# LIQUIDITY RISK MEASUREMENT AND MANAGEMENT

By

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I dedicate this dissertation to my family that helped and supported me through my graduate study.

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In this paper, we introduced the liquidity risk measures. The coherent risk measure introduced by Artzner[1] does not include the liquidity effects. That is the trading amount is so large that it will affect the market price. In this situation, the positive homogeneity axiom will no longer hold. So we set up a new axioms system to deal with liquidity effects. We call it the liquidity risk measure. In addition, we found the acceptance sets and established the relationship between the liquidity risk measures and the acceptance sets. We also developed appropriate liquidity risk measures for multi-asset portfolios and explored the corresponding optimal trading strategies. We further studied variations of liquidity risk measures and checked that the liquidity costs mentioned by Cetin et al.[7] is in fact a liquidity risk measure, but not coherent. We also gave an algorithm to calculate the optimal trading strategy under liquidity risk measures. Finally, we discussed some applications in risk allocations and tested liquidity risk measures in the real markets.

# CHAPTER 1
# INTRODUCTION

## 1.1 Financial Risk Environment

Over the past decade, the financial market has been so volatile that everyone realized the actual risk was much higher than they thought. Most investment managers now take risk into account for their performance. Instead of using excess returns, fund managers are using risk adjusted returns to manage their portfolios. There are three types of risks considered by Basel II: market risk, credit risk, and operational risk[23]. Market risk is the risk based on price volatility of an asset over a time period because of economic and market condition changes. Credit risk is the risk that a borrower may default by failing to repay part or all of the principal and interest at a predetermined date. Operational risk is the risk created by inefficient or failed process, people, systems or external events. It is closely related to everyday operation of a business[24].

We can divide investment assets into four broad categories: money market, stock market, bond market and derivatives[4]. Each category has different focus on different types of risk, but they have one in common: liquidity risk. Money market is typically short term investment, usually less than one year. The most common form is T-bills. The maturities for T-bills are 4 weeks, 3 months, 6 months and 1 year. And it is usually considered as risk-free instrument. Stock market is the most important and complex market in the world and controlling risk is critical in the market. There are all kinds of risks involved in the stock market. The most obvious one is market risk. Stock price raises and falls everyday and our investment highly depends on it. Credit risk is also common in the stock market. There are companies going to bankruptcy everyday and you may lose all or most of your investment from that company. Operational risk is also involved. If a company has a huge operational risk, the market may consider it as a potential loss and its stock price will fall. The bond market is primarily for long term investment. It is less volatile than the stock market and its major concern is credit

risk of the issuer. Its price is not affected by the performance of the company, but by the interest rate. The longer the maturity of the bond, the higher its liquidity risk. The derivatives market is highly volatile and hard to value. Its major concern is the market risk of the underlying assets. The longer the maturity of the derivatives, the higher the liquidity risk. For the purpose of research, we usually assume one month T-bills have no liquidity risk.

There are three stages dealing with risk: risk measurement, risk allocation and risk management. The most intuitive measure is the standard deviation. It is the most common measure of volatility in statistics. As a matter of fact, a lot of fund managers are still using it to estimate the risk of their portfolios. However, it can be affected by a lot of other factors like outliers. It also cannot give you the direction of the dispersion. In order to systematically study the risk, coherent risk measure was established by Artzner et al.[1]. It was the first time that risk was studied in a rigorous theoretical manner. Their work is so inspiring that many other risk measures have been created based on their theory. The coherent risk measure is based on four axioms that mentioned in the paper[1], and its follow-up works are conducted by modifying these axioms. The greatest advantage of coherent risk measure is that it has perfect theoretical results. However, it is not very useful in practice. There have been many researchers dedicated to this approach and proposed various modifications based on it. For example, the risk measure defined by Rockafellar[27] completely removed the positive homogeneity axiom. There are so many other factors we need to consider to make it applicable in the financial markets. One factor that is not considered is the liquidity effect, which is very important for big traders. Another factor is the volatility of interest rate, which is very dramatic during financial crisis. Figure 1-1 shows the demand curve for two assets with different liquidity properties in the same period of time. As we know, short-term T-bill is quite liquid and its price does not change much for different trading volume. On the other

hand, the stock UEIC is not as liquid as T-bill and shows up sloping demand curve. That is, price increases with large demand to buy.

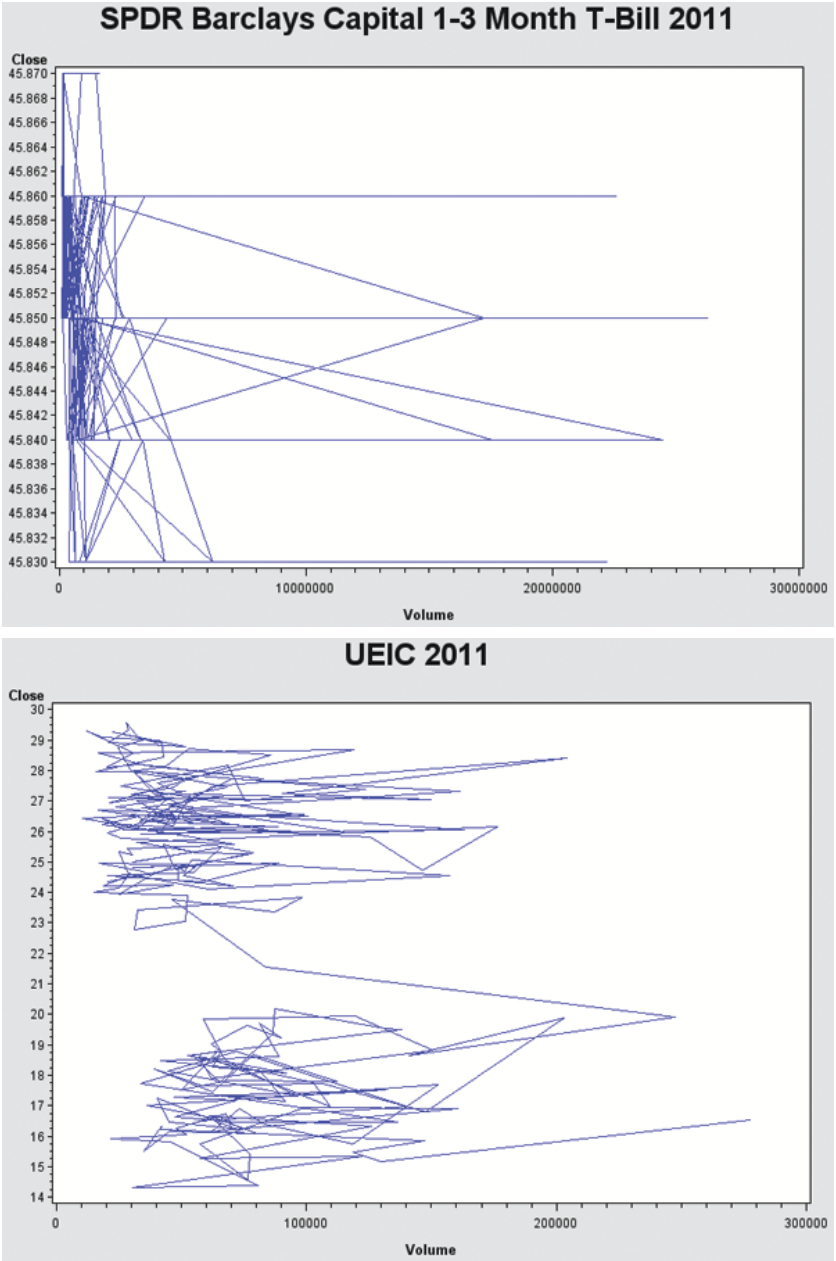


Figure 1-1. Liquidity Risk Comparison

## 1.2 Scope of This Paper

In this paper, we are going to introduce a new risk measure that fits better in an illiquid market with interest rate volatility. Liquidity effect has been studied for a long

time. Baum and Bank [2] described an illiquid financial market model where market prices can be decided by some large trading parties. Under geometric Brownian motion, Duffie and Ziegler [14] studied the correlation between bid-ask spread and mid-price and modeled them as stochastic processes. Cetin et al.[7] used stochastic supply curve to approximate liquidity effect and developed corresponding pricing method.

Interest rate is also an important factor in risk measures. The coherent risk measure assumes risky positions are discounted before applying the risk measure and the discounting process does not involve any additional volatility. However, when interest rate is volatile, this procedure does not separate the risk of the financial position and the risk associated with the discounting process[20]. Risk optimization has been thoroughly studied by many researchers. Uryasev established optimization techniques for C-Var[29], a coherent risk measure. Longstaff [22] studied the optimal trading strategies in an illiquid market as bounded variations. By considering liquidity effect, multiple researchers studied the optimal liquidation process[3, 16]. Pricing techniques have also been studies for coherent risk measures[5, 8].

The organization of this paper is as following: Chapter 2 reviews coherent risk measures and states the basic axioms for liquidity risk measures and acceptance sets. Chapter 3 studies the relationship between the axioms on acceptance sets and the axioms on liquidity risk measures. Chapter 4 defines specific liquidity risk measures for different scenarios. Chapter 5 introduces conditional diversification and some variations of liquidity risk measures. Chapter 6 proposes the optimal trading strategy problem and provides certain solutions including computational algorithm and numerical examples. Chapter 7 explores the application of liquidity risk measures in risk allocation problems. Finally, Chapter 8 provides some empirical study to support liquidity risk measure as a better alternative for practical uses.

# CHAPTER 2
# LIQUIDITY RISK MEASURES

## 2.1 Review on Coherent Risk Measures

In financial markets, risk is usually divided into three categories: market risk, credit risk and operational risk[23]. Market risk is the risk of a financial position due to the uncertainty of the market price or valuation of the containing assets. For example, a portfolio consists of several stocks or bonds is greatly exposed to price changes in stock and bond markets. Credit risk is also called default risk. It is the risk that the borrowers will not fully payback their loans. Any financial loans are subject to credit risk. For example, people may not be able to pay their auto loans and thus lenders suffer huge credit risk. The less studied one is the operational risk. It is the potential losses due to inefficiency or failure of internal process, systems or the management. For example, company's policy changes can result great uncertainty in operation and unstable cash flows. These three categories are not mutually exclusive. The boundaries are not clear and one risk can affect another. An axiomatic approach to model market risk is through coherent risk measures.

Coherent risk measure was first introduced by Artzner et al. [1] and developed and extended to broader situations[13, 17, 26]. It was established through four basic axioms described below. Before exploring the details of the axioms, we need to get a general understanding of what coherent means. By Merriam-Webster© dictionary, coherent means relating to or composed of waves having a constant difference in phase. This is just the key of this risk measure: no matter how the position changes, the risk will always change in proportion. It is a neat condition and we can get beautiful results under this condition, like the representation theorem. However, it is also a very strong assumption and in most cases it does not hold. Thus it calls for a weakening of the coherent assumption and that is the goal of this paper. We first carry out a brief study of the coherent risk measures in axiomatic settings.

One important concept used to introduce coherent risk measure is acceptance set, which is a set of acceptable future net worth[1]. The acceptance or rejection of a financial position can be based on whether it is in the acceptance set. Intuitively, all positive future net worths are acceptable, so they should be included in the acceptance set. And all strictly negative positions cannot be included in the acceptance set. It also assumes risk aversion which leads to the convexity of the set. Finally, to reflect the coherent property, the acceptance set is required to be positively homogeneous. The following is the list of all the axioms for acceptance set[1]:

Positive: The acceptance set $\mathcal{A}$ contains $L_+$, the cone of nonnegative elements.

Nonnegative: The set $\mathcal{A}$ does not intersect the set $L_{--} = \{X|X(\omega) < 0\}$.

Convexity: The acceptance set $\mathcal{A}$ is convex.

Coherence:The acceptance set $\mathcal{A}$ is a positively homogeneous cone.

After defining acceptance set, we can roughly measure the risk by simply determining whether it is acceptable or not. Strictly speaking, we can induce a risk measure based on the acceptance set. And luckily enough, the risk measure induced by the acceptance set defined above is coherent. We will discuss this relationship further in later sections. What we want to do next is to introduce coherent risk measure as an independent concept and study its properties.

If a position is described by the resulting discounted net worth at the end of a given period, defined as a real-valued function $X$ on some set $\Omega$ of possible scenarios, then a quantitative measure of risk is given by a mapping $\rho$ from a certain space $\mathcal{G}$ of functions on $\Omega$ to the real line. Formally speaking, we have the following definitions[1]:

**Definition 2.1** A measure of risk is a mapping from $\mathcal{G}$ into $\mathbb{R}$.

**Definition 2.2** A reference instrument is a commonly accepted instrument that is risk free.

Now assume $\Omega$ is finite and $r$ is the total return of a reference instrument.

**Definition 2.3** The coherent measure of risk $\rho : \mathcal{G} \to \mathbb{R}$ satisfies the following axioms:

$$\text{Monotonicity: If} \quad X \leq Y, \quad \text{then} \quad \rho(X) \geq \rho(Y) \tag{2–1}$$

$$\text{Translation Invariance: If} \quad m \in \mathbb{R}, \quad \text{then} \quad \rho(X + mr) = \rho(X) - m \tag{2–2}$$

$$\text{Positive Homogeneity: For any} \quad \lambda \geq 0, \rho(\lambda X) = \lambda \rho(X) \tag{2–3}$$

$$\text{Subadditivity: } \rho(X + Y) \leq \rho(X) + \rho(Y) \tag{2–4}$$

Monotonicity is just saying we prefer $X$ to be positive large numbers. Since $X$ is defined as the final net worth, we are better off when our positive net worth is great. That also means we have little risk of losing money. That is why the relationship is the reversed monotonicity. Translation invariance means any constant can be taken out of the risk measure. If you have a risky position $X$ and a risk-free position $m$ today, the risk of your whole portfolio is just the risk of your risky position minus the risk-free position. The risk-free position serves as an insurance in this case. That is why it lowers your risk. The only issue with this axiom is that interest rate is absolutely not constant and the equation does not hold under volatile interest rate. Positive Homogeneity is just a linear assumption needed for the representation theorem. For small $\lambda$ and $X$, it is approximately true. But for large positions and multipliers, this equation does not hold either. Subadditivity conveys risk diversifications. The risk of holding two positions separately will be greater than holding them together since sometime they may cancel out some risk. However, risk diversification is not always true. We will see some examples later.

The most important result of coherent risk measure is the representation theorem. Typically, a coherent measure of risk $\rho$ arises from some family $\mathcal{P}$ of probability measures on $\Omega$ by computing the expected loss under $P \in \mathcal{P}$ and then taking the

worst result as $P$ varies over $\mathcal{P}$:

$$\rho(X) = \sup_{P \in \mathcal{P}} E_P[-\frac{X}{r}] \tag{2–5}$$

for the case where $\Omega$ is finite and $r$ is the total return of a reference instrument. This is the representation theorem of coherent measure of risk given by Artzner[1]. It is a very strong theorem and the modern risk measure system is established on this ground. The problem of the theorem is that it has very strong assumptions which are not good estimates of the real market conditions. We will test the market conditions for the assumptions in Chapter 8.

When we think about the assumptions in the real market conditions, the Positive Homogeneity and Subadditivity are often not valid. Because in an illiquid market, which is true in most cases, trading large amount of assets can affect the price of these assets and then affect the benefits you will get. For example, if you want to trade $nX$, where $X$ is very large and $n > 1$, then the risk encountered should be bigger than $n\rho(X)$ because you also should consider the liquidity costs. Hence the Positive Homogeneity axiom is violated. Similarly, if $X = Y$ is very large, then $\rho(X + Y) = \rho(2X)$ should be bigger than $2\rho(X)$, which violates the Subadditivity axiom. Therefore, in the new system of measures of risk, we should replace these two axioms by others. One is the Convexity axiom and the other is the Liquidity axiom:

Convexity: For any $\lambda \in [0, 1]$, $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$ (2–6)

Liquidity: If $\lambda > 1$, then $\rho(\lambda X) \geq \lambda\rho(X)$; If $0 < \lambda \leq 1$, then $\rho(\lambda X) \leq \lambda\rho(X)$ (2–7)

Convexity means that diversification does not increase the risk, i.e., the risk of a diversified position $\lambda X + (1 - \lambda)Y$ is less than or equal to the weighted average of the individual risks. Liquidity means for a large position, the risk of trading the whole position is bigger than the risk of trading divided parts of the position separately.

## 2.2 Notations

First, we clarify the notations we will use throughout this paper. We define $\Omega$ to be the set of states of nature and assume it is finite. The final net worth of a position for each element of $\Omega$ is a random variable, denoted by $X$. Let $\mathcal{G}$ be the set of all risks, that is the set of all real-valued functions on $\Omega$. Since $\Omega$ is finite, $\mathcal{G}$ can be identified with $\mathbb{R}^n$, where $n = card(\Omega)$. Define $L_{--} = \{X|X < 0\}$. Let $\mathcal{A}$ be the sets of final net worth which are acceptable and we call it the acceptance set.

## 2.3 Axioms on Acceptance Sets

The acceptable sets under liquidity risk measures have the same meaning of the acceptance sets under coherent risk measures but with different conditions. Since we take out the coherent condition, the acceptance set is no longer a positively homogeneous cone, but we still need it to be convex. We adopt some axioms from coherent risk measures and add some new ones. We summarize them in the following:

**Axiom 1** The acceptance set $\mathcal{A}$ does not intersect the set $L_{--}$.

**Axiom 2** The acceptance set $\mathcal{A}$ is convex.

**Axiom 3** If $X \in \mathcal{A}$ and $Y \in \mathcal{G}$, then $\{\lambda \in [0,1]|\lambda X + (1-\lambda)Y \in \mathcal{A}\}$ is closed in $[0,1]$.

**Axiom 4** For $\lambda \geq 1$, if $X \notin \mathcal{A}$, then $\lambda X \notin \mathcal{A}$. For $0 < \lambda \leq 1$, if $X \in \mathcal{A}$, then $\lambda X \in \mathcal{A}$.

**Remark:** That means the acceptance set is a convex set, which reflects risk aversion on the part of the regulators. The Axiom 3 is not obvious, and we will see it is necessary for later use.

## 2.4 Axioms on Liquidity Risk Measures

We put the old and our new axioms together to get the following axiomatic system for liquidity risk measure:

**Axiom S** Separation. If $X < 0$, then $\rho(X) > 0$.

**Axiom T** Translation Invariance. If $m \in \mathbb{R}$, then $\rho(X + mr) = \rho(X) - m$

**Axiom C** Convexity. For any $\lambda \in [0,1]$, $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$

**Axiom L** Liquidity. If $\lambda > 1$, then $\rho(\lambda X) \geq \lambda\rho(X)$;If $\lambda \in (0,1]$, $\rho(\lambda X) \leq \lambda\rho(X)$

Now we come to the formal definition of a liquidity risk measure:

**Definition 2.1** A liquidity risk measure is a measure of risk satisfies Axioms S, T, and L. A convex liquidity risk measure is a risk measure satisfies Axioms S, T, C, and L.

We have explained Translation Invariance before. We do not have Monotonicity here. Because some assets may have large negative liquidity effect to make them less valuable than other assets. We will carefully examine this condition in Chapter 4. To understand Convexity, we can think about the term structure model for interest risk. The yield curve is usually concave, for example, in 30 years. You can always observe the pattern that the long term rates are usually greater than the short term rates. The increase in the rate is quick in the beginning and slow when maturity passes 5 years. So it is a concave increasing curve. The risk here is defined as the negative of the interest rate and hence it should be a convex curve. The Liquidity axiom is true for large transactions in most markets. The commodities markets are the least liquid markets and we can often see price increase due to large buying powers. The Liquidity axiom is a reasonable assumption and it contains the Positive Homogeneity condition. For small transactions, we can assume no liquidity effects and the equality holds.

For general risk measure, it can be interpreted as the cash amount we need to make our position risk free. When positive, the number $\rho(X)$ will be interpreted as the minimum extra cash needed to add to the risky position $X$ and invest in the reference instrument to make it acceptable. If it is negative, the cash amount $-\rho(X)$ can be withdrawn from the position.

# CHAPTER 3
# CORRESPONDENCE BETWEEN ACCEPTANCE SETS AND LIQUIDITY RISK MEASURES

## 3.1 Dual Definitions of Risk Measures and Acceptance Sets

As we mentioned in Chapter 2, the acceptance sets and risk measures are closely related to each other. In fact, we can derive one from the other. The followings are general definitions of risk measures and acceptance sets[1]. And we will see the definitions proposed in Chapter 2 are consistent with them.

**Definition 3.1** Given the total return $r$ on a reference instrument, the risk measure associated with the acceptance set $\mathcal{A}$ is the mapping from $\mathcal{G}$ to $\mathbb{R}$ denoted by $\rho_{\mathcal{A}}$ and defined by $\rho_{\mathcal{A}}(X) = \inf\{m|mr + X \in \mathcal{A}\}$.

Correspondingly, we can define the acceptance set $\mathcal{A}$ associated with a risk measure:

**Definition 3.2** The acceptance set associated with a risk measure $\rho$ is the set denoted by $\mathcal{A}_\rho$ and defined by $\mathcal{A}_\rho = \{X \in \mathcal{G}|\rho(X) \leq 0\}$.

The risk measure can be considered as the minimum cash we need to add to the the position to make it acceptable by investors. And the acceptance set is the collection of positions that with non-positive risks. They are perfectly correlated with each other based on the fact that a position is acceptable by investors if and only if the investor's risk measure of this position is non-positive. These definitions are true for any kinds of acceptance sets and any kinds of risk measures. The problem is that these definitions hold when considered separately but may conflict with each other when put together due to the inconsistence of the acceptance sets and risk measures defined by investors. Consistence here means $\rho_{\mathcal{A}} = \rho$ and $\mathcal{A} = \mathcal{A}_\rho$.

For example, in one dimensional case, we define $\mathcal{A} = \{X > 0\}$ and $\rho(X) = 5 - X$. So $\mathcal{A}_\rho = \{X|\rho(X) \leq 0\} = \{X \geq 5\}$. Clearly, $\mathcal{A} \neq \mathcal{A}_\rho$.

However, we will see in the case of convex liquidity risk measures, they are perfectly consistent with each other.

### 3.2 Correspondence Theorems

In this section, we will discuss the relationship between convex liquidity risk measures and the corresponding acceptance sets. We apply the above two definitions to convex liquidity risk measures and check the consistency inside the system. Given a convex liquidity risk measure $\rho$, the naturally induced set $\mathcal{A}_\rho$ is an acceptance set.

**Theorem 3.1** If $\rho$ is a convex liquidity risk measure, then $\mathcal{A}_\rho$ satisfies Axioms 1, 2, 3, and 4. In addition, we have $\rho_{\mathcal{A}} = \rho$.

Proof:

(1) For any $X \in L_{--}$ we have $X < 0$. By Axiom S, we get $\rho(X) > 0$. By the definition of acceptance set, $X \notin \mathcal{A}$. So Axiom 1 is satisfied.

(2) As $\rho$ satisfies Axiom C, $\rho$ is a convex function, then it is continuous. Hence, $\mathcal{A}_\rho = \{X|\rho(X) \leq 0\}$ is closed and convex.

(3) Define $F : \lambda \longmapsto \rho(\lambda X + (1-\lambda)Y)$. Then $F$ is continuous, as it is convex. Moreover, $\{\lambda \in [0,1]|\lambda X + (1-\lambda)Y \in \mathcal{A}\} = \{\lambda \in [0,1]|\rho(\lambda X + (1-\lambda)Y) \leq 0\}$ is closed.

(4) If $X \notin \mathcal{A}$ and $\lambda \geq 1$, then by definition, $\rho(X) > 0$. By Axiom L, we have $\rho(\lambda X) \geq \lambda\rho(X) > 0$. So $\lambda X \notin \mathcal{A}$.

If $X \in \mathcal{A}$ and $0 < \lambda \leq 1$, then $\rho(X) \leq 0$ and by Axiom L, $\rho(\lambda X) \leq \lambda\rho(X) \leq 0$. So $\lambda X \in \mathcal{A}$. So Axiom 4 is satisfied.

(5) For any $X$, $\rho_{\mathcal{A}}(X) = \inf\{m|mr + X \in \mathcal{A}\} = \inf\{m|\rho(mr + X) \leq 0\} = \inf\{m|\rho(X) - m \leq 0\} = \inf\{m|\rho(X) \leq m\} = \rho(X)$, by Axiom T. □

Correspondingly, we have the following theorem for the convex liquidity risk measure induced by an acceptance set.

**Theorem 3.2** If the set $\mathcal{A}$ satisfies Axioms 1, 2, 3, and 4, then the risk measure $\rho_{\mathcal{A}}$ is a convex liquidity risk measure. Moreover, we have $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$.

Proof:

(1) We have the equality $\inf\{p|X + (\alpha + p)r \in \mathcal{A}\} = \inf\{q|X + qr \in \mathcal{A}\} - \alpha$. So it implies $\rho_{\mathcal{A}}(X + \alpha r) = \rho_{\mathcal{A}}(X) - \alpha$. So Axiom T is satisfied.

(2) If $X < 0$, then $X \in L_{--}$. By Axiom 1, $L_{--} \cap \mathcal{A} = \phi$, then $X \notin \mathcal{A}$. By definition of acceptance set, $\rho(X) > 0$. So Axiom S is satisfied.

(3) Let $X_1, X_2 \in \mathcal{X}$ and $m_1, m_2 \in \mathbb{R}$, such that $X_i + m_i r \in \mathcal{A}$. As $\mathcal{A}$ is convex, for any $\lambda \in [0, 1]$, $\lambda(X_1 + m_1 r) + (1 - \lambda)(X_2 + m_2 r) \in \mathcal{A}$. Then by Axiom T proved above, $0 \geq \rho_{\mathcal{A}}(\lambda(X_1 + m_1 r) + (1 - \lambda)(X_2 + m_2 r)) = \rho_{\mathcal{A}}(\lambda X_1 + (1 - \lambda)X_2) - (\lambda m_1 + (1 - \lambda)m_2)$. That is $\rho_{\mathcal{A}}(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda m_1 + (1 - \lambda)m_2$. As this is true for any $m_1, m_2$, such that $X_i + m_i r \in \mathcal{A}$, it is also true that $\rho_{\mathcal{A}}(\lambda X_1 + (1 - \lambda)X_2) \leq \lambda \rho_{\mathcal{A}}(X_1) + (1 - \lambda)\rho_{\mathcal{A}}(X_2)$. So Axiom C holds.

(4) If $X \notin \mathcal{A}$, then $\rho_{\mathcal{A}}(X) > 0$. For any $m < \rho_{\mathcal{A}}(X)$, we have $\rho_{\mathcal{A}}(X + mr) > 0$. That means $X + mr \notin \mathcal{A}$. By Axiom 4, for any $\lambda \geq 1$, $\lambda(X + mr) \notin \mathcal{A}$. So $\rho_{\mathcal{A}}(\lambda(X + mr)) > 0$. That is $\rho_{\mathcal{A}}(\lambda X) \geq \lambda m$. As it is true for any $m \notin \{n | X + nr \in \mathcal{A}\}$, we can take supremum of $m$ which is the infimum of $\{n | X + nr \in \mathcal{A}\}$, and the inequality still holds. Hence, we have $\rho_{\mathcal{A}}(\lambda X) \geq \lambda \rho_{\mathcal{A}}(X)$. If $X \in \mathcal{A}$, then $\rho_{\mathcal{A}}(X) \leq 0$. For any $m \geq \rho_{\mathcal{A}}(X)$, we have $\rho_{\mathcal{A}}(X + mr) \leq 0$. For any $0 < \lambda \leq 1$, by Axiom 4, $\lambda(X + mr) \in \mathcal{A}$. So $\rho_{\mathcal{A}}(\lambda X) \leq \lambda m$. Similar as above, after taking infimum of $m$, we still get the inequality and have $\rho_{\mathcal{A}}(\lambda X) \leq \lambda \rho_{\mathcal{A}}(X)$.

So either $X \in \mathcal{A}$ or not, we have at least one inequality. And based on this inequality we can derive the other inequality by substituting $\frac{1}{\lambda}$. So Axiom L holds.

(5) For any $X \in \mathcal{A}$, $\rho_{\mathcal{A}}(X) \leq 0$, hence $X \in \mathcal{A}_{\rho_{\mathcal{A}}}$. So $\mathcal{A} \subseteq \mathcal{A}_{\rho_{\mathcal{A}}}$. Now assume $X \notin \mathcal{A}$. For any $m > 0$, $m \in \mathcal{A}$. By Axiom 3, exists $\lambda \in (0, 1)$ such that $\lambda m + (1 - \lambda)X \notin \mathcal{A}$. Thus, $\lambda m \leq \rho_{\mathcal{A}}((1 - \lambda)X) \leq (1 - \lambda)\rho_{\mathcal{A}}(X)$. That is $\rho_{\mathcal{A}}(X) \geq \frac{\lambda}{1-\lambda} m > 0$. Hence, $X \notin \mathcal{A}_{\rho_{\mathcal{A}}}$. Therefore, $\mathcal{A} = \mathcal{A}_{\rho_{\mathcal{A}}}$. $\square$

**Remark:** We can see Definition 3.1 and Definition 3.2 are dual to each other. And the convex liquidity risk measure we defined is consistent within this system.

# CHAPTER 4
# LIQUIDITY RISK MEASURES WITH DISCRETE THRESHOLDS

The risk measure defined above is quite general and hard to apply in the real market. In this chapter, we will define some specific liquidity risk measures that are ready to use. The basic assumption for such risk measures is the threshold structure of the asset prices. We will begin with some definitions and notations first.

## 4.1 Single Risky Asset Portfolio

To better model liquidity risk, we adopt the concept acceptable portfolio by Ku[21]. A portfolio is considered to be acceptable if it can be turned into an acceptable cash only position with positive future cash flow at some fixed date. Assume we have unlimited access to risk free assets in the market as well as risky assets. We denote these risky assets as $S^1, S^2, ..., S^N$. We assume these assets' prices $S^1, S^2, ..., S^N$ are adapted stochastic processes on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration.

**Definition 4.1** A trading strategy $\pi_t = (\pi_t^0, \pi_t^1, ..., \pi_t^N)$ is a (N+1)-dimensional $\{\mathcal{F}_t\}$ adapted process. $\pi_t^0$ is the number of units of the risk-free asset and $\pi_t^n$ is the number of shares of assets $S_t^n$ held at time $t$.

There is liquidity risk in the market because of two facts. The first is when you sell an asset, it is hard to find a counterparty to buy. And the second, even if you find some counterparty to buy, they may only be able to pay a price much lower than your asking price. So even if you are holding assets with great value, you may not be able to sell them at a fair price in a short period of time. In the bankruptcy case you must pay back your debt. There is not much time allowed for you to sell at a fair price. So if you only keep just the assets with fair value equal to your debt, you may not be able to cover all the liabilities and make the company insolvent. That is why liquidity risk is so important in risk regulation. In the words of risk measure, liquidity risk measures should always be greater than coherent risk measures. We call the difference between these two risk measures the liquidity spread.

**Definition 4.2** A liquidity spread is a function $s(X) = \rho_L(X) - \rho_C(X)$, where $\rho_L$ is a liquidity risk measure and $\rho_C$ is a coherent risk measure.

**Proposition 4.1** Any liquidity spread satisfies the Liquidity axiom. That is for any $\lambda \geq 1$, $s(\lambda X) \geq \lambda s(X)$.

**Proof:** By definition, for any $\lambda \geq 1$ $s(\lambda X) = \rho_L(\lambda X) - \rho_C(\lambda X) \geq \lambda\rho_L(X) - \lambda\rho_C(X) = \lambda(\rho_L(X) - \rho_C(X)) = \lambda s(X)$. $\square$

We begin with a coherent risk measure $\rho_C$ with a set of scenario probabilities $\{P^i, i \in I\}$. By the definition in Chapter 3, a random variable $X$ is called acceptable if $\rho_C(X) \leq 0$. That is equivalent to $E_{P^i}[X] \geq 0$ for all $i \in I$ using the representation theorem for coherent risk measures. According to Ku[21], a portfolio is said to be positive if it entails only non-negative cash flows in the future.

**Definition 4.3** A portfolio $X$ is acceptable if there is an admissible trading strategy $\pi_t$ and a date $T$ such that $X$ can be decomposed by trading into a cash only position $C$ and a positive portfolio by date $T$. That is,

(i) $\pi_T^{n,C} = 0$ for all $1 \leq n \leq N$, where $\pi_T^{n,C}$ denotes the number of shares (corresponding to cash only part C) of asset $S^n$ held at time $T$,

(ii) the random variable $e^{-rT}\pi_T^{0,C}$ satisfies $E_{P^i}[e^{-rT}\pi_T^{0,C}] \geq 0$ for all $i \in I$.

We denote the set of all acceptable portfolios to be $\mathcal{A}_p$. It is not the same as the acceptance set $\mathcal{A}$ of the corresponding coherent risk measure. Because $\mathcal{A}_p$ may not be positively homogeneous. For an acceptable portfolio $X$, it is possible that you may not have a trading strategy to liquidate larger positions like $\lambda X$, where $\lambda > 1$. This is the difference between liquidity risk measure and coherent risk measure.

For simplicity, we first only consider a portfolio with one risk-free asset and one risky asset $S$. We assume the price of the risky asset $S$ follows an $\{\mathcal{F}\}$ adapted geometric Brownian motion $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$. And the interest rate for the risk-free asset is $r$. The market has different prices for buyers and sellers. We assume the price for the buyers is

$P_t^+ = S_t + \frac{\alpha}{2} S_t$ and the price for the sellers is $P_t^- = S_t - \frac{\alpha}{2} S_t$. The $\alpha$ here is the bid-ask spread.

For a given short period of time, it is usually impossible to trade a large number of shares completely. Therefore we assume the number of shares that can be traded is $\epsilon$ multiplying the length of time interval, i.e. $\epsilon \Delta t$. Any trading strategy satisfying this condition is called an admissible trading strategy[21].

**Definition 4.4** A trading strategy $\pi_t$ is admissible if $|\pi_{t_1}^1 - \pi_{t_2}^1| \leq \epsilon |t_1 - t_2|$ for all $t_1, t_2 \geq 0$.

By the above definition, any admissible trading strategy is Lipschitz continuous in $t$. So $\pi_t^1$ is a process of bounded variation.

The total wealth at time $t$ is denoted by $X_t = \pi_t^0 + \pi_t^1 S_t$. We also need to assume admissible trading strategies are self-financing, i.e. besides trading no additional cash flow will be generated.

The initial wealth is $X_0 = \pi_0^0 + \pi_0^1 S_0$ with $\pi_0^1 \geq 0$. And for simplicity, we assume $X_0 = 0$. That means $\pi_0^0 = -\pi_0^1 S_0$. In order to study liquidity effect, we distinguish the wealth before and after liquidating all risky positions. The ending wealth before liquidating all shares of risky assets is $X_T^- = \pi_T^0 + \pi_T^1 S_T$, and the ending wealth after liquidating all shares of risky assets is $X_T^+ = \pi_T^{0,C}$. The difference between coherent risk measure and liquidity risk measure is that coherent risk measure calculates the change $X_T^- - X_0$ while liquidity risk measure calculates the change $X_T^+ - X_0$. So the implied liquidity effect is $X_T^+ - X_T^-$. According to Ku[21], the discounted wealth process before liquidation $X^-$ is a supermartingales under any risk neutral probability measure $Q$.

In order to derive a liquidity risk measure from the restriction on trading, we rewrite the Lipschitz condition to define a trading limit for a certain period of time. We define $\tau_{\Delta t} = \epsilon \Delta t$ to be the trading limit for the time period $\Delta t$. That means if the trading amount is less than $\tau$, then we can freely trade them without affecting the price. However, if the trading amount is greater than $\tau$, then we must pay a price premium if we buy or

sell at discount. That means the price is affected by a liquidity factor. If $|\pi_0^1| \leq \tau$, then $X_T^+ = e^{rT}\pi_0^0 + \pi_0^1 S_T$. If $|\pi_0^1| > \tau$, then $X_T^+ = e^{rT}\pi_0^0 + \tau S_T + (|\pi_0^1| - \tau)S'_T$.

**Definition 4.5** For any long position $\pi^1$, we define $\rho(X_T^+) = \sup_{i\in I} E_{P^i}[-\pi_0^0 - e^{-rT}\min\{\tau, \pi_0^1\}S_T - e^{-rT}\max\{\pi_0^1 - \tau, 0\}S'_T]$, with $S'_T \leq S_T$.

We shall prove that the risk measure defined above is a convex liquidity risk measure. It is more realistic because small trading rarely affects the price but large trading usually has big impact on the price. The following theorem verifies that this measure is a convex liquidity risk measure.

**Theorem 4.1** The risk measure defined by 4.5 is a convex liquidity risk measure.

**Proof:**

Translation Invariance. $\rho(X + me^{rT}) = -\inf_{i\in I} E_{P^i}[\pi_0^0 + m + e^{-rT}\min\{\tau, |\pi_0^1|\}S_T + e^{-rT}\max\{|\pi_0^1| - \tau, 0\}S'_T] = -\inf_{i\in I} E_{P^i}[\pi_0^0 + e^{-rT}\min\{\tau, |\pi_0^1|\}S_T + e^{-rT}\max\{|\pi_0^1| - \tau, 0\}S\prime_T] - m = \rho(X) - m$.

Liquidity. For $\lambda \geq 1$, we have $\rho(\lambda X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\min\{\tau, \lambda\pi_0^1\}S_T + e^{-rT}\max\{\lambda\pi_0^1 - \tau, 0\}S'_T]$.

If $\pi_0^1 \leq \tau$ and $\lambda\pi_0^1 \leq \tau$, then $\rho(\lambda X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\lambda\pi_0^1 S_T] = \lambda(-\inf_{i\in I} E_{P^i}[\pi_0^0 + e^{-rT}\pi_0^1 S_T]) = \lambda\rho(X)$.

If $\pi_0^1 \leq \tau$ and $\lambda\pi_0^1 \geq \tau$, then $\rho(\lambda X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\tau S_T + e^{-rT}(\lambda\pi_0^1 - \tau)S'_T]$ and $\lambda\rho(X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\lambda\pi_0^1 S_T]$. So $\rho(\lambda X) - \lambda\rho(X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\tau S_T + e^{-rT}(\lambda\pi_0^1 - \tau)S'_T - \lambda\pi_0^0 - e^{-rT}\lambda\pi_0^1 S_T] = -\inf_{i\in I} E_{P^i}[e^{-rT}\tau(S_T - S'_T) + e^{-rT}\lambda\pi_0^1(S'_T - S_T)] = e^{-rT}\sup_{i\in I} E_{P^i}[(S_T - S'_T)(\lambda\pi_0^1 - \tau)] \geq 0$. Therefore, $\rho(\lambda X) \geq \lambda\rho(X)$.

If $\pi_0^1 \geq \tau$, then $\lambda\pi_0^1 \geq \tau$. So $\rho(\lambda X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\tau S_T + e^{-rT}(\lambda\pi_0^1 - \tau)S'_T]$ and $\lambda\rho(X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + \lambda e^{-rT}(\pi_0^1 - \tau)S'_T]$. So $\rho(\lambda X) - \lambda\rho(X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\tau S_T + e^{-rT}(\lambda\pi_0^1 - \tau)S'_T] + \inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + \lambda e^{-rT}(\pi_0^1 - \tau)S'_T] = \inf_{i\in I} E_{P^i}[(\lambda - 1)e^{-rT}\tau S_T + (1-\lambda)e^{-rT}\tau S'_T] = (\lambda - 1)e^{-rT}\tau \inf_{i\in I} E_{P^i}[S_T - S'_T] \geq 0$. Hence, $\rho(\lambda X) \geq \lambda\rho(X)$.

Therefore, for any $\lambda$ and any $X$, we have $\rho(\lambda X) \geq \lambda\rho(X)$. And we can see from the above that the inequality is not trivial. For example, if $S_T > S'_T$, we have the strict inequality $\rho(\lambda X) > \lambda\rho(X)$ in the last case.

Convexity. For $0 \leq \lambda \leq 1$, with $X(\pi_0^0, \pi_0^1)$ and $Y(\mu_0^0, \mu_0^1)$, we have $\rho(\lambda X + (1-\lambda)Y) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + (1-\lambda)\mu_0^0 + e^{-rT}\min\{\tau, \lambda\pi_0^1 + (1-\lambda)\mu_0^1\}S_T + e^{-rT}\max\{\lambda\pi_0^1 + (1-\lambda)\mu_0^1 - \tau, 0\}S'_T]$

If $\lambda\pi_0^1+(1-\lambda)\mu_0^1 \leq \tau$, then $\lambda\pi_0^1 \leq \tau$ and $(1-\lambda)\mu_0^1 \leq \tau$. So we have $\rho(\lambda X+(1-\lambda)Y) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + (1-\lambda)\mu_0^0 + e^{-rT}(\lambda\pi_0^1 + (1-\lambda)\mu_0^1)S_T] = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + e^{-rT}\lambda\pi_0^1 S_T] - \inf_{i\in I} E_{P^i}[(1-\lambda)\mu_0^0 + e^{-rT}(1-\lambda)\mu_0^1 S_T] = \lambda\rho(X) + (1-\lambda)\rho(Y)$.

If $\lambda\pi_0^1 \geq \tau$ and $(1-\lambda)\mu_0^1 \geq \tau$, then $\lambda\pi_0^1 + (1-\lambda)\mu_0^1 \geq \tau$. Then $\rho(\lambda X + (1-\lambda)Y) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + (1-\lambda)\mu_0^0 + e^{-rT}\tau S_T + e^{-rT}(\lambda\pi_0^1 + (1-\lambda)\mu_0^1 - \tau)S'_T] = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + e^{-rT}\lambda(\pi_0^1 - \tau)S'_T] - \inf_{i\in I} E_{P^i}[(1-\lambda)\mu_0^0 + (1-\lambda)e^{-rT}\tau S_T + e^{-rT}(1-\lambda)(\mu_0^1 - \tau)S'_T] = \lambda\rho(X) + (1-\lambda)\rho(Y)$.

WOLOG, we assume $\lambda\pi_0^1 \leq \tau \leq (1-\lambda)\mu_0^1$, then $\lambda\pi_0^1 + (1-\lambda)\mu_0^1 \geq \tau$. So $\rho(\lambda X+(1-\lambda)Y) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + (1-\lambda)\mu_0^0 + e^{-rT}\tau S_T + e^{-rT}(\lambda\pi_0^1 + (1-\lambda)\mu_0^1 - \tau)S'_T] = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + e^{-rT}\lambda(\pi_0^1 - \tau)S'_T] - \inf_{i\in I} E_{P^i}[(1-\lambda)\mu_0^0 + (1-\lambda)e^{-rT}\tau S_T + e^{-rT}(1-\lambda)(\mu_0^1 - \tau)S'_T] = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + e^{-rT}\lambda(\pi_0^1 - \tau)S'_T] + (1-\lambda)\rho(Y)$. There are two cases for this situation. If $\pi_0^1 \geq \tau$, then $-\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + e^{-rT}\lambda(\pi_0^1 - \tau)S'_T] = \lambda\rho(X)$. So we get $\rho(\lambda X + (1-\lambda)Y) = \lambda\rho(X) + (1-\lambda)\rho(Y)$. The other case is $\pi_0^1 \leq \tau$. Then $-\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + e^{-rT}\lambda(\pi_0^1 - \tau)S'_T] - \lambda\rho(X) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + e^{-rT}\lambda(\pi_0^1 - \tau)S'_T - \lambda\pi_0^0 - \lambda e^{-rT}\pi_0^1 S_T] = \lambda e^{-rT}\sup_{i\in I} E_{P^i}[-\tau S_T - \pi_0^1 S'_T + \tau S'_T + \pi_0^1 S_T] = \lambda e^{-rT}\sup_{i\in I} E_{P^i}[(S'_T - S_T)(\tau - \pi_0^1)] \leq 0$. That means $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$.

The last situation is $\lambda\pi_0^1 \leq \tau$ and $(1-\lambda)\mu_0^1 \leq \tau$, but $\lambda\pi_0^1 + (1-\lambda)\mu_0^1 \geq \tau$. So $\rho(\lambda X+(1-\lambda)Y) = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + (1-\lambda)\mu_0^0 + e^{-rT}\tau S_T + e^{-rT}(\lambda\pi_0^1 + (1-\lambda)\mu_0^1 - \tau)S'_T] = -\inf_{i\in I} E_{P^i}[\lambda\pi_0^0 + \lambda e^{-rT}\tau S_T + e^{-rT}\lambda(\pi_0^1 - \tau)S'_T] - \inf_{i\in I} E_{P^i}[(1-\lambda)\mu_0^0 + (1-\lambda)e^{-rT}\tau S_T + e^{-rT}(1-\lambda)(\mu_0^1 - \tau)S'_T]$. For each part, there are two cases. One is $\pi_0^1 \geq \tau$ and

the other is $\pi_0^1 \leq \tau$. So this is the same situation as above. So We can conclude $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$ and the inequality is not trivial.

In summary, no matter which situation, we always have $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$.

Therefore, the risk measure defined in 4.5 is a convex liquidity risk measure. □

**Remark:** A liquidity risk measure may not have Monotonicity. Because with two positions, $X \leq Y$ with $\tau \leq \pi_0^1 \leq \mu_0^1$, we must have $S_T \geq e^{rT}S_0$ and $\pi_0^1(S_T - e^{rT}S_0) \leq \mu_0^1(S_T - e^{rT}S_0)$. So $\rho(X) - \rho(Y) = \sup_{i\in I} E_{P^i}[(\pi_0^1 - \mu_0^1)(S_0 - e^{-rT}S_T')]$. If $e^{-rT}S_T' \leq S_0$, then $\rho(X) \leq \rho(Y)$, which contradicts the Monotonicity axiom.

General diversification does not hold for liquidity risk measure either. General diversification says $\rho(X+Y) \leq \rho(X) + \rho(Y)$. However, there is no certain relation between $\rho(X+Y)$ and $\rho(X) + \rho(Y)$ for liquidity risk measures. Suppose $\pi_0^1 \geq \tau$ and $\mu_0^1 \geq \tau$, then $\rho(X+Y) - \rho(X) - \rho(Y) = \sup_{i\in I} E_{P^i}[(\pi_0^1 + \mu_0^1)S_0 - e^{-rT}\tau S_T - e^{-rT}(\pi_0^1 + \mu_0^1 - \tau)S_T' - \pi_0^1 S_0 + e^{-rT}\tau S_T + e^{-rT}(\pi_0^1 - \tau)S_T' - \mu_0^1 S_0 + e^{-rT}\tau S_T + e^{-rT}(\mu_0^1 - \tau)S_T'] = \sup_{i\in I} E_{P^i}[e^{-rT}\tau S_T - e^{-rT}\tau S_T'] \geq 0$. So $\rho(X+Y) \geq \rho(X) + \rho(Y)$.

On the other hand, if $\pi_0^1 \leq \tau \leq \mu_0^1$, we have $\pi_0^1 + \mu_0^1 \geq \tau$. So $\rho(X+Y) - \rho(X) - \rho(Y) = \sup_{i\in I} E_{P^i}[(\pi_0^1 + \mu_0^1)S_0 - e^{-rT}\tau S_T - e^{-rT}(\pi_0^1 + \mu_0^1 - \tau)S_T' - \pi_0^1 S_0 + e^{-rT}\pi_0^1 S_T - \mu_0^1 S_0 + e^{-rT}\tau S_T + e^{-rT}(\mu_0^1 - \tau)S_T'] = \sup_{i\in I} E_{P^i}[e^{-rT}\pi_0^1(S_T' - S_T)] \leq 0$. So $\rho(X+Y) \leq \rho(X) + \rho(Y)$. In addition, the inequalities are not trivial, so there is no certain relationship between $\rho(X+Y)$ and $\rho(X) + \rho(Y)$.

### 4.2 Bid-Ask Spread

All financial transactions are conducted between at least two parties: the buyer and the seller. Eventually, every transaction is a negotiation process. The sellers ask a price that they are willing to sell and the buyers bid a price that they are willing to buy. If the two prices are the same, then the transaction is completed and each party get exactly what they want. But usually, the market is not efficient enough to match buyers and sellers with the same bid and ask price. The gap between the highest bid price and

the lowest ask price is called bid-ask spread. It is one way to measure the liquidity of the market. For perfectly liquid market, the bid-ask spread is zero. The more illiquid, the greater the bid-ask spread. To be more specific, we can simply model liquidity effect by bid-ask spread. That means $S'_T = (1 - \frac{\alpha}{2})S_T$ and the liquidity risk measure becomes explicitly defined.

**Definition 4.6** For any long position $(\pi^0, \pi^1)$, we define the bid-ask spread risk measure as following:

$\rho(X_T^+) = \sup_{i \in I} E_{P^i}[\pi_0^1 S_0 - (\min\{\tau, \pi_0^1\} + (1 - \frac{\alpha}{2}) \max\{\pi_0^1 - \tau, 0\})e^{-rT} S_T]$.

**Theorem 4.2** If $S_t$ is a geometric Brownian motion then the bid-ask spread risk measure has a lower bound that does not depends on $S_T$.

**Proof:** Since $S_t$ is a geometric Brownian motion, $E[S_t] = e^{rt} S_0$. So $\rho(X_T^+) = \sup_{i \in I} E_{P^i}[\pi_0^1 S_0 - (\min\{\tau, \pi_0^1\} + (1 - \frac{\alpha}{2}) \max\{\pi_0^1 - \tau, 0\})e^{-rT} S_T] = \pi_0^1 S_0 - (\min\{\tau, \pi_0^1\} + (1 - \frac{\alpha}{2}) \max\{\pi_0^1 - \tau, 0\})e^{-rT} \inf_{i \in I} E_{P^i}[S_T] \geq \pi_0^1 S_0 - (\min\{\tau, \pi_0^1\} + (1 - \frac{\alpha}{2}) \max\{\pi_0^1 - \tau, 0\})S_0 = (\pi_0^1 - \min\{\tau, \pi_0^1\} - (1 - \frac{\alpha}{2}) \max\{\pi_0^1 - \tau, 0\})S_0$ □

We use an example to illustrate that bid-ask spread risk measure is more appropriate than coherent risk measure. Suppose in a market, we only have two parties, seller A and buyer B. And there are only two assets in the market: risk-free asset and stock $S$. Suppose at $t = 0$, the stock is trading at $S_0 = 10$. There is a decreasing demand for the stock $S$ and the fair price of the stock at the end of the holding period $t = T$ is predicted to be \$8 with probability 0.7 and \$5 with probability 0.3. But B is only willing to buy at most 5 shares at price \$8 and all other shares at price \$5. The estimated bid-ask spread coefficient in this market is $.4$. We do not consider interest rate in this case. The seller A is now holding 10 shares of the stock. His coherent risk is $\rho_c = 0.7 * 20 + 0.3 * 50 = 29$. But the bid-ask spread risk is $\rho_l = 10 * 10 - (5 + (1 - 0.2) * 5)E[S_T] = 100 - 9 * 7.1 = 36.1$. Now we can calculate the real risk. The gain to A is $5 * 8 + 5 * 5 = 65$. So the loss of A is $\$100 - \$65 = \$35$. So the coherent risk measure highly underestimates the risk and bid-ask spread risk measure does a better job in this illiquid market.

One key element in the bid-ask spread risk measure is the coefficient $\alpha$. It is usually a percentage of the stock price and it changes overtime. There are different ways to estimate the bid ask spread coefficient. Here we use the approach from Shane Corwin and Paul Schultz[11]. This approach simply uses daily high and low prices. The reason is that daily high prices are buyer initiated and daily low prices are seller initiated. The ratio of high-to-low reflects both the volatility and bid-ask spread of the stock. Let $H_t^A$, $L_t^A$ denote the actual high and low prices on day $t$ and $H_t^O$, $L_t^O$ denote the observed high and low prices on day $t$. We have the following relationship: $[\ln(\frac{H_t^O}{L_t^O})]^2 = [\ln(\frac{H_t^A(1+\alpha/2)}{L_t^A(1-\alpha/2)})]^2$. This is equivalent to $[\ln(\frac{H_t^O}{L_t^O})]^2 = [\ln(\frac{H_t^A}{L_t^A})]^2 + 2[\ln(\frac{H_t^A}{L_t^A})][\ln(\frac{2+\alpha}{2-\alpha})] + [\ln(\frac{2+\alpha}{2-\alpha})]^2$. The bid-ask spread estimator given in that paper is $\alpha = \frac{2(e^\gamma-1)}{1+e^\gamma}$, where $\gamma = \frac{\sqrt{2a}-\sqrt{a}}{3-2\sqrt{2}} - \sqrt{\frac{b}{3-2\sqrt{2}}}$. $a = E[\sum_{j=0}^{1}[\ln(\frac{H_{t+j}^O}{L_{t+j}^O})]^2]$ and $b = [\ln(\frac{H_{t,t+1}^O}{L_{t,t+1}^O})]^2$. Hence, we can calculate the coefficient $\alpha$ by using daily high and low prices.

### 4.3 Extension to Multiple Stages

Up to now, we have studied the simplest case of liquidity risk measures, that is only one threshold for price changes. But in reality, there are usually multiple thresholds for price changes and the greater the trading volume the greater the price changes. Suppose the thresholds are $\tau_1, \tau_2, ..., \tau_n$ and the prices after passing each threshold are $S_T^{(1)}, S_T^{(2)}, ..., S_T^{(n)}$. We have $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n$, $S_T = S_T^{(0)} \geq S_T^{(1)} \geq S_T^{(2)} \geq \cdots \geq S_T^{(n)}$ for sellers and $S_T = S_T^{(0)} \leq S_T^{(1)} \leq S_T^{(2)} \leq \cdots \leq S_T^{(n)}$ for buyers. For liquidity effect, discrete model is more appropriate than continuous model. That is because prices are usually affected by large amount and small increment hardly has any impact on prices.The number of stages $n$ depends on individual stock and market conditions. We have the following general definition for multiple stages risk measures.

**Definition 4.7** For an n-stage risk measure, first find $k$, such that $\tau_k \leq \pi_0^1 < \tau_{k+1}$. Then $\rho(X_T^+) = \sup_{i\in I} E_{P^i}[\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)S_T^{(k)}]$.

For some special $S^{(j)}$, we can have more specific definitions. For example, for bid-ask spread risk measure, if we have $S^{(j+1)} = (1-\frac{\alpha}{2})S^{(j)}$, then $S^{(j)} = (1-\frac{\alpha}{2})^j S$.

Based on this relationship, we can have bid-ask spread risk measure in the form $\rho(X) = \sup_{i\in I} E_{P^i}[\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)S_T^{(k)}] = \sup_{i\in I} E_{P^i}[\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})(1-\frac{\alpha}{2})^{j-1}S_T - e^{-rT}(\pi_0^1 - \tau_k)(1-\frac{\alpha}{2})^k S_T]$.

**Proposition 4.2** For long positions, the n-stage bid-ask spread risk measure has a lower bound that does not depend on $S_T$ if $S_T$ follows geometric Brownian motion.

**Proof:** We have $\rho(X) = \sup_{i\in I} E_{P^i}[\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})(1-\frac{\alpha}{2})^{j-1}S_T - e^{-rT}(\pi_0^1 - \tau_k)(1-\frac{\alpha}{2})^k S_T] = \pi_0^1 S_0 - e^{-rT}[\sum_{j=1}^{k}(\tau_j - \tau_{j-1})(1-\frac{\alpha}{2})^{j-1} + (\pi_0^1 - \tau_k)(1-\frac{\alpha}{2})^k]\sup_{i\in I} E_{P^i}[S_T] \geq \pi_0^1 S_0 - [\sum_{j=1}^{k}(\tau_j - \tau_{j-1})(1-\frac{\alpha}{2})^{j-1} + (\pi_0^1 - \tau_k)(1-\frac{\alpha}{2})^k]S_0 = [\pi_0^1 - \sum_{j=1}^{k}(\tau_j - \tau_{j-1})(1-\frac{\alpha}{2})^{j-1} - (\pi_0^1 - \tau_k)(1-\frac{\alpha}{2})^k]S_0$. □

Most of the case, it is convenient to assume infinite stages for the prices of the asset. Now we assume an infinite set of thresholds $\tau = \{\tau_0, \tau_1, \dots\}$ with $\tau_0 < \tau_1 < \cdots$. In addition, we want the threshold set to reflect some property of the liquidity condition in the market. We define the closest threshold from below to the trading volume $\pi$ to be the greatest lower bound (GLB) of $\pi$ in $\tau$. One property of an illiquid market is that the GLB threshold goes up more than proportional to the growth of trading size. For example, if the GLB threshold for $\pi$ is $100$, then the GLB threshold for $2\pi$ should be greater than $200$. This is the diminishing increase property of illiquid market. We also want the order of the GLB threshold to be bounded. Continue from the above example, if $\tau_k = 100$ and $\tau_m = 200$, then we require $m \leq 2k$. That is because we want the threshold set to be relatively evenly distributed. We summarize our key assumptions about the threshold set in the following:

**Threshold Assumption:** The threshold set $\tau = \{\tau_0, \tau_1, \dots\}$ is said to evenly reflect the market liquidity condition if for trading volume $\pi$ and $\lambda \geq 1$, we have the following

results:

(1) The GLB threshold for $\pi$ is $\tau_k$ and the GLB threshold for $\lambda\pi$ is $\tau_m$,

(2) $\tau_m \geq \lambda\tau_k$, and

(3) $m \leq \lambda k$.

**Remark:** Without specific announcement, we will assume the Threshold Assumption throughout this paper.

**Theorem 4.3** The risk measure in Definition 4.7 is a liquidity risk measure.

**Proof:** Translation Invariance.

$$\begin{aligned}
\rho(X + me^{rT}) &= \sup_{i\in I} E_{P^i}[-m + \pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)S_T^{(k)}] \\
&= \sup_{i\in I} E_{P^i}[\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)S_T^{(k)}] - m = \rho(X) - m
\end{aligned}$$

Liquidity. Suppose $\lambda \geq 1$ and the GLB threshold for $\lambda X$ is $\tau_m$. We have

$$\begin{aligned}
\rho(\lambda X) &= \sup_{i\in I} E_{P^i}[\lambda\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{m}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\lambda\pi_0^1 - \tau_m)S_T^{(m)}] \\
&\geq \sup_{i\in I} E_{P^i}[\lambda\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{m}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\lambda\pi_0^1 - \lambda\tau_k)S_T^{(k)}] \\
&\geq \sup_{i\in I} E_{P^i}[\lambda\pi_0^1 S_0 - e^{-rT}\lambda\sum_{j=1}^{k}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\lambda\pi_0^1 - \lambda\tau_k)S_T^{(k)}] \\
&= \lambda\sup_{i\in I} E_{P^i}[\pi_0^1 S_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})S_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)S_T^{(k)}] = \lambda\rho(X). \ \square
\end{aligned}$$

### 4.4 Extension to Two-Asset Portfolios

There are thousands of investment assets in the real world and typical portfolios usually contain more than just one risky asset. We begin the discussion with two assets case. The risk measure of two assets is not just the summation of the risk measures of the two assets separately. In other words, there is no linearity in different assets.

There are two factors we need to consider. One is the correlation between the prices of different assets. For example, if the prices of two different assets U and V are positively correlated and the portfolio contains long positions in both U and V. Then selling large portion of U will draw down the price $p_u$ of U. Since the price of V $p_v$ is positively correlated with $p_u$, $p_v$ will fall at the same time. So the loss of the portfolio is more than the loss of just U position. Therefore, it makes sense the corresponding risk measure should also be greater. The other factor is the impact on the liquidity of the other assets. Trading one asset could directly affect the supply and demand of the other assets and hence influence the liquidity spread and final price of the other assets. These two factors are different. Correlation is an exogenous factor which is determined by the market conditions while liquidity impact is caused by transactions and trading volumes. So it is better to model them separately in risk measures.

Now suppose a market contains one risk-free asset and two different risky assets U and V. A portfolio consists of the three assets and its net value is zero at $t = 0$. The shares of the risk-free asset, asset U and asset V are $\pi^0, \pi^1, \pi^2$ respectively. So we should have the relationship $\pi_0^0 = \pi_0^1 U_0 + \pi_0^2 V_0$. The thresholds for U and V are $\tau_1, ..., \tau_m$ and $\kappa_1, ..., \kappa_n$. In addition assume $\tau_k \leq \pi_0^1 < \tau_{k+1}$ and $\kappa_l \leq \pi_0^2 < \kappa_{l+1}$. The risk measure of the portfolio should be $\rho(U, V) = \rho(U) + \rho(V) + \beta Corr(U, V)(\pi_0^1 + \pi_0^2) + LI(U, V)$, where $Corr(U, V)$ is the corelation of U and V, and $LI(U, V)$ is the liquidity impact between $U$ and $V$. Here we define $LI(U, V) = \gamma_{U,V}\pi_T^1 + \gamma_{V,U}\pi_T^2$. $\gamma_{U,V}$ is the liquidity impact parameter of trading asset U on the value of asset V, and $\gamma_{V,U}$ is the liquidity impact parameter of trading asset V on the value of asset U. Since risk-free assets are not affected by any trading impact, we assume $\gamma_{U,0} = \gamma_{0,U} = 0$. And we also restrict that $\beta \geq 0$. We summarize the above definition formally in the following:

**Definition 4.8** The risk of two risky assets U and V can be measured by $\rho(U, V) = \sup_{i \in I} E_{P^i}[\pi_0^1 U_0 + \pi_0^2 V_0 - e^{-rT} \sum_{j=1}^{k} (\tau_j - \tau_{j-1}) U_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k) U_T^{(k)} - e^{-rT} \sum_{j=1}^{l} (\kappa_j -$

$\kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\pi_0^2 - \kappa_l)V_T^{(l)} + \beta Corr_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2]$. The notations and assumptions are the same as above.

**Theorem 4.4** The risk measure in Definition 4.8 is a liquidity risk measure.

**Proof:** Translation invariance: $\rho(U_T, V_T, me^{rT}) = \sup_{i\in I} E_{P^i}[-\pi_0^0 - m - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)U_T^{(k)} - e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\pi_0^2 - \kappa_l)T_T^{(l)} + \beta Cov_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2] = \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 + \pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)U_T^{(k)} - e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\pi_0^2 - \kappa_l)T_T^{(l)} + \beta Cov_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2] - m = \rho(U, V) - m$.

Liquidity: For simplicity, denote the function inside the expectation of the single asset risk measure to be $f(X)$. Then $\rho(X) = \sup_{i\in I} E_{P^i}[f(X)]$. For $\lambda \geq 1$, $\rho(\lambda U, \lambda V) = \sup_{i\in I} E_{P^i}[f(\lambda U) + f(\lambda V) + \beta Cov_{P^i}(U, V)(\lambda\pi_0^1 + \lambda\pi_0^2) + \gamma_{U,V}\lambda\pi_0^1 + \gamma_{V,U}\lambda\pi_0^2] \geq \sup_{i\in I} E_{P^i}[\lambda f(U) + \lambda f(V) + \lambda\beta Cov_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \lambda\gamma_{U,V}\pi_0^1 + \lambda\gamma_{V,U}\pi_0^2] = \lambda \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 + \pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)U_T^{(k)} - e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\pi_0^2 - \kappa_l)T_T^{(l)} + \beta Corr_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2] = \lambda\rho(U, V)$.
□

**Remark:** The liquidity impact is caused by inequilibrium of the market, i.e. instant demand or supply is not matched. The demand or supply surplus causes the price to increase or decrease respectively. This is also the source for liquidity cost. The liquidity impact between two different risky assets is formed in a similar situation. The demand surplus of one asset may cause a demand or supply surplus of the other asset. The interaction between the two assets is another source of liquidity cost. If the demand and supply curve of one asset is not affected by the other asset, we must have $\gamma_{U,V} = \gamma_{V,U} = 0$. More generally, we can define the situation with zero net impact between two assets.

**Definition 4.9** The trading strategy $(\pi_0^1, \pi_0^2)$ of two assets $U$ and $V$ is called a balanced trading strategy if $\gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2 = 0$.

The balanced trading strategy takes the impact between two assets out and leaves only the correlation effect between the two assets in the risk measure. Also, we should notice that balanced trading strategy may not always exist. For the cases such strategy does exist, we introduce a new risk measure in the Chapter 5, the conditional diversification risk measure. The balanced trading strategy simply says zero net trading effect on both assets. It is a way to pair the trading volumes of the two assets to eliminate any cross effect caused by each other. Here we assume we hold long positions of both assets. So $\pi_0^1 > 0$ and $\pi_0^2 > 0$. In order to reach balanced trading strategy, we must need $\gamma_{U,V}\gamma_{V,U} < 0$, or we can just assume $\gamma_{U,V} > 0$ and $\gamma_{V,U} < 0$. That means selling $U$ will cause the price of $V$ to drop and selling $V$ will cause the price of $U$ to rise.

### 4.5 Extension to Multiple Trading Periods

The final extension we need is considering multiple time periods. All the above discussions are conducted in one single time period. However, we often need to divide big transaction into smaller multiple transactions in consideration of trading limit or liquidity cost. We construct our model under the two risky assets assumption. For simplicity, we consider two periods at first. So the time line is from $t = 0$ to $t = T_1$ for the first transaction and to $t = T_2$ for the second and final transaction. Since we need to liquidate all risky assets in the end, we must have $\pi_2^1 = \pi_2^2 = 0$. So the trading volume in the first period is $\pi_0^1 - \pi_1^1$ and $\pi_0^2 - \pi_1^2$ respectively. And the trading volume in the second and final period is $\pi_1^1$ and $\pi_1^2$ respectively. To precisely calculate the risk of each position, we need to find the corresponding threshold for each transaction. We assume that $\tau_{k_1} \leq \pi_0^1 - \pi_1^1 < \tau_{k_1+1}, \kappa_{l_1} \leq \pi_0^2 - \pi_1^2 < \kappa_{l_1+1}$ and $\tau_{k_2} \leq \pi_1^1 < \tau_{k_2+1}, \kappa_{l_2} \leq \pi_1^2 < \kappa_{l_2+1}$. All other assumptions are the same as before. So we can define the risk measure for two time periods as follows:

**Definition 4.10** The risk of two risky assets U and V in two-period transaction can be measured by $\rho_2(U, V) = \sup_{i \in I} E_{P^i}[\pi_0^1 U_0 + \pi_0^2 V_0 - e^{-rT_1} \sum_{j=1}^{k_1} (\tau_j - \tau_{j-1}) U_{T_1}^{(j-1)} -$

$e^{-rT_1}(\pi_0^1 - \pi_1^1 - \tau_{k_1})U_{T_1}^{(k_1)} - e^{-rT_2}\sum_{j=1}^{k_2}(\tau_j - \tau_{j-1})U_{T_2}^{(j-1)} - e^{-rT_2}(\pi_1^1 - \tau_{k_2})U_{T_2}^{(k_2)} - e^{-rT_1}\sum_{j=1}^{l_1}(\kappa_j - \kappa_{j-1})V_{T_1}^{(j-1)} - e^{-rT_1}(\pi_0^2 - \pi_1^2 - \kappa_{l_1})V_{T_1}^{(l_1)} - e^{-rT_2}\sum_{j=1}^{l_2}(\kappa_j - \kappa_{j-1})V_{T_2}^{(j-1)} - e^{-rT_2}(\pi_1^2 - \kappa_{l_2})V_{T_2}^{(l_2)} + \beta Corr_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2]$.

This is a really long definition. But if we are familiar with previous definitions, we can clearly catch the structure here. We are just adding the second time period terms to the two risky assets definition. There is no change in the correlation term and liquidity impact term. That is because these two terms assume linear relation in trading volumes. That is, the sum of the two trading volumes will be the same as the initial holding volumes. So there is no need to change these terms. However, these two assets may have different correlation in different time periods and the coefficient $\beta$ could be different in different time periods. So the correlation term will change to $\sum_{t=1}^{T_2}\beta_t Corr_{P^i}(U_t, V_t)(\Delta\pi_t^1 + \Delta\pi_t^2)$. The same could occur to the liquidity impact term. That is the impact coefficients are different from time to time. So the liquidity impact term will change to $\sum_{t=1}^{T_2}(\gamma_{U_t,V_t}\Delta\pi_t^1 + \gamma_{V_t,U_t}\Delta\pi_t^2)$. We call different time periods consistent if the correlations and all coefficients are the same in every time period. So precisely speaking, Definition 4.10 is under consistent time periods assumption.

Now we can further extend the above risk measure to multi-period situations. Suppose the total number of time periods is $N$. So the time line is from $t = 0$ to $t = T_1$, $\cdots$, $t = T_N$ for the final transaction. After liquidating all assets, we have $\pi_T^1 = \pi_T^2 = 0$. So the trading volume for each transaction is $\pi_{T_{j-1}}^i - \pi_{T_j}^i$ for different assets. We also assume that $\tau_{k_j} \leq \pi_{T_{j-1}}^1 - \pi_{T_j}^1 < \tau_{k_j+1}$ and $\kappa_{l_j} \leq \pi_{T_{j-1}}^2 - \pi_{T_j}^2 < \kappa_{l_j+1}$. With all the same other assumptions, we have the extended definition:

**Definition 4.11** The risk of two risky assets U and V in N-period transaction can be measured by $\rho_N(U, V) = \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 + \pi_0^2 V_0 - \sum_{h=1}^{N}(e^{-rT_h}\sum_{j=1}^{k_h}(\tau_j - \tau_{j-1})U_{T_h}^{(j-1)} - e^{-rT_h}(\pi_{h-1}^1 - \pi_h^1 - \tau_{k_h})U_{T_h}^{(k_h)} - e^{-rT_h}\sum_{j=1}^{l_h}(\kappa_j - \kappa_{j-1})V_{T_h}^{(j-1)} - e^{-rT_h}(\pi_{h-1}^2 - \pi_h^2 - \kappa_{l_h})V_{T_h}^{(l_h)} + \beta_h Corr_{P^i}(U_{T_h}, V_{T_h})(\pi_{h-1}^1 - \pi_h^1 + \pi_{h-1}^2 - \pi_h^2) + \gamma_{U_{T_h},V_{T_h}}(\pi_{h-1}^1 - \pi_h^1) + \gamma_{V_{T_h},U_{T_h}}(\pi_{h-1}^2 - \pi_h^2))]$.

**Remark:** We add all the N time periods inside the supreme. That is because we assume the same market condition throughout the whole trading periods. More specifically, we assume the whole transaction will be carried out in the same scenario under probability expectation. That is why we can calculate all the risk and take the supreme of all scenarios.

# CHAPTER 5
# OTHER VARIATIONS OF LIQUIDITY RISK MEASURES

## 5.1 Conditional Diversification

When we talk about risk diversification, we always assume that the more the better. The reasoning behind this is that if we have two positions $X$ and $Y$, then we may be better off if they move to opposite directions. Assume $\rho$ is a risk measure. Then we would argue that $\rho(X+Y) \leq \rho(X)+\rho(Y)$, which means risk diversification. If the prices of X and Y go to the same direction, then we have no difference holding them together or separately. Actually, the equality will hold in this case. If the prices of $X$ and $Y$ go to opposite direction, we are better off holding them together. In this case, we have $\rho(X+Y) < \rho(X)+\rho(Y)$. The basic assumption of this reasoning is that changing the position of $X$ will not affect the price of $Y$, or vice versa. However, it is not the case when the market is illiquid[6].

If the market is illiquid and $X$ is positively correlated with $Y$, then we may have $\rho(X+Y) > \rho(X)+\rho(Y)$. If we want to sell $X$ and $Y$ separately in an illiquid market, the price of $X$ will drop by $\$1$ and the price of $Y$ will drop by $\$2$. So we have $\rho(X) = 1$ and $\rho(Y) = 2$. However, if we sell $X$ and $Y$ together, the price of $X$ will drop by $1 and the price of $Y$ will drop by $2. But at the same time, the downward pressure of $X$ will draw the price of $Y$ further and vice versa. So the actual price drop of $X$ could be $1.1 and the actual price drop of $Y$ could be $2.2. So selling them together will end up with risk $\rho(X+Y) = 3.3$ which is bigger than $\rho(X)+\rho(Y)$.

If the market is illiquid and $X$ is negatively correlated with $Y$, then we could have diversification $\rho(X+Y) < \rho(X)+\rho(Y)$. If we want to sell $X$ and $Y$ separately in the illiquid market, the price of $X$ will drop by $\$1$ and the price of $Y$ will drop by $\$2$. So we have $\rho(X) = 1$ and $\rho(Y) = 2$. However, if we sell $X$ and $Y$ together, the price of $X$ will drop by $1 and the price of $Y$ will drop by $2. But since they are negatively correlated with each other, the downward pressure of $X$ will push up the price of $Y$ a little bit and

vice versa. So the actual drop of $X$ could be \$0.9 and the actual drop of $Y$ could be \$1.8. That means $\rho(X+Y) < \rho(X) + \rho(Y)$.

Now we see two different results may happen with risk diversification in an illiquid market. So it is necessary to define a risk measure to specify this situation. We define this class of risk measures as conditional diversification risk measures. The formal definition is as follows:

**Definition 5.1** A risk measure is called a conditional diversification risk measure if it satisfies the following axioms:

**Axiom M** Monotonicity. If $X \leq Y$, then $\rho(X) \geq Y$.

**Axiom T** Translation Invariance. If $m \in \mathbb{R}$, then $\rho(X + mr) = \rho(X) - m$.

**Axiom L** Liquidity. If $\lambda > 1$, $\rho(\lambda X) \geq \lambda\rho(X)$; if $\lambda \in [0, 1]$, $\rho(\lambda X) \leq \lambda\rho(X)$.

**Axiom C** Conditional Diversification. If $corr(X, Y) < 0$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

The difference between conditional diversification risk measure and convex liquidity risk measure is that conditional diversification risk measure is not convex. In theory, we prefer a convex risk measure because the convexity can yield nice representation theorem. However, the evidences from financial markets do not always support convexity. For example, the yield curve for Mortgage Backed Securities and callable bonds usually have negative convexity. That means they are concave. To get a better understanding of what is liquidity risk measure and conditional diversification risk measure, we examine some specific examples later.

## 5.2 Power Liquidity Risk Measures

In addition, the Translation Invariance axiom does not take interest rate risk into consideration. The Translation Invariance says when $\frac{m}{r}$ dollars are subtracted from the current position, the capital requirement is subtracted by the amount $m$ dollars. It assumes a constant interest rate $r$. However, in the real world, as mentioned by Karoui and Ravanelli[20], any form of uncertainty in interest rates will violate the Translation Invariance. In an extreme case, when the money market is illiquid, the interest rates

may also depend on $m$. In order to be real-life applicable, we define some specific risk measures without Translation Invariance. Because we only care about the cases when we lose money, we define the risk measures only for the case $X \leq 0$. We can make the measure constant for any positive $X$.

**Definition 5.2** A liquidity factor is a function $L : \mathbb{R}^+ \to \mathbb{R}^+$, such that $L(\lambda) \geq \lambda$ when $\lambda \geq 1$.

Liquidity factor can more precisely describe how liquidity issue may affect our overall risk. Since different markets have different liquidity situations, it makes sense to use different liquidity factors for different markets. Also, the same market may have different liquidity during different periods of time. So we need to specify a certain liquidity factor to a specific period of time. In summary, liquidity factor may change over time and markets. The next question is how can we choose appropriate liquidity factors. We will define some simple liquidity factors and approximate general situations by the simple factors.

**Definition 5.3** Power factor is a liquidity factor such that $L(\lambda X) = \lambda^k L(X)$, for $k \in \mathbb{Z}^+$.

**Definition 5.4** A liquidity risk measure without Translation Invariance is called a power liquidity risk measure if it satisfies the following Power axiom:

**Power axiom:** $\rho(\lambda X) = \lambda^k \rho(X)$ for some $k \in \mathbb{Z}^+$.

Power liquidity risk measure is useful because it is easy to calculate and good to approximate other risk measures. The following theorem defines a risk measure based on the power liquidity risk measures.

**Theorem 5.1** The risk measure $\rho_p(X) = E[\sum_{i=1}^n a_i(-X)^{b_i}]$ for $X \leq 0$ is a convex liquidity risk measure without Translation Invariance if $a_i > 0$ and $b_i$ are positive integers.

**Proof:** For any $X < 0$, $\rho_p(X) = E[\sum_{i=1}^n a_i(-X)^{b_i}] > E[\sum_{i=1}^n a_i(0)^{b_i}] = 0$.

Since all $a_i(-X)^{b_i}$ are convex functions given $a_i > 0$, their summation is also a convex function. So it satisfies the Convexity axiom.

For any $\lambda > 1$, $\rho_p(\lambda X) = E[\sum_{i=1}^n a_i(-\lambda X)^{b_i}] = E[\sum_{i=1}^n a_i\lambda^{b_i}(-X)^{b_i}] = \lambda E[\sum_{i=1}^n a_i\lambda^{b_i-1}(-X)^{b_i}] \geq \lambda E[\sum_{i=1}^n a_i(-X)^{b_i}] = \lambda\rho_p(X)$ since $b_i - 1 \geq 0$.

It does not satisfy Translation Invariance because $(X-m)^2 \neq X^2 - m$. □

### 5.3 Some Examples

In order to illustrate the liquidity risk measure is more appropriate than coherent risk measure, we need to find a liquidity risk measure that is not coherent. It is not a trivial example and we need some preparations.

We consider the liquidity cost proposed by Cetin et al[7] as a candidate. For simplicity, we only have one period, $[0, T]$. We consider a stock market here, although the subsequent model applies equally well to bonds, commodities, foreign currencies, etc. And we assume that the spot rate of interest is zero. Let the supply curve $S_t(x)$ represent the stock price per share at time $t$ with $x$ shares traded. A positive order $(x > 0)$ represents a buy and a negative order $(x < 0)$ represents a sale. Normally we assume the same price for any order size. That is a horizontal supply curve $S_t(0)$. But in an illiquid market, the supply curve should be $S_t(x)$, an increase curve. Since the more we buy, the higher the price of the stock will be. We will adopt the assumption in [7], that the supply curve has the form $S_t(x) = e^{\alpha x}S_t(0)$ with $\alpha > 0$. And throughout this paper, we will assume $\alpha = S_0(0)$.

Now we assume at the beginning, $t = 0$, we have $x$ shares, so the initial wealth is $xS_0(0)$. If the market is liquid, at time $t = T$, the net worth $X = x(S_T(0) - S_0(0))$. But in an illiquid market, the final net worth is $X^l = x(S_T(-x) - S_0(0))$. Now we define

$$\rho_l(X) = \sup_{P\in\mathcal{P}}\{E_P[-X^l]\} = \sup_{P\in\mathcal{P}}\{E_P[xS_0(0) - xS_T(-x)]\}. \tag{5–1}$$

**Theorem 5.2** The risk measure $\rho_l$ defined in (5-1) is a convex liquidity risk measure.

**Proof**: We need to check $\rho_l$ satisfies Axioms S, T, C and L:

If we have another position, say Y, it means we have the same amount of money at the beginning, but invested in another stock.

So $yU_0(0) = xS_0(0)$, where $U$ is the supply curve of another stock and $x$ is the shares held. So $Y^l = y(U_T(-y) - U_0(0))$.

Axiom S: This one is trivial. If $X < 0$, then $\rho_l(X) < \sup_{P\in\mathcal{P}}\{E_P[-X]\} \leq 0$.

Axiom T: $\rho_l(X+\alpha) = \sup_{P\in\mathcal{P}}\{E_P[-(X^l+\alpha)]\} = \sup_{P\in\mathcal{P}}\{E_P[-X^l]\} - \alpha = \rho(X) - \alpha$.

Axiom C: for any $\lambda \in [0,1]$,

$$\begin{aligned}
\rho_l(\lambda X + (1-\lambda)Y) &= \sup_{P\in\mathcal{P}}\{E_P[-((\lambda X)^l + ((1-\lambda)Y)^l)]\} \\
&= \sup_{P\in\mathcal{P}}\{E_P[-(\lambda X)^l]\} + \sup_{P\in\mathcal{P}}\{E_P[-((1-\lambda)Y)^l]\} \\
&= \rho_l(\lambda X) + \rho_l((1-\lambda)Y) \\
&\leq \lambda\rho_l(X) + (1-\lambda)\rho_l(Y)
\end{aligned}$$

The last inequality comes from the liquidity property proved below.

Axiom L: for any $k > 1$, as $S_T(x)$ is strictly increasing in $x$, $S_T(-kx) < S_T(-x)$.

$$\begin{aligned}
\rho(kX) &= \sup_{P\in\mathcal{P}}\{E_P[-(kX)^l]\} = \sup_{P\in\mathcal{P}}\{E_P[-kx(S_T(kx) - S_0(0))]\} \\
&= k\sup_{P\in\mathcal{P}}\{E_P[-x(S_T(kx) - S_0(0))]\} \\
&> k\sup_{P\in\mathcal{P}}\{E_P[-x(S_T(x) - S_0(0))]\} = k\rho_l(X)
\end{aligned}$$

The case $0 \leq k \leq 1$ is similar. □

**Remark:** As we have strict inequality in Axiom L, the Positive Homogeneity does not hold. So it is not a coherent risk measure. And the situation constructed above is more realistic, since prices do get affected by large orders. Hence, liquidity risk measure is more appropriated in this case.

An example of conditional diversification risk measure is the following log-exponential risk measure, given $\lambda > 0$:

$$\rho(X) = \lambda \log(E[e^{-\frac{X}{\lambda}}]) \tag{5–2}$$

We will prove this risk measure satisfies the axioms of conditional diversification risk measure.

**Theorem 5.3** The risk measure defined in (5-2) is a conditional diversification risk measure but it is not coherent.

Proof: Clearly, the function $\rho$ is decreasing, so the Monotonicity axiom is satisfied.

For simplicity, we consider here $r = 1$. We calculate $\rho(X + m) = \lambda \log(E[e^{\frac{-(x+m)}{\lambda}}]) = \lambda \log(E[e^{\frac{-x}{\lambda}} e^{\frac{-m}{\lambda}}]) = \lambda \log(e^{\frac{-m}{\lambda}} E[e^{\frac{-x}{\lambda}}]) = \lambda(\frac{-m}{\lambda} + \log(E[e^{\frac{-x}{\lambda}}])) = \rho(X) - m$. So Translation Invariance is satisfied.

To test Liquidity, we let $\alpha > 1$, so $\rho(\alpha X) = \lambda \log(E[e^{\frac{-\alpha x}{\lambda}}]) = \lambda \log(E[(e^{\frac{-x}{\lambda}})^{\alpha}]) \geq \lambda \log(E[e^{\frac{-x}{\lambda}}])^{\alpha} = \alpha\lambda \log(E[e^{\frac{-x}{\lambda}}]) = \alpha\rho(X)$. We can get this relationship by applying Jensen's inequality. If $\alpha > 1$, then $X^{\alpha}$ is convex.

For Conditional Diversification, we assume $corr(X, Y) \leq 0$. Since we have $E[XY] = E[X]E[Y] + cov(X, Y)$, $E[XY] \leq E[X]E[Y]$. So $\rho(X + Y) = \lambda \log(E[e^{-\frac{x+y}{\lambda}}]) = \lambda \log(E[e^{-\frac{x}{\lambda}} e^{-\frac{y}{\lambda}}]) \leq \lambda \log(E[e^{-\frac{x}{\lambda}}]E[e^{-\frac{y}{\lambda}}]) = \lambda \log(E[e^{-\frac{x}{\lambda}}]) + \lambda \log(E[e^{-\frac{y}{\lambda}}]) = \rho(X) + \rho(Y)$.
□

**Remark:** If we take the weighted average of a series of conditional diversification risk measures, the result is also a conditional diversification risk measure.

For an example of power liquidity risk measure, we want a foreign stock in the US market, which is not very liquid. The stock we picked here is China Mass Media Corp.(CMM). Because it is very difficult to find historical trading data and volume of a certain party, we consider the market has only two participants, the buyer and seller. Below is a graph of how the adjusted price changes according to the trading volumes. The adjusted price is the close price adjusted to dividends.

For this stock, as shown in Figure 5-1, its risk measure cannot be linear, and power liquidity risk measure can do a better job. Either in small trading volume range (below 20,000 shares) or large trading volume range (above 80,000 shares), the price is driven up by buying powers. In the middle range (between 20,000 and 80,000

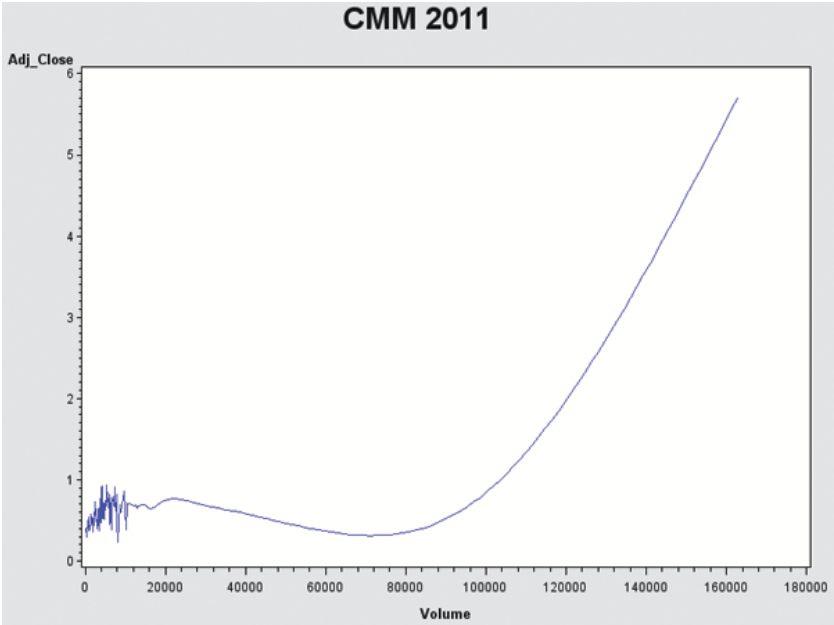


Figure 5-1. Non-Linear Risk Measure

shares), the price is driven by selling powers. And in each range, it is not a simple linear relationship. Another advantage of power liquidity risk measure is that it can be applied to approximate other risk measures in computation without explicit expressions.

# CHAPTER 6
# OPTIMAL BALANCED TRADING STRATEGY

Risk optimization problems have been widely studied by numerous researchers[12, 29]. Trading constraints are highly important in optimizing risk measures[15]. Liquidity risk can also be affected by trading strategies. A trading strategy has three major elements: the trading volume, the length of trading and the direction of trading. There is definitely a limit of the amount you can trade in a short period of time. Trading too much in a short time could cause either failure or huge liquidity cost. The total duration of trading also matters. We can model the duration by the number of steps of the trading. For example, there is only 5 buy or sell can be made in the period of time. The trading direction simply means either buying or selling in a transaction. We prefer a balanced trading strategy if it exists. That is because it eliminates the interaction between the trading assets during the transaction. In this section, we are going to find the balanced trading strategy that will minimize the overall risk of the transaction and we call such strategy the optimal balanced trading strategy, or OBTS. We will only consider two assets case here. We will adopt Definition 4.11 and assume consistent N-period transactions. So the risk measure is $\rho_N(U, V) = \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 + \pi_0^2 V_0 - \sum_{h=1}^{N}(e^{-rT_h}\sum_{j=1}^{k_h}(\tau_j - \tau_{j-1})U_{T_h}^{(j-1)} - e^{-rT_h}(\pi_{h-1}^1 - \pi_h^1 - \tau_{k_h})U_{T_h}^{(k_h)} - e^{-rT_h}\sum_{j=1}^{l_h}(\kappa_j - \kappa_{j-1})V_{T_h}^{(j-1)} - e^{-rT_h}(\pi_{h-1}^2 - \pi_h^2 - \kappa_{l_h})V_{T_h}^{(l_h)}) + \beta Corr_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2]$.

Under these assumptions, only a certain kind of transactions will have the OBTS. By definition, no matter how many steps of trading, there is only one requirement for existence of balanced trading strategy for each step. That is the total amount of trading of asset 1 $\pi_0^1 = -\frac{\gamma_{V,U}}{\gamma_{U,V}}\pi_0^2$, where $\pi_0^2$ is the total amount of trading of asset 2. We will discuss the simplest version of the problem first, i.e. $N = 2$. So the risk measure becomes $\rho_2(U, V) = \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 + \pi_0^2 V_0 - e^{-rT_1}\sum_{j=1}^{k_1}(\tau_j - \tau_{j-1})U_{T_1}^{(j-1)} - e^{-rT_1}(\pi_0^1 - \pi_1^1 - \tau_{k_1})U_{T_1}^{(k_1)} - e^{-rT_2}\sum_{j=1}^{k_2}(\tau_j - \tau_{j-1})U_{T_2}^{(j-1)} - e^{-rT_2}(\pi_1^1 - \tau_{k_2})U_{T_2}^{(k_2)} - e^{-rT_1}\sum_{j=1}^{l_1}(\kappa_j - \kappa_{j-1})V_{T_1}^{(j-1)} -$

$e^{-rT_1}(\pi_0^2-\pi_1^2-\kappa_{l_1})V_{T_1}^{(l_1)}-e^{-rT_2}\sum_{j=1}^{l_2}(\kappa_j-\kappa_{j-1})V_{T_2}^{(j-1)}-e^{-rT_2}(\pi_1^2-\kappa_{l_2})V_{T_2}^{(l_2)}+\beta Corr_{P^i}(U,V)(\pi_0^1+\pi_0^2)+\gamma_{U,V}\pi_0^1+\gamma_{V,U}\pi_0^2]$. And the optimization problem turns to be

$$
\begin{aligned}
\underset{\pi_1^1,\pi_1^2}{\text{minimize}} \quad & \sup_{i\in I}\{E_{P^i}[-e^{-rT_1}\sum_{j=1}^{k_1}(\tau_j-\tau_{j-1})U_{T_1}^{(j-1)}-e^{-rT_1}(\pi_0^1-\pi_1^1-\tau_{k_1})U_{T_1}^{(k_1)} \\
& -e^{-rT_2}\sum_{j=1}^{k_2}(\tau_j-\tau_{j-1})U_{T_2}^{(j-1)}-e^{-rT_2}(\pi_1^1-\tau_{k_2})U_{T_2}^{(k_2)}-e^{-rT_1}\sum_{j=1}^{l_1}(\kappa_j-\kappa_{j-1})V_{T_1}^{(j-1)} \\
& -e^{-rT_1}(\pi_0^2-\pi_1^2-\kappa_{l_1})V_{T_1}^{(l_1)}-e^{-rT_2}\sum_{j=1}^{l_2}(\kappa_j-\kappa_{j-1})V_{T_2}^{(j-1)}-e^{-rT_2}(\pi_1^2-\kappa_{l_2})V_{T_2}^{(l_2)}] \\
& +\beta Corr_{P^i}(U,V)(\pi_0^1+\pi_0^2)\} \\
\text{subject to} \quad & \gamma_{U,V}\pi_1^1+\gamma_{V,U}\pi_1^2=0, \\
& 0\le\pi_1^1\le\pi_0^1,\ 0\le\pi_1^2\le\pi_0^2.
\end{aligned}
$$

### 6.1 Separate Trading Strategy

To solve this optimization problem, we can first solve for $\pi_1^2$ using the first constraint and substitute it into the objective function to eliminate the variable $\pi_1^2$. From the constraint, $\pi_1^2=-\frac{\gamma_{U,V}}{\gamma_{V,U}}\pi_1^1$. So the optimization problem becomes

$$
\begin{aligned}
\underset{\pi_1^1}{\text{minimize}} \quad & \sup_{i\in I}\{E_{P^i}[-e^{-rT_1}\sum_{j=1}^{k_1}(\tau_j-\tau_{j-1})U_{T_1}^{(j-1)}-e^{-rT_1}(\pi_0^1-\pi_1^1-\tau_{k_1})U_{T_1}^{(k_1)} \\
& -e^{-rT_2}\sum_{j=1}^{k_2}(\tau_j-\tau_{j-1})U_{T_2}^{(j-1)}-e^{-rT_2}(\pi_1^1-\tau_{k_2})U_{T_2}^{(k_2)}-e^{-rT_1}\sum_{j=1}^{l_1}(\kappa_j-\kappa_{j-1})V_{T_1}^{(j-1)} \\
& -e^{-rT_1}(\pi_0^2+\frac{\gamma_{U,V}}{\gamma_{V,U}}\pi_1^1-\kappa_{l_1})V_{T_1}^{(l_1)}-e^{-rT_2}\sum_{j=1}^{l_2}(\kappa_j-\kappa_{j-1})V_{T_2}^{(j-1)} \\
& +e^{-rT_2}(\frac{\gamma_{U,V}}{\gamma_{V,U}}\pi_1^1+\kappa_{l_2})V_{T_2}^{(l_2)}]+\beta Corr_{P^i}(U,V)(\pi_0^1+\pi_0^2)\} \\
\text{subject to} \quad & 0\le\pi_1^1\le\pi_0^1.
\end{aligned}
$$

This is not a simple optimization problem because the objective function will change depending on the value of $\pi_1^1$. When $\pi_1^1$ changes, $k_1$ will change depending on $\pi_0^1-\pi_1^1$ and $k_2$ will change depending on $\pi_1^1$. These changes also depend on the structure of

the thresholds of the stochastic price processes. So there is no general explicit solution. To explore the possible trivial solutions, we prove below that trading separately is better than trading together and hence eliminates trivial solutions.

**Theorem 6.1** It makes sense to trade the whole portfolio separately under consistent multi-period assumption. More precisely, $\rho_2(U, V) \leq \rho(U, V)$.

**Proof:** We denote the term inside the expectation to be $f_2(U, V)$ and $f(U, V)$ respectively. Under consistent multi-period assumption, we have

$$
\begin{aligned}
& f(U, V) - f_2(U, V) \\
= \; & \pi_0^1 U_0 + \pi_0^2 V_0 - e^{-rT} \sum_{j=1}^{k} (\tau_j - \tau_{j-1}) U_T^{(j-1)} - e^{-rT} (\pi_0^1 - \tau_k) U_T^{(k)} \\
& - e^{-rT} \sum_{j=1}^{l} (\kappa_j - \kappa_{j-1}) V_T^{(j-1)} - e^{-rT} (\pi_0^2 - \kappa_l) V_T^{(l)} + \beta Corr_{P^i}(U, V)(\pi_0^1 + \pi_0^2) \\
& + \gamma_{U,V} \pi_0^1 + \gamma_{V,U} \pi_0^2 - (\pi_0^1 U_0 + \pi_0^2 V_0 - e^{-rT_1} \sum_{j=1}^{k_1} (\tau_j - \tau_{j-1}) U_{T_1}^{(j-1)} - e^{-rT_1} (\pi_0^1 - \pi_1^1 - \tau_{k_1}) U_{T_1}^{(k_1)} \\
& - e^{-rT_2} \sum_{j=1}^{k_2} (\tau_j - \tau_{j-1}) U_{T_2}^{(j-1)} - e^{-rT_2} (\pi_1^1 - \tau_{k_2}) U_{T_2}^{(k_2)} - e^{-rT_1} \sum_{j=1}^{l_1} (\kappa_j - \kappa_{j-1}) V_{T_1}^{(j-1)} \\
& - e^{-rT_1} (\pi_0^2 - \pi_1^2 - \kappa_{l_1}) V_{T_1}^{(l_1)} - e^{-rT_2} \sum_{j=1}^{l_2} (\kappa_j - \kappa_{j-1}) V_{T_2}^{(j-1)} - e^{-rT_2} (\pi_1^2 - \kappa_{l_2}) V_{T_2}^{(l_2)} \\
& + \beta Corr_{P^i}(U, V)(\pi_0^1 + \pi_0^2) + \gamma_{U,V} \pi_0^1 + \gamma_{V,U} \pi_0^2) \\
= \; & e^{-rT_1} \sum_{j=1}^{k_1} (\tau_j - \tau_{j-1}) U_{T_1}^{(j-1)} + e^{-rT_1} (\pi_0^1 - \pi_1^1 - \tau_{k_1}) U_{T_1}^{(k_1)} + e^{-rT_2} \sum_{j=1}^{k_2} (\tau_j - \tau_{j-1}) U_{T_2}^{(j-1)} \\
& + e^{-rT_2} (\pi_1^1 - \tau_{k_2}) U_{T_2}^{(k_2)} + e^{-rT_1} \sum_{j=1}^{l_1} (\kappa_j - \kappa_{j-1}) V_{T_1}^{(j-1)} + e^{-rT_1} (\pi_0^2 - \pi_1^2 - \kappa_{l_1}) V_{T_1}^{(l_1)} \\
& + e^{-rT_2} \sum_{j=1}^{l_2} (\kappa_j - \kappa_{j-1}) V_{T_2}^{(j-1)} + e^{-rT_2} (\pi_1^2 - \kappa_{l_2}) V_{T_2}^{(l_2)} - e^{-rT} \sum_{j=1}^{k} (\tau_j - \tau_{j-1}) U_T^{(j-1)} \\
& - e^{-rT} (\pi_0^1 - \tau_k) U_T^{(k)} - e^{-rT} \sum_{j=1}^{l} (\kappa_j - \kappa_{j-1}) V_T^{(j-1)} - e^{-rT} (\pi_0^2 - \kappa_l) V_T^{(l)}
\end{aligned}
$$

Since $\tau_{k_1} \leq \pi_0^1 - \pi_1^1 < \tau_{k_1+1}$, $\tau_{k_2} \leq \pi_1^1 < \tau_{k_2+1}$ and $\tau_k \leq \pi_0^1 < \tau_{k+1}$, by Threshold Assumption, we have $\tau_{k_1} + \tau_{k_2} \leq \tau_k$ and $k_1 + k_2 \geq k$.

$$
\begin{aligned}
E[f - f_2] \;\geq\;& E[e^{-rT_1}\frac{k_1}{k}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_{T_1}^{(j-1)} + e^{-rT_1}(\pi_0^1 - \pi_1^1 - \tau_{k_1})U_{T_1}^{(k)} \\
&+e^{-rT_2}\frac{k_2}{k}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_{T_2}^{(j-1)} + e^{-rT_2}(\pi_1^1 - \tau_{k_2})U_{T_2}^{(k)} + e^{-rT_1}\frac{l_1}{l}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_{T_1}^{(j-1)} \\
&+e^{-rT_1}(\pi_0^2 - \pi_1^2 - \kappa_{l_1})V_{T_1}^{(l)} + e^{-rT_2}\frac{l_2}{l}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_{T_2}^{(j-1)} + e^{-rT_2}(\pi_1^2 - \kappa_{l_2})V_{T_2}^{(l)} \\
&-e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)U_T^{(k)} - e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_T^{(j-1)} \\
&-e^{-rT}(\pi_0^2 - \kappa_l)V_T^{(l)}] \\
\geq\;& e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})E[U_T^{(j-1)}] + e^{-rT_1}(\pi_0^1 - \pi_1^1 - \tau_{k_1})E[U_{T_1}^{(k)}] \\
&+e^{-rT_2}(\pi_1^1 - \tau_{k_2})E[U_{T_2}^{(k)}] + e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})E[V_T^{(j-1)}] \\
&+e^{-rT_1}(\pi_0^2 - \pi_1^2 - \kappa_{l_1})E[V_{T_1}^{(l)}] + e^{-rT_2}(\pi_1^2 - \kappa_{l_2})E[V_{T_2}^{(l)}] \\
&-e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})E[U_T^{(j-1)}] - e^{-rT}(\pi_0^1 - \tau_k)E[U_T^{(k)}] \\
&-e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})E[V_T^{(j-1)}] - e^{-rT}(\pi_0^2 - \kappa_l)E[V_T^{(l)}] \\
\geq\;& (\tau_k - \tau_{k_1} - \tau_{k_2})E[U^(k)] + (\kappa - \kappa_{l_1} - \kappa_{l_2}) \geq 0.
\end{aligned}
$$

So $E[f(U,V)] \geq E[f_2(U,V)]$. That means $\rho_2(U,V) \leq \rho(U,V)$. □

## 6.2 Solution to A Simple Case

To illustrate the optimization problem has nontrivial solutions and the idea of separate trading, we can consider a simple example. We consider only one asset S and two trading periods $T_1$ and $T_2$. We suppose the beginning asset is $\pi_0^1 = 20$ shares and the thresholds $\tau_1 = 10$ and $\tau_2 = 20$. We need to liquidate all the 20 shares by the end of period 2. The asset prices at $T = 1$ are $S_{T_1}^{(1)} = 10$ and $S_{T_1}^{(2)} = 9$. And the asset prices at $T = 2$ are $S_{T_2}^{(1)} = 11$ and $S_{T_2}^{(2)} = 8$ as shown in Figure 6-1. We want to solve for the

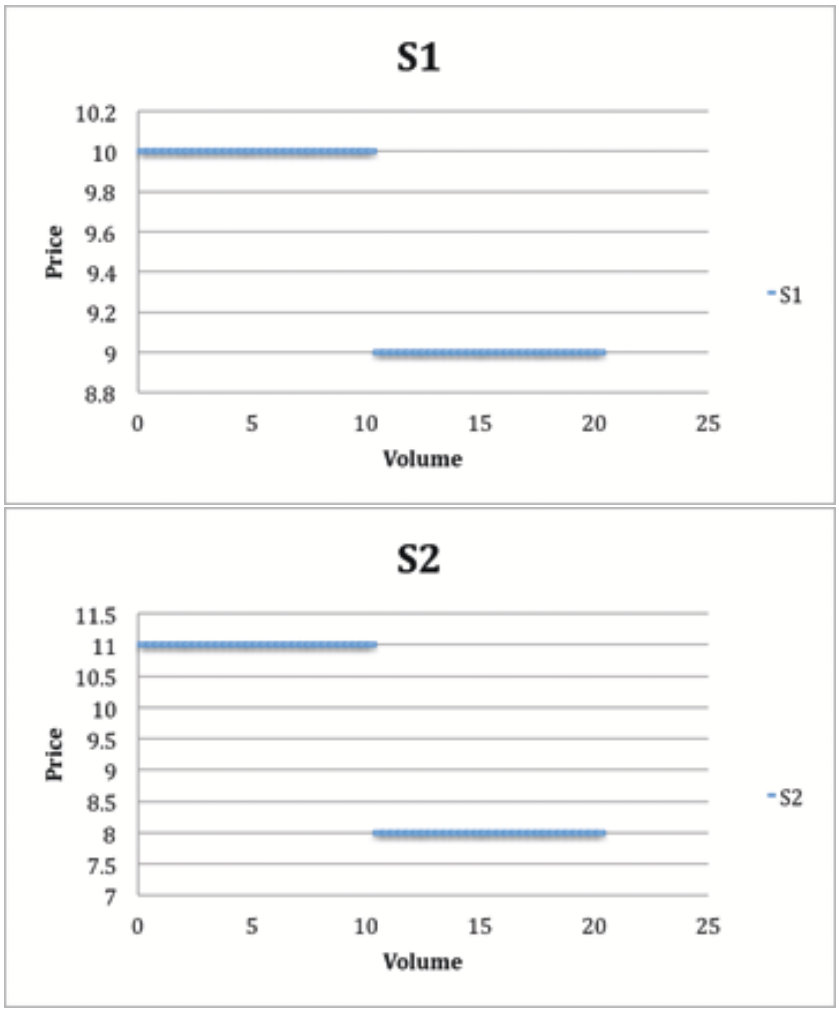


Figure 6-1. Asset Price Process

optimal trading strategy $\pi_1^1$ to maximize our gain. That is equivalent to minimize the risk measure.

If $\pi_1^1 \leq 10$, then our total gain is $G(\pi_1^1) = 10*10+9(20-\pi_1^1-10)+11\pi_1^1 = 190+2\pi_1^1$. It is a simple linear optimization. Its optimal solution is $\pi_1^1 = 10$ and $G = 210$.

If $\pi_1^1 \geq 10$, then $G(\pi_1^1) = 10(20-\pi_1^1)+11*10+8(\pi_1^1-10) = 230-2\pi_1^1$. The optimal solution to this linear problem is $\pi_1^1 = 10$ and $G = 210$.

In both cases, we get the same optimal solutions. That means $\pi_1^1 = 10$ is the optimal solution to the whole problem. It is a nontrivial trading strategy and it demonstrates separate trading is better than whole sales. For general situations, we

have more complex scenarios, and the solutions become more and more difficult to calculate. Since there is no general solution, it makes more sense to solve it using simulations.

### 6.3 Optimal Solutions Set

Generally speaking, there is no unique solution to this optimization problem. Instead, there are infinitely many solutions. In fact, this is a trivial problem for risk measures without liquidity effects. That is, no matter how many you are trading in each step, you will get absolutely the same price and hence the same total return. So the risk measure is the same. Every number between $\pi_0^1$ and 0 is a valid solution and also an optimal solution. So the optimal solution set is $[0, \pi_0^1]$. For liquidity risk measures, the difference is we have a decreasing supply function for the asset. But since this function is discrete, it does not matter how much it passes the threshold. The only thing matters is whether the supply passes the threshold. Therefore, the optimal solution set should be a subset of $[0, \pi_0^1]$.

To explore the solution sets, we begin with the simple case: assuming no interest rate and consistent multi-period. So the optimization problem changes to the following:

$$\begin{aligned}
\underset{\pi_1^1}{\text{minimize}} \quad & \sup_{i\in I}\{E_{P^i}[\sum_{j=1}^{k_1}(\tau_j-\tau_{j-1})U^{(j-1)}-(\pi_0^1-\pi_1^1-\tau_{k_1})U^{(k_1)} \\
& -\sum_{j=1}^{k_2}(\tau_j-\tau_{j-1})U^{(j-1)}-(\pi_1^1-\tau_{k_2})U^{(k_2)}-\sum_{j=1}^{l_1}(\kappa_j-\kappa_{j-1})V^{(j-1)} \\
& -(\pi_0^2+\frac{\gamma_{U,V}}{\gamma_{V,U}}\pi_1^1-\kappa_{l_1})V^{(l_1)}-\sum_{j=1}^{l_2}(\kappa_j-\kappa_{j-1})V^{(j-1)} \\
& +(\frac{\gamma_{U,V}}{\gamma_{V,U}}\pi_1^1+\kappa_{l_2})V^{(l_2)}]+\beta Corr_{P^i}(U,V)(\pi_0^1+\pi_0^2)\} \\
\text{subject to} \quad & 0\le\pi_1^1\le\pi_0^1.
\end{aligned}$$

We will show below that there is an explicit optimal solution set for this optimization problem. Before that, we will examine the assumptions first. We are considering two assets in two time periods. We first assume there exists optimal balanced trading

strategy, i.e. $\gamma_{U,V}\pi_0^1 + \gamma_{V,U}\pi_0^2 = 0$ and $\gamma_{U,V}\pi_1^1 + \gamma_{V,U}\pi_1^2 = 0$. So we must have $\gamma_{U,V}\gamma_{V,U} < 0$. Here we are studying liquidation in a short period of time. Therefore, the time value of money could be ignored. Strictly speaking, the trading in previous period may have price impact on the next period. But here we just assume the price distributions are the same in the two time periods. We will see how results change later for different price distributions. Under these assumptions, we have the following result:

**Theorem 6.2** Under optimal balanced trading strategy, no interest rate, and consistent multi-period assumptions, the above optimization problem has an optimal solution set $OSS = (\tau_k, \tau_{k+1}] \cap [\pi_0^1 - \tau_{k+1}, \pi_0^1 - \tau_k) \cap (-\frac{\gamma_{V,U}}{\gamma_{U,V}}\kappa_l, -\frac{\gamma_{V,U}}{\gamma_{U,V}}\kappa_{l+1}] \cap [\frac{\gamma_{V,U}}{\gamma_{U,V}}(\kappa_{l+1} - \pi_0^2), \frac{\gamma_{V,U}}{\gamma_{U,V}}(\kappa_l - \pi_0^2))$, where $\tau_k < \frac{\pi_0^1}{2} \leq \tau_{k+1}$ and $\kappa_l < \frac{\pi_0^2}{2} \leq \kappa_{l+1}$.

**Proof:** Since the price distribution of the same asset is the same in the two time period, the equally distributed trading amount $\pi_1^1 = \frac{\pi_0^1}{2}$ should be an optimal solution. If not, there should be another price strategy achieving a smaller value. Now assume the thresholds for $\frac{\pi_0^1}{2}$ are $\tau_k$ and $\tau_{k+1}$. Then $\tau_k < \frac{\pi_0^1}{2} \leq \tau_{k+1}$. To achieve smaller value, we want smaller $k$ value. To find a different strategy, we must have $\pi_1^1 < \tau_k$. That means $\pi_0^1 - \pi_1^1 > \tau_k$. This is not optimal because $\pi_1^1$ doesn't take the full advantage of the price stage $\tau_k$ and $\pi_0^1 - \pi_1^1$ may in the threshold that even further than $\tau_{k+1}$. So the result of this solution must be greater than $\pi_1^1 = \frac{\pi_0^1}{2}$. In addition, any trading volume between these two thresholds should be an optimal solution. Therefore, we must have $\tau_k < \pi_1^1 \leq \tau_{k+1}$ and $\tau_k < \pi_0^1 - \pi_1^1 \leq \tau_{k+1}$. So the solution is the set $(\tau_k, \tau_{k+1}] \cap [\pi_0^1 - \tau_{k+1}, \pi_0^1 - \tau_k)$.

We can conduct the same argument for the second asset. We find the thresholds $\kappa_l < \frac{\pi_0^2}{2} \leq \kappa_{l+1}$. Then the trading volume must satisfy

$$\kappa_l < -\frac{\gamma_{U,V}}{\gamma_{V,U}}\pi_1^1 \leq \kappa_{l+1}$$
$$\kappa_l < \pi_0^2 + \frac{\gamma_{U,V}}{\gamma_{V,U}}\pi_1^1 \leq \kappa_{l+1}$$

Solving the inequalities, we have the set $(-\frac{\gamma_{V,U}}{\gamma_{U,V}}\kappa_l, -\frac{\gamma_{V,U}}{\gamma_{U,V}}\kappa_{l+1}] \cap [\frac{\gamma_{V,U}}{\gamma_{U,V}}(\kappa_{l+1} - \pi_0^2), \frac{\gamma_{V,U}}{\gamma_{U,V}}(\kappa_l - \pi_0^2)$.

Since $\pi_1^1$ should satisfy both conditions, the optimal solution set is $OSS = (\tau_k, \tau_{k+1}] \cap [\pi_0^1 - \tau_{k+1}, \pi_0^1 - \tau_k) \cap (-\frac{\gamma_{V,U}}{\gamma_{U,V}}\kappa_l, -\frac{\gamma_{V,U}}{\gamma_{U,V}}\kappa_{l+1}] \cap [\frac{\gamma_{V,U}}{\gamma_{U,V}}(\kappa_{l+1} - \pi_0^2), \frac{\gamma_{V,U}}{\gamma_{U,V}}(\kappa_l - \pi_0^2)$. $\square$

**Remark:** This optimal solution set does not depend on the probability scenarios. That is because under any scenario, we have the same optimal solution. So the supremum on all scenarios does not affect the optimal solution set. This is not a general conclusion. We can see later that different scenarios do have different solutions.

### 6.4 General Solutions

Now we discuss the solutions to general cases. By general case, we mean two different risky assets with different price distributions for N time periods with nonzero interest rates. The only assumption we make here is that they have the same price impact factors over time, i.e. $\gamma$ dose not depends on time. As mentioned before, there is no explicit solutions for the general cases. However, we can design an algorithm to calculate the optimal solutions numerically. Before that, we will examine the conditions for the optimization problem to have an unique solution. We can write the general case optimization problem after taking out non-influential terms as below:

$$\begin{aligned}
\underset{\pi_1^1,\pi_1^2}{\text{minimize}} \quad & \sup_{i\in I}\{E_{P^i}[-\sum_{h=1}^{N}(e^{-rT_h}\sum_{j=1}^{k_h}(\tau_j - \tau_{j-1})U_{T_h}^{(j-1)} - e^{-rT_h}(\pi_{h-1}^1 - \pi_h^1 - \tau_{k_h})U_{T_h}^{(k_h)} \\
& - e^{-rT_h}\sum_{j=1}^{l_h}(\kappa_j - \kappa_{j-1})V_{T_h}^{(j-1)} - e^{-rT_h}(\pi_{h-1}^2 - \pi_h^2 - \kappa_{l_h})V_{T_h}^{(l_h)})] \\
& + \beta Corr_{P^i}(U, V)(\pi_0^1 + \pi_0^2)\} \\
\text{subject to} \quad & 0 \le \pi_h^1 \le \pi_0^1,\ 0 \le \pi_h^2 \le \pi_0^2 \text{ for } h = 1 \cdots N.
\end{aligned}$$

For the convenience on notations, we define the set of all expectations of discounted assets prices in the following:

**Definition 6.1** The expectation price set for asset U with N trading periods $\mathcal{U}_N$ is a set of all expectations of discounted asset prices. Specifically, $\mathcal{U}_N^i = \{e^{-rT_i}E_{P^i}U_j^h \mid j = 1, \cdots N, h \in \mathbb{Z}^+$.

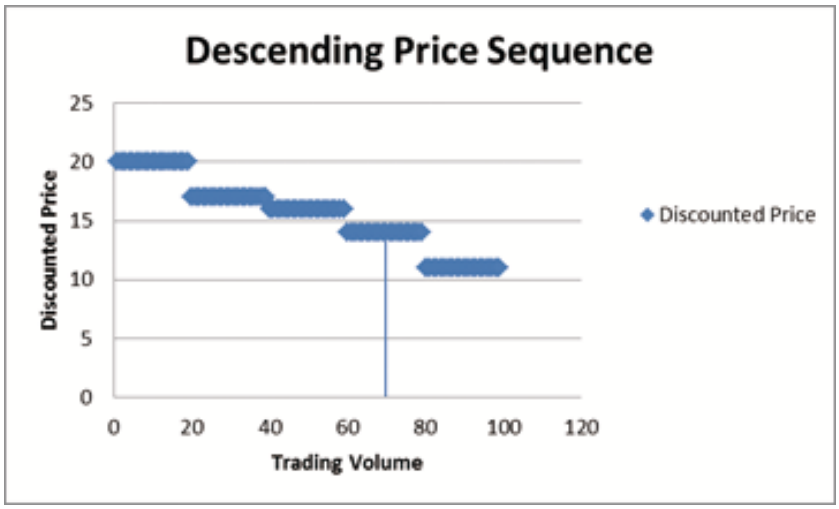


Figure 6-2. Descending Price Sequence with Trading Volume

**Theorem 6.3** The optimization problem has an unique optimal solution if there are no two elements in $\mathcal{U}_N^i$ and $\mathcal{V}_N^i$ are the same.

**Proof:** Since there are no two elements in $\mathcal{U}_N^i$ and $\mathcal{V}_N^i$ are the same, we can sort these two sets in strictly descending orders. And we can draw a graph with y-axis to be the elements in the set and x-axis to be the corresponding maximal trading volume(Figure 6-2). Then $\pi_0^1$ must falls in an interval says $(\tau_{m-1}^{T_h}, \tau_m^{T_h}]$. Then we get a set $M$ of intervals that on the left of $(\tau_{m-1}^{T_h}, \tau_m^{T_h}]$. Then the optimal solution for asset U is $\pi_0^1 - \pi_1^1 = \sum_M (\tau_j^{T_1} - \tau_{j-1}^{T_1}), \cdots, \pi_{h-1}^1 - \pi_h^1 = \sum_M (\tau_j^{T_1} - \tau_{j-1}^{T_1}) + (\pi_0^1 - \tau_{m-1}^{T_h}), \cdots$.

The above solution is unique and optimal. The area under the piecewise function is the negative of the risk measure. To minimize the risk, we need to maximize this area. It is optimal because if we exchange any interval in $M$ with some arbitrary intervals outside $M$, the area under the function will decrease. That is because the function is decreasing. This solution is unique because any other solution will decrease the area under the function, hence leading to suboptimal solutions. □

**Remark:** In the above proof, we found the optimal solution and write it in the form of summations of corresponding intervals. This result is so important that we will restate it separately in Theorem 6.4.

**Theorem 6.4** Under the same assumptions, if there are no two elements in $\mathcal{U}_N^i$ are the same then the optimal solution for asset U is $\pi_0^1 - \pi_1^1 = \sum_M (\tau_j^{T_1} - \tau_{j-1}^{T_1}), \cdots,$ $\pi_{h-1}^1 - \pi_h^1 = \sum_M (\tau_j^{T_1} - \tau_{j-1}^{T_1}) + (\pi_0^1 - \tau_{m-1}^{T_h}), \cdots.$

**Remark:** The proof is the same as Theorem 6.3. The result actually provides a computer algorithm to find the optimal solutions. The complete algorithm is in the following.

**Algorithm**

(1) Generate random variables $U_{T_h}^{(i)}$ and $V_{T_h}^{(j)}$ according to the joint distribution $f(U, V)$ for $h = 1$ to $N$, $i = 0$ to $k_h$ and $j = 0$ to $j_h$.

(2) Sort $e^{-rT_h} U_{T_h}^{(i)}$ and $e^{-rT_h} V_{T_h}^{(j)}$ according to descending order separately.

(3) Find the corresponding trading volumes sequences $M$ for U and $Q$ for V.

(4) Set $s = 0$ and $t = 0$.

(5) $s = s + (\tau_{i+1}^{T_h} - \tau_i^{T_h})$ in the order of the sequence in (2).

(6) $t = t + (\kappa_{i+1}^{T_h} - \kappa_i^{T_h})$ in the order of the sequence in (2).

(7) Repeat (5) until $s \geq \pi_0^1$ and record the interval $(\tau_{m-1}^{T_a}, \tau_m^{T_a}]$.

(8) Repeat (6) until $t \geq \pi_0^2$ and record the interval $(\kappa_{q-1}^{T_b}, \kappa_q^{T_b}]$.

(9) $\pi_{N-1}^1 = \sum_M (\tau_j^{T_{N-1}} - \tau_{j-1}^{T_{N-1}}), \pi_{N-2}^1 = \pi_{N-1}^1 + \sum_M (\tau_j^{T_{N-2}} - \tau_{j-1}^{T_{N-2}}), \cdots, \pi_a^1 =$ $\pi_{a+1}^1 + \sum_M (\tau_j^{T_{N-2}} - \tau_{j-1}^{T_{N-2}}) + (\pi_0^1 - \tau_{m-1}^{T_a}), \cdots.$

(10) $\pi_{N-1}^2 = \sum_Q (\kappa_i^{T_{N-1}} - \kappa_{i-1}^{T_{N-1}}), \pi_{N-2}^2 = \pi_{N-1}^2 + \sum_Q (\kappa_i^{T_{N-2}} - \kappa_{i-1}^{T_{N-2}}), \cdots, \pi_b^2 =$ $\pi_{b+1}^2 + \sum_Q (\kappa_i^{T_{N-2}} - \kappa_{i-1}^{T_{N-2}}) + (\pi_0^2 - \kappa_{q-1}^{T_b}), \cdots.$

(11) Calculate the value inside supremum according to the formula under scenario $i$ and denote the value as $G_i$.

(12) Repeat (1)-(11) until get all $G_i$ for $i \in I$.

(13) $\rho = \max G_i$.

(14) Repeat (1)-(13) for $R$ times.

(15) Pick the maximal $\rho$ in the $R$ times and the corresponding trading strategy is the optimal trading strategy.

## 6.5 Numerical Examples

In this section, we will apply the Algorithm to solve a real problem and compare the result with ordinary trading strategies. For simplicity, we consider the case with only one risky asset $S$ in three trading periods. We also assume that $S_t, S_t^{(1)}, S_t^{(2)}$ follow geometric Brownian motion. Recall that a stochastic process $S_t$ is a geometric Brownian motion if $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$, where $B_t$ is an ordinary Brownian motion. The initial price is known $S_0 = 10$. Assume they have the same deviation $\sigma = 0.2$. Their $\mu$ values are as follow: $\mu = 0.2, \mu^{(1)} = 0.15, \mu^{(3)} = 0.1$. We have the following thresholds $\tau_1^{(1)} = 10, \tau_2^{(1)} = 30, \tau_3^{(3)} = 100, \tau_1^{(2)} = 20, \tau_2^{(2)} = 70, \tau_3^{(2)} = 100, \tau_1^{(3)} = 10, \tau_2^{(3)} = 20, \tau_3^{(3)} = 100$. We also assume equal time periods, i.e. $T = 0, 1, 2, 3$. We also assume three different scenarios: log-normal distribution, exponential distribution and uniform distribution corresponding to $i = 1, 2, 3$.

We know the mean of geometric Brownian motion under log-normal distribution is $E_1[S_t] = S_0 e^{\mu t}$. But we need to simulate the expectations under the other two distributions. For numerical computation, we can assume the period interest rate is $r = 3\%$.

We run the program in Matlab® and get convergent results shown in Figure 6-3. The average optimal trading strategies are $8.88, 56.28, 34.84$ and the corresponding risk measure is $\rho = 25.41$ shown in Figure 6-4 and Figure 6-5. We also compare our optimal trading strategy with general trading strategies. The general trading strategy is to trade equal amount in each time period. In our case, the general trading strategy is trading $33.33$ in period 1, 2, and 3 and the corresponding risk measure is $\rho = 29.16$. Our result shows that the optimal trading strategy is always better than the general trading strategy. Therefore, our optimal trading strategy is a better way to trade large amount of assets in the real market.

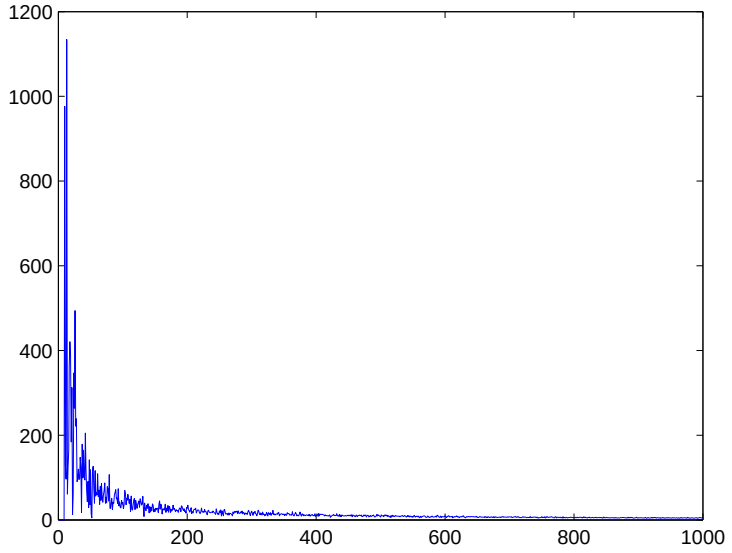


Figure 6-3. Convergence of the Variance

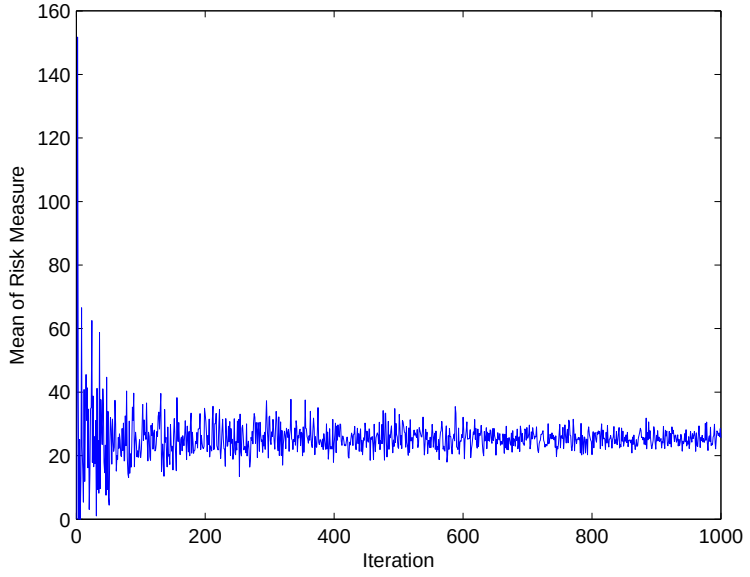


Figure 6-4. Convergence of the Mean

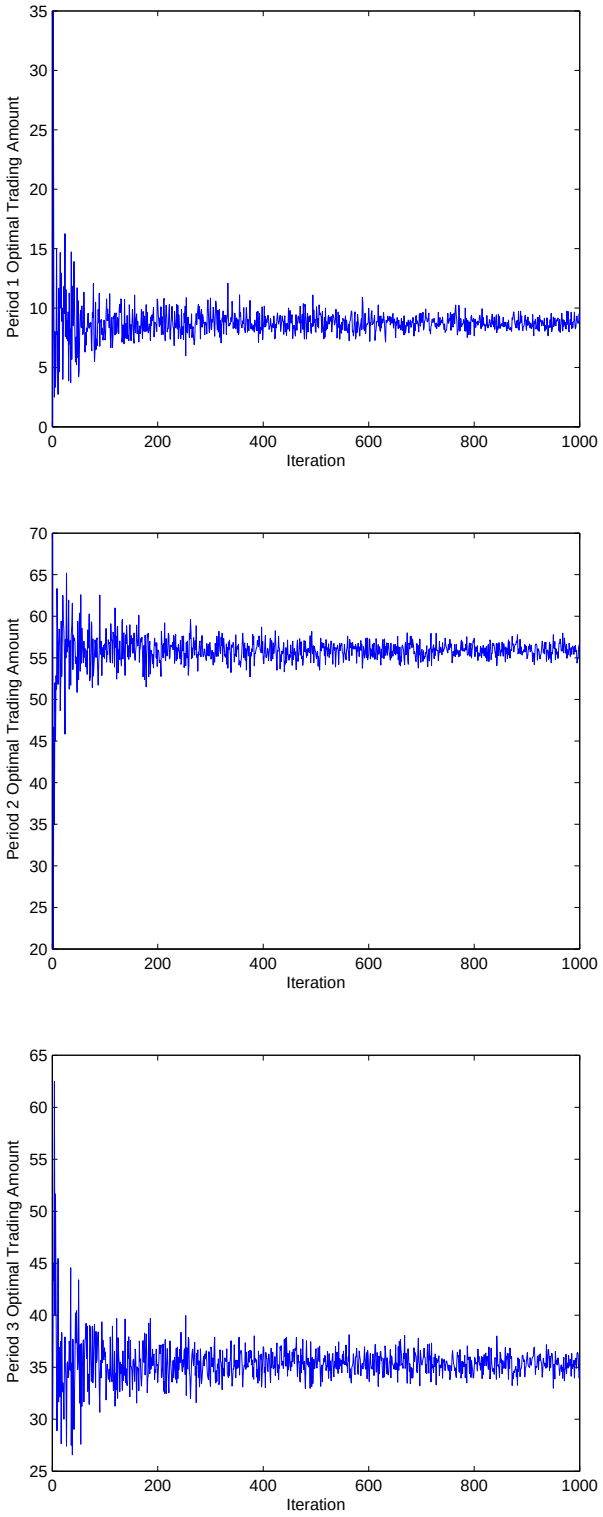


Figure 6-5. Optimal Trading Strategy

# CHAPTER 7
# LIQUIDITY RISK ALLOCATIONS

## 7.1 Basic Concept

One direct application of risk measures is to study the risk allocation problems[9]. The basic problem in risk management is how the risk of each component in a portfolio contributes to the risk of the whole portfolio. Risk allocation is generally used to describe such contributions. More specifically, let $X$ be a portfolio that consists of subportfolios $X_1, ..., X_n$. So $X = X_1 + \cdots + X_n$. In order to evaluate the risk, we need to assign a risk measure $\rho$ to the portfolio system. Then the total risk of the portfolio is denoted by $\rho(X)$. We want to study the allocation of the risk to the subportfolios $X_1, ..., X_n$. Clearly the choice of risk measure directly affects the properties of the risk allocation. As coherent risk measure reflects proportional change of assets, we can get linear risk allocations under this risk measure[10]. However, as we discussed earlier, liquidity risk measure does not have linearity. So the risk allocation induced by liquidity risk measure will no longer support linearity either. Conversely, when we allocate risk to each component, we also assign heavier weights on illiquid assets instead of allocating proportionally. Therefore, nonlinear risk allocations make sense for liquidity risk measures.

The widely accepted definition is the following[18]:

**Definition 7.1** Let $\rho : \mathcal{X} \to \mathbb{R}$ be any risk measure. A risk allocation with respect to $\rho$ is a function $\Lambda : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that for any $X \in \mathcal{X}$, $\Lambda(X, X) = \rho(X)$.

In addition, the risk allocation is called a liquidity risk allocation if for any $\lambda \geq 1$, $\Lambda(\lambda X, Y) \geq \lambda\Lambda(X, Y)$. For any risk measure $\rho$, we can define a new function

$$\Lambda(X, Y) = \lim_{\epsilon \to 0} \frac{\rho(Y + \epsilon X) - \rho(Y)}{\epsilon}. \tag{7–1}$$

If we take $\rho$ in the above to be a coherent risk measure, then we can get a linear risk contribution. The linearity partially comes from the Positive Homogeneity of the coherent risk measure. The definition for linear risk contribution is as follows[10]:

**Definition 7.2** A linear risk contribution is a function $\Lambda^l(X, Y)$ defined on $L^\infty \times L^\infty$ satisfying the axioms:

(1) (Linearity) $\Lambda^l(a_1X_1 + a_2X_2, Y) = a_1\Lambda^l(X_1, Y) + a_2\Lambda^l(X_2, Y)$ for $a_1, a_2 \in \mathbb{R}$

(2) (Diversification) $\Lambda^l(X, Y) \leq \rho(X)$

(3) (Consistency) $\Lambda^l(X, X) = \rho(X)$

(4) (Law Invariance) $\Lambda^l(X, Y)$ depends only on the joint law of $(X, Y)$

(5) (Continuity) If $|X_n| \leq 1$ and $X_n \to X$ in probability, then $\Lambda^l(X_n, Y) \to \Lambda^l(X, Y)$

For the liquidity risk measure, the linearity does not hold any more and it becomes more complex. The reason is that liquidating $X_1 + X_2$ could be more difficult than liquidating $X_1$ and $X_2$ separately. So you may end up liquidating at discount. But it does have the Consistency property and other special properties.

Notice that the limit defined in (7-1) may not always exist. So we need one more condition to make it a valid definition. That is[18]

$$\lim_{\epsilon \to 0} \Lambda(X, Y + \epsilon X) = \Lambda(X, Y) \tag{7–2}$$

**Proposition 7.1** If $\rho$ is a liquidity risk measure, then the limit in (7-1) exists if and only if the condition (7-2) is satisfied[18].

Utilizing the above proposition, we can show that the risk contribution defined in (7-1) using coherent risk measure satisfies the general definition of risk contribution.

**Theorem 7.1** $\Lambda(X, Y)$ induced by coherent risk measure is a valid risk allocation function.

**Proof**: By definition, we only need to verify that $\Lambda(X, X) = \rho(X)$ for any $X \in \mathcal{X}$. Let $F(X) = \Lambda(X, X) - \rho(X)$. We need to show that $F(X) \equiv 0$. Let $F_\epsilon(X) = \frac{\rho((1+\epsilon)X) - \rho(X)}{\epsilon} - \rho(X)$. When $\epsilon > 0$, $1 + \epsilon > 1$. By Liquidity axiom, $\rho((1 + \epsilon)X) \geq (1 + \epsilon)\rho(X)$. So $F_\epsilon(X) = \frac{\rho((1+\epsilon)X) - \rho(X)}{\epsilon} - \rho(X) \geq \frac{(1+\epsilon)\rho(X) - \rho(X)}{\epsilon} - \rho(X) = \rho(X) - \rho(X) = 0$.

When $\epsilon < 0$, $1+\epsilon < 1$. By Liquidity axiom, $\rho((1+\epsilon)X) \leq (1+\epsilon)\rho(X)$. Then $F_\epsilon(X) = \frac{\rho((1+\epsilon)X)-\rho(X)}{\epsilon} - \rho(X) \leq \frac{(1+\epsilon)\rho(X)-\rho(X)}{\epsilon} - \rho(X) = \rho(X) - \rho(X) = 0$.

Since the limit exists by Proposition 7.1, then for any X,

$$F(X) = \lim_{\epsilon\to 0} F_\epsilon(X) = \lim_{\epsilon\downarrow 0} F_\epsilon(X) = \lim_{\epsilon\uparrow 0} F_\epsilon(X) = 0. \ \square$$

### 7.2 Lower and Upper Bound

Risk contributions are most useful when they are bounded. Any risk allocation $\Lambda$ induced by a convex risk measure $\rho$ has a lower and upper bound. In this section, we derive a lower bound of $\Lambda$ if $\rho$ is continuous and an upper bound of $\Lambda$. They also give a way to estimate the risk allocation.

**Theorem 7.2** If $\Lambda$ is defined by (7-1) using a convex risk measure $\rho$, then $\Lambda(X, Y) \leq \rho(X+Y) - \rho(Y)$ for any $X, Y \in \mathcal{X}$.

**Proof**: Science $\rho$ is convex,

$$\begin{aligned}
\frac{\rho(Y+\epsilon X)-\rho(Y)}{\epsilon} &= \frac{\rho((1-\epsilon)Y+\epsilon(X+Y))-\rho(Y)}{\epsilon} \\
&\leq \frac{(1-\epsilon)\rho(Y)+\epsilon\rho(X+Y)-\rho(Y)}{\epsilon} \\
&= \frac{\epsilon\rho(X+Y)-\epsilon\rho(Y)}{\epsilon} = \rho(X+Y)-\rho(Y)
\end{aligned}$$

So

$$\Lambda(X, Y) = \lim_{\epsilon\to 0} \frac{\rho(Y+\epsilon X)-\rho(Y)}{\epsilon} \leq \rho(X+Y)-\rho(Y). \ \square$$

**Theorem 7.3** If $\Lambda$ is defined by (7-1) using a continuous convex risk measure $\rho$, then $\Lambda(X, Y) \geq \rho(Y) - \rho(Y-X)$ for any $X, Y \in \mathcal{X}$.

**Proof**: Science $\rho$ is convex, let $Z = Y + \epsilon X$ we have

$$\begin{aligned}
\frac{\rho(Y+\epsilon X)-\rho(Y)}{\epsilon} &= \frac{\rho(Z)-\rho(Z-\epsilon X)}{\epsilon} = \frac{\rho(Z)-\rho((1-\epsilon)Z+\epsilon(Z-X))}{\epsilon} \\
&\geq \frac{\rho(Z)-(1-\epsilon)\rho(Z)-\epsilon\rho(Z-X)}{\epsilon} = \frac{\epsilon\rho(Z)-\epsilon\rho(Z-X)}{\epsilon} \\
&= \rho(Y+\epsilon X)-\rho(Y+(\epsilon-1)X)
\end{aligned}$$

Since $\rho$ is continuous, we have

$$\begin{aligned}\Lambda(X, Y) &= \lim_{\epsilon\to 0}\frac{\rho(Y+\epsilon X)-\rho(Y)}{\epsilon}\\ &\geq \lim_{\epsilon\to 0}[\rho(Y+\epsilon X)-\rho(Y+(\epsilon-1)X)] = \rho(Y)-\rho(Y-X) \quad \square\end{aligned}$$

### 7.3 Risk Allocations Using Liquidity Risk Measures

In this section, we will derive the risk allocation functions for liquidity risk measures. We use the same basic definition for risk allocation function, but substitute the risk measure by a liquidity risk measure. Recall from Chapter 4, the liquidity risk measure of $U+\epsilon V$ is $\rho(U+\epsilon V) = \sup_{i\in I} E_{P^i}[\pi_0^1 U_0+\epsilon\pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j-\tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1-\tau_k)U_T^{(k)} - e^{-rT}\sum_{j=1}^{l}(\kappa_j-\kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\epsilon\pi_0^2-\kappa_l)V_T^{(l)} + \beta Corr_{P^i}(U, V)(\pi_0^1+\epsilon\pi_0^2) + \gamma_{U,V}\pi_0^1 + \gamma_{V,U}\epsilon\pi_0^2]$. If we assume existence of balanced trading strategy and there is no correlation effect, we will get $\rho(U+\epsilon V) = \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 + \epsilon\pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j-\tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)U_T^{(k)} - e^{-rT}\sum_{j=1}^{l}(\kappa_j-\kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\epsilon\pi_0^2-\kappa_l)V_T^{(l)}]$. Substitute this into the risk allocation function, we will get an explicit expression for the risk allocation function. Another thing we need to deal with is the probability distribution of each scenario $P_i$. Since we will focus on short term trading in the same market condition, all risk measures should be calculated at the same probability distribution and take supremum over the calculated results. That means we can take the outside operations into the supremum and still have the same results. Formally speaking, we have the following assumption:

**Same Market Assumption:** When we evaluate risk allocation, we assume all assets are traded in the same market condition. In other words, they have the same price distributions.

**Theorem 7.4** Under Same Market Assumption, for two assets U and V, if there is no correlation effect and there exists balanced trading strategy, then the risk allocation function induced by the liquidity risk measure in Definition 4.8 is a coherent risk measure of V, i.e. $\Lambda(V, U) = \sup_{i\in I} E_{P^i}[\pi_0^2 V_0 - e^{-rT}\pi_0^2 V_T] = \rho_c(V)$.

**Proof:** We first evaluate the numerator in the definition of risk allocation.

$$\begin{aligned}
\rho(U+\epsilon V)-\rho(U) &= \sup_{i\in I} E_{P^i}[\pi_0^1 U_0+\epsilon\pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j-\tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1-\tau_k)U_T^{(k)} \\
&\quad -e^{-rT}\sum_{j=1}^{l}(\kappa_j-\kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\epsilon\pi_0^2-\kappa_l)V_T^{(l)}] - \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 \\
&\quad -e^{-rT}\sum_{j=1}^{k}(\tau_j-\tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1-\tau_k)U_T^{(k)}] \\
&= \sup_{i\in I} E_{P^i}[\epsilon\pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{l}(\kappa_j-\kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\epsilon\pi_0^2-\kappa_l)V_T^{(l)}]
\end{aligned}$$

As $\epsilon$ goes to 0, $\epsilon\pi_0^2$ also approaches to 0. So the GLB threshold for $\epsilon\pi_0^2$ also goes to 0. For $\epsilon$ small enough, we must have $\tau_l = \tau_0 = 0$. So the middle term $e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_T^{(j-1)} = 0$. Therefore, $\rho(U+\epsilon V)-\rho(U) = \sup_{i\in I} E_{P^i}[\epsilon\pi_0^2 V_0 - e^{-rT}\epsilon\pi_0^2 V_T]$.

$$\begin{aligned}
\Lambda(V,U) &= \lim_{\epsilon\to 0}\frac{\rho(U+\epsilon V)-\rho(U)}{\epsilon} = \lim_{\epsilon\to 0}\frac{\sup_{i\in I} E_{P^i}[\epsilon\pi_0^2 V_0 - e^{-rT}\epsilon\pi_0^2 V_T]}{\epsilon} \\
&= \sup_{i\in I} E_{P^i}[\pi_0^2 V_0 - e^{-rT}\pi_0^2 V_T] = \rho_c(V). \ \square
\end{aligned}$$

The above result is the risk allocation function in which one asset is independent of the other asset. However, when we allocate risk, a lot of the assets will be correlated. Therefore, we need to introduce a risk allocation function in a much relaxed environment. When one asset is positively correlated with the whole portfolio, a loss in this asset will also bring down the value of the other assets that are positively correlated with it. So the total loss of the whole portfolio should be greater than the loss of the single asset. Since the extra loss is created by this single asset, we should allocate all or part of the extra loss to this trouble asset. Therefore, the risk allocation function should be greater than that without correlation effect.

**Theorem 7.5** Under the balanced trading strategy and Same Market Assumption, the risk allocation function of two assets U and V induced by a liquidity risk measure

is also a liquidity risk measure, i.e. $\Lambda(V,U) = \sup_{i\in I} E_{P^i}[\pi_0^2 V_0 - e^{-rT}\pi_0^2 V_T + \beta Corr(U,V)\pi_0^2] = \rho(V,0)$.

**Proof:** Since this is in the two-asset market, we should use the two-asset definition of liquidity risk measure.

$$\begin{aligned}
\rho(U,\epsilon V) - \rho(U,0) &= \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 + \epsilon\pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_T^{(j-1)} - e^{-rT}(\pi_0^1 - \tau_k)U_T^{(k)} \\
&\quad -e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\epsilon\pi_0^2 - \kappa_l)V_T^{(l)} \\
&\quad +\beta Corr(U,V)(\pi_0^1 + \epsilon\pi_0^2)] - \sup_{i\in I} E_{P^i}[\pi_0^1 U_0 - e^{-rT}\sum_{j=1}^{k}(\tau_j - \tau_{j-1})U_T^{(j-1)} \\
&\quad -e^{-rT}(\pi_0^1 - \tau_k)U_T^{(k)} + \beta Corr(U,V)\pi_0^1] \\
&= \sup_{i\in I} E_{P^i}[\epsilon\pi_0^2 V_0 - e^{-rT}\sum_{j=1}^{l}(\kappa_j - \kappa_{j-1})V_T^{(j-1)} - e^{-rT}(\epsilon\pi_0^2 - \kappa_l)V_T^{(l)} \\
&\quad +\beta Corr(U,V)\epsilon\pi_0^2]
\end{aligned}$$

Same argument as above, as $\epsilon$ goes to 0, we can have

$$\rho(U,\epsilon V) - \rho(U,0) = \sup_{i\in I} E_{P^i}[\epsilon\pi_0^2 V_0 - e^{-rT}\epsilon\pi_0^2 V_T + \beta Corr(U,V)\epsilon\pi_0^2].$$

Therefore,

$$\begin{aligned}
\Lambda(V,U) &= \lim_{\epsilon\to 0}\frac{\rho(U+\epsilon V) - \rho(U)}{\epsilon} = \lim_{\epsilon\to 0}\frac{\sup_{i\in I} E_{P^i}[\epsilon\pi_0^2 V_0 - e^{-rT}\epsilon\pi_0^2 V_T + \beta Corr(U,V)\epsilon\pi_0^2]}{\epsilon} \\
&= \sup_{i\in I} E_{P^i}[\pi_0^2 V_0 - e^{-rT}\pi_0^2 V_T + \beta Corr(U,V)\pi_0^2]. \ \square
\end{aligned}$$

**Remark:** The correlation term is the proportion of the extra risk created by V to the whole portfolio U allocated back to V. The back allocation will definitely increase the risk allocation function. However, we can see the risk allocated from V has nothing to do with the volume $\pi_0^1$ of U. This is because it does not allocation all the extra risk back to V.

### 7.4 Properties of Risk Allocations

In this section, we investigate the special properties of the risk allocation functions induced from liquidity risk measures due to the special Liquidity axiom. Recall that a

risk allocation is called a liquidity risk allocation if $\Lambda(\lambda X, Y) \geq \lambda\Lambda(X, Y)$ if $\lambda \geq 1$ and $\Lambda(\lambda X, Y) \leq \lambda\Lambda(X, Y)$ if $0 < \lambda \leq 1$.

**Theorem 7.6** The risk allocation induced by a liquidity risk measure in (7-1) is a liquidity risk allocation.

**Proof**: We want to show that for any $0 < \lambda \leq 1$, $\Lambda(\lambda X, Y) \leq \lambda\Lambda(X, Y)$.

$$\begin{aligned}
\Lambda(\lambda X, Y) - \lambda\Lambda(X, Y) &= \lim_{\epsilon\to 0} \frac{\rho(Y + \lambda\epsilon X) - \rho(Y) - \lambda\rho(Y + \epsilon X) + \lambda\rho(Y)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\rho(Y + \lambda\epsilon X) - \lambda\rho(Y + \epsilon X) + (\lambda - 1)\rho(Y)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\rho(Y + \lambda\epsilon X) - [\lambda\rho(Y + \epsilon X) + (1 - \lambda)\rho(Y)]}{\epsilon} \\
&\leq \lim_{\epsilon\to 0} \frac{\rho(Y + \lambda\epsilon X) - \rho(\lambda(Y + \epsilon X) + (1 - \lambda)Y)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{\rho(Y + \lambda\epsilon X) - \rho(Y + \lambda\epsilon X)}{\epsilon} = 0 \quad \square
\end{aligned}$$

From this, we can derive the case for $\lambda \geq 1$.

**Corollary** For $\lambda \geq 1$, we have $\Lambda(\lambda X, Y) \geq \lambda\Lambda(X, Y)$.

**Proof**: $\Lambda(X, Y) = \Lambda(\frac{1}{\lambda}\lambda X, Y) \leq \frac{1}{\lambda}\Lambda(\lambda X, Y) \quad \square$

The risk allocation does not necessarily diversify the risk of the portfolio. So the Diversification property of linear contribution does not hold for liquidity risk contribution. In some cases, it can even increase the risk. For example, we have two stocks $X$ and $Y$, and $Corr(X, Y) = 1$. If we sell $X$, then the price of $X$ falls by considering the liquidity effect. Since $X$ is perfectly correlated with $Y$, the price of $Y$ also falls. This means the risk $X$ contributed to the portfolio $X + Y$ is greater than the risk of $X$ alone. So $\Lambda(X, X + Y) > \Lambda(X, X) = \rho(X)$.

# CHAPTER 8
# EMPIRICAL STUDIES ON LIQUIDITY RISK MEASURES

Up to now, we have already established a lot of axioms, assumptions and models regarding liquidity risk. The rest we want to do is to test how this risk measure works in the real financial market and whether our assumptions are reasonable. Due to limitation of our resources, we are unable to test everything in the paper. Our major concern is whether the strict inequality in Liquidity axiom will hold in real transactions. This is the major distinction between the risk measure established in this paper and others. In order to apply the risk measure in the real market, we are also concerned about the estimation of coefficients in the formula. There are several computational studies concerning risk measures[28, 30].There are several different axioms proposed by researchers to build risk measurement systems. The most popular one is the Positive Homogeneity proposed by Artzner[1] in coherent risk measures. One way to see which axiom is better suited to the real world is to test the axioms in the real market. However, it is usually difficult to perform such test. Here we carry out a statistical test to study the Liquidity axiom. The Liquidity axiom states that when $\lambda \geq 1$, we have $\rho(\lambda X) \geq \lambda\rho(X)$. To distinct our axiom from coherent risk measure, we test the strict version of the Liquidity axiom. That means we want to test given $\lambda > 1$, we have $\rho(\lambda X) > \lambda\rho(X)$. So we have the following null hypothesis:

$$H_0: \quad \rho(\lambda X) - \lambda\rho(X) \leq 0 \tag{8–1}$$

In order to conduct statistical test, we need to gather the data in specific transactions and compare them with similar transactions with different sizes. Our test compares two transactions, one is the trading $X$ and the other is the trading $\lambda X$. These two transactions are carried out in the exactly same market condition at the same time. In a certain market, $\rho(\lambda X) = \Delta P_1 \lambda X$ and $\lambda\rho(X) = \lambda \Delta P_2 X$. So if we want to compare these two values, we only need to compare $\Delta P_1$ and $\Delta P_2$, where $\Delta P_1$ is the price change with

As of 1/20DELAYED Vol 750 Op 124.370 Hi 124.370 Lo 124.370 TRAC

Definitions | QR/QRM Options | Trade Recap | Page 1

Time : Source TRMT Min Vol Sprd To B Benchmark

Date 1/19 Price Range To High 124.375 Low 123.509

GOLDCORP INC GCN 2 08/01/14 124.370/124.370 (-6.72/-6.72) TRMT

| Time | Act | Price | Ind | Yield | RPS | Sprd | Benchmark | Size(M) | CC | Trd Time | Date |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 15:28:08 | | ↓123.509 | | -6.443 | S | -667.9 | T0 1⅛ 12/31/13 | 1000 | | 15:23:04 | 1/19/12 |
| 12:23:49 | | ↓123.651 | | -6.488 | B | -672.8 | T0 1⅛ 12/31/13 | 3500 | | 12:23:18 | 1/19/12 |
| 11:21:40 | | ↑124.375 | | -6.716 | S | -695.2 | T0 1⅛ 12/31/13 | 500 | | 11:16:40 | 1/19/12 |
| 10:31:32 | | ↓124.125 | | -6.638 | B | -686.5 | T0 1⅛ 12/31/13 | 400 | | 10:26:32 | 1/19/12 |
| 10:25:41 | | ↑124.375 | | -6.716 | S | -694.4 | T0 1⅛ 12/31/13 | 3500 | | 10:25:18 | 1/19/12 |
| 10:14:10 | | ↑124.081 | | -6.624 | B | -685.1 | T0 1⅛ 12/31/13 | 1000 | | 10:08:51 | 1/19/12 |

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SN 536551 EST GMT-5:00 H190-595-6 21-Jan-2012 00:02:36

Figure 8-1. GCN Trade Recap

large trading volume and $\Delta P_2$ is the price change with small trading volume. Due to limited information about trading, we have only two securities to study. One is GCN and the other is IBM. The data is extracted from Bloomberg Terminal™. To study the price impact, we only extracted the data with trading volume greater than 100 million. One snap shot of the trading data is displayed in Figure 8-1.

The first thing we want to test is that the price change due to large trading is statistically significant. That is to test the mean of the price change is significantly different from 0. The SAS® test result is shown in Figure 8-2.

The t-test rejected the null hypothesis, meaning the price change is statistically significant. This is what we expect: Liquidity axiom is a more realistic approach than Positive Homogeneity axiom. In addition, we get similar results from corporate bond of IBM in Figure 8-3.

The next test we can conduct is the correlation between price change and trading type, either buying or selling. For test purpose, we denote selling by 0 and buying by 1. The test result shows a significant positive correlation in Figure 8-4.

**GCN**

*The TTEST Procedure*

*Variable: change (change)*

| N | Mean | Std Dev | Std Err | Minimum | Maximum |
|---|---|---|---|---|---|
| 184 | 0.2805 | 0.4231 | 0.0312 | 0 | 2.8000 |

| Mean | 95% CL Mean | | Std Dev | 95% CL Std Dev | |
|---|---|---|---|---|---|
| 0.2805 | 0.2189 | 0.3420 | 0.4231 | 0.3838 | 0.4714 |

| DF | t Value | Pr > \|t\| |
|---|---|---|
| 183 | 8.99 | <.0001 |

Figure 8-2. T-Test for GCN

**IBM**

*The TTEST Procedure*

*Variable: change (change)*

| N | Mean | Std Dev | Std Err | Minimum | Maximum |
|---|---|---|---|---|---|
| 29 | 0.1550 | 0.2611 | 0.0485 | 0 | 1.0000 |

| Mean | 95% CL Mean | | Std Dev | 95% CL Std Dev | |
|---|---|---|---|---|---|
| 0.1550 | 0.0557 | 0.2543 | 0.2611 | 0.2072 | 0.3531 |

| DF | t Value | Pr > \|t\| |
|---|---|---|
| 28 | 3.20 | 0.0034 |

Figure 8-3. T-Test for IBM

**GCN Price Correlation with Trade**

*The CORR Procedure*

| 2 Variables: | change trade |
|---|---|

| Simple Statistics | | | | | | | |
|---|---|---|---|---|---|---|---|
| Variable | N | Mean | Std Dev | Sum | Minimum | Maximum | Label |
| change | 184 | -0.0008315 | 0.50803 | -0.15300 | -2.45000 | 2.80000 | change |
| trade | 205 | 0.26341 | 0.44156 | 54.00000 | 0 | 1.00000 | trade |

| Pearson Correlation Coefficients<br>Prob > \|r\| under H0: Rho=0<br>Number of Observations | | |
|---|---|---|
| | change | trade |
| change<br>change | 1.00000<br><br>184 | 0.19577<br>0.0077<br>184 |
| trade<br>trade | 0.19577<br>0.0077<br>184 | 1.00000<br><br>205 |

Figure 8-4. Correlation Between Price and Volume

This positive correlation shows that large amount of buying will increase the price and large amount of selling will decrease the price. This is also consistent with Liquidity axiom.

In summary, the markets are more likely to accept liquidity risk measures because any large enough trading will cause the market to become illiquid at least for a short period of time. So it is better for market participants to evaluate the risk including liquidity effect before making investment decisions. Accurately measuring the risk is the key to chose optimal trading strategies.

# CHAPTER 9
# CONCLUSION

In this paper, we have developed a new risk measurement system for illiquid markets and large tradings. It can help investment managers to more accurately assess their after-trading risk and actual returns. It is important to understand the change in supply and demand and to determine optimal trading strategies based on price impact. We have defined risk measures for multiple-asset portfolios and corresponding trading strategies to minimize the total risk. Although there is no general solution for complex optimal trading strategies, it is possible to calculate such strategies by computational method in practice. There are still numerous factors to be considered when defining risk measures. Due to limited time and resources, we cannot explore everything in this paper. But we believe more sophisticated risk systems will be developed by us or other researchers in the near future.

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## BIOGRAPHICAL SKETCH

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