hbar^2$ corrections present in the Lagrangian. Therefore, we have arrived at a novel, extremely natural phase-space path integral quantization for the motion of a general physical system on the group manifold of a general Lie group that is (a) more general than, (b) exact, and (c) free from the limitations present in the path integral quantizations for the motion of a free physical system on the group manifold of a general unimodular Lie group discussed in chapter 2. To illustrate the general theory we have then constructed the representation independent propagator for the affine group.

In chapter 5 we have discussed the classical limit of the representation independent propagator of a general Lie group $G$ and have shown that its classical limit refers indeed to the degrees of freedom associated with $G$. In sections 5.1 and 5.2 we have discussed in detail the classical limit of the coherent state propagator for compact Lie groups and non-compact Lie groups. In section 5.3 we have proved that
the equations of motion obtained from the action functional of the representation independent propagator for a general Lie group indeed imply the equations of motion obtained from the most general action functional of the coherent state propagator for a general Lie group (cf. Proposition 5.3.1).

We have focused our attention in this thesis on general Lie groups with square integrable, irreducible, unitary representations, since for this case the existence of a resolution of identity is guaranteed in general by Theorem 4.1.2 and we were able to construct a representation independent propagator rigorously. It would be interesting to see if the construction of the representation independent propagator presented in section 4.4.1 can be extended to Lie groups that do not possess square integrable, irreducible representations, such as the Euclidean group. The obstacle one has to overcome when one considers such groups is the introduction of a resolution of identity. This problem has recently been solved by Isham and Klauder [55] for the n-dimensional Euclidean group $E(n)$. In Ref. 55 attention is focused on reducible, square integrable representations of $E(n)$. In this case it becomes possible to introduce a set of coherent states, i.e. to establish a resolution of identity, and to introduce a coherent state propagator.

Therefore, if the problem of introducing coherent states for these groups can be solved, then one can use the following argument to introduce a fiducial vector independent propagator for these groups. Denote by $U$ a generic, continuous, unitary representation of a Lie group $G$ on some Hilbert space $H$, which does not need to be a square integrable, irreducible representation. For definiteness let us assume that we have parameterized the Lie group $G$ such that the representation $U$ is given by:

$$U_{g(l)} = \exp(-il^{1}\overline{X}_1)\ldots\exp(-il^{d}\overline{X}_d),$$

for some ordering, where the $\overline{X}_1, \ldots, \overline{X}_d$ form an integrable, representation of the associated Lie algebra $L$ of $G$ by essentially self-adjoint operators on some common dense invariant domain $\overline{D} \in H$ and where $l$ is an element of a $d$-dimensional parameter
space $\mathcal{G}$. Let us denote by $\eta(l)$ the coherent states associated with the representation $U_{g(l)}$ of $\mathcal{G}$, where $\eta \in \mathcal{H}$ is the fixed, normalized fiducial vector. Let us furthermore assume, that these states give rise to a resolution of identity

$$I_{\mathcal{H}} = \int_{\mathcal{G}} \eta(l) \langle \eta(l), \cdot \rangle d\mu(l),$$

where $d\mu(l)$ denotes the normalized, left invariant group measure given by

$$d\mu(l) = \frac{1}{|\mathcal{G}|} dg(l),$$

where $1/|\mathcal{G}|$ denotes the normalization; for the definition of $dg(l)$ see (4.4). We can now use this set of coherent states to give a continuous representation of $\mathcal{H}$. We define the map

$$C_\eta : \mathcal{H} \rightarrow L^2(\mathcal{G}, d\mu(l))$$

$$\psi \mapsto [C_\eta \psi](l) = \psi_\eta(l) \equiv \langle \eta(l), \psi \rangle.$$

Which as we know yields a representation of $\mathcal{H}$ by bounded, continuous, square integrable functions on the closed subspace $L^2_\eta(\mathcal{G}, d\mu(l))$ of $L^2(\mathcal{G}, d\mu(l))$.

We now introduce the *fiducial vector independent propagator* $K_{\mathcal{H}}(l'', t''; l', t')$ as follows, it is a single, (possibly generalized) function that is independent of any particular choice of the fiducial vector, which, nevertheless, propagates the $\psi_\eta$ correctly, i.e.,

$$\psi_\eta(l, t) = \int_{\mathcal{G}} K_{\mathcal{H}}(l, t; l', t') \psi_\eta(l', t') d\mu(l') \quad (6.1)$$

If equation (6.1) is to hold for arbitrary $\eta$, we must require that

$$\lim_{t \rightarrow t'} K_{\mathcal{H}}(l, t; l', t') = |\mathcal{G}| \delta_e(g^{-1}(l)g(l')), \quad (6.2)$$

where $\delta_e(g^{-1}(l)g(l'))$ is defined in (4.22).

An analysis of our results presented in chapter 4 shows that Lemma 4.1.4, Corollary 4.1.5 hold for *reducible*, square integrable representations. Even though we have
stated Lemma 4.1.4 and Corollary 4.1.5 for irreducible representations, this property of the representation is not used in the proofs of these results, hence these results also apply to the case when one considers reducible representations. Therefore, it is a direct consequence of Lemma 4.1.4 (i) that for any \( \psi \in \mathcal{D} \)

\[
\tilde{x}_k(-i\nabla l, l)[C_\eta \psi](l) = [C_\eta \bar{X}_k \psi](l), \quad k = 1, \ldots, d,
\]

holds independently of \( \eta \). Hence, we find that \( C_\eta \) intertwines the representation of the Lie algebra \( L \) associated with \( G \) on \( H \), with a subrepresentation of \( L \) by right invariant, essentially self-adjoint differential operators on any one of the reproducing kernel Hilbert spaces \( L^2_\eta(G, d\mu(l)) \).

Denote by \( \mathcal{H}(\bar{X}_k) \) the essentially self-adjoint Hamilton operator of a quantum system on \( H \). Then the continuous representation of Schrödinger's equation on \( H \),

\[
i\partial_t \psi(t) = \mathcal{H} \psi(t),
\]
takes on \( L^2_\eta(G, d\mu(l)) \), the following form

\[
i\partial_t \psi_\eta(l, t) = [C_\eta \mathcal{H}(\bar{X}_k) \psi(t)](l)
\]

\[
= \mathcal{H}(\tilde{x}_k(-i\nabla l, l))\psi_\eta(l, t).
\]

Using (6.1) we find that the fiducial vector independent propagator \( K_H \) is a solution to this Schrödinger equation, i.e.

\[
i\partial_t K_H(l, t; l', t') = \mathcal{H}(\tilde{x}_k(-i\nabla l, l))K_H(l, t; l', t').
\]

Therefore, together with equation (6.2), we find the following initial value problem

\[
i\partial_t K_H(l, t; l', t') = \mathcal{H}(\tilde{x}_k(-i\nabla l, l))K_H(l, t; l', t'),
\]

\[
\lim_{t \to t'} K_H(l, t; l', t') = |G|\delta_c(g^{-1}(l)g(l')). \tag{6.3}
\]

It was such an initial value problem that we have taken as our starting point for the path integral formulation of the representation independent propagator. Hence, we find that one can, using Proposition 4.4.4, introduce path integral representations for general Lie groups that have reducible square integrable representations. However,
observe that we can only introduce a *fiducial vector independent propagator* in this case. This program of has been explicitly carried out for the case of $E(2)$ by Tulsian and Klauder [102].

Observe that if one considers groups with *reducible*, square integrable representations one has to proceed on a case by case basis since a general theory in this case is lacking. It would therefore, be of some interest to see if the theory developed by Duflo and Moore [30] for locally compact groups with irreducible, square integrable, unitary representations can be extended to locally compact groups with *reducible*, square integrable, unitary representations.

In our opinion another interesting avenue to achieve the path integral quantization of the form (4.32) for general Lie groups that do not have square integrable, irreducible representations would be to start form the classical mechanics associated with the particular Lie group one considers and to try to derive the form of the action functional we have arrived at in Proposition 4.4.4. The quantization would then be achieved by postulating (4.32) as the path integral quantization for these kinds of Lie groups.

Furthermore, we believe that the representation independent propagator holds considerable interest for quantum field theory. We have used in this thesis the word representation independent in a dual meaning, its first meaning pertained to the fact that the representation independent propagator is independent of the choice of the fiducial vector and its second meaning to the fact that this propagator is also independent of the choice of the unitary, irreducible representation of the Lie group $G$. In the case of quantum field theory these two meanings of the word representation independent are inextricably related, since the *dynamics* chooses a representation for the basic kinematical variables, see for instance Haag [50, pp. 56–57] and Klauder and Skagerstam [68, pp. 82–83]. We therefore, believe that it would be a worthwhile task to extend our concept of a representation independent propagator into the realm of quantum field theory.
We have collected in this appendix some standard results from the fields of Algebra, Functional Analysis, and Representation Theory.

A.1 Algebra

Let $\mathcal{A} \neq \{0\}$ be a vector space over the complex numbers $\mathbb{C}$. $\mathcal{A}$ is called an associative algebra with unity over $\mathbb{C}$, if a product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}, (A, B) \mapsto AB$ is defined on $\mathcal{A}$ such that

\[
(AB)C = A(BC),
\]
\[
A(B + C) = AB + AC,
\]
\[
(A + B)C = AC + BC,
\]
\[
(\alpha A)B = A(\alpha B) = \alpha AB, \text{ for } \alpha \in \mathbb{C},
\]

and if there exists an element $I \in \mathcal{A}$ such that $IA = AI = A$ for all $A \in \mathcal{A}$.

A set $\mathcal{G}$ of elements of $\mathcal{A}$ is called a system of generators of $\mathcal{A}$ if the smallest closed subalgebra with unity containing $\mathcal{G}$ coincides with $\mathcal{A}$. The unity $I$ is not included in the system of generators.

Let us assume that the associative algebra with unity $\mathcal{A}$ is generated by $d$ elements $X_1, \ldots, X_d$, i.e. $\mathcal{G} = \{X_i\}_{i=1}^d$. Then each element of $\mathcal{A}$ can be written as

\[
A = a^0I + \sum_{i=1}^d a^i X_i + \sum_{i,j=1}^d a^{ij} X_i X_j + \ldots
\]

(A.1)

$a^0, a^i, a^{ij}, \ldots \in \mathbb{C}$. 

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We restrict the discussion to the case of a finite number of generators and we will not discuss topologies of $A$, i.e. it is assumed that the above sums for every $A$ are arbitrarily large but finite. Defining algebraic relations are relations among the generators

$$P(X_1, \ldots, X_d) = 0.$$  \hspace{1cm} (A.2)

where $P(x_1, \ldots, x_d)$ is a polynomial of $d$ variables with complex coefficients. Let $B \in A$ be represented by

$$B = b^0 I + \sum_{i=1}^d b^i X_i + \sum_{i,j=1}^d b^{ij} X_i X_j + \ldots$$ \hspace{1cm} (A.3)

If one can bring (A.3) into the same form as (A.1) with the same coefficients $a^0, a^i, \ldots$ by using the defining algebraic relations (A.2), then $B$ is equal to $A$.

A.2 Functional Analysis

A.2.1 Operators on Hilbert Space

We list here some definitions and properties of operators on Hilbert spaces which are used in the main body of the text. As in the previous section we denote by $C$ the set of complex numbers.

**Definition A.2.1** A complex vector space $H$ is called an inner product space if there exists a complex valued function $\langle \cdot, \cdot \rangle$ on $H \times H$ satisfying for all $\phi, \psi, \eta$ and $a, b \in C$:

$(i) \quad \langle a \psi + b \eta, \phi \rangle = a \langle \psi, \phi \rangle + b \langle \eta, \phi \rangle,$

$(ii) \quad \langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle},$

$(iii) \quad \langle \phi, \phi \rangle \geq 0 \text{ and } \langle \phi, \phi \rangle = 0 \text{ if and only if } \phi = 0.$

The function $\langle \cdot, \cdot \rangle$ is called an inner product.

One can show that every inner product space is a normed space with norm $|| \cdot || = \langle \cdot, \cdot \rangle^{1/2}$. A sequence $\{\phi_n\}$ in $H$ is called a Cauchy sequence if for every $\epsilon > 0$ there
exists a $N(\varepsilon)$, such that

$$\|\phi_n - \phi_m\| \leq \varepsilon \text{ for } n, m \geq N(\varepsilon).$$

An inner product space $H$ is called complete, if every Cauchy sequence in $H$ converges (that is has a limit in $H$). A complete inner product space is called a Hilbert space.

A Hilbert space is called separable if it contains a countable dense subset. One can show that a Hilbert space is separable if and only if it has a countable orthonormal bases.

Let $H$ and $H'$ be Hilbert spaces and let $T$ be a map from a linear subspace $D(T) \subset H$ to $H'$, such that

$$T(a\phi + b\psi) = aT\phi + bT\psi,$$

for all $\phi, \psi \in D(T)$ and all $a, b \in \mathbb{C}$, then $T$ is called an operator from $H$ to $H'$. The linear subspace $D(T)$ is called the domain of $T$. The set $R(T) = T(D(T))$ is called the range of $T$. An operator $T : H \to H'$ is called bounded if there exists some constant $C \geq 0$ such that $\|T\phi\|_{H'} \leq C\|\phi\|_H$, for all $\phi \in H$, otherwise it is called unbounded. Note that for bounded operators one has $D(T) = H$, hence bounded operators are defined on all of $H$. We denote the set of all bounded operators from $H$ to $H'$ by $\mathcal{L}(H,H')$, if $H = H'$ we write $\mathcal{L}(H)$. Let $T, S \in \mathcal{L}(H)$ then we define the product of $S$ and $T$ by $(ST)\phi = S(T\phi)$ for all $\phi \in H$. One can easily check that the set of all bounded operators $\mathcal{L}(H)$ forms an algebra.

Now let $T : D(T) \to H$ be a not necessarily bounded operator from $H$ to $H$. Let $T_1$ and $T$ be two operators on $H$. $T_1$ is called an extension of $T$ if and only if $D(T) \subset D(T_1)$ and $T_1\phi = T\phi$ for all $\phi \in D(T)$. If $T_1$ is an extension of $T$ we write $T \subset T_1$. Let $T : H \to H$ be a densely defined operator on $H$, i.e. $D(T)$ is a dense subset of $H$ and let $D(T^*)$ be the set of all $\psi \in H$ for which there is an $\eta \in H$ with

$$\langle T\phi, \psi \rangle = \langle \phi, \eta \rangle \text{ for all } \phi \in D(T).$$
For each such $\psi \in D(T^*)$ we define $T^*\psi = \eta$. Since $D(T)$ is dense in $H$, the vector $\eta$ is uniquely determined. $T^*$ is called the adjoint of $T$. If the domain of $T^*$ is dense in $H$, then we can define $T^{**} = (T^*)^*$.

For the moment let us concentrate on bounded operators on Hilbert spaces. An operator $T \in L(H)$ is called self-adjoint if $T = T^*$. We have

$$(S + T)^* = S^* + T^*, \quad (aS)^* = \overline{a}S^*, \quad (ST)^* = T^*S^*, \quad T^{**} = T.$$  

Thus we see that $L(H)$ can be regarded as an involutive $*$-algebra. Every subalgebra of $L(H)$ which is stable with respect to the adjoint operation is called a $*$-algebra.

Let $P \in L(H)$, if $P^2 = P$ and $P = P^*$, then $P$ is called an orthogonal projection. Two orthogonal projections are called mutually orthogonal if $P_1P_2 = P_2P_1 = 0$. We denote by $\pi(H)$ the set of all orthogonal projections.

An operator $U$ from $H$ onto $H'$ is called unitary if $\langle U\phi, U\psi \rangle_{H'} = \langle \phi, \psi \rangle_H$ for all $\phi, \psi \in H$. A unitary operator satisfies $U^*U = UU^* = I$. Two Hilbert spaces $H_1$ and $H_2$ are said to be isomorphic if there exists a unitary operator from $H_1$ onto $H_2$.

An operator $T \in L(H, H')$ is called compact if and only if for every bounded sequence $\{\phi_n\} \subset H$, the sequence $\{T\phi_n\} \subset H'$ has a subsequence that converges in $H'$. Let us note the following facts about nonzero self-adjoint compact operators in $L(H)$.

(i) Every nonzero self-adjoint compact operator has at least one eigenvector $\phi_\lambda$ which belongs to a nonzero eigenvalue $\lambda$. (see [1, pp. 124–126]).

(ii) From the Rellich-Hilbert-Schmidt Theorem [78, p. 42] we have the following spectral resolution for nonzero compact self-adjoint operators

$$T = \sum_{k=1}^\infty \lambda_k P_k, \quad \text{(A.4)}$$

where the $P_k$ are mutually orthogonal projections on the finite dimensional eigenspaces $H_k = P_kH$ and $|\lambda_k| \to 0$ as $k \to \infty$. Moreover, one has that
\[ H = \bigoplus_{k=1}^{\infty} H_k \bigoplus \ker(T), \text{ where } \ker(T) = \{ \phi \in H : T\phi = 0 \} \text{ is the kernel of } T. \] Hence, the Hilbert space \( H \) decomposes into a direct orthogonal sum of mutually orthogonal finite dimensional subspaces. If \( T \) is one-to-one, then \( \ker(T) \) consists only of the zero element.

An operator \( T \) is called trace class if and only if \( \text{tr}(\sqrt{T^*T}) < \infty \). If \( T \) is trace class and \( B \in \mathcal{L}(H) \), then \( TB \) and \( BT \) are trace class, furthermore, one has \( \text{tr}(TB) = \text{tr}(BT) \).

An operator is called Hilbert-Schmidt if and only if \( \text{tr}(T^*T) < \infty \). One can show that if \( T \) is trace class or Hilbert-Schmidt then \( T \) is compact.

We will now discuss unbounded operators on \( H \). An operator \( T \) on \( H \) is called closed if the relations

\[ \lim_{n \to \infty} \phi_n = \phi, \quad \lim_{n \to \infty} T\phi_n = \psi, \quad \{ \phi_n \} \subset \mathcal{D}(T) \]

imply \( \phi \in \mathcal{D}(T) \) and \( T\phi = \psi \). Here, the notation \( \lim_{n \to \infty} \eta_n = \eta \) means strong convergence and is shorthand for: For every \( \epsilon > 0 \) there exists a \( N(\epsilon) \) such that \( ||\eta_n - \eta|| < \epsilon \) for all \( n > N(\epsilon) \). Closedness is a weaker condition than continuity. If \( T \) is a continuous operator on \( H \), then \( \lim_{n \to \infty} \phi_n = \phi \) implies that the sequence \( \{ T\phi_n \} \) converges. On the other hand if \( T \) is only closed then the convergence of the sequence \( \{ \phi_n \} \subset \mathcal{D}(T) \) does not imply the convergence of the sequence \( \{ T\phi_n \} \). Nevertheless, if \( T \) is closed and the sequences \( \{ \phi_n \}, \{ \psi_n \} \subset \mathcal{D}(T) \) have the same limit, then the sequences \( \{ T\phi_n \} \) and \( \{ T\psi_n \} \) cannot have different limits. It is worth emphasizing that an operator is continuous if and only if it is bounded. Therefore, unbounded operators on \( H \) can not be continuous, however they can be closed.

If \( T \) is not closed, one can sometimes find a closed extension of \( T \). If a closed extension of \( T \) exists then \( T \) is called closeable, the smallest closed extension of \( T \) is called its closure, which is denoted by \( \overline{T} \). An operator \( T : H \to H' \) is closeable if and only if the following holds: If \( \{ \phi_n \} \) is a sequence in \( \mathcal{D}(T) \) with \( \lim_{n \to \infty} \phi_n = 0 \) and
\{T \phi_n\} \subset \mathcal{H}' \text{ is convergent, then } \lim_{n \to \infty} T \phi_n = 0. \text{ If } T \text{ is closeable, then}

\begin{align*}
\text{D}(\overline{T}) &= \{ \phi \in \mathcal{H} : \text{there exists a } \{\phi_n\} \text{ in } \text{D}(T) \text{ with } \lim_{n \to \infty} \phi_n = \phi, \\
&\text{such that } \{T \phi_n\} \text{ is convergent} \}, \\
\overline{T} \phi &= \lim_{n \to \infty} T \phi_n \text{ for } \phi \in \text{D}(\overline{T}).
\end{align*}

There exists a simple relationship between the notions of adjoint and closure. Let \( T \) be a densely defined operator on \( \mathcal{H} \). Then:

(i) \( T^* \) is closed.

(ii) \( T \) is closeable, if and only if \( \text{D}(T^*) \) is dense in \( \mathcal{H} \) in which case \( \overline{T} = T^{**} \).

(iii) If \( T \) is closeable, then \( (\overline{T})^* = T^* \).

(For a proof see Ref. 87, p. 253.)

Example of a non-closeable operator: Let \( \mathcal{H} = \ell^2 \), the space of all absolutely square summable sequences, i.e. \( \ell^2 = \{\{a_n\} : \sum_{n=1}^{\infty} |a_n|^2 < \infty\} \). Consider the following operator

\begin{align*}
T : \text{D}(T) &\to \ell^2 \\
\ell^2 \ni \{a_n\} &\mapsto \{ \sum_{n=1}^{\infty} na_n, 0, 0, \ldots \}.
\end{align*}

\[ \text{D}(T) = \{ \{a_n\} \in \ell^2 : \sum_{n=1}^{\infty} n^2 |a_n|^2 < \infty \} \]

One can easily see that \( \text{D}(T) \) contains the dense subspace \( \ell^2_0 \) of \( \ell^2 \), where \( \ell^2_0 = \{\{a_n\} : \{a_1, \ldots, a_n, 0, 0, \ldots\} \} \) is the space of all finite sequences, i.e. only finitely many entries of \( \{a_n\} \) are nonzero. Hence, \( T \) is densely defined, but \( T \) is non-closeable. This can be seen as follows. The adjoint of \( T \) is given by

\[ T^* \{a_n\} = \{a_1, 2a_1, 3a_1, \ldots, na_1, \ldots\} \]

Now for \( \{a_n\} \) to be in \( \text{D}(T^*) \) we must have that \( ||T^* \{a_n\}|| = \left( \sqrt{\sum_{n=1}^{\infty} n^2} \right) a_1 < \infty \), which implies that \( a_1 = 0 \). Therefore \( \text{D}(T^*) \) consists of all elements of \( \ell^2 \) whose
first entry is zero. Hence, the one dimensional subspace spanned by \{1,0,0,\ldots\} is orthogonal to \(D(T^*)\), which implies that \(T^*\) is not densely defined, and therefore, \(T\) is not closeable. ◇

A densely defined operator \(T\) on a Hilbert space \(H\) is called *symmetric* if \(T \subset T^*\), that is, if \(D(T) \subset D(T^*)\) and \(T\phi = T^*\phi\) for all \(\phi \in D(T)\). Equivalently, \(T\) is symmetric if and only if
\[
\langle T\phi, \psi \rangle = \langle \phi, T\psi \rangle \text{ for all } \phi, \psi \in D(T).
\]

An operator is called *self-adjoint* if and only if \(T\) is symmetric and \(D(T) = D(T^*)\). Note that symmetric operators are always closeable, since \(D(T^*) \supset D(T)\) is dense in \(H\). If \(T\) is a symmetric operator, then \(T^*\) is an extension of \(T\), so the smallest closed extension \(T^{**}\) of \(T\) has to be contained in \(T^*\). Hence, one has for symmetric operators
\[
T \subset T^{**} \subset T^*.
\]

For closed symmetric operators,
\[
T = T^{**} \subset T^*.
\]

And for self-adjoint operators,
\[
T = T^{**} = T^*.
\]

The distinction between closed symmetric operators and self-adjoint operators is very important. Only self-adjoint operators have a spectral resolution (see below) and only self-adjoint operators may be exponentiated to give one-parameter unitary groups which give the dynamics of a quantum system.

A symmetric operator is called essentially self-adjoint if its closure \(\overline{T}\) is self-adjoint. To show that an operator \(T\) is essentially self-adjoint it is necessary and sufficient to show that \(\ker(T^* \pm iI) = \{0\}\). In other words one has to show that the equation
\[
T^*\phi_{\pm} = \pm i\phi_{\pm}
\]
has no solutions in \( H \) other than \( \phi_\pm = 0 \). For self-adjoint operators one has the following spectral decomposition [87, p. 263]:

**Theorem A.2.2** (Spectral Theorem) *There is a one-to-one correspondence between self-adjoint operators \( T \) and the projection valued measures \( P^T(\cdot) \) on \( H \). This correspondence is given by

\[
T = \int_{-\infty}^{+\infty} \lambda P^T(d\lambda),
\]

where a projection-valued-measure is a map from the Borel measurable sets of \( \mathbb{R} \) into the set of all orthogonal projections \( \pi(H) \) satisfying the following conditions:

(i) \( P^T(\emptyset) = 0 \) and \( P^T(\mathbb{R}) = I \).

(ii) If \( \{E_i\}_{i \in \mathbb{N}^*}, N^* = \{1, 2, \ldots\}, \) is a sequence of mutually disjoint real Borel measurable sets, then

\[
P^T(\bigcup_{i \in \mathbb{N}^*} E_i) = \sum_{i \in \mathbb{N}^*} P^T(E_i).
\]

(iii) \( P^T(E)P^T(F) = P^T(E \cap F) \).

**A.2.2 Direct Integrals**

All Hilbert spaces in this section are separable. Let \( \Xi \) be a locally compact separable space and let \( \nu \) be a positive measure on \( \Xi \). For every \( \zeta \in \Xi \) let there exist a Hilbert space \( H_\zeta \) with inner product \( \langle \cdot, \cdot \rangle_\zeta \).

A *vector field* \( \psi(\cdot) \) is a map from \( \Xi \) to \( \prod_{\zeta \in \Xi} H_\zeta \), such that \( \Xi \ni \zeta \mapsto \psi_\zeta \in H_\zeta \).

A countable family of vector fields \( \{\psi_i(\cdot)\}_{i \in \mathbb{N}^*} \) is called a *fundamental family* if the following two conditions are fulfilled:

(i) All functions \( \Xi \ni \zeta \mapsto \langle \psi_i^j, \psi_j^i \rangle_\zeta \) are \( \nu \)-measurable for \( i, j \in \mathbb{N}^* \).

(ii) For all \( \zeta \in \Xi \), the family of vectors \( \{\psi_i^j\}_{i \in \mathbb{N}^*} \) spans the space \( H_\zeta \).

A vector field \( \psi(\cdot) \) is called *measurable* if all the functions \( \zeta \mapsto \langle \psi_\zeta, \psi_i^j \rangle_\zeta, i \in \mathbb{N}^* \), are \( \nu \)-measurable. Let us note the following facts about measurable vector fields (cf. [78, Lemma I.6.5])

(i) The measurable vector fields form a linear subspace of \( \prod_{\zeta \in \Xi} H_\zeta \).
(ii) If \( \psi(.) \) is a measurable vector field, then \( \|\psi(.)\|_\zeta \) is a \( \nu \)-measurable function.

(iii) If \( \psi(.) \) and \( \phi(.) \) are \( \nu \)-measurable vector fields, then \( \langle \psi_\zeta, \phi_\zeta \rangle_\zeta \) is a measurable function.

Using the Gram-Schmidt orthogonalization procedure, one can construct a complete orthonormal set of vector fields, i.e., a sequence \( \phi^1, \phi^2, \ldots \) of measurable vector fields such that

(i) If \( \dim H_\zeta = \infty \), then the set \( \{ \phi^i_\zeta \}_{i=1}^\infty \) spans \( H_\zeta \) and \( \langle \phi^i_\zeta, \phi^j_\zeta \rangle_\zeta = \delta_{ij} \), where \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) if \( i \neq j \).

(ii) If \( \dim H_\zeta = d < \infty \), then \( \phi^1_\zeta, \ldots, \phi^d_\zeta \) form an orthonormal basis of \( H_\zeta \), and \( \phi^j_\zeta = 0 \) for \( j > d \).

One calls a measurable vector field \( \phi(.) \) square integrable if

\[
\int_{\Xi} \|\phi_\zeta\|^2_\zeta d\nu(\zeta) < \infty.
\]

One calls two measurable vector fields equivalent, if they are equal \( \nu \)-almost everywhere on \( \Xi \).

**Definition A.2.3** *The space of equivalence classes of measurable square integrable vector fields \( \phi(.) \) equipped with the innerproduct

\[
\langle \phi, \psi \rangle \equiv \int_{\Xi} \langle \phi_\zeta, \psi_\zeta \rangle_\zeta d\nu(\zeta),
\]

is called the direct integral of the Hilbert spaces \( H_\zeta \). We denote this space by the symbol

\[
\int_{\Xi} H_\zeta d\nu(\zeta).
\]

Generalizing the arguments used in the proof of the Riesz-Fischer Theorem (cf. [88, p. 59]) one can show that \( \int_{\Xi} H_\zeta d\nu(\zeta) \) is complete. Hence, \( \int_{\Xi} H_\zeta d\nu(\zeta) \) is a Hilbert space, which we simply denote by \( H \).
Examples

(i) let $\Xi = \mathbb{N}_*$ and let $\nu(n) = 1$ for all $n \in \mathbb{N}_*$; then every vector field is measurable and

$$H = \int_{\Xi} H_\zeta d\nu(\zeta)$$

can be identified with $\bigoplus_{\zeta \in \mathbb{N}_*} H_\zeta$.

 Hence, for this case the direct integral reduces to the direct orthogonal sum.

(ii) If $\dim H_\zeta = 1$ for all $\zeta \in \Xi$, then one can choose the fundamental family in such a way that all vector fields are complex measurable functions. Hence,

$$H = \int_{\Xi} H_\zeta d\nu(\zeta)$$

can be identified with $L^2(\Xi, d\nu)$.

One can show that $H = \int_{\Xi} H_\zeta d\nu(\zeta)$ is separable if $\Xi$ is separable (cf. [78, Proposition I.6.8]).

A.2.2.1 Diagonal and Decomposable Operators

We call two function $f$ and $g$ equivalent if $f = g$ $\nu$-almost everywhere. We denote by $L^\infty(\Xi, d\nu)$ the space of all equivalence classes of measurable functions $f$ which are bounded except possibly on a set of measure zero. Then $L^\infty(\Xi, d\nu)$ is a linear space, and it becomes a normed linear space if we define

$$\|f\|_{\infty} = \text{ess sup}|f(\zeta)|,$$

where $\text{ess sup} f(\zeta)$ is the infimum of $\sup g(\zeta)$ as $g$ ranges over all functions which are equal to $f$ $\nu$-almost everywhere. Thus

$$\text{ess sup} f(\zeta) = \inf\{M : \nu(\{\zeta : f(\zeta) > M\}) = 0\}.$$ 

Let $f \in L^\infty(\Xi, d\nu)$ and let $I_\zeta$ be the identity operator on $H_\zeta$, then we call the operator field

$$\Xi \ni \zeta \mapsto f(\zeta)I_\zeta \in \mathcal{L}(H_\zeta)$$

a continuously diagonal operator in the Hilbert space $H = \int_{\Xi} H_\zeta d\nu(\zeta)$. One can associate with the operator field $\zeta \mapsto f(\zeta)I_\zeta$ the following bounded operator in $H$:

$$H \ni \phi \mapsto T(f)\phi \in H$$

where $(T(f)\phi)(\zeta) \equiv f(\zeta)\phi_\zeta$ for all $\zeta \in \Xi$. 
One can show that \( \|T(f)\| = \|f\|_{\infty} \), (cf. [78, Proposition I.6.9]). We call an operator field \( T(z) : \Omega \ni z \mapsto T_z \in \mathcal{L}(H_z) \) measurable if all functions \( z \mapsto \langle \phi^i_z, T_z \phi^j_z \rangle_z \), where \( \{\phi^i\} \) is a fundamental family of vector fields, are measurable. If \( \phi_z \) is a measurable vector field and \( T_z \) is a measurable operator field, then \( z \mapsto T_z \phi_z \in H_z \) is a measurable vector field. This can be seen as follows, since \( z \mapsto \langle \phi^i_z, T_z \phi^j_z \rangle_z = \langle T^*_z \phi^i_z, \phi^j_z \rangle_z \) is measurable, the vector field \( T^*_z \phi_z^i \) is measurable for every \( i \in \mathbb{N}_* \). Hence, \( z \mapsto \langle \phi^i_z, T_z \phi_z \rangle_z = \langle T^*_z \phi^i_z, \phi_z \rangle_z \) is measurable for every \( i \in \mathbb{N}_* \), since \( z \mapsto \langle \phi_z, \psi_z \rangle_z \) is measurable if \( \phi_z \) and \( \psi_z \) are measurable.

We now introduce the concept of decomposable operators. Let \( T(z) \) be a measurable operator field such that the function

\[
||T(z)|| = (z \mapsto ||T_z||_z) \in L^\infty(\Omega, d\nu).
\]

Define \( C = \text{ess sup} ||T(z)|| \). For every \( \phi_z \in H \), the vector field \( z \mapsto T_z \phi_z \) is measurable and one has

\[
||T_z \phi_z||_z \leq ||T_z||_z ||\phi_z||_z \leq C ||\phi_z||_z,
\]

\( \nu \)-almost everywhere. Therefore,

\[
\int_\Omega ||T_z \phi_z||^2 d\nu(z) \leq C^2 \int_\Omega ||\phi_z||^2 d\nu(z) \leq C^2 ||\phi||^2.
\]

If one denotes the vector field \( z \mapsto T_z \phi_z \) by \( T\phi \) then one has \( T\phi \in H \) and \( ||T\phi|| \leq C ||\phi|| \). Thus \( T \) is a bounded operator on \( H \). One calls the operator \( T \) a decomposable operator and we shall denote it by

\[
T = \int_\Omega T_z d\nu(z).
\]

Every diagonal operator is a decomposable operator. We will remark below on the relationship between these operators. One can show that, (see [78, Proposition I.6.9]).

\[
||\int_\Omega T_z d\nu(z)|| = \text{ess sup} ||T_z||_z.
\]
Let us note the following properties of decomposable operators:

(i) \( \int (S_\zeta + T_\zeta) d\nu(\zeta) = \int S_\zeta d\nu(\zeta) + \int T_\zeta d\nu(\zeta) \).

(ii) \( \int aT_\zeta d\nu(\zeta) = a \int T_\zeta d\nu(\zeta) \).

(iii) \( \int T_\zeta^* d\nu(\zeta) = (\int T_\zeta d\nu(\zeta))^* \).

(iv) \( \int S_\zeta T_\zeta d\nu(\zeta) = \int S_\zeta d\nu(\zeta) \circ \int T_\zeta d\nu(\zeta) \).

(v) If \( \nu \)-almost all \( U_\zeta \) are unitary, then \( \int U_\zeta d\nu(\zeta) \) is an unitary operator in \( \mathbf{H} \).

Hence, the decomposable operators form a \( \ast \)-algebra. Let \( \mathcal{A} \subset \mathcal{L}(\mathbf{H}) \) be a \( \ast \)-algebra, we denote by \( \mathcal{A}' \) the set of those elements of \( \mathcal{L}(\mathbf{H}) \) that commute with all elements of \( \mathcal{A} \). \( \mathcal{A}' \) is called the commutant of \( \mathcal{A} \) and \( (\mathcal{A}')' = \mathcal{A}'' \) is called the bicommutant of \( \mathcal{A} \). One has that \( \mathcal{A} \subset \mathcal{A}'' \), and that \( \mathcal{A} \subset \mathcal{B} \) implies \( \mathcal{B}' \subset \mathcal{A}' \).

**Definition A.2.4** A von Neumann algebra in \( \mathbf{H} \) is a \( \ast \)-subalgebra \( \mathcal{A} \) of \( \mathcal{L}(\mathbf{H}) \) such that \( \mathcal{A} = \mathcal{A}'' \).

Since \( \mathcal{A} \subset \mathcal{A}'' \) implies that \( (\mathcal{A}'')' \subset \mathcal{A}' \) and since we have that \( \mathcal{A}' \subset (\mathcal{A}'')'' \) we conclude that \( \mathcal{A}' = (\mathcal{A}'')'' \). Hence the commutant of every \( \ast \)-algebra is a von Neumann algebra. Other examples of von Neumann algebras are \( \mathcal{L}(\mathbf{H}) \) and its commutant the scalar operators \( \mathcal{C}_\mathbf{H} \). There exists the following relationship between diagonal and decomposable operators.

**Theorem A.2.5** (von Neumann)

(i) The algebra \( \mathcal{D} \) of diagonal operators is a commutative von Neumann algebra.

(ii) The commutant \( \mathcal{D}' \) of \( \mathcal{D} \) is the von Neumann algebra \( \mathcal{R} \) of decomposable operators in \( \mathbf{H} \), i.e.,

\[
\mathcal{D}' = \mathcal{R}, \quad \mathcal{R}' = \mathcal{D}.
\]

(For a proof see Ref. 78, Theorem I.6.24)
Let $\mathcal{A}$ be a von Neumann algebra, the subalgebra

$$\mathcal{Z} = \{X : XY = YX \text{ for every } Y \in \mathcal{A}\}$$

is called the center of $\mathcal{A}$. Note that $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$.

**Definition A.2.6** A von Neumann algebra is called a factor if and only if its center contains only the scalar operators.

**Examples:**

(i) The set $\mathcal{L}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$ is a factor.

(ii) A von Neumann algebra $\mathcal{A}$ which is isomorphic to $\mathcal{L}(\mathcal{H}')$ for some Hilbert space $\mathcal{H}'$, is a factor. Such a factor is said to be of type I. Below we shall give another definition of factors of type I.

(iii) If $U : G \ni g \mapsto U_g$ is an irreducible, unitary representation of a Lie group $G$ (cf. subsection A.5.2), then the von Neumann algebra generated by $U$ is a factor of type I.

(iv) If the von Neumann algebra $\mathcal{A}$ is a factor, then its commutant $\mathcal{A}'$ is also a factor.

**Definition A.2.7** A von Neumann algebra is of type I if it is isomorphic to a von Neumann algebra $\mathcal{B}$ which has an abelian commutant; $\mathcal{A} \cong \mathcal{B}$ and $\mathcal{B}'$ is commutative.

**Definition A.2.8** Let $\mathcal{A}$ be a von Neumann algebra, and $P$ a projection of $\mathcal{A}$. One says that $P$ is minimal (relative to $\mathcal{A}$) if $P \neq 0$ and every projection of $\mathcal{A}$ majorized by $P$ is equal to 0 or to $P$.

The following Theorem gives a characterization of type I factors:

**Theorem A.2.9** Let $\mathcal{A}$ be a factor in $\mathcal{H}$. Then the following are equivalent:

(i) $\mathcal{A}$ is of type I.
(ii) $A'$ is of type I.

(iii) $A$ possesses minimal projections.

(iv) $A'$ possesses minimal projections.

(v) There exist Hilbert spaces $H'$, $H''$, and an isomorphism of $H$ onto $H' \otimes H''$ which transforms $A$ into $L(H') \otimes C_{H''}$ and $A'$ into $C_{H'} \otimes L(H'')$.

(For a proof see Ref. 26, Corollary 1.8.2.3.)

There are also other types of factors such as factors of type II and type III, however we shall not be concerned with them. For very readable accounts of the classification of von Neumann algebras we refer the interested reader to the monographs by Emch [33] and Gaal [39].

A.3 The Nuclear Spectral Theorem

The Spectral Theorem A.2.2 for self-adjoint operators is an essential tool in many fields of Mathematics. Nevertheless, Theorem A.2.2 does not give the most convenient form of the spectral resolution of a self-adjoint operator in quantum mechanics. There is however a form of the spectral theorem called the Nuclear Spectral Theorem, conjectured by Dirac, which is especially convenient in quantum mechanics. The Nuclear Spectral Theorem is the analog to the spectral decomposition of a compact self-adjoint operator for a general self-adjoint operator defined on a nuclear-space. To do this Theorem any justice, one would have to devote a whole chapter or even an entire monograph to it. We refer the interested reader to the excellent monograph by Maurin [78] which discusses the Nuclear Spectral Theorem and many of its applications in Mathematics and Physics at length. For a more Physics oriented introduction to the Nuclear Spectral Theorem we refer the reader to Böhm [11]. Let us introduce some terminology and then simply state the Nuclear Spectral Theorem in a physics minded way.
A.3.1 Some Topological Notions

Definition A.3.1 Let $X$ be a set. A family of subsets $\mathcal{T}$ of $X$ is called a topology on $X$, if the following axioms hold:

$\text{Top}_1 \emptyset \in \mathcal{T}, \ X \in \mathcal{T}$.
$\text{Top}_2 \ O_1, O_2 \in \mathcal{T}$ implies that $O_1 \cap O_2 \in \mathcal{T}$.
$\text{Top}_3 \ \mathcal{W} \subset \mathcal{T}$ implies that $\bigcup_{W \in \mathcal{W}} W \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space. The elements of $X$ are called points of the topological space. The elements of $\mathcal{T}$ are called open sets in $(X, \mathcal{T})$.

A subset $A$ of a topological space $(X, \mathcal{T})$ is called closed if and only if its relative complement $X \sim A$ is open. The $\mathcal{T}$-closure of a subset $A$ of a topological space $(X, \mathcal{T})$ is the intersection of the members of the family of all closed sets containing $A$. We denote the closure of $A$ by $\overline{A}$. Since $\overline{A}$ is the intersection of closed sets it is always closed. Furthermore, since $\overline{A}$ is contained in every closed set containing $A$ it is the smallest closed set containing $A$. This yields an alternate definition of closedness: A set $A$ is closed if and only if $A = \overline{A}$.

Let $X$ be a set and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on $X$. If $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say $\mathcal{T}_1$ is a weaker topology than $\mathcal{T}_2$ and $\mathcal{T}_2$ is a stronger topology than $\mathcal{T}_1$. This terminology derives from the fact that fewer sequences converge in $\mathcal{T}_2$ than do in $\mathcal{T}_1$; so $\mathcal{T}_2$-convergence is a stronger notion than $\mathcal{T}_1$-convergence.

A.3.2 Nuclear Space

Let $\Psi \subset \mathcal{H}$ be a dense set of analytic vectors for an essentially self-adjoint, positive definite operator $\Delta$ on $\mathcal{H}$. For the definition of analytic vectors see subsection 3.1. We denote by $\mathcal{T}_\mathcal{H}$ the usual topology on $\mathcal{H}$ generated by the norm on $\mathcal{H}$. The open sets $\mathcal{O}$ of this topology are those sets, $O \subset \mathcal{H}$, with the property that for all $\phi \in O$ there exists an $r > 0$ such that the set $\{\psi : \|\psi - \phi\| < r\}$ is contained in $O$. The closure of $\Psi$ in the $\mathcal{T}_\mathcal{H}$-topology is $\mathcal{H}$. 
Let us now introduce another topology on $\Psi$ which we call $T_\Phi$. Take the inner product on $H$ and define a family of inner products and norms on $\Psi$. Let $\psi, \phi \in \Psi$ then we define
\[
\langle \psi, \phi \rangle_p = \langle \psi, (\Delta + I)^p \phi \rangle \quad \text{for } p = 0, 1, 2, \ldots
\]
\[
\|\phi\|_p = \langle \phi, \phi \rangle_p^{1/2}.
\]
Since $\Delta$ is symmetric and positive definite it is easy to see that $\langle \cdot, \cdot \rangle_p$ fulfills all the axioms in Definition A.2.1 for an inner product, furthermore one has that
\[
\|\phi\|_0 \leq \|\phi\|_1 \leq \|\phi\|_2 \leq \ldots
\]
We call a space with a countable number of inner products (norms) a countably inner product (countably normed) space. We now define $T_\Phi$ by:

**Definition A.3.2** A sequence \( \{\phi_n\}_{n=1}^{\infty} \) in $\Psi$ converges in the topology $T_\Phi$ to $\phi$ in $\Psi$ if, for each $p$,
\[
\lim_{n \to \infty} \|\phi_n - \phi\|_p = 0.
\]
If a sequence \( \{\phi_n\} \) converges to $\phi$ in the topology $T_\Phi$ then \( \{\phi_n\} \) also converges in the topology $T_H$ to $\phi$, but not vice versa. Hence, $T_\Phi$ is a stronger topology than $T_H$.

In analogy to the case of an ordinary inner product space we call a sequence \( \{\phi_n\} \) a $T_\Phi$-Cauchy sequence if for every $p$ and every $\epsilon > 0$ there exists an $N(\epsilon, p)$ such that $\|\phi_m - \phi_n\|_p < \epsilon$ for all $m, n > N(\epsilon, p)$.

We now complete the space $\Psi$ with respect to the topology $T_\Phi$, i.e. we add the limits of the $T_\Phi$-Cauchy sequences to $\Psi$. We call the linear space we have so obtained $\Phi$. Note that $\Psi \subset \Phi$ is dense in $\Phi$ in the $T_\Phi$ topology. We call $\Phi$ a countably Hilbert space. In Dirac's terminology the elements of this space are called ket vectors. Furthermore, since $T_\Phi$ is stronger than $T_H$ we have that
\[
\Psi \subset \Phi \subset H.
\]
Therefore, $\Phi$ is $T_H$-dense in $H$, since $\Psi$ is already $T_H$-dense in $H$. 
Let us consider a Lie algebra $L$ of symmetric operators on a Hilbert space $H$ which have a common dense invariant domain $D$. Let $X_1, \ldots, X_d$ be an operator basis for $L$ such that the Nelson operator $\Delta = \sum_{i=1}^{d} X_i^2$ is essentially self-adjoint. It then follows that $X_1, \ldots, X_d$ are essentially self-adjoint (cf. [82, Lemmas 5.2 & 6.2]). Since the $X_k$, $k = 1, \ldots, d$, are symmetric $\Delta$ is a positive definite operator. Furthermore, since $\Delta$ is essentially self-adjoint there exists on $H$ by Theorem 5 in Ref. 82 a unique unitary representation $U$ of the simply connected locally compact Lie group $G$ which has $L$ as its Lie algebra such that for all $X$ in $L$, $U(X) = \overline{X}$. Now let us denote by $A_U$ the dense set of analytic vectors for the representation $U$ of $G$. It is shown in Ref. 7, pp. 364–365, that the dense set $A_U \subset H$ of analytic vectors forms a common dense invariant domain for the basis $X_1, \ldots, X_d$ of $L$ and its enveloping algebra $\mathcal{E}(G)^1$. Therefore, every element of $A_U$ is in the set of analytic vectors for $\Delta$.

Let $\Psi$ be the dense set of analytic vectors $A_U$ for $U$, then we can as outlined above construct a countably Hilbert space $\Phi$. We now show that the elements of the enveloping algebra $\mathcal{E}(G)$ are continuous with respect to $T_\Phi$ and are therefore, uniquely defined on the whole space $\Phi$. To show this it is sufficient to show that the generators $X_1, \ldots, X_d$ are $T_\Phi$-continuous, since the sum and products of continuous operators are continuous (see [105, pp.63–64]). Let us consider $X_1, \ldots, X_d$ on $\Psi$.

We use Lemma 6.3 in Ref. 82: For every $\phi \in \Psi$ one has

$$\langle \phi, X_i (\Delta + I)^p X_i \phi \rangle \leq k \langle \phi, (\Delta + I)^{p+1} \phi \rangle,$$

(A.5)

where $k < \infty$ is some constant and $X_i$, $i = 1, \ldots, d$ is one of the generators. Let $\{\phi_n\}$ be a sequence converging to zero in the $T_\Phi$-topology, i.e., $\lim_{n \to \infty} \|\phi_n\|_p = 0$ for every $p$. This is equivalent to

$$\lim_{n \to \infty} \langle \phi_n, (\Delta + I)^p \phi_n \rangle = 0 \text{ for every } p. \quad (A.6)$$

Let $X_i$ be arbitrary, then to show that $X_i$ is a continuous operator it is sufficient to

\footnote{For the definition of an enveloping algebra of a Lie group $G$ see A.4.2}
show that
\[ \lim_{n \to \infty} \|X_i \phi_n\|_q = 0 \text{ for every } q, \] (A.7)
i.e., that
\[ \lim_{n \to \infty} \langle X_i \phi_n, (\Delta + I)^q X_i \phi_n \rangle = \lim_{n \to \infty} \langle \phi_n, X_i (\Delta + I)^q X_i \phi_n \rangle = 0 \text{ for every } q, \]
(cf. [87, Theorem I.6]). By (A.5) one has:
\[ \langle \phi_n, X_i (\Delta + I)^q X_i \phi_n \rangle \leq k \langle \phi_n, (\Delta + I)^{q+1} \phi_n \rangle \leq k \|\phi_n\|_{q+1}^2, \]
however, by (A.6) the right hand side converges to zero as \( n \to \infty \) for every \( q \), and therefore, the left hand side also converges to zero for every \( q \), this establishes (A.7).

Since \( X_i \) was arbitrary this shows that all generators are continuous operators. Since \( \Psi \) is a \( T_\Phi \) dense linear subspace of \( \Phi \) we can, using the B.L.T. Theorem (cf. [87, Theorem I.7]), uniquely extend the linear operators \( X_i, i = 1, \ldots, d \) on \( \Psi \) to operators on the whole space \( \Phi \). Note that since the operators \( X_i, i = 1, \ldots, d \) are continuous on \( \Phi \) they are defined everywhere on \( \Phi \), hence, domain questions do not arise.

We are now ready to give the definition of a nuclear space.

**Definition A.3.3** \( \Phi \) is a **nuclear space** if and only if there exists an essentially self-adjoint \( T_\Phi \)-continuous operator \( \Delta \in \mathcal{E}(G) \), whose inverse is Hilbert-Schmidt.

This definition uses a Theorem of Roberts [89, Theorem 1]. It has been shown that the enveloping algebra \( \mathcal{E}(G) \) of the following groups \( G \) have the property of nuclearity:

(i) \( G \) is nilpotent (cf. [57]). This is the case we are considering in chapter 4.4.2.

(ii) \( G \) is semisimple (cf. [10, Appendix B]).

Summarizing, if \( G \) is either nilpotent or semisimple, then the space \( \Phi \) we have constructed above is a **linear nuclear space** on which all elements of the enveloping algebra \( \mathcal{E}(G) \) are **continuous operators**.
A.3.3 Linear Functionals

A linear functional $L$ on a linear space $\Xi$ is a linear map from $\Xi$ to $C$ such that
\[ \Xi \ni \phi \mapsto L(\phi) \equiv \langle L | \phi \rangle \in C \] and
\[ L(a\phi + b\psi) = aL(\phi) + bL(\psi) \] for $\phi, \psi \in \Xi$ and $a, b \in C$.

We call $\langle \cdot | \cdot \rangle$ a generalized inner product. A linear functional is called $\mathcal{F}$-continuous if and only if for every $\epsilon > 0$ there exists an integer $q$ and a $\delta > 0$, such that
\[ \|\phi - \psi\|_q < \delta \text{ implies } |L(\phi) - L(\psi)| < \epsilon. \]

A second alternative is: if
\[ \lim_{n \to \infty} \|\phi_n - \phi\|_q = 0 \text{ for all } q \]
then
\[ \lim_{n \to \infty} |L(\phi_n) - L(\phi)| = 0. \]

We denote the space of all $\mathcal{F}$-continuous linear functionals acting on $\Phi$ by $\Phi'$. In Dirac's terminology the elements of this space are called bra vectors. Since $\Phi \subset H$ one finds that $H' \subset \Phi'$. Since $H$ is a Hilbert space one has that $H = H'$, this yields the following sequence of inclusions
\[ \Phi \subset H = H' \subset \Phi' \]

This triplet is called a Gel'fand triplet or a Rigged Hilbert space. This notion enables one to give a precise mathematical meaning to Dirac's bra and ket vector formalism. The bra vectors are linear $\mathcal{F}$-continuous functionals acting on the ket vectors which are the elements of the nuclear space $\Phi$. Since $\Phi \subset \Phi'$, there are more bra vectors then there are ket vectors.

For every continuous operator $T$ on $\Phi$ one can define the adjoint operator $T^\dagger$ on $\Phi'$ by
\[ \langle T^\dagger L | \phi \rangle \equiv (T^\dagger L)(\phi) = L(T\phi) = \langle L | T\phi \rangle, \]
for all $L \in \Phi'$, $\phi \in \Phi$. If $T$ is a continuous operator on $\Phi$ and $L$ is a continuous linear functional on $\Phi'$ then $T^\dagger$ is a continuous operator on $\Phi'$, i.e., one has that

$$T^\dagger L_n \to T^\dagger L \text{ for all } L_n \to L.$$ 

For every $T_\Phi$-continuous essentially self-adjoint operator $T$ we have in correspondence to the relation

$$\Phi \subseteq H = H' \subseteq \Phi'$$

between the spaces, the relation

$$T \subseteq \overline{T} = T^* \subseteq T^\dagger$$

between the operators.

A.3.4 Generalized Eigenvectors and the Nuclear Spectral Theorem

We have now collected all the tools we need to state the Nuclear Spectral Theorem.

**Definition A.3.4** We call a family of symmetric operators $\{X_i\}_{i=1}^d$ a system of commuting operators if and only if

(i) $[X_i, X_j] = 0$ for all $i \neq j$.

(ii) $\Delta = \sum_{k=1}^d X_k^2$ is essentially self-adjoint.

The family of operators $\{X_i\}_{i=1}^d$ is called a complete commuting system of operators if and only if there exists a vector $\phi \in \Phi$ such that the linear subspace

$$\mathcal{A}\phi = \{X\phi : X \in \mathcal{A} \equiv \text{algebra generated by the family } \{X_i\}_{i=1}^d\}$$

is dense in $\Phi$. The vector $\phi \in \Phi$ is called a cyclic vector for the $*$-algebra $\mathcal{A}$.

One can show that the $*$-algebras generated by the two commuting systems $\{\mathcal{L}^k\}_{k=1}^d$ and $\{P_{L^k}\}_{k=1}^d$ given in chapter 4.4.2 have cyclic vectors (cf. [11, p. 37]), hence, $\{\mathcal{L}^k\}_{k=1}^d$ and $\{P_{L^k}\}_{k=1}^d$ form two separate complete commuting systems of operators, respectively.
Definition A.3.5 Let \( \{X_k\}_{k=1}^d \) be a complete commuting system of operators on \( \Phi \).

A generalized eigenvector of the complete commuting system \( \{X_k\}_{k=1}^d \) is a linear functional \( L_\zeta \in \Phi' \) such that

\[
\langle L_\zeta | X_k \phi \rangle = \langle X_k^\dagger L_\zeta | \phi \rangle = \bar{\zeta}_k \langle L_\zeta | \phi \rangle
\]

holds for every \( \phi \in \Phi \), which may formally be written as:

\[
X_k^\dagger L_\zeta = \bar{\zeta}_k L_\zeta.
\]

The \( d \)-tupel of numbers \( \zeta = (\zeta_1, \ldots, \zeta_d) \) is called a generalized eigenvalue corresponding to the generalized eigenvector \( L_\zeta \equiv \langle \zeta_1, \ldots, \zeta_d \rangle \).

According to the next theorem there exists a complete system of generalized eigenvectors.

Theorem A.3.6 (Nuclear Spectral Theorem) Let \( \{X_k\}_{k=1}^d \) be a complete commuting system of \( \mathcal{T}_\Phi \)-continuous operators on the Gel'fand triplet \( \Phi \subset H \subset \Phi' \). Then there exist generalized eigenvectors \( \langle \zeta_1, \ldots, \zeta_d \rangle \in \Phi' \)

\[
\langle \zeta_1, \ldots, \zeta_d | X_k^\dagger \rangle = \zeta_k \langle \zeta_1, \ldots, \zeta_d \rangle,
\]

where \( \zeta_k \in \text{spec}(X_k) \subset \mathbb{R} \), such that for every \( \phi \in \Phi \) and some uniquely defined measure \( \nu \) on \( \Xi = \text{spec}(X_1) \times \ldots \times \text{spec}(X_d) \) one has the following spectral synthesis

\[
\phi = \int_{\Xi} |\zeta_1, \ldots, \zeta_d \rangle \langle \zeta_1, \ldots, \zeta_d | \phi \rangle d\nu(\zeta).
\]

The Nuclear Spectral Theorem can also be stated in a more general form using the direct integral (cf. [7, p. 658]), however, we have made use of a Theorem of von Neumann that states that for the case we are considering the direct integral is given by \( L^2(\Xi, d\nu) \) and that the spectrum of the commutative von Neumann algebra generated by the complete commuting system of operators \( \{X_k\}_{k=1}^d \) is multiplicity free (cf. [78, p. 59] and [87, Theorem VII.5]). The Nuclear Spectral Theorem gives a mathematically precise formulation of the famous Dirac conjecture.
Before we can give the definition of Lie groups we have to define the concept of a differentiable (complex) manifold:

**Definition A.4.1** A topological space $X$ is called **Hausdorff** if and only if for all $x$ and $y$ in $X$, $x \neq y$, there are open sets $O_1$ and $O_2$ such that $x \in O_1$, $y \in O_2$ and $O_1 \cap O_2 = \emptyset$.

**Definition A.4.2** Let $M$ be a topological space and $x \in M$. A coordinate chart about $x$ of dimension $d$ is a neighborhood $U$ of $x$ and a one-to-one continuous function $\phi : U \to G$ onto an open subset $G$ of $\mathbb{R}^d$. The pair $(U, \phi)$ is called a proper coordinate chart if and only if $\phi^{-1} : G \to U \subset M$ is continuous.

**Definition A.4.3** A topological space $M$ is called a (d-dimensional) differentiable manifold if and only if:

(i) $M$ is Hausdorff.

(ii) There is a collection $\mathcal{A}$ of coordinate charts $(U, \phi)$, called the atlas of $M$, such that

(a) For every $x \in M$ there exists a proper coordinate chart $(U, \phi)$ of dimension $d$ with $x \in U$.

(b) If $(U, \phi), (V, \psi) \in \mathcal{A}$ with $U \cap V \neq \emptyset$ then the mapping $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is infinitely often continuously differentiable (as a mapping between open subsets of $\mathbb{R}^d$).

(c) $\mathcal{A}$ is maximal with respect to conditions (a) and (b), i.e. $\mathcal{A}$ contains all possible charts with these properties.

If one replaces in this definition $\mathbb{R}^d$ by $\mathbb{C}^d$ and infinitely often continuously differentiable by holomorphic, one obtains the concept of a (d-dimensional) complex manifold. In practice one usually gives a collection of proper coordinate charts $(U_j, \phi_j)$ which
cover \( M \), i.e., \( M = \bigcup_j U_j \). Then there is a unique atlas determined which includes the collection of proper coordinate charts \((U_j, \phi_j)\). Note that it is possible to have two different atlases on a topological space \( M \), making \( M \) into a manifold in different ways, however we shall not consider such problems here.

Having introduced the concept of a complex manifold we are now in a position to define Lie groups.

**Definition A.4.4** A Lie group is a group \( G \) which is also a complex manifold such that the mapping \( G \times G \ni (g_1, g_2) \mapsto g_1 g_2^{-1} \in G \) is holomorphic.

We say that a Lie group \( G \) has a topological property if the Lie group \( G \) has this property when it is considered as a topological space. A topological space \((X, \mathcal{T})\) is called separable if and only if there exists a countable dense subset \( A \subset X \). A topological Hausdorff space \((X, \mathcal{T})\) is called compact if and only if every family of open sets, whose union covers \( X \), contains a finite subfamily, whose union covers \( X \), (i.e. if every open cover of \( X \) contains a finite subcover). The \( n \)-sphere \( S^n \), \( n = 1, 2, \ldots < \infty \) is a compact space. Moreover, a topological Hausdorff space \((X, \mathcal{T})\) is called locally compact if each point of \( X \) has a compact neighborhood. From this definition one clearly sees that every compact space is locally compact. The straight line \( \mathbb{R} \) is locally compact; this property follows form the Heine-Borel Theorem. Two subsets \( A \) and \( B \) of a topological space \( X \) are called separated if and only if \( \overline{A} \cap B \) and \( A \cap \overline{B} \) are both empty. A topological space \( X \) is called connected if it can not be represented as the union of two non-empty separated subsets. A topological space \( X \) is called simply connected if and only if every closed path in \( X \) can be continuously deformed in \( X \) into a point.

One can show that every Lie group \( G \) determines a Lie algebra \( L \) up to an isomorphism (cf. [7, Theorem 3.3.2]).

**Definition A.4.5** Let \( L \) be a finite dimensional vector space over the field \( K \) of real or complex numbers. The vector space \( L \) is called a Lie algebra over \( K \) if there exists
a product \( L \times L \ni (X, Y) \mapsto [X, Y] \in L \) on \( L \) satisfying the following axioms:

(i) \[ a) [aX + bY, Z] = a[X, Z] + b[Y, Z] \text{ for } a, b \in K \]
\[ b) [X, cY + dZ] = c[X, Y] + d[X, Z] \text{ for } c, d \in K \text{ (bilinearity).} \]

(ii) \[ [X, Y] = -[Y, X] \text{ for all } X, Y \in L \text{ (antisymmetry).} \]

(iii) \[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \text{ for all } X, Y, Z \in L \text{ (Jacobi Identity).} \]

If \( K \) is the field of real or complex numbers, then \( L \) is called a real or complex Lie algebra, respectively. Let \( A \) and \( B \) be two linear subspaces of \( L \). Then \([A, B]\) denotes the linear subspace spanned by the elements \([X, Y]\), where \( X \in A \) and \( Y \in B \). A subspace \( B \) is called a subalgebra of \( L \) if \([B, B] \subset B\), and an ideal, if \([L, B] \subset B\).

The set \( C = \{ X \in L : [X, Y] = 0, \text{ for all } Y \in L \} \) is called the center of \( L \); since \([L, C] \subset C\) the center is an ideal of \( L \). A Lie algebra \( L \) is called abelian if \([L, L] = \{0\}\), i.e. if the center of \( L \) is all of \( L \).

### A.4.1 Nilpotent, solvable, semisimple, and simple Lie algebras and Lie groups

One can show that if \( B \) is an ideal then \([B, B]\) is also an ideal. Since \([L, L] \subset L\) we have that \( L \) is an ideal of \( L\), and therefore, \([L, L]\) is also an ideal of \( L \) that may be smaller than \( L \). Now let us define the following sequences of ideals

\[
L^0 \equiv L, \quad L^1 \equiv [L^0, L^0], \ldots, \quad L^{k+1} \equiv [L^k, L^k],
\]

\[
L_0 \equiv L, \quad L_1 \equiv [L_0, L], \ldots, \quad L_{k+1} \equiv [L_k, L],
\]

A Lie algebra is called solvable if \( L^k = \{0\} \) for some finite \( k \), and nilpotent if \( L_k = \{0\} \) for some finite \( k \). Since \( L^k \subset L_k \), every nilpotent Lie algebra is solvable. However, the converse is not true. If \( L \) is solvable then \( L^k = \{0\} \) for some finite \( k \), hence, \([L^{k-1}, L^{k-1}] = 0\), and therefore, \( L^{k-1} \) is a commutative ideal of \( L \). Or every solvable Lie algebra contains a commutative ideal. The Lie algebra of the \( ax + b \) group is an example of a solvable Lie algebra, since \([X_1, X_2] = -iX_1\) implies that \( L^2 = \{0\}\). However, one can easily check the Lie algebra of the \( ax + b \) group is not nilpotent.
The familiar Heisenberg algebra $[P, Q] = iI$ is an example of a nilpotent Lie algebra since $L_2 = \{0\}$, and one easily verifies that the Heisenberg algebra is also solvable.

A Lie group $G$ is called is called solvable (nilpotent) if its Lie algebra is solvable (nilpotent).

A Lie algebra is called semisimple if it contains no nonzero abelian ideal. A Lie algebra is called simple if it does not contain any ideal other than $\{0\}$ and $L$, and if $[L, L] \neq \{0\}$. Every simple Lie algebra is semisimple, however the converse does not need to hold. The condition $[L, L] \neq \{0\}$ excludes one dimensional Lie algebras, which would be simple but not semisimple. Semisimple Lie algebras are in some sense the opposite to solvable Lie algebras. In fact one can show that every Lie algebra $L$ can be written as the semidirect sum of a maximal solvable ideal $N$ and a semisimple subalgebra $S$ (cf. [7, Theorem 1.3.5]).

A Lie group is called semisimple (simple) if its Lie algebra is semisimple (simple). Examples of Lie groups that are both simple and semisimple are given by $SU(2)$ and $SU(3)$ which do not contain any proper ideals. One can show that any semisimple Lie algebra can be written as the direct sum of simple ones (cf. [7, Theorem 1.3.6]).

A4.2 The Enveloping algebra of a Lie algebra

Let $L$ be the Lie algebra of a Lie group $G$ and let $X_1, \ldots, X_d$ be a finite dimensional basis of $L$, that satisfies the following commutation relations:

$$[X_i, X_j] = \sum_{k=1}^{d} c_{ij}^k X_k,$$

where $c_{ij}^k$ denote the structure constants. Then one can define the enveloping algebra of $L$ as follows.

**Definition A.4.6** The enveloping algebra $\mathcal{E}(G)$ of a Lie algebra $L$ is the associative algebra with generators $X_1, \ldots, X_d$ in which multiplication is defined by relations of the form

$$X_i X_j = X_j X_i + \sum_{k=1}^{d} c_{ij}^k X_k.$$
A.5 Some Basic Notions of the Theory of Group Representation

In this thesis attention is focused on square integrable unitary representations of real, separable, locally compact, connected and simply connected Lie groups on a separable Hilbert space.

Let $G$ be a locally compact Lie group and let $H$ be a separable Hilbert space. A map $U : G \ni g \mapsto U_g \in \mathcal{L}(H)$ is called a continuous representation if

(i) $U_{g_1 g_2} = U_{g_1} U_{g_2}$, $U_e = I$ (identity),

(ii) for every $\phi \in H$, the map $G \ni g \mapsto U_g \phi \in H$ is continuous.

A representation is called unitary if each $U_g \in \mathcal{L}(H)$ is a unitary operator on $H$, and trivial if $U_g = I$ for all $g \in G$. One of the most important unitary representations of $G$ is the left regular representation $\Lambda$ on $H = L^2(G, dg)$, where $dg$ is the left invariant measure of $G$. The left regular representation $\Lambda : G \ni g \mapsto \Lambda_g$ is defined by means of left translation, i.e.

$$ (\Lambda_{g_1} \phi)(g) = \phi(g_1^{-1} g), \quad \forall \phi \in L^2(G), \quad g_1, g \in G, \quad (A.8) $$

Claim A.5.1 The left regular representation $\Lambda$ of $G$ is a continuous unitary representation of $G$ on $L^2(G)$.

Proof: In our proof we follow Ref. 7, pp. 135–136. Clearly, every $\Lambda_g$ is a linear operator. Furthermore,

$$ \{\Lambda_{g_1} [\Lambda_{g_2} \phi](g) = [\Lambda_{g_2} \phi](g_1^{-1} g) = \phi(g_2^{-1} g_1^{-1} g) = \phi((g_1 g_2)^{-1} g) = (\Lambda_{g_1 g_2} \phi)(g), $$

i.e.,

$$ \Lambda_{g_1} \Lambda_{g_2} = \Lambda_{g_1 g_2} \quad \text{and} \quad \Lambda_e = 1. $$

Therefore, the map (A.8) defines a representation of $G$ in $L^2(G)$. Furthermore, since $dg$ is left invariant we have

$$ (\Lambda_g \phi, \Lambda_g \chi) = \int \overline{\phi(g^{-1} g_1)} \chi(g^{-1} g_1) dg_1 = \langle \phi, \chi \rangle, $$
hence, $A_g$ is isometric, and since the range of $A_g$ is all of $L^2(G)$, every $A_g$ is unitary. Strong continuity is established as follows, let $\chi \in C_0(G)$, where $C_0(G)$ is the set of all continuous functions with compact support on $G$, since every compactly supported continuous function on a Lie group $G$ is uniformly continuous on its support (cf. [7, Proposition 2.2.4]), we have

$$\sup |\chi(g^{-1}g_1) - \chi(g_1)| \leq \epsilon,$$

for $g \in V$, where $V$ is a neighborhood of the identity, $e$, of $G$. Moreover, since $\chi \in C_0(G)$ there exists a fixed compact set $K \subseteq G$, supporting $\chi$ and $A_g \chi$ for $g$ sufficiently close to $e$, such that

$$\|A_g \chi - \chi\|_2 = \left[ \int_G |\chi(g^{-1}g_1) - \chi(g_1)|^2 dg_1 \right]^{1/2} \leq \sup_{g_1 \in K} |\chi(g^{-1}g_1) - \chi(g_1)| M \leq M \epsilon,$$

where $M = \sqrt{\int_K dg_1}$. Now since $C_0(G)$ is dense in $L^2(G)$ there exists for each $\phi \in L^2(G)$ a $\chi \in C_0(G)$ such that $\|\phi - \chi\|_2 \leq \epsilon$. Hence,

$$\|A_g \phi - \phi\|_2 = \|A_g (\phi - \chi) + (A_g \chi - \chi) - (\phi - \chi)\|_2 \leq \|A_g (\phi - \chi)\|_2 + \|A_g \chi - \chi\|_2 + \|\phi - \chi\|_2,$$

using the fact that the Haar measure is invariant under left translations, we find

$$\|A_g \phi - \phi\|_2 \leq 2\epsilon + \|A_g \chi - \chi\|_2 \leq (2 + M)\epsilon \leq \epsilon.$$

Hence,

$$\|A_g \phi - \phi\|_2 \leq \epsilon \quad \text{for} \quad g \in V.$$

Therefore, we conclude that $G \ni g \mapsto A_g$ is continuous in $e$. To show that $G \ni g \mapsto A_g$ is continuous in every point we have to show that

$$\|A_{g_1} \phi - A_{g_2} \phi\|_2 \leq \epsilon \quad \text{for} \quad g_2^{-1}g_1 \in V.$$

Using again the left invariance of the Haar measure we conclude

$$\|A_{g_1} \phi - A_{g_2} \phi\|_2 = \|A_{g_2^{-1}g_1} \phi - \phi\|_2 \leq \epsilon \quad \text{for} \quad g_2^{-1}g_1 \in V.$$
Therefore, $G \ni g \mapsto \Lambda_g$ is continuous in every point. Hence, $\Lambda$ is a continuous unitary representation of $G$ on $L^2(G)$. □

A.5.1 Equivalence of Representations

Let $U^1$ and $U^2$ be two unitary representations of the same locally compact Lie group $G$ on the Hilbert spaces $H_1$ and $H_2$, respectively. Then $U^1$ is unitarily equivalent to $U^2$ if there exists a unitary operator $T : H_1 \to H_2$ such that

$$TU_g^1 = U_g^2T$$

for every $g \in G$. (A.9)

If $U^1$ is unitarily equivalent to $U^2$ then we write $U^1 \cong U^2$. A bounded operator $T$ from $H_1$ to $H_2$ is called an intertwining operator for $U^1$ and $U^2$ if $TU_g^1 = U_g^2T$ for all $g \in G$. The set of all intertwining operators forms a linear space, which is denoted by $\mathcal{R}(U^1, U^2)$. Hence, two unitary representations $U^1$ and $U^2$ are unitarily equivalent if $\mathcal{R}(U^1, U^2)$ contains a unitary operator from $H_1$ onto $H_2$. Observe that for $U^1 = U^2$, $\mathcal{R}(U^1, U^1)$ is the commutant of the $*$-algebra $A(U^1)$ generated by the representation $G \ni g \mapsto U_g^1$. Hence, $\mathcal{R}(U^1, U^1)$ is a von Neumann algebra.

A.5.2 Irreducibility of Representations

A subspace $H_1 \subset H$ is called invariant under a unitary representation $g \mapsto U_g$ if and only if $U_gH_1 \subset H_1$ for all $g \in G$. A unitary representation $g \mapsto U_g$ of a locally compact group is called irreducible if $U$ has no invariant subspaces other than $H_1 = H$ and $H_1 = \{0\}$. A unitary representation, which has proper invariant subspaces is called reducible. Let $H_1$ be a closed subspace of $H$, then one calls the restriction of $U$ to $H_1$ a unitary subrepresentation of $U$. A unitary representation is called completely reducible if it can be expressed as a direct sum of irreducible unitary subrepresentations. One can show that every finite-dimensional unitary representation of any Lie group is completely reducible (cf. [7, Corollary 5.3.2]).

The following theorem is of fundamental importance in the theory of group representations:
Theorem A.5.1 (Schur's Lemma - unitary case) Let $U^1$ and $U^2$ be irreducible, unitary representations of a Lie group $G$ on $H_1$ and $H_2$, respectively. If $T \in L(H_1, H_2)$ is such that

$$TU^1_g = U^2_gT \quad \text{for all } g \in G,$$

then, either $T$ is an unitary operator from $H_1$ onto $H_2$ (i.e. $U^1 \cong U^2$), or $T = 0$.

(For a proof see Ref. 7, pp. 143–144.)

Hence, the set of all irreducible unitary representations of a Lie group $G$ can be partitioned into equivalence classes. We denote by $\hat{G}$ the set of all equivalence classes of irreducible, unitary representations of a Lie group $G$. Schur's Lemma implies the following criterion of irreducibility for unitary representations:

Corollary A.5.2 For a unitary representation $U$ of a Lie group $G$ on a separable Hilbert space $H$ to be irreducible it is necessary and sufficient that the only operators that commute with all the $U_g$ are scalar multiples of the identity operator.

One can use Corollary A.5.2 to give a new definition of irreducibility. A unitary representation $U$ is called irreducible if the only operators that commute with all the $U_g$ are scalar multiples of the identity operator. One refers to this formulation of irreducibility as operator irreducibility of $U$.

A.6 Reducible Representations

One can classify reducible unitary representations $U$ according to the properties of the of the center $Z$ of the von Neumann algebra $\mathcal{R}(U, U)$ of intertwining operators. Let us start with the case when $\mathcal{R}(U, U)$ is a factor.

Definition A.6.1 A unitary representation $U$ of a Lie group $G$ is said to be a factor representation if and only if $\mathcal{R}(U, U)$ is a factor. Representations of this type are called primary representations.

Clearly, by Corollary A.5.2 every irreducible unitary representation is a factor representation.
Proposition A.6.2 A representation \( U \) is a factor representation of type I if and only if it is the discrete orthogonal sum of finitely or countably many equivalent irreducible representations, i.e. if \( U \) is the multiple of some irreducible representation.

(For a proof see Ref. 78, Proposition V.1.1.)

Another interesting class of unitary representations is obtained if the von Neumann Algebra \( \mathcal{R}(U,U) \) is abelian, i.e. if the center \( Z \) of \( \mathcal{R}(U,U) \) coincides with \( \mathcal{R}(U,U) \).

Definition A.6.3 A unitary representation \( U \) is said to be multiplicity free, if and only if \( \mathcal{R}(U,U) \) is abelian.

Observe that if \( U \) is both a factor representation and multiplicity free, then \( \mathcal{R}(U,U) = \{aI\} \), i.e. \( U \) is irreducible by the virtue of Corollary A.5.2. If \( U \) is completely reducible and multiplicity free, then

\[
U = \bigoplus_{\zeta=1}^{\infty} U^\zeta,
\]

where all \( U^\zeta \) are mutually inequivalent and irreducible. Multiplicity free unitary representations can be decomposed in an essentially unique way into irreducible unitary subrepresentations. If \( G \) is a type I group then one can decompose every unitary representation of \( G \) in an essentially unique way into irreducible unitary subrepresentations, as the following theorem shows:

Theorem A.6.4 Let \( (U, H) \) be a unitary type I representation of a Lie group \( G \). Then there exists a standard Borel measure \( \tilde{\nu} \) and a function \( \hat{n}(\cdot) \) such that one has the decomposition of the space \( H \) into a direct integral

\[
H = \int_{G} H_{\zeta} \hat{n}(\zeta) d\tilde{\nu}(\zeta), \quad U = \int_{G} U^\zeta \hat{n}(\zeta) d\tilde{\nu}(\zeta).
\]

(For a proof see Ref. 78, Theorem V.2.8.)

For the definition of a standard Borel measure see Appendix B.3. Note that one can show that any unitary representation can be written as a direct integral of irreducible
representations, however only for unitary type I representations is the decomposition \textit{essentially unique}. For a discussion of this point see for example Mackey [74, pp.54–63].
APPENDIX B
CONTINUOUS REPRESENTATION THEORY

B.1 Continuous Representation

In this Appendix we confine our attention to real, locally compact, separable, connected and simply connected Lie groups $G$ that have irreducible, square integrable unitary representations. Let us denote by $U$ a fixed continuous, irreducible, square integrable unitary representations of $G$ on the Hilbert space $H$.

Let $X_1, ..., X_d$ be an irreducible representation of the basis of the Lie algebra $L$ corresponding to $G$, by symmetric operators on $H$ satisfying Hypothesis (A), see section 3.1, then $L$ is integrable to a unique unitary representation of $G$ on $H$. Suppose there exists a parameterization of $G$ such that,

$$U_{g(l)} = \prod_{k=1}^{d} \exp(-il^k X_k) = \exp(-il^1 X_1) \cdots \exp(-il^d X_d);$$

where $l$ is an element of a $d$-dimensional parameter space $G$.

We defined in chapter 4 the set of coherent states for $G$, corresponding to a fixed square integrable unitary representation $U_{g(l)}$ as:

$$\eta(l) \equiv U_{g(l)} K^{1/2} \eta; \quad \eta \in D(K^{1/2}) \quad \text{and} \quad ||\eta|| = 1,$$

where $K$ is the unique self-adjoint, positive, semi-invariant operator with weight $\Delta^{-1}(g(l))$ given in Theorem 4.1.2.

**Claim B.1.1** The map $C_\eta : H \rightarrow L^2(G)$, defined for any $\psi \in H$ by:

$$[C_\eta(\psi)](l) = \psi(l) = \langle \eta(l), \psi \rangle$$

maps the elements of $H$ into complex, bounded, continuous, square integrable functions.
Proof: Let $\psi \in H$ be arbitrary, then

$$|\psi_\eta(l)| = |\langle \eta(l), \psi \rangle|$$

$$= |\langle U_g(l)K^{1/2}\eta, \psi \rangle|$$

$$\leq \|U_g(l)K^{1/2}\eta\|\|\psi\|$$

$$\leq \|U_g(l)\|\|K^{1/2}\eta\|\|\psi\|$$

$$\leq M, \quad \forall \ l \in G,$$

where we have used the Cauchy-Schwarz inequality in the third step and where $M$ is given by $M \equiv \|K^{1/2}\eta\|\|\psi\|$. Hence, the functions $\psi_\eta(l)$ are bounded. That the functions $\psi_\eta(l)$ are continuous follows from the fact that the unitary representation $U_g(l)$ is strongly continuous, and therefore, weakly continuous. To see that the functions $\psi_\eta(l)$ are square integrable, let $\psi \in H$ be arbitrary, then since $\eta \in D(K^{1/2})$ it follows that $K^{1/2}\eta \in D(K^{-1/2})$, therefore, by Theorem 4.1.2 (i) the function $\psi_\eta(l) = \langle U_g(l)K^{1/2}\eta, \psi \rangle$ is square integrable. Since $\psi \in H$ was arbitrary it follows that the functions $\psi_\eta(l)$ are square integrable (cf. remark 4.1.1). □

We denote by $L^2_\eta(G)$ the space spanned by the bounded, continuous, square integrable functions $\psi_\eta(l)$. The space $L^2_\eta(G)$ is clearly a subspace of $L^2(G)$, and is therefore, an inner product space. The inner product on $L^2_\eta(G)$ is given by the restriction of the inner product on $L^2(G)$ to $L^2_\eta(G)$. We denote the inner product and the norm on $L^2_\eta(G)$ by $\langle \cdot, \cdot \rangle_\eta$ and $\| \cdot \|_\eta \equiv (\langle \cdot, \cdot \rangle_\eta)^{1/2}$, respectively.

Claim B.1.2 The map $C_\eta$ defined in (B.1) is an isometric isomorphism from the Hilbert space $H$ onto the inner product space $L^2_\eta(G)$.

Proof: To show that the map $C_\eta$ is an isomorphism from $H$ to $L^2_\eta(G)$ we have to show that $C_\eta$ is linear, one to one, onto, and continuous (cf. [31, II.3.17]). By definition the map $C_\eta$ is clearly a linear operator from $H$ onto $L^2_\eta(G)$.
To see that $C_\eta$ is one to one we have to show that $[C_\eta \psi](l) = [C_\eta \phi](l) \ \forall \ l \in \mathcal{G}$ implies $\psi = \phi$. Since $C_\eta$ is linear we have

$$[C_\eta(\psi - \phi)](l) = \langle \eta(l), \psi - \phi \rangle = 0 \ \forall \ l \in \mathcal{G}. \quad \text{(B.2)}$$

Since the unitary representation $U_{\theta(l)}$ is irreducible the set $\mathbf{T} = \{ \eta(l) : l \in \mathcal{G} \}$ is a total set in $H$ (cf. [7, Proposition 5.4.2]). Therefore, (B.2) implies that $\psi = \phi$. Hence, the map $C_\eta$ is one to one.

We show next that the linear operator $C_\eta$ from $H$ onto $L^2_\eta(G)$ is continuous. To show that $C_\eta$ is continuous it is necessary and sufficient to show that $C_\eta$ is bounded (cf. [87, Theorem 1.6]). Let $\psi \in H$ be arbitrary then

$$\| [C_\eta \psi](l) \|_\eta = \left[ \int_\mathcal{G} |\psi_\eta(l)|^2 \, dg(l) \right]^{1/2} = \left[ \int_\mathcal{G} \langle \psi, \eta(l) \rangle \langle \eta(l), \psi \rangle \, dg(l) \right]^{1/2} = \| \psi \|,$$

where we have used (4.3). Hence, $C_\eta$ is a continuous linear operator. We therefore conclude that $C_\eta$ is an isomorphism from $H$ onto $L^2_\eta(G)$.

We now show that $C_\eta$ is isometric. Let $\phi, \psi \in H$ be arbitrary then

$$\langle [C_\eta \phi](l), [C_\eta \psi](l) \rangle_\eta = \int_\mathcal{G} \langle \phi, \eta(l) \rangle \langle \eta(l), \psi \rangle \, dg(l) = \langle \phi, \psi \rangle,$$

where we have used (4.3) in the last step. Equation (B.3) shows that $C_\eta$ is isometric. Therefore, we conclude that $C_\eta$ is an isometric isomorphism from $H$ onto $L^2_\eta(G)$. $\square$

Claim B.1.2 shows that the Hilbert space $H$ is isometrically isomorphic to the inner product space $L^2_\eta(G)$ and it follows from this that $L^2_\eta(G)$ is a Hilbert space (cf. [105, Theorem 4.9]). We call $L^2_\eta(G)$ the continuous representation of $H$. 
Reproducing Kernels and Reproducing Kernel Hilbert Spaces

A reproducing kernel is abstractly defined as follows (cf. [79, pp.42-43]):

**Definition B.2.1** Let $\mathcal{L}$ be a topological space and $\mathbf{R}$ be a Hilbert space whose elements are functions from $\mathcal{L}$ to $\mathbb{C}$, the set of complex numbers. We say that the function $K : \mathcal{L} \times \mathcal{L} \rightarrow \mathbb{C}$, $(l', l) \mapsto K(l'; l)$ is a reproducing kernel, if and only if

(i) $K(l'; l)$ belongs to $\mathbf{R}$ as a function of $l'$ for all $l$.
(ii) $K(l'; l)$ has the reproducing property:

$$\text{for all } \phi \text{ in } \mathbf{R}, \quad \phi(l) = \langle K(l'; l), \phi(l') \rangle_{\nu}$$

We say that $\mathbf{R}$ is a reproducing kernel Hilbert space if there is a function $K$ satisfying conditions (i) and (ii), i.e. if it has a reproducing kernel.

It follows from this definition that the reproducing kernel has the following properties:

$$K(l; l) \geq 0, \quad K(l'; l) = \overline{K(l; l')}$$

$$|K(l'; l)|^2 \leq K(l'; l') K(l, l).$$

To see that the inequality (B.5) holds we use the reproducing kernel property (B.4); since for fixed $l$ we have that $K(l'; l) \in \mathbf{R}$ one finds

$$K(l; l) = \langle K(l'; l), K(l'; l) \rangle_{\nu} = \|K(l'; l)\|^2 \geq 0.$$

From

$$\langle K(l''; l'), K(l''; l) \rangle_{\nu} = K(l'; l) \quad \langle K(l''; l), K(l''; l') \rangle_{\nu} = K(l; l')$$

and (ii) in Definition A.2.1 the second statement in (B.5) easily follows. (B.6) is established as follows:

$$|K(l'; l)|^2 = K(l'; l) \overline{K(l'; l)} = \langle K(l''; l'), K(l''; l) \rangle_{\nu} \langle K(l''; l'), K(l''; l) \rangle_{\nu}$$

$$= |\langle K(l''; l'), K(l''; l) \rangle_{\nu}|^2$$

$$\leq \|K(l'; l'')\|^2 \|K(l; l'')\|^2$$

$$\leq K(l'; l') K(l; l).$$
where we have used the Cauchy Schwarz inequality in the third step. One can generalize the inequality (B.5) as follows; let $\lambda_\nu$, $\nu = 1, 2, 3, \ldots, n$, be arbitrary complex numbers then

$$\sum_{\mu, \nu = 1}^{n} \lambda_\nu \lambda_\mu \mathcal{K}(l_\nu, l_\mu) \geq 0 \quad (l_\mu, l_\nu \in \mathcal{L}).$$

Using (B.4) and the linearity of the inner product we find

$$0 \leq \sum_{\nu = 1}^{n} \lambda_\nu \mathcal{K}(l'; l_\nu), \sum_{\mu = 1}^{n} \lambda_\mu \mathcal{K}(l'; l_\mu) \rangle_{\nu} = \sum_{\mu, \nu = 1}^{n} \lambda_\nu \lambda_\mu \mathcal{K}(l_\nu; l_\mu).$$

Hence, the reproducing kernel is a positive definite function.

**Claim B.2.1** A reproducing kernel Hilbert space can never have more than one reproducing kernel.

**Proof:** In our proof we follow Ref. 79, p. 43, suppose there exists another function $\mathcal{K}'(l'; l)$ which has the reproducing property (B.4), then

$$\|\mathcal{K}(l'; l) - \mathcal{K}'(l'; l)\|^2 = \langle \mathcal{K} - \mathcal{K}', \mathcal{K} - \mathcal{K}' \rangle_{\nu}$$

$$= \langle \mathcal{K}, \mathcal{K} - \mathcal{K}' \rangle_{\nu} - \langle \mathcal{K}', \mathcal{K} - \mathcal{K}' \rangle_{\nu}$$

$$= \mathcal{K}(l; l) - \mathcal{K}'(l;l) - \mathcal{K}(l;l) + \mathcal{K}'(l;l)$$

$$= 0,$$

since both kernels are 'reproducing'. □

We now show that the function $\mathcal{K}_\eta(l'; l)$ defined in chapters 2 and 4 as

$$\mathcal{K}_\eta(l'; l) = \langle \eta(l'), \eta(l) \rangle,$$

is a reproducing kernel. Repeating the proof of claim B.1.1 word by word, substituting $\mathcal{K}_\eta(l'; l) = \langle \eta(l'), \eta(l) \rangle$ with $l$ arbitrary but fixed for $\psi_\eta(l)$ everywhere, we see that $\mathcal{K}_\eta(l'; l)$ is a continuous, bounded, square integrable function of $l'$. Therefore, $\mathcal{K}_\eta(l'; l)$ belongs to $L^2_\eta(G)$ as a function of $l'$ for all $l$. From (4.6) we see that the function $\mathcal{K}_\eta(l'; l)$ satisfies the reproducing property. Hence, $\mathcal{K}_\eta(l'; l)$ is a reproducing kernel and therefore, by Definition B.2.1, $L^2_\eta(G)$, which is a Hilbert space of continuous functions, is a reproducing kernel Hilbert space.
Claim B.2.2 If $K(l'; l)$ is the reproducing kernel of a proper closed subspace $R'$ of a Hilbert space $H$, then

$$\phi'(l) = \langle K(l'; l), \phi(l') \rangle_{\nu},$$

is the projection of $\phi \in H$ onto $R'$.

Proof: In our Proof we follow Ref. 79, pp. 47-48, by the Projection Theorem, see Ref. 87, Theorem II.3, every element of $H$ can be written uniquely as

$$\phi(l') = \phi'(l') + \chi(l'),$$

where,

$$\phi' \in R', \text{ and } \langle \phi', \chi \rangle_{\nu} = 0 \text{ for all } \phi' \in R'.$$

Since $K(l'; l)$ belongs to $R'$ as a function of $l'$, we have

$$\langle K(l'; l), \chi(l') \rangle_{\nu} = 0,$$

and hence, we see that

$$\langle K(l'; l), \phi(l') \rangle_{\nu} = \langle K(l'; l), \phi'(l') + \chi(l') \rangle_{\nu} = \langle K(l'; l), \phi'(l') \rangle_{\nu} = \phi'(l').$$

Since, $K_n(l'; l)$ is the reproducing kernel of the proper closed subspace $L^2_n(G)$ of $L^2(G)$ it is the kernel of a projection operator from $L^2(G)$ onto $L^2_n(G)$.

B.3 Proof that Equation (4.22) is Well Defined

All measures considered in this section are positive and $\sigma$-finite. A measure $\nu$ on $X$ is called $\sigma$-finite if there is a sequence $\{X_n\}_{n=1}^{\infty}$ of measurable sets in the $\sigma$-algebra $\mathcal{B}$ such that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\nu(X_n) < \infty$. The Lebesgue measure $dx$ on $(-\infty, \infty)$ is an example of a $\sigma$-finite measure.
Definition B.3.1 A measure \( \nu \) on \( \hat{G} \) is a standard Borel measure if there exists a Borel set \( S \) such that \( \nu(S) = 0 \), and the subspace \( \hat{G} - S \) is a standard Borel space (it is isomorphic with a Borel subset of some complete metric space).

By Corollary 5.1 in Ref. 30 there exists a standard Borel measure \( \nu \) on \( \hat{G} \), a \( \nu \)-measurable decomposition \( (U_\zeta, H_\zeta)_{\zeta \in \hat{G}} \) of the type I part of the left regular representation of \( G \), and a measurable field \( (K_\zeta)_{\zeta \in \hat{G}} \) of nonzero, positive, self-adjoint operators such that \( K_\zeta \) is a semi-invariant operator of weight \( \Delta(g^{-1}) \) in \( H_\zeta \) for \( \nu \)-almost all \( \zeta \in \hat{G} \) such that

(i) For \( \alpha \in P_l[D(G)] \) the operator \( K_\zeta^{1/2}U_\zeta(\alpha)K_\zeta^{1/2} \) is trace class.

(ii) For \( \alpha, \beta \in P_l[D(G)] \), one has

\[
\delta_{\nu}(\alpha^* \star \beta) = \int_{\hat{G}} \text{tr}[K_\zeta^{1/2}U_\zeta(\alpha^* \star \beta)K_\zeta^{1/2}]d\nu(\zeta). \tag{B.7}
\]

Claim B.3.2 The equation (B.7) is well defined.

Proof. Suppose there exists another standard Borel measure \( \nu' \) and measurable fields \( (U'_\zeta, H'_\zeta)_{\zeta \in \hat{G}} \) and \( (K'_\zeta)_{\zeta \in \hat{G}} \) with the same properties as above, then by Theorem 5.2 in Ref. 30 the measures \( \nu \) and \( \nu' \) are equivalent, and one has, for \( \nu \)-almost all \( \zeta \in \hat{G} \)

\[
K'_\zeta = \left( \frac{d\nu(\zeta)}{d\nu'(\zeta)} \right) V_\zeta K_\zeta V_\zeta^{-1}, \tag{B.8}
\]

where \( V_\zeta \) is for all \( \zeta \in \hat{G} \) an operator intertwining \( U_\zeta \) and \( U'_\zeta \), i.e. \( V_\zeta U_\zeta = U'_\zeta V_\zeta \) and \( V_\zeta \) is bounded for all \( \zeta \in \hat{G} \). Therefore, using (B.8) one can write

\[
\int_{\hat{G}} \text{tr}[K_\zeta^{1/2}U_\zeta(\alpha^* \star \beta)K_\zeta^{1/2}]d\nu'(\zeta)
\]

\[
= \int_{\hat{G}} \text{tr}[K'_\zeta U'_\zeta(\Delta^{-1/2}\alpha^* \star \beta)]d\nu'(\zeta)
\]

\[
= \int_{\hat{G}} \text{tr}[V_\zeta K_\zeta V_\zeta^{-1}U_\zeta(\Delta^{-1/2}\alpha^* \star \beta)] \left( \frac{d\nu(\zeta)}{d\nu'(\zeta)} \right) d\nu'(\zeta)
\]

\[
= \int_{\hat{G}} \text{tr}[V_\zeta K_\zeta U_\zeta(\Delta^{-1/2}\alpha^* \star \beta)V_\zeta^{-1}] \left( \frac{d\nu(\zeta)}{d\nu'(\zeta)} \right) d\nu'(\zeta)
\]

\[
= \int_{\hat{G}} \text{tr}[K_\zeta U_\zeta(\Delta^{-1/2}\alpha^* \star \beta)] \left( \frac{d\nu(\zeta)}{d\nu'(\zeta)} \right) d\nu'(\zeta)
\]

\[
= \int_{\hat{G}} \text{tr}[K_\zeta^{1/2}U_\zeta(\alpha^* \star \beta)K_\zeta^{1/2}]d\nu(\zeta),
\]
where the fourth equality holds by \( \text{tr}(AB) = \text{tr}(BA) \), for \( A \) trace class and \( B \) bounded and the last equality holds by Theorem 32.B in Ref. 51, since \( \nu \) and \( \nu' \) are equivalent and \( \sigma \)-finite. Hence, we have shown that

\[
\int_{\hat{G}} \text{tr}[K^{1/2}_\zeta U^\zeta (\alpha \ast \beta) K^{1/2}_\zeta]d\nu'(\zeta) = \int_{\hat{G}} \text{tr}[K^{1/2}_\zeta U^\zeta (\alpha \ast \beta) K^{1/2}_\zeta]d\nu(\zeta),
\]

for any two decompositions of the type I part of the left regular representation of \( G \). Therefore, equation (B.7) is well defined. \( \square \)
In this Appendix we present the exact lattice calculation of the representation independent propagator for the affine group for the following two Hamilton operators:

(i) \( \mathcal{H}(X_1, X_2) = \frac{1}{2m} X_1^2 \),

(ii) \( \mathcal{H}(X_1, X_2) = \frac{1}{2m} X_1^2 + \omega X_2 \).

C.1 The Free Particle

Our first example is that of the Hamiltonian \( \mathcal{H}(p, x) = (pk)^2/2m \). Here \( p \) is restricted to be positive and the variables \( k, x \) are unrestricted. Let \( \bar{p}_j = (p_{j+1} + p_j)/2 \), \( \Delta p_j = p_{j+1} - p_j \), \( \bar{q}_j = (q_{j+1} + q_j)/2 \), and \( \Delta q_j = q_{j+1} - q_j \), then the representation independent propagator for the affine group becomes:

\[
K(p'', q'', t'', p', q', t') = 
\lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ k_{j+1/2} \left( \bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j \right) - x_{j+1/2} \frac{\Delta p_j}{\bar{p}_j} - \epsilon \frac{\left( \bar{p}_j k_{j+1/2} \right)^2}{2m} \right] \right\} 
\times \prod_{j=1}^{N} dq_j dp_j \prod_{j=0}^{N} dk_{j+1/2} dx_{j+1/2} \\
\lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ -\epsilon \frac{\left( \bar{p}_j k_{j+1/2} \right)^2}{2m} + k_{j+1/2} \left( \bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j \right) \right] 
- \frac{m}{2\epsilon \bar{p}_j^2} \left( \bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j \right)^2 + \frac{m}{2\epsilon \bar{p}_j^2} \left( \bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j \right)^2 - x_{j+1/2} \frac{\Delta p_j}{\bar{p}_j} \right\} 
\times \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} dk_{j+1/2} dx_{j+1/2} \\
\lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ \frac{-\epsilon \bar{p}_j^2}{2m} \left[ k_{j+1/2}^2 - \frac{m}{2\epsilon \bar{p}_j^2} \left( \bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j \right)^2 \right] \right] \right\}
\]
\[
+ \frac{m}{2\varepsilon p_j} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j)^2 - x_{j+1/2} \frac{\Delta p_j}{\bar{p}_j} \}
\]
\[
\prod_{j=1}^N dp_j dq_j \prod_{j=0}^N \frac{dk_{j+1/2} dx_{j+1/2}}{(2\pi)^2}.
\]

Now for all j we let \( \tilde{k}_{j+1/2} = k_{j+1/2} + \frac{m}{2\varepsilon p_j} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j) \) and carrying out the integrations over \( \tilde{k}_{j+1/2} \) we find:

\[
K(p'', q'', t''; p', q', t') = \lim_{N \to \infty} \left( \frac{2\pi m}{i\varepsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \sum_{j=0}^N \left[ \frac{m}{2\varepsilon} \left( \Delta q_j + \frac{\Delta p_j}{p_j} \bar{q}_j \right)^2 \right] \right\}
\]
\[
\times \prod_{j=1}^N dp_j dq_j \prod_{j=0}^N \frac{dx_{j+1/2}}{(2\pi)^2 p_j}.
\]

As a next step we carry out the integrations over the \( x_{j+1/2} \) and find:

\[
K(p'', q'', t''; p', q', t') = \lim_{N \to \infty} \left( \frac{2\pi m}{i\varepsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \sum_{j=0}^N \left[ \frac{m}{2\varepsilon} \left( \Delta q_j + \frac{\Delta p_j}{p_j} \bar{q}_j \right)^2 \right] \right\}
\]
\[
\times \prod_{j=0}^N \frac{1}{2\pi p_j} \delta \left( \frac{\Delta p_j}{p_j} \right) \prod_{j=1}^N dp_j dq_j
\]
\[
= \lim_{N \to \infty} \left( \frac{m}{2\pi i\varepsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \sum_{j=0}^N \left[ \frac{m}{2\varepsilon} \left( \Delta q_j + \frac{\bar{q}_j \Delta p_j}{\bar{p}_j} \right)^2 \right] \right\}
\]
\[
\times \prod_{j=0}^N \delta(\Delta p_j) \prod_{j=1}^N dp_jdq_j
\]
\[
= \lim_{N \to \infty} \left( \frac{m}{2\pi i\varepsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \frac{m}{2\varepsilon} \sum_{j=0}^N (\Delta q_j)^2 \right\}
\]
\[
\times \prod_{j=0}^N \delta(p_{j+1} - p_j) \prod_{j=1}^N dp_j dq_j.
\]

Making use of the identity

\[
\int \delta(x - y) \delta(y - z) dy = \delta(x - z),
\]

and carrying out the integrations over the \( p_j \) successively we find:
\[ K(p'', q'', t''; p', q', t') = \delta(p'' - p') \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left[ i \sum_{j=0}^{N} \frac{m}{2\epsilon} (q_{j+1} - q_j)^2 \right] \prod_{j=1}^{N} dq_j. \]

This last integral can be straightforwardly evaluated using the following definite integral:

\[ \int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}} \exp\left[-a(x-u)^2\right] \sqrt{\frac{b}{\pi}} \exp\left[-b(u-y)^2\right] du = \sqrt{\frac{ab}{\pi(a+b)}} \exp\left[-\frac{ab}{a+b}(x-y)^2\right]. \]

(C.1)

for \( \Re(a) > 0, \Re(b) > 0 \) extended to \( \Re(a) = 0 = \Re(b) \) as an improper integral.

This identity is easily proved by completing the square in the exponential. Using (C.1) the integral over \( q_1 \) is given by

\[ \left( \frac{m}{2\pi i \epsilon} \right) \int \exp \left[ \frac{im}{2\epsilon} (q_2 - q_1)^2 \right] \exp \left[ \frac{im}{2\epsilon} (q_1 - q_0)^2 \right] dq_1 \]

= \[ \sqrt{\frac{m}{2\pi i (2\epsilon)}} \exp \left[ \frac{im}{2(2\epsilon)} (q_2 - q')^2 \right], \]

where we have included two of the \( (m/2\pi i \epsilon)^{1/2} \) factors and have identified \( q_0 \) with \( q' \). The effect of the \( q_1 \)-integration is to change \( \epsilon \) to \( 2\epsilon \), both in the exponential and in the square root, and to replace the exponentials by a single exponential which depends only on \( q_2 - q' \). The \( q_1 \)-dependence has been integrated out. Proceeding in this manner carrying out all the \( q \)-integrations we find as our final result:

\[ K(p'', q'', t''; p', q', t') = \delta(p'' - p') \lim_{N \to \infty} \sqrt{\frac{m}{2\pi i (N+1) \epsilon}} \exp \left[ \frac{im}{2(N+1)\epsilon} (q'' - q')^2 \right] \]

= \[ \sqrt{\frac{m}{2\pi i (t'' - t')}} \delta(p'' - p') \exp \left[ \frac{im}{2(t'' - t') (q'' - q')^2} \right], \]

where we have made use of the following identities \( q_{N+1} = q'' \) and \( (t'' - t') = (N+1)\epsilon \).

Hence, our final result is given by

\[ K(p'', q'', t''; p', q', t') = \sqrt{\frac{m}{2\pi i T}} \delta(p'' - p') \exp \left[ \frac{im}{2T} (q'' - q')^2 \right], \]

where \( T \equiv t'' - t' \). Observe that this result agrees with the usual result for the free particle except that here \( p \) is restricted to be positive, i.e. \( p > 0 \).
C.2 The Hamilton Operator $\mathcal{H}(X_1, X_2) = \frac{1}{2m} X_1^2 + \omega X_2$

Let us now consider the Hamiltonian $\mathcal{H}(p, x) = (pk)^2/2m + \omega x$. With the notations of the previous section, the representation independent propagator for the affine group takes the following form:

$$K(p'', q'', t''; p', q', t') = \lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ k_{j+1/2} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j) ight. ight.$$ 

$$\left. - x_{j+1/2} \frac{\Delta p_j}{\bar{p}_j} - \frac{\left( \bar{p}_j k_{j+1/2} \right)^2}{2m} - \epsilon \omega x_{j+1/2} \right]\right\} \prod_{j=1}^{N} dq_j dp_j \prod_{j=0}^{N} \frac{dk_{j+1/2} dx_{j+1/2}}{(2\pi)^2}$$

$$= \lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ -\epsilon \frac{(\bar{q}_j k_{j+1/2})^2}{2m} + k_{j+1/2} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j) ight. ight.$$ 

$$\left. - \frac{m}{2\epsilon \bar{p}_j^2} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j)^2 + \frac{m}{2\epsilon \bar{p}_j^2} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j)^2 ight.$$ 

$$\left. - x_{j+1/2} \left( \frac{\Delta p_j}{\bar{p}_j} + \epsilon \omega \right) \right]\right\} \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} \frac{dk_{j+1/2} dx_{j+1/2}}{(2\pi)^2}$$

$$= \lim_{N \to \infty} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ -\frac{\epsilon \bar{p}_j^2}{2m} \left( k_{j+1/2} - \frac{m}{\epsilon \bar{p}_j^2} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j) \right)^2 ight. ight.$$ 

$$\left. + \frac{m}{2\epsilon \bar{p}_j^2} (\bar{q}_j \Delta p_j + \bar{p}_j \Delta q_j)^2 - x_{j+1/2} \left( \frac{\Delta p_j}{\bar{p}_j} + \epsilon \omega \right) \right]\right\} \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} \frac{dk_{j+1/2} dx_{j+1/2}}{(2\pi)^2} \times \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} \frac{dx_{j+1/2}}{(2\pi)^2 \bar{p}_j^j}.$$

Just as in the case of the previous section integration over $k_{j+1/2}$ yields:

$$K(p'', q'', t''; p', q', t') =$$

$$= \lim_{N \to \infty} \left( \frac{2\pi m}{i \epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ \frac{m}{2\epsilon} \left( \Delta q_j + \frac{\Delta p_j}{\bar{q}_j} \right)^2 ight. ight.$$ 

$$\left. - x_{j+1/2} \left( \frac{\Delta p_j}{\bar{p}_j} + \epsilon \omega \right) \right]\right\} \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} \frac{dx_{j+1/2}}{(2\pi)^2 \bar{p}_j^j}.$$

Next we carry out the integrations over the $x_{j+1/2}$ and find:
\[ K(p'', q'', t''; p', q', t') = \]
\[ = \lim_{N \to \infty} \left( \frac{2\pi m}{i\epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ \frac{m}{2\epsilon} \left( \Delta q_j + \frac{\Delta p_j}{\bar{p}_j} \right) \right]^2 \right\} \]
\[ \times \prod_{j=0}^{N} \frac{1}{2\pi\bar{p}_j} \delta \left( \frac{\Delta p_j}{\bar{p}_j} + \epsilon \omega \right) \prod_{j=1}^{N} dp_j dq_j \]
\[ = \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ \frac{m}{2\epsilon} \left( \Delta q_j + \frac{\Delta p_j}{\bar{p}_j} \right) \right]^2 \right\} \]
\[ \times \prod_{j=0}^{N} \delta (\Delta p_j + \epsilon \omega \bar{p}_j) \prod_{j=1}^{N} dp_j dq_j \]
\[ = \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left\{ i \sum_{j=0}^{N} \left[ \frac{m}{2\epsilon} (\Delta q_j - \epsilon \omega \bar{q}_j) \right]^2 \right\} \]
\[ \times \prod_{j=0}^{N} \delta \left[ \frac{1 + \epsilon \omega}{2} p_{j+1} - \left( 1 - \frac{\epsilon \omega}{2} \right) p_j \right] \prod_{j=1}^{N} dp_j dq_j. \]

Now using the identity
\[ \int \delta(ax - by) \delta(ay - bz) dy = \delta(a^2 x - b^2 z), \]
and carrying out the \( p_j \)-integrations we find:

\[ K(p'', q'', t''; p', q', t') = \]
\[ = \lim_{N \to \infty} \delta \left[ \left( 1 + \frac{\omega T/2}{N + 1} \right)^{N+1} p'' - \left( 1 - \frac{\omega T/2}{N + 1} \right)^{N+1} p' \right] \]
\[ \times \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left[ i \sum_{j=0}^{N} \frac{m}{2\epsilon} (\Delta q_j - \epsilon \omega \bar{q}_j)^2 \right] \prod_{j=1}^{N} dq_j \]
\[ = \delta \left( e^{\omega T/2} p'' - e^{-\omega T/2} p' \right) \]
\[ \times \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon} \right)^{(N+1)/2} \int \ldots \int \exp \left[ i \sum_{j=0}^{N} \frac{m}{2\epsilon} (\Delta q_j - \epsilon \omega \bar{q}_j)^2 \right] \prod_{j=1}^{N} dq_j \]
\[ = \delta \left( e^{\omega T/2} p'' - e^{-\omega T/2} p' \right) \]
\[ \times \mathcal{N} \int \ldots \int \exp \left[ i \int_0^T \frac{m}{2} (\ddot{q} - \omega q)^2 dt \right] \mathcal{D}q, \]

where \( T \equiv t'' - t' \).
The final path-integral we have to solve is a Lagrangian path integral for a quadratic Lagrangian. This final Lagrangian path integral can be done using extremal techniques which are exact in the case of a quadratic Lagrangian, see Ref. 95.

The action for this Lagrangian path integral is given by

\[ I_{cl} = \frac{m}{2} \int_0^T (\dot{q} - \omega q)^2 dt \]

variation of which yields the equation of motion

\[ \ddot{q} = \omega^2 q, \]

which has the general solution

\[ q(t) = A \sinh(\omega t) + B \cosh(\omega t). \]

Imposing the boundary conditions \( q(0) = q' \) and \( q(T) = q'' \) yields:

\[ q(t) = \frac{1}{\sinh(\omega T)} \{q' \sinh[\omega(T - t)] + q'' \sinh(\omega t)\}. \tag{C.2} \]

Therefore, using (C.2) one finds the evaluated classical action to be:

\[ S_{cl} = \frac{m\omega}{2\sinh(\omega T)} \{[(q'')^2 + (q')^2] \cosh(\omega T) - 2q''q' \} - \frac{m\omega}{2} [(q'')^2 - (q')^2]. \]

So that our final result for the representation independent propagator with this Lagrangian becomes

\[ K(p'', q'', t''; p', q', t') \]

\[ = \sqrt{\frac{2\pi m\omega}{i \sinh(\omega T)}} \delta \left( e^{\omega T/2} p'' - e^{-\omega T/2} p' \right) \]

\[ \times \exp \left( \frac{i m\omega}{2 \sinh(\omega T)} \{[(q'')^2 + (q')^2] \cosh(\omega T) - 2q''q' \} - \frac{i m\omega}{2} [(q'')^2 - (q')^2] \right). \tag{C.3} \]


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BIOGRAPHICAL SKETCH

Wolfgang Tomé was born in Ludwigsburg, Germany, on May 15, 1962. He was raised in Reutlingen, Germany, where he finished 9 years of elementary school in 1977, served an apprenticeship from 1977-1980 as a mechanical engineer at Burkhard und Weber KG, attended the Berufsaufbauschule from 1980-1981, and the Technische Gymnasium from 1981-1984.

In October 1984 he entered the Fakultät für Physik at the Eberhard Karls Universität in Tübingen where he received his Vordiplom in Physik with magna cum laude in October 1986.

The period from September 1987 to August 1989 he spent at the Departments of Physics and Mathematics at the University of Denver, Colorado, on a fellowship from the Studienstiftung des deutschen Volkes. He received his Master of Science with thesis in mathematical physics in August 1989.

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He has been married to Marie-Jacqueline Lamoth since August of 1990. They are the parents of Anne-Sophie Tomé.
I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

John R. Klauder, Chairman
Professor of Physics

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