OPTIMIZATION MODELS FOR INVENTORY SYSTEMS WITH PRICE-DEPENDENT SUPPLY

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2016

To my mother, Fatma, my father, Rıza, and my brother, Onur

ACKNOWLEDGMENTS

First and foremost I would like to thank my advisor Joseph Geunes. During my first year in ISE Ph.D. program I asked him, whether I could be his Ph.D. student. He did not hesitate to accept me. From the very beginning he trusted my capabilities and encouraged me to develop as a good researcher. I am very grateful for his unlimited support throughout my years at UF. I am determined to work persistently not to disappoint him with my future academic career.

I would also like to thank my dissertation committee members Jean-Philippe Richard, Yongpei Guan, and Anand Paul for their valuable contribution in the development of my dissertation.

I would also like to thank my former advisor Ali Tamer Ünal and my former co-advisor and colleague Z. Caner Taşkın at Boğaziçi University for their endless support and encouragement that lead me to pursue a Ph.D. degree and an academic career. Their contribution in my professional life is unquestionable.

I am very happy to be a part of the Gator Nation. I would like to thank everybody who took a part in my life during my years in Gainesville, Florida. I am very grateful to all my friends and colleagues for making this experience a happily memorable one.

Last but not least I would like to thank my parents Fatma and Rıza and my brother Onur for their endless love and support during all these years we have been apart. It was very tough for me and especially for them to be thousands of miles away for such a long period of time. There are no words to describe my appreciation for their patience. I would also like to thank my fiancé Mehmet for his consistently rising love and making me happy.

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May 2016

Chair: Joseph P. Geunes
Major: Industrial and Systems Engineering

In this dissertation we consider a class of production planning problems for a producer who procures an input component for production by offering a price to suppliers. The available supply quantity for the production input depends on the price the producer offers, and this supply level constrains production output. We consider this problem in several problem settings. First, we study the case where the producer seeks a time-phased production and supply-pricing plan that minimizes the cost incurred while meeting a set of demands over a finite horizon consisting of a discrete number of time periods. We model the problem as a finite-horizon, discrete-time production and component-supply-pricing planning problem with non-stationary costs, demands, and component supply levels. This leads to a class of two-level lot-sizing problems with objective functions that are neither concave nor convex. Although the most general version of the problem is NP-Hard, we provide polynomial-time algorithms for practical special cases. Motivated by this problem we also provide a polynomial-time algorithm for economic lot-sizing problems with convex costs in the production and inventory quantities. The resulting algorithm is based on a primal-dual approach that takes advantage of the problem's special structure.

Second, we consider the problem in an infinite-horizon setting with stationary production and inventory holding costs as well as stationary demand and supply rates. We analyze the behavior of the optimal replenishment and pricing policy, which depends on the economics of production and procurement costs and the prices associated with input

components and end-items. This analysis sheds light on how to deal with price-dependent supply in production planning, as well as on the value of supplier heterogeneity and information on the relationship between component price and supply.

Third, we consider the problem in a single planning period where the demand is random, which provides a generalization of the newsvendor problem. We characterize the optimal order quantity for cases in which the supply versus price relationship is either linear or nonlinear. We also examine cases where demand is also price sensitive, and analyze the behavior of the optimal profit margin.

CHAPTER 1
INTRODUCTION

In the century following the seminal paper by Harris [32], the economic order quantity (EOQ) model has been widely used to solve inventory planning problems under deterministic and stationary cost and demand assumptions to minimize long term average production and holding costs. The EOQ model has drawn the attention of numerous researchers because of its well-established, simple and easy-to-modify nature (see Nahmias [55] and Silver, Pyke, and Peterson [75]). Another segment of production planning and inventory management research focuses on dynamic lot-sizing models, which are used to solve production planning problems when dynamic effects exist in market demand and the economies of production (e.g., Wagner and Whitin [88]). This research stream considers production planning problems with a finite planning horizon divided into discrete time periods, where the production and inventory holding costs as well as the market demand change from period to period. Another research stream focuses on planning for items for which demand only arises in a single selling period or season (e.g., fashion goods, newspapers). The decision maker faces random demand and needs to determine how much to order/produce to minimize the expected cost of inventory holding (in case of overstocking) and the expected cost of losing customers (in case of understocking). This problem, which is called the newsvendor problem, was first introduced by Arrow, Harris and Marshak [5], and numerous variations have been studied by many researchers since then.

For each of the literature streams mentioned above, a segment of research exists which has recognized the relationship between pricing decisions and the demand levels that drive production requirements. Pricing decisions and price-dependent demand rates have been considered in the EOQ literature by many researchers over the past several decades starting with Whitin [89], who presented the first model in which price-dependent demand was incorporated within the traditional EOQ model (also see e.g., Arcelus and Sirinivasan

[4] and Ray, Gerchak, and Jewkes [62]). In the dynamic lot-sizing context the decision maker's goal is to determine the optimal pricing schedule in addition to production and inventory levels using a profit-maximizing approach (see, e.g., Thomas [80], Gilbert [26], Deng and Yano [14], Geunes, Romeijn, and Taaffe [25], van den Heuvel and Wagelmans [82], and Geunes, Merzifonluoğlu, and Romeijn [24]). The newsvendor problem with price-dependent demand has also been studied by many researchers. Examples include Whitin [89], Mills [53], Karlin and Carr [40], and Petruzzi and Dada [59].

Models for production planning and inventory management have typically taken the prices of production inputs as fixed. That is, the cost of producing a number of units typically equals some predefined (usually concave) and exogenously determined function of the quantity produced. In acquiring a production input component that is not a commonly available market commodity, a producer may in some cases have discretion over the cost of the input component via their willingness or ability to pay suppliers of the input. That is, for certain classes of production inputs, the price the producer offers to suppliers for the component may determine the quantity that suppliers are willing to provide. A higher price offered to suppliers will likely attract a greater quantity of the component. This phenomenon is particularly relevant for a remanufacturer who requires inputs from individual consumers who own a product that the remanufacturer wishes to acquire as a production input. Offering a higher price to consumers increases the number of consumers who will sell back the product to the producer. Beyond the remanufacturing context, however, producers routinely seek suppliers for customized production inputs; the price offered by the producer directly impacts the quantity they are able to acquire from potential suppliers.

Motivated by the relationship between supply price and quantity, we consider various inventory models where the available supply serving as input for production is price-dependent. Each final product requires an input component from an external supplier. For example, in the remanufacturing context, each final product is remanufactured using an old item returned by an end-user. To ensure availability of production inputs, the producer offers a price to suppliers, where the total supply of production inputs depends on the price offered by the producer/remanufacturer. In the problem settings we consider, the producer needs to decide on the price of production input as well as the production and inventory levels.

Chapter 2 presents the production planning problem where we consider a producer who wishes to meet a set of demands for a final product over a finite planning horizon divided into discrete time periods. The producer seeks to meet all final product demands while minimizing its production, inventory, and supply-input-related costs. The resulting model takes the form of a two-level dynamic lot sizing problem with diseconomies of scale in procurement costs and economies of scale in production. The special case of this problem class with zero fixed charges in production and procurement levels lead us to the study of dynamic lot-sizing problems with convex costs, for which we developed a polynomial time solution algorithm (see Chapter 3). Chapter 4 introduces a production planning problem in which an input component is required for production and for which all cost, demand and pricing parameters are assumed to be stationary. Chapter 5 introduces a newsvendor problem where again the availability of input components depends on the price offered to suppliers. In Chapter 6 we summarize the results of our study.

CHAPTER 2
PRODUCTION PLANNING WITH PRICE-DEPENDENT SUPPLY

2.1 Motivation

In this chapter, we consider a production planning problem where the producer seeks a time-phased production and supply-pricing plan that minimizes the cost incurred while meeting a set of demands over a finite horizon consisting of a discrete number of time periods. We model the problem as a finite-horizon, discrete-time production and component-supply-pricing planning problem with non-stationary costs, demands, and component supply levels. This leads to a class of two-level lot-sizing problems with objective functions that are neither concave nor convex. Although the most general version of the problem is $\mathcal{NP}$-Hard, we provide polynomial-time algorithms for practical special cases. We then apply the resulting algorithms heuristically to the more general problem version, and provide computational results that demonstrate the high performance quality of the resulting heuristic solution methods.

The rest of this chapter is organized as follows. Section 2.2 summarizes literature related to our work and discusses the contributions of this chapter. Section 2.3 introduces the notation we use throughout the chapter, states the modeling assumptions, provides a general model formulation, and demonstrates the $\mathcal{NP}$-Hardness of the general model. Section 2.4 considers practically relevant special cases of the general model and demonstrates the polynomial solvability of these special cases. Section 2.5 applies the resulting algorithms to demonstrate their performance characteristics, both as an exact approach for the special cases for which they were developed, and as a heuristic approach for more general problem categories.

2.2 Related Work and Contributions

This section provides a review of the literature related to our work, which falls within three main research streams: dynamic lot sizing in serial production systems, pricing in dynamic lot sizing, and dynamic lot sizing with convex costs. A natural application area of our model is within a remanufacturing setting where a remanufacturer wishes to collect used products from the market to serve as input to its production process. Therefore, we will review a segment of the remanufacturing/reverse logistics literature, focusing on those works which are most closely related to ours.

2.2.1 Dynamic Lot Sizing in Serial Production Systems

As we will later see, the model we consider results in a dynamic lot sizing problem with two stages in series, i.e., the input component procurement stage followed by the production stage. Zangwill [92, 93] provided seminal work on multistage production systems, characterizing extreme point solution properties and their optimality under concave costs. Love [50] subsequently focused on serial production systems and demonstrated the optimality of nested production schedules under nonincreasing production costs at a stage as a function of time and nondecreasing holding costs as we move downstream in the production system. A nested schedule implies that if production at a stage in a period equals zero, then production at the immediate predecessor also equals zero. Kaminsky and Simchi-Levi [39] studied a dynamic lot sizing problem in a two-stage serial capacitated production setting with fixed costs for transportation between stages. Under stationary capacity levels, they developed a polynomial time algorithm to solve the problem with so-called "non-speculative" costs (the nature of which we will discuss later) and time-invariant capacity levels. Van Hoesel et al. [85] and Sargut and Romeijn [72] developed polynomial time algorithms for two-level serial lot sizing problems with concave costs and stationary production capacities. In contrast to previous studies, Sargut and Romeijn [72] consider backlogging, outsourcing and overtime production options. Melo and Wolsey [52] considered the uncapacitated, two-level series dynamic lot sizing problem, and provided an $\mathcal{O}(T^2 \log T)$ dynamic programming algorithm, where T corresponds to the number of periods in the planning horizon. Solyali et al. [76] study the uncapacitated and capacitated versions of this problem, where backlogging is allowed, and they develop efficient shortest path formulations. For each of these problem classes, all costs are concave in the production and inventory levels. For the problem we consider, in contrast, the total cost consists of a sum of convex and concave cost functions.

2.2.2 Price-Dependent Production Planning

In the problem we consider, the available quantity of production inputs depends on the price the producer offers to suppliers for each unit input or component, i.e., supply is price-dependent. While a significant body of literature exists that considers production planning problems with pricing effects, to our knowledge, this literature focuses exclusively on the pricing impact on end-item demand (see, for example, Thomas [80], Kunreuther and Schrage [44], Pekelman [58], Kim and Lee [42], Zhao and Wang [95], Elmaghraby and Keskinocak [17], Deng and Yano [14], Ahn et al. [1], Haugen et al. [33], Onal and Romeijn [57], van den Heuvel et al. [81], Gilbert [26], Geunes, Romeijn, and Taaffe [25], van den Heuvel and Wagelmans [82], and Geunes, Merzifonluoğlu, and Romeijn [24]). For these problem classes, a given price, or price vector, determines the set of demands that must be satisfied in a production planning problem with concave costs.

2.2.3 Dynamic Lot Sizing with Convex Costs

As we will later see, the price-sensitivity of supply availability we consider leads to a class of two-level serial lot sizing problems whose cost is the sum of concave and convex functions, and where the supply availability limits production quantities. The literature on lot-sizing problems containing convex production costs is reasonably sparse, with a few notable exceptions. Veinott [86] considered a single-stage, dynamic lot sizing problem in which production and inventory costs are piecewise-linear and convex. He designed a parametric-programming-based procedure in which the solution for a problem with a fixed parameter set is built upon the solution of another problem with a similar parameter set. Veinott [86] assumed all parameters were integer valued, and his procedure resulted in a pseudo-polynomial solution algorithm. Florian et al. [22] mentioned Veinott's procedure as the most attractive approach to solve lot sizing problems with convex production and inventory costs and without fixed setup costs, even though the problem is demonstrated to be no harder than linear programming, which is polynomially solvable. Erenguc and Aksoy [18] considered a single-item, capacitated dynamic lot sizing problem with fixed production setup costs and linear inventory costs, while variable production costs were piecewise linear and convex in the production quantity in a period. They used a branch-and-bound algorithm for this problem, which contains neither a convex nor concave objective function. Shaw and Wagelmans [74] developed a pseudo-polynomial dynamic program to solve a capacitated single-item lot-sizing problem with piecewise linear production costs. Their algorithm can be utilized to solve problems with piecewise-linear and convex production costs, although it does not require any special structure for the piecewise-linear cost function. Feng et al. [19] developed an $\mathcal{O}(T \log T)$ algorithm for the single-item lot-sizing problem with constant capacity, convex inventory costs, and non-increasing fixed order costs. The work by Kian et al. [41] is closely related to ours, as they analyze a single-stage, uncapacitated economic lot sizing problem with fixed setup costs and variable costs in each period that are convex in the production quantity (taking the form of a polynomial function). They derive several key optimality conditions for this problem class, as well as a dynamic programming solution algorithm that is exponential in the length of the time horizon. They also provide several heuristic solution approaches, along with a comprehensive numerical test set that demonstrates the effectiveness of their proposed heuristics. The problem we consider, in contrast, is a two-level serial lot sizing problem, and we focus on reasonable cost assumptions that lead to polynomial solvability, as well as the use of the resulting algorithms as heuristic solution methods when these cost assumptions are violated.

2.2.4 Pricing and Dynamic Lot Sizing in Remanufacturing

Numerous factors, including potential cost savings and environmental concerns, have motivated manufacturers to emphasize reverse logistics in the past decade. As a result, reverse logistics and remanufacturing have received significant interest by researchers in recent years, and the literature is voluminous. Our discussion of the relevant literature focuses on those works in which dynamic lot sizing problems arise within the reverse logistics literature. Fleischmann et al. [20] provide a comprehensive overview of reverse logistics problems, classifying various research thrusts within the field.

Golany et al. [27] study a version of the single-item dynamic lot sizing problem in which disposal of remanufacturable items is an option. They show that the problem is generally $\mathcal{NP}$-Complete under concave costs and develop a polynomial-time algorithm for the case with linear costs. Teunter et al. [79] study a dynamic lot sizing problem in which remanufactured returns can be used for demand satisfaction along with newly manufactured products. They consider two versions of the problem; the first version assumes joint setup costs for both manufacturing and remanufacturing, and the second considers separate setup costs for each production mode. The first version turns out to be polynomially solvable, and the authors provide an exact algorithm, whereas the second version is $\mathcal{NP}$-Hard and is solved using heuristic methods. Schulz [73] focuses on improving the Silver-Meal heuristic for solving the case with separate setup costs for manufacturing and remanufacturing proposed by Teunter et al. [79]. Helmrich et al. [34] study the problem with separate and joint setup costs for manufacturing and remanufacturing options. They show that both versions are, in general, $\mathcal{NP}$-Hard, and provide tight mixed integer programming formulations that, on average, perform better than the well-known single item lot sizing formulation. Richter and Sombrutzki [65] consider the case where remanufacturing is triggered by the return of products from customers, which they call the purely reverse Wagner/Whitin model. Richter and Weber [66] extend this setting to a model in which multiple production modes (manufacturing and remanufacturing) are available. In each of the noted studies, the quantity of available production inputs for remanufacturing is determined exogenously and is known in advance.

Guide et al. [29] study a somewhat similar setting to the one we consider, in which prices are determined for production inputs with varying quality levels. They solve a price-matching problem for demand and supply in a single-period setting. Sun et al. [77] consider a case in which returns to a remanufacturer are also price-sensitive, but random. They formulate the problem as a multiperiod stochastic dynamic program where the decision variables include the price and remanufacturing quantity in each period. These studies are related to our work, as they consider price decisions for supply inputs. In contrast, we incorporate the idea of price-sensitive supply into classical production planning problems, where the goal is to minimize production and inventory related costs while satisfying deterministic demand during a multiperiod planning horizon.

2.2.5 Contributions of this Chapter

To the best of our knowledge, prior work does not exist that considers two-level lot sizing problems with convex component supply cost functions, along with the usual fixed production setup costs and standard linear holding costs. We show that this problem is $\mathcal{NP}$-Hard in general, even in the absence of explicit production capacity restrictions, and consider several practically relevant special cases that result in polynomial solvability. The resulting algorithms for these special cases serve two purposes. First, they demonstrate the polynomial solvability of practically relevant special cases of this class of nonconvex, nonlinear mixed integer optimization problems. Second, the algorithmic development provides insights on the structure of optimal solutions and suggests the use of the resulting algorithms as heuristic solution approaches for more general classes of problem instances. Our work also contributes to the reverse logistics literature, as it encompasses problem classes in which price-sensitive returns from customers serve as the supply source for production.

2.3 Problem Definition, Formulation, and Complexity

We consider a production process that requires the transformation of an input component into a final good for which customer demand exists. We refer to the owner of this production process as the producer. The availability of the required component depends on a unit price that the producer offers for the component. That is, those agents with the ability to supply the component are willing to supply an amount that depends on the unit price offered by the producer.

We consider a finite set T of planning periods, where $T = \{1, 2, \ldots, |T|\}$, such that the producer may offer a different unit price for the input component in each period. Let p_t and κ_t denote, respectively, the unit price the producer offers to suppliers in period t, and the total supply obtained by the producer at this price. We assume that κ_t is determined by a nonnegative, nondecreasing function $\kappa_t = K_t(p_t)$, such that $K_t(p_t)$ is invertible and a one-to-one mapping exists between price p_t and component supply level κ_t, i.e., $p_t = K_t^{-1}(\kappa_t)$. The amount the producer transfers to suppliers in period t then equals $p_t K_t(p_t) = \kappa_t K_t^{-1}(\kappa_t)$, and we assume $\lim_{p_t \to \infty} \kappa_t = \infty$, i.e., no finite bound exists on available supply. In addition to the cost incurred for obtaining components from suppliers, the producer may incur additional procurement related costs; therefore, we let $f^C(\kappa_t)$ denote a function equal to the total procurement-related cost incurred by the producer in period t, where $f^C(\kappa_t) \geq \kappa_t K_t^{-1}(\kappa_t)$, $t \in T$. We assume that production and component procurement lead times are negligible, although the model can accommodate finite and constant lead times as well.

For a final-goods production level of x_t in period t, the producer incurs a production cost of $f_t(x_t)$. We assume for convenience that one unit of the input component is required for each unit of the final good produced. The producer may hold components and final goods in inventory at a cost of h_t per unit of the final good held at the end of period t and a cost of h_t^C per unit of input component held at the end of period t, $t \in T$. Letting I_t and I_t^C denote, respectively, the final-goods and component inventory levels at the end of period t, the holding cost incurred at the end of period t equals $h_t I_t + h_t^C I_t^C$. We assume zero initial inventories for both final good and component levels.

The producer wishes to meet demand for the final good without shortages in each period of the planning horizon, where d_t denotes the number of units of the final good demanded in period t. The producer seeks to minimize the costs incurred in meeting this demand, where total cost is comprised of component procurement costs, final-goods production costs, and inventory holding costs. We formulate the general version of this problem, denoted as P(G), as follows.

$$\textbf{P(G):} \quad \text{Minimize} \quad \sum_{t \in T} \left\{ f_t(x_t) + f_t^C(\kappa_t) + h_t I_t + h_t^C I_t^C \right\} \tag{2–1}$$

$$\text{Subject to:} \quad I_{t-1} + x_t = d_t + I_t, \qquad \forall t \in T, \tag{2–2}$$

$$I_{t-1}^C + \kappa_t = x_t + I_t^C, \qquad \forall t \in T, \tag{2–3}$$

$$x_t,\ I_t,\ I_t^C,\ \kappa_t \geq 0, \qquad \forall t \in T. \tag{2–4}$$

The objective function minimizes total cost, while constraints (2–2) and (2–3) correspond to inventory balance constraints for the final goods and components, respectively. Constraint set (2–4) requires all variables to take nonnegative values. Observe that if $f_t(x_t)$ and $f_t^C(\kappa_t)$ are concave functions for all $t \in T$, then this problem falls into the class of uncapacitated two-level serial lot-sizing problems, which have been widely studied and are well solved in polynomial time as a function of $|T|$ (see, e.g., Zangwill [93], van Hoesel, Romeijn, Morales and Wagelmans [85], Sargut and Romeijn [72], and Melo and Wolsey [52]). For practical contexts, however, while we might expect concavity to hold for the production cost functions ($f_t(x_t)$, $t \in T$), it is unlikely to hold for the price-dependent component supply cost function $f_t^C(\kappa_t)$, $t \in T$, when supply quantity is increasing in price. In contrast, the convexity of this function appears to be a more reasonable assumption, as higher acquisition prices are likely to increase supply at an increasing rate. If $f_t(x_t)$ is concave for all $t \in T$ and $f_t^C(\kappa_t)$ is nondecreasing and convex for all $t \in T$, then problem P(G) becomes $\mathcal{NP}$-Hard, which we next demonstrate using a special case of P(G).

In order to reflect practical production planning contexts, we will generally assume that $f_t(x_t)$ takes the form of a fixed-charge plus linear cost function for each $t \in T$, i.e., $f_t(x_t) = F_t y_t + c_t x_t$, where y_t denotes a binary variable equal to one if $x_t > 0$ and equal to zero otherwise. This is a commonly used concave functional form for representing production costs that arise in the form of a fixed setup cost plus a variable cost for each unit of production. We further assume, except where specifically noted, that $f_t^C(\kappa_t)$ takes a particular functional form described as follows. Suppose that a minimum reservation price exists in each period t, denoted by p_t^0, such that for any price not exceeding this value, no supply is available. For prices exceeding this threshold, we assume that supply increases linearly in price at a rate of $\beta_t > 0$ per unit price change. As a result of these assumptions, the supply in period t, $\kappa_t(p_t) \geq 0$ is is a nonnegative and nondecreasing piecewise linear function of price, p_t, with one breakpoint at p_t^0, defined by

$$\kappa_t(p_t) = \begin{cases} \beta_t p_t - \alpha_t, & \text{if } p_t \geq p_t^0, \\ 0, & \text{otherwise,} \end{cases} \tag{2–5}$$

for $p_t \geq 0$, with $\alpha_t \geq 0$ for all $t \in T$ (see Figure 2-1).

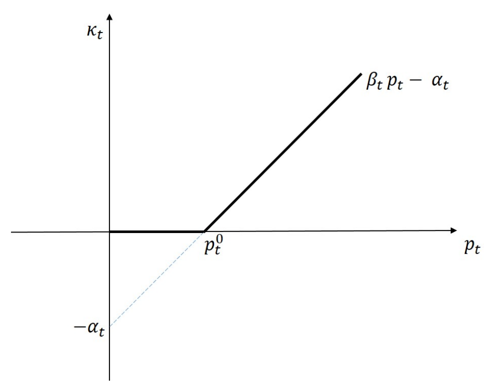


Figure 2-1. Supply vs. price function.

For each unit of the component procured, the producer pays p_t, which implies that the total amount paid to suppliers in period t equals $p_t\kappa_t = \beta_t p_t^2 - \alpha_t p_t = \kappa_t^2/\beta_t + \alpha_t\kappa_t/\beta_t$ (where we suppress the dependence of κ_t on p_t for convenience). In addition, we assume the producer incurs a cost of $F_t^C y_t^C + \gamma_t\kappa_t$ for a procurement level of κ_t in period t, where F_t^C denotes a fixed cost incurred for any positive level of procurement, and y_t^C denotes a binary variable equal to one if $\kappa_t > 0$ and equal to zero otherwise. Thus, a total procurement related cost of $F_t^C y_t^C + (p_t + \gamma_t)\kappa_t = F_t^C y_t^C + \gamma_t\beta_t p_t + \beta_t p_t^2 - \alpha_t p_t =$

$F_t^C y_t^C + \kappa_t^2/\beta_t + (\gamma_t + \alpha_t/\beta_t)\kappa_t$ is incurred in each period $t \in T$. We will refer to this version of the problem with fixed charges for both production and procurement as problem P(FC).

Proposition 2.1. *The production planning problem with price-dependent supply capacity is $\mathcal{NP}$-Hard.*

Proof. Please see Appendix A. □

In the proof of Proposition 2.1 in Appendix A, all fixed production setup costs equal one and all fixed procurement costs equal zero. This implies that even the special case in which no fixed procurement costs exist and all production setup costs are equal for all periods is $\mathcal{NP}$-Hard. We can also show that the special case in which $\beta_t = \beta$, $\alpha_t = \alpha$, and $\gamma_{t+1} = \gamma_t + h_t^C$ for all $t \in T$, i.e., when the price-supply functions are identical in every period and the unit component procurement cost increase from one period to the next equals the component holding cost in the prior period, is $\mathcal{NP}$-Hard as well. (Note that this latter condition implies that the variable procurement cost associated with a component in inventory in any period is independent of the period in which the component was procured.)

Proposition 2.2. *For the special case in which $\beta_t = \beta$, $\alpha_t = \alpha$, and $\gamma_{t+1} = \gamma_t + h_t^C$ for all $t \in T$, the production planning problem with price-dependent supply capacity remains $\mathcal{NP}$-Hard.*

Proof. Please see Appendix B. □

The following section analyzes key properties of optimal solutions for problem P(FC) and identifies practically relevant special cases that may be solved in polynomial time.

2.4 Analysis of P(FC) and Polynomially Solvable Cases

In order to formally analyze properties of optimal solutions for problem P(FC), we define a production (component) regeneration interval $RI_p(t, t')$ ($RI_c(t, t')$) as a partial solution for periods t through $t' - 1$, such that $I_{t-1} = I_{t'-1} = 0$ ($I_{t-1}^C = I_{t'-1}^C = 0$) and $I_j > 0$ ($I_j^C > 0$) for all $j = t, \ldots, t' - 2$. Note that any feasible solution for problem P(FC) can be characterized as a sequence of component regeneration intervals that provide input to production regeneration intervals. Moreover, an optimal solution for problem P(FC) exists consisting of sequences of these regeneration intervals. The following proposition characterizes an important property of an optimal component regeneration interval RI_c for problem P(FC).

Proposition 2.3. *If $RI_c(t,t')$ is part of an optimal solution, then for any pair of periods i_1, i_2 such that $t \leq i_1 < i_2 \leq t'-1$ and both κ_{i_1} and κ_{i_2} positive, $2\kappa_{i_1}/\beta_{i_1} + \gamma_{i_1} + \alpha_{i_1}/\beta_{i_1} + \sum_{j=i_1}^{i_2-1} h_j^C = 2\kappa_{i_2}/\beta_{i_2} + \gamma_{i_2} + \alpha_{i_2}/\beta_{i_2}$.*

Proof. Suppose we have an optimal solution containing the $RI_c(t,t')$ in which $2\kappa_{i_1}/\beta_{i_1} + \gamma_{i_1} + \alpha_{i_1}/\beta_{i_1} + \sum_{j=i_1}^{i_2-1} h_j^C < 2\kappa_{i_2}/\beta_{i_2} + \gamma_{i_2} + \alpha_{i_2}/\beta_{i_2}$. Let $\tilde{\kappa}_{i_1} = \kappa_{i_1} + \epsilon$ and let $\tilde{\kappa}_{i_2} = \kappa_{i_2} - \epsilon$ for some $\epsilon > 0$ and sufficiently small, and let $\tilde{I}_j^C = I_j^C + \epsilon$ for $j = i_1, \ldots, i_2 - 1$ (the values of all other variables remain unchanged). The value of the original solution minus the new solution can be written as

$$\Delta = \epsilon\left(\frac{2\kappa_{i_2}}{\beta_{i_2}} - \frac{2\kappa_{i_1}}{\beta_{i_1}} - \sum_{j=i_1}^{i_2-1} h_j^C + \gamma_{i_2} + \frac{\alpha_{i_2}}{\beta_{i_2}} - \gamma_{i_1} - \frac{\alpha_{i_1}}{\beta_{i_1}} - \epsilon\left(\frac{\beta_{i_1}+\beta_{i_2}}{\beta_{i_1}\beta_{i_2}}\right)\right).$$

For positive ϵ such that

$$\epsilon < \frac{\beta_{i_1}\beta_{i_2}}{\beta_{i_1}+\beta_{i_2}}\left(\frac{2\kappa_{i_2}}{\beta_{i_2}} - \frac{2\kappa_{i_1}}{\beta_{i_1}} - \sum_{j=i_1}^{i_2-1} h_j^C + \gamma_{i_2} + \frac{\alpha_{i_2}}{\beta_{i_2}} - \gamma_{i_1} - \frac{\alpha_{i_1}}{\beta_{i_1}}\right),$$

we have $\Delta > 0$, which contradicts the optimality of the original solution (note that the right-hand side of the above is strictly positive by assumption). A similar argument shows that a solution in which $2\kappa_{i_1}/\beta_{i_1} + \gamma_{i_1} + \alpha_{i_1}/\beta_{i_1} + \sum_{j=i_1}^{i_2-1} h_j^C > 2\kappa_{i_2}/\beta_{i_2} + \gamma_{i_2} + \alpha_{i_2}/\beta_{i_2}$ cannot be optimal. □

Observe that for a period i within a $RI_c(t,t')$ (i.e., $t \leq i \leq t'-1$), the marginal cost for supplying a component to period i that is procured in period $\tau < i$ (where $\tau \geq t$) equals $2\kappa_\tau/\beta_\tau + \gamma_\tau + \alpha_\tau/\beta_\tau + \sum_{j=\tau}^{i-1} h_j^C$. Suppose we have an optimal solution containing the $RI_c(t,t')$. Following the arguments in the preceding proof, for all periods

i within the regeneration interval (i.e., such that $t \leq i \leq t' - 1$), an optimal solution exists such that the marginal cost for supplying a component to period i is the same for all prior periods within $RI_c(t, t')$. A similar property arises in Deng and Yano [14] and Geunes, and Merzifonluoğlu, and Romeijn [24] in the context of production planning with price-sensitive demand, as well as in Kian, Gürler, and Berk [41] in single-stage lot sizing with convex production costs.

Remark. Suppose procurement costs take the form of a general convex function $f_t^C(\kappa_t)$, and let $\partial f_t^C(\kappa_t)/\partial \kappa_t$ denote the set of subgradients of f_t^C at κ_t for all $t \in T$. For this more general case, Proposition 2.3 can be generalized to state that if $RI_c(t, t')$ is part of an optimal solution, then for periods i_1 and i_2 such that $t \leq i_1 < i_2 \leq t' - 1$ with $\kappa_{i_1} > 0$ and $\kappa_{i_2} > 0$, we have $\partial f_t^C(\kappa_{i_1})/\partial \kappa_{i_1} + \sum_{j=i_1}^{i_2-1} h_j^C \cap \partial f_t^C(\kappa_{i_2})/\partial \kappa_{i_2} \neq \emptyset$. If each f_t^C is everywhere differentiable, then this condition becomes $df_t^C(\kappa_{i_1})/d\kappa_{i_1} + \sum_{j=i_1}^{i_2-1} h_j^C = df_t^C(\kappa_{i_2})/d\kappa_{i_2}$.

Despite these structural results, the problem P(FC) is, in general, $\mathcal{NP}$-Hard in the absence of any additional conditions imposed on the problem's parameters. We next impose a fairly mild assumption on the structure of production and holding costs that leads to the optimality of the well-known Zero-Inventory Production (ZIP) property, under which an optimal solution exists such that production in a period occurs if and only if no end-item inventory is held over from the prior period.

Assumption 1. *For every period $t = 1, \ldots, |T| - 1$, we assume $c_t + h_t \geq c_{t+1} + h_t^C$.*

Assumption 1 states that it is always at least as costly (in terms of variable production and holding costs) to produce and hold a product as it is to hold a component for later production (this assumption is consistent with the assumptions on production and holding costs in Love [50]; Kaminsky and Simchi-Levi [39] imposed a similar assumption on transportation and holding costs in their analysis of a two-level capacitated lot sizing problem with concave production and transportation costs). It is not uncommon in practice for holding costs to increase downstream in a multi-stage system, as value is added to a product; thus, we typically expect the condition $h_t \geq h_t^C$ to hold in each period. Observe that Assumption 1 does not require nonincreasing production costs; whenever the increase in production cost from period-to-period does not exceed the difference between downstream and upstream holding cost, this structural cost assumption will apply. (We note that the case in which $c_t + h_t < c_{t+1} + h_t^C$ for $t = 1, \ldots, |T| - 1$ is $\mathcal{NP}$-Hard as a corollary to Proposition 2.1, because the special case of infinite component holding costs satisfies this condition.)

Proposition 2.4. *Under Assumption 1, an optimal solution exists satisfying the zero-inventory production (ZIP) property, i.e., $x_t I_{t-1} = 0$ for $t = 1, \ldots, |T|$.*

Proof. Suppose the required condition $x_t I_{t-1} = 0$ for $t = 1, \ldots, |T|$ is violated in some period i in an optimal solution, which implies $x_i > 0$ and $I_{i-1} > 0$, and let $(\kappa, \mathbf{x}, \mathbf{I}^C, \mathbf{I})$ denote the vectors of component procurement, production, component inventory, and final product inventory variable values in the corresponding solution. Consider a unit of inventory in period $i-1$, and let $\tau < i$ denote the period in which this unit of inventory was produced. Because each unit produced in period τ required one unit of component, we can create a new solution $(\tilde{\kappa}, \tilde{\mathbf{x}}, \tilde{\mathbf{I}}^C, \tilde{\mathbf{I}})$ such that $\tilde{x}_\tau = x_\tau - 1$, $\tilde{x}_i = x_i + 1$, $\tilde{I}_t = I_t - 1$ for $t = \tau, \ldots, i-1$, and $\tilde{I}_t^C = I_t^C + 1$ for $t = \tau, \ldots, i-1$ (for all other variables $\tilde{\kappa} = \kappa$, $\tilde{x} = x$, $\tilde{I} = I$ and $\tilde{I}^C = I^C$). The cost of the original solution minus the new solution equals

$$c_\tau(x_\tau - \tilde{x}_\tau) + c_i(x_i - \tilde{x}_i) + \sum_{j=\tau}^{i-1}(h_j(I_j - \tilde{I}_j) + h_j^C(I_j^C - \tilde{I}_j^C))$$

$$= c_\tau + \sum_{j=\tau}^{i-1} h_j - \left(c_i + \sum_{j=\tau}^{i-1} h_j^C\right).$$

Observe that by assumption $c_\tau \geq c_{\tau+1} + h_\tau^C - h_\tau$, and this implies

$$c_\tau \geq c_{\tau+1} + h_\tau^C - h_\tau \geq c_{\tau+2} + h_{\tau+1}^C - h_{\tau+1} + h_\tau^C - h_\tau.$$

Substituting this inequality for successive values of $t = \tau + 2, \ldots, i$ gives

$$c_\tau \geq c_i + \sum_{j=\tau}^{i-1} h_j^C - \sum_{j=\tau}^{i-1} h_j,$$

which gives

$$c_\tau + \sum_{j=\tau}^{i-1} h_j \geq c_i + \sum_{j=\tau}^{i-1} h_j^C.$$

Thus, the new solution costs no more than the original solution. Repeating this argument for each unit of inventory in period $i - 1$ implies that we can either find a new optimal solution satisfying the desired property or we have a contradiction to the optimality of the original solution. In either case, an optimal solution exists satisfying the ZIP property. $\square$

The ZIP property implies that every production setup produces a quantity equal to a consecutive set of period demands, i.e., letting $D_{i,j} = \sum_{t=i}^{j} d_t$, in the worst case we only need to consider solutions such that $x_t \in \{0, D_{t,t},\ \ D_{t,t+1}, \ldots, D_{t,|T|}\}$ for $t = 1, \ldots, |T|$.

Next observe that any production requires component procurement, and the production in any period must draw its component supply from some $RI_c(t, t')$. Moreover, the $RI_c(t, t')$ can serve production in any period greater than or equal to t, and if the $RI_c(t, t')$ is contained in an optimal solution, then the final period of production this $RI_c(t, t')$ serves must occur in period $t' - 1$ (otherwise we have held component inventory at the end of period $t' - 1$ that is not used to satisfy production, violating the definition of $RI_c(t, t')$).

Proposition 2.5. *Under Assumption 1, if the $RI_c(t, t')$ is contained in an optimal solution, then the total procurement amount in periods $t, t+1, \ldots, t'-1$ equals $D_{u,v}$ for some u and v, where $u \geq t$ and $v \geq t' - 1$.*

Proof. Suppose $RI_c(t, t')$ serves production in periods $t_1, t_2, \ldots, t_n$ where $t \leq t_1 < t_2 < \ldots < t_n = t' - 1$. By the ZIP property, t_1 is the first period that the $RI_c(t, t')$ satisfies the demand for. Hence, $u = t_1$. The number of components procured in $RI_c(t, t')$ must satisfy all productions in periods $t_1, \ldots, t_n$, therefore the procurement amount in $RI_c(t,t')$ must be equal to the sum of productions in periods $t_1,\dots,t_n$. By the ZIP property, we know that the respective production amounts in periods $t_1,\dots,t_n$ are $D_{u,t_2-1}, D_{t_2,t_3-1},\dots,D_{t'-1,v}$ where $v \ge t'-1$. The sum of these production amounts are equal to $D_{u,v}$. Thus the component procurement amount in $RI_c(t,t')$ equals $D_{u,v}$. $\square$

When the periods with positive procurement within a component RI, say $RI_c(t,t')$, are known, we can compute the procurement levels for each of these periods using the results of Proposition 2.3. We define $\bar{\gamma}_t = \gamma_t + \alpha_t/\beta_t + \sum_{j=t}^{|T|} h_j^C$, and suppose there are m periods with positive procurement $t_1,t_2,\dots,t_m$ within $RI_c(t,t')$, and these components serve demand in periods u through v. By Proposition 2.3, we have

$$\bar{\gamma}_{t_1} + \frac{2\kappa_{t_1}}{\beta_{t_1}} = \bar{\gamma}_{t_2} + \frac{2\kappa_{t_2}}{\beta_{t_2}} = \dots = \bar{\gamma}_{t_m} + \frac{2\kappa_{t_m}}{\beta_{t_m}}.$$

Any procurement level κ_{t_j} for $j=2,\dots,m$ can be written in terms of κ_{t_1} as follows:

$$\kappa_{t_j} = \left(\bar{\gamma}_{t_1} - \bar{\gamma}_{t_j} + \frac{2\kappa_{t_1}}{\beta_{t_1}}\right)\frac{\beta_{t_j}}{2}. \tag{2–6}$$

Because $D_{u,v} = \sum_{j=1}^{m} \kappa_{t_j}$, we can thus compute κ_{t_1} as follows:

$$\kappa_{t_1} = \frac{D_{u,v} - \sum_{j=2}^{m} (\beta_{t_j}/2)(\bar{\gamma}_{t_1} - \bar{\gamma}_{t_j})}{1 + \sum_{j=2}^{m} \beta_{t_j}/\beta_{t_1}}. \tag{2–7}$$

We can now use the value of κ_{t_1} and Equation (2–6) to obtain each of the m values of κ_{t_j}. Unfortunately, for a given value of $m \le t'-t+1$, we have $2^m - 1$ unique choices of a subset of m periods in which positive procurement may occur within $RI_c(t,t')$. Thus, determining the best set of m positive procurement periods corresponds to a difficult combinatorial optimization problem in general. However, under additional assumptions on the behavior of procurement-related costs, it is possible to determine an optimal set of m positive procurement periods in polynomial time for any $m \le t'-t+1$.

The remainder of this section consists of two subsections, each of which considers a set of assumptions on the behavior of procurement costs over time. Section 2.4.1 first considers cases with stationary price-supply functions and variable procurement costs but permits arbitrary fixed charge values for both procurement and production. Section 2.4.2 then allows for slightly more general, nondecreasing price-supply functions and variable procurement costs over time, but requires nondecreasing fixed procurement costs as well.

2.4.1 P(FC) with Uniform Price-Supply Functions and Variable Procurement Costs

In many practical contexts, the response of suppliers to the price offered is not likely to be variable in the short run. Recall from Proposition 2.2 that even when the price-supply curve parameters are stationary and variable procurement costs are independent of the procurement period, i.e., $\beta_t = \beta$, $\alpha_t = \alpha$, and $\gamma_{t+1} = \gamma_t + h_t^C$, $t = 1, \ldots, |T|$, the problem remains $\mathcal{NP}$-hard. Under the nonspeculative cost structure of Assumption 1, however, the problem becomes solvable in polynomial time, as we next demonstrate.

Assumption 2. *For every period $t \in T$, we have $\beta_t = \beta$, $\alpha_t = \alpha$, and for every $t \in T \backslash \{|T|\}$, $\gamma_{t+1} = \gamma_t + h_t^C$.*

Corollary 1. *Under Assumption 2, Proposition 2.3 implies that the procurement levels in periods with positive procurement within an RI_c must be equal.*

Under Assumptions 1 and 2, we can solve the problem in polynomial time under arbitrary values of the fixed charges at both the procurement production levels. We first define a subplan by the quadruple (s, u, t, v), which indicates that the component regeneration interval $RI_c(s, t+1)$ serves production in periods u through v, which is used to satisfy all demand in periods u through v, where $s \le u \le t \le v$. Subplan (s, u, t, v) implies that we procure $D_{u,v}$ units of the component in $RI_c(s, t+1)$ and produce this amount of end-product in periods u through v, using a set of consecutive production regeneration intervals $RI_p(u, i_1), RI_p(i_1, i_2), \ldots, RI_p(i_n, v+1)$. Given a fixed number m of periods with positive procurement (for $m = 1, \ldots, t - s + 1$), we can easily compute the procurement amount in each period with positive procurement, as this must equal

$D_{u,v}/m$. However, the exact periods with positive procurement are unknown and must be determined.

Let $SPC(s,u,t,v)$ be the minimum cost of subplan (s,u,t,v), including the sum of the fixed and variable production costs and final product holding costs in periods u through v, plus the fixed and variable component procurement and holding costs in periods s through t. In order to compute $SPC(s,u,t,v)$, we need to determine the subset of periods from u through v in which production will occur. However, we also need to determine the periods with positive procurement from s through t. By the definition of subplan (s,u,t,v), we know that the last production setup period within the subplan must occur in period t, with a production quantity equal to $D_{t,v}$, as a result of the ZIP property from Proposition 2.4. Given the number of periods with positive procurement, m, and demand requirements in each period, we can determine the subset of periods u through v in which production occurs and the subset of periods s through t in which procurement occurs by solving a shortest path problem on a graph created as follows: Let $G(V,A)$ be a directed graph with vertex set V and arc set A (see Figure 2-2). The vertex set V consists of an origin vertex s and a destination vertex t^m. The intermediate vertices are denoted by i^l, where $i \in \{u, u+1, \dots t-1\}$, $1 \le l \le \min\{m, i-s+1\}$, and $m-l \le t-i$. Vertex i^l corresponds to production period i when the number of procurement setups up to period i equals l. The destination vertex t^m implies all m procurement setups occurred up to period t, which is also implied by the definition of the subplan (s,u,t,v). The arc set A consists of two subsets of arcs: A^1, and A^2, i.e., $A = A^1 \cup A^2$. The arc set A^1 includes arcs (s, u^l) where $1 \le l \le \min\{m, u-s+1\}$. An arc from s to u^l implies that there are l procurement setups from s through u. Arc set A^2 is defined as follows: $A^2 = \{(i^l, j^k) \mid i^l, j^k \in V,\ i < j,\ l \le k \text{ and } k-l \le j-i\}$. An arc from i^l to j^k then implies that a production setup occurs in period i to satisfy all demand in periods i through $j-1$, and the number of procurement setups from $i+1$ through j equals $k-l$.

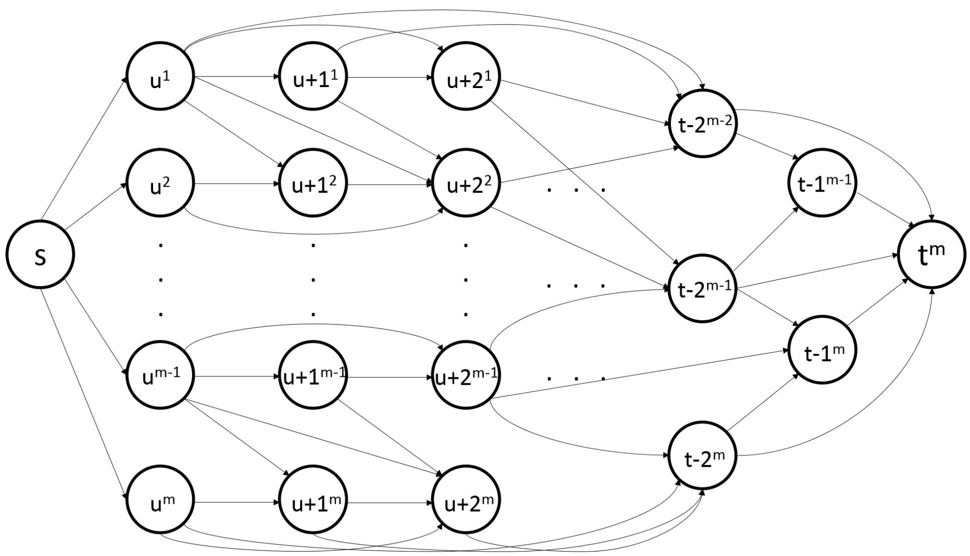


Figure 2-2. Partial representation of the directed graph $G(V, A)$.

Next we show how to compute the costs of arcs in A^1, and A^2, given that the number of procurement setup periods equals m. Note that the procurement amounts in all periods with positive procurement are equal, and let κ denote this procurement level, i.e., $\kappa = D_{u,v}/m$. Let c_a^m denote the cost of arc $a \in A$.

For $a \in A^1$, c_a^m is computed as follows, where $a = (s, u^l)$ and $1 \leq l \leq \min\{m, u-s+1\}$. We need to determine the l cheapest procurement periods from s through $u-1$. Let $F_{[j]}^C$ denote the j^{th} cheapest fixed procurement cost from s through $u-1$. Here $[j]$ returns the index of the period with the j^{th} cheapest fixed procurement cost. From s through $u-1$, the procurement levels, κ_i, are equal to κ for all $i = [1], \ldots, [l]$, and zero otherwise. Using Equation (2–8), we can compute the cost of arc a for all $a \in A^1$. We define $\mathcal{K}_{s,i}$ to denote the cumulative procurement amount from period s to period i, i.e., $\mathcal{K}_{s,i} = \sum_{k=s}^{i} \kappa_k$.

$$c_a^m = \sum_{j=1}^{l} \left(F_{[j]}^C + \gamma_{[j]}\kappa \right) + \sum_{k=s}^{u-1} h_k^C \mathcal{K}_{s,k} + \frac{l}{\beta}\kappa^2 + \frac{l\alpha}{\beta}\kappa. \tag{2–8}$$

For $a \in A^2$, c_a^m includes both production and procurement related costs, as well as inventory holding costs at both levels. Here a denotes an arc of the form (i^l, j^k) where $i < j$, $l \leq k$ and $k-l \leq j-i$. This latter condition implies that the number of procurement setups from $i+1$ through j cannot exceed the number periods within the time window. When $k-l < j-i$, we need to choose the $k-l$ cheapest procurement setup periods from period $i+1$ to j. Here again, let $[j]$ be the index of the period with the j^{th} cheapest fixed procurement cost from i through j. The procurement level in period i, κ_i, equals κ for $i = [1], \ldots, [k-l]$, and zero for other periods. Note that the arc (i^l, j^k) is only feasible when the number of components procured up to period i is sufficient to satisfy all production requirements up to period $j-1$. That is, in period i, we can produce $D_{i,j-1}$ units only if $\mathcal{K}_{s,i} \geq D_{u,j-1}$, where, at vertex i^l, we have $\mathcal{K}_{s,i} = l\kappa$ by the definition of vertex i^l and by Corollary 1. Thus,

$$c_a^m = \begin{cases} \infty, & \text{if } D_{u,j-1} > \mathcal{K}_{s,i}, \\ C(i^l, j^k) + P(i^l, j^k), & \text{otherwise,} \end{cases} \tag{2–9}$$

where

$$C(i^l, j^k) = \sum_{t=1}^{k-l} \left(F_{[t]}^C + \gamma_{[t]}\kappa \right) + \sum_{t=i}^{j-1} h_t^C (\mathcal{K}_{s,t} - D_{i,j-1}) + \frac{k-l}{\beta}\kappa^2 + \frac{(k-l)\alpha}{\beta}\kappa, \tag{2–10}$$

and

$$P(i^l, j^k) = F_i + c_i D_{i,j-1} + \sum_{k=i}^{j-2} h_k D_{k+1,j-1}. \tag{2–11}$$

Here $C(i^l, j^k)$ denotes the total fixed and variable procurement costs and component inventory holding costs from period i through j. $P(i^l, j^k)$ is the sum of fixed and variable production costs and end-item inventory holding costs from i through $j-1$.

Let $\delta_m(s, u, t)$ denote the cost of the shortest path in the graph $G(V, A)$ from s to t. The cost of subplan (s, u, t, v) is computed as follows:

$$SPC(s, u, t, v) = \min_{1 \leq m \leq t-s+1} \left\{ \delta_m(s, u, t) + F_t + c_t D_{tv} + \sum_{i=t}^{v-1} h_i D_{i+1,v} \right\}. \tag{2–12}$$

To solve the problem under Assumption 2, we can now apply a dynamic programming approach. Let $C_{s,u}$ be the minimum cost of satisfying demand from period u until the end of the horizon, given that all component procurement occurs in period s and later, where

$s \leq u$. $C_{s,u}$ can be computed using the following dynamic programming recursion:

$$C_{s,u} = \min_{v,t:u \leq t \leq v} \{SPC(s,u,t,v) + C_{t+1,v+1}\}. \tag{2–13}$$

The associated boundary conditions are as follows:

$$C_{|T|+1,i} = C_{i,|T|+1} = 0 \text{ for all } i = 1, \ldots, |T| + 1. \tag{2–14}$$

Here $C_{1,1}$ gives an optimal production and procurement plan for the entire planning horizon. To compute $C_{1,1}$, we need to compute $\mathcal{O}(|T|^4)$ values of $SPC(s,u,t,v)$. Computing $SPC(s,u,t,v)$ for every quadruple (s,u,t,v) requires solving m shortest path problems to determine $\delta_m(s,u,t)$ for each $m \in \{1, \ldots, t-s+1\}$. For each of these shortest path problems we need to construct a graph, which requires $\mathcal{O}(|T|^5)$ operations. Therefore, the time complexity to compute $\delta_m(s,u,t)$ for a given m equals $\mathcal{O}(|T|^5)$. Thus, the overall time complexity of the dynamic program is $\mathcal{O}(|T|^9)$.

Remark. Suppose that for every period $t \in T$, we have $F_t^C = 0$. Note that under this assumption and Assumption 2, there are no fixed charges at the procurement level, and the convex procurement cost functions are identical in every period. In this case, we can show that there exists an optimal solution where the procurement quantities are equal in all periods with positive procurement within an RI_c. We can also show that if an $RI_c(t,t')$ exists in an optimal solution with positive procurement in period i_1, then an optimal solution exists with positive procurement in each of the periods $i_1+1, i_1+2, \ldots, t'-1$. Because positive procurement must occur in the first period of an RI_c, the previous observations imply that positive procurement must occur in every period of the RI_c and that the procurement level must be the same in each period of the RI_c. Following these results, the special case of problem P(FC) under these more restrictive cost assumptions can be solved using a dynamic program that is designed in a similar manner to the dynamic program discussed in this section. Because positive procurement exists in each period of an RI_c with equal procurement quantities in each period of the RI_c, we need not consider different vectors of positive procurement periods, i.e., we only need to consider one procurement scenario for any subplan (s, u, t, v), and the time complexity of this dynamic program would therefore equal $\mathcal{O}(|T|^6)$.

2.4.2 P(FC) with Nondecreasing Procurement-Related Costs

This section considers a more general set of assumptions on the nature price-supply functions and variable procurement costs than in the prior subsection, at the expense of less generality in the nature of fixed procurement costs. In particular, in this section we assume

Assumption 3. *For every period $t = \{1, \ldots, |T| - 1\}$, we assume $F_t^C \leq F_{t+1}^C$, $\bar{\gamma}_t \leq \bar{\gamma}_{t+1}$, and $\beta_t \geq \beta_{t+1}$.*

Proposition 2.6. *Under Assumption 3, if an $RI_c(t, t')$ exists in an optimal solution with m positive procurement periods, then these periods are the first m periods of $RI_c(t, t')$.*

Proof. Suppose optimal procurement periods within $RI_c(t, t')$ are $t = t_1 \leq t_2 \leq \ldots \leq t_m$ where $t_m > t + m - 1$. Since $t_m > t + m - 1$, there exists a period i within first m periods of the RI with no procurement. We create a new solution by moving the procurement in period t_m, κ_{t_m}, to period i where everything else remains the same. Let Δ be the difference between costs of the original solution and the new solution.

$$\Delta = F_{t_m}^C + \bar{\gamma}_{t_m}\kappa_{t_m} + \frac{\kappa_{t_m}^2}{\beta_{t_m}} - F_i^C - \bar{\gamma}_i\kappa_{t_m} - \frac{\kappa_{t_m}^2}{\beta_i},$$

$$\Delta = F_{t_m}^C - F_i^C + (\bar{\gamma}_{t_m} - \bar{\gamma}_i)\kappa_{t_m} + \left(\frac{1}{\beta_{t_m}} - \frac{1}{\beta_i}\right)\kappa_{t_m}^2.$$

By Assumption 3, $F_{t_m}^C \geq F_i^C$, $\bar{\gamma}_{t_m} \geq \bar{\gamma}_i$, and $\beta_{t_m} \leq \beta_i$, thus $\Delta \geq 0$. If $\Delta > 0$ then this contradicts the optimality of the original solution. And if $\Delta = 0$ then alternative optimal solutions exist. Both cases lead to the desired result. $\square$

Proposition 2.7. *Under Assumption 3, if there exist two consecutive component regeneration intervals $RI_c(s, t)$ and $RI_c(s', t')$ in an optimal solution, where $s \leq t \leq s' \leq t'$, then $t = s'$.*

Proof. Suppose an optimal solution contains two consecutive RI_cs, $RI_c(i_1, i_2)$ and $RI_c(i_3, i_4)$ where $i_1 \leq i_2 < i_3 \leq i_4$, and suppose the number of periods with positive procurement in $RI_c(i_3, i_4)$ is m. Holding all other variables at their original levels, we create a new RI_c, $RI_c(i_2, i_4)$ replacing the original $RI_c(i_3, i_4)$ and moving all procurement to the first m periods of $RI_c(i_2, i_4)$. Feasibility is maintained because procurement is moved to earlier periods. Under Assumption 3, the cost of procurement in earlier periods is at least as low as the cost of procurement in later periods, as discussed by Proposition 2.6. Thus, the new solution has a cost that is less than or equal to the original solution, which implies either an alternative optimal solution or a contradiction of the optimality of the original solution. □

Under Assumptions 1 and 3, the problem can be solved optimally in polynomial time using dynamic programming. We define the subplan (s, u, t, v) as in the previous subsection. The component procurement quantities for this production are distributed among the periods $s, s+1, \ldots, s+m-1$ by Proposition 2.6, given that there are m periods with positive procurement. The procurement quantities κ_j for $j = s, s+1, \ldots, s+m-1$ can be computed using Equations (2–6) and (2–7).

Let $SPC'(s, u, t, v)$ denote the minimum cost of subplan (s, u, t, v), equal to the sum of the fixed and variable production costs and final product holding costs in periods u through v, plus the fixed and variable component procurement and holding costs in periods s through t. To compute $SPC'(s, u, t, v)$, we need to determine the subset of periods from u through v in which production will occur. By the definition of subplan (s, u, t, v), we know that the last production setup period within the subplan must occur in period t, and that the production quantity in period t must equal the demand in periods t through v, i.e., $D_{t,v}$, as a result of the ZIP property from Proposition 2.4. Given the number of periods with positive procurement and demand requirements in each period, we can determine the subset of periods u through v in which production occurs by solving a shortest path problem on a graph created as follows: Let $G'(V, A)$ be a directed graph with vertex set $V = \{u, u+1, \ldots, t-1, t\}$ and arc set $A = \{(i,j) \mid i, j \in V \text{ and } j > i\}$. An arc from i to j implies that a production setup occurs in period i that satisfies all demand in periods i through $j-1$.

A production setup in period i which will satisfy demands from period i to period $j-1$ is only feasible if the component procurement level is sufficient to cover the production in this period. Note that the production amount in period i equals $D_{i,j-1}$, and the sum of all productions up to and including period i equals $D_{u,j-1}$, by the ZIP property. If the total procurement amount up to and including period i is greater than the total production amount up to and including period i, i.e., $\mathcal{K}_{s,i} \geq D_{u,j-1}$, then it is feasible to have a production setup in period i to satisfy demand in periods i through $j-1$. Given that procurement occurs in the first m periods of the subplan for a given value of m, the arc costs, c_{ij}^m, for all $(i,j) \in A$, are computed as follows:

$$c_{ij}^m = \begin{cases} F_i + c_i D_{i,j-1} + \sum_{k=i}^{j-2} h_k D_{k+1,j-1} + \sum_{k=i}^{j-1} h_k^C (\mathcal{K}_{s,k} - D_{u,j-1}), & \text{if } \mathcal{K}_{s,i} \geq D_{u,j-1}, \\ \infty, & \text{otherwise.} \end{cases} \tag{2–15}$$

Let $\delta'_m(s,u,t)$ equal the length of the shortest path from u to t on the directed graph; $\delta'_m(s,u,t)$ corresponds to the minimum cost production plan from period u to period $t-1$, including component and end-item holding costs in these periods, given that component regeneration interval starts at period s and the number of periods with positive procurement equals m. By the construction of the shortest path network, the value of $\delta'_m(s,u,t)$ can be determined in $\mathcal{O}(|T|^2)$ time for any (s,u,t,m) quadruple.

The cost of subplan (s,u,t,v), $SPC'(s,u,t,v)$, is then computed as follows:

$$\begin{aligned} SPC'(s,u,t,v) = \min_{1 \leq m \leq t-s+1} \Bigg\{ & \sum_{i=s}^{s+m-1} \left(F_i^C + \frac{\kappa_i^2}{\beta_i} + \frac{\alpha_i \kappa_i}{\beta_i} + \gamma_i \kappa_i\right) + \sum_{i=s}^{u(s,m)} h_i^C \mathcal{K}_{s,i} \\ & + F_t + c_t D_{tv} + \sum_{i=t}^{v-1} h_i D_{i+1,v} + \delta'_m(s,u,t) \Bigg\}, \end{aligned} \tag{2–16}$$

where $u(s,m) = \min\{u-1, s+m-1\}$.

Let $C'_{s,u}$ denote the minimum cost of satisfying demand from period u until the end of the horizon, given that all component procurement occurs in period s and later, where $s \leq u$. We can compute $C'_{s,u}$ using the following dynamic programming recursion:

$$C'_{s,u} = \min_{t,v\,:\,u \leq t \leq v} \{SPC'(s,u,t,v) + C'_{t+1,v+1}\} \tag{2–17}$$

The boundary conditions for this dynamic program are written as

$$C'_{|T|+1,i} = C'_{i,|T|+1} = 0 \text{ for all } i = 1,\ldots,|T|+1. \tag{2–18}$$

Note that $C'_{1,1}$ gives the cost of an optimal plan for the entire planning horizon. The bottleneck operation for computing $C'_{1,1}$ arises in computing each of the $\mathcal{O}(|T|^4)$ values of $SPC'(s,u,t,v)$, where computing this value requires explicitly considering $\mathcal{O}(|T|)$ values of m in (2–16), as well as $\mathcal{O}(|T|^2)$ time to compute $\delta'_m(s,u,t)$ for each m. Thus, the worst-case complexity of this dynamic program equals $\mathcal{O}(|T|^7)$.

2.5 Computational Test Results

This section presents the results of a set of computational tests intended to assess the performance of the dynamic programs presented in Section 2.4. In Section 2.5.1 we compare the performance of the proposed dynamic programs (coded in C++) with the general purpose nonconvex mixed integer nonlinear programing solver Couenne[1] . For this performance analysis, we considered problems with various planning horizon lengths. The data for each problem instance were randomly generated using a set of uniform distributions; we let $U(a,b)$ denote a given uniform distribution on $[a,b]$, where $a < b$. Table 2-1 shows the a and b values for each parameter, with the exception of the fixed procurement cost (F_t^C) and demand (d_t) values. We created several scenarios using different combinations of distribution parameters for the values of F_t^C and d_t,

[1] Couenne is a software package that uses spatial branch-and-bound to solve non-convex MINLPs. Belotti et al. [8] explain the implementation details, such as bound tightening and branching techniques, developed for Couenne. Also see https://projects.coin-or.org/Couenne.

where all others remain as shown in Table 2-1. Table 2-2 provides the different scenarios considered for the F_t^C and d_t values. Our goals in generating the random data used in the experiments were twofold: (1) the data should lead to problem instances in which optimal solutions are not extreme or trivial, meaning that we do not have a single setup at the beginning of the horizon at either level; and (2) the data should reflect relative costs we might expect to find in practical settings. Achieving (1) required higher settings for procurement fixed costs than we initially expected, as the convexity of procurement-related costs leads to diseconomies of scale in procurement and, thus, more frequent procurement setups.

Table 2-1. Uniform distribution parameters.

Problem parameter description		a	b
F_t	Fixed production cost	500	1000
c_t	Variable production cost	1	5
β_t	Price-vs-procurement amount function slope	1	5
γ_t	Component handling cost	1	5
h_t	End item inventory holding cost	0.001	0.005
h_t^C	Component inventory holding cost	0.0005	0.002
p_t^0	Threshold price (α_t/β_t)	1	5

Table 2-2. Scenarios.

Scenario No.	F_t^C (a, b)	d_t (a, b)
1	(500, 1000)	(10, 100)
2	(500, 1000)	(100, 500)
3	(1000, 5000)	(100, 500)
4	(1000, 5000)	(10, 100)

Section 2.5.2 later discusses the performance of our dynamic program when used as a heuristic method for solving P(FC) without any specific restrictions on cost structures. All numerical studies were performed on a computer with 64-bit Windows operating system, a 2.67GHz Intel® Core™ 2 CPU, and 4 GB RAM.

2.5.1 Performance of the Dynamic Program

In this section we benchmark the performance of the dynamic programs proposed in Section 2.4 versus the nonconvex mixed integer nonlinear programming (MINLP) solver

Couenne. First we analyze the case with uniform price-supply functions and general fixed procurement charges, as discussed in Section 2.4.1 (for these cases, we randomly generate a single value of each of the parameters $\beta_t = \beta$, $p_t^0 = p^0$, and $\gamma_t = \gamma$). The time complexity of the dynamic program to solve this case is $\mathcal{O}(|T|^9)$ where $|T|$ is the planning horizon length. Recall that the bottleneck operation requires determining the cost of a subplan (s, u, t, v) where a shortest path problem is solved for each possible number of procurement setups, m, for $1 \leq m \leq t - s + 1$. For each subplan (s, u, t, v), we perform a preprocessing step in which we compute a lower bound and an upper bound on $SPC(s, u, t, v)$ for each m and eliminate those that cannot be part of an optimal solution. For the upper bound computation, we require the m procurement setups to occur in the first m periods of the $RI_c(s, t + 1)$, and then solve the lot-sizing problem at the production level. Note that if this solution is not feasible, then no feasible solution exists for subplan (s, u, t, v) with m setups. To compute a lower bound, we assume that the m procurement setups will take place in the m periods with the cheapest procurement setup costs (without considering infeasibility), and solve the uncapacitated lot-sizing problem at the production level, assuming an infinite supply of components. We then eliminate any m value from consideration that produces a higher lower bound than the minimum upper bound obtained in this process. Clearly, these eliminated cases cannot be part of an optimal solution. This preprocessing step significantly reduces the number of shortest path problems solved for each $SPC(s, u, t, v)$.

Table 2-3 compares the performance of the nonconvex MINLP solver Couenne versus the dynamic program proposed in Section 2.4.1. The values (in seconds) are the averages over 40 problem instances (10 instances for each scenario shown in Table 2-2). Here we set the time limit parameter for the solver to 3,600 seconds. We observed that for some of the problem instances, the solver stopped before finding an optimal solution because it reached the time limit (the number of instances for which this occurred is noted in the superscript in the second column of the table).

Table 2-3. Run time comparison: Uniform price-supply function case.

	Average run time		Standard dev. of run time	
$\|T\|$	Couenne	DP	Couenne	DP
4	0.326	<0.001	0.139	<0.001
5	0.534	<0.001	0.210	<0.001
6	0.975	0.001	0.406	0.004
7	1.524	0.002	0.718	0.006
8	2.231	0.002	1.485	0.006
9	3.723	0.005	2.421	0.008
10	6.988	0.012	5.381	0.010
11	11.533	0.020	11.067	0.012
12	27.391	0.032	26.885	0.020
13	51.913	0.061	64.900	0.027
14	151.608	0.082	178.799	0.038
15	329.919	0.135	413.806	0.062
16	1057.167$^{(5)}$	0.224	1386.404	0.114
17	1567.080$^{(10)}$	0.325	1580.380	0.156
18	1955.395$^{(12)}$	0.527	1603.575	0.261
19	1983.928$^{(13)}$	0.640	1660.719	0.349
20	2198.771$^{(9)}$	0.895	1594.985	0.495

$^{(*)}$ Superscripts show the number of instances in which Couenne reached the time limit.

Clearly, the dynamic program outperforms the general purpose MINLP solver, because it utilizes the structural characteristics of an optimal solution for this problem class. As the problem size grows, the computation time increases drastically for the MINLP solver, as does as the variation in total solution time. The average and standard deviation of run times remain much lower when using the dynamic programming method.

Here we note that another bottleneck operation in the dynamic programming algorithm is the generation of the shortest path graph to compute the cost of a subplan (s, u, t, v) for a given value of m (number of procurement setups in the subplan). The number of arcs of this graph is bounded by $\mathcal{O}(|T|^4)$, where we need $\mathcal{O}(|T|)$ operations to compute each arc's cost. Our numerical analysis shows that the average size is much smaller (see Table 2-4), and that the worst-case bounds on the number of vertices and arcs are quite loose. This, in fact, implies that our dynamic program has an average-case performance that is much better than the worst-case performance bound.

Table 2-4. Size of the set of vertices, V and arcs, A.

$\|T\|$	Average $\|V\|$	Maximum $\|V\|$	Average $\|A\|$	Maximum $\|A\|$
4	3.69	7	2.75	9
5	4.06	10	3.15	12
6	4.32	13	3.63	22
7	4.56	17	4.07	36
8	4.83	21	4.55	43
9	5.11	26	5.24	74
10	5.33	31	5.81	91
11	5.61	37	6.31	111
12	5.78	43	7.02	177
13	6.17	50	7.89	179
14	6.33	57	8.82	214
15	6.83	65	10.15	338
16	6.82	73	10.72	342
17	7.17	82	11.68	393
18	7.37	91	12.21	571
19	7.84	101	14.11	471
20	7.78	111	14.56	678

Table 2-5 shows similar results for cases with nondecreasing procurement related costs, as discussed in Section 2.4.2 (to generate these problem instances, we sort the $|T|$ randomly generated values of F_t^C, β_t, α_t, and γ_t; if $\bar{\gamma}_t > \bar{\gamma}_{t+1}$, we increase $\bar{\gamma}_{t+1}$ until $\bar{\gamma}_t = \bar{\gamma}_{t+1}$). Here again the values (in seconds) are the averages over 40 problem instances (10 instances for each scenario shown in Table 2-2; we did not go beyond $|T| = 18$ periods for this case, as the results and trends were very similar to the uniform procurement cost case, and the run times tend to be very high for Couenne).

As noted at the beginning of this section, the number of setups that occur in an optimal solution is typically unusually high when compared to problems in the absence of convex procurement costs. For each of the scenarios shown in Table 2-2, Table 2-6 shows the average fraction of periods in which procurement and production setups occurred in an optimal solution, for both the uniform procurement cost and nondecreasing procurement cost cases. Recall that Scenarios 3 and 4 correspond to higher fixed procurement costs, while Scenarios 2 and 3 correspond to higher demand levels. As a simple benchmark, under stationary demand, even with our lowest setting of fixed cost (500), highest level

Table 2-5. Run time comparison: Nondecreasing procurement related cost case.

	Average run time		Standard dev. of run time	
$\|T\|$	Couenne	DP	Couenne	DP
4	0.293	<0.001	0.146	<0.001
5	0.499	<0.001	0.213	0.002
6	0.837	<0.001	0.346	<0.001
7	1.351	0.001	0.513	0.003
8	1.876	0.001	0.745	0.004
9	2.688	0.002	1.046	0.005
10	4.522	0.003	2.360	0.006
11	7.017	0.006	4.229	0.008
12	11.155	0.008	6.596	0.008
13	21.697	0.011	15.668	0.007
14	45.090	0.017	41.735	0.004
15	96.253	0.024	90.841	0.008
16	177.110	0.035	190.259	0.007
17	454.353	0.048	501.630	0.005
18	1059.758	0.067	1373.065	0.007

of demand per period (500) and highest level of holding cost per period (0.005), we would not setup more frequently than once every 20 periods (using the EOQ formula, $T = \sqrt{2F/hD}$). Table 2-6 shows that an optimal solution sets up much more frequently than this (in some cases, we observe procurement in every period). These diseconomies of scale in procurement also lead to more frequent production setups (as a result of limiting the amount of procurement in any period), despite the absence of capacity limits.

Table 2-6. Average fraction of periods in which a setup occurs.

	Uniform costs		Nondecreasing costs	
Scenario No.	Production	Procurement	Production	Procurement
1	0.32	0.62	0.26	0.52
2	0.86	1.00	0.74	1.00
3	0.86	0.99	0.71	0.98
4	0.30	0.39	0.21	0.34

2.5.2 Performance as a Heuristic Solution Method

In this section we test the performance of the dynamic program presented in Section 2.4.1 when used as a heuristic method for problem instances that violate the cost assumptions in Section 2.4.1. The idea of this heuristic approach is to impose a requirement on the solution to procure equal amounts in all periods with positive procurement within a subplan. In addition, the heuristic will require production plans to satisfy the ZIP property, even when the cost structure does not imply that this property must hold in an optimal solution.

The heuristic solution method works as follows. We compute the cost of subplan (s, u, t, v), i.e., $SPC(s, u, t, v)$, as described in Section 2.4.1 with a few modifications. Given the number of periods with positive procurement, m, we create the shortest path graph $G(V, A)$, where the set of vertices V and arcs A are generated the same way as described before. The cost of arcs in the sets A_1 and A_2 is computed differently, in order to account for the time-varying procurement cost parameters. Here again, we let κ be the amount to be procured given m; hence, $\kappa = D_{u,v}/m$. Equations (2–19) and (2–20) present the arc cost computations for the arcs in sets A_1 and A_2, respectively.

$$c_a^m = \sum_{j=1}^{l} \left(F_{[j]}^C + \gamma_{[j]}\kappa + \frac{\kappa^2}{\beta_{[j]}} + \frac{\alpha_{[j]}}{\beta_{[j]}}\kappa \right) + \sum_{k=s}^{u-1} h_k^C \mathcal{K}_{s,k}, \quad \forall a \in A_1. \tag{2–19}$$

$$c_a^m = \begin{cases} \infty, & \text{if } D_{u,j-1} > \mathcal{K}_{s,i}, \\ C'(i^l, j^k) + P(i^l, j^k), & \text{otherwise}, \end{cases} \quad , \quad \forall a \in A_2, \tag{2–20}$$

where

$$C'(i^l, j^k) = \sum_{t=1}^{k-l} \left(F_{[t]}^C + \gamma_{[t]}\kappa + \frac{\kappa^2}{\beta_{[t]}} + \frac{\alpha_{[t]}}{\beta_{[t]}}\kappa \right) + \sum_{t=i}^{j-1} h_t^C (\mathcal{K}_{s,t} - D_{i,j-1}), \tag{2–21}$$

and $P(i^l, j^k)$ is computed as in equation (2–11). Here $[j]$ returns the index of the period with the j^{th} cheapest fixed procurement cost. The computation of $SPC(s, u, t, v)$ and formulation of the dynamic program follow as in Section 2.4.1.

We performed two sets of tests to benchmark the performance of the dynamic programming algorithm as a heuristic method. In the first set, we created instances for which Assumption 1 and, therefore, the ZIP property, holds with a general cost structure for remaining problem parameters. In the second set, we created instances with general costs structures, i.e., none of the problem parameters are required to follow any of the specialized cost assumptions stated in Section 2.4.

Table 2-7 demonstrates the performance of the dynamic program as a heuristic, where the optimal solution was obtained in each case by allowing Couenne to run until an optimal solution was obtained. The average optimality gap achieved by the heuristic was reasonably small, even for the problem instances where the cost structure does not guarantee the optimality of a ZIP solution. The maximum optimality gap did not exceed 5.5% for instances where a ZIP solution is not guaranteed to be among the optimal solutions, and it was smaller than 3.9% for instances where a ZIP solution is optimal. This clearly shows that the dynamic program can be used as an effective heuristic approach to solve general instances of P(FC). Moreover, this analysis indicates that, on average, the cost penalty for imposing a ZIP requirement on production and equal procurement batch sizes with component regeneration intervals is generally small.

Table 2-7. Performance of the dynamic program as a heuristic.

	with ZIP		without ZIP	
$\lvert T\rvert$	Average %Gap	Standard dev.	Average %Gap	Standard dev.
4	0.623	0.859	0.930	1.287
6	1.026	1.042	1.192	1.150
8	1.023	0.877	1.159	0.695
10	0.871	0.610	1.425	1.050
12	1.111	0.760	1.268	0.711
14	1.286	0.744	1.357	0.538
16	1.389	0.815	1.402	0.737

CHAPTER 3
A POLYNOMIAL TIME ALGORITHM FOR CONVEX COST LOT SIZING PROBLEMS

3.1 Model and Related Work

We consider the classic discrete-time, finite-horizon economic lot-sizing problem with nondecreasing and convex costs in the production quantities and inventory levels in each period. This problem considers a set of consecutive demand periods and seeks to meet deterministic demand for a product in each period $t = 1, 2, \ldots, T$ without lost sales or backlogging, at a minimum total cost over the planning horizon of length T. The total cost incurred over the horizon consists of those costs associated with producing the product in any period as well as the cost of holding inventory of the product between periods. At the beginning of each period, the production quantity in the period is added to remaining inventory from the prior period with zero lead time, and the sum of these two quantities must be at least as great as the demand in the period (with the difference comprising the amount of inventory that will remain at the end of the period). Because all demands and costs are assumed deterministic, all production decisions may be made in advance of the first planning period, and the model can in principle accommodate any finite, deterministic production planning lead time. We assume throughout that no capacity limit exists on the production quantity or the inventory held in any period.

Of particular importance in approaching the solution of problems in this class are the nature and structure of the production and inventory holding costs. Because a wide range of practical settings exist with economies of scale in production, the cost of production in a period is often modeled as a nondecreasing and concave function of the production quantity, while conventional approaches typically treat the cost of holding inventory as a linear function of the inventory amount at the end of the period. This leads to a total cost function that may be expressed as the sum of concave functions of the production quantities, implying a concave total cost function. Conversely, production processes and inventory costs with diseconomies of scale may often be modeled using convex functions of the production quantities and inventory levels. This chapter considers this latter class of problems with nondecreasing convex cost functions.

Our main contribution lies in providing a polynomial-time algorithm for the lot-sizing problem we have described with general nondecreasing convex production and holding cost functions. We discuss two ways to implement our approach. The first is an iterative numerical approach that runs in $\mathcal{O}(T^2 \max\{\log T, \log \mathcal{S} \log \mathcal{M}/\epsilon\})$ time. Here T denotes the number of time periods and $\mathcal{S}$ provides an upper bound on the number of non-differentiable points of the cost to supply demand in period s using production in period t for any (t, s) pair with $1 \leq t \leq s \leq T$. The value of $\mathcal{M}$ is an upper bound on the marginal production cost in any period, and ϵ denotes a stopping criterion for a bisection search routine in the proposed algorithm. As we will see in Section 3.2, for the case in which all of the cost functions are piecewise linear and convex, this iterative approach requires $\mathcal{O}(T^2 \max\{\log T, (\log \mathcal{S})^2\})$ time. The second implementation we provide requires repeated solution of a system of equations with at most $T + 1$ equations and $T + 1$ variables. Assuming this system of equations can be solved in $\mathcal{O}(\phi(T))$ time, where $\phi(T)$ is a polynomial function of T, this solution approach requires $\mathcal{O}(T \max\{T \log T, \phi(T)\})$ time. When all costs are differentiable and quadratic, then $\mathcal{O}(\phi(T)) = \mathcal{O}(T)$, and this expression becomes $\mathcal{O}(T^2 \log T)$. To the best of our knowledge, the literature does not contain a special-purpose algorithm for this general problem class that runs in polynomial time in the worst case. Although the resulting problem is a convex optimization problem, the application of a general purpose nonlinear programming solver to problems with special structure may result in unnecessarily long solution times, and the associated running time would be weakly polynomial, regardless of the structure of the cost functions. The algorithm we provide is based on a primal-dual solution approach derived from analyzing the special structure of the problem's generalized Karush Kuhn-Tucker (KKT) conditions, which are necessary and sufficient for optimality. We specialize the resulting algorithm for application to a production planning context involving price-dependent supply components, where component supply is linearly increasing in price. The resulting problem is a special case of the general lot-sizing problem with nondecreasing and convex production costs, where the production costs take a quadratic form and the associated complexity is $\mathcal{O}(T^2 \log T)$.

Veinott [86] considered a single-stage, dynamic lot sizing problem in which production and inventory costs are piecewise-linear and convex. He designed a parametric-programming-based procedure in which the solution for a problem with a fixed parameter set is built upon the solution of another problem with a similar parameter set. Veinott [86] assumed all parameters were integer valued, and his procedure resulted in a pseudo-polynomial solution algorithm. The time complexity the algorithm is $\mathcal{O}(TD)$, where $D = \sum_{t=1}^{T} d_t$ denotes the sum of all demands over the planning horizon. Florian et al. [22] mentioned Veinott's procedure as the most attractive approach to solve lot sizing problems with convex production and inventory costs and without fixed setup costs, even though the problem is demonstrated to be no harder than linear programming, which is polynomially solvable.

The work by Kian et al. [41] is also closely related to ours, as they analyzed a single-stage, uncapacitated economic lot-sizing problem with fixed setup costs and variable costs in each period that are convex in the production quantity (taking the form of a polynomial function of the production quantity). They derived several key optimality conditions for this problem class, as well as a dynamic programming solution algorithm that is exponential in the length of the time horizon. They presented an exact solution algorithm for the lot-sizing problem with zero setup costs and production costs taking the form of polynomial convex functions, and stated that the worst-case time complexity of such an algorithm would be $\mathcal{O}(T^2)$. In Section 3.3, we will explain why this bound applies to the number of subplans that must be considered, and not to the problem's overall worst-case complexity.

The problem we consider falls into the class of the single-stage, dynamic lot-sizing problems, which has been studied by many researchers, starting with the seminal work of Wagner and Whitin [88], who first considered the problem with concave production costs. Brahimi et al. [9] provided an extensive review of uncapacitated and capacitated versions of the single-stage, dynamic lot-sizing problem. The uncapacitated version of the classical single-stage, dynamic lot-sizing problem with concave costs is polynomially solvable, whereas the general capacitated version of the problem with concave costs is $\mathcal{NP}$-Hard, although the uniform-capacity version can be solved in polynomial time via dynamic programming (see Florian and Klein [21]). Many studies have considered variations of the single-stage, dynamic lot-sizing problem (see e.g. Van Hoesel and Wagelmans [84], and Van den Heuvel and Wagelmans [83]). In these studies, all costs are assumed to be concave in the production and inventory levels.

The literature on lot-sizing problems containing convex production costs is reasonably sparse, with a few notable exceptions. Erenguc and Aksoy [18] considered a single-item, capacitated dynamic lot sizing problem with fixed production setup costs and linear inventory costs, while variable production costs were piecewise linear and convex in the production quantity in a period. They used a branch-and-bound algorithm for this problem, which contains neither a convex nor concave objective function. Shaw and Wagelmans [74] developed a pseudo-polynomial dynamic program to solve a capacitated single-item lot-sizing problem with piecewise linear production costs. Their algorithm can be utilized to solve problems with piecewise-linear and convex production costs, although it does not require any special structure for the piecewise-linear cost function. Feng et al. [19] developed an $\mathcal{O}(T \log T)$ algorithm for the single-item lot-sizing problem with constant capacity, convex inventory costs, and non-increasing fixed order costs.

The rest of this chapter is organized as follows. Section 3.2 provides a general model formulation, description of the Karush Kuhn-Tucker (KKT) conditions for the problem, and development of a polynomial-time algorithm, along with a comparison of this approach to the one provided by Veinott in [86]. Section 3.3 discusses the production planning problem with price-dependent supply, and illustrates the application of the model to a practical special case that falls within the general problem class we consider. Section ?? contains brief concluding remarks.

3.2 Problem Formulation and Solution Method

The lot-sizing problem requires meeting a set of demands d_t, for $t = 1, \ldots, T$ without shortages at a minimum total production and inventory holding cost over the horizon of length T. Letting x_t denote the production quantity in period t, it will be convenient and useful to keep track of the amount produced in period t in order to meet demand in period τ, $x_{t\tau}$, where $x_t = \sum_{\tau=t}^{T} x_{t\tau}$ for $t = 1, \ldots, T$. We assume that the cost to produce x_t units in period t is a nondecreasing convex function $f_t(x_t)$. In addition, $h_t(i_t)$ denotes a nondecreasing and convex inventory holding cost function, which depends on the inventory at the end of period t, denoted by i_t. The inventory remaining at the end of period t can be equivalently written as $i_t = \sum_{\tau=1}^{t} \sum_{i=\tau}^{T} x_{\tau i} - \sum_{\tau=1}^{t} d_\tau$; thus, we can alternatively write h_t as a function of the production variables using the expression $h_t\left(\sum_{\tau=1}^{t} \sum_{i=\tau}^{T} x_{\tau i}\right)$, where we have suppressed the dependence of h_t on cumulative demand up to period t for notational convenience. We assume that each of the functions f_t and h_t is everywhere locally Lipschitz continuous. The convex cost lot-sizing problem may then be formulated as follows.

$$\text{P:} \quad \text{Minimize} \quad \sum_{t=1}^{T} \left\{ f_t\left(\sum_{\tau=t}^{T} x_{t\tau}\right) + h_t\left(\sum_{\tau=1}^{t} \sum_{i=\tau}^{T} x_{\tau i}\right) \right\} \tag{3–1}$$

$$\text{Subject to:} \quad \sum_{t=1}^{\tau} x_{t\tau} = d_\tau, \quad \tau = 1, \ldots, T, \tag{3–2}$$

$$x_{t\tau} \geq 0, \quad \tau = 1, \ldots, T, \text{ and } t \leq \tau. \tag{3–3}$$

As the sum of convex functions, the objective function of problem P is convex; this combined with the linear constraint set implies that the generalized KKT conditions are necessary and sufficient for optimality (see Hiriart-Urruty [36]).

3.2.1 Generalized KKT Conditions

To characterize the generalized KKT conditions, let $\partial f_t(X^t)$ denote the generalized gradient of f_t at X^t, where $X^t = (x_{tt}, x_{tt+1}, \ldots, x_{tT})$. If f_t is differentiable at X^t, then $\partial f_t(X^t)$ consists of a singleton equal to the partial derivative with respect to any element of X^t at X^t; otherwise, $\partial f_t(X^t)$ corresponds to the set of subgradients at X^t. Similarly, let $\bar{X}^t = (X^1, X^2, \ldots, X^t)$, and let $\partial h_t(\bar{X}^t)$ denote the generalized gradient of h_t at $\bar{X}^t$. Note that the sum over all variables in the set $\bar{X}^t$ gives the cumulative production through the end of period t. We also note that a given variable $x_{t\tau}$ ($\tau = 1, \ldots, T$ and $t \leq \tau$) appears in the argument of the function f_t and in the argument of each function h_i for $i \geq t$.

Let μ and $\mathbf{v}$ denote vectors of KKT multipliers associated with constraints (3–2) and (3–3), respectively. The following provide the generalized KKT conditions for problem P, separated into dual feasibility and complementary slackness conditions.

- Dual feasibility conditions:

$$0 \in \partial f_t(X^t) + \sum_{i=t}^{T} \partial h_i(\bar{X}^i) - \mu_\tau - v_{t\tau}, \quad \tau = 1, \ldots, T \text{ and } t \leq \tau, \tag{3–4}$$

$$v_{t\tau} \geq 0 \quad \tau = 1, \ldots, T \text{ and } t \leq \tau. \tag{3–5}$$

- Complementary slackness conditions:

$$v_{t\tau} x_{t\tau} = 0 \quad \tau = 1, \ldots, T \text{ and } t \leq \tau. \tag{3–6}$$

3.2.2 Solution Approach

Let P_{s-1} denote the subproblem obtained by eliminating periods $s, \ldots, T$ from problem P (for some positive $s \leq T$), and suppose we have an optimal solution for P_{s-1}. Let $x_{t\tau}^{s-1}$ denote the value of $x_{t\tau}$ in this optimal solution for $\tau = 1, \ldots, s-1$ and $t \leq \tau$, and let μ_τ^{s-1} and $v_{t\tau}^{s-1}$ denote the corresponding optimal KKT multipliers. Because the generalized KKT conditions are necessary and sufficient, (3–2) through (3–6) must hold for this solution. Next, consider problem P_s, and let $x_{t\tau}^s = x_{t\tau}^{s-1}$, $v_{t\tau}^s = v_{t\tau}^{s-1}$, and $\mu_\tau^s = \mu_\tau^{s-1}$ for $\tau = 1, \ldots, s-1$ and $t \leq \tau$. We will create a solution for problem P_s by determining values of x_{ts}^s and v_{ts}^s for $t = 1, \ldots, s$, as well as μ_s^s, while leaving all remaining primal variables unchanged. We do this by first initializing each of these new variables at zero, and then determining the optimal values of these new variables. Observe that by condition (3–6), if $x_{ts}^s > 0$ for some $t \leq s$, then we must have $v_{ts}^s = 0$. If we let $\mathbb{T}$ denote the set of periods t from 1 through s such that $x_{ts}^s > 0$, then we must have

$$\mu_s^s \in \partial f_t(X^t) + \textstyle\sum_{i=t}^s \partial h_i(\bar{X}^i), \qquad \forall t \in \mathbb{T}, \tag{3–7}$$

at a generalized KKT point for problem P_s. Our goal is to find a value of μ_s^s and a corresponding set $\mathbb{T}$, with associated x_{ts}^s values, such that (3–7) holds and $\sum_{t \in \mathbb{T}} x_{ts}^s = d_s$. We first characterize a key property of the relationship between μ_s^s and x_{ts}^s in (3–7). To characterize this property, let us define $X_{-s}^t = (x_{tt}^s, x_{t,t+1}^s, \ldots, x_{t,s-1}^s)$, and $\bar{X}_{-(t,s)}^i = \bar{X}^i \backslash \{x_{ts}^s\}$.

Property 3.1. *For any $t \in \mathbb{T}$ and for any fixed X_{-s}^t and a fixed $\bar{X}_{-(t,s)}^i$, $i = t, \ldots, s$, the minimum value of x_{ts}^s that satisfies (3–7) is monotonically increasing in μ_s^s.*

Property 3.1 follows from the convexity of f_t and each h_t. We can replace the word "minimum" in this property with "maximum" and the property continues to hold (if f_t and all h_t functions are strictly convex, then the word "minimum" may be removed from the statement; the "minimum" or "maximum" qualifier accounts for the possibility of linear segments of the function, wherein a given value of μ_s^s does not map to a unique value of x_{ts}^s in (3–7)).

For clarity, we write $\partial f_t(X^t)$ for a fixed $X_{-s}^t = (x_{tt}^s, x_{t,t+1}^s, \ldots, x_{t,s-1}^s)$ as $\partial f_t(X_{-s}^t, x_{ts}^s)$, where $\partial f_t(X_{-s}^t, 0)$ corresponds to the generalized gradient at $x_{ts}^s = 0$. Similarly, for a fixed $\bar{X}_{-(t,s)}^i$, we write $\partial h_t(\bar{X}_{-(t,s)}^i, x_{ts}^s)$. We also use the notation $\partial^l f_t$ and $\partial^l h_t$ ($\partial^u f_t$ and $\partial^u h_t$) to denote the minimum (maximum) subgradient value of the production and holding cost function in period t at a point.

Property 3.2. *Suppose μ_s^s corresponds to an optimal multiplier at a generalized KKT solution. If $\mu_s^s < \partial^l f_t(X_{-s}^t, 0) + \sum_{i=t}^s \partial^l h_i(\bar{X}_{-(t,s)}^i, 0)$ for some $t \leq s$, then the corresponding generalized KKT point must have $v_{ts}^s > 0$ and $x_{ts}^s = 0$.*

Property 3.2 follows from (3–4) through (3–6), the monotonicity property 3.1, and the nonnegativity of x_{ts}^s. Note also that if $\mu_s^s \in \partial^l f_t(X_{-s}^t, 0) + \sum_{i=t}^s \partial^l h_i(\bar{X}_{-(t,s)}^i, 0)$ for some $t \leq s$, then we may set $x_{ts}^s = v_{ts}^s = 0$ to obtain a corresponding KKT point. If $\mu_s^s > \partial^u f_t(X_{-s}^t, 0) + \sum_{i=t}^s \partial^u h_i(\bar{X}_{-(t,s)}^i, 0)$ for some $t \leq s$, however, then we can satisfy condition (3–7) by increasing x_{ts}^s. Assume that for a given μ_s^s, we can determine an interval for x_{ts}^s such that $\mu_s^s \in \partial f_t(X_{-s}^t, x_{ts}^s) + \sum_{i=t}^s \partial h_i(\bar{X}_{-(t,s)}^i, x_{ts}^s)$, i.e., such that (3–7) holds. Let $[l_{ts}(\mu_s^s), u_{ts}(\mu_s^s)]$ denote the corresponding interval. Next, define $\mathbb{T}(\mu_s^s)$ as the set of periods t such that $\mu_s^s > \partial f_t^u(X_{-s}^t, 0) + \sum_{i=t}^s \partial h_i^u(\bar{X}_{-(t,s)}^i, 0)$. In order to find a generalized KKT point, we seek a feasible solution to the following system:

$$x_{ts}^s \in [l_{ts}(\mu_s^s), u_{ts}(\mu_s^s)], \quad t \in \mathbb{T}(\mu_s^s), \tag{3–8}$$

$$\textstyle\sum_{t \in \mathbb{T}(\mu_s^s)} x_{ts}^s = d_s. \tag{3–9}$$

The above system defines a polyhedron, and we can determine whether a feasible solution exists by evaluating the left-hand side of (3–9) at $x_{ts}^s = l_{ts}(\mu_s^s)$ for each $t \in \mathbb{T}(\mu_s^s)$ and at $x_{ts}^s = u_{ts}(\mu_s^s)$ for each $t \in \mathbb{T}(\mu_s^s)$. If the former value is less than or equal to d_s and the latter value is greater than or equal to d_s, then a feasible solution exists and can be constructed in $\mathcal{O}(T)$ time. If the left-hand side of (3–9) evaluated at $x_{ts}^s = u_{ts}(\mu_s^s)$ is strictly less than d_s, then the given value of μ_s^s cannot correspond to an optimal KKT multiplier; in this case, by Property 3.1, the optimal value of μ_s for problem P_s must exceed the chosen value of μ_s^s. On the other hand, if the left-hand side of (3–9) evaluated at $x_{ts}^s = l_{ts}(\mu_s^s)$ is strictly greater than d_s, then the given value of μ_s^s cannot correspond to an optimal KKT multiplier; in this case, by Property 3.1, the optimal value of μ_s for problem P_s must be less than the chosen value of μ_s^s. As a result of the monotonicity property (3.1) and the continuity of each f_t and h_t, we can perform a bisection search on

μ_s in order to find a value of μ_s^s such that a feasible solution exists for (3–8) and (3–9) to within a chosen tolerance $\epsilon > 0$ (note that each μ_s^s evaluated in the bisection search requires explicitly defining the set $\mathbb{T}(\mu_s^s)$). We call such a solution an ϵ-approximate generalized KKT point. The resulting solution determines μ_s^s and x_{ts}^s for $t \in \mathbb{T}(\mu_s^s)$. After finding a solution to (3–8) and (3–9) via bisection search on μ_s, for each $t \notin \mathbb{T}(\mu_s^s)$ we set v_{ts}^s equal to an element of $\partial f_t(X^t) + \sum_{i=t}^s \partial h_i(\bar{X}^i)$.

Proposition 3.1. *The solution obtained after performing a bisection search on μ_s as described provides an ϵ-approximate generalized KKT point for conditions (3–2) through (3–6) for $\tau = s$ and $t \leq s$.*

Proof. The solution is primal feasible with a tolerance of ϵ as a result of satisfying (3–9) to within ϵ for $\tau = s$ and all $t \leq s$ (and the fact that no primal variables $x_{t\tau}$ were changed for $\tau < s$). By construction, (3–4) through (3–6) hold for all $\tau = s$ and $t \leq \tau$ as well (within ϵ), after setting $v_{ts}^s \in \partial f_t(X^t) + \sum_{i=t}^s \partial h_i(\bar{X}^i)$ for any t such that $x_{ts}^s = 0$. By the optimality of the initial solution for P_{s-1}, this initial solution satisfied the generalized KKT conditions for problem P_{s-1}. Increasing the production quantity in some period $t < s$, however, may result in a violation of (3–4) for some (t, τ) pair with $t \leq \tau < s$, as a result of $\partial f_t(X^t)$ having increased due to increased production in period t, as well as an increase in $\partial h_i(\bar{X}^i)$ in periods from t through s. Note that this implies $x_{ts}^s > 0$. We consider two cases.

Case I: $x_{t\tau}^s = 0$. In this case, an increase in $v_{t\tau}^s$ can absorb any increase in $\partial f_t(X^t) + \sum_{i=t}^s \partial h_i(\bar{X}^i)$.

Case II: $x_{t\tau}^s > 0$. If $x_{t\tau}^s > 0$, then suppose we increase μ_τ^s to $\partial f_t^l(X^t) + \sum_{i=t}^s \partial h_i^l(\bar{X}^i)$; note that $\mu_\tau^s \leq \mu_s^s \in \partial f_t(X^t) + \sum_{i=t}^s \partial h_i(\bar{X}^i)$. Suppose this increase in μ_τ^s causes a violation in a constraint of the form (3–4) for some period $r \leq \tau$, which may only occur if $x_{rs}^s = 0$ (otherwise, (3–7) would have to hold at $t = r$). This constraint violation implies that $\mu_\tau^s + v_{r\tau}^s > \partial f_r^u(X^r) + \sum_{i=r}^s \partial h_i^u(\bar{X}^i)$. If $x_{r\tau}^s > 0$ then this implies $v_{r\tau}^s = 0$ and $\mu_\tau^s > \partial f_r^u(X^r) + \sum_{i=r}^s \partial h_i^u(\bar{X}^i)$; however, because $\mu_s^s \geq \mu_\tau^s$, this implies $\mu_s^s >$

$\partial f_r^u(X^r) + \sum_{i=r}^s \partial h_i^u(\bar{X}^i)$, which would have led to $r \in \mathbb{T}(\mu_s^s)$, which in turn would have led to a solution in which $\mu_s^s \in \partial f_r(X^r) + \sum_{i=r}^s \partial h_i(\bar{X}^i)$, creating a contradiction. Therefore, it must be the case that $v_{r\tau}^s > 0$ and $x_{r\tau}^s = 0$. In this case, we can reduce $v_{r\tau}$ until the constraint is satisfied or until it hits zero. If the latter occurs first, then we are back at the case in which $\mu_s^s \geq \mu_\tau^s > \partial f_r^u(X^r) + \sum_{i=r}^s \partial h_i^u(\bar{X}^i)$, leading to a contradiction.

As a result, we have constructed a solution that satisfies the generalized KKT conditions for problem P_s to within ϵ, and therefore, an ϵ-approximate generalized KKT solution. □

It is a simple matter to solve P_1, by setting $x_{11}^1 = d_1$, $\mu_1^1 = \partial f_1(d_1)$, and $v_{11}^1 = 0$. Algorithm 3.1 then solves P_s for $s = 2, \ldots, T$.

Algorithm 3.1. *Given optimal solution for P_{s-1}, solve P_s.*

Input: Optimal solution for P_{s-1}: $x_{t\tau}^{s-1}$ for $\tau = 1, \ldots, s-1$ and $t \leq \tau$.
Output: Solution for P_s: $x_{t\tau}^s$ for $\tau = 1, \ldots, s$ and $t \leq \tau$.
Initialization: $i = 0$, $x_{t\tau}^s = x_{t\tau}^{s-1}$ for $\tau = 1, \ldots, s-1$ and $t \leq \tau$; $x_{ts}^s = 0$ for $t = 1, \ldots, s$.
Sort $\partial f_t^u(X_{-s}^t, 0) + \sum_{i=t}^s \partial h_i^u(\bar{X}_{-(t,s)}^i, 0)$ values in nondecreasing order, $t = 1, \ldots, s$.
while $x_{ts}^s = 0$ for all $t = 1, \ldots, s$ **do** Bisection search on μ_s^s
    Given μ_s^s, let $\mathbb{T}(\mu_s^s) = \{t \leq s | \mu_s^s > \partial f_t^u(X_{-s}^t, 0) + \sum_{i=t}^s \partial h_i^u(\bar{X}_{-(t,s)}^i, 0)\}$.
    For $t \in \mathbb{T}(\mu_s^s)$ determine interval $[l_{ts}(\mu_s^s), u_{ts}(\mu_s^s)]$ using binary search.
    Compute $\sum_{t \in \mathbb{T}(\mu_s^s)} l_{ts}(\mu_s^s) = L$ and $\sum_{t \in \mathbb{T}(\mu_s^s)} u_{ts}(\mu_s^s) = U$.
    if $L \leq d_s \leq U$ **then**
        Find ϵ-feasible solution to $x_{ts}^s \in [l_{ts}(\mu_s^s), u_{ts}(\mu_s^s)], t \in \mathbb{T}(\mu_s^s); \sum_{t=1}^s x_{ts}^s = d_s$.
    else
        Adjust μ_s^s value in bisection search and continue.
    end if
end while

The following proposition characterizes the worst-case complexity of Algorithm 3.1. In stating this proposition, we let $\mathcal{S}$ denote a bound on the maximum number of points of non-differentiability of any of the functions of the form $f_t + \sum_{i=t}^{s} h_i$, where we have suppressed the dependence of these functions on the $x_{t\tau}$ variables for notational convenience.

Proposition 3.2. *Algorithm 3.1 requires* $\mathcal{O}(T^2 \max\{\log T, \log \mathcal{S} \log \mathcal{M}/\epsilon\})$ *time.*

Proof. Characterizing the worst-case complexity of Algorithm 3.1 requires some assumptions on the characteristics of the functions contained in the objective function. We require the characterization of $\mathcal{O}(T^2)$ functions of the form $F_t \equiv f_t + \sum_{i=t}^{s} h_i$. To this end, suppose the function F_t is composed of S_t segments, such that the function is differentiable on the interior of each segment, the union of the segments equals $\mathbb{R}_+$, and no two segments intersect (the endpoints of each segment correspond to the function's "breakpoints," where l_t^j and u_t^j denote the lower and upper breakpoints of segment j). Let $\mathcal{S} = \max_{t=1,\dots,T}\{S_t\}$. For each function F_t and each interval of the function, we need to characterize the generalized gradient function ∂F_t. For the j^{th} segment of F_t, denoted as s_t^j for $j = 1, \dots, S_t$, let $\underline{m}_t^j$ and $\overline{m}_t^j$ denote the minimum and maximum slopes (derivatives) for the segment. If a segment is linear, then $\underline{m}_t^j = \overline{m}_t^j$; otherwise $\overline{m}_t^j > \underline{m}_t^j$. The set of subgradients at the lower (upper) breakpoint of segment j is given by $[\overline{m}_t^{j-1}, \underline{m}_t^j]$ ($[\overline{m}_t^j, \underline{m}_t^{j+1}]$). Within the interior of each segment, the function's derivative provides the subgradient value.

Although Step 3 of Algorithm 3.1 requires the initialization of $\mathcal{O}(T^2)$ variables, this step is not strictly necessary, as all $x_{t\tau}$ variables for $\tau < s$ will retain their former values. Step 4 involves evaluating $\mathcal{O}(T)$ functions and sorting them, which can be done in $\mathcal{O}(T \log T)$ time. We next characterize the worst-case complexity of the binary search routine for a given value of μ_s^s. Step 6 involves the evaluation of $\mathcal{O}(T)$ values of ∂F_t in order to create the set $\mathbb{T}(\mu_s^s)$. In Step 7, for each $t \in \mathbb{T}(\mu_s^s)$, we need to determine an interval of F_t such that μ_s^s is a subgradient on the interval (unless μ_s^s equates to the slope of a linear segment, this interval will correspond to a point). Determining the appropriate interval requires an $\mathcal{O}(\log \mathcal{S})$ binary search among ordered set of minimum and maximum slope values associated with each segment of the function; after identifying the appropriate segment, we assume that the corresponding interval may be determined in constant time. Because this is required for $\mathcal{O}(T)$ functions F_t, this step of the algorithm requires $\mathcal{O}(T \log \mathcal{S})$ operations. As discussed previously, Steps 8 through 10 require $\mathcal{O}(T)$ operations.

The while loop (steps 5-14) performs a bisection search on μ_s^s. If $\mathcal{M}$ denotes an upper bound on the slope (derivative) value that needs to be evaluated for all of our convex functions and ϵ denotes the stopping criterion of the bisection method, the while loop requires $\mathcal{O}(\log \mathcal{M}/\epsilon)$ iterations for each of the functions F_t; thus, the while loop requires a total of $\mathcal{O}(T \log \mathcal{S} \log \mathcal{M}/\epsilon)$ iterations.

An algorithm for determining an optimal solution to problem P_s, given an optimal solution to problem P_{s-1}, requires two steps and a subroutine. The overall complexity of solving P_s is then $\mathcal{O}(T \max\{\log T, \log \mathcal{S} \log \mathcal{M}/\epsilon\})$ given a solution for problem P_{s-1}. In order to solve problem $\mathrm{P} = \mathrm{P}_T$, we apply Algorithm 3.1 for $s = 2, \ldots, T$. The complexity associated with solving P therefore equals $\mathcal{O}(T^2 \max\{\log T, \log \mathcal{S} \log \mathcal{M}/\epsilon\})$. □

Observe that for the piecewise-linear cost case, we can perform a binary search on μ_s^s instead of a bisection search. Because $\mathcal{S}$ denotes an upper bound on the number of segments of the piecewise-linear cost function in any period, the while loop requires a total of $\mathcal{O}(T(\log \mathcal{S})^2)$ iterations, which leads to an overall complexity of $\mathcal{O}(T^2 \max\{\log T, (\log \mathcal{S})^2\})$.

3.2.3 Alternative Implementation

We next briefly describe an alternative implementation of Algorithm 3.1, which eliminates the need for bisection search on μ_s^s and provides an exact solution. In Step 4 of the algorithm, we sort the values of $\partial f_t^u(X_{-s}^t, 0) + \sum_{i=t}^s \partial h_i^u(\bar{X}_{-(t,s)}^i, 0)$ in nondecreasing order for $t = 1, \ldots, s$. Let $g_t^s = \partial f_t^u(X_{-s}^t, 0) + \sum_{i=t}^s \partial h_i^u(\bar{X}_{-(t,s)}^i, 0)$, and let $g_{[\tau]}^s$ correspond to the τ^{th} such value after sorting in nondecreasing order, e.g., $g_{[1]}^s \leq g_{[2]}^s \leq \cdots \leq g_{[s]}^s$. Next note that the definition of the set $\mathbb{T}(\mu_s^s)$ is the same for any μ_s^s value in the interval $(g_{[\tau-1]}^s, g_{[\tau]}^s]$, for $\tau = 1, \ldots, s+1$ (where $g_{[0]}^s = 0$ and $g_{[s+1]} = \infty$); let $\mathbb{T}_\tau$ denote the corresponding set definition for the interval ending at $g^s_{[\tau]}$. If we can find some value of the free variable μ on this interval such that

$$\mu \in \partial f_t(X^t) + \textstyle\sum_{i=t}^{s} \partial h_i(\bar{X}^i), \qquad \forall t \in \mathbb{T}_\tau, \tag{3–10}$$

$$\textstyle\sum_{t \in \mathbb{T}_\tau} x^s_{ts} = d_s, \tag{3–11}$$

we will have found a generalized KKT point. As a result, instead of performing the binary search on μ^s_s, we may attempt to determine whether a solution to the above system exists directly for each interval of μ^s_s on which the definition of $\mathbb{T}_\tau$ is invariant. The complexity associated with finding such a solution will depend on the structure of the f_t and h_t functions and their subgradients. Let $\phi(T)$ denote a function that characterizes the number of operations associated with finding the $\mathcal{O}(T)$ solutions to (3–10)–(3–11) for each definition of $\mathbb{T}_\tau$, $\tau = 1, \ldots, s$. Thus, we replace the while loop in Algorithm 3.1 with a procedure that runs in $\mathcal{O}(\phi(T))$ time, with a resulting overall complexity of $\mathcal{O}(T \max\{T \log T, \phi(T)\})$ time. If, for example, these functions are everywhere differentiable and quadratic, then the gradient functions will be linear functions, and Equation (3–10) takes the form of a linear equation. This permits expressing each x^s_{ts} as a linear function of μ for each $t \in \mathbb{T}_\tau$, substituting the resulting expression for x^s_{ts} into Equation (3–11), and solving directly for μ for each set $\mathbb{T}_\tau$, $\tau = 1, \ldots, s$. In this case, $\phi(T) = \mathcal{O}(T)$, and the worst-case complexity for solving problem P becomes $\mathcal{O}(T^2 \log T)$. We discuss a practical application with quadratic and differentiable cost functions in Section 3.3.

3.2.4 Relationship to Veinott's Approach

We next describe how the algorithm we propose for lot sizing with convex costs compares to Veinott's approach ([86]) for lot sizing with piecewise-linear and convex production and holding costs and integer parameter values. Veinott's method is a parametric approach that proceeds forward in time, one demand unit at a time. Beginning in period 1, for the first unit of demand, he determines the least cost option for production of this unit (in the case without backlogging, there is only one option for producing this unit, as well as all units of period 1 demand). After satisfying all demand in period 1, he moves to period 2. For each unit of demand in period 2, he determines whether it is cheaper to produce the unit in period 1 or 2, based on the incremental cost of producing the unit, which in turn depends on the partial solution determined at previous iterations. If it is cheaper to allocate the unit to period 1, then he increases period 1 production by one unit; otherwise, he increases period 2 production by one unit. For a unit of demand in period t, given the partial solution created at the previous iteration, he determines the least cost solution among solutions that produce an additional unit in period 1, 2, $\ldots$, $t-1$, or t. In the worst-case, therefore, the running time of Veinott's algorithm is a function of the total demand. In fact, one can show that the time complexity of this algorithm is $\mathcal{O}(TD)$, where D equals the sum of all demands over the planning horizon.

Our algorithm, on the other hand, permits handling multiple demands at each iteration, instead of considering one demand unit at a time. This is enabled by searching (via subgradient search) for a subgradient value such that the sum of the production levels with the corresponding subgradient value (for a subset of periods) equals demand. For the case of piecewise-linear and convex production cost functions, this requires keeping track of the slopes and breakpoints associated with the piecewise-linear cost function for each production/demand period pair. Based on our prior analysis, the worst-case complexity of this approach is $\mathcal{O}(T^2 \max\{\log T, (\log \mathcal{S})^2\})$, where $\mathcal{S}$ corresponds to the maximum number of linear segments associated with the cost to satisfy demand in period s using production in period t, $1 \leq t \leq s \leq T$.

3.3 Production Planning with Price-Dependent Supply

This section describes an application of the lot-sizing problem with convex production costs, which results in a reduction in the worst-case running time of Algorithm 3.1 in general. Consider a production planning setting for a product such that the product's manufacturing process requires an input component for which the producer offers a unit price, p_t, in each period t to suppliers of that production input. One such example would correspond to a reseller of a refurbished product who needs to acquire the used item from consumers. (Assume for convenience that one unit of the input component is required for each unit of the end product.) The amount that suppliers are willing to provide depends on the price the producer offers. Suppose that the producer refurbishes each item in the period in which it is received, which implies that the production amount in any period t, x_t, is a function of the price offered to suppliers. To characterize this function, suppose that a minimum reservation price exists in each period t, denoted by p_t^0, such that for any price not exceeding this value, no supply is available. For prices exceeding this threshold, we assume that supply increases linearly in price at a rate of $\beta_t > 0$ per unit price change. As a result of these assumptions, the supply in period t is a nonnegative and nondecreasing piecewise linear function of price, p_t, with one breakpoint at p_t^0, defined by

$$x_t = \begin{cases} \beta_t p_t - \alpha_t, & \text{if } p_t \geq p_t^0, \\ 0, & \text{otherwise,} \end{cases} \tag{3–12}$$

for $p_t \geq 0$, with $\alpha_t \geq 0$ for all $t = 1, \ldots, T$. The parameter $\beta_t > 0$ corresponds to the price elasticity of supply in period t, for $t = 1, \ldots, T$. For each unit of the component procured, the producer pays p_t, which implies that the total amount paid to suppliers in period t equals $p_t x_t = \beta_t p_t^2 - \alpha_t p_t = (x_t^2 + \alpha_t x_t)/\beta_t$. In addition, we assume the producer incurs a cost of $\gamma_t x_t$ for a procurement level of x_t in period t, where γ_t denotes a material handling cost per unit component. Thus, a total procurement related cost of $(p_t + \gamma_t)x_t = \beta_t p_t^2 - \alpha_t p_t + \gamma_t(\beta_t p_t - \alpha_t) = (x_t^2 + \alpha_t x_t)/\beta_t + \gamma_t x_t$ is incurred in each period $t = 1, \ldots, T$. An additional inventory holding cost of h_t is assessed against each unit remaining in inventory at the end of period t. Letting $\bar{c}_t = c_t + \gamma_t + (\alpha_t/\beta_t) + \sum_{j \geq t} h_j$, the resulting formulation, P′, is written as follows.

$$\text{P}': \quad \text{Minimize} \quad \sum_{t=1}^{T} \bar{c}_t \sum_{\tau=t}^{T} x_{t\tau} + \sum_{t=1}^{T} \frac{\left(\sum_{\tau=t}^{T} x_{t\tau}\right)^2}{\beta_t} \tag{3–13}$$

$$\text{Subject to:} \quad \sum_{t=1}^{\tau} x_{t\tau} = d_\tau, \quad \tau = 1, \ldots, T, \tag{3–14}$$

$$x_{t\tau} \geq 0, \quad \tau = 1, \ldots, T \text{ and } t \leq \tau. \tag{3–15}$$

Note that we have omitted the constant term $-\sum_{t=1}^{T} d_t \sum_{\tau=t}^{T} h_\tau$ from the objective in this formulation. Constraint set (3–14) ensures satisfying demand in each period, whereas constraints (3–15) enforce the nonnegativity requirements on the $x_{t\tau}$ variables. Problem P′ is a special case of problem P in which each h_t function equals zero and each f_t function is everywhere differentiable and quadratic. As a result, problem P′ can be solved in $\mathcal{O}(T^2 \log T)$ time using Algorithm 3.1.

Kian et al. [41] study the uncapacitated dynamic lot-sizing problem with convex costs that take the form of a polynomial function of the production quantities. They provide a forward dynamic program to solve the version of the problem with fixed charges, which they modify in order to solve for the problem with zero fixed charges, where they state the complexity of this algorithm as $\mathcal{O}(T^2)$. Similar to our method, their solution approach uses the marginal costs to determine the production amounts for each period. Their approach solves a set of $\mathcal{O}(T^2)$ subproblems in a dynamic programming recursion. This constructive algorithm requires solving a system of polynomial equations for each subproblem in order to compute the production quantity for each of $\mathcal{O}(T)$ periods (as well as the cost of the associated subproblem, which contains $\mathcal{O}(T)$ periods). Thus, a worst-case complexity of $\mathcal{O}(T^2)$ is only achievable when each of these subproblems is assumed to be performed in constant time.

CHAPTER 4
AN EOQ MODEL WITH PRICE-DEPENDENT SUPPLY AND DEMAND

4.1 Motivation

In the century following the seminal paper by Harris [32], the economic order quantity (EOQ) model has been widely used to solve inventory planning problems under deterministic and stationary cost and demand assumptions. The EOQ model has drawn the attention of numerous researchers because of its well-established, simple and easy-to-modify nature (see Nahmias [55] and Silver, Pyke, and Peterson [75]). This chapter considers this classic problem in the presence of a broader set of factors that may influence the economics of production in practice.

More specifically, we study a production planning problem where an input component is required to produce a particular type of end-item. The cost of the input component depends on the price offered to suppliers by the producer of the end-item. As the price offered by the producer increases, the aggregate number of units of the component available to the producer increases as well. This phenomenon follows a fundamental principle in economics that the available supply of a good in a market increases in price. As in the classical EOQ model, a production process is required to transform the input component to an end-item. This production process involves a fixed cost for producing a batch of end-items, as well as a variable (per-unit) production cost and inventory holding costs for components and end-items. Demand for the end-item is a nonincreasing function of its selling price. Thus, the producer wishes to determine the price offered to suppliers for the input component, the selling price of the end-item, and its periodic production lot size, in order to maximize average profit per unit time. We model this production planning and pricing problem and characterize the corresponding optimal decisions. In addition, we analyze the model in order to better understand the way in which price-dependent supply affects pricing and inventory planning decisions.

This work contributes to the literature by considering how the pricing of inputs (components) and outputs (end-items) influences the economics and profitability of production. In the process, we identify some interesting relationships between the model's parameters and the characteristics of optimal pricing and production planning decisions, as well as the optimal profit per unit time and production profit margins. The rest of this chapter is organized as follows. Section 4.2 summarizes the related work in the literature. Section 4.3 defines the problem under consideration, and formalizes the EOQ model that will serve as the basis for our analysis. Section 4.4 provides the bulk of our analysis. We first propose a functional relationship between supply and selling prices in equilibrium. Section 4.4.1 characterizes optimal supply and selling prices for our model, as well as the way in which key parameter values affect these decision variables at optimality. Section 4.4.2 next explores the way in which the nature of the component supply-price curve influences profitability and pricing decisions. Section 4.4.3 then demonstrates how a lack of accounting for the component supply-price relationship can reduce profitability.

4.2 Related Literature

This section provides a review of the literature closely related to the work presented in this chapter. Our work is most closely related to prior work on EOQ models with price-dependent demand and EOQ models with procurement quantity discounts. In addition, a natural application area of our model lies within a remanufacturing setting, where a remanufacturer wishes to collect used items from a market in order to provide input to its production process. Therefore, we will discuss the relation of our work to a segment of the remanufacturing/reverse logistics literature, focusing on those works which are most closely related to ours.

4.2.1 Price-Dependent EOQ Models

Pricing decisions and price-dependent demand rates have been considered in the inventory control literature by many researchers over the past several decades. Whitin [89] presented the first model in which price-dependent demand was incorporated within the traditional EOQ model. Arcelus and Sirinivasan [4] subsequently studied a deterministic EOQ model in which demand was price dependent, with a goal of characterizing an optimal pricing and replenishment policy. In their model, the selling price was set according to a markup on the item's unit cost, where the markup rate acted as a decision variable, and they considered the objectives of maximizing profit, return on investment, and residual income. Ray et al. [62] characterized profit-maximizing solutions under a price-sensitive demand rate for two classes of policies. In the first policy class, price was an independent decision variable, while in the second, price was set according to a markup on unit cost. In our work, both the input to the production process (component) and the output (end-item) are price-dependent. Additional work on generalizations of the EOQ model with price-dependent demand accounts for perishability and backlogging (Sana [68]), a demand rate that depends on price and promotional effort (De and Sana [13]), and demand that depends on these factors as well as on-hand inventory (Sana and Panda [71]).

4.2.2 EOQ Models with Discounts

A number of studies exist in the literature that deal with EOQ models in which the purchase cost varies depending on the quantity that the buyer/producer is willing to purchase. This price variation typically comes in the form of discounts as the purchase quantity increases. Monahan [54] devised an optimal quantity discounting scheme for a supplier in a single-supplier, single-retailer system, assuming the supplier follows a lot-for-lot policy in response to the retailer's orders. Lee and Rosenblatt [47] generalized Monahan's [54] model to allow the supplier to deviate from a lot-for-lot policy. Matsuyama [51] studied cases in which the purchase price is a nonincreasing step function of the order quantity. Arcelus et al. [3] determined a profit-maximizing strategy for a retailer/buyer when the vendor proposes either a discount on the purchase price or a delay in payments. Viswanathan and Wang [87] considered the effectiveness of quantity discounts and volume discounts in a single-vendor, single-buyer system with price-sensitive demand faced by the buyer. They characterized an equilibrium point for determining optimal pricing and discount options in a Stackelberg game where both players aim to maximize profit. Qin et al. [60] considered price discounts and franchise fees as coordination mechanisms in a system consisting of a vendor and a buyer. In a manner similar to the approach of Viswanathan and Wang [87], they formulated the problem as a Stackelberg game where the vendor is the leader (first setting its price to the buyer) and the buyer is the follower, who subsequently sets the market price and, hence, determines the market demand for the commodity. Sana and Chaudhuri [70] considered the impact of delayed payments to the supplier, together with quantity discounts. In these contexts, wholesale pricing is used to facilitate coordination between a supplier and buyer, or to provide incentives to the buyer to purchase goods from the supplier. In the problem we consider, the producer attracts suppliers by establishing the price it is willing to pay for each unit of supply. In such a setting, when higher prices are offered to suppliers, the number of suppliers and the number of units that the suppliers are willing to provide increase.

4.2.3 EOQ Models for Remanufacturing

Price-dependent supply is a particularly relevant phenomenon in reverse logistics settings, where the inputs required by a remanufacturer are owned by individual consumers who may be willing to sell their products back to the remanufacturer, depending on the price offered. The existing operations literature contains a significant number of studies in which remanufacturing is considered as an option for demand satisfaction within an EOQ framework. Examples include Richter [63], Richter and Dobos [64], Teunter [78], Dobos and Richter [15], and El Saadany and Jaber [16]. Guide et al. [29] consider a setting where return and demand rates are price dependent. They solve a single-period problem and seek the profit maximizing acquisition and selling prices for the remanufactured product. Their model, however, does not include operations costs such as inventory holding and setup costs. To the best of our knowledge, no existing study in the literature incorporates supply and demand pricing decisions, and the implications of these decisions on the economics of production, with a goal of characterizing an optimal replenishment and pricing policy.

4.3 Production Planning and Pricing Model

We consider a production planning problem in which an input component is required for production of an end-item, and for which all cost, demand and pricing parameters are assumed to be stationary. The cost of the input component depends on the price the producer offers to suppliers. Let $p_s \geq 0$ denote the unit purchase price the producer offers to its suppliers, and suppose that the supply rate of components for production is a stationary, nondecreasing, nonnegative, and differentiable function of p_s denoted by $K(p_s)$. The cost per unit time for procuring components is therefore $p_s K(p_s)$. We assume that the producer must meet end-item demand (without shortages), where the demand rate is price sensitive and deterministic, and equals $D(p_c)$ units per unit time at the price level p_c, which corresponds to the selling price offered to customers. We also assume that the time required to convert a batch of input components to end-items is negligible.

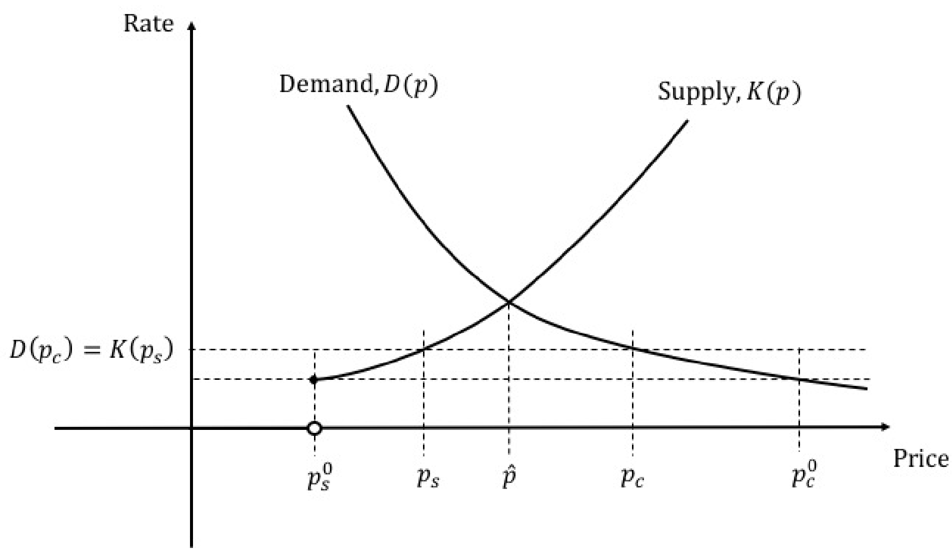


Figure 4-1. Supply and demand curves.

Consider Figure 4-1, which shows the behavior of the supply rate function, $K(p)$, and the demand rate function, $D(p)$, as a function of price. The supply rate function is a nondecreasing function of the price offered to suppliers where, for any price offered that is below a specified price limit, p_s^0, the supply rate is zero, i.e., no supplier in the market has a lower reservation price than p_s^0. We let p_c^0 denote the selling price at the corresponding demand level. The demand rate function is a nonincreasing function of the selling price offered to customers for a unit of the end-item. As the selling price approaches infinity, the demand rate approaches zero. The intersection of the curves, at the price $\hat{p}$, provides a lower bound on the price to customers, p_c, and an upper bound on the price offered to suppliers, p_s (if p_c is below this price or if p_s is above this price then we will require setting $p_s > p_c$, which implies a negative profit level). Given a value of $p_c \geq \hat{p}$, we assume that a unique price p_s exists such that the supply rate equals the demand rate. Thus, some function exists such that p_s is a decreasing function of p_c (or, equivalently, we could express p_c as a decreasing function of p_s).

Suppose that $K(p_s)$ is the supply rate, expressed as a function of the supply price, and $D(p_c)$ is the demand rate, expressed as a function of the end-item price offered to customers. We consider a production process that converts the supplied components to end items, where we assume without loss of generality that each end-item requires one unit of the component (alternatively, if each end-item requires multiple components per unit of end-item, we can scale the demand rate and associated costs accordingly, so that the end-item demand rate is expressed in units of the component). We assume that operating this process requires incurring some fixed cost, denoted by F, which leads to production batching. Components arrive to the production stage at a rate of $K(p_s)$, and are held in inventory until being converted into end-items via the batch production process. End-items are depleted from end-item inventory according to the demand rate $D(p_c)$. Because the producer must meet end-item demands without shortages, this implies that we require $K(p_s) \geq D(p_c)$. Moreover, because the supply rate is nondecreasing in price, this implies that, at optimality, $K(p_s) = D(p_c)$.[1] Thus, if Q denotes the production batch size, then in steady state, whenever component inventory reaches Q units, these components are converted to end-items via production of a batch of size Q. Let c denote the cost of converting one unit of supply to demand, and let h_s denote the unit cost per unit time for holding supplied components, while h_e is the holding cost per unit per unit time for end items. If both supply and demand occur at a fixed rate, then for given prices p_c and p_s and a given batch size Q, Figure 4-2 illustrates the behavior of component and end-item inventory over time.

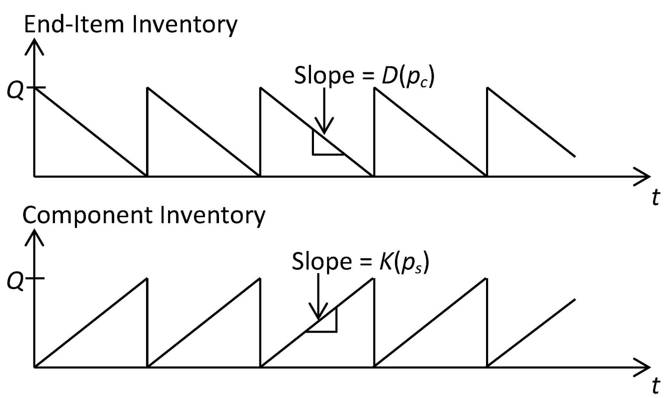


Figure 4-2. Component and end-item inventory levels with time with a batch size of Q.

Given the inventory levels of components and end-items illustrated in Figure 4-2, the average inventory level for both components and end-items equals $Q/2$, leading to an average inventory holding cost per unit time of $(h_s + h_e)Q/2$. As in the standard EOQ model, because a fixed cost of F is incurred for every production batch, and we have an average number of $D(p_c)/Q$ production batches per unit time, the average fixed cost per unit time equals $FD(p_c)/Q$. As a result, for $Q \geq 0$, $p_s^0 \leq p_s \leq \hat{p} \leq p_c \leq p_c^0$, the average

[1] Given any solution with prices p_s' and p_c' such that $K(p_s') > D(p_c')$, note that a solution exists with prices p_s'' and p_c' such that $K(p_s'') = D(p_c')$ and $p_s'' < p_s'$, with $p_s''K(p_s'') < p_s'K(p_s')$. The solution with supply price p_s'' meets all demand at lower component inventory and procurement costs than the solution at price p_s'.

cost per unit time can be expressed as

$$AC(Q, p_s, p_c) = \frac{FD(p_c)}{Q} + \frac{(h_s + h_e)Q}{2} + p_s K(p_s) + cD(p_c). \tag{4–1}$$

The average annual cost is the sum of the average fixed cost, component and end-item inventory holding cost, and purchasing and production cost per unit time.

Letting $h = h_s + h_e$, we note that for any price p_c we have an average annual cost minimizing solution for Q: $Q^*(p_c) = \sqrt{2FD(p_c)/h}$ [2]. We can characterize the average annual profit equation for $p_s^0 \leq p_s \leq \hat{p} \leq p_c \leq p_c^0$ as

$$\pi(p_s, p_c) = (p_c - p_s - c)D(p_c) - \sqrt{2FhD(p_c)}. \tag{4–2}$$

As mentioned previously, the supply price p_s can be characterized as a decreasing function of the selling price p_c, e.g., $p_s(p_c)$. Thus, the average annual profit can be written as a function of p_c only for $\hat{p} \leq p_c \leq p_c^0$, i.e.,

$$\pi(p_c) = (p_c - p_s(p_c) - c)D(p_c) - \sqrt{2FhD(p_c)}. \tag{4–3}$$

In the following section, we characterize the supply and selling prices that maximize the average annual profit, where we assume specific functional forms for $D(p_c)$ and $p_s(p_c)$. We also analyze the behavior of the optimal prices, profit, and batch size with respect to key problem parameters.

4.4 Optimal Supply and Selling Prices

In this section, we characterize the optimal solution when the demand rate function takes an iso-elastic form. That is, we assume that $D(p_c) = ap_c^{-b}$, where $a, b > 0$ and $p_c \geq \hat{p}$. As noted in the previous section, we assume the supply price p_s can be characterized as a decreasing function of p_c. In particular, we assume that this relationship may

[2] Note that $AC(Q, p_s, p_c)$ is convex in Q. Thus, solving the first order condition yields the optimal solution for Q, which is $Q^*(p_c) = \sqrt{2FD(p_c)/h}$.

be characterized by the equation $p_s(p_c) = \hat{p} - (k-1)(p_c - \hat{p})$, where $k > 1$ and $p_s^0 \leq p_s \leq \hat{p} \leq p_c \leq p_c^0$. The parameter $k-1$ characterizes the response of the equilibrium supply price[3] to changes in the end-customer selling price. For this reason, we will sometimes refer to $k-1$ as the price-to-price response. While this relationship between prices is characterized by a simple linear function, observe that the implied supply rate function in price becomes $K(p_s) = a(k-1)^b(k\hat{p} - p_s)^{-b}$, which permits considerable flexibility in characterizing a range of nonlinear supply curves that are increasing in price (e.g., see Figure 4-3).

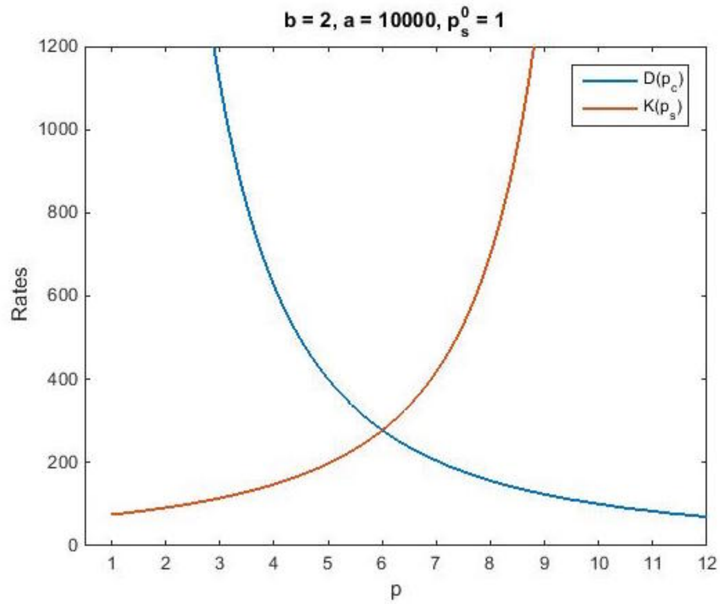


Figure 4-3. Supply and demand curves for linear supply-price-selling-price relationship ($k = 1.9, \hat{p} = 6$).

[3] The equilibrium supply price corresponds to the supply price such that, for a given selling price, the corresponding supply rate equals the demand rate.

Using the relationship between supply and selling prices, we can write the average annual profit as a function of p_c only, for $\hat{p} \leq p_c \leq p_c^0$:

$$\begin{aligned}
\pi(p_c) &= (p_c - (\hat{p} - (k-1)(p_c - \hat{p})) - c)ap_c^{-b} - \sqrt{2Fhap_c^{-b}}, \\
&= (k(p_c - \hat{p}) - c)ap_c^{-b} - \sqrt{2Fhap_c^{-b}}. \qquad (4\text{–}4)
\end{aligned}$$

Observe that maximizing $\pi(p_c)$ is the same as maximizing $\hat{\pi}(p_c)$ which is equal to $\pi(p_c)/k$. Thus,

$$\hat{\pi}(p_c) = (p_c - c')ap_c^{-b} - \sqrt{2F'hap_c^{-b}}, \qquad (4\text{–}5)$$

where $c' = c/k + \hat{p}$ and $F' = F/k^2$. The functional form of this average annual profit function is equivalent to the one presented in Ray et al. [62], who considered the EOQ model with price-dependent demand. Thus, for a fixed value of k, the results of Ray et al. [62] permit characterizing the profit function and the optimal selling price. The limiting special case[4] of the profit equation (4–4) in which the price-to-price response $(k-1)$ equals zero, and the variable production/procurement cost equals $c + \hat{p}$, is precisely the model provided by Ray et al. [62]. Our interest thus lies primarily in exploring and understanding how generalizing the model to permit $k > 1$ affects optimal pricing decisions and the economics of supply, demand, and production decisions. In order to do this, the following section first characterizes optimal pricing and production decisions for our model for a fixed value of k when $k > 1$, largely drawing on the results of Ray et al. [62]. Following this, in Section 4.4.2, we explore the way in which the price-to-price response affects the model's results. Section 4.4.3 then considers how a lack of consideration of the supply-price relationship can impact decisions and profit performance.

[4] Strictly speaking, our model assumes $k > 1$, which provides a functional relationship between the supply price and the selling price. We refer to the case of $k = 1$ as the limiting special case as $k \to 1$ from above. At $k = 1$, the functional relationship between the supply and selling prices breaks down, and supply price becomes independent of selling price.

4.4.1 Optimal Decisions under a Fixed Price-to-Price Response

As noted earlier, for a fixed price-to-price response $(k-1)$, our average annual profit equation (4–5) is mathematically equivalent to that considered by Ray et al. [62]. We wish to maximize this profit equation over p_c such that $\hat{p} \leq p_c \leq p_c^0$, where $\hat{p}$ corresponds to the minimum price such that nonnegative profit is possible, and p_c^0 corresponds to the maximum price level such that the demand level ensures that some positive supply is available at the corresponding equilibrium supply level. We can thus use the results from Ray et al. [62] to characterize the optimal selling price, p_c^{opt}, as well as how this price and the corresponding optimal batch size and profit margin are influenced by various problem parameters.

As in Ray et al. [62], we have that for $0 < b \leq 1$, it is possible to show that $\hat{\pi}'(p_c)$ is always positive; therefore, the optimal choice of p_c equals p_c^0, which is the maximum price that can be offered to customers while ensuring positive profit. Because of this, throughout the rest of this chapter we will assume that $b > 1$. Proposition 4.1 shows that, for $b > 1$, a stationary point solution exists which is the unconstrained maximizer of $\hat{\pi}(p_c)$, and that the optimal selling price is the minimum between this stationary point and p_c^0.

Proposition 4.1. *For $b > 1$, the optimal selling price is the minimum of the smallest positive stationary point solution for $\hat{\pi}(p_c)$ and p_c^0.*

Proof. See Appendix C. □

For $b = 2$, we obtain the following closed-form stationary point solution, p_c^*:

$$p_c^* = \frac{2a(c + k\hat{p})}{ak - \sqrt{2Fha}}. \tag{4–6}$$

Note that the optimal price Ray et al. [62] obtain for the case of $b = 2$ equals the above equation with $k = 1$. The optimal selling price, p_c^{opt}, can be characterized as:

$$p_c^{opt} = \min\left\{p_c^*, p_c^0\right\}, \tag{4–7}$$

where $p_c^0 = (k\hat{p} - p_s^0)/(k-1)$. The corresponding optimal supply price, p_s^{opt}, is given by $\hat{p} - (k-1)(p_c^{opt} - \hat{p})$. For $b = 2$, we can obtain the following closed-form solution for p_s^{opt} as well:

$$p_s^{opt} = \max\{p_s^0, p_s^*\}, \tag{4–8}$$

where

$$p_s^* = k\hat{p} - \frac{2a(k-1)(c+k\hat{p})}{ak - \sqrt{2Fha}}. \tag{4–9}$$

Figures 4-4 and 4-5 illustrate example curves for the average annual profit as a function of price ($\pi(p_c)$) for different parameter sets.

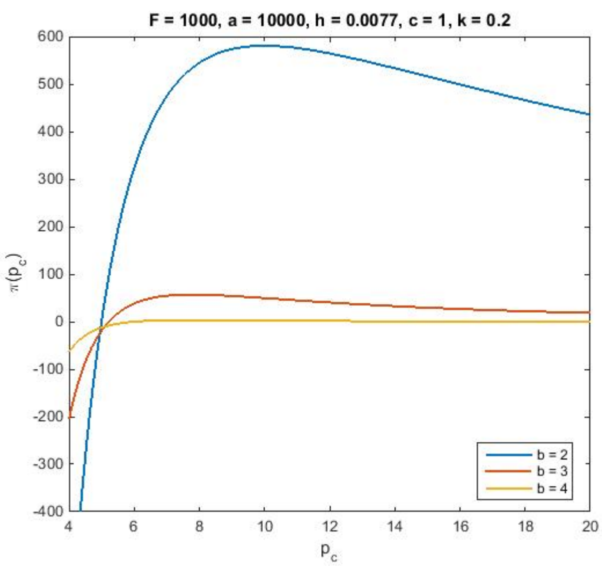


Figure 4-4. Profit function for different values of b, where $\hat{p} = 4$.

We next investigate the behavior of the optimal selling price, optimal supply price, and optimal batch size with respect to key problem parameters, where k is assumed to be fixed and $b > 1$.

Proposition 4.2. *For $b > 1$ and fixed k, the optimal selling price, $p_c^{opt} \uparrow$ as $a \downarrow$, $F \uparrow$, $c \uparrow$, $h \uparrow$, $\hat{p} \uparrow$, when $p_c^{opt} = p_c^*$. When $p_c^{opt} = p_c^0$, $p_c^{opt} \uparrow$ as $\hat{p} \uparrow$ and $p_s^0 \downarrow$.*

Proof. See Appendix D. □

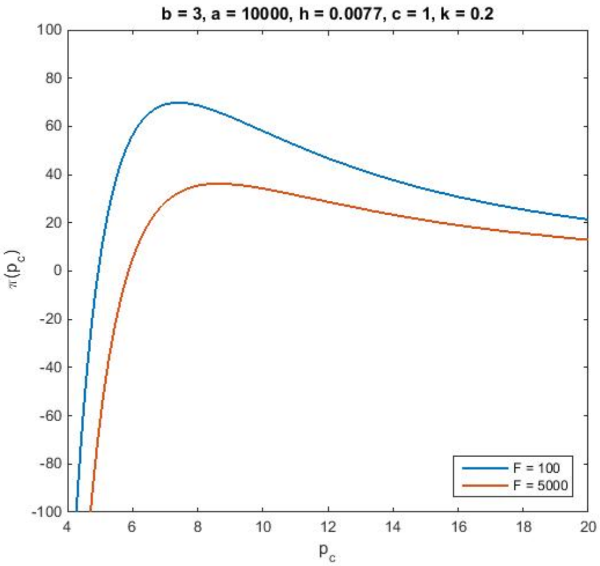


Figure 4-5. Profit function for different values of F, where $\hat{p} = 4$.

When $p_c^{opt} = p_c^*$, the optimal selling price corresponds to a stationary point solution, which is affected by the problem parameters in the same way as in the price-dependent EOQ model of Ray et al. [62]. Note, however, that when p_c^{opt} hits the boundary value of p_c^0, its value is no longer affected by the operations cost or demand parameter values, only by the equilibrium supply and demand parameters. Figure 4-6 shows some examples of p_c^{opt} as a function of F for different values of a. Note that as F increases, p_c^{opt} increases until it reaches its maximum possible value, i.e., p_c^0.

Proposition 4.3. *For $b > 1$ and fixed k, the optimal supply price, $p_s^{opt} \uparrow$ as $a \uparrow$, $F \downarrow$, $c \downarrow$, $h \downarrow$, when $p_c^{opt} = p_c^*$ (otherwise $p_s^{opt} = p_s^0$ which is constant). As $\hat{p} \uparrow$, p_s^{opt} first decreases and then increases.*

Proof. See Appendix E. $\square$

As the proposition shows, the effects of operations cost and demand parameter values on the optimal supply price act in the opposite direction of the selling price, which follows because a higher selling price implies a lower supply price in equilibrium (where the demand rate equals the supply rate).

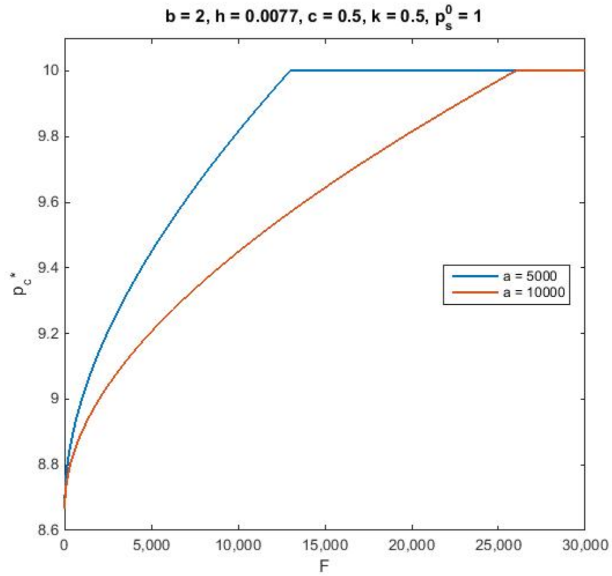


Figure 4-6. p_c^* vs F for different values of a.

Proposition 4.4. *For $b > 1$ and fixed k, the optimal batch size, $Q^{opt} \uparrow$ as $a \uparrow$, $c \downarrow$, $h \downarrow$, $\hat{p} \downarrow$ when $p_c^{opt} = p_c^*$. When $p_c^{opt} = p_c^*$, as $F \uparrow$, Q^{opt} first increases and then decreases. When $p_c^{opt} = p_c^0$, $Q^{opt} \uparrow$ as $a \uparrow$, $F \uparrow$, $h \downarrow$, $\hat{p} \downarrow$, and $p_s^0 \uparrow$.*

Proof. See Appendix F. □

When $p_c^{opt} = p_c^*$, the way in which operations and demand parameters affect the optimal batch size is the same as in the EOQ model with price-dependent demand of Ray et al. [62]. Observe that in this case, the optimal batch size starts off increasing in the fixed cost F and later decreases in F, as an increasing F leads to an increased selling price (and thus a decreased demand rate). Recall that $Q^{opt} = \sqrt{2FD(p_c)/h}$; at lower values of F, the impact of an increase in F on the production batch size outweighs the corresponding decrease in demand, while at higher values of F, the reduction in demand level outweighs the impact of a corresponding increase in F on the batch size. When $p_c^{opt} = p_c^0$, the impact of operations and demand parameters on the optimal batch size is the same as in the standard EOQ model without price-dependent demand, because the price p_c^0 is independent of the operations cost and demand parameters.

Net revenue per unit. We define $\delta = p_c - p_s - c$ as the net revenue earned per unit. When both selling and supply prices are decision variables, the behavior of the net revenue per unit is not as straightforward as in the case where the selling price is the only decision variable. Therefore it is interesting for our case to investigate the net revenue per unit at optimality as a performance measure of the system. At optimality, the net revenue per unit, δ^{opt}, equals $p_c^{opt} - p_s^{opt} - c = kp_c^{opt} - k\hat{p} - c$. Note that $\delta^{opt} = kp_c^* - k\hat{p} - c$ when the stationary point solution is feasible (and therefore optimal). Otherwise, $\delta^{opt} = k(\hat{p} - p_s^0)/(k-1) - c$. For $b = 2$, we can express δ^{opt} as follows.

$$\delta^{opt} = \begin{cases} \frac{ak+\sqrt{2Fha}}{ak-\sqrt{2Fha}}\left(c + k\hat{p}\right), & \text{if } p_c^* < p_c^0, \\ \frac{k\hat{p}-p_s^0}{k-1} - c, & \text{if } p_c^* \geq p_c^0. \end{cases} \tag{4–10}$$

Proposition 4.5. *For $b > 1$ and fixed k, the optimal net revenue per unit, $\delta^{opt} \uparrow$ as $a \downarrow$, $F \uparrow$, $c \uparrow$, $h \uparrow$, and $\hat{p} \uparrow$, when $p_c^{opt} = p_c^*$. When p_c^0 is the optimal selling price, $\delta^{opt} \uparrow$ as $\hat{p} \uparrow$, $c \downarrow$, and $p_s^0 \downarrow$.*

Proof. See Appendix G. □

When $p_c^{opt} = p_c^*$, as any of the operations cost parameters increases (or as the demand parameter a decreases), the optimal selling price p_c^{opt} increases as well, while the optimal supply price p_s^{opt} decreases, leading to a higher-margin, lower volume solution. In contrast, when $p_c^{opt} = p_c^0$, reducing the unit production cost, c, serves to only increase the net revenue per unit, as the operations cost and demand parameters do not affect p_c^0.

4.4.2 Effects of the Supply-Price Relationship

This section explores the effects of price-sensitive supply on optimal pricing decisions, as well as on the optimal profit and the optimal net revenue per unit. We are particularly interested in how the price-to-price response (via the parameter k) affects the behavior of the optimal selling price, p_c^{opt}, the optimal profit, $\pi(p_c^{opt})$, and the optimal net revenue per unit, δ^{opt} (see Proposition 4.6)[5] . Proposition 4.6 indicates that p_c^{opt} is decreasing in k. Thus, as the price-to-price response increases, we offer a lower price to customers in order to capture greater demand volume.

Proposition 4.6. *For $b > 1$ as $k \uparrow$, $p_c^{opt} \downarrow$ and $Q^{opt} \uparrow$. When $p_c^{opt} = p_c^0$, p_s^{opt} is constant (p_s^0), and $\delta^{opt} \downarrow$ as $k \uparrow$. When p_c^* is the optimal selling price and $b = 2$, $p_s^{opt} \downarrow$, and a threshold value of k exists, $\tilde{k}$, such that δ^{opt} is decreasing in k for $k < \tilde{k}$ and increasing in k for $k > \tilde{k}$ (the value of $\tilde{k}$ is provided in the appendix).*

Proof. See Appendix H. □

The following proposition shows that, despite this reduced selling price, the average annual profit is increasing in the price-to-price response when the optimal price occurs at the stationary point p_c^*.

Proposition 4.7. *For $b > 1$, the optimal average annual profit, $\pi(p_c^{opt})$, is increasing in the price-to-price response parameter k, when $p_c^{opt} = p_c^*$.*

Proof. See Appendix I. □

Because p_c^0 is decreasing in k, it is not possible to show that $\pi(p_c^0)$ is monotonically increasing or decreasing in k in general. However, when $b = 2$, we can establish conditions on the problem parameters that guarantee that when $p_c^{opt} = p_c^0$, a threshold value for k exists, k', such that $\pi(p_c^0)$ is increasing in k for $k < k'$ and decreasing in k for $k > k'$ (these conditions are provided in Appendix G). We can gain some insight on the implications of Proposition 4.7 by examining extreme values of k and what these extreme values imply in terms of the characteristics of the supplier base.

Consider the extreme cases for k, i.e., the cases where $k = 1$ and $k = \infty$. As discussed earlier, when $k = 1$, $p_s(p_c)$ becomes equal to $\hat{p}$, which is independent of p_c. That

[5] Note that as we increase k, we hold p_s^0 and $\hat{p}$ fixed in order to ensure a nonnegative supply price corresponding to every feasible selling price.

is, when $k = 1$, the model reduces to the EOQ model where only the demand rate is price-dependent, which was well characterized by Ray et al. in [62]. Although the supply rate is independent of p_s, we can envision a price-supply curve that equals zero for all $p_s < \hat{p}$ and equals ∞ for all $p_s \geq \hat{p}$. Under such a price-supply curve, a rational producer will thus set $p_s = \hat{p}$, but will purchase from suppliers only the amount necessary to meet demand at the optimal selling price, which is exactly what occurs in the model of Ray et al. [62].

Recall that our model assumes that $K(p_s)$ is a nondecreasing function of p_s. The economic reasoning underlying this relationship assumes that a heterogeneous collection of suppliers exists, each with a minimum reservation price required for selling their supply to the producer. In other words, we can think of this curve as representing, for any given p_s, the number of suppliers, each possessing an individual unit, with a reservation price less than or equal to p_s. As p_s increases, more and more suppliers exist with a reservation price less than or equal to p_s. The curve we described corresponding to the special case of $k = 1$ then represents zero supplier heterogeneity, or a uniformly homogeneous supplier base, where all supply has a reservation price of $\hat{p}$. As we increase the price-to-price response $(k - 1)$ from zero, the supply base becomes increasingly heterogeneous, and we would like to understand how this heterogeneity in the supply base influences decision making and profitability.

At the opposite extreme, as $k \to \infty$, observe that the supply rate $K(p_s)$ approaches the fixed value of $a\hat{p}^{-b}$ for any price value p_s, i.e., the price-supply curve becomes flat. Thus, for any possible supply price p_s such that $p_s^0 \leq p_s \leq \hat{p}$, the resulting supply rate equals $a\hat{p}^{-b}$, which is the demand rate when $p_c = \hat{p}$. So, as the price-to-price response approaches infinity, for any supply price the producer obtains a fixed supply. Thus, $\hat{p}$ becomes the only rational choice for the selling price, while the optimal supply price equals p_s^0.

Given any p_c then, which must be greater than or equal to $\hat{p}$, for any finite $k > 1$, the supply-price curve permits matching the supply level to the demand level using a price less than or equal to $\hat{p}$. This is possible because of the heterogeneity of supplier reservation prices. In other words, there is increasing value in the heterogeneity of the supplier base, which is consistent with Proposition 4.7.

We can gain some additional insight into the influence of the price-to-price response parameter k by considering the case in which $b = 2$. When $b = 2$, the functional form of the average annual profit equation permits obtaining closed-form expressions for the optimal selling and supply prices. We can show that p_c^{opt} is strictly decreasing in k whether it equals the stationary point solution p_c^* or the boundary value p_c^0. However, the rate of decrease in p_c^{opt} is different in these two cases, and a value of k exists, denoted by $\bar{k}$, at which the optimal p_c^{opt} switches from p_c^* to p_c^0. In Figure 4-7, which illustrates p_c^{opt} as a function of k for an example parameter set, this point can be clearly seen as a breakpoint between two smooth functions. The value of this breakpoint, $\bar{k}$, corresponds to the value of k at which p_c^* equals p_c^0. Hence, for $b = 2$, we have

$$\bar{k} = \frac{-\alpha + \sqrt{\alpha^2 + 4a\hat{p}(2ac + p_s^0\sqrt{2Fha})}}{2a\hat{p}}, \tag{4–11}$$

where $\alpha = ap_s^0 + \hat{p}\sqrt{2Fha} + 2a(c + \hat{p})$.[6] In the example shown in Figures 4-7, 4-8, and 4-9, the value of $\bar{k}$ equals 1.686.

The average annual profit takes the following form for $b = 2$:

$$\pi(p_c^{opt}) = \begin{cases} \frac{(ak-\sqrt{2Fha})^2}{4a(c+k\hat{p})}, & \text{if } p_c^* < p_c^0, \\ a\left(\frac{k(\hat{p}-p_s^0)}{k-1} - c\right)\left(\frac{k-1}{k\hat{p}-p_s^0}\right)^2 - \frac{k-1}{k\hat{p}-p_s^0}\sqrt{2Fha} & \text{if } p_c^* \geq p_c^0. \end{cases} \tag{4–12}$$

By Proposition 4.7, the optimal average annual profit is increasing in k. This behavior is illustrated in Figure 4-8 for $b = 2$. When $k < \bar{k}$, the optimal selling price equals p_c^* and

[6] $\bar{k}$ is the positive root of the resulting quadratic equation when solving for $p_c^* - p_c^0 = 0$.

the optimal profit takes the form of the top equation in (4–12). When $k \geq \bar{k}$, the optimal selling price is p_c^0, and the optimal profit takes the form of the bottom equation in (4–12). As k approaches infinity, the optimal selling price approaches $\hat{p}$. Hence, we have

$$\lim_{k\to\infty} \pi(p_c^0) = \frac{a(\hat{p} - p_s^0 - c)}{\hat{p}^2} - \frac{\sqrt{2Fha}}{\hat{p}}. \tag{4–13}$$

For the example depicted in Figure 4-8, the above limiting value equals approximately 1103.75.

Recall that we defined δ^{opt} as equal to $p_c^{opt} - p_s^{opt} - c$ (δ^{opt} is not, therefore, the maximum value of $\delta = p_c - p_s - c$, as the solutions that give the maximum average annual profit and the maximum net revenue values are not likely to coincide). As noted in Proposition 4.6, we can show that δ^{opt} is decreasing in k when the optimal selling price equals p_c^0. In Figure 4-9, observe that δ^{opt} is decreasing for $k \geq \bar{k}$ (to the right of the breakpoint), where p_c^{opt} equals p_c^0. In this example, δ^{opt} is increasing when $p_c^{opt} = p_c^*$, i.e., when $1 < k < \bar{k}$. The optimal supply price p_s^{opt} equals p_s^0 when $k \geq \bar{k}$, and equals p_s^*, otherwise. Note that p_c^* is decreasing in k (from Proposition 4.6), even though the associated optimal net revenue per unit is increasing in k for $1 < k \leq \bar{k}$. This can only occur when p_s^* is also decreasing in k, and doing so at a higher rate than the decrease in p_c^* (we can also show that when $b = 2$, p_s^* is decreasing in k). We can obtain a lower bound on δ^{opt} for any k as the minimum between the δ^{opt} values at the extremes of $k = 1$ and $k = \infty$. At $k = 1$, the value of δ^{opt} equals

$$\frac{2a(c + \hat{p})}{a - \sqrt{2Fha}} - \hat{p} - c, \tag{4–14}$$

while, as k approaches infinity, the optimal net revenue per unit approaches $\hat{p} - p_s^0 - c$. Thus, if

$$p_s^0 < 2\hat{p} - \frac{2a(c + \hat{p})}{a - \sqrt{2Fha}}. \tag{4–15}$$

the quantity $\hat{p} - p_s^0 - c$ provides a lower bound on δ^{opt} for any k. Otherwise, (4–14) serves as the corresponding lower bound on δ^{opt}.

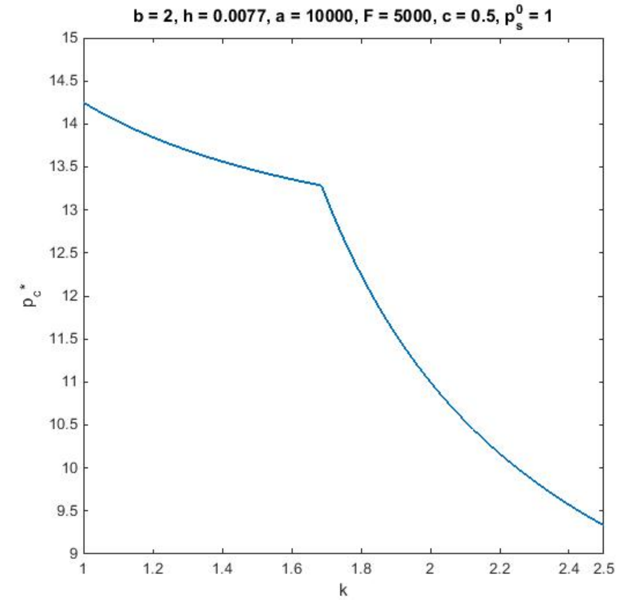


Figure 4-7. Optimal selling price with respect to k.

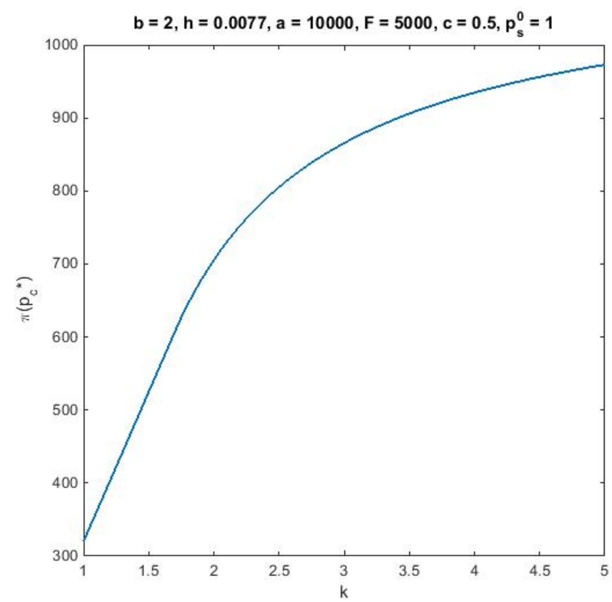


Figure 4-8. Optimal annual profit with respect to k.

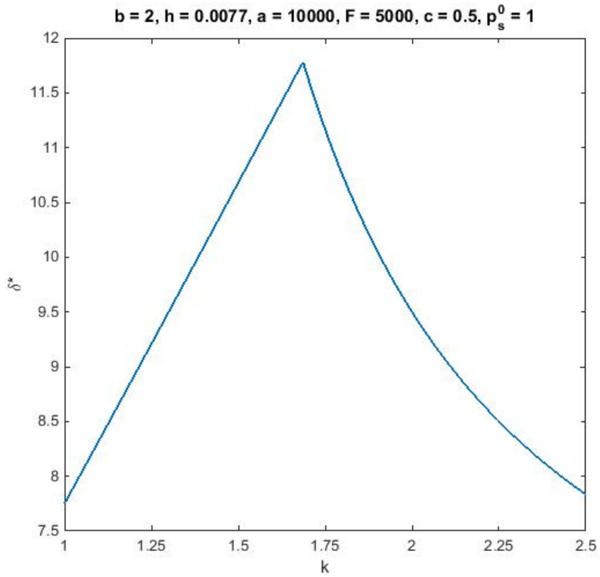


Figure 4-9. Optimal net revenue per unit with respect to k.

4.4.3 Decision Making without Accounting for Supply-Price Relationship

This section analyzes the case in which the supply-price relationship is not accounted for by the producer (e.g, this relationship is either unknown or ignored). Because existing models do not account for price-dependent supply, we would like to explore how this lack of a formalized methodological approach can impact profit performance. Thus, we assume that the producer chooses a supply price without knowledge of how it affects the available supply quantity. The producer then determines the end-customer price, given its chosen supply price, using an existing model that accounts for price-dependent demand (Ray et al. [62]). As a result, we can expect a mismatch between the resulting supply and demand rates. This would lead to unanticipated costs that were not accounted for in determining the selling price. We wish to quantify these additional costs in order to characterize the value of information regarding the supply-price relationship.

Suppose the supply price, p_s, is chosen heuristically by the producer without knowledge of the supply-price relationship. This pre-determined supply price p_s implies a supply rate of $K(p_s)$, that is initially unknown to the producer. The producer subsequently determines the selling price p_c that maximizes the average annual profit given in (4–2) for the given supply price p_s, i.e., via the EOQ model with price-dependent demand from Ray et al. [62], assuming a variable cost of $p_s + c$. One of three possible scenarios results: (1) $K(p_s) > D(p_c)$, i.e., the resulting supply rate exceeds the demand rate, in which case we can expect that the producer would collect only an amount of supplied components sufficient to satisfy demand (and, perhaps, turn away a subset of suppliers with a reservation price lower than the chosen supply price); (2) $K(p_s) < D(p_c)$, i.e., the resulting demand rate exceeds the supply rate, in which case the producer can only satisfy a portion of its end-customer demand with the available supply; (3) $K(p_s) = D(p_c)$, i.e., the chosen selling price may lead to a demand rate that exactly matches the supply rate, and the demand can be fully satisfied.

When the chosen prices imply $K(p_s) > D(p_c)$, presumably the producer would only buy a sufficient quantity of input components from suppliers to meet its end-customer demand. Hence, the average amount paid by the producer to suppliers would equal $p_s D(p_c)$. However, a supply price p'_s exists that provides a supply rate $K(p'_s)$ such that $K(p'_s) = D(p_c)$. Note that $p'_s < p_s$, as the supply rate function is increasing in price, and $p'_s = K^{-1}(D(p_c))$. All else being equal, the producer could have chosen a supply price of p'_s, and the average amount paid to suppliers would have been equal to $p'_s D(p_c)$. Thus, the reduction in average annual profit that results from not accounting for the supply-price relationship equals $(p_s - p'_s)D(p_c)$.

If the selling price p_c results in a case where $K(p_s) < D(p_c)$, the producer cannot meet the demand rate at the price p_c. Thus, the availability of input components would restrict the rate of satisfied demand to $K(p_s)$. However, a selling price p'_c exists such that $D(p'_c) = K(p_s)$. Since the demand rate is a decreasing function of the selling price, we have $p'_c > p_c$. At a selling price of $p'_c = D^{-1}(K(p_s))$, however, the producer could achieve a higher profit margin. All else being equal, the resulting average annual profit would increase by $(p'_c - p_c)K(p_s)$, which provides a lower bound on the cost penalty for not accounting for the price-supply relationship.

Suppose $K(p_s) = D(p_c)$, and note that this can only occur if $D(p_c) = ap_c^{-b} = K(p_s) = a(k-1)^b(k\hat{p} - p_s)^{-b}$. This only results if the value of p_c that maximizes (4–2) at the chosen value of p_s happens to equal $(k\hat{p} - p_s)/(k-1)$, which is highly unlikely in the absence of information about the supply-price relationship (and, therefore, without knowing k). When $b = 2$, for example, the closed-form expression for the value of p_c that maximizes (4–2) for a given p_s is equal to

$$p_c^* = \frac{2a(c + p_s)}{a - \sqrt{2Fha}}. \tag{4–16}$$

In this case, the only value of p_s that leads to $K(p_s) = D(p_c^*)$ is the unique value

$$\bar{p}_s \doteq \frac{k\hat{p}(a - \sqrt{2Fha}) - 2a(k-1)c}{a(2k-1) - \sqrt{2Fha}}. \tag{4–17}$$

Although $K(\bar{p}_s) = D(p_c^*)$, the pair $(\bar{p}_s, p_c^*)$ does not maximize the true average annual profit Equation (4–4), as p_c^* does not equal the optimal value from Equation (4–6). Thus, using the EOQ model with price-sensitive demand only, it is impossible for the producer to choose a pair of prices (p_s, p_c) such that the supply rate equals the demand rate and the true average annual profit Equation (4–4) is maximized.

The following example illustrates the cost penalty incurred when the supply-price relationship is not properly taken into account. In this example, we assume that $D(p_c) = ap_c^{-b}$, with $b = 2$, $a = 10000$, $F = 5000$, $h = 0.0077$, $c = 0.5$ and $p_s^0 = 1$. The price-to-price response $(k-1)$ and lower bound on selling price $(\hat{p})$ equal 0.6 and 6, respectively, and are unknown to the producer. Here again we assume that the relationship between the supply and selling prices is given by $p_s(p_c) = \hat{p} - (k-1)(p_c - \hat{p})$, which results in a supply rate function equal to $K(p_s) = a(k-1)^b(k\hat{p} - p_s)^{-b}$.

Suppose the producer first determines a supply price p_s (which must be greater than or equal to p_s^0), and then determines the selling price, p_c^* using (4–16). The blue curve in Figure 4-10 shows p_c^* as a function of p_s. (Note that p_c^* is a nondecreasing linear function of p_s.) The anticipated average annual profit, which we denote as $\bar{\pi}(p_c^*)$, computed using Equation (4–2), is shown by the blue curve in Figure 4-11. This profit level, however, is not always achievable, because the supply level implied by the initial choice of p_s may be too low to satisfy the demand rate associated with the selling price p_c^*. Figure 4-12 shows the supply curve as a function of p_s and the resulting demand rate when p_s is chosen as the supply price. This figure shows that up to a particular supply price value ($\bar{p}_s$) the demand rate is higher than the supply rate, and it is lower for any p_s value that exceeds $\bar{p}_s$.

When $p_s < \bar{p}_s$, $D(p_c^*) > K(p_s)$, and the best the producer can do is to satisfy a demand rate equal to the supply rate $K(p_s)$. Therefore, the actual profit will be lower than the initially anticipated profit level. The red curve in Figure 4-11 shows the actual profit level. We can clearly see that the actual profit is lower than the anticipated level when $p_s < \bar{p}_s$, and they are equal when $p_s \geq \bar{p}_s$. When $p_s \geq \bar{p}_s$, however, the supply rate exceeds the demand rate, and the producer may choose to collect only a supply level necessary to satisfy demand.

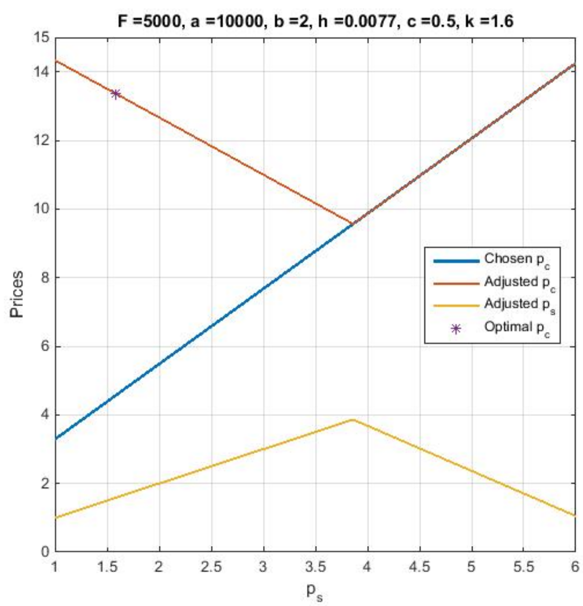


Figure 4-10. Comparison of the optimal choice of p_c with respect to predetermined supply price p_s.

If the supply-price relationship was known to the producer, however, she could adjust the selling and/or supply price to ensure that the demand and supply rates equalize.

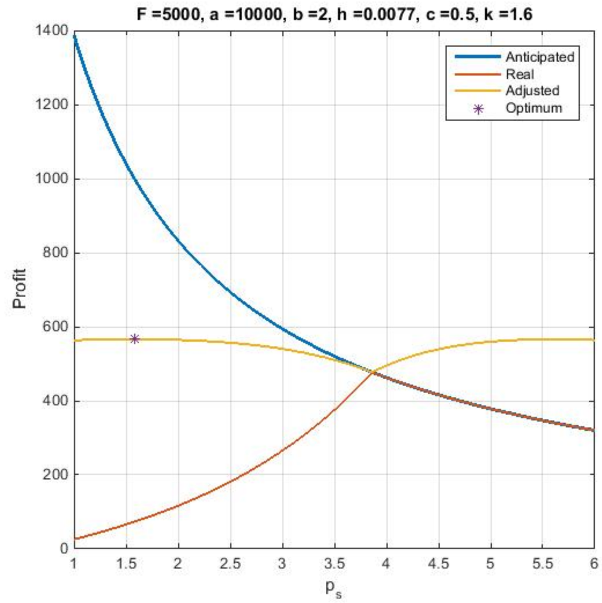


Figure 4-11. Comparison of the optimal choice of p_c with respect to predetermined supply price p_s via the effects on profit.

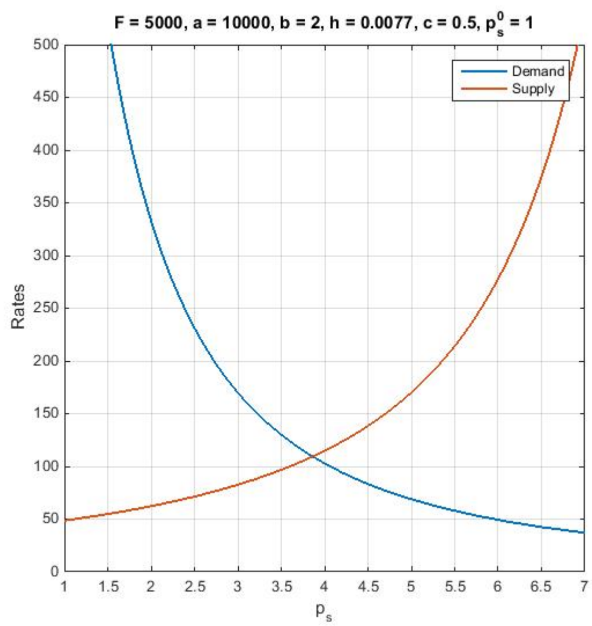


Figure 4-12. Demand and supply curves with respect to p_s.

When $p_s < \bar{p}_s$, the available supply is insufficient to satisfy the entire demand when the selling price equals p_c^*. In this case, the producer could increase the selling price to a level such that $D(p_c) = K(p_s)$. The adjusted value of p_c must be higher than p_c^*, and all else being equal, the profit level after this price adjustment would increase. In Figure 4-10, the red curve shows the adjusted selling price for $p_s < \bar{p}_s$ and the associated profit levels are shown in Figure 4-11 by the yellow curve. When $p_s > \bar{p}_s$, the supply price results in a supply rate that exceeds the demand rate. The producer could reduce the supply price to a level which would ensure $D(p_c^*) = K(p_s)$. The adjusted supply prices are shown in Figure 4-10 by the yellow curve. The supply price would thus decrease in order to match the demand rate when $p_s > \bar{p}_s$. The corresponding profit level is shown in Figure 4-11 by the yellow curve for $p_s > \bar{p}_s$. The difference between the yellow and red curves in Figure 4-11 corresponds to the potential increase in profit if either the supply price or selling price is subsequently adjusted to ensure that supply equals demand, as a function of the initially selected supply price p_s. The difference in the value of the yellow curve at the optimal point (denoted by the asterisk, $*$) and the value of the yellow curve at any other p_s gives the reduction in profit performance associated with using a suboptimal supply price, while adjusting either the selling price or the supply price to ensure that supply equals demand. Thus, the penalty for using a suboptimal supply price is generally quite small relative to the penalty for not ensuring equal supply and demand rates.

When the supply price is chosen such that $p_s = \bar{p}_s$, the resulting demand and supply rates happen to equal one another; hence, no supply imbalance exists in this case (this corresponds to the point at which the three curves coincide in Figure 4-11). Even though the supply and demand rates turn out to match nicely, and there is not a penalty at this point for not ensuring equal supply and demand rates, this point does not correspond to the optimal decision that the producer would make when properly accounting for the supply-price relationship. Moreover, this solution corresponds to the maximum penalty for using a suboptimal supply price. The optimal selling and supply prices are denoted in Figure 4-10 by $*$; $p_s^* = 1.5855$ and $p_c^* = 13.3576$. The optimal profit for this system is $\pi(p_s^*, p_c^*) = 566.0646$ (see Figure 4-11 denoted by $*$).

It is interesting to consider what would happen in this example problem if the producer had chosen the optimal supply price ($p_s^* = 1.5855$) and then solved the EOQ model with price-dependent demand only of Ray et al. [62]. This model prescribes a selling price equal to $2a(c + p_s)/(a - \sqrt{2Fha}) = 4.5722$ and a corresponding demand rate of 478.3542 units per unit time, with an average annual profit value of 997.6033. However, this value of supply price implies a supply rate of only 56.0466 units per unit time, which is insufficient to meet the demand rate. As a result, using this pair of prices ($p_s = 1.5855, p_c = 4.5722$) while meeting a demand rate of 56.0466 units per unit time will lead to an annual profit of only 73.678. If the producer were to adjust its selling price to the optimal value prescribed by our model ($p_c^{opt} = p_c^* = 13.3576$), it could then extract the optimal profit level of 566.0646. This example illustrates how accounting for the impacts of price-dependent supply can drastically affect profitability.

CHAPTER 5
A NEWSVENDOR PROBLEM WITH PRICE-DEPENDENT SUPPLY AND DEMAND

5.1 Motivation and Related Literature

The newsvendor problem considers planning for items for which the demand arises only in one selling season. The producer faces a random demand with a known distribution, and aims to determine the order quantity such that the expected profit is maximized for the corresponding planning period. In the standard version of the problem, all unit cost and revenue parameters are known with certainty, and the optimal ordering policy is determined by the balance between unit cost of understocking and overstocking.

In this chapter, we consider the planning problem of a producer for a single period where the end-item demand is random and the supply quantity depends on the unit price offered by the producer to suppliers. We first study the case where only supply is price-dependent. In particular, we characterize the optimal supply-pricing decisions for cases in which the supply quantity is defined by a linear or an isoelastic function of the supply price. We then investigate the differences in the optimal policies between the standard newsvendor problem and our model. We also present some results for the case where the supply is price-dependent and random. Later in Section 5.3 we study the case where the demand is also price-dependent.

Our work in this chapter is related to several research streams consisting of various extensions of the newsvendor problem. The most relevant work to our study includes newswendor problems with price-dependent demand and problems with supply discounts. As a practical application area we consider the remanufacturing setting, and we review the existing works in this area which are most related to our study in this chapter.

5.1.1 Newsvendor Problem with Price-Dependent Demand

The effect of pricing decisions on demand values has received a significant interest within the production planning and inventory management research. The concept of price-dependent demand was first introduced within the newsvendor setting by Whitin in [89]. In newsvendor problems with price-dependent demand the selling price is also a decision variable in addition to the order quantity. Some examples include Mills [53], Karlin and Carr [40], Zabel [91], Lau and Lau [46], and Petruzzi and Dada [59]. Petruzzi and Dada [59] recapitulate the existing results while introducing insightful formulations of the problem for various demand models. In contrast to these studies, Sana [69] considers the newsvendor problem with price-dependent demand where the selling price is a random variable.

In all of the aforementioned studies, the unit cost of the supply is exogenously determined, and the producer can acquire an infinite amount of supply by paying the same amount for each unit of supply. In our study, the unit cost of supply is given by the unit price offered by the producer to suppliers, and it is a decision variable for the producer.

5.1.2 Newsvendor Problem with Supplier Discounts

The literature on newsvendor problems contains a significant number of research articles addressing the relationships between the order quantity and discount offers initiated by the supplier. There are two types of discount schemes studied widely in the literature; all-unit discounts and incremental discounts. In the case of all-unit discounts, the producer pays a higher unit price to the suppliers unless the order quantity exceeds a predetermined threshold. If the producer orders a quantity greater than the threshold, the unit cost drops. Some studies considering an all-unit discount scheme include Jucker and Rosenblatt [38], Burnetas, Gilbert, and Smith [10], Haji, Haji, and Darabi [31], Altintas, Erhun, and Tayur [2], and Zhang [94]. In contrast, when the producer pays a lower unit price only for those units ordered in excess of the threshold quantity, this is referred to as an incremental discount scheme. Examples from the literature in a newsvendor setting include Lin and Kroll [49], Guder, Zydiak, and Chaudhry [28], and Ji and Shoa [37]. Lin and Kroll [49] study both types of quantity discounts in an effort to determine the ordering policy that maximizes expected profit and also guarantees that the probability of reaching a certain profit level exceeds a target level.

In all of these studies the supplier provides an incentive to the producer to buy more by reducing the price as the order quantity increases. In our model, the producer wishes to attract suppliers by offering a unit purchase price. Clearly a higher price results in a higher supplied quantity. The relationship between the supply quantity and supply cost we consider runs counter to that considered in the existing works in this particular research stream.

5.1.3 Remanufacturing Problems in Newsvendor Setting

The literature on production planning with a remanufacturing option contains a significant number of studies where this planning problem is considered within a newsvendor setting. Some examples include Robotis, Bhattacharya, and Van Wassenhove [67], Xanthopoulos, Vlachos, and Iakovou [90], and Li, Li, and Cai [48]. In all of these studies the unit cost of inputs for remanufacturing processes is known in advance.

As we pointed out in the earlier chapters, the concept of price-dependent supply is particularly relevant in reverse logistics settings. In such settings, the inputs required by a remanufacturer are owned by individual consumers who may be willing to sell their products to the remanufacturer, depending on the price offered. Some studies in the literature characterize the relationship between the quantity of the used-items available for remanufacturing and the acquisition price offered by the remanufacturer to used-item suppliers, and analyze the effects on remanufacturing decisions (see for example Guide and van Wassenhove [30], Klausner and Hendrickson [43], and Ray, Boyaci, and Aras [61]). The study of Bakal and Akcali [6] is closely related to ours, where they consider price-dependent acquisition of used vehicles as a source of part remanufacturing. Their model includes a random yield for the remanufacturable parts and a price-dependent demand. In contrast to the study of Bakal and Akcali [6], we first consider cases where end-item demand is random but not price-dependent, and later we study optimal supply-pricing decisions where the end-item demand is both price-dependent and random.

5.2 Price-Dependent Supply

We consider a single period planning problem for a producer who faces random demand for a single item. We denote the end-item demand as a random variable D with a known distribution. The probability density function and the cumulative distribution function of the demand distribution are denoted by $f(\cdot)$ and $F(\cdot)$, respectively, whereas μ_D and σ_D are, respectively, the mean and the standard deviation of this distribution. The producer obtains components from the suppliers and converts them to end-items via a production process. The number of components obtained from suppliers, equivalently the order quantity, depends on the price that the producer offers to its suppliers. Let c denote the price offered to suppliers by the producer. We assume that the order quantity is a nonnegative, nondecreasing and convex function of c, which we denote by $Q(c)$. The supplied components must be processed at a unit cost v to convert components into sellable end-items. Each end-item can be sold for a unit price denoted by p. At the end of the planning period the end-items in inventory can be salvaged for an amount of s per unit. When the end-item is out-of-stock, i.e., the producer is unable to satisfy a customer demand, a penalty cost, g, is incurred per unit unsatisfied demand to account for the loss of goodwill.

The producer seeks to determine the supply price that maximizes its expected profit. Let $\pi(c, D)$ denote the total profit of the producer when the supply price equals c and the demand is equal to D. We can write $\pi(c, D)$ as

$$\pi(c,D) = \begin{cases} pQ(c) - g[D - Q(c)] - (c+v)Q(c), & \text{if } D \geq Q(c), \\ pD + s[Q(c) - D] - (c+v)Q(c), & \text{if } D < Q(c). \end{cases} \tag{5–1}$$

Note that when the demand is greater than the supplied quantity, lost sales occur and the producer incurs a loss of goodwill cost; this is the case of understocking. When the demand is less than the quantity supplied, the producer ends up with excess inventory, which can be salvaged at the end of the planning period; this is the case of overstocking.

Let $\Pi(c)$ denote the expected profit, i.e., $\Pi(c) = E[\pi(c, D)]$. We can write this expected profit as

$$\Pi(c) = \int_0^{Q(c)} (px + s[Q(c) - x])f(x)dx + \int_{Q(c)}^{\infty} (pQ(c) - g[x - Q(c)])f(x)dx \\ -(c+v)Q(c). \tag{5–2}$$

The producer's goal is to determine a unit purchase price, c, such that its expected profit is maximized. To determine the optimal supply price, c^*, we need to consider c values that satisfy the first and second order optimality conditions, i.e., we require that $\Pi'(c^*) = 0$ and $\Pi''(c^*) < 0$. The first and second derivatives of $\Pi(c)$ are as follows.

$$\Pi'(c) = (p + g - s)Q'(c)[1 - F(Q(c))] - (c - s)Q'(c) - Q(c). \tag{5–3}$$

$$\Pi''(c) = (p + g - s)\{Q''(c)[1 - F(Q(c))] - f(Q(c))(Q'(c))^2\} \\ -(c + v - s)Q''(c) - 2Q'(c). \tag{5–4}$$

Solving the first order condition we obtain the equality (5–5). Any c satisfying (5–5) is a stationary point of the expected profit function $\Pi(c)$. Note that $\Pi(c)$ is strictly concave when $\Pi''(c) < 0$ for all c. Note also that this depends on both the characteristics of the demand distribution and the form of the supply-price function $Q(c)$. When $\Pi(c)$ is concave, a positive c value satisfying (5–5) is a maximizer of $\Pi(c)$.

$$F(Q(c)) = \frac{p + g - c - v - \frac{Q(c)}{Q'(c)}}{p + g - s}. \tag{5–5}$$

Here $F(Q(c))$ gives the probability that the customer demand will be fully satisfied when the order quantity equals $Q(c)$ or, equivalently, when the price offered to the suppliers by the producer equals c. This probability of satisfying the demand is also known as the service level of the producer. The value of c that makes the service level equal to the right-hand side of (5–5), which can be referred as the critical fractile, is a stationary point of $\Pi(c)$. Note that if this problem were the standard newsvendor problem, where the unit purchasing price, c, is fixed and an infinite amount of supply is available at that price, the critical fractile would be equal to $(p+g-c-v)/(p+g-s)$, where the unit overstocking cost, c_o, is $c+v-s$ and the understocking cost, c_u, is $p+g-c-v$. Hence the solution to the standard newsvendor problem is given by

$$F(Q^*) = \frac{c_u}{c_u + c_o}. \tag{5–6}$$

Note that in the standard newsvendor problem, the marginal cost of the next unit to be supplied is equal to the fixed unit supply cost, c. As shown in Proposition 5.1, this marginal cost is equal to $c + Q(c)/Q'(c)$ when the supply quantity depends on the unit price offered by the producer. Hence, in our model, the marginal understocking cost, c_u, and the marginal overstocking cost, c_o, are $p+g-v-c-Q(c)/Q'(c)$ and $c+Q(c)/Q'(c)+v-s$, respectively, when the price offered to suppliers equals c. Consequently, this shows that the interpretation of the critical fractile remains the same as in the standard newsvendor problem where the order quantity is chosen based on the balance between overstocking and understocking costs. Thus, (5–6) still holds where the optimal order quantity Q^* is given by $Q(c^*)$ in our model, and where c^* denotes the optimal supply price.

Proposition 5.1. *Given that the current choice of supply price is c, the marginal cost of supplying the next unit equals $c + Q(c)/Q'(c)$.*

Proof. Let $c(q)$ denote the unit supply price that the producer needs to offer to suppliers to receive an amount of supply of q. That is $c(q)$ denotes the inverse of the function $Q(c)$ given a supply quantity level q, i.e., $c(q) = Q^{-1}(q)$. For a total supply quantity of q the producer spends $c(q)q$. The marginal cost of the next unit is given by

$$\frac{d(c(q)q)}{dq} = c(q) + qc'(q).$$

By the inverse function theorem, we can write

$$c'(q) = \frac{1}{Q'(c(q))}.$$

Selecting q to be the quantity received by setting the supply price at c, we obtain

$$\left.\frac{d(c(q)q)}{dq}\right|_{q=Q(c)} = c + Q(c)/Q'(c).$$

$\square$

Note that the expected profit function $\Pi(c)$ is concave in c when $\Pi''(c) \leq 0$. This concavity condition of $\Pi(c)$ is satisfied when

$$\frac{Q''(c)}{Q'(c)^2} - \left(\frac{1}{Q'(c)^2\bar{F}(Q(c))}\right)\frac{(c+v-s)Q''(c)+2Q'(c)}{p+g-s} \leq \frac{f(Q(c))}{\bar{F}(Q(c))}, \tag{5–7}$$

where $\bar{F}(\cdot) = 1 - F(\cdot)$. Note that the right-hand side of the inequality (5–7) is the failure rate of the demand distribution and therefore, always positive.

Next we consider two special cases for the supply-price function. First we assume that the relationship between the supply quantity and price is linear (see Section 5.2.1). Then in Section 5.2.2 we assume we have an isoelastic supply-price function. Later in Section 5.2.4 we consider the case where the supply quantity is both price-dependent and random.

5.2.1 Linear Supply-Price Function Case

In this section we analyze a special case where the price-supply function is linear. Here we assume that $Q(c) = \beta c - \alpha$, where $\beta > 0$ and $\alpha \geq 0$. Let c_0 be the maximum price where $Q(c_0) = 0$. We refer to c_0 as the threshold price, where the producer can not purchase any supply if the price offered to suppliers is below this threshold price c_0. Clearly, $c_0 = \alpha/\beta$ for this particular supply-price relationship case. Note that for a linear supply-price function, $Q''(c) = 0$ and the right hand side of the inequality (5–7) is negative, implying that the expected profit function is strictly concave for all c for the case of a linear supply-price relationship. Thus, c^* satisfying (5–5) is the unique maximizer of $\Pi(c)$. Note that this result is independent of the demand distribution. Thus, we have

$$\begin{aligned} F(Q(c^*)) &= \frac{p+g-c^*-v-\frac{\beta c^*-\alpha}{\beta}}{p+g-s}, \\ &= \frac{p+g-c^*-v-(c^*-c_0)}{p+g-s}. \end{aligned} \tag{5–8}$$

Figure 5-1 illustrates an example of the expected profit function $\Pi(c)$. Here the problem parameters are chosen as follows: $p = 10$, $g = 5$, $s = 3$, and $v = 1$. The demand is assumed to follow a normal distribution with $\mu_D = 2000$ and $\sigma_D = 100$. The parameters β and α of $Q(c)$ are 500 and 1000, respectively. Here the threshold price, c_0 equals 2, and the optimal supply price c^* is 5.921.

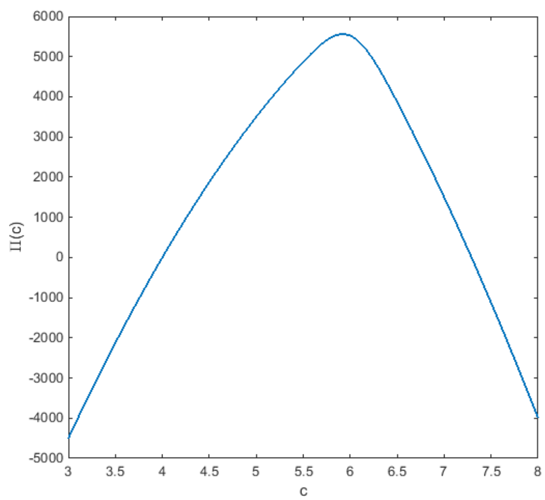


Figure 5-1. Expected profit function $\Pi(c)$ with respect to c, where $Q(c)$ is linear with $\beta = 500$ and $\alpha = 1000$.

5.2.2 Isoelastic Supply-Price Function Case

In this section, we assume that $Q(c)$ takes the following isoelastic form: $Q(c) = \alpha c^{\beta}$ where $\alpha > 0$ and $\beta > 1$. In this case we have $Q'(c) = \alpha\beta c^{\beta-1}$ and $Q''(c) = \alpha\beta(\beta-1)c^{\beta-2}$. For this supply-price relationship, the stationary point condition (5–5) becomes the following.

$$F(Q(c)) = \frac{p + g - v - \frac{\beta+1}{\beta}c}{p + g - s}. \tag{5–9}$$

Next we examine the concavity condition of the expected profit $\Pi(c)$, i.e. $\Pi''(c) \leq 0$. For analytical purposes, we multiply both sides of the inequality (5–7) with $Q(c)$. On the left-hand side of the inequality we obtain the generalized failure rate of the demand distribution as a function of c. Hence, inequality (5–7) can be written as follows.

$$Q(c)\frac{Q''(c)}{Q'(c)^2} - \left(\frac{Q(c)}{Q'(c)^2\bar{F}(Q(c))}\right)\frac{(c + v - s)Q''(c) + 2Q'(c)}{p + g - s} \leq G(Q(c)), \tag{5–10}$$

where

$$G(Q(c)) = \frac{Q(c)f(Q(c))}{\bar{F}(Q(c))}. \tag{5–11}$$

Note that $G(Q(c))$ is the generalized failure rate of the demand distribution. For this particular supply-price function case, we have $Q(c)\frac{Q''(c)}{Q'(c)^2} = (\beta - 1)/\beta$. Therefore (5–10) reduces to

$$\frac{\beta - 1}{\beta} - \left(\frac{1}{\bar{F}(Q(c))}\right) \frac{\frac{\beta+1}{\beta}c + \frac{\beta-1}{\beta}(v - s)}{p + g - s} \leq G(Q(c)). \tag{5–12}$$

Proposition 5.2 shows that when the demand distribution has an increasing generalized failure rate, there exists a value of c, denoted by $\underline{c}$, such that $\Pi(c)$ is concave for any $c \geq \underline{c}$. Furthermore, the stationary point satisfying (5–9) is the only stationary point and it is greater than $\underline{c}$, which implies that it is the maximum of $\Pi(c)$.

Proposition 5.2. *Given that the demand distribution has an increasing generalized failure rate (IGFR), then there exists a* $\underline{c}$, *such that for any* $c \geq \underline{c}$, $\Pi(c)$ *is concave. Furthermore, the stationary point satisfying* (5–9) *is the unique maximizer of* $\Pi(c)$.

Proof. By our initial assumption, $Q(c)$ is increasing in c and equals zero when $c = 0$. Since the demand distribution has an IGFR,[1] the right-hand side of inequality (5–12) increases with c and is always positive. We can rewrite the left-hand side of (5–12) as follows:

$$\frac{\beta - 1}{\beta}\left(1 - \frac{v - s}{\bar{F}(Q(c))(p + g - s)}\right) - \frac{\frac{\beta+1}{\beta}c}{\bar{F}(Q(c))(p + g - s)}. \tag{5–13}$$

The term $(\beta - 1)/\beta$ on the left-hand side is a positive constant less than 1 and it is multiplied with a term that decreases as c increases. This is because

$$\lim_{c \to \infty} \bar{F}(Q(c)) = 0.$$

[1] The class of distributions with IGFR covers a wide range of distributions including some of the distributions which do not have an increasing failure rate [45]. Examples include some widely used distributions such as the normal, uniform, gamma, and Weibull distributions.

The second term also increases with c, which implies that the term in (5–13) decreases as c increases. That implies that as c increases, the left-hand side of (5–12) decreases and the right-hand side of (5–12) increases. If at $c = 0$ the inequality (5–12) is satisfied, then it is satisfied for all positive c values and $\Pi(c)$ is concave for all $c \geq 0$. If at $c = 0$ the inequality (5–12) is not satisfied, there exists a positive c value, $\underline{c}$, where (5–12) is satisfied, i.e., the profit function is concave in the interval $[\underline{c}, \infty)$. We have

$$\lim_{c \to 0} \Pi(c) = -g\mu, \quad \text{and} \quad \lim_{c \to 0} \Pi'(c) = 0.$$

We can also show that, as c approaches to infinity, $\Pi(c)$ and $\Pi'(c)$ go to negative infinity. And since there is only one inflection point $\underline{c}$, $\Pi(c)$ is unimodal. We next show that the stationary point is the maximizer of $\Pi(c)$ using the identity (5–9). From (5–9), we obtain

$$\bar{F}(Q(c)) = \frac{\frac{\beta+1}{\beta}c + v - s}{p + g - s}.$$

After substituting $\bar{F}(Q(c))$ in (5–12) we obtain

$$\left(-\frac{1}{\beta}\right)\frac{\beta+1}{\beta}c < Q(c)f(Q(c))(p + g - s), \tag{5–14}$$

which is satisfied for all $c > 0$, since the left-hand side of the inequality is always negative, whereas the right-hand side of the inequality is positive. Thus the stationary point satisfies the second order optimality condition and therefore it is the maximizer of $\Pi(c)$. $\square$

Figure 5-2 illustrates an example of the expected profit function $\Pi(c)$ when $Q(c) = 50c^2$. Here the problem parameters are chosen as follows: $p = 10$, $g = 5$, $s = 3$, and $v = 1$. The demand is assumed to follow a normal distribution with $\mu_D = 2000$ and $\sigma_D = 100$. For this case, the optimal supply price c^* is equal to 6.277.

5.2.3 Comparison with the Standard Newsvendor Problem

Next we compare our model's outcomes with the standard newsvendor problem. In the standard version of the problem, the newsvendor faces random demand following some

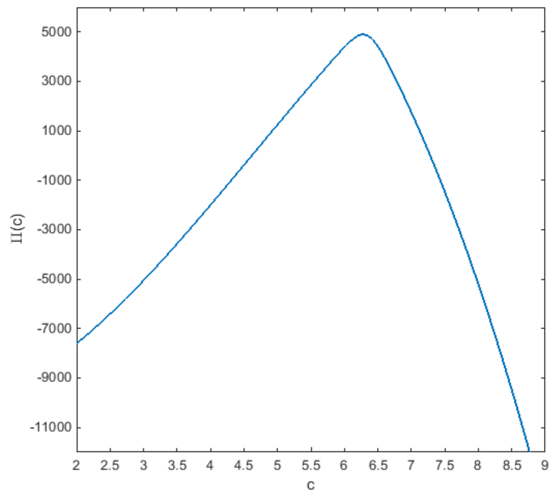


Figure 5-2. Expected profit function $\Pi(c)$ with respect to c, where $Q(c)$ is isoelastic with $\beta = 2$ and $\alpha = 50$.

distribution $F(\cdot)$ in a single period. Given the cost parameters (unit cost c, salvage value s, loss of goodwill cost g, and unit production cost v) and the unit selling price p, the optimal order quantity Q^* equates the service level to the well-known critical ratio given by the balance between average overstocking and understocking costs. More specifically, Q^* is the quantity that satisfies the following condition.

$$F(Q^*) = \frac{p+g-v-c}{p+g-s}. \tag{5–15}$$

Note that leaving all other parameters constant, the optimal order quantity Q^* can be expressed as a function of the supply price c. Hence, with a slight abuse of notation, we can write

$$Q^*(c) = F^{-1}\left(\frac{p+g-v-c}{p+g-s}\right). \tag{5–16}$$

Similarly, the optimal expected profit for this case, $\Pi(Q^*)$, can be reformulated as a function of the supply price c. We denote this optimal expected profit function as $\Pi_s(c)$.

Next we illustrate a comparison between the standard newsvendor problem solution and our model's solution using some examples, where the problem parameters are chosen as follows: $p = 10$, $g = 5$, $s = 3$, and $v = 1$. The demand is assumed to follow a normal distribution with $\mu_D = 2000$ and $\sigma_D = 100$. Table 5-1 shows our selection of various supply-price functions used in this comparison. Here first five scenarios include linear supply-price functions and the last five include isoelastic supply-price functions.

Table 5-1. Supply-price relationship examples.

Scenario No.	Supply-price function	α	β
1	$Q(c) = \beta c - \alpha$	1000	500
2	$Q(c) = \beta c - \alpha$	1500	750
3	$Q(c) = \beta c - \alpha$	1000	750
4	$Q(c) = \beta c - \alpha$	500	750
5	$Q(c) = \beta c - \alpha$	1000	1000
6	$Q(c) = \alpha c^{\beta}$	100	1.5
7	$Q(c) = \alpha c^{\beta}$	50	2
8	$Q(c) = \alpha c^{\beta}$	100	2
9	$Q(c) = \alpha c^{\beta}$	150	2
10	$Q(c) = \alpha c^{\beta}$	100	3

Figure 5-3 shows the optimal solutions for the scenarios in Table 5-1. The optimal expected profit values are all strictly below the $\Pi_s(c)$ curve. This implies that given the optimal supply price c^* of our model, the expected profit obtained in the standard newsvendor solution is always higher than the expected profit obtained in our model, where the supply quantity depends on the supply price (also see Table 5-3). This difference in the optimal expected profit of the two models arises from the fact that, in our model, the available supply quantity depends on the price offered to the suppliers, and the producer can receive more supply only if she offers a higher price for each supply unit to be purchased. However, in the standard newsvendor setting, it is assumed that there is an infinite amount of supply at the given unit cost, c. For the same unit supply price, the standard newsvendor can obtain more supply and satisfy a higher proportion of the demand, and hence, achieve a higher expected profit. Therefore, the expected profit level of the standard newsvendor is not achievable by the producer in our problem setting.

Table 5-2 show the optimal supply prices and the marginal costs of supplying an additional unit for the scenarios presented in Table 5-1. A higher marginal supply cost actually indicates that the corresponding scenario leads to a low service level and therefore

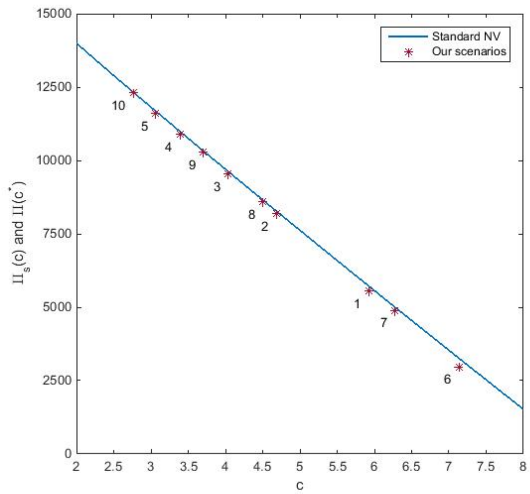


Figure 5-3. Optimal expected profit function $\Pi_s(c)$ and optimal expected profit values for price-dependent supply case.

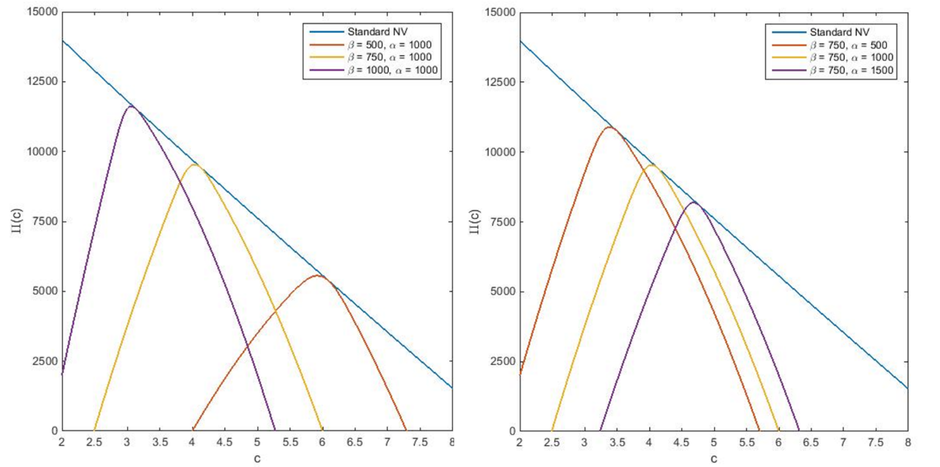
a low volume operation. The resulting optimal expected profit is also lower for the cases with higher marginal supply costs.

Table 5-2. Marginal cost of procuring next unit of supply at c^*.

Sc. #	c^*	$c^* + \frac{Q(c^*)}{Q'(c^*)}$
1	5.921	9.842
2	4.684	7.369
3	4.036	6.738
4	3.388	6.108
5	3.064	5.128
6	7.137	21.411
7	6.277	12.554
8	4.502	9.003
9	3.700	7.400
10	2.762	4.144

Figures 5-4 and 5-5 illustrate the $\Pi_s(c)$ function and $\Pi(c)$ functions for the various scenarios presented in Table 5-1. In those figures, we can observe that for any c value $\Pi_s(c)$ is always greater than or equal to the corresponding $\Pi(c)$ for all scenarios. Here $\Pi_s(c)$ forms an upper envelope for all cases of $\Pi(c)$.

Figures 5-4 and 5-5 also demonstrate how the supply market conditions affect the expected profit of the producer. Figure 5-4 illustrates the scenarios with linear

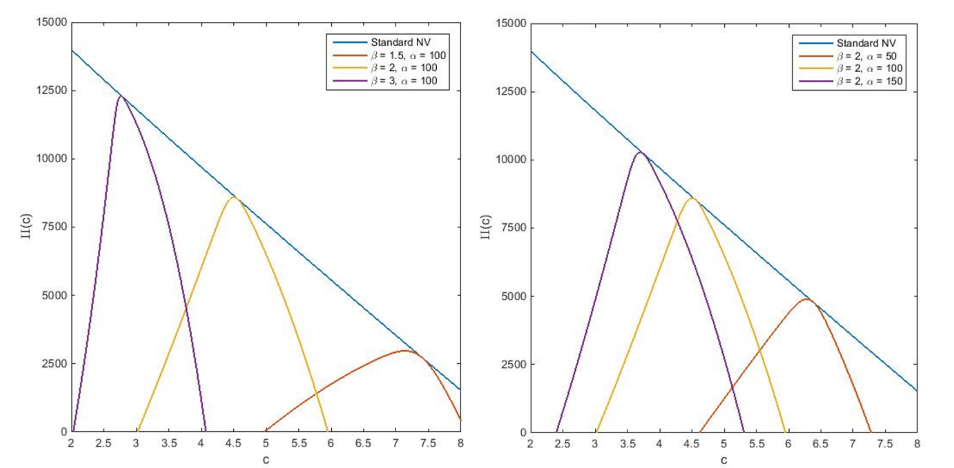


(a) Linear $Q(c)$ with $\alpha = 1000$ (b) Linear $Q(c)$ with $\beta = 750$

Figure 5-4. Optimal expected profit function $\Pi_s(c)$ and expected profit functions $\Pi(c)$ with respect to c when $Q(c)$ is linear.

supply-price functions. Figure 5-4a shows the expected profit functions for scenarios 1, 3, and 5, where β, the slope of the function $Q(c)$, takes values 500, 750 and 1000, respectively, and α stays constant at level of 1000. Here scenario 1 has the lowest rate of increase in supply when the supply price increases, whereas scenario 5 has the highest increase rate in supply. Hence, for the same amount of supply, we need to offer a higher price in scenario 1 compared to the others. Scenario 1 is also the case with the highest threshold price. This results in an optimal supply price for scenario 1 that is higher than in the other two scenarios. Note that the highest difference between the standard newsvendor's optimal expected profit and the expected profit in our model occurs in scenario 1. As the slope of the supply price increases, the optimal expected profit of our model converges to the standard newsvendor's optimal expected profit. (Assuming that there exists a finite positive threshold price, as $\beta \to \infty$, the producer could access an infinite amount of supply for a unit price equal to the threshold price. This is practically the case of a standard newsvendor problem, where the unit cost of supply equals the threshold price.) Note also that as β increases the function $\Pi(c)$ gets steeper, which means any small change in the supply price c results in bigger changes in the expected profit. When the producer offers a slightly higher price than the optimal supply price, the decrease in the expected profit is highest in scenario 5. Here the total cost of supply increases faster compared with other scenarios, since the same amount of increase in the supply price results in a higher increase in the supply quantity. Hence, as the responsiveness of the supply market to the increase in the offered supply price increases, the sensitivity of the expected profit to the changes in the supply price increases as well.

Figure 5-4b shows the expected profit functions for scenarios 2, 3, and 4, where β, the slope of the function $Q(c)$, stays constant at level of 750, and α takes values 1500, 1000 and 500, respectively. While the slope of the supply-price function stays constant, the shape of the function $\Pi(c)$ stays the same among these three scenarios. The optimal supply prices for scenarios 2 and 3 and for scenarios 3 and 4 differ almost by the same amount which is equal to the difference in the threshold prices (threshold prices differ by 2/3). Here the marginal response of the supply market to the changes in supply price is the same for the three scenarios. We obtain a lower optimal supply price as the threshold price decreases.



(a) Isoelastic $Q(c)$ with α= 100 (b) Isoelastic $Q(c)$ with β = 2

Figure 5-5. Optimal expected profit function $\Pi_s(c)$ and expected profit functions $\Pi(c)$ with respect to c when $Q(c)$ is isoelastic.

Figure 5-5 illustrates the scenarios with isoelastic supply-price functions. In Figure 5-5a, we illustrate the expected profit functions for scenarios 6, 8, and 10, where β takes values 1.5, 2 and 3, respectively, and α stays constant at a level of 100. Scenario 6, with the lowest β value, has the highest optimal supply price. As β increases, to obtain the same amount of supply we need pay the lowest unit price in scenario 8 compared to scenarios 6 and 7. Figure 5-4b shows the expected profit functions for scenarios 7, 8, and 9, where β stays constant at level of 2, and α takes the values 50, 100 and 150, respectively. Unlike the case of the linear supply-price function, the parameter α affects the rate of increase in the supply quantity as the supply price increases. Therefore, changes in α also change the shape of $\Pi(c)$, and as it increases $\Pi(c)$ becomes steeper. We can observe that, as α increases, the optimal supply price c^* decreases, and the optimal expected profit increases. Note also that as β and/or α increase, the responsiveness of the supply market to changes in the supply price increases as well. Therefore, in cases with higher β or higher α values we obtain expected profit functions that are more sensitive to changes in the unit supply price.

Table 5-3. Optimal solutions to the scenarios presented in Table 5-1 compared with standard newsvendor solutions.

					Standard NV		
Sc. #	c^*	$Q(c^*)$	$\Pi(c^*)$	SL(c^*)	Q^*	$\Pi(Q^*)$	SL*
1	5.921	1960.50	5560.28	0.347	2044.89	5725.16	0.673
2	4.684	2013.23	8192.34	0.553	2075.98	8272.69	0.776
3	4.036	2026.70	9538.96	0.605	2095.56	9625.56	0.830
4	3.388	2040.63	10894.68	0.658	2119.72	10991.18	0.884
5	3.064	2064.10	11614.34	0.739	2134.90	11679.07	0.911
6	7.137	1906.66	2972.10	0.175	2018.13	3255.07	0.572
7	6.277	1969.97	4894.31	0.382	2036.81	4998.82	0.644
8	4.502	2026.35	8593.93	0.604	2081.18	8652.66	0.792
9	3.700	2053.61	10284.40	0.704	2107.28	10330.54	0.858
10	2.762	2107.95	12307.16	0.860	2152.58	12325.72	0.937

Figure 5-6 illustrates the optimal order quantity Q^* of the standard newsvendor problem as a function of c, i.e. $Q^*(c)$ (see blue curve), and the optimal supply quantity $Q(c^*)$ with its corresponding optimal supply price c^* for every scenario presented in Table

5-1. Here again we can observe that the $(c^*, Q(c^*))$ points corresponding to the optimal solutions of our model lie below the $Q^*(c)$ curve. In Table 5-3 we can see that given the optimal choice of the supply price, the optimal order quantity in the standard newsvendor solution is always greater. This result can also be observed in the critical fractile terms of both models. The numerator of the critical fractile of the price-dependent supply case has an additional term, $-Q(c)/Q'(c)$. This term is always negative, and therefore, for the same c value, the critical fractile of our newsvendor model is always less than the standard newsvendor's critical fractile. This leads our newsvendor to operate with a lower service level (also see Figure 5-7) and hence at a lower volume. This can be clearly observed in Table 5-3, where we compute the standard newsvendor's optimal order quantity, Q^* and optimal service level SL* given the optimal supply price c^* of the models with price-dependent supply.

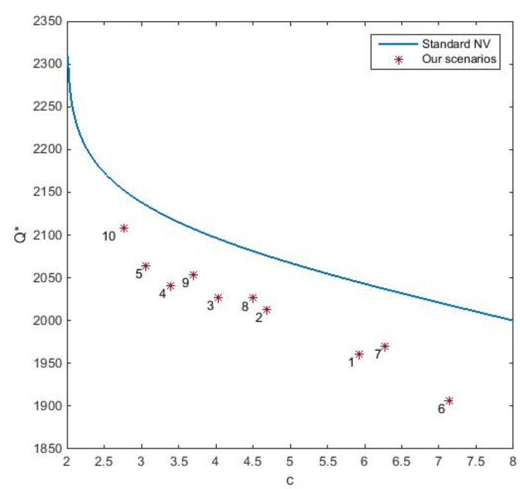


Figure 5-6. Optimal order quantity function $Q^*(c)$ and supply-price function $Q(c)$.

The difference between our model and the standard newsvendor model arises from the fact that in our model a greater supply quantity can be only attracted by offering a higher price to the suppliers. Hence, to reach to the optimal order quantity level of the standard newsvendor, our producer has to pay a higher unit price, which is clearly not the

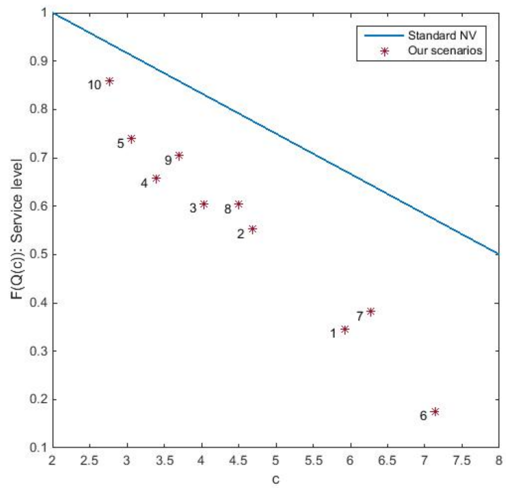


Figure 5-7. Optimal service level function $\mathrm{SL}^*(c)$ and optimal service level values for price-dependent supply case.

optimal behavior. Thus the gap between our optimal expected profit and the standard newsvendor's is caused by the increasing marginal cost of the supply.

Remark. In Figures 5-4 and 5-5, we can observe that there exists a supply price value where $\Pi_s(c) = \Pi(c)$. This supply price value corresponds to the point where the optimal order quantity of the standard newsvendor equals the price-dependent supply of our model $Q(c)$. We denote this supply price value by $\hat{c}$. Note that $\hat{c}$ is the value satisfying the following condition:

$$F(Q(c)) = \frac{p+g-v-c}{p+g-s}. \tag{5–17}$$

For a given c, the right-hand side of (5–17) is the standard newsvendor's critical fractile, and it is higher than the critical fractile obtained in our newsvendor model with price-dependent supply. Therefore, solving for the value of c satisfying (5–17) would lead to a higher order quantity and hence, to a higher supply price value c. Clearly $\hat{c} > c^*$, where c^* is the optimal choice of the supply price. The results when solving (5–17) are presented in Table 5-4. The results presented in this table clearly show that when the increasing marginal cost is ignored while choosing the optimal supply price, the volume of operation will be higher; however the optimal expected profit and the optimal service level cannot be achieved. Furthermore, depending on the supply market, the solution obtained can lead to an expected profit and a service level which have arbitrarily large optimality gaps.

Table 5-4. $\hat{c}$ values and corresponding solutions for the scenarios presented in Table 5-1.

					% Difference			
Sc. #	$\hat{c}$	$Q(\hat{c})$	$\Pi(\hat{c})$	SL$(\hat{c})$	c	$Q(c)$	$\Pi(c)$	SL(c)
1	6.082	2041.20	5395.41	0.320	2.726	4.116	-2.965	-7.763
2	4.765	2073.75	8105.25	0.539	1.723	3.006	-1.063	-2.425
3	4.124	2092.70	9441.27	0.591	2.181	3.257	-1.024	-2.429
4	3.487	2115.55	10779.63	0.641	2.949	3.672	-1.056	-2.524
5	3.132	2131.50	11535.29	0.728	2.200	3.265	-0.681	-1.515
6	7.399	2012.57	2727.05	0.139	3.670	5.555	-8.245	-20.753
7	6.379	2034.52	4791.19	0.369	1.625	3.276	-2.107	-3.350
8	4.560	2079.45	8530.75	0.597	1.302	2.621	-0.735	-1.209
9	3.747	2105.55	10232.60	0.698	1.257	2.529	-0.504	-0.838
10	2.781	2151.28	12285.27	0.858	0.681	2.056	-0.178	-0.244

5.2.4 Random Supply Quantity

The standard newsvendor problem assumes that, when the producer places an order, she receives the exact same amount that she orders. In this chapter, we generally assume that the supply quantity obtained by the producer depends on the price that the producer offer to its suppliers, and the response of suppliers to a given price is known with certainty. We also assume that the producer receives the exact amount given by this deterministic supply-price relationship. In this section, we study a particular case where the supply quantity is price-dependent and also random. The literature on the newsvendor problem contains some studies where the order quantity is assumed to be random. The randomness is incorporated in various ways. Some studies assume that the supplied amount follows a distribution with a mean equal to the ordered quantity (examples include Ciarallo, Akella, and Morton [11], and Okyay, Karaesmen, and Özekici [56]). There are some others which assume random yield, which means that only a random fraction of the ordered quantity arrives, or is useful in production (examples include Henig and Gerchak [35], Bassok and

Akella [7], and Dada, Petruzzi, and Schwarz [12]). To our knowledge, no study exists in the literature which assumes that the supply quantity is price-dependent and also random.

We consider the case where the supply quantity follows a distribution whose mean depends on the supply price c. We assume that the total quantity supplied given an offered price c is a nonnegative random variable with mean $\mu(c)$ and standard deviation σ. Here, $\mu(c)$ is a nonnegative nondecreasing function of c. The standard deviation of the supply is constant and known, while $h(\cdot)$ and $H(\cdot)$ are the probability density function and the cumulative distribution function of the supply distribution, respectively. Given a supply price c, $Q(c)$ gives the random supply quantity. The profit function is formulated in (5–18), where $Q(c)$ and D are both random variables.

$$\pi(c, D) = p\ \min\{Q(c), D\} - g\ [D - Q(c)]^+ + s\ [Q(c) - D]^+ - c\ Q(c) \tag{5–18}$$

Using the identity $\min\{Q(c), D\} = Q(c) - [Q(c) - D]^+$, we obtain

$$\pi(c, D) = (p - c)Q(c) - g[D - Q(c)]^+ - (p - s)[Q(c) - D]^+. \tag{5–19}$$

Our goal is to determine the supply price c such that the expected profit is maximized. The expected profit $\Pi(c)$ for this case is as follows:

$$\Pi(c) = (p - s)\mu_D - (c - s)\mu(c) - (p + g - s)E[(D - Q(c))^+]. \tag{5–20}$$

We next assume that D and $Q(c)$ are normally distributed. We define a new random variable W, where $W = D - Q(c)$. The expected profit function becomes

$$\Pi(c) = (p - s)\mu_D - (c + v - s)\mu(c) - (p + g - s)\int_0^\infty w f_W(w) dw. \tag{5–21}$$

Here $f_W(\cdot)$ is the normal density function with the mean $\mu_D - \mu(c)$ and variance $\sigma^2 + \sigma_D^2$. Let L denote the standard normal loss function $L(z) = \int_z^\infty (x - z)\phi(x)dx$, where ϕ is the standard normal density function. We denote the corresponding cumulative distribution function by Φ, and $\bar{\Phi}(x) = 1 - \Phi(x)$. We have $L(z) = \phi(z) - z\bar{\Phi}(z)$ and $L'(z) = -\bar{\Phi}(z)$.

Hence, we can reformulate the expected profit $\Pi(c)$ using the standard normal loss function.

$$\Pi(c) = (p-s)\mu_D - (c-s)\mu(c) - (p+g-s)L(z(c))\sqrt{\sigma^2+\sigma_D^2}. \tag{5–22}$$

where $z(c) = (\mu(c) - \mu_D)/\sqrt{\sigma^2+\sigma_D^2}$. Noting that $dL(z(c))/dc = -z'(c)\bar{\Phi}(z(c))$ and $z'(c) = \mu'(c)/\sqrt{\sigma^2+\sigma_D^2}$, we next present the first and second derivatives of $\Pi(c)$.

$$\Pi'(c) = -\mu(c) - (c-s)\mu'(c) + (p+g-s)\left(\sqrt{\sigma^2+\sigma_D^2}\right)z'(c)\bar{\Phi}(z(c)). \tag{5–23}$$

$$\Pi''(c) = -2\mu'(c) - (c-s)\mu''(c) + (p+g-s)\left[\mu''(c)\bar{\Phi}(z(c)) - \frac{(\mu'(c))^2}{\sqrt{\sigma^2+\sigma_D^2}}\phi(z(c))\right]. \tag{5–24}$$

Solving for the first order optimality condition yields equation (5–25). The value of c satisfying (5–25) is a stationary point of $\Pi(c)$. Here $\Phi(z(c))$ gives the probability that the supply is greater than the demand. Note that the right-hand side of this equation is very similar to the ratio we obtained in Equation (5–5) at the beginning of this section. The ratio $Q(c)/Q'(c)$ for the deterministic case is replaced by $\mu(c)/\mu'(c)$ in the random supply case. In this case, the marginal understocking cost, c_u, is given by $p+g-c-v-\mu(c)/\mu'(c)$, and the marginal overstocking cost, c_o equals $c+\mu(c)/\mu'(c)+v-s$.

$$\Phi(z(c)) = \frac{p+g-c-v-\frac{\mu(c)}{\mu'(c)}}{p+g-s}. \tag{5–25}$$

$\Pi(c)$ is concave in c when $\Pi''(c) \leq 0$ for all $c > 0$. The second order optimality condition is satisfied when the following inequality holds.

$$\frac{\mu''(c)}{\mu'(c)^2} - \left(\frac{1}{\mu'(c)^2\bar{\Phi}(z(c))}\right)\frac{(c-s)\mu''(c)+2\mu'(c)}{p+g-s} \leq \frac{\phi(z(c))}{\bar{\Phi}(z(c))\sqrt{\sigma^2+\sigma_D^2}}. \tag{5–26}$$

Note that the condition (5–26) depends on the form of the function $\mu(c)$ and the characteristics of the distribution of the random variable W. When the function $\mu(c)$ takes a linear form as described in Section 5.2.1, $\Pi(c)$ is concave in c independent of the distribution of W. When the function $\mu(c)$ takes an isoelastic form as described in Section 5.2.2, concavity of $\Pi(c)$ also depends on the distribution of W. By Proposition 5.2 in

Section 5.2.2, we know that when the demand distribution has an increasing generalized failure rate, we can obtain a unique maximizer of the expected profit. Since W follows a normal distribution and the normal distribution has an increasing generalized failure rate, there exists a unique stationary point c^* that satisfies the inequality (5–26).

The results in this section show that, in a single period setting, the random supply quantity does not affect the optimal policy that is characterized by the balance between marginal overstocking and understocking costs. Our findings here coincide with the ones in the study of Ciarallo, Akella, and Morton [11]. That is, the producer has to decide on a supply price assuming that she will receive the expected supply quantity.

5.3 Price-Dependent Supply and Demand

We next study the case where the demand is also price-sensitive. In this model, the selling price p is also a decision variable in addition to the supply price c. Here we consider two demand models: an additive model ($D(p) = y(p) + \epsilon$ where $y(p) = a - bp$ for $a, b > 0$ and ϵ is the random component) and a multiplicative model ($D(p) = y(p)\epsilon$ where $y(p) = ap^{-b}$ for $a > 0$, $b > 1$ and ϵ is the random component). We assume that ϵ is a random variable between A and B, where $B > A$. For the additive demand case, we require $A > -a$, and for the multiplicative demand case, we require $A > 0$. The probability density function and the cumulative distribution function of ϵ are denoted by $f(\cdot)$ and $F(\cdot)$, respectively. We also let μ and σ represent the mean and the variance of ϵ, respectively.

The demand models considered in this section are the same as those described in the study of Petruzzi and Dada [59], who consider the newsvendor problem where only demand is price-dependent. Our goal is to investigate the differences caused by the price-dependence of the supply on the optimal pricing decisions compared to the case where only demand is price-sensitive.

5.3.1 Additive Demand Model

In this section we consider the following demand model. The price-dependent and random demand denoted by $D(p, \epsilon)$ takes the form $y(p) + \epsilon$ where $y(p) = a - bp$ $(a, b > 0)$. For a given realization of ϵ and given supply and selling prices c and p, we can write the profit function as follows:

$$\pi(c, p, \epsilon) = \begin{cases} pQ(c) - g(D(p, \epsilon) - Q(c)) - (c + v)Q(c), & \text{if } D(p, \epsilon) > Q(c), \\ pD(p, \epsilon) + s(Q(c) - D(p, \epsilon)) - (c + v)Q(c), & \text{if } D(p, \epsilon) < Q(c). \end{cases} \tag{5–27}$$

For mathematical convenience, we define $z = Q(c) - y(p)$. Next we reformulate the profit as a function of z and p. Note that c can be expressed as a function of z and p as well. Hence, we write the supply price as a function of z and p, i.e., $c(z, p) = Q^{-1}(z + y(p))$. After necessary substitutions, the profit function can be reformulated as follows.

$$\pi(z, p, \epsilon) = \begin{cases} p(y(p) + z) - g(\epsilon - z) - (c(z, p) + v)(y(p) + z), & \text{if } \epsilon > z, \\ p(y(p) + \epsilon) + s(z - \epsilon) - (c(z, p) + v)(y(p) + z), & \text{if } \epsilon < z. \end{cases} \tag{5–28}$$

Our goal is to determine the values of z and p such that the expected profit is maximized. Once optimal z and p values (z^* and p^*) are determined, we can compute the optimal supply price c^* via $Q^{-1}(z^* + y(p^*))$. We next formulate this optimization problem where $\Pi(z, p)$ denotes the expected profit as a function of z and the selling price p.

$$\max_{z,p} \; \Pi(z, p) = \Psi(z, p) - L(z, p), \tag{5–29}$$

where

$$\Psi(z, p) = (p - c(z, p) - v)(y(p) + \mu), \tag{5–30}$$

and

$$L(z, p) = (p + g - c(z, p) - v)\Theta(z) + (c(z, p) + v - s)\Lambda(z). \tag{5–31}$$

Similar to the definitions of Petruzzi and Dada [59], here $\Psi(z, p)$ is called the riskless profit and $L(z, p)$ is the loss function. $\Theta(z)$ gives the expected shortages and $\Lambda(z)$ gives the expected leftovers, i.e.,

$$\Theta(z) = \int_z^B (x - z) f(x) dx, \tag{5–32}$$

$$\Lambda(z) = \int_A^z (z - x) f(x) dx. \tag{5–33}$$

Next consider the first and second derivatives of $\Pi(z, p)$ with respect to z.

$$\frac{\partial \Pi(z,p)}{\partial z} = -\frac{\partial c(z,p)}{\partial z}(y(p) + z) - (c(z,p) + v - s) + (p + g - s)(1 - F(z)). \tag{5–34}$$

$$\frac{\partial^2 \Pi(z,p)}{\partial z^2} = -\frac{\partial^2 c(z,p)}{\partial z^2}(y(p) + z) - 2\frac{\partial c(z,p)}{\partial z} - (p + g - s) f(z). \tag{5–35}$$

Note that for a given p value, the concavity of the expected profit function $\Pi(z, p)$ with respect to z depends on the supply-price relationship. In this section we will assume that the supply-price function $Q(c)$ takes the linear form described in Section 5.2.1, i.e., $Q(c) = \beta c - \alpha$ where $\beta, \alpha > 0$. When the supply-price relationship is linear, we have $c(z, p) = (1/\beta)(z + \alpha + a - bp)$, $\partial c(z, p)/\partial z = 1/\beta$, and $\partial^2 c(z, p)/\partial z^2 = 0$. Therefore for this case, $\Pi(z, p)$ is concave in z for a given p. Note also that when $\Pi(z, p)$ is concave in z, the z value satisfying the following condition maximizes $\Pi(z, p)$ for a given value of p.

$$F(z) = \frac{p + g - c(z,p) - v - \frac{\partial c(z,p)}{\partial z}(y(p) + z)}{p + g - s}. \tag{5–36}$$

Similar to Equation (5–5) in Section 5.2, Equation (5–36) shows that, at optimality, the probability that the uncertain portion of the demand is satisfied equals the well-known critical fractile, $c_u/(c_u + c_o)$. Note that $y(p) + z = Q(c(z, p))$ and by the inverse function theorem, $\partial c(z, p)/\partial z = 1/Q'(c(z, p))$, which shows that we obtain the same critical fractile behavior as discussed in Section 5.2. That is, the marginal understocking cost, c_u, equals $p + g - c(z, p) - v - \frac{\partial c(z,p)}{\partial z}(y(p) + z)$, and the marginal overstocking cost, c_o, equals $c(z, p) + \frac{\partial c(z,p)}{\partial z}(y(p) + z) + v - s$.

We next consider the first and second derivatives of $\Pi(z, p)$ with respect to the selling price p.

$$\frac{\partial \Pi(z,p)}{\partial p} = \left(1 - \frac{\partial c(z,p)}{\partial p}\right) y(p) + \mu - b(p - c(z,p) - v) - \Theta(z) - z\frac{\partial c(z,p)}{\partial p}. \tag{5–37}$$

$$\frac{\partial^2 \Pi(z,p)}{\partial p^2} = -b\left(2 - \frac{\partial c(z,p)}{\partial p}\right) - \frac{\partial^2 c(z,p)}{\partial p^2}(z + y(p)). \tag{5–38}$$

In the additive demand case considered in Petruzzi and Dada [59], where the unit price of supply is fixed and the supply quantity does not depend on this price, the expected profit function is concave in the selling price p for a fixed z. Equation (5–38) shows that the concavity of $\Pi(z, p)$ with respect to p depends on the form of the function $c(z, p)$. This implies that in our problem setting, the relationship between the price offered to suppliers and the supply quantity that the producer receives in response to the offered price affects the behavior of the expected profit function $\Pi(z, p)$ with respect to the selling price p. When the supply-price relationship is linear as discussed above, we have $\partial c(z, p)/\partial p = -b/\beta$ and $\partial^2 c(z, p)/\partial p^2 = 0$. Hence, when the supply-price relationship is linear, the expected profit $\Pi(z, p)$ is strictly concave in p for a given z, and for a given z the selling price p that maximizes $\Pi(z, p)$ can be written as a function of z.

$$p^*(z) = p^0(z) - \frac{(\frac{b}{\beta})(\mu + z) + \Theta(z)}{2b(1 + \frac{b}{\beta})}, \tag{5–39}$$

where

$$p^0(z) = \frac{(1 + \frac{b}{\beta})(\mu + a) + bv + (\frac{b}{\beta})(a + \alpha + z)}{2b(1 + \frac{b}{\beta})}. \tag{5–40}$$

Here $p^0(z)$ is the value of p that maximizes the riskless profit $\Psi(z, p)$ for a given z. If $(\frac{b}{\beta})(\mu + z) + \Theta(z)$ is positive, we have $p^*(z) \leq p^0(z)$. This relationship was also shown in [59] by Petruzzi and Dada for the case where only demand is price-dependent and the additive demand model is assumed. They did not, however, need any further assumptions on the problem parameters to show this result, which was first demonstrated by Mills in [53]. Note that if we assume that ϵ is a random variable with a positive mean (i.e., if $\mu > 0$), then the same result also holds for our case. Note also that both $p^0(z)$ and $p^*(z)$ are increasing in z, and as z goes to infinity $p^0(z)$ goes to infinity as well. However, $p^*(z)$ approaches a finite value as shown in (5–41). In the case where only demand is price-dependent, the price p^0 that maximizes the riskless profit is independent of z, and the optimal selling price, $p^*(z)$ approaches p^0 as z goes to infinity.

$$\lim_{z\to\infty} p^*(z) = \frac{(1+\frac{b}{\beta})a + \mu + b(v + (\alpha + a)/\beta)}{2b(1+\frac{b}{\beta})}. \tag{5–41}$$

For any given z value we know that $p^*(z)$ in (5–39) maximizes the expected profit. Using (5–39), we can reduce the number of decision variables of our maximization problem to one. In the reformulated problem, we want to determine the value of z such that $\Pi(z, p^*(z))$ is maximized. This approach is also presented by Petruzzi and Dada [59] for the case where only demand is price-dependent. Next we consider the first and second derivatives of $\Pi(z, p^*(z))$.

$$\frac{d\Pi(z, p^*(z))}{dz} = \frac{-(y(p^*(z)) + z)}{\beta} - (c(z, p^*(z)) + v - s) + (p^*(z) + g - s + \frac{z}{\beta + b})\bar{F}(z). \tag{5–42}$$

$$\frac{d^2\Pi(z, p^*(z))}{dz^2} = \frac{\bar{F}(z)}{2b(1+\frac{b}{\beta})}\left(\frac{4b}{\beta} + \bar{F}(z)\right) - \frac{2}{\beta} - f(z)\left(p^*(z) + g - s + \frac{z}{\beta + b}\right). \tag{5–43}$$

Solving for the first order condition yields the following. Any z value satisfying the condition (5–44) is a stationary point of the function $\Pi(z, p^*(z))$.

$$F(z) = \frac{p^*(z) + g - c(z, p^*(z)) - v + \frac{z}{\beta+b} - \frac{1}{\beta}(y(p^*(z)) + z)}{p^*(z) + g - s + \frac{z}{\beta+b}}. \tag{5–44}$$

Proposition 5.3. *Given that the price-dependent demand follows an additive model and the supply-price relationship is linear, and assuming that the distribution of ϵ has an increasing failure rate, there exists a $\underline{z}$ such that the function $\Pi(z, p^*(z))$ is concave for all $z \geq \underline{z}$, and the value of z satisfying (5–44) is the maximizer of $\Pi(z, p^*(z))$.*

Proof. For concavity, we require $d^2\Pi(z, p^*(z))/dz^2 < 0$, and this condition is satisfied when condition (5–45) is satisfied. (We obtain condition (5–45) using Equation (5–43).)

$$\frac{\frac{4b/\beta+\bar{F}(z)}{2b(1+b/\beta)} - \frac{2}{\beta\bar{F}(z)}}{p^*(z) + g - s + \frac{z}{\beta+b}} < \frac{f(z)}{\bar{F}(z)}. \tag{5–45}$$

When the distribution of ϵ has an increasing failure rate, the right-hand side of (5–45) increases in z. We have $dp^*(z)/dz > 0$, hence the left-hand side of (5–45) decreases in z. This proves that a z value, $\underline{z}$, must exist such that for all $z \geq \underline{z}$ condition (5–45) is satisfied, i.e., $\Pi(z, p^*(z))$ is concave for all $z \geq \underline{z}$.

As z approaches $-a$ (this is the minimum value that z can take), $\Pi(z, p^*(z))$ approaches a value, i.e., it does not tend to infinity. And as z approaches to infinity, $\Pi(z, p^*(z))$ goes to negative infinity. And since there is only one inflection point $\underline{z}$, $\Pi(z, p^*(z))$ is unimodal, and the z value satisfying (5–44) is its maximizer. $\square$

Remark. Appendix J shows the derivation of first order necessary optimality conditions for a solution that maximizes the expected profit. Here we formulate the profit function in terms of the selling price p and the supply price c. Our analysis shows that any optimal solution (c^*, p^*) must satisfy the following equality (see Appendix J for the derivation).

$$p^* - c^* - v - \frac{Q(c^*)}{Q'(c^*)} = \frac{y(p^*) + \mu - \Theta(c^*, p^*)}{b}. \tag{5–46}$$

On the right-hand side of (5–46), the numerator equals the expected demand minus the expected shortages, which gives the expected sales at optimality. On the left-hand side of (5–46) we have the net revenue that can be generated by the next unit to be purchased at optimality. This equality shows that, at optimality, the marginal net revenue equals the expected sales divided by the price elasticity of demand.

5.3.2 Multiplicative Demand Model

In this section we consider the case where the demand $D(p, \epsilon)$ equals $y(p)\epsilon$, where $y(p) = ap^{-b}$ ($a > 0$ and $b > 1$). Given a supply price and a selling price, and a realization of ϵ, we can formulate the profit as follows.

$$\pi(c,p,\epsilon) = \begin{cases} pQ(c) - g(D(p,\epsilon) - Q(c)) - (c+v)Q(c), & \text{if } D(p,\epsilon) > Q(c), \\ pD(p,\epsilon) + s(Q(c) - D(p,\epsilon)) - (c+v)Q(c), & \text{if } D(p,\epsilon) < Q(c). \end{cases} \tag{5–47}$$

For mathematical convenience, we define $z = Q(c)/y(p)$, and we reformulate the profit as a function of z and p. Here again c can be expressed as a function of z and p. Hence we write the supply price as a function of z and p, i.e., $c(z,p) = Q^{-1}(y(p)z)$. After necessary substitutions, the profit function can be reformulated as follows.

$$\pi(z,p,\epsilon) = \begin{cases} py(p)z - gy(p)(\epsilon - z) - (c(z,p)+v)y(p)z, & \text{if } \epsilon > z, \\ py(p)\epsilon + sy(p)(z-\epsilon) - (c(z,p)+v)y(p)z, & \text{if } \epsilon < z. \end{cases} \tag{5–48}$$

Our goal is to determine the values of z and p such that the expected profit is maximized. We next formulate this optimization problem where $\Pi(z,p)$ denotes the expected profit as a function of z and the selling price p.

$$\max_{z,p} \Pi(z,p) = \Psi(z,p) - L(z,p), \tag{5–49}$$

where

$$\Psi(z,p) = (p - c(z,p) - v)y(p)\mu, \tag{5–50}$$

and

$$L(z,p) = y(p)[(p + g - c(z,p) - v)\Theta(z) + (c(z,p) + v - s)\Lambda(z)]. \tag{5–51}$$

Similar to the previous section, $\Psi(z,p)$ is the riskless profit and $L(z,p)$ is the loss function. $\Theta(z)$ and $\Lambda(z)$ give the expected shortages and the expected leftovers, respectively.

We next consider the first and second derivatives of $\Pi(z,p)$ with respect to z. Here again we assume that the supply-price function takes the linear form introduced in Section 5.2.1, that is $Q(c) = \beta c - \alpha$. Using the definition of z, we can express the supply price c as a function of z and p. Hence we have $c(z,p) = (zy(p) + \alpha)/\beta$ and $\partial c(z,p)/\partial z = y(p)/\beta$

$$\frac{\partial \Pi(z,p)}{\partial z} = y(p)\left[-\frac{y(p)z}{\beta} - (c(z,p) + v - s) + (p + g - s)(\bar{F}(z))\right]. \tag{5–52}$$

$$\frac{\partial^2\Pi(z,p)}{\partial z^2} = y(p)\left[-\frac{2y(p)z}{\beta} - (p+g-s)f(z)\right]. \tag{5–53}$$

Note that for a given p value, $\Pi(z,p)$ is concave in z. Note also that when $\Pi(z,p)$ is concave in z, the z value satisfying the following condition maximizes $\Pi(z,p)$ for a given value of p.

$$F(z) = \frac{p+g-c(z,p)-v-\frac{y(p)z}{\beta}}{p+g-s}. \tag{5–54}$$

Note that $y(p)z = Q(c(z,p))$ and $y(p)z/\beta = Q(c(z,p))/Q'(c(z,p))$, which shows that we obtain the same critical fractile behavior as discussed in Section 5.2. Similar to the additive demand model, Equation (5–54) shows that, at optimality, the probability that the uncertain portion of the demand is satisfied equals the well-known critical fractile, $c_u/(c_u+c_o)$. The marginal understocking cost, c_u, equals $p+g-c(z,p)-v-\frac{y(p)z}{\beta}$, and the marginal overstocking cost, c_o, equals $c(z,p)+\frac{y(p)z}{\beta}+v-s$.

Now we consider the first and second derivatives of $\Pi(z,p)$ with respect to p.

$$\frac{\partial\Pi(z,p)}{\partial p} = y'(p)\left[(b-1)(\mu-\Theta(z))p - z\left(2\frac{zap^{-b}}{\beta}+\frac{\alpha}{\beta}+v\right) - g\Theta(z)+s\Lambda(z)\right]. \tag{5–55}$$

$$\begin{aligned}\frac{\partial^2\Pi(z,p)}{\partial p^2} = {} & y''(p)\left[(b-1)(\mu-\Theta(z))p - z\left(2\frac{zap^{-b}}{\beta}+\frac{\alpha}{\beta}+v\right) - g\Theta(z)+s\Lambda(z)\right] \\ & + y'(p)\left[(b-1)(\mu-\Theta(z)) + \frac{2}{\beta}z^2bap^{-b-1}\right].\end{aligned} \tag{5–56}$$

For a given z value, we can compute the values of p satisfying (5–57). Any p value obtained by solving (5–57) is a stationary point of $\Pi(z,p)$ given z.

$$(b-1)(\mu-\Theta(z))p - z\left(2\frac{zap^{-b}}{\beta}+\frac{\alpha}{\beta}+v\right) - g\Theta(z)+s\Lambda(z) = 0. \tag{5–57}$$

Note that at a positive stationary point p, we have $\partial^2\Pi(z,p)/\partial p^2 < 0$. Therefore, at most one such stationary point can exist. Assuming that there exists a positive stationary point p given z, it is the global maximizer of $\Pi(z,p)$ given z.

We next propose a sequential procedure which stops at a point where the gradient of $\Pi(z,p)$ equals zero. We start with an initial value of p and solve for the corresponding

z value that satisfies (5–54). Clearly, this value of z maximizes the expected profit given the initial p. Then, using the z value, we solve (5–57) to obtain this p value which will maximize the expected profit for the fixed z. If the corresponding value of p equals the initially selected value, we stop. Otherwise, we continue the procedure by solving (5–54) for the next z value. This procedure is outlined in Algorithm 5.1. The algorithm stops when the z and p values obtained by solving (5–54) and (5–57) do not change anymore.

Algorithm 5.1. *Find a stationary point of* $\Pi(z, p)$*.*

1: **Input**: An initial value of p, p^0.
2: **Output**: A stationary point solution (z^*, p^*).
3: Set $t = 1$.
4: Solve (5–54) for z where $p = p^{t-1}$. The solution is z^t.
5: Solve (5–57) for p where $z = z^t$. The solution is p^t.
6: **if** $p^t = p^{t-1}$ **then**
7:     Stop. $(z^*, p^*) = (z^t, p^t)$.
8: **else**
9:     Set $t = t + 1$. Go to 4.
10: **end if**

By the development of the algorithm, we know that $\nabla\Pi(z^*, p^*) = 0$. When Algorithm 5.1 stops, it clearly stops at a stationary point solution of $\Pi(z, p)$. If the solution (z^*, p^*) obtained by Algorithm 5.1 satisfies the second order sufficient optimality conditions, then we can conclude that, (z^*, p^*) is a local maximum. Hence, (z^*, p^*) is a local maximum when $\nabla^2\Pi(z^*, p^*) \prec 0$. From (5–53) and (5–56), we know that the trace of the Hessian matrix $\nabla^2\Pi(z^*, p^*)$ is negative. We must have $\det(\nabla^2\Pi(z^*, p^*)) > 0$ to conclude that (z^*, p^*) is a local maximum. Note that

$$\det(\nabla^2\Pi(z^*, p^*)) = \nabla^2_{1,1}\Pi(z^*, p^*)\nabla^2_{2,2}\Pi(z^*, p^*) - (\nabla^2_{1,2}\Pi(z^*, p^*))^2,$$

and $\nabla^2_{1,2}\Pi(z^*, p^*) = \nabla^2_{2,1}\Pi(z^*, p^*) = \frac{\partial^2 \Pi(z,p)}{\partial p \partial z}$. We have

$$\frac{\partial^2 \Pi(z, p)}{\partial p \partial z} = y'(p)\left[((b-1)p + g - s)\bar{F}(z) - \frac{4zy(p)}{\beta} - \frac{\alpha}{\beta} - v + s\right]. \tag{5–58}$$

Hence, the stationary point (z^*, p^*) is a local maximum of $\Pi(z, p)$, if it satisfies condition (5–59).

$$\left(\frac{2z^* y(p^*)p^*}{b\beta} + \frac{(p^* + g - s)f(z^*)p^*}{b}\right)\left((b-1)(\mu - \Theta(z^*)) - \frac{2(z^*)^2 y'(p^*)}{\beta}\right)$$
$$> ((b-1)p^* + g - s)\bar{F}(z^*) - \frac{4z^* y(p^*)}{\beta} - \left(\frac{\alpha}{\beta} + v - s\right). \tag{5–59}$$

CHAPTER 6
SUMMARY AND CONCLUSIONS

In this study we considered inventory models where the quantity of input available for production depends on the price that the producer offers to its suppliers. This is a particularly relevant phenomenon in a reverse logistics setting where the inputs required by a remanufacturer are owned by individual consumers who may be willing to sell their products back to the remanufacturer depending on the price offered. Supply pricing is also relevant beyond this context, where the price offered directly impacts the quantity made available by potential suppliers. This motivated the consideration of various models for a single-item production planning problem with price-sensitive supply.

We first studied the finite-horizon, discrete-time production and component-supply-pricing planning problem with non-stationary costs, demands,and component supply levels. The producer seeks a production and supply-pricing plan that minimizes the cost incurred while meeting a set of demands over a discrete-time finite planning horizon. The price-sensitivity of supply availability led to a class of two-level lot sizing problems whose cost is the sum of concave and convex functions, and where the supply availability limits production quantities. We showed that the resulting problem is $\mathcal{NP}$-Hard in general, and considered several practically relevant special cases that permit polynomial solvability. Our work provides contributions to the class of convex-cost lot sizing problems and to the reverse logistics literature, as we studied problem classes in which price-sensitive returns from end-users serve as input for production. Directions for future research in this context include developing solution approaches for handling problems requiring multiple components and multiple products. The impact of pricing decisions on customer demand levels, which directly affect production requirements, has received significant attention in the literature. Thus, another high-value direction would incorporate price-sensitive demands together with price-sensitive supply.

Motivated by the zero-fixed-charged version of the two-level dynamic lot sizing problem, in Chapter 3, we developed a polynomial time algorithm for dynamic lot-sizing problems with convex costs in production and inventory quantities. The class of convex-cost dynamic lot-sizing problems we considered permitted convex functions of the production and inventory quantities in each period over a finite horizon, which is the class of problems Veinott [86] considered in his classic paper on the topic. Here a promising direction for future research would be the development of a solution method for a model that permits convex holding costs that explicitly depend on the production and demand period (e.g., $h_{t\tau}(x_{t\tau})$ for $\tau \geq t$). Another worthwhile research direction may account for both price-sensitive supply and price-dependent demand in production planning.

We next studied the infinite-horizon version of the problem where the production and inventory holding costs as well as the pricing parameters are stationary. Chapter 4 provided a novel generalization of the deterministic EOQ model that permits using pricing on both the supply side and the demand side in order to optimally match supply and demand rates in an inventory system. Here we assumed that the supply for production arrives with a rate depending on the price offered to suppliers. We also assumed that the demand rate depends on the price offered to customers by the producer. We characterized optimal solutions for pricing decisions and investigated their behavior with respect to changing problem parameters. While past work has accounted for the ways in which pricing influences demand levels, our work also accounts for the important impact of price-dependent supply of input components. We first examined the relationship between the price-dependent demand and supply rates, and showed that these rates must be equal in equilibrium. We then formulated the profit maximization problem in terms of the end-item selling price, which implies an underlying equilibrium component supply price. Under an iso-elastic functional form for the demand rate, we characterized the profit-maximizing lot size and prices.

Our model and analysis permitted demonstrating the way in which price-dependent supply impacts optimal decisions and profitability. In particular, this analysis showed the value of heterogeneity in the supply base, as well as the importance of understanding and accounting for the relationship between the supply price and the quantity supplied, in terms of profitability. Future research in this setting may consider generalizing our model to account for shortages (in the form of lost sales or backorders) at an associated cost. Additionally, while our model considered an iso-elastic price-demand function and a linear price-to-price response function, extensions of this model may consider different classes of price-demand and price-to-price response functions.

In the last chapter, we considered a single period planning problem where the end-item demand faced by the producer was random, and the supply quantity depended on the unit price offered by the producer to suppliers. This planning problem corresponds to a newsvendor problem where the available supply quantity is price-dependent. First we studied the case where only the supply is price-dependent. We characterized the optimal supply pricing policies where the supply versus price relationships were linear and isoelastic. We showed that the linear supply-price relationship led to an expected profit function which is concave in the supply price. This result was independent of the demand distribution. When the supply-price relationship is isoelastic, we showed that the expected profit function is unimodal and the stationary point solution maximizes the expected profit when the demand distribution has an increasing generalized failure rate. In both of these cases, we showed that the well-known critical fractile (underage cost divided by the sum of overage and underage costs) gives the probability that the end-item demand is satisfied.

We investigated the differences in the optimal policies between the standard newsvendor problem and our model. We showed that, for the same supply price, the standard newsvendor solution results in a higher service level, and consequently, in a higher optimal order quantity. We also showed that the optimal expected profit obtained by the standard newsvendor is always greater than the optimal expected profit of our model. This difference arises from the fact that, in the standard newsvendor setting, it is assumed that there is an infinite amount of supply at the given unit cost. However, in our model, the available supply quantity depends on the price offered to the suppliers, and the producer can purchase more supply only if she offers a higher price to suppliers. For the same unit supply price, the standard newsvendor can obtain more supply and satisfy a higher proportion of the demand, and hence, achieve a higher expected profit.

We also presented the optimal supply-pricing policy for the case where the supply is price-dependent and also random. In particular, we considered the case where the supply quantity follows a normal distribution with a price-dependent mean and a known variance. Our analytical results showed that the producer's optimal supply-pricing policy is not different from the deterministic price-dependent supply case.

We then studied the case where the demand is also price-sensitive and analyzed the optimal pricing decisions. We considered two demand models, additive and multiplicative, which were also presented in the study of Petruzzi and Dada in [59]. We also assumed that the supply-price relationship is linear. For the additive demand case, we obtained closed-form solutions for the optimal selling and supply prices given that the end-item demand follows a distribution with an increasing failure rate. Unlike the case where only demand is price-dependent, we could not obtain closed-form solutions for the multiplicative demand model. Therefore, we proposed a sequential procedure to obtain stationary point solutions for the expected profit function and provided conditions that a local maximum has to satisfy.

APPENDIX A
PROOF OF PROPOSITION 2.1

We begin with a special case of P(FC) in which $F_t^C = 0$ and $h_t^C = \infty$ for all $t \in T$, i.e., no fixed charges exist for procurement and no component inventory is held (note that for this special case, $f^C(\kappa_t)$ is a convex function of κ_t for each $t \in T$). Let SP1 denote this special case of the problem with zero fixed costs for component procurement and where component inventory holding is not permitted. To formulate problem SP1, let x_{ti} denote the amount produced in period t to satisfy demand in period i, where $t, i \in T$ and $t \leq i$. To ensure that production in every period does not exceed available component supply, we require

$$\sum_{i=t}^{|T|} x_{ti} \leq \kappa_t, \quad \forall t \in T. \tag{A–1}$$

In fact, because no component inventory holding is permitted, it is straightforward to show that an optimal solution exists such that we procure a number of components in a period equal to the production amount in the period, i.e., such that (A–1) is satisfied at equality. Thus, we can formulate problem SP1 as follows:

$$\textbf{SP1:}\quad \text{Minimize} \quad \sum_{t\in T} F_t y_t + \sum_{t\in T} \bar{c}_t \sum_{i=t}^{|T|} x_{ti} + \sum_{t\in T} \frac{1}{\beta_t}\left(\sum_{i=t}^{|T|} x_{ti}\right)^2 \tag{A–2}$$

$$\text{Subject to:}\quad \sum_{t=1}^{i} x_{ti} = d_i, \qquad \forall i \in T, \tag{A–3}$$

$$\sum_{i=t}^{|T|} x_{ti} \leq M y_t, \quad \forall i \in T, \tag{A–4}$$

$$x_{ti} \geq 0, \qquad \forall t, i \in T \text{ and where } t \leq i, \tag{A–5}$$

$$y_t \in \{0, 1\}, \qquad \forall t \in T, \tag{A–6}$$

where $\bar{c}_t = c_t + \sum_{i=t}^{|T|} h_i + \gamma_t + \alpha_t/\beta_t$ and M corresponds to a big-M value (a large positive number that we can set to $D_{t,T} = \sum_{i=t}^{T} d_i$ without loss of optimality). Here, constraint set (A–3) corresponds to the demand satisfaction requirements. Constraint set (A–4) forces y_t to equal one if any production occurs in period t, while (A–5) and (A–6) provide nonnegativity and binary restrictions.

Noting that $\sum_{i=t}^{|T|} x_{ti} = x_t$ for all $t \in T$, we define the function $P_t(x_t)$ as the sum of the production and procurement costs in period t, where

$$P_t(x_t) = \begin{cases} F_t + \bar{c}_t x_t + \frac{x_t^2}{\beta_t}, & \text{if } x_t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

We show that problem SP1 is $\mathcal{NP}$-Hard using an approach similar to the one used by Florian, et al. [22] for capacitated lot sizing. Clearly one can evaluate the cost of any feasible solution in polynomial time, which implies that the recognition version of problem SP1 is in $\mathcal{NP}$. We can show that the SUBSET SUM problem, which is known to be $\mathcal{NP}$-Complete [23], is reducible to a special case of SP1.

SUBSET SUM: Given positive integers $a_1, \ldots, a_t$, A, does there exist a subset $S \subset T = \{1, \ldots, t\}$ such that $\sum_{i \in S} a_i = A$?

Given any instance of SUBSET SUM, we define the following instance of SP1. Let $T = \{1, \ldots, t\}$ denote the set of periods and let

$$\begin{aligned} d_i &= 0, \quad i = 1, \ldots, t-1, \\ d_t &= A, \\ \beta_i &= a_i^2, \quad i \in T, \\ \bar{c}_i &= 1 - \frac{2}{a_i}, \quad i \in T, \\ F_i &= 1, \quad i \in T. \end{aligned}$$

We claim that SUBSET SUM has a solution if and only if there exists a feasible solution to the corresponding instance of SP1 with total cost of at most A. Observe that $P_i(0) = 0$ and $P_i(a_i) = a_i$ for all $i \in T$. We can show that $P_i(x_i) > x_i$ if $x_i > 0$ and $x_i \neq a_i$ and, in fact, $P_i(x_i) - x_i$ attains its minimum at $x_i = a_i$. We define $g_i(x_i) = P_i(x_i) - x_i$ for $x_i > 0$, where

$$g_i(x_i) = \frac{1}{a_i^2}x_i^2 + \left(1 - \frac{2}{a_i}\right)x_i + 1 - x_i$$
$$= \frac{1}{a_i^2}x_i^2 - \frac{2}{a_i}x_i + 1,$$

and note that

$$g_i'(x_i) = \frac{2}{a_i^2}x_i - \frac{2}{a_i}.$$

Observe that $g_i(x_i)$ is a strictly convex function of x_i, and $g_i'(x_i) = 0$ for $x_i = a_i$, which is the minimizer of $g_i(x_i)$. In fact, $g_i(a_i) = 0$, which shows that $g_i(x_i) > 0$, i.e., $P_i(x_i) > x_i$ for any $x_i \neq a_i$, and $x_i > 0$. For all $i \in T$, we have

$$P_i(x_i) = x_i \quad \text{if } x_i = 0 \text{ or } x_i = a_i,$$
$$P_i(x_i) > x_i \quad \text{otherwise.}$$

Hence, any feasible production plan has total cost of at least A, i.e.,

$$\sum_{i \in T} P_i(x_i) \geq \sum_{i \in T} x_i = A.$$

The cost of the production plan will be exactly equal to A if and only if $x_i \in \{0, a_i\}$ for all $i \in T$, i.e., if and only if there exists a subset $S \subset T$ such that $\sum_{i \in S} a_i = A$.

Observe that the capacitated lot sizing problem can be viewed as a limiting case of the problem with fixed setup costs and piecewise linear and convex variable costs with a single breakpoint in each time period (where the breakpoint occurs at the capacity level in each period, and as the slope of the linear piece to the right of the breakpoint tends to infinity). The proof of $\mathcal{NP}$-Completeness by Florian, et al. [22] is thus equivalent to using a production cost function defined by

$$P_t(x_t) = \begin{cases} F_t + \bar{c}_t^1 x_t, & \text{if } 0 < x_t \leq a_t, \\ F_t + \bar{c}_t^1 a_t + \bar{c}_t^2 (x_t - a_t), & \text{if } x_t > a_t, \end{cases}$$

with $F_t = 1$, $\bar{c}_t^1 = (1 - 1/a_t)$, and any $\bar{c}_t^2 > 1$ for all $t \in T$ (the value of a_t becomes the capacity limit in period t as $\bar{c}_t^2 \to \infty$).

APPENDIX B
PROOF OF PROPOSITION 2.2

Here again, we can show that SUBSET SUM problem is reducible to a special case of P(FC). Given any instance of SUBSET SUM with positive integers $a_1, \ldots, a_t, A$, suppose without loss of generality that the a_i values are sorted in nonincreasing order for $i = 1, \ldots, t$. We create an instance of a special case of P(FC) as follows. Let $T = \{1, \ldots, t\}$ denote the set of planning periods, with (as in the previous proof) $d_i = 0, i = 1, \ldots, t-1$ and $d_t = A$. We consider the special case of P(FC) such that $F_t = h_t = h_t^C = 0$ for all $t \in T$. By assumption we have $\beta_t = \beta$, $\alpha_t = \alpha$, and $\gamma_t = \gamma$ for all $t \in T$. In addition, we set

$$\bar{c}_i = 1 - \frac{2a_i}{\beta}, \quad i \in T,$$

$$F_i^C = \frac{a_i^2}{\beta}, \quad i \in T.$$

Note that $\bar{c}_i = 1 - 2a_i/\beta$ implies $c_i = 1 - 2a_i/\beta - \alpha/\beta - \gamma$ (and we can select values of α, β, and γ that ensure all $c_i \geq 0$). We claim that SUBSET SUM has a solution if and only if there exists a feasible solution to this special case of the production planning problem with total cost of at most A. We first note that for this special case, an optimal solution exists such that any unit procured in period i is used in production in period i (this results from the fact that all holding costs and fixed production costs equal zero, variable production costs are nondecreasing in time as a result of the a_i values having been sorted in nonincreasing order, and the above $\bar{c}_i$ values). Let x_i denote the procurement (and production) amount in period i and define the associated cost function

$$\tilde{P}_t(x_t) = \begin{cases} F_t^C + \bar{c}_t x_t + \frac{x_t^2}{\beta}, & \text{if } x_t > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\tilde{P}_i(0) = 0$ and $\tilde{P}_i(a_i) = a_i$ for all $i \in T$. Additionally, $\tilde{P}_i(x_i) > x_i$ for any $x_i > 0$ such that $x_i \neq a_i$. Following the same steps as in the proof of Proposition 2.1, we can show that the cost of any feasible solution equals at least A, and the production plan cost will exactly equal A if and only if $x_i \in \{0, a_i\}$ for all $i \in T$, i.e., if and only if there exists a subset $S \subset T$ such that $\sum_{i \in S} a_i = A$.

APPENDIX C
PROOF OF PROPOSITION 4.2

The first order condition yields Equation (C–1). Any p_c satisfying this equation is a stationary point of $\hat{\pi}(p_c)$.

$$c'ab = (b-1)ap_c - \frac{b}{2}\sqrt{2F'ha}\, p_c^{b/2}. \tag{C–1}$$

Observe that for $1 < b \leq 2$, the right hand side of (C–1) is increasing as p_c increases. Therefore, Equation (C–1) is only satisfied at a single p_c value. Let p_c^* denote this stationary point. We need to show that $\hat{\pi}''(p_c^*)$ is strictly negative to conclude that p_c^* is the maximum of $\hat{\pi}(p_c^*)$. That is, we want to show that

$$(b+1)c'ab > (b-1)bap_c^* - \frac{b}{2}\left(\frac{b}{2}+1\right)\sqrt{2F'ha}\,(p_c^*)^{b/2}. \tag{C–2}$$

Equation (C–1) implies that

$$\frac{b}{2}\sqrt{2F'ha}\,(p_c^*)^{b/2} = (b-1)ap_c^* - c'ab.$$

Thus, we can substitute $\frac{b}{2}\sqrt{2F'ha}\,(p_c^*)^{b/2}$ in the inequality (C–2). We then have

$$\left(\frac{b}{2}+1\right)((b-1)ap_c^* - c'ab) > (b-1)bap_c^* - (b+1)c'ab,$$

which implies

$$\left(1-\frac{b}{2}\right)(b-1)ap_c^* > -\frac{b}{2}c'ab.$$

For $1 < b \leq 2$, the left hand side of the above inequality is nonnegative and the right hand side is strictly negative. This proves that there is only one stationary point for the case where $1 < b \leq 2$, and this point corresponds to the maximum of $\hat{\pi}(p_c)$. For $b > 2$, the right hand side of (C–1) increases in p_c until the point p_c^x defined in (C–3) and then decreases. This implies that there are two solutions to (C–1), one of which is in the interval $[0, p_c^x]$

and the other of which is in $[p_c^x, \infty]$.

$$p_c^x = \left(\frac{(b-1)a}{\frac{b^2}{4}\sqrt{2F'ha}} \right)^{\frac{2}{b-2}}. \tag{C–3}$$

The right hand side of inequality (C–2) increases in p_c until the point p_c^{xx} defined in (C–4) and then decreases. This implies that $\hat{\pi}(p_c)$ is first concave, then convex, and then concave again.

$$p_c^{xx} = \left(\frac{(b-1)ba}{\frac{b^2}{4}\left(\frac{b}{2}+1\right)\sqrt{2F'ha}} \right)^{\frac{2}{b-2}}. \tag{C–4}$$

Note that $p_c^x < p_c^{xx}$ when $b > 2$. Let p_c^* denote the smaller stationary point, that is $p_c^* \in [0, p_c^x]$. Thus, $p_c^* \leq p_c^x < p_c^{xx}$, showing that $\hat{\pi}(p_c)$ is strictly concave at p_c^*. This proves that the smaller of the two stationary points is the maximum of $\hat{\pi}(p_c)$ for the case where $b > 2$.

When the maximum point p_c^* is greater than p_c^0, then the optimal choice of selling price is p_c^0.

$$p_c^{opt} = \min\{p_c^*, p_c^0\}. \tag{C–5}$$

APPENDIX D
PROOF OF PROPOSITION 4.3

For p_c^{opt}, when taking the stationary point solution p_c^*, we will show the result for F. The rest follows similarly. Taking the partial derivative of $\partial\hat{\pi}(p_c)/\partial p_c = 0$ with respect to F, we obtain

$$\frac{\partial(\partial\hat{\pi}(p_c)/\partial p_c)}{\partial F} + \frac{\partial(\partial\hat{\pi}(p_c)/\partial p_c)}{\partial p_c}\frac{\partial p_c}{\partial F} = 0. \tag{D–1}$$

Therefore

$$\left.\frac{\partial p_c}{\partial F}\right|_{p_c=p_c^*} = -\left.\frac{\frac{\partial(\partial\hat{\pi}(p_c)/\partial p_c)}{\partial F}}{\frac{\partial^2\hat{\pi}(p_c)}{\partial p_c^2}}\right|_{p_c=p_c^*}. \tag{D–2}$$

Note that the denominator of the right hand side of (D–2) is negative, since $\hat{\pi}(p_c)$ is strictly concave at p_c^*. And we have

$$\frac{\partial(\partial\hat{\pi}(p_c)/\partial p_c)}{\partial F} = \frac{b}{4}\sqrt{\frac{2ha}{F}}\frac{1}{k}(p_c^*)^{-b/2-1}, \tag{D–3}$$

which is clearly positive. Thus, $\frac{\partial p_c^*}{\partial F}$ is positive, which shows the result. When $p_c^* > p_c^0$, the optimal selling price p_c^{opt} equals p_c^0, which only depends on $\hat{p}$ and p_s^0 for a fixed k. Note that $p_c^0 = (k\hat{p} - p_s^0)/(k-1)$, $\partial p_c^0/\partial\hat{p} = k/(k-1)$ and $\partial p_c^0/\partial p_s^0 = -1/(k-1)$. This shows the desired result.

APPENDIX E
PROOF OF PROPOSITION 4.4

When $p_c^{opt} = p_c^0$, the optimal choice for the supply price equals p_s^0 which is a constant determined by the market. We denote the optimal supply price when $p_c^{opt} = p_c^*$ as p_s^*. Note that $p_s^* = k\hat{p} - (k-1)p_c^*$. We have

$$\frac{\partial p_s^*}{\partial \hat{p}} = k - (k-1)\frac{\partial p_c^*}{\partial \hat{p}}. \tag{E–1}$$

By Proposition 4.2, we know that $\partial p_c^*/\partial \hat{p} > 0$. We can also show that $\partial^2 p_c^*/\partial \hat{p}^2 < 0$, which implies that $\partial^2 p_s^*/\partial \hat{p}^2 > 0$, because $\partial^2 p_s^*/\partial \hat{p}^2 = -(k-1)\partial^2 p_c^*/\partial \hat{p}^2$. Therefore p_s^* is convex in $\hat{p}$ and as $\hat{p}$ increases, it decreases until the $\hat{p}$ value is reached where $k/(k-1) = \partial p_c^*/\partial \hat{p}$. After this point, p_s^* starts increasing again.

For the other parameters a, F, c and h, p_s^* behaves in the opposite direction as p_c^*. This is because for any of these parameters (e.g., for x), we have $\partial p_s^*/\partial x = -(k-1)(\partial p_c^*/\partial x)$.

APPENDIX F
PROOF OF PROPOSITION 4.5

The optimal batch size can be computed given p_c^{opt}, where $Q^{opt} = \sqrt{2Fa/h}(p_c^{opt})^{-b/2}$. First we will show the result when $p_c^{opt} = p_c^*$. To show the result for a, we take the derivative of Q^{opt} with respect to a:

$$\frac{dQ^{opt}}{da} = \frac{\partial Q^{opt}}{\partial a} + \frac{\partial Q^{opt}}{\partial p_c^*}\frac{\partial p_c^*}{\partial a}.$$

From Proposition 4.2 we know that the optimal selling price p_c^* is decreasing in a, hence, $\partial p_c^*/\partial a$ is negative. We have $\partial Q^{opt}/\partial a = (1/2)\sqrt{\frac{2F}{ha}}(p_c^*)^{-b/2}$, which is positive, and $\partial Q^{opt}/\partial p_c^* = -\frac{b}{2}\sqrt{\frac{2Fa}{h}}(p_c^*)^{-b/2-1}$, which is negative. Therefore, $dQ^{opt}/da > 0$. The proof of the results for other parameters (h, c, and $\hat{p}$) follows similarly, except for F. A value of F, $\bar{F}$, exists where $dQ^{opt}/dF = 0$, and $dQ^{opt}/dF > 0$ for $F \leq \bar{F}$ and $dQ^*/dF < 0$ for $F < \bar{F}$. $\bar{F}$ is the value which satisfies

$$\frac{1}{2}\sqrt{\frac{2a}{Fh}} = \frac{b}{2}\sqrt{\frac{2Fa}{h}}(p_c^*)^{-b/2-1}\frac{\partial p_c^*}{\partial F}, \tag{F–1}$$

or, equivalently,

$$\frac{p_c^*}{bF} = \frac{\partial p_c^*}{\partial F}. \tag{F–2}$$

For $b = 2$, we have

$$\bar{F} = \frac{2a}{h}\frac{k^2}{(b+2)^2}. \tag{F–3}$$

When $p_c^{opt} = p_c^0$, Q^{opt} does not respond to the changes in c, however, it changes as p_s^0 changes. We have

$$\frac{dQ^{opt}}{dp_s^0} = -\frac{b}{2}\sqrt{\frac{2Fa}{h}}(p_c^0)^{-b/2-1}\frac{dp_c^0}{dp_s^0}. \tag{F–4}$$

By Proposition 4.2, we know that $dp_c^0/dp_s^0 < 0$, therefore $dQ^{opt}/dp_s^0 > 0$. The results for the parameters a, F, h, and $\hat{p}$ follow similarly.

APPENDIX G
PROOF OF PROPOSITION 4.6

When the optimal selling price equals p_c^0, the net revenue per unit only depends only on $\hat{p}$, p_s^0, and c for fixed k. In this case, we have $\frac{d\delta^{opt}}{d\hat{p}} = \frac{k}{k-1}$, $\frac{d\delta^{opt}}{dp_s^0} = -\frac{k}{k-1}$, and $\frac{d\delta^{opt}}{dc} = -1$, hence the result follows.

When the optimal selling price equals p_c^*, δ^{opt} behaves the same way as p_c^* with respect to the parameters a, F, and h.[1] The derivative with respect to $\hat{p}$ is as follows:

$$\frac{d\delta^{opt}}{d\hat{p}} = k\frac{\partial p_c^*}{\partial \hat{p}} - k. \tag{G–1}$$

If $\partial p_c^*/\partial\hat{p} > 1$ for any $\hat{p}$, then we have the result. This condition is satisfied when

$$p_c^* > \frac{\frac{b^2}{2}(c/k + \hat{p})}{\frac{b^2}{2} - \frac{b}{2} + 1}. \tag{G–2}$$

p_c^* must be greater than $c/k + \hat{p}$ in order to obtain positive profit. The multiplier term of $c/k + \hat{p}$, $0.5b^2/(0.5b^2 - 0.5b + 1)$, is less than or equal to 1 when $1 < b \leq 2$. Hence the result clearly follows for this range of b values. When $b > 2$, the term is slightly greater than 1. It takes its maximum value (8/7) when $b = 4$ and it approaches 1 as b increases. The inequality (G–1) is likely to hold for $b > 2$, since the chosen selling price must also account for the operations costs. Similarly, if $\partial p_c^*/\partial c > 1/k$ for any c, we have the result for c. The conditions discussed above hold for c as well.

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[1] For example, $d\delta^{opt}/da = k(dp_c^*/da)$. By Proposition 4.2, we know that $dp_c^*/da < 0$, therefore $d\delta^{opt}/da < 0$, as well.

APPENDIX H
PROOF OF PROPOSITION 4.7

When p_c^{opt} equals p_c^*, we can determine the derivative with respect to k as follows.

$$\left.\frac{\partial p_c}{\partial k}\right|_{p_c=p_c^*} = -\left.\frac{\frac{\partial(\partial\hat{\pi}(p_c)/\partial p_c)}{\partial k}}{\frac{\partial^2\hat{\pi}(p_c)}{\partial p_c^2}}\right|_{p_c=p_c^*}. \tag{H–1}$$

Noting that the second derivative of $\hat{\pi}(p_c)$ at p_c^* is negative and

$$\left.\frac{\partial(\partial\hat{\pi}(p_c)/\partial p_c)}{\partial k}\right|_{p_c=p_c^*} = -\frac{cab(p_c^*)^{-b-1}}{k^2} - \frac{1}{k^2}\frac{b}{2}\sqrt{2Fha}(p_c^*)^{-b/2-1} < 0, \tag{H–2}$$

and we can conclude that p_c^* is decreasing in k. When $p_c^{opt} = p_c^0$, we have

$$\frac{dp_c^0}{dk} = -\frac{\hat{p} - p_s^0}{(k-1)^2}. \tag{H–3}$$

Since $\hat{p} > p_s^0$, $\frac{dp_c^0}{dk} < 0$, hence p_c^0 is also decreasing in k. Note also that p_c^0 approaches $\hat{p}$ as $k \to \infty$.

When $p_c^{opt} = p_c^0$, $p_s^{opt} = p_s^0$, and p_s^0 is a constant. For $p_c^{opt} = p_c^*$, we have

$$\frac{dp_s^{opt}}{dk} = \hat{p} - p_c^* - (k-1)\frac{\partial p_c^*}{\partial k}. \tag{H–4}$$

Since $\frac{\partial p_c^*}{\partial k} < 0$, it is not clear whether $\frac{dp_s^{opt}}{dk}$ is positive or negative. For $b = 2$, we can use the closed form expression for p_c^* to show that p_s^{opt} is decreasing[1] .

For Q^{opt}, we have

$$\frac{dQ^{opt}}{dk} = \sqrt{\frac{2Fa}{h}}\left(-\frac{b}{2}\right)(p_c^{opt})^{-b/2-1}\frac{\partial p_c^{opt}}{\partial k}. \tag{H–5}$$

We have seen that $\frac{\partial p_c^{opt}}{\partial k} < 0$, which shows $\frac{dQ^{opt}}{dk} > 0$.

[1] The proof requires that $\sqrt{2Fh/a} < 1$. $\sqrt{2Fh/a}$ is the unit operations cost when the demand rate equals a. This condition is also required for the limiting case where $k = 1$ to have a nonnegative optimal selling price.

When $p_c^{opt} = p_c^0$, $\delta^{opt} = p_c^0 - p_s^0 - c$, and for this case δ^{opt} is clearly decreasing in k. When $p_c^{opt} = p_c^*$, $\delta^{opt} = kp_c^* - k\hat{p} - c$. We have

$$\frac{d\delta^{opt}}{dk} = p_c^* - \hat{p} + k\frac{\partial p_c^*}{\partial k}. \tag{H–6}$$

Since $\frac{\partial p_c^*}{\partial k} < 0$, it is not clear whether $\frac{d\delta^{opt}}{dk}$ is positive or negative. We can show that δ^{opt} is increasing in k for $b = 2$ when the following condition holds:

$$a\hat{p}(k^2 - \epsilon^2 - 2\epsilon k) > 2c, \tag{H–7}$$

where $\epsilon = \sqrt{2Fh/a}$, and ϵ is assumed to be less than 1. This implies that δ^{opt} is increasing in k for

$$k > \tilde{k} \doteq \frac{2\hat{p}\sqrt{2Fha} + \sqrt{4\hat{p}(2Fha) + 4a\hat{p}(2c + \hat{p}2Fh)}}{2a\hat{p}}. \tag{H–8}$$

APPENDIX I
PROOF OF PROPOSITION 4.8

We need to show that $d\pi(p_c^{opt})/dk > 0$. We have

$$\frac{d\pi(p_c^{opt})}{dk} = \frac{\partial\pi(p_c^{opt})}{\partial k} + \frac{\partial\pi(p_c^{opt})}{\partial p_c^{opt}}\frac{\partial p_c^{opt}}{\partial k}. \tag{I–1}$$

Note that $p_c^{opt} = \min\{p_c^*, p_c^0\}$. The partial derivative of the optimal profit with respect to k is

$$\frac{\partial\pi(p_c^{opt})}{\partial k} = (p_c^{opt} - \hat{p})a(p_c^{opt})^{-b}. \tag{I–2}$$

Since $p_c^{opt} \geq \hat{p}$, $\partial\pi(p_c^{opt})/\partial k$ is always nonnegative. When $p_c^{opt} = p_c^*$, $\partial\pi(p_c^{opt})/\partial p_c^{opt}$ is zero. Therefore, the result holds when the stationary point solution p_c^* is optimal. When $p_c^{opt} = p_c^0$, we have

$$\frac{\partial p_c^0}{\partial k} = -\frac{\hat{p} - p_s^0}{(k-1)^2}, \tag{I–3}$$

and we know that $\frac{\partial\pi(p_c^0)}{\partial p_c^0} > 0$, since $p_c^0 < p_c^*$. We need to determine whether the following is positive:

$$\frac{d\pi(p_c^0)}{dk} = (p_c^0 - \hat{p})a(p_c^0)^{-b} - \frac{\partial\pi(p_c^0)}{\partial p_c^0}\frac{\hat{p} - p_s^0}{(k-1)^2}. \tag{I–4}$$

We can show that $\frac{d\pi(p_c^0)}{dk}$ is positive when $k - 1 > \frac{\partial\pi(p_c^0)}{\partial p_c^0}$. For $b = 2$, this condition is satisfied for any k value when

$$1 - \sqrt{\frac{2Fh}{a}} > 2\frac{p_s^0 + c}{\hat{p}}. \tag{I–5}$$

If (I–5) is not satisfied then there exists a k value, k', such that $\frac{d\pi(p_c^0)}{dk} = 0$, $\frac{d\pi(p_c^0)}{dk} > 0$ for $k < k'$, and $\frac{d\pi(p_c^0)}{dk} < 0$ for any $k > k'$. k' is computed as follows:

$$k' = \frac{(1 + \sqrt{2Fh/a})p_s^0 + 2c}{2(p_s^0 + c) - (1 - \sqrt{2Fh/a})\hat{p}}. \tag{I–6}$$

APPENDIX J
FIRST ORDER NECESSARY OPTIMALITY CONDITIONS FOR $\Pi(C, P)$

In this appendix, we present a part of the analysis for the case where the demand is also price-dependent and it follows an additive model as described in Section 5.3.1. Here we present the profit function as a function of the supply price c and the selling price p. For mathematical convenience, we define $z(c,p) = Q(c) - y(p)$. After necessary substitutions, the profit function can be reformulated as follows.

$$\pi(c,p,\epsilon) = \begin{cases} p(y(p) + z(c,p)) - g(\epsilon - z(c,p)) - (c+v)(y(p) + z(c,p)), & \text{if } \epsilon > z(c,p), \\ p(y(p) + \epsilon) + s(z(c,p) - \epsilon) - (c+v)(y(p) + z(c,p)), & \text{if } \epsilon < z(c,p). \end{cases} \tag{J–1}$$

Our goal is to determine the supply price c and the selling price p such that the expected profit is maximized. We next formulate this problem where $\Pi(c,p)$ denotes the expected profit as a function of the supply price c and the selling price p.

$$\max_{c,p \geq 0} \Pi(c,p) = \Psi(c,p) - L(c,p), \tag{J–2}$$

where the riskless profit

$$\Psi(c,p) = (p - c - v)(y(p) + \mu), \tag{J–3}$$

and the loss function

$$L(c,p) = (p + g - c - v)\Theta(c,p) + (c + v - s)\Lambda(c,p). \tag{J–4}$$

Here $\Theta(c,p)$ gives the expected shortages and $\Lambda(c,p)$ gives the expected leftovers.

$$\Theta(c,p) = \int_{z(c,p)}^{B} (x - z(c,p))f(x)dx. \tag{J–5}$$

$$\Lambda(c,p) = \int_{A}^{z(c,p)} (z(c,p) - x)f(x)dx. \tag{J–6}$$

We next consider the first and second derivatives of $\Pi(c,p)$ with respect to c.

$$\frac{\partial \Pi(c,p)}{\partial c} = -Q(c) + (p + g - c - v)Q'(c) - (p + g - s)Q'(c)F(z(c,p)). \tag{J–7}$$

$$\frac{\partial^2 \Pi(c,p)}{\partial c^2} = (p+g-s)\left[Q''(c)\bar{F}(z(c,p)) - (Q'(c))^2 f(z(c,p))\right] - (c+v-s)Q''(c) - 2Q'(c). \quad \text{(J–8)}$$

Note that the concavity of $\Pi(c,p)$ depends on the relationship between the supply quantity and the price offered to suppliers as well as the distribution of the random variable ϵ. However, by the first order necessary optimality condition, we know that if c^* maximizes the expected profit, it must satisfy the following condition for any given p.

$$F(z(c,p)) = \frac{p+g-c-v-\frac{Q(c)}{Q'(c)}}{p+g-s}. \quad \text{(J–9)}$$

Next consider the first and second derivatives of $\Pi(c,p)$ with respect to p.

$$\frac{\partial \Pi(c,p)}{\partial p} = y(p) + \mu - gy'(p) + (p+g-s)y'(p)F(z(c,p)) - \Theta(c,p). \quad \text{(J–10)}$$

$$\frac{\partial^2 \Pi(c,p)}{\partial p^2} = (Q'(c) - b)\bar{F}(z(c,p)) - b^2(p+g-s)f(z(c,p)). \quad \text{(J–11)}$$

From (J–11) we can observe that the concavity of $\Pi(c,p)$ with respect to p depends on the distribution of the random variable ϵ and to the supply-price relationship. However, by the first order necessary optimality condition, we know that if p^* maximizes the expected profit, it must satisfy the following condition for any given c.

$$F(z(c,p)) = \frac{\frac{y(p)+\mu-\Theta(c,p)}{b} + g}{p+g-s}. \quad \text{(J–12)}$$

Thus, if we obtain the optimal selling price p^* and the optimal supply price c^*, both conditions (J–9) and (J–12) must be satisfied. That implies that at optimality, we have

$$p^* - c^* - v - \frac{Q(c^*)}{Q'(c^*)} = \frac{y(p^*) + \mu - \Theta(c^*, p^*)}{b}. \quad \text{(J–13)}$$

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BIOGRAPHICAL SKETCH

Zehra Melis Teksan was born in 1986 in Istanbul, Turkey. After she received her high school degree in 2005 from German High School Istanbul (Deutsche Schule Istanbul), she started her undergraduate studies in the Industrial Engineering Department at Boğaziçi University. She received her B.S. and M.S. degrees in the same department in 2009 and 2011, respectively. During her master's studies, she also worked in several positions for ICRON Technologies, including senior consultant, project manager, and R&D software product developer. In August 2011, she joined the Ph.D. program in the Industrial and Systems Engineering Department at the University of Florida. Her main research focus lies in the field of production planning and inventory theory, and, in general, she is interested in research problems that are relevant to real-life industry practice. She received her Ph.D. degree in Spring 2016.