ON LUSTERNIK SCHNIRELMANN CATEGORY OF CONNECTED SUMS

By

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I dedicate this dissertation to my parents, Bob and Mary Newton.

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This dissertation uses techniques from algebraic topology to place bounds on the Lusternik-Schnirelmann category of a quotient space with sufficient conditions. The first chapter contains a quick summary on the origins of the theory and some mention on current activity within the field.

The second chapter provides the definitions and constructions used in this study. There will be examples of the Lusternik-Schnirelmann category for some basic spaces. The techniques from algebraic topology that are used will be discussed here. The chapter concludes with some integration of the algebera and topology.

In the third chapter we get to the main ideas in this dissertation. We prove the main theorem and describe some very quick results that follow from the theorem. The dissertation concludes with ideas for future work.

CHAPTER 1
BRIEF, SELECTED HISTORY OF LUSTERNIK-SCHNIRELMANN CATEGORY

In 1929 Lazar Lusternik and Lev Schnirelmann solved the Poncaré conjecture that every Riemannian metric on the sphere S^2 possesses at least 3 closed non-self-intersected geodesics, [28], for greater detail see [1, 29]. To this aim, Lusternik and Schnirelmann introduced a topological invariant, $\operatorname{cat} M$ of a (closed and smooth) manifold M, called later the Lusternik-Schnirelmann, or more simply LS, category and proved that $\operatorname{cat} M$ estimates from below the number of critical points a smooth function $M \to \mathbb{R}$. For the definition of the LS category, see Chapter 2 and [5].

In 1936 Karol Borsuk [4] noticed that one can consider the LS category not only of manifolds but also for general spaces. Soon after, Ralph Fox [13] began to integrate more algebra into the study of Lusternik-Schnirelmann category, and was able to tie the category of a space to its covering dimension. Grossman [18] extended the last inequality $\operatorname{cat} X \leq \dim X$ by proving that the category is bounded from above by the dimension/connectivity ratio. A short, but influential paper [7] related the category of a space with the homology properties of the fundamental group of the space. Tudor Ganea [16] fueled more research in this subject with his list of unsolved problems related to Lusternik-Schnirelmann category. In particular, the famous question asking if $\operatorname{cat}(X \times S^n) = \operatorname{cat} X + 1$ (named later as Ganea conjecture) appeared here.

In late 1950s and early 1960s researchers found purely homotopy-theoretical description of the number $\operatorname{cat} X$. G.Whitehead [41] proved the following: let $T^n X$ denote the fat n-wedge of X and let $j_n : T^n X \to X^n$ be the obvious inclusion. The $\operatorname{cat} X$ is the smallest n such that the diagonal $d_{n+1} : X \to X^{n+1}$ passes through $T^{n+1}X$ with respect to j_{n+1}, uniquely up to homotopy. Similarly, Ganea [15] and Schwarz [40] constructed a fibration $p_n = p_n^X : P_n X \to X$ (using different but homotopy equivalent construction) such that $\operatorname{cat} X$ is the smallest n such that p_{n+1} has a section, see Chapter 2. In this way

Schwarz introduced a powerful abstraction, today called Schwarz Genus. Also notice that p_n is a fibrational substitute of the map induced from j_n by d_n.

Algebraic input to Lusternik-Schnirelmann theory started with the paper [14]. It was proven that the cup-length of a space estimates Lusternik-Schnirelmann category from below, see (Chapter 2). Given a commutative ring R, the cup-length of a space (X) is the maximal length $cl_R(X)$ of non-trivial cup-product in reduced R-cohomology of X. Notice, however, that the cohomology language appeared a few years later after the paper [14]. So, as we can expect, Froloff and Elsholz stated the Poincaré dual claim in terms of intersection of homology classes. This technique allows us to evaluate the Lusternik-Schnirelmann category for certain important spaces, like tori and projective spaces.

More powerful algebraic tools were developed in studying the Ganea-Schwarz fibration. Michael Ginzburg [17] related differentials of the Eilenberg-Moore spectral sequence of the fibrations to the existence of a section to this one. In a similar way, given a ring R, Graham Toomer described a numerical invariant $e_R(X)$ such that $cl_R(X) \leq e_R(X) \leq \operatorname{cat} X$, [38]), see Chapter 3.

In late 1980s we had a renaissance of the Lusternik-Schnirelmann theory. First, we mention an influential survey by Ian James [25], and see also the previous version [24]. The progress developed in several directions.

Dennis Sullivan's theory of rational spaces [37] was used to study Lusternik-Schnirelmann category of rational spaces. In this way Yves Felix and Stephen Halperin [10] suggested a purely algebraic model for the Ganea-Schwartz fibration in a rational context. Based on this, Kathryn Hess [21] and Barry Jessup [26] proved the Ganea conjecture $\operatorname{cat}(X \times S^n) = \operatorname{cat} X + 1$ for rational simply connected spaces.

Furthermore, Yves Felix, Stephen Halperin and Jean-Michel Lemaire prove that, for rational manifolds, the Toomer invariant is equal to the LS category, i.e $\operatorname{cat}(M_Q) = e_Q(M)$ for a simply connected closed manifold M, [11].

In 1992 Ed Fadell and Sufian Husseini [8] developed the cup-length estimate by introducing the notion of category weight. However, their construction was not a homotopy invariant. The homotopy invariant version of category weight was suggested by Yuli Rudyak and Jeff Strom [32, 36]. Now this concept has many versions and analogs and turns out to be one of the most powerful tools in LS theory. One example of its usefulness is in obtaining a partial solution of the well-known Arnold conjecture on symplectic fix points, [31, 34]. Note also the paper [6] that relates the fundamental group of a closed manifold M, the dimension of M, and the LS category of M. For example, if a closed manifold has category 2 then its fundamental group is free.

In recent history, Norio Iwase provided a counterexample to the Ganea Conjecture for non-rational cases [22] using Hopf invariants introduced by Israel Berstein and Peter Hilton in their discussion of suspensions, [3].

The study of Lusternik-Schnirelmann category continues to provide value, in particular there is a strong relationship between Lusternik-Schnirelmann category and what Michael Farber [9] has described as the topology of robot motion planning. Yuliy Baryshnikov and Rob Ghrist have demonstrated that Lusternik-Schirelmann category has applications outside topology and geometry in studying topological statistics, [2]. Steve Smale [35] applied LS theory to complexity of algorithms, see also [39]. There are still interesting problems to be solved and examples to be found within the field itself as well.

CHAPTER 2
BACKGROUND

2.1 Basic Definitions

As with any field, sometimes the best way to understand something is to break it up into smaller pieces. Lusternik-Schnirelmann category is a means for decomposing a topological space into more manageable pieces. The basic definition reads as such:

Definition 2.1.1. The Lusternik-Schnirelmann category (LS category) of a space X is the smallest nonnegative integer n such that there exists $\{A_0, A_1, ..., A_n\}$, an open cover of X with each A_i contractible in X. This is denoted by $\operatorname{cat} X$, and such a cover is called categorical.

It is important to note that our definition of category is normalized, that is we count down one from the number of sets when stating the category. The original definition didn't count down by one, but now this is a common convention. Also for spaces X where no such integer exists, we say $\operatorname{cat} X = \infty$.

Example 2.1.2. *Following this definition, spaces with LS category 0 are contractible, and conversely spaces of category 0 are contractible..*

Example 2.1.3. *The LS-category of S^n is 1 as a 2-set categorical cover can be constructed by extending the northern and southern hemispheres.*

2.2 Properties of LS-Category

As with any topological invariant, there is a laundry list of properties we would like the invariant to satisfy. We include some of the basic properties Lusternik-Schnirelmann category satisfies, particularly ones that will be used to obtain results in this disseration. Also we are going to assume throughout the dissertation that all of our spaces are CW-complexes.

Remark 2.2.1. *These claims are well-known, see [5]; we list them here for reference.*

(1) $\operatorname{Cat}(X \vee Y) = \max\{\operatorname{cat} X, \operatorname{cat} Y\}$.

(2) $\operatorname{Cat}(X \cup Y) \leq \operatorname{cat}(X) + \operatorname{cat}(Y) + 1$.

(3) $\mathrm{Cat}\, X \cup CA \leq \mathrm{cat}\, X + 1$.

(4) $\mathrm{Cat}(X/A) - 1 \leq \mathrm{cat}\, X$. *This follows from the fact that* X/A *has homotopy type of* $X \cup CA$*, the union of* X *with the cone over* A*, and item (3).*

(5) If $f : X \to Y$ *has a right homotopy inverse, then* $\mathrm{cat}\, Y \leq \mathrm{cat}\, X$.

(6) $\mathrm{Cat}\, X \leq \dim X$*, where* $\dim X$ *is the covering dimension of* X *for path-connected* X.

As another word on mapping cones, it should be mentioned that Berstein and Hilton explored the changes in category of a space via attaching cones in [3]. They used generalized Hopf invariants to describe more interesting cases where the category does not increase by one after the cone is attached.

Definition 2.2.2. *Let R be a commutative ring and X a space. We define the cup-length of* X *with coefficients in* R*, denoted by* $cup_R(X)$*, to be the least integer* k *such that all* $(k+1)$*-fold cup products vanish in the reduced cohomology,* $\tilde{H}(X; R)$.

Proposition 2.2.3. *We have the inequality* $cup_R(X) \leq \mathrm{cat}\, X$ *[5, 14].*

Example 2.2.4. *Let* T^2 *denote the 2-torus. Then* $T^2 \setminus pt \cong S^1 \vee S^1$ *and* $\mathrm{cat}\, T^2 \leq 2$*. We observe that there are 2 unique elements in* $\tilde{H}_1(T^2)$ *with nontrivial cup product. Thus* $2 \leq cup T^2$ *and* $\mathrm{cat}\, T^2 = 2$.

The example with the torus is one with no difference between the cup-length and the Lusternik-Schnirelmann category, but [5] gives plenty of treatment to cases where these two invariants do not coincide. The Heisenberg group provides an example where category exceeds cup-length.

As mentioned in the history, one of the significant early results in the field was an inequality that relates the number of critical points a function on a manifold to its Lusternik-Schnirelmann category given sufficient conditions on the function and manifold. Critical points of smooth functions can be very interesting, and this result is a very beautiful relation between geometry and calculus.

Theorem 2.2.5. *(**Lusternik-Schnirelmann Theorem**). Let* M *be a paracompact* C^2*-Banach manifold and* $Crit(M)$ *the minimum number of critical points for any* C^2 *function*

with a certain boundedness property from M to R. When M is finite-dimensional and closed, we can remove the boundedness condition. Then $\mathrm{cat}(M) + 1 \leq Crit(M)$, [5] [40].

2.3 LS Category-Constructions

Definition 2.3.1. *For a path-connected space X with basepoint x_0, we define PX to be the set of all continuous functions $\gamma : I \to X$ satisfying $\gamma(0) = x_0$ topologized by the compact-open topology.*

Definition 2.3.2. *We define $p : PX \to X$ given by $p(\gamma) = \gamma(1)$. It can be proven that this yields a fibration with base space X and fiber that is homotopy equivalent to $\Omega(X, x_0)$, the loop space of X.*

Definition 2.3.3. *Given $f : Y \to X$ and $g : Z \to X$ we can define the fiberwise join, $Y *_X Z$ of Y and Z over X, as follows: $Y *_X Z = \{(y, z, t) \in Y \times Z \times I | f(y) = g(z)\}/\sim$ where $(y, z_1, 0) \sim (y, z_2, 0)$ and $(y_1, z, 1) \sim (y_2, z, 1)$.*

Remark 2.3.4. *Recall the join of topological spaces Y and Z, $Y * Z$, is defined as $Y * Z = (Y \times Z \times I)/\sim$ where $(y, z_1, 0) \sim (y, z_2, 0)$ and $(y_1, z, 1) \sim (y_2, z, 1)$.*

Definition 2.3.5. *From this, we define P_nX to be the iterated fiberwise join of n copies of PX over X via fibers of $p : PX \to X$ defined above and denote the fiberwise map with*

$$p_n^X : P_nX \to X. \tag{2–1}$$

Note that the homotopy fiber of p_n^X is $\Omega(X, x_0)^{*n}$, the n-fold join of $\Omega(X, x_0)$.

The following theorem of Schwarz, see [5, 40] ties this construction to Lusternik-Schnirelmann Category

Theorem 2.3.6. *The inequality $\mathrm{cat}(X) \leq n$ holds iff there exists a section $s : X \to P_{n+1}X$ to $p - n + 1 : P_{n+1}X \to X$.*

Schwarz actually developed a broad generalization of Lusternik-Schnirelman category [40] as what he calls genus. The biggest advantage to being able to use these ideas is that the task of computing the category of a space now takes on the flavor of other questions in algebraic topology, namely, investigating when a section to

a particular map exists. Sometimes describing open or closed sets can be less intuitive (particularly for really bad spaces), and this gives us a tool to get around that.

Later in the dissertation, when we go to construct a section, we'll find it necessary to inspect certain homotopy groups.

2.4 Toomer Invariant

As mentioned earlier, there are spaces where the inequality between cup-length and category is strict. The Toomer invariant is a tool that sits between cup-length and category.

Definition 2.4.1. *Fix a commutative ring R. The Toomer invariant of X with coefficients in R, $e_R(X)$, is the least integer k for which the map $p_n^* : H^*(X; R) \to H^*(P_n(X); R)$ is injective, see [5].*

Proposition 2.4.2. *It follows that $e_R(X) \leq \operatorname{cat} X$ for any choice of R.*

As with cup-length earlier, it should be mentioned that [5, 2.9] contains examples where the inequalities are strict.

Definition 2.4.3. *For $f : X \to Y$, where X and Y are closed, connected, and oriented n-dimensional manifolds, with $[x]$ and $[y]$ generators for $H_n(X)$ and $H_n(Y)$ respectively, we define the degree of f, denoted by $deg(f)$ to be the integer such that $f_*([x]) = deg(f)[y]$.*

Theorem 2.4.4. *Let $f : X \to Y$ be a map of degree 1, where X and Y are closed, orientable manifolds. Then $f_* : H_*(X) \to H_*(Y)$ is an epimorphism, and $f^* : H^*(Y) \to H^*(X)$ is a monomorphism.*

Rudyak asked if the existence of a map $f : M \to N$, of degree 1, implies the inequality $\operatorname{cat} M \geq \operatorname{cat} N$ [32], [5, Open problem 2.48]. While not achieving the full result, he was able to prove some partial results. In particular it follows from the injective property of f^* that $e_R(M) \geq e_R(N)$, when such a map exists [32].

Remark 2.4.5. *We know $e_R(M \times S^n) \geq e_R(M) + 1$, and there exist examples where $\operatorname{cat}(M \times S^n) = \operatorname{cat} M$ for suitable M and N, ([22, 23]).*

2.5 Rationalization

Rationalizations became quite useful for answering questions in Lusternik-Schnirelmann category. As mentioned before, rationalizations provide a setting where the Ganea Conjecture holds true.

Definition 2.5.1. *The rational n-sphere is defined to be the complex*

$$S^n_{\mathbb{Q}} = (\bigvee_{k\geq 1} S^n_k) \bigcup (\coprod_{k\geq 2} D^{n+1}_k)$$

where D^{n+1}_{k+1} is attached to $S^n_k \vee S^n_{k+1}$ by a map representing $[S^n_k] - m[S^n_{k+1}]$ for every integer m.

Definition 2.5.2. $D^{n+1}_{\mathbb{Q}} = S^n_{\mathbb{Q}} \times I / S^n_{\mathbb{Q}} \times 0$ *is called the rational* $(n+1)$ *disk.*

Definition 2.5.3. *A rationalization of a simply connected space, X is a map $\phi : X \to X_{\mathbb{Q}}$ to a simply connected rational space $X_{\mathbb{Q}}$ such that ϕ induces an isomorphism*

$$\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to X_{\mathbb{Q}}$$

Remark 2.5.4. *There is a natural inclusion $(D^{n+1}, S^n) \to (D^{n+1}_{\mathbb{Q}}, S^n_{\mathbb{Q}})$, and this allows us to construct target spaces for rationalizations.*

Throughout the section we assume X to be simply connected and denote by $X_{\mathbb{Q}}$ the rationalization of X, see [12, 37]. We define $e_{\mathbb{Q}}(X)$ to be the least integer n such that the nth fibration $P_nX \to X$ induces an injection in cohomology with coefficients in $\mathbb{Q}$. For X simply connected and of finite type, we have that $e_{\mathbb{Q}}(X) = e(X_{\mathbb{Q}})$, [5].

Proposition 2.5.5. *For simply connected CW spaces X and Y, we have $(X \cup CY)_{\mathbb{Q}} \cong X_{\mathbb{Q}} \cup CY_{\mathbb{Q}}$, [5]. In particular, $(X \vee Y)_{\mathbb{Q}} \cong X_{\mathbb{Q}} \vee Y_{\mathbb{Q}}$.*

Proof. In the following diagram, the map l is the localization map of $X \vee Y$, and k is given by the wedge of localization maps on X and Y. The map j exists by the universal property of $(X \vee Y)_{\mathbb{Q}}$, and induces isomorphisms in homology. Hence $X_{\mathbb{Q}} \vee Y_{\mathbb{Q}} \cong (X \vee Y)_{\mathbb{Q}}$

$$\begin{tikzcd} X \vee Y \arrow[r, "l"] \arrow[dr, "k"'] & (X \vee Y)_{\mathbb{Q}} \arrow[d, dotted, "j"] \\ & X_{\mathbb{Q}} \vee Y_{\mathbb{Q}} \end{tikzcd}$$

□

Remark 2.5.6. *In [11] it is shown that for a closed, simply connected manifold M, $e_{\mathbb{Q}}(M) = \mathrm{cat}(M_{\mathbb{Q}})$, and hence $\mathrm{cat}\, M_{\mathbb{Q}} \leq \mathrm{cat}\, M$.*

Returning to Rudyak's question on a possible relation between degree and category, we can settle it in the rational context.

Proposition 2.5.7. *For closed and simply connected m-manifolds M and N with $f : M \to N$ of nonzero degree, we have $\mathrm{cat}\, M_{\mathbb{Q}} \geq \mathrm{cat}\, N_{\mathbb{Q}}$.*

Proof. It suffices to show $e_{\mathbb{Q}}(M) \geq e_{\mathbb{Q}}(N)$ based on the remark above. That is, suppose $e_{\mathbb{Q}}(M) \leq n$ and so $p^* : H^*(M; \mathbb{Q}) \to H^*(P_n(M)\mathbb{Q})$ in the following diagram is injective.

$$\begin{tikzcd} H^*(P_n(M); \mathbb{Q}) & H^*(P_n(N); \mathbb{Q}) \arrow[l] \\ H^*(M; \mathbb{Q}) \arrow[u, "p^*"] & H^*(N; \mathbb{Q}) \arrow[l, "f^*"'] \arrow[u, "p^*"'] \end{tikzcd}$$

By [33, V.2.13], the map f^* is injective. Since p^* and f^* are injective, the composition $p^* \circ f^*$ is injective, and $p^* : H^*(N; \mathbb{Q}) \to H^*(P_n(N); \mathbb{Q})$ is injective. Thus $e_{\mathbb{Q}}(N) \leq n$. □

CHAPTER 3
CATEGORY OF CONNECTED SUMS AND SOME DISCUSSION

3.1 Category of Quotient Spaces

Theorem 3.1.1. *Suppose X is an n-dimensional space with m-connected subcomplex A, with $2 \leq \operatorname{cat}(X/A) \leq k$, and $k + m - 1 \geq n$. Then $\operatorname{cat} X \leq k$.*

Proof. For sake of simplicity, put $p = p_{k+1}^{X/A}$ and $p^{\#} = p_{k+1}^{X}$, cf. (2–1). As $\operatorname{cat}(X/A) \leq k$, and by Theorem 2.3.6, there exists the following section s with $ps = 1_{X/A}$.

$$\begin{tikzcd} P_{k+1}(X/A) \arrow[d, "p"'] \\ X/A \arrow[u, dotted, bend right, "s"'] \end{tikzcd}$$

We consider the collapsing map $q : X \to X/A$, and get the fiber-pullback diagram.

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & P_{k+1}(X/A) \arrow[d, "p"] \\ X \arrow[r] & X/A \end{tikzcd}$$

Now consider $P_{k+1}(X)$. We already have $p^{\#} : P_{k+1}(X) \to X$, and the collapsing map $q : X \to X/A$ induces a map $q^{\#} : P_{k+1}(X) \to P_{k+1}(X/A)$. Since $pq^{\#} = qp^{\#}$ and the square is the pull-back diagram, we get a map $h : P_{k+1}(X) \to X$ such that the following diagram commutes.

Recall that our goal is to prove $\operatorname{cat} X \leq k$. Because of Schwarz's Theorem 2.3.6, it suffices to construct a section of $p^{\#}$. To do this, it suffices in turn to construct a section of the map $h : P_{k+1}(X) \to E$. Moreover, since $\dim X = n$, it suffices to construct a section of h over the n-skeleton $E^{(n)}$ of E, i.e., to construct a map $\phi : E^{(n)} \to P_{k+1}(X)$ with $h\phi = 1_E$.

By homotopy excision [20, Prop. 4.28], and because A is m-connected, the quotient map $q : X \to X/A$ induces isomorphisms $q_* : \pi_i(X) \to \pi_i(X/A)$ for $i \leq m$ and epimorphism for $i = m + 1$. So, $\pi_i(\Omega X) \to \pi_i(\Omega(X/A))$ is an isomorphism for $i \leq$

$(m-1)$ and epimorphism for $i = m$. Therefore $\pi_i(\Omega X)^{*(k+1)} \to \pi_i(\Omega(X/A))^{*(k+1)}$ is an isomorphism for $i \leq m+k$ because of [6, Prop. 5.7].

The long exact sequence of homotopy groups for the fibration $(\Omega X)^{*(k+1)} \to P_{k+1}(X) \to X$, where the map from $P_{k+1}(X)$ to X is the map in 2–1 yields the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & \pi_i((\Omega X)^{*(k+1)}) \arrow[r] \arrow[d, "\cong"] & \pi_i(P_{k+1}X) \arrow[r] \arrow[d, dotted, "h_*"] & \pi_i(X) \arrow[r] \arrow[d, "\cong"] & \dots \\
\dots \arrow[r] & \pi_i((\Omega(X/A))^{*(k+1)}) \arrow[r] & \pi_i(E) \arrow[r] & \pi_i(X) \arrow[r] & \dots
\end{tikzcd}$$

By the 5-lemma, the map h_* is an isomorphism for $i \leq (m+k-1)$ and epimorphism for $n = m+k$. So by Whitehead's theorem, there exists a map $\phi : E^{(n)} \to P_{k+1}X$. Now, the composition $(\phi \circ s^{\#})$ is a section to $p^{\#} : P_{k+1} \to X$. Thus $\operatorname{cat} X \leq k$. $\square$

Combining this with the previous Remark 2.2.1 gives the following inequality:

$$\operatorname{cat}(X/A) - 1 \leq \operatorname{cat}(X) \leq \operatorname{cat}(X/A)$$

under the dimension-connectivity conditions from Theorem 3.1.1.

3.2 Preliminaries on Connected Sums

Definition 3.2.1. *Let M and N be n-dimensional manifolds. Define D_M and D_N to be n-disks in M and N, and let $M^* = M \setminus D_M$ and $N^* = N \setminus D_N$. Fix a diffeomorphism, $f : \partial D_M \to \partial D_N$ and define the connected sum of M and N, $M\#N = M^* \cup_f N^*$. Note that the homotopy type of $M\#N$ depends only on the homotopy class of f.*

Remark 3.2.2. *We mention that if M and N are oriented n-dimensional manifolds, then there is a canonically oriented connected sum, such that the map f reverses orientation.*

Also for any connected sum, $M\#N$, there exists a map $f : M\#N \to M$ collapsing N^* into an arbitrarily small ball in M, fixing most of M^*, and that this map has degree one.

Proposition 3.2.3. *For closed and oriented manifolds M and N, $e(M\#N) \geq \max\{e(M), e(N)\}$.*

Proof. Consider $f : M\#N \to M$ the collapsing map onto M. Then we have the following diagram.

$$\begin{tikzcd} H^*(P_nM) \arrow[r] & H^*(P_n(M\#N)) \\ H^*(M) \arrow[u] \arrow[r, tail, "f^*"] & H^*(M\#N) \arrow[u] \end{tikzcd}$$

The map f has degree 1, and so $f^* : H^*(M) \to H^*(M\#N)$ is injective [33, Theorem V, 2.13]. Also suppose $p_n^* : H^*(M\#N) \to H^*(P_n(M\#N)$ is injective. Consider $u \in H^*(M)$. As f^* and p_n^* are injective, $p_n^*(u) \in H^*(P_nM)$ is nonzero, and so $p_n^* : H^*(M) \to H^*(P_nM)$ is injective, and similarly for N. And so $e(M\#N) \geq \max\{e(M), e(N)\}$. □

Proposition 3.2.4. *For closed and oriented manifolds M and N, if $\operatorname{cat} M = e(M)$ and $\operatorname{cat} N = e(N)$, then $\operatorname{cat}(M\#N) = \max\{\operatorname{cat} M, \operatorname{cat} N\}$.*

Proof. Combining the assumptions $e(M) = \operatorname{cat} M$ and $e(N) = \operatorname{cat} N$ with the inequality $\max\{e(M), e(N)\} \leq e(M\#N) \leq \operatorname{cat}(M\#N) \leq \max\{\operatorname{cat} M, \operatorname{cat} N\}$, we have the claim. □

Proposition 3.2.5. *The inequality*

$$\max\{\operatorname{cat} M, \operatorname{cat} N\} - 1 \leq \operatorname{cat}(M\#N) \leq \max\{\operatorname{cat} M, \operatorname{cat} N\}$$

holds whenever $\max\{\operatorname{cat} M, \operatorname{cat} N\} \leq 1$

Proof. If $\operatorname{cat} M = 1 = \operatorname{cat} N$ then M and N are homotopy spheres, and so $M\#N$ is. Conversely, if $\operatorname{cat} M\#N = 1$ then M and N must be homotopy spheres. Thus, we proved that $\operatorname{cat}(M\#N) = \max\{\operatorname{cat} M, \operatorname{cat} N\}$ if $\max\{\operatorname{cat} M, \operatorname{cat} N\} \leq 1$. □

3.3 Applications to Connected Sums

Let M and N be two closed n-dimensional manifolds.

Corollary 3.3.1. *Suppose that either* $\max\{\operatorname{cat} M, \operatorname{cat} N\} \geq 3$ *or* $\max\{\operatorname{cat} M, \operatorname{cat} N\} = 1$ *. Then there is a double inequality*

$$\max\{\operatorname{cat} M, \operatorname{cat} N\} - 1 \leq \operatorname{cat}(M\#N) \leq \max\{\operatorname{cat} M, \operatorname{cat} N\}.$$

Proof. Consider the case of Theorem 3.1.1 where $X = M\#N$, the connected sum of M and N, and $A = S^{n-1}$ is the separating sphere between M and N. Then $X/A = M \vee N$,

and $\mathrm{cat}(X/A) = \max\{\mathrm{cat}\, M, \mathrm{cat}\, N\}$. The case $\max\{\mathrm{cat}\, M, \mathrm{cat}\, N\} = 1$ is covered by Proposition 3.2.5. So, assume that $\mathrm{cat}(X/A) =\geq 3$. So as A is $(n-2)$-connected and $(n-2)+3-1 \geq n$, we are in the case of Theorem 3.1.1 and get the right-hand inequality. Note that we are using item (4) of Remark 2.2.1. □

Remark 3.3.2. *We note that the Corollary 3.3.1 above doesn't cover the case when* $\mathrm{cat}\, M = \mathrm{cat}\, N = 2$ *while* $\mathrm{cat}(M\#N) = 3$*. We do not know if such a case is possible, however. If* $M\#N$ *is simply connected, then so is* $M \vee N$*. In that case, the isomorphisms from homotopy excision in Theorem 3.1.1 extend to at least one dimension higher, and the result follows. In the non-simply connected case though, the situation is unresolved.*

Remark 3.3.3. *In [19], an upper bound is given for the LS category of a double mapping cylinder. If we consider the connected sum of n-manifolds M and N as such a double mapping cylinder, then the following inequality is obtained:*

$$\mathrm{cat}\, M\#N \leq min\{1 + \mathrm{cat}\, M^* + \mathrm{cat}\, N^*, 1 + max\{\mathrm{cat}\, M^*, catN^*\}\}.$$

Here M^* *and* N^* *are* $M\setminus$ *pt, and* $N\setminus$ *pt, respectively. Rivadeneyra proved in [30] that the category of a manifold without boundary sometimes does not increase when a point is removed. If the categories of* M *and* N *do decrease by one when a point is removed, then Hardie's result yields the same information as proven here. However in [27], a closed manifold is constructed so that the category remains unchained after the deletion of a point, and our result gives an improvement of the category in such a case.*

Also when to reconsider the case of rationalizations of connected sums, we obtain the following result.

Proposition 3.3.4. *For* M *and* N*, closed and simply connected manifolds,* $\mathrm{cat}(M\#N)_{\mathbb{Q}} = \max\{\mathrm{cat}\, M_{\mathbb{Q}}, \mathrm{cat}\, N_{\mathbb{Q}}\}$*, provided* $\mathrm{cat}(M\#N)_{\mathbb{Q}} \geq 3$.

Proof. As M and N are closed and simply connected, $M\#N$ is closed and simply connected, and $e_{\mathbb{Q}}(M\#N) = \mathrm{cat}(M\#N)_{\mathbb{Q}}$. Combining Remark 2.5.6 and Proposition 3.2.3

establishes on the left hand side,

$$\max\{\operatorname{cat} M_{\mathbb{Q}}, \operatorname{cat} N_{\mathbb{Q}}\} = \max\{e_{\mathbb{Q}}(M), e_{\mathbb{Q}}(N)\} \leq e_{\mathbb{Q}}(M\#N) = \operatorname{cat}(M\#N)_{\mathbb{Q}}.$$

While on the right hand side we have,

$$\operatorname{cat}(M\#N)_{\mathbb{Q}} = \operatorname{cat}(M_{\mathbb{Q}}\#N_{\mathbb{Q}}) \leq \operatorname{cat}(M_{\mathbb{Q}} \vee N_{\mathbb{Q}}) = \max\{\operatorname{cat} M_{\mathbb{Q}}, \operatorname{cat} N_{\mathbb{Q}}\},$$

where the inequality $\operatorname{cat}(M_{\mathbb{Q}}\#N_{\mathbb{Q}}) \leq \operatorname{cat}(M_{\mathbb{Q}} \vee N_{\mathbb{Q}})$ is obtained via Corollary 3.3.1 and item (1) in Remark 2.2.1. The inequality $\operatorname{cat}(M\#N)_{\mathbb{Q}} \leq \operatorname{cat}(M \vee N)_{\mathbb{Q}}$ can be proved using the previous result. However since dim $(M\#N)_{\mathbb{Q}} = n+1$, we need the restriction on $\operatorname{cat}(M\#N)_{\mathbb{Q}}$. $\square$

There are plenty of examples where the category does not increase when the connected sum of two manifolds is formed, but the inequality leaves open the possibliity that a connected sum of two manifolds could actually have a category 1 less than either of the composant manifolds. Such an example has not been revealed, and represents an area for future work.

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BIOGRAPHICAL SKETCH

Robert Newton was born in Glens Falls, New York. His father was an active duty member of the US Army, he spent his childhood moving around, including stops in California, Texas, and Germany. He earned his B.A. and M.A. both in Mathematics from SUNY Potsdam in 2007. To further his career in mathematics, and partially to escape the cold weather, he began graduate school at the University of Florida in August of 2007. In the summer of 2013 Robert graduated with his Ph.D. in Mathematics. Upon graduation, he will pursue teaching high school mathematics at Trinity School in Manhattan.