I dedicate this dissertation to my family and friends.
ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. Joseph Hartman, for the constant support and optimism that this program could be completed in three years. Our weekly meetings not only provided me guidance on this research but also provided invaluable insights into academia. I would also like to thank my committee members Dr. Joseph Geunes, Dr. Jean Philippe Richard, and Dr. Joseph Alba. Your responsiveness and flexibility to support my compressed timeline was critical to completing this dissertation.

Next, to my classmates and friends. Without your support and encouragement, I never would have made it through the last three years. From game nights to current event discussions, time with you was always a welcome break from the intense school schedule. Without our general exam study group, I would have never made it past the first year. For all you have done, I am grateful and look forward to continuing these lifelong friendships.

And finally, to my family. Thank you to my parents who provided constant words of encouragement. To my brother, although miles apart, we could commiserate as we both pursued advanced degrees. And to my wife and son. You are the daily reminder of what life is all about. Through your unlimited support and sacrifice, we were able to accomplish this goal. Words cannot express how blessed and thankful I am to have you. Without you, nothing is possible, and with you, everything is perfect.
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This dissertation studies the use of contracts that can be used by a manufacturer to affect consumer behavior by providing protection against poor product performance. We first define the concept of a performance based warranty (PBW) that reimburses a consumer if a product performs below a certain threshold and then show how these warranties can be used to entice consumers to make purchases sooner than planned. We show in a finite horizon equipment replacement problem with a single challenger that a manufacturer can increase their net revenue by offering PBWs at certain times. Next, we consider another finite horizon problem where there are multiple challengers. In this scenario, we model performance money back guarantees (MBG) that influence a consumer’s expectation of future product performance. Specifically, we show how an incumbent manufacturer can use performance MBGs to entice the consumer to return for a follow-on purchase instead of taking their business to another company. Finally, we study the practicality of using extended performance based warranties (EPBWs) when the product’s performance is based on the consumption of a given commodity. We model a renewable, deferrable EPBW that compensates a consumer according to the market price of the underlying commodity. In a semi-empirical example, we show the actual effect on revenue of offering EPBWs and the potential effect on revenue if the manufacturer can exploit the consumer’s full willingness-to-pay.
CHAPTER 1
INTRODUCTION

A warranty (or guarantee) is a contractual mechanism to provide a consumer a level of compensation from the manufacturer when a product does not operate as promised. For the consumer, this protection may lower, or at least limit, certain periodic costs while the product is under warranty. For the manufacturer, the warranty may be a signal of quality to the consumer, generate a source of additional revenue, or increase consumer loyalty.

The designs of a warranty can vary significantly as there are many different factors, such as price, length, and compensation rate, that must be specified for each warranty. Murthy and Blischke [1, 2] provide an extensive analysis of warranties that are predicated on the failure characteristics of the product, which we refer to as traditional warranties. These warranties are modeled and designed to address the condition of whether an item is able to perform its intended function properly or not, and thus they impact the consumer's maintenance costs. Another major total life cost component that is not considered by traditional warranties is operating costs.

A performance based warranty (PBW) guarantees that a product operates to a pre-defined level of performance over a certain length of time. A PBW allows a customer to control the periodic operating costs associated with the product just as a traditional warranty is used to control periodic maintenance costs. Similarly, the design of a PBW has a wide range of factors that must be specified and the conditions of the warranty must be adjusted to the unique nature of operating costs.

There is extensive research on traditional warranties. Conversely, the study of performance based warranties is relatively sparse although there are examples in the commercial sector where they have been applied:
• Boeing and Airbus include performance guarantees in their contracts with airline companies. An example performance guarantee is on the maximum level of fuel consumption.

• Photovoltaic (PV) cell manufacturers, like Schott or Solon, provide performance guarantees for the power output of their PV modules.

• LG enacted a post facto guarantee for some refrigerator models to bring the energy efficiency level up to that advertised on its Energy Guide Label.

• Energy producer, Raser, sued its power generator provider, UTC Power, when the generators did not produce the power output advertised.

These examples are instructive regarding the type of product and business environments where PBWs would be most suitable. The central component of the operating costs in these instances is energy, whether in the form of electricity or fuel. Because there can be other inputs to operating costs such as labor, cases where a product’s operating costs are heavily dependent on energy consumption are a natural fit for PBWs. As for the business environments, there are the retail side with individual consumers and the industrial side with businesses and government agencies. On the retail side, products that are minimally affected by a consumer’s operating habits, such as a refrigerator, significantly increase the feasibility of using a PBW. As for the industrial side, there is a much greater latitude to craft operating guidance that can be monitored to ensure the product is used within a certain operating envelope. Clearly, there are many factors that affect fuel consumption for an aircraft that are independent of the engines fuel efficiency, but the structured nature of operating an aircraft makes compliance to a PBW possible. Additionally, the significant involvement of the customer in the manufacturer’s development process can help define an enforceable PBW.

Besides the type of products and business environments, these practical examples exist in an equipment replacement environment. The services of the products are
required on a continuous basis and therefore the customers must plan the replacement
schedule of the products to ensure the service is always available. Because of the
possibility of a follow-on customer purchase, it is less likely that the manufacturer will
exploit a PBW in the short-run at the risk of alienating a customer with regards to
making a future purchase. In this environment, warranties can play a strategic role
that affect not only the item that is currently under warranty but the future product
replacement.

Within warranty analysis, there is often an associated equipment replacement
problem. Hartman and Laksana [3] analyze the use of warranties when there is only
one challenger at a time but subsequent challengers have lower maintenance costs.
Mamer [4] considers the total expected discounted and average per unit costs of a
warranty using renewal theory when there is one stationary challenger. Hartman and
Laksana [5] also consider one stationary challenger and use a dynamic program model
to allow for the additional decision to purchase an extended warranty. Additionally, there
are equipment replacement problems which are well suited for the analysis of a PBW
as well as other types of performance guarantee contracts. Nair and Hopp [6] explore
the optimal replacement policy when the currently owned item is obsolete and the
single challenger is a technological improvement. Oakford et. al [7] consider equipment
replacement policies with multiple, non-stationary challengers.

In this dissertation, we study the use of various performance guarantee contracts in
a variety of equipment replacement problems. In Chapter 2, our first model uses a PBW
to entice a consumer to replace their obsolete product sooner than they had originally
planned. Then in Chapter 3, we model the use of a performance guarantee to motivate
a consumer to make a follow-on purchase from the incumbent manufacturer in the
presence of competition. Finally, in Chapter 4, we introduce the concept of an extended
performance based warranty (EPBW) that allows a consumer to obtain a warranty at any
time the product is owned, not just at the time of purchase. In each of these chapters,
we use dynamic programming (DP) to model the consumer's decision making process and determine their optimal purchase policy over a finite horizon. DP is well suited for these type of sequential decision problems, and using the consumer's purchase policies, the best contract design and associated effect on a manufacturer's revenue can be determined.
2.1 Motivation and Literature Review

Traditional warranties are generally structured around the operational status of a product (i.e., the binary states of working or not working) and are, therefore, predicated on the reliability of the item. While traditional warranties address failed products, there is little literature on warranties that address the operations of a product. A performance based warranty (PBW) provides the consumer insurance against a product not performing to its advertised specifications.

Similar to other warranties, a PBW provides a signal of confidence that the item will perform as advertised. This aspect may be important when a consumer is deciding between a base model or a more expensive, technologically advanced model or between models from competing manufacturers. A PBW also provides the potential to generate revenue and build consumer loyalty for the manufacturer, much like a traditional warranty. A good example to illustrate the use of a PBW would be in the sale of standard household appliances. For instance, if a company is promoting a new, high efficiency refrigerator, they can offer a performance warranty that will reimburse the consumer for energy usage above a specified level or energy efficiency below some given level. In 2008, LG essentially enacted an ex post facto PBW in response to a downgrade in the energy efficiency rating by the Department of Energy (ConsumerReports.org and lg.com, 2008). LG compensated consumers with an initial payment to cover the substandard performance through the point in time of the downgrade and agreed to pay annual installments thereafter to bring the consumer’s cost down to the advertised performance level. Similar examples exist for industrial equipment such as farm combines and aircraft. Commercial aircraft manufacturers often provide guarantees on engine performance, measured in fuel consumption or fuel burn, to airlines that purchase their product [8].
For a consumer, a PBW limits the risk that a product will not perform at its promised level. Note that a PBW implies that the product is functioning sufficiently to perform the task for which it was purchased; protection against a lack of functionality is addressed by a traditional warranty. A PBW has the potential to lower the total expected costs for a consumer or minimize the probability that costs will exceed a certain threshold. Also, since better performance generally comes at a higher purchase cost, a PBW can lower the payback period for the higher investment by lowering the expected operating costs.

As with traditional warranties, the manufacturer risks incurring additional costs to pay for any performance warranty claims. The additional risk can provide additional revenue or generate revenue earlier. A PBW may also help change a consumer’s purchasing habits, which might support other company goals. For example, a company may need to meet a non-revenue driven goal, such as adherence to government regulation (Corporate Average Fuel Economy (CAFE) standards for example) and thus need to increase sales of certain products at certain times. Or, a company may attempt to enter a market and increase its credibility with a PBW, such as a solar panel company entering the residential energy market [9].

There is extensive literature on warranty analysis and in particular, on the structure and design of warranties that address whether a product performs the function for which it was intended. Although the term “function” inherently includes a reasonable level of performance, it is normally reserved for the binary states of failed or operating, and therefore focuses on the failure characteristics of the product. The warranty literature provides a wide range of warranty designs that provide a level of protection against the failure of an item. Murthy and Blischke [1, 2] define numerous warranty policies that use one or more of three primary types of warranties: 1) Free-Replacement Warranty (FRW), 2) Pro-Rata Warranty (PRW), and 3) Reliability Improvement Warranty (RIW). The FRW repairs or replaces the item at no charge to the customer while under warranty, whereas the PRW provides a partial reimbursement for the unused portion of the warranty. A
RIW is primarily used with products where the customer has a significant role in product development, such as military procurement, and the manufacturer's contract incentive is based on improving the reliability of the product. Several variations of these policies exist based on the dimensionality of the warranty, number of items covered (single item or lot), and whether the warranty is renewable or not. Additionally, various models with a wide range of failure characteristics, repair rectification actions, and repair costs are provided.

An extended warranty (EW) is a warranty that can be purchased once the base warranty that comes with the product purchase expires. Various designs for extended warranties have been proposed and studied. Jack and Murthy [10] develop an EW where the consumer has the flexibility to determine the length and start time of the warranty and the manufacturer then prices it accordingly. Hartman and Laksana [5] show that a menu of different EW designs can increase a manufacturer's profit versus a stand alone warranty by capitalizing on the various risk preferences of customers. Gallego et al. [11] study a variation of a PRW warranty called a Residual Value Warranty (RVW) where the reimbursement is based on the number of claims over the warranty period.

Besides varying the design of the warranties, different objective functions and consumer differentiation methods are addressed throughout the literature. Mamer [4] analyzes FRW and PRW warranties using total discounted costs/profits and per unit costs/profits. The optimal policies for an extended warranty when the consumer has different cost criteria, total expected costs versus average per period costs, are explored by Lam and Lam [12]. Chun and Tang [13] study the effects of consumer risk preferences on warranty pricing. In fact, they also look at the effect of producer risk preferences which suggests there are alternative manufacturer objectives from which to study warranty structures. Padmanabhan [14] differentiates between consumers based

We contribute to the warranty analysis literature by first developing and designing performance based warranties in the context of equipment replacement decisions. Similar to Bellman [16], we consider equipment replacement in two cases: 1) no technological improvement so any replacement occurs with a similar model, and 2) technological improvement where a replacement must be with an upgraded product. While we are unaware of any previous research of PBWs, our models build on some design characteristics of traditional warranties such as limiting when and how often a user may purchase a warranty. Second, we show how providing a PBW with the purchase of a product, at no cost, can increase a manufacturer's revenue by incentivizing a consumer to purchase a replacement earlier than expected, regardless of whether the consumer is optimistic or pessimistic about the expected performance of the item. Third, we develop sufficient conditions for determining when to charge for a PBW and present a polynomial-time algorithm to determine the optimal warranty design (defined by a price, length, and performance threshold). Based on these conditions, we find that a manufacturer can charge a pessimistic consumer for a PBW and increase total revenue by creating an additional revenue source.

This chapter has the following structure. In section 2.2, we define a general PBW and discuss the two specific types that are analyzed in the remainder of the chapter. Then, in section 2.3, we define our method of consumer differentiation based on the consumer’s belief about a product’s performance. Section 2.4 identifies the optimal warranty to increase the manufacturer’s revenue for an infinite horizon equipment replacement problem with no technological change. In section 2.5, we present an algorithm to solve for the optimal warranty in a finite horizon equipment replacement problem where the existing product becomes obsolete and must be replaced by a technologically advanced product.
2.2 Performance Based Warranty Definition

Before defining a PBW, we introduce notation with regards to an equipment replacement problem. As this is a sequential decision problem, we define the time horizon by $T$ which represents the total number of equal length time periods over which the problem is solved. An asset can be purchased at price $P_t$ at time $t \in \{1, \ldots, T\}$. In each period of the item’s useful life $N$, the equipment has operating costs $C_t(a)$ where $a \in \{1, \ldots, N\}$ denotes the age of the asset at the beginning of period $t$. It is expected that the operating costs of the asset are uncertain and thus follow a distribution. To protect the consumer against high operating costs, the manufacturer may offer to cap the operating costs that the consumer will pay each period for a certain amount of time at a predetermined price. This is the essence of a performance based warranty (PBW).

First, consider the continuous case. Let $f(x; a, t)$ represent the probability density function of operating costs of a product of age $a$ at time $t$. Also let $l_{a,t}$ and $u_{a,t}$ be the lowest and highest possible operating costs, respectively. We assume all costs are positive and finite so the support of $f(x; a, t)$ must be a finite segment of the positive real number line (see Figure 2-1).

![Figure 2-1. Continuous Operating Cost Distribution](image)

The expected operating cost for an asset of age $a$ at time $t$ is:
Now assume a manufacturer offers a PBW such that operating costs are capped at $b_{a,t} \leq u_{a,t}$ for an $a$-period old asset in period $t$. This is known as the cost cap or coverage level. The expected operating cost for a consumer with a PBW for a product of age $a$ at time $t$ with a cost cap of $b_{a,t}$ is:

$$
\tilde{\mu}_{a,t} = \int_{b_{a,t}}^{u_{a,t}} xf(x; a, t)dx + b_{a,t}(1 - F(b_{a,t}; a, t)),
$$

such that, by definition, $\tilde{\mu}_{a,t} \leq \mu_{a,t}$. Let $W$ be the number of periods that operating costs are capped. Also, let $\gamma$ be the periodic discount factor in order to account for the time value of money. We can now express an upfront fee or warranty cost, $P_{w_t}$, that the consumer is willing to pay at time $t$ for limiting periodic operating costs as:

$$
P_{w_t} = \sum_{i=1}^{W} \gamma^i (\mu_{a+i,t+i} - \tilde{\mu}_{a+i,t+i})
= \sum_{i=1}^{W} \gamma^i (\int_{b_{a+i,t+i}}^{u_{a+i,t+i}} xf(x; a + i, t + i)dx - \int_{b_{a+i,t+i}}^{u_{a+i,t+i}} xf(x; a + i, t + i)dx - b_{a+i,t+i}(1 - F(b_{a+i,t+i}; a + i, t + i)))
= \sum_{i=1}^{W} \gamma^i (\int_{b_{a+i,t+i}}^{u_{a+i,t+i}} xf(x; a + i, t + i)dx - b_{a+i,t+i}(1 - F(b_{a+i,t+i}; a + i, t + i))). \quad (2-1)
$$

Note that this assumes that the customer is risk neutral (minimizes expected discounted costs) and has full knowledge of all costs and parameters.

In the discrete case, let $L_a$ denote the number of cost realizations for a product of age $a$. Similarly, this can be defined as $L_a$ operational levels, each with an associated operating cost. Then $c_{a,t_j} : j \in \{1, \ldots, L_a\}$ represents the possible cost realizations at time $t$ with probability $p_{a,t_j}$. The expected operating cost for an asset of age $a$ in period $t$ is:
\[ \mu_{a,t} = \sum_{j=1}^{L_a} p_{a,t_j} c_{a,t_j}. \]

The expected operating cost under a PBW for an asset of age \(a\) in period \(t\) is:

\[ \bar{\mu}_{a,t} = \sum_{j=1}^{L_a} p_{a,t_j} \min(b_{a,t_j}, c_{a,t_j}). \]

Now assume, without loss of generality, that for each \(a \in W\) and \(t \in T\), the costs are in decreasing order such that \(c_{a,t_j} > c_{a,t_{j+1}}\) for \(j \in \{1, \ldots, L_a - 1\}\). Then we can define \(\bar{L}_a\) as the number of cost realizations where \(b_{a,t_j} < c_{a,t_j}\) for \(j = 1, \ldots, L_a\) (i.e., \(\bar{L}_a = \arg\min_j c_{a,t_j} = \{j : c_{a,t_j} < b_{a,t}\}\)). The warranty cost the consumer is willing to pay at time \(t\) is:

\[
P_{w_i} = \sum_{i=1}^{W} \gamma^i (\mu_{a+i,t+1} - \bar{\mu}_{a+i,t+i}) \]

\[ = \sum_{i=1}^{W} \gamma^i (\sum_{j=1}^{L_a} p_{a+i,t_j+i} c_{a+i,t_j+i} - \sum_{j=1}^{L_a} p_{a+i,t_j+i} \min(b_{a+i,t_j+i}, c_{a+i,t_j+i})) \]

\[ = \sum_{i=1}^{W} \gamma^i (\sum_{j=1}^{\bar{L}_a} p_{a+i,t_j+i} (c_{a+i,t_j+i} - b_{a+i,t+i})). \]  

(2–2)

Note, the use of \(i\) as the index in discounting instead of \(t\) is due to analyzing a subset of the entire time horizon \(T\). Additionally, the \(t\) subscripts can be omitted under the assumption of stationarity. For the remainder of the chapter, we assume stationarity of costs with respect to time.

From Equations 2–1 and 2–2, we can define the concept of a warranty frontier which is annotated by \(H\). A warranty frontier defines the combinations of cost (\(P_w\)) and cap levels \((b_a : a \in \{1, \ldots, W\}\)) for a fixed warranty length \(W\), where the consumer is indifferent between warranties over the same period of time. For example, if \(W\) is fixed, then a risk-neutral consumer with full information will require a lower cost cap for a larger warranty price in order to be indifferent between two PBWs. All possible
warranties (defined by a warranty price, a warranty coverage level for each period of the warranty, and the length of the warranty) for which the consumer is indifferent defines the consumer’s warranty frontier. The warranty frontier further defines if an arbitrary warranty makes a consumer better or worse off by assessing the warranty’s relationship to the warranty frontier.

The warranty design \((P_w, b_1, \ldots, b_W, W)\) is general. Here, we examine the case where operating costs for the asset are expected to grow from \((l_a, u_a)\) to \((l_{a+1}, u_{a+1})\) at rate \(g\) per period (as the product ages). We can now define specific types of PBWs by providing a structure to the truncation levels, \(b_a\). If we consider truncation levels that increase at a constant rate, \(r\), over the length of the warranty (i.e., \(b_{a+1} = (1 + r)b_a, \forall a\)), then we can define all truncation levels with respect to the initial truncation level \(b_1\). In order to simplify notation, we drop the subscript and let \(b\) refer to the initial truncation level. Although there are no restrictions on \(r\), we will closely examine two cases, \(r = g\) and \(r = 0\). Let us formally define two types of PBWs: (1) a constant performance (CP) warranty \((r = g)\), and (2) a constant cost (CC) warranty \((r = 0)\).

A CP warranty allows the cost cap to increase at the same growth rate as the expected periodic operating costs and the design of the warranty is defined as the triple \((P_w, b, W)_{CP}\) where \(P_w\) is the purchase price of the warranty, \(b\) is the initial cap level in the first period of owning a product, and \(W\) is the length of the warranty. Assuming certain restrictions on the parameters of the distribution over time (such as constant shape parameters in a four parameter beta distribution), when the cost cap increases at the same growth rate as the periodic costs, the CP warranty always protects the consumer from the same percentage of cost realizations in any period the warranty is in effect.

Similar to the CP warranty, the CC design is defined as the triple \((P_w, b, W)_{CC}\), but the cost cap remains constant as the periodic costs grow as a product ages. Thus,
the CC warranty protects the consumer from the same or a higher percentage of cost realizations in each subsequent period the warranty is in effect.

### 2.3 Consumer Differentiation

As discussed in the literature review, there are alternative ways to differentiate consumers, but ultimately, differentiation is an attempt to characterize how different individuals make decisions. We are interested in capturing a consumer’s preconceptions about product performance as it is expected these preconceptions would influence a consumer’s decision regarding the purchase of a PBW. Thus, we introduce the concept of consumer beliefs where the consumer may modify the manufacturer’s operating cost distribution. Effectively, these modifications are an indication of the consumer’s willingness to believe or trust the manufacturer under the assumption of symmetric information. The manufacturer’s advertised initial operating costs distribution is considered symmetric information and the consumer’s belief is modeled by modifying the probabilities of this distribution. For example, an automobile manufacturer may display an expected gas mileage and range. Different consumers may “believe” these numbers differently.

Using the concept of a warranty frontier, we can specify different types of warranty frontiers based on consumer beliefs. If a consumer is neutral, she accepts the manufacturer’s operating cost distribution exactly and will have a resulting warranty frontier $H_{CP}^n$ for a constant performance warranty. We use the superscript $n$ to denote a consumer with neutral beliefs. If the consumer is pessimistic (denoted with a superscript $p$), the probabilities are modified such that his warranty frontier is greater than or equal to the neutral warranty frontier, i.e., $H_{CP}^P \geq H_{CP}^n \forall (P_w, b, W)$ and $H_{CP}^P > H_{CP}^n$ for some $(P_w, b, W)$. If the consumer is optimistic (denoted with a superscript $o$), the probabilities are modified such that the consumer’s warranty frontier is always less than or equal to the neutral frontier, i.e., $H_{CP}^O \leq H_{CP}^n \forall (P_w, b, W)$ and $H_{CP}^O < H_{CP}^n$ for some $(P_w, b, W)$.
Equivalent definitions exist for a constant cost warranty and are indicated with a $CC$ subscript.

As alluded to in the previous section, the consumer’s objective is to minimize total expected discounted costs. Similarly, the manufacturer seeks to maximize total expected discounted revenue. Thus, the analysis in this chapter implicitly addresses only risk neutrality, but utility functions could be incorporated into the methodology to analyze other risk preferences.

Additionally, our characterization of consumer belief has an interesting relationship to risk preference. Baker [17] uses a method of inflating maintenance costs to indicate risk aversion. As shown later, a pessimistic consumer has higher expected operating costs, an optimistic consumer has lower expected operating costs, and a neutral consumer has the same expected operating costs. Therefore, the consumer belief as a measure of consumer risk is compatible with Baker’s method of characterizing risk preference. In fact, using the proposed definition of consumer belief as a model of risk allows for an even higher level of discrimination between different risk preferences because expectation alone does not fully describe a consumer’s risk preference.

The use of a distribution for the operating costs would also incorporate risk through the variance of the distribution. Conceptually, this is similar to Markowitz’s [18] mean-variance approach as well as more modern risk characterizations, such as conditional value at risk (CVaR), where variance is a measure of risk. Ultimately, the purpose of attempting to define a consumer’s risk is to differentiate between consumers based on the unknown, inherent beliefs a consumer holds that influence their decision making. Therefore, the pessimistic, optimistic, and neutral consumer beliefs could equivalently describe risk averse, risk taking, and risk neutral, respectively. In this research, we treat consumer beliefs as a measure of the consumer’s belief in the credibility of the manufacturer and leave open the possibility to incorporate risk through the application of utility theory [19].
2.4 Motivation Through Stationary Analysis

Let us consider a stationary, infinite horizon equipment replacement problem without technological change and assume the product has three levels of performance: poor, average, and good with initial costs \( c_1 > c_2 > c_3 \) and probabilities \( p_1^m, p_2^m, \) and \( p_3^m \) respectively. The \( m \) superscript denotes the manufacturer’s probabilities for the product’s performance. The initial costs increase with the age of the product at the rate \( g \) per period. The consumer’s beliefs of the performance costs are \( p_1^c, p_2^c, \) and \( p_3^c \). In this context, the consumer replaces the product at its economic life.

2.4.1 Constant Performance Warranty

The manufacturer is interested in offering a CP PBW to the consumer. In the absence of the warranty, the consumer’s initial expected operating costs are \( C^c = p_1^c c_1 + p_2^c c_2 + p_3^c c_3 \). Recall the subscript \( t \) can be dropped due to stationarity. Using the growth rate \( g \), the product’s maximum useful life \( N \), and discount factor \( \gamma \), the asset’s economic life \( N^* \) (age which minimizes discounted annualized life-cycle costs) can be determined. The consumer’s optimal policy without a PBW is to replace the item after \( N^* \) periods. Under a PBW, the consumer’s expected periodic costs of a one period old product is \( \bar{C}^c = p_1^c \min(c_1, b) + p_2^c \min(c_2, b) + p_3^c \min(c_3, b) \). Setting the annual equivalent costs (AEC) with a PBW equal to the AEC without a PBW, results in (assuming discrete, end-of-period cash flows):

\[
P + P_w + \bar{C}^c \sum_{j=1}^{W} \gamma^j (1 + g)^{j-1} + C^c \sum_{j=W+1}^{N^*} \gamma^j (1 + g)^{j-1} =
\]

\[
P + C^c \sum_{j=1}^{W} \gamma^j (1 + g)^{j-1} + C^c \sum_{j=W+1}^{N^*} \gamma^j (1 + g)^{j-1} \tag{2-3}
\]

\[
P_w + \bar{C}^c \sum_{j=1}^{W} \gamma^j (1 + g)^{j-1} = C^c \sum_{j=1}^{W} \gamma^j (1 + g)^{j-1}
\]
Define \( G(v) = \sum_{j=1}^{\nu} \gamma^j (1 + g)^j \) to simplify the notation.

To solve Equation 2–4, we need to consider three levels of performance:

if \( c_1 \geq b \geq c_2 \), then Equation 2–4 is:

\[
p_1^c c_1 + p_2^c c_2 + p_3^c c_3 = p_1^c c_1 + p_2^c c_2 + p_3^c c_3 - \frac{P_w}{G(W)}
\]

\[
b = c_1 - \frac{P_w}{p_1^c G(W)}.
\]

(2–5)

if \( c_2 \geq b \geq c_3 \), then Equation 2–4 is:

\[
p_1^c c_1 + p_2^c c_2 + p_3^c c_3 = p_1^c c_1 + p_2^c c_2 + p_3^c c_3 - \frac{P_w}{G(W)}
\]

\[
b = \frac{p_1^c c_1 + p_2^c c_2}{p_1^c + p_2^c} - \frac{P_w}{(p_1^c + p_2^c) G(W)}.
\]

(2–6)

if \( c_3 \geq b \), then Equation 2–4 is:

\[
p_1^c c_1 + p_2^c c_2 + p_3^c b = p_1^c c_1 + p_2^c c_2 + p_3^c c_3 - \frac{P_w}{G(W)}
\]

\[
b = C - \frac{P_w}{G(W)}.
\]

(2–7)

Equations 2–5,2–6, and 2–7 form the CP warranty frontier for a consumer with beliefs \((p_1^c, p_2^c, p_3^c)\). The frontier is piecewise linear and the segments can be generalized to the form:

\[
b = \frac{\sum_{j=1}^{k} p_j^c c_j}{\sum_{j=1}^{k} p_j^c} - \frac{P_w}{G(W) \sum_{j=1}^{k} p_j^c} \quad \text{for} \quad c_k \geq b \geq c_{k+1}.
\]

(2–8)

We present a numerical example later, and the CP warranty frontiers for that example can be seen in Figure 2–2.

Note that for a given CP warranty, \((P_w, b, W)\), the effect on the manufacturer’s total discounted revenue, \(\Delta R\), depends on the consumer’s warranty frontier and can be found
by subtracting the discounted expected liability of providing the warranty from the extra revenue provided by the sale of the warranty.

$$\Delta R_{CP} = P_w - G(W) \sum_{j=1}^{k} p_j^m (c_j - b) \quad (2-9)$$

By substituting the generalized warranty frontier formula (Equation 2–8) into Equation 2–9, we can then generalize $\Delta R$ as:

$$\Delta R_{CP} = P_w \left(1 - \frac{\sum_{j=1}^{k} p_j^m}{\sum_{j=1}^{k} p_j^c}\right) - G(W)\left(\sum_{j=1}^{k} p_j^m c_j - \sum_{j=1}^{k} p_j^c c_j + \sum_{j=1}^{k} \frac{p_j^m}{p_j^c}\right) \quad (2-10)$$

The manufacturer’s change in revenue is also a piecewise linear function. The transition points in the warranty frontier occur at the initial periodic operating costs $c_1, c_2, \ldots, c_L$, therefore the transition points on the change in revenue curve also correspond to $c_1, c_2, \ldots, c_L$. Thus, a PBW, if revenue increasing, will always be offered such that $b \in \{c_1, c_2, \ldots, c_L\}$.

**Theorem 2.1.** If one of the three following conditions hold, then a customer is indifferent (or better off) by accepting a CP warranty and a manufacturer can increase its revenue:

i.) $p_1^c > p_1^m$

ii.) $H_{CP}^p \geq H_{CP}^m \forall (P_w, b, W)$ and $H_{CP}^p > H_{CP}^m$ for some $(P_w, b, W)$

iii.) $C^c(1) > C^m(1)$

**Proof.**

i.) From Equation 2–5, we see that every $H_{CP}$ begins with the warranty $(0, c_1, W) \forall W$. Evaluating Equation 2–9 at this point equates to a change of revenue of zero so every change in revenue curve goes through (0,0). From Equation 2–10, the slope of the revenue curve at this point is $(1 - \frac{p_1^m}{p_1^c})$. Since $p_1^c > p_1^m$, then $(1 - \frac{p_1^m}{p_1^c}) > 0$
when \( c_1 \leq b \leq c_2 \). Therefore \( \Delta R_{CP} > 0 \) and the manufacturer can offer a valid warranty on this segment of the \( H_{CP} \) and increase its revenue.

ii.) If a customer is pessimistic, then his warranty frontier (\( H^p_{CP} \)) is point-wise larger than or equal to the warranty frontier of the neutral consumer (\( H^n_{CP} \)), i.e., \( H^p_{CP} \geq H^n_{CP} \) \( \forall b \in [0, c_1] \) and \( H^p_{CP} \geq H^p_{CP} \) for some \( b \in [0, c_1] \). For a neutral consumer, \( p^c_j = p^m_j \) \( \forall j \). Therefore, from Equation 2–10, \( \Delta R^p_{CP} = 0 \) for any warranty offered on the neutral frontier. Let \( b^+ \) represent the set of cap levels where \( P^p_w > P^n_w \) and since there exists at least one point where the pessimistic warranty frontier is strictly larger than the neutral frontier, then \( b^+ \) is non-empty. For any \( b \in b^+ \), \( P^n_w - G(W) \sum_{j=1}^{k} p^m_j (c_j - b) = 0 \) and for that same \( b \), \( P^p_w - G(W) \sum_{j=1}^{k} p^m_j (c_j - b) > 0 \) since \( P^p_w > P^n_w \) and thus the manufacturer’s revenue is increased.

iii.) The probabilities for both the consumer and manufacturer sum to one. Then, Equation 2–10 simplifies to \( \Delta R_{CP} = -G(W)(\sum_{j=1}^{L} p^m_j c_j - \sum_{j=1}^{L} p^c_j c_j) = G(W)(C^c(1) - C^m(1)) \) when offering a valid warranty with a cost cap at \( c_L \).

Therefore, when \( C^c(1) > C^m(1) \), we must have \( \Delta R_{CP} > 0 \). Thus, the manufacturer can increase its revenue.

\[ \square \]

**Example:** To illustrate the effect of a consumer’s beliefs on warranty frontiers and the manufacturer’s change in revenue, we use an example where a 3-period CP warranty is offered for a product with five different performance levels (\( c_1 > c_2 > c_3 > c_4 > c_5 \)). Table 2-1 provides the parameters for the consumer and manufacturer, and Table 2-2 lists the performance beliefs for three different types of consumers: neutral, pessimistic, and optimistic. Note that neutral beliefs coincide with the manufacturer’s probabilities.
Table 2-1. Parameters for Performance Based Warranty (PBW) Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Purchase Price (P)</td>
<td>10000</td>
</tr>
<tr>
<td>Warranty Length (W)</td>
<td>3</td>
</tr>
<tr>
<td>Product Useful Life (N)</td>
<td>10</td>
</tr>
<tr>
<td>Discount Rate ($\gamma$)</td>
<td>.9</td>
</tr>
<tr>
<td>Cost Increase Rate (g)</td>
<td>.15</td>
</tr>
<tr>
<td>Initial Cost for Low Performance ($c_1$)</td>
<td>1500</td>
</tr>
<tr>
<td>Initial Cost for Medium-Low Performance ($c_2$)</td>
<td>1350</td>
</tr>
<tr>
<td>Initial Cost for Medium Performance ($c_3$)</td>
<td>1200</td>
</tr>
<tr>
<td>Initial Cost for Medium-High Performance ($c_4$)</td>
<td>1100</td>
</tr>
<tr>
<td>Initial Cost for High Performance ($c_5$)</td>
<td>1000</td>
</tr>
</tbody>
</table>

Table 2-2. Consumer Belief Vectors

<table>
<thead>
<tr>
<th>Probability</th>
<th>Neutral/Manufacturer (N)</th>
<th>Pessimistic (P)</th>
<th>Optimistic (O)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1$</td>
<td>.2</td>
<td>.3</td>
<td>.1</td>
</tr>
<tr>
<td>$p_2$</td>
<td>.2</td>
<td>.25</td>
<td>.15</td>
</tr>
<tr>
<td>$p_3$</td>
<td>.2</td>
<td>.2</td>
<td>.2</td>
</tr>
<tr>
<td>$p_4$</td>
<td>.2</td>
<td>.15</td>
<td>.25</td>
</tr>
<tr>
<td>$p_5$</td>
<td>.2</td>
<td>.1</td>
<td>.3</td>
</tr>
</tbody>
</table>

| Expected Initial Cost (C) | 1230 | 1292.5 | 1167.5 |

Figure 2-2 illustrates the CP warranty frontiers for consumers with each belief characterization. The pessimistic consumer warranty frontier lies above the neutral frontier and the optimistic consumer frontier falls below the neutral curve as expected. The squares indicate where the slopes change (at initial operating costs $c_1, c_2, \ldots, c_L$).

![Figure 2-2. Constant Performance Warranty Frontiers](image)
Figure 2-3 shows the manufacturer’s increase in revenue from offering CP warranties at each point on the consumer warranty frontiers in Figure 2-2. There is no warranty that can be offered to a neutral or optimistic consumer that will increase revenue. Since any warranty on a neutral consumer’s warranty frontier will have no effect on the manufacturer’s revenue, the manufacturer can only generate additional revenue by offering warranties when a consumer’s frontier is above the neutral warranty frontier, e.g., a pessimistic consumer. For the pessimistic consumer in this example, the manufacturer will maximize revenue by offering the warranty with a cost cap at $c_L$.

![Figure 2-3. Constant Performance Warranty Change in Revenue Curves](image)

The squares on a revenue curve in Figure 2-3 again define the linear segments of the graph and correspond to the squares on the associated warranty frontier. The horizontal portions of the revenue curves illustrate that revenues cannot be increased further by relaxing the restriction that the warranty coverage level be greater than or equal to the lowest operating cost.

### 2.4.2 Constant Cost Warranty

We are also interested in evaluating the warranty frontier for a CC warranty such that $r = 0$. From a practical perspective, this may be attractive to a consumer
because of its simplicity in that the cap does not change from period to period.

Again, $P_w$ represents the price of a CC warranty with cap level $b$, and $(c_1, c_2, \ldots, c_L)$ represents the initial periodic costs of the different performance levels. We must also introduce the parameters $(q_1, q_2, \ldots, q_L)$ where $q_i$ represents the first period that a cost cap will cover the performance levels associated with $c_1, c_2, \ldots, c_L$, i.e., $q_i(b) = \arg\min_i (1 + g)^{i-1} c_j = \{i : (1 + g)^{i-1} c_j \geq b\}$. Note that $q_i(b) \leq q_{i+1}(b) \forall i$.

Under the condition $c_1 \geq b \geq c_L$ (i.e., the initial cap level is in the range of the initial costs), then $q_1(b) = 1$. Therefore $q_i(b)$ is a function of the cap level $b$, but for notational convenience, we use $q_i$ with the understanding it is dependent on the choice of the cap level. Beginning with Equation 2–3, we determine the constant cap warranty frontier by discounting the terms where the warranty cap is in effect:

$$
(p^c_1 c_1 + p^c_2 c_2 + p^c_3 c_3) \sum_{i=1}^{W} \gamma^i (1 + g)^{i-1} = P_w + p^c_1(b \sum_{i=1}^{W} \gamma^i) +
$$

$$
p^c_2(c_2 \sum_{i=1}^{q_2-1} \gamma^i (1 + g)^{i-1} + b \gamma^{q_2-1} \sum_{i=1}^{W-q_2+1} \gamma^i) +
$$

$$
p^c_3(c_3 \sum_{i=1}^{q_3-1} \gamma^i (1 + g)^{i-1} + b \gamma^{q_3-1} \sum_{i=1}^{W-q_3+1} \gamma^i).
$$

Solving for the warranty coverage level $b$, we obtain:

$$
b = \frac{C^c G(W) - p^c_2 c_2 G(q_2 - 1) - p^c_3 c_3 G(q_3 - 1) - P_w}{p^c_1 \gamma^{q_1-1} U(W - q_1 + 1) + p^c_2 \gamma^{q_2-1} U(W - q_2 + 1) + p^c_3 \gamma^{q_3-1} U(W - q_3 + 1)}.
$$

Define $U(v) = \sum_{j=1}^{V} \gamma^j$ to simplify the notation.

In generalized form for any number of performance levels, the warranty frontier is:

$$
b = \frac{C^c G(W) - \sum_{j=2}^{L} p^c_j c_j G(q_j - 1) - P_w}{\sum_{j=1}^{L} p^c_j \gamma^{q_j-1} U(W - q_j + 1)}.
$$

(2–11)

For a given CC warranty $(P_w, b, W)$, the effect on the manufacturer’s revenue is:

$$
\Delta R_{CC} = P_w - \sum_{j=1}^{L} p^m_j ((1 + g)\gamma)^{q_j-1} G(W - q_j + 1) - b \gamma^{q_j-1} U(W - q_j + 1))
$$

(2–12)
Substituting the consumer’s warranty frontier (Equation 2–11) into Equation 2–12 yields the following change in revenue curve for the manufacturer:

\[
\Delta R_{CC} = P_w(1 - \frac{\sum_{k=1}^{L} p_k^m q_k - 1 U(W - q_k + 1)}{\sum_{j=1}^{L} p_j^c q_j - 1 U(W - q_j + 1)}) \\
+ \sum_{k=1}^{L} p_k^m q_k - 1 U(W - q_k + 1)(\frac{C_c^c G(W) - \sum_{j=2}^{L} p_j^c G(q_j - 1)}{\sum_{j=1}^{L} p_j^c q_j - 1 U(W - q_j + 1)}) \\
- \sum_{k=1}^{L} p_k^m c_k((1 + g)\gamma)^{q_k - 1} G(W - q_k + 1). 
\]

(2–13)

Using Equation 2–13, we can determine the CC warranty frontiers for the same parameters and consumers used for the CP example. Figure 2-4 shows the CC warranty frontiers, and the associated changes in revenue are shown in Figure 2-5.

![Figure 2-4. Constant Cost Warranty Frontiers](image)

The squares on the graphs represent the minimum and maximum warranty prices necessary to ensure the warranty level is contained in the range of the initial periodic operating costs \(c_1 \geq b \geq c_L\). Each CC warranty frontier and revenue curve include warranty prices that correspond to warranty levels greater than the highest possible initial operating cost; this is to show the convergence of the curves to the point (0,0). The CC warranty frontiers and revenue curve are also piecewise linear. The number
of segments defining the curves is larger in comparison to the CP warranty. For this example, there are twice as many segments so the graphs appear to be smooth.

![Graph showing change in revenue vs. warranty cost with three lines representing Neutral, Pessimistic, and Optimistic scenarios.]

Figure 2-5. Constant Cost Warranty Change in Revenue Curves

### 2.5 Finite Horizon with Technology Upgrade

Now assume a consumer owns an asset and will require the services of this asset (or similar asset) for a finite period of time $T$. At the beginning of the horizon, a technologically advanced asset is introduced and the existing version is obsolete and no longer available for purchase. The consumer may keep the original item, up to its useful life, then purchase a replacement. We are interested in studying how a warranty might be structured to persuade the consumer to upgrade earlier. Therefore, the customer only receives the warranty if they replace their original product sooner than planned. This provides an economic incentive to the manufacturer to offer the warranty.

We model the problem as a dynamic program with the following parameters: $n_e$ is the age of the existing asset owned at time zero, $N_e$ is the maximum useful life of the current item, $N'_e$ is the planned replacement age when a warranty is not offered (i.e., $N'_e \leq N_e$), $N_u$ is the maximum useful life of the upgraded item, and $W$ is length of the warranty.
The state of the process is the duple \((n, w)\) where \(n\) is the age of the asset the consumer owns and \(w\) is the remaining number of periods on the warranty. Let \(g\) be the growth rate of the periodic operating costs and \(r\) is the growth rate of the warranty level as \(n\) increases. \(P\) is the cost of the upgraded asset (which we assume is constant over time) and \(c'_j: j' = 1, \ldots, L\) represents the costs associated with the \(L\) performance levels of the new item. Again, we maintain \(c'_j > c'_{j+1}: j' = 1, \ldots, L\). For a consumer with beliefs \(p^c_j : j \in \{1, \ldots, L\}\), we define \(C_e = \sum_{i=1}^{L} p^c_j c_j\) to be the expected initial cost of the existing asset, \(C_u = \sum_{i=1}^{L} p^c_j c'_j\) to be the expected initial cost of the upgraded asset, and \(\bar{C}_u(n) = \sum_{i=1}^{L} p^c_j \min(b(1+r)^{n-1}, c'_j(1+g)^{n-1})\) to be the expected initial cost of the upgraded asset under a PBW. Note, the term initial expected cost refers to the operating cost of a product in its first year of operation. The periodic discount rate \(\gamma\) is used to account for the time value of money. The consumer’s decision each period is to keep or replace their currently owned product, modeled as:

\[
v_t(n, 0) = \min \left\{ \begin{array}{l}
\text{Keep : } \gamma(C_e(1+g)^n + v_{t+1}(n+1, 0)) \\
\text{Replace : } P + \gamma(\bar{C}_u(1) + v_{t+1}(1, W - 1))
\end{array} \right\}, \quad n_e \leq n < N'_e, \quad n > t \quad (2-14)
\]

\[
v_t(n, 0) = \min \left\{ \begin{array}{l}
\text{Keep : } \gamma(C_e(1+g)^n + v_{t+1}(n+1, 0)) \\
\text{Replace : } P + \gamma(C_u + v_{t+1}(1, 0))
\end{array} \right\}, \quad N'_e \leq n < N_e, \quad n > t \quad (2-15)
\]

\[
v_t(n, w) = \min \left\{ \begin{array}{l}
\text{Keep : } \gamma(\bar{C}_u(n) + v_{t+1}(n+1, w - 1)) \\
\text{Replace : } P + \gamma(C_u + v_{t+1}(1, 0))
\end{array} \right\}, \quad n \leq t, \quad 1 \leq w \leq W \quad (2-16)
\]

\[
v_t(n, 0) = \min \left\{ \begin{array}{l}
\text{Keep : } \gamma(C_u(1+g)^n + v_{t+1}(n+1, 0)) \\
\text{Replace : } P + \gamma(C_u + v_{t+1}(1, 0))
\end{array} \right\}, \quad n \leq t \quad (2-17)
\]

\[
v_t(N_e, 0) = \min \left\{ \begin{array}{l}
\text{Replace : } P + \gamma(C_u + v_{t+1}(1, 0))
\end{array} \right\}, \quad N_e > t \quad (2-18)
\]

\[
v_t(N_u, w) = \min \left\{ \begin{array}{l}
\text{Replace : } P + \gamma(C_u + v_{t+1}(1, 0))
\end{array} \right\}, \quad N_u \leq t, \quad \forall w. \quad (2-19)
\]

\[
v_{t+1}(n, w) = 0, \quad \forall n, w. \quad (2-20)
\]
With \( n < N_e' \), the consumer may keep the product or replace it with the new technology under a PBW as in Equation 2–14. The condition \( n > t \) indicates that the current item is the obsolete technology. Equation 2–15 defines the consumer’s choice to keep the original product through the originally planned replacement age such that no warranty is offered. Equation 2–16 shows the consumer’s choice when the product is under warranty. Note that the warranty is only offered with the first early upgrade. Equation 2–17 defines the consumer’s choice when the upgrade is not under warranty. Equations 2–18, 2–19, and 2–20 represent the replacement of the obsolete item at its maximum useful life, the replacement of the improved item at its maximum useful life, and the boundary condition for the DP, respectively. Note that we assume the product has no resale/salvage value, but this is easily incorporated.

To further illustrate the process, let us consider the following example. Suppose the product in Table 2-1 is the existing item, \( n_e = 3 \), and \( T = 20 \). At the start of the time horizon (which corresponds to the existing item being three periods old), a new product is introduced with a purchase price of \( P = 10500 \). Using Equations 2–14–2–20 and letting \( \tilde{C}_u(n) = C_o(1 + g)^{n-1} \), we can determine a consumer’s optimal replacement policy when no warranty is offered. Table 2-3 shows the optimal policy for the consumers from Table 2-2 when the replacement item is one of six different technologies. The technologies are numbered 0-5 in the first column, and the second column represents the percent reduction in initial periodic operating costs compared to the initial operating costs of the old technology (listed in Table 2-1). The actual costs are listed in column 3. The triple \((x, y, z)\) in columns 4-6 represents the consumers purchase policy: \(x\) is the number of periods the obsolete item is kept, \(y\) is the number of periods the first upgrade is kept, and \(z\) is the number of periods the final upgrade is kept. Note, the value of \(x\) for a given technology defines the window for which a warranty would be offered, i.e. \(N_e' = x\) in the DP formulation. Since \(x\) represents the planned replacement with no warranty,
then \(x - 1\) represents the number of periods that the warranty would be made available to the consumer.

Table 2-3. Consumer Policies for Various Levels of Technology Improvement \((n_e = 3)\)

<table>
<thead>
<tr>
<th>Upgrade Version</th>
<th>% Reduction in (c_i)</th>
<th>Costs ((c'_1, c'_2, c'_3, c'_4, c'_5))</th>
<th>Neutral</th>
<th>Pessimistic</th>
<th>Optimistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>((1500,1350,1200,1100,1000))</td>
<td>(6,8,6)</td>
<td>(6,8,6)</td>
<td>(6,8,6)</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>((1425,1282.5,1140,1045,950))</td>
<td>(5,8,7)</td>
<td>(6,8,6)</td>
<td>(6,8,6)</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>((1350,1215,1080,990,900))</td>
<td>(5,8,7)</td>
<td>(5,8,7)</td>
<td>(5,8,7)</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>((1275,1147.5,1020,935,850))</td>
<td>(5,8,7)</td>
<td>(5,8,7)</td>
<td>(5,9,6)</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>((1200,1080,960,880,800))</td>
<td>(5,9,6)</td>
<td>(5,7,6)</td>
<td>(5,9,6)</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>((1125,1012.5,900,825,750))</td>
<td>(4,9,7)</td>
<td>(4,9,7)</td>
<td>(5,9,6)</td>
</tr>
</tbody>
</table>

For clarity, let us look at upgrade version 3 (third row) in Table 2-3 where the new product has 15% lower periodic operating costs than the existing asset. The neutral or pessimistic consumer will keep the original product through the fifth period (the age of the item is eight), replace it at the beginning of period six, and then replace it once again at the beginning of period 14. For the same technology, the optimistic consumer will replace the existing item at the same time but will hold the first upgrade one period longer, thus making the second upgrade a period later.

To motivate an earlier replacement of the original item and generate additional revenue, the manufacturer considers offering the consumer either a CP or CC PBW when the obsolete product is replaced earlier than the planned replacement when no PBW is offered.

2.5.1 Constant Performance Warranty

For a constant performance PBW, the expected growth rate of the warranty level is set equal to the growth rate of the operating costs, i.e., \(r = g\). In the DP formulation for the CP warranty, the expected periodic operating costs can be simplified to \(\bar{C}_w(n) = \sum_{i=1}^{L} p_i \min(b, c'_i)\). Table 2-4 shows the effects of offering a no-cost CP PBW on the consumer’s purchase policy and the manufacturer’s revenue for the five different technology improvements defined in Table 2-3 (recall that the goal is to change the consumer’s purchase policy). The warranty levels listed in column 2 and the associated warranty length in column 3 define the no-cost warranty which generates the greatest
revenue increase from the baseline (no warranty offered) solution. Column 4 represents the new consumer purchase policy when the warranty is offered; the notation is the same as Table 2-3. Column 5 indicates the manufacturer’s total discounted revenue over the entire problem horizon, \( T \), and column 6 shows the percentage change in revenue relative to the total discounted revenue when no warranty is offered.

Table 2-4. Optimal Zero-Cost CP Warranties \((n_e = 3)\)

<table>
<thead>
<tr>
<th>Upgrade Version</th>
<th>( b )</th>
<th>( W )</th>
<th>New Policy</th>
<th>Revenue with PBW</th>
<th>( \Delta ) Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Neutral</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1131.45</td>
<td>5</td>
<td>(4,8,8)</td>
<td>9567.58</td>
<td>7.88%</td>
</tr>
<tr>
<td>2</td>
<td>1197.01</td>
<td>9</td>
<td>(4,9,7)</td>
<td>9348.64</td>
<td>5.41%</td>
</tr>
<tr>
<td>3</td>
<td>1128.08</td>
<td>6</td>
<td>(4,9,7)</td>
<td>9429.33</td>
<td>6.32%</td>
</tr>
<tr>
<td>4</td>
<td>1134.80</td>
<td>6</td>
<td>(4,9,7)</td>
<td>9507.57</td>
<td>10.52%</td>
</tr>
<tr>
<td>5</td>
<td>779.38</td>
<td>3</td>
<td>(3,9,8)</td>
<td>10316.35</td>
<td>7.93%</td>
</tr>
<tr>
<td><strong>Pessimistic</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1274.68</td>
<td>8</td>
<td>(4,8,8)</td>
<td>9685.51</td>
<td>9.21%</td>
</tr>
<tr>
<td>2</td>
<td>1217.01</td>
<td>7</td>
<td>(4,8,8)</td>
<td>9732.38</td>
<td>9.73%</td>
</tr>
<tr>
<td>3</td>
<td>1212.50</td>
<td>8</td>
<td>(4,9,7)</td>
<td>9491.20</td>
<td>7.01%</td>
</tr>
<tr>
<td>4</td>
<td>1112.95</td>
<td>1</td>
<td>(4,9,7)</td>
<td>9547.73</td>
<td>7.65%</td>
</tr>
<tr>
<td>5</td>
<td>1003.00</td>
<td>9</td>
<td>(3,9,8)</td>
<td>10441.11</td>
<td>9.24%</td>
</tr>
<tr>
<td><strong>Optimistic</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1422.52</td>
<td>4</td>
<td>(5,8,7)</td>
<td>8867.99</td>
<td>11.10%</td>
</tr>
<tr>
<td>2</td>
<td>933.31</td>
<td>3</td>
<td>(4,9,7)</td>
<td>9227.21</td>
<td>4.04%</td>
</tr>
<tr>
<td>3</td>
<td>866.33</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9388.79</td>
<td>8.56%</td>
</tr>
<tr>
<td>4</td>
<td>876.77</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9410.71</td>
<td>9.40%</td>
</tr>
<tr>
<td>5</td>
<td>853.75</td>
<td>1</td>
<td>(4,9,7)</td>
<td>9501.77</td>
<td>10.46%</td>
</tr>
</tbody>
</table>

Under the warranties presented in Table 2-4, the consumer changes their behavior when presented with a warranty that is at least as good as what is listed. For example, when the technology improvement is version 3, a pessimistic consumer will accept any CP warranty that is eight periods or longer with a warranty coverage level less than or equal to 1212.50. Note that the optimal values were found under the restriction that the initial warranty level was within the range of initial operating costs \((c_1' \geq b \geq c_2')\). Within this range, there are some warranty lengths that are not feasible and thus cannot change the consumer’s behavior. A feasible warranty is defined as a warranty that does not increase the consumer’s total discounted expected costs. In these cases, the increase in performance is not large enough to offset the consumer’s increased cost of purchasing an upgrade earlier, or the performance increase is so large that the
availability of the upgrade alone caused the consumer to upgrade early enough that a warranty would have no effect. The optimal warranty length corresponds to the minimum warranty length a consumer would accept at a given cost cap because if a consumer is willing to accept warranty \((0, b, W)\), he would surely accept \((0, b, W + 1)\).

The effect on revenue of offering a PBW such that the consumer makes the first upgrade earlier than planned is found for any feasible warranty. Figure 2-6 shows all feasible warranties for a pessimistic consumer when the upgrade offers a 15% reduction in operating costs. The red horizontal line represents the manufacturer’s revenue when no warranty is offered so any warranty above the line would increases the manufacturer’s revenue.

![Figure 2-6. Feasible Constant Performance Warranties for Pessimistic Consumers](image)

The effect of the level of technology improvement can have a significant effect on the manufacturer’s revenue. Figure 2-7 shows the maximum revenue for a pessimistic consumer for the five different technological improvements listed in Table 2-4. Notice the curve for technology version 3 (which corresponds to a 15% reduction in expected periodic operating costs) can be obtained by finding the maximum envelop of the curves.
in Figure 2-6, i.e., the shortest length warranty for a given cost cap where the warranty would still change the consumer’s behavior. The horizontal portions of these graphs represent the manufacturer’s revenue when no warranty is offered or equivalently, when a warranty is offered that does not change the consumers behavior. Note that the effect of technology on the manufacturer’s revenue is not monotonic. In this example, the manufacturer maximizes revenue with the most technologically improved product, but the product with a 10% reduction in operating costs provides a larger increase in revenue than either the 15% or 20% technology.

Figure 2-7. Maximum Revenue Curves for Constant Performance Warranties for Different Levels of Technology Improvement

To obtain these optimal no-cost warranties, the DP was solved $N_w$ times with $b = c_5'$ and $W \in \{1, \ldots, N_w\}$. This established an initial point on each of the revenue curves corresponding to a fixed value for $W$. Then, based on the consumer’s spending level under the warranty relative to spending when no warranty is offered, the remainder of the warranty curve can be derived. If the consumer’s spending at this initial point is greater than their spending when no warranty is offered, then no feasible warranty with
length $W$ will be acceptable to the consumer since increasing the warranty level only increases spending. If the spending level under the warranty is less than or equal to the spending with no warranty, then the warranty coverage level can be increased until the spending level under the warranty equals the spending with no warranty. Note that the change from the initial warranty point to where the spending levels are equivalent is piecewise linear due to the fact that changes in the warranty coverage level affect which levels of performance are covered by the warranty. The maximum envelop of these $N_i$ curves form the maximum revenue curve.

We can see there is a wide range of warranties that can be offered to increase the manufacturer’s revenue. In determining the optimal warranty, we found the largest revenue value of the local maxima on the maximum revenue curve (Figure 2-7). For a neutral consumer, all the local maxima are the same since the consumer and manufacturer share the same performance beliefs. For a non-neutral consumer, this is not the case and each local maximum needs to be evaluated separately. Note the graphs are actually discontinuous piecewise linear functions (the vertical segments are byproducts of connecting the points of discontinuity). Each segment corresponds to a warranty length as shown in Figure 2-6. Therefore, evaluating the local maximum would be required for at most $W$ points.

In the DP formulation, Equations 2–14–2–20, the cost of the warranty was zero under the premise that the purpose of the warranty was to change consumer behavior. Using the results from the infinite horizon model and the revenue curves for each $W \in \{1, \ldots, N_a\}$, we can determine if the manufacturer can charge for the warranty and thus increase revenue further while still maintaining the consumer’s behavior change. We begin to answer this question by looking at the maximum revenue curve. From the results in Table 2-4, we see that the maximum revenue for a pessimistic consumer when the improved technology reduces initial operating costs by 15% is achieved by offering a $(0, 1212.5, 8)$ CP warranty (this maximum is actually achieved at every "peak" for
this pessimistic consumer when the cost cap associated with the maximum revenue is between \(c_1\) and \(c_5\). For this warranty, the manufacturer’s revenue is \(R = 9491.20\).

If we define the consumer’s spending for a given warranty \((P_w, b, W)\) as \(S_{(P_w,b,W)}\), then the consumer’s total spending associated with the manufacturer’s revenue is \(S_{(0,1212.5,8)}\). This spending must necessarily equal the consumer’s spending when no warranty is offered \((S_0)\). Otherwise, the manufacturer could raise the cost cap level until \(S_{(0,1212.5,8)} = S_0\) which would decrease the manufacturer’s liability and increase his revenue. Since \(S_{(0,1212.5,8)} = S_0\), the consumer still accepts this offer. On a given segment of the maximum revenue curve, \(W\) is fixed and \(S_0\) is the decision making criteria (instead of annual equivalent cost with no warranty that was used in the infinite horizon model). Using the CP warranty frontier equation from the infinite horizon model, we can determine the relationship between the warranty price and cost cap level to ensure a warranty remains acceptable to the consumer as the price increases from zero.

Differentiating Equation 2–8 with respect to warranty price, we have:

\[
\frac{\partial P_w}{\partial b} = -G(W) \sum_{j=1}^{k} p_j^c, \quad \text{for } c_k \geq b \geq c_{k+1}.
\] (2–21)

Therefore, for every unit increase in the warranty price, the cost cap must decrease by the magnitude of Equation 2–21. For the manufacturer, total differentiation and the linear relationship among the variables in Equation 2–9 allows us to determine the relationship between warranty price and cost cap that keeps the revenue unchanged for the manufacturer:

\[
\frac{\partial \Delta R}{\partial b} = \frac{\partial P_w}{\partial c} = -G(W) \sum_{j=1}^{k} p_j^m, \quad \text{for } c_k \geq b \geq c_{k+1}.
\] (2–22)

For the manufacturer, \(b\) must decrease by the magnitude of Equation 2–22 for every unit increase in warranty price for the revenue to remain unchanged. If the change in \(b\) is less than Equation 2–22, then the manufacturer’s revenue increases. By taking the
ratio between Equations 2–21 and 2–22, we can state the conditions for charging for the warranty on a particular segment of a warranty revenue curve:

$$\frac{\sum_{j=1}^{k} p_j^c}{\sum_{j=1}^{k} p_j^m} = 1 : \text{Revenue does not change by charging,}$$

$$\frac{\sum_{j=1}^{k} p_j^c}{\sum_{j=1}^{k} p_j^m} > 1 : \text{Charging for warranty increases revenue,}$$

$$\frac{\sum_{j=1}^{k} p_j^c}{\sum_{j=1}^{k} p_j^m} < 1 : \text{Charging for warranty decreases revenue.}$$

The charging condition ratios represent the change in warranty level needed for the consumer to remain at the same spending level as when the warranty was zero cost compared to the change in warranty level needed for the manufacturer to remain revenue neutral given a unit change in the price of the warranty. Furthermore, the difference between the partial derivatives from Equations 2–21 and 2–22, adjusted by the discount factor, represents the change in discounted revenue for the manufacturer for every unit change in the warranty cost cap while controlling for the corresponding change in warranty price, i.e., \( \left. \frac{\partial R}{\partial b} \right|_{P_w} = d^h G(W)(\sum_{j=1}^{k} p_j^c - \sum_{j=1}^{k} p_j^m) \) where \( h \) represents the number of periods the existing item is held. Note that the range of the rules depends not only on the values of the cost cap that define the segment for a particular value of \( W \) but also on the range \( c_k \geq b \geq c_{k+1} \). If we define \( \tilde{b}_w \) and \( \overline{b}_w \) as the lower and upper bounds, respectively, of the maximum warranty curve for a given \( W \), then the range of the conditions above is \([\max(\tilde{b}_w, c_k), \overline{b}_w]\). Since the cash flow factor \( G(W) \) is an element in the change of revenue rate and is increasing in \( W \), the revenue changes at a faster rate on lower revenue curves.

This insight provides an algorithm to solve for the unrestricted (relaxing \( P_w = 0 \)), optimal warranty \((P_w, b, W)\). First, determine the restricted \((P_w = 0)\) revenue curves for each possible warranty length as shown in Figure 2-6. This can be done in polynomial
time in the input, \( O( T^2 N_u^2 + N_u L) \). The steps to build the revenue curves are provided below:

1) Solve the DP formulation (Equations 2–14–2–20) for the \((0, b, W)\) warranty, with \( b = c_L \), for each \( W \in \{1, \ldots, N_u\} \). The solution of one DP can be completed in \( O( T^2 N_u) \). Initialize \( j = L \).

2) For each \((0, b, W)\) point, increase \( b \) from \( c_j \) until \( b = c_{j-1} \) or \( S_{(0,b,W)} = S_0 \), whichever occurs first, through linearly extrapolation. This is \( O(1) \).

3) If \( b = c_1 \) or \( S_{(0,b,W)} = S_0 \), the revenue curve is complete for that particular \( W \). Otherwise, if \( b = c_{j-1} \neq c_1 \) is the stopping condition for increasing \( b \) in step 2, let \( j = j-1 \) and return to step 2. Note that the extrapolation now used in step 2 will be different from the previous extrapolation because the warranty level is in a different range of performance costs. Building the full revenue curves for each \( w \in \{1, \ldots, N_u\} \) from the \( N_u \) points in step 1 is \( O( N_u L) \).

To illustrate this process, let us consider point A in Figure 2-6. Step 1 determines the revenue at this point and requires the DP to be solved once for the warranty \((0, b, 4)\) where \( b = c_L \). Since \( S_{(0,c_L,4)} < S_0 \) at point A, \( b \) can be increased. In this case, it can be increased to \( c_{L-1} \), represented by point B. Since \( S_{(0,c_{L-1},4)} < S_0 \) at point B, \( b \) can again be increased. This process is repeated until point C is reached. Note that at point C, \( b = 1143.83 \) and could not be increased to the next \( c_j \) because \( S_{(0,1143.83,5)} = S_0 \).

With the restricted revenue curves complete, we determine if the manufacturer can increase revenues by charging for the warranty. For a given \( W \in \{1, \ldots, N_u\} \), evaluate the manufacturer’s revenue at the point of maximum revenue on the associated revenue curve. If the condition for charging is satisfied at that point, decrease the cost cap to the largest performance cost that is less than the warranty coverage level. Associated with the decrease in the cost cap is an increase in revenue and price. If the charging condition is not satisfied, evaluate the manufacturer’s revenue with \( P_w = \left( \frac{S_0 - S_{(0,x_{\text{max}},w)}}{d^w} \right) \) at
the the largest performance level cost \(c_j\) that is less than the cost cap associated with the maximum revenue cost cap \(b_{\text{max}}\). Repeat this process moving from \(c_j\) to \(c_{j+1}\) until \(c_L\) has been reached. Note that the first iteration moves from \(b_{\text{max}}\) to the closest, smaller \(c_j\) and every subsequent move is from \(c_j\) to \(c_{j+1}\). Implementing the algorithm over all possible values of \(W\) can be completed in \(O(N_uL)\). Therefore the total time to find the optimal warranty is \(O(T^2N_u^2 + N_uL)\).

Table 2-5 shows the effect on the warranty design and manufacturer’s revenue for a CP warranty if the manufacturer is willing to charge for the warranty. The percentage change in column 6 is calculated with respect to the maximum revenue generated when the optimal no-cost CP PBW is offered.

Table 2-5. Optimal Unrestricted-Cost CP Warranties \((n_e = 3)\)

<table>
<thead>
<tr>
<th>Upgrade Version</th>
<th>(P_w)</th>
<th>(b)</th>
<th>(W)</th>
<th>Revenue</th>
<th>(\Delta) Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutral</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1131.45</td>
<td>5</td>
<td>9567.58</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1197.01</td>
<td>9</td>
<td>9348.64</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1128.08</td>
<td>6</td>
<td>9429.33</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1134.80</td>
<td>6</td>
<td>9507.57</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>779.38</td>
<td>3</td>
<td>10316.35</td>
<td>0</td>
</tr>
<tr>
<td>Pessimistic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1880.40</td>
<td>950</td>
<td>8</td>
<td>9920.43</td>
<td>2.43%</td>
</tr>
<tr>
<td>2</td>
<td>1865.24</td>
<td>900</td>
<td>8</td>
<td>9971.94</td>
<td>2.46%</td>
</tr>
<tr>
<td>3</td>
<td>2452.23</td>
<td>850</td>
<td>10</td>
<td>9821.42</td>
<td>3.48%</td>
</tr>
<tr>
<td>4</td>
<td>2444.04</td>
<td>800</td>
<td>10</td>
<td>9888.28</td>
<td>3.57%</td>
</tr>
<tr>
<td>5</td>
<td>1904.73</td>
<td>750</td>
<td>10</td>
<td>10704.35</td>
<td>2.52%</td>
</tr>
<tr>
<td>Optimistic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1422.52</td>
<td>4</td>
<td>8867.99</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>933.31</td>
<td>3</td>
<td>9227.21</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>866.33</td>
<td>2</td>
<td>9338.79</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>876.77</td>
<td>2</td>
<td>9410.71</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>853.75</td>
<td>1</td>
<td>9501.77</td>
<td>0</td>
</tr>
</tbody>
</table>

The pessimistic consumer we’ve used in the previous example is a special case. In general, when the consumer modifies their beliefs, a new performance cost distribution is defined such that it has the same range as the manufacturer’s cost distribution. After the distribution modification, if the consumer’s distribution of performance costs is first
order stochastic dominant to the manufacturer’s distribution, then the following is true:

\[
\sum_{j=1}^{k} p_j^c \geq \sum_{j=1}^{k} p_j^m \quad \forall \ k \in \{1, \ldots, L\}.
\]  

(2–23)

In this case, the optimal cost cap occurs at \( c_L \) for some \( W \) (see the pessimistic consumer section of Table 2-5). This can be seen from the fact that the charging condition is met along every segment of a revenue curve which drives the warranty level to \( c_L \). The length of the warranty will tend to be as large as possible because the rate of change in revenue is greater for larger values of \( W \). This won’t always drive \( W \) to \( N_u \) (or even ensure that the warranty length is longer than the optimal solution for a no-cost CP warranty) because a larger warranty length might induce a different consumer policy than \( W - 1 \), or the maximum revenue for a no-cost warranty of length \( W \) may be significantly less than a no-cost warranty of length \( W - 1 \).

Similarly, a special case of the optimistic consumer is if the consumer’s distribution is first-order stochastic dominated by the manufacturer’s distribution. This leads to the condition:

\[
\sum_{j=1}^{k} p_j^m \geq \sum_{j=1}^{k} p_j^c \quad \forall \ k \in \{1, \ldots, L\}.
\]  

(2–24)

Under this condition, the optimal warranty will be the optimal no-cost warranty. To see this, compare the optimistic consumer section of Table 2-5 to the optimistic consumer section of Table 2-4. When Equation 2–24 holds, the charging condition will never be met and thus there is no range of warranty levels over which charging for the warranty will increase revenue. This is not surprising since we would expect an optimistic consumer to have faith in the product and therefore, be unwilling to pay for the warranty.

### 2.5.2 Effect of Mean and Variance

Using the previous algorithm, we can easily show the effect of modeling operating costs with a distribution as opposed to expected values. Table 2-6 shows the effect
on revenue if the manufacturer offers a CP warranty under the same conditions as
the example in section 2.5.1, except that the manufacturer and consumer beliefs
are different. In the seven combinations of beliefs in the table, the expected periodic
operating costs for both parties are the same, but the variances of the costs differ.
The first entry shows the manufacturer’s revenue when both parties have the same
deterministic belief of operating costs. The next three entries show how the manufacturer’s
revenue increases as the variance of the consumer’s beliefs increases. The next
three entries show how the manufacturer’s revenue decreases as the variance of the
operating costs increases.

Table 2-6. Effect of Variance on Revenue with 15% Technology Improvement \((n_e = 3)\)

<table>
<thead>
<tr>
<th>Manufacturer Belief</th>
<th>Consumer Belief</th>
<th>Revenue</th>
<th>(P_w)</th>
<th>(b)</th>
<th>(W)</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0,1,0,0)</td>
<td>(0,0,1,0,0)</td>
<td>9415.67</td>
<td>0</td>
<td>962.81</td>
<td>4</td>
<td>–</td>
</tr>
<tr>
<td>(0,0,1,0,0)</td>
<td>(0,24,4,36,0)</td>
<td>9593.37</td>
<td>59.88</td>
<td>1020</td>
<td>8</td>
<td>9000</td>
</tr>
<tr>
<td>(0,0,1,0,0)</td>
<td>(0,4,0,6,0)</td>
<td>9691.50</td>
<td>226.07</td>
<td>1020</td>
<td>8</td>
<td>15000</td>
</tr>
<tr>
<td>(0,0,1,0,0)</td>
<td>(.4,0,0,0,.6)</td>
<td>9963.83</td>
<td>641.54</td>
<td>1020</td>
<td>8</td>
<td>60000</td>
</tr>
<tr>
<td>(0,.24,.4,.36,0)</td>
<td>(0,0,1,0,0)</td>
<td>9415.67</td>
<td>0</td>
<td>901.54</td>
<td>2</td>
<td>9000</td>
</tr>
<tr>
<td>(0,.4,.0,.6,0)</td>
<td>(0,0,1,0,0)</td>
<td>9415.67</td>
<td>0</td>
<td>901.54</td>
<td>2</td>
<td>15000</td>
</tr>
<tr>
<td>(.4,0,0,0,.6)</td>
<td>(0,0,1,0,0)</td>
<td>9394.94</td>
<td>46.93</td>
<td>850</td>
<td>1</td>
<td>60000</td>
</tr>
</tbody>
</table>

Additionally, we can demonstrate the effect of different expectations on manufacturer
revenue. Table 2-7 shows how the manufacturer’s revenue varies when the expected
operating costs of the two parties are different. In the five belief combinations, the
variance of both the manufacturer’s and consumer’s beliefs is zero. The first entry
provides the revenue when the manufacturer and consumer have the same expected
operating costs. The second and third entries show how the manufacturer’s revenue
increases as the consumer’s expectation of performance costs increases. The fourth
and fifth entries show that the manufacturer’s revenue decreases as the consumer’s
expectation of high performance (i.e., lower costs) increases. Notice that when the
consumer’s expectation is at the lowest possible cost, the manufacturer can’t even offer
a warranty that the consumer will accept.
Table 2-7. Effect of Mean on Revenue with 15% Technology Improvement ($n_e = 3$)

<table>
<thead>
<tr>
<th>Manufacturer Belief</th>
<th>Consumer Belief</th>
<th>Revenue</th>
<th>$P_w$</th>
<th>$b$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0,1,0,0)</td>
<td>(0,0,1,0,0)</td>
<td>9415.67</td>
<td>0</td>
<td>962.81</td>
<td>4</td>
</tr>
<tr>
<td>(0,0,1,0,0)</td>
<td>(0,1,0,0,0)</td>
<td>10193.94</td>
<td>1076.96</td>
<td>1020</td>
<td>9</td>
</tr>
<tr>
<td>(0,0,1,0,0)</td>
<td>(1,0,0,0,0)</td>
<td>11709.70</td>
<td>1660.87</td>
<td>1020</td>
<td>8</td>
</tr>
<tr>
<td>(0,0,1,0,0)</td>
<td>(0,0,0,1,0)</td>
<td>9142.95</td>
<td>0</td>
<td>853.23</td>
<td>4</td>
</tr>
<tr>
<td>(0,0,1,0,0)</td>
<td>(0,0,0,0,1)</td>
<td>7741.99</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

2.5.3 Constant Cost Warranty

As demonstrated in the infinite horizon problem, CC warranties affect consumers differently than CP warranties, and thus, they can affect the manufacturer's revenue differently. For a constant cost warranty, the warranty level does not grow so $r = 0$.

Thus $\bar{C}_d(n) = \sum_{i=1}^{n} p_i \min(b, (1 + g)^{-1} c_i^f)$ in the DP formulation. Table 2-8 below shows the maximum revenue generated from a no-cost CC warranty for a given consumer for the technology improvements in Table 2-3. Note that the CP warranty generates more revenue for the optimistic consumer but the CC warranty generates more revenue for the pessimistic consumer. As would be expected, there is no difference between the revenues for a neutral consumer although the warranty designs are different.

Table 2-8. Optimal Zero-Cost CC Warranties on Manufacturer Revenue ($n_e = 3$)

<table>
<thead>
<tr>
<th>Upgrade Version</th>
<th>$b$</th>
<th>$W$</th>
<th>New Policy</th>
<th>Revenue with PBW</th>
<th>$\Delta$ Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutral</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1202.12</td>
<td>3</td>
<td>(4,8,8)</td>
<td>9567.58</td>
<td>7.88%</td>
</tr>
<tr>
<td>2</td>
<td>1013.55</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9348.64</td>
<td>5.41%</td>
</tr>
<tr>
<td>3</td>
<td>1047.10</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9429.33</td>
<td>6.32%</td>
</tr>
<tr>
<td>4</td>
<td>937.64</td>
<td>1</td>
<td>(4,9,7)</td>
<td>9507.57</td>
<td>10.52%</td>
</tr>
<tr>
<td>5</td>
<td>908.53</td>
<td>3</td>
<td>(3,9,8)</td>
<td>10316.35</td>
<td>7.93%</td>
</tr>
<tr>
<td>Pessimistic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1306.03</td>
<td>3</td>
<td>(4,8,8)</td>
<td>9667.45</td>
<td>9.00%</td>
</tr>
<tr>
<td>2</td>
<td>1298.34</td>
<td>3</td>
<td>(4,8,8)</td>
<td>9720.89</td>
<td>9.60%</td>
</tr>
<tr>
<td>3</td>
<td>1145.88</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9487.10</td>
<td>6.97%</td>
</tr>
<tr>
<td>4</td>
<td>1112.95</td>
<td>1</td>
<td>(4,9,7)</td>
<td>9547.73</td>
<td>7.65%</td>
</tr>
<tr>
<td>5</td>
<td>994.10</td>
<td>3</td>
<td>(3,9,8)</td>
<td>10419.55</td>
<td>9.01%</td>
</tr>
<tr>
<td>Optimistic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1414.56</td>
<td>1</td>
<td>(5,8,7)</td>
<td>8867.99</td>
<td>11.10%</td>
</tr>
<tr>
<td>2</td>
<td>920.31</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9257.90</td>
<td>4.38%</td>
</tr>
<tr>
<td>3</td>
<td>936.73</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9342.06</td>
<td>8.60%</td>
</tr>
<tr>
<td>4</td>
<td>952.88</td>
<td>2</td>
<td>(4,9,7)</td>
<td>9414.44</td>
<td>9.44%</td>
</tr>
<tr>
<td>5</td>
<td>853.75</td>
<td>1</td>
<td>(4,9,7)</td>
<td>9501.77</td>
<td>10.46%</td>
</tr>
</tbody>
</table>
Figure 2-8 shows the feasible CC warranties for a pessimistic consumer when the upgrade offers a 15% reduction in operating costs. Unlike the CP warranty in Figure 2-6, the CC warranty cannot increase revenue with a warranty of any length; only shorter length warranties will increase revenue.

![Figure 2-8. Feasible Constant Cost Warranties for Pessimistic Consumers](image)

Figure 2-9 shows the maximum revenue curves when a pessimistic consumer is offered a zero-cost CC PBW for the five different technologies in Table 2-8. The relative relationship of the different technology versions are the same as for the CP warranty as shown in Figure 2-7.
We can now determine the optimal unrestricted CC warranty by modifying the algorithm from section 2.5.1. For the CC warranty, the relationship between the warranty price and cost cap level is found by differentiating Equation 2–11 with respect to warranty price. Recall that $q$ is a function of the warranty cap, $b$, therefore the change in warranty price with respect to the warranty coverage level is only valid over the range of $b$ such that the $q$ function is constant. Define $b_k$ and $b_{k+1}$ as the range over which the given $q$ function does not change.

\[
\frac{\partial P_w}{\partial b} = - \sum_{j=1}^{L} p_j^c d_{q_j-1} U( W - q_j + 1 ) \quad \text{for } b_k \geq b \geq b_{k+1}
\] (2–25)

Therefore, for every unit increase in the warranty price, the cost cap must decrease by the magnitude of Equation 2–25. Similar to the CP warranty, Equation 2–12 allows us to determine the relationship between warranty price and cost cap that keeps the revenue unchanged for the manufacturer:
For the manufacturer, \( b \) must decrease by the magnitude of Equation 2–26 for every unit increase in warranty price for the revenue to remain unchanged. Clearly, \( b \) cannot decrease out of the range defined by the performance costs and the \( q \) function. If the change in \( b \) is less than Equation 2–26, then the manufacturer's revenue increases. Again, the ratio between Equations 2–25 and 2–26 provides the conditions for charging for the warranty on a particular segment of a warranty revenue curve:

\[
\frac{\sum_{j=1}^{L} p_j^m d^{q_j-1} U(W - q_j + 1)}{\sum_{j=1}^{L} P_j^m d^{q_j-1} U(W - q_j + 1)} = 1 : \text{Revenue does not change by charging,}
\]

\[
\frac{\sum_{j=1}^{L} p_j^c d^{q_j-1} U(W - q_j + 1)}{\sum_{j=1}^{L} P_j^c d^{q_j-1} U(W - q_j + 1)} > 1 : \text{Charging for warranty increases revenue,}
\]

\[
\frac{\sum_{j=1}^{L} p_j^c d^{q_j-1} U(W - q_j + 1)}{\sum_{j=1}^{L} P_j^c d^{q_j-1} U(W - q_j + 1)} < 1 : \text{Charging for warranty decreases revenue.}
\]

Conceptually, the charging condition ratios are the same as for the CP warranty. Since the warranty can cover different performance levels over the life of the warranty, the probabilities alone do not characterize the charging conditions as before. With the CC warranty, the probabilities are multiplied by the discount factors which are not constant for all \( b \). The difference between the partial derivatives from Equations 2–25 and 2–26, adjusted by the discount factor, represents the change in revenue for the manufacturer for every unit change in the warranty cost cap while controlling for the corresponding change in warranty price, i.e.,

\[
\frac{\partial \Delta R}{\partial b} |_{w} = d^h \sum_{j=1}^{k} d^{q_j-1} U(W - q_j + 1) (p_j^c - p_j^m).
\]

The algorithm to solve for the optimal CC warranty with no restriction on charging is the same as for the CP warranty with one major change. Instead of moving between...
initial performance level costs, $c_j$ and $c_{j+1}$, to determine if the price of the warranty can be increased, the algorithm moves between performance level growth costs $b_k$ and $b_{k+1}$. Formally, we define the performance level growth costs as the ordered set $K$ such that $b \in K$ when \( \{b = c_j(1 + g)^i \text{ for some } j \in \{1, \ldots, L\}, i \in \mathbb{N}_+: c_1 \geq b_k \geq c_L \} \). The number of performance level growth costs are \( O(L + L \frac{\ln(\frac{1}{\epsilon})}{\ln(1 + g)}) \). Note that the initial performance level costs are a subset of $K$.

Table 2-9 shows the effect on the warranty design and manufacturer’s revenue for a CC warranty when the warranty price is unrestricted. The percentage change in revenue is with respect to the maximum revenue when a no-cost CC PBW is offered. For the neutral and optimistic consumer, there is no value that can be charged for the warranty that will increase the manufacturer’s revenue more than offering an optimal no-cost warranty. Note, feasible warranties with a price greater than zero do exist that are acceptable to the neutral consumer, but the revenue remains the same as the zero-cost warranty. As for the pessimistic consumer, revenues can be increased above the level of the optimal zero-cost warranty by charging for the warranty. These revenue increases are not as large as the revenue increases achieved by motivating the consumer to purchase the asset earlier. Additionally, the optimal warranty prices are extremely large as a percentage of the purchase price. While acceptable to the consumer from a minimum discount expected cost perspective, the practicality of such a large "additional cost" is questionable. The manufacturer, therefore, could offer a sub-optimal CC warranty with a lower cost that still increases revenue over the no-cost warranty, but not to the level listed in column 6 of Table 2-9.
Table 2-9. Optimal Unrestricted-Cost CC Warranties ($n_e = 3$)

<table>
<thead>
<tr>
<th>Upgrade Version</th>
<th>$P_w$</th>
<th>$b$</th>
<th>$W$</th>
<th>Revenue</th>
<th>$\Delta$ Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>Neutral</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1022.12</td>
<td>3</td>
<td>9567.58</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1013.55</td>
<td>2</td>
<td>9348.64</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1047.10</td>
<td>2</td>
<td>9429.33</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>937.64</td>
<td>1</td>
<td>9507.57</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>908.53</td>
<td>3</td>
<td>10316.35</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Pessimistic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>4750.09</td>
<td>950</td>
<td>9920.43</td>
<td>2.62%</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4583.89</td>
<td>900</td>
<td>9971.94</td>
<td>2.58%</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5413.01</td>
<td>850</td>
<td>9783.05</td>
<td>3.12%</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4747.01</td>
<td>800</td>
<td>9848.71</td>
<td>3.15%</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>4547.27</td>
<td>750</td>
<td>10673.76</td>
<td>2.44%</td>
<td></td>
</tr>
<tr>
<td>Optimistic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1414.56</td>
<td>1</td>
<td>8867.99</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>920.31</td>
<td>2</td>
<td>9257.90</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>936.73</td>
<td>2</td>
<td>9342.06</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>952.88</td>
<td>2</td>
<td>9414.44</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>853.75</td>
<td>1</td>
<td>9501.77</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

2.6 Implementing Performance Based Warranties

Clearly, there are challenges with regards to implementing PBWs because operating costs can be dependent on inputs outside the control of the asset being covered. As structured in this chapter, the performance levels are measured in periodic costs. Such a PBW could be implemented by predefining the usage of the product such as miles driven or hours used per period. Defining product usage in a business environment might be much more realistic than in a personal use environment. For instance, aircraft operators (whether airline companies or the government) can craft contracts that reference usage amounts and conditions, but a family’s ability to commit to pre-defined car usage could be more tenuous. The tracking of actual usage over the problem horizon could be implemented relatively easily through usage trackers, such as odometers in automobiles, engine hour gauges in aircraft, or smart grid technology for home appliances. Progressive Insurance actually uses a device to track usage, including type of usage; technologies like this can improve implementation of PBWs.

Next, performance metrics, such as miles per gallon or kilowatts per hours, would be applied to the usage to get the total operating commodity used (e.g., gasoline or
electricity). Then using mutually agreed to commodity rates (e.g., price per gallon or price per kilowatt hour), the periodic operating costs could be obtained. This is implicitly the method used in this chapter where the range of the distribution is the periodic cost. By using cost instead of the performance metric, the relationship between the performance metric and underlying commodity cost is confounded. We use an inflation rate on the costs to model decreasing performance over time, but the cost increase could be from either a performance decrease or commodity price increase.

Even with agreed to usage and commodity costs, implementing a process to track the periodic costs would be difficult. The consumer could collect receipts or be issued a special payment card but the possibly of fraud, especially with the retail consumer, might be difficult to control. For the retail consumer, it would be possible though with modern technologies, like the ability to require near real-time updates of purchases on a website, which would allow the manufacturer to identify fraudulent activity. In a business-to-business environment, this issue would not be as difficult because of the accounting and auditing standards that require accurate tracking of costs. Overall, the implementation strategy of reconciling a PBW would need to be considered carefully.

2.7 Summary

When customers purchase durable goods such as a car or appliance, they not only incur the initial purchase cost but also periodic operating and maintenance costs. The vast field of warranty analysis has focused on warranty designs that address the failure characteristics of the product and thus help the consumer control maintenance costs. Likewise, consumer’s have an interest in controlling operating costs and are thus concerned about product performance levels. Performance based warranties address this concern and offer a mechanism for consumers to control their periodic operating costs. The benefits to a manufacturer offering PBWs include additional revenue or increased consumer loyalty.
We introduce the concept of a performance based warranty (PBW) that compensates a consumer if a product does not perform at a certain level. Specifically, two types of PBWs are analyzed: a constant performance (CP) warranty that increases the cost cap at the same expected growth rate as the operating costs and a constant cost (CC) warranty that keeps the operating cost cap constant over the life of the warranty. Using discounted cash flows to model infinite horizon equipment replacement problems with no technological change, it is shown that PBWs can increase a manufacturer’s total discounted revenue for consumers with a pessimistic belief about the credibility of a product’s advertised performance. Warranty frontiers, which show the combinations of warranty prices and periodic cost caps for which a consumer is indifferent, and the associated manufacturer change in revenue curves from offering warranties on the warranty frontiers are determined for consumers with different belief characterizations for each type of warranty.

We also analyze the use of PBWs for a finite horizon equipment replacement problem with technological improvement using dynamic programming. It is shown that no-cost PBWs can be used to influence the consumer’s optimal purchase policy and increase the manufacturer’s total revenue. Additionally, if the manufacturer is willing to charge for the warranty, revenues can be increased further for a pessimistic consumer. The specific design of the warranty depends on the type of warranty, the level of product improvement, and the consumer’s belief of the product’s expected performance.
3.1 Motivation and Literature Review

Consumers are provided information about the products that are available to purchase when considering the replacement of equipment. For example, when purchasing a car, consumers are informed of the estimated miles per gallon rating before the purchase. This information helps define a consumer’s expectation of the product’s performance. After a product has been purchased, the consumer gains additional information by observing the product’s actual performance. As there are many ways in which the consumer may be influenced, the consumer’s expectations may change over time. Furthermore, the performance of a product will surely influence a consumer’s perception and expectation of future product performance from the same manufacturer, which presumably will influence their future product purchase decisions.

In a competitive market, the evolution of a consumer’s expectation of a manufacturer’s product line can have a significant impact on the probability that the consumer returns to the same manufacturer for a follow-on purchase. Therefore, it is in the interest of a manufacturer to evaluate ways in which to ensure a consumer is satisfied with a product. While there are a wide range of product attributes that could constitute meeting expected performance, Swan and Combs [20] state that the actual attributes that determine a consumer’s satisfaction with performance are limited to those that are affected by the consumer’s perceptions. For instance, a routine consumer may be unaware that front-wheel drive and rear-wheel drive vehicles behave differently in wet conditions while an automobile expert understands the difference. For the routine consumer, this attribute is irrelevant while it does matter for the car expert, therefore each is affected by their perception. Additionally, Myers and Alpert [21] and Alpert [22] discuss that only a limited number of attributes are important with regards to purchasing a product and post-purchase performance satisfaction.
In our problem, we consider the case of a single performance attribute that determines consumer satisfaction. Specifically in the context of an equipment replacement problem with multiple potential replacement options (challengers), the attribute of interest is operating cost, that for many products is highly dependent on the economical use of an underlying commodity. For example, the commodity may be gas for a car or electricity for a refrigerator. It is assumed here that the only differentiator in products is operating cost. One example would be the consideration of a traditional sedan and its hybrid version such that the only (assumed) differentiation is the operating cost of each vehicle. In this chapter, we examine the use of a performance guarantee (with respect to the single attribute of operating costs) to increase a manufacturer’s revenue in a finite equipment replacement problem.

Sequential decision problems, such as equipment replacement problems, are often solved with dynamic programming. In the case of adjusting the consumer’s performance expectation over time, a DP is well suited since it provides an opportunity to update expected performance each period based on observed performance. The process of updating a consumer’s information using a Bayesian framework is well studied. Canan and Ulu [23] describe a general Bayesian updating strategy where the consumer receives a new piece of information each period about the potential benefit of a new technology that helps determine if they want to purchase the product or wait for more information. Erdem and Keane [24] use previous usage and current advertising to update a consumer’s product expectation and analyze the effect on the probability of buying from a given company. Mehta et al. [25] present a model where the consumer updates their price beliefs based on diverse information and then develops the consideration set of possible competitors. The use of Bayesian updating in warranty analysis has been used to show how a manufacturer updates product failure characteristics in designing a warranty to optimize profits. Huang and Zhuo [26] update the time between failures in a Bayesian nature to determine the optimal warranty. Fang
and Huang [27] update the level of product deterioration based on expert opinion and limited data in order to make optimal decisions with respect to price, production, and warranty policy.

While considering a consumer's equipment replacement policy, we are ultimately interested in designing contracts that a manufacturer can use in order to improve a customer's satisfaction with a product and thus improve the likelihood of a future purchase. A natural contract to consider is a performance warranty or guarantee. The use of warranties and guarantees to influence consumer behavior is diverse and has been studied extensively. Generally, warranties can be differentiated from guarantees in that a warranty covers a longer period and provides protection against certain sub-standard product attributes that cannot be quickly determined, such as reliability. Guarantees, specifically money-back guarantees (MBG), are traditionally much shorter term and essentially cover unlimited undesired attributes as can be seen with labels such as "no questions asked". Both warranties and guarantees can be used to signal product quality to the consumer. Moorthy and Srinivasan [28] and Shieh [29] show how a MBG is used to signal quality. The use of a warranty as a signal of product quality is studied by Lutz [30] when there is a potential moral hazard by the consumer's choice of product maintenance. Gal-Or [31] examines the effects of a warranty as a signal in competitive markets.

Another common purpose for guarantees and warranties is to screen customers in order to optimize profits based on a heterogenous population. Kubo [32] demonstrates the use of a menu of product prices, with and without a guarantee, to screen consumers based on income. The menu concept is further developed by Padmanaban [14] and Hartman and Laksana [5] for extended warranties where consumers are differentiated based on product usage and risk preference, respectively. The concept of a warranty to both screen and signal simultaneously is studied by Soberman [33].
In this chapter, we present the concept of a performance guarantee that differentiates consumers based on their belief of the product’s performance. We use the term guarantee over warranty for several semantic reasons. First, warranties often cover explicit product characteristics, most often reliability. In the case of using operating costs as the performance metric, we assume that true performance is not observed and performance is only indicated by the realized costs. Therefore, there may be a component to the guarantee that insures the consumer against an operating cost factor outside of the manufacturer’s control. In our particular model, the underlying commodity cost is this component, e.g. electricity prices are not controlled by the manufacturer for a guarantee on the energy efficiency of a household appliance. Second, a warranty, in many cases, involves a conscious decision by the consumer to purchase the warranty. The guarantee we design is inherent in the product price, as MBGs often are, such that there is no consumer action to acquire it. This is equivalent to the case where there is a single purchase price that includes a base warranty.

Our contribution to the literature includes a proposed method for modeling consumer beliefs about expected product performance and how the beliefs are updated as the consumer observes periodic operating costs. We show under this model that: 1) any feasible guarantee, where feasibility is defined as the consumer having lower (or equal) total expected discounted costs than without the guarantee, will be accepted by the consumer, and the probability of the manufacturer getting a follow-on purchase increases in the length of the guarantee; 2) a manufacturer can increase revenue by increasing consumer satisfaction such that the probability of a follow-on purchase increases; 3) the optimal guarantee is dependent not only on the value of the consumer’s belief but the confidence of that belief.

The chapter has the following structure. In section 3.2, we present a model for updating consumer expectation of product performance with respect to expected periodic operating costs. We also present a structure for performance guarantees that
can be offered to consumers. Then in section 3.3, we present a dynamic program that solves for the consumer’s optimal policy and minimum total discounted expected costs when a specific performance guarantee is offered. We conduct a scenario analysis [34] to decrease the number of potential guarantees that must be evaluated. Section 3.4 presents the results from an example problem and provides sensitivity analysis.

3.2 Bayesian Updating Methodology

Our design and analysis of performance guarantees are considered in a finite horizon equipment replacement problem with multiple challengers and technological change. Let \( T \) represent the total time horizon where the consumer has the choice at each time \( t \in \{0, 1, \ldots, T - 1\} \) to purchase a product from any manufacturer or keep the product they currently own for another period. Let \( P_{i,j} \) represent the purchase price of the \( i^{th} \) manufacturer’s \( j^{th} \) technology version. Each of these products has a maximum useful life \( N_{i,j} \) and a periodic operating cost distribution \( f_{i,j,n} \) where \( n \) is the age of the product.

We use the same consumer differentiation method from Chapter 2 whereby the consumer modifies the manufacturer’s cost distributions \( f_{i,j,n} \), which are assumed to be accurate (represents the true performance distribution) and symmetric (the manufacturer provides the distribution to the consumer). We annotate the consumer’s operating cost distribution, after modification, as \( \bar{f}_{i,j,n} \). Correspondingly, \( F_{i,j,n} \) and \( \bar{F}_{i,j,n} \) represent the cumulative distributions for the manufacturer and consumer, respectively. The practical justification for the modification stems from many different sources, such as the consumer’s past experience with the manufacturer, the manufacturer’s general reputation, or the inherent nature of the consumer to be optimistic or pessimistic (or neutral) in their decision making. The only limitation imposed on the consumer modifications is that the modified distribution must have the same support as the true distribution. Note, the requirement of identical supports does not impose the requirement of the same distribution. Consider the example of the manufacturer
claiming an exponential distribution and the consumer modifying this to a chi-squared
distribution. These distributions are defined over the positive real number line but are
different distributions. The same support assumption essentially translates to agreement
between consumer and manufacturer on the range of possible operating costs.

Our choice of consumer differentiation fits well into the concept of having the
consumer update their beliefs on product performance after each time period. The
preconceptions that a consumer has prior to purchasing an item can change after
observing a product’s performance. We use Bayesian updating as the basis for the
progression of the consumer’s modification of the manufacturer’s advertised operating
costs. Although there are other ways to update the beliefs (such as exponential
smoothing), there is an intuitive connection between our problem and the Bayesian
framework.

Recall in Bayesian statistics that we have an initial distribution \( \pi(\theta) \) (called the prior)
for a given parameter \( \theta \). Then given a data set \( X \), we use the likelihood function \( L(X|\theta) \)
and Bayes formula to develop a new probability distribution using the relationship:

\[
\pi(\theta|X) \propto L(X|\theta)\pi(\theta). \tag{3–1}
\]

Note, the proportional relationship exists since \( L(X|\theta) \) is a function of \( \theta \) and therefore
\( L(X|\theta)\pi(\theta) \) may not integrate to one. Thus a constant may be needed to normalize the
right-hand side such that \( \pi(\theta|X) \) is a valid distribution function. In the updating process,
\( \pi(\theta|X) \) becomes the new prior that is updated when the next set of data is observed
(see [35] or [36] for further details on Bayesian statistics). For example, suppose a data
set \( Y \) is observed after determining \( \pi(\theta|X) \). Then we can update this probability in the
following way:

\[
\pi(\theta|Y) \propto L(Y|\theta)\pi(\theta|X). 
\]

In terms of our problem, \( \theta \) is some vector of parameters that describes the periodic
operating costs for a product of a given age \( n \). The consumer has some initial belief
before observing any performance, \( \bar{r}_{i,j,n} \), which is the initial prior in Equation 3–1.

After owning a product for a period, the consumer can update any subset of possible operating costs distributions \((\bar{r}_{i,j,n}, \forall i, j, n)\) depending on the consumer’s specific strategy. For example, if the observed performance is from the \( j^{th} \) technology of manufacturer \( i \), the consumer may choose to update only the distributions for this \( \{i,j\} \) product, for all of manufacturer \( i \)’s products, or for all products in the equipment replacement problem. The strategy for updating the cost distributions is determined by the consumer’s particular application of observations to the field of products as well as the structure of the likelihood function \( L(X|\theta) \). At this point, the nature of the data, \( X \), has not been explicitly stated. Since the goal is to design a performance guarantee and we are using periodic operating costs as a measure of performance, then we let \( X \) represent a measurement of performance costs for the product that is currently owned.

As there are multiple competitors, the observed performance and subsequent updating of performance distributions could have an effect on a consumer’s future decision to keep or replace a product. The use of a performance guarantee might help a manufacturer retain the business of a consumer in the event of substandard performance which in turn could increase the manufacturer’s total revenue. Therefore we define a performance guarantee by the duple \( (W, L) \) where \( W \) is the guarantee length and \( L \) is the level of performance that is guaranteed. Note that \( W \) and \( L \) may not be scalars. For example \( W \) could be a binomial coefficient (i.e., \( \binom{5}{3} \)) which indicates that 3 out of 5 periods are guaranteed or \( L \) could be a vector representing different guaranteed levels of performance for different ages of a product.

As the proposed updating process suggests, a natural method for characterizing periodic costs given a cost distribution is to use the expected value of operating costs for a given product. Due to the sequential decision process of this problem, minimizing total expected discounted costs is a reasonable optimization criteria for the consumer for the entire problem, and dynamic programming is well suited to solve this problem.
Since there is uncertainty in the performance of the products and the distributions that characterize this uncertainty are conditional on previous performance, a stochastic dynamic program is formulated.

3.2.1 Beta-Bernoulli Model with Two Competitors

The general model above allows for an unlimited range of competitors, technologies, updating relationships, and guarantee designs. Therefore we will study the design and effect of a performance guarantee for a specific, tractable model. Consider the case of two competitors where each offers one technology at a time. With only two competitors, we can use a single integer parameter, $p$, to designate both the manufacturer ($i$) and technology ($j$). When $p$ is odd, the product is from manufacturer 1, and when $p$ is even the product is from manufacturer 2. This relationship is mathematically described by the relationship:

$$i = [(p + 1) \mod 2] + 1 \text{ for } p \in \{1, 2, \ldots, R\}.$$ 

Additionally, the technology version is described by:

$$j = \lceil \frac{p}{2} \rceil \text{ for } p \in \{1, 2, \ldots, R\}.$$ 

If we assume that each manufacturer offers a product each period then $R = 2T$. This does not preclude a manufacturer from offering the same product in consecutive periods, i.e. parameters and functions subscripted with $p$ and $p + 2$ are not required to be different. This notation implies (given $p$ is odd and thus from manufacturer 1) the following: $p + 1$ represents manufacturer 2’s product when product $p$ was offered, $p + 2$ represents manufacturer 1’s product the period after offering product $p$, and $p + 3$ represents manufacturer 2’s product the period after offering product $p$. A similar relationship can be made when $p$ is assumed to be even. The parameters $f$, $P$, and $N$ for the general model can now be subscripted with $p$ instead of $\{i, j\}$.

Next, we define our updating relationship. We use a Beta-Bernoulli process similar to that used by McCardle [37] to describe the gathering of information about the benefit of a product. The beta distribution is well suited to model a random variable where
the minimum, maximum, and most likely values are known. In our case, we treat all minimum and maximum values as universally known and accepted parameters, and the most likely value is determined by each individual manufacturer and consumer. We actually use a four-parameter beta model that includes the standard $\alpha_{p,n}$ and $\beta_{p,n}$ shape parameters for product $p$ at age $n$ as well as $a_{p,n}$ and $b_{p,n}$ which define the minimum and maximum potential operating costs, respectively. We represent performance degradation by using a universal periodic rate of increase $g$ for the operating costs so that the minimum and maximum values for the cost distribution of product $p$ can be defined with respect to the initial maximum and minimum values and the age of the product, i.e. $a_{p,n} = (1 + g)^{n-1}a_{p,1}$ and $b_{p,n} = (1 + g)^{n-1}b_{p,1}$. Note the age subscript may be dropped in which case $a_p$ or $b_p$ are assumed to be for a product that is one period old.

The manufacturer’s cost distribution for product $p$ at age $n$ is:

$$f_{p,n} = Beta(\alpha_{p,n}, \beta_{p,n}, (1 + g)^{n-1}a_p, (1 + g)^{n-1}b_p)$$

Whereas the manufacturer has a defined cost distribution for a product of each age, the consumer only defines initial cost distributions for each product. The Bayesian updating process defines the consumer’s cost distributions for a product when $n > 1$. Therefore the consumer’s cost distribution for product $p$ before being purchased is:

$$\bar{f}_{p,1} = Beta(\bar{\alpha}_{p,1}, \bar{\beta}_{p,1}, a_p, b_p)$$

Besides the suitability of the beta distribution to describe performance costs, it also provides a conjugate prior when the likelihood function is a Bernoulli distribution. This guarantees that the posterior distributions, and thus consumer’s cost distributions for $n > 1$, are also beta distributions. For the general Beta-Bernoulli Bayesian update, given a prior distribution $Beta(\alpha, \beta)$ and likelihood function where $X$ is distributed as a Bernoulli random variable, the posterior distribution is $Beta(\alpha + x, \beta + 1 - x)$ where $x$ is a single realization of the random variable $X$.

Typically, this Bayesian model is used to determine the distribution of an unknown probability. The standard beta distribution has support $[0, 1]$ so after each set of
observations, which are drawn from a Bernoulli distribution, the new beta distribution is also over the support \([0, 1]\). In our model though, the beta distributions are on supports of the form \([a_{p,n}, b_{p,n}]\). Additionally, we are interested in the distribution of costs, not of an unknown probability. More specifically we are interested in the expected value of operating costs. Under the four-parameter beta distribution the expected value is:

\[
a + \frac{\alpha}{\alpha + \beta}(b - a)
\]

The term \(\frac{\alpha}{\alpha + \beta}\) is always between 0 and 1 and can be interpreted as the percentage of the distance between the points \(a\) and \(b\) where the expected value lies. Said another way, \(\frac{\alpha}{\alpha + \beta}\) is a level of belief for where the expected costs occur. For our problem, \(\frac{\bar{a}_{p,n}}{\bar{a}_{p,n} + \bar{b}_{p,n}}\) is the consumer’s current belief. Note that a similar argument could be made where the mode is a measure of the consumer’s belief for product \(p\) at age \(n\). Since a consumer defines the shape parameters for each product initially, the choice of these parameters reveals the consumer’s initial belief of the performance of a given product before observing any relevant performance information. If \(\frac{\bar{a}_{p,n}}{\bar{a}_{p,n} + \bar{b}_{p,n}}\) is close to one, the consumer expects costs to be near the maximum level. Conversely, if \(\frac{\bar{a}_{p,n}}{\bar{a}_{p,n} + \bar{b}_{p,n}}\) is near zero, the consumer expects performance to be close to the minimum. Based on the performance observed after owning product, the consumer’s belief for the expected costs will change depending on the performance compared to their prior beliefs.

Therefore, we define our likelihood function in terms of a Bernoulli distribution where a success is defined as performance costs greater than what the consumer expects and a failure as performance costs lower than expected. Using the manufacturer’s distribution as the true distribution for operating costs, the underlying Bernoulli distribution is:

\[
X_{p,n} = \begin{cases} 
0 & \text{with probability } F_{p,n}(\frac{\bar{a}_{p,n}}{\bar{a}_{p,n} + \bar{b}_{p,n}}) \\
1 & \text{with probability } (1 - F_{p,n})(\frac{\bar{a}_{p,n}}{\bar{a}_{p,n} + \bar{b}_{p,n}}) 
\end{cases}
\]

(3–2)

While there is no explicit limitation on the performance guarantee design given the proposed updating relationship, there is a natural structure to the guaranteed
performance level that comes to mind from the definition of $X_{p,n}$. Recall we described a performance guarantee by the duple $(W, L)$ where $L$ is a vector for performance guarantee levels. Based on Equation 3–2, we design guarantee levels based on the consumer's initial belief and subsequent product performance.

### 3.3 Dynamic Program Formulation

Using the specific model from the previous section, we can now formulate a stochastic dynamic program (SDP) that allows us to determine the equipment replacement policy when no guarantee is offered. We first must define one additional component of the updating process in order to generate the SDP. Recall that in the general model the currently owned product’s performance could be used to update different consumer cost distributions (e.g., just the currently owned product, all products from the current manufacturer, or all products regardless of the manufacturer). Here, we assume that product performance only affects the beliefs of products from that same manufacturer. While it is clear that past performance affects the beliefs of the current item, the fact that past performance affects future products from the same manufacturer is realistic since consumers develop a general perception of a manufacturer based on personal experience. We can now define the state space of the SDP by $(p, n, m, s_1, s_2)$ where $p$ is the product currently owned, $n$ is the age of the currently owned product at the beginning of the period, $m$ is the number of periods that any product from manufacturer 1 has been owned ($m$ may not equal $n$), $s_1$ is the number of periods that a product from manufacturer 1 has been owned and performed worse than expected, and $s_2$ is the number of periods that a product from manufacturer 2 has been owned and performed worse than expected.

We let $t$ represent time where $t \in \{0, 1, \ldots, T\}$, and $v_t(p, n, m, s_1, s_2)$ is the cost-to-go function at time $t$ from the state $(p, n, m, s_1, s_2)$. The cost-to-go function is from the consumer’s perspective and thus represents the expected minimum discounted cost of making optimal decisions through the remaining time horizon.
We account for the time value of money using the periodic discount rate $\gamma$. Given that a consumer’s expectation of performance of a product $p$ is based upon the initial belief of the product’s performance as well as the previous performance of all products from the $i^{th}$ manufacturer, we can annotate the consumer’s expectation of performance as the function $E_p(m, s_1)$ for the first manufacturer and $E_p(t - m, s_2)$ for the second manufacturer. Similarly, we represent the expected periodic operating cost of product $p$ at age $n$ with the function $C_p(n, m, s_1)$ for a product from manufacturer 1 and $C_p(n, t - m, s_2)$ for a product from manufacturer 2. Note that the time subscript is used in the arguments of $E_p$ and $C_p$ when $p$ is even in order to track the number of periods that a product from manufacturer 2 is owned. Therefore it is not really time dependent as we could have just added an additional state to the state space. Also, recall from the previous section that $F_{p,n}$ is the true distribution of product $p$ at age $n$.

The inclusion of uncertainty in costs adds a stochastic component to the rewards but does not necessarily require an SDP in and of itself. Since the model has the consumer update their beliefs based on previous performance, the transitions are uncertain and thus a stochastic dynamic program formulation follows. The consumer’s decisions each period are to keep the item they have (K), replace it with the current product offered by manufacturer 1 (R1), or replace it with the current product offered by manufacturer 2 (R2). Clearly the decision to keep is not available if the product has reached its maximum useful life.

$$v_t(p, n, m, s_1, s_2) =$$

$$\min \begin{cases} 
R1 : P_{p'} + \gamma(C_{p'}(1, m, s_1) + F_{p',n}(E_{p'}(m, s_1))v_{t+1}(p', 1, m + 1, s_1, s_2) \\
+ (1 - F_{p',n}(E_{p'}(m, s_1)))v_{t+1}(p', 1, m, s_1 + 1, s_2)) \\
\end{cases}$$

$$R2 : P_{p'+1} + \gamma(C_{p'+1}(1, t - m, s_2) + F_{p'+1,n}(E_{p'+1}(t - m, s_2))v_{t+1}(p' + 1, 1, m, s_1, s_2) \\
+ (1 - F_{p'+1,n}(E_{p'+1}(t - m, s_2)))v_{t+1}(p' + 1, 1, m, s_1, s_2 + 1))$$

for $n = 0, N_p$ \hspace{1cm} (3–3)
\( v_t(\rho, n, m, s_1, s_2) = \)
\[
\begin{align*}
K : & \gamma(\mathcal{C}_\rho(n, m, s_1)) + F_{\rho,n}(E_\rho(m, s_1))v_{t+1}(\rho, n + 1, m + 1, s_1, s_2) \\
& + (1 - F_{\rho,n}(E_\rho(m, s_1)))v_{t+1}(\rho, n + 1, m + 1, s_1 + 1, s_2)) \\
R1 : & P_\rho + \gamma(\mathcal{C}_\rho(n, m, s_1)) + F_{\rho',n}(E_{\rho'}(m, s_1))v_{t+1}(\rho', 1, m + 1, s_1, s_2) \\
& + (1 - F_{\rho',n}(E_{\rho'}(m, s_1)))v_{t+1}(\rho', 1, m + 1, s_1 + 1, s_2)) \\
R2 : & P_{\rho'+1} + \gamma(\mathcal{C}_{\rho'+1}(n, t - m, s_2)) + F_{\rho'+1,n}(E_{\rho'+1}(t - m, s_2))v_{t+1}(\rho' + 1, 1, m, s_1, s_2) \\
& + (1 - F_{\rho'+1,n}(E_{\rho'+1}(t - m, s_2)))v_{t+1}(\rho' + 1, 1, m, s_1 + 1, s_2 + 1))
\end{align*}
\]

when \( \rho \) is odd: for \( \rho > 0, N_\rho > n > 0, 0 \leq s_1 + s_2 \leq t, s_1 \leq m \leq t \)  
(3–4)

\( v_t(\rho, n, m, s_1, s_2) = \)
\[
\begin{align*}
K : & \gamma(\mathcal{C}_\rho(n, t - m, s_2)) + F_{\rho,n}(E_\rho(t - m, s_2))v_{t+1}(\rho, n + 1, m + 1, s_1, s_2) \\
& + (1 - F_{\rho,n}(E_\rho(t - m, s_2)))v_{t+1}(\rho, n + 1, m + 1, s_1 + 1, s_2) \\
R1 : & P_\rho' + \gamma(\mathcal{C}_\rho'(n, m, s_1)) + F_{\rho',n}(E_{\rho'}(m, s_1))v_{t+1}(\rho', 1, m + 1, s_1, s_2) \\
& + (1 - F_{\rho',n}(E_{\rho'}(m, s_1)))v_{t+1}(\rho', 1, m + 1, s_1 + 1, s_2) \\
R2 : & P_{\rho'+1} + \gamma(\mathcal{C}_{\rho'+1}(n, t - m, s_2)) + F_{\rho'+1,n}(E_{\rho'+1}(t - m, s_2))v_{t+1}(\rho' + 1, n + 1, m, s_1, s_2) \\
& + (1 - F_{\rho'+1,n}(E_{\rho'+1}(t - m, s_2)))v_{t+1}(\rho' + 1, n + 1, m, s_1 + 1, s_2 + 1))
\end{align*}
\]

when \( \rho \) is even: for \( \rho > 0, N_\rho > n > 0, 0 \leq s_1 + s_2 \leq t, s_1 \leq m \leq t \)  
(3–5)

\( v_T(\rho, n, m, s_1, s_2) = 0, \quad \forall \rho, n, m, s_1, s_2 \)  
(3–6)

Note that we use \( \rho' \) to represent the product offered at time \( t \) from manufacturer 1 so \( \rho' > \rho \). The formulation above is without a manufacturer guarantee. Under no guarantee, we annotate the initial state value \( v_0(0, 0, 0, 0, 0) \) as \( v_0^{NG} \). If the manufacturer introduces a guarantee, then \( v_0^G \leq v_0^{NG} \) (where \( v_0^G \) is the consumer’s total expected discounted cost with the guarantee) must hold for the guarantee to be beneficial to the consumer. When a guarantee is offered, the consumer’s periodic expected operating costs are reduced, and we define these costs as \( \tilde{\mathcal{C}}_\rho(n, m, s_1) \) and \( \tilde{\mathcal{C}}_\rho(n, t - m, s_2) \).
for product $p$ at states $(p, n, m, s_1, -)$ and $(p, n, m, -, s_2)$ when $p$ is odd and even, respectively. Equations 3–3–3–6 are updated with these new costs accordingly. Since we already defined the guarantee coverage $L$ with respect to the consumer’s expected performance, we are interested in determining the warranty length $(W)$ that maximizes the manufacturer’s revenue.

### 3.3.1 Scenario Analysis

The SDP in the previous section is for a generalized equipment replacement model, therefore solving for an optimal guarantee length, $W$, could be prohibitive based on the large number of possible designs. For example, suppose the first manufacturer is seeking to offer a performance guarantee the first time their product is purchased in order to improve the possibility the consumer returns to them for the follow-on purchase. The maximum useful life of this first product is $N_1$, therefore there are $2^{N_1}$ possible guarantee length designs if there are no limitations on which periods the product performance is guaranteed.

With the number of guarantee designs being exponential, let us consider the impact of different assumptions on the possible number of guaranteed periods. First if we only consider guarantees over consecutive periods, the number of guarantee designs becomes polynomial with $\frac{N_1(N_1+1)}{2}$ possible designs. While this requirement seems natural in a retail consumer environment, we will quickly address an area where the consumer may have an interest in non-consecutive guarantees. In government procurement, minimum cost is certainly a desired outcome. In the context of a restrictive budgeting process though, the trade-off of being within budget in certain years at the cost of increased total program costs is often made. In this environment, the desire for a non-consecutive guarantee for the government is realistic. Regardless, we make the consecutive period assumption here.

Next, consider the further assumption that the consecutive periods either start at the beginning of the purchase or end at the maximum useful life. The guarantee that
starts with the purchase is easily marketed as guaranteed to meet performance for at least $W$ number of periods. Conversely, and only slightly more cumbersome to market, a $W$ period guarantee that ends at the useful life is marketed as covering all unexpected costs for the remainder of the product’s life. One could argue that it is more difficult for a manufacturer to market and a retail consumer to accept a consecutive guarantee that lies in the middle of the product’s life cycle. This assumption reduces the possible number of guarantee designs to $2N_1 - 1$.

With the possible number of guarantee designs now linear, let us consider problems with a special characteristic: the subsequent available technologies (challengers) are significant technology upgrades. We define significant as having a large enough purchase price such that the item’s predecessor will be held to its maximum useful life before purchasing the new technology. There are several mathematical ways to ensure that the technology improvement is significant which are sufficient, but not necessary, conditions. Consider two products $p$ and $p'$ where $p'$ is one of the next available technologies after having purchased $p$. We say that $p'$ is a significant technological change over $p$ if one of the following conditions holds:

1. The maximum annual equivalent cost (AEC) of $p$ is less than the minimum AEC for $p'$, then $p'$ is a significant technology change.

2. The AEC for $p'$ over any length of time beginning with the purchase of $p'$ is greater than the AEC for $p$ over any equivalent length of time (not necessarily starting with the purchase).

While the impact of the significant technology change assumption does not produce a reduction in the number of guarantee designs, the solution methodology and analysis is simplified under this characteristic. Without a significant technology change, replacements may happen at different time periods based on the previous performance. The process and insights we gain from analyzing the significant change problem though can be extended to more general cases. There are many examples where a significant
technology change is common. Consider the migration from a compact car to a large SUV or moving from an obsolete tube styled TV to a new LED flat screen television. The military is often faced with the scenario in moving from one generation of aircraft to the next generation.

3.3.2 Guarantee Effect

With the assumptions from the previous section, we can analyze the effect a performance guarantee has on a manufacturer’s revenue as well as presenting the methodology for solving for the optimal guarantee length. With the significant change assumption, the replacement of a product will occur at the end of the current product’s maximum useful life. Figure 3-1 shows the transition network for a product with $N_p = 4$ and a consumer’s initial belief defined by $\alpha_{p,1} = 3$ and $\beta_{p,1} = 2$, which equates to a consumer believing costs will be skewed towards the high end. We annotate the consumer’s initial belief with $E_0$. Note $F$ is the true cost distribution of the currently owned product.

Based on the performance of the item, the consumer will be at one of the five nodes in the far right column (which we call terminal nodes) after experiencing the variable costs of the product over four periods. The number of paths to each terminal node is annotated on the graph. The total probability of being at a terminal node is the sum of the probabilities of all possible paths, and the nodes from top to bottom are in increasing order based on the number of periods of below expected performance, i.e. $s_1$ in increasing. The decision at any given terminal node is not affected by the path to get to the node since the decision at that state is only dependent on the number of successes, and the number of successes at any given terminal node is the same regardless of when the success occurred. We can find a relationship between the terminal nodes. Specially, we can show that the remaining costs-to-go for a consumer at any time $t$ are non-increasing in the number of periods of substandard performance.
Figure 3-1. Network Paths Without Performance Guarantee \((N_p = 4)\)

**Theorem 3.1.** \(v_t(p, n, m, s_1, s_2) \leq v_t(p, n, m, s_1 + 1, s_2)\) when a Beta-Bernoulli process is followed.

**Proof.**

Given the state \((p, n, m, s_1, s_2)\), there is a corresponding optimal policy at time \(t\) that is annotated by \(D_t(p, n, m, s_1, s_2)\) where:

\[
D_t(p, n, m, s_1, s_2) = \begin{cases} 
1 & \text{if consumer replaces with manufacturer } 1\text{'s product} \\
2 & \text{if consumer replaces with manufacturer } 2\text{'s product} \\
3 & \text{if consumer keeps manufacturer } 1\text{'s product}
\end{cases}
\]

In defining \(D_t(\cdot)\), we assume the incumbent product is from manufacturer 1 but the proof can easily be applied to the case where the incumbent product is from manufacturer 2. Note that the time subscript on the decision function tracks the
number of periods that a product from manufacturer 2 is owned, therefore it is not really time dependent as we could have just added an additional state to the state space. The non-discounted expected periodic operating cost for the next period if the current item (from manufacturer 1) is kept or replaced by another product from manufacturer 1 is \( C_p(n, m, s_1) = q_p(a_{n'} + \frac{a_1 + s_1}{a_1 + \beta_1 + m}(b_{n'} - a_{n'})) \) where \( n' \) represents the age of item, \( q_p \) is the multiplicative factor for the change in operating costs from product 1 to product \( p \), and \((a_n, b_n)\) for \( n \in \{1, 2, \ldots, N\} \) represent the minimum and maximum operating costs for an initial product of age \( n \) from manufacturer 1. Since the age at the current time \( t \) is \( n \), then \( n' \in \{1, n\} \). Similarly, given the state \((p, n, m, s_1 + 1, s_2)\) at time \( t \), \( C_p(n, m, s_1 + 1) = q_p(a_{n'} + \frac{a_1 + s_1 + 1}{a_1 + \beta_1 + m}(b_{n'} - a_{n'})). \)

Therefore,
\[
C_p(n, m, s_1 + 1) > C_p(n, m, s_1)
\]  
\[(3-7)\]

By the same rationale, \( C_p(n, m, s_1) > C_p(n, m + 1, s_1) \). Let \( C_p(n, m, s_2) \) represent the expected operating cost of the item from manufacturer 2 if the product is replaced where \( C_p = (n, m, s_2) = q_p(a_{n'} + \frac{a_2 + s_2}{a_2 + \beta_2 + (t - m)}(b_{n'} - a_{n'})) \). Note that \( C_p(\cdot) \) does not include the purchase price of the item.

Recall that \( \nu_T(p, n, m, s_1, s_2) = 0 \ \forall \ p, n, m, s_1, s_2 \) is the boundary condition. Now consider the possible decisions at time \( T-1 \) for the state \((p, n, m, s_1 + 1, s_2)\).

Case 1: Replace with new product from manufacturer 1

If \( D_{T-1}(p, n, m, s_1 + 1, s_2) = 1 \), then
\[
\nu_{T-1}(p, n, m, s_1 + 1, s_2) = P_{p'} + \gamma C_{p'}(1, m, s_1 + 1)
\]

If \( D_{T-1}(p, n, m, s_1, s_2) = 1 \), then
\[
\nu_{T-1}(p, n, m, s_1, s_2) = P_{p'} + \gamma C_{p'}(1, m, s_1 + 1)
\]
By Equation 3–7, \( v_{T-1}(p, n, m, s_1, s_2) < v_{T-1}(p, n, m, s_1 + 1, s_2) \). If \( D_{T-1}(p, n, m, s_1, s_2) \neq 1 \), then \( v_{T-1}(p, n, m, s_1, s_2) \leq v_{T-1}(1, n, m, s_1 + 1, s_2) \) because if this wasn’t true, the customer would have a lower cost-to-go function by switching to manufacturer 2 or keep the product from manufacturer 1.

Case 2: Replace with new product from manufacturer 2

If \( D_{T-1}(p, n, m, s_1 + 1, s_2) = 2 \), then

\[
v_{T-1}(p, n, m, s_1 + 1, s_2) = P_{p'+1} + \gamma C_{p'+1}(1, m, s_2)
\]

If \( D_{T-1}(p, n, m, s_1, s_2) = 2 \) also, then

\[
v_{T-1}(p, n, m, s_1, s_2) = P_{p'+1} + \gamma C_{p'+1}(1, m, s_2)
\]

Therefore \( v_{T-1}(p, n, m, s_1, s_2) = v_{T-1}(p, n, m, s_1 + 1, s_2) \). If \( D_{T-1}(p, n, m, s_1, s_2) \neq 2 \), then \( v_{T-1}(p, n, m, s_1, s_2) \leq v_{T-1}(p, n, m, s_1 + 1, s_2) \) because if this wasn’t true, the customer would have a lower cost to go function by switching to manufacturer 2.

Case 3: Keep product from manufacturer 1

If \( D_{T-1}(p, n, m, s_1 + 1, s_2) = 3 \), then

\[
v_{T-1}(p, n, m, s_1 + 1, s_2) = \gamma C_p(n, m, s_1 + 1)
\]

If \( D_{T-1}(p, n, m, s_1, s_2) = 3 \), then

\[
v_{T-1}(p, n, m, s_1, s_2) = v_{T-1}(p, n, m, s_1, s_2) = \gamma C_p(n, m, s_1)
\]

By Equation 3–7, \( v_{T-1}(p, n, m, s_1, s_2) < v_{T-1}(p, n, m, s_1 + 1, s_2) \). If \( D_{T-1}(p, n, m, s_1, s_2) \neq 3 \), then \( v_{T-1}(p, n, m, s_1, s_2) \leq v_{T-1}(p, n, m, s_1 + 1, s_2) \) because if this wasn’t true, the customer would have a lower cost-to-go function by purchasing from either manufacturer 1 or 2.
Therefore,

\[ \nu_{T-1}(p, n, m, s_1, s_2) \leq \nu_{T-1}(p, n, m, s_1 + 1, s_2) \]  

(3–8)

Now let us consider the decisions at time \( T-2 \). Since there is a future decision to make, the analysis of possible decisions now includes uncertain transitions and not just uncertain rewards. Again we must consider 3 cases.

Case 1: Replace with new product from manufacturer 1

If \( D_{T-2}(p, n, m, s_1 + 1, s_2) = 1 \), then \( \nu_{T-2}(p, n, m, s_1 + 1, s_2) = \)

\[
P_{p'} + \gamma(C_p(1, m, s_1 + 1) + F_{p',n}(\frac{\alpha_{p'} + s_1 + 1}{\alpha_{p'} + \beta_{p'} + m})\nu_{T-1}(p', 1, m + 1, s_1 + 1, s_2) \\
+ (1 - F_{p',n}(\frac{\alpha_{p'} + s_1 + 1}{\alpha_{p'} + \beta_{p'} + m}))\nu_{T-1}(p', 1, m + 1, s_1 + 2, s_2))
\]

If \( D_{T-2}(p, n, m, s_1, s_2) = 1 \), then \( \nu_{T-2}(p, n, m, s_1, s_2) = \)

\[
P_{p'} + \gamma(C_p(1, m, s_1) + F_{p',n}(\frac{\alpha_{p'} + s_1}{\alpha_{p'} + \beta_{p'} + m})\nu_{T-1}(p', 1, m + 1, s_1, s_2) \\
+ (1 - F_{p',n}(\frac{\alpha_{p'} + s_1}{\alpha_{p'} + \beta_{p'} + m}))\nu_{T-1}(p', 1, m + 1, s_1 + 1, s_2))
\]

To determine the relationship between \( \nu_{T-2}(p, n, m, s_1 + 1, s_2) \) and \( \nu_{T-2}(p, n, m, s_1, s_2) \) let:

\[
r = F_{p',n}(\frac{\alpha_{p'} + s_1 + 1}{\alpha_{p'} + \beta_{p'} + m}) \\
s = F_{p',n}(\frac{\alpha_{p'} + s_1}{\alpha_{p'} + \beta_{p'} + m}) \\
x = \nu_{T-1}(p', 1, m + 1, s_1, s_2) \\
y = \nu_{T-1}(p', 1, m + 1, s_1 + 1, s_2) \\
z = \nu_{T-1}(p', 1, m + 1, s_1 + 2, s_2)
\]

We know that \( 0 < s < r < 1 \) since \( F_{p',n} \) is a beta distribution and \( x \leq y \leq z \) from Equation 3–7 so using a convexity argument we have:

\[
sx + (1 - s)y \leq sy + (1 - s)y = ry + (1 - r)y \leq ry + (1 - r)z \]  

(3–9)
Since the purchase prices \( P_{p'} \) cancel and \( C_{p'}(1, m, s_1 + 1) > C_{p'}(1, m, s_1) \), we have shown that \( \nu_{T-2}(p, n, m, s_1, s_2) < \nu_{T-2}(p, n, m, s_1 + 1, s_2) \). If \( D_{T-2}(p, n, m, s_1, s_2) \neq 1 \), then \( \nu_{T-2}(p, n, m, s_1, s_2) \leq \nu_{T-2}(p, n, m, s_1 + 1, s_2) \) because if this wasn’t true, the customer would have a lower cost-to-go function by switching to manufacturer 2 or keep the product from manufacturer 1.

Case 2: Replace with new product from manufacturer 2

If \( D_{T-2}(p, n, m, s_1 + 1, s_2) = 2 \), then \( \nu_{T-2}(p, n, m, s_1 + 1, s_2) = \)

\[
P_{p'+1} + \gamma(C_{p'+1}(1, m, s_2) \\
+ F_{p'+1,1}(\frac{\alpha_{p'+1} + s_2}{\alpha_{p'+1} + \beta_{p'+1} + (T - 2 - m)})\nu_{T-1}(p' + 1, n, m, s_1 + 1, s_2 + 1) \\
+ (1 - F_{p'+1,1}(\frac{\alpha_{p'+1} + s_2}{\alpha_{p'+1} + \beta_{p'+1} + (T - 2 - m)}))\nu_{T-1}(p' + 1, n, m, s_1 + 1, s_2 + 1))
\]

If \( D_{T-2}(p, n, m, s_1, s_2) = 2 \) also, then \( \nu_{T-2}(p, n, m, s_1, s_2) = \)

\[
P_{p'+1} + \gamma(C_{p'+1}(1, m, s_2) \\
+ F_{p'+1,1}(\frac{\alpha_{p'+1} + s_2}{\alpha_{p'+1} + \beta_{p'+1} + (T - 2 - m)})\nu_{T-1}(p' + 1, n, m, s_1, s_2 + 1) \\
+ (1 - F_{p'+1,1}(\frac{\alpha_{p'+1} + s_2}{\alpha_{p'+1} + \beta_{p'+1} + (T - 2 - m)}))\nu_{T-1}(p' + 1, n, m, s_1, s_2 + 1))
\]

Using Equation 3–8, we have \( \nu_{T-2}(p, n, m, s_1, s_2) \leq \nu_{T-2}(p, n, m, s_1 + 1, s_2) \). If \( D_{T-2}(p, n, m, s_1, s_2) \neq 2 \), then \( \nu_{T-2}(p, n, m, s_1, s_2) \leq \nu_{T-2}(p, n, m, s_1 + 1, s_2) \) because if this wasn’t true, the customer would have a lower cost-to-go function by purchasing from manufacturer 1 or keeping the product from manufacturer 1.

Case 3: Keep product from manufacturer 1
If $D_{T-2}(p, n, m, s_1 + 1, s_2) = 3$, then $v_{T-2}(p, n, m, s_1 + 1, s_2) =$

$$\gamma(C_p(n, m, s_1 + 1) + F_{\rho,s}(\frac{\alpha_p + s_1 + 1}{\alpha_p + \beta_p + m})v_{T-1}(p, n + 1, m + 1, s_1 + 1, s_2)$$

$$+ (1 - F_{\rho,s}(\frac{\alpha_p + s_1 + 1}{\alpha_p + \beta_p + m}))v_{T-1}(p, n + 1, m + 1, s_1 + 2, s_2))$$

If $D_{T-2}(p, n, m, s_1, s_2) = 3$, then $v_{T-2}(p, n, m, s_1, s_2) =$

$$\gamma(C_p(n, m, s_1) + F_{\rho,s}(\frac{\alpha_p + s_1}{\alpha_p + \beta_p + m})v_{T-1}(p, n + 1, m + 1, s_1, s_2)$$

$$+ (1 - F_{\rho,s}(\frac{\alpha_p + s_1}{\alpha_p + \beta_p + m}))v_{T-1}(p, n + 1, m + 1, s_1 + 1, s_2))$$

Using Equations 3–7 and 3–9, we have $v_{T-2}(p, n, m, s_1, s_2) < v_{T-2}(p, n, m, s_1 + 1, s_2)$. If $D_{T-2}(p, n, m, s_1, s_2) \neq 3$, then $v_{T-2}(p, n, m, s_1, s_2) \leq v_{T-2}(p, n, m, s_1 + 1, s_2)$ because if this wasn’t true, the customer would have a lower cost-to-go function by purchasing from manufacturer 1 or manufacturer 2.

By induction, $v_t(1, n, m, s_1, s_2) \leq v_t(1, n, m, s_1 + 1, s_2) \forall t$. Note, further induction steps follow the same formulas used in the T-1 step.

Using this theorem, we know that for any terminal node in Figure 3-1 that has a policy of changing manufacturers, every terminal node below it will also result in the decision to change manufacturers. This fact provides a benefit in further reducing the number of dynamic programs needed to find the optimal guarantee length. Without Theorem 3.1 and to determine the optimal guarantee, the initial DP with no guarantee is solved to determine the manufacturer’s baseline revenue. Then two DPs (one for a guarantee starting at purchase and the other for a guarantee expiring at replacement) for each possible guarantee length (from one up to the maximum useful life) are solved. Any guarantee with a revenue larger than the baseline could be offered to increase revenue. From the baseline DP though, we can use the policy matrix from the baseline DP to further reduce the possible guarantee lengths. Note the policy matrix (array from
the initial DP that stores the consumer’s decision at any given state) describes the consumer’s decision at all possible states.

For a given product \( p \), we determine the maximum value of \( s_1 \) such that the consumer does not change manufacturers which is found by searching the states \((N_p, p, N_p, N_p, s_1, 0) : s_1 \in \{0, 1, \ldots, N\}\). If \( i \) is the maximum number of successes allowed in order to remain with the current manufacturer, then it will not be necessary to offer a guarantee with more periods than \( N_p - i \). Therefore the number of additional DPs to solve after the baseline is \( 2(N_p - i) - 1 \). Note, in Figure 3-1, \( i = 2 \) was used as an example. Theorem 3.1 provides the following corollary for the existence of a threshold.

**Corollary 1.** At any time \( t \) that a replacement must be made, there exists a threshold, \( i \), such that if \( s_1 \leq i \) the replacement is made from manufacturer 1 and for \( s_1 > i \) the replacement is made from manufacturer 2.

**Proof.**

The proof follows directly from Theorem 3.1. For the largest value of \( s_1 \) (which we annotate as \( i \)) such that a replacement is made from manufacturer 1, all values of \( s_1 < i \) must lead to a replacement from manufacturer 1. Otherwise, the consumer would have been better off by making a replacement from manufacturer 2 when \( s_1 = i \).

Note, the threshold may exist at the endpoints 0 or \( N \) such that the replacement is either from manufacturer 2 or manufacturer 1, respectively, regardless of performance.

While changing a consumer’s expectation of future performance (by limiting the number of underperforming periods) lowers their cost-to-go at replacement time, we must determine the impact on the probabilities of reaching a given state.
Theorem 3.2. The probability of replacing with the incumbent manufacturer’s product is increasing as the length of guarantee increases.

Proof.

Given a consumer belief on the open interval $(0, 1)$ at time zero and since any given product is held to its maximum useful life by assumption, then a network with $N + 1$ terminal nodes can be constructed to represent the probabilities of reaching the states $(t, p, n, m, s_1, s_2)$ for $s_1 \in \{0, 1, \ldots, N\}$ at which a replacement will occur. At each non-terminal node in the network there are two possible outcomes: success (item performs worse than expected) and failure (item performs better than expected), both with positive probability. Therefore the total number of paths to the terminal nodes is $2^{N_p}$.

The $2^{N_p}$ paths to the $N_p + 1$ terminal nodes can be further characterized by the binomial coefficients such that $2^{N_p} = \sum_{j=0}^{N_p} \binom{N_p}{j}$. The combinations in this summation represent the number of paths associated with $j$ successes. Let $R_j$ be the sum of the probabilities of all possible paths to the terminal node represented by $j$ successes which we will refer to as the $j^{th}$ terminal node. Furthermore since there are $\binom{N_p}{j}$ paths to the $j^{th}$ terminal node, we can represent the sum of the probabilities as $R_j = \sum_{k=1}^{\binom{N_p}{j}} r_{j,k}$ where $r_{j,k}$ is the $k^{th}$ path to terminal node $j$. Since a terminal node must be reached, $\sum_{j=0}^{N_p} R_j = 1$.

Recall that $i$ represents the maximum number of successes to remain with the current manufacturer. Therefore the $R_j$’s can be partitioned into two sets: $R_j$ for $j \in \{0, \ldots, i\}$ for the paths that lead to staying with the same manufacturer such that $B_1 = \sum_{j=0}^{i} R_j$ and $R_j$ for $j \in \{i + 1, \ldots, N_p\}$ for the paths that lead to changing manufacturers such that $B_2 = \sum_{j=i+1}^{N_p} R_j$. $B_1$ and $B_2$ represent the total
probabilities of buying from manufacturer 1 and manufacturer 2, respectively. Now let us consider the effect of offering a single period back-end guarantee.

By offering a single period back-end guarantee, the manufacturer eliminates all \( N_p \) success arcs in the \( N_p^{th} \) period. Elimination means deleting the path on an arc and moving the probability to the failure arc coming from the same source node. Before elimination, we have:

\[
\begin{align*}
R_0 &= r_{0,1} \\
R_1 &= r_{1,1} + \ldots + r_{1}\left(\binom{N_p}{1}\right) \\
&\vdots \\
R_{N_p-2} &= r_{N_p-2,1} + \ldots + r_{N_p-2}\left(\binom{N_p}{N_p-2}\right) \\
R_{N_p-1} &= r_{N_p-1,1} + \ldots + r_{N_p-1}\left(\binom{N_p}{N_p-1}\right) \\
R_{N_p} &= r_{N_p,1}
\end{align*}
\]

Consider eliminating the success arcs one at a time beginning at the bottom of the network. Deleting the bottom arc \( (N_p^{th} \text{ arc in the } N_p^{th} \text{ period}) \) results in decreasing the number of paths by \( \binom{N_p}{N_p-1} = 1 \) but increases the chances of reaching terminal node \( N_p - 1 \) by \( r_{N_p+1,1} \). Therefore the new probabilities to the remaining terminal nodes are:

\[
\begin{align*}
R_0 &= r_{0,1} \\
R_1 &= r_{1,1} + \ldots + r_{1}\left(\binom{N_p}{1}\right) \\
&\vdots \\
R_{N_p-2} &= r_{N_p-2,1} + \ldots + r_{N_p-2}\left(\binom{N_p}{N_p-2}\right) \\
R_{N_p-1} &= r_{N_p-1,1} + \ldots + r_{1}\left(\binom{N_p}{N_p-1}\right) + r_{N_p,1}
\end{align*}
\]
Note in the case of eliminating the bottom arc, a terminal node is also no longer feasible. Now by eliminating the next success arc \( (N_{p-1}^{th} \text{ arc in the } N_p^{th} \text{ period}) \), the \( {\binom{n_p-1}{n_p-2}} \) remaining paths to terminal node \( N_{p-1} \) are removed and absorbed by terminal node \( N_{p-2} \) therefore the new probabilities to the terminal nodes are:

\[
\begin{align*}
R_0 &= r_{0,1} \\
R_1 &= r_{1,1} + \ldots + r_{1,}\left(\binom{n_p}{n_p}\right) \\
&\quad \vdots \\
R_{n_p-2} &= r_{n_p-2,1} + \ldots + r_{n_p-2,}\left(\binom{n_p}{n_p-2}\right) + r_{n_p-1,1} + \ldots + r_{n_p-1,}\left(\binom{n_p-1}{n_p-2}\right) \\
R_{n_p-1} &= r_{n_p-1,}\left(\binom{n_p-1}{n_p-2}\right)+1 + n_{n_p,1}
\end{align*}
\]

Continuing this process, any single path probability, \( r_{j,k} \), remains in the terminal node probability, \( R_j \) that it was originally in or it moves to the \( R_{j-1} \) terminal node. Therefore, \( B_1 \) and \( B_2 \) only change values when the \( i^{th} \) success arc is eliminated. After the \( i + 1, i + 2, \ldots, n_p \) success arcs have been eliminated, we know:

\[
\begin{align*}
R_i &= r_{i,1} + \ldots + r_{i,}\left(\binom{n_p}{i}\right) \\
R_{i+1} &= r_{i+1,1} + \ldots + r_{i+1,}\left(\binom{n_p}{i+1}\right) + r_{i+2,1} + \ldots + r_{i+2,}\left(\binom{n_p-1}{i+1}\right)
\end{align*}
\]

If we now eliminate the \( i + 1 \) success arc, the affected terminal probabilities become:

\[
\begin{align*}
R_i &= r_{i,1} + \ldots + r_{i,}\left(\binom{n_p}{i}\right) + r_{i+1,1} + \ldots + r_{i+1,}\left(\binom{n_p-1}{i+1}\right) \quad (3-10) \\
R_{i+1} &= r_{i+1,}\left(\binom{n_p-1}{i+1}\right)+1 + \ldots + r_{i+1,}\left(\binom{n_p}{i+1}\right) + r_{i+2,1} + \ldots + r_{i+2,}\left(\binom{n_p-1}{i+1}\right) \quad (3-11)
\end{align*}
\]

Let \( \hat{B}_1 \) and \( \hat{B}_2 \) represent the new probabilities of buying the replacement from the first and second manufacturer, respectively. Using Equations 3–10 and 3–11, we know \( \hat{B}_1 = B_1 + r_{i+1,1} + \ldots + r_{i+1,}\left(\binom{n_p-1}{i+1}\right) \) therefore \( \hat{B}_1 > B_1 \) and the probability of remaining with manufacturer 1 has increased since all \( r_{j,k} > 0 \) by assumption.
Note, after eliminating all success arcs in the $N_p^{th}$ period, the new network will have $N_p$ terminal nodes with $2^{N_p} - 1$ probability paths. The same process used above is applied iteratively to account for a multi-period back-end guarantee. For example, the two-period guarantee would use the same procedure on the new network where the $N_p - 1$ period success arcs are eliminated. Figure 3-2 shows what the new network would look like after applying a one period back-end guarantee when $N_p = 4$. The horizontal arcs with probability 1 are displayed to show the network still accounts for $N_p$ time periods; they could be deleted so the network is structured as before. Figure 3-2 illustrates that the number of performance paths decreases (compared to the network in Figure 3-1) when guaranteeing the performance in the final period. Note that in this example the number of paths leading to changing manufacturers is reduced from five to one (the total number of paths is reduced from 16 to eight).

Figure 3-2. Network Paths After Back-End Performance Guarantee ($N_p = 4$)
Now let us consider a front-end warranty. For a given performance network with $N_p$ time periods, without a warranty, there are $N_p + 1$ terminal nodes. Additionally, at any time $0 \leq t < N_p$, there are $t$ possible nodes which the performance path can go through at that time. Note, if the performance network does not start at time $0$, the time index can be adjusted so that the performance network is indexed over the range $[0, N_p + 1]$. 

For any arbitrary threshold $i$, we define $P(y, z)$ as the probability of reaching any terminal node above the threshold from any node $(y, z)$ in the network. Note $(y, z)$ represents the row and column index for a given node and above refers to the physical position of the terminal node (above the dotted line in Figure 3-1). Technically, the threshold is defined by $i = \arg\max_y P(y, N_p + 1) = 1$. 

For terminal nodes, we have $P(y, N_p + 1)$ for $y \leq i$ and $P(y, N_p + 1) = 0$ for $y > i$. Notice then that the terminal nodes are non-increasing in $y$. We now claim that the nodes in any given column are non-increasing in the row index, $y$. Using induction and this initial case, we can show that the claim is true for all columns.

For any arbitrary node $(y, z)$, let $q_{y,z}$ represent the probability of moving to node $(y, z + 1)$ and $(1 - q_{y,z})$ represent moving to node $(y + 1, z + 1)$. Similarly, consider node $(y + 1, z)$ where $q_{y+1,z}$ represents moving to node $(y + 1, z + 1)$ and $(1 - q_{y+1,z})$ represents moving to node $(y + 2, z + 1)$. By assumption, we know $P(y, z + 1) \geq P(y + 1, z + 1) \geq P(y + 2, z + 1)$. 

Therefore $P(y, z) \geq P(y + 1, z)$ since any convex combination of $P(y, z + 1)$ and $P(y + 1, z + 1)$ is at least as large as any convex combination of $P(y + 1, z + 1)$ and $P(y + 2, z + 1)$. Thus the probabilities of reaching a terminal node above the threshold are non-increasing for any column.
Now, if we consider the first node in the network before a warranty is offered, then
\[ P(1, 1) = q_{1,1} P(1, 2) + (1 - q_{1,1}) P(2, 2). \] Let \( \bar{P}(1, 1) \) represent the probability of reaching a terminal above the threshold after offering a single period front-end warranty. In this case \( q_{1,1} = 1 \), so \( \bar{P}(1, 1) \geq P(1, 1) \) since \( P(1, 2) \geq P(2, 2) \).

Thus, a single period front-end warranty increases the probability of returning to the incumbent manufacturer for a follow-on purchase. This proof is easily extended to a front-end warranty of any length.

Theorems 3.1 and 3.2 show that any guarantee offered will make the consumer at least as well as off with to respect total discounted expected cost. Therefore, finding the optimal guarantee is a tradeoff between the manufacturer’s cost increase in the periods when a guarantee is offered versus the increased probability that the consumer makes their next purchase from the same manufacturer. When a guarantee is provided, the manufacturer incurs a liability only in the periods that the guarantee is active. Offering the guarantee increases the probability that the consumer will purchase from the same manufacturer. Thus the goal is to find the optimal length guarantee such that the change in probability of purchasing from the same manufacturer (multiplied by the purchase price) is larger than the sum of the liabilities incurred over each period of the guarantee.

### 3.4 Results

We now illustrate the effects of offering a performance guarantee through an example. Let us consider the scenario where both manufacturers have identical parameters. In this case the total expected discounted cost for the consumer would be the same for each manufacturer, so we arbitrarily choose manufacturer 1 to be the preferred choice for the initial purchase. Table 3-1 lists the parameters. Note that the consumer in this case is slightly pessimistic with an initial expectation of .6 for each product compared to an expectation of .5 for the manufacturers. Additionally, the periodic operating costs of the significant technology change are the same as the
preceding technology. This captures the concept that improved technologies are often more expensive to purchase while reducing operating costs (such as purchasing a hybrid or electric vehicle); even though the operating costs are the same, the time value of money in effect makes the operating costs of the later technology an improvement.

Table 3-1. Parameters for Bayesian Updating Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Horizon ($T$)</td>
<td>15</td>
<td>–</td>
</tr>
<tr>
<td>Purchase Price ($P_p$)</td>
<td>20000</td>
<td>$p = 1, 2$</td>
</tr>
<tr>
<td>Purchase Price ($P_p$)</td>
<td>30000</td>
<td>$p = 3, 4, \ldots, 2T$</td>
</tr>
<tr>
<td>Product Useful Life ($N_p$)</td>
<td>10</td>
<td>$p = 1, \ldots, 2T$</td>
</tr>
<tr>
<td>Cost Increase Rate ($g$)</td>
<td>.15</td>
<td>–</td>
</tr>
<tr>
<td>Initial Cost of Lowest Performance ($a_p$)</td>
<td>1500</td>
<td>$p = 1, 2, \ldots, 2T$</td>
</tr>
<tr>
<td>Initial Cost of Highest Performance ($b_p$)</td>
<td>1700</td>
<td>$p = 1, 2, \ldots, 2T$</td>
</tr>
<tr>
<td>Consumer’s Initial Performance Expectation ($\alpha_p, \beta_p$)</td>
<td>2.4,1.6</td>
<td>$p = 1, 2, 3, 4, \ldots, 2T$</td>
</tr>
<tr>
<td>Manufacturer’s Performance Distribution ($\alpha_p, \beta_p$)</td>
<td>2,2</td>
<td>$p = 1, 2, 3, 4, \ldots, 2T$</td>
</tr>
<tr>
<td>Discount Rate ($\gamma$)</td>
<td>.9</td>
<td>–</td>
</tr>
</tbody>
</table>

Evaluating an example with the same parameters for each manufacturer attempts to isolate the effect of the guarantee on the performance updating process of the consumer from any difference in parameters. In solving the baseline DP with no guarantee, manufacturer 1’s revenue is 28664 while manufacturer 2’s is 1796. Additionally, $i = 5$ so if the first product has six or more periods of below expected performance, then the consumer will switch manufacturers. Recall, that we assumed that a guarantee is not only consecutive but starts at purchase (referred to as a front-end guarantee) or ends with replacement (referred to as a back-end guarantee). Figure 3-3 shows the effect on the first manufacturer’s revenue for each possible guarantee length.
Notice that the effect on revenue is not monotonic in the length of the guarantee. The maximum revenue is achieved with a back-end three period performance guarantee and increases revenue to $30367$, approximately a 6% increase. Also, note that neither type of guarantee, front-end or back-end, dominates the other for each guarantee length. While this example shows the uncertainty in making generalized statements about the optimal guarantee length or type, we can use sensitivity analysis to gain further insight into the role of the relationship between the consumer’s performance expectations and the true product performance.

Under the condition that a product will be held to its maximum useful life, the maximum available revenue between the two manufacturers can be calculated. In the previous example, the product is replaced after ten periods so the maximum available revenue after discounting is $30460$. This provides an upper bound on the effect of a guarantee. Any guarantee can increase revenue toward this upper bound, but because of the liability, it cannot be obtained. If we now vary the consumer’s initial expectation and the manufacturer’s advertised cost distribution, we see the impact.
on the manufacturer’s revenue and the structure of the optimal guarantee. All other
dparameters are the same as listed in Table 3-1. Table 3-2 shows the effect on the
manufacturer’s revenue, in terms of percentage change, when the optimal guarantee is
offered. Note the maximum available revenue is the same upper bound as previously
stated since the performance expectation levels do not affect this bound.

Table 3-2. Manufacturer Revenue with Optimal Performance Guarantee

<table>
<thead>
<tr>
<th>α/(α+β)</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
<th>.6</th>
<th>.7</th>
<th>.8</th>
<th>.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>15.22</td>
<td>4.44</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<tr>
<td>.1</td>
<td>48.27</td>
<td>41.02</td>
<td>31.46</td>
<td>4.28</td>
<td>0.07</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>.2</td>
<td>48.19</td>
<td>43.13</td>
<td>35.38</td>
<td>26.52</td>
<td>3.52</td>
<td>1.65</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>.3</td>
<td>50.82</td>
<td>50.48</td>
<td>35.48</td>
<td>27.06</td>
<td>18.84</td>
<td>1.82</td>
<td>0.72</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>.4</td>
<td>50.35</td>
<td>50.10</td>
<td>48.58</td>
<td>45.34</td>
<td>18.23</td>
<td>11.53</td>
<td>0.64</td>
<td>0.18</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>.5</td>
<td>49.67</td>
<td>50.48</td>
<td>50.48</td>
<td>44.36</td>
<td>38.68</td>
<td>10.63</td>
<td>5.94</td>
<td>2.88</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>.6</td>
<td>48.91</td>
<td>50.01</td>
<td>50.16</td>
<td>50.12</td>
<td>49.06</td>
<td>30.18</td>
<td>22.12</td>
<td>2.43</td>
<td>0.89</td>
<td>0.26</td>
<td>0.00</td>
</tr>
<tr>
<td>.7</td>
<td>48.09</td>
<td>49.42</td>
<td>50.32</td>
<td>50.43</td>
<td>49.03</td>
<td>46.63</td>
<td>21.50</td>
<td>14.03</td>
<td>7.95</td>
<td>0.13</td>
<td>0.00</td>
</tr>
<tr>
<td>.8</td>
<td>47.24</td>
<td>48.75</td>
<td>49.86</td>
<td>50.04</td>
<td>50.69</td>
<td>47.27</td>
<td>43.44</td>
<td>14.51</td>
<td>8.21</td>
<td>3.74</td>
<td>1.32</td>
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<tr>
<td>.9</td>
<td>46.37</td>
<td>48.02</td>
<td>49.31</td>
<td>49.54</td>
<td>50.46</td>
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<td>42.38</td>
<td>11.07</td>
<td>5.09</td>
<td>1.50</td>
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<tr>
<td>1</td>
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<td>52.71</td>
<td>53.43</td>
<td>53.69</td>
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<td>54.17</td>
<td>54.04</td>
<td>54.15</td>
<td>53.65</td>
<td>43.79</td>
<td>27.46</td>
</tr>
</tbody>
</table>

Reviewing Table 3-2, there are some trends that may seem surprising. First, for
a given level of true performance, the ability to affect manufacturer revenue with a
guarantee increases as the consumer becomes more optimistic. We might expect
the opposite since pessimistic consumers, which are akin to risk averse consumers
in that they have inflated expected values, are often more responsive to warranty and
guarantee type contracts.

The second observation is that the revenue changes decrease faster as the
consumer gets more pessimistic when the product offered by the manufacturer has
better performance. This may stem from the fact that when the true distribution is
skewed to the lower costs, the probability of a success is small when consumer
pessimism is high since the threshold for a success would be in the “tail” of the beta
distribution when either of the shape parameters is larger than 1. This leads to a small
probability of making a follow-on purchase from the second manufacturer. Additionally,
when the consumer is pessimistic, the manufacturer has to guarantee a larger range of costs. Therefore, there is a limited benefit of using a guarantee to change the consumer’s satisfaction since there is a small amount of probability to ”move” from a follow-on purchase from the second manufacturer to the first manufacturer while simultaneously having the potential for a large liability when the guarantee is active.

Of importance to note, the examination of Table 3-2 does not imply revenue is maximized when the consumer has high expectations or the performance of the product is very good. It only shows the areas where the guarantee is most effective. As a matter of fact, if we consider the total maximum discounted expected revenue, we find that for a given product cost distribution, the manufacturer’s revenue increases when offering the optimal guarantee as the the consumer becomes more pessimistic.

Figures 3-4 and 3-5 show the manufacturer’s total revenue with the optimal guarantee and with no guarantee. The consumer’s expectation is shown on the horizontal axis and the manufacturer’s expectation is listed in parentheses in the legend. The graphs show that the revenue curves are increasing in the consumer’s expectation and that the optimal guarantee curves are significantly greater than the no guarantee curves for more optimistic consumers. As the consumer becomes more pessimistic though, the no guarantee curves approach the revenue upper bound. Also notice that the revenue with no guarantee approaches the revenue with the optimal guarantee quicker when a product has good performance. In this case, quicker means for a more pessimistic consumer. Note that Figure 3-4 shows revenue curves for a product with better performance than Figure 3-5.
Besides trying to understand trends in manufacturer revenue based on the consumer’s and manufacturer’s expectation, we can also analyze trends in the structure of the optimal guarantee. Table 3-3 shows the optimal guarantee for the same combinations of consumer and manufacturer expectations as before.
Table 3-3. Optimal Performance Guarantee Designs

<table>
<thead>
<tr>
<th>Consumer Initial Belief ($E_0$)</th>
<th>0</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
<th>.6</th>
<th>.7</th>
<th>.8</th>
<th>.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\alpha}{\alpha + \beta}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>(10,B)</td>
<td>(10,B)</td>
<td>(9,F)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
</tr>
<tr>
<td>.1</td>
<td>(9,B)</td>
<td>(9,B)</td>
<td>(9,F)</td>
<td>(6,B)</td>
<td>(2,B)</td>
<td>(1,B)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
</tr>
<tr>
<td>.2</td>
<td>(8,B)</td>
<td>(8,B)</td>
<td>(8,B)</td>
<td>(7,B)</td>
<td>(4,B)</td>
<td>(4,B)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
</tr>
<tr>
<td>.3</td>
<td>(8,B)</td>
<td>(8,B)</td>
<td>(7,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(3,B)</td>
<td>(2,B)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
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<tr>
<td>.4</td>
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<td>(7,B)</td>
<td>(7,B)</td>
<td>(7,B)</td>
<td>(5,B)</td>
<td>(4,B)</td>
<td>(2,B)</td>
<td>(1,B)</td>
<td>(___)</td>
<td>(___)</td>
<td>(___)</td>
</tr>
<tr>
<td>.5</td>
<td>(8,B)</td>
<td>(7,B)</td>
<td>(7,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(4,B)</td>
<td>(3,B)</td>
<td>(3,B)</td>
<td>(1,F)</td>
<td>(___)</td>
<td>(___)</td>
</tr>
<tr>
<td>.6</td>
<td>(8,B)</td>
<td>(7,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(4,B)</td>
<td>(4,B)</td>
<td>(2,B)</td>
<td>(2,B)</td>
<td>(1,B)</td>
<td>(1,F)</td>
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<td>.7</td>
<td>(8,B)</td>
<td>(7,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(5,B)</td>
<td>(5,B)</td>
<td>(3,B)</td>
<td>(3,B)</td>
<td>(3,B)</td>
<td>(1,B)</td>
<td>(1,F)</td>
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<tr>
<td>.8</td>
<td>(8,B)</td>
<td>(7,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(5,B)</td>
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<td>(4,B)</td>
<td>(3,B)</td>
<td>(2,B)</td>
<td>(3,F)</td>
<td>(1,B)</td>
</tr>
<tr>
<td>.9</td>
<td>(8,B)</td>
<td>(7,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(5,B)</td>
<td>(4,B)</td>
<td>(4,B)</td>
<td>(3,B)</td>
<td>(2,F)</td>
<td>(2,F)</td>
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<td>(7,B)</td>
<td>(6,B)</td>
<td>(6,B)</td>
<td>(5,B)</td>
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<td>(3,B)</td>
<td>(3,B)</td>
<td>(2,B)</td>
<td>(1,B)</td>
<td>(1,B)</td>
</tr>
</tbody>
</table>

The structure of the guarantee is represented by the duple $(x, y)$ where $x$ represents the length of the guarantee and $y$ represents whether the guarantee is front-end (F) or back-end (B). For some combination of expectations, offering no guarantee produces the maximum revenue. When the optimal guarantee is offered, the back-end guarantee, in this example, is usually preferred to the front-end guarantee. Note, 94 of the 121 scenarios show that the manufacturer's revenue can be increased by offering the optimal guarantee. Of these 94, the optimal guarantee is a front-end guarantee in only 9 of them. Although this might suggest a front-end guarantee plays a meaningful role in this problem (almost 10% of optimal guarantees are front-end), further analysis might suggest otherwise. If we look at the 9 scenarios where the front-end guarantee is optimal and compare it to the best back-end guarantee, we'll find that the difference in the effect on manufacturer revenue is negligible. Table 3-4 shows the percentage difference in the change in manufacturer revenue by offering the optimal (front-end) guarantee as opposed to offering the best back-end guarantee. The absolute difference in manufacturer revenue is also provided.
Table 3-4. Analysis of Optimal Front-End Guarantees

<table>
<thead>
<tr>
<th>$(E_0, \frac{\alpha}{\alpha + \beta})$</th>
<th>% Difference</th>
<th>Absolute Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(.2,0)</td>
<td>.0005</td>
<td>.1406</td>
</tr>
<tr>
<td>(.2,.1)</td>
<td>.0091</td>
<td>2.0907</td>
</tr>
<tr>
<td>(.8,.5)</td>
<td>.0189</td>
<td>5.7457</td>
</tr>
<tr>
<td>(.8,.9)</td>
<td>.0012</td>
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</tr>
<tr>
<td>(.9,.8)</td>
<td>.0052</td>
<td>1.5259</td>
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<tr>
<td>(.9,.9)</td>
<td>.0855</td>
<td>24.7560</td>
</tr>
<tr>
<td>(1,.6)</td>
<td>.0824</td>
<td>25.1040</td>
</tr>
<tr>
<td>(1,.7)</td>
<td>.0448</td>
<td>13.6310</td>
</tr>
<tr>
<td>(1,.9)</td>
<td>.0212</td>
<td>6.3582</td>
</tr>
</tbody>
</table>

The results in Table 3-4 suggest for this example that when the optimal guarantee is on the front-end, offering the best back-end guarantee does not make the manufacturer significantly worse off. The absolute differences suggest that the differences are not due to rounding or other possible numerical anomalies. The purpose of this analysis is to provide a counter-example against the hypothesis that one type of guarantee might dominate the other.

The outliers, in this example, occur near the margin where offering a guarantee does not increase revenue and where the consumer’s belief is not that different from the true distribution. These results suggest that for a given problem with predefined parameters, one type of guarantee may be preferred to the other in a general sense. Additionally, for a given consumer expectation, the length of the guarantee tends to increase as the true operating costs increase.

Using this example, an interesting characteristic regarding the consumer’s expectation for fractional equivalents can be observed. Consider the case where the true distribution is defined by the shape parameters $\alpha_{p,1} = 2$ and $\beta_{p,1} = 2$ and the consumer has an initial belief of .6. Under this expectation, the consumer’s parameters are $\tilde{\alpha}_{p,1} = 2.4$ and $\tilde{\beta}_{p,1} = 1.6$. After ten periods of observed costs, there is the possibility that the consumer’s expectation remains unchanged. If the consumer realizes six periods of worse than expected performance and four periods of better than expected
performance, the consumer’s expectation of manufacturer 1’s next product \((p' = p + 20)\) is defined by the parameters \(\bar{\alpha}_{p',1} = 8.4\) and \(\bar{\beta}_{p',1} = 5.6\). Since no information has been observed on the performance of a product from manufacturer 2, the consumer maintains their original expectation of \(0.6\) defined by \(\bar{\alpha}_{p'+1,1} = 2.4\) and \(\bar{\beta}_{p'+1,1} = 1.6\). In this instance, the consumer has the same expectation for each of the manufacturer’s products. The consumer’s decision though is to purchase manufacturer 2’s product. Although the expectation for each product is the same, the larger value of the shape parameters for manufacturer 1’s product means that a success or failure has less impact on the consumer’s future expectation. This equates to the consumer being more confident in the product’s periodic operating costs. For manufacturer 1, this confidence has a negative impact on the consumer’s choice at the margin where product expectations are the same for each manufacturer. This result can be interpreted as the pessimism of the consumer has been reinforced by the realized costs whereas the pessimism towards manufacturer 2’s product is unproven and thus is more likely to be updated.

3.5 Summary

There are many types of products that consumers continually need and technology improvements for these products will command a larger purchase price. Automobiles and television sets are two common examples. Therefore it is in the best interest of manufacturers to consider ways to enhance consumer satisfaction with their product in order to improve the chances a consumer returns for a follow-on purchase.

In this chapter, we specifically consider the case where consumers are strongly influenced by product performance as measured by periodic operating costs. When costs are the dominant attribute such that the other variable attributes are considered insignificant or the set of products under consideration share the same qualities on all attributes other than costs, a consumer’s satisfaction with a product’s performance can be modeled based on their initial belief of a product’s cost along with the observed product performance as indicated by the observed periodic operating costs. We present
a Bayesian updating strategy where the consumer has an initial belief about a product's cost based on any range of inputs such as previous experience with the manufacturer, and they update their expectation every period based on that period's realized costs.

Under this model and assuming two identical manufacturers, we show how a performance guarantee can be structured to increase the total expected discounted revenue of a manufacturer over a fixed horizon. With a renewed emphasis on product performance with respect to commodity usage, the type of contractual instruments presented in this chapter can go a long way towards improving the probability of a consumer returning for a future purchase. If properly structured, these guarantees can also increase manufacturer revenue.
4.1 Motivation and Literature Review

A performance based warranty (PBW) guarantees that a product will perform at some minimum level (i.e. the warranty coverage level) or else the warranty providing agent must compensate the consumer. When the performance level of a product is related to the use of a commodity, such as fuel for a PBW on gas mileage for a car, the value of a PBW is non-stationary over time since the price of the commodity changes, often dramatically, over time. For example, consider a product, such as a household appliance, that has an advertised level of energy usage. If a manufacturer guarantees a minimum level of energy efficiency, the value of the guarantee to the consumer can vary greatly depending on the underlying cost of energy. If electricity or gas rates are low, the value of the warranty may be minimal to the consumer whereas if the energy costs are inflated, the value to the consumer might be significant. Similarly, when a PBW reimburses a consumer for costs incurred from substandard product performance relative to the use of a commodity, the price a manufacturer charges for a PBW can also be time-dependent.

To illustrate the importance of non-stationarity when modeling PBWs, let us compare a traditional warranty and performance based warranty. Suppose we evaluate the price changes of key inputs to maintenance and operating costs in the automobile industry. Figure 4-1 compares the average wage of an auto mechanic and the average gasoline price for the years 1999-2011. The average mechanic wage is relatively flat whereas the price of gasoline is highly volatile. Although both types of warranties could claim non-stationarity in this case, it is clear that gas costs are much more dependent on time than maintenance wages, and thus pricing a PBW based on gas costs would be more sensitive to time. Note that the data is from the Bureau of Labor Statistics (BLS).
and the Department of Energy (DOE) with data points for each year based on May measurements and 1999 used as the baseline year in order to generate the index.

Figure 4-1. Mechanic Wages vs. Gasoline Prices

The non-stationarity of the value of a fixed length warranty to the consumer and the manufacturer’s price of a PBW of the same length are therefore dependent on estimates of the associated commodity costs from both the consumer and manufacturer perspectives. It is not only likely that the consumer and manufacturer will have different estimates but also reasonable that they have no systematic relationship.

In Chapters 2 and 3, performance was modeled as a single random variable representing operating costs. While mathematically simple, these models confound the effects of various factors on operating costs. Additionally, the manufacturer may limit the consumer’s realized operating costs due to factors outside their control instead of as a result of the product’s substandard performance. Under simplifying assumptions about operations, in order to make the model tractable (e.g. constant manpower requirements to operate any product), we can decompose the operating costs into a finite number of factors by modeling the costs as a function of multiple random variables or parameters. This method allows us to account for the unique characteristics of a PBW, such as non-stationary inputs. Using this decomposition of operating costs, we consider the
problem of how a manufacturer can use a PBW to increase revenue in the presence of uncertainty in the underlying commodity costs.

Specifically, we study an equipment replacement problem with a single challenger where the manufacturer offers a warranty each period over a finite time horizon. The consumer may purchase the PBW in any period regardless of their history of previous PBW purchases. Since a warranty is offered every period and there are no limitations on the consumer’s choice to purchase it, the PBW is both renewable and deferrable. Additionally, the length and level of the performance guarantee are fixed for the entire time horizon. Conversely, the warranty price may be dynamic over time since the warranty can be priced according to the manufacturer’s forecast of future commodity costs (as well as product performance). Also, the consumer’s expected operating costs are not stationary since they depend on the consumer’s forecast of future commodity costs. Both the manufacturer and consumer update their forecasts every period after observing actual commodity costs.

In this chapter, we use dynamic programming (DP) to analyze the effect of a manufacturer offering PBWs in each period over a finite time horizon when the consumer and manufacturer view periodic operating costs as a product of multiple random variables. Different estimation techniques are used by the manufacturer and consumer for the commodity cost parameters. We assume the manufacturer uses advanced time-series forecasting techniques to make sophisticated predictions. The consumer on the other hand is assumed to use more rudimentary estimation techniques. While the models and computational results that follow are directly related to the estimation techniques chosen, the primary intent is to differentiate between the level of sophistication of the estimation technique used by each agent. In using DPs to model this sequential decision problem, it is relatively straightforward to allow the manufacturer and consumer to periodically update their commodity cost estimates by iteratively solving a sequence of DPs. Our proposed strategy of iteratively solving DPs
also simultaneously allows for the inclusion of uncertain transition characteristics, such as future performance dependence on current performance (Markovian performance), seen in stochastic dynamic programs.

As previously mentioned, the price of the commodity, such as gasoline, used by a product, such as a car, can have a significant impact on the value a consumer assigns to a PBW, such as a guaranteed minimum level of gas mileage. Therefore, the consumer’s cost estimation method is vitally important. We might expect that consumers often lack the statistical knowledge and tools to build a sophisticated forecast model or conduct in-depth analysis. Turrentine and Kurani [38] discuss consumer behavior as it pertains to estimating gasoline prices when considering fuel economy and car purchases. They find that consumers often use rudimentary estimates such as the current price of gas to estimate future prices. In a National Bureau of Economic Research (NBER) report, Kellogg et al. [39] also found that consumers, on average, use the current price of gasoline as an estimate of future prices. They further note that there is much heterogeneity among consumers, and the technique of using current prices to estimate the future prices is not observed during large financial shocks. Therefore, even with simple estimation techniques, the consumer’s forecast relative to actual prices can vary dramatically. Greene [40] conducts a survey of over two dozen statistical studies to determine how consumer’s value fuel economy. He states that in some models consumers underestimate the value of fuel savings and in others overestimate fuel savings, but a common assumption in all the models is consumers use the current price as an estimate for future prices.

For the manufacturer, it would be expected that they could use time-series techniques and statistical software to forecast commodity prices. Although the use of a more sophisticated technique does not ensure that the manufacturer’s estimate is better than the consumer’s, we hypothesize that the manufacturer can capitalize on this asymmetry. While the estimation techniques are important, our goal is not to provide an
in-depth analysis of methods to predict commodity prices. We only attempt to provide a few reasonable examples that clearly differentiate a sophisticated manufacturer estimate from a relatively unsophisticated consumer estimate and determine if the manufacturer can increase revenue by offering PBWs based on their estimates.

Regardless of the estimation technique for either the manufacturer or consumer, we would expect both parties to update their estimates based on new information. This could be accomplished by using a parametric distribution for future costs or merely updating the current price of the commodity and treating it as a deterministic estimate for future costs. Therefore, we propose solving a sequence of dynamic programs that allow both the manufacturer and consumer to update their estimates each period. Powell and Spivey [41] discuss solving a dynamic assignment problem by solving a sequence of static assignment problems which allows decisions to be be made using the most up-to-date information. Applying this same method to a sequence of dynamic programs allows the consumer and manufacturer to incorporate the most current cost data in making pricing and purchase decisions for PBWs.

For traditional warranties, a non-homogeneous poisson process is often used to model the number of failures over time [42–44]. With the property of independent increments, the number of failures in the current period does not affect the number of failures in the next period. Conversely, when modeling product performance in terms of a performance metric such as energy usage, we might expect performance to be non-increasing over the age of the product, i.e. a property analogous to independent increments would not hold. For example, a household appliance like a refrigerator would not be expected to use a lot of energy early on and then become more efficient as it ages. Rather, the opposite may be true. To model this dependency, a stochastic dynamic program (SDP) is used to model the uncertain transitions based on the current performance level. Under certain assumptions about how the consumer models
non-increasing performance with age, the SDP can be replaced with a sequence of DPs that can be solved iteratively.

Our contributions to the literature include proposing a framework to model PBWs that are tied to market commodity costs and a dynamic programming formulation to determine the consumer’s optimal purchase policy for a set of dynamically priced warranties. Using actual time series data for gasoline prices from the Department of Energy and nominal product parameters that represent an economy class automobile, we show that the magnitude of a manufacturer’s revenue loss from offering performance warranties is relatively small. Additionally, we find that if the manufacturer can fully exploit the consumer’s willingness-to-pay for a PBW, then the manufacturer can increase total revenue by offering the PBWs. Finally, we incorporate non-increasing performance into the model and show that when a naive consumer does not account for this practical characteristic, the manufacturer’s revenue decreases, on average, by less than one percent for our data set.

This chapter has the following structure. In section 4.2, we present a methodology for pricing PBWs and determine the effect on manufacturer revenue by solving a dynamic program iteratively for a semi-empirical example. We compare the effect of different commodity cost estimates, for both the manufacturer and consumer, on manufacturer revenue. Next, in section 4.3 we present a stochastic dynamic program (SDP) that models performance as non-increasing. This SDP is also solved iteratively to determine the change in revenue from offering PBWs. Because each solution is only for a specific performance history over the time horizon, we conduct a Monte Carlo simulation to determine the average impact on revenue when the consumer and manufacturer use the same commodity cost estimates. Then, in section 4.4 we combine the disparate cost estimates from section 4.2 with the non-increasing performance assumption from section 4.3 to determine the effect on manufacturer revenue.
4.2 Model Formulation with Updated Cost Estimates

Operating costs are a significant input when making equipment replacement decisions based on minimizing costs. When performance based warranties are offered that potentially limit the consumer’s operating costs, it is important for both the manufacturer and consumer to account for the factors that most influence these costs, and clearly, there are many factor that affect them. While it is not reasonable to account for every possible variable, it is important to capture the effect of the critical factors. By using a function of the most relevant factors, we can provide a more realistic model of operating costs and incorporate specific consumer behavior tendencies into the model.

4.2.1 Uncertainty in the Model

We consider the case where the variation in operating costs is primarily due to the consumption of a specific commodity. With total product usage, a unit commodity price, and a performance metric for the amount of usage per unit of the commodity, we model the operating costs as a product of multiple random variables. Define a random variable $M_n$ with distribution $f_n$ to represent the true performance level of a product of age $n$. The consumer may modify the true performance distribution based on their own beliefs to derive their own performance distribution, which we annotate as $M_n^c$. The only limitation on this modification is the support for $M_n$ and $M_n^c$ must be the same.

We assume the performance metric is positive such that the common support is some subset of the positive real number line. This is similar to our method of a consumer modifying operating cost distributions in Chapters 2 and 3. For the commodity costs, define $\hat{c}_t$ as the manufacturer’s estimate of the unit commodity cost at time $t$. Similarly, let $\hat{c}_t^c$ represent the consumer’s estimated unit commodity cost at time $t$. The consumer’s and manufacturer’s cost estimates are not necessarily random variables depending on their estimation technique (which we address later). Notice that $\hat{c}_t$ and $\hat{c}_t^c$ are time variant random variables (or parameters) whereas $M_n$ and $M_n^c$ are time invariant. Let $U$
represent the total periodic usage of a product which we assume to be constant. While the model could be extended to allow for usage to vary, we are limiting our analysis to the case where $U$ is fixed. We assume that both $t$ and $n$ represent discrete, equally spaced periods which means they can be indexed over a subset of non-negative integers. Specifically, let $T$ be the maximum number of time periods and $N$ be the maximum useful life of the product such that $t \in \{0, \ldots, T\}$ and $n \in \{1, \ldots, N\}$.

The consumer’s expected periodic operating costs, $O$, then at time $t$ for a product of age $n$ are:

$$O_t(n) = E\left[\frac{U \hat{\xi}_t}{M^n_t}\right] = UE\left[\hat{\xi}_t\right] E\left[\frac{1}{M^n_t}\right].$$  

(4–1)

Equation 4–1 describes the total periodic cost as the product of periodic usage (e.g. miles per year), unit commodity cost (e.g. price per gallon of gas), and a performance metric (e.g. miles per gallon).

Now, we are interested in determining if a manufacturer can increase revenue by offering performance based warranties. Specifically, we introduce the idea of an extended performance based warranty (EPBW) defined by a length $\mathcal{W}$, minimum performance level $\mathcal{X}$, and a price $P$. The term EPBW is used to denote that these contracts can be obtained apart from the purchase of a product, similar to the definition of an extended warranty or service contract [45]. For an EPBW that guarantees a minimum performance level $X$, the effective performance, for the manufacturer, for a product of age $n$ under a warranty is defined by the following random variable:

$$\tilde{M}_n = \min(M_n, X).$$  

(4–2)

The manufacturer’s liability is derived from the difference between effective performance, $\tilde{M}_n$, and actual performance, $M_n$. Similarly, we define $\tilde{M}_n^c$ as the random variable capturing the consumer’s anticipated effective performance of a product of age $n$ when under warranty:
With Equation 4–3, we can now define the consumer’s expected operating costs under an EPBW at time $t$ and age $n$ as:

$$\bar{M}_n^c = \max(M_n^c, X).$$  (4–3)

In Equations 4–1 and 4–4, $\hat{c}_t^c$ and $M_n^c$ are independent. We will use the more general assumption throughout this chapter that states that all commodity price estimates and performance random variables are independent for both the consumer and manufacturer. Also, since a random variable is in the denominator, there are some implied restrictions on the distributional form of $M_n$ and $M_n^c$. These random variables cannot be from a distribution such that the expected value of the reciprocal is undefined. For example, if either of these are an exponential distribution, then the reciprocal is an inverse-gamma distribution with an undefined mean.

While the warranty parameters $W$ and $X$ are similar to our previous performance warranties and guarantees, there is a significant difference under the multiple random variable model. The warranty level is defined on a performance metric instead of operating costs therefore the compensation to the consumer is not fully defined by the warranty. Under this model, there must be some function to translate substandard performance to consumer reimbursement. Additionally, we have not yet stated any pricing conditions on the EPBW.

Suppose the manufacturer uses the actual, current commodity price to determine any reimbursement to the consumer. While this exposes the manufacturer to the risk of an exogenous factor in determining their potential liability, there exists the possibility that the manufacturer could structure EPBWs that capitalize on the difference between
both the consumer’s performance distribution and commodity cost estimate and the manufacturer’s (true) performance distribution and commodity cost estimate.

With the reimbursement structure defined, the price of the EPBW can now be determined. In many warranty analysis problems, the price of the warranty is often a stationary parameter. With our chosen reimbursement methodology, the warranty price is dynamic. Using a discount factor, $\gamma$, to account for the time value of money, the warranty price can be calculated by setting the price equal to the discounted expected liability over the length of the warranty:

$$P_{t,n} = \sum_{i=1}^{W} \gamma^i E\left[\frac{\hat{c}_{t+i}}{M_n}\right] = U \sum_{i=1}^{W} \gamma^i E\left[\hat{c}_{t+i}\right] E\left[\frac{1}{M_n}\right].$$

(4–5)

Over this $W$ period segment starting at time $t$ and with the currently owned product of age $n$, the manufacturer is willing to offer an EPBW to a consumer at price $P_{t,n}$ with the understanding the realized liability is dependent on the actual commodity prices at the end of each time period. Let $c_t$ represent the actual commodity unit cost at time $t$. Under this warranty, the consumer would purchase the warranty under the condition:

$$P_{t,n} + \sum_{i=1}^{W} \gamma^i \hat{O}_{t+i}(n + i) \leq \sum_{i=1}^{W} \gamma^i O_{t+i}(n + i).$$

Conversely, the manufacturer realizes a net increase in revenue for this warranty under the condition:

$$P_{t,n} \geq U \sum_{i=1}^{W} \gamma^i E[c_{t+i}] E\left[\frac{1}{M_n}\right].$$

(4–6)

Interestingly, if we consider the entire period of the warranty when Equation 4–6 is satisfied, the change in manufacturer revenue is not fully characterized by the difference in the left-hand side and right-hand side of the inequality. Theoretically, it is possible for...
the actual and estimated commodity costs to vary over the life of the warranty such that a warranty is replaced before it has expired.

4.2.2 Cost Estimates

From the previous section, the roles of consumer and manufacturer cost estimates in determining whether a warranty is purchased are extremely important. There are many models that could be used to estimate a time series like gasoline prices. One such model is a multiplicative decomposition that can isolate long-term trend effects from shorter-term business cycle effects [46]. While many other time series models are reasonable, such as an autoregressive model, the multiplicative decomposition has a nice intuitive simplicity that is beneficial when comparing it to the consumer’s estimate.

Let us first consider a classical time series decomposition model for the manufacturer’s estimate of the commodity cost. We assume a multiplicative model of the form:

\[ c_t = T_t S_t B_t I R_t \]

where:

\[ c_t = \text{commodity cost} \]
\[ T_t = \text{trend component} \]
\[ S_t = \text{seasonal component} \]
\[ B_t = \text{business cyclic component} \]
\[ I R_t = \text{irregular component} \]

In conducting the decomposition, we assume \( T_t, S_t, B_t, \) and \( I R_t \) are independent such that \( E(c_t) = E(T_t)E(S_t)E(B_t)E(I R_t) \). For the trend component, an ordinary least squares (OLS) method is used with the following linear form:

\[ T'_t = \beta_0 + \beta_1 t' + \epsilon, \quad \epsilon \sim N(0, \Sigma). \]
A quadratic fit could also be applied to the trend component in which case:

\[ T'_t = a_2 (t')^2 + a_1 t' + a_0 + \epsilon, \quad \epsilon \sim N(0, \Sigma). \]

Note \( t' \) is a function of the time period \( t \) since the data used for the estimate begins before the problem time period starts. To forecast the trend, the point estimate could be used or a prediction interval could be determined and the upper limit used. Using the upper limit of the prediction interval indicates a manufacturer who wishes to take on less risk of underestimating the trend effect. We annotate the size of the prediction interval by the parameter \( \tau \), i.e., a \( 100(1 - \tau) \) prediction interval.

For the seasonal component, \( S_t \), we make the simplifying assumption that the time periods are yearly; thus \( S_t = 1 \) because each period covers the full range of seasonal effects, i.e., a period is a year and thus covers the seasonal effect of all 12 months. This simplification is reasonable for two reasons. First, an annual warranty reconciliation (determining the reimbursement to the consumer) seems practical for both the manufacturer and consumer. Second, the predictability and repetitive nature of the seasonal impacts on certain commodities may be well-known by both parties.

The random component \( B_t \) represents the business cycle variation and we assume that \( B_t \sim G(\cdot) \) where \( G \) is some general distribution. Likewise, we assume \( iR_t \sim \tilde{G}(\cdot) \) where \( \tilde{G} \) is some general distribution with mean one. Under these set of assumptions the manufacturer might use the estimate:

\[ \hat{e}_t = (\hat{\lambda}_0 + \hat{\lambda}_1 t') \hat{B}. \] (4–7)

In Equation 4–7, \( \hat{B} \) is an estimate of the business cycle effect, and we consider three different estimators for this component. The first is the sample mean of the cyclic effects calculated from the observed commodity values. We call this method the average business cycle (ABC). The second is a quadratic fit to the last three business cycle values. The justification for this estimate is the Juglar fixed investment cycle which
hypothesizes that business cycles are between seven and eleven years long [47], and thus we call this method the Juglar business cycle (JBC). Last we consider that the business cycle has no positive or negative effect (NBC) so $\hat{B} = 1$.

While the previous manufacturer estimates are not supposed to be an authoritative model for predicting commodity prices, we do consider them to be sophisticated in comparison to the following consumer models. We consider three different methods for the consumer estimates. If $t$ is the current time, then we define the methods mathematically as follows:

i. **Constant**: $\hat{c}_{t+i} = c_t, \forall i \in \{1, \ldots, T-t\}$

ii. **Linear**: $\hat{c}_{t+i} = i(c_t - c_{t-1}), \forall i \in \{1, \ldots, T-t\}$

iii. **Geometric**: $\hat{c}_{t+i} = (1 + \omega)^t c_t, \forall i \in \{1, \ldots, T-t\}, \omega \in (0, 1)$

The three consumer estimate methods are based on the concepts that they are simple and do not incorporate much more than the current data. In fact, only the linear model uses the previous period’s cost along with the current period. The constant method is based on Turrentine and Kurani’s [38] finding while the linear and geometric methods allow for the consumer to account for cost inflation in a basic way. Note that $\omega$ is the inflation factor for the geometric estimate.

The model presented thus far is a subset of a larger equipment replacement problem. We are interested in the use of warranties over the entire time horizon $T$ where the consumer has the options to replace the product and/or purchase an EPBW during any period in the horizon.

### 4.2.3 Dynamic Program

Based on the variables in the previous section, we define a dynamic program (DP) to determine the consumer’s replacement policy and the associated effect on the manufacturer’s revenue. Let $P$, without any subscripts, be the purchase price of the product which is stationary over the finite time horizon, $T$. The state space for the DP
has two dimensions: \( n \) is the age of the current item and \( w \) is the amount of time left on the warranty. \( P_{t,n} \) is the warranty price that the manufacturer offers for a product of age \( n \) at time \( t \). For each feasible state \((n, w)\) where \( 0 \leq n \leq N, 0 \leq w \leq W \), we define \( v_t(n, w) \) as the consumer's cost-to-go function. Equations 4–8–4–11 define a DP where the consumer's expected operating costs and the manufacturer's warranty prices are determined at time zero. Equation 4–8 represents the initial purchase at the beginning of the time horizon or a mandatory replacement because the product has reached its maximum useful life. In this case the consumer can replace the item without a warranty \( (R) \) or replace it with a warranty \( (R+PBW) \). When the consumer does not have a warranty, Equation 4–9 allows for the additional choices of keeping the item \( (K) \) or buying a warranty \( (K+PBW) \). Equation 4–10 represents the case where the consumer has a warranty and thus has the same choices as in Equation 4–9. Equation 4–11 is the terminal condition.

\[
v_t(n, w) = \min \begin{cases} 
R : P + \gamma(O_t(1) + v_{t+1}(2, 0)) \\
R + PBW : P + P_{t,n} + \gamma(\bar{O}(1) + v_{t+1}(2, W - 1)) 
\end{cases}, \quad (4–8)
\]

\[
v_t(n, 0) = \min \begin{cases} 
K : \gamma(O_t(n) + v_{t+1}(n + 1, 0)) \\
K + PBW : P_{t,n} + \gamma(\bar{O}(n) + v_{t+1}(n + 1, W - 1)) \\
R : P + \gamma(O_t(1) + v_{t+1}(2, 0)) \\
R + PBW : P + P_{t,n} + \gamma(\bar{O}(1) + v_{t+1}(2, W - 1)) 
\end{cases}, \quad (4–9)
\]

\[
v_t(n, w) = \min \begin{cases} 
K : \gamma(\bar{O}_t(n) + v_{t+1}(n + 1, 0)) \\
K + PBW : P_{t,n} + \gamma(\bar{O}(n) + v_{t+1}(n + 1, w - 1)) \\
R : P + \gamma(O_t(1) + v_{t+1}(2, 0)) \\
R + PBW : P + P_{t,n} + \gamma(\bar{O}(1) + v_{t+1}(2, w - 1)) 
\end{cases}, \quad (4–10)
\]
This formulation represents the scenario where the cost estimates are determined upfront and never updated. One way to modify the DP to allow the consumer to update their cost estimates and account for warranty price changes is to add another dimension to the state space. This third dimension would capture the current commodity price, but since the transition to the next state is also uncertain, the DP would be stochastic. Additionally, the range for the commodity price would be continuous and thus there would be an infinite number of states so the transitions would need to be discretized, which could still dramatically increase the state space. Also, the policy of a stochastic dynamic program (SDP) is dependent on knowing the states over the horizon of the problem. Finally, using this method requires the consumer to develop a distribution which is a much more complicated estimation process than a simple extrapolation of the current commodity cost. Using an SDP is possible to model this problem, but we want to use a method that better captures the consumer behavior with regards to commodity cost estimation.

Therefore, we solve the deterministic DP iteratively and, at each iteration, update the consumer's and manufacturer's cost estimates. This will require a total of $T$ DPs to be solved where the time horizon of the subsequent iterations decreases by one. Using this method maintains the same limitation as the SDP in that any policy is dependent on knowing the states over the time horizon, but it allows for the consumer and manufacturer to update their parameters and thus eliminate the issue of having a large number of states when the commodity cost is part of the state space. It also better reflects the reality of how we expect the entities to act in the fact that they will incorporate the most recent information. Ultimately, we are interested in determining if it is possible to capitalize on the difference in cost estimates, and this iterative method is sufficient for that purpose. We use a numerical example with an empirical component
for the commodity costs and the cost estimation techniques from the previous section to see if pricing and offering EPBWs, as described, can in fact improve manufacturer revenue in a realistic environment.

4.2.4 Numerical Example

The conjecture that a manufacturer can increase revenue through EPBWs by capitalizing on its more sophisticated commodity cost estimates is difficult to test. The estimation of the commodity costs alone is an extremely complex endeavor by itself. The importance of the proposed estimation technique is not the specific estimate itself, but the disparity in the sophistication of the estimation method versus the consumer’s estimate. Additionally, due to the uncertainty and nature of commodity costs, it is near impossible to make any generalized comment on the ability to affect revenue when the reimbursement is tied to actual costs. Therefore, our best effort is to present a numerical example that incorporates actual commodity cost data to determine the feasibility of using EPBWs to positively affect revenue.

The empirical data set used for the commodity costs comes from the U.S. Department of Energy (DOE) and entails the weekly gasoline prices for the U.S. from January 1992 to December 2011. The product data on the other hand is hypothetical but roughly modeled on an economy class automobile. Table 4-1 below lists the problem parameters.

For this example, we use a discrete uniform distribution for product performance so the five different performance levels occur with equal probability. Additionally, we account for performance degradation by defining a periodic rate, $\delta$, by which the performance changes with each period increase in age, i.e. $M_{n+1} = \delta M_n$. Since we are trying to determine the effect of the estimation techniques in the presence of actual costs, we assume the consumer has neutral beliefs about product performance and accepts the manufacturer’s distribution. The warranty coverage level is determined
a priori by the manufacturer. While this assumption could be relaxed and the optimal coverage level could be determined, we do not study this aspect.

Table 4-1. Parameters for EPBW Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Horizon ((T))</td>
<td>12</td>
</tr>
<tr>
<td>Maximum Useful Life ((N))</td>
<td>12</td>
</tr>
<tr>
<td>Purchase Price ((P))</td>
<td>20000</td>
</tr>
<tr>
<td>Low Performance Level</td>
<td>26</td>
</tr>
<tr>
<td>Medium Low Performance Level</td>
<td>29</td>
</tr>
<tr>
<td>Medium Performance Level</td>
<td>32</td>
</tr>
<tr>
<td>Medium High Performance Level</td>
<td>35</td>
</tr>
<tr>
<td>High Performance Level</td>
<td>38</td>
</tr>
<tr>
<td>Performance Degradation ((\delta))</td>
<td>.99</td>
</tr>
<tr>
<td>Discount Rate ((\gamma))</td>
<td>.9</td>
</tr>
<tr>
<td>Warranty Coverage Level ((X))</td>
<td>32</td>
</tr>
</tbody>
</table>

Note that the length of the problem horizon is 12 periods but the length of the empirical time series is 20 periods. The DP time horizon starts with year 2000 so the previous eight years of data provide the initial basis for the commodity cost estimates. For the iterative DP method, the next period data is incorporated in the next set of estimates. Also, the weekly data is averaged to provide annual values. Table 4-2 shows the results for warranty lengths up to six periods and the different combinations of consumer and manufacturer estimation methods. The estimates that use the upper bound of the prediction interval are annotated with \(Pr\).

The entries in Table 4-2 provide the percentage changes in revenue from always offering EPBW\'s of a given length for the different combinations of consumer and manufacturer commodity cost estimates. The first entry of the duple is the change in actual revenue. It represents the realized revenue when the manufacturer prices the warranties in accordance with Equation 4–5 and reimburses the consumer based on the actual commodity costs when the consumer owns a warranty. The second term represents the maximum possible revenue if the manufacturer had priced the warranties at the maximum willingness-to-pay of the consumer. The results can be analyzed in many different ways, and we limit our comments to two particular points.
First, in summary, the average change in actual revenue of all instances is \(-.41\) with a range of \([-1.18, .05]\) when averages are grouped by warranty length. Note, the baseline revenue when warranties are not offered is 20000 since the economic life of the product is 12 periods regardless of which performance level is realized. The change in maximum possible revenue has an overall average of .73 with a range of \([- .25, 1.33]\) for the different warranty lengths. These summary statistics suggest that the downside risk of tying a warranty to actual costs may be acceptable as the decrease in revenue is less than a half-percent on average. This slight decrease in revenue has the
potential to offer other benefits such as the consumer returning to the purchase location annually to reconcile the warranty which could be tied to a marketing strategy to sell periodic maintenance functions (such as an oil change in the case of an automobile). Additionally, the warranty prices using our 12 estimation models leave revenue on the table so the prices in general are not capturing the full willingness-to-pay of the consumer. While the manufacturer’s estimation techniques are more sophisticated than the consumer, our implementations are formulaic and do not reflect the true ability of a manufacturer to make more flexible adjustments, such as varying the number of periods used to determine the business factor, \( \hat{B} \). The consumer’s estimates, especially for shorter length warranties, do a fairly good job of capturing the business cycle effect since it is based on the most current known price. The ABC and NBC techniques for estimating the business cycle may not actually do as well as the consumer in this case. The JBC, or a similar type estimate, has the potential to explain the business cycle effect better when it is flexible (such as allowing the number of data points used in the calculation to vary). Our formulation also assumes the manufacturer will always offer a warranty whereas in reality, based on the model estimates, the manufacturer may choose not to offer an EPBW at certain times.

Second, there is no systematic relationship between the results when grouped by the consumer’s cost estimation technique. We might expect the actual revenue and maximum possible revenue to increase as the consumer’s estimation technique moves from constant to linear and from linear to geometric. This is not always the case. For example, a constant estimate might lead to a lower willingness-to-pay (WTP) for a consumer which may inhibit the consumer from purchasing a warranty at a certain point in time. If the subsequent commodity prices rise significantly, then the manufacturer could have incurred a liability larger than the warranty purchase price if the constant consumer would have purchased the warranty. A consumer who uses a geometric estimate though has a higher willingness-to-pay, and may have purchased the same
warranty that the constant consumer did not. With the sharp rise in commodity prices, the manufacturer might realize the large liability and thus there is a negative net effect on the manufacturer's revenue. Therefore, no generalization can be made regarding the ability of a manufacturer to capitalize on a consumer overestimating future commodity prices.

4.3 Markovian Performance

In Chapters 2 and 3, the method used to model performance uncertainty contained the implied assumption that future performance is not dependent on previous performance. For traditional warranties, this assumption is reasonable, especially when failures follow a poisson process. However, we would expect performance to be non-increasing and thus future performance depends on the current level of performance. To model performance in this manner we propose a Markov chain to describe the transition of product performance from one period to the next.

4.3.1 Model Formulation with Non-Increasing Performance

Up to this point, distributions describing product performance were not limited to being discrete. With the inclusion of non-increasing performance, we limit the analysis to discrete product distributions in order to utilize discrete space Markov chains. Note the modeling and analysis could be expanded to continuous state Markov chains or a continuous distribution could be discretized to an acceptable number of performance levels. Therefore we begin by defining a product with $L$ performance levels. For $l \in \{1, \ldots, L\}$, we assume that performance levels are indexed in increasing order such that the performance at level $l + 1$ is better than the performance at $l$. We define the initial performance distribution by the $L \times 1$ vector $\pi$ such that the probability of a product performing at level $i$ in its first period of operation is defined by $\pi(i)$. Next we define the subsequent transitions by the following one-step transition matrix:
The diagonal structure of $P^T$ is a by-product of non-increasing performance and the first diagonal element is always one. Using this new probability structure, we must include an additional index on the performance levels, the consumer’s periodic operating costs, and the manufacturer’s warranty prices in order to account for the current performance level. Let $M_{n,k}$ and $M_{n,k}$ be the random variables representing the manufacturer’s and consumer’s performance distribution, respectively, where $n$ is the age and $k$ is the current operating level of the product. As before, the consumer modifies $\pi$ and $P^T$ to construct $M_{n,k}^c$ and $M_{n,k}^c$ as well as the consumer’s cost with and without a warranty as $O_t(n, k)$ and $\bar{O}_t(n, k)$.

Using the new definitions, the warranty price under non-increasing performance can be calculated as:

$$P_{t,n,k} = \sum_{i=1}^{W} \sum_{j=1}^{k} \gamma^i E \left[ \frac{U_{\hat{c}_{t+i}}}{M_{n,j}} \right]$$

$$= U \sum_{i=1}^{W} \sum_{j=1}^{k} \gamma^i E[\hat{c}_{t+i}] E[\frac{1}{M_{n,j}}].$$

Under this performance structure, the consumer would purchase the warranty under the following condition:

$$P_{t,n,k} + \sum_{i=1}^{W} \sum_{j=1}^{k} \gamma^i \bar{O}_{t+i}(n + i, j) \leq \sum_{i=1}^{W} \sum_{j=1}^{k} \gamma^i O_{t+i}(n + i, j).$$
Conversely, the manufacturer realizes a net increase in revenue for this warranty under the following condition:

\[ P_{t,n,k} \geq U \sum_{i=1}^{W} \sum_{j=1}^{k} \gamma^i E[c_{t+i}]E\left[\frac{1}{M_{n,j}}\right]. \]

A specific case under the non-increasing performance model that is interesting to highlight is when the consumer believes current performance will continue indefinitely. In this case, the consumer is provided the initial product performance distribution, \( \pi \). As in Chapters 2 and 3, the consumer may modify \( \pi \) in such a way that they are characterized as optimistic, neutral, or pessimistic. The transition matrix is not revealed, though, because the consumer is assumed to exclude this performance characteristic from their decision making. When this is the case, the consumer uses the identity matrix as their one-step transition matrix. Whatever performance level a consumer observes at time \( t \), they in effect expect the product to perform at this level until replacement. Note, this situation, although mathematically equivalent, is not the same as a customer who is provided the transition matrix and chooses to modify it to be the identity matrix. We consider this special case because it is a reasonable assumption regarding how consumers make decisions. Just as we evaluated unsophisticated cost estimates for the consumer in the previous section, this scenario represents an unsophisticated outlook on performance. For this special case, if we assume the manufacturer and consumer use the same cost estimates and initial distribution, then a consumer will never purchase a warranty under non-increasing performance.

**Theorem 4.1.** A neutral or optimistic consumer who does not account for non-increasing performance will never purchase a warranty if \( \hat{c}_t \geq c_t^c \forall t \) and \( P^T \neq I \).

**Proof.**

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Given the state \((n, w, k)\) at time \(t\), the price of a warranty is:

\[
P_{t,n,k} = \sum_{i=1}^{w} \frac{\gamma^i U_{t+i}}{E[M_{n+i,k} - M_{n+i,k}]} = \sum_{i=1}^{w} \frac{\gamma^i U_{t+i}}{E[M_{n+i,k} - M_{n+i,k}]}.
\]

(4–12)

Similarly, the price the consumer is willing to pay, denoted by \(P_{t,n,k}^c\) is:

\[
P_{t,n,k}^c = \sum_{i=1}^{w} \frac{\gamma^i U_{t+i}}{E[M_{n+i,k}^c - M_{n+i,k}]} = \sum_{i=1}^{w} \frac{\gamma^i U_{t+i}}{E[M_{n+i,k}^c - M_{n+i,k}]}.
\]

(4–13)

Since we are only considering discrete distributions, let \(p_i : i \in \{1, \ldots, L\}\) represent the mutually accepted performance levels for the product in increasing order, i.e. \(p_{i+1} > p_i\).

Thus for the fractional component of Equation 4–12 we have:

\[
\frac{1}{E[M_{n+i,k} - M_{n+i,k}]} = \frac{1}{\sum_{j=1}^{k} p_j P_{k,j}^T}.
\]

For the consumer, the fractional component of Equation 4–13 is:

\[
\frac{1}{E[M_{n+i,k}^c - M_{n+i,k}^c]} = \frac{1}{\sum_{j=1}^{k} p_j l_{k,j}}.
\]

Since \(P^T\) is a lower diagonal Markov transition matrix (MTM), then \(\sum_{j=1}^{k} p_j P_{k,j}^T < \sum_{j=1}^{k} p_j l_{k,j}\) so:

\[
\frac{1}{E[M_{n+i,k}^c - M_{n+i,k}^c]} < \frac{1}{\sum_{j=1}^{k} p_j l_{k,j}}.
\]

Since \(U, \gamma, \widehat{c}_t, \) and \(\widehat{c}_t^c\) are positive and \(\widehat{c}_t \geq \widehat{c}_t^c \forall t\) then \(P_{t,n,k} > P_{t,n,k}^c\) and therefore the consumer will not purchase a warranty.

Note the result above only considers warranties for products that are already owned.

When a warranty is purchased at the same time a product is purchased, the result still
holds because the fractional component under the first period of operation is the same for the manufacturer and consumer when the consumer is neutral. When the consumer is optimistic, the consumer’s fractional component for the first period is less than the manufacturer’s component in the first period. For a neutral or optimistic consumer, the following condition holds:

\[
\frac{1}{E[\bar{M}_{1,k} - M_{n+i,k}]} \geq \frac{1}{E[\bar{M}^c_{1,k} - M^c_{n+i,k}]}.
\]

Therefore, whether the warranty is concurrent with a product purchase or not, the theorem is valid.

\[\square\]

4.3.2 Dynamic Program

With the inclusion of the assumption of non-increasing performance, the recursion in Equations 4–8–4–11 must be updated. Besides increasing the state space from two to three dimensions to account for the current performance level, the uncertain transitions also require the use of a stochastic dynamic program (SDP). Equations 4–14–4–17 define the SDP under the assumption of non-increasing product performance. This recursion is generalized to allow for the consumer to incorporate non-increasing performance such that the consumer’s MTM used to define transitions is denoted by \( P^c \). Conceptually, \( P^c \) could be set to the manufacturer’s MTM or otherwise modified like the consumer does with their initial belief. When \( P^c = I \), the consumer ignores non-increasing performance. Although the formulation is different from Equations 4–8–4–11, the decision space is the same as the deterministic model when future product performance was not influenced by the current performance level.

\[
v_t(n, w, k) = \min \left\{ \begin{array}{l}
R : P + \gamma(O_t(1, k) + \sum_{i=1}^{L} \pi^c(i)\nu_{t+1}(2, 0, i)) \\
R + PBW : P + P_{t,n,k} + \gamma(\tilde{O}(1, k) + \sum_{i=1}^{L} \pi^c(i)\nu_{t+1}(2, W - 1, i))
\end{array} \right\},
\]

\[n = 0, N, 0 \leq w \leq W, 1 \leq k \leq L\] (4–14)
Performance vectors that can be realized. Figure 4-2 shows an example of a transition network for a problem with three different performance levels (High (Hi), Medium (Md), and Low (Lo)).

An SDP does not have a single optimal policy like a deterministic dynamic program, so our method of iteratively solving a set of DPs to determine the effect on a manufacturer’s revenue has the same limitation as the SDP. In both cases, a vector of actual performance is needed to determine if offering the EPBWs increases revenue. In the previous example, the purpose of solving the DPs iteratively was to account for updated consumer estimates. Once an update was incorporated into the DP and a decision was made, the transition was deterministic. That is not the case when the DP incorporates non-increasing performance.

In order to characterize the effect on revenue for the non-increasing performance problem, we use a Monte Carlo simulation to provide an estimate of the average effect. To understand the scope of the simulation, let us consider the number of possible performance vectors that can be realized. Figure 4-2 shows an example of a transition network for a problem with three different performance levels (High (Hi), Medium (Md), and Low (Lo)).
Assuming a general non-increasing Markovian network with $L$ performance levels and $T$ time periods, we define $H(L, T)$ as the total number of possible performance paths such that:

$$H(L, T) = 1 + (1 + (T - 1)) + \sum_{i=3}^{L} [1 + (T - 1) + \sum_{j=2}^{T} \sum_{k=j+1}^{T} h_{i-1,k}]$$

$$= 1 + (L - 1)T + \sum_{i=3}^{L} \sum_{j=2}^{T} \sum_{k=j+1}^{T} h_{i-1,k}. \quad (4-18)$$

For Equation 4–18 above, $h_{i',j'}$ represents the total number of performance paths from node $(i', j')$ where $i'$ represents the current performance level and $j'$ represents the time period. Note $i'$ and $j'$ are used as generic indexing. The $h$ parameters are determined by the following recursion:

$$h_{1,j'} = 1 \text{ for } j' = 2, \ldots , T, \quad (4-19)$$

$$h_{2,j'} = 1 + T - j' \text{ for } j' = 2, \ldots , T, \quad (4-20)$$

$$h_{i',j'} = 1 + T - j' + \sum_{q=1}^{j'} h_{a,j'+1} \text{ for } i' = 3, \ldots , L; j' = 2, \ldots , T. \quad (4-21)$$

Equation 4–19 represents the single performance path once the product has performed at the lowest level. For the second-to-lowest performance level, Equation 4–20 represents the paths of constant performance for the remainder of the time horizon and progressing to the lowest performance level at any future time period. In Equation 4–21, the first term represents constant performance and moving directly to the lowest
performance in a future time period (similar to Equation 4–20); the summation term represents all other possible paths from the given node. Applying this recursion and Equation 4–18 to the example in Table 4-1 with $T = 12$ and $L = 5$, the total number of performance paths is 1720.

4.3.3 Numerical Example

With Theorem 4.1, it was shown that a neutral or optimistic consumer would not purchase a warranty when they do not account for non-increasing performance and cost estimates are assumed to be the same for both the manufacturer and consumer. Under this assumption, it is possible though that non-increasing performance, if not accounted for by the consumer, could offset the manufacturer's pricing advantage that comes from a pessimistic consumer.

In order to study the counter effects of a pessimistic consumer that does not account for non-increasing performance, we assume the consumer and manufacturer use the same commodity cost estimate. Consider the same problem parameters given in Table 4-1 but instead of the consumer sharing the same discrete uniform distribution as the manufacturer, it is modified to a pessimistic outlook. Table 4-3 shows four different pessimistic consumer beliefs (in addition to the neutral consumer) to be analyzed.

<table>
<thead>
<tr>
<th>Consumer 1</th>
<th>Consumer 2</th>
<th>Consumer 3</th>
<th>Consumer 4</th>
<th>Consumer 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability $k = 1$</td>
<td>.2</td>
<td>.3</td>
<td>.33</td>
<td>.5</td>
</tr>
<tr>
<td>Probability $k = 2$</td>
<td>.2</td>
<td>.25</td>
<td>.33</td>
<td>.5</td>
</tr>
<tr>
<td>Probability $k = 3$</td>
<td>.2</td>
<td>.2</td>
<td>.33</td>
<td>0</td>
</tr>
<tr>
<td>Probability $k = 4$</td>
<td>.2</td>
<td>.15</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Probability $k = 5$</td>
<td>.2</td>
<td>.1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Furthermore, define the manufacturer’s MTM as:
Obviously, the choice of $P^T$ affects the results as much as the selection of any parameters, but the particular MTM chosen here has practical characteristics.

First, the diagonal elements are larger than 0.5 so the most likely transition is to the same performance level. Next, the off diagonal elements in a given row decrease in magnitude as the column index decreases. This represents the characteristic that large performance changes are less likely. Using this transition matrix, Table 4-4 shows the average actual effect and average maximum possible effect, in percentages, on the manufacturer’s revenue for the customer beliefs listed in Table 4-3 for one-period warranties.

Table 4-4. Single-Period Warranty Effect on Revenue for Markovian Model

<table>
<thead>
<tr>
<th></th>
<th>Consumer 1</th>
<th>Consumer 2</th>
<th>Consumer 3</th>
<th>Consumer 4</th>
<th>Consumer 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear ABC</td>
<td>(-0.05, 0.05)</td>
<td>(-0.64, 0.55)</td>
<td>(-0.64, 0.47)</td>
<td>(-0.64, 0.35)</td>
<td>(-0.63, 0.23)</td>
</tr>
<tr>
<td>Linear ABC-P</td>
<td>(-0.07, 0.07)</td>
<td>(-0.65, 0.56)</td>
<td>(-0.62, 0.44)</td>
<td>(-0.65, 0.37)</td>
<td>(-0.64, 0.24)</td>
</tr>
<tr>
<td>Linear JBC</td>
<td>(-0.07, 0.07)</td>
<td>(-2.42, 2.41)</td>
<td>(-2.42, 2.40)</td>
<td>(-2.40, 2.38)</td>
<td>(-2.42, 2.39)</td>
</tr>
<tr>
<td>Linear JBC-P</td>
<td>(0.15, -0.15)</td>
<td>(-2.19, 2.18)</td>
<td>(-2.19, 2.18)</td>
<td>(-2.18, 2.15)</td>
<td>(-2.18, 2.14)</td>
</tr>
<tr>
<td>Linear NBC</td>
<td>(-0.03, 0.03)</td>
<td>(-0.60, 0.50)</td>
<td>(-0.59, 0.41)</td>
<td>(-0.59, 0.31)</td>
<td>(-0.60, 0.19)</td>
</tr>
<tr>
<td>Linear NBC-P</td>
<td>(-0.05, 0.05)</td>
<td>(-0.60, 0.51)</td>
<td>(-0.60, 0.42)</td>
<td>(-0.60, 0.31)</td>
<td>(-0.60, 0.19)</td>
</tr>
<tr>
<td>Quadratic ABC</td>
<td>(-0.70, 0.70)</td>
<td>(-0.71, 0.62)</td>
<td>(-0.70, 0.54)</td>
<td>(-0.70, 0.44)</td>
<td>(-0.71, 0.32)</td>
</tr>
<tr>
<td>Quadratic ABC-P</td>
<td>(0.00, 0.00)</td>
<td>(-2.36, 2.35)</td>
<td>(-2.36, 2.34)</td>
<td>(-2.36, 2.34)</td>
<td>(-2.36, 2.33)</td>
</tr>
<tr>
<td>Quadratic JBC</td>
<td>(0.20, -0.20)</td>
<td>(-0.27, 0.17)</td>
<td>(-0.26, 0.07)</td>
<td>(-0.26, -0.03)</td>
<td>(-0.27, -0.15)</td>
</tr>
<tr>
<td>Quadratic JBC-P</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
</tr>
<tr>
<td>Quadratic NBC</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
</tr>
<tr>
<td>Quadratic NBC-P</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
<td>(0.00, 0.00)</td>
</tr>
<tr>
<td>Averages</td>
<td>(-0.05, -0.05)</td>
<td>(-0.87, -0.82)</td>
<td>(-0.87, -0.77)</td>
<td>(-0.87, -0.72)</td>
<td>(-0.87, -0.66)</td>
</tr>
</tbody>
</table>

If we evaluate the averages at the bottom of Table 4-4, we see that as the consumer becomes more pessimistic, the maximum possible revenue increases which is to be expected since the pessimistic consumer values a warranty more than a neutral or
optimistic consumer and is willing to pay more. Note, this characteristic may not always be true even though it is intuitive and what we expect to observe. A consumer with a higher willingness-to-pay does not necessarily generate more revenue since the commodity prices could rise sufficiently such that there is a net negative effect on revenue over the life of the warranty.

As for the actual changes in revenue, they stay consistent. This may be due to the large values on the diagonal of the transition matrix or due to the short length of the warranty. Therefore, we should compare the averages over various length warranties and with different transition matrices. Table 4-5 shows the average effects (over all manufacturer estimation techniques) on actual revenue and maximum possible revenue for warranty lengths up to six periods and for three different transition matrices. The first MTM (labeled 1 in the table) is the transition matrix given in 4.3.3; the second MTM (labeled 2 in the table) doubles all the off-diagonal elements and reduces all but the first diagonal element to $0.8$; and the third MTM triples the off-diagonal elements and reduces the diagonal elements (except the first) to $0.7$.

Reviewing the entries in Table 4-5, there are several trends that appear. First as the structure of the MTM changes such that the likelihood of moving to a lower performance level in the future increases (i.e., moving from MTM 1 to MTM 3), the manufacture experiences a greater decrease in revenue. In effect, the price of the warranty is increasing relative to the consumer’s willingness-to-pay (WTP) so there are fewer chances that the consumer will purchase a warranty and when they do, it is more expensive. Note that all the averages are negative (with the exception of one 0.00 entry). Since we assumed the consumer used the same cost estimation technique as the manufacturer, we expect them to be negative because the naivety of the consumer not accounting for non-increasing performance generally keeps their WTP below the manufacturer’s warranty price. It is possible, though, that future commodity costs could move in such a way to overcome this effect, but we do not see this result on average.
Another trend we see is that the larger decreases in revenue occur with a long warranty length or an extremely pessimistic consumer. The effect of the longer warranty period decreasing revenue more than shorter lengths might be expected since the manufacturer cannot adjust for changes in commodity costs as often as with shorter length warranties. As for the larger decreases in revenue for the extremely pessimistic consumer, this may be a by product of a high WTP that causes this consumer to purchase a warranty in the initial period when other consumers would not. Depending on the movement of future commodity prices, this initial purchase may not have a positive net effect on revenue. These explanations highlight the difficulty in making generalizations for these models. Results vary depending on the actual realizations of commodity prices and when the finite horizon starts. In our example, the six period warranty decreases revenue more than other warranties, but it is easy to foresee a scenario when this may not hold. Consider a consumer who purchases a warranty when commodity costs are elevated and then they subsequently drop and remain relatively lower over the life of the warranty. In this case, the longer warranty would be
advantageous to the manufacturer because the consumer was locked in when prices were high.

### 4.4 Combined Model

We now consider a model that combines the characteristics of distinct commodity cost estimates and non-increasing performance. Note the consumer’s choice of commodity cost estimate could counter their lack of acknowledgement of non-increasing performance. In a general sense, when a consumer’s commodity cost estimate is high, they essentially are taking a pessimistic outlook on costs because they project higher costs relative to observed costs. Conversely, when a consumer does not account for non-increasing performance, they are in effect taking an optimistic perspective because they will have, all other factors being equal, lower expected operating costs than under the true performance process described by $P^T$. Using the parameters in Table 4-1 where the consumer and manufacturer share a uniform discrete distribution and the MTM from 4.3.3, we can determine the effect on the manufacturer’s revenue under different combinations of the warranty length and the consumer’s cost estimate. Table 4-6 show the results for warranty lengths up to six periods for each of the three consumer cost estimates. Similar to the previous tables, the values represent the average (over all manufacturer estimates) percent change in actual revenue and maximum possible revenue.

Table 4-6 shows two significant results from our model. First the average actual effects on revenue for all the combinations of warranty length, consumer initial belief, and consumer commodity estimate are in the range of $[-2.43, -1.98]$. This is a fairly small range given all the variable components in the model. To explain this surprising result would require a much more in-depth analysis of the data so we offer one possible explanation. Note that in Table 4-5 when the consumer used the same estimates as the manufacturer, the magnitude of the decreases were smaller. Therefore, the consumer’s use of the most current commodity price as a basis of estimates may be outperforming
the manufacturer’s estimates. Also, since both estimates may be tracking the actual prices reasonably well, there is a tight range for the effect on actual revenue.

Table 4-6. Warranty Effect on Manufacturer Revenue for Combined Model

<table>
<thead>
<tr>
<th>W</th>
<th>Consumer Estimate</th>
<th>Consumer 1</th>
<th>Consumer 2</th>
<th>Consumer 3</th>
<th>Consumer 4</th>
<th>Consumer 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Constant</td>
<td>-2.16, 0.35</td>
<td>-2.34, 0.24</td>
<td>-2.37, 0.26</td>
<td>-2.39, 0.38</td>
<td>-2.35, 0.47</td>
</tr>
<tr>
<td>1</td>
<td>Linear</td>
<td>-2.36, 0.45</td>
<td>-2.43, 0.61</td>
<td>-2.42, 0.70</td>
<td>-2.39, 0.79</td>
<td>-2.34, 0.87</td>
</tr>
<tr>
<td>1</td>
<td>Geometric</td>
<td>-2.37, 1.06</td>
<td>-2.36, 1.15</td>
<td>-2.44, 1.34</td>
<td>-2.38, 1.39</td>
<td>-2.43, 1.60</td>
</tr>
<tr>
<td>2</td>
<td>Constant</td>
<td>-2.08, 0.25</td>
<td>-2.10, 0.30</td>
<td>-2.13, 0.32</td>
<td>-2.27, 0.34</td>
<td>-2.25, 0.46</td>
</tr>
<tr>
<td>2</td>
<td>Linear</td>
<td>-2.13, 0.68</td>
<td>-2.17, 0.64</td>
<td>-2.25, 0.75</td>
<td>-2.26, 0.88</td>
<td>-2.27, 1.03</td>
</tr>
<tr>
<td>2</td>
<td>Geometric</td>
<td>-2.27, 1.43</td>
<td>-2.28, 1.52</td>
<td>-2.28, 1.61</td>
<td>-2.27, 1.75</td>
<td>-2.28, 1.89</td>
</tr>
<tr>
<td>3</td>
<td>Constant</td>
<td>-2.16, 0.30</td>
<td>-2.11, 0.34</td>
<td>-2.13, 0.37</td>
<td>-2.24, 0.49</td>
<td>-2.22, 0.63</td>
</tr>
<tr>
<td>3</td>
<td>Linear</td>
<td>-2.18, 0.99</td>
<td>-2.12, 0.99</td>
<td>-2.15, 1.06</td>
<td>-2.21, 1.28</td>
<td>-2.25, 1.41</td>
</tr>
<tr>
<td>3</td>
<td>Geometric</td>
<td>-2.15, 1.80</td>
<td>-2.16, 2.18</td>
<td>-2.18, 2.35</td>
<td>-2.19, 2.45</td>
<td>-2.15, 2.58</td>
</tr>
<tr>
<td>4</td>
<td>Constant</td>
<td>-2.26, 0.27</td>
<td>-2.26, 0.30</td>
<td>-2.25, 0.34</td>
<td>-2.27, 0.42</td>
<td>-2.26, 0.62</td>
</tr>
<tr>
<td>4</td>
<td>Linear</td>
<td>-2.21, 1.17</td>
<td>-2.16, 1.19</td>
<td>-2.21, 1.26</td>
<td>-2.20, 1.48</td>
<td>-2.23, 1.59</td>
</tr>
<tr>
<td>4</td>
<td>Geometric</td>
<td>-2.22, 2.15</td>
<td>-2.22, 2.32</td>
<td>-2.24, 2.76</td>
<td>-2.17, 2.92</td>
<td>-2.23, 3.10</td>
</tr>
<tr>
<td>5</td>
<td>Constant</td>
<td>-2.10, 0.19</td>
<td>-2.12, 0.26</td>
<td>-2.13, 0.28</td>
<td>-2.11, 0.37</td>
<td>-2.06, 0.64</td>
</tr>
<tr>
<td>5</td>
<td>Linear</td>
<td>-2.14, 1.08</td>
<td>-2.13, 1.14</td>
<td>-2.15, 1.22</td>
<td>-2.11, 1.43</td>
<td>-2.09, 1.56</td>
</tr>
<tr>
<td>5</td>
<td>Geometric</td>
<td>-2.12, 2.24</td>
<td>-2.13, 2.28</td>
<td>-2.13, 2.44</td>
<td>-2.10, 3.09</td>
<td>-2.05, 3.23</td>
</tr>
<tr>
<td>6</td>
<td>Constant</td>
<td>-2.15, 0.17</td>
<td>-2.17, 0.20</td>
<td>-2.20, 0.23</td>
<td>-2.21, 0.33</td>
<td>-2.13, 0.61</td>
</tr>
<tr>
<td>6</td>
<td>Linear</td>
<td>-2.18, 1.30</td>
<td>-2.18, 1.39</td>
<td>-2.17, 1.44</td>
<td>-2.05, 2.13</td>
<td>-1.98, 2.24</td>
</tr>
<tr>
<td>6</td>
<td>Geometric</td>
<td>-2.16, 2.67</td>
<td>-2.09, 2.61</td>
<td>-2.19, 2.96</td>
<td>-2.06, 4.09</td>
<td>-1.98, 4.19</td>
</tr>
</tbody>
</table>

The second result that deserves attention is the positive values for the maximum possible revenue. The range ([.17, 4.19]) is wider and the magnitude of maximum change is larger than the actual effect on revenue. Therefore, if the manufacturer could take better advantage of the consumer’s willingness-to-pay, the warranties could have a positive effect on revenue. Again, the manufacturer estimates presented, while more sophisticated, are rigid in their application of a formula for identifying the business and trend effects. In reality, a manufacturer could use a much more robust process to analyze the data and change the model as needed. While the consumer’s estimates are also rigid, the literature suggests that this is indeed how consumers behave, and thus we would not expect them to apply the same flexibility in adjusting their estimates. Additionally, if the manufacturer was not limited to always offering warranties of the same length, they may be able to increase revenue even above the levels indicated by the maximum possible revenue averages.
4.5 Summary

The value of a performance based warranty to a consumer will vary based on the underlying cost of the commodity that is consumed by operating the product. Since the cost of the commodity is not known a priori, the value of the PBW will be dependent on the consumer’s estimate of future commodity costs. Under the assumption that the consumer updates future cost estimates each period, the warranty value is non-stationary. Additionally, when a manufacturer reimburses the consumer based on the actual commodity costs, the warranty price will be dynamically priced according to the manufacturer’s current projection of future commodity costs as well as expected product performance. Since it is reasonable to assume there is no systematic relationship between the manufacturer’s and consumer’s estimates, it is difficult to make generalizations about the effect on revenue when a manufacturer offers PBWs. The results are ultimately dependent on the true commodity costs which are unknown to each agent.

In this chapter, we used a semi-empirical model with an actual time series to study the effects on manufacturer revenue of offering extended performance based warranties (EPBWs). Assuming that consumers use unsophisticated commodity cost estimates, we hypothesized that the manufacturer could use more sophisticated estimate techniques to better estimate future commodity costs and thus increase revenue by offering EPBWs. Using various manufacturer cost estimates, we found that a manufacturer on average would decrease revenue by offering EPBWs but that the magnitude of the decrease was less than one percent. This result suggests that the risk to a manufacturer of offering EPBWs where the liability is dependent on an exogenous factor, like gasoline prices, may be acceptable in a practical example. With the modest decrease in revenue, the EPBW gives the manufacturer a tool to ensure a return visit from the consumer periodically to reconcile the warranty, and thus it could be a strong marketing tool to sell periodic maintenance services for the product.
We also determined the maximum possible revenue the manufacturer could achieve if the warranties were priced at the willingness-to-pay level of the consumer and found that offering EPBWs at this level increased the manufacturer’s revenue. This shows the potential for an EPBW to be a additional revenue source in addition to being a marketing tool to entice return visits.

Finally, we incorporated the characteristic of non-increasing performance into the model. Often models for reliability use stochastic processes that possess independent increments so the current levels of failures do not influence future failures. On the other hand we would expect a product’s performance, such as energy consumption, to always decline (or remain constant) with age. We studied the non-increasing performance model under the assumption that the consumer does not account for it and uses the naive perspective that current performance will continue indefinitely. The effect of modeling performance in this way generates a lower willingness-to-pay from the consumer which may not always equate to a decrease in revenue.

In general, the results are difficult to generalize because they are dependent not only on different estimation techniques but most importantly on an unknown and hard to predict time series. The most meaningful result comes from observing the magnitude of the effects on revenue. The fact that the revenue decreases were small and that there is untapped willingness-to pay provides motivation for further research.
CHAPTER 5
CONCLUSION AND FUTURE RESEARCH

Traditional warranties have been studied extensively, but there is relatively little research regarding performance based warranties and guarantees. In this dissertation, we provide a framework for studying performance guarantee contracts. We propose different models for various equipment replacement scenarios and show the potential for a manufacturer to affect consumer behavior in a way that benefits the manufacturer.

In Chapter 2, we introduce the concept of a performance based warranty (PBW) that compensates a consumer in the event a product performs below the level stated in the warranty. Within this framework, we also propose a method of consumer differentiation (which is used throughout the dissertation) that describes a consumer’s expectation of product performance. While our method of modeling PBWs is general, we specifically evaluate the use of two types of PBWs: constant performance (CP) and constant cost (CC) warranties. We consider the use of CP and CC PBWs in the context of a finite horizon equipment replacement problem with a single challenger. We show that when these warranties are properly designed, they can entice a consumer to upgrade to a more technologically advanced product earlier and thus increase the manufacturer’s revenue. Additionally, for consumers that believe a product will perform at a level below the manufacturer’s advertised performance, the manufacturer can use a PBW to generate an additional revenue source.

Next, we study the use of a performance guarantee in a competitive environment in Chapter 3. We extend the concept of a consumer’s pre-existing belief about a product’s performance to include periodic updates of this belief based on the product’s previous performance. We model this extension using a Bayesian updating methodology. Assuming a competitive environment with two challengers, we show that a performance guarantee can be used to increase the probability that a consumer returns to the incumbent manufacturer for a follow-on purchase and thus increase the manufacturer's
revenue. When the purchase price is large compared to periodic operating costs, a manufacturer has a strong motivation to satisfy consumer expectations about product performance in order to retain the consumer’s business.

In Chapter 4, we introduce the concept of a specific type of PBW called an extended performance based warranty (EPBW) in which the warranty is offered at any time the product is owned, not just with the purchase of the product. We use an example to determine the actual risk to revenue that a manufacturer may experience when warranties are offered and product performance is linked to the use of an underlying commodity. Although our results cannot be generalized due to the reliance on the observed commodity prices, this model highlights the unique nature of performance based warranties in comparison to traditional warranties, i.e. the high dependency on the cost of the underlying commodity. This dependency influences the consumer’s willingness-to-pay (WTP) and makes the WTP non-stationary with respect to time due to a reliance on the consumer’s commodity cost estimate. When the manufacturer ties reimbursement to the observed commodity prices, they absorb the risk involved with the volatility of the commodity. We show that this risk is reasonable and that with an effective commodity estimate and warranty strategy, revenue can be increased by offering EPBWs.

This research can be extended in the future in several important ways. First, the rigid nature of the manufacturer’s estimation techniques in Chapter 4 do not fully capture the ability of a manufacturer to estimate future commodity costs. While the estimation techniques were relatively more sophisticated than those of the consumer, they nevertheless were not of the extent and level that would be available from a forecasting expert. Additionally, we assumed the warranties were of fixed length and that they would be offered every period. In reality, the warranty length, and even the periods in which they are offered, could vary. Our simplistic approach did not account for a manufacturer’s ability to evaluate current market conditions to decide if offering a
warranty at a given time is in their best interest. Also, we fixed the warranty level which could also be studied to determine which minimum performance provides the greatest increase in revenue. As the warranty length can be treated as variable, the warranty level could also be variable, especially as the product reveals its true performance.

The models presented in Chapters 2 and 3 used the simplified method of modeling performance with a single random variable representing operating costs. While this made the analysis more tractable, it eliminated a critical aspect of performance based warranties and guarantees. Decomposing operating costs into its components for these models, as in Chapter 4, would allow for a more practical analysis. Also, since the true benefit of a PBW can only be seen when accounting for commodity costs, extending the analysis of those models to include an empirical study would be beneficial. Additionally, the contracts studied in Chapters 2 and 3 were akin to basic warranties—warranties where the coverage cannot be extended beyond the initial terms. It would be interesting to see the effect on revenue of offering these warranties under the operating cost structure of Chapter 4 because there is only one price that must be determined. This may allow the manufacturer to better capitalize on the consumer’s unsophisticated estimation techniques.

In Chapter 2, the warranty designs focused on the structurally simple, single-term constant performance (CP) and constant cost (CC) PBWs. Future research could investigate more complex designs such as allowing renewals if an initial PBW is purchased or a variable warranty that changes as the obsolete product approaches its economic life. The literature on traditional warranties offer many warranty designs that could be applied to PBWs. Additionally, the decision to offer a CP or CC warranty is only considered in the context of a consumer that seeks to minimize costs. There are consumer implications from this decision that were not modeled, such as the consumer’s preference for the simplicity of a CC warranty or the willingness of the consumer to spend extra money on a warranty on the same day as making such a large purchase.
Accounting for consumer psychology issues like these (outside of minimizing total cost) would be an interesting addition to the model.

We also discussed the implementation of PBWs in Chapter 2. Further study of ways to effectively implement these new types of contracts would increase the likelihood of them being applied in practice. There are simplicities to these types of warranties, such as a lack of repair infrastructure, but there are also complexities, such as consumer behavior that affects product performance, that make implementing PBWs a challenge. Exploring ways to efficiently address these complexities would be an important extension of this research.

Finally, the inclusion of budget constraints within any of our models would be insightful to see how these effect consumer behavior and manufacturer revenue. Businesses and government agencies often have budget and planning processes that put a premium on low cost variability. For customers that seek some method of setting a maximum periodic cost threshold, a PBW could be a desirable contract type. This would provide a higher level of possible revenue for the manufacturer in exchange for taking on more risk. The manufacturer could then attempt to mitigate this risk with hedges on the commodity costs.
REFERENCES


BIOGRAPHICAL SKETCH

Clay Koschnick completed his Bachelor of Science in Operations Research in 1998 and a Master of Science in Operations Research in 2007. In 2009, he entered the Doctor of Philosophy program in the Industrial Systems and Engineering Department at the University of Florida. In the summer of 2012, he graduated with his PhD in industrial and system engineering. His research interests include solving practical business problems using dynamic programming and integrating econometric estimates and forecasts into dynamic optimization models.