COMPOSITION OPERATORS ON HARDY SPACES OF THE DISK AND HALF-PLANE

By

KRISTIN LUERY

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I would like to dedicate this dissertation to my family.

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Kristin Luery

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In his work, J. H. Shapiro provided an integral formula for the Nevanlinna counting function and used it to prove many results for composition operators on the Hardy space of the disk. We derive an integral formula for a counting function in the upper half-plane and use it to provide a function theoretic proof that composition operators are bounded above on the Hardy space of the upper half-plane. We also derive a new tool, the reproducing kernel thesis, to show that composition operators have closed range on the Hardy space of the disk. With it we are able to provide a direct geometric equivalence between the criterion of Cima, Thomson, and Wogen and the one of Zorboska for composition operators to have closed range on the Hardy space of the disk.

CHAPTER 1
INTRODUCTION

The study of composition operators is a beautiful subject that utilizes techniques from all parts of analysis to prove results: operator theory, functional analysis, measure theory, and analytic function theory to name a few. The concept of a composition operator is relatively simple and natural to study; given a function φ we define the composition operator C_φ on a space of analytic functions by $C_\varphi f = f \circ \varphi$ where f is in the function space. Study of composition operators as such truly began in the 1960s with Nordgren, and has steadily grown since then. They have been studied on several different function spaces of various domains in the complex plane as well as in higher dimensions. It has been found that many properties of the operator C_φ can be extracted from function theoretic properties of the function φ. In the simplest and most intuitive case, the Hardy space H^2 of the unit disk in the complex plane, many problems have already been solved such as conditions for C_φ to be compact [22], have closed range [5, 11, 26], be an isometry [16], or similar to an isometry [2]. Another interesting setting, and one of a lot of recent activity, is the Hardy space H^2 of the upper half of the complex plane. Though the two Hardy spaces are isomorphic, composition operators on the two spaces behave very differently. For example, there are no compact composition operators on H^2 of the upper half plane [14], and though every composition operator is bounded on H^2 of the disk not all are bounded on H^2 of the upper half-plane. Here we will be focusing on composition operators on the Hardy space of the disk and of the upper half of the complex plane.

1.1 Hardy space of the Disk

First, we define some notation. We denote the open unit disk of the complex plane by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and its boundary, the unit circle by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. There are several ways to define the Hardy space H^2 of the disk. In its most basic sense, $H^2(\mathbb{D})$ is defined to be the space of all analytic functions on the disk whose Taylor series

coefficients are square-summable, that is

$$H^2(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}. \tag{1–1}$$

This leads to the definition of the inner product on $H^2(\mathbb{D})$; let $f, g \in H^2(\mathbb{D})$ where $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$, we define

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b}_n.$$

The norm is then

$$\|f\|^2_{H^2(\mathbb{D})} = \langle f, f \rangle_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} |a_n|^2 . \tag{1–2}$$

These definitions give a direct correspondence between the Hilbert space $H^2(\mathbb{D})$ and the Hilbert space of square-summable infinite sequences $l^2(\mathbb{N})$. Another definition for $H^2(\mathbb{D})$ relates it to the Hilbert space of functions $L^2(\mathbb{T})$. In this case we define

$$H^2(\mathbb{D}) = \left\{ f \text{ analytic on } \mathbb{D} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, dm(\zeta) < \infty \right\}$$

where m denotes normalized Lebesgue measure on the circle $\mathbb{T}$. Here the inner product is

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \sup_{0<r<1} \int_{\mathbb{T}} f(r\zeta) \overline{g(r\zeta)} dm(\zeta),$$

and the norm is

$$\|f\|^2_{H^2(\mathbb{D})} = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^2 \, dm(\zeta). \tag{1–3}$$

The two definitions for $H^2(\mathbb{D})$ yield the same space of analytic functions on the disk, so will be used interchangeably, with the integral definition perhaps being used more often. Another Hardy space of the disk that will be of use to us is the space $H^\infty(\mathbb{D})$, which is simply the space of bounded analytic functions on $\mathbb{D}$ and has norm

$$\|f\|_{H^\infty(\mathbb{D})} = \sup_{z \in \mathbb{D}} |f(z)| .$$

Several of the results given, especially in Chapter 2, apply more generally to H^p spaces of the disk. The space $H^2(\mathbb{D})$ is actually a special case of an $H^p(\mathbb{D})$ space where $p = 2$. For $0 < p < \infty$ we define

$$H^p(\mathbb{D}) = \left\{ f \text{ analytic on } \mathbb{D} : \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) < \infty \right\}$$

where m denotes normalized Lebesgue measure on the circle, with norm

$$\|f\|^p_{H^p(\mathbb{D})} = \sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta).$$

1.2 Hardy space of the Half-Plane

The notation we will be using for the upper half-plane is $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$, which has boundary the real line, $\mathbb{R} = \{z \in \mathbb{C} : \Im z = 0\}$. A very similar situation holds for the Hardy space of the upper half-plane $H^2(\mathbb{H})$, though it is usually defined only by integral means. Let $z = x + iy \in \mathbb{C}$, then we define

$$H^2(\mathbb{H}) = \left\{ f \text{ analytic on } \mathbb{H} : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dm(x) < \infty \right\},$$

with norm

$$\|f\|^2_{H^2(\mathbb{H})} = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 \, dm(x)$$

where m denotes Lebesgue measure on the real line $\mathbb{R}$. The space $H^\infty(\mathbb{H})$ is defined similarly to that of the disk; it is the space of bounded analytic functions on $\mathbb{H}$ and has norm

$$\|f\|_{H^\infty(\mathbb{H})} = \sup_{z \in \mathbb{H}} |f(z)| .$$

We can define a similar generalization of the $H^2(\mathbb{H})$ space to an $H^p(\mathbb{H})$ space. For $0 < p < \infty$,

$$H^p(\mathbb{H}) = \left\{ f \text{ analytic on } \mathbb{H} : \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p \, dm(x) < \infty \right\},$$

with norm

$$\|f\|^p_{H^p(\mathbb{H})} = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p \, dm(x)$$

where m denotes Lebesgue measure on the real line $\mathbb{R}$.

Throughout this paper, we will be suppressing the subscripts on the norms, as it should be easily understood from context which function space the given function is in.

1.3 Composition Operators

At this time we give a formal definition of a composition operator on a Hardy space.

Definition 1.3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. We define the composition operator C_φ on $H^p(\mathbb{D})$ for $0 < p \leq \infty$ by*

$$C_\varphi f = f \circ \varphi$$

for $f \in H^p(\mathbb{D})$.

Composition operators on $H^p(\mathbb{H})$ are defined similarly.

Definition 1.3.2. *Let $\varphi : \mathbb{H} \to \mathbb{H}$ be analytic. We define the composition operator C_φ on $H^p(\mathbb{H})$ for $0 < p \leq \infty$ by*

$$C_\varphi f = f \circ \varphi$$

for $f \in H^p(\mathbb{H})$.

An underlying assumption in both of these definitions is that φ is nonconstant. This does not in any way inhibit generality as the case where φ is a constant is quite trivial. The map φ is frequently called a self-map of $\mathbb{D}$ or $\mathbb{H}$.

CHAPTER 2
PRELIMINARIES

The purpose of this chapter is to reproduce some well-known results in the field, so that they will be on hand to reference in later chapters.

2.1 Inner-Outer Factorization

Now that we have defined the Hardy spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{H})$ for $0 < p \leq \infty$ in the introduction, we will gather some important facts about Hardy spaces. One of the most important theorems concerning Hardy spaces is the inner-outer factorization. It allows us to factor any function f in a Hardy space as a product of two inner functions and an outer function. The terms inner and outer are emblematic of the properties of each of these functions. Inner functions encode all of the information about f inside of the domain $\mathbb{D}$ or $\mathbb{H}$; for example, the Blaschke product encodes all of the information about the zeros of f in the interior of the domain. On the other hand, outer functions encode all of the information about f at the boundary of the domain; we will see that the outer factor has the same boundary limits as f almost everywhere. Both Garnett [9, Chapter II] and Hoffman [10] provide full accounts and proofs of the inner-outer factorization on both the disk and the half-plane, and all the crucial points necessary for this paper are summarized below.

2.1.1 Disk Factorization

First, we will focus our account on the disk, and then we will move to the upper half-plane. The first important result we will discuss is a theorem by Fatou; it simply states that boundary limits exist Lebesgue almost everywhere.

Theorem 2.1.1 (Fatou)**.** *If f is a function in $H^p(\mathbb{D})$ for $0 < p \leq \infty$, then its boundary limits exist m-almost everywhere on $\mathbb{T}$ and we write*

$$f(\zeta) = lim_{r \to 1^-} f(r\zeta)$$

for m-almost every $\zeta \in \mathbb{T}$.

Now that we have determined that boundary values exist almost everywhere, we can define precisely what it means for a function to be inner or outer.

Definition 2.1.2. *An inner function is an analytic function $g : \mathbb{D} \to \mathbb{C}$ such that $|g(z)| \leq 1$ and $\left|g(e^{i\theta})\right| = 1$ m-a.e. $\theta \in \mathbb{T}$.*

An outer function is an analytic function $F : \mathbb{D} \to \mathbb{C}$ of the form

$$F(z) = \lambda \exp\left[\int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} k(\theta) dm(\theta)\right]$$

where k is a real-valued integrable function on the circle and $\lambda \in \mathbb{C}$ has modulus 1.

When F is an outer function in $H^p(\mathbb{D})$ for $0 < p \leq \infty$, then $k(\theta) = \log\left|F(e^{i\theta})\right|$ m-a.e.

Theorem 2.1.3. *[10, p. 62] Let F be a non-zero function in $H^p(\mathbb{D})$ for $0 < p \leq \infty$, then the following are equivalent:*

(a) *F is an outer function;*

(b) *if f is any function in H^p such that $|f| = |F|$ m-almost everywhere on $\mathbb{T}$, then $|F(z)| \geq |f(z)|$ for every point $z \in \mathbb{D}$;*

(c) $\log|F(0)| = \int_{\mathbb{T}} \log\left|F(e^{i\theta})\right| dm(\theta)$.

As mentioned above, a Blaschke product is an inner function that encodes the information about the zeros of a function in the disk, and is defined as the possibly infinite product below.

Theorem 2.1.4. *[10, p. 64] Let $\{a_n\}$ be a sequence of points in $\mathbb{D}$. Then the infinite product*

$$\prod_{n=1}^{\infty} \frac{a_n}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z}$$

converges uniformly on compact subsets of $\mathbb{D}$ if and only if

$$\sum (1 - |a_n|) < \infty. \tag{2–1}$$

Definition 2.1.5. *We define the Blaschke product to be the infinite product*

$$z^m \prod_{n=1}^{\infty} \frac{a_n}{|a_n|} \frac{a_n - z}{1 - \overline{a_n} z} \tag{2–2}$$

where m is a nonnegative integer and $\{a_n\}$ is a nonzero sequence that satisfies (2–1).

The condition on the sequence $\{a_n\}$, (2–1), is usually called the Blaschke condition. For convenience, we will also call a function that is a unimodular constant λ times $B(z)$ a Blaschke product.

A couple of observations about the Blaschke product:

- $B(z)$ is in $H^\infty(\mathbb{D})$;
- the zeros of $B(z)$ are precisely $\{a_n\}$ if $m = 0$ or $\{0\} \cup \{a_n\}$ if $m > 0$;
- $|B(z)| \leq 1$ and $\left|B(e^{i\theta})\right| = 1$ m-almost everywhere on $\mathbb{T}$, i.e. $B(z)$ is an inner function.

The following condition is equivalent to a function being a Blaschke product and will be very useful when we encounter Frostman's theorem in Section 2.3.

Theorem 2.1.6. *[9, p. 54] Let f be analytic in $\mathbb{D}$, then the following are equivalent:*

(a) *f is a Blaschke product, i.e. $f(z) = \lambda B(z)$, where λ is a unimodular constant and $B(z)$ is a Blaschke product.*

(b) $\lim_{r\to 1^-} \int_{\mathbb{T}} \log\left|f(re^{i\theta})\right| dm(\theta) = 0.$

Finally, we define the singular inner factor. It encodes all the information about the zeros that accumulate on the boundary of the disk.

Definition 2.1.7. *A singular function is a function of the form*

$$S(z) = \exp\left[-\int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right]$$

where the measure μ is positive and singular to Lebesgue measure.

Note that the singular function is an inner function that has no zeros inside of the disk. Now that we have these definitions, we can write any function in H^p of the disk as a product of each type of function, a Blaschke product, a singular inner function, and an outer function.

Theorem 2.1.8. *Let $f \in H^p(\mathbb{D})$ for $0 < p \leq \infty$ such that $f \neq 0$, then f can be written uniquely as the product BSF, where B is a Blaschke product, S is a singular function, and F is an outer function.*

2.1.2 Half-Plane Factorization

A similar factorization theorem holds for $H^p(\mathbb{H})$ for $0 < p \leq \infty$, with slight variations in the definitions of the Blaschke product, singular function, and outer function. We omit many of the details here, but they are very similar to the case of the disk. Hoffman has an in depth discussion in [10, Chapter 8]. First of all, Fatou's theorem works exactly the same way, but it is important to note that the boundary of the upper half-plane is simply the real line $\mathbb{R}$.

Theorem 2.1.9. *If f is a function in $H^p(\mathbb{H})$ for $0 < p \leq \infty$, then its boundary limits exist m-almost everywhere on $\mathbb{R}$ and we write*

$$f(x) = lim_{y \to 0^+} f(x + iy)$$

for m-almost every $x \in \mathbb{R}$.

An inner function is then defined to be an analytic functions whose boundary limits $f(x)$ are real m-almost everywhere.

The Blaschke product with zero sequence $\{b_n\} \subset \mathbb{H}$ is

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_{b_n \neq i} \frac{|b_n^2+1|}{b_n^2} \frac{z-b_n}{z-\overline{b}_n},$$

and converges if and only if

$$\sum \frac{\Im b_n}{1+|b_n|^2} < \infty.$$

The singular function is

$$S(z) = e^{\rho z} \exp\left[-\int_{\mathbb{R}} i \frac{tz+1}{z-t} d\mu(t)\right]$$

where μ is a finite singular positive measure on the real line and ρ is a non-negative real number. The factor $e^{\rho z}$ accounts for any necessary information at the boundary point ∞.

Finally, the outer function is given by

$$F(z) = \exp\left[\frac{1}{\pi}\int_{\mathbb{R}} \log|f(t)|\, i\frac{tz+1}{z-1}\frac{dt}{1+t^2}\right].$$

Therefore, we can factor any function f in $H^p(\mathbb{H})$ for $0 < p \leq \infty$ as a product BSF, as we did in $H^p(\mathbb{D})$.

2.2 Möbius Transformations

Several Möbius transformations will be important in our study. First are the conformal automorphisms of the disk,

$$\alpha_a(z) = \lambda\frac{a-z}{1-\overline{a}z}$$

where $|\lambda| = 1$ and $a \in \mathbb{D}$. If $\lambda = 1$, these maps are self-inverse, that is $\alpha_a = \alpha_a^{-1}$. It is also important to note that α_a maps $\mathbb{D} \to \mathbb{D}$ and $\mathbb{T} \to \mathbb{T}$. These maps are useful because they exchange the point a with the point 0. Often it is easier to check a property at the point $z = 0$, so these maps enable us to check a property at $z = a$ by transporting a to 0, checking at 0, then transporting it back to a. A useful identity involving automorphisms of the disk is

$$1 - |\alpha_a(z)|^2 = \frac{(1-|z|^2)(1-|a|^2)}{|1-\overline{a}z|^2}. \tag{2–3}$$

We will also make extensive use of the the Cayley transform. The Cayley transform conformally maps the unit disk to the upper half-plane by

$$J(z) = i\left(\frac{1+z}{1-z}\right) \qquad z \in \mathbb{D},$$

and has an inverse given by

$$J^{-1}(z) = i\left(\frac{1+z}{1-z}\right) \qquad z \in \mathbb{H}.$$

Some things to note about the Cayley transform are that 0 maps to i; 1 maps to ∞; and it maps the boundary to the boundary, i.e. $\mathbb{T}$ to $\mathbb{R}$. Its utility lies in the fact that many properties of composition operators are well-known on $H^2(\mathbb{D})$. Consequently, much of

the work being done on $H^2(\mathbb{H})$ relies on transporting quantities to the disk, exploiting properties already known there, and then transporting back to the half-plane. A thorough discussion of Möbius transformations can be found in any complex analysis book, for example in [6] or [20].

2.3 Frostman's Theorem

Before we get to Frostman's theorem, we need to understand a little bit about logarithmic capacity. This brief discussion is from Garnett [9, Section II.6], and a more robust account can be found in Ransford [18, Chapter 5].

Definition 2.3.1. *[9, p. 75] A compact set K in $\mathbb{C}$ has positive logarithmic capacity if there is a positive measure $\sigma \neq 0$ on K such that the logarithmic potential*

$$U_\sigma(z) = \int_K \log \frac{1}{|w - z|} d\sigma(w)$$

is bounded on some neighborhood of K.

If K is a subset of $\mathbb{D}$, then K has positive capacity if and only if K supports a positive measure σ for which Green's potential

$$G_\sigma(z) = \int_K \log \left| \frac{1 - \overline{w}z}{w - z} \right| d\sigma(w)$$

is bounded on $\mathbb{D}$.

An arbitrary set E is said to have positive capacity if there is a compact subset of E that has positive capacity.

Theorem 2.3.2 (Frostman)**.** *[9, p. 75] Let $f(z)$ be a nonconstant inner function on the unit disk, then for all $w \in \mathbb{D}$, except possibly for a set of capacity zero, the function*

$$f_w(z) = \frac{w - f(z)}{1 - \overline{w} f(z)}$$

is a Blaschke product.

For a proof of Frostman's theorem, see Garnett [9, p. 76].

Notice that the function f_w is simply equal to $\alpha_w \circ f$, hence it is frequently called the Frostman transform of f. The wonderful thing about the Frostman transform of f is that $f^{-1}(w) = f_w^{-1}(0)$. This proves incredibly useful in our discussion of the Nevanlinna counting function in Chapter 4.

The condition "for all $w \in \mathbb{D}$, except possibly for a set of capacity zero" used in Frostman's theorem is frequently worded as "for quasi every $w \in \mathbb{D}$" or "q.e. $w \in \mathbb{D}$" similar to the condition almost every or almost everywhere used in measure theory. In fact, quasi everywhere implies almost everywhere with respect to Lebesgue area measure.

In [19], Walter Rudin generalized Frostman's theorem to apply to all H^p functions, rather than just nonconstant inner functions. In this case, the conclusion is simply that f_w has no singular inner factor quasi everywhere, or in other words that $f_w = BF$ for q.e. $w \in \mathbb{D}$.

Theorem 2.3.3 (Rudin)**.** *If $f \in H^p(\mathbb{D})$ for $0 < p \leq \infty$, then for all $w \in \mathbb{D}$, except possibly for a set of capacity zero, f_w has no singular inner factor.*

2.4 Reproducing Kernel Hilbert Spaces

One important feature of $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$ is that they are both examples of reproducing kernel Hilbert spaces. The set of reproducing kernels for each space forms a subset that has dense linear span in that space, providing us with an exceptionally well behaved subsets of $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$ to work with. Utilizing these sets of kernel functions can greatly simply proofs as we will see in Chapter 5. This discussion is taken largely from Paulsen's manuscript [17].

Given a set X, let us denote the vector space of functions from X to $\mathbb{C}$ by $\mathscr{F}(X, \mathbb{C})$.

Definition 2.4.1. *[17] Given a set X, we will say that $\mathscr{H}$ is a reproducing kernel Hilbert space on X over $\mathbb{C}$, provided that*

(i) *$\mathscr{H}$ is a vector subspace of $\mathscr{F}(X, \mathbb{C})$;*

(ii) *$\mathscr{H}$ is endowed with an inner product $\langle \cdot, \cdot \rangle$, making it into a Hilbert space;*

(iii) *for every $y \in X$, the linear evaluation functional, $E_y : \mathscr{H} \to \mathbb{C}$, defined by $E_y(f) = f(y)$ is bounded.*

Since every bounded linear functional in a reproducing kernel Hilbert space $\mathscr{H}$ is given by the inner product with a unique vector in $\mathscr{H}$, we have that for every $y \in X$, there is a unique vector $k_y \in \mathscr{H}$, such that $f(y) = \langle f, k_y \rangle$ for every $f \in \mathscr{H}$.

Definition 2.4.2. *The function k_y is called the reproducing kernel at the point y. The two variable function defined by*

$$K(x, y) = k_y(x)$$

is called the reproducing kernel for $\mathscr{H}$.

Also note that

- $K(x, y) = k_y(x) = \langle k_y, k_x \rangle$ for $x, y \in X$, and
- $\|E_y\|^2 = \|k_y\|^2 = \langle k_y, k_y \rangle = K(y, y)$ for $y \in X$.

The first property above is called the reproducing property of the kernel functions.

One of the reasons reproducing kernel functions are so useful is because they have dense linear span in their associated Hilbert spaces.

Proposition 2.4.3. *[17] Let $\mathscr{H}$ be a reproducing kernel Hilbert space on the set X with kernel K. Then the linear span of the functions $k_y(\cdot) = K(\cdot, y)$ is dense in $\mathscr{H}$.*

We require one more fact about kernel functions, it is used in the proof the a composition operator is bounded above on $H^2(\mathbb{H})$. This proposition states that if there are two positive kernels over the same set X, then their product is also a positive kernel.

Proposition 2.4.4. *[17] Let X be a set and let $K_i : X \times X \to \mathbb{C}$, $i = 1, 2$ be two positive definite kernels, then their product, $P : X \times X \to \mathbb{C}$ given by*

$$P(x, y) = K_1(x, y)K_2(x, y)$$

is a positive definite kernel.

In this paper we will be concerned with two reproducing kernel Hilbert spaces in particular, $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$. The reproducing kernels for the Hardy space on the disk

$H^2(\mathbb{D})$ are the Szegő kernels $k_w(z) = \dfrac{1}{1-\overline{w}z}$ for $w, z \in \mathbb{D}$. The norm of the Szegő kernels is given by

$$\|k_w\|^2 = K(w,w) = \frac{1}{1-|w|^2}.$$

Because of this, rather than using the traditional Szegő kernels, we will be using the normalized Szegő kernels, $\tilde{k}_w(z) = \dfrac{\sqrt{1-|w|^2}}{1-\overline{w}z}$, which are normalized to have norm one. As for $H^2(\mathbb{H})$, the reproducing kernels are the functions $k_w(z) = \dfrac{1}{z+\overline{w}}$ for $w, z \in \mathbb{H}$.

The reproducing kernels of a reproducing kernel Hilbert space interact very nicely with the adjoint of a composition operator, as can be seen in [7]. This property will be used in the original proof that C_φ is bounded above on $H^2(\mathbb{H})$ in Chapter 3. Recall the definition of an adjoint:

Definition 2.4.5. *Let $\mathscr{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and consider a bounded linear operator $A : \mathscr{H} \to \mathscr{H}$, then there exists a unique bounded linear operator $A^* : \mathscr{H} \to \mathscr{H}$ such that*

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \qquad \text{for all } x, y \in \mathscr{H}.$$

The operator A^* is called the adjoint of A.

We will simply denote the adjoint of a composition operator C_φ by C_φ^*. Now let $\mathscr{H}$ be a reproducing kernel Hilbert space on X over $\mathbb{C}$ with reproducing kernels $\{k_x\}_{x \in X}$.

Theorem 2.4.6. *[7, p. 4] If A is a bounded linear operator mapping a reproducing kernel Hilbert space $\mathscr{H}$ to itself, then A is a composition operator if and only if the set $\{k_x\}_{x \in X}$ is invariant under A^*. In this case, $A = C_\varphi$ where φ and A are related by $A^*(k_x) = k_{\varphi(x)}$.*

Proof. [7, p. 4] If $A = C_\varphi$, then for each function f

$$(A^*(k_x))(f) = k_x(Af) = k_x(f \circ \varphi) = f(\varphi(x)) = k_{\varphi(x)} f$$

so $A^*(k_x) = k_{\varphi(x)}$, and the set of point evaluation linear functionals is invariant under A^*.

Conversely, if the set of point evaluation linear functionals is invariant under A^*, then define the map φ on X by $A^*(k_x) = k_{\varphi(x)}$. This defines φ since the vectors in a Hilbert spaces separate the points of X. Then

$$(Af)(x) = k_x(Af) = (A^*(k_x))(f) = k_{\varphi(x)}(f) = f(\varphi(x)),$$

so $A = C_\varphi$. □

2.5 Nevanlinna Counting Function

The Nevanlinna counting function is a tool that is intimately linked with composition operators. For example, in [22] Joel Shapiro used the counting function in a characterization of the essential norm of a composition operator and a criterion for composition operators to be compact.

Definition 2.5.1. *For a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$, define the Nevanlinna counting function of φ by*

$$N_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \log \frac{1}{|z|}$$

for all $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, where $\varphi^{-1}(w)$ denotes the set of φ-preimages of w counted according to their multiplicity, and $N_\varphi(w) = 0$ if $w \notin \varphi(\mathbb{D})$.

Since $\log \frac{1}{|z|} \approx 1 - |z|$ for $|z|$ close enough to 1, the Nevanlinna counting function provides a measure of the affinity that φ has for the value w by weighting each preimage of w by the product of its multiplicity and a weight that is essentially its distance from the boundary of the circle $\mathbb{T}$.

We wish to define a similar function on the upper half-plane. In this case the boundary is the real line $\mathbb{R}$, so our goal is to construct a function that weights each φ-preimage by its distance to $\mathbb{R}$. This is relatively simple as $\Im z$ gives precisely the distance from $z \in \mathbb{H}$ to $\mathbb{R}$.

Definition 2.5.2. *For a holomorphic map $\varphi : \mathbb{H} \to \mathbb{H}$, define the counting function of φ on $\mathbb{H}$ by*

$$N_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \Im z$$

for all $w \in \varphi(\mathbb{H})$, where $\varphi^{-1}(w)$ denotes the set of φ-preimages of w counted according to their multiplicity, and $N_\varphi(w) = 0$ if $w \notin \varphi(\mathbb{H})$.

This counting function weights each φ-preimage by the product of its multiplicity and its distance to the boundary of the upper half-plane $\mathbb{R}$.

2.6 Change of Variables Formula

In Chapter 1, we discussed the main formulations of the norm on H^p spaces, specifically $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$. There is one more formulation of the norm that we prove exceedingly useful to us, it is the Littlewood-Paley identity.

Proposition 2.6.1 (Littlewood-Paley Identity)**.** *If f is holomorphic on $\mathbb{D}$, then*

$$\|f\|^2 = 2\int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z) + |f(0)|^2 \tag{2–4}$$

where $\|\cdot\|$ denotes the $H^2(\mathbb{D})$ norm, and $\|f\| = \infty$ means that $f \notin H^2(\mathbb{D})$.

For a proof see Garnett [9, Chapter IV.3].

As we will be focusing our study on composition operators, we would like a simple formula for the norm of C_φ acting on a function $f \in H^2(\mathbb{D})$. When we use the Littlewood-Paley identity to find $\|C_\varphi f\|$ an interesting thing happens, the Nevanlinna counting function appears in the integral.

Proposition 2.6.2 (Change of Variables Formula)**.** *Suppose φ is holomorphic on $\mathbb{D}$, then*

$$\|C_\varphi f\|^2 = 2\int_{\mathbb{D}} |f'(z)|^2 N_\varphi(z) dA(z) + |f(\varphi(0))|^2 . \tag{2–5}$$

The reason why the Nevanlinna counting function appears is because we do not require φ to be univalent. Typically a change of variables requires the transformation function to be univalent, however that would limit the study of composition operators significantly. So, we divide $\varphi^{-1}(\mathbb{D})$ into regions where φ is one-to-one, then apply

the usual change of variables on each of these regions individually. The Nevanlinna counting function appears when we sum over all of these regions. It follows that the main step in proving the Change of Variables Formula is proving the changes of variables within the integral of (2–5) [23, Section 10.3].

Lemma 2.6.3. *[23, p. 186] Let g be a nonnegative measurable function on $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic, then*

$$\int_{\mathbb{D}} g(\varphi(z)) \, |\varphi'(z)|^2 \log \frac{1}{|z|} dA(z) = \int_{\mathbb{D}} g(z) N_\varphi(z) dA(z).$$

Proof. [23, p. 186] Here we make explicit use of the underlying assumption that φ is not a constant function. As such, its derivative φ' vanishes on an at most countable subset of $\mathbb{D}$ that has no limit point in $\mathbb{D}$; let us call this set Z. Around each point in $\mathbb{D} \setminus Z$ there is an open set on which φ is a homeomorphism, so we can rewrite $\mathbb{D} \setminus Z$ as an at most countable disjoint union of "semi-closed" polar rectangles R_n such that φ is univalent on each one. Now we can apply the usual change of variables formula on each R_n individually; for ease of notation, let ψ_n denote the inverse of $\varphi\big|_{R_n}$. Therefore,

$$\int_{R_n} g(\varphi(z)) \, |\varphi(z)|^2 \log \frac{1}{|z|} dA(z) = \int_{\mathbb{D}} g(w) \chi_{\varphi(R_n)}(w) \log \frac{1}{|\psi_n(w)|} dA(w).$$

Summing both sides over n yields

$$\int_{\mathbb{D}} g(\varphi(z)) \, |\varphi(z)|^2 \log \frac{1}{|z|} dA(z) = \int_{\mathbb{D}} g(w) \left[\sum_n \chi_{\varphi(R_n)}(w) \log \frac{1}{|\psi_n(w)|} \right] dA(w). \tag{2–6}$$

Now, if $w \in \varphi(\mathbb{D}) \setminus \varphi(Z)$, every point of $\varphi^{-1}(\{w\})$ has multiplicity one, so

$$\sum_n \chi_{\varphi(R_n)}(w) \log \frac{1}{|\psi_n(w)|} = N_\varphi(w)$$

for almost every $w \in \varphi(\mathbb{D})$. On the other hand, if $w \notin \varphi(\mathbb{D})$, then by definition $N_\varphi(w) = 0$ and the bracketed term in (2–6) is zero. □

We have a similar formulation for the H^2 norm in the upper half-plane. Here the term $\log \frac{1}{|z|}$ is replaced by $\Im z$, which represents the distance to the boundary in the upper half-plane.

Proposition 2.6.4 (Littlewood-Paley Identity for the Half-Plane)**.** *If f is holomorphic on $\mathbb{D}$, then*

$$\|f\|^2 = 2\int_{\mathbb{D}} |f'(z)|^2 \Im z dA(z) \tag{2–7}$$

where $\|\cdot\|$ denotes the $H^2(\mathbb{H})$ norm, and $\|f\| = \infty$ means that $f \notin H^2(\mathbb{H})$.

2.7 Aleksandrov Clark Measures

Another set of tools that are intimately tied with composition operators are the Aleksandrov-Clark measures. Similar to how the Nevanlinna counting function $N_\varphi(w)$ measures the affinity φ has with the value $w \in \mathbb{D}$, the singular part of an Aleksandrov measure μ_α measures the affinity that φ has for the boundary value $\alpha \in \mathbb{T}$. In fact, Nieminen and Saksman showed that in a certain sense the Nevanlinna counting function actually converges weak* to an Aleksandrov-Clark measure [15, Theorem 1.1]. Consequently, we find that the Aleksandrov-Clark measures provide much valuable information about the boundary behavior φ. See Cima, Matheson, and Ross [4, Chapter 9] for a more in depth study.

The definition of Aleksandrov-Clark measures is based off of Herglotz's theorem:

Theorem 2.7.1 (Herglotz)**.** *If u is a non-negative harmonic function on $\mathbb{D}$, then there is a unique positive Borel measure μ such that*

$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta),$$

the Poisson integral of μ.

Definition 2.7.2. *[4, p. 201] For an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ and a point $\alpha \in \mathbb{T}$, the function*

$$\Re\left(\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}\right) = \frac{1-|\varphi(z)|^2}{|\alpha - \varphi(z)|^2}$$

is positive and harmonic on $\mathbb{D}$. By Herglotz's theorem

$$\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2}=\int_{\mathbb{T}}\frac{1-|z|^2}{|\zeta-z|^2}d\mu_\alpha(\zeta)$$

for some unique positive Borel measure μ_α on $\mathbb{T}$. The set $\{\mu_\alpha\}_{\alpha\in\mathbb{T}}$ is the set of Aleksandrov measures, or Clark measures if the function φ is inner.

Now, we will enumerate several important quantities associated with Aleksandrov-Clark meaures: the total variation, absolutely continuous part, and carriers.

Proposition 2.7.3. *[4, p. 204] If μ_α is an Aleksandrov measure, then its total variation is given by*

$$\|\mu_\alpha\|=\frac{1-|\varphi(0)|^2}{|\alpha-\varphi(0)|^2}. \tag{2–8}$$

Definition 2.7.4. *[4, p. 205] For an Aleksandrov measure, μ_α, let*

$$d\mu_\alpha=h_\alpha dm+d\mu_\alpha^s,\quad h_\alpha\in L^1,\quad \mu_\alpha^s\perp m, \tag{2–9}$$

be the Lebesgue decomposition of μ_α with respect to m. Observe that $h_\alpha=\frac{d\mu_\alpha}{dm}$ m-a.e.

Proposition 2.7.5. *[4, p. 205] For m-a.e. $\zeta\in\mathbb{T}$,*

$$h_\alpha(\zeta)=\frac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2}. \tag{2–10}$$

It follows from the previous proposition that if φ is inner, then the resulting Clark measure is singular to Lebesgue measure.

Corollary 2.7.6. *[4, p. 205] If φ is an inner function, then $\mu_\alpha\perp m$ for every $\alpha\in\mathbb{T}$.*

A carrier of a measure is a generalization of the support of a measure. The support for a measure is always a carrier, however a carrier does not need to be closed.

Definition 2.7.7. *[4, p. 16] For a Borel measure μ, a Borel set $H\subset\mathbb{T}$ for which $\mu(H\cap A)=\mu(A)$ for all Borel subsets $A\subset\mathbb{T}$ is called a carrier of μ.*

The carrier for the absolutely continuous part of an Aleksandrov measure is apparent from (2–10), it is simply the set where $|\varphi(\zeta)|<1$. The carrier for the singular part of an Aleksandrov measure is defined below.

Proposition 2.7.8. *[4, p. 207]*

(1) Let G be the Borel set for which $\lim_{r\to 1^-}\varphi(r\zeta)$ exists and define $\varphi^*(\zeta)=\lim_{r\to 1^-}\chi_G(\zeta)\varphi(r\zeta)$. Then for an Aleksandrov measure μ_α the set $(\varphi^*)^{-1}(\{\alpha\})$ is a carrier for μ_α^s.

(2) For μ_α^s-a.e. $\zeta\in\mathbb{T}$, $\varphi(\zeta)=\alpha$.

An important theorem concerning Aleksandrov measures is Aleksandrov's disintegration theorem.

Theorem 2.7.9 (Aleksandrov's disintegration theorem)**.** *[4, p. 216] For $f\in L^1$*

$$\int_{\mathbb{T}} f(\zeta)dm(\zeta)=\int_{\mathbb{T}}\left(\int_{\mathbb{T}} f(\zeta)d\mu_\alpha\right)dm(\alpha).$$

The most important result about Aleksandrov measures that we will be using is their relationship to the norm of C_φ on the reproducing kernels of $H^2(\mathbb{D})$. In [3], Cima and Matheson found that the radial limit of $\left\|C_\varphi\tilde{k}_{r\alpha}\right\|$ is equal to the total variation of the singular part of the Aleksandrov measure μ_α.

Proposition 2.7.10. *[3] Let $\varphi:\mathbb{D}\to\mathbb{D}$ be holomorphic and $\alpha\in\mathbb{T}$, then*

$$\|\mu_\alpha^s\|=\lim_{r\to 1^-}\left\|C_\varphi\tilde{k}_{r\alpha}\right\|^2.$$

Proof. [3] By the Lebesgue decomposition of the Aleksandrov measure μ_α,

$$\|\mu_\alpha^s\|=\|\mu_\alpha\|-\int_{\mathbb{T}}h_\alpha(\zeta)dm(\zeta)=\Re\left(\frac{\alpha+\varphi(0)}{\alpha-\varphi(0)}\right)-\int_{\mathbb{T}}\frac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2}dm(\zeta).$$

On the other hand, simply computing the norm of $C_\varphi\tilde{k}_{r\alpha}$ yields

$$\begin{aligned}\left\|C_\varphi\tilde{k}_{r\alpha}\right\|^2&=\int_{\mathbb{T}}\frac{1-r^2}{|\alpha-r\varphi(\zeta)|^2}dm(\zeta)\\&=\int_{\mathbb{T}}\frac{1-r^2|\varphi(\zeta)|^2}{|\alpha-r\varphi(\zeta)|^2}dm(\zeta)-r^2\int_{\mathbb{T}}\frac{1-|\varphi(\zeta)|^2}{|\alpha-r\varphi(\zeta)|^2}dm(\zeta)\\&=\Re\left(\frac{\alpha+r\varphi(0)}{\alpha-r\varphi(0)}\right)-r^2\int_{\mathbb{T}}\frac{1-|\varphi(\zeta)|^2}{|\alpha-r\varphi(\zeta)|^2}dm(\zeta).\end{aligned}$$

Since $|\varphi(0)| < 1$, $\Re\left(\dfrac{\alpha + r\varphi(0)}{\alpha - r\varphi(0)}\right)$ converges to $\Re\left(\frac{\alpha+\varphi(0)}{\alpha-\varphi(0)}\right)$ uniformly in α as r increases to 1. Now if $0 < r < s \leq 1$ and $|w| \leq 1$ by a geometric argument,

$$\frac{r^2}{|1 - rw|^2} < \frac{s^2}{|1 - sw|^2}.$$

It follow that $r^2 \dfrac{1 - |\varphi(\zeta)|^2}{|\alpha - r\varphi(\zeta)|^2}$ increases monotonically to $\dfrac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2}$ for almost every $\zeta \in \mathbb{T}$, and so

$$\lim_{r\to 1^-} r^2 \int_{\mathbb{T}} \frac{1 - |\varphi(\zeta)|^2}{|\alpha - r\varphi(\zeta)|^2} dm(\zeta) = \int_{\mathbb{T}} \frac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2} dm(\zeta).$$

Hence, $\|\mu_\alpha^s\| = \lim\limits_{r\to 1^-} \left\| C_\varphi \tilde{k}_{r\alpha} \right\|^2$. □

2.8 Carleson Regions

Carleson regions are squares, triangles, or circles that are close to the boundary of a domain. For example, if the domain is the unit disk of the complex plane, one possible Carleson region is found by making a circle centered at a point on the boundary $\mathbb{T}$ and intersecting it with $\mathbb{D}$:

$$S(\xi, h), = \{z \in \mathbb{D} : |z - \xi| < h\} \qquad \text{for } 0 < h < 1.$$

The Carleson square or window, which is typically used in defining Carleson measures is

$$W(\xi, h) = \left\{ z \in \mathbb{D} : 1 - h < |z| < 1 \text{ and } \frac{z}{|z|} \in S(\xi, h) \right\},$$

see Figure 2-1. The importance of these regions comes in to play in inequalities relating the behavior of an $H^p(\mathbb{D})$ function in the disk with its behavior on the boundary of the disk. These inequalities are frequently called Carleson inequalities, and the underlying measures, Carleson measures.

Theorem 2.8.1 (Carleson)**.** *[7, p. 37] For a positive finite Borel measure μ on $\mathbb{D}$ and $0 < p < \infty$, the following are equivalent:*

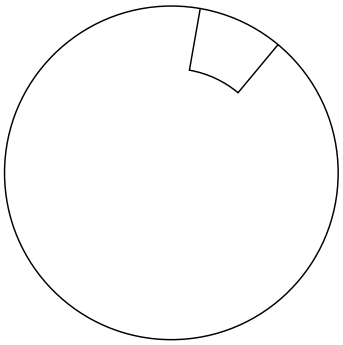

Figure 2-1. Carleson window

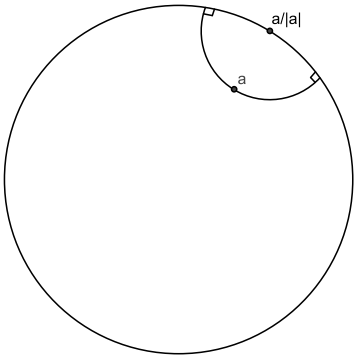

Figure 2-2. Carleson circle

(1) *there is a constant* $K < \infty$ *such that* $\mu(W(\xi, h)) < Kh$ *for* $|\xi| = 1$ *and* $0 < h < 1$*, and*

(2) *there is a constant* C *so that*

$$\int_{\mathbb{D}} |f|^p \, d\mu \leq C \, \|f\|^p$$

for all $f \in H^p(\mathbb{D})$*.*

A measure μ that satisfies Theorem 2.8.1 is called a Carleson measure.

In this theorem, a measure μ is a Carleson measure if the μ-area of the Carleson window is less than a constant times its height. The Carleson regions we will be using in this paper are the hyperbolic Carleson circles S_a. These are regions are hyperbolic circles with apex at a point $a \in \mathbb{D}$ and intersect $\mathbb{T}$ at right angles, see Figure 2-2.

More information about Carleson regions and theorems can be found in Garnett [9]; Cowen and MacCluer [7] provide a more in depth discussion of how they relate to composition operators.

2.9 Non-tangential Limits and Angular Derivatives

The ideas of radial limits and derivatives are well-known. In this section, we briefly discuss a generalization of those ideas to non-tangential limits and angular derivatives.

Definition 2.9.1. *For $\zeta \in \mathbb{T}$ and $\alpha > 1$ we define a non-tangential approach region at ζ by*

$$\Gamma(\zeta, \alpha) = \{z \in \mathbb{D} : |z - \zeta| < \alpha(1 - |z|)\}.$$

Note that $\Gamma(\zeta, \alpha)$ is a teardrop shaped region with its vertex at ζ, hence the name non-tangential: if $z \to a$ from within $\Gamma(\zeta, \alpha)$, then z cannot approach ζ along a curve that is tangent to the circle at ζ. A non-tangential limit is simply a limit where z approaches ζ from within a non-tangential approach region, or more formally:

Definition 2.9.2. *[7, p. 50] A function f is said to have a non-tangential limit A at ζ if $\lim_{z\to\zeta} f(z) = A$ in each non-tangential region $\Gamma(\zeta, \alpha)$, written*

$$\angle \lim_{z\to\zeta} f(z) = A.$$

Now we can define the angular derivative as z approaches a point $\zeta \in \mathbb{T}$. The only difference between the angular derivative and the traditional derivative is that the area of approach is now a non-tangential approach region rather than a radius.

Definition 2.9.3. *For an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ and a point $\zeta \in \mathbb{T}$, we say that φ has an angular derivative at $\zeta \in \mathbb{T}$ if for some $\eta \in \mathbb{T}$,*

$$\angle \lim_{z\to\zeta} \frac{\varphi(z) - \eta}{z - \eta}$$

exists and is finite. We denote the above limit whenever it exists by $\varphi'(\zeta)$.

One of the most important theorems concerning angular derivatives is the Julia-Carathéodory theorem, it provides conditions for their existence and uniqueness. A

proof can be found in Cowen, MacCluer [7] or Shapiro [23], and a reformulation utilizing reproducing kernels can be found in Sarason [21].

Theorem 2.9.4 (Julia-Carathéodory)**.** *[4, p. 28] For an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ and $\zeta \in \mathbb{T}$ the following statements are equivalent:*

(1) $\liminf_{z\to\zeta} \dfrac{1-|\varphi(z)|}{1-|z|} = \delta < \infty,$

(2) $\angle \lim_{z\to\zeta} \dfrac{\varphi(z)-\eta}{z-\zeta} = \varphi'(\zeta)$ *exists for some* $\eta \in \mathbb{T}$,

(3) $\angle \lim_{z\to\zeta} \varphi'(\zeta)$ *exists and* $\angle \lim_{z\to\zeta} \varphi(z) = \eta \in \mathbb{T}$.

Furthermore,

(a) $\delta > 0$ *in* (1)*;*

(b) the points η *in* (2) *and* (3) *are the same;*

(c) $\varphi'(\zeta) = \overline{\zeta}\eta\delta$ *and* $\angle \lim_{z\to\zeta} \varphi'(z) = \varphi'(\zeta)$*;*

(d) if any of the above equations hold, then $\delta = \angle \lim_{z\to\zeta} \dfrac{1-|\varphi(z)|}{1-|z|}$.

In the case of the upper half-plane, the definitions and theorems work as expected for non-tangential limits and angular derivatives on $\mathbb{R}$, however some care needs to be taken with the boundary point at ∞. The definitions and theorem below are due to Elliott and Jury in [8].

Definition 2.9.5. *[8] A sequence of points $z_n = x_n + iy_n$ in $\mathbb{H}$ approaches ∞ non-tangentially if $x_n \to \infty$ and the ratios $\dfrac{|y_n|}{x_n}$ are uniformly bounded.*

We say a map $\varphi : \mathbb{H} \to \mathbb{H}$ fixes infinity non-tangentially if $\varphi(z_n) \to \infty$ whenever $z_n \to \infty$ non-tangentially and we write $\varphi(\infty) = \infty$.

If $\varphi(\infty) = \infty$, we say that φ has finite angular derivative at ∞ if the non-tangential limit

$$\lim_{z\to\infty} \frac{z}{\varphi(z)} \quad z \in \mathbb{H} \tag{2–11}$$

exists and is finite, and we write $\varphi'(\infty)$.

In (2–11), the quotient may appear to be upside-down, usually when we take derivatives the term involving the function is in the numerator. However, in this case remember that $z \to \infty$ rather than zero, so this quotient is exactly what we want.

If $\psi : \mathbb{D} \to \mathbb{D}$ is the map that is conjugate to φ via the Cayley transform, that is $\psi = J^{-1}\varphi J$, then the existence of the limit in (2–11) is equivalent to the existence of the limit

$$\angle \lim_{z\to 1} \frac{1-\psi(z)}{1-z} \qquad z \in \mathbb{D}.$$

By the Julia-Carathéodory Theorem 2.9.4, this limit is equal to $\angle \lim_{z\to 1} \psi'(z)$. We have the following Julia-Carathéodory theorem for angular derivatives at ∞ in the half-plane.

Theorem 2.9.6 (Julia-Carathéodory in $\mathbb{H}$)**.** *[8] For an analytic function $\varphi : \mathbb{H} \to \mathbb{H}$, the following are equivalent:*

(1) $\varphi(\infty) = \infty$ *and* $\varphi'(\infty)$ *exists,*

(2) $\sup_{z\in\mathbb{H}} \dfrac{\Re z}{\Re \varphi(z)} < \infty$*,*

(3) $\limsup_{z\to\infty} \dfrac{\Re z}{\Re \varphi(z)} < \infty$*.*

Moreover, the quantities in (2) *and* (3) *are both equal to the angular derivative* $\varphi'(\infty)$*.*

CHAPTER 3
ORIGINAL PROOFS OF BOUNDED ABOVE

In this chapter we will review the original proofs that composition operators are bounded above on $H^2(\mathbb{D})$ and $H^2(\mathbb{H})$. Though Littlewood originally proved the upper bound for composition operators $H^2(\mathbb{D})$, the proof here is taken from Shapiro in [23, Section 1.3]. The essence of the proof is the same, however Shapiro uses the modern notation for objects. The proof that composition operators are bounded above on $H^2(\mathbb{H})$ is very recent, from 2010 by Elliott and Jury in [8]; the proof in this chapter is taken directly from their paper.

3.1 Bounded Above on the Disk

It has been known for a long time that composition operators are bounded above on $H^2(\mathbb{D})$. Though composition operators were not studied as such until about the 1960s, the proof of their boundedness harkens back to Littlewood's theorem from 1925 [12].

Theorem 3.1.1 (Littlewood's Theorem)**.** *[23, p. 16] Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic, then C_φ is bounded on $H^2(\mathbb{D})$ and*

$$\|C_\varphi\| \leq \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

To prove this theorem, we need to start with the simpler case where $\varphi(0) = 0$. This is an assumption we will make several times throughout this paper, and is not fatal. In most cases we can lift to the more general case where $\varphi(0) = a$ for some value $a \in \mathbb{D}$ by conjugating with the conformal automorphism of the disk α_a, as mentioned in Section 2.2.

Theorem 3.1.2 (Littlewood's Subordination Principle)**.** *[23, p. 13] Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $\varphi(0) = 0$, then for each $f \in H^2(\mathbb{D})$, $C_\varphi f \in H^2(\mathbb{D})$ and*

$$\|C_\varphi f\| \leq \|f\|.$$

This proof of Littlewood's subordination principle is taken from Shapiro [23]. In addition to the composition operator, this proof will require two additional operators on $H^2(\mathbb{D})$. First is the multiplication operator:

Definition 3.1.3. *For an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$, we define the multiplication operator M_φ on $H^2(\mathbb{D})$ by*

$$M_\varphi f = \varphi f \qquad f \in H^2(\mathbb{D}).$$

The only property we will need of multiplication operators with symbol $\varphi : \mathbb{D} \to \mathbb{D}$ is that they are contractive, in other words that

$$\|M_\varphi f\| \leq \|f\| \qquad \text{for } f \in H^2(\mathbb{D}). \tag{3–1}$$

Additionally, we will need the backward shift operator on $H^2(\mathbb{D})$. Recall from (1–1) that for $f \in H^2(\mathbb{D})$, we can write f as a power series $\sum_{n=0}^{\infty} a_n z^n$ where $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

Definition 3.1.4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function in $H^2(\mathbb{D})$. The backward shift operator B on $H^2(\mathbb{D})$ is defined by*

$$(Bf)(z) = \sum_{n=0}^{\infty} a_{n+1} z^n \qquad f \in H^2(\mathbb{D}).$$

The name comes from the fact that B shifts the power series coefficients of f to the left one unit and drops the constant term. In fact, we can rewrite the function f as

$$f(z) = f(0) + zBf(z) \qquad z \in \mathbb{D}. \tag{3–2}$$

From the definition it is obvious that B is a contraction on $H^2(\mathbb{D})$, $\|Bf\| \leq \|f\|$ for every $f \in H^2(\mathbb{D})$. We would expect this to be important in the proof, since we are proving that C_φ is a contraction when $\varphi(0) = 0$, however this is not the case.

Proof of Littlewood's Subordination Principle. [23, p. 13] The main tool in this proof is the backward shift.

We begin with the simplifying assumption that f is a polynomial. Then $f \circ \varphi$ is bounded on $\mathbb{D}$, so it is in $H^2(\mathbb{D})$; this leaves us to prove the norm estimate. To this end, substitute $\varphi(z)$ for z in (3–2) to obtain

$$f(\varphi(z)) = f(0) + \varphi(z)(Bf)(\varphi(z)) \qquad \text{for } z \in \mathbb{D}.$$

We can rewrite this equation in terms of the operators C_φ, M_φ, and B to get

$$C_\varphi f = f(0) + M_\varphi C_\varphi B f. \tag{3–3}$$

If we write φ as a power series, all the terms must have a common factor of z since $\varphi(0) = 0$, therefore all the terms in $M_\varphi C_\varphi B f$ do as well. It follows that $M_\varphi C_\varphi B f$ is orthogonal to the constant function $f(0)$ in $H^2(\mathbb{D})$. Thus, computing the norm of $C_\varphi f$ gives

$$\|C_\varphi f\|^2 = |f(0)|^2 + \|M_\varphi C_\varphi B f\|^2.$$

Since M_φ is contractive (3–1), $\|M_\varphi C_\varphi B f\| \leq \|C_\varphi B f\|$ and we have

$$\|C_\varphi f\| \leq |f(0)|^2 + \|C_\varphi B f\|. \tag{3–4}$$

Next, successively apply (3–4) to $Bf, B^2 f, B^3 f, \ldots$ to obtain the following string of inequalities

$$\begin{aligned}
\|C_\varphi B f\|^2 &\leq |Bf(0)|^2 + \left\|C_\varphi B^2 f\right\|^2 \\
\left\|C_\varphi B^2 f\right\|^2 &\leq \left|B^2 f(0)\right|^2 + \left\|C_\varphi B^3 f\right\|^2 \\
&\vdots \\
\|C_\varphi B^n f\|^2 &\leq |B^n f(0)|^2 + \left\|C_\varphi B^{n+1} f\right\|^2.
\end{aligned}$$

Putting all these together with (3–4), we get

$$\|C_\varphi f\|^2 \leq \sum_{k=0}^{n} \left|(B^k f)(0)\right|^2 + \left\|C_\varphi B^{n+1} f\right\|^2 \tag{3–5}$$

for every nonnegative integer n.

By assumption f is a polynomial, so let n denote the degree of f. If we shift f $n+1$ times we will get zero, hence $B^{n+1}f = 0$. Therefore (3–5) reduces to

$$\|C_\varphi f\|^2 \leq \sum_{k=0}^{n} \left|(B^k f)(0)\right|^2.$$

Note that $(B^k f)(0) = a_k$, so from the previous inequality we have that

$$\|C_\varphi f\|^2 \leq \sum_{k=0}^{n} |a_k|^2 = \|f\|^2.$$

This shows that C_φ is a contraction on the vector subspace of polynomials of $H^2(\mathbb{D})$. The fact that this holds for all functions $f \in H^2(\mathbb{D})$ follows from the fact that convergence in the $H^2(\mathbb{D})$ norm implies uniform convergence on compact subsets of $\mathbb{D}$, see [23, pg. 15] for details. □

The proof of Littlewood's theorem follows easily from Littlewood's subordination principle. To take care of the assumption $\varphi(0) = 0$, we will first look at the norm of a composition operator with symbol α_w, one of our automorphisms of the disk. Then the self-inverse property of α_w plus some basic operator theory results will yield the conclusion.

Proof of Littlewood's Theorem. [23, p. 16] Suppose that $\varphi(0) = w$ for some $w \in \mathbb{D}$. First we need to find the norm of C_{α_w} where $\alpha_w(z) = \dfrac{w - z}{1 - \overline{w}z}$. If $f \in H^2(\mathbb{D})$ is holomorphic in a neighborhood of the closed unit disk, say $r'\mathbb{D}$ for some $r' > 1$, then

$$\|f\|^2 = \lim_{r \to 1^-} \int_{\mathbb{T}} |f(r\zeta)|^2 \, dm(\zeta) = \int_{\mathbb{T}} |f(\zeta)|^2 \, dm(\zeta).$$

Applying the composition operator C_{α_w} to f we have

$$\|C_{\alpha_w} f\|^2 = \|f \circ \alpha_w\|^2 = \int_{\mathbb{T}} |f(\alpha_w(\zeta))|^2 \, dm(\zeta),$$

and using the change of variables $\xi = \alpha_w(\zeta)$ yields

$$\begin{aligned} \|C_{\alpha_w} f\|^2 &= \int_{\mathbb{T}} |f(\xi)|^2 \, |\alpha_w'(\xi)| \, dm(\xi) \\ &= \int_{\mathbb{T}} |f(\xi)|^2 \, \frac{1-|w|^2}{|1-\overline{w}\xi|^2} dm(\xi) \\ &\leq \frac{1-|w|^2}{(1-|w|)^2} \int_{\mathbb{T}} |f(\xi)|^2 \, dm(\xi) \\ &= \frac{1+|w|}{1-|w|} \|f\|^2 . \end{aligned} \tag{3–6}$$

Therefore, the desired result holds for all functions analytic in $r'\mathbb{D}$, and hence for polynomials. This holds for all functions $f \in H^2(\mathbb{D})$ by the same argument as in the proof of Littlewood's subordination principle.

Finally, consider the function $\psi = \alpha_w \circ \varphi$. Since α_w is a self-inverse automorphism of the disk, we also have that $\varphi = \alpha_w \circ \psi$, hence $C_\varphi = C_{\alpha_w} C_\psi$. Under this definition $\psi(0) = 0$, so by Littlewood's subordination principle 3.1.2, $\|C_\psi\| \leq 1$. Combining this with equation (3–6) yields

$$\|C_\varphi\| \leq \|C_{\alpha_w}\| \, \|C_\psi\| \leq \sqrt{\frac{1+|w|}{1-|w|}}.$$

□

3.2 Bounded Above on the Half-Plane

Results about composition operators on $H^2(\mathbb{H})$ are relatively recent, due to Elliott and Jury in 2010 [8]. In the half-plane case, boundedness of the composition operator C_φ is characterized by the angular derivative of φ at the point ∞. This discussion is from Elliott and Jury's paper [8].

Theorem 3.2.1 (Elliott, Jury)**.** *[8] Let $\varphi : \mathbb{H} \to \mathbb{H}$ be holomorphic. The composition operator C_φ is bounded on $H^2(\mathbb{H})$ if and only if φ has finite angular derivative $0 < \lambda < \infty$ at infinity, in which case $\|C_\varphi\| = \sqrt{\lambda}$*

The proof of Elliott and Jury hinges on restating the existence of the angular derivative in terms of the positivity of a kernel. Recall from the Julia-Carathéodory in

the half-plane 2.9.6 that $\varphi'(\infty)$ exists if and only if $\sup_{z\in\mathbb{H}} \frac{\Re z}{\Re\varphi(z)} < \infty$, and in that case $\varphi'(\infty) = \sup_{z\in\mathbb{H}} \frac{\Re z}{\Re\varphi(z)}$. Nevanlinna's theorem from Nevanlinna-Pick interpolation on the half-plane provides the key ingredient for framing this in terms of kernels.

Proposition 3.2.2 (Nevanlinna)**.** *[8] A holomorphic function $\varphi : \mathbb{H} \to \mathbb{H}$ has positive real part if and only if the kernel*

$$\frac{\varphi(z) + \overline{\varphi(z)}}{z + \overline{w}}$$

is positive.

If $\varphi'(\infty)$ exists and is equal to λ, then from the discussion above the function $\Re(\varphi(z) - \lambda^{-1}z)$ must be positive on $\mathbb{H}$. By Nevanlinna's theorem, the kernel

$$\frac{(\varphi(z) - \lambda^{-1}z) + \overline{(\varphi(w) - \lambda^{-1}w)}}{z + \overline{w}} \tag{3–7}$$

must also be positive.

Proof of Theorem 3.2.1. [8] First, suppose that φ fixes ∞ and $\varphi'(\infty) = \lambda$. It suffices to show that

$$\lambda \langle k_w, k_z\rangle - \left\langle C_\varphi^* k_w, C_\varphi^* k_z\right\rangle \tag{3–8}$$

is a positive kernel. If this is true then the densely defined operator $C_\varphi^* : k_w \to k_{\varphi(w)}$ is bounded by $\sqrt{\lambda}$, since the linear span of the kernel functions k_w is dense. Therefore it has a unique bounded extension to $H^2(\mathbb{H})$, and this extension must be the adjoint of the composition operator C_φ. Expanding the inner products in (3–8), we have that

$$\lambda \langle k_w, k_z\rangle - \left\langle C_\varphi^* k_w, C_\varphi^* k_z\right\rangle = \frac{\lambda}{z + \overline{w}} - \frac{1}{\varphi(z) + \overline{\varphi w}}. \tag{3–9}$$

Dividing through by λ and rearranging, the right hand side becomes

$$\frac{1}{\varphi(z) + \overline{\varphi(w)}} \left(\frac{(\varphi(z) - \lambda^{-1}z) + \overline{(\varphi(w) - \lambda^{-1}w)}}{z + \overline{w}} \right) \tag{3–10}$$

This is the product of two kernels, so by Proposition 2.4.4 we only need to show that each kernel is positive individually to show the product is positive. The kernel $(\varphi(z) +$

$\overline{\varphi(w)})^{-1}$ is positive since it factors as $\langle k_{\varphi(w)}, k_{\varphi(z)}\rangle$. The kernel

$$\frac{(\varphi(z) - \lambda^{-1}z) + \overline{(\varphi(w) - \lambda^{-1}w)}}{z + \overline{w}}$$

is positive by (3–7). Thus $\lambda \langle k_w, k_z\rangle - \left\langle C_\varphi^* k_w, C_\varphi^* k_z\right\rangle$ is a positive kernel, as desired.

For the other direction, suppose that C_φ is bounded above by M. Then for each $z \in \mathbb{H}$,

$$\|C_\varphi k_z\|^2 \leq M^2 \|k_z\|^2 \tag{3–11}$$

However, $\|k_z\|^2 = \dfrac{1}{2\Re z}$, so

$$\|C_\varphi k_z\|^2 = \|k_{\varphi(z)}\|^2 = \frac{1}{2\Re\varphi(z)}.$$

Plugging these into (3–11) we have that

$$\frac{1}{2\Re\varphi(z)} \leq M^2 \frac{1}{2\Re z}.$$

This implies that $\dfrac{\Re z}{\Re\varphi(z)}$ is bounded by M^2 on $\mathbb{H}$, therefore φ fixes ∞ and has finite angular derivative by Theorem 2.9.6.

Finally, note that the first part of the proof shows that $\|C_\varphi\| \leq \sqrt{\lambda}$, and the second part of the proof shows that $\|C_\varphi\| \geq \sqrt{\lambda}$. We conclude that $\|C_\varphi\| = \sqrt{\lambda}$. □

3.3 Remarks

Looking at the proof of Littlewood's subordination principle 3.1.2, we see that it depends heavily on the Hilbert space properties of $H^2(\mathbb{D})$. In the proof, we invoke two operators in addition to the composition operator: the multiplication operator M_φ and the backward shift operator B. The proof is very technical and gives little to no insight as to why composition operators are contractive if $\varphi(0) = 0$. Many properties of composition operators can be characterized by the geometric and function theoretic properties of the map φ, so we would expect these properties to appear with greater importance in the proof.

A similar observation can be made about the proof of boundedness from above in the upper half-plane, Theorem 3.2.1. Though, Elliott and Jury do utilize the angular derivative of φ at ∞, which is a purely function theoretic property, they go about proving that it is bounded using Hilbert space techniques. The action of the adjoint of a composition operator on kernel functions has surprising importance, and proving that the angular derivative is bounded reduces to showing a kernel is positive.

Our goal in the next chapter is to provide purely function theoretic proofs of both of these results by utilizing the Nevanlinna counting function. The key is expressing the counting function as a integral that is easy to bound. Much of the work on the disk has already been done by Joel Shapiro in [22] and [25], however the integral formula for the half-plane counting function and its results are new.

CHAPTER 4
BOUNDED ABOVE AND THE COUNTING FUNCTION

In this chapter we will develop an integral formula for the Nevanlinna counting function on the half-plane (2.5.2) following a similar method to Shapiro's from [24]. We will use this new formula to provide a purely function theoretic proof that composition operators are bounded above on $H^2(\mathbb{H})$.

4.1 Nevanlinna Counting Function on the Disk

First, we will review Shapiro's construction of an integral formula for the Nevanlinna counting function on the disk from [22, 24]. His formula proved to be incredibly useful; it enabled him to provide a function theoretic proof that composition operators are bounded above on $H^2(\mathbb{D})$ and to characterize when equality occurs in Littlewood's inequality.

Recall from Section 2.5 the Nevanlinna counting function:

Definition 4.1.1. *For a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$, define the Nevanlinna counting function of φ by*

$$N_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \log \frac{1}{|z|}$$

for all $w \in \varphi(\mathbb{D}) \setminus \{\varphi(0)\}$, where $\varphi^{-1}(w)$ denotes the set of φ-preimages of w counted according to their multiplicity, and $N_\varphi(w) = 0$ if $w \notin \varphi(\mathbb{D})$.

An important auxiliary function we will need is the Frostman transform of φ from Section 2.3, $\varphi_w = \alpha_w \circ \varphi$ for $w \in \mathbb{D}$. The nice thing about the Frostman transform is that $\varphi_w^{-1}(0) = \varphi^{-1}(w)$. As we saw in Chapter 3, sometimes it is easier to prove properties under the assumption that $\varphi(0) = 0$ first and then lift them to the more general case. The Frostman transform eases this transition since if $\varphi(0) = w$ for some $w \in \mathbb{D}$, then $\varphi_w(0) = 0$.

Definition 4.1.2. *Given a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$, for every $w \in \mathbb{D}$ define its Frostman transform by*

$$\varphi_w(z) = \frac{w - \varphi(z)}{1 - \overline{w}\varphi(z)}$$

for $z \in \mathbb{D}$.

Now we can state and prove Shapiro's integral formula; it originally appeared in [22].

Proposition 4.1.3 (Shapiro, 1987)**.** *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be a holomorphic function, then*

$$N_\varphi(w) = \lim_{r \to 1^-} \int_{\mathbb{T}} \log |\varphi_w(r\zeta)| \, dm(\zeta) - \log |\varphi_w(0)|$$

where m *denotes normalized Lebesgue on the circle* $\mathbb{T}$.

This version of Shapiro's proof is taken from his unpublished manuscript [24].

Proof. [24] First, suppose that $\varphi(0) \neq 0$. Arrange the φ-preimages of 0 in a sequence in order of increasing moduli and repeated according to their multiplicity. We denote this sequence by $\varphi^{-1}(0) = (z_1, z_2, z_3, \ldots)$. The sequence $\varphi^{-1}(0)$ may be infinite, so first we will consider preimages in a circle of radius $r < 1$. To this end, let $n(r)$ be the number of preimages with multiplicity counted that have modulus $\leq r$. Using Jensen's formula [20, p. 307], we have that

$$\sum_{j=1}^{n(r)} \log \frac{r}{|z_j|} = \int_{\mathbb{T}} \log |\varphi(r\zeta)| \, dm(\zeta) - \log |\varphi(0)| \, .$$

We take the limit as $r \nearrow 1$ of this equation. Since $\log |\varphi|$ is subharmonic on the disk, the integral on the right hand side increases as r increases to 1, but stays finite because φ is bounded on $\mathbb{D}$. The sum on the left hand side increases to $\sum_{j=1}^{\infty} \log \frac{1}{|z_j|}$ as r increases to one by monotone convergence. Therefore, we have that

$$\sum_{j=1}^{\infty} \log \frac{1}{|z_j|} = \lim_{r \to 1^-} \int_{\mathbb{T}} \log |\varphi(r\zeta)| \, dm(\zeta) - \log |\varphi(0)| \, . \tag{4–1}$$

If $0 \notin \varphi(\mathbb{D})$ we define the empty sum on the left hand side of (4–1) to be zero, as in the definition of N_φ. In this case the integral on the right hand side is harmonic, so is equal to zero by the mean value theorem. Hence, equality holds if $0 \notin \varphi(\mathbb{D})$. If $\varphi(0) = 0$, both the left and right hand sides can be interpreted as being $+\infty$; therefore equality also

holds if $\varphi(0) = 0$. This gives us that

$$N_\varphi(0) = \lim_{r\to 1^-} \int_{\mathbb{T}} \log|\varphi(r\zeta)|\, dm(\zeta) - \log|\varphi(0)|\,.$$

Finally, to get the integral formula for the preimage of any point w in the disk, simply apply the integral formula to φ_w rather than φ. Note that $\varphi_w^{-1}(0) = \varphi^{-1}(w)$, so $N_\varphi(w) = N_{\varphi_w}(0)$. Hence,

$$N_\varphi(w) = N_{\varphi_w}(0) = \lim_{r\to 1^-} \int_{\mathbb{T}} \log|\varphi_w(r\zeta)|\, dm(\zeta) - \log|\varphi_w(0)|\,.$$

□

Littlewood's inequality is an easy consequence of Proposition 4.1.3.

Theorem 4.1.4 (Littlewood's Inequality)**.** *For a holomorphic function $\varphi : \mathbb{D} \to \mathbb{D}$,*

$$N_\varphi(w) \le \log \frac{1}{|\varphi_w(0)|} = \log \left|\frac{1-\overline{w}\varphi(0)}{w-\varphi(0)}\right| \tag{4–2}$$

for all $w \in \mathbb{D}$.

Proof. In Proposition 4.1.3, note that the integrand on the right hand side is nonpositive, therefore

$$N_\varphi(w) \le \log \frac{1}{|\varphi_w(0)|} = \log \left|\frac{1-\overline{w}\varphi(0)}{w-\varphi(0)}\right|.$$

□

Shapiro's result also provides a purely function theoretic proof of Littlewood's Theorem 3.1.1. As in Chapter 3, we begin with Littlewood's subordination principle, and then use automorphisms of the disk to get the general result.

Theorem 4.1.5 (Littlewood's Subordination Principle)**.** *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function, with $\varphi(0) = 0$, then for each $f \in H^2(\mathbb{D})$, $C_\varphi f \in H^2(\mathbb{D})$ and*

$$\|C_\varphi f\| \le \|f\|\,.$$

Proof. For every $f \in H^2(\mathbb{D})$ by the change of variables formula with $\varphi(0) = 0$, we have that

$$\|C_\varphi f\|^2 = \|f \circ \varphi\|^2 = 2\int_{\mathbb{D}} |f'(z)|^2 N_\varphi(z) dA(z) + |f(0)|^2$$

where A denotes normalized Lebesgue area measure on the disk $\mathbb{D}$. Applying Littlewood's inequality (4–2) and the Littlewood-Paley identity (2–4) yields

$$\|C_\varphi f\|^2 \leq 2\int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|} dA(z) + |f(0)|^2 = \|f\|^2 .$$

□

This can easily be extended to Littlewood's theorem:

Theorem 4.1.6 (Littlewood's Theorem)**.** *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic, then C_φ is bounded on $H^2(\mathbb{D})$ and*

$$\|C_\varphi\| \leq \sqrt{\frac{1 + |\varphi(0)|}{1 - |\varphi(0)|}}.$$

Proof. This proof works identically to that in Section 3.1. □

As a consequence, in [25] Shapiro was able to provide a characterization inner functions based on properties of the Nevanlinna counting function N_φ.

Theorem 4.1.7 (Shapiro, 1999)**.** *For a holomorphic self-map φ of $\mathbb{D}$, the following are equivalent:*

(a) φ *is inner.*

(b) $N_\varphi(w) = \log \dfrac{1}{|\varphi_w(0)|}$ *for some* $w \in \mathbb{D}$*.*

(c) $N_\varphi(w) = \log \dfrac{1}{|\varphi_w(0)|}$ *for quasi-every* $w \in \mathbb{D}$*.*

Proof. [24] We begin with the integral formula from Proposition 4.1.3,

$$N_\varphi(w) = \lim_{r \to 1^-} \int_{\mathbb{T}} \log |\varphi_w(r\zeta)| \, dm(\zeta) - \log |\varphi_w(0)| .$$

The integrand on the right hand side is nonpositive, so by Fatou's lemma,

$$N_\varphi(w) \leq \int_{\mathbb{T}} \log|\varphi_w(\zeta)|\, dm(\zeta) - \log|\varphi_w(0)| \tag{4–3}$$

where $\varphi_w(\zeta)$ indicates the radial limit of φ_w at ζ.

Suppose that $N_\varphi(w) = \log\dfrac{1}{|\varphi_w(0)|}$ for some $w \in \mathbb{D}$. Then for that w,

$$\int_{\mathbb{T}} \log|\varphi_w(\zeta)|\, dm(\zeta) = 0.$$

Since the integrand is nonpositive, it must vanish almost everywhere on $\mathbb{D}$, that is $\log\dfrac{1}{|\varphi_w(\zeta)|} = 0$ for almost every $\zeta \in \mathbb{T}$. This occurs when $|\varphi_w(\zeta)| = 1$ for almost every $\zeta \in \mathbb{T}$. Therefore, φ_w must be an inner function, and we can conclude that φ is also an inner function. This proves the implication *(b)⇒(a)*.

To prove *(a)⇒(c)*, suppose that φ is inner. By Frostman's Theorem 2.3.2, φ_w is a Blaschke product for quasi-every $w \in \mathbb{D}$. Hence,

$$\lim_{r\to 1^-} \int_{\mathbb{T}} \log|\varphi_w(r\zeta)|\, dm(\zeta) = 0$$

for quasi-every $w \in \mathbb{D}$ by Theorem 2.1.6. Therefore, by Proposition 4.1.3,

$$N_\varphi(w) = \log\frac{1}{|\varphi_w(0)|}$$

for quasi-every $w \in \mathbb{D}$.

Finally, the implication *(c)⇒(b)* is obvious; if equality holds for quasi-every $w \in \mathbb{D}$, then it certainly holds for one $w \in \mathbb{D}$. □

4.2 Counting Function in the Half-Plane

We wish to employ a similar method as Shapiro's to obtain an integral formula for a counting function on the upper half-plane. We will use the counting function for the half-plane defined in Section 2.5. As with the Nevanlinna counting function for the disk, this counting function weights each φ-preimage by the product of its multiplicity and its distance to the boundary of the upper half-plane $\mathbb{R}$.

Definition 4.2.1. *For a holomorphic map $\varphi : \mathbb{H} \to \mathbb{H}$, define the counting function of φ on $\mathbb{H}$ by*

$$N_\varphi(w) = \sum_{z \in \varphi^{-1}(w)} \Im z$$

for all $w \in \varphi(\mathbb{H})$, where $\varphi^{-1}(w)$ denotes the set of φ-preimages of w counted according to their multiplicity, and $N_\varphi(w) = 0$ if $w \notin \varphi(\mathbb{H})$.

Next, we need to define a Frostman transform of the half-plane. In the disk case, the Frostman transform of a function f was simply $f_w = \alpha_w \circ f$, an automorphism of the disk composed with f. We hope to achieve something similar here. Möbius transformations that are automorphisms of the upper half-plane can only dilate a value $z \in \mathbb{H}$ by a positive real number a or shift it right or left by adding a real constant b. That is, automorphisms of the upper half-plane look like

$$az + b \qquad \text{where } a > 0 \text{ and } b \in \mathbb{R}.$$

Definition 4.2.2. *Given a holomorphic map $\varphi : \mathbb{H} \to \mathbb{H}$, for every $w \in \mathbb{H}$ define its Frostman Transform by*

$$\varphi_w(z) = \frac{1}{\Im w}\varphi(z) - \frac{\Re w}{\Im w}$$

for $z \in \mathbb{H}$.

Note that for $w \in \mathbb{H}$, $\dfrac{1}{\Im w}$ is a positive real number and $\dfrac{\Re w}{\Im w} \in \mathbb{R}$, hence the Frostman transform φ_w does in fact map $\mathbb{H}$ to itself.

Since Jensen's formula [20, p. 307] does not apply on the upper half-plane, we need a different tool to acquire the integral formula for the half-plane. The tool we will use is a theorem by Ahern and Clark from [1]. Recall from Section 2.1.2 that we can factor a bounded holomorphic function f on $\mathbb{D}$ as BSF. Define a measure on $\mathbb{T}$ by $d\mu = -\log|f|\, dm + d\sigma$, where f is taken to be the radial limit of its corresponding function in the disk and σ is the measure from the singular inner factor of f.

Theorem 4.2.3 (Ahern, Clark, 1974)**.** *[1] Let f be a holomorphic function on $\mathbb{D}$ bounded in modulus by 1, then for all $\zeta \in \mathbb{T}$,*

$$|f'(\zeta)| = \sum_n \frac{1-|a_n|^2}{|\zeta - a_n|^2} + 2\int_{\mathbb{T}} \frac{1}{|\lambda - \zeta|^2} d\mu(\lambda),$$

where $\{a_n\}$ is the Blaschke sequence for f and μ is defined as above.

When we transport Ahern and Clark's condition to the upper half-plane, something interesting happens. The angular derivative term reduces to the imaginary part of a number $w \in \mathbb{H}$, and the Blaschke product term reduces to $\sum_{z\in\varphi^{-1}(w)} \Im z$, the Nevalinna counting function for the half-plane.

Theorem 4.2.4. *Let $\varphi : \mathbb{H} \to \mathbb{H}$ be holomorphic such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) = 1$, then for q.e. $w \in \mathbb{H}$*

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}(w)} \Im z = \Im w + \frac{1}{2\pi}\int_{\mathbb{R}} \log|\varphi_w(t)|\, dt.$$

Note that we are assuming φ fixes ∞ and $\varphi'(\infty) = \infty$. Recall that these were the conditions necessary for a composition operator to be bounded above on $H^2(\mathbb{H})$ by 3.2.1. To prove this theorem, we will make extensive use of the Cayley tranform from Section 2.2:

$$J(z) = i\left(\frac{1+z}{1-z}\right),$$

which conformally maps the unit disk to the upper half plane, and its inverse

$$J^{-1}(z) = \frac{z-i}{z+i},$$

which conformally maps the upper half-plane to the unit disk. We begin with a simple computation; let $b \in \mathbb{D}$, then

$$\begin{aligned}
\Im J(b) &= \frac{1}{2i}(J(b) - \overline{J(b)}) \\
&= \frac{1}{2i}\left(i\left(\frac{1+b}{1-b}\right) - -i\left(\frac{1+\overline{b}}{1-\overline{b}}\right)\right) \\
&= \frac{1}{2}\left(\frac{1+b}{1-b} + \frac{1+\overline{b}}{1-\overline{b}}\right) \\
&= \frac{1}{2}\left(\frac{(1+b)(1-\overline{b}) + (1+\overline{b})(1-b)}{|1-b|^2}\right) \\
&= \frac{1}{2}\left(\frac{1-\overline{b}+b-|b|^2+1-b+\overline{b}-|b|^2}{|1-b|^2}\right) \\
&= \frac{1-|b|^2}{|1-b|^2}.
\end{aligned} \tag{4–4}$$

In this proof we will use a modified version of the Frostman transform from the disk. We insert an additional unimodular factor $\frac{1-\overline{b}}{b-1}$ to ensure that $\psi_b(z) = 1$ whenever $\psi(z) = 1$.

Definition 4.2.5. *Given a holomorphic map $\psi : \mathbb{D} \to \mathbb{D}$, for every $b \in \mathbb{D}$ define its modified Frostman transform by*

$$\psi_b(z) = \frac{1-\overline{b}}{b-1}\frac{b-\psi(z)}{1-\overline{b}\psi(z)}$$

for $z \in \mathbb{D}$.

Lemma 4.2.6. *Suppose that $\varphi : \mathbb{H} \to \mathbb{H}$ is holomorphic, and let ψ denote the function in the disk that is conjugate to $\mathbb{H}$ via the Cayley transform, $\varphi = J\psi J^{-1}$, then $\varphi_w = J\psi_b J^{-1}$ where $b = J^{-1}(w)$.*

Proof. Let $z \in \mathbb{H}$, then

$$
\begin{aligned}
(J\psi_b J^{-1})(z) &= J\left(\frac{1-\overline{b}}{b-1}\frac{b-\psi(J^{-1}(z))}{1-\overline{b}\psi(J^{-1}(z))}\right) \\
&= i\left(\frac{1+\frac{1-\overline{b}}{b-1}\frac{b-\psi(J^{-1}(z))}{1-\overline{b}\psi(J^{-1}(z))}}{1-\frac{1-\overline{b}}{b-1}\frac{b-\psi(J^{-1}(z))}{1-\overline{b}\psi(J^{-1}(z))}}\right)\frac{(b-1)(1-\overline{b}\psi(J^{-1}(z)))}{(b-1)(1-\overline{b}\psi(J^{-1}(z)))} \\
&= i\left(\frac{(b-1)(1-\overline{b}\psi(J^{-1}(z)))+(1-\overline{b})(b-\psi(J^{-1}(z)))}{(b-1)(1-\overline{b}\psi(J^{-1}(z)))-(1-\overline{b})(b-\psi(J^{-1}(z)))}\right) \\
&= i\left(\frac{b-|b|^2\psi(J^{-1}(z))-1+\overline{b}\psi(J^{-1}(z))+b-\psi(J^{-1}(z))-|b|^2+\overline{b}\psi(J^{-1}(z))}{b-|b|^2\psi(J^{-1}(z))-1+\overline{b}\psi(J^{-1}(z))-b+\psi J^{-1}+|b|^2-\overline{b}\psi(J^{-1}(z))}\right) \\
&= i\left(\frac{-|b|^2+b-1+\overline{b}-\overline{b}+b-|b|^2\psi(J^{-1}(z))+\overline{b}\psi(J^{-1}(z))}{-|b|^2\psi(J^{-1}(z))-1+\psi(J^{-1}(z))+|b|^2}\right) \\
&\quad + i\left(\frac{-\psi(J^{-1}(z))+b\psi(J^{-1}(z))-b\psi(J^{-1}(z))+\overline{b}\psi(J^{-1}(z))}{-|b|^2\psi(J^{-1}(z))-1+\psi(J^{-1}(z))+|b|^2}\right) \\
&= i\left(\frac{-|1-b|^2(1+\psi(J^{-1}(z)))+(b-\overline{b})(1-\psi(J^{-1}(z)))}{(1-|b|^2)(\psi(J^{-1}(z))-1)}\right) \\
&= i\left(\frac{1+\psi(J^{-1}(z))}{1-\psi(J^{-1}(z))}\right)\left(\frac{|1-b|^2}{1-|b|^2}\right) - i\frac{b-\overline{b}}{1-\overline{b}^2}\frac{1-\psi(J^{-1}(z))}{1-\psi(J^{-1}(z))} \\
&= (J\psi J^{-1})(z)\frac{1}{\Im J(b)} - i\frac{b-\overline{b}}{1-|b|^2} \\
&= \frac{1}{\Im J(b)}\varphi(z) - i\frac{b-\overline{b}}{b-|b|^2} \\
&= \frac{1}{\Im J(b)}\varphi(z) - \frac{\Re J(b)}{\Im J(b)} \\
&= \frac{1}{\Im w}\varphi(z) - \frac{\Re w}{\Im w} \\
&= \varphi_w(z).
\end{aligned}
$$

□

Finally, we provide a computation for the angular derivative of ψ_b.

Lemma 4.2.7. *Suppose that $\psi : \mathbb{D} \to \mathbb{D}$ is holomorphic with $\psi(1) = 1$ and finite angular derivative at 1. Let $b \in \mathbb{D}$ and define $w = J(b)$, then*

$$\psi_b'(1) = \frac{1-|b|^2}{|1-b|^2}\psi'(1) = \Im w\,\psi'(1)$$

Proof. Since $\psi(1) = 1$, by the definition of the modified Frostman transform, $\psi_b(1) = 1$. Calculating the angular derivative we have

$$\begin{aligned}
\psi_b'(1) &= \angle \lim_{z\to 1} \frac{\psi_b(1) - \psi_b(z)}{1-z} \\
&= \angle \lim_{z\to 1} \frac{1 - \frac{1-\overline{b}}{b-1}\frac{b-\psi(z)}{1-\overline{b}\psi(z)}}{1-z} \\
&= \angle \lim_{z\to 1} \frac{(b-1)(1-\overline{b}\psi(z)) - (1-\overline{b})(b-\psi(z))}{(b-1)(1-\overline{b}\psi(z))(1-z)} \\
&= \angle \lim_{z\to 1} \frac{b - |b|^2\psi(z) - 1 + \overline{b}\psi(z) - b + \psi(z) + |b|^2 - \overline{b}\psi(b)}{(b-1)(1-\overline{b}\psi(z))(1-z)} \\
&= \angle \lim_{z\to 1} \frac{\psi(z)(1-|b|^2) - (1-|b|^2)}{(b-1)(1-\overline{b}\psi(z))(1-z)} \\
&= \angle \lim_{z\to 1} \frac{-(1-\psi(z))(1-|b|^2)}{(b-1)(1-\overline{b}\psi(z))(1-z)} \\
&= \frac{1-|b|^2}{1-b} \angle \lim_{z\to 1} \frac{1-\psi(z)}{1-z}\frac{1}{1-\overline{b}\psi(z)} \\
&= \frac{1-|b|^2}{1-b}\psi'(1)\frac{1}{1-\overline{b}} \\
&= \frac{1-|b|^2}{|1-b|^2}\psi'(1) \\
&= \Im J(b)\psi'(1) \\
&= \Im w\, \psi'(1).
\end{aligned}$$

□

Finally we come to the proof of the integral formula for the half-plane counting function.

Proof of Theorem 4.2.4. The proof of this integral formula hinges on transporting all the requisite quantities in the half-plane to the unit disk, applying the theorem of Ahern and Clark 4.2.3, and then transporting back to the half-plane. To this end, define ψ to be the function conjugate to φ in the disk, $\psi = J^{-1}\varphi J$. Note that under the Cayley transform, $J(1) = \infty$ and $J^{-1}(\infty) = 1$, so that the hypothesis $\varphi(1) = 1$ implies $\psi(1) = 1$. Therefore, $\psi : \mathbb{D} \to \mathbb{D}$ with $\psi(1) = 1$. Two other quantities need to be defined; let $b = J^{-1}(w)$ and for

$z \in \mathbb{H}$ define $u = J^{-1}(z)$. Now, we apply Ahern-Clark to ψ_b at the value $\zeta = 1$:

$$|\psi_b'(1)| = \sum_{\psi_b(u)=0} \frac{1-|u|^2}{|1-u|^2} + 2\int_{\mathbb{T}} \frac{1}{|\lambda-1|^2} d\mu(\lambda).$$

From the simple computation above, $\dfrac{1-|u|^2}{|1-u|^2} = \Im J(u) = \Im z$, so that

$$\sum_{\psi_b(u)=0} \frac{1-|u|^2}{|1-u|^2} = \sum_{\psi(u)=b} \frac{1-|u|^2}{|1-u|^2} = \sum_{\varphi(J(u))=J(b)} \Im J(u) = \sum_{\varphi(z)=w} \Im z.$$

We rescale the function ψ so that $|\psi'(1)| = 1$, then from Lemma 4.2.7 we have that $|\psi_b'(1)| = \Im w$. The integral is the most difficult piece to consider. First for the integrand, let $t = J(\lambda) \in \mathbb{R}$ since the Cayley transform maps the boundary of $\mathbb{D}$ to the boundary of $\mathbb{H}$, then

$$\begin{aligned}
|\lambda-1|^1 &= \left|J^{-1}(t)-1\right|^2 \\
&= \left|\frac{t-i}{t+i}-1\right|^2 \\
&= \left|\frac{t-i-(t+1)}{t+1}\right|^2 \\
&= \left|\frac{-2i}{t+i}\right|^2 \\
&= \frac{4}{(t-i)(t+i)} \\
&= \frac{4}{t^2+1},
\end{aligned}$$

so $\dfrac{1}{|\lambda-1|^2} = \dfrac{1}{4}(t^2+1)$. Next, consider the measure $d\mu = -\log|\psi_b|\, dm + d\sigma$. By the generalization of Frostman's theorem due to Rudin, Theorem 2.3.3, ψ_b has no singular inner factor, so $d\mu = -\log|\psi_b|\, dm$. The equivalent measure in the half-plane is found by conjugating ψ_b to the half-plane and noting that normalized Lebesgue measure on the circle corresponds to the measure $\dfrac{1}{\pi}\dfrac{dt}{1+t^2}$ on the real line [10, Chapter 8]. Putting all of

this together yields

$$\begin{aligned}\Im w &= \sum_{\varphi(z)=w} \Im z + 2\int_{\mathbb{R}} \frac{1}{4}(t^2+1)(-1)\log|\varphi_w(t)|\,\frac{1}{\pi}\frac{dt}{1+t^2} \\ &= N_\varphi(w) - \frac{1}{2\pi}\int_{\mathbb{R}} \log|\varphi_w(t)|\,dt,\end{aligned}$$

and rearranging we have the desired result. □

Corollary 4.2.8. *Let $\varphi : \mathbb{H} \to \mathbb{H}$ be holomorphic such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) = 1$, then*

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}(w)} \Im z \leq \Im w$$

for all $w \in \mathbb{H}$.

Theorem 4.2.9. *Let $\varphi : \mathbb{H} \to \mathbb{H}$ be holomorphic such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) = 1$, then C_φ is bounded above on $H^2(\mathbb{H})$ and*

$$\|C_\varphi f\| \leq \|f\|$$

for all $f \in H^2(\mathbb{H})$.

Proof. Let $f \in H^2(\mathbb{H})$, then from the Littlewood-Paley identity for the half-plane (2–7),

$$\|C_\varphi f\|^2 = 2\int_{\mathbb{H}} |f'(z)|^2 N_\varphi(z)\,dA(z).$$

By the Littlewood-style inequality for the half-plane (4.2.8),

$$\|C_\varphi f\| \leq 2\int_{\mathbb{H}} |f'(z)|^2 \Im z\,dA(z) = \|f\|^2.$$

□

CHAPTER 5
BOUNDED BELOW

We will introduce three theorems for a composition operator on $H^2(\mathbb{D})$ to have closed range in this chapter. The original proofs of these theorems are completely disjoint; though all of these theorems provide conditions for C_φ to have closed range, it is not apparent from their proofs that they are related. We will produce a direct geometric equivalence between the criterion of Cima, Thomson, and Wogen [5] and that of Zorboska [26] using the Reproducing Kernel Thesis.

5.1 Closed Range

One of the earliest conditions for a composition operator C_φ to have closed range on $H^2(\mathbb{D})$ is from Cima, Thomson, and Wogen in 1974 [5]. They gave a condition that depends only on the behavior of the function φ on the boundary of the disk $\mathbb{T}$. To understand their condition, we must define some notation. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be holomorphic. First, we identify $\varphi(\zeta)$ where $\zeta \in \mathbb{T}$ as the boundary limit of points in the disk, that is $\varphi(\zeta) = \lim_{r\to 1^-} \varphi(r\zeta)$; this limit exists almost everywhere with respect to Lebesgue measure by Fatou's Theorem 2.1.1. Next, we define a measure on Borel sets $E \subset \mathbb{T}$ by $\nu(E) = m(\varphi^{-1}(E))$. This measure ν is absolutely continuous to Lebesgue measure on $\mathbb{T}$ and its Radon-Nikodym derivative $\frac{d\nu}{dm}$ is in $L^\infty(\mathbb{T})$.

Theorem 5.1.1 (Cima, Thomson, Wogen, 1974)**.** *[5] Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and nonconstant. Then C_φ has closed range if and only if $\frac{d\nu}{dm}$ is essentially bounded away from zero.*

At the end of their paper, Cima, Thomson, and Wogen said "[i]t would be interesting to characterize the composition operators with closed range by considering the range of the mapping on $\mathbb{D}$ rather than on $\mathbb{T}$". In 1994, Nina Zorboska did exactly that; in [26] she provided a criterion for C_φ to have closed range on $H^2(\mathbb{D})$ based upon properties of φ on Carleson regions inside the disk. Her theorem utilizes the Nevanlinna counting function for the disk in the guise of the function $\tau_\varphi(z) = \frac{N_\varphi(z)}{\log \frac{1}{|z|}}$ for $z \in \mathbb{D} \setminus \{\varphi(0)\}$.

Theorem 5.1.2 (Zorboska, 1994)**.** *[26] An operator C_φ on $H^2(\mathbb{D})$ has closed range if and only if there exists a positive constant c such that the set $G_c = \{z \in \mathbb{D} : \tau_\varphi(z) > c\}$ satisfies the following condition:*

(PB) There exists a constant $\delta > 0$ such that

$$A(G_c \cap S_a) > \delta A(S_a)$$

for all $a \in \mathbb{D}$ where A denotes normalized Lebesgue area measure on $\mathbb{D}$.

The condition (PB) is known as a "reverse Carleson inequality". We call this condition "(PB)" in reference to spreading peanut butter on a piece of toast. If the area the peanut butter is spread on is represented by G_c and the toast by S_a, then this condition tells us that the peanut butter, the preimages of τ_φ are spread thick enough over some positive fraction of the toast.

In 2010, Lefèvre, Li, Queffélec, and Rodríguez-Piazza gave another condition for C_φ to have closed range on $H^2(\mathbb{D})$ [11]. Their condition requires the average mass of the Nevanlinna Counting Function to be bounded below on Carleson regions.

Theorem 5.1.3 (Lefèvre, Li, Queffélec, Rodríguez-Piazza, 2010)**.** *[11] Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a non-constant analytic self map. Then the composition operator $C_\varphi : H^2(\mathbb{D}) \to H^2(\mathbb{D})$, has closed range if and only if there is a constant $c > 0$ such that*

$$\frac{1}{A(S_a)} \int_{S_a} N_\varphi(z) dA(z) \geq c(1 - |a|)$$

for all $a \in \mathbb{D}$.

Though Zorboska and Lefèvre, Li, Queffélec, and Rodríguez-Piazza reference previous contributions to the study of closed range properties of composition operators in their papers, the conditions remain unrelated. The proof of each theorem about closed range is completely independent of the proofs of each of the other theorems, and it is unclear how these conditions all give rise to the same property.

5.2 The Reproducing Kernel Thesis

We have developed a new tool for proving closed range properties of composition operators on $H^2(\mathbb{D})$, the Reproducing Kernel Thesis; it has allowed us to provide a direct geometric equivalence between the criterion of Cima, Thomson, and Wogen and Zorboska's criterion. The advantage of the Reproducing Kernel Thesis is that rather than having to verify a condition for C_φ to have closed range on all function in $H^2(\mathbb{D})$, it is only necessary verify the condition on a exceptionally well-behaved subest of $H^2(\mathbb{D})$, the set of Szegő kernels.

Theorem 5.2.1 (Reproducing Kernel Thesis)**.** *Given an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, C_φ is bounded below on $H^2(\mathbb{D})$ by $c > 0$ if and only if C_φ is bounded below on the set of normalized Szegő kernel functions $\{\tilde{k}_a(z)\}_{a\in\mathbb{D}}$ by c, that is*

$$\left\| C_\varphi \tilde{k}_a \right\|^2 \geq c \left\| \tilde{k}_a \right\|^2 = c.$$

In proving the Reproducing Kernel Thesis, we start with the Cima, Thomson, and Wogen criterion 5.1.1, then pass through several other equivalent conditions for C_φ to be bounded below on $H^2(\mathbb{D})$ before arriving at the result. The first thing we need to do is restate Cima, Thomson, and Wogen's criterion using tools that will help us move from their property on the boundary of the disk to a property on the reproducing kernels. The tools we need are the Aleksandrov-Clark measures. Recall from Section 2.7:

Definition 5.2.2. *For an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ and a point $\alpha \in \mathbb{T}$, the function*

$$\Re\left(\frac{\alpha+\varphi(z)}{\alpha-\varphi(z)}\right) = \frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2}$$

is positive and harmonic on $\mathbb{D}$. By Herglotz's theorem

$$\frac{1-|\varphi(z)|^2}{|\alpha-\varphi(z)|^2} = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta-z|^2} d\mu_\alpha(\zeta)$$

for some unique positive Borel measure μ_α on $\mathbb{T}$. The set $\{\mu_\alpha\}_{\alpha\in\mathbb{T}}$ is the set of Aleksandrov measures, or Clark measures if the function φ is inner.

The Lebesgue decomposition of μ_α is

$$d\mu_\alpha = h_\alpha dm + d\mu_\alpha^s$$

where $h_\alpha(\zeta) = \dfrac{1-|\varphi(\zeta)|^2}{|\alpha-\varphi(\zeta)|^2}$ for m-a.e. $\zeta \in \mathbb{T}$. Recall from Theorem 2.7.8 that μ_α^s is carried by the set $\{\zeta \in \mathbb{T} : \varphi(\zeta) = \alpha\}$.

Lemma 5.2.3. *Let μ_α^s denote the singular part of the Aleksandrov-Clark measure μ_α on $\mathbb{T}$, then*

$$\frac{d\nu}{dm} = \|\mu_\alpha^s\| \quad m-a.e.\ \alpha \in \mathbb{T}.$$

Proof. Fix a Borel set $E \subset \mathbb{T}$. By the definition of ν,

$$\nu(E) = \int_E d\nu(\zeta) = \int_{\mathbb{T}} \chi_E d(m\varphi^{-1})(\zeta) = \int_{\mathbb{T}} \chi_{\varphi^{-1}(E)} dm(\zeta). \tag{5–1}$$

Applying the Aleksandrov Disintegration Theorem yields

$$\nu(E) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \chi_{\varphi^{-1}(E)} d\mu_\alpha(\zeta) \right) dm(\alpha).$$

For $\alpha \in \mathbb{T}$, consider the absolutely continuous part of the measure μ_α, $h_\alpha dm = \dfrac{1-|\varphi|^2}{|\alpha-\varphi|^2} dm$. This measure is carried by the set $\{\zeta \in \mathbb{T} : |\varphi(\zeta)| < 1\}$, so that $\chi_{\varphi^{-1}(E)}(\zeta) = 0$ for $h_\alpha dm$-almost everywhere. Hence,

$$\int_{\mathbb{T}} \chi_{\varphi^{-1}(E)}(\zeta) h_\alpha(\zeta) dm(\zeta) = 0.$$

Now consider the singular part of the measure μ_α, μ_α^s. Note that from Proposition 2.7.8 for μ_α^s-a.e. $\zeta \in \mathbb{T}$, $\varphi(\zeta) = \alpha$. Hence, if $\alpha \in E$, then μ_α^s is carried by the set $\varphi^{-1}(\{\alpha\})$, which is contained in $\varphi^{-1}(E)$. Therefore, $\chi_{\varphi^{-1}(E)}(\zeta) = 1$ μ_α^s-almost everywhere and

$$\int_{\mathbb{T}} \chi_{\varphi^{-1}(E)}(\zeta) d\mu_\alpha^s(\zeta) = \|\mu_\alpha^s\|.$$

On the other hand, if $\alpha \notin E$, then $\varphi^{-1}(\{\alpha\}) \cap \varphi^{-1}(E) = \emptyset$ so that $\chi_{\varphi^{-1}(E)}(\zeta) = 0$ μ_α^s-almost everywhere and

$$\int_{\mathbb{T}} \chi_{\varphi^{-1}(E)}(\zeta) d\mu_\alpha^s(\zeta) = 0.$$

Putting this together, we have that

$$\int_{\mathbb{T}} \chi_{\varphi^{-1}(E)}(\zeta) d\mu_\alpha(\zeta) = \int_{\mathbb{T}} \chi_{\varphi^{-1}(E)}(\zeta) d\mu_\alpha^s(\zeta) = \|\mu_\alpha^s\| \chi_E(\alpha).$$

Plugging this result into equation (5–1) yields

$$\nu(E) = \int_{\mathbb{T}} \left(\int_{\mathbb{T}} \chi_{\varphi^{-1}(E)} d\mu_\alpha(\zeta) \right) dm(\alpha) = \int_{\mathbb{T}} \|\mu_\alpha^s\| \chi_E(\alpha) dm(\alpha) = \int_E \|\mu_\alpha^s\| dm(\alpha).$$

We conclude that $\frac{d\nu}{dm}(\alpha) = \|\mu_\alpha^s\|$ for Lebesgue almost every $\alpha \in \mathbb{T}$. □

This lemma leads to our first condition that is equivalent to the Cima, Thomson, Wogen condition for C_φ to be bounded below.

Proposition 5.2.4. *C_φ is bounded below by $c > 0$ on $H^2(\mathbb{D})$ if and only if* $\operatorname{ess\,inf}_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| \geq c > 0$.

This result can be further refined since in fact $\operatorname{ess\,inf}_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| = \inf_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\|$.

Lemma 5.2.5. $\operatorname{ess\,inf}_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| = \inf_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\|$

Proof. Suppose that $\operatorname{ess\,inf}_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| = c$ where c is some constant greater than zero. Then from Lemma 5.2.3, $\frac{d\nu}{dm}(\alpha) \geq c$ for m-a.e. $\alpha \in \mathbb{T}$. Let $f \in H^2(\mathbb{D})$ and consider

$$\int_{\mathbb{T}} |f(\varphi(\zeta))|^2 dm(\zeta) \geq \int_{\mathbb{T}} |f(\zeta)|^2 \frac{d\nu}{dm} dm(\zeta) \geq c \int_{\mathbb{T}} |f(\zeta)|^2 dm(\zeta),$$

which is true if and only if $\|C_\varphi f\|^2 \geq c \|f\|^2$, in other words if C_φ is bounded below.

Let $\tilde{k}_{r\alpha}(z)$ denote the normalized Szegő kernel function $\frac{\sqrt{1-|r\alpha|^2}}{1-\overline{r\alpha}z}$ for $\alpha \in \mathbb{T}$ and $0 < r < 1$. Given that $\|C_\varphi f\|^2 \geq c \|f\|^2$ for any $f \in H^2(\mathbb{D})$, then certainly $\left\| C_\varphi \tilde{k}_{r\alpha}(z) \right\|^2 \geq c \left\| \tilde{k}_{r\alpha}(z) \right\|^2 = c$. Recall from Proposition 2.7.10 that for $\alpha \in \mathbb{T}$, $\|\mu_\alpha^s\| = \lim_{r\to 1^{-1}} \left\| C_\varphi \tilde{k}_{r\alpha} \right\|^2$. Then we have that

$$\|\mu_\alpha^s\| = \lim_{r\to 1^-} \left\| C_\varphi \tilde{k}_{r\alpha} \right\|^2 \geq \lim_{r\to 1^-} c \left\| \tilde{k}_{r\alpha} \right\|^2 = c.$$

Since this inequality holds for every $\alpha \in \mathbb{T}$, $\inf_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| \geq c$. Therefore

$$c \leq \inf_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| \leq \operatorname{ess\,inf}_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| = c,$$

so $\inf_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| = \operatorname{ess\,inf}_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\|$. □

Proposition 5.2.6. *C_φ is bounded below by $c > 0$ on $H^2(\mathbb{D})$ if and only if $\inf_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| \geq c > 0$.*

Proof. This follows directly from Proposition 5.2.4 and Lemma 5.2.5. □

This brings us to the culmination of this section, the proof of the Reproducing Kernel Thesis.

Proof of Reproducing Kernel Thesis 5.2.1. The forward direction is trivial: if C_φ is bounded below on $H^2(\mathbb{D})$, then C_φ is certainly bounded below on $\{\tilde{k}_a(z)\}_{a\in\mathbb{D}}$, which is a subset of $H^2(\mathbb{D})$.

For the reverse direction, suppose that C_φ is bounded below on the set of kernel functions, $\{\tilde{k}_a(z)\}_{a\in\mathbb{D}}$, then there exists a constant $c > 0$ such that $\left\| C_\varphi \tilde{k}_{r\alpha} \right\|^2 \geq c$ for every $r > 0$ and every $\alpha \in \mathbb{T}$. Applying Proposition 2.7.10, $\|\mu_\alpha^s\| = \lim_{r\to 1^-} \left\| C_\varphi \tilde{k}_{r\alpha} \right\|^2 \geq c$. Since this holds for every $\alpha \in \mathbb{T}$, $\inf_{\alpha\in\mathbb{T}} \|\mu_\alpha^s\| \geq c > 0$. By Proposition 5.2.6, C_φ is bounded below. □

5.3 A New Proof of Zorboska's Condition

Zorboska's original proof hinged on Luecking's Theorem from 1981 [13]; she used it to show the equivalence of the geometric condition (PB) and boundedness from below.

Theorem 5.3.1 (Luecking, 1981)**.** *Let G be a measurable subset of $\mathbb{D}$ and $p > 0$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbb{D}} |f|^p \, dA \leq C \int_G |f|^p \, dA$$

for $f \in A^p$ if and only if there is a constant $\delta > 0$ such that $A(G \cap S_a) > \delta A(S_a)$ for all $a \in \mathbb{D}$.

Here A^p refers to the unweighted Bergman space; see Luecking [13] or Cowen, MacCluer [7] for more about Bergman spaces. Theorem 5.3.1 is easily restated in our setting of the Hardy space H^2 as can be seen in Zorboska's paper [26]. The forward

direction of Luecking's theorem is relatively easy to prove and very geometric in nature. We will employ many of the same sorts of techniques he used in this implication to prove the next proposition. However, the reverse implication is difficult to prove, Luecking even stated so himself in [13]. He employs three lemmas, the main purpose of which are to gain control over the size of f on the Carleson regions S_a. By utilizing reproducing kernels, we bypass this issue entirely. The behavior of the kernel function $\tilde{k}_a$ is well-known on the region S_a, and in fact they work together very nicely as will we see in Lemma 5.3.4.

Theorem 5.3.2. *C_φ is bounded below on the normalized Szegő kernel functions, $\{\tilde{k}_a(z)\}_{a\in\mathbb{D}}$, if and only if there exists a positive constant c such that the set $G_c = \{z \in \mathbb{D} : \tau_\varphi(z) > c\}$ satisfies the following condition:*

(PB) There exists a constant $\delta > 0$ such that

$$A(G_c \cap S_a) > \delta A(S_a)$$

for all $a \in \mathbb{D}$.

In order prove this theorem, a few lemmas are required. For a function $f \in H^2(\mathbb{D})$, recall from Section 2.6 the Littlewood-Paley identity for the norm

$$\|f\|^2 = 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z) + |f(0)|^2$$

and the change of variables formula, which we rewrite in terms of τ_φ:

$$\begin{aligned}\|C_\varphi f\|^2 = \|f\circ\varphi\|^2 &= 2\int_{\mathbb{D}} |f'(z)|^2 N_\varphi(z) dA(z) + |f(\varphi(0))|^2 \\ &= 2\int_{\mathbb{D}} |f'(z)|^2 \frac{N_\varphi(z)}{\log\frac{1}{|z|}} \log\frac{1}{|z|} dA(z) + |f(\varphi(0))|^2 \\ &= 2\int_{\mathbb{D}} |f'(z)|^2 \tau_\varphi(z) \log\frac{1}{|z|} dA(z) + |f(\varphi(0))|^2 .\end{aligned}$$

The first lemma relates the condition $\|C_\varphi f\| \geq c\|f\|$ on the norms of $C_\varphi f$ and f to a condition on the integrals in the change of variables and Littlewood-Paley identities.

Lemma 5.3.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic with $\varphi(0) = 0$. The operator C_φ is bounded below on the set kernel functions $\{\tilde{k}_a(z)\}_{a\in\mathbb{D}}$ if and only if there exists $0 < r < 1$ such that*

$$\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log\frac{1}{|z|} dA(z) \geq \lambda \int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \log\frac{1}{|z|} dA(z)$$

for some $\lambda > 0$ and all $|a| > r$.

Proof. Let $\varphi(0) = 0$. Consider the Littlewood-Paley identity and the change of variables formula on an arbitrary kernel function $\tilde{k}_a(z)$, where $a \in \mathbb{D}$.

$$\left\|\tilde{k}_a\right\|^2 = 2\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \log\frac{1}{|z|} dA(z) + \left|\tilde{k}_a(0)\right|^2$$
$$\left\|C_\varphi \tilde{k}_a\right\| = 2\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log\frac{1}{|z|} dA(z) + \left|\tilde{k}_a(0)\right|^2.$$

For ease of notation, let

$$I_C = \int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log\frac{1}{|z|} dA(z)$$
$$I_k = \int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \log\frac{1}{|z|} dA(z)$$

For the forward direction, suppose that C_φ is bounded below on the set of kernel functions $\{\tilde{k}_a(z)\}$, that is there exists $c > 0$ such that $\left\|C_\varphi \tilde{k}_a\right\|^2 \geq c\left\|\tilde{k}_a\right\|^2$. In the new notation, using the Littlewood-Paley identity and the change of variables formula, the bounded below condition becomes

$$2I_C + \left|\tilde{k}_a(0)\right|^2 \geq c\left(2I_k + \left|\tilde{k}_a(0)\right|^2\right),$$

and rearranging yields

$$I_C \geq cI_k - \frac{1}{2}(1-c)\left|\tilde{k}_a(0)\right|^2.$$

Since $\dfrac{|a|^2}{1-|a|^2}$ is an increasing function of $|a|$, we can choose $r_1 > 0$ so that for $|a| > r_1$,

$$\frac{|a|^2}{1-|a|^2} \geq \frac{2(1-c)}{c}.$$

Rearranging we have that

$$\begin{aligned}
\frac{|a|^2}{1-|a|^2} &\geq \frac{2(1-c)}{c} \\
\frac{c}{2}|a|^2 &\geq (1-c)(1-|a|^2) \\
c|a|^2 - \frac{c}{2}|a|^2 &\geq (1-c)(1-|a|^2) \qquad (5\text{–}2) \\
c|a|^2 - (1-c)(1-|a|^2) &\geq \frac{c}{2}|a|^2 \\
c\frac{1}{2}|a|^2 - \frac{1}{2}(1-c)(1-|a|^2) &\geq \frac{c}{2}\left(\frac{1}{2}|a|^2\right).
\end{aligned}$$

Note that

$$\left|\tilde{k}_a(0)\right|^2 = \left|\frac{\sqrt{1-|a|^2}}{1-\overline{a}\cdot 0}\right|^2 = 1-|a|^2,$$

and as $\left\|\tilde{k}_a\right\| = 1$,

$$I_k = \frac{1}{2}\left(\left\|\tilde{k}_a\right\|^2 - \left|\tilde{k}_a(0)\right|\right) = \frac{1}{2}|a|^2. \qquad (5\text{–}3)$$

Substituting I_k in for $\frac{1}{2}|a|^2$ in the previous inequality and applying (5–2) gives

$$I_C \geq cI_k - \frac{1}{2}(1-c)\left|\tilde{k}_a(0)\right|^2 \geq \frac{c}{2}I_k$$

Letting $\lambda = \frac{c}{2}$ yields the desired integral condition.

For the reverse direction, suppose that

$$I_C \geq \lambda I_k. \qquad (5\text{–}4)$$

The idea of the proof is similar to the above. Choose $r_2 > 0$ so that for $|a| > r_2$,

$$\frac{|a|^2}{1-|a|^2} \geq \frac{\lambda-2}{\lambda}.$$

Again, rearranging, we have that

$$\frac{|a|^2}{1-|a|^2} \geq \frac{\lambda-2}{\lambda}$$

$$\lambda |a|^2 \geq (\lambda-2)(1-|a|^2)$$

$$2\lambda |a|^2 + 2(1-|a|^2) \geq \lambda |a|^2 + \lambda(1-|a|^2)$$

$$\lambda\frac{1}{2}|a|^2 + \frac{1}{2}(1-|a|^2) \geq \frac{\lambda}{2}\left(\frac{1}{2}|a|^2 + \frac{1}{2}(1-|a|^2\right).$$

Using the substitutions $1-|a|^2 = \left|\tilde{k}_a(0)\right|^2$ and $\frac{1}{2}|a|^2 = I_k$,

$$\lambda I_k + \frac{1}{2}\left|\tilde{k}_a(0)\right|^2 \geq \frac{\lambda}{2}\left(I_k + \frac{1}{2}\left|\tilde{k}_a(0)\right|^2\right).$$

Applying the integral condition (5–4),

$$I_C + \frac{1}{2}\left|\tilde{k}_a(0)\right|^2 \geq \lambda I_k + \frac{1}{2}\left|\tilde{k}_a(0)\right|^2 \geq \frac{\lambda}{2}\left(I_k + \frac{1}{2}\left|\tilde{k}_a(0)\right|^2\right),$$

hence $\left\|C_\varphi \tilde{k}_a\right\|^2 \geq \frac{\lambda}{2}\left\|\tilde{k}_a\right\|^2$.

Finally, by taking $r = \max\{r_1, r_2\}$ we ensure that for $|a| > r$, C_φ is bounded below on the set kernel functions, $\{\tilde{k}_a(z)\}_{a\in\mathbb{D}}$ if and only if

$$\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log\frac{1}{|z|} dA(z) \geq \lambda \int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \log\frac{1}{|z|} dA(z)$$

for some $\lambda > 0$. □

By composing with an automorphism of the disk, Lemma 5.3.3 can be easily generalized to the case where φ is any analytic map of $\mathbb{D}$ to itself.

The second lemma provides a lower bound for the mass of a kernel function over its matching Carleson region. The kernel function $\tilde{k}_a$ and the Carleson region S_a play exceedingly well together since the maximum value of $\tilde{k}_a$ is found at $\frac{a}{|a|}$ on the boundary, $\mathbb{T}$, see Figure 2-2.

Lemma 5.3.4. *There exist constants $r > 0$ and $C > 0$ such that for all $r \leq |a| < 1$,*

$$\int_{S_a} \left|\tilde{k}_a'\right|^2 \log \frac{1}{|z|} dA(z) \geq C.$$

Proof. Note that for z close enough to the boundary of $\mathbb{D}$, we can make the estimate

$$\int_{S_a} \left|\tilde{k}_a{}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \approx \int_{S_a} \frac{|a|^2 (1-|a|^2)}{|1-\bar{a}z|^4} (1-|z|^2) dA(z)$$

Let $\alpha_a(z) = \dfrac{|a|}{a} \dfrac{a-z}{1-\bar{a}z}$ denote the automorphism of the disk that maps S_a conformally to the left half of the unit disk and maps a to 0, that is $\alpha_a(S_a) = \{z \in \mathbb{D} : \Re z \leq 0\}$. Note that $\alpha_a^{-1}(w) = \dfrac{a}{|a|} \dfrac{|a|-w}{1-|a|\,w}$. Apply the change of variables $w = \alpha_a(z)$. This transformation has Jacobian $\left|(\alpha_a^{-1})'(w)\right|^2 = \dfrac{(1-|a|^2)^2}{|1-|a|\,w|^4}$.

$$\begin{aligned}
&\int_{S_a} \frac{|a|^2 (1-|a|^2)}{|1-\bar{a}z|^4} (1-|z|^2) dA(z) \\
&\qquad = \int_{\alpha_a(S_a)} \frac{|a|^2 (1-|a|^2)}{\left|1-\bar{a}\alpha_a^{-1}(w)\right|^4} (1-\left|\alpha_a^{-1}(w)\right|^2) \left|(\alpha_a^{-1})'(w)\right|^2 dA(w) \\
&\qquad = \int_{\alpha_a(S_a)} \frac{|a|^2 (1-|a|^2)}{\left|1-\bar{a}\left(\frac{a}{|a|}\frac{|a|-w}{1-|a|w}\right)\right|^4} (1-\left|\left(\frac{a}{|a|}\frac{|a|-w}{1-|a|\,w}\right)\right|^2) \left|(\alpha_a^{-1})'(w)\right|^2 dA(w) \\
&\qquad = \int_{\alpha_a(S_a)} \frac{|a|^2 (1-|a|^2)}{\left|1-\bar{a}\left(\frac{a}{|a|}\frac{|a|-w}{1-|a|w}\right)\right|^4} (1-\left|\left(\frac{a}{|a|}\frac{|a|-w}{1-|a|\,w}\right)\right|^2) \frac{(1-|a|^2)^2}{|1-|a|\,w|^4} dA(w)
\end{aligned}$$

Now apply the identity (2–3), $1 - \left|\dfrac{|a|-w}{1-|a|\,w}\right|^2 = \dfrac{(1-|a|^2)(1-|w|^2)}{|1-|a|\,w|^2}$ so that

$$\begin{aligned}
&\int_{S_a} \frac{|a|^2 (1-|a|^2)}{|1-\bar{a}z|^4} (1-|z|^2) dA(z) \\
&\qquad = \int_{\alpha_a(S_a)} \frac{|a|^2 (1-|a|^2) |1-|a|\,w|^4}{\left|1-|a|\,w-|a|^2+|a|\,w\right|^4} \frac{(1-|a|^2)(1-|w|^2)}{|1-|a|\,w|^2} \frac{(1-|a|^2)^2}{|1-|a|\,w|^4} dA(w) \\
&\qquad = \int_{\alpha_a(S_a)} \frac{|a|^2 (1-|w|^2)}{|1-|a|\,w|^2} dA(w)
\end{aligned}$$

By hypothesis, $|a| \geq r$, and as $\Im(|a|) = 0$ and $\Re w \leq 0$ on the left half of the unit disk, we have that $2 > |1-|a|\,w| \geq \Re(1-|a|\,w) > 1$. Finally, choose $|w| \leq \eta < 1$, then the

estimate becomes

$$\int_{\alpha_a(S_a)} \frac{|a|^2(1-|w|^2)}{|1-|a|\,w|^2} dA(w) \geq \frac{1}{2} r^2(1-\eta^2).$$

Letting $C = \frac{1}{2} r^2(1-\eta^2)$ yields the desired result. □

One last lemma is necessary before the proof of Theorem 5.3.2. In general, the function $\tau_\varphi(z) = \frac{N_\varphi(z)}{\log \frac{1}{|z|}}$ is not bounded on the unit disk. However, if we restrict τ_φ to an annulus that does not contain $\varphi(0)$, then the resulting function is bounded on $\mathbb{D}$. With this in mind, choose $R > 0$ so that $\varphi(0) \notin \{z \in \mathbb{D} : R < |z| < 1\}$ and define

$$\tau_\varphi^R(z) = \begin{cases} \tau_\varphi(z) & |z| > R \\ 0 & |z| \leq R \end{cases}, \tag{5–5}$$

then τ_φ^R is bounded on $\mathbb{D}$.

Lemma 5.3.5. *There exist constants $c > 0$ and $k > 0$ such that*

$$\int_{G_c} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi^R(z) \log \frac{1}{|z|} dA(z) \geq \frac{1}{k} \int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z),$$

where $G_c = \{z \in \mathbb{D} : \tau_\varphi > c\}$.

The negation of this lemma is: for every constant $c > 0$ or for every constant $k > 0$,

$$\int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) > k \int_{G_c} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi^R(z) \log \frac{1}{|z|} dA(z).$$

Proof. By way of contradiction, let $\{c_n\}$ be a sequence converging to 0 such that

$$\int_{G_{c_n}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi^R(z) \log \frac{1}{|z|} dA(z) \longrightarrow 0.$$

Note that by (5–5),

$$\int_{G_{c_n}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi^R(z) \log \frac{1}{|z|} dA(z) = \int_{G_{c_n} \cap \{R<|z|<1\}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z).$$

For ease of notation, let $A_n = G_{c_n} \cap \{R < |z| < 1\}$, then rewriting our hypothesis we have that

$$\int_{A_n} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \longrightarrow 0.$$

Consider the integral $\int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z)$, since the integrand is positive, we can expand it as follows

$$\begin{aligned}
&\int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \\
&\qquad = \int_{A_n} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) + \int_{\mathbb{D}\setminus A_n} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \\
&\qquad \leq \int_{A_n} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) + \int_{\{z\in\mathbb{D}:\tau_\varphi(z)\leq c_n\}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \\
&\qquad\qquad + \int_{\{z\in\mathbb{D}:|z|\leq R\}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z).
\end{aligned} \tag{5–6}$$

We need to find an upper bound for the last integral above,

$$\int_{\{z\in\mathbb{D}:|z|\leq R\}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z).$$

To this end, consider

$$\begin{aligned}
\int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) = &\int_{\{z\in\mathbb{D}:R<|z|<1\}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \\
&+ \int_{\{z\in\mathbb{D}:|z|\leq R\}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z).
\end{aligned}$$

Since $\int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z)$ is finite and nonnegative, we can write

$$\int_{\{z\in\mathbb{D}:|z|\leq R\}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) = C \int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z)$$

for some constant $0 < C < 1$. Applying these estimates to (5–6), we have that

$$\begin{aligned}
&\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \\
&\qquad \leq \int_{A_n} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) + c_n \int_{\{z\in\mathbb{D}:\tau_\varphi(z)\leq c_n\}} \left|\tilde{k}_a'(z)\right|^2 \log \frac{1}{|z|} dA(z) \\
&\qquad\qquad + C \int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z),
\end{aligned}$$

and rearranging yields

$$\begin{aligned}
&\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \\
&\qquad \leq \frac{1}{1-C} \left(\int_{A_n} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) + c_n \int_{\{z\in\mathbb{D}:\tau_\varphi(z)\leq c_n\}} \left|\tilde{k}_a'(z)\right|^2 \log \frac{1}{|z|} dA(z) \right).
\end{aligned}$$

Taking $c_n \to 0$, we see that

$$\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \longrightarrow 0.$$

However, all of the factors in the integrand above are strictly positive, therefore

$$\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) > 0,$$

leading to a contradiction. □

Finally we come to the proof of the main proposition of this section.

Proof of Theorem 5.3.2. First, suppose that C_φ is bounded below on the set of kernel function $\{\tilde{k}_a(z)\}_{a\in\mathbb{D}}$, then by Lemma 5.3.3 there exists $0 < r_1 < 1$ such that

$$\int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \geq \lambda \int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \log \frac{1}{|z|} dA(z)$$

for some $\lambda > 0$ and for all $|a| \geq r_1$. Applying Lemma 5.3.5 to this result, there exist constants $k > 0$ and $c > 0$ such that

$$\int_{G_c} \left|\tilde{k}_a'(z)\right|^2 \tau_\varphi^R(z) \log \frac{1}{|z|} dA(z) \geq \frac{\lambda}{k} \int_{\mathbb{D}} \left|\tilde{k}_a'(z)\right|^2 \log \frac{1}{|z|} dA(z).$$

From Littlewood's inequality 4.1.4, under the assumption that $\varphi(0) = 0$,

$$\tau_\varphi^R(z) = \frac{N_\varphi(z)}{\log \frac{1}{|z|}} \leq \frac{\log \frac{1}{|z|}}{\log \frac{1}{|z|}} = 1,$$

so that

$$\int_{G_c} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \geq \frac{\lambda}{k} \int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z).$$

Note that, in general, $\int_{G_c} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \leq \int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z)$ since $G_c \subset \mathbb{D}$, hence the constant $\frac{\lambda}{k}$ must be strictly less than 1. Recall from equation (5–3)

$$\int_{\mathbb{D}} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) = \frac{1}{2} |a|^2,$$

and substituting this into the last inequality yields

$$\int_{G_c} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \geq \frac{|a|^2 \lambda}{2k}.$$

Consider

$$\begin{aligned}\int_{G_c \cap S_a} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) &= \int_{G_c} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) - \int_{G_c \setminus S_a} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \\ &\geq \int_{G_c} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) - \int_{\mathbb{D} \setminus S_a} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z).\end{aligned}$$

By Lemma 5.3.4, there exists $0 < r_2 < 1$ so that for $|a| > r_r$, $\int_{S_a} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \geq \frac{1}{2} r_2^2 (1 - \eta^2)$. From the proof of Lemma 5.3.4, there is some freedom in the choice of η, so choose $\eta > \sqrt{1 - \frac{\lambda}{k}}$, which is less than 1 since $0 < \frac{\lambda}{k} < 1$. Then we have

$$\int_{G_c \cap S_a} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \geq \frac{r_1^2 \lambda}{2k} - \frac{1}{2} r_2^2 (1 - \eta^2).$$

Setting $r = \max\{r_1, r_2\}$, for all $r < |a| < 1$ this simplifies to

$$\int_{G_c \cap S_a} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \geq \frac{r^2}{2} \left(\frac{\lambda}{k} - 1 + \eta^2\right), \tag{5–7}$$

where the choice of η above ensures that the constant on the right hand side is strictly positive. Now that we have the integral $\int_{G_c \cap S_a} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z)$ bounded below by an absolute constant, we need to find an upper bound for its integrand. Recall from the proof of Lemma 5.3.4 that

$$\left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \approx \frac{|a|^2 (1-|a|^2)}{|1-\bar{a}z|^4}(1-|z|^2)$$

for $|z|$ close enough to 1. On the Carleson region S_a, $|a| \leq |z| < 1$, so using the reverse triangle inequality on the denominator above gives

$$\begin{aligned} |1-\bar{a}z|^4 &\geq |1-|\bar{a}z||^4 \\ &= (1-|a|\,|z|)^4 \\ &\geq (1-|a|)^4. \end{aligned}$$

Therefore

$$\begin{aligned} \left|\tilde{k_a}'(z)\right|^2 \log \frac{1}{|z|} &\approx \frac{|a|^2 (1-|a|^2)}{|1-\bar{a}z|^4}(1-|z|^2) \\ &\leq \frac{|a|^2 (1-|a|^2)(1-|a|^2)}{(1-|a|)^4} \\ &\leq \frac{|a|^2}{(1-|a|)^4}(1-|a|^2)^2 \\ &= \frac{|a|^2 (1-|a|)^2(1+|a|)^2}{(1-|a|)^4} \\ &= \frac{|a|^2 (1+|a|)^2}{(1-|a|)^2} \\ &= \frac{|a|^2 (1+|a|)^2}{(1-|a|)^2}\frac{(1+|a|)^2}{(1+|a|)^2} \\ &= \frac{|a|^2 (1+|a|)^4}{(1-|a|^2)^2} \\ &\leq \frac{16}{(1-|a|^2)^2} \end{aligned} \tag{5–8}$$

where the last estimate uses the fact that $|a| < 1$. Putting together this upper bound on the integrand (5–8) with the lower bound on the integral (5–7) yields

$$\frac{16}{(1-|a|^2)^2} A(G_c \cap S_a) \geq \int_{G_c \cap S_a} \left|\tilde{k}_a{}'(z)\right|^2 \log \frac{1}{|z|} dA(z) \geq \frac{r^2}{2}\left(\frac{\lambda}{k} - 1 + \eta^2\right).$$

We also need an estimate on $A(S_a) = \int_{S_a} dA(z)$, the normalized Lebesgue area measure of S_a, in order to complete the proof. To this end, apply the change of variables $w = \alpha_a(z)$ to the integral as in Lemma 5.3.4,

$$\begin{aligned} A(S_a) &= \int_{\alpha_a(S_a)} \frac{(1-|a|^2)^2}{|1-|a|\, w|^4} dA(w) \\ &\leq (1-|a|^2)^2 \int_{\alpha_a(S_a))} dA(z) \\ &= (1-|a|^2)^2 \frac{1}{2} A(\mathbb{D}) \\ &= \frac{1}{2}(1-|a|^2)^2. \end{aligned}$$

So

$$\begin{aligned} A(G_c \cap S_a) &\geq \frac{r^2}{2}\left(\frac{\lambda}{k} - 1 + \eta^2\right) \frac{(1-|a|^2)^2}{16} \\ &\geq \frac{r^2}{2}\left(\frac{\lambda}{k} - 1 + \eta^2\right) \frac{1}{8} A(S_a) \end{aligned}$$

where $\frac{r^2}{2}\left(\frac{\lambda}{k} - 1 + \eta^2\right)\frac{1}{8}$ is some constant strictly greater than zero that does not depend on a, thus proving the forward direction for $|a| > r$.

For the other direction, given a parameter $0 < t < 1$, define the set

$$E_t(a) = \left\{ z \in S_a : \left|\tilde{k}_a'(z)\right|^2 (1-|z|^2) \geq t \left|\tilde{k}_a'(a)\right|^2 (1-|a|^2) \right\}.$$

Note that as $t \to 0$, $E_t(a) \nearrow S_a$. By hypothesis there exist constants $c > 0$ and $\delta > 0$ such that for all $a \in \mathbb{D}$, $A(G_c \cap S_a) > \delta A(S_a)$. Let $G_{c,a} = G_c \cap S_a$ and choose t so that

$A(E_t(a)) > (1 - \frac{\delta}{2})A(S_a)$. Consider

$$\begin{aligned} A(E_t(a) \cap G_{c,a}) &= \frac{1}{2}[A(E_t(a)) + A(G_{c,a}) - A(E_t(a) \triangle G_{c,a})] \\ &\geq \frac{1}{2}\left[\left(1 - \frac{\delta}{2}\right) A(S_a) + \delta A(S_a) - A(S_a)\right] \\ &= \frac{\delta}{4} A(S_a). \end{aligned}$$

Now, using the lemmas we can make several reductions to the conclusion that C_φ is bounded below on the Szegő kernels. From Lemma 5.3.3 it suffices to show that there exists $\lambda > 0$ such that

$$\int_{\mathbb{D}} \left|\tilde{k}'_a(z)\right|^2 \tau_\varphi(z) \log \frac{1}{|z|} dA(z) \geq \lambda \int_{\mathbb{D}} \left|\tilde{k}'_a(z)\right|^2 \log \frac{1}{|z|} dA(z)$$

for all $|a|$ larger than some value $r_1 > 0$. Since

$$\int_{\mathbb{D}} \left|\tilde{k}'_a(z)\right|^2 \log \frac{1}{|z|} dA(z) = \frac{|a|^2}{2} \geq \frac{r_1^2}{2}$$

and $G_c \cap S_a \subset \mathbb{D}$, it then suffices to show there exists $\delta' > 0$ such that

$$\int_{G_c \cap S_a} \left|\tilde{k}'_a(z)\right|^2 (1 - |z|^2) dA(z) \geq \delta'.$$

To this end, consider

$$\begin{aligned} \int_{G_c \cap S_a} \left|\tilde{k}'_a(z)\right|^2 (1 - |z|^2) dA(z) &\geq \int_{G_c \cap E_t(a)} \left|\tilde{k}'_a(z)\right|^2 (1 - |z|^2) dA(z) \\ &\geq \int_{G_c \cap E_t(a)} t \left|\tilde{k}'_a(a)\right|^2 (1 - |a|^2) dA(z) \\ &= t \left|\tilde{k}'_a(a)\right|^2 (1 - |a|^2) A(G_c \cap E_t(a)) \\ &\geq t \left|\tilde{k}'_a(a)\right|^2 (1 - |a|^2) \frac{\delta}{4} A(S_a) \\ &\geq t \frac{|a|^2}{(1 - |a|^2)^2} \frac{\delta}{4} (1 - |a|^2)^2 \\ &\geq t r^2 \frac{\delta}{4}. \end{aligned}$$

The constant, $tr^2\frac{\delta}{4}$ is strictly positive and independent of a, so applying our reductions above, we conclude C_φ is bounded below on kernel functions.

To generalize to all values of a in $\mathbb{D}$, simply note that all the regions S_a and the kernel functions $\tilde{k}_a$ are conformally equivalent for every $a \in \mathbb{D}$. The assumption that $\varphi(0) = 0$ is just as easily taken care of by conjugating with the disk automorphism α_a. □

5.4 Future Work

In Sections 5.2 and 5.3 we provided a direct geometric equivalence between Cima, Thomson, and Wogen's criterion 5.1.1 and Zorboska's criterion 5.1.2 for a composition operator to have closed range on $H^2(\mathbb{D})$. We hope to provide a proof of the criterion by Lefèvre, Li, Queffélec, and Rodríguez-Piazza 5.1.3 using the Reproducing Kernel Thesis as well.

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BIOGRAPHICAL SKETCH

Kristin Luery was born in Falls Church, VA in 1981, part of the Washington, D.C. suburbs. She comes from a background where math was always part of the equation; her father is a statistician, her mother was a computer programmer, her grandfather was an engineering, and her sister is now an accountant. So it is no surprise that Kristin went on to obtain her doctorate in mathematics. She attended high school at a science and technology magnet school, Thomas Jefferson High School for Science and Technology, focusing her studies on chemistry, physics, and mathematics. Deciding she wanted to major in physics, Kristin entered college in 1999 at Virginia Tech, then transferred after her first year to the University of Virginia. In 2004, she received a B.A. in physics and mathematics from the University of Virginia. Kristin then took several years off from school, trying to decide whether to pursue further study in physics or mathematics. She eventually decided on mathematics and entered graduate school in August of 2007 at the University of Florida. She received her M.S. in mathematics in 2009, then continued on to complete her Ph.D. in 2013.