P-ADIC THEORY OF EXPONENTIAL SUMS ON THE AFFINE LINE

By

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To my parents, Hitoshi Morofushi and Toko Morofushi
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In this work, we investigate p-adic theory of certain exponential sums on the affine line. Andrea Pulita introduced an F-isocrystal on the affine line which corresponds to a character of absolute Galois group of \( k((t^{-1})) \) of p-power order. We study the exponential sums of this F-isocrystal by examining the Newton polygon of the L-function of the exponential sums.

We first compute the degree of the L-function by using the formula by Philippe Robba. Then, we replace the Frobenius of Pulita’s F-isocrystal by a new Frobenius with larger radius of convergence to obtain better estimates for the Newton polygon. Finally, we find a lower bound of the Newton polygon of the L-function of the F-isocrystal.
CHAPTER 1
INTRODUCTION AND BASIC DEFINITIONS

In this chapter, we review the basic definitions, facts, and a brief history we will need for the rest of the dissertation.

1.1 Definitions

1.1.1 Algebraic Varieties

Let \( k \) be an algebraically closed field. Let \( \mathbb{A}^n_k \) be an affine \( n \)-space over \( k \), the set of all \( n \)-tuples of elements of \( k \). \( Y \subseteq \mathbb{A}^n_k \) is an algebraic set if there exists \( T \subseteq \mathbb{A}^n_k \) such that \( Y = \{ P \in \mathbb{A}^n_k | f(P) = 0 \text{ for all } f \in T \} \). An affine variety is an irreducible algebraic set \( X \subseteq \mathbb{A}^n_k \). \( Y \) is a quasi-affine variety if \( \mathbb{A}^n_k - Y \) is an algebraic set.

Let \( \mathbb{P}^n_k \) be a projective \( n \)-space over \( k \), the set of equivalence classes of \( n+1 \)-tuples of elements of \( k \), not all zero, under the equivalence relation given by \( (a_0, \cdots, a_n) \sim (\lambda a_0, \cdots, \lambda a_n) \) for all \( \lambda \in k, \lambda \neq 0 \). \( Y \subseteq \mathbb{P}^n_k \) is an algebraic set if there exists a set \( T \subseteq k[x_1, \cdots, x_n] \) of homogeneous elements such that \( Y = \{ P \in \mathbb{A}^n_k | f(P) = 0 \text{ for all } f \in T \} \). A projective variety is an irreducible algebraic set in \( \mathbb{P}^n_k \). \( Y \) is a quasi-projective variety if \( \mathbb{P}^n_k - Y \) is an algebraic set.

An algebraic variety is any affine, quasi-affine, projective, or quasi-projective variety.

1.1.2 Frobenius Structure

Let \( \mathcal{V} \) be a complete discrete valuation ring, \( k \) be the residue field of \( \mathcal{V} \), perfect of characteristic \( p \), and \( K \) be the fraction field of \( \mathcal{V} \). The Robba ring at \( \infty \), \( \mathcal{R}_K \), is defined by

\[
\mathcal{R}_K = \{ f(T) := \sum_{i \in \mathbb{Z}} a_i T^i | a_i \in K, \exists \rho > 1 \text{ such that } f(T) \text{ converges for } 1 < |T|_\rho < \rho \}.
\]

An absolute Frobenius on \( K \) is a \( \mathbb{Q}_p \)-endomorphism \( \sigma : K \to K \) such that \( |\sigma(x) - x^p| < 1 \) for all \( x \in \mathcal{O}_K \). An absolute Frobenius on \( \mathcal{R}_K \) is a continuous endomorphism of rings \( \phi : \mathcal{R}_K \to \mathcal{R}_K \) extending \( \sigma \) such that \( \phi(T) - T^p = \sum a_i(\phi) T^i \), with \( |a_i(\phi)| < 1 \) for all \( i \in \mathbb{Z} \), \( a_i (\phi) \in K \).
1.1.3 F-Isocrystals

An isocrystal on $\mathcal{R}$ is a pair $(M, \nabla)$ consisting of a finite free $\mathcal{R}$-module $M$ and a connection $\nabla : M \to M \otimes \Omega^1_{\mathcal{R}/K}$.

An F-isocrystal on $\mathcal{R}$ is a triple $(M, \nabla, F)$ where $(M, \nabla)$ is an isocrystal on $\mathcal{R}$ and $F$ is a Frobenius structure, i.e., a $\phi$-linear isomorphism $F : \phi^* M \to M$ commuting with $\nabla$.

1.1.4 Witt Vectors

We follow [15] for the notations concerning the ring of Witt vectors. Let $p$ be a prime and $\mathcal{R}$ be a ring. The Witt polynomials are defined by

$$\phi_n(X_0, \ldots, X_n) := X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^nX_n.$$ 

**Theorem 1.1.** Let $X = (x_0, \ldots, x_n, \ldots)$ and $Y = (y_0, \ldots, y_n, \ldots)$ be sequences of indeterminates. Then for each $\Phi \in \mathbb{Z}[X, Y]$, there is a unique sequence $(\varphi_0, \ldots, \varphi_n, \ldots)$ of elements of $\mathbb{Z}[x_0, \ldots; y_0, \ldots]$ such that

$$\phi_n(\varphi_0, \ldots, \varphi_n, \ldots) = \Phi(\phi_n(x_0, \ldots), \phi_n(y_0, \ldots))$$

for all $n \geq 0$.

Let $\Phi_1(X, Y) = X + Y$, $\Phi_2(X, Y) = XY$. Then we can find unique sequences $(S_0, \ldots)$ and $(P_0, \ldots)$ of elements of $\mathbb{Z}[x_0, \ldots; y_0, \ldots]$ such that

$$\phi_n(S_0, \ldots) = \phi_n(x_0, \ldots) + \phi_n(y_0, \ldots)$$

$$\phi_n(P_0, \ldots) = \phi_n(x_0, \ldots)\phi_n(y_0, \ldots)$$

for all $n \geq 0$.

Now for $\alpha, \beta \in R^\mathbb{N}$, define

$$\alpha + \beta = (S_0(\alpha, \beta), \ldots)$$

$$\alpha\beta = (P_0(\alpha, \beta), \ldots)$$
Then $R^N$ is a commutative ring with 1 under these operations. Call this ring the ring of Witt vectors of $R$, and denote it by $W(R)$.

For all $\nu := (\nu_0, \nu_1, \ldots) \in W(R)$, we call $\phi_j(\nu_0, \nu_1, \ldots) \in R$ the $j$-th phantom component of $\nu$.

The ring of Witt vectors of finite length is defined by $W_m(R) := W(R)/V^{m+1}W(R)$, where $V : W(R) \to W(R), (\nu_0, \nu_1, \ldots) \mapsto (0, \nu_0, \nu_2, \ldots)$, is the Verschiebung morphism. Define a morphism $\bar{F} : W_m(R) \to W_m(R)$ by $\bar{F}(\nu_0, \ldots, \nu_m) = (\nu_0^p, \ldots, \nu_m^p)$.

**Example 1.2.** $S_0(\alpha, \beta) = \alpha_0 + \beta_0$, and $P_0(\alpha, \beta) = \alpha_0 \cdot \beta_0$. So one has $W_0(R) = R$.

**Example 1.3.** If $\alpha, \beta \in W_1(R)$, then $\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1 + \frac{\alpha_0^p + \beta_0^p - (\alpha_0 + \beta_0)^p}{p})$ and $\alpha \cdot \beta = (\alpha_0 \cdot \beta_0, \beta_0^p \alpha_1 + \beta_1 \alpha_0^p + p \alpha_1 \beta_1)$.

**Theorem 1.4.** $W(F_p) = \mathbb{Z}_p$ and $W_n(F_p) = \mathbb{Z}/p^n\mathbb{Z}$ via the map $(\alpha_0, \ldots) \mapsto \sum_{i=0}^{\infty} \alpha_i^p - p^i$.

### 1.2 History

#### 1.2.1 Zeta Function and A. Weil’s Conjectures

**Definition 1.** Let $X$ be a non-singular $n$-dimensional projective algebraic variety over a field $\mathbb{F}_q$ with $q$ elements. The zeta function of $X$ defined over $\mathbb{F}_q$ is

$$\zeta(X, t) := \exp \sum_{m=0}^{\infty} N_m \frac{t^m}{m},$$

where $N_m$ is the number of points of $X$ defined over $\mathbb{F}_{q^m}$.

In 1949 A. Weil [20] stated his conjectures:

- (Rationality) $\zeta(X, t) = \frac{P(t)}{Q(t)}$ where $P(t), Q(t) \in \mathbb{Q}[t]$.
- (Functional Equation) $\zeta(X, \frac{1}{q^t}) = \pm q^{n \chi/2} \chi \zeta(X, t)$ where $\chi = \Delta \cdot \Delta$(Euler characteristic).
- (Riemann Hypothesis)

$$\zeta(X, t) = \frac{P_1(t) \cdot P_2(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}$$
\[ P_0(t) = 1 - t, \ P_{2n}(t) = 1 - q^n t, \ \text{and for } 1 \leq r \leq 2n - 1, \]
\[ P_r(t) = \prod_{j=1}^{\beta_r} (1 - \alpha_{r,j} t) \]

where \( \alpha_{r,j} \) are algebraic integers of absolute value \( q^{r/2} \).

Note that when \( X \) is a curve, \( n=1 \) and the Zeta function becomes
\[ \zeta(X, t) = \frac{P(t)}{(1-t)(1-qt)}, \]
where \( P(t) = \prod_{i=1}^{2g} (1 - \alpha_i t), \ g = \) genus of the curve, and \( \alpha_i \) are algebraic integers of absolute value \( q^{1/2} \). If we let \( t = q^{-s} \), then the statement that the roots of \( P(t) \) all have norm equal to \( q^{1/2} \) is equivalent to the Riemann hypothesis.

### 1.2.2 B.Dwork’s Work

In 1960 B.Dwork [4] proved that the zeta function of a variety was rational by p-adic analysis. He showed that

- the zeta function is meromorphic in \( \mathbb{C}_p \) i.e., a quotient of two entire power series;
- the zeta function has a non-zero radius of convergence in the complex plane.

Then he concluded that the zeta function is a rational function.

### 1.2.3 Cohomological Interpretation

Weil pointed out that given a suitable “Weil cohomology theory” \( H^*(X) \), we have
\[ \zeta(X, t) = \prod_{i=0}^{2d} \det (I - TF|H^i(X))^{(-1)^{i+1}}, \]
where \( F : X \to X \) is the qth power Frobenius.

When \( X \) is a projective smooth variety, we have a "Weil cohomology" \( H^*(X) \).

When \( X/\mathbb{F}_q \) is smooth and proper, there are several such cohomologies:

- l-adic étale cohomology (Grothendieck)
- Crystalline cohomology (Berthelot)
When $X/\mathbb{F}_q$ is separable and of finite type, we have

$$\zeta(X, t) = \prod_{i=0}^{2d} \det \left( I - TF|H^i_c(X) \right)^{(-1)^i+1},$$

where $H^i_c(X)$ is a cohomology with compact support, and

- $l$-adic cohomology with support
- rigid cohomology with support

### 1.2.4 Zeta Functions and the L-Functions

In order to generalize the zeta functions, we need, for each algebraic variety $X$, to define a category of coefficients $E$ on $X$ and associate to $E$ some number $S(E, x)$ at each closed point $x \in X$. Then define

$$L(X, E, T) := \exp \left( \sum_{i=1}^{\infty} S_i(X, E) T^i \right),$$

where $X(\mathbb{F}_{q'})$ is the set of points of $X$ over $\mathbb{F}_{q'}$.

Given an $F$-isocrystal $(E, \nabla, \Phi)$ over $X/K$, we can define the L-function of the $F$-isocrystal as follows.

Define the trace function associated to the $F$-isocrystal by

$$S_i(X, E) = \sum_{x \in X(\mathbb{F}_{q'})} Tr(\Phi_x^{\deg x}).$$

where $\Phi_x$ is a Frobenius on $E_x = \text{inverse image of } E \text{ on } x$. Here $S(E, X) = Tr(\Phi_x^{\deg x})$.

Since for any matrix $\beta$ we have $\det(I - \beta t) = \exp(-\sum_{i=1}^{\infty} \frac{T^i}{i} Tr(\beta^i))$, we get

$$L(X, E, T) = \prod_{x \in X^0} \frac{1}{\det_K(1 - T^{\deg x} \Phi_x^{\deg x})},$$

where $X^0$ is the set of closed points of $X$.

By the Lefschetz Trace Formula, we have $S_i(X, E) = \sum (-1)^i Tr \Phi^{i} | H^i_c(X, E)$. Hence

$$L(X, E, T) = \prod_{i=0}^{2d} \det \left( I - T \Phi^i | H^i_c(X, E) \right)^{(-1)^i+1},$$

and the L-functions are rational. Note that when $E$ is rank one, the trace functions are exponential sums.
1.2.5 Exponential Sums Associated to an F-Isocrystal

Here are some examples of the exponential sums associated to an F-isocrystals.

Example 1.5. In [6], B.Dwork studied the exponential sums defined by

$$
\sum_{x \in \mathbb{F}_p^*} \zeta_p^{\Tr(x - \bar{a})},
$$

where $\bar{a} = a$ in characteristic zero, $\bar{a}$ is the image of $a$ in $\mathbb{F}_p$. Then the corresponding Frobenius is defined by $\alpha := \psi \circ F$, where $\psi$ is given by $\psi(\sum B_n X^n) = \sum B_{pn} X^n$ and

$$
F(x, t) = \exp \left( \frac{\pi (t + \frac{a^p}{t})}{\exp \left( \frac{\pi (tp + \frac{a}{tp})} \right)} \right),
$$

$\pi = (-p)^{1/p - 1}$.

Example 1.6. In [19] and [21], D.Wan and H.J Zhu studied the exponential sums given by

$$
S_i(f \otimes \mathbb{F}_p) = \sum_{x \in \mathbb{F}_p^*} \zeta_p^{\Tr(f(x) \otimes \mathbb{F}_p)},
$$

where $f \in (\mathbb{Z}_p \cap \mathbb{Q})[x]$ is of degree $d$. For $1 \leq i \leq d$, let $a_i$ be the coefficients of $f$ with $a_d = 1$, $\bar{a}_i$ be the reduction of $a_i$ at $p$, and $\bar{a}_i$ be the Teichmüller lifting of $\bar{a}_i$. Let $\theta(x) = AH(\pi_0 x)$, where $AH(x)$ is the Artin-Hasse exponential function and $\pi_0$ is a root of $\log AH(x)$ with $\text{ord}_p(\pi_0) = \frac{1}{p - 1}$. Then the corresponding Frobenius $\alpha$ is defined by $\alpha = \psi \circ G(X)$, where $\psi$ is given by $\psi(\sum B_n X^n) = \sum B_{pn} X^n$, and $G(X) = \prod_{i=1}^d \theta(\bar{a}_i X^i)$.

Example 1.7. S. Sperber treated the Kloosterman exponential sums over a finite field of $q = p^a$ elements in [16]. The Kloosterman exponential sums are defined by

$$
S_m(f_a) = \sum \psi_m(x_1 + \cdots + x_n + \frac{a}{x_1 \cdots x_n}),
$$

where $a \in \mathbb{F}_q$, $\psi_m : \mathbb{F}_q^m \to \mathbb{C}^*$ is an additive character, and the sum ranges over all elements $(x_1, \cdots, x_n) \in (\mathbb{F}_q^*)^n$. Then the corresponding Frobenius $\bar{\alpha}_x$ is defined, on the chain level, by $\alpha_x = \psi \circ F(x, t)$, where $F(x, t) = \hat{F}(x, t) / \hat{F}(x^p, t^p)$, $\hat{F}(x, t) = \hat{F}(x, t)$.
\[\exp\left(\pi(t_1 + \cdots + t_n + x/t_1 t_2 \cdots t_n)\right)\] and \(\psi\) is linear and defined on monomials by
\[\psi(t^{\alpha}) = \begin{cases} 
\frac{t^{\alpha}}{p} & \text{if } p \mid \alpha_i \text{ for all } i, \\
0 & \text{otherwise}
\end{cases}\]

1.2.6 Unit-Root F-Isocrystals

**Definition 2.** A unit-root F-crystal on \(\mathbb{A}^1\) is a triple \((M, \nabla, \Phi)\) where

- \(M\) is a locally free \(V\{T\}\)-module, where \(V\{T\}\) is a p-adic completion of \(V[T]\).
- \(\nabla : M \to M \otimes \Omega^1\) is an integrable connection, i.e., \(\nabla\) is an additive map satisfying the Leibnitz rule: \(\nabla(fs) = f\nabla(s) + s \otimes df\), and the curvature \(\nabla^2\) is zero.
- \(\Phi : M \to \phi^* M\) is a \(\phi\)-linear isomorphism, i.e., \(\Phi(fg) = \phi(f)\Phi(g)\) if \(f \in V\{T\}\) and \(g \in M\).
- \(\Phi\) is horizontal, i.e., \(\nabla\) and \(\Phi\) commute.

A unit-root F-isocrystal is \(M_0 \otimes K\), where \(M_0\) is a unit-root F-crystal.

Let \(K^\sigma\) be the subset of \(K\) fixed by \(\sigma\), and \(\text{Rep}_{K^\sigma}(\pi_1(\mathbb{A}^1))\) be the category of continuous representations of \(\pi_1(\mathbb{A}^1)\) in finite dimensional \(K^\sigma\)-vector spaces.

**Theorem 1.8.** (R.Crew, N.Tsuzuki)[2, 17] There is an equivalence of categories

\[
G : \text{Rep}_{K^\sigma}(\pi_1(\mathbb{A}^1)) \simeq (\text{Unit-root F-isocrystals on } \mathbb{A}^1).
\]

In fact this theorem is true for any smooth curve.

1.2.7 L-Functions of the Exponential Sums

For every \(l \geq 1\), let \(S_l\) be exponential sums. Then the L-function of the exponential sum is defined by
\[
L(T) := \exp\left(\sum_{l=1}^{\infty} S_l \frac{T^l}{l}\right).
\]

It is well-known that L-functions are polynomials. So we have
\[
\exp\left(\sum_{l=1}^{\infty} S_l \frac{T^l}{l}\right) = \prod_{i=1}^{n}(1 - a_i T)
\]
for some $n$. By logarithmic differentiation we have $S_i = -a'_1 - \cdots - a'_n$. So understanding the exponential sums $S_i$ is reduced to understanding the reciprocal zeros of the $L$-function.

In this work, we investigate certain kinds of exponential sums introduced by A.Pulita in [12] by studying the corresponding $L$-functions. We compute an upper bound of the degree of the $L$-function using the formula by P.Robba [11, 13]. We then compute the $p$-adic absolute values of the zeros of the $L$-function using the Newton polygon of the $L$-function.
CHAPTER 2
ANDREA PULITA’S F-ISOCRYSTALS

Now we are going to review certain kinds of unit-root F-isocrystals on $\mathbb{A}^1$ that were introduced by A. Pulita in [12], and the corresponding exponential sums and the L-functions.

2.1 $\pi$-Exponentials

The Artin-Hasse exponential is defined by

$$AH(T) := \exp \left( T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots \right).$$

It is well-known that $AH(T)$ converges when $\text{ord}_p(T) > 0$ (see [8]).

Let $P(X) = pX + X^p$. Let $\tilde{\pi}_0 \in \mathbb{C}_p$ be a root of $P(X) = 0$, and let $\tilde{\pi}_i \in \mathbb{C}_p$ be such that $P(\tilde{\pi}_i) = \tilde{\pi}_{i-1}$ for $i \geq 1$. Define

$$E_m(T) := \exp \left( \tilde{\pi}_m T + \frac{\tilde{\pi}_{m-1} T^p}{p} + \cdots + \frac{\tilde{\pi}_0 T^{p^m}}{p^m} \right).$$

Pulita showed in [12] that the radius of convergence of $E_m(T)$ is 0, that is, $E_m(T)$ converges when $\text{ord}_p(T) > 0$.

Let $B$ be a $\mathbb{Z}_p[\tilde{\pi}_m]$-algebra. Let $\lambda = (\lambda_0, \ldots, \lambda_m) \in W_m(B)$, and let $\langle \phi_0, \ldots, \phi_m \rangle \in B^{m+1}$ be its phantom vector. Fix $n, m, d \in \mathbb{N}$ such that $d = np^m > 0$ and $(n, p)=1.$

**Definition 3.** The $\pi$-exponential attached to $\lambda$, $e_d(\lambda, T) \in 1 + \tilde{\pi}_m TB[[T]],$ is defined to be

$$e_d(\lambda, T) := \exp \left( \tilde{\pi}_m \phi_0 T^n + \frac{\tilde{\pi}_{m-1} \phi_1 T^{np}}{p} + \cdots + \frac{\tilde{\pi}_0 \phi_m T^{d}}{p^m} \right).$$

Here are some theorems by Pulita about $\pi$-exponentials (see [12]).

**Theorem 2.1.** For all $\lambda \in W_m(B)$, we have

$$e_d(\lambda, T) = \prod_{j=0}^{m} E_{m-j}(\lambda_j T^{np^j}).$$
Theorem 2.2. The $\pi$-exponential $e_d(\lambda, T)$ is over-convergent (i.e., convergent for $\text{ord}_p(T) > -\varepsilon$, with $\varepsilon > 0$) if and only if $\text{ord}_p(\lambda_i) > 0$ for all $0 \leq i \leq m$.

Theorem 2.3. Set

$$\theta_m(T) = \frac{E_m(T)}{E_m(T^p)}.$$ 

Then, $\theta_m(T)$ is over-convergent for all $m \geq 0$.

### 2.2 Pulita’s F-Isocrystals

Let $K$ be a complete valued field containing $\mathbb{Q}_p$. Set $K_s := K(\pi_s)$ and let $k_s$ be its residue field.

Let $f(T) \in W_s(T \mathcal{O}_{K_s}[T])$ and let $\overline{f}(t) \in W_s(t \mathcal{O}_{k_s}[t])$ be its reduction. A. Pulita proved the following theorems in [12].

Theorem 2.4. Let

$$L := \partial_T - \partial_{T, \log}(e_p(f(T), 1)),$$

where $\partial_T = T \frac{d}{dT}$. Then, $e_p(f(T), 1)$ is a solution of $Lu = 0$.

Theorem 2.5. If $\overline{f}_f(T) \in W_s(T \mathcal{O}_{K_s}[T])$ is an arbitrary lifting of $\overline{f}(f(t))$, then

$$\frac{e_p(f(T), 1)}{e_p(\overline{f}_f(T), 1)}$$

is over-convergent

Note that when $m = 0$, this is the Dwork exponential $\exp(\pi_0(f - f^p))$. This theorem implies the following.

Let

- $f(T) \in W_m(T \mathcal{O}_{K_m}[T])$;
- $M = \mathcal{R}_K$;
- $\nabla = \partial_T + \partial_{T, \log}(e^{p_m}(f(T), 1))$ where $\partial_T = T \frac{d}{dT}$ and $\partial_{T, \log}(f) = \frac{\partial_T(f)}{f}$;
- $\Phi_f : M \to M$ is given by $\Phi_f(x) = \frac{e^{p_m}(f(T), 1)}{e^{p_m}(\overline{f}(T), 1)} \phi(x)$. 

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Then \((M, \nabla, \Phi_T)\) is a unit-root F-isocrystal on \(\mathbb{A}^1\). Note that it’s the most general unit-root F-isocrystal on \(\mathbb{A}^1\) corresponding to a character of \(\pi_1(\text{Spec } k((t^{-1})))\), absolute Galois group of \(k((t^{-1}))\), of \(p\)-power order.

### 2.3 Exponential Sums and F-Isocrystals

Let \(f(X) = (f_0(X), \cdots, f_m(X)) \in W_m(\mathbb{F}_p[X])\). Let \(q = p^i\). Define \(\text{Tr}_{W_m(\mathbb{F}_q)/W_m(\mathbb{F}_p)} : W_m(\mathbb{F}_q) \to W_m(\mathbb{F}_p)\) by

\[
\text{Tr}_{W_m(\mathbb{F}_q)/W_m(\mathbb{F}_p)}(\alpha_0, \cdots, \alpha_m) = \sum_{\sigma \in \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)} (\sigma(\alpha_0), \cdots, \sigma(\alpha_m)) = \sum_{i=0}^{l-1} (\alpha_0^{\sigma^i}, \cdots, \alpha_m^{\sigma^i}).
\]

Elements in \(W_m(\mathbb{F}_p)\) can be identified with elements in \(\mathbb{Z}_p/\mathbb{Z}_p^{m+1}\) via the map

\[
(\alpha_0, \cdots, \alpha_m) \mapsto \sum_{i=0}^{m} \hat{\alpha}_i^{p^{-i}} p^i.
\]

So define \(\chi_i : W_m(\mathbb{F}_q) \to \mathbb{C}_p\) by

\[
\chi_i((\alpha_0, \cdots, \alpha_m)) = \zeta_p^{\sum_{i=0}^{l-1} \sum_{d=0}^{m} (\hat{\alpha}_j^{p^d}) d^i p^i},
\]

where \(\zeta_p^{m+1}\) is a primitive \(p^{m+1}\)-th root of unity. Then \(\chi_i\) is an additive character. Since \(\sum_{i=0}^{l-1} (\alpha_0^{p^d}, \cdots, \alpha_m^{p^d}) = \sum_{i=0}^{l-1} (\alpha_0^{p^d m^{(i)}}, \cdots, \alpha_m^{p^d m^{(i)}})\) and \(\hat{\alpha}_j^{p^d} = \hat{\alpha}_j\) get

\[
\chi_i((\alpha_0, \cdots, \alpha_m)) = \zeta_p^{\sum_{i=0}^{l-1} \hat{\alpha}_0^{p^d} + \hat{\alpha}_1^{p^{d+1}} + \cdots + \hat{\alpha}_m^{p^d m}}.
\]

Define the exponential sums of \(f\) by

\[
S_i((f_0, \cdots, f_m)) = \sum_{x \in \mathbb{F}_p^i} \chi_i((f_0(x), \cdots, f_m(x))).
\]

My goal is to understand these exponential sums for all \(m \geq 0\).

The \(L\)-function of the exponential sums is defined by

\[
L((f_0, \cdots, f_m); T) = \exp \left( \sum_{i=1}^{\infty} S_i((f_0, \cdots, f_m)) \frac{T^i}{i} \right).
\]

This \(L\)-function is the \(L\)-function of Pulita’s F-isocrystal.
CHAPTER 3
DEGREES OF THE L-FUNCTIONS

In this chapter, we compute the degrees of the L-functions using the formula by

P. Robba [11, 13].

Let $L : V \to V$ be a linear map. When kernel and cokernel are finite, then the index of $L$ on $V$ is defined to be

$$\chi(L, V) = \dim \ker L - \dim \text{coker} L.$$

Let $k$ be an algebraically closed field of characteristics 0. Let $A = B(0, 1^+)$ and $A_{\varepsilon} = B(0, (1 + \varepsilon)^+)$. Define

$$\mathcal{H}^\dagger(A) := \bigcup_{\varepsilon > 0} H(A_{\varepsilon}),$$

where $H(A_{\varepsilon})$ is the set of analytic elements of $A_{\varepsilon}$ with coefficients in $k$.

**Theorem 3.1.** (P. Robba)[13] Let $t$ be a generic point on the circumference $C(0, r)$ with $r > 1$. Let $x = t + y$. Let $D$ be a differential operator of first order, and $u$ be a solution of $D = 0$ near $t$. Suppose that the radius of convergence of $u$ is $1/r^N$. Then, $-\chi(D, \mathcal{H}^\dagger(A)) = N$.

**Theorem 3.2.** (P. Robba)[13] Define

$$D := T \frac{d}{dT} - \frac{T \frac{d}{dT} F}{F};$$

and

$$\alpha := F^{-1} \circ \psi_p \circ F.$$

Then the $L$-function given by $L(f; T) = \det (1 - T\alpha) / \det (1 - tp\alpha)$ has a degree $-\chi(D, \mathcal{H}^\dagger(A))$.

Let $D = \partial_T - \partial_{T, \log}(e_p^\gamma(\mathcal{f}(T), 1))$. Then Theorem 3.1 and Theorem 3.2 implies that the $L$-function of Pulita’s F-isocrystal $L((f_0, \cdots, f_m); T)$ has degree $-\chi(D, \mathcal{H}^\dagger(A)) = N$.

**3.1 m=1 Case**

Let $K$ be a field and $\mathcal{O}$ be the ring of integers of $K$. Let $(f, g)$ be an element of $W_1(T \mathcal{O}[T])$ such that $\deg(f) = d$ and $\deg(g) = e$. Assume that $p \nmid d$ and $p \nmid e$. 

Write \( f(T) = \sum_{i=1}^{d} a_i T^i \) and \( g(T) = \sum_{i=1}^{e} b_i T^i \), and assume that \( a_d = 1 \) and \( b_e = 1 \).

**Lemma 1.** Suppose that \( p \nmid n \). Then \( \exp(\pi_0 (t^n - (t + y)^n)) \) converges if and only if \( |y| < 1/r^{n-1} \), where \( r = |t + y| > 1 \).

**Proof.** Let \( h(t) = t^n \). Then using the Taylor series we obtain

\[
t^n - (t + y)^n = h'(t + y)(-y) + \frac{h''(t + y)}{2}(-y)^2 + \cdots
\]

\[
= (t + y)^{n-1}y\left(-n + \frac{n(n-1)(y)}{2(t + y)} + \cdots\right).
\]

Since \( \exp(\pi_0 (t^n - (t + y)^n)) \) converges if and only if \( |t^n - (t + y)^n| < 1 \), it converges if and only if \( |y| < \frac{1}{r^{n-1}|K|} \), where \( K = (-n + \frac{n(n-1)(y)}{2(t + y)} + \cdots) \). Note that \( |n| = 1 \) as \( p \nmid n \).

Since \( |K| \leq \sup \{|-n|, \frac{n(n-1)(t-t_0)}{2t_0}, \cdots\} \), when \( |y| < |t + y| \) we have \( |K| = 1 \). Thus \( \exp(\pi_0 (t^n - (t + y)^n)) \) converges if and only if \( |y| < 1/r^{n-1} \).

**Lemma 2.** \( \exp(\pi_1 (t - (t + y)) + \frac{\pi_0}{p}(t^p - (t + y)^p)) \) converges for \( |y| < 1/r^{p-1} \).

**Proof.** First suppose that \( p=2 \). Then,

\[
\exp \left( \pi_1 (t - (t + y)) + \frac{\pi_0}{2}(t^2 - (t + y)^2) \right)
= \exp \left( \pi_1 (-y) + \frac{\pi_0}{2}(-y)^2 \right) \exp \left( -\frac{\pi_0}{2}(2ty - 2y^2) \right).
\]

Note that

\[
\exp \left( \pi_1 (-y) + \frac{\pi_0}{2}(-y)^2 \right) = E_1(-y)
\]

converges if and only if \( |y| < 1 \), and

\[
\exp \left( -\frac{\pi_0}{2}(2ty - 2y^2) \right) = \exp(-\pi_0 y(t + y))
\]

converges if and only if \( |y(t + y)| < 1 \), i.e., \( |y| < 1/r \). So the lemma follows when \( p=2 \).

Now, suppose that \( p \neq 2 \). Let \( x = t + y \). Then,
\[
\exp \left( \pi_1(t - x) + \frac{\pi_0}{p} (t^p - x^p) \right) \\
= \exp \left( \pi_1(t - x) + \frac{\pi_0}{p} (t - x)^p - \frac{\pi_0}{p} \sum_{i=1}^{p-1} \left( \frac{p}{i} \right) t^{p-i}(-x)^i \right).
\]

Note that
\[
\exp \left( \pi_1(t - x) + \frac{\pi_0}{p} (t - x)^p \right) = E_1(-y)
\]
converges if and only if \(|y| < 1|.

Now consider
\[
\exp \left( -\frac{\pi_0}{p} \sum_{i=1}^{p-1} \left( \frac{p}{i} \right) t^{p-i}(-x)^i \right).
\]

Note that
\[
\sum_{i=1}^{p-1} \frac{-1}{p} \left( \frac{p}{i} \right) t^{p-i}(-x)^i = \sum_{i=1}^{p-1} (-1)^{i+1} \left( \frac{p}{i} \right) (t^{p-i}x^i - t^i x^{p-i}) \\
= \sum_{i=1}^{p-1} c_i (tx)^i (t^{p-2i} - x^{p-2i}),
\]
where \(|c_i| = 1|.

For each \(1 \leq i \leq \frac{p-1}{2}|, \exp (\pi_0 c_i (tx)^i (t^{p-2i} - x^{p-2i}) \) converges if and only if
\[
|c_i (tx)^i (t^{p-2i} - x^{p-2i})| < 1
\]
if and only if
\[
|t^{p-2i} - x^{p-2i}| < |tx|^{-i}.
\]
We have \(|x| = |t + y| \leq \sup \{|t|, |y|\} \) with equality if and only if \(|t| \neq |y|\|. As \(t\) is generic, we get \(|x| = |t|\). So,
\[
|t^{p-2i} - x^{p-2i}| < 1/|x|^{2i}.
\]
Hence, by the proof of Lemma 1,
\[ \exp \left( -\frac{\pi_0}{p} \sum_{i=1}^{p-1} \left( \frac{p}{i} \right) t^{p-i} x^i \right) \]
converges for \(|y| < 1/r^{p-1}|. \]

**Lemma 3.** \( \exp \left( \frac{\pi_1}{p} (t^n - (t + y)^n) + \frac{\pi_0}{p} (t^{p^n} - (t + y)^{p^n}) \right) \) converges for \(|y| < 1/r^{p^n-1}|. \)

**Proof.** Let \(x = t + y|.

Suppose that \(p = 2|. Then,
\[
\exp \left( \frac{\pi_1}{2} (t^n - (t + y)^n) + \frac{\pi_0}{2} (t^{2^n} - (t + y)^{2^n}) \right) 
= \exp \left( \frac{\pi_1}{2} (t^n - x^n) + \frac{\pi_0}{2} (t^{2^n} - x^{2^n}) \right) 
= \exp \left( \frac{\pi_1}{2} (t^n - x^n) + \frac{\pi_0}{2} (t^n - x^n)^2 \right) \exp \left( \pi_0 (t^n x^n - x^{2^n}) \right) 
= E_1(t^n - x^n)E_0(t^n x^n - x^{2^n}). \] (3–1)

\(E_1(t^n - x^n)\) converges if and only if \(|t^n - (t + y)^n| = |t^n - x^n| < 1|. By the proof of Lemma 1, \(E_1(t^n - x^n)\) converges if and only if \(|y| < 1/r^{n-1}|.

\(E_0(t^n x^n - x^{2^n})\) converges if and only if \(|(t + y)^n||t^n - (t + y)^n| = |t^n x^n - x^{2^n}| < 1|. Hence, by the proof of Lemma 1, \(E_0(t^n x^n - x^{2^n})\) converges if and only if \(|y| < 1/r^{2^n-1}|.

Thus, Equation 3–1 converges if and only if \(|y| < 1/r^{2^n-1}, and the lemma is true when \(p = 2|.

Now, suppose that \(p \neq 2|. Then,
\[
\exp \left( \frac{\pi_1}{p} (t^n - (t + y)^n) + \frac{\pi_0}{p} (t^{p^n} - (t + y)^{p^n}) \right) 
= \exp \left( \frac{\pi_1}{p} (t^n - x^n) + \frac{\pi_0}{p} (t^n - x^n)^p - \frac{\pi_0}{p} \sum_{i=1}^{p-1} \left( \frac{p}{i} \right) t^{n(p-i)} (-x)^{ni} \right) 
= E_1(t^n - x^n) \exp \left( -\frac{\pi_0}{p} \sum_{i=1}^{p-1} \left( \frac{p}{i} \right) t^{n(p-i)} (-x)^{ni} \right). \] (3–2)

\(E_1(t^n - x^n)\) converges if and only if \(|t^n - (t + y)^n| = |t^n - x^n| < 1| if and only if \(|y| < 1/r^{n-1}|.

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By the proof of Lemma 2, we have

\[
\exp \left( - \frac{\pi_0}{p} \sum_{i=1}^{p-1} \binom{p}{i} t^{n(p-i)} (-x)^n i \right) = \sum_{i=1}^{p-1} \exp \left( \pi_0 c_i (t^n x^n)^i (t^n (p-2i) - x^n (p-2i)) \right),
\]

where \(|c_i| = 1\). For each \(1 \leq i \leq \frac{p-1}{2}\), \(\exp \left( \pi_0 c_i (t^n x^n)^i (t^n (p-2i) - x^n (p-2i)) \right)\) converges if and only if \(|c_i (t^n x^n)^i (t^n (p-2i) - x^n (p-2i))| < 1\). Hence, by the same argument as the one in Lemma 2,

\[
\sum_{i=1}^{\frac{p-1}{2}} \exp \left( \pi_0 c_i (t^n x^n)^i (t^n (p-2i) - x^n (p-2i)) \right)
\]

converges when \(|y| < 1/r^{p-1}\). Thus, Equation 3–2 converges when \(|y| < 1/r^{p-1}\) and this proves the lemma.

\[\square\]

**Example 3.3.** Let \(p=5\). Then

\[
\exp \left( \pi_1 (t - (t + y)) + \frac{\pi_0}{5} (t^5 - (t + y)^5) \right)
\]

\[\begin{align*}
= & \exp \left( \pi_1 (-y) + \frac{\pi_0}{5} (-y)^5 - \frac{\pi_0}{5} \sum_{i=1}^{4} \binom{5}{i} t^{p-i} (-y)^i \right) \\
= & \exp \left( \pi_1 (-y) + \frac{\pi_0}{5} (-y)^5 \right)
\end{align*}\]

converges when \(|y| < 1/r\).

\[
\exp \left( - \frac{\pi_0}{5} \sum_{i=1}^{4} \binom{5}{i} t^{p-i} (-y)^i \right)
\]

\[\begin{align*}
= & \exp \left( \pi_0 (t^4 (t + y) - t(t + y)^4) - 2\pi_0 (t^3 (t + y)^2 - t^2 (t + y)^3) \right) \\
= & \exp \left( \pi_0 t(t + y)(t^3 - (t + y)^3) + 2\pi_0 t^2 (t + y)^2 (t - (t + y)) \right),
\end{align*}\]

and \(\exp \left( \pi_0 t(t + y)(t^3 - (t + y)^3) \right)\) converges when

\[|t^3 - (t + y)^3| < 1/|t(t + y)| = 1/r^2,
\]

i.e., when \(|y| < |t_0|^4\), and \(\exp \left( 2\pi_0 t^2 (t + y)^2 (t - (t + y)) \right)\) converges when

\[|y| < 1/|t(t + y)|^2,
\]

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i.e., when $|y| < 1/r^A$.

Lemma 4.

$$e_p((f(T), g(T), 1) = \prod_{i=1}^{d} \exp(\pi_1 a_i T^i + \frac{\pi_0}{p} a_i^p T^{pi}) \prod_{i=1}^{\max\{\phi-1\cdot e\}} \exp(\pi_0 \tilde{b}_i T^i),$$

where $|\tilde{b}_i| \leq 1$.

Proof. Note that if $(a_0, a_1)$ and $(b_0, b_1)$ are elements of $W_2(K)$, then $(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 + \frac{a_0^p + b_0^p - (a_0 + b_0)^p}{p})$. So, $(f(T), g(T)) = (\sum_{i=1}^{d} a_i T^i, 0) + \sum_{i=1}^{\phi} (0, b_i T^i)$, and I will show that $(f(T), 0) = \sum_{i=1}^{d} (a_i T^i, 0) + \sum_{i=p+1}^{\phi-1} (0, c_i T^i)$ with $|c_i| \leq 1$.

We have

$$(a_1 T + a_2 T^2, 0) = (a_1 T, 0) + (a_2 T^2, 0) + (0, \sum_{i=1}^{\frac{\phi-1}{p}} a_1^{2p-2} a_2^{p-2} (a_1 T^i))$$

$$= (a_1 T, 0) + (a_2 T^2, 0) + \sum_{i=p+1}^{2p-1} (0, \tilde{a}_i T^i),$$

where $|\tilde{a}_i| = \left|\frac{a_1^{p-2} a_2^{p-2}}{2}\right| \leq 1$.

If we have

$$(a_1 T + \cdots + a_{d-1} T^{d-1}, 0) = \sum_{i=1}^{d-1} (a_i T^i, 0) + \sum_{i=p+1}^{2(d-1)-1} (0, \tilde{a}_i T^i),$$

with $|\tilde{a}_i| \leq 1$, then we have

$$(f(T), 0) = (a_1 T + \cdots + a_{d-1} T^{d-1}, 0) + (a_d T^d, 0)$$

$$+(0, \sum_{i=1}^{\frac{\phi-1}{p}} (a_1 T + \cdots + a_{d-1} T^{d-1})^{\phi-2} (a_d T^d)^{\phi-2-i})$$

$$= \sum_{i=1}^{d} (a_i T^i, 0) + \sum_{i=p+1}^{2(d-1)-1} (0, \tilde{a}_i T^i) + \sum_{i=p+d+1}^{\phi-1} (0, \tilde{a}_i T^i)$$

$$= \sum_{i=1}^{d} (a_i T^i, 0) + \sum_{i=p+1}^{d\phi-1} (0, c_i T^i),$$

where $|c_i| \leq 1$. 
So
\[ (f(T), g(T)) = \sum_{i=1}^{d} (a_i T^i, 0) + \sum_{i=1}^{\max \{dp-1, e\}} (0, \tilde{b}_i T^i) \]
and hence
\[ e_p((f(T), g(T)), 1) = \prod_{i=1}^{d} \exp(\pi_1 a_i T^i + \frac{\pi_0}{p} a_i^p T^{pi}) \prod_{i=p+1}^{\max \{dp-1, e\}} \exp(\pi_0 \tilde{b}_i T^i). \]

**Theorem 3.4.** Let \( x = t + y \). Then
\[
\prod_{i=1}^{d} \exp(\pi_1 a_i T^i - x^i) + \frac{\pi_0}{p} a_i^p (T^{pi} - x^{pi}) \prod_{i=p+1}^{\max \{dp-1, e\}} \exp(\pi_0 \tilde{b}_i (T^i - x^i)) \quad (3–3)
\]
converges when \(|y| < 1/r^{\max \{dp-1, e-1\}}\).

Equation 3–3 can be written as
\[
\frac{e_p((f(t), g(t)), 1)}{e_p((f(t+y), g(t+y)), 1)},
\]
which is a solution of \( Lu = 0 \) near \( t \), where
\[
L := \partial_T - \partial_{T,kg}(e_p(f(T), 1)).
\]

Thus, using Theorem 3.1 and Theorem 3.2 by Robba, the degree of the L-function of Pulita’s F-isocrystal when \( m=1 \) is less than or equal to \( \max \{dp-1, e-1\} \), where \( d = \deg(f) \) and \( e = \deg(g) \).

**3.2 General Case**

**Lemma 5.** Let \( R \) be a ring and \((a_0, 0, \cdots, 0), \cdots, (0, \cdots, 0, a_m) \in W_m(R)\). Then
\[
(a_0, 0, \cdots, 0) + \cdots + (0, \cdots, 0, a_m) = (a_0, \cdots, a_m).
\]

**Proof.** If \( \alpha \) and \( \beta \) are in \( W_m(R) \), then \( \alpha + \beta = (S_0, S_1, \cdots, S_m) \), where \( S_0 = \alpha_0 + \beta_0 \), and for \( k \geq 1 \),
\[
S_k = \alpha_k + \beta_k + \alpha_{k-1}^p + \beta_{k-1}^p - S_{k-1}^p + \cdots + \frac{\alpha_0^p + \beta_0^p - S_0^p}{p^k}.
\]
So, if \( \alpha \) and \( \beta \) are such that \( \beta_i = 0 \) whenever \( \alpha_i \neq 0 \), and \( \alpha_i = 0 \) whenever \( \beta_i \neq 0 \), then
\[
\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_m + \beta_m).
\]
Thus,
\[
(a_0, 0, \ldots, 0) + \cdots + (0, \ldots, 0, a_m) = (a_0, \ldots, a_m).
\]

\[\square\]

**Lemma 6.** Let \( f(T) \) be an element of \( TO[T] \) of degree \( d \geq 2 \), and let \( (f, 0, \cdots, 0) \) be an element of \( W_m(TO[T]) \). Then
\[
e_p^m((f, 0, \cdots, 0), 1) = \prod_{i=1}^{d} E_m(\tilde{\alpha}_0, T^i) \prod_{i=1}^{d-1} E_{m-1}(\tilde{\alpha}_1, T^i) \cdots \prod_{i=1}^{d^m-1} E_0(\tilde{\alpha}_m T^i),
\]
where \(|\tilde{\alpha}_{ji}| \leq 1\).

**Proof.** Since \( e_p^m(\lambda + \mu, T) = e_p^m(\lambda, T)e_p^m(\mu, T) \) and \( e_p^m(\lambda, T) = \prod_{j=0}^{m} E_{m-j}(\lambda_j T^{p^j}) \), it suffices to show that
\[
(f, 0, \cdots, 0) = \sum_{i=1}^{d} (\tilde{\alpha}_0, T^i, 0, \cdots, 0) + \sum_{i=1}^{d-1} (0, \tilde{\alpha}_1, T^i, 0, \cdots, 0) + \cdots + \sum_{i=1}^{d^{m-1}} (0, \cdots, 0, \tilde{\alpha}_m T^i).
\]

Note that if \((a, 0, \cdots, 0)\) and \((b, 0, \cdots, 0)\) are in \( W_m(K) \), then \((a, 0, \cdots, 0) + (b, 0, \cdots, 0) = (S_0, \cdots, S_m)\), where \( S_0 = a + b, S_1 = \frac{a^p + b^p - S_0^p}{p} \), and
\[
S_k = -\frac{S_{k-1}^p}{p} - \frac{S_{k-2}^p}{p^2} - \cdots - \frac{S_{k-1}^p}{p^{k-1}} + \frac{a^{p^k} + b^{p^k} - S_0^{p^k}}{p^k}.
\]

Hence if \( a = g(T) \) and \( b = h(T) \) with \( \deg g(T) = i < j = \deg h(T) \), then
\[
(g(T) + h(T), 0, \cdots, 0) = (g(T), 0, \cdots, 0) + (h(T), 0, \cdots, 0) + (0, -S_1, -S_2, \cdots, -S_m),
\]
and the degree of \( S_k \) is \( jp^k + (i - j) \).

We will prove this lemma by induction on \( m \) and \( d \). The lemma is clearly true when \( m = 0 \) for all \( d \). So assume that the lemma is true for the case \( m - 1 \) for all \( d \), and consider the case \( m \). Let \( f(T) = a_1 T + a_2 T^2 \). Then
\[
(a_1 T + a_2 T^2, 0, \cdots, 0) = (a_1 T, 0, \cdots, 0) + (a_2 T^2, 0, \cdots, 0) + (0, -S_1, -S_2, \cdots, -S_m),
\]
where \( \deg S_k = 2^p - 1 \).

\[
(0, -S_1, -S_2, \ldots, -S_m) = (0, -S_1, 0, \ldots, 0) + \cdots + (0, \ldots, 0, -S_m),
\]

and by induction we have

\[
(0, \ldots, 0, -S_k, 0, \ldots, 0) = \sum_{i=1}^{2^{p+1}-1} (0, \ldots, 0, b_{ki} T^i, 0, \ldots, 0) \\
+ \sum_{i=1}^{2^{p+1} - 2^{p} - 1} (0, \ldots, 0, b_{k+1,i} T^i, 0, \ldots, 0) \\
+ \cdots + \sum_{i=1}^{2^{p+m} - 2^{p} - 1} (0, \ldots, 0, b_{m,i} T^i).
\]

Thus the lemma is true when \( d = 2 \). Suppose that the lemma is true for the case \( d - 1 \).

Let \( f(T) = a_1 T + \cdots + a_{d-1} T^{d-1} + a_d T^d \). Then,

\[
(f(T), 0, \ldots, 0) = (a_1 T + \cdots + 1_{d-1} T^{d-1}, 0, \ldots, 0) \\
+ (a_d T^d, 0, \ldots, 0) + (0, -S_1, -S_2, \ldots, -S_m),
\]

where \( \deg S_k = dp^k - 1 \). Using induction and a power of the Verschieburg morphism \( V \), we have

\[
(0, \ldots, 0, -S_k, 0, \ldots, 0) = \sum_{i=1}^{dp^k - 1} (0, \ldots, 0, b_{ki} T^i, 0, \ldots, 0) \\
+ \sum_{i=1}^{dp^k + 1 - p - 1} (0, \ldots, 0, b_{k+1,i} T^i, 0, \ldots, 0) \\
+ \cdots + \sum_{i=1}^{dp^{m} - p^{m} - 1} (0, \ldots, 0, b_{m,i} T^i)
\]
and
\[
(a_1 T + \cdots + a_{d-1} T^{d-1}, 0, \ldots, 0) = \sum_{i=1}^{d-1} (\tilde{a}_0 T^i, 0, \ldots, 0) \\
+ \sum_{i=1}^{(d-1)p-1} (0, \tilde{a}_1 T^i, 0, \ldots, 0) \\
+ \cdots + \sum_{i=1}^{(d-1)p^{m-1}-1} (0, \ldots, 0, \tilde{a}_m T^i).
\]

Hence
\[
(f, 0, \ldots, 0) = \sum_{i=1}^{d} (\tilde{a}_0 T^i, 0, \ldots, 0) + \sum_{i=1}^{dp-1} (0, \tilde{a}_1 T^i, 0, \ldots, 0) + \cdots + \sum_{i=1}^{dp^{m-1}-1} (0, \ldots, 0, \tilde{a}_m T^i)
\]
and the lemma holds. \(\square\)

Let \((f_0, \ldots, f_m)\) be an element of \(W_m(\mathcal{O}[T])\) such that \(\deg(f_i) = d_i\). Write
\[
f_k(T) = \sum_{i=1}^{d_i} a_{ki} T^i.
\]

**Proposition 3.1.** Let \(\tilde{d}_k = \max\{d_k, d_{k-1} p - 1, d_{k-2} p^2 - 1 \cdots, d_0 p^k - 1\}\). Then
\[
e_{p^m}((f_0, \cdots, f_m), 1) = \prod_{i=1}^{\tilde{d}_0} E_m(\tilde{a}_0 T^i) \prod_{i=1}^{\tilde{d}_1} E_{m-1}(\tilde{a}_1 T^i) \cdots \prod_{i=1}^{\tilde{d}_m} E_0(\tilde{a}_m T^i).
\]

**Proof.** Since \((f_0, \ldots, f_m) = (f_0, 0, \ldots, 0) + \cdots + (0, \ldots, 0, f_m)\), by the previous lemma, the proposition follows. \(\square\)

**Lemma 7.**
\[
\exp(\pi_k(t - (t + y)) + \frac{\pi_{k-1}}{\rho} (t^p - (t + y)^p) + \cdots + \frac{\pi_0}{\rho^k} (t^{p^k} - (t + y)^{p^k}))
\]
converges for \(|y| < 1/r^{p^k-1}\).

**Proof.** Put \(x = t + y\). Let \((t, 0, \cdots)\) and \((-x, 0, \cdots)\) be elements of Witt vectors, and let \(S_i\) be such that \((t, 0, \cdots) + (-x, 0, \cdots) = (S_0, S_1, \cdots)\). So, we have \(S_0 = t - x\),
\[
S_1 = \frac{t^p - x^p - S_0^p}{\rho},
\]
and
\[
S_k = -\frac{S_{k-1}^p}{\rho} - \frac{S_{k-2}^p}{\rho^2} - \cdots - \frac{S_0^{p^k-1}}{\rho^{k-1}} + \frac{t^{p^k} - x^{p^k} - S_0^{p^k}}{\rho^k}.
\]}
Then,

$$E_k(S_0)E_{k-1}(S_1) \cdots E_0(S_k)$$

$$= \exp \left( \pi_k S_0 + \frac{\pi_{k-1} S_0^p}{p} + \cdots + \frac{\pi_0 S_0^{p^k}}{p^{k-1}} \right)$$

$$\exp \left( \pi_{k-1} S_1 + \frac{\pi_{k-2} S_1^p}{p} + \cdots + \frac{\pi_0 S_1^{p^{k-1}}}{p^{k-1}} \right) \cdots \exp \left( \pi_0 S_k \right)$$

$$= \exp \left( \pi_k S_0 + \frac{\pi_{k-1} S_0^p}{p} + \cdots + \frac{\pi_0 S_0^{p^k}}{p^{k-1}} \right)$$

$$\exp \left( \pi_{k-1} \left( \frac{t - x}{p} - \frac{S_0^p}{p} \right) + \frac{\pi_{k-2} S_1^p}{p} + \cdots + \frac{\pi_0 S_1^{p^{k-1}}}{p^{k-1}} \right)$$

$$\exp \left( \pi_{k-2} \left( -\frac{S_0^p}{p^2} + \frac{t^p - x^p}{p} \right) + \frac{\pi_{k-3} S_2^p}{p} + \cdots + \frac{\pi_0 S_2^{p^{k-2}}}{p^{k-1}} \right)$$

$$\cdots \exp \left( \pi_0 \left( -\frac{S_0^p}{p^k} + \frac{t^p - x^p - S_0^p}{p^k} \right) \right)$$

$$= \exp \left( \pi_k (t - x) + \frac{\pi_{k-1} (t^p - x^p)}{p} + \cdots + \frac{\pi_0 (t^p - x^p)}{p^k} \right).$$

As $E_{k-i}(S_i)$ converges when $|S_i| < 1$, it suffices to show that $|S_i| < 1$ when $|y| < 1/r^{d-1}$.

We have $|S_0| = | - y |$. So, $|y| < 1$ if and only if $|S_0| < 1$. For $i \geq 1$, we have $S_i = \sum_{j=1}^{d-j} c_j t^{d-j}(-x)^j$, where $c_j = c_{d-j}$ and $|c_j| \leq 1$. Hence,

$$|S_i| \leq \sum_{j=1}^{d-j} |c_j| |t^{d-j}x^j - t^i x^{d-j}|$$

$$= \sum_{j=1}^{d-j} |c_j| |tx^j| |(t^{d-2j} - x^{d-2j})|.$$  

By Lemma 1, $|t^{d-2j} - x^{d-2j}| < 1/r^{2j}$ if and only if $|y| < 1/r^{d-1}$. Hence if $|y| < 1/r^{d-1}$, then

$$|S_i| \leq \sum_{j=1}^{d-j} |c_j| |tx^j| |(t^{d-2j} - x^{d-2j})| < 1.$$
Theorem 3.5. Let $x = t + y$. Then

$$\prod_{i=1}^{\tilde{a}_0} \exp(\pi_m \tilde{a}_0(t^i - x^i) + \cdots + \frac{\pi_0}{p^m} \tilde{a}_{0i}^m(t^i - x^i)^m) \cdots \prod_{i=1}^{\tilde{a}_m} \exp(\pi_0 a_{mi}(t^i - x^i))$$

(3–5)

converges when $|y| < 1/r^\alpha$, where

$$\alpha = \max\{d_0 p^m - 1, d_1 p^{m-1} - 1, \ldots, d_{m-1} p - 1, d_m - 1\}.$$ 

Equation 3–5 is a solution of $Lu = 0$ near $t$, where

$$L := \partial_T - \partial_{T, \log}(e_p^m(\mathfrak{f}(T), 1)).$$

Thus, using Theorem 3.1 and Theorem 3.2 by Robba, the degree of the L-function of Pulita’s F-isocrystal when $m > 1$ is less than or equal to

$$\max\{d_0 p^m - 1, d_1 p^{m-1} - 1, \ldots, d_{m-1} p - 1, d_m - 1\},$$

where $d_i = \text{deg}(f_i)$. 

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CHAPTER 4
NEWTON POLYGONS OF THE L-FUNCTIONS

In this chapter, we study the exponential sums of Pulita’s F-isocrystals by studying the Newton polygons of the L-functions.

**Definition 4.** The Newton Polygon of \[ f(X) = a_0 + \cdots + a_nX^n + \cdots \] is the lower convex hull of the points \((j, \operatorname{ord}_pf_j)\) in \(\mathbb{R}^2\) for \(j \geq 0\).

**Theorem 4.1.** To each finite side of the Newton Polygon of \( f \) there correspond \( i \) zeros \( \alpha \) of \( f \) where \( i \) is the length of the horizontal projection of the side. If \( \lambda \) is the slope of the side, then \( \alpha = -\lambda \). Conversely if \( \alpha \) is a root then \( -\alpha \) is the slope of a side.

**Example 4.2.** Let \( f(X) = X + \frac{x^n}{p} + \frac{x^{n+1}}{p^2} + \cdots \). Then the slopes of its Newton polygon are \(-\frac{1}{p(p-1)}\) for \( j \geq 0 \). Thus \( f(X) \) has roots \( \pi_j \) of order \( \frac{1}{p(p-1)} \).

D.Wan and H.J.Zhu studied the exponential sums of Pulita’s F-isocrystals when \( m = 0 \) by studying the Newton polygons of the \( L \)-functions [19, 21]. Their result is the following.

**Theorem 4.3.** A lower bound of the Newton polygon of the \( L \)-function of Pulita’s F-isocrystal when \( m = 0 \) is the lower convex hull of points \((n, \frac{n(n+1)}{2d})\) for \( 0 \leq n \leq d - 1 \), i.e., the slopes are \( \frac{1}{d} \), \( \frac{2}{d} \), \( \cdots \), \( \frac{d-1}{d} \).

I will discuss about the exponential sums of Pulita’s F-isocrystals when \( m \geq 1 \) by studying the Newton polygon of the corresponding L-functions.

### 4.1 Replacing Pulita’s Frobenius

Let \((M, \nabla, \Phi)\) be a rank one F-isocrystal over \( \mathcal{R}_K \). Let \( \{u\} \) be a basis of \( M \), and let \( \Phi(u) = \Phi(T)\phi(u) \). Suppose that there are over-convergent \( \tilde{\Phi}(T) \) and \( g(T) \) such that

\[
\frac{\phi(T)}{\phi(G(T))} = \frac{g(T)}{\tilde{\phi}(G(T))}.
\]

Define \( \tilde{\Phi}(x) = \tilde{\Phi}(T)\phi(x) \). Then the map \( M \rightarrow M, u \mapsto g(T)u \), is an isomorphism which commutes with Frobenius. In fact,

\[
\Phi(g(T)u) = \phi(g(T))\Phi(u) = \phi(g(T))\Phi(T)\phi(u)
\]

\[
= \tilde{\Phi}(T)g(T)\phi(u) = g(T)\tilde{\Phi}(u).
\]

So we have an isomorphism \( (M, \nabla, \Phi) \simeq (M, \tilde{\nabla}, \tilde{\Phi}) \).
Note that the exponential sums of F-isocrystal $(M, \nabla, \Phi)$ is equal to the one of $(M, \nabla, \Phi)$. Since the larger the radius of convergence of Frobenius, the better the estimates for the Newton polygon, I’d like to replace the Pulita’s Frobenius by a new Frobenius with better radius of convergence.

**Theorem 4.4.** Let $\pi_m$ be a root of $\log AH(X)$ of order $\frac{1}{p^m(p-1)}$. For $m = 0, 1$, there exist over-convergent functions $g_m(T)$ such that

$$\frac{\theta_m(T)}{AH(\pi_m T)} = \frac{g_m(T)}{g_m(T^p)}.$$

**Proof.** We can write $AH(\pi_m T)$ as

$$AH(\pi_m T) = \frac{\exp (\pi_m T + (\pi_m + \frac{\pi_m}{p}) T^p + \cdots)}{\exp (\pi_m T^p + (\pi_m + \frac{\pi_m}{p}) T^{p^2} + \cdots)}.$$

Since $\pi_m$ is a root of $\sum \frac{x^j}{p^j} = 0$ of order $1/p^m(p - 1)$, $ord_p(\pi_m + \frac{\pi_m}{p} + \cdots + \frac{\pi_m^s}{p^s}) = \frac{p^{s+1} - k}{p - 1} - k - 1$ and $\exp \{ (\pi_m + \frac{\pi_m}{p} + \cdots + \frac{\pi_m^s}{p^s}) T^p \}$ is over-convergent when $k \geq m + 1$.

Let $m = 0$. We can write

$$\frac{\theta_0(T)}{AH(\pi_0 T)} = \frac{g_0(T)}{g_0(T^p)},$$

where $g_0(T) = \exp \left( (\pi_0 - \pi_0) T - (\pi_0 + \frac{\pi_0}{p}) T^p - \cdots \right)$. So to show $g_0(T)$ is over-convergent, it suffices to show that $\exp \left( (\pi_0 - \pi_0) T \right)$ is over-convergent.

Since $ord_p(\pi_0) = ord_p(\tilde{\pi}_0) = \frac{1}{p-1}$, write $\pi_0 = a \tilde{\pi}_0$, where $ord_p(a) = 0$. Let $Y = a - 1$.

Then,

$$0 = \sum_{i=0}^{\infty} \frac{(a\tilde{\pi}_0)^i}{p^i} = \sum_{i=0}^{\infty} \frac{((1 + Y)\tilde{\pi}_0)^i}{p^i} = \left( \tilde{\pi}_0 + \frac{\tilde{\pi}_0^p}{p} + \frac{\tilde{\pi}_0^{p^2}}{p^2} + \cdots \right) + \left( \tilde{\pi}_0 + \tilde{\pi}_0^p + \tilde{\pi}_0^{p^2} + \cdots \right) Y + \cdots.$$
and \( Y \) is a root of \( \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \cdots = 0 \) of order \( \geq 0 \), where \( \alpha_0 = \tilde{\alpha}_0 + \frac{\tilde{\alpha}_0^p}{p^2} + \cdots \), \( \alpha_1 = \tilde{\alpha}_0 + \tilde{\alpha}_0^p + \tilde{\alpha}_0^{p^2} + \cdots \). We have \( \text{ord}_p(\alpha_0) = \frac{p^2 - 2p + 2}{p - 1} \), \( \text{ord}_p(\alpha_1) = \frac{1}{p - 1} \), and \( \text{ord}_p(\alpha_k) \geq \frac{1}{p - 1} \) for \( k \geq 2 \). So, \( \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \cdots = 0 \) has a root of order \( p - 1 \). Choose \( \pi_0 \) and \( \tilde{\pi}_0 \) such that \( \text{ord}(a - 1) = p - 1 \). Then \( \text{ord}(\pi_0 - \tilde{\pi}_0) = \text{ord}(\tilde{\pi}_0(a - 1)) = \frac{1}{p - 1} + p - 1 \) and \( \exp((\tilde{\pi}_0 - \pi_0)T) \) converges for \( \text{ord}_p(T) > -p + 1 \).

Let \( m = 1 \). Write

\[
\frac{\theta_1(T)}{AH(\pi_1 T)} = \frac{g_1(T)}{g_1(T^p)},
\]

where \( g_1(T) = \exp \left( (\tilde{\pi}_1 - \pi_1)T + \left( \frac{\tilde{\pi}_0}{p} - \pi_1 - \frac{\pi_0^p}{p} \right) T^p - (\pi_1 + \frac{\pi_0}{p} + \frac{\pi_0^p}{p^2}) T^{p^2} - \cdots \right) \). So \( g_1(T) \) is over-convergent if \( \exp((\tilde{\pi}_1 - \pi_1)T) \) and \( \exp((\frac{\tilde{\pi}_0}{p} - \pi_1 - \frac{\pi_0^p}{p}) T^p) \) are. We have

\[
0 = \sum_{i=0}^{\infty} \frac{(a\tilde{\pi}_1)^p}{p^i} = \sum_{i=0}^{\infty} \frac{((1 + Y)\pi_1)^p}{p^i} = \left( \tilde{\pi}_1 + \frac{\pi_1^p}{p} + \frac{\pi_1^{p^2}}{p^2} \cdots \right) + \left( \pi_1 + \frac{\pi_0}{p} + \frac{\pi_0^p}{p^2} \cdots \right) Y + \cdots,
\]

where \( \pi_1 = a\tilde{\pi}_1 \), with \( \text{ord}_p(a) = 0 \) and \( Y = a - 1 \). So \( Y \) is a root of \( \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots = 0 \) of order \( \geq 0 \), where \( \beta_0 = \tilde{\pi}_1 + \frac{\pi_1^p}{p} + \frac{\pi_1^{p^2}}{p^2} \cdots \) and \( \beta_1 = \tilde{\pi}_0 + \tilde{\pi}_0^p + \tilde{\pi}_0^{p^2} + \cdots \). We have \( \text{ord}_p(\beta_0) = 1 + \frac{1}{p(p - 1)} \), \( \text{ord}_p(\beta_1) = \frac{1}{p(p - 1)} \), and

\[
\text{ord}_p(\beta_i) \geq \begin{cases} 
\frac{1}{p(p - 1)} & \text{if } 2 \leq i \leq p - 1 \\
\frac{2 - p}{p - 1} & \text{if } i \geq p
\end{cases}
\]

Hence \( \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots = 0 \) has a root of order \( 1 \). Choose \( \pi \) and \( \tilde{\pi}_1 \) so that \( \text{ord}(a - 1) = 1 \). Then \( \text{ord}(\pi_1 - \tilde{\pi}_1) = \text{ord}(\tilde{\pi}_1(a - 1)) = \frac{1}{p(p - 1)} + 1 \) and \( \exp((\tilde{\pi}_1 - \pi_1)T) \) converges for \( \text{ord}_p(T) > -2\frac{p^2 + 1}{p(p - 1)} \). Since \( \text{ord}_p(\pi_0 - \pi_1 - \frac{\pi_0^p}{p}) \geq \text{ord}_p(\tilde{\pi}_1 - \pi_1) \), \( \exp((\frac{\pi_0}{p} - \pi_1 - \frac{\pi_0^p}{p}) T^p) \) converges for \( \text{ord}_p(T) > -2\frac{p^2 + 1}{p^2(p - 1)} \).
Let \( f(T) = \sum_{i=1}^{k}(\lambda_{0i} T^{i}, \lambda_{1i} T^{i}) \in W_1(T \otimes \mathbb{k}_2 \langle T \rangle) \). Then Pulita’s Frobenius can be written as

\[
\frac{e_p(f(T), 1)}{e_p(f_T(T), 1)} = \prod_{i=1}^{k} \frac{e_p((\lambda_{0i} T^{i}, \lambda_{1i} T^{i}), 1)}{e_p((\lambda_{0i}^p T^{\rho p}, \lambda_{1i}^p T^{\rho p}), 1)}
= \prod_{i=1}^{k} \frac{E_1(\lambda_{0i} T^{i}) E_0(\lambda_{1i} T^{i})}{E_1(\lambda_{0i}^p T^{\rho p}) E_0(\lambda_{1i}^p T^{\rho p})}
= \prod_{i=1}^{k} \theta_1(\lambda_{0i} T^{i}) \theta_0(\lambda_{1i} T^{i}).
\]

Hence there is an over-convergent function \( g(T) \) such that

\[
\frac{e_p(f(T), 1)}{e_p(f_T(T), 1)} = \frac{g(T)}{\phi(g(T))} \prod_{i=1}^{k} \Lambda H(\pi_1 \lambda_{0i} T^{i}) \Lambda H(\pi_0 \lambda_{1i} T^{i}).
\]

So \( e_p(f(T), 1)/e_p(f_T(T), 1) \) can be replaced by \( \prod_{i=1}^{k} \Lambda H(\pi_1 \lambda_{0i} T^{i}) \Lambda H(\pi_0 \lambda_{1i} T^{i}). \)

4.2 Newton Polygon of the L-Function

Note that by the proof of Lemma 4, if \( f(T) = (f_0(T), f_1(T)) \in W_1(\mathbb{F}_p \langle T \rangle) \) with \( d \) and \( e \) degrees of \( f_0(T) \) and \( f_1(T) \), respectively, then \( f(T) \) can be written as

\[
\sum_{i=1}^{k}(\lambda_{0i} T^{i}, \lambda_{1i} T^{i}),
\]

where \( k = \max\{e, dp - 1\}. \)

**Lemma 8.** Define

\[
G(X) = \prod_{i=1}^{k} \Lambda H(\pi_1 \hat{\lambda}_{0i} X^{i}) \Lambda H(\pi_0 \hat{\lambda}_{1i} X^{i}).
\]

Then \( G(X) = \sum_{n=0}^{\infty} G_n(\hat{\lambda}) X^n \), where \( \text{ord}_p(G_n(\hat{\lambda})) \geq n/kp(p - 1). \)

**Proof.** Write \( \Lambda H(\pi_0 X) = \sum_{m=0}^{\infty} \alpha_m X^m \) and \( \Lambda H(\pi_1 X) = \sum_{m=0}^{\infty} \beta_m X^m \). Then,

\[
\text{ord}_p(\alpha_m) \geq \frac{m}{p - 1}; \quad \text{ord}_p(\beta_m) \geq \frac{m}{p(p - 1)};
\]

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and for \( 0 \geq m \geq p - 1 \) we have

\[
\alpha_m = \frac{\pi_m^m}{m!} \quad \text{and} \quad \text{ord}_p(\alpha_m) = \frac{m}{p - 1};
\]

\[
\beta_m = \frac{\pi_1^m}{m!} \quad \text{and} \quad \text{ord}_p(\beta_m) = \frac{m}{p(p - 1)}.
\]

Let \( \bar{\lambda} = (\lambda_0, \ldots, \lambda_k, \lambda_1, \ldots, \lambda_{1k}) \). Then,

\[
G(X) := \prod_{i=1}^k AH(\pi_1 \lambda_0 X^i)AH(\pi_0 \lambda_k X^i)
\]

\[
= \left( \sum_{m_1=0}^{\infty} \beta_{m_1} \lambda_{01}^{m_1} X^{m_1} \right) \cdots \left( \sum_{m_k=0}^{\infty} \beta_{m_k} \lambda_{0k}^{m_k} X^{m_k} \right)
\]

\[
\left( \sum_{\tilde{m}_1=0}^{\infty} \alpha_{\tilde{m}_1} \lambda_{11}^{\tilde{m}_1} X^{\tilde{m}_1} \right) \cdots \left( \sum_{\tilde{m}_k=0}^{\infty} \alpha_{\tilde{m}_k} \lambda_{1k}^{\tilde{m}_k} X^{\tilde{m}_k} \right)
\]

\[
= \sum_{n=0}^{\infty} G_n(\bar{\lambda})X^n,
\]

where

\[
G_n(\bar{\lambda}) = \sum_{\sum i=0}^{\infty} \beta_{m_1} \cdots \beta_{m_k} \alpha_{\tilde{m}_1} \cdots \alpha_{\tilde{m}_k} \bar{\lambda}^{\bar{m}};
\]

\( \bar{m} = (m_1, \ldots, m_k; \tilde{m}_1, \ldots, \tilde{m}_k) \) and \( \bar{\lambda}^{\bar{m}} = \lambda_{01}^{m_1} \cdots \lambda_{0k}^{m_k} \lambda_{11}^{\tilde{m}_1} \cdots \lambda_{1k}^{\tilde{m}_k} \). As \( k(m_1 + \cdots + m_k + \tilde{m}_1 + \cdots + \tilde{m}_k) \geq \sum_{i=1}^{k} i(m_i + \tilde{m}_i) = n \), get

\[
\text{ord}_p G_n(\bar{\lambda}) \geq \min \left\{ \frac{m_1 + \cdots + m_k}{p(p - 1)} + \frac{\tilde{m}_1 + \cdots + \tilde{m}_k}{p - 1} \right\}
\]

\[
\geq \min \left\{ \frac{m_1 + \cdots + m_k + \tilde{m}_1 + \cdots + \tilde{m}_k}{p(p - 1)} \right\}
\]

\[
\geq \frac{n}{kp(p - 1)}.
\]

\[
\psi_p(X^\nu) = \begin{cases} 
X^{\nu/p} & \text{if } p \mid \nu \\
0 & \text{otherwise}
\end{cases}
\]

Now define
Let $\alpha := \psi_p \circ G(X)$. An argument using Poincaré duality [5] shows that $\alpha$ is the linear dual of Frobenius. Let $F$ be a matrix representation of $\alpha$. Then,

$$L^*((\mathfrak{f}_0, f_1); T) = \frac{\det(I - FT)}{\det(I - FP)}.$$ 

Note that $F = \{G_{p^{-j}}(\lambda)\}_{i,j \geq 0}$. Let $C_0(\lambda) = 1$, and for every $n \geq 1$ let

$$C_n(\lambda) := \sum_{1 \leq u_1 < u_2 < \cdots < u_n} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} G_{p^{u_i} - u_{s(i)}}(\lambda).$$

Then we have

$$\det(I - FT) = (1 - T) \det(I - \{G_{p^{-j}}(\lambda)\}_{i,j \geq 0}) = (1 - T) \sum_{n=0}^{\infty} (-1)^n C_n(\lambda) T^n.$$ 

Thus, we have

$$(1 - T)L((\mathfrak{f}_0, f_1); T) = L^*((\mathfrak{f}_0, f_1); T)$$

$$= \frac{(1 - T) \sum_{n=0}^{\infty} (-1)^n C_n(\lambda) T^n}{(1 - pT) \sum_{n=0}^{\infty} (-1)^n C_n(\lambda) p^n T^n}.$$ 

Hence the Newton polygon of the L-function $L((\mathfrak{f}_0, f_1); T)$ coincides with an initial segment of the Newton polygon of $1 - C_1 T + \cdots + (-1)^i C_i T^i + \cdots$. Note that $\text{ord}_p(C_n) \geq \frac{n(n+1)}{2kp}$. Thus, combining with the result of Chapter 3, we have

**Theorem 4.5.** Let $d = \text{deg}(f_0)$, $e = \text{deg}(f_1)$, and $k = \max\{dp - 1, e\}$. A lower bound of the Newton polygon of the L-function of the F-isocrystal when $m = 1$ is the lower convex hull of points $(n, \frac{n(n+1)}{2kp})$ for $0 \leq n \leq \max\{dp - 1, e - 1\}$.

### 4.3 $m > 1$ Case

Let $f(T) = (f_0, \cdots, f_m)$ and $\text{deg}(f_i) = d_i$. By Proposition 3.1, A Pulita’s Frobenius

$$e_p(f(T), 1)$$

$$e_p(f_F(T), 1)$$
Lemma 9. \( \sum_{i=1}^{\tilde{d}_m} E_m(\tilde{\alpha}_0i, T^i) \sum_{i=1}^{\tilde{d}_1} E_{m-1}(\tilde{\alpha}_1i, T^i) \cdots \sum_{i=1}^{\tilde{d}_m} E_0(\tilde{\alpha}_mi, T^i) \),

where \( \tilde{d}_k = \max \{ d_k, d_{k-1}p - 1, d_{k-2}p^2 - 1 \ldots, d_0 p^k - 1 \} \).

When \( m \geq 2 \), it is not known whether the statement of Theorem 4.4 is true. However, we can still replace this Frobenius by

\[
\prod_{i=1}^{\tilde{d}_0} E_m(\tilde{\alpha}_0i, T^i) \cdots \prod_{i=1}^{\tilde{d}_{m-2}} E_2(\tilde{\alpha}_{m-2}i, T^i) \prod_{i=1}^{\tilde{d}_{m-1}} AH(\pi_1 a_{m-1}i, T^i) \prod_{i=1}^{\tilde{d}_m} AH(\pi_0 a_{mi}, T^i).
\]

Consider the case \( m = 2 \). Suppose that \( p > 3 \).

Lemma 9. \( \prod_{i=1}^{2} \exp \left( \frac{(\tilde{\pi}_2^p - \tilde{\pi}_2^{p+1}) T^{p+1}}{p^{i+1}} \right) \)

converges when \( \text{ord}(T) > -\frac{p-3}{p^2} \).

Proof. We have

\[
\prod_{i=1}^{2} \exp \left( \frac{(\pi_2^p - \pi_2^{p+1}) T^{p+1}}{p^{i+1}} \right) = \exp \left( ((p \tilde{\pi}_2 + \pi_2^p - \pi_2^p) \frac{T^{p^2}}{p^2} + ((p \pi_1 + \pi_2^p) - \pi_2^p) \frac{T^{p^3}}{p^3} \right)
\]

\[
= \exp \left( \sum_{i=0}^{p-1} \left( \frac{p}{i} \right) (p \tilde{\pi}_2^p)^{p-i} (\pi_2^p)^i \frac{T^{p^2}}{p^2} \right) \exp \left( \sum_{i=0}^{p-1} \left( \frac{p^2}{i} \right) (p \pi_1)^{p-i} (\pi_2^p)^i \frac{T^{p^3}}{p^3} \right)
\]

\[
= \exp \left( \sum_{i=0}^{p-2} \left( \frac{p}{i} \right) (p \tilde{\pi}_2^p)^{p-i} (\pi_2^p)^i \frac{T^{p^2}}{p^2} \right) \exp \left( \sum_{i=0}^{p-2} \left( \frac{p^2}{i} \right) (p \pi_1)^{p-i} (\pi_2^p)^i \frac{T^{p^3}}{p^3} \right)
\]

\[
\exp \left( \frac{\pi_1 + p(p-1) T^{p^2}}{p} + \frac{\pi_1 + p(p-1) T^{p^3}}{p^2} \right) \exp \left( \sum_{i=0}^{p^2-1} \left( \frac{p^2}{i} \right) (p \tilde{\pi}_2^p)^{p-i} (\pi_2^p)^i \frac{T^{p^3}}{p^3} \right).
\]
Since
\[ \text{ord} \left( \sum_{i=0}^{p-2} \binom{p}{i} (p\tilde{n}_2) p^{-i} (\tilde{n}_2^p i) p^{-2} \right) \geq 1 + \frac{2 + p(p - 2)}{p^2 (p - 1)} \]
and
\[ \text{ord} \left( \sum_{i=0}^{p-2} \binom{p}{i} (p\tilde{n}_1) p^{-i} (\tilde{n}_1^p i) p^{-3} \right) \geq \frac{2 + p(p - 2)}{p(p - 1)}, \]
\[ \exp \left( \sum_{i=0}^{p-2} \binom{p}{i} (p\tilde{n}_2) p^{-i} (\tilde{n}_2^p i) \frac{T p^3}{p^3} \right) \exp \left( \sum_{i=0}^{p-2} \binom{p}{i} (p\tilde{n}_1) p^{-i} (\tilde{n}_1^p i) \frac{T p^3}{p^3} \right) \]
converges when \( \text{ord}(T) > -\frac{p^2 - 2}{p^2} \). Also as
\[ \text{ord} \left( \sum_{i=0}^{p^2 - 1} \binom{p^2}{i} (p\tilde{n}_2) p^{2-i} (\tilde{n}_2^p i) p^{-3} \right) \geq \frac{1 + p(p^2 - 1)}{p^2(p - 1)}, \]
\[ \exp \left( \sum_{i=0}^{p^2 - 1} \binom{p^2}{i} (p\tilde{n}_2) p^{2-i} (\tilde{n}_2^p i) \frac{T p^3}{p^3} \right) \]
converges when \( \text{ord}(T) > -\frac{p^2 - 1}{p^2} \). Finally, we have
\[ \exp \left( \frac{\tilde{n}_2^{1+p(p-1)} T p^3 + \tilde{n}_1^{1+p(p-1)} T p^3}{p} \right) \]
\[ = E \left( \tilde{n}_2^{1+p(p-1)} T p^3 \right) \exp \left( \left( \tilde{n}_1^{1+p(p-1)} - \tilde{n}_2^{p(p-1)} \right) \frac{T p^3}{p} \right) \prod_{i=2}^{\infty} \exp \left( - \left( \tilde{n}_2^{1+p(p-1)} T p^3 \right)^i \right). \]
\( E \left( \tilde{n}_2^{1+p(p-1)} T p^3 \right) \) converges when \( \text{ord}(T) > -\frac{1+p(p-1)}{p^2(p-1)}, \) and \( \prod_{i=2}^{\infty} \exp \left( - \left( \frac{\tilde{n}_2^{1+p(p-1)} T p^3}{p} \right)^i \right) \)
converges when \( \text{ord}(T) > -\frac{p^2 - p - 1}{p^2} \). Since
\[ \exp \left( \left( \tilde{n}_1^{1+p(p-1)} - \tilde{n}_2^{p(p-1)} \right) \frac{T p^3}{p} \right) \]
\[ = \exp \left( - \sum_{i=0}^{p(p-1)} \binom{p(p-1)}{i} \left( 1 + p(p - 1) \right) \tilde{n}_1^i (-p\tilde{n}_2)^{p(p-1)+1-j} \frac{T p^3}{p} \right), \]
it converges when \( \text{ord}(T) > -\frac{p^2 - p - 1}{p^2} \).
Hence
\[ \prod_{i=1}^{2} \exp \left( \left( \tilde{n}_2^p - \tilde{n}_2^{p+1} \right) \frac{T p^{i+1}}{p^{i+1}} \right) \]

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converges when

\[ \text{ord}(T) > \max \left\{ -\frac{p^2 - 2}{p^4}, -\frac{p^2 - 1}{p^5}, -\frac{p^2 - p + 1}{p^4(p - 1)}, -\frac{p^2 - p + 1}{p^5} \right\} = -\frac{p^2 - p + 1}{p^5}, \]

and the lemma holds. \( \square \)

**Proposition 4.1.** \( E_2(T)/E_2(T^p) \) converges when \( \text{ord}_p(T) > -\frac{p-3}{p^4} \).

**Proof.** Set \( \tilde{n}_{-1} = 0 \). Recall that \( \tilde{n}_{-i} - p\tilde{n}_{-i} = \frac{\tilde{n}_p}{\tilde{n}_{-i}} \) for \( 0 \geq i \geq 2 \). Then we have

\[
\frac{E_2(T)}{E_2(T^p)} = \exp \left( \sum_{i=0}^{2} \frac{\tilde{n}_{-i}}{p^i} T^{p^i} - \sum_{i=0}^{2} \frac{\tilde{n}_{-i}}{p^i} T^{p^i+1} \right) = \exp \left( \tilde{n}_2 T \right) \exp \left( \sum_{i=0}^{2} \left( \frac{\tilde{n}_{-i} - p\tilde{n}_{-i}}{p^i+1} \right) \right) = \exp \left( \tilde{n}_2 T + \sum_{i=0}^{2} \frac{\tilde{n}_p}{\tilde{n}_{-i}} \frac{T^{p^i+1}}{p^{i+1}} \right) = E(\tilde{n}_2 T) \prod_{i=1}^{2} \exp \left( \left( \frac{\tilde{n}_p}{\tilde{n}_{-i}} \frac{T^{p^i+1}}{p^{i+1}} \right) \right) \prod_{i=3}^{\infty} \exp \left( -\frac{(\tilde{n}_2 T)^{p^i+1}}{p^{i+1}} \right).
\]

\[ E(\tilde{n}_2 T) \text{ converges if and only if } \text{ord}(T) > -\text{ord}(\tilde{n}_2) = -\frac{1}{p^{i+1}}, \]

\[ \exp \left( -\frac{(\tilde{n}_2 T)^{p^i+1}}{p^{i+1}} \right) \text{ converges if and only if } \frac{p^{i+1}}{p^{i+1}} \text{ord}(\tilde{n}_2 T) - (i + 1) > \frac{1}{p-1} \text{ if and only if } \]

\[ \text{ord}(T) > \frac{1}{p-1} + \frac{i + 1}{p^{i+1}} - \frac{1}{p^2(p - 1)}. \]

Since the right-hand side is an increasing function for \( i \geq 1 \), we have

\[ \frac{1}{p-1} + \frac{i + 1}{p^{i+1}} - \frac{1}{p^2(p - 1)} \geq \frac{1}{p-1} + \frac{3}{p^3} - \frac{1}{p^3(p - 1)} = -\frac{p + 3}{p^4}. \]

Hence \( \prod_{i=3}^{\infty} \exp \left( -\frac{(\tilde{n}_2 T)^{p^i+1}}{p^{i+1}} \right) \) converges if and only if \( \text{ord}(T) > -\frac{p-3}{p^4} \).

By the previous lemma, the radius of convergence of

\[ \prod_{i=1}^{2} \exp \left( \left( \frac{\tilde{n}_p}{\tilde{n}_{-i}} \frac{T^{p^i+1}}{p^{i+1}} \right) \right) \]

is less than \(-\frac{p-3}{p^4}\). Thus the proposition follows. \( \square \)
Lemma 10. Let \( k = \max \{ \delta_0, \delta_1, \delta_2 \} \). Define
\[
G(X) = \prod_{i=1}^{\delta_0} \frac{E_2(\lambda_0 X^i)}{E_2(\lambda_0 X^i)} \prod_{i=1}^{\delta_1} \frac{\alpha_{11} X^i}{E_2(\alpha_{11} X^i)} \prod_{i=1}^{\delta_2} \frac{\beta_0 X^i}{E_2(\beta_0 X^i)}
\]
Then \( G(X) = \sum_{n=0}^{\infty} G_n(\lambda) X^n \), where \( \text{ord}_p(G_n(\lambda)) \geq n(p-3)/kp^4 \).

Proof. Write \( AH(\pi_0 X) = \sum_{m=0}^{\infty} \alpha_m X^m \), \( AH(\pi_1 X) = \sum_{m=0}^{\infty} \beta_m X^m \) and \( E_2(X)/E_2(X^p) = \sum_{m=0}^{\infty} \gamma_m X^m \). Then,
\[
\text{ord}_p(\alpha_m) \geq \frac{m}{p-1};
\]
\[
\text{ord}_p(\beta_m) \geq \frac{m}{p(p-1)};
\]
\[
\text{ord}_p(\gamma_m) \geq \frac{m(p-3)}{p};
\]
and for \( 0 \leq m \leq p-1 \) we have
\[
\alpha_m = \frac{\pi_0^m}{m!} \text{ and } \text{ord}_p(\alpha_m) = \frac{m}{p-1};
\]
\[
\beta_m = \frac{\pi_1^m}{m!} \text{ and } \text{ord}_p(\beta_m) = \frac{m}{p(p-1)};
\]
\[
\gamma_m = \frac{\pi_2^m}{m!} \text{ and } \text{ord}_p(\gamma_m) = \frac{m}{p^2(p-1)}.
\]
Let \( \lambda = (\lambda_0, \ldots, \lambda_{\delta_0}, \lambda_{11}, \ldots, \lambda_{1\delta_1}, \lambda_{21}, \ldots, \lambda_{2\delta_2}) \). Then,
\[
G(X) = \prod_{i=1}^{\delta_0} \frac{E_2(\lambda_0 X^i)}{E_2(\lambda_0 X^i)} \prod_{i=1}^{\delta_1} \frac{\alpha_{11} X^i}{E_2(\alpha_{11} X^i)} \prod_{i=1}^{\delta_2} \frac{\beta_0 X^i}{E_2(\beta_0 X^i)}
\]
\[
= \left( \sum_{i=0}^{\infty} \gamma_i^{1 \lambda_0 1 X^i} \right) \cdots \left( \sum_{i_0=0}^{\infty} \gamma_i^{1 \lambda_0 1 X^i} \right)
\]
\[
\left( \sum_{m_1=0}^{\infty} \beta_m^{1 \lambda_{11} 1 X^{m_1}} \right) \cdots \left( \sum_{m_{\delta_2}=0}^{\infty} \beta_m^{1 \lambda_{11} 1 X^{m_{\delta_2}}} \right)
\]
\[
\left( \sum_{m_1=0}^{\infty} \alpha_m^{1 \lambda_{21} 1 X^{m_1}} \right) \cdots \left( \sum_{m_{\delta_2}=0}^{\infty} \alpha_m^{1 \lambda_{21} 1 X^{m_{\delta_2}}} \right)
\]
\[
= \sum_{n=0}^{\infty} G_n(\lambda) X^n, \quad (4-1)
\]
Theorem 4.6. Since 
\[
\sum_{i,m_i,m_{i2} \geq 0} \gamma_i \cdots \gamma_{i0} \beta_{m_i} \cdots \beta_{m_{i2}} \alpha_{m_i} \cdots \alpha_{m_{i2}} \mathcal{X}^\mathcal{X}_{m_i} ;
\]
where the sum is taken over \(\sum_{i=1}^{\tilde{a}} i \tilde{l}_i + \sum_{i=1}^{\tilde{a}} i \tilde{m}_i + \sum_{i=1}^{\tilde{a}} i \tilde{m}_{i2} = n\) and 
\[
\tilde{m} = (l_1, \ldots, l_{\tilde{a}}, m_1, \ldots, m_{\tilde{a}}, \tilde{m}_1, \ldots, \tilde{m}_{\tilde{a}}); 
\]
\[
\mathcal{X}^\mathcal{X} = \lambda_{01}^{l_1} \cdots \lambda_{0\tilde{a}}^{l_{\tilde{a}}} \lambda_{11}^{m_1} \cdots \lambda_{1\tilde{a}}^{m_{\tilde{a}}} \lambda_{21}^{\tilde{m}_1} \cdots \lambda_{2\tilde{a}}^{\tilde{m}_{\tilde{a}}}. 
\]
Then we get 
\[
\text{ord}_p G_n(\tilde{\lambda}) \geq \min \left\{ \frac{(p-3)(l_1 + \cdots + l_{\tilde{a}})}{p^4} + \frac{m_1 + \cdots + m_{\tilde{a}} + \tilde{m}_1 + \cdots + \tilde{m}_{\tilde{a}}}{p(p-1)} \right\} 
\]
\[
\geq \left\{ \frac{(p-3) \min \{l_1 + \cdots + l_{\tilde{a}} + m_1 + \cdots + m_{\tilde{a}} + \tilde{m}_1 + \cdots + \tilde{m}_{\tilde{a}}\}}{p^4} \right\} 
\]
\[
\geq \frac{n(p-3)}{kp^4}. 
\]

Now using the same argument as the \(m = 1\) case, we obtain 
\[
(1 - T)L((f_0, f_1, f_2); T) = L^*(((f_0, f_1, f_2); T) 
\]
\[
= \frac{(1 - T) \sum_{n=0}^{\infty} (-1)^n C_n(\tilde{\lambda}) T^n}{(1 - T) \sum_{n=0}^{\infty} (-1)^n C_n(\tilde{\lambda}) T^n}, 
\]
where \(C_0(\tilde{\lambda}) = 1\), and for every \(n \geq 1\) 
\[
C_n(\tilde{\lambda}) := \sum_{1 \leq u_1 < u_2 < \cdots < u_n} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} G_{\mu_i - u_{\sigma(i)}}(\tilde{\lambda}). 
\]
Since \(\text{ord}_p(C_n) \geq \frac{n(n+1)(p-1)(p-3)}{2kp^4}\), we have 

**Theorem 4.6.** Let \(d_i = \deg(f_i)\) and \(k = \max\{d_0, d_1, d_2\}\). A lower bound of the Newton polygon of the L-function of the F-isocrystal when \(m = 2\) is the lower convex hull of points \((n, \frac{n(n+1)(p-1)(p-3)}{2kp^4})\) for \(0 \leq n \leq \max\{d_0, d_1, d_2\}\).
CHAPTER 5
CONCLUSIONS

In this work, we investigated certain kinds of exponential sums introduced by A. Pulita by studying the corresponding L-functions. In chapter 2, we reviewed the most general unit-root F-isocrystal on $\mathbb{A}^1$ corresponding to a character of $\pi_1(\text{Spec } k((t^{-1})))$, absolute Galois group of $k((t^{-1}))$, of $p$-power order introduced by Pulita. Then the corresponding exponential sums and the L-functions were introduced.

In chapter 3, we computed a lower bound of the degree of Pulita’s F-isocrystal. If $f(X) = (f_1(X), \cdots, f_m(X)) \in W_m(\mathbb{F}_p[X])$ and $d_i = \text{deg}(f_i)$, then the degree of the L-function introduced in chapter 2 has degree less than or equal to $\alpha$, where

$$\alpha = \max \{d_0 p^m - 1, d_1 p^{m-1} - 1, \cdots, d_{m-1} p - 1, d_m - 1\}.$$ 

In chapter 4, we estimated the p-adic absolute values of the zeros of the L-function. To obtain better estimates, we first replaced the Frobenius introduced by Pulita by a new Frobenius with larger radius of convergence. Using a new Frobenius, we estimated the p-adic absolute values of the zeros of the L-function. When $m = 1$, a lower bound of the Newton polygon of the L-function of the F-isocrystal is the lower convex hull of points $(n, \frac{n(n+1)}{2kp})$ for $0 \leq n \leq \max \{d_0 p - 1, d_1 - 1\}$, where $k = \max \{d_0 p - 1, d_1\}$. When $m = 2$, a lower bound of the Newton polygon of the L-function of the F-isocrystal is the lower convex hull of points $(n, \frac{n(n+1)(p-1)(p-3)}{2kp^4})$ for $0 \leq n \leq \max \{d_0 p^2 - 1, d_1 p - 1, d_2 - 1\}$, where $k = \max \{d_0 p^2 - 1, d_1 p - 1, d_2\}$. 

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REFERENCES


BIOGRAPHICAL SKETCH

Yuri Morofushi was born in 1978 in Japan. She earned her Bachelor of Arts degree in mathematics from University of Arizona in 2002. Yuri then started graduate school at the University of Florida to continue her studies in mathematics. She earned a Master of Science degree in 2004, and completed her Ph.D. in 2010.