GEOMETRIC MODULAR ACTION
IN 5-DIMENSIONAL MINKOWSKI SPACE

By

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I dedicate this work to all who supported me throughout the process of its preparation.

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The Condition of Geometric Modular Action is applied to a state on a net of local C^*-algebras of observables associated with wedge-like regions in 4-dimensional de Sitter space and in 5-dimensional Minkowski space, inducing, along with other suitable conditions, transformations of the set of the wedge-like regions that are necessarily induced by point transformations in the symmetry group of the underlying spacetime. Transitive action of the group formed by these point transformations on the set of the wedge-like regions is studied and alternative proofs are given to some of the theorems found in the earlier works on the subject. A unitary representation of the symmetry group is constructed for each of the two spacetimes, the construction being based on reflection maps. The unitary representation acts covariantly on the net of algebras of observables and its action leaves the state invariant.

CHAPTER 1
INTRODUCTION

The fundamental observation of Bisognano and Wichmann [3, 4] that modular objects associated with von Neumann algebras of local observables in wedge-like regions of Minkowski space and the vacuum state in finite-component quantum field theories satisfying Wightman axioms have geometrical interpretation has motivated studies on the subject by a number of authors (see [5, 7] for an extensive reference list). Buchholz and Summers [10], see also [7], proposed the following condition of geometric modular action (CGMA) as a means of algebraic characterization of the most symmetric physical states in a quantum theory with a net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ of local C^*-subalgebras of a C^*-algebra $\mathcal{A}$ of observables, where $\mathcal{W}$ is a collection of suitable open regions in the underlying spacetime manifold $\mathcal{M}$ and a state ω on $\mathcal{A}$. The condition is formulated in terms of GNS representation $(\mathcal{H},\pi,\Omega)$ of $\mathcal{A}$ where the vector Ω is assumed to be cyclic and separating for each von Neumann algebra $\mathcal{R}(W) := \pi(\mathcal{A}(W))''$, $W \in \mathcal{W}$ and J_W, $\{\Delta_W^{it}\}_{t\in\mathbb{R}}$, respectively, denote the modular involution and the modular unitary group assigned to $\mathcal{R}(W)$ by Tomita-Takesaki theory ([6]):

Condition of Geometric Modular Action. The pair $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}},\Omega)$ satisfies CGMA if the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is stable under the adjoint action of the modular involution J_W associated with the pair $(\mathcal{R},\Omega)$, for all $W\in\mathcal{W}$, i.e. for every pair $W_1, W_2\in\mathcal{W}$ there is some region $W_1\circ W_2\in\mathcal{W}$ such that

$$J_{W_1}\mathcal{R}(W_2)J_{W_2} = \mathcal{R}(W_1\circ W_2). \tag{1–1}$$

In the works [7] and [13] the authors investigated CGMA and its consequences for quantum field theories on a spacetime manifold $(\mathcal{M},g)$ and discussed in detail concrete examples of 4-dimensional Minkowski space $\mathcal{M}_4$ and 3-dimensional de Sitter space dS^3. They gave some natural requirements of the regions $W\in\mathcal{W}$ calling $\mathcal{W}$ admissible if

(a) for each $W\in\mathcal{W}$ the causal complement W' of W is also contained in $\mathcal{W}$,
(b) the set $\mathcal{W}$ is stable under the action of the group of isometries of $(\mathcal{M},g)$,

(c) all regions $W \in \mathcal{W}$ are contractible,

and proposed the following formulation of CGMA ([7], Chapter 3).

CGMA on $(\mathcal{M}, g)$**.** Let $\mathcal{W}$ be an admissible family of regions in the spacetime $(\mathcal{M}, g)$, let $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ be a net of C^*-algebras indexed by $\mathcal{W}$, let ω be a state on a $C^* - algebra\ \mathcal{A} \supset \mathcal{A}(W)$, $W \in \mathcal{W}$ and let $(\mathcal{H}, \pi, \Omega)$ be the GNS-triple associated with $(\mathcal{A}, \omega)$. The CGMA is fulfilled if the corresponding net $\{\mathcal{R}(W) = \pi(\mathcal{A}_W)''\}_{W\in\mathcal{W}}$ satisfies:

(i) $W \mapsto \mathcal{R}(W)$ is an order-preserving bijection,

(ii) for $W_1, W_2 \in \mathcal{W}$, if $W_1 \cap W_2 \neq \emptyset$, then Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$,

(iii) for $W_1, W_2 \in \mathcal{W}$, if Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$, then $\overline{W_1} \cap \overline{W_2} \neq \emptyset$,

(iv) for each $W \in \mathcal{W}$, the adjoint action of modular involutions J_W associated with $(\mathcal{R}(W), \Omega)$ leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant.

These conditions imply the existence of a family $\{\tau_W\}_{W\in\mathcal{W}}$ of order-preserving bijections of $\mathcal{W}$ (partially ordered by inclusion) such that for $W_1, W_2 \in \mathcal{W}$ one has

$$J_{W_1}\mathcal{R}(W_2)J_{W_2} = \mathcal{R}(\tau_{W_1}(W_2)).$$

1.0.1 Lemma (See Lemma 2.1 in [7])**.** *The group $\mathcal{T}$ generated by $\{\tau_W\}_{W\in\mathcal{W}}$ has the following properties:*

(1) For each $W \in \mathcal{W}$, $\tau_W^2 = \iota$, where ι is the identity map on $\mathcal{W}$.

(2) For every $\tau \in \mathcal{T}$ one has $\tau\tau_W\tau^{-1} = \tau_{\tau(W)}$.

(3) If $\tau \in \mathcal{T}$ and $\tau(W) = W$ for some $W \in \mathcal{W}$, then $\tau\tau_W = \tau_W\tau$.

(4) One has $\tau_W(W) = W$, for some $W \in \mathcal{W}$, if and only if the algebra $\mathcal{R}(W)$ is maximally abelian. If $\mathcal{T}$ acts transitively on $\mathcal{W}$, then $\tau_W(W) = W$, for some $W \in \mathcal{W}$, if and only if $\tau_W(W) = W$, for all $W \in \mathcal{W}$. Moreover, if $\tau_W(W) = W$, for some $W \in \mathcal{W}$, then W is an atom of $\mathcal{W}$, i.e. if $W_0 \in \mathcal{W}$ and $W_0 \subseteq W$, then $W_0 = W$.

(5) If $W_1 \subseteq W_2 \subseteq W_3 \subseteq W_4$, then $\tau_{W_1}(W_2) \supseteq \tau_{W_4}(W_3)$.

In the concrete examples of 4-dimensional Minkowski space $\mathcal{M}_4$ and 3-dimensional de Sitter space dS^3 with the $\mathcal{W}$ consisting of wedgelike regions it was shown in [7] that

(1) each τ_W, $W \in \mathcal{W}$ is induced by a point transformation g_W in the isometry group of the underlying spacetime such that $g_W W_0 = \tau_W(W_0)$ (note that CGMA above contains no a priori assumption regarding the isometry group of $\mathcal{M}$ opening a possibility to use it as a selection criterion for physically interesting states in theories on manifolds with small or trivial isometry group),

(2) the subgroup $\mathcal{G}$ of the isometry group generated by the collection $\{g_W\}_{W\in\mathcal{W}}$ is sufficiently large, in fact, with an additional assumption that the action of the modular involutions on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is transitive the group $\mathcal{G}$ contains the identity component $\mathcal{G}_i$ of the isometry group (what other subgroups can occur in the absence of transitivity has been investigated in [13]),

(3) under the assumption of transitivity and on an additional "net continuity condition" (see [7], Section 4.3) there exists a strongly continuous unitary representation $V : \mathcal{G}_i \to \mathcal{J}$, where $\mathcal{J}$ is the subgroup of the unitary group of the Hilbert space $\mathcal{H}$ generated by the modular involutions $\{J_W\}_{W\in\mathcal{W}}$, such that $V(\mathcal{G}_i)$ acts covariantly on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and leaves the state vector Ω invariant.

We will refer to (1)-(3) as the "CGMA program". The authors of [7] also investigated in the context of the proposed CGMA geometric action of modular groups $\{\Delta_W^{it}\}_{t\in\mathbb{R}}$, $W \in \mathcal{W}$, the condition of modular covariance, and the relativistic spectrum condition proposing the "modular stability condition" (see [7], Section 5, and also Section 2.3 and Section 3.3 below) as a physical stability condition which is formulated entirely in the algebraic terms of the net and the state.

In the works [8] and [11] the technical assumptions used in [7] were dropped and simpler arguments relying on the geometry of wedges in Minkowski space were used to construct strongly continuous unitary representation of the translation group ([8]) and that of the Poincaré group in $\mathcal{M}_4$ ([11]).

In the present work we intend to revisit CGMA on 4-dimensional de Sitter space dS^4, first investigated by Florig in [13], in light of the more recent, simpler and geometrically more lucid work [11]. In Section 2.1 we list Florig's results establishing the point (1) of the CGMA program, while in Section 2.2 we attempt to prove an analogue valid in dS^4 (and in $\mathcal{M}_5$, as well) of the following Proposition 4.2 of [7].

1.0.2 Proposition (Proposition 4.2 in [7])**.** *Any subgroup $\mathcal{G}$ of the Lorentz group $\mathcal{L}$ which is generated by a collection of involutions, intersects at most two connected components* *of $\mathcal{L}$ and acts transitively on the set $\mathcal{W}_0$ of wedges in $\mathcal{M}_4$ whose edges contain the origin, must contain $\mathcal{L}_+^\uparrow$-the identity component of $\mathcal{L}$.*

This is a result of a group-action theoretic nature not relying explicitly on CGMA. Its analogue in higher dimensional Minkowski space would hopefully be a step toward a more general statement on subgroups of isometry groups of Lorentzian manifolds generated by involutions and their homogeneous spaces. In Section 2.2 we prove such an analogue with an additional restrictive condition on the set of the generating involutions. Noting that the condition is implied by CGMA we reproduce Proposition 4.2.4 of [13] which establishes part (2) of the CGMA program as well as the concrete form of the involutions $g_W \in \mathcal{G} \leq \mathcal{L}$.

In Section 2.3 we construct a strongly continuous unitary representation of the de Sitter group (which is isomorphic to the proper Lorentz group $SO(1,4)$ of the ambient space $\mathcal{M}_5$) with the properties described in part (3) of the CGMA program using the properties of reflection maps introduced in [11]. Compared to the construction in the case of $\mathcal{M}_4$ and dS^3 the present construction requires some extra care because of the presence of the double rotations in $SO(1,4)$.

We also examine CGMA in 5-dimensional flat Minkowski space $\mathcal{M}_5$ (Chapter 3), establishing parts (1)-(3) of the CGMA program by modifying the results of Section 4.1 of [7], and using the results obtained in Chapter 2 as well as the construction of a strongly continuous unitary representation of the translation group given in [8].

CHAPTER 2
GEOMETRIC MODULAR ACTION IN 4-DIMENSIONAL DE SITTER SPACE

We will represent the 4-dimensional de Sitter space as the subset

$$dS^4 = \{(x_0, x_1, x_2, x_3, x_4) \in \mathbb{R}^5 | x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -1\},$$

of the 5-dimensional Minkowski space $\mathcal{M}_5$ with the metric and the causal structure induced by the Minkowski metric of the ambient space. The isometry group of dS^4, the de Sitter group, is isomorphic to the proper Lorentz group $\mathcal{L}_+ = SO(1,4)$ of $\mathcal{M}_5$.

Following the discussion of CGMA and its implications in [7] for the theories in 4-dimensional Minkowski space and in 3-dimensional de Sitter space, CGMA was formulated and its consequences were studied in [13] also for the case of 4-dimensional de Sitter space. The (directed) set $\mathcal{W}$ of admissible (see the Introduction) regions has been chosen to be formed by the intersections of dS^4 with the wedges in $\mathcal{M}_5$ whose edges contain the origin. The usual geometric action of the Lorentz group on $\mathcal{M}_5$ is transitive on $\mathcal{W}$ (since it is transitive on the set $\mathcal{W}_0$ of all wedges in $\mathcal{M}_5$ whose edges contain the origin.) In what follows we will use the notation such that for any wedge

$$\begin{aligned}\tilde{W}(\ell, \ell') =& \{\alpha\ell - \beta\ell' + \gamma\ell_\perp, \quad \alpha > 0, \beta > 0, \quad \gamma \in \mathbb{R}, \\ & \ell \cdot \ell_\perp = 0, \quad \ell' \cdot \ell_\perp = 0\} \in \mathcal{W}_0,\end{aligned}$$

where $\ell, \ell' \in \mathcal{M}_5$ is a pair of non-parallel, future-oriented lightlike vectors, we put $W(\ell, \ell') := \tilde{W}(\ell, \ell') \cap dS^4$.

2.1 Wedge Transformations are Induced by Linear Isometries

In this section we will give the CGMA as stated in [7] and highlight the results of [7] and [13] leading to the completion of the first stage of the CGMA program in dS^4, namely that the bijections of $\mathcal{W}$ implied by CGMA are induced by point transformations belonging to the Lorentz group $\mathcal{L}$ of $\mathcal{M}_5$ ([13], Section 4.1).

Strong Condition of Geometric Modular Action. A theory complies with the strong form of the CGMA if the pair $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$ satisfies

(i) $W \mapsto \mathcal{R}(W)$ is an order preserving bijection,

(ii) Ω is cyclic and separating for $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$ if and only if $W_1 \cap W_2 \neq \emptyset$, for $W_1, W_2 \in \mathcal{W}$,

(iii) for any $W_0, W_1, W_2 \in \mathcal{W}$ with $W_1 \cap W_2 \neq \emptyset$ there holds

$$\mathcal{R}(W_1) \cap \mathcal{R}(W_2) \subset \mathcal{R}(W_0) \text{ if and only if } W_1 \cap W_2 \subset W_0$$

(iv) for each $W \in \mathcal{W}$, the adjoint action of J_W leaves the set $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ invariant.

We remark that (i) above implies that for $W_1, W_2 \in \mathcal{W}$, $W_1 \subseteq W_2$ implies $W_1 = W_2$, so all the net algebras are atoms. An examination of Lemma 6.8 in [7] shows that it is applicable for the present case as well, yielding the following important property of any wedge transformation τ provided that the above strong CGMA holds:

2.1.1 Lemma (see Lemma 6.8 in [7]). *If the strong CGMA for the case of dS^4 holds, then for each $W \in \mathcal{W}$ the associated involution $\tau_W : \mathcal{W} \to \mathcal{W}$ satisfies*

$$W_1 \cap W_2 = W_3 \cap W_4 \Leftrightarrow \tau_W(W_1) \cap \tau_W(W_2) = \tau_W(W_3) \cap \tau_W(W_4), \tag{2–1}$$

for arbitrary pairs W_1, W_2 and W_3, W_4 in $\mathcal{W}$.

Consequently the above relation is satisfied by any bijection τ in the group $\mathcal{T}$ generated by the involutions τ_W, $W \in \mathcal{W}$. The bijections satisfying this relation have been shown in [7] to be induced by Lorentz transformations of $\mathcal{M}_4$. Combining of the results of [7] with those of [13] yields an analogous conclusion for the case of dS^4. We list the crucial steps of that process, noting that the proofs given in the cited references are valid in the present case as well:

2.1.2 Lemma (see Lemma 6.2 in [7]). *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection satisfying (2–1) and let ℓ_0 be a fixed future-directed lightlike vector in $\mathcal{M}_5$. The bijection τ maps* *the collections of wedges*

$$\{W(\ell_0, \ell)|\ell \text{ lightlike, } \ell \cdot \ell_0 > 0\},$$
$$\{W(\ell, \ell_0)|\ell \text{ lightlike, } \ell \cdot \ell_0 > 0\}$$

onto sets of the same form. Furthermore,

$$W_1 \cap W_2 = \emptyset \Leftrightarrow \tau(W_1) \cap \tau(W_2) = \emptyset, \text{ for any } W_1, W_2 \in \mathcal{W}, \tag{2–2}$$
$$\tau(W') = \tau(W)' \text{ for any } W \in \mathcal{W}. \tag{2–3}$$

In what follows the notation $H_0(\ell) := \{x \in \mathcal{M}_5 | x \cdot \ell = 0\}$ is used for the characteristic hyperplane determined by a lightlike vector ℓ.

2.1.3 Corollary (see Corollary 6.1 in [7])**.** *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection satisfying* (2–1)*. The bijection τ induces a bijection of characteristic hyperplanes such that for a pair of non-parallel future-directed lightlike vectors ℓ_1, ℓ_2 the images $\tau(H_0(\ell_1))$ and $\tau(H_0(\ell_2))$ are characteristic hyperplanes determined by $\tau(W(\ell_1, \ell_2))$.*

This new map, denoted again by τ, is defined by $\tau(H_0(\ell_0)) := H_0(\ell'_0)$, where $H_0(\ell'_0)$ is uniquely determined by one of the relations

$$\{\tau(W(\ell_0, \ell))|\ell \cdot \ell_0 > 0\} = \{W(\ell'_0, \ell)|\ell \cdot \ell'_0 > 0\},$$
$$\{\tau(W(\ell_0, \ell))|\ell \cdot \ell_0 > 0\} = \{W(\ell, \ell'_0)|\ell \cdot \ell'_0 > 0\}$$

which follows by Lemma 2.1.2. Lemma 4.1.11 and its Corollaries 4.1.12 and 4.1.13 in [13] are now applicable to define the map (also denoted by τ) on one-dimensional spacelike subspaces of $\mathcal{M}_5$, as follows:

2.1.4 Lemma (Lemma 4.1.11 in [13])**.** *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection satisfying* (2–1)*. If $\ell_1, \ell_2, \ell_3, \ell_4$ are linearly dependent future-directed lightlike vectors such that any two of them are linearly independent, then*

$$\bigcap_{i=1}^{4} \tau(H_0(\ell_i)) = \bigcap_{i \neq k} \tau(H_0(\ell_i)) \text{ for } k = 1, 2, 3, 4.$$

2.1.5 Corollary (Corollary 4.1.12 in [13]). *Let* $\tau : \mathcal{W} \to \mathcal{W}$ *be a bijection satisfying* (2–1). *If* $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ *are linearly dependent future-directed lightlike vectors such that any two of them are linearly independent, and such that*

$$\begin{aligned} span\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\} &= span\{\ell_1, \ell_2, \ell_3, \ell_4\}, \\ \textit{then} \quad \bigcap_{i=1}^{5} \tau(H_0(\ell_i)) &= \bigcap_{i=1}^{4} \tau(H_0(\ell_i)). \end{aligned}$$

2.1.6 Corollary (Corollary 4.1.13 in [13]). *Let* $\tau : \mathcal{W} \to \mathcal{W}$ *be a bijection satisfying* (2–1) *and let* $x \in \mathcal{M}_5$ *be spacelike. The intersection*

$$\bigcap_{\{\ell | x \in H_0(\ell)\}} \tau(H_0(\ell))$$

is one-dimensional and spacelike. Hence τ *induces a bijection*

$$\mathbb{R}x \mapsto \bigcap_{\{\ell | x \in H_0(\ell)\}} \tau(H_0(\ell))$$

on the set of one-dimensional spacelike subspaces of $\mathcal{M}_5$*.*

This new map will be denoted by τ, as well. Finally, Lemma 4.1.15 in [13] implies that τ induces a bijective point transformation of dS^4, as follows:

2.1.7 Lemma (Lemma 4.1.15 in [13]). *Let* $\tau : \mathcal{W} \to \mathcal{W}$ *be a bijection satisfying* (2–1), *and let* $x \in \mathcal{M}_5$ *be spacelike. If* $x \in W_1 \cap W_2$ *for* $W_1, W_2 \in \mathcal{W}$*, then*

$$\begin{aligned} span\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\} &= span\{\ell_1, \ell_2, \ell_3, \ell_4\}, \\ \emptyset \neq \tau(\mathbb{R}x) \cap \tau(W_1) &= \tau(\mathbb{R}x) \cap \tau(W_2). \end{aligned}$$

This implies that

$$\tau(\mathbb{R}x) \cap \left(\bigcap_{\substack{W \in \mathcal{W} \\ x \in W}} \tau(W) \right)$$

is nonempty, with the spacelike line $\mathbb{R}x$ intersecting dS^4 in a unique point $\delta(x) \in dS^4$ which leads to the following result.

2.1.8 Proposition (see Proposition 6.1 in [7]). *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection satisfying* (2–1)*. There exists a bijection $\delta : dS^4 \to dS^4$ such that*

$$\tau(W) = \{\delta(x) | x \in W\}, \text{ for all } W \in \mathcal{W}.$$

Suppose now that $x_1, x_2 \in dS^4$ are such that $\ell := x_2 - x_1$ is lightlike and future directed. In this case $x_1 \in H_0(\ell)$ and $x_2 \in H_0(\ell)$ so that $\delta(x_2) \in \tau(H_0(\ell)) := H_0(\ell')$ and $x_2 \in H_0(\ell')$. If $\delta(x_2) - \delta(x_1)$ is spacelike, then so is $\tau^{-1}(\mathbb{R}(\delta(x_2) - \delta(x_1)))$. Since the latter one-dimensional space contains x_1 and x_2, it follows that $\delta(x_2) - \delta(x_1)$ is lightlike (and parallel to ℓ'). Therefore the bijection $\delta : dS^4 \to dS^4$ maps lightlike separated points (in the metric of $\mathcal{M}_5$) to lightlike separated points (also in the metric of $\mathcal{M}_5$). The following theorem of Lester [15] shows that such a bijection must be induced by a Lorentz transformation.

2.1.9 Lemma (see the Theorem in [15]). *If $\phi : dS^4 \to dS^4$ is a bijection such that lightlike separated points (in the metric of $\mathcal{M}_5$) are mapped to lightlike separated points (also in the metric of $\mathcal{M}_5$), then there exists a Lorentz transformation Λ of the ambient Minkowski space $\mathcal{M}_5$ such that $\phi(x) = \Lambda x$, for all $x \in dS^4$.*

The above results are summarized in the following dS^4-analogue of Theorem 6.1 in [7].

2.1.10 Theorem. *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection satisfying* (2–1)*, and let $\delta : dS^4 \to dS^4$ be the associated bijection. Then there exists a Lorentz transformation of the ambient Minkowski space $\mathcal{M}_5$ such that $\delta(x) = \Lambda x$, for all $x \in dS^4$, and $\tau(W) = \Lambda W$, for all $W \in \mathcal{W}$.*

Note that since dS^4 with its points viewed as vectors contains a basis of $\mathcal{M}_5$ the action of the linear transformation Λ is completely determined by the action of the bijection δ.

2.2 Wedge Transformations Generate de Sitter Group

We have seen that a theory complying with CGMA gives rise to a set $\{\tau_W\}_{W\in\mathcal{W}}$ of order preserving involutive bijections of the set $\mathcal{W}$. If $\mathcal{T}$ is the group generated by $\{\tau_W\}_{W\in\mathcal{W}}$, then properties of any $\tau \in \mathcal{T}$ are listed in Lemma 1.0.1. When the strong CGMA in dS^4 holds Theorem 2.1.10 shows that each transformation τ_W, $W \in \mathcal{W}$, induces a unique Lorentz transformation g_W. If $\mathcal{G} \leq \mathcal{L}$ is the group generated by $\{g_W\}_{W\in\mathcal{W}}$, then the properties listed in Lemma 1.0.1 also apply to any $g \in \mathcal{G}$.

In this section we explore the action of the group $\mathcal{G}$ on $\mathcal{W}_0$ of all the wedges in $\mathcal{M}_5$ whose edges contain the origin (recall that the set $\mathcal{W}$ of admissible regions for dS^4 is formed by intersections of this space with wedges in $\mathcal{W}_0$) with the Strong CGMA containing the additional condition that

(v) the adjoint action of the group generated by modular conjugations $\{J_W\}_{W\in\mathcal{W}}$ on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is transitive,

implying that so is the action of $\mathcal{G}$ on $\mathcal{W}_0$.

Motivated by the approach of [7], we attempt to prove the following analogue of Proposition 4.2 of that work:

2.2.1 Proposition. *Any subgroup $\mathcal{G}$ of the Lorentz group $\mathcal{L}$ which is generated by a collection of involutions, intersects at most two connected components of $\mathcal{L}$ and acts transitively on the set $\mathcal{W}_0$ of wedges in $\mathcal{M}_5$ whose edges contain the origin, must contain $\mathcal{L}_+^\uparrow$-the identity component of $\mathcal{L}$.*

This would be a result of group action-theoretic nature, whose hypothesis does not rely directly on CGMA. In this attempt we give alternative proofs (based on similar ideas, but using transitivity of the action of $\mathcal{G}$ on $\mathcal{W}_0$ as the more essential ingredient) of Lemmas 4.2.2 and 4.2.3 of [13], leading to reproducing the following result of that work:

2.2.2 Theorem (Proposition 4.2.4 in [13])**.** *If the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ and Ω satisfy the CGMA (including the condition (v)) in dS^4, then the group $\mathcal{G}$ contains the identity* *component $\mathcal{L}_+^\uparrow$ of the Lorentz group $\mathcal{L}$ of $\mathcal{M}_5$, and for each $W \in \mathcal{W}_0$ the involution g_W is the reflection about the edge of the wedge W.*

Proof. First we establish the following notation: let $\mathbf{e}_i$, $i = 0, 1, 2, 3, 4$, denote the (column) vector in $\mathcal{M}_5$ whose j-th component is δ_{ij}, let W_i, $i = 1, 2, 3, 4$, denote the wedge

$$W[l_{i+}, l_{i-}] := \{\alpha l_{i+} - \beta l_{i-} + \gamma l_\perp | \alpha, \beta > 0, \gamma \in \mathbb{R}, l_\perp \cdot l_{i\pm} = 0\} = \{x \in \mathcal{M}_5 | \mp x \cdot l_{i\pm} > 0\},$$

where $l_{i\pm} = \mathbf{e}_0 \pm \mathbf{e}_i$, and let $W_i' := W[l_{i-}, l_{i+}]$, the causal complement of W_i. Let further $diag(a, b, c, d, e)$ denote the 5x5 diagonal matrix with the entries a, b, c, d, e on the main diagonal, in that order, let i_{mn} denote the 5x5 diagonal matrix with entries equal to 1 at all positions except at the positions mm and nn, where the entries are -1, e.g. $i_{34} = diag(1, 1, 1, -1, -1)$, $i_{01} = diag(-1, -1, 1, 1, 1)$, etc., let l_{mn} denote a matrix representing a reflection with respect to a line in the plane spanned by the vectors $\mathbf{e}_m$ and $\mathbf{e}_n$, and let R_{mn} denote a matrix representing a rotation in the plane spanned by the vectors $\mathbf{e}_m$ and $\mathbf{e}_n$, $m, n \in \{1, 2, 3, 4\}$. Finally, let the symbol $[g, h]$ denote the group commutator, $[g, h] := ghg^{-1}h^{-1}$.

Recall a well known fact that in a (fixed) basis of $\mathcal{M}_5$ each $\Lambda \in \mathcal{L}_+^\uparrow$ has the unique polar decomposition $\Lambda = RB$, where R is a rotation in 4-dimensional Euclidean subspace $\mathbb{E}_4$ of $\mathcal{M}_5$, and B is a transformation with a symmetric positive matrix representing a lorentzian boost in $\mathcal{M}_5$. If the direction of the boost is given by a unit vector $\mathbf{n} \in \mathbb{E}_4$, then the boost fixes each vector in the 3-dimensional subspace $\{\mathbf{n}\}^\perp$ of $\mathbb{E}_4$, and has two 1-dimensional eigenspaces spanned by the eigenvectors $e_0 \pm \mathbf{n}$, respectively, corresponding to the eigenvalues $e^{\pm\alpha}$ for some real number $\alpha \neq 0$.

Now we prove a number of lemmas needed to establish the claim of the Theorem.

2.2.3 Lemma. *Let $\mathcal{G} \leq \mathcal{L}$ be a subgroup acting transitively on $\mathcal{W}_0$. Let $Inv_H(W_1)$ denote the subgroup containing all the elements of a group $H \leq \mathcal{L}$ that leave the right wedge*

W_1 *invariant. If* $\mathcal{G}$ *contains a non-trivial element* $g_c \neq diag(-1,-1,-1,-1,-1)$ *of the centralizer of* $Inv_{\mathcal{L}_+^\uparrow}(W_1)$ *in* $\mathcal{L}$*, then* $\mathcal{G}$ *contains* $\mathcal{L}_+^\uparrow$*, the identity component of* $\mathcal{L}$*.*

Proof. Note first that $g_c = jB$, where B is a (possibly trivial) boost in the direction parallel to $\mathbf{e}_1$, and

$$j \in \{diag(1,1,1,1,1), diag(1,1,-1,-1,-1), diag(-1,-1,1,1,1)\}.$$

Since $\mathcal{G}$ acts transitively on $\mathcal{W}_0$, for each Λ in $\mathcal{L}_+^\uparrow$ there is a g in $\mathcal{G}$ and a $\tilde{\Lambda}$ in $Inv_{\mathcal{L}}(W_1)$, such that $\Lambda = g\tilde{\Lambda}$. Note that if $\tilde{\Lambda}$ is an orthochronous transformation, or if $B = 1$, then $\tilde{\Lambda} g_c \tilde{\Lambda}^{-1} = g_c$, otherwise $\tilde{\Lambda} g_c \tilde{\Lambda}^{-1} = g_c^{-1}$. It then follows that, for each Λ in $\mathcal{L}_+^\uparrow$, the element $g_c \Lambda g_c^{-1} \Lambda^{-1} = g_c g g_c^{\pm 1} g^{-1}$ belongs to $\mathcal{G}$. All such elements generate a non-trivial connected normal subgroup of $\mathcal{L}_+^\uparrow$ and the normal subgroup is contained in $\mathcal{G}$. Since $\mathcal{L}_+^\uparrow$ is simple (see [2] Section 1.4, Section 3.7, and [14]), one must have $\mathcal{L}_+^\uparrow \leq \mathcal{G}$. □

2.2.4 Lemma (Lemma 4.2.1 in [13].)**.** *Let* $\mathcal{G} \leq \mathcal{L}$ *be a subgroup acting transitively on* $\mathcal{W}_0$*. If* $\mathcal{G}$ *contains the element* $\eta = diag(1,-1,-1,-1,-1)$ *or the element* $\bar{\eta} = diag(-1,1,1,1,1)$*, or one of their conjugates, then* $\mathcal{G}$ *contains* $\mathcal{L}_+^\uparrow$*, the identity component of* $\mathcal{L}$*.*

Proof. If $\mathcal{G}$ contains a conjugate of η or a conjugate of $\bar{\eta}$, then choose the basis in $\mathcal{M}_5$ (by conjugating by an appropriate boost), so that $\eta \in \mathcal{G}$, or $\bar{\eta} \in \mathcal{G}$. Since $\mathcal{G}$ acts transitively on $\mathcal{W}_0$, there must be an element g in $\mathcal{G}$, having the form $g = BRj$, where j is either $\bar{\eta}$ or the identity, $R \in O(4)$, and $B \neq 1$ is a non-trivial boost, because the action of $O(4) \cup jO(4)$ on $\mathcal{W}_0$ is not transitive. If $\eta \in \mathcal{G}$, then $B^2 = BR\eta R^{-1} B^{-1} \eta \in \mathcal{G}$. Similarly, if $\bar{\eta} \in \mathcal{G}$, then again $B^2 = BR\bar{\eta} R^{-1} B^{-1} \bar{\eta} \in \mathcal{G}$. In either case, Lemma 2.2.3 implies the claim. □

2.2.5 Lemma (See Lemma 4.2.2 in [13])**.** *Let* $\mathcal{G} \leq \mathcal{L}_+^\uparrow$ *be a subgroup generated by a collection* $\mathcal{C} \subset \mathcal{G}$*, of involutions, acting transitively on* $\mathcal{W}_0$*, and satisfying the following condition:*

***(C1)** If $i \in \mathcal{C}$ is an involution and $g \in \mathcal{G}$ is such that $i \neq gig^{-1}$,and both i and gig^{-1} leave a wedge $W \in \mathcal{W}_0$ invariant, then there exists an involution $j \in \mathcal{G}$ such that $j = hih^{-1}$ for some $h \in \mathcal{G}$, j commutes with both i, and gig^{-1}, and $jW \neq W$.*

Then $\mathcal{G} = \mathcal{L}_+^\uparrow$.

Proof. Choose and fix an involution $i \in \mathcal{C}$. If $i = \eta = diag(1, -1, -1, -1, -1)$, or one of the conjugates of η, then Lemma 2.2.4 implies the claim. Suppose i equals neither η nor any conjugate of η. Since $\mathcal{G} \leq \mathcal{L}_+^\uparrow$, we may choose the basis of $\mathcal{M}_5$ such that $i = i_{34}$, so $iW_1 = W_1$. Since the action of $\mathcal{G}$ on $\mathcal{W}_0$ is transitive, there exists an element $g_{41} \in \mathcal{G}$ such that $g_{41}W_1 = W_4$, so, with $i_4 = g_{41}ig_{41}^{-1}$ one has $i_4W_4 = W_4$. Note that i_4 is necessarily a rotation. Applying a suitable rotation in the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, we choose a basis in $\mathcal{M}_5$ such that $i_{34} \in \mathcal{G}$, and $\mathcal{G} \ni i_4 = diag(1, -1, 1, 1, 1)l_{23}$, for some reflection l_{23}.

First suppose $[i, i_4] = 1$. In this case, either $l_{23} = diag(1, 1, -1, 1, 1)$, or $l_{23} = diag(1, 1, 1, -1, 1)$. The former case yields $\mathcal{G} \ni ii_4 = \eta$, and Lemma 2.2.4 implies $\mathcal{G} = \mathcal{L}_+^\uparrow$. The latter case means $i_4 = i_{13}$. Since both i and i_4 leave W_2 invariant, (C1) implies the existence of an involution $j_2 \in \mathcal{G}$, such that j_2 commutes with both i and i_4, and $j_2W_2 \neq W_2$. We can choose the basis of $\mathcal{M}_5$ (by applying the possibly nontrivial boost part of j_2, which commutes with i and i_4), so there is no loss of generality in assuming that j_2 is a rotation. It follows that either $j_2 = i_{24}$, or $j_2 = i_{23}$, or $j_2 = i_{12}$. The case $j_2 = i_{24}$ yields $\mathcal{G} \ni i_4j_2 = i_{13}i_{24} = \eta$, the case $j_2 = i_{23}$ yields $\mathcal{G} \ni ii_4j_2 = i_{34}i_{13}i_{23} = \eta$, and the last case yields $\mathcal{G} \ni ij_2 = i_{34}i_{12} = \eta$, so again Lemma 2.2.4 implies $\mathcal{G} = \mathcal{L}_+^\uparrow$.

Now suppose $[i, i_4] \neq 1$. Let $i_1 = i_4ii_4$. Then $i_1 \neq i$, $i_1W_1 = W_1$, and $i_1 = diag(1, 1, 1, 1, -1)l'_{23}$, where $l'_{23} = l_{23}diag(1, 1, 1, -1, 1)l_{23}$. The condition (C1) implies that there exists an involution j_1, conjugate to i, and such that $[j_1, i] = 1 = [j_1, i_1]$ and $j_1W_1 \neq W_1$. We can again assume the choice of the basis of $\mathcal{M}_5$ such j_1 is a rotation. If l'_{23} is diagonal, then it must be $i_1 = i_{24}$, leaving the possibilities $j_1 = i_{12}$, $j_1 = i_{13}$, $j_1 = i_{14}$. In each case it follows $\eta \in \mathcal{G}$ and Lemma 2.2.4 implies $\mathcal{G} = \mathcal{L}_+^\uparrow$.

If l'_{23} is not diagonal, then $j_1 = i_{14}$. Since both i and j_1 leave W_2 invariant,and since j_1 is conjugate to i, the condition (C1), in a way analogous to that in the case $[i, i_4] = 1$ analysis, implies that necessarily $\eta \in \mathcal{G}$, and Lemma 2.2.4 completes the proof. $\square$

2.2.6 Remark. In the proof of the above lemma, we have used the condition (C1) to prove that, under the stated hypothesis, the group $\mathcal{G}$ contains an isomorphic copy of the abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (having three mutually commuting generators, each of order 2). The rest of the proof uses Lemma 2.2.4 to prove the less trivial direction of the following assertion:

$\mathcal{G} \leq \mathcal{L}_+^\uparrow$, generated by a collection of involutions in $\mathcal{G}$ acting transitively on $\mathcal{W}_0$, is equal to $\mathcal{L}_+^\uparrow$ if and only if $\mathcal{G}$ contains an isomorphic copy of the abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

One could use the CGMA and the conclusions of Lemma 1.0.1 directly to prove the same, namely:

If the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ and Ω satisfy the CGMA, then the group $\mathcal{G}$ contains an isomorphic copy of the abelian group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

This has, effectively, been done in Lemma 4.2.2 of [13]. In Lemma 2.2.5 we have imposed (C1) as a weaker condition, implied by the claims of Lemma 1.0.1, in an attempt to provide a proof under the hypothesis merely that $\mathcal{G} \leq \mathcal{L}_+^\uparrow$ is generated by a collection of involutions in $\mathcal{G}$ and the action of $\mathcal{G}$ on $\mathcal{W}_0$ is transitive. The effort to find such a proof has been so far without success.

2.2.7 Lemma (See Lemma 4.2.3 and the proof of Proposition 4.2.4 in [13])**.** *Let $\mathcal{G} \leq \mathcal{L}_+^\uparrow \cup \mathcal{L}_+^\downarrow$ be a subgroup generated by a collection $\mathcal{C} \subset \mathcal{G}$ of involutions, acting transitively on $\mathcal{W}_0$, and such that $\mathcal{G} \cap \mathcal{L}_+^\downarrow \neq \emptyset$. Then $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.*

Proof. Since $\mathcal{G}$ intersects at most two connected components of $\mathcal{L}$, there must be an involution $i \in \mathcal{C}$ which belongs to $\mathcal{L}_+^\downarrow$, so i has an eigenspace corresponding to the eigenvalue -1 spanned by either one timelike and one spacelike eigenvector, or by one timelike and three spacelike eigenvectors. Assume the former. Choose a basis in $\mathcal{M}_5$

such that i is diagonal. Then $i \in \{i_{01}, i_{02}, i_{03}, i_{04}\}$, which are reflections about the edge of the wedges W_1, W_3, W_3, W_4, respectively. The transitivity of the action of $\mathcal{G}$ on $\mathcal{W}_0$ then implies that, actually, $i_{01} \in \mathcal{G}$. Since i_{01} belongs to the centralizer of $Inv_{\mathcal{L}_+^\uparrow}(W_1)$ in $\mathcal{L}$, Lemma 2.2.3 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.

If the eigenspace of i corresponding to the eigenvalue -1 is spanned by one timelike and three spacelike eigenvectors, then choose the basis of $\mathcal{M}_5$ so that $i = diag(-1, 1, -1, -1, -1)$. Note that $iW_1 = W_1$, and that i commutes with every rotation that leaves W_1 invariant. By the transitivity of the action of $\mathcal{G}$ on $\mathcal{W}_0$, there exists an element $g_{21} \in \mathcal{G}$, such that $g_{21}W_1 = W_2$, and such that , with $g_{21}ig_{21}^{-1}$ denoted by j_2, the latter has the form $j_2 = B_2 diag(-1, -1, 1, -1, -1)B_2^{-1}$, where B_2 is a (possibly trivial) boost in the direction parallel to $\mathbf{e}_2$. Applying, if necessary, the boost B_2^{-1} to change the basis of $\mathcal{M}_5$, we may assume that $\mathcal{G}$ contains both $i = diag(-1, 1, -1, -1, -1)$ (note $[i, B_2] = 1$), and $j_2 = diag(-1, -1, 1, -1, -1)$. Then, applying the transitivity again, we conclude that there is an element $g_{31} \in \mathcal{G}$, such that $g_{31}W_1 = W_3$, and such that , with $g_{31}ig_{31}^{-1}$ denoted by j_3, the latter has the form $j_3 = B_3 diag(-1, -1, -1, 1, -1)B_3^{-1}$, where B_3 is a (possibly trivial) boost in the direction parallel to $\mathbf{e}_3$. Applying now again, if necessary, the boost B_3^{-1} to change the basis of $\mathcal{M}_5$, we may assume that $\mathcal{G}$ contains $i = diag(-1, 1, -1, -1, -1)$ (note $[i, B_2] = 1$), $j_2 = diag(-1, -1, 1, -1, -1)$ (note $[i, B_3] = 1 = [j_2, B_3]$), and $j_3 = diag(-1, -1, -1, 1, -1)$, so $\mathcal{G}$ contains $ij_2j_3 = diag(-1, 1, 1, 1, -1)$, which is the reflection about the edge of W_4. As in the first case above, the transitivity of the action of $\mathcal{G}$ on $\mathcal{W}_0$ implies that $i_{01} \in \mathcal{G}$, and Lemma 2.2.3 completes the proof. □

2.2.8 Remark. Note that to prove Lemma 2.2.7 no additional condition like (C1) was needed. The conditions that $\mathcal{G}$ acts transitively on $\mathcal{W}_0$, is generated by involutions, and intersects only the connected components $\mathcal{L}_+^\uparrow$ and $\mathcal{L}_+^\downarrow$ of $\mathcal{L}$, were sufficient.

2.2.9 Lemma (See Lemma 4.2.2 in [13]). *Let $\mathcal{G} \leq \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\uparrow$ be a subgroup generated by a collection $\mathcal{C} \subset \mathcal{G}$, of involutions, acting transitively on $\mathcal{W}_0$, and such that $\mathcal{G} \cap \mathcal{L}_-^\uparrow \neq \emptyset$. Then $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.*

Proof. As in the proof of Lemma 2.2.7, we conclude that there must be an involution $i \in \mathcal{C}$ which belongs to $\mathcal{L}_-^\uparrow$, so i has an eigenspace corresponding to the eigenvalue -1 spanned by either three spacelike eigenvectors, or by a single spacelike eigenvector. Assuming the former, choose a basis in $\mathcal{M}_5$ such that i is diagonal. Then

$$i \in \{diag(1,1,-1,-1,-1), diag(1,-1,1,-1,-1),$$
$$diag(1,-1,-1,1,-1), diag(1,-1,-1,-1,1)\},$$

which belong to the centralizers of $Inv_{\mathcal{L}_+^\uparrow}(W_i)$, $i = 1,2,3,4$, respectively. The transitivity of the action of $\mathcal{G}$ on $\mathcal{W}_0$ then implies that, actually, $\mathcal{G}$ contains a nontrivial element of the centralizer of $Inv_{\mathcal{L}_+^\uparrow}(W_1)$ in $\mathcal{L}$, and Lemma 2.2.3 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.

If the eigenspace of i corresponding to the eigenvalue -1 is spanned by a single spacelike eigenvector, then choose the basis of $\mathcal{M}_5$ so that $i = diag(1,-1,1,1,1)$. Note that, again, i commutes with every rotation that leaves W_1 invariant. Using the transitivity of the action of $\mathcal{G}$ on $\mathcal{W}_0$ and suitable choice of the basis in $\mathcal{M}_5$ as in the proof of Lemma 2.2.7, we observe that $\mathcal{G}$ contains $i = diag(1,-1,1,1,1)$, $j_2 = diag(1,1,-1,1,1,1)$ and $j_3 = diag(1,1,1,-1,1)$, so $\mathcal{G}$ contains $ij_2j_3 = diag(1,-1,-1,-1,1)$, which is an element of the centralizer of $Inv_{\mathcal{L}_+^\uparrow}(W_4)$ in $\mathcal{L}$. The transitivity of the action of $\mathcal{G}$ on $\mathcal{W}_0$ implies that $\mathcal{G}$ contains a nontrivial element of the centralizer of $Inv_{\mathcal{L}_+^\uparrow}(W_1)$ in $\mathcal{L}$, and Lemma 2.2.3 completes the proof. $\square$

2.2.10 Remark. Note that, to prove Lemma 2.2.9, again no additional condition like (C1) was needed. The conditions that $\mathcal{G}$ acts transitively on $\mathcal{W}_0$, is generated by involutions, and intersects only the connected components $\mathcal{L}_+^\uparrow$ and $\mathcal{L}_-^\uparrow$ of $\mathcal{L}$, were sufficient.

2.2.11 Remark. For the case $\mathcal{G} \leq \mathcal{L}_+^\uparrow \cup \mathcal{L}_-^\downarrow$, $\mathcal{G} \cap \mathcal{L}_-^\downarrow \neq \emptyset$ we have not found an analog to the condition (C1) that would follow easily from the CGMA, resp. from Lemma 1.0.1. However, the method we use below in the proof of Theorem 2.2.2 to show that in this case it also follows that $\mathcal{G} \geq \mathcal{L}_+^\uparrow$ will be used later in proving the analogous theorem for CGMA in the flat 5-dimensional Minkowski space.

We now continue the proof of Theorem 2.2.2. We note (see [7], Lemma 6.9) that for every generating involution g_W, $W \in \mathcal{W}_0$, one must have $g_W W \neq W$, otherwise it would follow from CGMA (iii) and Lemma 1.0.1 (4), and by transitivity of the action of $\mathcal{G}$ upon $\mathcal{W}_0$, that for a pair W_1, W_2 of non-equal wedges with nonempty intersection, the algebra $\mathcal{R}(W_1) \cap \mathcal{R}(W_2)$ is maximally abelian and a proper subalgebra of a maximally abelian algebra $\mathcal{R}(W_1)$, a contradiction.

Suppose now $g_{W_1} \in \mathcal{L}_+^\uparrow$. Since by Lemma 1.0.1 (2) all involutions $\{g_W\}_{W \in \mathcal{W}_0}$ are conjugate in $\mathcal{G}$, it follows that $\mathcal{G} \leq \mathcal{L}_+^\uparrow$. Since the condition (C1) follows easily from Lemma 1.0.1, the conditions of Lemma 2.2.5 are satisfied and it implies that $\mathcal{G} = \mathcal{L}_+^\uparrow$.

If $g_{W_1} \in \mathcal{L}_+^\downarrow$ or $g_{W_1} \in \mathcal{L}_-^\uparrow$, then Lemma 2.2.7, or Lemma 2.2.9, respectively, implies that $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.

It remains to consider the possibility $g_{W_1} \in \mathcal{L}_-^\downarrow$. If, for some W, $g_W = \bar{\eta} = diag(-1,1,1,1,1)$, or one of its conjugates, then Lemma 2.2.4 implies that $\mathcal{G} \geq \mathcal{L}_+^\uparrow$. If $g_W = diag(-1,-1,-1,-1,-1)$ for some W, then $\mathcal{G}$ could not act transitively on $\mathcal{W}_0$.

Suppose then that the eigenspace of g_W, for each W, corresponding to the eigenvalue -1, is spanned by two spacelike vectors and a timelike vector. Choose a basis in $\mathcal{M}_5$ such that $\mathcal{G} \ni i = diag(-1,-1,-1,1,1)$. Note that $iW_4 = W_4$, so, by Lemma 1.0.1 (3), it must be $[g_{W_4}, i] = 1$, and also $g_{W_4} W_4 \neq W_4$, by the above observation. Therefore one has

$$\text{either } g_{W_4} = diag(-1,1,1,-1,-1)B \quad \text{or } g_{W_4} = \bar{\eta} I_{12} I_{34} B',$$

where B, B' are (possibly trivial) boosts commuting with i. The former case means that $\mathcal{G}$ contains a conjugate of η and Lemma 2.2.4 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$. Suppose the latter case and choose the basis by applying the boost $B'^{\frac{1}{2}}$ and a suitable rotation allowing one to assume that

$$\mathcal{G} \ni i = diag(-1,-1,-1,1,1), \quad \mathcal{G} \ni g_{W_4} = diag(-1,-1,1,1,-1).$$

Note that $g_{W_4}W_3 = W_3 = iW_3$, so one must have $[g_{W_3}, g_{W_4}] = 1 = [g_{W_3}, i]$, implying that

$$g_{W_3} \in \{diag(-1,1,-1,-1,1)B_1, diag(-1,1,1,-1,-1)B_1', diag(-1,-1,1,-1,1)\},$$

where B_1, B_1' are (possibly trivial) boosts in the direction parallel to $\mathbf{e}_1$. Again, the first two possibilities mean that $\mathcal{G}$ contains a conjugate of η and Lemma 2.2.4 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$, so assume

$$g_{W_3} = diag(-1,-1,1,-1,1).$$

Note that $g_{W_3}W_2 = W_2 = g_{W_4}W_2$, so one must have $[g_{W_2}, g_{W_4}] = 1 = [g_{W_2}, g_{W_3}]$, implying that

$$g_{W_2} \in \{diag(-1,1,-1,-1,1)B_1, diag(-1,1,-1,1,-1)B_1', diag(-1,-1,-1,1,1)\},$$

where B_1, B_1' are (possibly trivial) boosts in the direction parallel to $\mathbf{e}_1$. Again, the first two possibilities mean that $\mathcal{G}$ contains a conjugate of η and Lemma 2.2.4 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$, so assume

$$g_{W_2} = diag(-1,-1,-1,1,1) = i.$$

Notice that

$$g_{W_2}g_{W_3} = diag(1,1,-1,-1,1),$$

$$g_{W_2}g_{W_4} = diag(1,1,-1,1,-1),$$

$$g_{W_3}g_{W_4} = diag(1,1,1,-1,-1),$$

all of which are elements of $Inv_{\mathcal{L}_+^\uparrow}(W_1)$, so they must all commute with g_{W_1}. This implies that

$$g_{W_1} \in \{diag(-1,-1,-1,1,1), diag(-1,-1,1,-1,1), diag(-1,-1,1,1,-1)\}.$$

Suppose $g_{W_1} = diag(-1,-1,-1,1,1) = g_{W_2}$. Since the action of $\mathcal{G}$ upon $\mathcal{W}_0$ is transitive, there must be $W \in \mathcal{W}_0$ such that $g_W = \bar{\eta}jB$, where $B \neq 1$ is a boost, and $j \in \mathcal{L}_+^\uparrow$ is an involutive rotation. If the direction of B is parallel to $\vec{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 + n_4\mathbf{e}_4$, then the action of g_W leaves the wedge $W[l_+, l_-]$, where $l_\pm = \mathbf{e}_0 \pm \vec{n}$, invariant. The transitivity also implies that there exists an element $g_{n1} = kR_{n1}\tilde{R}_1\tilde{B}_1$, where R_{n1} is a rotation such that $R_{n1}\mathbf{e}_1 = \vec{n}$, $\tilde{R}_1, \tilde{B}_1 \in Inv_{\mathcal{L}_+^\uparrow}(W_1)$ are a rotation and a boost, respectively, and $k = diag(\pm 1, 1, 1, 1, 1)$, depending on which connected component of $\mathcal{L}$ the element g_{n1} belongs to. Then

$$g_{n1}^{-1} g_W g_{n1} = \tilde{B}_1^{-1} \bar{\eta} i_1 B_R \tilde{B}_1,$$

where $1 \neq B_R \in Inv_{\mathcal{L}_+^\uparrow}(W_1)$ is a boost, and $i_1 \in Inv_{\mathcal{L}_+^\uparrow}(W_1)$ is an involutive rotation. Since $\tilde{B}_1$ commutes with all elements of $Inv_{\mathcal{L}_+^\uparrow}(W_1)$, as well as with g_{W_1}, we can choose a basis in $\mathcal{M}_5$ (by applying $\tilde{B}_1$), allowing us to assume that $\mathcal{G}$ contains the elements

$$\begin{aligned} &g_{W_2}g_{W_3} = diag(1, 1, -1, -1, 1), \quad g_{W_2}g_{W_4} = diag(1, 1, -1, 1, -1), \\ &g_{W_3}g_{W_4} = diag(1, 1, 1, -1, -1), \quad g_{W_1} = diag(-1, -1, -1, 1, 1) = g_{W_2}, \\ &\text{and} \quad i_R = \bar{\eta} i_1 B_R. \end{aligned}$$

Since $i_R W_1 = W_1$, i_1 must commute with g_{W_1}. Note also that

$$diag(-1, -1, -1, -1, -1) = g_{W_2}g_{W_3}g_{W_4} \in \mathcal{G},$$

so, with $j_1 = \bar{\eta} i_1 diag(-1, -1, -1, -1, -1)$, one has

$$j_R = j_1 B_R = i_R diag(-1, -1, -1, -1, -1) \in \mathcal{G}.$$

Since $j_R W_1 = W_1'$ and $[j_R, g_{W_1}] = 1$, it follows that either $j_1 = diag(1, -1, -1, 1, 1)$ or $j_1 = diag(1, -1, 1, 1, 1)l_{34}$ for some reflection l_{34}. The former possibility implies that $\eta B_R = j_R g_{W_3} g_{W_4} \in \mathcal{G}$, so Lemma 2.2.4 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$, hence we assume $j_1 = diag(1, -1, 1, 1, 1)l_{34}$. Since we have $g_{W_2} = g_{W_1}$, it follows from Lemma 1.0.1 that if $g_{21}W_1 = W_2$ for some $g_{21} \in \mathcal{G}$, then $[g_{21}, g_{W_1}] = 1$. This implies that there is an element

$g_{21} \in \mathcal{G}$ such that $g_{21} = R_{21} R_1 B_1 k_1$, where R_{21} is the (fixed) block diagonal matrix

$$R_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$B_1, R_1 \in Inv_{\mathcal{L}_+^\uparrow}(W_1)$, are a boost and a rotation, respectively, such that $[R_1, g_{W_1}] = 1$, and $k_1 = diag(\pm 1, 1, 1, 1, 1)$, depending again on which connected component of $\mathcal{L}$ the element g_{21} belongs to. It follows that either $R_1 = R_{34}$ or $R_1 = diag(1, 1, -1, 1, 1) I_{34}$, for some reflection I_{34}. In either case, denoting $g_{W_2} g_{W_3} j_R$ by g_1 and $[g_{W_2} g_{W_3}, g_{21}]$ by h_1, we get

$$g_1 = diag(1, -1, -1, 1, 1) R'_{34} B_R$$
$$h_1 = diag(1, -1, -1, 1, 1) R''_{34},$$

for some rotations R'_{34}, R''_{34}, implying $\mathcal{G} \ni [h_1, g_1] = B_R^2$, i.e. $\mathcal{G}$ contains a nontrivial boost in the direction parallel to $\mathbf{e}_1$, so Lemma 2.2.3 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.

The cases $g_{W_1} = g_{W_3}$ and $g_{W_1} = g_{W_4}$ can be treated in an analogous way: in the former case we construct j_R and g_{31}, $g_{31} W_1 = W_3$, such that $[j_R, g_{W_1}] = 1 = [g_{31}, g_{W_1}]$, and obtain

$$g_3 = g_{W_3} g_{W_4} j_R = diag(1, -1, 1, -1, 1) R'_{24} B_3$$
$$h_3 = [g_{W_3} g_{W_4}, g_{31}] = diag(1, -1, 1, -1, 1) R''_{24},$$

with some nontrivial boost $B_3 \in Inv_{\mathcal{L}_+^\uparrow}(W_1)$, so that $\mathcal{G} \ni [h_3, g_3] = B_3^2$, and in the remaining case again we construct j_R and g_{41}, $g_{41} W_1 = W_4$, such that $[j_R, g_{W_1}] = 1 = [g_{41}, g_{W_1}]$, and obtain

$$g_4 = g_{W_2} g_{W_4} j_R = diag(1, -1, 1, 1, -1) R'_{23} B_4$$
$$h_4 = [g_{W_2} g_{W_4}, g_{41}] = diag(1, -1, 1, 1, -1) R''_{23},$$

with some nontrivial boost $B_4 \in Inv_{\mathcal{L}_+^\uparrow}(W_1)$, so that $\mathcal{G} \ni [h_4, g_4] = B_4^2$. In both cases, Lemma 2.2.3 implies $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.

We have thus shown that the CGMA and the assumption of transitivity of the action of $\mathcal{G}$ upon $\mathcal{W}_0$ imply $\mathcal{G} \geq \mathcal{L}_+^\uparrow$. It follows from Lemma 1.0.1 that $[g_{W_1}, B] = 1 = [g_{W_1}, R]$, for each boost B and each rotation R in $Inv_{\mathcal{L}_+^\uparrow}(W_1) \leq \mathcal{G}$, hence one must have

$$g_{W_1} = i_{01} = diag(-1, -1, 1, 1, 1),$$

i.e. the reflection with respect to the edge of W_1. The transitivity then implies that for each $W \in \mathcal{W}_0$, the involution g_W is the reflection about the edge of W. □

2.3 Continuous Unitary Representation of de Sitter Group via Reflection Maps

Reflection maps were introduced in [11], Definition 2.3, as the maps

$$J : \mathcal{R} \to \mathcal{J},$$

from the set $\mathcal{R}$ of all reflections over the edges of wedges in the 4-dimensional Minkowski space $\mathcal{M}_4$ to a topological group $\mathcal{J}$, such that for every $\lambda \in \mathcal{R}$ the element $J(\lambda) \in \mathcal{J}$ is an involution and

$$J(\lambda_1)J(\lambda_2)J(\lambda_1) = J(\lambda_1\lambda_2\lambda_1) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathcal{R}.$$

It has been established in [11] that any continuous reflection map extends to a unique continuous homomorphism of the proper Poincaré group $\mathcal{P}_+$ into $\mathcal{J}$. This fact was subsequently used to construct a unitary representation of $\mathcal{P}_+$, associated with the pair $(\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}, \Omega)$, satisfying CGMA in $\mathcal{M}_4$, where $\tilde{\mathcal{W}}$ is the set of all wedges in $\mathcal{M}_4$. The representation preserves the vector Ω and acts covariantly on the net $\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}$. This representation has been constructed also in [13] and [7] using the Moore cohomology theory, but the construction in [11] is simpler and geometrically much more lucid.

It is our intention to prove analogs of Proposition 2.8 and Theorem 4.1 of [11] for the case of CGMA in the 4-dimensional de Sitter spacetime dS^4, to obtain a unitary representation of the de Sitter group $\mathcal{L}_+$ (i.e. the proper Lorentz group in 5-dimensional Minkowski space $\mathcal{M}_5$), associated with the pair $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$, satisfying CGMA in this spacetime, so that the unitary representation preserves the vector Ω and acts covariantly on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$. The construction, like that given in [11], is based on reflection maps and yields a simpler and geometrically more lucid version of what has been established in [13] using the more abstract method of Moore's cohomology theory.

We have shown in Section 2.2 that the involutive transformations of $\mathcal{M}_5$ which are induced by the geometric modular action on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ generate the whole $\mathcal{L}_+$, and are actually reflections over the edge of the wedges whose edges contain the origin of $\mathcal{M}_5$. Following [11], we will consider reflection maps on the set $\mathcal{R}$ of all these reflections:

2.3.1 Definition. Let $\mathcal{R}$ be the set of all reflections about the edges of the wedges whose edge contains the origin in $\mathcal{M}_5$, and let $\mathcal{J}$ be a topological group. A map $J : \mathcal{R} \to \mathcal{J}$ is called a reflection map if

$$J(\lambda_1)J(\lambda_2)J(\lambda_1) = J(\lambda_1\lambda_2\lambda_1) \qquad \text{for all } \lambda_1, \lambda_2 \in \mathcal{R}. \tag{2–4}$$

We will now make some more detailed observations about how each element of $\mathcal{L}_+$ is expressed as a composition of reflections. If a reflection λ_0 is fixed, then each transformation $\Lambda' \in \mathcal{L}_+$ that does not belong to the identity component $\mathcal{L}_+^\uparrow$ has the form $\Lambda' = \lambda_0\Lambda$ where $\Lambda \in \mathcal{L}_+^\uparrow$. Recall that in a (fixed) basis of $\mathcal{M}_5$ each $\Lambda \in \mathcal{L}_+^\uparrow$ has the unique polar decomposition $\Lambda = RB$, where R is a rotation in 4-dimensional Euclidean subspace $\mathbb{E}_4$ of $\mathcal{M}_5$, and B is a transformation representing a lorentzian boost in $\mathcal{M}_5$ whose direction is given by a unit vector $\mathbf{n} \in \mathbb{E}_4$ so that B fixes each vector in the 3-dimensional subspace $\{\mathbf{n}\}^\perp$ of $\mathbb{E}_4$, and B has two 1-dimensional eigenspaces spanned by the eigenvectors $e_0 \pm \mathbf{n}$, respectively, corresponding to the eigenvalues $e^{\pm\alpha}$ for some real number $\alpha \neq 0$.

There are two types of rotations in 4-dimensional Euclidean space ([16], Art. 101, Theorem 1, also see Art. 81):

- simple rotations: each of these transformations fixes every point of a (fixed) plane.
- double rotations: each of these transformations is a composition of two nontrivial simple rotations R_1 and R_2 such that the plane of points fixed by R_1 is the orthogonal complement (in the 4-dimensional Euclidean space) of the plane of points fixed by R_2.

Just like in the case of $\mathcal{M}_4$ (Lemma 2.1 in [11]), in $\mathcal{M}_5$ each boost B may be expressed as $B = (B\lambda_e)\lambda_e$, where $\mathbf{e}$ is a unit vector orthogonal to the direction of the boost B and λ_e is a reflection over the edge of the wedge

$$W_e = \{(x_0, \mathbf{x}) \in \mathcal{M}_5 | |x_0| \leq \mathbf{x} \cdot \mathbf{e}\}.$$

In this case $\lambda_e B = B^{-1}\lambda_e$, so $B\lambda_e = B^{\frac{1}{2}}\lambda_e B^{-\frac{1}{2}}$ is also a reflection (over the edge of the wedge $B^{\frac{1}{2}} W_e$). Similarly, each simple rotation R may be expressed as $R = (R\lambda_e)\lambda_e$, where $\mathbf{e}$ is a unit vector orthogonal to the plane of fixed points of R and λ_e is a reflection over the edge of W_e. In this case $\lambda_e R = R^{-1}\lambda_e$, so $R\lambda_e = R^{\frac{1}{2}}\lambda_e R^{-\frac{1}{2}}$ is also a reflection (over the edge of the wedge $R^{\frac{1}{2}} W_e$).

A double rotation $\tilde{R}$ is not a composition of any two reflections, since in that case it would possess a plane of fixed points, contrary to its definition. If $\tilde{R} = RR^{\perp}$ and R fixes each point of the plane p and $R^{\perp}$ fixes each point of the plane $p^{\perp}$, then $\tilde{R} = (R\lambda_{e_{\perp}})\lambda_{e_{\perp}}(R^{\perp}\lambda_e)\lambda_e$, where $\mathbf{e} \in p$, $\mathbf{e}^{\perp} \in p^{\perp}$ are unit vectors, and $\lambda_{e_{\perp}}$ and λ_e are reflections over the edge of $W_{e_{\perp}}$ and W_e, respectively, so that each double rotation is a composition of four reflections. The following lemmas characterize the decomposition of a transformation $\Lambda \in \mathcal{L}_+^{\uparrow}$.

2.3.2 Lemma. *If $\Lambda = RB$, where R is a rotation and B is a boost, then R is simple if and only if Λ fixes a spacelike vector whose time component is 0.*

Proof. If B is trivial, then the claim follows immediately, since each simple rotation fixes each vector of a 2-dimensional subspace of $\mathbb{E}_4$.

Suppose that B is nontrivial. If R is simple, then it fixes each vector in a 2-dimensional subspace S_R of the 4-dimensional Euclidean space $\mathbb{E}_4 \subset \mathcal{M}_5$, and, if $\mathbf{e}$ is the direction of the boost B, then B fixes each vector in the 3-dimensional subspace $S_B = \{\mathbf{e}\}^{\perp}$ of $\mathbb{E}_4$. Thus $S_R \cap S_B$ has dimension at least 1.

Conversely, if $\mathbf{e}$ is a unit vector in $\mathbb{E}_4 \subset \mathcal{M}_5$, then $RB\mathbf{e} = \mathbf{e}$ implies $\mathbf{e} - R^{-1}\mathbf{e} = \mathbf{e} - B\mathbf{e}$, so that $\mathbf{e} - B\mathbf{e}$, and hence also $B\mathbf{e}$ has zero time component, which is the case only if $B\mathbf{e} = \mathbf{e}$. Therefore $R\mathbf{e} = \mathbf{e}$, so R is a simple rotation. □

2.3.3 Lemma. *A transformation $\Lambda \in \mathcal{L}_+^{\uparrow}$ can be expressed as a composition of two reflections if and only if Λ fixes a spacelike vector.*

Proof. If λ_1, λ_2 are reflections, then each of them fixes each vector in a 3-dimensional subspace S_1, S_2 of $\mathcal{M}_5$, respectively, consisting of spacelike vectors, so that $S_1 \cap S_2$ has dimension at least one. Therefore if $\Lambda = \lambda_1\lambda_2$, then Λ fixes a spacelike vector.

Conversely, if $\Lambda\mathbf{v} = \mathbf{v}$ for some spacelike vector $\mathbf{v}$, then there is a boost B such that $B\mathbf{v}$ has zero time component, so that $B\Lambda B^{-1}$ has a simple rotation in its polar decomposition, according to the preceding lemma. Therefore $B\Lambda B^{-1}$ and, consequently, Λ may be expressed as a composition of two reflections. □

2.3.4 Lemma. *A transformation $\Lambda \in \mathcal{L}_+^{\uparrow}$ is a composition of at least four reflections if and only if Λ is conjugate (in $\mathcal{L}_+^{\uparrow}$) to a double rotation.*

Proof. If Λ is conjugate to a double rotation $\tilde{R} = RR^{\perp}$, such that R and $R^{\perp}$ are simple rotations fixing each point in the planes p and $p^{\perp}$ in $\mathbb{E}_4$, respectively, then there are vectors $\mathbf{e}_1 \in p^{\perp}$, $\mathbf{e}_2 \in p$, and the reflections λ_{e_1}, λ_{e_2} over the edges of the wedges W_{e_1}, W_{e_2} such that $\tilde{R} = \lambda_{\mathrm{e}_1}(\lambda_{\mathrm{e}_1}R)\lambda_{\mathrm{e}_2}(\lambda_{\mathrm{e}_2}R^{\perp})$ is a composition of four reflections. If Λ were expressible as a composition of two reflections, then, by Lemma 2.3.3, Λ would fix a spacelike vector, and, consequently, $\tilde{R}$ would fix a spacelike vector with zero time component, implying that $\tilde{R}$ is a simple rotation by Lemma 2.3.2, a contradiction.

Conversely, if $\Lambda = \tilde{R}B$ is a composition of at least four reflections, then $\tilde{R}$ is a double rotation, as a consequence of Lemma 2.3.2, so $\tilde{R} = RR^{\perp}$, such that R fixes each point in the plane p in $\mathbb{E}_4$, and $R^{\perp}$ fixes each point in the plane $p^{\perp}$. If $B = 1$, then the claim follows trivially. If B is a nontrivial boost in the direction $\mathbf{n}$, then it fixes each vector in the 3-dimensional subspace $S_B = \{\mathbf{n}\}^{\perp}$ of $\mathbb{E}_4$, so that the dimension of the subspaces $p \cap S_B$ and $p^{\perp} \cap S_B$ is at least 1. Choosing the vectors $\mathbf{e}_1 \in p \cap S_B$, $\mathbf{e}_{1\perp} \in p^{\perp} \cap S_B$, the reflections λ_{e_1}, $\lambda_{\mathrm{e}_{1\perp}}$ over the edges of the wedges W_{e_1}, $W_{\mathrm{e}_{1\perp}}$, and vectors $\mathbf{e}_2 \in p \cap \{\mathbf{e}_1, \mathbf{e}_{1\perp}\}^{\perp}$, $\mathbf{e}_{2\perp} \in p^{\perp} \cap \{\mathbf{e}_1, \mathbf{e}_{1\perp}\}^{\perp}$, we have $\mathbf{n} \in \mathit{span}\{\mathbf{e}_2, \mathbf{e}_{2\perp}\}$, and

$$\Lambda = \tilde{R}B = RR^{\perp}(\lambda_{\mathrm{e}_2}\lambda_{\mathrm{e}_{2\perp}})^2 B = (\tilde{R}^{\frac{1}{2}}\lambda_{\mathrm{e}_2}\lambda_{\mathrm{e}_{2\perp}}\tilde{R}^{-\frac{1}{2}})(B^{-\frac{1}{2}}\lambda_{\mathrm{e}_2}\lambda_{\mathrm{e}_{2\perp}}B^{\frac{1}{2}}) = i_R i_B,$$

where $i_R = (\tilde{R}^{\frac{1}{2}}\lambda_{\mathrm{e}_2}\lambda_{\mathrm{e}_{2\perp}}\tilde{R}^{-\frac{1}{2}})$, $i_B = (B^{-\frac{1}{2}}\lambda_{\mathrm{e}_2}\lambda_{\mathrm{e}_{2\perp}}B^{\frac{1}{2}})$ are involutions, each having a 2-dimensional eigenspace of spacelike vectors corresponding to the eigenvalue -1, and a 3-dimensional eigenspace corresponding to the eigenvalue 1. Therefore there is an eigenvector $v \in \mathcal{M}_5$ which is fixed by Λ. By Lemma 2.3.3 v cannot be spacelike, and it cannot be lightlike, since no $\Lambda \in \mathcal{L}_+^{\uparrow}$ fixes a lightlike vector, unless it fixes both its time component and space component. Therefore v is timelike implying that Λ is conjugate to a rotation, which cannot be a simple one, since then Λ would fix a spacelike vector, and Lemma 2.3.3. would imply a contradiction. □

Next we turn our attention to the ambiguity in the representation of a transformation $\Lambda \in \mathcal{L}_+^{\uparrow}$ as a composition of two or four reflections. It will be necessary to have control over this ambiguity during the construction of a unitary representation of $\mathcal{L}_+$ based on a reflection map.

We first focus on the transformations that can be represented as a composition of two reflections. Following the discussion after Lemma 2.1 in [11] we observe that if $\Lambda \in \mathcal{L}_+^{\uparrow}$ and $\Lambda = \lambda_1\lambda_2$, where λ_1 and λ_2 are reflections, then $\lambda_2 = \lambda_1\Lambda$, and if $\Lambda = \lambda_1'\lambda_2'$, where λ_1', λ_2' is another pair of reflections, then again $\lambda_2' = \lambda_1'\Lambda$, and $\lambda_1' = \lambda_1\Lambda'$, where $\Lambda' \in \mathbf{\Lambda'}(\Lambda)$, and

$$\mathbf{\Lambda'}(\Lambda) = \{\Lambda' \in \mathcal{L}_+^\uparrow | \Lambda\Lambda' = \Lambda'\Lambda, \quad \lambda_1\Lambda' = \Lambda'^{-1}\lambda_1, \text{ and } (\lambda_1\Lambda')\Lambda \text{ are reflections}\}. \tag{2–5}$$

We will now find the sets $\mathbf{\Lambda'}$ for a simple rotation and for a boost.

Suppose first that R_0 is a simple rotation such that $R_0^2 \neq 1$, and that R_0 fixes each point of the plane $p_0 \subset \mathbb{E}_4$. Then also $p_0^\perp \subset \mathbb{E}_4$ is an invariant plane of R_0. If $\Lambda \in \mathcal{L}_+^\uparrow$ has the polar decomposition $\Lambda = RB$ and if $R_0\Lambda R_0^{-1} = \Lambda$, then the uniqueness of the polar decomposition implies $R_0RR_0^{-1} = R$ and $R_0BR_0^{-1} = B$. Therefore the planes p_0 and $p_0^\perp$ are invariant planes of R, so it must be $R = R_1R_2$, where R_1 fixes each point of p_0 and $R_1p_0^\perp = p_0^\perp$, and R_2 fixes each point of $p_0^\perp$ and $R_2p_0 = p_0$. Note that $R_1|_{p_0^\perp}$, and $R_2|_{p_0}$ cannot both be reflections, since $R_0^2 \neq 1$ (so the reflection R_1 would not commute with R_0), hence they are both rotations. Since $R_0BR_0^{-1} = B$ implies that the direction vector of B lies in p_0, it follows that the centralizer of R_0 in $\mathcal{L}_+^\uparrow$ when $R_0^2 \neq 1$ is isomorphic to $SO(2) \times SO(1,2)$.

If $R_0^2 = 1$, then $R_1|_{p_0^\perp}$, and $R_2|_{p_0}$ can both be rotations or they can both be reflections, so that in this case the centralizer of R_0 in $\mathcal{L}_+^\uparrow$ is isomorphic to $(O(2) \times O(1,2)) \cap \mathcal{L}_+^\uparrow$.

Suppose λ_1, λ_2 are reflections such that $R_0 = \lambda_1\lambda_2$, i.e. $\lambda_2 = \lambda_1R_0$. The condition that $\lambda_1\Lambda'$ and $\lambda_1\Lambda'\Lambda$ be reflections puts further constraint on Λ' in the centralizer of R_0 in order for Λ' to belong to $\mathbf{\Lambda}'$:

if $\mathbf{e} \in p_0^\perp$ is a unit vector, then $\lambda_1 = \lambda_e$ and $\lambda_2 = \lambda_eR_0$ are reflections, such that $\lambda_1\lambda_2 = R_0$. If $R_0^2 \neq 1$, in order for $(\lambda_e\Lambda')^2 = 1$, with $\Lambda' = R_1R_2B$ in the centralizer of R_0, as above, it follows that $R_2 = 1$, and, since λ_eR_1B, as well as $\lambda_eR_1BR_0$ are reflections for any R_1 fixing each point of p_0 and any boost B whose direction vector lies in p_0, one finally has (with $B(\mathbf{n})$ being a boost in the direction $\mathbf{n}$):

$$\mathbf{\Lambda}'(R_0) = \{\Lambda' = RB(\mathbf{n}) | \mathbf{n} \in p_0, R \text{ fixes each point of } p_0\}. \tag{2–6}$$

If $R_0^2 = 1$, then for $\Lambda' = R_1 R_2 B$ in the centralizer of R_0, $R_1|_{p_0^\perp}$, and $R_2|_{p_0}$ can be reflections, but if they are, then at least one of $\lambda_e \Lambda'$ and $\lambda_e \Lambda' R_0$ fails to be a reflection, implying that the set $\boldsymbol{\Lambda}'$ is the same as in case $R_0^2 \neq 1$.

Suppose now that $B_0 \neq 1$ is a boost in the direction $\mathbf{n}$, and that Λ' in the centralizer of B_0 in $\mathcal{L}_+^\uparrow$ has the polar decomposition $\Lambda' = RB$. Then $B^{-1}R^{-1}B_0RB = B_0$, so that $BB_0B^{-1} = R^{-1}B_0R$ is a boost (i.e. it is represented by a symmetric matrix, as are B, B^{-1}, and B_0), so $BB_0B^{-1} = B^{-1}B_0B$, implying that B_0 commutes with B, and also with R (hence also R commutes with B, and R fixes $\mathbf{n}$).

If $\mathbf{e} \in \mathbb{E}_4$ is a unit vector orthogonal to $\mathbf{n}$, then $\lambda_1 = \lambda_e$ and $\lambda_2 = \lambda_e B_0$ are reflections, such that $B_0 = \lambda_1\lambda_2$. If $\Lambda' = RB$ is in the centralizer of B_0 and $\lambda_e RB$ is a reflection, then $\lambda_e R$ must be an involution, and hence R must be fixing some vector $\mathbf{v}$ orthogonal to both $\mathbf{n}$ and $\mathbf{e}$. One therefore has (with $B(\mathbf{n})$ being a boost in the direction $\mathbf{n}$):

$$\boldsymbol{\Lambda}'(B_0) = \{\Lambda' = RB(\mathbf{n}) | R \text{ fixes each point of } span\{\mathbf{n}, \mathbf{v}\} \text{ for some } \mathbf{v} \in \{\mathbf{n}, \mathbf{e}\}^\perp\}. \qquad (2\text{–}7)$$

Note that unlike in the case of 4-dimensional Minkowski spacetime, the sets $\boldsymbol{\Lambda}'(R_0)$ and $\boldsymbol{\Lambda}'(B_0)$ are not groups, nevertheless, characterizing them gives one an insight into the ambiguity of the representation of these elements as composition of reflections.

Let now $\mathcal{R}$ denote the set of all the reflections over the edge for wedges in $\mathcal{M}_5$ whose edge contains the origin of the (fixed) coordinate system, and let $\mathcal{J}$ be a topological group. Suppose that there is a reflection map $J : \mathcal{R} \to \mathcal{J}$. Given a simple rotation R and a boost B, we follow [11] and choose a unit vector $\mathbf{e}$ which is orthogonal to the plane whose each point is fixed by R and a unit vector $\mathbf{u}$ which is orthogonal to the direction of B, and define

$$V_e(R) = J(\lambda_e)J(\lambda_e R), \qquad V_u(B) = J(\lambda_u)J(\lambda_u B).$$

Since J is a reflection map, it follows for any $\lambda \in \mathcal{R}$ that

$$V_{\mathrm{e}}(R)J(\lambda)V_{\mathrm{e}}(R)^{-1} = J(\lambda_{\mathrm{e}})J(\lambda_{\mathrm{e}}R)J(\lambda)J(\lambda_{\mathrm{e}}R)J(\lambda_{\mathrm{e}}) = J(R\lambda R^{-1})$$
$$V_{\mathrm{u}}(B)J(\lambda)V_{\mathrm{u}}(B)^{-1} = J(\lambda_{\mathrm{u}})J(\lambda_{\mathrm{u}}B)J(\lambda)J(\lambda_{\mathrm{u}}B)J(\lambda_{\mathrm{u}}) = J(B\lambda B^{-1}),$$

since $V_{\mathrm{e}}(R)^{-1} = J(\lambda_{\mathrm{e}}R)J(\lambda_{\mathrm{e}})$, and $V_{\mathrm{u}}(B)^{-1} = J(\lambda_{\mathrm{u}}B)J(\lambda_{\mathrm{u}})$. Note that for a boost B any vector $\mathbf{u}$ suitable for the above definition can be obtained by an action of a suitable rotation, which fixes the direction of B, on one fixed suitable vector $\mathbf{u}_0$, and that all such rotations commute with B. Also, for a simple rotation R any vector $\mathbf{e}$ suitable for the above definition can be obtained by an action of a suitable rotation, which fixes each point in the same plane as R does, on one fixed suitable vector $\mathbf{e}_0$. This means the proofs of Lemmas 2.4, 2.5, 2.6 in [11] apply also to the simple rotations and boosts in the de Sitter group. We now list these results:

2.3.5 Lemma (Lemma 2.4 in [11]). *The elements $V_{\mathrm{u}}(B)$, $V_{\mathrm{e}}(R)$ defined above do not depend on the choice of the vectors $\mathbf{u}$ and $\mathbf{e}$ within the above limitations.*

Therefore we may write $V(B) = V_{\mathrm{u}}(B)$, $V(R) = V_{\mathrm{e}}(R)$, for any suitable vectors $\mathbf{u}$, $\mathbf{e}$. Note that Lemma 2.3.5 implies that

$$V(R^{-1}) = J(\lambda_{\mathrm{e}})J(\lambda_{\mathrm{e}}R^{-1})J(\lambda_{\mathrm{e}})^2 = J(\lambda_{\mathrm{e}}R)J(\lambda_{\mathrm{e}}) = V(R)^{-1}$$
$$V(B^{-1}) = J(\lambda_{\mathrm{u}})J(\lambda_{\mathrm{u}}B^{-1})J(\lambda_{\mathrm{u}})^2 = J(\lambda_{\mathrm{u}}B)J(\lambda_{\mathrm{u}}) = V(B)^{-1} \qquad (2–8)$$

2.3.6 Lemma (Lemma 2.5 in [11]). *The elements $V(B)$ and $V(R)$ depend continuously on the boosts B and rotations R, respectively.*

2.3.7 Lemma (Lemma 2.6 in [11]). *If B is a boost and R is a simple rotation, then*

$$V(R)V(B)V(R)^{-1} = V(RBR^{-1}) \qquad (2–9)$$

and $V(\cdot)$ defines a true representation of every one-parameter subgroup of boosts or rotations.

In a way analogous to that of [11] we now define for $\Lambda \in \mathcal{L}_+^\uparrow$, such that $\Lambda = RB$ with R being a simple rotation,

$$\begin{aligned} V(\Lambda) &:= V(R)V(B) = V_{\mathrm{e}}(R)V_{\mathrm{e}}(B) \\ &= J(\lambda_{\mathrm{e}})J(\lambda_{\mathrm{e}}R)J(\lambda_{\mathrm{e}})J(\lambda_{\mathrm{e}}B) = J(R\lambda_{\mathrm{e}})J(\lambda_{\mathrm{e}}B), \end{aligned} \tag{2–10}$$

where the vector $\mathbf{e}$ has been chosen orthogonal to both the direction of B and to the plane of the points fixed by R, and the fact that J is a reflection map has been used. We will now use the sets $\mathbf{\Lambda}'(R)$ for a simple rotation R, and $\mathbf{\Lambda}'(B)$ for a boost B (see (2–5) - (2–7)) to show that, for the transformations $\Lambda = \Lambda_1 R \Lambda_1^{-1}$ or $\Lambda = \Lambda_1 B \Lambda_1^{-1}$, with $\Lambda_1 \in \mathcal{L}_+^\uparrow$ having a simple rotation in its polar decomposition, the above definition of V does not depend on the particular choice of the reflections λ_1 and λ_2, such that $\Lambda = \lambda_1\lambda_2$. This follows in the same manner as in the proof of Lemma 2.7 of [11], except that in the present case the class of transformations which the variant of this lemma applies to is restricted, because of the different structure of the sets $\mathbf{\Lambda}'(R)$ and $\mathbf{\Lambda}'(B)$. However, the class is large enough (it contains all the simple rotations and boosts) for one to proceed to extend V to a continuous homomorphism of $\mathcal{L}_+^\uparrow$ to $\mathcal{J}$. One has the following lemma.

2.3.8 Lemma. *If $\Lambda = \Lambda_1 B \Lambda_1^{-1}$ where B is a boost or $\Lambda = \Lambda_1 R \Lambda_1^{-1}$, where R is a simple rotation, and $\Lambda_1 = R_1 B_1$, with R_1 a simple rotation, is the polar decomposition of Λ_1, then the assignment $V(\Lambda) = J(\lambda_1)J(\lambda_2)$, where λ_1, λ_2 are reflections such that $\lambda_1\lambda_2 = \Lambda$, does not depend on the particular choice of these reflections.*

Proof. Since for any reflection λ, Λ_1 as in the hypothesis, and V defined above one has

$$V(\Lambda_1)J(\lambda)V(\Lambda_1)^{-1} = J(\Lambda_1\lambda\Lambda_1^{-1}), \tag{2–11}$$

it suffices to show the independence for the cases when Λ is a simple rotation R or a simple boost B. Suppose R is a simple rotation. Then $R = \lambda_{\mathrm{e}}\lambda_{\mathrm{e}}R$, for a suitable vector $\mathbf{e}$ perpendicular to the plane of fixed points of R, and $\lambda_{\mathrm{e}}R$ is a reflection. If λ_1, λ_2 are other two reflections such that $\lambda_1\lambda_2 = R$, then $\lambda_1 = \Lambda'^{\frac{1}{2}}\lambda_{\mathrm{e}}\Lambda'^{-\frac{1}{2}}$ and $\lambda_2 = \Lambda'^{\frac{1}{2}}\lambda_{\mathrm{e}}R\Lambda'^{-\frac{1}{2}}$ for some

$\Lambda' = R'B' \in \mathbf{\Lambda}'(R)$, so that using the equalities (2–8) - (2–11), one obtains

$$
\begin{aligned}
J(\lambda_1)J(\lambda_2) &= J(\lambda_1)J(\lambda_1 R) \\
&= J(\Lambda'^{\frac{1}{2}}\lambda_e\Lambda'^{-\frac{1}{2}})J(\Lambda'^{\frac{1}{2}}\lambda_e R\Lambda'^{-\frac{1}{2}}) \\
&= V(\Lambda'^{\frac{1}{2}})J(\lambda_e)V(\Lambda'^{-\frac{1}{2}})V(\Lambda'^{\frac{1}{2}})J(\lambda_e R)V(\Lambda'^{-\frac{1}{2}}) \\
&= V(\Lambda'^{\frac{1}{2}})J(\lambda_e)J(\lambda_e R)V(\Lambda'^{-\frac{1}{2}}) \\
&= V(\Lambda'^{\frac{1}{2}})J(\lambda_e)J(\lambda_e R)V(\Lambda'^{-\frac{1}{2}}) \\
&= V(R'^{\frac{1}{2}})V(B'^{\frac{1}{2}})J(\lambda_e)J(\lambda_e R)V(B'^{-\frac{1}{2}})V(R'^{-\frac{1}{2}}) \\
&= V(R'^{\frac{1}{2}})V(B'^{\frac{1}{2}})V(R)V(B'^{-\frac{1}{2}})V(R'^{-\frac{1}{2}}) = V(R),
\end{aligned}
$$

where the last equality follows by Lemma 2.3.7 and the fact that R commutes with B' and with R'. The proof for a boost B is done the same way, since if $\Lambda' = R'B' \in \mathbf{\Lambda}'(B)$, then both R' and B' commute with B. □

To be able to proceed to extending V to a homomorphism (of $\mathcal{L}_+^\uparrow$ to $\mathcal{J}$) one must also define $V(\tilde{R})$ for a double rotation $\tilde{R}$ and show that this definition is independent on the representation of $\tilde{R}$ as a composition of four reflections (see Lemma 2.3.4).

Recall that a double rotation $\tilde{R} = RR^\perp = R^\perp R$, where R fixes each point of the plane p, and $R^\perp$ fixes each point of the plane $p^\perp$, the orthogonal complement of p in $\mathbb{E}_4$. These planes are invariant planes of $\tilde{R}$. We wish to examine the question of whether a pair of invariant planes is determined uniquely by a double rotation $\tilde{R}$. Let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\mathbb{E}_4$, and let $\mathbf{u} \cdot \mathbf{v}$ denote their euclidean scalar product.

Suppose p, $p^\perp$ are fixed invariant planes of $\tilde{R}$. A unit vector $\mathbf{e}$ lies in an invariant plane of $\tilde{R}$ if and only if the vectors $\mathbf{e}$, $\tilde{R}\mathbf{e}$, and $\tilde{R}^2\mathbf{e}$ lie in the same plane, i.e. if and only if $\mathbf{e}$ satisfies

$$\mathbf{e} - (\mathbf{e} \cdot \tilde{R}\mathbf{e})\tilde{R}\mathbf{e} = \pm(\tilde{R}^2\mathbf{e} - (\mathbf{e} \cdot \tilde{R}\mathbf{e})\tilde{R}\mathbf{e}).$$

One therefore has two cases: (1) $\mathbf{e} = \tilde{R}^2\mathbf{e}$, and (2) $\mathbf{e} + \tilde{R}^2\mathbf{e} = 2(\mathbf{e} \cdot \tilde{R}\mathbf{e})\tilde{R}\mathbf{e}$. Case (1) implies that $\tilde{R}^2$ is a simple rotation. If $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$, with $\mathbf{e}_1 \in p$, $\mathbf{e}_2 \in p^\perp$, then it follows that $\mathbf{e}_2 = R^{\perp}\mathbf{e}_2$, and $\mathbf{e}_1 = R\mathbf{e}_1$. If $\mathbf{e}_1 \neq 0$, and $\mathbf{e}_2 \neq 0$, then one must have $\tilde{R} = diag(-1,-1,-1,-1)$ (as a transformation in $\mathbb{E}_4$). Therefore, in this case, either $\tilde{R} = diag(-1,-1,-1,-1)$, or $\mathbf{e}$ lies in one of the planes p, $p^{\perp}$.

In case (2), if again $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2$, with $\mathbf{e}_1 \in p$, $\mathbf{e}_2 \in p^{\perp}$, then, if $\mathbf{e}_1 \neq 0$, and $\mathbf{e}_2 \neq 0$, it follows that

$$\frac{(\mathbf{e}_2 \cdot R\mathbf{e}_2)}{\|\mathbf{e}_2\|^2} = \frac{(\mathbf{e}_1 \cdot R^{\perp}\mathbf{e}_1)}{\|\mathbf{e}_1\|^2},$$

i.e. with θ_1, θ_2 being the angles between $\mathbf{e}_1$ and $R^{\perp}\mathbf{e}_1$ and between $\mathbf{e}_2$ and $R\mathbf{e}_2$, respectively, one has either $\theta_1 = \theta_2$ or $\theta_1 = -\theta_2$. Such double rotations $\tilde{R}$ are called isocline rotations (see [16], Art. 112). We sum up the above observations:

2.3.9 Lemma. *If $\tilde{R} = RR^{\perp}$ is a double rotation, then either $\tilde{R}$ is isocline, or p and $p^{\perp}$ are the only invariant planes of $\tilde{R}$.*

We therefore define V_p on a double rotation $\tilde{R} = RR^{\perp}$ as follows:

$$V_p(\tilde{R}) := V(R)V(R^{\perp}) = J(\lambda_{\mathrm{e}_{\perp}})J(\lambda_{\mathrm{e}_{\perp}}R)J(\lambda_{\mathrm{e}})J(\lambda_{\mathrm{e}}R^{\perp}) = V(R^{\perp})V(R),$$

where $\mathbf{e} \in p$, $\mathbf{e}_{\perp} \in p^{\perp}$ are unit vectors and λ_{e}, $\lambda_{\mathrm{e}_{\perp}}$ are the corresponding reflections, which commute. If $\tilde{R}$ is isocline, then it has infinitely many invariant planes ([16], Art. 112), which form series of isocline planes ([16], Art. 105). Next we will show that the above definition is actually independent of the choice of invariant planes for an isocline rotation. Suppose that $\tilde{R} = RR^{\perp}$ is isocline, and that for a given series of isocline invariant planes of $\tilde{R}$, an orthonormal basis of $\mathbb{E}_4$ is chosen such that $p = span\{\mathbf{e}_1, \mathbf{e}_2\}$, and $p^{\perp} = span\{\mathbf{e}_3, \mathbf{e}_4\}$. All the invariant planes in the series are isocline to p in the same sense ([16], Art. 112), i.e., with the above choice of the basis, if q $q^{\perp}$ are invariant planes of $\tilde{R}$, then there is an isocline rotation $\tilde{S} = SS^{\perp} = S^{\perp}S$, such that S fixes each point of $span\{\mathbf{e}_2, \mathbf{e}_4\}$, $S^{\perp}$ fixes each point of $span\{\mathbf{e}_1, \mathbf{e}_3\}$, $\tilde{S}p = q$, $\tilde{S}p^{\perp} = q^{\perp}$, and $\tilde{S}\tilde{R}\tilde{S}^{-1} = \tilde{R}$.

Since $\tilde{S}R\tilde{S}^{-1}$, $\tilde{S}R^{\perp}\tilde{S}^{-1}$ are simple rotations, one has

$$V(\tilde{S}R\tilde{S}^{-1}) = J(\tilde{S}\lambda_{e_\perp}\tilde{S}^{-1})J(\tilde{S}\lambda_{e_\perp}R\tilde{S}^{-1})$$
$$V(\tilde{S}R^{\perp}\tilde{S}^{-1}) = J(\tilde{S}\lambda_{e}\tilde{S}^{-1})J(\tilde{S}\lambda_{e}R^{\perp}\tilde{S}^{-1})$$

Since $\tilde{S} = SS^{\perp}$, the relation (2–11) implies

$$J(\tilde{S}\lambda_{e}\tilde{S}^{-1}) = V(S)V(S^{\perp})J(\lambda_e)V(S^{\perp})^{-1}V(S)^{-1},$$

as well as analogous relations for the rest of the reflections in the above equations, yielding

$$V(\tilde{S}R\tilde{S}^{-1})V(\tilde{S}R^{\perp}\tilde{S}^{-1}) = V(S)V(S^{\perp})V(R)V(R^{\perp})V(S)^{-1}V(S^{\perp})^{-1}.$$

Since $\lambda_{e_\perp}\lambda_e$ and $\lambda_{e_\perp}R\lambda_e R^{\perp} = \tilde{R}^{-\frac{1}{2}}\lambda_{e_\perp}\lambda_e\tilde{R}^{\frac{1}{2}}$ are both simple rotations, one has

$$\begin{aligned}V(R)V(R^{\perp}) &= J(\lambda_{e_\perp})J(\lambda_{e_\perp}R)J(\lambda_e)J(\lambda_e R^{\perp})\\ &= J(\lambda_{e_\perp})J(\lambda_e)J(\lambda_{e_\perp}R)J(\lambda_e R^{\perp}) = V(\lambda_{e_\perp}\lambda_e)V(\lambda_{e_\perp}R\lambda_e R^{\perp}).\end{aligned}$$

The choice $\lambda_{e_\perp} = \lambda_{e_3}$, $\lambda_e = \lambda_{e_1}$ yields that the rotation $\lambda_{e_3}\lambda_{e_1}$ commutes with both S and $S^{\perp}$, so that one finally has

$$\begin{aligned}V(\tilde{S}R\tilde{S}^{-1})V(\tilde{S}R^{\perp}\tilde{S}^{-1}) &= V(S)V(S^{\perp})V(R)V(R^{\perp})V(S)^{-1}V(S^{\perp})^{-1}\\
&= V(S)V(S^{\perp})V(\lambda_{e_3}\lambda_{e_1})V(\tilde{R}^{-\frac{1}{2}}\lambda_{e_3}\lambda_{e_1}\tilde{R}^{\frac{1}{2}})V(S)^{-1}V(S^{\perp})^{-1}\\
&= V(\lambda_{e_3}\lambda_{e_1})V(S)V(S^{\perp})V(\tilde{R}^{-\frac{1}{2}}\lambda_{e_3}\lambda_{e_1}\tilde{R}^{\frac{1}{2}})V(S)^{-1}V(S^{\perp})^{-1}\\
&= V(\lambda_{e_3}\lambda_{e_1})V(S)V(S^{\perp})J(\tilde{R}^{-\frac{1}{2}}\lambda_{e_3}\tilde{R}^{\frac{1}{2}})J(\tilde{R}^{-\frac{1}{2}}\lambda_{e_1}\tilde{R}^{\frac{1}{2}})V(S)^{-1}V(S^{\perp})^{-1}\\
&= J(\lambda_{e_3})J(\lambda_{e_1})J(\tilde{S}\tilde{R}^{-\frac{1}{2}}\lambda_{e_3}\tilde{R}^{\frac{1}{2}}\tilde{S}^{-1})J(\tilde{S}\tilde{R}^{-\frac{1}{2}}\lambda_{e_1}\tilde{R}^{\frac{1}{2}}\tilde{S}^{-1})\\
&= V(\lambda_{e_3}\lambda_{e_1})V(\tilde{S}\tilde{R}^{-\frac{1}{2}}\lambda_{e_3}\lambda_{e_1}\tilde{R}^{\frac{1}{2}}\tilde{S}^{-1})\\
&= V(\lambda_{e_3}\lambda_{e_1})V(\tilde{R}^{-\frac{1}{2}}\lambda_{e_3}\lambda_{e_1}\tilde{R}^{\frac{1}{2}}) = V(R)V(R^{\perp}).\end{aligned}$$

Therefore, we may drop the subscript p in the above definition and put

$$V(\tilde{R}) := V(R)V(R^{\perp}) = J(\lambda_{e_{\perp}})J(\lambda_{e_{\perp}}R)J(\lambda_{e})J(\lambda_{e}R^{\perp}),$$

where $\mathbf{e} \in p$, $\mathbf{e}_{\perp} \in p^{\perp}$ for some pair p, $p^{\perp}$ of invariant planes of $\tilde{R}$.

We will now prove that the map $V : \mathcal{L}_{+}^{\uparrow} \to \mathcal{J}$, defined by

$$V(\Lambda) := V(R)V(B),$$

where $\Lambda = RB$ is the polar decomposition of Λ, is a homomorphism. First we follow [11] to show

$$V(B_1B_2) = V(B_1)V(B_2), \quad V(R_1R_2) = V(R_1)V(R_2), \tag{2–12}$$

where B_1, B_2 are boosts, and R_1, R_2, as well as R_1R_2 are simple rotations. If $\mathbf{e} \in \mathbb{E}_4$ is a unit vector orthogonal to the directions of both B_1 and B_2, then

$$V(B_1)V(B_2) = J(B_1\lambda_e)J(\lambda_e)J(\lambda_e)J(\lambda_eB_2) = J(B_1\lambda_e)J(\lambda_eB_2) = V(B_1B_2),$$

where the last equality follows from the fact that $B_1B_2 = B_1^{\frac{1}{2}}(B_1^{\frac{1}{2}}B_2B_1^{\frac{1}{2}})B_1^{-\frac{1}{2}}$, i.e. B_1B_2 is conjugate to the boost $B_1^{\frac{1}{2}}B_2B_1^{\frac{1}{2}}$ by the boost $B_1^{\frac{1}{2}}$, so Lemma 2.3.8 applies to it.

If R_1, R_2 and R_1R_2 are simple rotations, then the planes p_1 p_2 of points fixed by R_1 and R_2, respectively, intersect in a line ([16], Art. 102, Theorem 2), so there is a unit vector $\mathbf{e} \in \mathbb{E}_4$, orthogonal to both p_1 and p_2. Then

$$V(R_1)V(R_2) = J(R_1\lambda_e)J(\lambda_e)J(\lambda_e)J(\lambda_eR_2) = J(R_1\lambda_e)J(\lambda_eR_2) = V(R_1R_2),$$

where the last equality again follows by Lemma 2.3.8.

If R_1 and R_2 are nontrivial simple rotations such that R_1R_2 is a double rotation, then $p_1 \cap p_2 = \{0\}$, and $R_1R_2 = RR^{\perp}$, where R and $R^{\perp}$ are rotations fixing each point in the invariant planes p, $p^{\perp}$, respectively.

If $p_2 = p_1^\perp$, then we may put $R_1 = R$, $R_2 = R^\perp$, and $p_1 = p$, $p_2 = p^\perp$. Then, by the definition, one has

$$V(R_1R_2) = V(R_1)V(R_2).$$

If $p_2 \neq p_1^\perp$, then there exists a unit vector $\mathbf{e}_1 \in p_1$ which does not lie in p or in $p^\perp$. If $\mathbf{u} \in p$ and $\mathbf{u}_\perp \in p^\perp$ are unit vectors such that $\mathbf{e}_1 = \alpha\mathbf{u} + \beta\mathbf{u}_\perp$, ($\alpha \neq 0$, $\beta \neq 0$), then for the reflections λ_{e_1}, λ_{u}, $\lambda_{\mathrm{u}_\perp}$ it follows $R\lambda_{\mathrm{u}} = \lambda_{\mathrm{u}}R$, $R\lambda_{\mathrm{u}_\perp} = \lambda_{\mathrm{u}_\perp}R^{-1}$, $R^\perp\lambda_{\mathrm{u}_\perp} = \lambda_{\mathrm{u}_\perp}R^\perp$, $R^\perp\lambda_{\mathrm{u}} = \lambda_{\mathrm{u}}R^{\perp-1}$, $\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp} = \lambda_{\mathrm{u}_\perp}\lambda_{\mathrm{u}}$, and that $\lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp}$ is a reflection. Then, since $R_1R_2 = RR^\perp$, one has, with $\mathbf{e}_2$ being a unit vector in $p_2^\perp$:

$$\lambda_{\mathrm{e}_1}R_1R_2\lambda_{\mathrm{e}_2} = \lambda_{\mathrm{e}_1}RR^\perp\lambda_{\mathrm{e}_2} = \lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}^2\lambda_{\mathrm{u}_\perp}^2RR^\perp\lambda_{\mathrm{e}_2} = (\lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp})(\lambda_{\mathrm{u}_\perp}R)(\lambda_{\mathrm{u}}R^\perp)\lambda_{\mathrm{e}_2}.$$

Since $(\lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp})$ is a reflection, the rotations $(\lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp})(\lambda_{\mathrm{u}_\perp}R)$ and $(\lambda_{\mathrm{u}}R^\perp)\lambda_{\mathrm{e}_2}$ are simple, as is their composition, $\lambda_{\mathrm{e}_1}R_1R_2\lambda_{\mathrm{e}_2}$, so (2–12) applies, yielding

$$\begin{aligned}
V(R_1)V(R_2) &= J(\lambda_{\mathrm{e}_1})J(\lambda_{\mathrm{e}_1}R_1)J(R_2\lambda_{\mathrm{e}_2})J(\lambda_{\mathrm{e}_2}) \\
&= J(\lambda_{\mathrm{e}_1})V(\lambda_{\mathrm{e}_1}R_1R_2\lambda_{\mathrm{e}_2})J(\lambda_{\mathrm{e}_2}) \\
&= J(\lambda_{\mathrm{e}_1})V(\lambda_{\mathrm{e}_1}RR^\perp\lambda_{\mathrm{e}_2})J(\lambda_{\mathrm{e}_2}) \\
&= J(\lambda_{\mathrm{e}_1})V((\lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp})(\lambda_{\mathrm{u}_\perp}R))V((\lambda_{\mathrm{u}}R^\perp)\lambda_{\mathrm{e}_2})J(\lambda_{\mathrm{e}_2}) \\
&= J(\lambda_{\mathrm{e}_1})J(\lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp})J(\lambda_{\mathrm{u}_\perp}R)J(\lambda_{\mathrm{u}}R^\perp)J(\lambda_{\mathrm{e}_2})^2.
\end{aligned}$$

Since $\lambda_{\mathrm{e}_1}(\lambda_{\mathrm{e}_1}\lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp}) = \lambda_{\mathrm{u}}\lambda_{\mathrm{u}_\perp}$ is a simple rotation, we finally have:

$$\begin{aligned}
V(R_1)V(R_2) &= J(\lambda_{\mathrm{u}})J(\lambda_{\mathrm{u}_\perp})J(\lambda_{\mathrm{u}_\perp}R)J(\lambda_{\mathrm{u}}R^\perp) \\
&= J(\lambda_{\mathrm{u}_\perp})J(\lambda_{\mathrm{u}_\perp}R)J(\lambda_{\mathrm{u}})J(\lambda_{\mathrm{u}}R^\perp) = V(R)V(R^\perp) = V(R_1R_2).
\end{aligned}$$

Next assume that R_1 is a simple rotation and $\tilde{R}_2 = R_2R_2^\perp$ is a double rotation, with R_2, $R_2^\perp$ fixing each point of p_2, $p_2^\perp$, respectively. If $R_3 := R_1\tilde{R}_2$ is simple, then R_1^{-1} and R_3 are simple rotations whose composition is the double rotation $\tilde{R}_2$, so

$$V(\tilde{R}_2) = V(R_1^{-1})V(R_3) = V(R_1)^{-1}V(R_3), \text{ implying that}$$
$$V(R_1\tilde{R}_2) = V(R_3) = V(R_1)V(\tilde{R}_2).$$

If $\tilde{R}_3 := R_1\tilde{R}_2$ is a double rotation, then let p_1 be the plane whose each point is fixed by R_1, and suppose $dim(p_1 \cap p_2) > 0$ or $dim(p_1 \cap p_2^{\perp}) > 0$. The former case implies that R_1R_2 is simple. Since its composition with the simple rotation $R^{\perp}$ is the double rotation $\tilde{R}_3$, one has

$$V(R_1)V(\tilde{R}_2) = V(R_1)V(R_2)V(R_2^{\perp}) = V(R_1R_2)V(R_2^{\perp}) = V(\tilde{R}_3) = V(R_1R_2).$$

Since $R_2R_2^{\perp} = R_2^{\perp}R_2$, the latter case yields the same result in an analogous way.

Now suppose $dim(p_1 \cap p_2) = 0 = dim(p_1 \cap p_2^{\perp})$, and choose unit vectors $\mathbf{e}_1 \in p_1^{\perp}$ and $\mathbf{u}_2 \in p_2$, $\mathbf{u}_{2\perp} \in p_2^{\perp}$, such that $\mathbf{e}_1 \in span\{\mathbf{u}_2, \mathbf{u}_{2\perp}\}$. Then $R_1\lambda_{\mathrm{u}_2}\lambda_{\mathrm{u}_{2\perp}}$ is a simple rotation, and one has

$$\begin{aligned} V(\tilde{R}_3) = V(R_1\tilde{R}_2) &= V((R_1\lambda_{\mathrm{u}_2}\lambda_{\mathrm{u}_{2\perp}})(\lambda_{\mathrm{u}_2}R_2^{\perp}\lambda_{\mathrm{u}_{2\perp}}R_2)) \\ &= V(R_1\lambda_{\mathrm{u}_2}\lambda_{\mathrm{u}_{2\perp}})V((\lambda_{\mathrm{u}_2}R_2^{\perp})(\lambda_{\mathrm{u}_{2\perp}}R)) \\ &= V(R_1)J(\lambda_{\mathrm{u}_2})J(\lambda_{\mathrm{u}_{2\perp}})J(\lambda_{\mathrm{u}_2}R_2^{\perp})J(\lambda_{\mathrm{u}_{2\perp}}R) = V(R_1)V(\tilde{R}_2). \end{aligned}$$

Since $V(R^{-1}) = V(R)^{-1}$ for any rotation, one also has $V(\tilde{R}_2R_1) = V(R_1^{-1}\tilde{R}_2^{-1})^{-1} = (V(R_1)^{-1}V(\tilde{R}_2^{-1}))^{-1} = V(\tilde{R}_2)V(R_1)$.

If both $\tilde{R}_1 = R_1R_1^{\perp}$and $\tilde{R}_2$ are double, then, since $R_1^{\perp}\tilde{R}_2$ is either simple, or double, one has

$$V(\tilde{R}_1\tilde{R}_2) = V(R_1(R_1^{\perp}\tilde{R}_2)) = V(R_1)V(R_1^{\perp}\tilde{R}_2) = V(R_1)V(R_1^{\perp})V(\tilde{R}_2) = V(\tilde{R}_1)V(\tilde{R}_2).$$

Therefore, the map V restricted to rotations is a homomorphism. We now define for any element $\Lambda \in \mathcal{L}_+^\uparrow$ with the polar decomposition $\Lambda = RB$

$$V(\Lambda) := V(R)V(B)$$

and show that this map is a homomorphism: if $\Lambda_1, \Lambda_2 \in \mathcal{L}_+^\uparrow$ have the polar decomposition $\Lambda_1 = R_1 B_1$, $\Lambda_2 = R_2 B_2$, and if $R_2^{-1} B_1 R_2 B_2 = R_3 B_3$, then

$$\begin{aligned}
V(\Lambda_1 \Lambda_2) &= V(R_1 R_2 (R_2^{-1} B_1 R_2 B_2)) \\
&= V(R_1 R_2 R_3 B_3) \\
&= V(R_1) V(R_2) V(R_3) V(B_3) \\
&= V(R_1) V(R_2) V(R_3 B_3) \\
&= V(R_1) V(R_2) V((R_2^{-1} B_1 R_2) B_2) \\
&= V(R_1) V(R_2) V((R_2^{-1} B_1 R_2)) V(B_2) \\
&= V(R_1) V(R_2) V(R_2^{-1}) V(B_1) V(R_2) V(B_2) \\
&= V(R_1) V(B_1) V(R_2) V(B_2) = V(\Lambda_1) V(\Lambda_2).
\end{aligned}$$

This, in particular, implies that for any two reflections λ_1, λ_2 with polar decomposition $i_1 B_1$ and $i_2 B_2$, respectively, one has

$$\begin{aligned}
V(\lambda_1 \lambda_2) &= V(B_1^{-1} i_1 i_2 B_2) \\
&= V(B_1^{-1}) V(i_1 i_2) V(B_2) = V(B_1^{-1}) J(i_1) J(i_2) V(B_2) \\
&= J(B_1^{-1} i_1) J(i_1)^2 J(i_2)^2 J(i_2 B_2) = J(\lambda_1) J(\lambda_2).
\end{aligned}$$

Proceeding to extend the definition of V to the transformations in $\mathcal{L}_+^\downarrow$, we follow [11], fix a reflection $\lambda_0 \in \mathcal{L}_+^\downarrow$, so that each $\Lambda' \in \mathcal{L}_+^\downarrow$ has the form $\Lambda' = \lambda_0 \Lambda$, for some $\Lambda \in \mathcal{L}_+^\uparrow$, and define

$$V(\Lambda') := J(\lambda_0) V(\Lambda).$$

Since for any $\Lambda \in \mathcal{L}_+^\uparrow$ one has $J(\lambda_0)V(\Lambda)J(\lambda_0) = V(\lambda_0\Lambda\lambda_0)$, it follows that

$$\begin{aligned}
V(\lambda_0\Lambda)V(\Lambda_1) &= J(\lambda_0)V(\Lambda)V(\Lambda_1) = V(\lambda_0\Lambda\Lambda_1),\\
V(\Lambda_1)V(\lambda_0\Lambda) &= V(\Lambda_1)J(\lambda_0)V(\Lambda) = J(\lambda_0)^2V(\Lambda_1)J(\lambda_0)V(\Lambda)\\
&= J(\lambda_0)V(\lambda_0\Lambda_1\lambda_0)V(\Lambda) = J(\lambda_0)V(\lambda_0\Lambda_1\lambda_0\Lambda) = V(\Lambda_1\lambda_0\Lambda),\\
V(\lambda_0\Lambda_1)V(\lambda_0\Lambda) &= J(\lambda_0)V(\Lambda_1)J(\lambda_0)V(\Lambda) = V(\lambda_0\Lambda_1\lambda_0)V(\Lambda) = V(\lambda_0\Lambda_1\lambda_0\Lambda),
\end{aligned}$$

for any $\Lambda, \Lambda_1 \in \mathcal{L}_+^\uparrow$, and that, for any reflection $\lambda \in \mathcal{L}_+$,

$$V(\lambda) = V(\lambda_0\lambda_0\lambda) = J(\lambda_0)V(\lambda_0\lambda) = J(\lambda_0)J(\lambda_0)J(\lambda) = J(\lambda).$$

Thus every continuous reflection map on $\mathcal{L}_+$ extends to a unique homomorphism (since the reflections generate $\mathcal{L}_+$). Summarizing, we have the following analogue of Proposition 2.8 of the work [11]:

2.3.10 Proposition. *Let J be a continuous reflection map from the set $\mathcal{R}$ of all reflections in the de Sitter group $\mathcal{L}_+$ (acting on $\mathcal{M}_5$) to a topological group $\mathcal{J}$. Then J is the restriction to $\mathcal{R}$ of a unique continuous homomorphism of $\mathcal{L}_+$ to $\mathcal{J}$.*

In the remainder of this section it will be shown that CGMA imposed on the pair $(\Omega, \{\mathcal{R}(W)\}_{W\in\mathcal{W}})$ in dS^4 together with certain additional condition (which is satisfied by a large class of quantum field theories) imply the existence of a continuous reflection map

$$J : \mathcal{R} \to \mathcal{J},$$

from the set $\mathcal{R}$ of all the reflections over the edges of wedges in the 4-dimensional de Sitter space dS^4 to the group $\mathcal{J}$ generated by anti-unitary operators on the Hilbert space $\mathcal{H}$, endowed with the strong*-topology.

In Proposition 4.6 of the work [7] it has been shown for the case of 4-dimensional Minkowski space $\mathcal{M}_4$ that the map

$$\tilde{\mathcal{W}} \ni W \mapsto J_W \in \mathcal{J},$$

where J_W is the modular involution associated with $(\Omega, \mathcal{R}(W))$, is continuous when $\tilde{\mathcal{W}}$ is, as an orbit of a transitive action of the identity component $\mathcal{P}_+^\uparrow$ of the Poincaré group, endowed with the quotient topology of $\mathcal{P}_+^\uparrow/\mathcal{P}_0$ with $\mathcal{P}_0$ being the invariance subgroup of $\mathcal{P}_+^\uparrow$ for a fixed wedge W_0, the group $\mathcal{J}$ is endowed with the strong*-topology and the following Net Continuity Condition [7] is satisfied by the net $\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}$:

Net Continuity Condition. For any $W \in \tilde{\mathcal{W}}$ and any continuous collection $\{W_\delta\}_{\delta>0}$ converging to W (as $\delta \to 0$), the net $\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}$ satisfies

$$\mathcal{R}(W) = \left(\bigcup_{\varepsilon>0} \mathcal{R}(I_\varepsilon)\right)'' = \bigcap_{\varepsilon>0} \mathcal{R}(A_\varepsilon),$$

where, for $\varepsilon > 0$,

$$A_\varepsilon := \bigcup_{0\le\delta<\varepsilon} W_\delta, \qquad I_\varepsilon := \bigcap_{0\le\delta<\varepsilon} W_\delta,$$
$$\mathcal{R}(A_\varepsilon) := (\bigcup_{0\le\delta<\varepsilon} \mathcal{R}(W_\delta))'', \qquad \mathcal{R}(I_\varepsilon) := \bigcap_{0\le\delta<\varepsilon} \mathcal{R}(W_\delta),$$

with $W_0 := W$. Moreover, there exists an $\varepsilon_0 > 0$ such that Ω is cyclic for the algebras $\mathcal{R}(I_\varepsilon)$, with $0 < \varepsilon < \varepsilon_0$.

In [11] it has been shown that the Net Continuity Condition is satisfied if the net $\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}$ satisfies the wedge duality, i.e. $\mathcal{R}(W') = \mathcal{R}(W)'$, for each $W \in \tilde{\mathcal{W}}$, and is locally generated in the following sense:

2.3.11 Definition. The net $\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}$ is said to be locally generated if there is a family $\mathcal{C}$ (called a generating family) of regions in $\mathbb{R}^4$ satisfying

(a) each $C \in \mathcal{C}$ can be approximated from the outside by wedges, i.e. $C = \bigcap_{W\Supset C} W$, where $W \Supset C$ means that there is an open neighborhood of W in $\tilde{\mathcal{W}}$ all of whose elements contain C,

(b) each wedge $W \in \tilde{\mathcal{W}}$ can be approximated from the inside by elements of $\mathcal{C}$, i.e. $W = \bigcup_{C\Subset W} C$,

(c) the family $\mathcal{C}$ is stable under the action of $\mathcal{P}_+^\uparrow$

and

$$\mathcal{R}(W) = \bigvee_{C \Subset W} \mathcal{R}(C), \qquad W \in \tilde{\mathcal{W}}$$

where $\mathcal{R}(C) := \bigwedge_{W \Supset C} \mathcal{R}(W)$ and the vector Ω is cyclic for each $\mathcal{R}(C)$, $C \in \mathcal{C}$.

Nets of wedge algebras associated with quantum field theories in Minkowski space satisfying Wightman axioms were shown to be locally generated in [17] with the generating family $\tilde{\mathcal{C}}$ consisting of closed double cones. Since in the present case of dS^4 the set $\mathcal{W}$ of wedges consists of the regions $W = dS^4 \cap W_0$, where W_0 is a wedge in the ambient 5-dimensional Minkowski space $\mathcal{M}_5$ such that the edge of W_0 contains the origin, the family $\mathcal{C} := \{K \cap dS^4 | K \in \tilde{\mathcal{C}}\}$ is a generating family for $\mathcal{W}$, so the results of [17] yield an example of locally generated nets of algebras associated with quantum field theories in dS^4.

If the set $\mathcal{W}$ of wedges in dS^4 is endowed with the quotient topology of $\mathcal{L}_+^\uparrow / \mathcal{L}_0$, where $\mathcal{L}_0$ is the invariance subgroup of $\mathcal{L}_+^\uparrow$ for a fixed wedge W_0, then an examination of Proposition 4.6 in [7], Proposition 3.1 and Corollary 3.2 in [11] shows that the arguments given in their proofs also apply in the present case of dS^4 (with obvious modifications). One therefore has the following analogues.

2.3.12 Proposition (Proposition 3.1 in [11]). *If $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ is a locally generated net satisfying the wedge duality, i.e. $\mathcal{R}(W') = \mathcal{R}(W)'$ for all $W \in \mathcal{W}$, then the map $W \mapsto J_W$ from the wedges $W \in \mathcal{W}$ to the modular conjugations J_W corresponding to $(\mathcal{R}(W), \Omega)$ is continuous.*

2.3.13 Corollary (Corollary 3.2 in [11]). *If $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ is a locally generated net satisfying the wedge duality, i.e. $\mathcal{R}(W') = \mathcal{R}(W)'$ for all $W \in \mathcal{W}$, then the map $\lambda \mapsto J(\lambda)$ from the reflections $\lambda \in \mathcal{L}_+$ to the modular conjugations $J(\lambda)$ corresponding to $(\mathcal{R}(W), \Omega)$, hence (by the wedge duality) also to $(\mathcal{R}(W'), \Omega)$, is continuous, the wedges W, W' being fixed by the condition $\lambda W = W'$.*

Since the CGMA implies the wedge duality as well as the relation (2–4), the above results imply that $\lambda \mapsto J(\lambda)$ is a continuous reflection map and combined with

Theorem 2.2.2 and Proposition 2.3.10 they yield the following analogue of Theorem 4.1 in [11]:

2.3.14 Theorem. *If $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is a locally generated net and Ω is a state vector satisfying CGMA, then the net satisfies the wedge duality and there is a continuous (anti)unitary representation V of the de Sitter group $\mathcal{L}_+$ which leaves Ω invariant and acts covariantly on the net. For any given wedge W and the reflection λ about its edge, $V(\lambda)$ is the modular involution J_W corresponding to the pair $(\mathcal{R}(W), \Omega)$.*

It has also been shown in [11], Lemma 3.3 that, in the presence of a generating family $\mathcal{C}$ such that Ω is cyclic for each $\mathcal{R}(C)$, $C \in \mathcal{C}$, the net $\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}$ is locally generated provided that $(\Omega, \{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}})$ satisfies CGMA and the Modular Stability Condition (CMS):

Modular Stability Condition. For any $W \in \tilde{\mathcal{W}}$, the elements Δ_W^{it}, $t \in \mathbb{R}$, of the modular group corresponding to $(\mathcal{R}(W), \Omega)$ are contained in the group generated by the modular involutions $\{J_W\}_{W\in\tilde{\mathcal{W}}}$.

The modular stability obtains (along with the geometric action of the modular objects on the net) in quantum field theories satisfying Wightman axioms, as demonstrated by Bisognano and Wichmann in [3]. It is shown in [7], Theorem 5.1. that for quantum field theories satisfying CGMA in $\mathcal{M}_4$ with transitive adjoint action of the modular conjugations J_W, $W \in \tilde{\mathcal{W}}$ on the net $\{\mathcal{R}(W)\}_{W\in\tilde{\mathcal{W}}}$ the modular stability implies that the unitary representation U of the translation group (generated by suitable products of modular conjugations) satisfies the relativistic spectrum condition, i.e. that the joint spectrum $sp(U)$ is in the forward closed lightcone, suggesting that CMS be used as a stability condition on spacetimes with no timelike Killing vector. With this in mind and noting that the arguments in the proof of Lemma 3.3 of [11] also apply to the case of de Sitter space, we consider the following analogue of this lemma (see the cited reference for the proof):

2.3.15 Lemma (Lemma 3.3 in [11]). *Let $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$ be a pair satisfying CGMA and CMS. If there is a generating family $\mathcal{C}$ of regions such that for each $C \in \mathcal{C}$ the vector Ω is cyclic for $\mathcal{R}(C)$, then the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ is locally generated.*

The above lemma may be used to reformulate Theorem 2.3.14 and obtain an analogue of Theorem 4.2 of [11]:

2.3.16 Theorem. *Let $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be a net and Ω a state vector satisfying CGMA and CMS, and let $\mathcal{C}$ be some generating family of regions such that Ω is cyclic for the algebras $\mathcal{R}(C)$, $C \in \mathcal{C}$. The net satisfies the wedge duality and there is a continuous (anti)unitary representation V of the de Sitter group $\mathcal{L}_+$ which leaves Ω invariant and acts covariantly on the net. For any given wedge W and the reflection λ about its edge, $V(\lambda)$ is the modular involution J_W corresponding to the pair $(\mathcal{R}(W), \Omega)$.*

CHAPTER 3
GEOMETRIC MODULAR ACTION IN 5-DIMENSIONAL MINKOWSKI SPACE

We will study CGMA and its implications in 5-dimensional Minkowski space $\mathcal{M}_5$. Although this space is not of a direct physical interest, physically interesting 4-dimensional manifolds are embedded in it (e.g. de Sitter space, anti-de Sitter space). We will use results of Chapter 2 as well as those of Section 4.1 in [7] and [8] to establish the results (1)-(3) of the CGMA program (see the Introduction) for $\mathcal{M}_5$.

3.1 Wedge Transformations are Induced by Poincaré Transformations

We will use the formulation of the CGMA on $(\mathcal{M}, g)$ stated in the Introduction. Choosing the set of all wedges in $\mathcal{M}_5$ for the set of admissible regions $\mathcal{W}$ we use the following notation (see [7]):

$$W(\ell, \ell', d) = \{\alpha\ell - \beta\ell' + \gamma\ell_\perp + d, \quad \alpha > 0, \beta > 0, \quad \gamma \in \mathbb{R},$$
$$\ell \cdot \ell_\perp = 0, \quad \ell' \cdot \ell_\perp = 0\} \in \mathcal{W},$$

where ℓ, ℓ' are two non-parallel future-oriented lightlike vectors and $d \in \mathcal{M}_5$ is a translation vector. For a given future-oriented lightlike vector $\ell \in \mathcal{M}_5$ and a given real number p

$$H_p(\ell)^\pm := \{x \in \mathcal{M}_5 | \pm (x \cdot \ell - p) > 0\}$$

will be called characteristic half-spaces determined by ℓ and p. The boundary of either of these half-spaces

$$H_p(\ell) := \partial H_p(\ell)^\pm = \overline{H_p(\ell)^+} \cap \overline{H_p(\ell)^-} = \{x \in \mathcal{M}_5 | x \cdot \ell = p\}$$

is the characteristic hyperplane determined by ℓ and p. Any lightlike vector parallel to $H_p(\ell)$ is parallel to ℓ and any vector parallel to $H_p(\ell)$ and not parallel to ℓ is spacelike. For two nonzero, non-parallel lightlike vectors ℓ_1, ℓ_2 and $p_1, p_2 \in \mathbb{R}$, the intersection $H^+_{p_1}(\ell_1) \cap H^+_{p_2}(\ell_2)$ is a wedge and every wedge has that form. Note also that for each wedge $W \in \mathcal{W}$ there are future-directed lightlike vectors (unique up to a positive multiple) ℓ_+, ℓ_- such that $W \pm \ell_\pm \subset W$, and that the half-spaces $H^\pm$ such that $H^+ \cap H^- = W$ can be expressed as follows:

$$H^\pm = \bigcup_{\lambda \in \mathbb{R}} (W + \lambda \ell_\pm).$$

The families $\mathcal{F}^\pm := \{W + \lambda \ell_\mp | \lambda \in \mathbb{R}\}$ have the following properties:

(i) $\mathcal{F}^\pm$ is linearly ordered, i.e. if $W_1, W_2 \in \mathcal{W}$, then either $W_1 \subset W_2$ or $W_2 \subset W_1$.

(ii) $\mathcal{F}^\pm$ is maximal in the sense that if $W_1, W_2 \in \mathcal{W}$ satisfy $W_1 \subset W_2$ and there is a wedge $W \in \mathcal{W}$ such that $W_1 \subset W \subset W_2$, then $W \in \mathcal{F}^\pm$.

(iii) $\mathcal{F}^\pm$ has no upper or lower bound in the partially ordered set $(\mathcal{W}, \subset)$.

Lemma 4.1 in [7] shows that families of wedges satisfying (i)-(iii) (called characteristic families) have the same form as $\mathcal{F}^\pm$ above:

3.1.1 Lemma (Lemma 4.1 in[7])**.** *Every characteristic family of wedges $\mathcal{F}$ has the form*

$$\mathcal{F} := \{W + \lambda \ell | \lambda \in \mathbb{R}\},$$

for some wedge $W \in \mathcal{W}$ and some future-directed lightlike vector ℓ with the property that $W + \ell \subset W$ or $W - \ell \subset W$.

This lemma relates the order (by inclusion) properties of wedges implied by CGMA to the geometrical description of wedges (in terms of lightlike vectors, half-spaces and hyperplanes) and is a first step in the construction of the Poincaré transformations inducing the wedge transformations τ_W, $W \in \mathcal{W}$ (see below).

It was shown in Section 4.1 in [7] that under CGMA in $\mathcal{M}_4$ for any wedge W in $\mathcal{M}_4$ the associated involution τ_W (see the Introduction) is induced by an element of Poincaré group. An examination of the arguments provided there shows that many of them are also applicable in the present case. In a way similar to that of Section 2.1 for the case of dS^4 we list here the crucial steps in the construction of the Poincaré transformation inducing τ_W, $W \in \mathcal{W}$ pointing out where non-obvious modifications had to be made for $\mathcal{M}_5$. We omit the proofs of the analogues of statements of [7] where transfer of the argument from $\mathcal{M}_4$ to $\mathcal{M}_5$ is immediate. For those proofs see Section 4.1 in [7].

CGMA implies the following properties of the involutions τ_W, $W \in \mathcal{W}$:

3.1.2 Proposition (see Proposition 3.1 in [7])**.** *If* $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$ *satisfies the CGMA, then the involutions* τ_W, $W \in \mathcal{W}$ *satisfy, with* $W_1, W_2 \in \mathcal{W}$,

$$\overline{W_1} \cap \overline{W_2} = \emptyset \Rightarrow \tau_W(W_1) \cap \tau_W(W_2), \tag{3–1}$$

and

$$W_1 \subset W_2 \Leftrightarrow \tau_W(W_1) \subset \tau_W(W_2). \tag{3–2}$$

The following statements were proved for bijections (not necessarily involutive) on $\mathcal{W}$ satisfying (3–1) and (3–2).

3.1.3 Lemma (Lemma 4.2 in [7])**.** *Let* $\tau : \mathcal{W} \to \mathcal{W}$ *be a bijective map with the property* (3–2)*. Then* τ *maps every characteristic family* $\mathcal{F}$ *of wedges onto a characteristic family* $\tau(\mathcal{F}) := \{\tau(W)|W \in \mathcal{F}\}$*. In fact, if* $\mathcal{F}_1 = \{W_1 + \lambda\ell_1|\lambda \in \mathbb{R}\}$*, for some wedge* $W_1 \in \mathcal{W}$ *and some future-directed lightlike vector* ℓ_1 *with the property that* $W_1 + \ell_1 \subset W_1$ *or* $W_1 - \ell_1 \subset W_1$*, and if* $\tau(W_1) = W_2$*, then* $\tau(W_1 + \lambda\ell_1) = W_2 + f(\lambda)\ell_2$*, where* $f : \mathbb{R} \to \mathbb{R}$ *is continuous monotonic bijection,* $f(0) = 0$*, and* ℓ_2 *is some future-directed lightlike vector with the property that* $W_2 + \ell_2 \subset W_2$ *or* $W_2 - \ell_2 \subset W_2$*.*

Recall that for a characteristic family $\mathcal{F}$ of wedges $\cup_{W\in\mathcal{F}} W$ is a characteristic half-space. Verbatim analogues of Corollary 4.1 and Lemma 4.3, Corollary 4.2 and Lemma 4.6 in [7] then lead to defining the following map on the set $\tilde{\mathcal{H}}$ of characteristic half-spaces in $\mathcal{M}_5$.

3.1.4 Lemma (see Lemma 4.7 in [7])**.** *Let* $\tau : \mathcal{W} \to \mathcal{W}$ *be a bijection with the properties* (3–1) *and* (3–2)*. If* $H \in \tilde{\mathcal{H}}$*, then the assignment*

$$H \mapsto \bigcup_{W\in\mathcal{F}} \tau(W),$$

with $\mathcal{F}$ being any characteristic family generating H, is a well-defined map from $\tilde{\mathcal{H}}$ to $\tilde{\mathcal{H}}$ with the following properties:

(1) *τ is bijective on $\tilde{\mathcal{H}}$;*

(2) *if $H^c = \mathcal{M}_5 \setminus \overline{H}$, then $\tau(H^c) = \tau(H)^c$, for all $H \in \tilde{\mathcal{H}}$;*

(3) *for $H_1, H_2 \in \tilde{\mathcal{H}}$, $H_1 \cap H_2 = \emptyset$ if and only if $\tau(H_1) \cap \tau(H_2) = \emptyset$; moreover, $H_1 \subset H_2$ if and only if $\tau(H_1) \subset \tau(H_2)$;*

(4) *for a given $H \in \tilde{\mathcal{H}}$ and every element $a \in \mathcal{M}_5$ there exists an element $b \in \mathcal{M}_5$ (and vice versa) such that $\tau(H + a) = \tau(H) + b$;*

(5) *for any $W \in \mathcal{W}$, $W = H^+ \cap H^-$ if and only if $\tau(W) = \tau(H^+) \cap \tau(H^-)$;*

(6) *either $\tau(\tilde{\mathcal{H}}^\pm) = \tilde{\mathcal{H}}^\pm$ or $\tau(\tilde{\mathcal{H}}^\pm) = \tilde{\mathcal{H}}^\mp$, where $\tilde{\mathcal{H}}^\pm \subset \tilde{\mathcal{H}}$ denote the set of all future-directed and past-directed half-spaces $H^\pm$, respectively.*

Noting that the common boundary hyperplane for the half-spaces $H_p(\ell)^\pm$ is $\overline{H_p(\ell)^+} \cap \overline{H_p(\ell)^-}$, one has the following map on the set of all characteristic hyperplanes in $\mathcal{M}_5$.

3.1.5 Corollary (see Corollary 4.3 in [7])**.** *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection with the properties (3–1) and (3–2) and let $H\tau : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ be the associated mapping of characteristic half-spaces. The assignment*

$$H_p(\ell) \mapsto \overline{\tau(H_p(\ell)^+)} \cap \overline{\tau(H_p(\ell)^-)}$$

is a mapping of characteristic hyperplanes onto characteristic hyperplanes. This map, which will again be denoted by τ, has the following properties:

(1) *τ is bijective on the set of all characteristic hyperplanes of in $\mathcal{M}_5$;*

(2) *for given hyperplane $H_p(\ell)$ and every element $a \in \mathcal{M}_5$ there exists an element $b \in \mathcal{M}_5$ (and vice versa) such that $\tau(H_p(\ell) + a) = \tau(H_p(\ell)) + b$;*

(3) *τ maps distinct parallel characteristic hyperplanes onto distinct parallel characteristic hyperplanes.*

Using the notation of Section 2.2, putting $\ell_{i\pm} := \mathbf{e}_0 \pm \mathbf{e}_i$, $i = 1 \dots 4$ as well as $W_R := W(\ell_{1+}, \ell_{1-}, 0)$ for the right wedge and examining Lemma 4.4 and Lemma 4.5 in [7] one concludes that their claims are both valid also for the wedges W_R and

$W := W(\ell_{2+}, \ell, d)$, where $\ell = (1, a, b, c, f)$, $a^2 + b^2 + c^2 + f^2 = 1$, $b \neq 1$, with an additional condition that $f = 0$ (see also Lemma 4.1.1 in [13]). Consequently, Lemma 4.8 in [7] is valid as well as its corollary

3.1.6 Lemma (see also Corollary 4.1.12 in [13]). *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection with the properties* (3–1) *and* (3–2)*. If $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5$ are linearly dependent future-directed lightlike vectors such that any two of them are linearly independent, and such that*

$$span\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\} = span\{\ell_1, \ell_2, \ell_3, \ell_4\},$$

$$\textit{then} \quad \bigcap_{i=1}^{5} \tau(H_0(\ell_i)) = \bigcap_{i=1}^{4} \tau(H_0(\ell_i)).$$

Noting that the intersection of five characteristic hyperplanes $H_{p_i}(\ell_i)$, $i = 1 \dots 5$ where $\ell_1 \dots \ell_5$ are linearly independent is a singleton set we can use the preceding lemma to modify the proof of Lemma 4.9 in [7] to get its following analogue.

3.1.7 Lemma. *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection with the properties* (3–1) *and* (3–2) *and let τ be the associated mapping of the characteristic hyperplanes. The intersection $\cap\tau(H_0(\ell))$ taken over all future-directed lightlike vectors ℓ in $\mathcal{M}_5$ is a singleton set.*

Proof. By Corollary 3.1.5 τ maps parallel characteristic hyperplanes onto parallel characteristic hyperplanes, so there exist pairwise linearly independent future-directed lightlike vectors $\tilde{\ell}_1$, $\tilde{\ell}_2$, $\tilde{\ell}_3$, $\tilde{\ell}_4$, $\tilde{\ell}_5$ and real numbers c_1, c_2, c_3, c_4, c_5 such that $\tau(H_0(\ell_{1+})) = H_{c_1}(\tilde{\ell}_1)$, $\tau(H_0(\ell_{1-})) = H_{c_2}(\tilde{\ell}_2)$, $\tau(H_0(\ell_{2+})) = H_{c_3}(\tilde{\ell}_3)$, $\tau(H_0(\ell_{3+})) = H_{c_4}(\tilde{\ell}_4)$, $\tau(H_0(\ell_{4+})) = H_{c_5}(\tilde{\ell}_5)$. By part (2) of the Corollary 3.1.5 there are real numbers b_1, b_2, b_3, b_4, b_5 such that $\tau(H_{b_1}(\ell_{1+})) = H_0(\tilde{\ell}_1)$, $\tau(H_{b_2}(\ell_{1-})) = H_0(\tilde{\ell}_2)$, $\tau(H_{b_3}(\ell_{2+})) = H_0(\tilde{\ell}_3)$, $\tau(H_{b_4}(\ell_{3+})) = H_0(\tilde{\ell}_4)$, $\tau(H_{b_5}(\ell_{4+})) = H_0(\tilde{\ell}_5)$. If $\{\tilde{\ell}_i\}_{i=1\dots5}$ is linearly dependent, then Lemma 3.1.6 (if $span\{\tilde{\ell}_i\}_{i=1\dots5}$ is 4-dimensional) or Lemma 4.8 of [7] (if $span\{\tilde{\ell}_i\}_{i=1\dots5}$ is 3-dimensional) applied to τ^{-1} implies that also $\{\ell_{1+}, \ell_{1-}, \ell_{2+}, \ell_{3+}, \ell_{4+}\}$ is linearly dependent, a contradiction. Hence $\{\tilde{\ell}_i\}_{i=1\dots5}$ is linearly independent.

Since an arbitrary lightlike vector $\ell \neq 0$ is a linear combination of ℓ_{4+} and two other linearly independent lightlike vectors ℓ_1, ℓ_2 from $span\{\ell_{1+}, \ell_{1-}, \ell_{2+}, \ell_{3+}\}$, the relation

$$\tau(H_0(\ell_1)) \cap \tau(H_0(\ell_2)) \cap \tau(H_0(\ell_{4+})) \subset \tau(H_0(\ell))$$

follows by Lemma 4.8 of [7] if ℓ_1, ℓ_2, ℓ are pairwise linearly independent, or trivially otherwise (since in that case ℓ would be a multiple of one of the other two vectors). Since Lemma 3.1.6 implies

$$\tau(H_0(\ell_{1+})) \cap \tau(H_0(\ell_{1-})) \cap \tau(H_0(\ell_{2+})) \cap \tau(H_0(\ell_{3+})) \subset \tau(H_0(\ell_i))$$

for $i = 1, 2$, it follows for an arbitrary future-directed lightlike vector ℓ that

$$\bigcap_{i=1}^{5} H_{c_i}(\tilde{\ell}_i) = \tau(H_0(\ell_{1+})) \cap \tau(H_0(\ell_{1-})) \cap \tau(H_0(\ell_{2+})) \cap \tau(H_0(\ell_{3+})) \cap \tau(H_0(\ell_{4+})) \subset \tau(H_0(\ell)),$$

proving the claim. □

This leads to the definition of the following point transformation on $\mathcal{M}_5$.

3.1.8 Proposition (see Proposition 4.1 in [7]). *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection with the properties* (3–1) *and* (3–2) *and let τ be the associated mapping of the characteristic hyperplanes. For each $x \in \mathcal{M}_5$ and a hyperplane H let $T_x H := H + x$. The map $\delta : \mathcal{M}_5 \to \mathcal{M}_5$, defined by*

$$\delta(x) := \bigcap_{\ell} \tau(T_x H_0(\ell)) \text{ for } x \in \mathcal{M}_5,$$

where the intersection is over all future-directed lightlike vectors ℓ, is a bijection such that

$$\tau(W) = \{\delta(x) | x \in W\} \text{ for all } W \in \mathcal{W}.$$

Finally, showing that the bijection $\delta : \mathcal{M}_5 \to \mathcal{M}_5$ and its inverse map spacelike separated points onto spacelike separated points (Lemma 4.10 of [7]), and using the result of Alexandrov ([1]) stating that such a bijection is an element of the Poincaré group extended by dilations one obtains the following theorem.

3.1.9 Theorem (Theorem 4.1 in [7]). *Let $\tau : \mathcal{W} \to \mathcal{W}$ be a bijection with the properties* (3–1) *and* (3–2) *. Then there exists an element δ of the extended Poincaré group such that*

$$\tau(W) = \{\delta(x) | x \in W\} \textit{ for all } W \in \mathcal{W}.$$

Since the wedge transformations τ_W, $W \in \mathcal{W}$ induced by the CGMA are involutions satisfying (3–1) and (3–2) the preceding theorem implies

3.1.10 Theorem (see Corollary 4.4 in [7]). *If the pair $(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega)$ satisfies CGMA, then for every involution $\tau_{W_0} : \mathcal{W} \to \mathcal{W}$, $W_0 \in \mathcal{W}$ there exists an involutive Poincaré transformation g_{W_0} such that*

$$\tau_{W_0}(W) = \{g_{W_0}(x) | x \in W\} \textit{ for all } W \in \mathcal{W}.$$

Since the action of a g_W is completely determined by the action of τ_W, the assignment $\tau_W \mapsto g_W$, $W \in \mathcal{W}$ extends to an isomorphism of the group $\mathcal{T}$ generated by $\{\tau_W\}_{W \in \mathcal{W}}$ and a subgroup $\mathcal{G}$ of the Poincaré group $\mathcal{P}$ of $\mathcal{M}_5$ generated by $\{g_W\}_{W \in \mathcal{W}}$.

3.2 Wedge Transformations Generate Poincaré Group

Throughout this section we will consider CGMA on $\mathcal{M}_5$ with the additional condition (v) of transitivity of the adjoint action of the modular involutions $\{J_W\}_{W \in \mathcal{W}}$ on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ which implies that the action of the corresponding subgroup $\mathcal{G}$ of the Poincaré group $\mathcal{P}$ on the set $\mathcal{W}$ of all wedges in $\mathcal{M}_5$ is also transitive. Consequently all the generating involutions $\{g_W\}_{W \in \mathcal{W}}$ of $\mathcal{G}$ are conjugate in $\mathcal{G}$ so they all lie in the same connected component of $\mathcal{P}$. Since an even product of these involutions lies in the identity component $\mathcal{P}_+^\uparrow$ of $\mathcal{P}$ it follows that $\mathcal{G}$ intersects at most two connected components of $\mathcal{P}$. We will use the results of Section 2.2 to prove the following analogue of Proposition 4.4 in [7].

3.2.1 Theorem. *Let the pair $(\{\mathcal{R}(W)\}_{W \in \mathcal{W}}, \Omega)$ satisfy CGMA on $\mathcal{M}_5$ and let the adjoint action of the modular involutions $\{J_W\}_{W \in \mathcal{W}}$ on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ be transitive. If $\{g_W\}_{W \in \mathcal{W}}$ is the set of involutive transformations in the Poincaré group $\mathcal{P}$ induced by* *this action, and if $\mathcal{G}$ is the subgroup of $\mathcal{P}$ generated by these transformations, then for the right wedge $W_1 := W(\ell_{1+}, \ell_{1-}, 0)$ with $g_{W_1} = (\Lambda_{W_1}, d_{W_1})$ one has $d_{W_1} = 0$ and $\Lambda_{W_1} = diag(-1, -1, 1, 1, 1)$, and consequently for every wedge $W \in \mathcal{W}$ the corresponding g_W is the reflection through the edge of W. Moreover, $\mathcal{G} = \mathcal{P}_+$, the proper Poincaré group and every element of the identity component $\mathcal{P}_+^\uparrow$ of $\mathcal{P}$ can be obtained as a product of an even number of involutions g_W, $W \in \mathcal{W}$.*

Proof. First we prove the following preparatory lemma.

3.2.2 Lemma. *Let the hypothesis of Theorem 3.2.1 hold, let $\mathcal{G} \ni g = (\Lambda, d)$ and let $\sigma : \mathcal{G} \to \mathcal{L}$ be the projection homomorphism defined by $\sigma(g) = \Lambda$. Then $\sigma(\mathcal{G})$ contains the identity component $\mathcal{L}_+^\uparrow$ of the Lorentz group $\mathcal{L}$.*

Proof. Suppose first that $\mathcal{G} \leq \mathcal{P}_+^\uparrow$. Then $\sigma(\mathcal{G}) \leq \mathcal{L}_+^\uparrow$ is generated by the collection of involutions $\mathcal{C} = \{\sigma(g_W)\}_{W \in \mathcal{W}}$ and it has a transitive action on the set $\mathcal{W}_0$ of wedges whose edges contain the origin. If $i \in \mathcal{C}$ and $\Lambda \in \sigma(\mathcal{G})$ are such that $j := \Lambda i \Lambda^{-1} \neq i$ and such that they both preserve a wedge $W \in \mathcal{W}_0$, then the involutions $g_i := (i, d_i)$ and $g_j := (j, d_j)$ also preserve W, since the translation vectors d_i, d_j satisfy $id_i = -d_i$, $jd_j = -d_j$ and the eigenvectors i and j corresponding to eigenvalue -1 lie in the edge of W. Lemma 1.0.1 then implies, with $g_W := (i_W, d_W)$, that $[i_W, i] = 1 = [i_W, j]$, that i_W is conjugate to i in $\sigma(\mathcal{G})$, and that $i_W W \neq W$ (if $i_W W = W$, then $g_W W = W$, contradicting Lemma 1.0.1 (4) and the fact that the set $\mathcal{W}$ has no atoms). Hence the group $\sigma(\mathcal{G})$ satisfies the hypothesis of Lemma 2.2.5, implying that $\sigma(\mathcal{G}) = \mathcal{L}_+^\uparrow$.

If $\mathcal{G} \leq \mathcal{P}_+^\uparrow \cup \mathcal{P}_+^\downarrow$ and $\mathcal{G} \cap \mathcal{P}_+^\downarrow \neq \emptyset$, then $\sigma(\mathcal{G})$ satisfies the hypothesis of Lemma 2.2.7 so in this case it follows $\sigma(\mathcal{G}) \geq \mathcal{L}_+^\uparrow$.

If $\mathcal{G} \leq \mathcal{P}_+^\uparrow \cup \mathcal{P}_-^\uparrow$ and $\mathcal{G} \cap \mathcal{P}_-^\uparrow \neq \emptyset$, then $\sigma(\mathcal{G})$ satisfies the hypothesis of Lemma 2.2.9 so in this case it also follows $\sigma(\mathcal{G}) \geq \mathcal{L}_+^\uparrow$.

Finally, suppose $\mathcal{G} \leq \mathcal{P}_+^\uparrow \cup \mathcal{P}_-^\downarrow$ and $\mathcal{G} \cap \mathcal{P}_-^\downarrow \neq \emptyset$. We may choose the basis $\{\mathbf{e}_i\}_{i=0\ldots4}$ and the origin in $\mathcal{M}_5$ so that one of the generating involutions (i, d) has $d = 0$ and

$i = diag(-1, 1, -1, -1, 1)$. Using again the notation $W_i := W(\ell_{i+}, \ell_{i-}, 0)$, $i = 1, 2, 3, 4$ (see Section 2.2) note that $iW_1 = W_1$, so by Lemma 1.0.1 (2) one has $[(i, 0), g_{W_1}] = 1$ implying that, with $g_{W_1} = (\Lambda_1, d_1)$, also $[i, \Lambda_1] = 1$ and $d_1 \in span\{\mathbf{e}_1, \mathbf{e}_4\}$. Let $\Lambda_1 = I_1 B_1$, where $I_1 \in \mathcal{L}_-^\downarrow$, $I_1^2 = 1$, and B_1 is a boost. It follows that $[i, I_1] = 1 = [i, B_1]$. We may choose the basis (by applying $B_1^{\frac{1}{2}}$) so that $\Lambda_1 = I_1 \in \mathcal{L}_-^\downarrow$. Since $[i, I_1] = 1$, I_1 must leave the coordinate planes 14 and 23 invariant. Since $g_{W_1} W_1 \neq W_1$, one has either $\Lambda_1 = diag(-1, -1, 1, 1, -1)$ or, choosing a basis by applying a suitable rotations in the planes 14 and 23, $\Lambda_1 = diag(-1, -1, -1, 1, 1)$. The former possibility implies that $\sigma(\mathcal{G}) \ni i\Lambda_1 = \eta$ and Lemma 2.2.4 implies the claim.

Assuming $\Lambda_1 = diag(-1, -1, -1, 1, 1)$ implies $\Lambda_1 i = diag(1, -1, 1, -1, 1)$, and since $g_{W_1}(i, 0) = (\Lambda_1 i, d_1)$ is an involution, one must have $d_1 \in span\{\mathbf{e}_1, \mathbf{e}_3\}$ and consequently (since $d_1 \in span\{\mathbf{e}_1, \mathbf{e}_4\}$), d_1 must be parallel to $\mathbf{e}_1$. Since now both g_{W_1} and $(i, 0)$ preserve the wedge W_4, one must have $[g_{W_4}, (i, 0)] = 1 = [g_{W_4}, g_{W_1}]$ implying that, with $g_{W_4} = (\Lambda_4, d_4)$, also $[i, \Lambda_4] = 1 = [\Lambda_1, \Lambda_4]$, and that $d_4 \in span\{\mathbf{e}_1, \mathbf{e}_4\}$. Let $\Lambda_4 = I_4 B_4$, where $I_4 \in \mathcal{L}_-^\downarrow$, $I_4^2 = 1$, and B_4 is a boost. It follows that $[i, I_4] = 1 = [i, B_4]$ and $[\Lambda_1, B_4] = 1 = [\Lambda_1, I_4]$, so the direction of B_4 is parallel to $\mathbf{e}_2$ and one must have

$$\Lambda_4 \in \{diag(-1, -1, 1, 1, -1)B_4, diag(-1, 1, 1, -1, -1)B_4, diag(-1, 1, -1, 1, -1)B_4\}.$$

The first two possibilities imply that $\sigma(\mathcal{G}) \ni i\Lambda_4 = \eta B_4$ and $\sigma(\mathcal{G}) \ni \Lambda_1\Lambda_4 = \eta B_4$, respectively, so Lemma 2.2.4 implies the claim in those cases.

Assuming $I_4 = diag(-1, 1, -1, 1, -1)$ implies $B_4 = 1$ and $\Lambda_4 i = diag(1, 1, 1, -1, -1)$, so that, since $g_{W_4}(i, 0) = (\Lambda_4 i, d_4)$ is an involution, one must have $d_1 \in span\{\mathbf{e}_4, \mathbf{e}_3\}$ and consequently (since $d_4 \in span\{\mathbf{e}_1, \mathbf{e}_4\}$), d_4 must be parallel to $\mathbf{e}_4$. Since now both g_{W_1} and g_{W_4} preserve the wedge W_3, one must have $[g_{W_3}, g_{W_1}] = 1 = [g_{W_3}, g_{W_4}]$ implying that, with $g_{W_3} = (\Lambda_3, d_3)$, also $[\Lambda_3, \Lambda_1] = 1 = [\Lambda_3, \Lambda_4]$. Let $\Lambda_3 = I_3 B_3$, where $I_3 \in \mathcal{L}_-^\downarrow$, $I_3^2 = 1$, and B_3 is a boost. It follows that $[I_3, \Lambda_4] = 1 = [I_3, \Lambda_1]$ and $[\Lambda_1, B_3] = 1 = [\Lambda_4, B_3]$, so the direction of B_3 is also parallel to $\mathbf{e}_2$ and one must have

$$\Lambda_3 \in \{diag(-1,1,1,-1,-1)B_3, diag(-1,-1,1,-1,1)B_3, diag(-1,1,-1,-1,1)B_3\}.$$

The first two possibilities imply that $\sigma(\mathcal{G}) \ni \Lambda_1\Lambda_3 = \eta B_3$ and $\sigma(\mathcal{G}) \ni \Lambda_4\Lambda_3 = \eta B_3$, respectively, so Lemma 2.2.4 implies the claim in those cases.

Assuming $I_3 = diag(-1,1,-1,-1,1) = i$ implies $B_3 = 1$ and $\Lambda_3 d_1 = d_1$, as well as $\Lambda_3 d_4 = d_4$. Since $g_{W_3}g_{W_4} = g_{W_4}g_{W_3}$, one has

$$(\Lambda_3\Lambda_4, \Lambda_3 d_4 + d_3) = (\Lambda_4\Lambda_3, \Lambda_4 d_3 + d_4),$$

implying $\Lambda_4 d_3 = d_3$, so $d_3 \in span\{\mathbf{e}_1, \mathbf{e}_3\}$. Since $g_{W_3}g_{W_1} = g_{W_1}g_{W_3}$, one shows similar way that $d_3 \in span\{\mathbf{e}_3, \mathbf{e}_4\}$, implying that d_3 is parallel to $\mathbf{e}_3$.

We therefore have $g_{W_i} = (\Lambda_i, d_i)$, with d_i parallel to $\mathbf{e}_i$, $i = 1,3,4$ and

$$\begin{aligned}
\Lambda_1 &= diag(-1,-1,-1,1,1),\\
\Lambda_3 &= i = diag(-1,1,-1,-1,1),\\
\Lambda_4 &= diag(-1,1,-1,1,-1).
\end{aligned} \tag{3–3}$$

Since the involutions $g_{W_1}g_{W_4}$, $g_{W_4}i$, $g_{W_1}i$ preserve the wedge W_2, they must all commute with $g_{W_2} := (\Lambda_2, d_2)$. Since

$$\begin{aligned}
g_{W_1}g_{W_4} &= (diag(1,-1,1,1,-1), d_1 + d_4),\\
g_{W_1}(i,0) &= (diag(1,-1,1,-1,1), d_1),\\
g_{W_4}(i,0) &= (diag(1,1,1,-1,-1), d_4),
\end{aligned}$$

it follows that

$$\begin{aligned}
\Lambda_2 \in \{&diag(-1,-1,-1,1,1) = \Lambda_1,\\
&diag(-1,1,-1,-1,1) = \Lambda_3,\\
&diag(-1,1,-1,1,-1) = \Lambda_4\}.
\end{aligned} \tag{3–4}$$

Note also that $\mathcal{G} \ni g_{W_3} i = (1, d_3)$, a translation along the coordinate axis 3. Suppose $d_3 \neq 0$. By transitivity there is an element $\mathcal{G} \ni g_{31} = (\Lambda_{31}, d_{31})$ such that $\Lambda_{31} W_1 = W_3$ and the translation $(1, d_{31})$ leaves W_3 invariant. Then $\Lambda_{31} = R_{31} j R_1 B_1$ where R_{31} is a fixed rotation such that $R_{31}\mathbf{e}_1 = \mathbf{e}_3$, R_1 is a rotation and B_1 a boost such that they both leave W_1 invariant, and $j = diag(\pm 1, 1, 1, 1, 1)$, depending on what connected component of $\mathcal{L}$ the element Λ_{31} belongs to. One then has

$$\mathcal{G} \ni g_{31}^{-1}(1, d_3) g_{31} = (1, \Lambda_{31}^{-1} d_3) = (1, B_1^{-1} u_1),$$

where $u_1 := R_1^{-1} j R_{31}^{-1} d_3$ is a nonzero vector parallel to $\mathbf{e}_1$, and also (recalling (3–3))

$$\mathcal{G} \ni g_{W_3}^{-1}(1, B_1^{-1} u_1) g_{W_3} = (1, B_1 u_1),$$

implying that the non-trivial translation $T_1 := (1, v_1) = (1, B_1 u_1)(1, B_1^{-1} u_1)$, parallel to the coordinate axis 1, belongs to $\mathcal{G}$.

Finding a $g_{34} \in \mathcal{G}$ such that $g_{34} W_4 = W_3$ in an analogous way, one constructs in $\mathcal{G}$ a nontrivial translation T_4 along the coordinate axis 4. Putting $T_3 := (1, d_3)$ implies that the nontrivial translations $T_1, T_3, T_4 \in \mathcal{G}$ all commute with g_{W_2}, and, consequently, with Λ_2. Recalling (3–3) gives a contradiction, implying that $d_3 = 0$. Transitivity of $\mathcal{G}$ on $\mathcal{W}$ then implies $g_W W = W'$, for all $W \in \mathcal{W}$, i. e. the wedge duality, which in turn implies $d_1 = 0$, $d_4 = 0$. Consequently, with $-id := diag(-1, -1, -1, -1, -1)$, one has $(-id, 0) = g_{W_1} g_{W_3} g_{W_4} \in \mathcal{G}$. Since for every $W \in \mathcal{W}_0$, one has $-id W = W'$, the wedge duality implies that if $g_W = (\Lambda_W, d_W)$, then $d_W = 0$, because $[(-id, 0), g_W] = 1$. Hence $g_W = (\Lambda_W, 0)$, for all $W \in \mathcal{W}_0$.

The transitivity of $\mathcal{G}$ on $\mathcal{W}$ implies that there is an involution $k_B \in \mathcal{G}$ such that $k_B = (kB, d_B)$, where $B \neq 1$ is a boost, and $k \in \mathcal{L}_-^{\downarrow}$ is an involution. Let $\mathbf{n}$ be the direction of B and put $\ell_\pm = \mathbf{e}_0 \pm \mathbf{n}$. Then $k\mathbf{n} = \mathbf{n}$, the involution kB leaves the wedge $W_0 := W(\ell_+, \ell_-, 0)$ invariant, and k_B leaves the wedge $W_B := W_0 + \frac{d_B}{2}$ invariant. By transitivity there is $g_{2B} \in \mathcal{G}$ such that $g_{2B} W_B = W_2$ so that $g_{2B} k_B g_{2B}^{-1} W_2 = W_2$. Note that g_{2B} has the form $g_{2B} = (\Lambda_{20}, d_0)$ where $\Lambda_{20} = \tilde{R}_2 \tilde{B}_2 j R_{20}$ with R_{20} a fixed rotation such that $R_{20}\mathbf{n} = \mathbf{e}_2$, $\tilde{R}_2$ is a rotation and $\tilde{B}_2$ is a boost, both leaving W_2 invariant, and $j = diag(\pm 1, 1, 1, 1, 1)$, depending on which component of $\mathcal{L}$ the element Λ_{20} belongs to. One then has $\Lambda_{20} k B \Lambda_{20}^{-1} = \tilde{B}_2 j_2 B_2^{\pm 1} \tilde{B}_2^{-1}$, where $B_2 \neq 1$ is a boost with the direction parallel to $\mathbf{e}_2$ and $j_2 = \tilde{R}_2 R_{20} k R_{20}^{-1} \tilde{R}_2^{-1} \in \mathcal{L}_-^{\downarrow}$ with $j_2 W_2 = W_2$. We may choose a basis by applying $\tilde{B}_2^{-1}$ so that (3–3) still holds and $\Lambda_{20} k B \Lambda_{20}^{-1} = j_2 B_2^{\pm 1}$. It follows that

$$j_2 \in \{diag(-1, 1, 1, -1, -1), diag(-1, -1, 1, 1, 1) I_{34}\},$$

The former case implies that $\sigma(\mathcal{G}) \ni j_2 \Lambda_1 = \eta$, so Lemma 2.2.4 implies the claim. Recalling the notation established in Section 2.2 for involutions in $\mathcal{L}$ one observes that the latter case implies that $\sigma(\mathcal{G})$ contains the following elements (recall that $g_{W_i} = (\Lambda_i, 0)$, $i = 1 \dots 4$)

$$\Lambda_1 \Lambda_3 = i_{13} = diag(1, -1, 1, -1, 1),$$

$$\Lambda_1 \Lambda_4 = i_{14} = diag(1, -1, 1, 1, -1),$$

$$\Lambda_3 \Lambda_4 = i_{34} = diag(1, 1, 1, -1, -1),$$

$$(-id) j_2 B_2^{\pm 1} = (-id) i_{01} I_{34} B_2^{\pm 1} = diag(1, 1, -1, 1, 1) I'_{34} B_2^{\pm 1}.$$

Next recall (3–4) and suppose $\Lambda_2 = \Lambda_1$. Then $g_{W_2} = g_{W_1}$ and there must be $g_{12} \in \mathcal{G}$ such that $g_{12} W_2 = W_1$ and $g_{12} g_{W_2} g_{12}^{-1} = g_{W_2}$. If $g_{12} = (\Lambda_{12}, d_{12})$, then $\Lambda_{12} = R_{12} j R'_2 B'_2$, where again $j = diag(\pm 1, 1, 1, 1, 1)$, the rotation R'_2 and the boost B'_2 both preserve W_2 and

$$R_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Since $[\Lambda_2, \Lambda_{12}] = 1$, one must have $R'_2 \in \{R_{34}, diag(1, -1, 1, 1, 1) I'_{34}\}$ where R_{34} is a rotation in the plane spanned by $\mathbf{e}_3, \mathbf{e}_4$. In a similar way as at the end of proof of Theorem 2.2.2 in both cases one obtains $\Lambda_{12} i_{13} \Lambda_{12}^{-1} = diag(1, 1, -1, 1, 1) \tilde{I}_{34}$ for some reflection $\tilde{I}_{34}$ in the plane spanned by $\mathbf{e}_3, \mathbf{e}_4$, so that one has, for some rotations $R_{34}^{(1)}$ and $R_{34}^{(2)}$ in the same plane

$$[i_{14}\Lambda_{12}i_{13}\Lambda_{12}^{-1}, i_{14}(-id)j_2B_2^{\pm 1}] = [diag(1,-1,-1,1,1)R_{34}^{(1)}, diag(1,-1,-1,1,1)R_{34}^{(2)}B_2^{\pm 1}]$$
$$= B_2^{\pm 2} \in \sigma(\mathcal{G}),$$

which is a non-trivial boost lying in the centralizer of the invariance group of W_2, so Lemma 2.2.3 implies the claim.

Since $g_{2B}k_Bg_{2B}^{-1}W_2 = W_2$, the element $g_{2B}k_Bg_{2B}^{-1}$ commutes with g_{W_2}, so that $[j_2, \Lambda_2] = 1$. The cases $\Lambda_2 = \Lambda_3$ and $\Lambda_2 = \Lambda_4$ then imply

$$j_2 \in \{diag(-1,-1,1,-1,1), diag(-1,-1,1,1,-1)\},$$

so $j_2\Lambda_1\Lambda_2 = \bar{\eta}$ and Lemma 2.2.4 completes the proof of the Lemma. □

We now continue the proof of Theorem 3.2.1. Since $\sigma(\mathcal{G}) \geq \mathcal{L}_+^\uparrow$, we can use a very slight modification of the proof of Proposition 4.3 in [7]: if there exists $\Lambda \in \mathcal{L}$ and $a, b \in \mathcal{M}_5, a \neq b$ such that both (Λ, a) and (Λ, b) belong to $\mathcal{G}$, then so does $(\Lambda, a)(\Lambda, b)^{-1} = (1, a-b)$. For each $\Lambda \in \mathcal{L}_+^\uparrow$ there is $c \in \mathcal{M}_5$ such that $(\Lambda, a) \in \mathcal{G}$. Hence

$$\mathcal{G} \ni (\Lambda, c)(1, a-b)(\Lambda, c)^{-1} = (1, \Lambda(a-b)).$$

Since $(1, c) \in \mathcal{G}$ implies $(1, -c) \in \mathcal{G}$, one has that $(1, x) \in \mathcal{G}$ for all $x \in \mathcal{M}_5$ such that $x \cdot x = \kappa = (a-b)\cdot(a-b)$. Since every vector in $\mathcal{M}_5$ can be expressed as a sum of vectors with the fixed Minkowski product, and since $(1, x) \in \mathcal{G}$ and $(1, y) \in \mathcal{G}$ implies $(1, x+y) \in \mathcal{G}$, it follows that $(1, x) \in \mathcal{G}$ for all $x \in \mathcal{M}_5$. If $\Lambda \in \mathcal{L}_+^\uparrow$ and $(\Lambda, c) \in \mathcal{G}$, then also $(\Lambda, 0) = (\Lambda, c)(1, -\Lambda^{-1}c)$ belongs to $\mathcal{G}$, implying that $\mathcal{G} \geq \mathcal{L}_+^\uparrow$.

If for every $\Lambda \in \sigma(\mathcal{G})$ there is exactly one $a(\Lambda) \in \mathcal{M}_5$ such that $(\Lambda, a(\Lambda)) \in \mathcal{G}$, then

$$a(\Lambda\Lambda') = a(\Lambda) + \Lambda a(\Lambda'). \tag{3–5}$$

Consider the subgroup

$$\mathcal{G}_0 := \{(\Lambda, a(\Lambda)) | \Lambda \in \sigma(\mathcal{G}), \Lambda W_1 = W_1\} \leq \mathcal{G}$$

of transformations in $\mathcal{G}$ that translate the right wedge W_1. Note that $x \in \mathcal{M}_5$ may be uniquely expressed as $x = x_1 + x_2$, where $x_1 \in span\{\mathbf{e}_0, \mathbf{e}_1\}$ and $x_2 \in span\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. Consider a boost $B \neq 1$ in the direction $\mathbf{e}_1$ and $\Lambda \in \mathcal{L}_+^\uparrow$ such that $(\Lambda, a(\Lambda)) \in \mathcal{G}$. Since $\Lambda B = B\Lambda$ the relation (3–5) implies

$$(1 - B)a(\Lambda) = (1 - \Lambda)a(B). \tag{3–6}$$

Expressing $a(B) = a_1(B)+a_2(B)$, $a(\Lambda) = a_1(\Lambda)+a_2(\Lambda)$, where $a_1(B), a_1(\Lambda) \in span\{\mathbf{e}_0, \mathbf{e}_1\}$ and $a_2(B), a_2(\Lambda) \in span\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and noting that these two subspaces are invariant for any $\Lambda \in \mathcal{L}_+^\uparrow$ that leaves W_1 invariant, the restriction of the equation (3–6) to these subspaces yields, with B_1, B_2 and Λ_1, Λ_2 being the restrictions of B and Λ, respectively,

$$\begin{aligned}(1 - B_1)a_1(\Lambda) &= (1 - \Lambda_1)a_1(B)\\ (1 - \Lambda_2)a_2(B) &= 0.\end{aligned}$$

Since $(1 - B_1)$ is invertible one can put $a := (1 - B_1)^{-1}a_1(B)$ to get

$$a_1(\Lambda) = (1 - \Lambda_1)a. \tag{3–7}$$

If there exists an element $(\Lambda_0, a(\Lambda_0)) \in \mathcal{G}_0$ such that $\Lambda_0 \in \mathcal{L} \setminus \mathcal{L}_+^\uparrow$, then every element of $\mathcal{G}_0 \setminus \mathcal{L}_+^\uparrow$ has the form $(\Lambda\Lambda_0, a(\Lambda\Lambda_0)), \Lambda \in \mathcal{L}_+^\uparrow, \Lambda W_1 = W_1$ with

$$\begin{aligned}a(\Lambda\Lambda_0) &= a(\Lambda) + \Lambda a(\Lambda_0)\\ &= a_1(\Lambda) + \Lambda a_1(\Lambda_0) + \Lambda a_2(\Lambda_0) + a_2(\Lambda)\\ a_1(\Lambda\Lambda_0) &= a + \Lambda_1(a_1(\Lambda_0) - a),\end{aligned}$$

where again Λ_1 is the restriction of Λ to $span\{\mathbf{e}_0, \mathbf{e}_1\}$. Since a and $a_1(\Lambda_0)$ are fixed vectors in $span\{\mathbf{e}_0, \mathbf{e}_1\}$ the last equation together with (3–7) implies that the elements of $\mathcal{G}_0$ translate the edge of W_1 only along a hyperbola or a lightlike line, contradicting the transitivity of $\mathcal{G}$ on $\mathcal{W}$. Therefore $\mathcal{G}$ must contain elements (Λ, a) and (Λ, b) for some $\Lambda \in \mathcal{L}$ and $a \neq b$, and, consequently, $\mathcal{G} \geq \mathcal{P}_+^\uparrow$.

By Lemma 1.0.1, g_{W_1} commutes with all elements in $\mathcal{P}_+^\uparrow$ that leave W_1 invariant. If $g_{W_1} = (i_1, d_1)$, then the fact that g_{W_1} commutes with any boost B in the direction $\mathbf{e}_1$ implies that $[i_1, B] = 1$ and that $d_1 \in span\{\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. The fact that g_{W_1} commutes with any rotation R fixing the direction $\mathbf{e}_1$ implies $d_1 = 0$ and $[R, i_1] = 1$. The last commutation relation and $[i_1, B] = 1$ together with $g_{W_1} W_1 \neq W_1$ (a consequence of the fact that $\mathcal{W}$ has no atoms, so the algebras $\mathcal{R}(W)$, $W \in \mathcal{W}$ are non-abelian) imply that $g_{W_1} = diag(-1, -1, 1, 1, 1)$. By transitivity it then follows that for all $W \in \mathcal{W}$ the involution g_W is the reflection through the edge of W, and, consequently, $g_W = W'$, completing the proof of the theorem. □

3.3 Continuous Unitary Representation of Poincaré Group via Reflection Maps

Assuming CGMA in $\mathcal{M}_5$ for a pair $(\{\mathcal{R}(W)\}_{W\in\mathcal{W}}, \Omega)$ on $\mathcal{M}_5$ and the transitivity of the action of the modular involutions $\{J_W\}_{W\in\mathcal{W}}$ on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$, we wish to show, in light of the results of Section 2.3 and the work [11], that there exists a unitary representation of the proper Poincaré group $\mathcal{P}_+$ which acts covariantly on the net and leaves the state vector Ω invariant. We obtain the following analogue of the Theorem 4.1 in [11].

3.3.1 Theorem. *Let $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be a locally generated net and let Ω be a state vector complying with CGMA on $\mathcal{M}_5$. The net satisfies wedge duality and there is a continuous (anti-) unitary representation U of $\mathcal{P}_+$ which leaves Ω invariant and acts covariantly on the net. Moreover, for any given wedge W and the reflection λ about its edge, $U(\lambda)$ is the modular involution corresponding to the pair $(\mathcal{R}(W), \Omega)$.*

Proof. Recalling Definition 2.3.11 (of a locally generated net), an examination of the proof of Proposition 3.1 and Corollary 2.3 in [11] shows that they both remain valid also in case of $\mathcal{M}_5$, yielding the continuous map

$$J : (\lambda, x) \mapsto J(\lambda, x), \quad (\lambda, x) \in \mathcal{R}_5,$$

where $\mathcal{R}_5$ is the set of all $(\lambda, x) \in \mathcal{P}_+$ such that (λ, x) is a reflection about the edge of some wedge $W \in \mathcal{W}$, $J(\lambda, x) = J_W = J_{W'}$ is the modular involution associated with the pair $(\mathcal{R}(W), \Omega)$, and the group generated by $\{J(\lambda, x)\}_{(\lambda,x)\in\mathcal{R}_5}$ is equipped with the strong*-topology. CGMA and the modular theory imply that the map J has the property

$$J((\lambda, x)(\lambda_0, y)(\lambda, x)) = J(\lambda, x)J(\lambda_0, y)J(\lambda, x), \quad (\lambda_0, y), (\lambda, x) \in \mathcal{R}_5,$$

i.e. J is a continuous reflection map. Proposition 2.3.10 then yields a continuous (anti-)unitary representation V of $\mathcal{L}_+ \leq \mathcal{P}_+$ such that $V(\lambda, 0) = J(\lambda, 0)$ for all $(\lambda, 0) \in \mathcal{R}_5$, and

$$V(\Lambda, 0) = J(\lambda_1, 0)J(\lambda_2, 0), \quad (\lambda_1, 0), (\lambda_2, 0) \in \mathcal{R}_5, \Lambda = \lambda_1\lambda_2$$

when Λ is a product of two reflections, or

$$V(\Lambda, 0) = J(\lambda_1, 0)J(\lambda_2, 0)J(\lambda_3, 0)J(\lambda_4, 0),$$
$$\Lambda = \lambda_1\lambda_2\lambda_3\lambda_4, \quad (\lambda_i, 0) \in \mathcal{R}_5, i = 1 \dots 4$$

when Λ is a product of four reflections. CGMA implies that $V(\mathcal{L}_+)$ acts covariantly on the net and leaves Ω invariant.

Note that the arguments leading in [11] to extending V to a unitary representation of $\mathcal{P}_+$ with the desired properties also apply in case of $\mathcal{M}_5$. We list below the crucial steps of this process. For details see [11], Section 4.

Let $x \in \mathcal{M}_5$ be a timelike vector and let $\lambda \in \mathcal{L}_+$ be a reflection such that $\lambda x = -x$ (such vectors lie in the 2-dimensional eigenspace of λ that corresponds to the eigenvalue -1). One has the following relations:

$$(\lambda, x) = (1, x/2)(\lambda, 0)(1, -x/2)$$
$$(1, x) = (\lambda, x)(\lambda, 0),$$

so (λ, x) is also a reflection. It was shown in [7], Section 4.3, and in [8] (in the latter work solely based on CGMA, without any additional conditions, and both arguments also apply to the case of $\mathcal{M}_5$) that

$$U_\lambda(1, x) = J(\lambda, x)J(\lambda, 0)$$

defines a continuous unitary representation of the 2-dimensional subgroup of translations which acts covariantly on the net and leaves Ω invariant. Since x is timelike, its stability group in $\mathcal{L}_+$ is conjugate to the group of all rotations. The arguments used in [11] to prove Lemma 2.4 also apply here to yield

$$U_{S\lambda S^{-1}}(1, x) = U_\lambda(1, x)$$

for all $S \in \mathcal{L}_+$ such that $Sx = x$. Defining, for a given timelike x and any $\lambda \in \mathcal{L}_+$ such that $\lambda x = -x$,

$$U(1, x) = U_\lambda(1, x) = J(\lambda, x)J(\lambda, 0),$$

and noting that for any future-directed timelike vectors x, y there is a reflection $\lambda \in \mathcal{L}_+$ such that $\lambda x = -x$ and $\lambda y = -y$, one has (since J is a reflection map)

$$U(1, x)U(1, y) = U(1, x + y),$$
$$U(1, x)^{-1} = U(1, -x). \tag{3–8}$$

Since any $z \in \mathcal{M}_5$ may be expressed as $z = x - y$ for some future-directed timelike vectors x, y, one defines

$$U(1, z) := U(1, x)U(1, -y).$$

As a consequence of (3–8) this definition does not depend on the choice of x, y such that $z = x - y$, and it yields a continuous unitary representation of the translation subgroup $\mathbb{R}^4$ of $\mathcal{P}_+$.

Since for any $\Lambda \in \mathcal{L}_+$ and $x \in \mathcal{M}_5$ one has

$$U(\Lambda, 0)U(1, x)U(\Lambda, 0)^{-1} = U(1, \Lambda x),$$

by defining $U(\Lambda, x) := U(1, x)U(\Lambda, 0)$ for any $(\Lambda, x) \in \mathcal{P}_+$ one obtains a continuous unitary representation of $\mathcal{P}_+$ with the property $U(\lambda, x) := J(\lambda, x)$ for every reflection $(\lambda, x) \in \mathcal{P}_+$, i.e. U extends the reflection map J.

Since for every $(\Lambda, x) \in \mathcal{P}_+$ the operator $U(\Lambda, x)$ is a product of modular involutions associated with wedge algebras, one has $U(\Lambda, x)\Omega = \Omega$ for every $(\Lambda, x) \in \mathcal{P}_+$ and CGMA implies that the net satisfies wedge duality (by Theorem 3.2.1) and the representation $U(\mathcal{P}_+)$ acts covariantly on the net. $\square$

Recalling the Modular Stability Condition (CMS, see Section 2.3, before Theorem 2.3.16) and the definition of a generating family $\mathcal{C}$ of regions (see Definition 2.3.11), and noting that the arguments used to prove Lemma 3.3 in [11] apply also in case of $\mathcal{M}_5$, one obtains the following analogue of Theorem 4.2 of [11].

3.3.2 Theorem. *Let $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ be a net, let Ω be a state vector satisfying the CGMA and the CMS on $\mathcal{M}_5$, and let $\mathcal{C}$ be a generating family of regions such that Ω is cyclic for the algebras $\mathcal{R}(C)$, $C \in \mathcal{C}$. The net satisfies wedge duality and there is a continuous (anti-) unitary representation U of $\mathcal{P}_+$ with the properties described in the preceding theorem and such that the joint spectrum $sp(U)$ of the generators of translations satisfies either $sp(U) \subset \overline{V}_+$ or $sp(U) \subset -\overline{V}_+$, where $\overline{V}_+$ denotes the closed forward lightcone.*

We conclude this section with the following observation (see [8]): since neither CGMA nor CMS contain any information about the direction of the time, and since the set $\mathcal{W}$ is invariant with respect to the action of time translations, one can choose a coordinate system in $\mathcal{M}_5$ so that $sp(U) \subset \overline{V}_+$, i.e. that the relativistic spectrum condition is satisfied. Proposition 5.2 of [7] (which is also applicable in case of $\mathcal{M}_5$) then allows one to conclude that $U(\mathbb{R}^4)$ is the only continuous unitary representation of the translation group which satisfies the spectrum condition, acts covariantly on the net $\{\mathcal{R}(W)\}_{W \in \mathcal{W}}$ and leaves the state vector Ω invariant.

CHAPTER 4
CONCLUDING REMARKS

In this study we have examined an application of the Condition of Geometric Modular Action (proposed in [7, 10]) to a state ω on a net $\{\mathcal{A}(W)\}_{W\in\mathcal{W}}$ of local C^*-algebras of observables associated with wedge-like regions in 4-dimensional de Sitter space dS^4 and in 5-dimensional Minkowski space $\mathcal{M}_5$. We have addressed the points (1)-(3) of the CGMA program (see the Introduction) for these two spacetimes.

In case of dS^4 (first studied in [13]) we gave a construction based on reflection maps (introduced in [11]) of a continuous unitary representation of the de Sitter group which acts covariantly on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ (of von Neumann algebras generated by the GNS-representations of the algebras $\mathcal{A}(W)$, $W \in \mathcal{W}$, corresponding to the state ω) and leaves the corresponding state vector Ω invariant (Section 2.3).

As a part in the process of establishing (2) of the CGMA program with the additional condition of transitivity of the adjoint action of modular involutions $\{J_W\}_{W\in\mathcal{W}}$ on the net $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ we have attempted to prove an interesting group-action-theoretical statement analogous to Proposition 4.2 in [7]. We have proved it only with an additional restrictive condition, which is implied by the CGMA (see Lemma 2.2.5).

To complete the discussion of the CGMA on dS^4 we have listed the relevant results of [7] and [13] establishing the point (1) of the CGMA program.

In case of $\mathcal{M}_5$ we have proved Theorem 3.2.1 to establish the point (2) of the CGMA program, and using our construction of the unitary representation of the de Sitter group as well as the relevant results of [11] we have established the point (3) of this program. We have also listed the results of [7] whose direct analogues establish the point (1) for $\mathcal{M}_5$.

In future investigations it would be interesting to examine the CGMA program also on other 4-dimensional spacetime manifolds, especially on anti-de Sitter space, to supplement the studies [9, 12].

It would also be interesting to further study the group-theoretic aspects of the CGMA program with the transitivity assumption to attempt to establish a more general statement on transitive actions of the resulting transformation groups and their homogeneous spaces.

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BIOGRAPHICAL SKETCH

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