ANALYSIS OF THREE DEGREE OF FREEDOM 6 × 6 TENSEGRITY PLATFORM

By

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A THESIS PRESENTED TO THE GRADUATE SCHOOL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENT FOR THE DEGREE OF MASTER OF SCIENCE

UNIVERSITY OF FLORIDA

2005
ACKNOWLEDGEMENTS

I would like to thank Dr. Carl Crane for being the chairperson of my committee and providing his technical expertise during my time as a master’s degree student. With his in-depth knowledge, we have overcome many obstacles that hindered the solution of this analysis. In addition, I would also like to thank Dr. John Schueller, Dr. John Ziegert and Dr. Brian Mann for serving on my committee and providing me with valuable insight. I thank my parents, Clinton and Lorraine, for their love and support. Last but not least, I thank my brother for keeping my head on straight and keeping me grounded.
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The mechanism studied in this paper is a three degree of freedom $6 \times 6$ tensegrity mechanism. A tensegrity structure is one that balances internal (pre-stressed) forces of tension and compression. These structures have the unique property of stabilizing themselves if subjected to certain types of disturbances. The structure analyzed in this paper consists of two rigid bodies (platforms) connected by a total of six members. Three of the members are noncompliant constant-length struts and the other three members consist of springs. For typical parallel mechanisms, if the bottom platform is connected to ground and the top platform is connected to the base by six leg connectors, the top platform will have six degrees of freedom relative to the bottom platform. However, because three of the six members connecting two platforms are noncompliant constant-length struts, the top platform has only three degrees of freedom. The orientation of the top platform is not known relative to the bottom platform.
The primary contribution of this thesis is the analysis of the three degree of freedom tensegrity platform. Specifically, given the location of the connector points on the base and top platforms, the lengths of the three noncompliant constant-length struts, and the desired location of a point embedded in the top platform measured with respect to a coordinate system attached to the base, all possible orientations of the top platform are determined. The thesis presents the development of three equations in three variables that are based on the analysis of three spherical four bar mechanisms that are present within the tensegrity mechanism. These three equations are then manipulated in order to obtain a single sixteenth degree polynomial in one of the variables. Unique corresponding values for the remaining two variables are then obtained and thus it is shown that a maximum of sixteen possible orientation solutions can exist.

The thesis also presents a static force/torque analysis of the device in order to determine the external wrench that must be applied to the top platform to maintain equilibrium when the point embedded in the top platform is positioned as desired. Twists that are reciprocal to this wrench are identified as these represent motion of the top platform upon which the external wrench performs no work.
The word *tensegrity* is a combination of the words *tension* and *integrity*. Tensegrity describes a structural relationship principle in which structural shape is guaranteed by the closed, continuous, tensional behaviors of the system and not by the discontinuous compressional member behaviors. Tensegrity provides the ability of a structure to yield increasingly without ultimately breaking.  

Houses can and do collapse when a sufficient horizontal (shearing) force is applied above the foundation. The walls act as levers, creating torque. The house deals with this torque by destroying itself. Beams and cables have strength in compression and tension respectively, but both are poor in resisting torque.

Because torque is the cause of failure in many structures, the design challenge was to come up with a structure that can support loads without generating torque. This design criterion is how tensegrity structures were born. Tensegrity structures are able to support a load in the vertical and horizontal directions without creating any torque or moment throughout the structure.

Tensegrity is the case when compression and tension are said to be in balance with each other. When these two forces are in balance, the structure is said to be optimally strong. An example of tensegrity that everyone is familiar with includes the muscular-skeletal system. For this case the muscles are in tension, while the skeletal system is in compression. Other examples include air balloons or automobile tires. The
air molecules provide a discontinuous push while the rubber provides continuous tension. A final example of a tensegrity structure includes geodesic domes. These structures are shown in Figure 1-1.

Figure 1-1. Air Balloon and Geodesic Dome as Tensegrity Examples

This paper will present an analysis of the geometric properties of platforms which incorporate tensegrity principles. A platform is described as any device that has multiple legs connecting a moving (top) platform to a bottom (base) platform. It was only recently that the forward position solution of a $3 \times 3$ platform was formulated by Duffy and Griffis. In this analysis, all positions and orientations of the top platform are determined based on given lengths of the six leg connectors. This $3 \times 3$ platform (see Figure 1-2) was the simplest of the geometries to solve. The formulation of these solutions yielded an eighth degree polynomial in the square of one defining parameter. Later, this solution technique was applied to a $6 \times 3$ platform (see Figure 1-3). A
forward displacement analysis of a general $6 \times 6$ platform (see Figure 1-4) showed that
the solution was in the form of a $40^{th}$ degree polynomial.\textsuperscript{7}

Figure 1-2.  $3 \times 3$ Platform

Figure 1-3.  $6 \times 3$ Platform-Also Called a Stewart Platform
The tensegrity structure analyzed in this paper consists of a special $6 \times 6$ platform. This geometry was designed by Griffis and Duffy to include the benefits of a $3 \times 3$ platform and a general $6 \times 6$ platform. This special platform makes the analysis comparable to a $3 \times 3$ platform while eliminating mechanical interference associated with the $3 \times 3$ platform. The $6 \times 6$ platform analyzed in this paper consists of two rigid bodies connected together by three constant length noncompliant struts and three compliant ties that each consists of a spring in series with a non-compliant tie where the length of the non-compliant tie can be controlled. The legs are connected to the platforms with ball and socket joints. For this analysis, the bottom platform is connected to the ground and the top platform has three degrees of freedom relative to the bottom platform. The special $6 \times 6$ tensegrity platform is shown in Figure 1-5.
If the three struts for this mechanism were able to change in length, the position and orientation of the top platform could be determined when given the six leg connector lengths as was done by Duffy and Griffis.\textsuperscript{5} This device would have six degrees of freedom. However, for the case presented in this paper, the struts are noncompliant and are of constant length. This reduces the system to a three degree of freedom device. Chapter 2 discusses how the orientations of the top platform are determined with respect to a coordinate system that is connected to the ground when given the desired position of a single point in the top platform. All orientations that satisfy the length condition associated with the constant length struts are determined.
Another important aspect of this paper revolves around the concept of reciprocal screws. For any position and orientation of the top platform, there exists certain rotations and/or translations (twists) that the top platform can be moved along to eliminate any work being performed by the external force/moment (wrench) that holds the top platform in static equilibrium. This analysis will be performed using the concept of reciprocal screws and will be discussed in Chapter 3.

Although beyond the scope of this thesis, the future goal of this analysis is to determine if the point embedded in the top platform can be moved to a desired goal point along a path where the instantaneous twist of motion is always instantaneously reciprocal to the applied external wrench that maintains static equilibrium. By moving along this path, the energy required to move the top platform would be minimized.
CHAPTER 2
ORIENTATION OF TOP PLATFORM

This chapter presents the solution of how to find all possible orientations of the top platform with respect the bottom platform when a point embedded in the top platform is positioned at a desired location. The model used for this analysis is shown in Figure 2-1. Points B₁, B₂, B₃, and T₁, T₂, T₃ correspond to the centers of the spherical joints at the bottom and top ends of the three constant length noncompliant struts which are numbered 2, 5, and 8 in the figure. Point P is a point that is embedded in the top platform.

Two coordinate systems are defined for this problem. The first is attached to the base platform with its origin at point B₁ and X axis through B₂. Point B₃ lies in the XY plane such that the Z axis is defined as parallel to the cross product of the vector along the X axis with the vector from B₁ to B₃. This first coordinate system is shown in the figure. The second coordinate system is attached to the top platform. The origin is at point T₁ and its X axis passes through point T₂. Point T₃ is in the XY plane and the direction of the Z axis is defined in a similar manner as in the prior case. The precise problem statement is now presented as follows:
Figure 2-1. Model Used for Analysis of 6×6 Platform
Given

- all dimensions of the top and bottom platforms, i.e. the lengths $L_{10}$, $L_{11}$, $L_{12}$, $L_{13}$, $L_{14}$, and $L_{15}$, where the notion $L_i$ refers to the length of bar $i$,
- the lengths of the three constant length noncompliant struts, $L_2$, $L_5$, and $L_8$,
- the coordinates of point $P$ in the 2nd (top) coordinate system, which implies that the lengths $L_3$, $L_6$, and $L_9$ are known,
- the coordinates of point $P$ in the 1st (base) coordinate system, which implies that the lengths $L_1$, $L_4$, and $L_7$ are known

Find

- all possible orientations of the top platform, which can be represented by the $3 \times 3$ rotation matrix $R$. 

It is important to note that based on the problem statement, the lengths of all fifteen line segments shown in Figure 2-1 are known. Also it is helpful to visualize the problem as that of having two tetrahedrons, one defined by points $B_1$, $B_2$, $B_3$, and $P$ and the other by points $T_1$, $T_2$, $T_3$, and $P$, that share the common point $P$. The problem can be thought of as that of determining all the possible relative orientations of the two tetrahedrons such that the three distance constraints associated with the constant length struts are satisfied.
As stated in the given information, the location of point P in the top platform is known in both the first and second coordinate system. However, the orientation of the top platform with respect to the bottom platform is not known.

The analysis begins by identifying three major planes that are fixed with respect to the ground coordinate system. These three planes consist of lines (1-7-12), (4-7-11), and (1-4-10). Because these planes all have vertices at points known in the first coordinate system, the equations of these planes can be readily determined.

Three new angles will now be defined, \( \theta_1 \), \( \theta_2 \), and \( \theta_3 \). \( \theta_1 \) is defined as the angle between planes formed by the lines 1-3 and 1-7 (see Figure 2-2). \( \theta_2 \) is defined as the angle between planes formed by the lines 9-7 and 1-7 (see Figure 2-3). \( \theta_3 \) is defined as the angle between planes formed by the lines 4-6 and 4-7 (see Figure 2-4).

Figure 2-2. Angle \( \theta_1 \); Figure on right shows view looking along line 1 from point P to point B₁
Figure 2-3. Angle $\theta_2$; Figure on right shows view looking along line 7 from point P to point B3.

Figure 2-4. Angle $\theta_3$; Figure on right shows view looking along line 4 from point P to point B2.
For the analysis it is possible to model the system as three spherical four bar mechanisms. These spherical four bar mechanisms are composed of lines (1-3-9-7), (7-9-6-4) and (4-6-3-1) (see Figure 2-1). When a sphere of unit radius is centered at point P, then the four planes passing through the unit circle cut it into four arcs. These four arcs form the spherical quadrilateral that will be used for this analysis. Also by choosing the spherical four bars in this fashion, the output angle of one spherical four bar mechanism will be the input angle of another causing the equations of the three spherical four bars to be interrelated.

2.1 Defining the Equations of a Generic Spherical Four Bar

A generic spherical four bar mechanism is shown if Figure 2-5. Imagine the a unit sphere is drawn with its center originating at an arbitrary point O. The unit vectors $S_i$ will meet this sphere at a series of points, $i=1,2,3,4$ as shown in Figure 2-5. Links can be drawn on the unit sphere to join adjacent points, 12, 23, 34 and 41. For these links the angle between them is defined as $\alpha_{ij}$. For example, the angle between $S_1$ and $S_2$ is defined as $\alpha_{12}$. The angle $\theta_j$ is defined as the angle between the links $\alpha_{ij}$ and $\alpha_{jk}^{10}$. 
Sets of sine, sine-cosine, and cosine laws can be generated for the spherical four bar mechanism which contain the link angles, $\alpha$’s, and the joint angles, $\theta$’s. One such spherical cosine law that relates the angles $\theta_4$ and $\theta_1$, but does not contain the angles $\theta_2$ or $\theta_3$ can be written as

\[ Z_{41} = c_{23} \quad (2.1) \]

where $Z_{41}$ is defined as

\[ Z_{41} = s_{12} (X_4 s_1 + Y_4 c_1) + c_{12} Z_4. \quad (2.2) \]

The terms $X_4$, $Y_4$, and $Z_4$ are defined as

\[ X_4 = s_{34} s_4, \]
\[ Y_4 = - (s_{41} c_{34} + c_{41} s_{34} c_4), \]
\[ Z_4 = c_{41} c_{34} - s_{41} s_{34} c_4. \quad (2.3) \]

and where the terms $s_i$, $c_i$, $s_{ij}$, and $c_{12}$ are defined as
\[ s_i = \sin(\theta_i), \]
\[ c_i = \cos(\theta_i), \]
\[ s_{ij} = \sin(\alpha_{ij}), \]
\[ c_{ij} = \cos(\alpha_{ij}) . \]  

(2.4)

The following tan-half-angle trigonometric identity formulas are used to transform the spherical cosine law from a transcendental to an algebraic equation.

\[ c_i = \frac{1 - x_i^2}{1 + x_i^2}, \]
\[ s_i = \frac{2x_i}{1 + x_i^2} \]  

(2.5)

where \( x_i = \tan \frac{\theta_i}{2} \).

Substituting equations (2.3) through (2.5) into (2.2) and collecting terms gives

\[ H_1 x_4^2 x_1^2 + H_2 x_1^2 + H_3 x_4^2 + H_4 x_4 x_1 + H_5 = 0 \]  

(2.6)

where

\[ H_1 = (-c_{4i} s_{34} + c_{4i} s_{34}) c_{12} + (c_{4i} c_{34} + s_{4i} s_{34}) c_{12} - c_{23} \]
\[ H_2 = (s_{4i} c_{34} + c_{4i} s_{34}) s_{12} + (c_{4i} c_{34} + s_{4i} s_{34}) c_{12} - c_{23} \]
\[ H_3 = (-s_{4i} c_{34} + c_{4i} s_{34}) s_{12} + (c_{4i} c_{34} + s_{4i} s_{34}) c_{12} - c_{23} \]
\[ H_4 = 4s_{12} s_{34} \]
\[ H_5 = (-s_{4i} c_{34} - c_{4i} s_{34}) s_{12} + (c_{4i} c_{34} - s_{4i} s_{34}) c_{12} - c_{23} \]  

(2.7)

Equation (2.6) will be applied to the three spherical four bar mechanisms defined previously. All the link angles, \( \alpha \)'s, for all three of the spherical four bar mechanisms can
be determined in terms of the known line segment lengths shown in Figure 2-1. For example, given a triangle of sides (L₁, L₂ and L₃), the cosine law of a planar triangle returns the interior angle opposite of the side of interest. For example the interior angle opposite of side L₃ can be obtained as

\[
\theta = \arccos\left(\frac{L_1^2 + L_2^2 - L_3^2}{2L_1L_2}\right)
\]  

(2.8)

Therefore, for each of the three spherical four bar mechanisms, an equation corresponding to (2.6) can be written where the coefficients H₁ through H₅ can be determined.

2.2 Defining the Equations of the Three Spherical Four Bars

Now that the equation that defines a generic spherical quadrilateral has been presented, the goal is to now obtain the specific equations for the three spherical four bars present in the mechanism. Figure 2-6 shows the first spherical quadrilateral that is defined by lines (1-7-9-3). By using an appropriate exchange of subscripts in (2.6), the following equation which relates the defined angles \( \theta_1 \) and \( \theta_2 \) is obtained

\[
a_1x_2^2x_1^2 + a_2x_2^2 + a_3x_1^2 + a_4x_2x_1 + a_5 = 0
\]  

(2.10)
where

\[
\begin{align*}
    a_1 &= (-c_7 s_{31} + s_7 c_{31}) s_{79} + (c_7 c_{31} + s_7 s_{31}) c_{79} - c_{93} \\
    a_2 &= (s_7 c_{31} + c_7 s_{31}) s_{79} + (c_7 c_{31} + s_7 s_{31}) c_{79} - c_{93} \\
    a_3 &= (-s_7 c_{31} + c_7 s_{31}) s_{79} + (c_7 c_{31} + s_7 s_{31}) c_{79} - c_{93} \\
    a_4 &= 4s_{79}s_{31} \\
    a_5 &= (-s_7 c_{31} - c_7 s_{31}) s_{79} + (c_7 c_{31} - s_7 s_{31}) c_{79} - c_{93}
\end{align*}
\]

(2.11)

All the terms in (2.11) are expressed in terms of known parameters.

The same procedure is used to develop an equation relating the input/output relation for the second spherical quadrilateral (see Figure 2-7) which is defined by lines (7-4-6-9). However for this case, the ‘input’ angle is defined by the angle $\varphi_2$ (later it will be shown how the angle $\varphi_2$ can be expressed in terms of $\theta_2$). By using an appropriate exchange of subscripts in (2.6) the expression that relates the angles $\varphi_2$ and $\theta_3$ can be written as Equation (2.12).

\[
A \cos(\varphi_2) \cos(\theta_3) + B \sin(\varphi_2) \sin(\theta_3) + D \cos(\varphi_2) + E \cos(\theta_3) + F = 0
\]

(2.12)
Lastly, this procedure will be used to develop an equation relating the input/output relation for the third spherical quadrilateral. This third spherical quadrilateral (see Figure 2-8) consists of lines (4-1-3-6). By substituting these lines into the coefficients of the generic spherical quadrilateral (2.6), the coefficients for the third spherical quadrilateral are obtained.
Defining $\varphi_3$ and $\varphi_1$ as the input/output angles. The general equation for the third spherical quadrilateral is shown in Equation (2.13).

$$A \cos(\varphi_3) \cos(\varphi_1) + B \sin(\varphi_3) \sin(\varphi_1) + D \cos(\varphi_3) + E \cos(\varphi_1) + F = 0$$ (2.13)

We now have three general equations that relate the input and output angles for the three spherical quadrilaterals (2.10, 2.12 and 2.13). To summarize, Table 2-1 shows the parameter substitutions that were used for the three spherical quadrilaterals.
Table 2-1. Relationships Between Generic Spherical Quadrilateral to Three Cases

<table>
<thead>
<tr>
<th>Generic Quadrilateral</th>
<th>$\theta_4$</th>
<th>$\theta_1$</th>
<th>$\alpha_{4-1}$</th>
<th>$\alpha_{1-2}$</th>
<th>$\alpha_{2-3}$</th>
<th>$\alpha_{3-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Quadrilateral</td>
<td>$\theta_1$</td>
<td>$\theta_2$</td>
<td>$\alpha_{1-7}$</td>
<td>$\alpha_{7-9}$</td>
<td>$\alpha_{9-3}$</td>
<td>$\alpha_{3-1}$</td>
</tr>
<tr>
<td>2nd Quadrilateral</td>
<td>$\phi_2$</td>
<td>$\phi_3$</td>
<td>$\alpha_{7-4}$</td>
<td>$\alpha_{4-6}$</td>
<td>$\alpha_{6-9}$</td>
<td>$\alpha_{9-7}$</td>
</tr>
<tr>
<td>3rd Quadrilateral</td>
<td>$\phi_3$</td>
<td>$\phi_1$</td>
<td>$\alpha_{4-1}$</td>
<td>$\alpha_{1-3}$</td>
<td>$\alpha_{3-6}$</td>
<td>$\alpha_{6-4}$</td>
</tr>
</tbody>
</table>

2.3 Relationships Between $\theta_i$ and $\phi_i$

Now that equations for the three spherical quadrilaterals are known, a relationship between the angles $\theta_i$ and $\phi_i$ must be determined. By revisiting Figures 2-6 through 2-8 the relationship between $\theta_i$ and $\phi_i$ can be written as the following. For example, a visual of $\gamma_2$ will be shown in Figure 2-9.

\[
\phi_1 + \gamma_1 + \theta_1 = 0 \quad (2.14)
\]

\[
\phi_2 + \gamma_2 + \theta_2 = 0 \quad (2.15)
\]

\[
\phi_3 + \gamma_3 + \theta_3 = 0 \quad (2.16)
\]
Figure 2-9. Visual Representation of angle $\gamma_2$

The angles $\gamma_i$ can be obtained in terms of known quantities. Using the properties of dot products and cross products, the angle $\gamma_i$ can be determined. Shown below is a guide to finding these angles. The angle $\gamma_2$ will be chosen as an example for clarity.

1. Find a Vector that passes through the axis of rotation. For $\gamma_2$ this vector will be (Point P- Point B3).
2. Unitize this Vector.
3. Find a vector perpendicular to a plane containing the axis of rotation and the first point of interest. For \( \gamma_2 \), this first point of interest would be Point B2.

4. Unitize this Vector.

5. Find a second vector perpendicular to a plane containing the axis of rotation and the second point of interest. For \( \gamma_2 \), this second point of interest would be Point B1.

6. Unitize this Vector.

7. Because all of the vectors are of unit value, the cosine of \( \gamma_2 \) will be the dot product of 4 & 6.

8. The sine of \( \gamma_2 \) will be \( 2 \) dot \( (4 \times 6) \).

9. \( \gamma_2 = \text{atan2} (\sin \gamma_2, \cos \gamma_2) \).

### 2.4 Solution of the Three Spherical Equations

Now that an expression \( \gamma_1 \) has been obtained, it will be substituted into the three spherical equations (2.10), (2.12) and (2.13). After using the tan-half angle formulas and collecting terms, we obtain three equations in the parameters \( \theta_1, \theta_2, \) and \( \theta_3 \) as follows

\[
\begin{align*}
(A_9 x_1^2 + A_8 x_1 + A_7) x_2^2 + (A_6 x_1^2 + A_5 x_1 + A_4) x_2 + (A_3 x_1^2 + A_2 x_1 + A_1) &= 0 \quad (2.17) \\
(B_9 x_1^2 + B_8 x_1 + B_7) x_2^2 + (B_6 x_1^2 + B_5 x_1 + B_4) x_2 + (B_3 x_1^2 + B_2 x_1 + B_1) &= 0 \quad (2.18) \\
(D_9 x_1^2 + D_8 x_1 + D_7) x_3^2 + (D_6 x_1^2 + D_5 x_1 + D_4) x_3 + (D_3 x_1^2 + D_2 x_1 + D_1) &= 0. \quad (2.19)
\end{align*}
\]
Bezout’s and Sylvester’s methods will now be employed to solve these three equations. The coefficients of Equations (2.17) through (2.19), which are all defined in terms of known quantities, are presented in the Appendix.

First, let

\[ L_1 = A_6x_1^2 + A_8x_1 + A_7 \]
\[ L_2 = B_6x_3^2 + B_8x_3 + B_7 \]
\[ M_1 = A_6x_1^2 + A_5x_1 + A_4 \]  \hspace{1cm} (2.20)
\[ M_2 = B_6x_3^2 + B_5x_3 + B_4 \]
\[ N_1 = A_3x_1^2 + A_2x_1 + A_1 \]
\[ N_2 = B_3x_3^2 + B_2x_3 + B_1 \]

Substituting equations (2.20) into (2.17) and (2.18) yields

\[ L_1x_2^2 + M_1x_2 + N_1 = 0 \]  \hspace{1cm} (2.21)
\[ L_2x_2^2 + M_2x_2 + N_2 = 0 \]  \hspace{1cm} (2.22)

The condition equations (2.21) and (2.22) have a common root for \( x_2 \) is the following:

\[
\begin{bmatrix} L_1 & M_1 \\ L_2 & M_2 \end{bmatrix} \begin{bmatrix} M_1 & N_1 \\ M_2 & N_2 \end{bmatrix} - \begin{bmatrix} L_1 & N_1 \\ L_2 & N_2 \end{bmatrix}^2 = 0
\]  \hspace{1cm} (2.23)

Expanding equation (2.23) and collecting terms gives

\[
\left( P_4x_1^4 + P_3x_1^3 + P_2x_1^2 + P_1x_1 + P_0 \right)x_3^4 + \left( Q_4x_1^4 + Q_3x_1^3 + Q_2x_1^2 + Q_1x_1 + Q_0 \right)x_3^3 + \left( R_4x_1^4 + R_3x_1^3 + R_2x_1^2 + R_1x_1 + R_0 \right)x_3^2 - \left( S_4x_1^4 + S_3x_1^3 + S_2x_1^2 + S_1x_1 + S_0 \right)x_3 + \left( T_4x_1^4 + T_3x_1^3 + T_2x_1^2 + T_1x_1 + T_0 \right) = 0
\]  \hspace{1cm} (2.24)
where the terms $P_i$, $Q_i$, $R_i$, $S_i$, and $T_i$ ($i=1..4$) are expressed in terms of known quantities.

For simplification, equation (2.24) is rewritten as

$$V_4x_3^4 + V_3x_3^3 + V_2x_3^2 + V_1x_3 + V_0 = 0$$

(2.25)

where

$$V_4 = P_4x_1^4 + P_3x_1^3 + P_2x_1^2 + P_1x_1 + P_0 ,$$

$$V_3 = Q_4x_1^4 + Q_3x_1^3 + Q_2x_1^2 + Q_1x_1 + Q_0 ,$$

$$V_4 = R_4x_1^4 + R_3x_1^3 + R_2x_1^2 + R_1x_1 + R_0 ,$$

$$V_4 = S_4x_1^4 + S_3x_1^3 + S_2x_1^2 + S_1x_1 + S_0 ,$$

$$V_4 = T_4x_1^4 + T_3x_1^3 + T_2x_1^2 + T_1x_1 + T_0 .$$

Equation (2.19) can be written as the following:

$$W_2x_2^2 + W_1x_2 + W_0 = 0$$

(2.27)

where,

$$W_2 = D_6x_1^2 + D_8x_1 + D_7$$

$$W_1 = D_6x_1^2 + D_8x_1 + D_4$$

(2.28)

$$W_0 = D_3x_1^2 + D_2x_1 + D_1$$

Multiplying Equation (2.25) by $x_3$ and multiplying Equation (2.27) by $x_3^2$, gives six equations in six unknowns which can be written in matrix format as
The set of six ‘homogeneous’ equations will have a solution only if the determinant of the coefficient matrix equals zero. Thus

\[
\begin{vmatrix}
0 & 0 & 0 & W_2 & W_1 & W_0 \\
0 & 0 & W_2 & W_1 & W_0 & 0 \\
0 & W_2 & W_1 & W_0 & 0 & 0 \\
W_2 & W_1 & W_0 & 0 & 0 & 0 \\
0 & V_4 & V_3 & V_2 & V_1 & V_0 \\
V_4 & V_3 & V_2 & V_1 & V_0 & 0 \\
\end{vmatrix} \begin{bmatrix} x_3^1 \\ x_3^2 \\ x_3^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\] (2.29)

The terms \( V_i \) and \( W_i \) are polynomials in the term \( x_1 \). Expanding this determinant yields a 16th degree polynomial in \( x_1 \). Each value of \( x_1 \) obtained by the polynomial can be substituted back into Equation (2.29) to obtain corresponding values for \( x_3 \). Once the corresponding values of \( x_3 \) have been determined for each value of \( x_1 \), Equations (2.31) and (2.32) are used to determine the corresponding values of \( x_2 \) as

\[
x_2 = -\begin{vmatrix}
M_1 & N_1 \\
M_2 & N_2 \\
L_1 & N_1 \\
L_2 & N_2 \\
\end{vmatrix}
\] (2.31)

or
25

\[ x_2 = \begin{vmatrix} L_1 & N_1 \\ L_2 & N_2 \\ L_1 & M_1 \\ L_2 & M_2 \end{vmatrix} \]  

(2.32)

2.5 Numerical Conformation of Results

To confirm that the results are correct, a model was constructed using CAD software (see Figure 2-1). After this model was constructed, all of the dimensions for the model were measured. The following information was used as input for the numerical case:

Given:

Point P in 1\textsuperscript{st} Coordinate System: \( (6.2486, 10, -3.4732) \)

Point P in 2\textsuperscript{nd} Coordinate System: \( (1.7590, 2, -1.9251) \)

Point B\textsubscript{1} in First Coordinate System: \( (0, 0, 0) \)

Point B\textsubscript{2} in First Coordinate System: \( (10, 0, 0) \)

Point B\textsubscript{3} in First Coordinate System: \( (6.75, 0, -5.6292) \)

Point T\textsubscript{1} in Second Coordinate System: \( (0, 0, 0) \)

Point T\textsubscript{2} in Second Coordinate System: \( (4.9500, 0, 0) \)

Point T\textsubscript{3} in Second Coordinate System: \( (0.2658, 0, -5.5675) \)

Constant Strut Lengths: \( L_2=11.035, L_5=9.970, L_8=9.455 \)
Table 2-2 shows the calculated lengths of all fifteen of the line segments shown in Figure 2-1. Table 2-3 presents the calculated values for the coefficients of equations (2.18) through (2.20). For this case, four of the sixteen solutions of equation (2.30) were real. Table 2-4 presents the calculated values for the parameters $\theta_1$, $\theta_2$, and $\theta_3$ for these four cases. The table also shows that one of the solutions matches that of the CAD model.

Table 2-2. Leg Lengths

<table>
<thead>
<tr>
<th>Leg</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.292</td>
</tr>
<tr>
<td>2</td>
<td>11.035</td>
</tr>
<tr>
<td>3</td>
<td>3.286</td>
</tr>
<tr>
<td>4</td>
<td>11.231</td>
</tr>
<tr>
<td>5</td>
<td>9.970</td>
</tr>
<tr>
<td>6</td>
<td>4.232</td>
</tr>
<tr>
<td>7</td>
<td>10.242</td>
</tr>
<tr>
<td>8</td>
<td>9.455</td>
</tr>
<tr>
<td>9</td>
<td>4.415</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>6.500</td>
</tr>
<tr>
<td>12</td>
<td>8.789</td>
</tr>
<tr>
<td>13</td>
<td>5.573</td>
</tr>
<tr>
<td>14</td>
<td>4.954</td>
</tr>
<tr>
<td>15</td>
<td>7.278</td>
</tr>
</tbody>
</table>
Table 2-3. Coefficients of Equations (2.18) through (2.20)

<table>
<thead>
<tr>
<th></th>
<th>$A_3$</th>
<th>$A_8$</th>
<th>$A_7$</th>
<th>$A_6$</th>
<th>$A_5$</th>
<th>$A_4$</th>
<th>$A_3$</th>
<th>$A_2$</th>
<th>$A_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_9$</td>
<td>0.1541</td>
<td>0</td>
<td>0.8172</td>
<td>0</td>
<td>3.2010</td>
<td>0</td>
<td>0.6450</td>
<td>0</td>
<td>-0.964</td>
</tr>
<tr>
<td>$B_9$</td>
<td>0.6988</td>
<td>1.6103</td>
<td>0.5053</td>
<td>-0.8319</td>
<td>-0.4803</td>
<td>1.811</td>
<td>0.8229</td>
<td>-1.6103</td>
<td>0.2352</td>
</tr>
<tr>
<td>$D_9$</td>
<td>0.7258</td>
<td>-0.7830</td>
<td>-0.4451</td>
<td>-0.4650</td>
<td>-0.4520</td>
<td>1.6870</td>
<td>-0.3038</td>
<td>1.7045</td>
<td>0.0223</td>
</tr>
</tbody>
</table>

Table 2-4. Results of Analysis

<table>
<thead>
<tr>
<th></th>
<th>CAD Model Angle (degrees)</th>
<th>CAD Model Tan-Half Angle</th>
<th>Analysis Results (1)</th>
<th>Analysis Results (2)</th>
<th>Analysis Results (3)</th>
<th>Analysis Results (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_1$</td>
<td>100.749</td>
<td>1.2077</td>
<td>1.2077</td>
<td>-2.307</td>
<td>4.1559</td>
<td>4.6202</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>210.122</td>
<td>-3.716</td>
<td>-3.716</td>
<td>0.3637</td>
<td>-1.057</td>
<td>-2.153</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>243.046</td>
<td>-1.630</td>
<td>-1.630</td>
<td>1.0044</td>
<td>0.5439</td>
<td>0.1679</td>
</tr>
</tbody>
</table>
2.6 Orientation of Top Platform Using Quaternions

Now that the angles $\theta_1$, $\theta_2$, and $\theta_3$ have been determined, a procedure follows that shows how to determine the coordinates of points $T_1$, $T_2$, and $T_3$ in terms of the ground coordinate system.

Determining the Coordinate of Point $T_1$

1. Determine the absolute distance from Point $B_1$ to Point $P$.

\[
d = \left| P - B_1 \right| \tag{2.33}
\]

2. Obtain a unit vector going from Point $B_1$ to Point $P$.

\[
\lambda = \frac{P - B_1}{d} \tag{2.34}
\]

3. If two spheres of radii $L_2$ and $L_3$ are centered on Points $B_1$ and $P$, respectively, their intersection would be a circle centered on line $\lambda$. Also, this circle would lie on a plane perpendicular to line $\lambda$. The distance from point $B_1$ to the center of this circle is given by Equation (2.35) and the radius of this circle is given by Equation (2.36).

\[
D = \frac{d^2 - L_3^2 + L_2^2}{2d} \tag{2.35}
\]

\[
r = \frac{1}{2d} \sqrt{4d^2L_2^2 - (d^2 - L_3^2 + L_2^2)^2} \tag{2.36}
\]
4. \( S \) is defined as a vector perpendicular to the plane formed by the lines 1-7.

\[
S = \frac{1}{|P_{-1}B_3|} \times \lambda
\]  

(2.37)

5. \( \mu \) is defined as a vector perpendicular to the line \( \lambda \) that lies in the plane formed by the lines 1-7.

\[
\mu = \frac{\lambda \times S}{|\lambda \times S|}
\]

(2.38)

6. Now define point \( E_1 \) as shown in Figure 2-10.

\[
P_{E_1} = P_{E_1} + D \ast \lambda + r \ast \mu
\]

7. To find the point \( T_1 \) we simply use the quaternion method for rotating the point \( E_1 \) about line 1 by the angle \( \theta_1 \).

\[
q = \cos \left( \frac{\theta_1}{2} \right) + \sin \left( \frac{\theta_1}{2} \right) \lambda
\]

(2.39)

\[
P_{T_1} = q(P_{E_1})q^{-1}
\]

(2.40)
The same method can be used to find Points $T_2$ and $T_3$. Once the coordinates of points $T_1$, $T_2$, and $T_3$ are known in the fixed coordinate system, it is a straightforward task to determine the relative rotation matrix $\frac{1}{2} R$.

Figure 2-10. Definition of Point E1

For our particular analysis, we obtained 4 real orientations of the top platform.

Therefore, the analysis will have to be repeated if all four orientations are to be determined. These four orientations are shown in Figure 2-11 through 2-14.
Figure 2-11. Orientation from Analysis Results 1

Figure 2-12. Orientation from Analysis Results 2
Figure 2-13. Orientation from Analysis Results 3

Figure 2-14. Orientation from Analysis Results 4
CHAPTER 3
MOTION ANALYSIS USING RECIPROCAL SCREWS

Now that all of the possible orientations of the top platform have been established; the next goal is to find a way to minimize the work required in moving the top platform to another specified location $P_{time,2}$. In other words, consider that the $6 \times 6$ platform supports an external force/moment combination (wrench) on the top platform (see Figure 3-1) to maintain static equilibrium. The goal is to find a way to move the top platform while minimizing any work produced by this external wrench.

Figure 3-1. $6 \times 6$ Platform Supporting External Wrench
Before undertaking any analysis, it is necessary to define two terms. The first of the terms are Plücker line coordinates. Plücker line coordinates can be used to describe a line connected by two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. The equations for the Plücker line coordinates are shown in Equation (3.1). An attractive property of Plücker line coordinate is that they are homogeneous. Homogeneous coordinates imply that if all the quantities shown in Equation (3.1) are multiplied by the same scalar, the coordinates would define the same line in space.

\[
\begin{align*}
I &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
m &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \\
n &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
p &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \\
q &= \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\
r &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}
\end{align*}
\]

This chapter will also discuss screws. A screw can be thought of as a line with a pitch “h”. A screw $s$ can be written as in equation (3.2). Screws have two unique properties: 1. A screw multiplied by an angular velocity magnitude describes the motion of one body relative to another. 2. A screw multiplied by a force becomes a force/moment combination acting on a body. This force/moment combination is called a wrench.

Also discussed in this chapter is the concept of reciprocal screws. For this study, a reciprocal screw can be thought of as any motion(s) that will not cause work to be done by an applied force/moment combination (wrench). For this particular platform, screws will be determined that are reciprocal to the external wrench shown in Figure 3-1.
The problem is stated below.

Given

- All Constant Geometrical Parameters
- Position & Orientation of Top Platform
- External Wrench Applied to the Top Platform

Find

- Motions (twists) of Top Platform to Eliminate Work Caused by External Wrench

Outline of Solution

The solution of this problem will be outlined from Crane and Duffy.\(^\text{10}\) If we consider an external wrench acting on the top platform “\(f_{\text{ext}} \ S_{\text{ext}}\)”; any screw that is reciprocal to this external wrench must satisfy Equation (3.3).

\[
L_{\text{ext}} P_{\text{recip.}} + M_{\text{ext}} Q_{\text{recip.}} + N_{\text{ext}} R_{\text{recip.}} + L_{\text{recip}} P_{\text{ext}} + M_{\text{recip}} Q_{\text{ext}} + N_{\text{recip}} R_{\text{ext}} = 0
\]  

The values of \(L_{\text{ext}}...R_{\text{ext}}\) are of known quantities which represent the six coordinates of the wrench or twist. Equation (3.3) is a homogeneous equation that consists of terms \(L_r...R_r\). These terms consist of five independent parameters. In order to find the values of

\[
S = \{l, m, n; p + lh, q + mh, r + nh\}
\]  

(3.2)
$L_{\text{recip}}...R_{\text{recip}}$, we simply select five of these values and compute the six value from Equation (3.3).
CHAPTER 4
CONCLUSION

This study successfully analyzed a three degree of freedom tensegrity platform that incorporates the special $6 \times 6$ platform geometry. This analysis shows that if a coordinate system attached to the bottom platform is specified, and a predetermined location of a point attached to the top platform is also specified, then the orientation of the top platform with respect to the coordinate system of the bottom platform can be determined. This analysis focused on a closed form algebraic solution of the platform. The analysis proceeding by defining three dependent spherical quadrilaterals. Although iterative methods would have also given the orientation of the top platform, these methods would leave out critical information. By using iterative procedures, one would never know how many orientations are possible of the top platform for a given position $P$. For example, a numerical case shown in this thesis identified four possible orientations of the top platform for the given point $P$. Using an iterative procedure, the solution of the orientation of the top platform could possibly converge at any one of the four orientations. The solution of the iterative method is almost entirely based on the initial values given to the method. Given this analysis is much more rigorous than iterative methods, the information obtained about the system makes up for the additional effort. It was learned that there are sixteen possible orientations of the top platform.

This analysis also shows once the orientation of the top platform is determined, screws that are reciprocal to the external wrench can be determined. These reciprocal
screws form the basis of continued analysis on ways to minimize the energy required to move the top platform.
APPENDIX

Coefficients of First Spherical Quadrilateral

\[ A_9 = a_1 \]
\[ A_7 = a_2 \]
\[ A_5 = a_4 \]
\[ A_3 = a_3 \]
\[ A_1 = a_5 \]
\[ A_8 = A_6 = A_4 = A_2 = 0 \]

Coefficients of Second Spherical Quadrilateral

\[ B_9 = -S \cos(\gamma_2) - T + Q \cos(\gamma_2) + U \]
\[ B_8 = 2R \sin(\gamma_2) \]
\[ B_7 = T - Q \cos(\gamma_2) - S \cos(\gamma_2) + U \]
\[ B_6 = 2Q \sin(\gamma_2) - 2S \sin(\gamma_2) \]
\[ B_5 = -4R \cos(\gamma_2) \]
\[ B_4 = -2Q \sin(\gamma_2) - 2S \sin(\gamma_2) \]
\[ B_3 = S \cos(\gamma_2) - Q \cos(\gamma_2) + U - T \]
\[ B_2 = -2R \sin(\gamma_2) \]
\[ B_1 = Q \cos(\gamma_2) + S \cos(\gamma_2) + T + U \]

where,

\[ Q = -s_{4-6}c_{7-4}s_{9-7} \]
\[ R = s_{4-6}s_{9-7} \]
\[ S = -c_{4-6}s_{7-4}s_{9-7} \]
\[ T = -s_{4-6}s_{7-4}c_{9-7} \]
\[ U = c_{4-6}c_{7-4}c_{9-7} - c_{6-9} \]
Coefficients of Third Spherical Quadrilateral

\[
D_0 = W \sin(\gamma_3) \sin(\gamma_1) + V \cos(\gamma_3) \cos(\gamma_1) - X \cos(\gamma_3) - Y \cos(\gamma_1) + Z
\]
\[
D_4 = -2W \sin(\gamma_3) \cos(\gamma_1) - 2Y \sin(\gamma_1) + 2V \cos(\gamma_3) \sin(\gamma_1)
\]
\[
D_8 = -X \cos(\gamma_3) - W \sin(\gamma_3) \sin(\gamma_1) - V \cos(\gamma_3) \cos(\gamma_1) + Y \cos(\gamma_1) + Z
\]
\[
D_6 = 2V \sin(\gamma_3) \cos(\gamma_1) - 2X \sin(\gamma_3) - 2W \cos(\gamma_3) \sin(\gamma_1)
\]
\[
D_{10} = 4V \sin(\gamma_3) \sin(\gamma_1) + 4W \cos(\gamma_3) \cos(\gamma_1)
\]
\[
D_4 = -2X \sin(\gamma_3) + 2W \cos(\gamma_3) \sin(\gamma_1) - 2V \sin(\gamma_3) \cos(\gamma_1)
\]
\[
D_8 = -Y \cos(\gamma_1) + X \cos(\gamma_3) - W \sin(\gamma_3) \sin(\gamma_1) - V \cos(\gamma_3) \cos(\gamma_1) + Z
\]
\[
D_{12} = -2Y \sin(\gamma_1) - 2V \cos(\gamma_3) \sin(\gamma_1) + 2W \sin(\gamma_3) \cos(\gamma_1)
\]
\[
D_4 = Y \cos(\gamma_1) + W \sin(\gamma_3) \sin(\gamma_1) + V \cos(\gamma_3) \cos(\gamma_1) + X \cos(\gamma_3) + Z
\]

where,

\[
V = -s_{1\rightarrow 3} c_{4\rightarrow 1} s_{6\rightarrow 4}
\]
\[
W = s_{1\rightarrow 3} s_{6\rightarrow 4}
\]
\[
X = -c_{1\rightarrow 3} s_{4\rightarrow 1} s_{6\rightarrow 4}
\]
\[
Y = -s_{1\rightarrow 3} c_{4\rightarrow 1} c_{6\rightarrow 4}
\]
\[
Z = c_{1\rightarrow 3} c_{4\rightarrow 1} c_{6\rightarrow 4} - c_{3\rightarrow 6}
\]
REFERENCES


BIOGRAPHICAL SKETCH

Antoin Lenard Baker was born on 1981 in Portsmouth, Virginia. In 2003, he received his Bachelor of Science in Mechanical Engineering from the University of Florida. After graduating, he joined the Army Reserves while deciding to continue his education in mechanical engineering at the University of Florida.