SOME TOPICS IN q-SERIES AND PARTITIONS

By

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To my Mom...

and to Gainesville, FL

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By

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This dissertation presents various studies in the theory of partitions. We present classical and weighted partition identities and related q-series results. This work is divided into four chapters. The chapters are mostly independent. The first chapter only introduces the objects of study. It can be seen as a reference guide of definitions and notations. The rest of the chapters are this work's independent main parts.

In the first part, we discuss our generalization of the work of Stanley–Boulet. We later focus on the implications of these extensions for the theory of partitions. We generalize a result due to Savage–Sills after studying partitions with fixed number of odd and even indexed odd parts. We prove generating functions for the number of partitions with a prescribed BG-rank and show the implications of these on Rogers–Szegő polynomials and some q-hypergeometric series. We finish this section by studying our companion of the Capparelli's identities.

In the second part, we utilize false theta function results of Nathan Fine to discover four new partition identities involving weights. These relations connect Göllnitz–Gordon type partitions and partitions with distinct odd parts, partitions into distinct parts and ordinary partitions, and partitions with distinct odd parts where the smallest positive integer that is not a part of the partition is odd and ordinary partitions subject to some initial conditions, respectively. We finish this chapter by applying our methods to a false theta relation of Ramanujan. We interpret the related result as a weighted partition identity.

The last part comprises of weighted partition identities related with the alternating weights related to the parity of the smallest part of a partition. We show a relation between weighted partition counts with an alternating weight and the number of representation of numbers as sums of two squares. We discuss more weighted identities relating partitions with distinct even parts and the triangular numbers. Similar to the previous part, we finish this chapter with an interpretation of an identity of Ramanujan and get interesting weighted partition identities.

CHAPTER 1
INTRODUCTION

The first notable account on integers partitions is due to Leibniz [46]. In his letter to Bernoulli, he raised the question of finding the number of all the essentially different representations of a given positive integer n as the sum of positive integers, such as representing 3 as 3 only, $2+1$, or $1+1+1$. Although, Leibniz is the pioneer of what we today call integer partitions, the Theory of Partitions as it is understood nowadays can be said to excel with the great Euler [40].

The simple question of representing positive numbers as a sum of positive integers proves to be connected with many branches not only in mathematics, but also in computer science, theoretical physics, statistics and such. Standing in the middle of Number Theory, Analysis and Combinatorics, the Theory of Partitions is a rich field that caught interest of many.

For more on the subject, its history, and a great excursion in the field, the interested reader is invited to examine Andrews' encyclopaedia [17].

As we move on, we want to note that, the general field of partitions roughly has three interlacing subfields due to the nature of questions: counting problems, congruence problems, and asymptotic problems. This account and author's recent work is more geared towards the counting problems of partitions.

1.1 Integer Partitions

A *partition* π is a non-increasing finite sequence $\pi = (\lambda_1, \lambda_2, \dots)$ of positive integers. The elements λ_i that appear in the sequence π are called *parts* of π. For positive integers i, we call λ_{2i-1} odd-indexed parts, and λ_{2i} even indexed parts of π.

Some widely used statistics on partitions are as follows:

$$\nu_e(\pi) := \textit{number of even parts in } \pi, \tag{1–1}$$

$$\nu_o(\pi) := \textit{number of odd parts in } \pi, \tag{1–2}$$

$$\nu(\pi) := \textit{number of parts in } \pi, = \nu_e(\pi) + \nu_o(\pi), \tag{1–3}$$

$$\nu_d(\pi) := \textit{number of different parts in } \pi, \tag{1–4}$$

$$|\pi| := \textit{norm of the partition } \pi = \sum_{i=1}^{\nu(\pi)} \lambda_i, \tag{1–5}$$

$$s(\pi) := \textit{smallest part of the partition } \pi = \lambda_{\nu(\pi)}, \tag{1–6}$$

$$l(\pi) := \textit{largest part of the partition } \pi = \lambda_1.. \tag{1–7}$$

We call π a partition of n if $|\pi| = n$. Conventionally, the empty sequence is considered as the unique partition of zero. We will abide by this definition as well. We define the number of parts, smallest part and the largest part of the empty sequence all to be 0 for completion.

As an example, $\pi = (10, 9, 5, 5, 4, 1, 1)$ is a partition of 35. The number of even parts of π, $\nu_e(\pi)$, is 2. and the rest of the defined statistics are $\nu_o(\pi) = 5$, $\nu(\pi) = 7$, $|\pi| = 35$, $s(\pi) = 1$, and $l(\pi) = 10$.

An equivalent definition of partitions is the *frequency notation* [17]. We write a partition π in the frequency notation as $\pi = (1^{f_1}, 2^{f_2}, \dots)$, where the exponents, $f_i(\pi)$ of i, are non-negative integers and they denote the number of appearances of the part i in π. We abuse the notation and write f_i, frequency of i, when the partition is understood from the context. Similarly, we can drop the zero frequencies in our notation to keep the notations neater. A zero frequency may still be used to emphasize an integer not being a part of a partition.

The example partition $\pi = (10, 9, 5, 5, 4, 1, 1)$ can be represented in the frequency notation as $(1^2, 2^0, 3^0, 4^1, 5^2, 6^0, 7^0, 8^0, 9^1, 10^1, 11^0, \dots) = (1^2, 4, 5^2, 7^0, 9, 10)$. Here π is a partition, where the frequency of 1: $f_1(\pi) = f_1 = 2$, $f_4 = 1$, $f_5 = 2 \dots$ and the integer 7 is not a part of the partition π.

Previously defined statistics can as easily be defined in this notation. We will define the statistics once again to underline the equivalence of the definitions.

$$\nu_e(\pi) := \sum_{i \geq 0} f_{2i},$$

$$\nu_o(\pi) := \sum_{i \geq 0} f_{2i+1},$$

$$\nu(\pi) := \sum_{i\geq 0} f_i,$$

$$\nu_d(\pi) := \sum_{i\geq 0}(1-\delta_{f_i,0}),$$

$$|\pi| := \sum_{i\geq 0} f_i \cdot i,$$

$$s(\pi) := \min_{i\geq 0}\{i, f_i \neq 0\},$$

$$l(\pi) := \max_{i\geq 0}\{i, f_i \neq 0\},$$

where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise is the Kronecker δ function.

1.2 Visual Representations

1.2.1 Young Diagrams

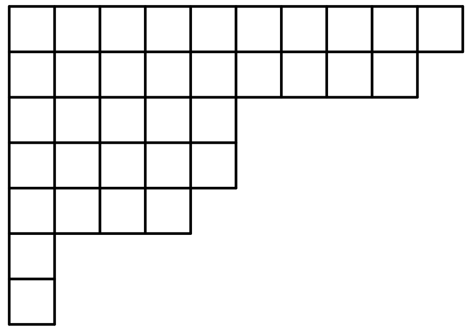
For visualization purposes partitions can be represented graphically in multiple ways. We are going to focus on representing partitions using Young diagrams and some of its derivatives. A Young diagram of a partition $\pi = (\lambda_1, \lambda_2, \dots)$ is a table of left-aligned rows of boxes, which has λ_i boxes on its i-th row. Whence, the number of boxes on i-th row gives the size of the part λ_i.

Another representation with dots instead of boxes are also prominent in partition theory literature and those diagrams are called Ferrers diagrams. Author is not against the use of either name in the context and both would mean a table drawn with boxes, but for consistency this work will stick with the name Young diagrams.

There is a one-to-one correspondence between Young diagrams and partitions. The words *partition* and *Young diagram* can be used interchangeably. An example of a Young diagram is given in Figure 1-1.

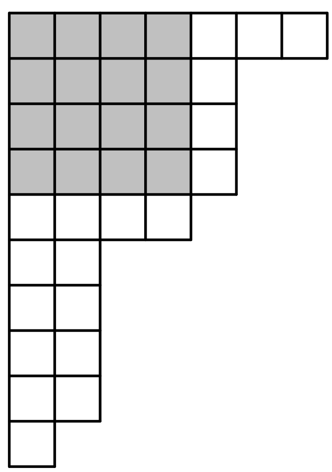
Given a partition, it is easy to see that there is a unique largest inscribed square, called *Durfee square*, with one vertex on the top left vertex of the Young diagram. Instead of reading a Young diagram of a partition π row-wise one can it column-wise a partition. This partition is called the *conjugate partition* of π. Another way of visualizing the conjugation is to reflect

Figure 1-1. The Young Diagram of the partition $\pi = (10, 9, 5, 5, 4, 1, 1)$.



the Young diagram of the partition over the Durfee square's diagonal, which passes through the very top left of the Young diagram. It should be noted that the Durfee square itself does not get affected by the conjugation. If a partition is equal to its conjugate then that partition is called a *self-conjugate* partition. The conjugate of the partition $\pi = (10, 9, 5, 5, 4, 1, 1)$ is $\pi* = (7, 5, 5, 5, 4, 2, 2, 2, 2, 1)$. The Young diagram of $\pi*$ if given in Figure 1-2 where the Durfee square is shaded.

Figure 1-2. The Conjugate Young Diagram of $\pi = (10, 9, 5, 5, 4, 1, 1)$, where the Durfee square is also identified.



We will now introduce three derivatives of the classical Young diagrams, which will be useful in our studies later.

1.2.2 2-modular Young Diagrams

We define the 2-modular Young diagram similar to the Young diagrams. Let $\lceil x \rceil$ denote the smallest integer $\geq x$. For a given partition $\pi = (\lambda_1, \lambda_2, \dots)$, we draw $\lceil \lambda_i/2 \rceil$ many boxes at the i-th row. We decorate the boxes on the i-th row with 2's with the option of having a 1 at the right most box of the row, such that the sum of the numbers in the boxes of the i-th row becomes λ_i. Figure 1-3 is an example of a 2-modular Young diagram.

Figure 1-3. The 2-Modular Young Diagram of $\pi = (10, 9, 5, 5, 4, 1, 1)$.

2	2	2	2	2
2	2	2	2	1
2	2	1		
2	2	1		
2	2			
1				
1				

It should be noted that the conjugation of partitions does not carry over to 2-modular Young diagrams. The conjugation of the Young Diagram of Figure 1-3 would have two rows of boxes where more than one box includes a 1 on the inside. In general, only the partitions with distinct odd parts (which may still include even parts with repetition) yield admissible conjugates for their 2-modular Young Diagrams.

1.2.3 4-decorated Young Diagrams

For later generalizations of the subject matter, here we define the 4-decorated Young diagrams. Introduced by Stanley [52] and implemented by Boulet [33], these diagrams has been at the heart of many results of the author.

We can decorate any Young diagram of a partition with variables a, b, c, and d. We fill the boxes on the odd-indexed rows with alternating variables a and b starting from a, and boxes on the even-indexed rows filled with alternating variables c and d starting from c. These diagrams are called *4-decorated Young Diagrams*. One example of 4-decorated Young diagram is given in Figure 1-4 for our running example $\pi = (10, 9, 5, 5, 4, 1, 1)$.

Figure 1-4. The 4-Decorated Young Diagram of $\pi = (10, 9, 5, 5, 4, 1, 1)$.

a	b	a	b	a	b	a	b	a	b
c	d	c	d	c	d	c	d	c	
a	b	a	b	a					
c	d	c	d	c					
a	b	a	b						
c									
a									

One can define a weight of a 4-decorated diagram π as

$$\omega_\pi(a, b, c, d) = a^{\#a} b^{\#b} c^{\#c} d^{\#d}, \tag{1–8}$$

where "$\#a$" means the *number of a's in the diagram* of π. It is easy to see that, this weight with the choice of $a = b = c = d = q$ takes partitions to $q^{|\pi|}$.

1.2.4 BG-rank and 2-residue Diagrams

Another important statistic used for partitions is the BG-rank [24]. The BG-rank of a partition π—denoted $BG(\pi)$—is defined as

$$BG(\pi) := i - j,$$

where i is the number of odd-indexed odd parts and j is the number of even-indexed odd parts [24]. Another equivalent representation of BG-rank of a partition π comes from 2-residue Young diagrams. The 2-*residue Young diagram* of partition π is given by taking the ordinary Young diagram with filled boxes using alternating 0's and 1's starting from 0 on odd-indexed parts and 1 on even -indexed parts. We can exemplify 2-residue diagrams with $\pi = (10, 9, 5, 5, 4, 1, 1)$ in Figure 1-5. With this definition, one can show that

$$BG(\pi) = r_0 - r_1,$$

Figure 1-5. The 2-Residue Young Diagram of $\pi = (10, 9, 5, 5, 4, 1, 1)$.

0	1	0	1	0	1	0	1	0	1
1	0	1	0	1	0	1	0	1	
0	1	0	1	0					
1	0	1	0	1					
0	1	0	1						
1									
0									

where r_0 is the number of 0's in the 2-residue diagram of π, and r_1 is the number of 1's in the 2-residue diagram. BG-rank of the partition $\pi = (10, 9, 5, 5, 4, 1, 1)$ in the example of Table 1-5 is equal to -1.

One can view a 2-residue diagram as the 4-decorated diagram with the choice $(a, b, c, d) = (1, 0, 0, 1)$.

1.3 Generating Functions

This section includes a basic ground work for the analytical treatment of partitions and includes examples of some generating functions to increase the familiarity of the reader with the subject. More involved analytical results to be used are collected under Appendix A. We start the section with an abstract definition.

Definition 1. *Given a sequence $\{a_n\}_{n=0}^{\infty}$, the formal series*

$$\sum_{n\geq 0} a_n q^n, \tag{1–9}$$

is the generating function of this sequence.

Generating functions is a bookkeeping tool and an enveloping counting objects from a set that are organized by some statistics. Generating functions can be written in different formats. Abstractly, a generating function for the numbers of objects a from a set $\mathcal{A}$ counted with respect to some statistic $\Lambda : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ can be written as

$$\sum_{a\in\mathcal{A}} q^{\Lambda(a)}, \tag{1–10}$$

where q is the formal variable keeping the index as its exponent. For the convergence of this generating function, $\Lambda(a) = n$ must have finitely many solutions for any $n \in \mathbb{Z}_{\geq 0}$.

The choice of statistics (as far as the author knows) classically has been the norm of partitions. This statistics satisfies all the requirements for yielding a well defined/convergent power series for the number of partitions. Interested reader can check the recent work [54] of the author for an example of generating functions grouped with respect to a different partition statistics.

Let $\mathcal{U}$ be the set of all partitions. The generating function for the number of partitions grouped with respect to their norms can be written as

$$\sum_{\pi \in \mathcal{U}} q^{|\pi|}. \tag{1–11}$$

Let $p(n)$ be the number of partitions of n, then the series

$$\sum_{n \geq 0} p(n) q^n \tag{1–12}$$

is an equivalent representation of (1–11). In general, this type of series, where the coefficient of q^n is specified, is called the *enumerative* form of generating function. By some simple exploration, it is easy to see that

$$\sum_{\pi \in \mathcal{U}} q^{|\pi|} = \sum_{n \geq 0} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \dots, \tag{1–13}$$

where the partitions for the first five non-zero norms are given in Table 1-1.

Table 1-1. List of all the partitions with norms 1, 2, 3, 4, and 5.

(1)
(2), (1,1)
(3), (2,1), (1,1,1)
(4), (3,1), (2,2), (2,1,1), (1,1,1,1)
(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1)

It is desirable to be able to write an explicit formula the series (1–13), or for any generating function for that matter. For this task we define the *q-Pochhammer symbol* (also called *rising q-factorials*):

$$(a;q)_L := \prod_{i=0}^{L-1}(1-aq^i),$$

where a, q are variables and L is some non-negative integer. If $|q|<1$ the limit

$$(a)_L := (a;q)_\infty := \lim_{L\to\infty}(a;q)_L$$

exists.

Let $\mathcal{U}_m$ be the set of partitions into parts $\leq\ m$, and let $p(n,m)$ be the number of partitions of n into parts $\leq m$. One can show that

$$\sum_{\pi\in\mathcal{U}_m} q^{|\pi|} = \sum_{n\geq 0} p(n,m)q^n = \frac{1}{(q;q)_m}$$

by looking at the multiplication of the terms of type

$$\frac{1}{1-q^k} = 1 + q^k + q^{2\cdot k} + q^{3\cdot k} + \dots,$$

where k is some positive integer. One observation, by thinking about the conjugate partitions that, $(q;q)_m^{-1}$ is also the generating function for the number of partitions into $\leq\ m$ parts. Another observation is that, assuming $|q|<1$, it is evident that

$$\sum_{\pi\in\mathcal{U}} q^{|\pi|} = \sum_{n\geq 0} p(n) = \frac{1}{(q;q)_\infty}. \tag{1–14}$$

Let $\mathcal{D}_m$ be the set of partitions into distinct parts $\leq m$, and let $p_d(n,m)$ be the number of partitions of n into distinct parts $\leq m$. The generating function

$$\sum_{\pi\in\mathcal{D}_m} q^{|\pi|} = \sum_{n\geq 0} p_d(n,m)q^n = (-q;q)_m$$

This closed form can be understood as every factor $(1+q^k)$ of the product representing the statements "is the part k not a part of the partition?" (represented as multiplication by 1) or "is the part k a part of the partition?" (represented as multiplication by q^k) for part sizes $k \leq m$. Here it should be noted that since the conjugation of partitions into distinct parts do not necessarily yield partitions with distinct parts, one cannot change the constraint on the size of the parts to a constraint on the number of parts as in the previous example. From this discussion we also get

$$\sum_{\pi \in \mathcal{D}} q^{|\pi|} = \sum_{n \geq 0} p_d(n) = (-q; q)_\infty,$$

where $\mathcal{D}$ is the set of partitions into distinct parts, and $p_d(n)$ is the number of partitions of n into distinct parts.

We would also like to define the *q-binomial coefficients* (also called *Gaussian polynomials*). The generating function for the number of partitions into parts *leqn* where every part is $\leq m$ is given by

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q := \frac{(q)_{n+m}}{(q)_n (q)_m} = \frac{(q^{m+1})_n}{(q)_n}.$$

The q-binomial coefficients are symmetric in n and m. This symmetry is consistent with the conjugation of partitions and the generating function interpretation reflects that. Gaussian polynomials are the q-analog of the binomial coefficients and $q \to 1$ they converge to the binomial coefficient $\binom{n+m}{n}$.

CHAPTER 2
STANLEY–BOULET WEIGHTS, GENERALIZATIONS AND IMPLICATIONS

This chapter is devoted to an excursion of the papers [26] and [27]. The published works are organized differently, and have more results that will not be included here. The interested reader is invited to examine these papers. Here the main objective will be to give an introduction to the Stanley–Boulet weights and results that can be attained as their direct implications.

Section 2.1 goes over the introduction of the related literature and presents some mainlines of our extension of some of these results. In Section 2.2, we focus on the generating functions for the number of partitions where we fix the number of odd-indexed and even-indexed odd parts. Also included in this section, we will cover some partition identities including the generalization of a result due to Savage and Sills. We continue with Section 2.3; we discuss partitions, and generating functions for the number of partitions with a fixed value of BG-rank. In Section 2.4, we will discuss some implications of the introduced work on Rogers–Szegő polynomials and q-hypergeometric series. The last section of this Chapter, introduces Capparelli's identities, their companion and our refinements of these results.

2.1 Partitions with Stanley–Boulet Weights and Generating Functions

In [52], Stanley made a suggestion for suitable weights to be used on diagrams. Boulet, in [33], extensively utilized this suggestion on a four-variable decoration of Young diagram of a partition as in Section 1.2. Recall that one can define a weight on these four-variable decorated diagrams as

$$\omega_\pi(a, b, c, d) = a^{\#a} b^{\#b} c^{\#c} d^{\#d},$$

where $\#a$ denotes the number of boxes decorated with variable a in partition π's four-variable decorated diagram, etc.

The generating function for the weighted count of 4-decorated Young diagrams of partitions from a set S with weight $\omega_\pi(a, b, c, d)$ is

$$\sum_{\pi \in S} \omega_\pi(a, b, c, d).$$

Let ρ_N be the obvious bijective map $\mathcal{U}_N \mapsto \mathcal{D}_N \times \mathcal{E}_N$, where one sets aside the greatest amount of even repetitions of a part in a partition π leaving a partition into distinct parts. Let $\mathcal{E}_N$ be this set of partitions into (set aside) parts less than or equal to N, where every part repeats an even number of times. Let $\rho : \mathcal{U} \mapsto \mathcal{D} \times \mathcal{E}$, where $\rho := \lim_{N\to\infty} \rho_N$ and $\mathcal{E} := \lim_{N\to\infty} E_N$.

In [33], Boulet proved identities for the generating functions for weighted count of partitions with 4-decorated Young diagrams from the sets $\mathcal{D}$ and $\mathcal{U}$.

Theorem 2.1 (Boulet)**.** *For variables a, b, c, and d and $Q := abcd$, we have*

$$\Psi(a, b, c, d) := \sum_{\pi \in \mathcal{D}} \omega_\pi(a, b, c, d) = \frac{(-a, -abc; Q)_\infty}{(ab; Q)_\infty}, \tag{2–1}$$

$$\Phi(a, b, c, d) := \sum_{\pi \in \mathcal{U}} \omega_\pi(a, b, c, d) = \frac{(-a, -abc; Q)_\infty}{(ab, ac, Q; Q)_\infty}. \tag{2–2}$$

The transition from (2–2) to (2–1) can be done with the aid of the bijective map ρ. The generating function for the weighted count of four-variable Young diagrams which exclusively have an even number of rows of the same length is

$$\frac{1}{(ac, Q; Q)_\infty}.$$

Hence, we get

$$\Phi(a, b, c, d) = \frac{\Psi(a, b, c, d)}{(ac, Q; Q)_\infty}. \tag{2–3}$$

It should be noted that these generating functions are consistent with the ordinary generating functions for number of partitions, as in Section 1.3, when all variables are selected to be q. For example,

$$\Phi(q, q, q, q) = \frac{1}{(q; q)_\infty},$$

which is nothing but (1–14).

Define the generating functions

$$\Psi_N(a,b,c,d) := \sum_{\pi \in \mathcal{D}_N} \omega_\pi(a,b,c,d), \tag{2–4}$$

$$\Phi_N(a,b,c,d) := \sum_{\pi \in \mathcal{U}_N} \omega_\pi(a,b,c,d), \tag{2–5}$$

which are finite analogues of Boulet's generating functions for the weighted count of four-variable decorated Young diagrams. In [47], Ishikawa and Zeng write explicit formulas for (2–4) and (2–5).

Theorem 2.2 (Ishikawa, Zeng)**.** *For a non-zero integer N, variables a, b, c, and d, we have*

$$\Psi_{2N+\nu}(a,b,c,d) = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_Q (-a;Q)_{N-i+\nu}(-c;Q)_i(ab)^i, \tag{2–6}$$

$$\Phi_{2N+\nu}(a,b,c,d) = \frac{1}{(ac;Q)_{N+\nu}(Q;Q)_N} \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_Q (-a;Q)_{N-i+\nu}(-c;Q)_i(ab)^i, \tag{2–7}$$

where $\nu \in \{0,1\}$ and $Q = abcd$.

We want to point out that the case of (2–7) with $(a,b,c,d) = (qzy, qy/z, qz/y, q/zy)$ was first discovered and proven by Andrews in [12].

Similar to (2–3), the connection between (2–6) and (2–7) can be obtained by means of the bijection ρ_N. In this way we have

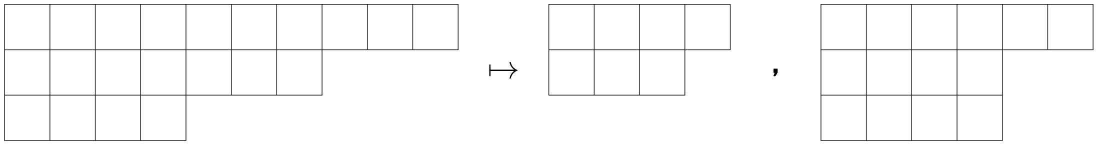
$$\Phi_{2N+\nu}(a,b,c,d) = \frac{\Psi_{2N+\nu}(a,b,c,d)}{(ac;Q)_{N+\nu}(Q;Q)_N}, \tag{2–8}$$

where N is a non-negative integer and $\nu \in \{0,1\}$.

In the rest of this section we will extend Boulet's approach to the weighted partitions with bounds on the number of parts and largest parts, Theorem 2.1.

Let $\widetilde{D}_N$ be the set of partitions into parts $\leq N$, where the difference between an odd-indexed part and the following non-zero even-indexed part is ≤ 1. Let $\widetilde{\mathcal{E}}_N$ be the set of partitions into parts $\leq N$, where the Young diagrams of these partitions, exclusively, have odd-height columns, and every present column size repeats an even number of times. Define $\widetilde{\mathcal{D}}$ and $\widetilde{E}$ similar to the sets $\widetilde{\mathcal{D}}_N$ and $\widetilde{\mathcal{E}}_N$, where we remove the restriction on the largest part.

Figure 2-1. An example of the map ρ^*.



$$\rho^*((10,7,4)) = ((4,3),(6,4,4))$$

Let $\rho^* : \mathcal{U} \mapsto \widetilde{\mathcal{D}} \times \widetilde{\mathcal{E}}$ ($\rho^*_N : \mathcal{U}_N \mapsto \widetilde{\mathcal{D}}_N \times \widetilde{\mathcal{E}}_N$) be the similar map to ρ (ρ_N). Let $\widetilde{\pi}$ be a partition in U_N. The image $\rho^*_N(\widetilde{\pi}) = (\widetilde{\pi}`, \widetilde{\pi}^*)$ is obtained by extracting even number of odd height columns from $\widetilde{\pi}$'s Young diagram repeatedly, until there are no more repetitions of odd height columns in the Young diagram of $\widetilde{\pi}$. We put these extracted columns in $\widetilde{\pi}^*$, and the partition that is left after extraction is $\widetilde{\pi}`$. An example of this map is $\rho^*((10,7,4)) = (\widetilde{\pi}`, \widetilde{\pi}^*) = ((4,3),(6,4,4))$, as demonstrated in Figure 2-1.

With the definition of the bijection ρ^*_N, we can finitize Boulet's combinatorial approach. Let $\widetilde{\pi}$ be a fixed partition with largest part less than or equal to $2N+\nu$ for a non-negative integer N and $\nu \in \{0,1\}$. We look at $\rho^*_{2N+\nu}(\widetilde{\pi}) = (\widetilde{\pi}`, \widetilde{\pi}^*)$. Here $\widetilde{\pi}`$ is a partition in $\mathcal{D}_{2(N-k)+\nu}$ with the specified difference conditions. In this construction, k is half the number of parts in $\widetilde{\pi}^*$ which is a partition in $\mathcal{E}_{2k}$. The generating function for the weighted count of four-variable decorated Young diagrams of such a partition $\widetilde{\pi}`$ is

$$\frac{(-a;Q)_{N-k+\nu}(-abc;Q)_{N-k}}{(ac;Q)_{N-k+\nu}(Q;Q)_{N-k}}. \tag{2–9}$$

Similarly, the generating function for the weighted count of four-variable decorated Young diagrams of such $\widetilde{\pi}^*$ is

$$\frac{(ab)^k}{(Q;Q)_k}. \tag{2–10}$$

Hence, for $\nu \in \{0,1\}$, the generating function $\Phi_{2N+\nu}(a,b,c,d)$ for the weighted count of partitions with parts less than or equal to $2N+\nu$, is the sum over k of the product of two functions in (2–9) and (2–10). In this way, we arrive at

Theorem 2.3. *For a non-zero integer N, variables a, b, c, d, and $Q = abcd$, we have*

$$\Phi_{2N+\nu}(a,b,c,d) = \frac{1}{(Q;Q)_N}\sum_{i=0}^{N}\begin{bmatrix} N \\ i \end{bmatrix}_Q \frac{(-a;Q)_{i+\nu}(-abc;Q)_i}{(ac;Q)_{i+\nu}}(ab)^{N-i}, \tag{2–11}$$

$$\Psi_{2N+\nu}(a,b,c,d) = \sum_{i=0}^{N}\begin{bmatrix} N \\ i \end{bmatrix}_Q (-a;Q)_{i+\nu}(-abc;Q)_i\frac{(ac;Q)_{N+\nu}}{(ac;Q)_{i+\nu}}(ab)^{N-i}, \tag{2–12}$$

where $\nu \in \{0,1\}$.

We remark that we use (2–8) to derive (2–12). Observe that Theorem 2.3 is a perfect companion to Theorem 2.2. However, our derivation of Theorem 2.3, unlike Theorem 2.2, is completely combinatorial.

Next we rewrite (2–6) and (2–12) using hypergeometric notations as

$$\Psi_{2N+\nu}(a,b,c,d) = (ab)^N(-c;Q)_N(1+\nu a)\ {}_2\phi_1\left(\begin{matrix} Q^{-N},\ -aQ^{\nu} \\ -\frac{Q^{1-N}}{c}\end{matrix};Q,-d\right), \tag{2–13}$$

and

$$\begin{aligned}\Psi_{2N+\nu}(a,b,c,d) = (-a^2b)^N Q^{N\nu}(1+\nu a)&\frac{(-c,bdQ^{-N-\nu};Q)_N}{(-\frac{Q^{1-N}}{c};Q)_N}\times \\ &{}_3\phi_1\left(\begin{matrix} Q^{-N},\ -aQ^{\nu},\ -abc \\ acQ^{\nu}\end{matrix};Q,\frac{Q^N}{ab}\right).\end{aligned} \tag{2–14}$$

Comparing (2–13) and (2–14) we get

$$\begin{aligned}{}_2\phi_1\left(\begin{matrix} Q^{-N},\ -aQ^{\nu} \\ -\frac{Q^{1-N}}{c}\end{matrix};Q,-d\right) = \\ (-aQ^{\nu})^N\frac{(bdQ^{-N-\nu};Q)_N}{(-\frac{Q^{1-N}}{c};Q)_N}{}_3\phi_1\left(\begin{matrix} Q^{-N},\ -aQ^{\nu},\ -abc \\ acQ^{\nu}\end{matrix};Q,\frac{Q^N}{ab}\right).\end{aligned} \tag{2–15}$$

It is easy to check that (2–15) is nothing else but [42, (III.8)] with the choice of variables $q \mapsto Q$, $b \mapsto -aQ^{\nu}$, $c \mapsto -Q^{1-N}/c$, and $z \mapsto -d$ for $\nu \in \{0,1\}$.

In [12], Andrews notably finds the (2–11) representation of $\Phi_N(qzy,\ qy/z,\ qz/y,\ q/zy)$. He sees that there is a connection between both representations using ${}_3\phi_2$ transformation [42, (III.13)]. This empirical discovery is a direct consequence of (2–11) with the above shown variable choices.

We can further generalize (2–11) of Theorem 2.3 by putting bounds on the number of parts in the given partitions. Let $\Phi_{N,M}(a,b,c,d)$ be the generating function of partitions with weights, where every part is less than or equal to N, and the number of non-zero parts is less than or equal to M.

Theorem 2.4. *Let $\nu \in \{0,1\}$ and $2N+\nu$, M be positive integers, then*

$$\Phi_{2N+\nu,2M}(a,b,c,d) = \sum_{l=0}^{N} \begin{bmatrix} N-l+M-1 \\ N-l \end{bmatrix}_Q (ab)^{N-l} \sum_{m_2=0}^{l} (abc)^{m_2} Q^{\binom{m_2}{2}} \begin{bmatrix} l \\ m_2 \end{bmatrix}_Q \times$$
$$\sum_{m_1=0}^{l+\nu} a^{m_1} Q^{\binom{m_1}{2}} \begin{bmatrix} l+\nu \\ m_1 \end{bmatrix}_Q \sum_{n=0}^{M-m_1-m_2} \begin{bmatrix} M+l-n-m_1-m_2 \\ M-n-m_1-m_2 \end{bmatrix}_Q \frac{(Q^{l+\nu};Q)_n}{(Q;Q)_n} (ac)^n. \tag{2–16}$$

Proof of Theorem 2.4 utilizes the same maps ρ_N and ρ_N^* as in the proof of Theorem 2.3. To account for the bounds on the number of parts, the previously used generating functions are being replaced with appropriate q-binomial coefficients. Summing over the possibilities as we did in the proof of Theorem 2.3 yields Theorem 2.4. For example, we replace product in (2–10) with

$$\begin{bmatrix} M+k-1 \\ k \end{bmatrix}_Q (ab)^k.$$

This is the generating function for the number of partitions into even parts such that the total number of parts is odd and $\leq 2M-1$, with the largest part being exactly $2k$, where we count these partitions with Boulet-Stanley weights.

Fixing $(a,b,c,d) = (q,q,q,q)$, we get

$$\Phi_{N,2M}(q,q,q,q) = \begin{bmatrix} N+2M \\ 2M \end{bmatrix}_q. \tag{2–17}$$

So it makes sense to view the sum in (2–16) as a four-parameter generalization of the q-binomial coefficient (2–17).

We can combinatorially see that $\Phi_{2N+\nu,2M}(a,b,c,d)$ satisfies similar recurrence relations to q-binomial coefficients. Using these recurrences, we can write the $\Phi_{N,M}(a,b,c,d)$ for an odd M. We have the relations

$$\Phi_{2N+\nu,2M+1}(a,b,c,d) = \frac{\Phi_{2N+\nu,2(M+1)}(c,d,a,b) - \Phi_{2N-1+\nu,2(M+1)}(c,d,a,b)}{c^{\nu}(cd)^{N}}, \qquad (2\text{–}18)$$

where N, M are non-negative integers and $\nu \in \{0,1\}$. This gives us the full list of possibilities for the bounds of $\Phi_{N,M}(a,b,c,d)$.

Yee in [55], wrote a generating function for some weighted count of partitions with bounds on the largest part and the number of parts. We can also remove Yee's restrictions on weights. In other words, Yee's combinatorial study can be generalized to deal with the four-variable decorated Young diagrams.

Theorem 2.5. *Let $2N+\nu$ and $2M+\mu$ be positive integers. Then*

$$\begin{aligned}\Phi_{2N+\nu,2M+\mu}(a,b,c,d) = \sum_{k=0}^{M}(ac)^{k}\begin{bmatrix}N+k-1+\nu\\ k\end{bmatrix}_Q \sum_{j=0}^{N}(ab)^{N-j}\times \\ \left(\sum_{m_1=0}^{j}(1+\nu\mu a)a^{m_1}Q^{\binom{m_1}{2}+\nu\mu m_1}\begin{bmatrix}M-k+\mu-\nu\\ m_1\end{bmatrix}_Q\begin{bmatrix}M-k+j-m_1\\ j-m_1\end{bmatrix}_Q\right)\times \\ \left(\sum_{m_2=0}^{N-j}c^{m_2}Q^{\binom{m_2}{2}}\begin{bmatrix}M-k\\ m_2\end{bmatrix}_Q\frac{(Q^{M-k+\mu};Q)_{N-j-m_2}}{(Q;Q)_{N-j-m_2}}\right)\end{aligned} \qquad (2\text{–}19)$$

for $\nu,\ \mu \in \{0,1\}$, where $(\nu,\mu) \neq (1,0)$ and $Q = abcd$.

Theorem 2.4 and Theorem 2.5 give different expressions for the same function. Also note that the idea behind (2–18) can be applied to Theorem 2.5 for the missing combination of bounds.

2.2 Fixed Number of Even-indexed and Odd-indexed Odd Parts

For positive integers j and n and a function f of multiple variables $x_1, x_2, \ldots$ we define $\llbracket x_j^n \rrbracket f$ as the coefficient of the x_j^n term of the power series expansion of f with respect to x_j.

By the variable selection $(a,b,c,d)=(az,b/z,ct,c/t)$ in Theorem 2.1 it is easy to see that we get

$$\Psi(az,b/z,ct,d/t):=\sum_{\pi\in\mathcal{D}}\omega_\pi(a,b,c,d)z^{\#O(\pi)}t^{\#E(\pi)}, \tag{2–20}$$

$$\Phi(az,b/z,ct,d/t):=\sum_{\pi\in\mathcal{U}}\omega_\pi(a,b,c,d)z^{\#O(\pi)}t^{\#E(\pi)}, \tag{2–21}$$

where $\#O(\pi)$ is the number of odd-indexed odd parts of π and $\#E(\pi)$ is the number of even-indexed odd parts of π. With this type of control we can fix the number of odd parts and their location in a partition.

Let $P_N(i,j,q)$ be the generating function for the number of partitions into distinct parts $\leq\ N$ where there are i odd-indexed and j even-indexed odd parts. Observe that $\omega_\pi(qz,q/z,qt,q/t)=q^{|\pi|}z^{\#O(\pi)}t^{\#E(\pi)}$. Therefore,

$$\Psi_N(qz,q/z,qt,q/t)=\sum_{i,j\geq 0}P_N(i,j,q)t^iz^j. \tag{2–22}$$

Then it is obvious that

$$\llbracket z^it^j\rrbracket\Psi_N(qz,q/z,qt,q/t), \tag{2–23}$$

where

$$\begin{aligned}\llbracket z^it^j\rrbracket\Psi_N(qz,q/z,qt,q/t):&=\llbracket z^i\rrbracket\llbracket t^j\rrbracket\Psi_N(qz,q/z,qt,q/t)\\ &=\llbracket t^j\rrbracket\llbracket z^i\rrbracket\Psi_N(qz,q/z,qt,q/t)\end{aligned}$$

denotes the coefficient of z^it^j in the power series of the function $\Psi_N(qz,q/z,qt,q/t)$. Let $p_N(i,j,n)$ be the number of partitions of n into distinct parts $\leq N$ where i odd-indexed and j even-indexed odd parts. Then we have

$$P_N(i,j,q)=\sum_{n\geq 0}p_N(i,j,n)q^n.$$

Also let

$$P(i,j,q) = \lim_{N\to\infty} P_N(i,j,q),$$

and

$$p(i,j,n) = \lim_{N\to\infty} p_N(i,j,n).$$

We have the first result coming from (2–23):

Theorem 2.6. *Let N, i, and j be non-negative integers and q be a variable then*

$$P_{2N}(i,j,q) = q^{2i^2-i+2j^2+j}(-q^2;q^2)_{N-i-j}\begin{bmatrix} N \\ i,\ j\end{bmatrix}_{q^4}, \tag{2–24}$$

$$P_{2N+1}(i,j,q) = q^{2i^2-i+2j^2+j}(-q^2;q^2)_{N-i-j+1}\begin{bmatrix} N+1 \\ i,\ j\end{bmatrix}_{q^4}\frac{1-q^{2(N+i-j+1)}}{1-q^{4(N+1)}}, \tag{2–25}$$

where we define the q-trinomial coefficients as

$$\begin{bmatrix} n \\ m,\ k\end{bmatrix}_q := \begin{bmatrix} n \\ m\end{bmatrix}_q\begin{bmatrix} n-m \\ k\end{bmatrix}_q.$$

These two identities coming from the extraction can be formally proven by q-theoretic techniques as well. We prove this assertion using recurrence relations. Using the definitions of $P_N(i,j,q)$ recurrence relations are easily attained by extracting the largest parts of partitions counted by these generating functions.

Lemma 1. *Let N, i, and j be non-negative integers, $\nu \in \{0,1\}$. Then*

$$P_{2N+\nu}(i,j,q) = P_{2N+\nu-1}(i,j,q) + q^{2N+\nu}\chi(i\geq\nu)P_{2N+\nu-1}(j,i-\nu,q), \tag{2–26}$$

$$P_{2N}(i,j,q) = P_{2N-1}(i,j,q) + q^{2N}P_{2N-1}(j,i,q), \tag{2–27}$$

$$P_{2N+1}(i,j,q) = P_{2N}(i,j,q) + q^{2N+1}\chi(i>0)P_{2N}(j,i-1,q), \tag{2–28}$$

where $N \geq 1-\nu$. The initial conditions are

$$P_0(i,j,q) = \delta_{i,0}\delta_{j,0},$$

where

$$\chi(\textit{statement}) := \begin{cases} 1, & \textit{if statement is true,} \\ 0, & \textit{otherwise,} \end{cases} \tag{2–29}$$

and $\delta_{i,j} := \chi(i=j)$, the Kronecker delta function.

Proof. Let $\nu \in \{0,1\}$, $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition counted by $p_{2N+\nu}(i,j,n)$ for some k. If $\lambda_1 < 2N+\nu$, then π is also counted by $p_{2N+\nu-1}(i,j,q)$. If $\lambda_1 = 2N+\nu$, then by extracting λ_1 we get a new partition $\pi^* = (\lambda_2, \lambda_3, \ldots, \lambda_k)$ into distinct parts with largest part $\leq 2N+\nu-1$. If $\nu = 0$ then the number of odd parts stay the same, but the indexing of those parts switch. If $\nu = 1$, then on top of the change of parities of odd numbers, the number of odd parts in π^* is one less than the number of odd parts in π. Therefore, π^* is a partition that is counted by $p_{2N+\nu-1}(j, i-\nu, q)$. This proves the lemma. □

We need to show that the right-hand sides of (2–24) and (2–25) satisfy the same recurrence relations of Lemma 1 to prove Theorem 2.6. The right-hand side of (2–24) can be rewritten for this purpose:

$$\begin{aligned}
&q^{2i^2-i+2j^2+j}(-q^2;q^2)_{N-i-j}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^4} \\
&\qquad = q^{2i^2-i+2j^2+j}(-q^2;q^2)_{N-i-j}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^4}\frac{1-q^{2(N+i-j)}+q^{2(N+i-j)}(1-q^{2(N-i+j)})}{1-q^{4N}} \\
&\qquad = q^{2i^2-i+2j^2+j}(-q^2;q^2)_{N-i-j}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^4}\frac{1-q^{2(N+i-j)}}{1-q^{4N}} \\
&\qquad\qquad + q^{2i^2+i+2j^2-j+2N}(-q^2;q^2)_{N-i-j}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^4}\frac{1-q^{2(N-i+j)}}{1-q^{4N}}.
\end{aligned}$$

This proves that the right-hand side (2–24) satisfies the recurrence relation of Lemma 1 for $\nu = 0$. Recurrence relation of (2–25) can be shown in the same manner. Moreover, the initial condition of Lemma 1 is obviously true for the right-hand side of (2–24) and (2–25). This finishes the proof of Theorem 2.6.

One point to highlight here is that $P_{2N+1}(i, j, q)$ is a polynomial in q for all choices of N, i, and j from is definition. It is also easy to see that the right-hand side of the (2–25) needs to be a polynomial by combining (2–24), and (2–26) with $\nu = 1$. Yet, in its current form the right-hand side of (2–25) comes with a non-trivial rational term with no obvious cancellation. Another way of seeing this is by directly combining (2–6) and (2–22)

Moreover, $N \to \infty$ in either line of Theorem 2.6 proves:

Theorem 2.7. *For non-negative integers i, j, and n*

$$p(i, j, n) = p`(i, j, n),$$

where $p(i, j, n)$ is the number of partitions of n into distinct parts with i odd-indexed odd parts and j even-indexed odd parts and $p`(i, j, n)$ is the number of partitions of n into distinct parts with i parts that are congruent to 1 modulo 4, and j parts that are congruent to 3 modulo 4.

Lastly, let k be a fixed non-negative integer. Taking the limit $N \to \infty$ in (2–24) and/or (2–25), setting $j = k$ ($i = k$), summing over i (j), and finally using q-binomial theorem (A–53), we get Theorem 2.8.

Theorem 2.8. *Let k be a fixed non-negative integer, then*

$$\sum_{i\geq 0} P(i, k, q) = (-q^2; q^2)_\infty \frac{q^{2k^2+k}}{(q^4; q^4)_k} \sum_{i\geq 0} \frac{q^{2i^2-i}}{(q^4; q^4)_i} = (-q^2; q^2)_\infty (-q; q^4)_\infty \frac{q^{2k^2+k}}{(q^4; q^4)_k}, \tag{2–30}$$

$$\sum_{j\geq 0} P(k, j, q) = (-q^2; q^2)_\infty \frac{q^{2k^2-k}}{(q^4; q^4)_k} \sum_{j\geq 0} \frac{q^{2j^2+j}}{(q^4; q^4)_j} = (-q^2; q^2)_\infty (-q^3; q^4)_\infty \frac{q^{2k^2-k}}{(q^4; q^4)_k}. \tag{2–31}$$

Next, we can compare combinatorial interpretations of extremes of (2–30) and (2–31). The sum on the left-hand side of the identity (2–30) (or (2–31)) gives us the generating function for number of partitions into distinct parts with k even-indexed (or odd-indexed) odd parts. On the right-hand side of (2–30) (or (2–31)) we have the generating function for the number of partitions into distinct parts with exactly k parts congruent to 1 (or 3) modulo 4. We rewrite these interpretations together.

Theorem 2.9. *For a fixed non-zero integer k, the number of partitions of n into distinct parts where there are k odd-indexed (even-indexed) odd parts is equal to the number of partitions of n into distinct parts where there are exactly k parts congruent to 1 (3) modulo 4.*

Special case $k = 0$ in the Theorem 2.8 and Theorem 2.9 is well known in literature. Setting $k = 0$, we can rewrite products in (2–30) and (2–31) as

$$(-q^2; q^2)_\infty(-q\ ; q^4)_\infty = \frac{(-q; q^4)_\infty(q; q^4)_\infty}{(q^2; q^4)_\infty(q; q^4)_\infty} = \frac{(q^2; q^8)_\infty}{(q, q^2, q^5, q^6; q^8)_\infty} = \frac{1}{(q, q^5, q^6; q^8)_\infty}, \tag{2–32}$$

and

$$(-q^2; q^2)_\infty(-q^3; q^4)_\infty = \frac{1}{(q^2, q^3, q^7; q^8)_\infty}, \tag{2–33}$$

where we use the Euler Theorem. Far right products of (2–32) and (2–33) appear in the little[1] Göllnitz identities [43]. Little Göllnitz identities have the combinatorial counterparts.

Theorem 2.10 (Göllnitz)**.** *The number of partitions of n into parts (greater than 1) differing by at least 2, and no consecutive odd parts appear in the partitions is equal to the number of partitions of n into parts congruent to 1, 5 or 6 modulo 8 (2, 3 or 7 modulo 8).*

Comparing Theorem 2.9 with $k = 0$ and (2–32), (2–33) we arrive at the "new little Göllnitz" theorems of Savage and Sills [50].

[1] This terminology was introduced by Alladi [2]. It is used to distinguish between (little) Göllnitz and (big) Göllnitz partition theorems.

Theorem 2.11 (Savage, Sills)**.** *The number of partitions of n into distinct parts where odd-indexed (even-indexed) parts are even is equal to the number of partitions of n into parts congruent to 2, 3 or 7 modulo 8 (1, 5 or 6 modulo 8).*

2.3 Partitions with fixed value of BG-rank

The explicit generating function formulas for $P_N(i, j, q)$, given in Theorem 2.6, opens the door to various applications and interesting interpretations. We can start by setting $i = j + k$ for any integer k. $P_N(j + k, j, q)$ is the generating function for number of partitions into distinct parts $\leq N$ with j even-indexed odd parts and BG-rank equal to k. By summing these functions over j, we lift the restriction on the even-indexed odd parts. Let $B_N(k, q)$ denote the generating function for number of partitions into distinct parts less than or equal to N with BG-rank equal to k. Then,

$$B_N(k, q) := \sum_{j \geq 0} P_N(j + k, j, q). \tag{2–34}$$

We have a new combinatorial result.

Theorem 2.12. *Let N be a non-negative integer, k be any integer. Then*

$$B_{2N+\nu}(k, q) = q^{2k^2-k} \begin{bmatrix} 2N + \nu \\ N + k \end{bmatrix}_{q^2}, \tag{2–35}$$

where $\nu \in \{0, 1\}$.

Proof. This identity is a consequence of the q-Gauss identity (A–54). We can outline the proof as follows. Using the definition of $P_{2N+\nu}(j + k, j, q)$, (I.10), (I.25) in [42] as needed we come to the following

$$B_{2N+\nu}(k, q) = q^{2k^2-k} \frac{(q^4; q^4)_N (1 - \nu q^{2(N+k+1)})}{(q^4; q^4)_k (q^2; q^2)_{N-k+\nu}} \times \\ {}_2\phi_1 \left(\begin{matrix} q^{-2(N-k+\nu)}, \ q^{-2(N-k+\nu-1)} \\ q^{4(k+1)} \end{matrix} ; q^4, \ q^{4(N+\nu)+2} \right).$$

Applying the q-Gauss identity (A–54) and rewriting infinite products in a non-trivial fashion using (I.5) in [42] repeatedly yields

$$B_{2N+\nu}(k,q) = q^{2k^2-k}\begin{bmatrix}2N+\nu\\ N+k\end{bmatrix}_{q^2}.$$

□

Theorem 2.12 can also be used to prove the similar result for partitions (not necessarily in distinct parts) with the same type of bounds on the largest part and fixed BG-rank. Let $\widetilde{B}_N(k,q)$ be the generating function for number of partitions into parts less than or equal to N with BG-rank equals k.

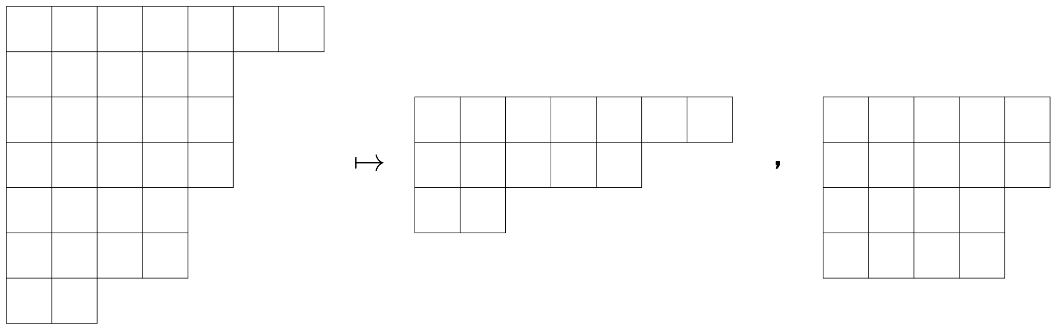
Theorem 2.13. *Let N be a non-negative integer, k be any integer. Then*

$$\widetilde{B}_{2N+\nu}(k,q) = \frac{q^{2k^2-k}}{(q^2;q^2)_{N+k}(q^2;q^2)_{N-k+\nu}},$$

where $\nu \in \{0,1\}$.

The proof of Theorem 2.13 comes from the combinatorial bijection of extracting doubly repeating parts to partitions. Let $\mathcal{U}_{N,k}$ be the set of partitions with parts less than or equal to N and BG-rank equal to k. Let $\mathcal{D}_{N,k}$ be the set of partitions into distinct parts less than or equal to N with BG-rank being equal to k. Recall that $\mathcal{E}_N$ is the set of partitions with parts less than or equal to N, whose parts appear an even number of times. Define bijection $\rho_{N,k} : \mathcal{U}_{N,k} \to \mathcal{D}_{N,k} \times \mathcal{E}_N$ where $\rho_{N,k}(\pi) = (\pi‘,\pi^*)$, where this bijection is established as the row extraction of two parts of same size at once from a given partition repeatedly until there are no more repeating parts in the outcome partition $\pi‘$. The extracted parts are collected in the partition π^*. Note that the number of odd parts in π and $\pi‘$ might be different $BG(\pi) = BG(\pi‘)$. Table 2-2 is an example of this map with $\pi = (7,5,5,5,4,4,2)$, so $\rho_{N,k}(\pi) = (\pi‘,\pi^*) = ((7,5,2),(5,5,4,4))$ with $BG(\pi) = BG(\pi‘) = 0$ for any $N \geq 7$ and $k = 0$.

Figure 2-2. An example of the map $\rho_{N,k}$.



$$\rho_{N,k}((7,5,5,5,4,4,2)) = ((7,5,2),(5,5,4,4))$$

The generating function for number of partitions from the set $\mathcal{E}_{2N+\nu}$, where all parts appear an even number of times and are less than or equal to $2N+\nu$ is

$$\frac{1}{(q^2;q^2)_{2N+\nu}},$$

where $\nu \in \{0,1\}$. Therefore, keeping the bijection $\rho_{2N+\nu,k}$ above in mind, the generating function for number of partitions with BG-rank equal to k and the bound on the largest part being $2N+\nu$ is the product

$$\widetilde{B}_{2N+\nu}(k,q) = \frac{B_{2N+\nu}(k,q)}{(q^2;q^2)_{2N+\nu}}, \tag{2–36}$$

proving Theorem 2.13.

The combinatorial interpretation coming from (2–35) was previously unknown. Recall that the expression

$$q^{2k^2-k}\begin{bmatrix} 2N+\nu \\ N+k \end{bmatrix}_{q^2},$$

is the generating function for number of partitions with BG-rank equal to k and the largest part $\leq 2N+\nu$, where $\nu \in \{0,1\}$. This relation gives a combinatorial explanation of a well known identity:

Theorem 2.14. *Let $\nu \in \{0,1\}$, and let $N+\nu$ be a positive integer. Then,*

$$\sum_{k=-N}^{N+\nu} q^{2k^2-k} \begin{bmatrix} 2N+\nu \\ N+k \end{bmatrix}_{q^2} = (-q;q)_{2N+\nu}. \tag{2–37}$$

It is clear that summing $B_N(k,q)$, defined in (2–34), over all possible BG-ranks yield the generating function for the number of partitions into distinct parts $\leq N$, yielding (2–37).

Identity (2–37) was discussed in [13, 4.2] by Andrews. He showed that (2–37) is equivalent to an identity for Rogers–Szegő polynomials. The Rogers–Szegő polynomials are defined as

$$H_N(z,q) := \sum_{l=0}^{N} \begin{bmatrix} N \\ l \end{bmatrix}_q z^l. \tag{2–38}$$

We have the identity [13]

$$H_{2N+\nu}(q,q^2) = (-q;q)_{2N+\nu}, \tag{2–39}$$

where $\nu \in \{0,1\}$. In oder to show the equivalence of (2–37) and (2–39) we use

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_{q^{-1}} = q^{-nm} \begin{bmatrix} n+m \\ n \end{bmatrix}_q, \tag{2–40}$$

where n and m are positive integers.

In (2–37), first we let $q \mapsto q^{-1}$. Then we change the q-binomial term using (2–40) on the left-hand side, and rewrite the right-hand side as

$$(-1/q;1/q)_{2N+\nu} = q^{-(N+\nu)(2N+1)}(-q;q)_{2N+\nu}.$$

Multiplying both sides with $q^{(N+\nu)(2N+1)}$ and changing the summation variable in the equation (2–37) with $k \mapsto N+(-1)^\nu k+\nu$. In $\nu = 1$ case, in addition, we change the order of summation. In this way we arrive at (2–39).

Theorem 2.13 and Theorem 2.14 is enough to prove the following, combinatorially anticipated, corollary.

Corollary 1. *Let N be a non-negative integer. Then*

$$\sum_{k=-N}^{N+\nu} \frac{q^{2k^2-k}}{(q^2;q^2)_{N+k}(q^2;q^2)_{N-k+\nu}} = \frac{1}{(q;q)_{2N+\nu}}, \tag{2–41}$$

where $\nu \in \{0,1\}$.

2.4 Some Implications on Rogers–Szegő Polynomials and q-Hypergeometric Series

We start with the following special case of Theorem 2.2:

$$\Psi_{2N+\nu}(qz,qz,q/z,q/z) = \sum_{i=0}^{N} \begin{bmatrix} N \\ i \end{bmatrix}_{q^4} (-zq;q^4)_{N-i+\nu}(-q/z;q^4)_i (zq)^{2i}. \tag{2–42}$$

The sum on the right-hand side of (2–6) was seen in the literature before. In fact, Berkovich and Warnaar [30] found that sum in in junction with Rogers–Szegő polynomials.

Theorem 2.15 (Berkovich, Warnaar)**.** *Let N be a non-negative integer, then the Rogers–Szegő polynomials can be expressed as*

$$H_{2N+\nu}(zq,q^2) = \sum_{l=0}^{N} \begin{bmatrix} N \\ l \end{bmatrix}_{q^4} (-zq;q^4)_{N-l+\nu}(-q/z;q^4)_l (zq)^{2l} \tag{2–43}$$

for $\nu \in \{0,1\}$.

We would also like to mention that Cigler provided a new proof of (2–43) in [38].

Let $\pi = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition, and $\gamma(\pi) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots + (-1)^{k+1}\lambda_k$, the alternating sum of parts of the partition π.

Our observation:

$$H_N(zq,q^2) = \Psi_N(zq,zq,q/z,q/z), \tag{2–44}$$

and some interpretation of the Rogers–Szegő polynomials is enough to get some weighted partition theorems.

Clearly, $\Psi_N(zq,zq,q/z,q/z)$ is the generating function for the number of partitions into distinct parts $\leq N$, where exponent of z is the alternating sum of the parts of partition. That is

$$\Psi_N(zq,zq,q/z,q/z) = \sum_{\pi\in\mathcal{D}_N} q^{|\pi|}\, z^{\gamma(\pi)}.$$

Therefore, extraction of the coefficient of z^k would give us the generating function for the number of partitions with the alternating sums of parts equal to k. Using definition for $H_N(zq, q^2)$, given in (2–38), one can easily extract the coefficient of z^k for a fixed non-negative integer k. This way we get

$$q^k \begin{bmatrix} N \\ k \end{bmatrix}_{q^2} = \sum_{\substack{\pi \in \mathcal{D}_N, \\ \gamma(\pi)=k}} q^{|\pi|}. \tag{2–45}$$

This can be interpreted as

Theorem 2.16. *Let N, n, and k be non-negative integers. The number of partitions of n into solely k odd parts $\leq 2N-2k+1$ is equal to the number of partitions of n into distinct parts $\leq N$ with the alternating sum of parts being equal to k.*

This result can also combinatorially be proven using *the Sylvester bijection* ([34], p.52, ex. 2.2.5). Theorem 2.15 may be interpreted as an analytical proof of Theorem 2.16.

Moreover, the connection (2–8) used in (2–45) yields

$$\lfloor\!\lfloor z^k \rfloor\!\rfloor \Phi_N(zq, zq, q/z, q/z) = \frac{q^k}{(q^2;q^2)_k (q^2;q^2)_{N-k}}, \tag{2–46}$$

where $\lfloor\!\lfloor z^k \rfloor\!\rfloor \Phi_N(zq, zq, q/z, q/z)$ denotes the coefficient of z^k term in the power series expansion of $\Phi_N(zq, zq, q/z, q/z)$. This leads us to a new combinatorial theorem.

Theorem 2.17. *Let N, n, and k be non-negative integers. Then*

$$\mathcal{A}_N(n,k) = \mathcal{B}_N(n,k),$$

where $\mathcal{A}_N(n,k)$ is the number of partitions of n into no more than N parts, where exactly k parts are odd, and $\mathcal{B}_N(n,k)$ is the number of partitions of n into parts $\leq N$, where alternating sum of parts is equal to k.

We illustrate Theorem 2.17 in Table 2-1.

It is clear that the partitions counted by $\mathcal{A}_N(n,k)$ and $\mathcal{B}_N(n,k)$ are conjugates of each other. This follows easily from the observation that the number of odd parts in a partition turns into the alternating sum of parts in the conjugate of this partition.

Table 2-1. $\mathcal{A}_3(10,2)$ and $\mathcal{B}_3(10,2)$ with respective partitions for Theorem 2.17.

$$\mathcal{A}_3(10,2)=9: \quad \begin{array}{c}(9,1),\ (8,1,1),\ (7,3),\ (7,2,1),\ (6,3,1),\\ (5,5),\ (5,4,1),\ (5,3,2),\ (4,3,3).\end{array}$$

$$\mathcal{B}_3(10,2)=9: \quad \begin{array}{c}(3,3,3,1),\ (3,3,2,1,1),\ (3,2,2,2,1),\\ (3,2,2,1,1,1),\ (3,2,1,1,1,1,1),\ (3,1,1,1,1,1,1,1),\\ (2,2,2,2,2),\ (2,2,2,1,1,1,1),\ (2,1,1,1,1,1,1,1,1).\end{array}$$

Another important result comes from knowing the combinatorial generating function interpretations of the function $\Psi_N(a,b,c,d)$. By looking at different choices of the variables, we can find q-series identities that were not combinatorially interpreted before. One finding of this type is a q-hypergeometric identity of Berkovich and Warnaar, [30, (3.30)], and it's analogue.

For $\nu \in \{0,1\}$, it is obvious that $\Psi_{2N+\nu}(aq, q/a, aq, q/a)$ is the generating function of partitions into distinct parts less than or equal to $2N+\nu$, where the exponent of a counts the number of odd parts in the partitions. Clearly,

$$\Psi_{2N+\nu}(aq, q/a, aq, q/a) = (-aq;q^2)_{N+\nu}(-q^2;q^2)_N. \tag{2–47}$$

Using (2–6) yields

$$\Psi_{2N+\nu}(aq, q/a, aq, q/a) = (-aq;q^4)_{N+\nu}\ {}_2\phi_1\left(\begin{matrix} q^{-4N},\ -aq \\ -(aq)^{-1}q^{-4N-\nu}\end{matrix}; q^4, -q^{1+4\delta_{\nu,0}}/a\right). \tag{2–48}$$

Comparing (2–47) and (2–48) gives

Theorem 2.18. *For a non-negative integer N and variables a and q*

$${}_2\phi_1\left(\begin{matrix} q^{-4N},\ -aq \\ -(aq)^{-1}q^{-4(N-\delta_{\nu,0})}\end{matrix}; q^4, -q^{1+4\delta_{\nu,0}}/a\right) = \frac{(-q^2;q^2)_N(-aq^3;q^2)_{N-1+\nu}}{(-aq^5;q^4)_{N-1+\nu}},$$

where $\nu \in \{0,1\}$ and $\delta_{i,j}$ is the Kronecker delta function.

The case $\nu = 1$, with $N \mapsto n$, $a \mapsto -a/q$ and $q^2 \mapsto q$ in Theorem 2.18, is [30, (3.30)] and $\nu = 0$ is the easy analogue we get using (2–47).

2.5 Generalization of Fixed number of odd and even indexed odd parts; Companion Identity to Capparelli's Theorem

In 1988, In his thesis S. Capparelli [36] conjectured a partition identity, which was later partially proven by G. E. Andrews [16] in 1992. Capparelli also showed his identities in 1994 in [37].

Let $C_m(n)$ be the number of partitions of n into distinct parts where no part is congruent to $\pm m \mod 6$. Define $D_m(n)$ to be the number of partitions of n into distinct parts $\neq m$ where the difference between consecutive parts is ≥ 4 unless consecutive parts are either $3l \pm 1$ for a positive integer l (giving a difference of 2) or are both multiples of 3 (yielding a difference of 3).

First Capparelli's identity was translated to an equivalent form and proven by Andrews [16]. In 1995, Alladi, Andrews and Gordon, in their article [7], improved on this result by giving a refinement of Capparelli's conjecture with restriction on the largest part and number of occurrences of parts with certain congruence conditions.

Theorem 2.19 (Alladi, Andrews, Gordon 1995)**.** *For non-negative integer n and $m \in \{1, 2\}$,*

$$C_m(n) = D_m(n).$$

We exemplify Theorem 2.19 in Table 2-2.

Table 2-2. An example of Theorem 2.19 with $|\pi| = 19$ and $m = 1$.

$C_1(19) = 10$:	$(16, 3),\ (15, 4),\ (14, 3, 2),\ (12, 4, 3),\ (10, 9),$ $(10, 6, 3),\ (10, 4, 3, 2),\ (9, 8, 2),\ (9, 6, 4),\ (8, 6, 3, 2).$
$D_1(19) = 10$:	$(19),\ (17, 2),\ (16, 3),\ (15, 4),\ (14, 5),$ $(13, 6),\ (13, 4, 2),\ (12, 7),\ (11, 6, 2),\ (10, 6, 3).$

Let $A_m(n)$ be the number of partitions $\pi = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ where $\lambda_{2i+r} \not\equiv 3-m+(-1)^m r \mod 3$, and $\lambda_{2i+r} - \lambda_{2i+1-r} > \lfloor m/2 \rfloor + (-1)^{m-1} r$ for $r \in \{0, 1\}$ and $1 \leq i \leq k-1$.

The following is a new companion to Theorem 2.19, the classical Capparelli's conjecture:

Theorem 2.20. *Let n be a non-negative integer and $m \in \{1, 2\}$,*

$$A_m(n) = C_m(n).$$

We will prove a refinement of this result (Theorem 2.21) in this section. Table 2-3 with Table 2-2 gives an example of Theorem 2.20.

Table 2-3. The $A_1(19)$ value and the respective partitions with $|\pi| = 19$.

$A_1(19) = 10$:	$(18,1),\ (15,4),\ (14,3,2),\ (12,7),\ (12,4,3),$ $(11,6,2),\ (11,4,3,1),\ (9,7,3),\ (9,6,3,1),\ (8,6,5).$

Suppose $m \in \{1,2\}$, N, i, and j are non-negative integers where $N \geq i,\ j$. Let $\lceil\cdot\rceil$ be the ceil, $\lfloor\cdot\rfloor$ be the floor, and $\{\{\cdot\}\}$ be the fractional part function defined on real numbers.

Definition 2. *Define $F_{m,N}(i,j,q)$ to be the generating function for the number of partitions $\pi = (\lambda_1, \lambda_2, \dots, \lambda_k)$ where*

i. the largest part of π, λ_1, is $\leq 3\lceil N/2\rceil - 2m\{\{N/2\}\}$,

ii. the number of parts $\equiv 2 \mod 3$ is i,

iii. the number of parts$\equiv 1 \mod 3$ is j,

iv. $\lambda_{2i+r} \not\equiv 3 - m + (-1)^m r \mod 3$, and $\lambda_{2i+r} - \lambda_{2i+1-r} > \lfloor m/2 \rfloor + (-1)^{m-1} r$ for $r \in \{0,1\}$ and $1 \leq i \leq k-1$.

Let $A_{m,N}(n,i,j)$ be the number of partitions of n satisfying the conditions i.–iv. of Definition 2. From these definitions, it is easy to see that

$$\lim_{N\to\infty} \sum_{i,j=0}^{\infty} F_{m,N}(i,j,q) = \lim_{N\to\infty} \sum_{i,j=0}^{\infty} \sum_{n=0}^{\infty} A_{m,N}(n,i,j) q^n = \sum_{n=0}^{\infty} A_m(n) q^n, \tag{2–49}$$

where

$$A_m(n) = \lim_{N\to\infty} \sum_{i,j=0}^{\infty} A_{m,N}(n,i,j). \tag{2–50}$$

The series in identity 2–50 is a finite sum as $A_{m,N}(n,i,j) = 0$ for all i and $j \geq N$.

Similar to the generating functions $P_N(i,j,q)$ in Section 2.2, $F_{m,N}(i,j,q)$ generating functions are directly related to $\Psi_N(a,b,c,d)$ (2–4). Observe,

$$\Psi_N(q^2 z, q/z, qt, q^2/t) = \sum_{i,j,\geq 0} F_{1,N}(i,j,q) z^i t^j, \tag{2–51}$$

$$\Psi_N(qt, q^2/t, q^2 z, q/z) = \sum_{i,j,\geq 0} F_{2,N}(i,j,q) z^i t^j. \tag{2–52}$$

This is enough to prove Theorem 2.20 as N tends to ∞ using Boulet's original product formula (2–1).

Definition 3. *For $m = 1, 2$, let $Q_{m,N}(i, j, q)$ be the generating function for the number of partitions into distinct parts where*

i. no part is congruent to $\pm m \pmod 6$,

ii. there are exactly i parts $\equiv m + (-1)^{m+1} \pmod 6$ and these parts are all $\leq 6N - (3 + m)$,

iii. there are exactly j parts $\equiv 3 + m \pmod 6$ and these parts are all $\leq 6(N - i) - (m + (-1)^{m+1})$,

iv. all parts that are $0 \pmod 3$ are bounded by $3(N - i - j)$.

It is clear from Definition 3 that the bounds on the parts depend on the congruence classes modulo 6. We proceed by formulating the main generalization of the companion result to Capparelli's identities. Let $C_{m,N}(n, i, j)$ be the number of partitions of n satisfying the conditions i–iv of Definition 3; explicitly,

$$Q_{m,N}(i, j, q) = \sum_{n=0}^{\infty} C_{m,N}(n, i, j) q^n.$$

The refinement of Theorem 2.20 is the following theorem:

Theorem 2.21. *For $N, n, i, j \in \mathbb{Z}_{\geq 0}$ and $m \in \{1, 2\}$,*

$$A_{m,2N}(n, i, j) = C_{m,N}(n, i, j),$$

and its equivalent analytical form is

$$F_{m,2N}(i, j, q) = Q_{m,N}(i, j, q).$$

To prove Theorem 2.21 we formulated $\llbracket x^i t^j \rrbracket \Psi_N(a, b, c, d)$, formally proven it using q-series techniques and later did the interpretation of these generating functions as $Q_{m,N}(i, j, q)$ showing the generating function equality First of we have:

Theorem 2.22. *For non-negative integers N, i, j where $N \geq i, j$, and $m = 1, 2$, we have*

$$F_{m,2N}(i,j,q) = q^{\omega(m,i,j)} \begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6} (-q^3;q^3)_{N-i-j}, \tag{2–53}$$

$$F_{m,2N+1}(i,j,q) = q^{\omega(m,i,j)} \frac{1-q^{3(N+1+i+(-1)^m j)}}{1-q^{6(N+1)}} \begin{bmatrix} N+1 \\ i,\ j \end{bmatrix}_{q^6} (-q^3;q^3)_{N+1-i-j}, \tag{2–54}$$

where $\omega(m,i,j) := (3i + (-1)^m m)i + (3j + (-1)^{m+1} m)j$.

In order to prove this theorem we need the following recurrence relations:

Lemma 2. *For $N,\ i,\ j, n$ defined as before,*

$$F_{1,2N+1}(i,j,q) = F_{1,2N}(i,j,q) + \chi(i > 1)q^{3N+2}F_{2,2N}(i-1,j,q), \tag{2–55}$$

$$F_{1,2N+2}(i,j,q) = F_{1,2N+1}(i,j,q) + q^{3(N+1)}F_{2,2N+1}(i,j,q), \tag{2–56}$$

$$F_{2,2N+1}(i,j,q) = F_{2,2N}(i,j,q) + \chi(j > 1)q^{3N+1}F_{1,2N}(i,j-1,q), \tag{2–57}$$

$$F_{2,2N+2}(i,j,q) = F_{2,2N+1} + q^{3(N+1)}F_{1,2N+1}(i,j,q). \tag{2–58}$$

Lemma 2 along with the initial conditions $F_{m,0}(i,j,q) = \delta_{i,0}\delta_{j,0}$ for $m = 1,\ 2$ uniquely specifies these generating functions. Here the Kronecker delta function $\delta_{i,j} = 1$ if $i = j$, and 0, otherwise. Similar to Definition 2 we define generating functions for the number of partitions for a particular refinement of Capparelli-type congruence conditions.

Proof. Let $m = 1$, and $N,\ i$, and j be non-negative integers satisfying $N \geq i,\ j$. The first recursion, (2–55), comes from elementary observations. Let $\pi = (\pi_1, \pi_2, \dots, \pi_k)$ be a partition satisfying the conditions in Definition 2 with $m = 1$, and $N \mapsto 2N+1$. If π_1 less than $3N+2$ then π must also satisfy the conditions for $F_{1,2N}(i,j,q)$ because the only difference between $F_{m,2N+1}(i,j,q)$ and $F_{m,2N}(i,j,q)$ is in the bounds on the largest parts. If $\pi_1 = 3N+2$ (which implicitly requires $i > 0$) we can extract this part from π and get a new partition. The leftover partition $\pi` = (\pi_2, \pi_3, \dots, \pi_k) = (\pi`_1, \pi`_2, \dots, \pi`_{k-1})$ has one less count of 2 modulo 3 parts ($i \mapsto i-1$) and the largest part of $\pi`$, $\pi`_1$, is bounded by $3N$. Lastly the congruence conditions ii and iii in Definition 2 for $F_{2,2N}(i-1,j,q)$ are satisfied by $\pi`$ as the extraction of the largest part from π alters the parities of the indices of parts. Hence, we get the recurrence $F_{1,2N+1}(i,j,q) = F_{1,2N}(i,j,q) + \chi(i>0)q^{3N+2}F_{2,2N}(i-1,j,q)$.

The recurrences (2–56), (2–57), and (2–58) can similarly be established by examining the partitions satisfying the conditions for their respective definitions. □

In order to prove Theorem 2.22, we need to show that both sides of the equations (2–53) and (2–54) satisfy the same recurrences (2–55)–(2–58) with the same initial conditions. The recurrences of the left-hand side of the equations in Theorem 2.22 are handled in Lemma 2. Next, we show that the right-hand side of the equations in Theorem 2.22 satisfy the recurrences of Lemma 2.

Proof. We will start with the right-hand side of (2–53). Let $m = 1$, N, i, and j be non-negative integers satisfying $N \geq i+j$. Then,

$$q^{(3i-1)i+(3j+1)j}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6}(-q^3;q^3)_{N-i-j} \tag{2–59}$$

$$= q^{(3i-1)i+(3j+1)j}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6}(-q^3;q^3)_{N-i-j}\frac{1-q^{3(N+i-j)}+q^{3(N+i-j)}+q^{6N}}{1-q^{6N}}$$

$$= \left(q^{(3i-1)i+(3j+1)j}\frac{1-q^{3(N+i-j)}}{1-q^{6N}}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6}(-q^3;q^3)_{N-i-j}\right) \tag{2–60}$$

$$+\, q^{3N}\left(q^{(3i+2)i+(3j-2)j}\frac{1-q^{3(N-i+j)}}{1-q^{6N}}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6}(-q^3;q^3)_{N-i-j}\right).$$

Comparison between (2–59) and (2–60) shows that (2–59) satisfies the same recursion relation as $F_{1,2N}(i,j,q)$ given in (2–56). Similarly the right-hand side of (2–54) with $m=1$ satisfies the recurrence (2–55). Here the $i=0$ case is obvious as the recurrences reduce down to q-binomial recurrences. Suppose that $i \geq 1$:

$$q^{(3i-1)i+(3j+1)j}\frac{1-q^{3(N+1+i-j)}}{1-q^{6N}}\begin{bmatrix} N+1 \\ i,\ j \end{bmatrix}_{q^6}(-q^3;q^3)_{N+1-i-j} \tag{2–61}$$

$$= q^{(3i-1)i+(3j+1)j}\begin{bmatrix} N+1 \\ i,\ j \end{bmatrix}_{q^6}(-q^3;q^3)_{N+1-i-j}$$

$$\times \left(\frac{1 - q^{3(N+1-i-j)} + q^{3(N+1-i-j)} + q^{3(N+1+i-j)}}{1 - q^{6(N+1)}} \right)$$

$$= \left(q^{(3i-1)i+(3j+1)j} \begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6} (-q^3; q^3)_{N-i-j} \right) \tag{2–62}$$

$$+ q^{3N+2} \left(q^{(3i-1)(i-1)+(3j-2)j} \begin{bmatrix} N+1 \\ i-1,\ j \end{bmatrix}_{q^6} (-q^3; q^3)_{N+1-i-j} \right).$$

Analogous proofs can be easily given for the right-hand sides of (2–53) and (2–54) with $m = 2$ in Theorem 2.22. Picking $N = i = j = 0$ in the right-hand side of the (2–53) we see that these functions have the same initial conditions as $F_{m,N}(i, j, q)$, which finishes the proof of Theorem 2.22. □

Now, we suppose that $m \in \{1, 2\}$, N, i, and j are non-negative integers where $N \geq i,\ j$. The proof of Theorem 2.21 follows from showing that $F_{m,2N}(i, j, q)$, and $Q_{m,N}(i, j, q)$ are equal. We focus our attention on the product representation (2–53) of the generating functions $F_{m,2N}(i, j, q)$. For $m = 1$, he expression

$$q^{(3i-1)i+(3j+1)j} \begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6} (-q^3; q^3)_{N-i-j},$$

can be rewritten in terms of q-binomial coefficients as

$$(q^6)^{\binom{i+1}{2}} \begin{bmatrix} N \\ i \end{bmatrix}_{q^6} (q^6)^{\binom{j+1}{2}} \begin{bmatrix} N-i \\ j \end{bmatrix}_{q^6} (-q^3; q^3)_{N-i-j}\ q^{-4i-2j}.$$

The factor

$$(q^6)^{\binom{i+1}{2}} \begin{bmatrix} N \\ i \end{bmatrix}_{q^6} \tag{2–63}$$

is the generating function for the number of partitions into i distinct multiples of 6 less than or equal to $6N$. Multiplying (2–63) with the term q^{-4i} can be interpreted as taking off 4 from each and every one of the i parts. Consequently,

$$q^{-4i}(q^6)^{\binom{i+1}{2}} \begin{bmatrix} N \\ i \end{bmatrix}_{q^6}$$

is the generating function for the number of partitions into i distinct parts less than or equal to $6N-4$ where every part is congruent to 2 modulo 6. Similarly,

$$q^{-2j}(q^6)^{\binom{j+1}{2}}\begin{bmatrix} N-i \\ j \end{bmatrix}_{q^6}$$

is the generating function for the number of partitions into j distinct parts less than or equal to $6(N-i)-2$ where every part is congruent to 4 modulo 6. Finally, we also know that $(-q^3;q^3)_{N-i-j}$ is the generating function for number of partitions into distinct parts less than or equal to $3(N-i-j)$. Discussion of the $m=2$ case can be given along the similar lines. Therefore we get the proof of Theorem 2.21 as follows:

Proof. The above construction shows that

$$F_{1,2N}(i,j,q) = q^{(3i-1)i+(3j+1)j}\begin{bmatrix} N \\ i,\ j \end{bmatrix}_{q^6}(-q^3;q^3)_{N-i-j} = Q_{1,N}(i,j,q),$$

thus giving us the refined companion to Capparelli's identity. $F_{2,2N}(i,j,q) = Q_{2,N}(i,j,q)$ can be shown in the same manner. $\square$

CHAPTER 3
WEIGHTED PARTITION IDENTITIES INSPIRED BY THE WORK OF NATHAN FINE

In this chapter we will introduce the work that came after [54]. The results here appeared in literature in [28].

While the study of the classical partition identities goes back to Leibniz and Euler, the study of weighted partition identities is relatively new with many important consequences to be discovered. In 1997, Alladi [6] began a systematic study of weighted partition identities. Among many interesting results, he proved that

Theorem 3.1 (Alladi, 1997)**.**

$$\frac{(a(1-b)q;q)_n}{(aq;q)_n} = \sum_{\pi\in\mathcal{U}_n} a^{\nu(\pi)} b^{\nu_d(\pi)} q^{|\pi|}. \tag{3–1}$$

Theorem 3.1 provides a combinatorial interpretation for the left-hand side product of (3–1) as a weighted count of ordinary partitions with a restriction on the largest part. In [39], Corteel and Lovejoy elegantly interpreted (3–1) with $a=1$ and $b=2$,

$$\frac{(-q;q)_n}{(q;q)_n} = \sum_{\pi\in\mathcal{U}_n} 2^{\nu_d(\pi)} q^{|\pi|}, \tag{3–2}$$

in terms of *overpartitions*.

Also in [6], Alladi discovered and proved a weighted partition identity relating unrestricted partitions and the Rogers–Ramanujan partitions. Let $\mathcal{U}$ be the set of all partitions, and let $\mathcal{RR}$ be the set of partitions with difference between parts ≥ 2.

Theorem 3.2 (Alladi, 1997)**.**

$$\sum_{\pi\in\mathcal{RR}} \omega(\pi) q^{|\pi|} = \sum_{\pi\in\mathcal{U}} q^{|\pi|}, \tag{3–3}$$

where

$$\omega(\pi) := \lambda_{\nu(\pi)} \cdot \prod_{i=1}^{\nu(\pi)-1} (\lambda_i - \lambda_{i+1} - 1), \tag{3–4}$$

and the weight of the empty sequence is considered to be the empty product and is set equal to 1.

Theorem 3.2 is extended in [54].

Note that it is clear that $\mathcal{RR}$ is a proper subset of $\mathcal{U}$. Therefore, computationally, calculating the sum on the left is much easier than calculating the sum on the right.

In (3–3) the set $\mathcal{RR}$ can be replaced with $\mathcal{D}$.

$$\sum_{\pi\in\mathcal{RR}}\omega(\pi)q^{|\pi|}=\sum_{\pi\in\mathcal{D}}\omega(\pi)q^{|\pi|}. \tag{3–5}$$

The weight $\omega(\pi)$ of (3–4) for a partition π becomes 0 if the gap between consecutive parts of π is ever 1.

Section 3.1 discusses weighted partition identities connecting Göllnitz-Gordon type partitions and partitions with distinct odd parts. Some combinatorial connections between $\mathcal{D}$ and $\mathcal{U}$ will be presented in Section 3.2. In Section 3.3, we use a known identity from Ramanujan's lost notebook to discover and prove a new striking weighted partition identity, Theorem 3.14.

3.1 Weighted partition identities involving Göllnitz–Gordon type partitions

We start by reminding the reader of the well-known Göllnitz–Gordon identities of 1960's.

Theorem 3.3 (Slater, 1952)**.** *For $i\in\{1,2\}$*

$$\sum_{n\geq 0}\frac{q^{n^2+2(i-1)n}(-q;q^2)_n}{(q^2;q^2)_n}=\frac{1}{(q^{2i-1};q^8)_\infty(q^4;q^8)_\infty(q^{9-2i};q^8)_\infty}. \tag{3–6}$$

These analytic identities in (3–6), though commonly referred as Göllnitz–Gordon identities, were proven a decade before Göllnitz and Gordon by Slater [51, (34) & (36), p. 155]. It should be noted that both cases of (3–6) were known to Ramanujan [19, (1.7.11–12), p. 37] before any known proof emerged.

For a lot of authors, including both Göllnitz and Gordon, the combinatorial interpretations of (3–6) have been of interest. For $i=1$ or 2, let $\mathcal{GG}_i$ be the set of partitions into parts $\geq 2i-1$ with minimal difference between parts ≥ 2 and no consecutive even numbers appear as parts. Let $C_{i,8}$ be the set of partitions into parts congruent to $\pm(2i-1)$, and 4 mod 8. Then Theorem 3.3 can be rewritten in its combinatorial form [43], [44]:

Theorem 3.4 (Göllnitz–Gordon, 1967 & 1965)**.** *For $i = 1$ or 2, the number of partitions of n from $\mathcal{GG}_i$ is equal to the number of partitions of n from $C_{i,8}$.*

$$\sum_{\pi \in \mathcal{GG}_i} q^{|\pi|} = \sum_{\pi \in C_{i,8}} q^{|\pi|}. \tag{3–7}$$

We now present some analytical identities that will later be interpreted in terms of the Göllnitz–Gordon type partitions. This discussion will yield to the first set of weighted partition identities of this paper.

Theorem 3.5.

$$\sum_{n\geq 0} \frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n^2} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{3–8}$$

$$\sum_{n\geq 0} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n^2} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n^2+n}}{(-q;q^2)_{n+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{j\geq 0} q^{3j^2+2j}(1-q^{2j+1}). \tag{3–9}$$

Proof. The left-hand side of (3–8) and (3–9) can be rewritten as

$$\lim_{\rho\to\infty} {}_2\phi_1\left(\begin{matrix} -q,\ \rho \\ q^2 \end{matrix}; q^2, -\frac{q}{\rho}\right) \text{ and } \lim_{\rho\to\infty} {}_2\phi_1\left(\begin{matrix} -q,\ \rho q^2 \\ q^2 \end{matrix}; q^2, -\frac{q}{\rho}\right), \text{ respectively.}$$

Then it is easy to show that (3–8) is a limiting case of q-Gauss summation (A–54). An equivalent form of the identity (3–8) is also present in Ramanujan's lost notebooks [19, 4.2.6, p. 84].

The first equality of (3–9) is an application of the Heine transformation (A–55) with $a = \rho q^2$, and the second equality is due to Fine [41, (26.91–97), p. 62] with $q \mapsto -q$, and Rogers [48, (4), p. 333] with $q \mapsto -q$. Another equivalent proof and an alternative representation of the second equality in (3–9) is present in Ramanujan's lost notebooks [18, §9.5]. □

Note that the minuscule change: $q^{n^2} \mapsto q^{n^2+2n}$, on the left side of (3–9) yields increase in complexity on the right of (3–9) which involves a false theta function.

Similar to the situation in Theorem 3.3, analytic identities (3–8) and (3–9) can be interpreted combinatorially. In fact, the interpretation of (3–8) was discussed in [4]. For the sake of completeness we will slightly paraphrase this discussion below.

We can easily interpret the product on the right-hand side of (3–8). The expression $(-q;q^2)_\infty$ is the generating function for the number of partitions into distinct odd parts and $1/(q^2;q^2)_\infty$ is the generating function for the number of partitions into even parts. These two generating functions' product is the generating function for the number of partitions with distinct odd parts (even parts may be repeating). This is clear as the parity of a part in a partition completely identifies which generating function it is coming from.

The generating function interpretation of the left-hand side of (3–8) needs us to identify weights on partitions. For a positive integer n, the reciprocal of the q-Pochhammer symbol

$$\frac{1}{(q^2;q^2)_n} \tag{3–10}$$

is the generating function for the number of partitions into $\leq n$ even parts. The expression

$$\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n} \tag{3–11}$$

can be interpreted as the generating function for the number of partitions into exactly n parts from $\mathcal{GG}_1$, [2, (8.2), p. 173]. We can represent the partitions counted by (3–11) as 2-modular graphs. There are four possible patterns that can appear at the end of consecutive parts of these 2-modular Young diagrams. All these possible endings of consecutive parts λ_i and λ_{i+1}, where $i < \nu(\pi)$ of a partition π, are demonstrated in Table 3-1.

Table 3-1. Ends of consecutive parts of Göllnitz-Gordon partitions.

$$\begin{array}{ll}
\begin{array}{l}\lambda_i\ \boxed{2}\cdots\boxed{2}\overbrace{\boxed{2}\cdots\boxed{2}}^{\frac{\lambda_i-\lambda_{i+1}-2}{2}\geq 0}\boxed{1}\\ \lambda_{i+1}\boxed{2}\cdots\boxed{1}\end{array}, &
\begin{array}{l}\lambda_i\ \boxed{2}\cdots\boxed{2}\overbrace{\boxed{2}\cdots\boxed{2}}^{\frac{\lambda_i-\lambda_{i+1}-3}{2}\geq 0}\boxed{2}\boxed{1}\\ \lambda_{i+1}\boxed{2}\cdots\boxed{2}\end{array}, \\
\begin{array}{l}\lambda_i\ \boxed{2}\cdots\boxed{2}\overbrace{\boxed{2}\cdots\boxed{2}\,\boxed{2}}^{\frac{\lambda_i-\lambda_{i+1}-3}{2}\geq 0}\\ \lambda_{i+1}\boxed{2}\cdots\boxed{1}\end{array}, &
\begin{array}{l}\lambda_i\ \boxed{2}\cdots\boxed{2}\overbrace{\boxed{2}\cdots\boxed{2}}^{\frac{\lambda_i-\lambda_{i+1}-4}{2}\geq 0}\boxed{2}\boxed{2}\\ \lambda_{i+1}\boxed{2}\cdots\boxed{2}\end{array}.
\end{array}$$

The labelled gaps on Table 3-1 are the number of non-essential number of boxes between consecutive parts for the partition to be in $\mathcal{G}\mathcal{G}_1$. These differences can be equal to zero. In general, the number of the non-essential boxes of the 2-modular Young diagram for a partition in $\mathcal{G}\mathcal{G}_1$ is given by the formula

$$\frac{\lambda_i - \lambda_{i+1} - \delta_{\lambda_i,e} - \delta_{\lambda_{i+1},e}}{2} - 1, \tag{3–12}$$

where

$$\delta_{n,e} := \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{3–13}$$

Later we will need

$$\delta_{n,o} := 1 - \delta_{n,e}. \tag{3–14}$$

The product of (3–10) and (3–11) is the generating function for the number of partitions from $\mathcal{G}\mathcal{G}_1$ into exactly n parts where the non-essential boxes in their 2-modular Young diagram representation come in two colors. Let $\pi`$ be a partition counted by (3–10) in one color and let $\pi^*$ be a partition counted by (3–11) in another color. Then we insert columns of the 2-modular Young diagram representation of $\pi`$ in the 2-modular Young diagram representation of π^*. In doing so, we insert the different colored columns all the way left those columns can be inserted, without violating the definition of 2-modular Young diagrams. One example of the insertion of this type for $n = 4$ is presented in Table 3-2.

Table 3-2. Insertion of the columns of $\pi` = (6, 4, 4, 4)$ in $\pi^* = (12, 8, 3, 1) \in \mathcal{G}\mathcal{G}_1$.

π^*

2	2	2	2	2	2
2	2	2	2		
2	1				
1					

,

$\pi`$

2	2	2
2	2	
2	2	
2	2	

$\rightarrow$

2	2	2	2	2	2	2	2	2
2	2	2	2	2	2			
2	2	2	1					
2	2	1						

The inserted columns from $\pi`$ are all non-essential for the outcome partition to lie in $\mathcal{G}\mathcal{G}_1$, though those are not the only non-essential columns.

This insertion changes the number of non-essential boxes of a partition $\pi \in \mathcal{GG}_1$, and it does not effect any essential structure of the 2-modular Young diagrams. There are a total of

$$\frac{\lambda_i - \lambda_{i+1} - \delta_{\lambda_i,e} - \delta_{\lambda_{i+1},e}}{2} \tag{3–15}$$

many different possibilities for the coloration of the non-essential boxes that appear from the part λ_i to λ_{i+1}. Similarly, there are

$$\frac{\lambda_n + \delta_{\lambda_n,o}}{2}$$

many coloration possibilities for a the smallest part of a partition that gets counted by the summand of (3–8), where $\delta_{n,o}$ is defined in (3–14).

Hence, combining all the possible number of colorations, there are

$$\omega_1(\pi) := \frac{\lambda_{\nu(\pi)} + \delta_{\lambda_{\nu(\pi)},o}}{2} \cdot \prod_{i=1}^{\nu(\pi)-1} \frac{\lambda_i - \lambda_{i+1} - \delta_{\lambda_i,e} - \delta_{\lambda_{i+1},e}}{2} \tag{3–16}$$

total number of colorations of a partition $\pi \in \mathcal{GG}_1$. The far right partition in Table 3-2 is one of the possible colorations of the partition $(18, 12, 7, 5)$, and the total number of colorations via (3–16) is $\omega_1(18, 12, 7, 5) = 3 \cdot 2 \cdot 2 \cdot 1 = 12$.

The above discussion yields the weighted partition identity:

Theorem 3.6 (Alladi, 2012)**.**

$$\sum_{\pi \in \mathcal{GG}_1} \omega_1(\pi) q^{|\pi|} = \sum_{\pi \in \mathcal{P}_{do}} q^{|\pi|}, \tag{3–17}$$

where ω_1 is defined as in (3–16) and $\mathcal{P}_{do}$ is the set of partitions with distinct odd parts.

Theorem 3.6 is essentially [4, Theorem 3] with $a = b = 1$ with minor corrections for the weight associated with the smallest part of Göllnitz–Gordon partitions. The set of partitions $\mathcal{P}_{do}$, partitions with distinct odd parts, has also been studied in [5] and [25]. We give an example of Theorem 3.6 in Table 3-3.

Formally, let $\lambda_{\nu(\pi)+1} := 0$ for a partition π. Following the same construction (3–10)–(3–16) step-by-step for the $\mathcal{GG}_2$ type partitions, we see that the left-hand side of (3–9) can be

Table 3-3. Example of Theorem 3.6 with $|\pi| = 12$.

$\pi \in \mathcal{GG}_1$	ω_1	$\pi \in \mathcal{P}_{do}$	$\pi \in \mathcal{P}_{do}$
(12)	6	(12)	$(6,4,2)$
$(11,1)$	5	$(11,1)$	$(6,3,2,1)$
$(10,2)$	3	$(10,2)$	$(6,2,2,2)$
$(9,3)$	6	$(9,3)$	$(5,4,3)$
$(8,4)$	2	$(9,2,1)$	$(5,4,2,1)$
$(8,3,1)$	2	$(8,4)$	$(5,3,2,2)$
$(7,5)$	3	$(8,3,1)$	$(5,2,2,2,1)$
$(7,4,1)$	1	$(8,2,2)$	$(4,4,4)$
		$(7,5)$	$(4,4,3,1)$
		$(7,4,1)$	$(4,4,2,2)$
		$(7,3,2)$	$(4,3,2,2,1)$
		$(7,2,2,1)$	$(4,2,2,2,2)$
		$(6,6)$	$(3,2,2,2,2,1)$
		$(6,5,1)$	$(2,2,2,2,2,2)$

The summation of all $\omega_1(\pi)$ values for $\pi \in \mathcal{GG}_1$ with $|\pi| = 12$ equals 28 as the number of partitions from $\mathcal{P}_{do}$ with the same norm.

interpreted as a weighted generating function for the number of partitions

$$\sum_{\pi \in \mathcal{GG}_2} \omega_2(\pi) q^{|\pi|},$$

where

$$\omega_2(\pi) := \prod_{i=1}^{\nu(\pi)} \frac{\lambda_i - \lambda_{i+1} - \delta_{\lambda_i,e} - \delta_{\lambda_{i+1},e}}{2}. \tag{3–18}$$

This weight ω_2, unlike ω_1, is uniform on every pair of consecutive parts with our customary definition $\lambda_{\nu(\pi)+1} = 0$.

We rewrite the sum in the middle term of (3–9) as

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n^2+n}}{(-q;q^2)_{n+1}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \left(\sum_{j\geq 0} \frac{q^{4j^2+2j}(1-q^{4j+2})}{(-q;q^2)_{2j+1}} + \sum_{j\geq 1} \frac{q^{4j^2+2j-1}}{(-q;q^2)_{2j}} \right). \tag{3–19}$$

Clearly (3–19) amounts to

$$\sum_{j\geq 0} \frac{q^{4j^2+2j}}{(-q;q^2)_{2j+1}} - \sum_{j\geq 1} \frac{q^{4j^2-2j}}{(-q;q^2)_{2j}} = \sum_{j\geq 0} \frac{q^{4j^2+2j}(1-q^{4j+2})}{(-q;q^2)_{2j+1}} + \sum_{j\geq 1} \frac{q^{4j^2+2j-1}}{(-q;q^2)_{2j}}, \tag{3–20}$$

where we split the sum on the left of (3–19) into two sub-sums according to the parity of the summation variable and changing the variable name n to j. After cancellations, (3–20) turns into

$$\sum_{j\geq 1} \frac{q^{4j^2-2j}(1+q^{4j-1})}{(-q;q^2)_{2j}} = \sum_{j\geq 0} \frac{q^{4j^2+6j+2}}{(-q;q^2)_{2j+1}}. \tag{3–21}$$

The equation (3–21) can be easily established by simplifying the fraction on the left and shifting the summation variable $j \mapsto j+1$.

Let $\mathcal{P}_{rdo}$ be the set of partitions with distinct odd parts with the additional restrictions that the smallest part is > 1, and if the smallest part of a partition π is 2, then π starts either as

$$\pi = (2^{f_2}, 4^{f_4}, 6^{f_6}, \dots, (4j-2)^{f_{4j-2}}, (4j-1)^1, \dots), \tag{3–22}$$

where $f_2,\ f_4,\ \dots,\ f_{4j-2}$ are all positive, or as

$$\pi = (2^{f_2}, 4^{f_4}, 6^{f_6}, \dots, (4j)^{f_{4j}}, (4j+1)^0, (4j+2)^0, \dots), \tag{3–23}$$

where $f_2,\ f_4,\ \dots,\ f_{4j}$ are all positive, for any positive j. We now claim that the middle term of (3–9) is the generating function for the number the partitions from the set $\mathcal{P}_{rdo}$. We demonstrate this with the aid of (3–19). Using distribution on the right of (3–19), we get

$$\begin{aligned}\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}&\left(\sum_{j\geq 0}\frac{q^{4j^2+2j}(1-q^{4j+2})}{(-q;q^2)_{2j+1}} + \sum_{j\geq 1}\frac{q^{4j^2+2j-1}}{(-q;q^2)_{2j}}\right)\\ &= \sum_{j\geq 0} q^{4j^2+2j}(-q^{4j+3};q^2)_\infty \frac{1-q^{4j+2}}{(q^2;q^2)_\infty} + \sum_{j\geq 1} q^{4j^2+2j-1}\frac{(-q^{4j+1};q^2)_\infty}{(q^2;q^2)_\infty}.\end{aligned}$$

Thus we have for the right-hand side of (3–19)

$$\frac{(-q^3;q^2)_\infty}{(q^4;q^2)_\infty} + \sum_{j\geq 1} q^{4j^2+2j-1}\frac{(-q^{4j+1};q^2)_\infty}{(q^2;q^2)_\infty} + \sum_{j\geq 1} q^{4j^2+2j}(-q^{4j+3};q^2)_\infty \frac{1-q^{4j+2}}{(q^2;q^2)_\infty}. \tag{3–24}$$

For positive integers j, we can write

$$4j^2 + 2j - 1 = 2 + 4 + 6 + \cdots + (4j - 2) + (4j - 1),$$
$$4j^2 + 2j = 2 + 4 + \cdots + 4j.$$

The above implies the initial conditions in (3–22) and (3–23), respectively. The presence of the distinct odd parts and the (possibly repeated) even parts is clear from the shifted q-factorials. This proves,

Theorem 3.7.

$$\sum_{\pi \in \mathcal{GG}_2} \omega_2(\pi) q^{|\pi|} = \sum_{\pi \in \mathcal{P}_{rdo}} q^{|\pi|}, \tag{3–25}$$

where ω_2 *as in* (3–18).

The second equality in (3–9) connects an order 3 false theta function with the combinatorial objects we have interpreted above (3–25). This false theta function can also be interpreted as a generating function for the number of partitions on a set after some modification. It is easy to see that

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{j \geq 0} q^{3j^2+2j}(1 - q^{2j+1}) = \sum_{j \geq 0} q^{3j^2+2j} \frac{(-q;q^2)_j}{(q^2;q^2)_{2j}} \frac{(-q^{2j+3};q^2)_\infty}{(q^{4j+4};q^2)_\infty}. \tag{3–26}$$

Let $\mathcal{A}$ denote the set of partitions where for any partition

i. the first integer that is not a part is odd,

ii. the double of the first missing part is also missing,

iii. each even part less than the first missing part appears at least twice,

iv. each odd part less than the first missing part appears at most twice,

v. each odd larger than the first missing part is not repeated.

The expression (3–26) can be interpreted —as G. E. Andrews did [15]— as the generating function for the number of partitions from the set $\mathcal{A}$. This interpretation can be seen after the clarification that $3j^2 + 2j = 1 + 2 + 2 + 3 + 4 + 4 + \cdots + (2j - 1) + 2j + 2j$.

The set used in this interpretation does not consist of distinct odd parts necessarily, and therefore gets out of the scope of the identity of Theorem 3.7. Nevertheless, this observation finalizes the discussion of the combinatorial version of (3–9):

Theorem 3.8.

$$\sum_{\pi \in \mathcal{GG}_2} \omega_2(\pi) q^{|\pi|} = \sum_{\pi \in \mathcal{P}_{rdo}} q^{|\pi|} = \sum_{\pi \in \mathcal{A}} q^{|\pi|}.$$

where ω_2 as in (3–18).

We demonstrate Theorem 3.8 in Table 3-4.

Table 3-4. Example of Theorem 3.8 with $|\pi| = 12$.

$\pi \in \mathcal{GG}_2$	$\omega_2(\pi)$	$\pi \in \mathcal{P}_{rdo}$	$\pi \in \mathcal{A}$
(12)	5	(12)	(12)
$(9,3)$	3	$(9,3)$	$(9,3)$
$(8,4)$	1	$(8,4)$	$(8,4)$
$(7,5)$	2	$(7,5)$	$(7,5)$
		$(7,3,2)$	$(7,2,2,1)$
		$(6,6)$	$(6,6)$
		$(5,4,3)$	$(5,4,3)$
		$(5,3,2,2)$	$(5,2,2,2,1)$
		$(4,4,4)$	$(4,4,4)$
		$(4,4,2,2)$	$(4,2,2,2,1,1)$
		$(4,2,2,2,2)$	$(2,2,2,2,2,1,1)$

The summation of all $\omega_2(\pi)$ values for $\pi \in \mathcal{GG}_2$ with $|\pi| = 12$ equals 11 as the number of partitions from $\mathcal{P}_{rdo}$ and $\mathcal{A}$ with the same norm.

Recall that $\mathcal{RR}$ is the set of partitions into distinct parts with difference between parts ≥ 2. We also note that, similar to (3–5), the choice of the set $\mathcal{GG}_2$ in Theorem 3.8 can be replaced with a superset such as $\mathcal{GG}_1$ or $\mathcal{RR}$. The weight $\omega_2(\pi)$ would vanish for a partition $\pi \in \mathcal{RR} \setminus \mathcal{GG}_2$. In particular, we have

$$\sum_{\pi \in \mathcal{GG}_2} \omega_2(\pi) q^{|\pi|} = \sum_{\pi \in \mathcal{RR}} \omega_2(\pi) q^{|\pi|}.$$

3.2 Weighted partition identities relating partitions into distinct parts and unrestricted partitions

We start with two identities that will yield weighted partition identities between the sets $\mathcal{D}$, partitions into distinct parts, and $\mathcal{U}$, the set of all partitions.

Theorem 3.9.

$$\sum_{n\geq 0} \frac{q^{(n^2+n)/2}(-1;q)_n}{(q;q)_n^2} = \frac{(-q;q)_\infty}{(q;q)_\infty}, \tag{3–27}$$

$$\sum_{n\geq 0} \frac{q^{(n^2+n)/2}(-q;q)_n}{(q;q)_n^2} = \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+\sum_{n\geq 1}\frac{(-1)^n q^{2n-1}}{(-q;q^2)_n}\right) \tag{3–28}$$

$$= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{j\geq 0} q^{(3j^2+j)/2}(1-q^{2j+1}). \tag{3–29}$$

Proof. We note that the left-hand sides of (3–27) and (3–28) are

$$\lim_{\rho\to\infty} {}_2\phi_1\left(\begin{matrix}-1,\ \rho\\ q\end{matrix};q,-\frac{q}{\rho}\right) \text{ and } \lim_{b\to -1}\lim_{\rho\to\infty} {}_2\phi_1\left(\begin{matrix}\rho,\ qb\\ qb^2\end{matrix};q,\frac{qb}{\rho}\right), \text{ respectively.}$$

Similar to the case of Theorem 3.5, equation (3–27) is a special case of the q-Gauss identity (A–54). This identity has also been previously proven in the work of Starcher [53, (3.7), p. 805].

Identity (3–28) is more involved. To establish the equality of (3–28), we apply the Heine transformation (A–55) with $a=\rho$ which yields

$$\sum_{n\geq 0} \frac{q^{(n^2+n)/2}(-q;q)_n}{(q;q)_n^2} = \lim_{b\to -1}\lim_{\rho\to\infty} \frac{(b;q)_\infty(q^2b^2/\rho;q)_\infty}{(qb^2;q)_\infty(qb/\rho;q)_\infty}\sum_{n\geq 0}\frac{(qb;q)_n}{(q^2b^2/\rho;q)_n}b^n. \tag{3–30}$$

After the limit $\rho\to\infty$, the sum on the right of (3–30) turns into

$$\sum_{n\geq 0} \frac{q^{(n^2+n)/2}(-q;q)_n}{(q;q)_n^2} = \lim_{b\to -1}\frac{(bq;q)_\infty}{(qb^2;q)_\infty}(1-b)F(b,0;b), \tag{3–31}$$

where in Fine's notation [41, (1.1)]

$$F(a,b;t) := {}_2\phi_1\left(\begin{matrix}q,\ aq\\ bq\end{matrix};q,t\right).$$

We have three explicit formulas for the expression $\lim_{b\to -1}(1-b)F(b,0;b)$ coming from Fine's work:

$$\lim_{b\to -1}(1-b)F(b,0;b) = \sum_{j\geq 0} q^{(3j^2+j)/2}(1-q^{2j+1}) \tag{3–32}$$

$$= 1 + \sum_{n\geq 1} \frac{(-1)^n q^{2n-1}}{(-q;q^2)_n} \tag{3–33}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n^2+n)/2}}{(-q;q)_n}. \tag{3–34}$$

These identities are [41, (7.7), p. 7], [41, (23.2), p. 45], and [41, (6.1), p. 4] with $a = t \to -1$, respectively. Formulas (3–32) and (3–33) in comparison with (3–31) prove both (3–29) and (3–28), respectively. □

Similar to the situation of Theorem 3.5, the small change on the left side of (3–28): $(-1;q)_n \mapsto (-q;q)_n$, yields increase in complexity on the right side of (3–29) which again involves a false theta function.

Note that the equality of the right sides of the identities (3–32)–(3–34) can be proved in a purely combinatorial manner with the aid of Sylvester's bijection [34] and Franklin's involution [17]. The equality of (3–32) and (3–34) will be used later in the proof of the Theorem 3.13.

We remark that identity (3–27) was further studied in [39]. There the identity was combinatorially interpreted as a relation between generalized Frobenius symbols and overpartitions.

Now we will move on to our discussion of combinatorial interpretations of the analytic identities of Theorem 3.9. We have already pointed out that the product on the right side (3–27) is a special case of Alladi's (3–1) with $a = 1$, $b = 2$ and $n \to \infty$. This can be interpreted as the weighted sum on the set of partitions $\mathcal{U}$:

$$\frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{\pi\in\mathcal{U}} 2^{\nu_d(\pi)} q^{|\pi|}, \tag{3–2}$$

where $\nu_d(\pi)$ is the number of different parts of π.

The left-hand side of (3–27) can also be interpreted as a weighted sum. In order to derive the weights involved, we dissect the summand on the left. For a positive integer n, we have

$$\frac{q^{(n^2+n)/2}(-1;q)_n}{(q;q)_n^2} = \frac{q^{(n^2+n)/2}}{(q;q)_n}\frac{2}{1-q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}. \tag{3–35}$$

The first expression on the right

$$\frac{q^{(n^2+n)/2}}{(q;q)_n} \tag{3–36}$$

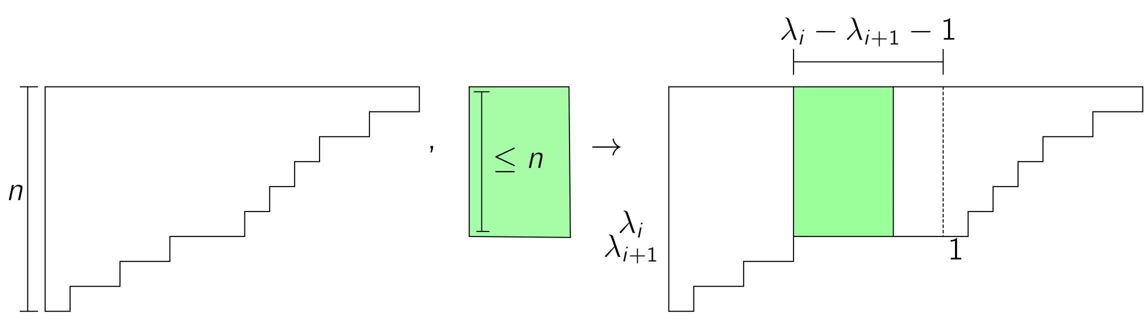
is the generating function for the number of partitions into exactly n distinct parts [17]. We will think of these partitions to have the base color of white. The rational factor

$$\frac{2}{1-q^n} = 2 + 2q^n + 2q^{2n} + 2q^{3n} + \dots \tag{3–37}$$

is the generating function for the number of partitions into parts each of size n each time counted with weight 2, regardless of occurrence. We combine Young diagrams of partitions enumerated by (3–36) and the conjugate of partitions counted by (3–37) using column insertions. This yields the generating function for the number of partitions into exactly n disctinct parts, where part λ_n is counted with weight $2\lambda_n$.

The column insertion is similar to the case in the 2-modular Young diagrams as we exemplified in Table 3-2. We embed a colored column from a conjugate of a colored partition counted by (3–37) all the way left inside a Young diagram counted by (3–36) without violating the definition of a partition. An example of column insertion is given in Table 3-5.

Table 3-5. Illustration of the column insertion.



The expression

$$\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}$$

is the generating function for the number of partitions into parts $\leq n-1$, where every different sized part is counted with weight 2. After conjugating these partitions and inserting its columns to partitions into n distinct parts, we see that there are $2(\lambda_i - \lambda_{i+1} - 1) + 1$ possible colorations between consecutive parts, where at least one secondary color appears for $1 \leq i \leq n-1$. To be more precise, there are $\lambda_i - \lambda_{i+1} - 1$ columns coloring the space between λ_{i+1} and $\lambda_i - 1$ and each coloring comes with weight 2. This way we have the weight $2(\lambda_i - \lambda_{i+1} - 1) + 1$ where the extra 1 comes from the option of not having a colored column at all. Again these column insertions are demonstrated in Table 3-5.

Hence, for a partition $\pi = (\lambda_1, \lambda_2, \dots)$, we have

$$\sum_{n\geq 0} \frac{q^{(n^2+n)/2}(-1;q)_n}{(q;q)_n^2} = \sum_{\pi\in\mathcal{D}} \widetilde{\omega}_1(\pi) q^{|\pi|}, \tag{3–38}$$

where

$$\widetilde{\omega}_1(\pi) := 2\lambda_{\nu(\pi)} \cdot \prod_{i=1}^{\nu(\pi)-1} (2\lambda_i - 2\lambda_{i+1} - 1). \tag{3–39}$$

Similar to (3–18), we can change the product of $\widetilde{\omega}_1$ into a uniform product over the parts of a partition. With the custom choice that $\lambda_{\nu(\pi)+1} := -1/2$, we have

$$\widetilde{\omega}_1(\pi) = \prod_{i=1}^{\nu(\pi)} (2\lambda_i - 2\lambda_{i+1} - 1). \tag{3–40}$$

Combining (3–27), (3–2), and (3–38) yields

Theorem 3.10.

$$\sum_{\pi\in\mathcal{D}} \widetilde{\omega}_1(\pi) q^{|\pi|} = \sum_{\pi\in\mathcal{U}} \omega_1`(\pi) q^{|\pi|},$$

where $\widetilde{\omega}_1(\pi)$ *is as in* (3–40) *and* $\omega_1`(\pi) = 2^{\nu_d(\pi)}$.

This is the first example of a weighted partition identity connecting $\mathcal{D}$ and $\mathcal{U}$ with strictly positive weights. The combinatorial interpretation of (3–28) is going to provide a second example of a connection between $\mathcal{D}$ and $\mathcal{U}$ making use of a new partition statistic.

The left side of (3–28) can be interpreted similar to (3–27). The weights associated with this case differ from the weight $\widetilde{\omega}_1$ only at the last part. For a partition $\pi = (\lambda_1, \lambda_2, \dots)$ with the custon definition that $\lambda_{\nu(\pi)+1} := 0$ we define the new weight uniformly as in (3–40),

$$\widetilde{\omega}_2(\pi) = \prod_{i=1}^{\nu(\pi)} (2\lambda_i - 2\lambda_{i+1} - 1). \tag{3–41}$$

With this definition, we have the identity similar to (3–38),

$$\sum_{n\geq 0} \frac{q^{(n^2+n)/2}(-q;q)_n}{(q;q)_n^2} = \sum_{\pi\in\mathcal{D}} \widetilde{\omega}_2(\pi) q^{|\pi|}. \tag{3–42}$$

In order to get the weights for the right side of (3–28), we modify that expression. We rewrite $(-q;q^2)_\infty$, the generating function for number of partitions into distinct odd parts, as

$$(-q;q^2)_\infty = 1 + \sum_{n\geq 1} q^{2n-1}(-q^{2n+1};q^2)_\infty. \tag{3–43}$$

Note that the summands in (3–43) are generating functions for the number of partitions into distinct odd parts with the smallest part being equal to $2n-1$.

The right-hand side expression of the identity (3–28) directly yields

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + \sum_{n\geq 1} \frac{(-1)^n q^{2n-1}}{(-q;q^2)_n}\right) = \frac{(-q^2;q^2)_\infty}{(q;q)_\infty}\left((-q;q^2)_\infty + \sum_{n\geq 1}(-1)^n q^{2n-1}(-q^{2n+1};q^2)_\infty\right). \tag{3–44}$$

Employing (3–43), combining sums and changing the summation indices $n \mapsto n+1$ on the right side of (3–44), we get

$$\begin{aligned}
&= \frac{(-q^2;q^2)_\infty}{(q;q)_\infty}\left(1 + 2\sum_{n\geq 0} q^{4n+3}(-q^{4n+5};q^2)_\infty\right)\\
&= \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(\frac{1}{(q;q^2)_\infty} + \sum_{n\geq 0} \frac{1}{(q;q^2)_{2n+1}} \frac{2q^{4n+3}}{1-q^{4n+3}} \frac{(-q^{4j+5};q^2)_\infty}{(q^{4j+5};q^2)_\infty}\right).
\end{aligned} \tag{3–45}$$

We can interpret (3–45) as a combinatorial weighted identity over the set of unrestricted partitions, $\mathcal{U}$. Let $\pi = (\lambda_1, \lambda_2, \dots)$ be a partition. Let $\nu_{de}(\pi)$ be the number of different even parts. Let $\mu_{n,o}(\pi)$ denote the new partition statistic, defined as the number of different odd parts (without counting repetitions) $\geq n$ of π, for some integer n. We define

$$\omega_2`(\pi) = 2^{\nu_{de}(\pi)}\left(1 + \sum_{i\geq 0}\chi((4i+3)\in\pi)2^{\mu_{4i+3,o}(\pi)}\right), \tag{3–46}$$

where χ is defined as in (2–29). With these definitions and keeping (3–45) in mind, we have the weighted identity

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1 + \sum_{n\geq 1}\frac{(-1)^n q^{2n-1}}{(-q;q^2)_n}\right) = \sum_{\pi\in\mathcal{U}}\omega_2`(\pi)q^{|\pi|}. \tag{3–47}$$

The emergence of this weight can be explained in two parts. The front factor of (3–45) (identity (3–2) with $q\mapsto q^2$) yields the weight $2^{\nu_{de}(\pi)}$. This is easy to see as in the combined partition all of the parts coming from (3–2) with $q\mapsto q^2$ can be thought of as even parts. The summation part of the weight (3–46) comes from the respective summation in (3–45)

$$\frac{1}{(q;q^2)_\infty} + \sum_{n\geq 0}\frac{1}{(q;q^2)_{2n+1}}\frac{2q^{4n+3}}{1-q^{4n+3}}\frac{(-q^{4j+5};q^2)_\infty}{(q^{4j+5};q^2)_\infty}.$$

The first term is the generating function for the number of partitions into odd parts where we count every partition once. The right summation is the weighted count of partitions into odd parts. For a non-negative integer n the summand

$$\frac{1}{(q;q^2)_{2n+1}}\frac{2q^{4n+3}}{1-q^{4n+3}}\frac{(-q^{4j+5};q^2)_\infty}{(q^{4j+5};q^2)_\infty}$$

is the generating function for the number of partitions, where $4n+3$ appears as a part, every odd part less than $4n+3$ is counted once, and every different odd part $\geq 4n+3$ is counted with the weight 2. This yields the weight $2^{\mu_{4n+3,o}(\pi)}$ for a partition π.

Above observations (3–42) and (3–47) combined with (3–28) provide another new example of a relation between partitions into distinct parts and partitions into unrestricted parts with non-vanishing weights.

Theorem 3.11.

$$\sum_{\pi\in\mathcal{D}} \widetilde{\omega}_2(\pi) q^{|\pi|} = \sum_{\pi\in\mathcal{U}} {\omega_2}`(\pi) q^{|\pi|},$$

where $\widetilde{\omega}_2(\pi)$ *is as in* (3–41) *and* ${\omega_2}`(\pi)$ *as in* (3–46).

We would like to exemplify Theorem 3.11 in Table 3-6.

Table 3-6. Example of Theorem 3.11 with $|\pi| = 10$.

$\pi\in\mathcal{U}$	${\omega_2}`$	$\pi\in\mathcal{U}$	${\omega_2}`$	$\pi\in\mathcal{U}$	${\omega_2}`$	$\pi\in\mathcal{D}$	$\widetilde{\omega}_2$
(10)	2	(5, 3, 2)	10	(3, 3, 3, 1)	3	(10)	19
(9, 1)	1	(5, 3, 1, 1)	5	(3, 3, 2, 2)	6	(9, 1)	15
(8, 2)	4	(5, 2, 2, 1)	2	(3, 3, 2, 1, 1)	6	(8, 2)	33
(8, 1, 1)	2	(5, 2, 1, 1, 1)	2	(3, 3, 1, 1, 1, 1)	3	(7, 3)	35
(7, 3)	7	(5, 1, 1, 1, 1, 1)	1	(3, 2, 2, 2, 1)	6	(6, 4)	21
(7, 2, 1)	6	(4, 4, 2)	4	(3, 2, 2, 1, 1, 1)	6	(6, 3, 1)	15
(7, 1, 1, 1)	3	(4, 4, 1, 1)	2	(3, 2, 1, 1, 1, 1, 1)	6	(5, 4, 1)	5
(6, 4)	4	(4, 3, 3)	6	(3, 1, 1, 1, 1, 1, 1, 1)	3	(5, 3, 2)	9
(6, 3, 1)	6	(4, 3, 2, 1)	12	(2, 2, 2, 2, 2)	2	(4, 3, 2, 1)	1
(6, 2, 2)	4	(4, 3, 1, 1, 1)	6	(2, 2, 2, 2, 1, 1)	2		
(6, 2, 1, 1)	4	(4, 2, 2, 2)	4	(2, 2, 2, 1, 1, 1, 1)	2		
(6, 1, 1, 1, 1)	2	(4, 2, 2, 1, 1)	4	(2, 2, 1, 1, 1, 1, 1, 1)	2		
(5, 5)	1	(4, 2, 1, 1, 1, 1)	4	(2, 1, 1, 1, 1, 1, 1, 1, 1)	2		
(5, 4, 1)	2	(4, 1, 1, 1, 1, 1, 1)	2	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	1		

The summation of all ${\omega_2}`(\pi)$, or all $\widetilde{\omega}_2(\pi)$ for $|\pi| = 10$ are the same and the sum equals 162.

In literature, there are many examples of partition identities with multiplicative weights. This is no different from the previous parts of this paper, such as Theorem 3.2, 3.4, 3.6, 3.8. and 3.10. Theorem 3.11 is interesting not only because it gives a weighted connection between the sets $\mathcal{D}$ and $\mathcal{U}$, but also because of the appearance of the unusual additive weights.

The expression (3–29), which involves an order 3/2 false theta function, can be interpreted as a generating function for a weighted count of the ordinary partitions. The interpretation of the similar expression (3–9), which has an order 3 false theta function, required us to depart from the set of partitions with distinct odd parts $\mathcal{P}_{do}$ to an unexpected set $\mathcal{A}$ (with partitions not necessarily having distinct odd parts) with trivial weight 1 for each partition. Now we have a different situation. We stay with the set of all partitions $\mathcal{U}$, but the weights become non-trivial and, occasionally, zero.

Recall that in frequency notation, a partition $\pi = (1^{f_1}, 2^{f_2}, \dots)$, where $f_i(\pi) = f_i$ is the number of occurrences of i in π. Let

$$\omega_2^*(\pi) = (1 - \chi(f_1(\pi) \geq 2)) \prod_{n\geq 2} 2^{\chi(f_n(\pi)\geq 1)} + \tag{3–48}$$
$$\sum_{j\geq 1} \left(\chi(f_{2j+1}(\pi) \leq 1)\chi(f_j(\pi) \geq 2)2^{\chi(f_j(\pi)\geq 3)} \prod_{i=1}^{j-1} \chi(f_i(\pi) \geq 3)2^{\chi(f_i(\pi)\geq 4)} \prod_{\substack{n>j,\\ n\neq 2j+1}} 2^{\chi(f_n(\pi)\geq 1)} \right),$$

where χ is defined in (2–29). We remark that the sum in $\omega_2^*(\pi)$ is finite as partitions are finite, and so $\chi(f_i(\pi) \geq 3)$ vanishes for any value of i greater than the largest part of π. Then we have

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{j\geq 0} q^{(3j^2+j)/2}(1 - q^{2j+1}) = \sum_{\pi\in\mathcal{U}} \omega_2^*(\pi) q^{|\pi|}. \tag{3–49}$$

This can be proven by doing cancellations with the front factor of the false theta function (3–50):

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{j\geq 0} q^{(3j^2+j)/2}(1 - q^{2j+1}) = (-q;q)_\infty \sum_{j\geq 0} \frac{q^{(3j^2+j)/2}}{(q;q)_{2j}(q^{2j+2};q)_\infty}. \tag{3–50}$$

The expression (3–50) is the generating function of partitions with weights ω_2^*. The front factor $(-q;q)_\infty$ is the generating function for the number of partitions into distinct parts. Therefore, for our interpretation, every part can appear at least once. For a non-negative integer j the summand is the generating function for the number of partitions, where $2j+1$ does not appear as a part, every number up to $j-1$ appears at least 3 times, and j appears at least 2 times, as $(3j^2+j)/2 = 1+1+1+2+2+2+\cdots+(j-1)+(j-1)+(j-1)+j+j$.

This weight is also non-trivial and a sum of multiplicative terms. This is exemplified in Table 3-7.

Hence, we get the similar result to Theorem 3.8:

Theorem 3.12.

$$\sum_{\pi\in\mathcal{D}} \widetilde{\omega}_2(\pi) q^{|\pi|} = \sum_{\pi\in\mathcal{U}} \omega_2`(\pi) q^{|\pi|} = \sum_{\pi\in\mathcal{U}} \omega_2^*(\pi) q^{|\pi|},$$

where weights $\widetilde{\omega}_2(\pi)$, $\omega_2`(\pi)$, and ω_2^* are as in (3–41), (3–46) and (3–48), respectively.

Table 3-7. Example of Theorem 3.11 with $|\pi| = 10$.

$\pi \in \mathcal{U}$	ω_2^*	$\pi \in \mathcal{U}$	ω_2^*	$\pi \in \mathcal{U}$	ω_2^*
(10)	2	(5, 3, 2)	8	(3, 3, 3, 1)	2
(9, 1)	2	(5, 3, 1, 1)	2	(3, 3, 2, 2)	4
(8, 2)	4	(5, 2, 2, 1)	4	(3, 3, 2, 1, 1)	0
(8, 1, 1)	2	(5, 2, 1, 1, 1)	8	(3, 3, 1, 1, 1, 1)	0
(7, 3)	4	(5, 1, 1, 1, 1, 1)	4	(3, 2, 2, 2, 1)	4
(7, 2, 1)	4	(4, 4, 2)	4	(3, 2, 2, 1, 1, 1)	6
(7, 1, 1, 1)	4	(4, 4, 1, 1)	2	(3, 2, 1, 1, 1, 1, 1)	4
(6, 4)	4	(4, 3, 3)	4	(3, 1, 1, 1, 1, 1, 1, 1)	2
(6, 3, 1)	4	(4, 3, 2, 1)	8	(2, 2, 2, 2, 2)	2
(6, 2, 2)	4	(4, 3, 1, 1, 1)	4	(2, 2, 2, 2, 1, 1)	2
(6, 2, 1, 1)	4	(4, 2, 2, 2)	4	(2, 2, 2, 1, 1, 1, 1)	8
(6, 1, 1, 1, 1)	4	(4, 2, 2, 1, 1)	4	(2, 2, 1, 1, 1, 1, 1, 1)	6
(5, 5)	2	(4, 2, 1, 1, 1, 1)	8	(2, 1, 1, 1, 1, 1, 1, 1, 1)	4
(5, 4, 1)	4	(4, 1, 1, 1, 1, 1, 1)	4	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	2

The summation of all $\omega_2^*(\pi)$ values for $|\pi| = 10$ equals 162, as in the values of Table 3-6.

3.3 A Weighted Partition Identity Related to $\frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty} q^{(3j^2+j)/2}(1-q^{2j+1})$

In Section 3.1, we have proven Theorems 3.6 and 3.7 involving partitions with distinct odd parts counted with trivial weights. In this section we will derive another partition identity involving partitions with distinct odd parts, this time with non-trivial weights. To this end we prove the following theorem.

Theorem 3.13.

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty} q^{(2n+1)n}(-q^{2n+2};q)_\infty = \frac{1}{(q;q)_\infty}\sum_{j=0}^{\infty} q^{(3j^2+j)/2}(1-q^{2j+1}). \tag{3–51}$$

Proof. This theorem amounts to manipulating the equality of (3–32) and (3–34). We point out that doing the even–odd index split of the summand of (3–34) and using

$$\frac{q^{(2n+1)n}}{(-q;q)_{2n}} - \frac{q^{(2n+1)n+(2n+1)}}{(-q;q)_{2n+1}} = \frac{q^{(2n+1)n}}{(-q;q)_{2n+1}}$$

yields

$$\sum_{n=0}^{\infty}\frac{q^{(2n+1)n}}{(-q;q)_{2n+1}} = \sum_{j=0}^{\infty} q^{(3j^2+j)/2}(1-q^{2j+1}). \tag{3–52}$$

The identity (3–52) appears in the Ramanujan's lost notebooks [18, (9.4.4), p. 233]. Multiplying both sides of (3–52) with

$$\frac{(-q;q)_\infty}{(q^2;q^2)_\infty} = \frac{1}{(q;q)_\infty},$$

and doing the necessary simplifications on the left, we arrive at (3–51). $\square$

Next we define two sets of partitions. Let $\mathcal{P}_{dom}$ be the set of partitions with distinct odd parts, where the smallest positive integer that is not a part is odd, and let $\mathcal{U}_{ic}$ be the set of ordinary partitions subject to the initial condition that if $2j+1$ is the smallest positive odd number that is not a part of the partition, then every even natural number $\leq j$ appears as a part, and all the odd natural numbers $\leq j$ appear at least twice in this partition. We rewrite (3–51) suggestively as

$$\sum_{n=0}^{\infty} \frac{q^{(2n+1)n}}{(q^2;q^2)_n} \frac{(-q^{2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty} (-q^{2n+3};q^2)_\infty = \sum_{j=0}^{\infty} \frac{q^{(3j^2+j)/2}}{(q;q)_{2j}(q^{2j+2};q)_\infty} \tag{3–53}$$

to show that the left and the right sides of (3–51) are related to counts for the partitions from the sets $\mathcal{P}_{dom}$ and U_{ic}, respectively. Observe that

$$(2n+1)n = 1+2+\cdots+2n$$

and

$$\frac{q^{(2n+1)n}}{(q^2;q^2)_n}$$

is the generating function for the number of partitions with distinct odd parts where every part is $\leq 2n$ and every integer $\leq 2n$ appears at least once. The factor

$$\frac{(-q^{2n+2};q^2)_\infty}{(q^{2n+2};q^2)_\infty}$$

is the generating function for the number of partitions into even parts $\geq 2n+2$ where each different even part is counted with weight 2. Putting the factors in the left-hand summand of (3–53) together, we see that the left side sum is a weighted count of partitions from $\mathcal{P}_{dom}$.

Also note that

$$(3j^2+j)/2 = (1+2+3+\cdots+j)+(1+3+5+\cdots+(2j-1)),$$

which is enough to see that the right side of (3–53) is the generating function for the number of partitions from $\mathcal{U}_{ic}$. These observations prove the following

Theorem 3.14.

$$\sum_{\pi\in\mathcal{P}_{dom}} 2^{\tau(\pi)} q^{|\pi|} = \sum_{\pi\in\mathcal{U}_{ic}} q^{|\pi|},$$

where, for a partition π, $\tau(\pi)$ is the number of different even parts of π larger than the smallest positive odd integer that is not a part of π.

We conclude with an example of this result in Table 3-8.

Table 3-8. Example of Theorem 3.14 with $|\pi| = 8$.

$\pi \in \mathcal{P}_{dom}$	$2^{\tau(\pi)}$	$\pi \in \mathcal{U}_{ic}$
(8)	2	(8)
(6, 2)	4	(6, 2)
(5, 3)	1	(6, 1, 1)
(5, 2, 1)	1	(5, 3)
(4, 4)	2	(5, 1, 1, 1)
(4, 2, 2)	4	(4, 4)
(2, 2, 2, 2)	2	(4, 2, 2)
		(4, 2, 1, 1)
		(4, 1, 1, 1, 1)
		(3, 3, 2)
		(3, 2, 1, 1, 1)
		(2, 2, 2, 2)
		(2, 2, 2, 1, 1)
		(2, 2, 1, 1, 1, 1)
		(2, 1, 1, 1, 1, 1, 1)
		(1, 1, 1, 1, 1, 1, 1, 1)

The sum of the weights $2+4+1+1+2+4+2 = 16$ is the same as the number of partitions from $\mathcal{U}_{ic}$ with $|\pi| = 8$.

CHAPTER 4
WEIGHTED PARTITION IDENTITIES WITH THE EMPHASIS ON THE SMALLEST PART

The results of [29] is presented in this chapter. In Section 4.1, we look at the partitions with alternating weights due to the parity of the smallest part of the partitions. The relations of a weighted partition identities of a positive integer and its relation with sums of squares are in (4.2). Section 4.3 has the discussion of some weighted partition identities for partitions with distinct even parts relating these partitions with triangular numbers. We finish the chapter with the weighted partition interpretation of an analytic identity due to Ramanujan.

4.1 Weighted Identities with respect to the Smallest Part of a Partition

Let $\mathcal{U}`$ be the set of non-empty partitions and let $\mathcal{U}^*$ be the subset of $\mathcal{U}`$ such that for every $\pi \in \mathcal{U}^*$, $f_1(\pi) \equiv 1 \mod 2$. Next, we introduce a new partition statistic $t(\pi)$ to be the number defined by the properties

i. $f_i \equiv 1 \mod 2$, for $1 \leq i \leq t(\pi)$,

ii. and $f_{t(\pi)+1} \equiv 0 \mod 2$.

Note that for any $\pi \in \mathcal{U}`$ with an even frequency of 1 (which could be 0) we have $t(\pi) = 0$. Then we have the weighted partition identity between the set of ordinary partitions and its subset $\mathcal{U}^*$ as follows.

Theorem 4.1.

$$\sum_{\pi \in \mathcal{U}`} (-1)^{s(\pi)+1} q^{|\pi|} = \sum_{\pi \in \mathcal{U}^*} t(\pi) q^{|\pi|}. \tag{4–1}$$

The left side identity is the weighted count of partitions of a given norm n where every partition with an odd smallest part gets counted with $+1$ and the partitions of n with an even smallest part gets counted with -1. There are 42 partitions of 10 in total. From this number, 9 partitions, (2^5), $(2^3, 4)$, $(2^2, 3^2)$, $(2^2, 6)$, $(2, 3, 5)$, $(2, 4^2)$, $(2, 8)$, $(4, 6)$, (10), have an even smallest part. Therefore, from the count of the left-hand side of (4–1), the coefficient of the q^{10} is $24 = 42 - 2 \cdot 9$. The right-hand side count and the weights can be found in Table 4-1.

The proof of Theorem 4.1 will be given as the combinatorial interpretation of the following analytic identity.

Table 4-1. Example of Theorem 4.1 with $|\pi| = 10$.

$\pi \in \mathcal{U}^*$	$t(\pi)$	$\pi \in \mathcal{U}^*$	$t(\pi)$
$(1, 2, 3, 4)$	4	$(1, 4, 5)$	1
$(1, 2^3, 3)$	3	$(1, 2^2, 5)$	1
$(1^5, 2, 3)$	3	$(1^5, 5)$	1
$(1, 2, 7)$	2	$(1^3, 3, 4)$	1
$(1^3, 2, 5)$	2	$(1, 3^3)$	1
$(1, 9)$	1	$(1^3, 2^2, 3)$	1
$(1^3, 7)$	1	$(1^7, 3)$	1
$(1, 3, 6)$	1		

The sum of the weights is 24, which is the same as the count of partitions with the altering sign with respect to their smallest part's parity.

Theorem 4.2.

$$\sum_{n\geq 1} \frac{q^n}{1+q^n}\frac{1}{(q;q)_{n-1}} = \sum_{n\geq 1} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n(q^{n+1};q)_\infty}. \tag{4–2}$$

Proof. Recall that $(0;q)_n = 1$ for any integer $n \geq 0$. Also note that

$$\frac{1+q}{1+q^n} = \frac{(-q;q)_{n-1}}{(-q^2;q)_{n-1}}, \tag{4–3}$$

for positive n. We start by writing the left–hand side of (4–2) as a q-hypergeometric function. Multiplying and dividing by $1+q$ and using (4–3), shifting the sum with $n \mapsto n+1$, and finally factoring out $q/(1+q)$ yields

$$\sum_{n\geq 1} \frac{q^n}{1+q^n}\frac{1}{(q;q)_{n-1}} = \frac{q}{1+q}\,{}_2\phi_1\left(\begin{matrix}0, -q\\ -q^2\end{matrix}; q, q\right). \tag{4–4}$$

We now apply Jackson's transformation (A–56) to (4–4). This gives us

$$\frac{q}{1+q}\,{}_2\phi_1\left(\begin{matrix}0, -q\\ -q^2\end{matrix}; q, q\right) = \frac{q}{1+q}\frac{1}{(q;q)_\infty}\,{}_2\phi_2\left(\begin{matrix}0,\ q\\ -q^2,\ 0\end{matrix}; q, -q^2\right). \tag{4–5}$$

Distributing the front factor to each summand on the right-hand side of (4–5), doing the necessary simplifications, and finally shifting the summation index $n \mapsto n-1$ finishes the proof. $\square$

Theorem 4.2 is the analytical version of Theorem 4.1. We will now move on to the generating function interpretations of both sides of (4–2). This study will in-turn prove Theorem 4.1.

We start with the left-hand side sum

$$\sum_{n\geq 1}\frac{q^n}{1+q^n}\frac{1}{(q;q)_{n-1}}, \tag{4–6}$$

of (4–2). For a positive integer n, the summand

$$\frac{q^n}{1+q^n}=\sum_{k\geq 1}(-1)^{k+1}q^{nk}=q^n-q^{2n}+q^{3n}\ldots \tag{4–7}$$

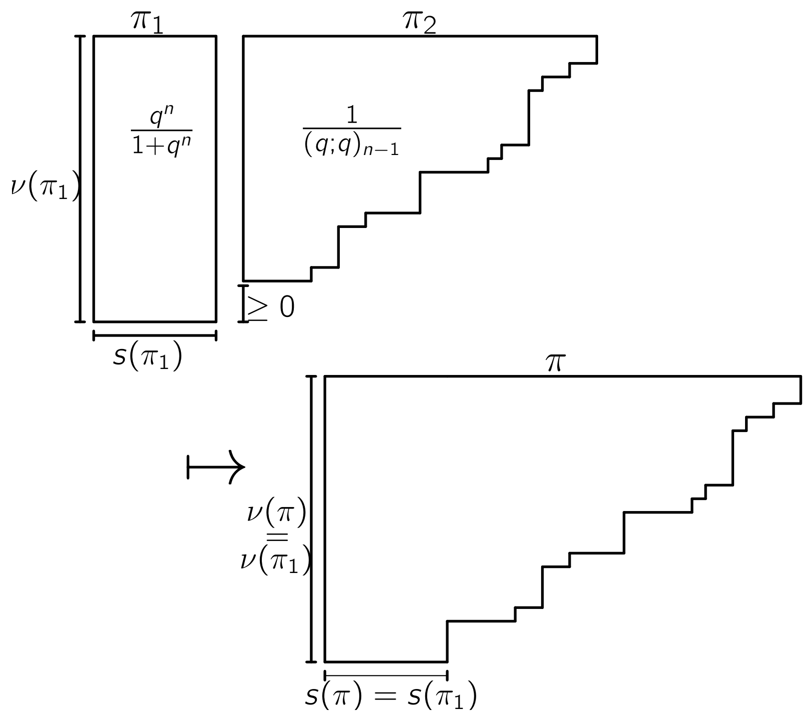
is the generating function for the number of partitions of the form (k^n) where the partition gets counted with the weight $+1$ if the part k is odd and it gets counted with the weight -1 if the part k is even. The factor

$$\frac{1}{(q;q)_{n-1}} \tag{4–8}$$

is the generating function for the number of partitions into parts less than n. With conjugation in mind, another equivalent interpretation of (4–8) is that it is the generating function for the number of partitions into less than n parts.

We put the partitions counted by the factors in the summand into a single partition bijectively by part-by-part addition. For the same positive integer n, let π_1 be a partition counted by (4–7) and a partition π_2 counted by (4–8). We know that $\pi_1=(k^n)$ for some positive integer k. Starting from the largest part of π_2, we add a part of π_2 to a part of π_1 and put the outcome as a part of a new partition π. Recall that a part of a partition is a positive integer that is an element of that partition. The partition π_2 has less than n parts. Therefore, there is at least one part of π_1 that does not get anything added to it. We add these leftover parts of π_1 to π after the additions. This way we know that the new partition π has exactly n parts, where the smallest part is exactly k. This can be easily demonstrated using Young diagrams in Figure 4-1.

Figure 4-1. Demonstration of putting together partitions in the summand of (4–6).



Moreover, the partition π gets counted with the weight $+1$ if the smallest part is odd and it gets counted with the weight -1 if the smallest part is even. The sum of all these terms gives us the generating function for the weighted count of ordinary partitions from $\mathcal{U}$‘. Hence,

$$\sum_{n\geq 1} \frac{q^n}{1+q^n} \frac{1}{(q;q)_{n-1}} = \sum_{\pi\in\mathcal{U}‘} (-1)^{s(\pi)+1} q^{|\pi|}, \tag{4–9}$$

where $s(\pi)$ is the smallest part of the partition π.

The right-hand side summation

$$\sum_{n\geq 1} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n (q^{n+1};q)_\infty} \tag{4–10}$$

of (4–2) can also be interpreted as a weighted count of partitions. For some positive integer n, the term $q^{n(n+1)/2}$ can be thought of as the generating function of the partition $\pi_1^* = (1, 2, 3, 4, \ldots, n)$ where every part less than or equal to n appears exactly one time. The factor

$$\frac{1}{(q^2;q^2)_n} \tag{4–11}$$

is the generating function for partitions into parts $\leq n$ where every part appears with an even frequency. Let π_2^* be a partition counted by (4–11). By adding the frequencies of π_1^* and π_2^* we get another partition

$$\pi^* = (1^{f_1}, 2^{f_2}, \dots, n^{f_n}),$$

where all $f_i \equiv 1 \mod 2$. The quotient

$$\frac{1}{(q^{n+1};q)_\infty} \tag{4–12}$$

is the generating function for the number of partitions into parts $> n$. Therefore, for a partition π‘ that is counted by (4–12) one can put together π^* and π‘ without the need of adding any frequencies. Call the outcome partition of merging π^* and π‘, π.

With this interpretation, the partitions counted by (4–10) have the frequency restriction that $f_1(\pi) \equiv 1 \mod 2$. Also, let i be the first positive integer where $f_i(\pi)$ is even (maybe zero). It is obvious that the partition π might be the final outcome of the merging procedure explained above for any summand in (4–10) as long as the index of the summand is $< i$. Therefore, the partition π is weighted by the number of the parts in its initial chain of odd frequencies of parts. This proves

$$\sum_{n\geq 1} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n (q^{n+1};q)_\infty} = \sum_{\pi\in\mathcal{U}^*} t(\pi) q^{|\pi|}, \tag{4–13}$$

where $t(\pi)$ is as defined in Theorem 4.1. The identities (4–9) and (4–13) together prove Theorem 4.1.

Now we move on to another analytical identity similar to (4–2). This identity will later prove a weighted partition identity for overpartitions.

Theorem 4.3.

$$\sum_{n\geq 1} \frac{2q^n}{1+q^n} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n} \frac{2q^{n+1}}{1-q^{2(n+1)}} \frac{(-q^{n+2};q)_\infty}{(q^{n+2};q)_\infty} \tag{4–14}$$

Proof. Multiply and divide the left-hand side of (4–14) by $(1+q)$, use (4–3), and write it as a q-hypergeometric series:

$$\sum_{n\geq 1}\frac{2q^n}{1+q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}=\frac{2q}{1+q}{}_2\phi_1\left(\begin{matrix}-q,-q\\-q^2\end{matrix};q,q\right). \tag{4–15}$$

Now we apply the transformation (A–56) to (4–15). This yields,

$$\frac{2q}{1+q}{}_2\phi_1\left(\begin{matrix}-q,-q\\-q^2\end{matrix};q,q\right)=\frac{2q}{1+q}\frac{(-q^2;q)_\infty}{(q;q)_\infty}{}_2\phi_2\left(\begin{matrix}-q,q\\-q^2,-q^2\end{matrix};q,-q^2\right). \tag{4–16}$$

Distributing the front factor to each summand, doing the necessary simplifications, and regrouping terms shows that the right-hand sides of identities (4–14) and (4–16) are equal. □

Identities (4–2) and (4–14) are $z=1$ and 2 special cases of the more general result, respectively.

Theorem 4.4.

$$\sum_{n\geq 1}\frac{q^n}{1+q^n}\frac{((1-z)q;q)_{n-1}}{(q;q)_{n-1}}=\frac{((1-z)q;q)_\infty}{(q;q)_\infty}\sum_{n\geq 1}\frac{q^{n(n+1)/2}}{(-q)_n(1-(1-z)q^n)}.$$

This identity can be proven using the same Jackson transformation (A–56) with $(a,\ b,\ c,\ q,\ z)\mapsto((1-z)q,\ -q,\ -q^2,\ q,\ q)$.

The combinatorial interpretation of (4–14) is similar to the one of (4–2). Consider the left-hand side sum

$$\sum_{n\geq 1}\frac{2q^n}{1+q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}$$

of (4–14). For a given n the summand factor

$$\frac{2q^n}{1+q^n}$$

is the generating function of the number of overpartitions into exactly n parts of the same size, where the partitions are counted with weight $+1$ if the part is odd and with -1 if the part is even. In other words, it is the generating function for the number of partitions (k^n) and $(\bar{k}^n)$ for any integer $k\geq 1$, where these partitions are counted with the weight $(-1)^{k+1}$. The other factor

$$\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}, \tag{4–17}$$

(by (3–2)) is the generating function for the number of overpartitions with strictly less than n parts. As we did in the proof of Theorem 4.1, we put the parts of these partitions together. This part-by-part addition gives an overpartition in exactly n parts with the smallest part k. And coming from the first factor we count these partitions with weight $+1$ if the smallest part k is odd and with weight -1 if k is even. Hence,

$$\sum_{n\geq 1} \frac{2q^n}{1+q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \sum_{\pi\in\mathcal{O}} (-1)^{s(\pi)+1} q^{|\pi|}, \tag{4–18}$$

where $\mathcal{O}$ is the set of non-empty overpartitions. We have not explicitly defined overpartitions due to the fact that overpartition identities in our cases can be turned into ordinary partition identities with weights where the conversion weights comes from (3–2). We now start doing this mentioned conversion to get a weighted partition identity over $\mathcal{U}$‘ rather than $\mathcal{O}$.

The right-hand side of (4–14) can be interpreted in a way similar to that of (4–10). For some non-negative integer n, the factor

$$\frac{q^{n(n+1)/2}}{(q;q)_n} \tag{4–19}$$

is the generating function for number of partitions of the type $(1^{f_1}, 2^{f_2}, \dots, n^{f_n})$, where $f_i \geq 1$ for all $1 \leq i \leq n$, as $n(n+1)/2 = 1+2+\dots+n$. The rest of the factors

$$\frac{2q^{n+1}}{1-q^{2(n+1)}}\frac{(-q^{n+2};q)_\infty}{(q^{n+2};q)_\infty} \tag{4–20}$$

can be interpreted as the generating function for the number of overpartitions where the smallest part (which definitely appears in the partition) is $n+1$ and that part has an odd frequency.

There is no overlapping in the size of the parts in the partitions counted by (4–19) and (4–20) for a fixed n. One can merge these partitions into a single partition without any need of non-trivial addition of frequencies. On the other hand, an outcome overpartition may be coming from different merged couples of partitions/overpartitions. Given an outcome overpartition, there is no clean cut point that would indicate where the overpartition counted by (4–20) started. The only indication is the odd frequency of the smallest part of overpartitions. Also, we know that every part below the smallest part of overpartition in the combined partition is coming from a partition counted by the generating function (4–19). In particular, 1 appears as a part in any outcome of this merging process. Therefore, we need to keep account of all these possible connection points when we are finding the count of a partition coming from the right-hand side of (4–14). By going through only the odd frequencies in a given partition and counting the number of larger parts with the overpartition weights, we can find the total count of combinations that would yield the same merged overpartition images.

Given a partition π, let $m(\pi)$ be the smallest positive integer that is not a part of π. Let $\nu_d(\pi, n)$ be the number of different parts $\geq n$ in partition π. Then, the right-hand side of (4–14) can be written as a weighted count of partitions as

$$\sum_{n\geq 0} \frac{q^{n(n+1)/2}}{(q;q)_n} \frac{2q^{n+1}}{1-q^{2(n+1)}} \frac{(-q^{n+2};q)_\infty}{(q^{n+2};q)_\infty} = \sum_{\pi \in \mathcal{U}`} \tau(\pi) q^{|\pi|}, \tag{4–21}$$

where

$$\tau(\pi) = \sum_{i=1}^{m(\pi)} \chi(f_i \equiv 1 (\text{mod } 2)) 2^{\nu_d(\pi,i)}. \tag{4–22}$$

This study proves the combinatorial version of Theorem 4.3. We put (4–18) and (4–21) together, and get the following theorem.

Theorem 4.5.

$$\sum_{\pi \in \mathcal{O}} (-1)^{s(\pi)+1} q^{|\pi|} = \sum_{\pi \in \mathcal{U}`} \tau(\pi) q^{|\pi|}, \tag{4–23}$$

where $\tau(\pi)$ is defined as in (4–22).

There are 100 overpartitions of 8. There are 18 overpartitions of 8 with an even smallest part. Hence, in the weighted count of the left-hand side of (4–23) the coefficient of q^8 term is

$100 - 2 \cdot 18 = 64$. We exemplify the right-hand side weights of Theorem 4.5 for the same norm in Table 4-2.

Table 4-2. Example of Theorem 4.5 with $|\pi| = 8$.

$\pi \in \mathcal{U}'$	$\tau(\pi)$	$\pi \in \mathcal{U}'$	$\tau(\pi)$
$(1^3, 2, 3)$	$8+4+2=14$	$(1^2, 2, 4)$	4
$(1, 2, 5)$	$8+4=12$	$(1^3, 5)$	4
$(1, 2^2, 3)$	$8+2=10$	$(1, 7)$	4
$(1, 3, 4)$	8	$(1^6, 2)$	2
$(1^5, 3)$	4	$(1^2, 2^3)$	2

The sum of the weights is 64, which is the same as the count of overpartitions with the alternating sign with respect to their smallest part's parity.

4.2 A Weighted Identity with respect to the Smallest Part and the Number of Parts of a Partition in relation with Sums of Squares

We start with a short proof of an analytic identity.

Lemma 3.

$$\sum_{n\geq 1} \frac{(-1)^n q^{n(n+1)/2}}{(1+q^n)(q;q)_n} = \sum_{n\geq 1} (-1)^n q^{n^2}. \tag{4–24}$$

Proof. It is easy to see that

$$1 + 2\sum_{n\geq 1} \frac{(-1)^n q^{n(n+1)/2}}{(1+q^n)(q;q)_n} = \lim_{\rho\to\infty} {}_2\phi_1 \left(\begin{matrix} -1, & \rho q \\ & -q \end{matrix} ; q, 1/\rho \right) = \frac{(q;q)_\infty}{(-q;q)_\infty},$$

where we used q-Gauss sum (A–54). Rewiriting the sum in (A–57) as

$$1 + 2\sum_{n\geq 1} (-1)^n q^{n^2} \tag{4–25}$$

proves the claim. □

The identity (4–24) is a special case of a more general identity of Ramanujan [19, E. 1.6.2, p. 25] which even has a combinatorial proof [32]. But, more relevant to this paper, Alladi [3, Thm 2, p. 330] is the first one to give a combinatorial interpretation to the left-hand side of Lemma 3 in the spirit of the Euler pentagonal number theorem. In his study, he interpreted the left-hand side sum as the number of partitions into distinct parts with smallest part being odd weighted with $+1$ or -1 depending on the number of parts of the partition being even or odd, respectively. In our notations:

Theorem 4.6 (Alladi [3], 2009). *Let N be a positive integer. Then,*

$$\sum_{\substack{\pi \in \mathcal{D}_o, \\ |\pi| = N}} (-1)^{\nu(\pi)} = (-1)^N \chi(N = \square),$$

where $\mathcal{D}_o$ is the set of non-empty partitions into distinct parts where the smallest part is odd, χ is as defined in (2–29), and $\square$ represents the statement "a perfect integer square."

It is easy to check that

$$|\pi| \equiv \nu_o(\pi) \mod 2,$$

for any partition π. Hence,

$$\nu(\pi) - |\pi| \equiv \nu_e(\pi) \mod 2. \tag{4–26}$$

This enables us to rewrite Theorem 4.6 as in [31].

Theorem 4.7 (Bessenrodt, Pak [31], 2004). *Let N be a positive integer. Then,*

$$\sum_{\substack{\pi \in \mathcal{D}_o, \\ |\pi| = N}} (-1)^{\nu_e(\pi)} = \chi(N = \square).$$

There, they also discussed a refinement of Theorem 4.7.

Theorem 4.8 (Bessenrodt, Pak [31], 2004).

$$\sum_{\substack{\pi \in \mathcal{D}_o, \\ |\pi| = N, \\ \nu_o(\pi) = k}} (-1)^{\nu_e(\pi)} = \chi(N = k^2).$$

Theorems 4.6 – 4.8 connect the weighted count of the partitions into distinct parts, where the smallest part is necessarily odd and the number of representation of integer as a perfect square. Our next theorem will be connecting the weighted count of partitions and the number of representations of a number as a sum of two squares.

Theorem 4.9.

$$\sum_{n\geq 1}\frac{(-1)^n 2q^n}{1+q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \varphi(-q)^2-\varphi(-q) \tag{4–27}$$

Proof. Similar to the proof of Theorem 4.2, we would like to write the left-hand side of (4–27) as a hypergeometric function first. On the left-hand side of (4–27) we multiply and divide the summand by $(1+q)$, use (4–3), factor out the terms $-2q/(1+q)$, and finally shift the summation variable $n\mapsto n+1$ to write the expression as a ${}_2\phi_1$ hypergeometric series. Applying the Jackson's transformation (A–56) to this expression yields

$$\frac{-2q}{1+q}{}_2\phi_1\left(\begin{matrix}-q,-q\\ -q^2\end{matrix};q,-q\right) = \frac{-2q}{1+q}\frac{(q^2;q)_\infty}{(-q;q)_\infty}{}_2\phi_2\left(\begin{matrix}-q,q\\ -q^2,q^2\end{matrix};q,q^2\right).$$

Writing the ${}_2\phi_2$ explicitly, distributing the factor $q/(1+q)$, performing the simple cancellations, shifting the summation variable $n\mapsto n-1$ and multiplying and dividing with $1-q$ we get

$$\frac{-2q}{1+q}\frac{(q^2;q)_\infty}{(-q;q)_\infty}{}_2\phi_2\left(\begin{matrix}-q,q\\ -q^2,q^2\end{matrix};q,q^2\right) = 2\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n\geq 1}\frac{(-1)^n q^{n(n+1)/2}}{(1+q^n)(q;q)_n}. \tag{4–28}$$

Applying Lemma 3 to the right-hand side of (4–28) and rewriting the identity we see that

$$\sum_{n\geq 1}\frac{(-1)^n 2q^n}{1+q^n}\frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = 2\frac{(q;q)_\infty}{(-q;q)_\infty}\left(-1+\sum_{n\geq 0}(-1)^n q^{n^2}\right). \tag{4–29}$$

Using observations (4–25) and (A–57) on the right-hand side of (4–29) we complete the proof. $\square$

The combinatorial interpretation of Theorem 4.9 combines a weighted partition count with a representation of numbers by sum of two squares. Moreover, we can provide an explicit formula for the weighted count of partitions with respect to the norm.

Let n be a positive integer. The summand

$$\frac{(-1)^n 2q^n}{1+q^n}$$

of (4–27) is the generating function for the number of partitions of the form (k^n) (keeping (4–7) in mind) gets counted with the weight $(-1)^{k+n+1}2$. Here it should be noted that k is the smallest part and n is the number of parts of this partition. After the needed addition of partitions (similar to the ones we did for Theorems (4.1) and (4.5)) these two variables are going to stay the same for the outcome partition. To have a uniform notation, recall that $\nu(\pi)$ denotes the number of parts, and $\nu_d(\pi)$ is the number of different parts of a partition π. The second summand

$$\frac{(-q;q)_{n-1}}{(q;q)_{n-1}} \tag{4–17}$$

that appears in (4–27) is the generating function for the number of overpartitions into strictly less than n parts as mentioned before. We know that this is the same as counting the number of ordinary partitions π in less than n parts counted with the weight $2^{\nu_d(\pi)}$ by (3–2).

Putting together the partition $\pi_1 = (k^n)$ and a partition π_2 counted by (4–17) (similar to the way we did in Figure 4-1) gives us an outcome overpartition π, which we will treat as a partition and count with the related weight at first. The partition π has the properties $s(\pi) = k$, $\nu(\pi) = n$, and $\nu_d(\pi) = \nu_d(\pi_1) + \nu_d(\pi_2) = \nu_d(\pi_2) + 1$. This partition is counted with the weight

$$\omega(\pi) := (-1)^{s(\pi)+\nu(\pi)+1} 2^{\nu_d(\pi)}, \tag{4–30}$$

(the multiplication of weights of π's generators) by the right-hand side of (4–27). This proves

$$\sum_{n\geq 1} \frac{(-1)^n 2q^n}{1+q^n} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} = \sum_{\pi\in\mathcal{U}`} \omega(\pi) q^{|\pi|}. \tag{4–31}$$

On the other side of the equation (4–27) we have the difference of two theta series. The summation of (A–57) is enough to see that

$$\varphi(-q)^2 - \varphi(-q) = \sum_{x,y\in\mathbb{Z}} (-1)^{x+y} q^{x^2+y^2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}. \tag{4–32}$$

Let $r_2(N)$ be the number of representations of N as a sum of two squares. Any positive integer N has the unique prime factorization

$$N = 2^e \prod_{i\geq 1} p_i^{v_i} \prod_{j\geq 1} q_j^{w_j},$$

where e, v_i, and w_j are non-negative integers, and p_i and q_j are primes 1 and 3 mod 4, respectively. It is known [49, Thm 14.13, p. 572] that

$$r_2(N) = 4\prod_{i\geq 1}(1+v_i)\prod_{j\geq 1}\frac{1+(-1)^{w_j}}{2}.$$

Writing the first series organized with respect to r_2, rewriting the second series, and finally cancelling the constant terms of both series we get

$$\sum_{x,y\in\mathbb{Z}}(-1)^{x+y}q^{x^2+y^2} - \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \sum_{N\geq 1}(-1)^N r_2(N)q^N - 2\sum_{n\geq 1}(-1)^n q^{n^2}. \tag{4–33}$$

On the right-hand side of (4–33) one can collect the terms with respect to the exponents of q. Writing the two series together with the use of a truth function (2–29) and comparing (4–32) and (4–33) yields the identity

$$\varphi(-q)^2 - \varphi(-q) = \sum_{N\geq 1}(-1)^N(r_2(N) - 2\chi(N=\square)\,)q^N, \tag{4–34}$$

where χ is defined as in (2–29) and $\square$ represents "a perfect integer square."

Now we put the right-hand sides of (4–30), (4–31) and (4–34) together and get an explicit expression for the sum of weights $\omega(\pi)$ of partitions for a fixed positive norm N:

$$\sum_{\substack{\pi\in\mathcal{U}',\\ |\pi|=N}}(-1)^{s(\pi)+\nu(\pi)+1}2^{\nu_d(\pi)} = (-1)^N(r_2(N) - 2\chi(N=\square)\,). \tag{4–35}$$

We can employ the observation (4–26) to simplify (4–35).

Theorem 4.10.

$$\sum_{\substack{\pi\in\mathcal{U}',\\ |\pi|=N}}\omega^*(\pi) = r_2(N) - 2\chi(N=\square),$$

where

$$\omega^*(\pi) = (-1)^{s(\pi)+\nu_e(\pi)+1}2^{\nu_d(\pi)}.$$

Two examples of Theorem 4.10 are given in Table 4-3.

Table 4-3. Examples of Theorem 4.10 with $|\pi| = 4$ and 5.

$\pi \in \mathcal{U}'$, $|\pi| = 4$	$\omega^*(\pi)$	$\pi \in \mathcal{U}'$, $|\pi| = 5$	$\omega^*(\pi)$
(4)	2	(5)	2
(2^2)	-2	$(2,3)$	2^2
$(1,3)$	2^2	$(1,4)$	-2^2
$(1^2,2)$	-2^2	$(1^2,3)$	2^2
(1^4)	2	$(1,2^2)$	2^2
		$(1^3,2)$	-2^2
		(1^5)	2
Total:	2		8

and the explicit formula of (4–35) suggests:

$$(r_2(4) - 2 \cdot 1) = 4 - 2 = 2,$$
$$(r_2(5) - 2 \cdot 0) = 8.$$

Another equivalent statement of Theorem 4.10 can be given over the set of overpartitions by evaluating (4–30) and (3–2).

Theorem 4.11.

$$\sum_{\substack{\pi \in \mathcal{O}, \\ |\pi| = N}} (-1)^{s(\pi)+\nu_e(\pi)+1} = r_2(N) - 2\chi(N = \square).$$

4.3 Some Weighted Identities for Partitions with Distinct Even Parts

Let $\mathcal{P}$ denote the set of non-empty partitions with distinct even parts. A partition $\pi \in \mathcal{P}$ may still have repeated odd parts. This set has been studied before in [4, § 5], [14] and [20].

We start with the analytic identity:

Theorem 4.12.

$$\sum_{n \geq 1} \frac{(-1)^n q^{2n}}{1 - q^{2n}} \frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}} q^{n-1} = \psi(-q) - \frac{1}{1+q}. \tag{4–36}$$

Proof. We multiply both sides of (4–36) with $1 + q$ and add 1. The resulting identity becomes a special case of the q-binomial theorem (A–53) with $(a,\ q,\ z) = (-1/q,\ q^2,\ -q^3)$ provided that we use (A–58) with $q \mapsto -q$. $\square$

The combinatorial interpretation of the left-hand side summand,

$$\frac{(-1)^n q^{2n}}{1-q^{2n}} \frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}} q^{n-1}, \tag{4–37}$$

for some positive n is really similar to the previous constructions. The main difference is the use of 2-modular Young diagrams, which has been introduced in Section 1.2, instead. We will be following similar steps that we followed in finding the combinatorial interpretation of Theorem 4.2.

Let n be a fixed positive integer. The factor

$$\frac{(-1)^n q^{2n}}{1-q^{2n}} = \sum_{k\geq 1} (-1)^n q^{2kn}$$

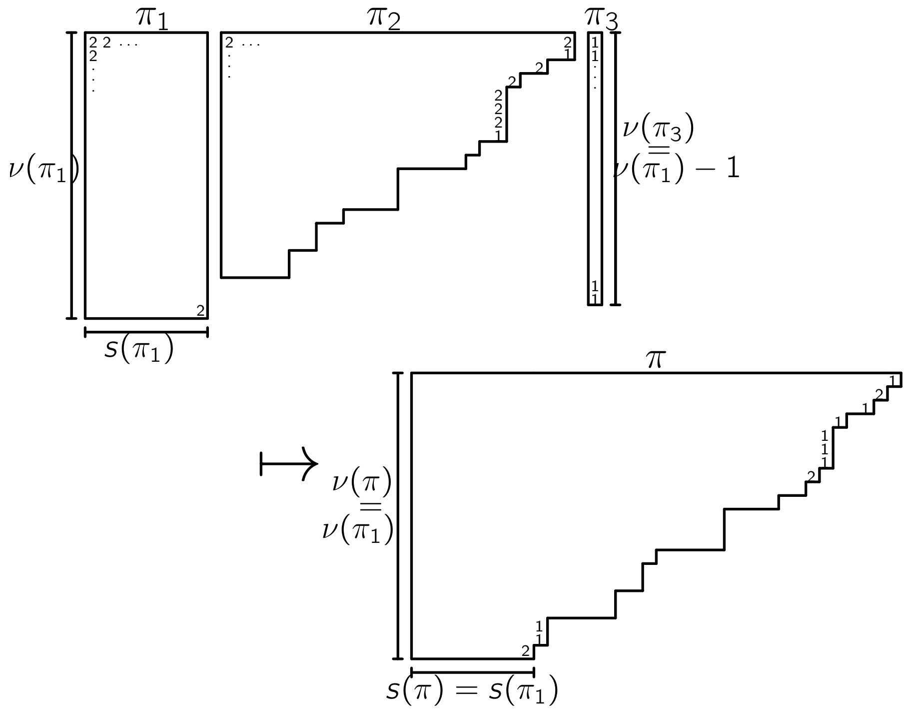
is the generating function of partitions of the type $\pi_1 = (\,(2k)^n\,)$ for some positive integer k, where these partitions get counted with a weight $+1$ if the number of parts of the partition n is even and with -1 if n is odd. The second factor

$$\frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}}$$

is the generating function for the number of partitions with distinct odd parts $\leq 2n-2$. We can express these partitions in 2-modular Young diagrams and take their conjugates. The outcome would show that the same factor is the generating function for the number of partitions π_2 with distinct odd parts where the number of parts is $< n$. Finally, the term q^{n-1} can be thought as the generating function of the partitions $\pi_3 = (1^{n-1})$.

We would like to add the partitions π_1, π_2, and π_3 to make up a new partition. This will be done similar to the example of Figure 4-1. We start by putting partitions π_1, π_2, and π_3 and add them up row-wise. When doing so, the possible boxes filled with 1's coming from π_2 are combined with the 1's of π_3 and turned into a row ending of a box with a 2 in it. There being $n-1$ parts in π_3 and the row-wise addition of these partitions also makes sure that the outcome partition is a partition π with distinct even parts where the smallest part is necessarily even. An illustration is given in Figure 4-2.

Figure 4-2. Demonstration of putting together partitions in the summand of (4–37).



Let $\mathcal{P}_e$ be the subset of $\mathcal{P}$ where the smallest part is necessarily a positive even integer. The above construction proves that the left hand side of (4–36) is the generating function for the weighted count of partitions from $\mathcal{P}_e$ counted by the weight $+1$ or -1 depending on the number of parts in the partition being even or odd, respectively:

$$\sum_{\pi \in \mathcal{P}_e} (-1)^{\nu(\pi)} q^{|\pi|} = \sum_{n \geq 1} \frac{(-1)^n q^{2n}}{1 - q^{2n}} \frac{(-q; q^2)_{n-1}}{(q^2, q^2)_{n-1}} q^{n-1}. \tag{4–38}$$

The right-hand side of (4–36), by looking at the geometric series, can easily be interpreted combinatorially. This study proves

$$\sum_{\substack{\pi \in \mathcal{P}_e, \\ |\pi| = N}} (-1)^{\nu(\pi)} = (-1)^{N+1} \chi(N \neq \triangle). \tag{4–39}$$

where N is a positive integer and $\triangle$ represents “a triangular number.” The simple observation (4–26) can be used on (4–39) to simplify the equation.

Theorem 4.13. *Let N be a positive integer. Then,*

$$\sum_{\substack{\pi \in \mathcal{P}_e, \\ |\pi|=N}} (-1)^{\nu_e(\pi)+1} = \chi(N \neq \triangle).$$

where $\triangle$ represents "a triangular number."

Moreover, it is easy to see that the generating function for the weighted count of partitions from $\mathcal{P}$ counted by the weight $+1$ or -1 depending on the number of parts is clearly

$$\sum_{\pi \in \mathcal{P}} (-1)^{\nu(\pi)} q^{|\pi|} = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} - 1 = \psi(-q) - 1. \tag{4–40}$$

Hence, (4–36), (4–38), and (4–40) together yields

$$\sum_{\pi \in \mathcal{P}_o} (-1)^{\nu(\pi)} q^{|\pi|} = \frac{1}{1+q} - 1,$$

where $\mathcal{P}_o$ is the subset of $\mathcal{P}$ where the smallest part is necessarily a positive odd integer.

We note that the above study can be easily generalized by inserting an extra parameter z. The identities (4–36) and (4–38) turn into

$$\sum_{n \geq 1} \frac{(-1)^n q^{2n}}{1-q^{2n}} \frac{(-q/z;q^2)_{n-1}}{(q^2,q^2)_{n-1}} (zq)^{n-1} = \frac{(q^2;q^2)_\infty}{(-qz;q^2)_\infty} - \frac{1}{1+zq}$$

and

$$\sum_{\pi \in \mathcal{P}_e} (-1)^{\nu(\pi)} z^{\nu_o(\pi)} q^{|\pi|} = \frac{(q^2;q^2)_\infty}{(-qz;q^2)_\infty} - \frac{1}{1+zq}, \tag{4–41}$$

respectively. We also get the generalization of (4–40)

$$\sum_{\pi \in \mathcal{P}} (-1)^{\nu(\pi)} z^{\nu_o(\pi)} q^{|\pi|} = \frac{(q^2;q^2)_\infty}{(-qz;q^2)_\infty} - 1. \tag{4–42}$$

Combining (4–41) and (4–42) and replacing z by $-z$ we get the result

$$\sum_{\pi \in \mathcal{P}_o} (-1)^{\nu_e(\pi)} z^{\nu_o(\pi)} q^{|\pi|} = \frac{1}{1-zq} - 1, \tag{4–43}$$

which can also be found in [31, Cor 4, p.1146]. The equation (4–43) implies

Theorem 4.14.

$$\sum_{\substack{\pi\in\mathcal{P}_o,\\ |\pi|=N,\\ \nu_o(\pi)=k}} (-1)^{\nu_e(\pi)} = \chi(N=k).$$

We can step up our study on the set $\mathcal{P}$ by putting more restrictive conditions on the smallest part. Let $\mathcal{P}_{2,4}$ be the subset of $\mathcal{P}_e$ where the smallest part of a partition is necessarily 2 mod 4. Knowing the argument behind the generating function interpretation for $\mathcal{P}$, the generating function of $\mathcal{P}_{2,4}$ with the ± 1 weight with respect to the number of parts can easily be written as

$$\sum_{\pi\in\mathcal{P}_{2,4}} (-1)^{\nu(\pi)} q^{|\pi|} = \sum_{n\geq 1} \frac{(-1)^n q^{2n}}{1-q^{4n}} \frac{(-q;q^2)_{n-1}}{(q^2,q^2)_{n-1}} q^{n-1}. \tag{4–44}$$

We write the related analytic equality.

Theorem 4.15.

$$\sum_{n\geq 1} \frac{(-1)^n q^{2n}}{1-q^{4n}} \frac{(-q;q^2)_{n-1}}{(q^2;q^2)_{n-1}} q^{n-1} = \frac{1}{1-q}\sum_{n\geq 0}(-1)^n q^{n^2} - \frac{1}{1-q^2}. \tag{4–45}$$

Proof. By multiplying both sides of (4–45) with $2(1+q)$ and adding 1 to both sides, we see that one can apply the q-Gauss sum (A–54) where $(a,\ b,\ c,\ q,\ z) = (-1,\ -1/q,\ -q^2,\ q^2,\ -q^3)$ to the left-hand side. Showing the equality of the right-hand side to the outcome product of the q-Gauss sum is a simple task of combining like terms and using the Gauss identity (A–57). □

The right-hand side of (4–45) can be studied further to get exact formulas.

$$\begin{aligned}\frac{1}{1-q}\sum_{n\geq 0}(-1)^n q^{n^2} &= \sum_{k\geq 0} q^k \sum_{n\geq 0}(-1)^n q^{n^2}\\ &= 1+q^4+q^5+q^6+q^7+q^8+q^{16}+q^{17}+q^{18}+q^{19}\ldots\\ &= 1+\sum_{N\geq 1}\sum_{j\geq 1}\chi((2j)^2\leq N<(2j+1)^2)q^N, \end{aligned}\tag{4–46}$$

where χ is as defined in (2–29). Also from the geometric series

$$\frac{1}{1-q^2} = 1+q^2+q^4+q^6+q^8+q^{10}+\ldots. \tag{4–47}$$

Therefore, combining (4–46) and (4–47), we get

$$\begin{aligned}\frac{1}{1-q}\sum_{n\geq 0}(-1)^n q^{n^2} - \frac{1}{1-q^2} &= -q^2+q^5+q^7-q^{10}-q^{12}-q^{14}+q^{17}+\ldots \\ &= \sum_{N\geq 1}\sum_{j\geq 1}(\chi(N \text{ is odd})\chi((2j)^2<N<(2j+1)^2) \\ &\qquad - \chi(N \text{ is even})\chi((2j-1)^2<N<(2j)^2)\,)q^N.\end{aligned} \tag{4–48}$$

Combining (4–44), (4–45), and (4–48) we get the interesting explicit formula for the weighted count of partitions from the set $\mathcal{P}_{2,4}$.

Theorem 4.16.

$$\begin{aligned}\sum_{\substack{\pi\in\mathcal{P}_{2,4},\\ |\pi|=N}}(-1)^{\nu(\pi)} &= \sum_{j\geq 1}(\chi(N \textit{ is odd})\chi((2j)^2<N<(2j+1)^2) \\ &\qquad - \chi(N \textit{ is even})\chi((2j-1)^2<N<(2j)^2)) \\ &= \begin{cases} 1, & \textit{if } N \textit{ is odd and in between an even square} \\ & \qquad \textit{and the following odd square,} \\ -1, & \textit{if } N \textit{ is even and in between an odd square} \\ & \qquad \textit{and the following even square,} \\ 0, & \textit{otherwise.}\end{cases}\end{aligned}$$

Let $\mathcal{P}_{3,4}$, similar to $P_{2,4}$, be the subset of $\mathcal{P}_o$ where the smallest part of a partition is necessarily 3 mod 4. Adding a single 1 to the smallest part of a partition from $\mathcal{P}_{2,4}$ is a bijective map from the set $\mathcal{P}_{2,4}$ to $\mathcal{P}_{3,4}$. Therefore, writing the analogous generating function of weighted count of partitions from $\mathcal{P}_{3,4}$ is rather easy and only requires multiplying (4–44) with and extra q. This proves the following theorem.

Theorem 4.17.

$$\begin{aligned}\sum_{\substack{\pi\in\mathcal{P}_{3,4},\\ |\pi|=N}}(-1)^{\nu(\pi)} &= \sum_{j\geq 1}(\chi(N \textit{ is even})\chi((2j)^2<N<(2j+1)^2) \\ &\qquad - \chi(N \textit{ is odd})\chi((2j-1)^2<N<(2j)^2)\,)\end{aligned}$$

$$= \begin{cases} 1, & \textit{if } N \textit{ is even and in between an even square} \\ & \qquad \textit{and the following odd square}, \\ -1, & \textit{if } N \textit{ is odd and in between an odd square} \\ & \qquad \textit{and the following even square}, \\ 0, & \textit{otherwise}. \end{cases}$$

The combination of the weighted generating functions accounts for every number that is not a perfect square. This interesting relation can be represented as follows.

Theorem 4.18.

$$\sum_{\substack{\pi \in \mathcal{P}_{3,4}, \\ |\pi| = N}} (-1)^{\nu(\pi)} - \sum_{\substack{\pi \in \mathcal{P}_{2,4}, \\ |\pi| = N}} (-1)^{\nu(\pi)} = (-1)^N \chi(N \neq \square). \tag{4–49}$$

This result, in a sense, is complementary to Alladi's identity, Theorem 4.6. Also, the right-hand side formula also appears in the recent study of Andrews and Yee [22, Thm 3.2, p.10] as the same weighted count with respect to the number of parts of bottom-heavy partitions (a specific subset of overpartitions). The interested reader is invited to examine the relation between the set of bottom-heavy partitions, $\mathcal{P}_{2,4}$, and $\mathcal{P}_{3,4}$.

Once again one can simplify the argument of (4–49) with the observation (4–26).

Theorem 4.19.

$$\sum_{\substack{\pi \in \mathcal{P}_{3,4}, \\ |\pi| = N}} (-1)^{\nu_e(\pi)} - \sum_{\substack{\pi \in \mathcal{P}_{2,4}, \\ |\pi| = N}} (-1)^{\nu_e(\pi)} = \chi(N \neq \square).$$

4.4 Partitions with no parts divisible by 3

In this section we treat the weighted interpretation of an identity of Ramanujan [19, E. 4.2.8, p. 85]. We write this identity in an equivalent form for the ease of interpretation purposes.

Theorem 4.20 (Ramanujan [19])**.**

$$\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty} - 1 = \sum_{n \geq 1} \frac{(-q;q)_{n-1}}{(q;q)_{n-1}} \frac{2q^n}{1-q^n} \frac{q^{n^2-n}}{(q;q^2)_n}. \tag{4–50}$$

Identity (4–50) also appears in the Slater's list [51, 6, p. 152] with a misplaced exponent type typo.

It is clear that the left-hand side of (4–50) is the generating function for the number of overpartitions where no part is 0 mod 3. Let $\mathcal{C}$ be the set of all non-empty partitions with no parts divisible by 3. We focus our interest in the combinatorial interpretation of the right-hand side of (4–50). Let n be a fixed positive integer. The factors

$$\frac{(-q;q)_{n-1}}{(q;q)_{n-1}}\frac{2q^n}{1-q^n}$$

of the right-hand side of (4–50) is the generating function for the number of overpartitions, $\bar{\pi}_1$, into parts $\leq n$ where the part n appears at least once. When counting the total number of overpartitions of this type of partitions, we can instead count the partitions π_1 into parts $\leq n$ where the part n appears at least once with the weight $2^{\nu_d(\pi_1)}$ by (3–2). The remaining factor

$$\frac{q^{n^2-n}}{(q;q^2)_n}$$

can be split into two in the interpretation. The term q^{n^2-n} is the generating function of the partitions of type $\pi_2 = (2,4,6,\dots,2(n-1))$ in frequency notation as n^2-n is double a triangle number. The term $(q;q^2)_n^{-1}$ is the generating function for the number of partitions, π_3, into odd parts $\leq 2n-1$.

It is clear that among the parts of π_1, π_2, and π_3 the largest possible part-size is $2n-1$. Even if $2n-1$ is not a part of π_3, the second largest possible part $2n-2$ is a part of π_2. Therefore, given π_1, π_2, and π_3 we can directly find the respective n. We merge (add the parts' frequencies of) these three partitions into a new partition and look at the number of possible sources for different part sizes. The partition π has one appearance of all the even parts $\leq 2n-2$ coming from π_2, any extra appearance of an even number (which is necessarily $\leq n$) must be coming from the partition π_1 and should be counted with the overpartition weights. The odd parts $\leq n$ can either be coming from the partition π_1 or π_3. These parts need to be counted with both the overpartition weights and normally to account for both possibilities. All the other parts' source partitions can uniquely be identified so they would be counted with trivial weight 1.

Let $\mathcal{R}$ be the set of partitions, where

i. all parts $\leq 2n-1$ for some integer $n > 0$,

ii. all even integers $\leq 2n-2$ appears as parts,

iii. n appears with the frequency $f_n \geq 1 + \chi(n \text{ is even})$,

iv. no even part $> n$ repeats.

Clearly,

$$n := n(\pi) = \frac{\text{largest even part of } \pi}{2} + 1.$$

Define the statistics

$$\delta(\pi) = \sum_{j=1}^{n-1} \chi(f_{2j} > 1),$$

$$\gamma(\pi) = (\chi(n \text{ is even}) + 2 \cdot f_n \cdot \chi(n \text{ is odd})) \prod_{2j+1<n} (2f_{2j+1} + 1),$$

and

$$\mu(\pi) = 2^{\delta(\pi)} \cdot \gamma(\pi),$$

for $\pi \in \mathcal{R}$. We have the following identity.

Theorem 4.21.

$$\sum_{\pi \in \mathcal{C}} 2^{\nu_d(\pi)} q^{|\pi|} = \sum_{\pi \in \mathcal{R}} \mu(\pi) q^{|\pi|}.$$

One example of Theorem 4.21 will be given in Table 4-4.

In Ramanujan's entry [19, E. 4.2.9, p. 86],

$$\frac{(-q;q^3)_\infty(-q^2;q^3)_\infty}{(q;q^3)_\infty(q^2;q^3)_\infty} = \sum_{n\geq 0} \frac{q^{n^2}(-q;q)_n}{(q;q)_n(q;q^2)_{n+1}}, \tag{4–51}$$

we see the same product of (4–50). The sum on the right-hand side of (4–51) can also be interpreted as a weighted partition count for a special subset of partitions. This is rather analogous to $\mathcal{R}$. Let the set $\mathcal{Q}$ be the set of partitions π, where

i. the largest part is $= 2n-1$ for some integer $n > 0$,

ii. all odd integers $\leq 2n-1$ appear as a part,

iii. and no even parts $> n$ appear.

Clearly here

$$n := \frac{\text{largest part of } \pi + 1}{2}.$$

A similar weight to μ can be defined on $\mathcal{Q}$ as follows

$$\eta(\pi) := 2^{\nu_{d,e}(\pi)}(\chi(n \text{ is even}) \cdot (1 + \chi(f_n = 0)\,) + 2f_n \cdot \chi(n \text{ is odd})\,) \prod_{2j+1<n}(2f_{2j+1} - 1),$$

where $\nu_{d,e}(\pi)$ is the number of different even parts of π. Hence, we have the identity

Theorem 4.22.

$$\sum_{\pi\in\mathcal{C}} 2^{\nu_d(\pi)} q^{|\pi|} = \sum_{\pi\in\mathcal{R}} \mu(\pi) q^{|\pi|} = \sum_{\pi\in\mathcal{Q}} \eta(\pi) q^{|\pi|}.$$

The example of this result is included in Table 4-4. From that table, it appears that there exists a weight, norm, and n-value preserving bijection from $\mathcal{R}$ to $\mathcal{Q}$. We would like to leave the discovery of this bijection for a motivated reader.

Table 4-4. Example of Theorem 4.22 with $|\pi| = 7$.

s	$\pi \in \mathcal{C}$	$2^{\nu_d(\pi)}$	$\pi \in \mathcal{R}$	n	$\mu(\pi)$	$\pi \in \mathcal{Q}$	n	$\eta(\pi)$
	$(1,2,4)$	2^3	(1^7)	1	14	(1^7)	1	14
	$(2,5)$	2^2	$(1^3,2^2)$	2	14	$(1^4,3)$	2	14
	$(1^2,5)$	2^2	$(1,2^3)$	2	6	$(1^2,2,3)$	2	6
	$(1^3,4)$	2^2	$(2^2,3)$	2	2	$(1,3^2)$	2	2
	$(1^5,2)$	2^2						
	$(1^3,2^2)$	2^2						
	$(1,2^3)$	2^2						
	(7)	2						
	(1^7)	2						
Total:		36			36			36

APPENDIX:
q-HYPERGEOMETRIC SERIES

This appendix is a short list of results that are used and referred to in this work. We define the basic q-hypergeometric series as they appear in [42]. Let r and s be non-negative integers and $a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s, q$, and z be variables. Then

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right) := \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n \ldots (a_r;q)_n}{(q;q)_n(b_1;q)_n \ldots (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1-r+s} z^n. \quad \text{(A–52)} $$

Let a, b, c, q, and z be variables. The q-binomial theorem [42, II.4, p. 236] is

$$ {}_1\phi_0\left(\begin{matrix} a \\ - \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad \text{(A–53)} $$

and the q-Gauss sum [42, II.8, p. 236] is

$$ {}_2\phi_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; q, c/ab\right) = \frac{(c/a;q)_\infty(c/b;q)_\infty}{(c;q)_\infty(c/ab;q)_\infty}. \quad \text{(A–54)} $$

One of the three Heine's transformations [42, III.2, p. 241]

$$ {}_2\phi_1\left(\begin{matrix} a, \ b \\ c \end{matrix}; q, z\right) = \frac{(\frac{c}{b};q)_\infty(bz;q)_\infty}{(c;q)_\infty(z;q)_\infty} {}_2\phi_1\left(\begin{matrix} \frac{abz}{c}, \ b \\ bz \end{matrix}; q, \frac{c}{b}\right) \quad \text{(A–55)} $$

The Jackson ${}_2\phi_1$ to ${}_2\phi_2$ transformation [42, III.4, p. 241] is

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} {}_2\phi_2\left(\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz\right). \quad \text{(A–56)} $$

We would also like to recall the definition of the classical theta functions φ and ψ

$$ \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \text{ and } \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. $$

The Gauss identities [17, Cor 2.10, p. 23] for these functions will be of use:

$$ \varphi(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}, \quad \text{(A–57)} $$

$$ \psi(q) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \quad \text{(A–58)} $$

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