ADVANCES IN GLOBAL PSEUDOSPECTRAL METHODS FOR OPTIMAL CONTROL

By

DIVYA GARG

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2011
To my parents, Anil and Anita and brother, Ankur
ACKNOWLEDGMENTS

The journey of earning this Ph.D. degree has been one of the most important and fulfilling learning experiences of my life. It has provided me with an immense sense of accomplishment. I have come a long way in terms of my professional and personal growth. My advisor, Dr. Anil V. Rao has played the most instrumental role in my success as a doctoral candidate. His attention for detail in evaluating my work has taught me to never take anything for granted and that all good things in life are earned with great efforts. Without his help, insight, and guidance this work would not have been possible.

I would also like to thank the members of my committee: Dr. William Hager, Dr. Warren Dixon, and Dr. Prabir Barooah. Dr. Hager’s feedback as a mathematician have been extremely helpful in providing the mathematical elegance to my research of which I am particularly proud. His experience and wisdom have given me new perspectives on how to continue to grow as a researcher. I am very thankful to Dr. Dixon and Dr. Barooah for their invaluable suggestions and for taking an active interest in my research.

To my fellow members of VDOL: Chris Darby, Camila Françolin, Mike Patterson, Pooja Hariharan, Darin Toscano, Brendan Mahon, and Begum Senses, I would like to say thank you for all the good times. I am confident that when I will look back on these days, I will fondly remember each one of you and the role you played in this journey. Chris Darby, you are not only an ideal colleague but also a very dear friend. Your ever willingness to answer the questions I had related to research and unparalleled work ethics are inspiring. Thank you Chris and Camila for being my sounding boards whenever I was feeling low. I would like to thank Mike for the contributions he has made towards my research.

Lastly, I cannot stress enough upon the importance of the role played by my parents, my brother Ankur, my friends, and Manoj. My parents and Ankur have always been confident of my capabilities, even at the times when I was in doubt. So much of
what I have accomplished is because of you. Thank you so much for being there to support me and standing by me at all times. Thank you Manoj for being my family away from family. Every time I felt like giving up, you were there to encourage me and helped me get through this.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>4</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>9</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>12</td>
</tr>
<tr>
<td>CHAPTER</td>
<td></td>
</tr>
<tr>
<td>1 INTRODUCTION</td>
<td>14</td>
</tr>
<tr>
<td>2 MATHEMATICAL BACKGROUND</td>
<td>24</td>
</tr>
<tr>
<td>2.1 Optimal Control</td>
<td>25</td>
</tr>
<tr>
<td>2.1.1 Calculus of Variations and Necessary Conditions</td>
<td>26</td>
</tr>
<tr>
<td>2.1.2 Pontryagin’s Principle</td>
<td>30</td>
</tr>
<tr>
<td>2.2 Numerical Optimization</td>
<td>31</td>
</tr>
<tr>
<td>2.2.1 Unconstrained Optimization</td>
<td>32</td>
</tr>
<tr>
<td>2.2.2 Equality Constrained Optimization</td>
<td>33</td>
</tr>
<tr>
<td>2.2.3 Inequality Constrained Optimization</td>
<td>35</td>
</tr>
<tr>
<td>2.3 Finite-Dimensional Approximation</td>
<td>37</td>
</tr>
<tr>
<td>2.3.1 Polynomial Approximation</td>
<td>37</td>
</tr>
<tr>
<td>2.3.1.1 Approximation error</td>
<td>39</td>
</tr>
<tr>
<td>2.3.1.2 Family of Legendre-Gauss points</td>
<td>41</td>
</tr>
<tr>
<td>2.3.2 Numerical Solution of Differential Equations</td>
<td>46</td>
</tr>
<tr>
<td>2.3.2.1 Time-marching methods</td>
<td>46</td>
</tr>
<tr>
<td>2.3.2.2 Collocation</td>
<td>48</td>
</tr>
<tr>
<td>2.3.3 Numerical Integration</td>
<td>50</td>
</tr>
<tr>
<td>2.3.3.1 Low-order integrators</td>
<td>51</td>
</tr>
<tr>
<td>2.3.3.2 Gaussian quadrature</td>
<td>53</td>
</tr>
<tr>
<td>3 MOTIVATION FOR THE RADAU PSEUDOSPECTRAL METHOD</td>
<td>59</td>
</tr>
<tr>
<td>3.1 Scaled Continuous-Time Optimal Control Problem</td>
<td>62</td>
</tr>
<tr>
<td>3.2 Lobatto Pseudospectral Method</td>
<td>67</td>
</tr>
<tr>
<td>3.2.1 NLP Formulation of the Lobatto Pseudospectral Method</td>
<td>68</td>
</tr>
<tr>
<td>3.2.2 Necessary Optimality Conditions</td>
<td>71</td>
</tr>
<tr>
<td>3.3 Gauss Pseudospectral Method</td>
<td>75</td>
</tr>
<tr>
<td>3.3.1 NLP Formulation of the Gauss Pseudospectral Method</td>
<td>76</td>
</tr>
<tr>
<td>3.3.2 Necessary Optimality Conditions</td>
<td>79</td>
</tr>
<tr>
<td>3.4 Summary</td>
<td>85</td>
</tr>
<tr>
<td>4 RADAU PSEUDOSPECTRAL METHOD</td>
<td>87</td>
</tr>
<tr>
<td>4.1 NLP Formulation of the Radau Pseudospectral Method</td>
<td>88</td>
</tr>
<tr>
<td>4.2 Necessary Optimality Conditions</td>
<td>92</td>
</tr>
</tbody>
</table>
8 CONCLUSION ................................................................. 227

8.1 Dissertation Summary .............................................. 227
8.2 Future Work ............................................................ 230
  8.2.1 Convergence Proof for Gauss and Radau Pseudospectral Method 230
  8.2.2 Costate Estimation Using Lobatto Pseudospectral Method .... 230

REFERENCES ................................................................. 231

BIOGRAPHICAL SKETCH .................................................. 238
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1</td>
<td>An extremal curve $y^*(t)$ and a comparison curve $y(t)$.</td>
<td>27</td>
</tr>
<tr>
<td>2-2</td>
<td>Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using uniform discretization points.</td>
<td>40</td>
</tr>
<tr>
<td>2-3</td>
<td>Schematic diagram of Legendre points.</td>
<td>42</td>
</tr>
<tr>
<td>2-4</td>
<td>Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 11 and 41 LG discretization points.</td>
<td>43</td>
</tr>
<tr>
<td>2-5</td>
<td>Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 11 and 41 LGR discretization points.</td>
<td>44</td>
</tr>
<tr>
<td>2-6</td>
<td>Error vs. number of discretization points</td>
<td>45</td>
</tr>
<tr>
<td>2-7</td>
<td>Four interval trapezoid rule approximation.</td>
<td>52</td>
</tr>
<tr>
<td>2-8</td>
<td>Error vs. number of intervals for trapezoid rule approximation</td>
<td>53</td>
</tr>
<tr>
<td>2-9</td>
<td>Error vs. number of Gaussian quadrature points for approximation</td>
<td>58</td>
</tr>
<tr>
<td>3-1</td>
<td>Relationship between KKT conditions and first-order optimality conditions</td>
<td>60</td>
</tr>
<tr>
<td>3-2</td>
<td>Multiple-interval implementation of LGL, LG, and LGR points.</td>
<td>61</td>
</tr>
<tr>
<td>3-3</td>
<td>Discretization and collocation points for Lobatto pseudospectral method.</td>
<td>70</td>
</tr>
<tr>
<td>3-4</td>
<td>Discretization and collocation points for Gauss pseudospectral method.</td>
<td>78</td>
</tr>
<tr>
<td>4-1</td>
<td>Discretization and collocation points for Radau pseudospectral method.</td>
<td>91</td>
</tr>
<tr>
<td>4-2</td>
<td>Discretization and collocation points for flipped Radau pseudospectral method.</td>
<td>102</td>
</tr>
<tr>
<td>4-3</td>
<td>Relationship between KKT conditions and first-order optimality conditions</td>
<td>110</td>
</tr>
<tr>
<td>6-1</td>
<td>The GPM solution for Example 1.</td>
<td>138</td>
</tr>
<tr>
<td>6-2</td>
<td>The RPM solution for Example 1.</td>
<td>139</td>
</tr>
<tr>
<td>6-3</td>
<td>The f-RPM solution for Example 1.</td>
<td>140</td>
</tr>
<tr>
<td>6-4</td>
<td>The LPM solution for Example 1.</td>
<td>141</td>
</tr>
<tr>
<td>6-5</td>
<td>LPM costate error and null space for Example 1.</td>
<td>142</td>
</tr>
<tr>
<td>6-6</td>
<td>Solution errors vs. number of collocation points for Example 1</td>
<td>143</td>
</tr>
<tr>
<td>6-7</td>
<td>The GPM solution for Example 2.</td>
<td>146</td>
</tr>
<tr>
<td>6-8</td>
<td>The RPM solution for Example 2.</td>
<td>147</td>
</tr>
<tr>
<td>6-9</td>
<td>The f-RPM solution for Example 2.</td>
<td>148</td>
</tr>
</tbody>
</table>
6-10 The LPM solution for Example 2. .............................................. 149
6-11 The solution for modified Example 2. ........................................ 151
6-12 Solution errors vs. number of collocation points for Example 2 .......... 152
6-13 The GPM solution for Example 3. .............................................. 154
6-14 The RPM solution for Example 3. .............................................. 155
6-15 The LPM solution for Example 3. .............................................. 156
6-16 The GPM solution for Example 4. .............................................. 158
6-17 The RPM solution for Example 4. .............................................. 159
6-18 The f-RPM solution for Example 4. ........................................... 160
6-19 The LPM solution for Example 4. .............................................. 161
6-20 The GPM solution for Example 5. .............................................. 164
6-21 The RPM solution for Example 5. .............................................. 165
6-22 The f-RPM solution for Example 5. ........................................... 166
6-23 The LPM solution for Example 5. .............................................. 167
6-24 Solution errors vs. number of collocation points for Example 5 .......... 169
6-25 The GPM solution for Example 6. .............................................. 171
6-26 The RPM solution for Example 6. .............................................. 172
6-27 The f-RPM solution for Example 6. ........................................... 173
6-28 The LPM solution for Example 6. .............................................. 174
6-29 Solution errors vs. number of collocation points for Example 6 .......... 175
7-1 Growth in $\phi(\tau)$ at 40 LG points. ........................................... 180
7-2 Growth of $y(t)$ and location of 40 collocation points using $\phi_b(\tau)$. . 181
7-3 The GPM solution using $\phi_a(\tau)$ for infinite-horizon 1-dimensional problem. . 204
7-4 The RPM solution using $\phi_a(\tau)$ for infinite-horizon 1-dimensional problem. . 205
7-5 The GPM solution using $\phi_b(\tau)$ for infinite-horizon 1-dimensional problem. . 206
7-6 The RPM solution using $\phi_b(\tau)$ for infinite-horizon 1-dimensional problem. . 207
7-7 The GPM solution using $\phi_c(\tau)$ for infinite-horizon 1-dimensional problem. . 208
7-8 The RPM solution using $\phi_c(\tau)$ for infinite-horizon 1-dimensional problem. . . . 209
7-9 State errors for infinite-horizon 1-dimensional problem . . . . . . . . . . . . . . 210
7-10 Control errors for infinite-horizon 1-dimensional problem . . . . . . . . . . . . . 211
7-11 Costate errors for infinite-horizon 1-dimensional problem . . . . . . . . . . . . . 212
7-12 The GPM solution using $\phi_a(\tau)$ for infinite-horizon LQR problem. . . . . . . 215
7-13 The RPM solution using $\phi_a(\tau)$ for infinite-horizon LQR problem. . . . . . . 216
7-14 The GPM solution using $\phi_b(\tau)$ for infinite-horizon LQR problem. . . . . . . 217
7-15 The RPM solution using $\phi_b(\tau)$ for infinite-horizon LQR problem. . . . . . . 218
7-16 The GPM solution using $\phi_c(\tau)$ for infinite-horizon LQR problem. . . . . . . 219
7-17 The RPM solution using $\phi_c(\tau)$ for infinite-horizon LQR problem. . . . . . . 220
7-18 First component of state errors for infinite-horizon LQR problem . . . . . . . . . . . 221
7-19 Second component of state errors for infinite-horizon LQR problem . . . . . . . . . 222
7-20 Control errors for infinite-horizon LQR problem . . . . . . . . . . . . . . . . . . . 223
7-21 First component of costate errors for infinite-horizon LQR problem . . . . . . . . . 224
7-22 Second component of costate errors for infinite-horizon LQR problem . . . . . . . 225
ADVANCES IN GLOBAL PSEUDOSPECTRAL METHODS FOR OPTIMAL CONTROL

By

Divya Garg

August 2011

Chair: Anil V. Rao
Major: Mechanical Engineering

A new pseudospectral method that employs global collocation at the Legendre-Gauss-Radau (LGR) points is presented for direct trajectory optimization and costate estimation of finite-horizon optimal control problems. This method provides accurate state and control approximations. Furthermore, transformations are developed that relate the Karush-Kuhn-Tucker (KKT) multipliers of the discrete nonlinear programming problem (NLP) to the costate and the Lagrange multipliers of the continuous-time optimal control problem. More precisely, it is shown that the transformed KKT multipliers of the NLP correspond to the Lagrange multipliers of the continuous problem and to a pseudospectral approximation of costate that is approximated using polynomials one degree smaller than that used for the state. The relationship between the differentiation matrices for the state equation and for the costate equation is established.

Next, a unified framework is presented for the numerical solution of optimal control problems based on collocation at the Legendre-Gauss (LG) and the Legendre-Gauss-Radau (LGR) points. The framework draws from the common features and mathematical properties demonstrated by the LG and the LGR methods. The framework stresses the fact that even though LG and LGR collocation appear to be only cosmetically different from collocation at Legendre-Gauss-Lobatto (LGL) points, the LG and the LGR methods are, in fact, fundamentally different from the LGL method. Specifically, it is shown that the LG and the LGR differentiation matrices are non-square and full rank whereas the
LGL differentiation matrix is square and singular. Consequently, the LG and the LGR schemes can be expressed equivalently in either differential or integral form, while the LGL differential and integral forms are not equivalent. Furthermore, it is shown that the LG and the LGR discrete costate systems have a unique solution while the LGL discrete costate system has a null space. The LGL costate approximation is found to have an error that oscillates about the exact solution, and this error is shown by example to be due to the null space in the LGL discrete costate system. Finally, it is shown empirically that the discrete state, costate and control obtained by the LG and the LGR schemes converge exponentially as a function of the number of collocation points, whereas the LGL costate is potentially non-convergent.

Third, two new direct pseudospectral methods for solving infinite-horizon optimal control problems are presented that employ collocation at the LG and the LGR points. A smooth, strictly monotonic transformation is used to map the infinite time domain $t \in [0, \infty)$ onto the interval $\tau \in [-1, 1)$. The resulting problem on the interval $\tau \in [-1, 1)$ is then transcribed to a NLP using collocation. The proposed methods provide approximations to the state and the costate on the entire horizon, including approximations at $t = +\infty$. These infinite-horizon methods can be written equivalently in either a differential or an implicit integral form. In numerical experiments, the discrete solution is found to converge exponentially as a function of the number of collocation points. It is shown that the mapping $\phi : [-1, +1) \to [0, +\infty)$ can be tuned to improve the quality of the discrete approximation.
CHAPTER 1
INTRODUCTION

Optimal control is a subject that arises in many branches of engineering including aerospace, chemical, and electrical engineering. Particularly in aerospace engineering, optimal control is used in various applications including trajectory optimization, attitude control, and vehicle guidance. As defined by Kirk [1], “The objective of an optimal control problem is to determine the control signals that will cause a process to satisfy the physical constraints and at the same time minimize (or maximize) some performance index”. Possible performance indices include time, fuel consumption, or any other parameter of interest in a given application.

Except for special cases, most optimal control problems cannot be solved analytically. Consequently, numerical methods must be employed. Numerical methods for solving optimal control problem fall into two categories: indirect methods and direct methods, as summarized by Stryk et al., Betts, and Rao [2–4]. In an indirect method, the calculus of variations [1, 5] is applied to determine the first-order necessary conditions for an optimal solution. Applying the calculus of variations transforms the optimal control problem to a Hamiltonian boundary-value problem (HBVP). The solution to the HBVP is then approximated using one of the various numerical approaches. Commonly used approaches for solving the HBVP are shooting, multiple shooting [6, 7], finite difference [8], and collocation [9, 10]. Although using an indirect method has the advantage that a highly accurate approximation can be obtained and that the proximity of the approximation to the optimal solution can be established, indirect methods have several disadvantages. First, implementing an indirect method requires that the complicated first-order necessary optimality conditions be derived. Second, the indirect methods require that a very good initial guess on the unknown boundary conditions must be provided. These guesses include a guess for the costate which is a mathematical quantity inherent to the HBVP. Because the costate is a non-intuitive and non-physical
quantity, providing such a guess is difficult. Third, whenever a problem needs to be modified (e.g., adding or removing a constraint), the necessary conditions need to be reformulated. Lastly, for problems whose solutions have active path constraints, a priori knowledge of the switching structure of the path constraints must be known.

In a direct method, the continuous functions of time (the state and/or the control) of the optimal control problem are approximated and the problem is transcribed into a finite-dimensional nonlinear programming problem (NLP). The NLP is then solved using well developed algorithms and software [11–14]. In the case where only the control is approximated, the method is called a control parameterization method. When both the state and the control are approximated, the method is called a state and control parameterization method. Direct methods overcome the disadvantages of indirect methods because the optimality conditions do not need to be derived, the initial guess does not need to be as good as that required by an indirect method, a guess of the costate is not needed, and the problem can be modified relatively easily. Direct methods, however, are not as accurate as indirect methods, require much more work to verify optimality, and many direct methods do not provide any information about the costate.

Many different direct methods have been developed. The two earliest developed direct methods for solving optimal control problem are the direct shooting method and the direct multiple-shooting method [15–17]. Both direct shooting and direct multiple-shooting methods are control parameterization methods where the control is parameterized using a specified functional form and the dynamics are integrated using explicit numerical integration (e.g., a time-marching algorithm). A direct shooting method is useful when the problem can be approximated with a few number of variables. As the number of variables used in a direct shooting method grows, the ability to successfully use a direct shooting method declines. In the direct multiple-shooting method, the time interval is divided into several subintervals and then the direct shooting method is used over each interval. At the interface of each subinterval, the state
continuity condition is enforced and the state at the beginning of each subinterval is a parameter in the optimization. The direct multiple-shooting method is an improvement over the standard direct shooting method as the sensitivity to the initial guess is reduced because integration is performed over significantly smaller time intervals. Both the direct shooting method and the direct multiple-shooting method, however, are computationally expensive due to the numerical integration operation and require a priori knowledge of the switching structure of inactive and active path constraints. Well-known computer implementation of direct shooting methods are POST [18] and STOPM [19].

Another approach is that of direct collocation methods [20–50], where both the state and the control are parameterized using a set of trial (basis) functions and a set of differential-algebraic constraints are enforced at a finite number of collocation points. In contrast to indirect methods and direct shooting methods, a direct collocation method does not require a priori knowledge of the active and inactive arcs for problems with inequality path constraints. Furthermore, direct collocation methods are much less sensitive to the initial guess than either the aforementioned indirect methods or direct shooting methods. Some examples of computer implementations of direct collocation methods are SOCS [51], OTIS [52], DIRCOL [53], DIDO [54] and GPOPS [55, 56]. The two most common forms of direct collocation methods are local collocation [20–30] and global collocation [31–50].

In a direct local collocation method, the time interval is divided into subintervals and a fixed low-degree polynomial is used for approximation in each subinterval. The convergence of the numerical discretization is achieved by increasing the number of subintervals. Two categories of discretization have been used for local collocation: (a) Runge-Kutta methods [20–25] that use piecewise polynomials; (b) orthogonal collocation methods [26–30] that use orthogonal polynomials. Direct local collocation leads to a sparse NLP with many of the constraint Jacobian entries as zero. Sparsity in the NLP greatly increases the computational efficiency. However, the convergence
to the exact solution is at a polynomial rate and often an excessively large number of subintervals are required to accurately approximate the solution to an optimal control problem resulting in a large NLP with often tens of thousands of variables or more. In a direct global collocation method, the state and the control are parameterized using global polynomials. In contrast to local methods, the class of direct global collocation methods uses a small fixed number of approximating intervals (often only a single interval is used). Convergence to the exact solution is achieved by increasing the degree of polynomial approximation in each interval.

In recent years, a particular class of methods that has received a great deal of attention is the class of pseudospectral or orthogonal collocation methods. In a pseudospectral method, the basis functions are typically the Chebyshev or the Lagrange polynomials and the collocation points are obtained from very accurate Gaussian quadrature rules. These methods are based on spectral methods and typically have faster convergence rates (exponential) than the traditional methods for a small number of discretization points [57–59]. Spectral methods were applied to optimal control problems in the late 1980’s using Chebyshev polynomials by Vlassenbroeck et al. in Ref. [31, 32], and later a Legendre-based pseudospectral method using Lagrange polynomials and collocation at Legendre-Gauss-Lobatto (LGL) points was developed by Elnagar et al. in Ref. [33–36]. An extension of the Legendre-based pseudospectral method was performed by Fahroo et al. in Ref. [37] to generate costate estimates. This method later came to be known as the Lobatto pseudospectral method (LPM) [37–47]. At the same time, another Legendre-based pseudospectral method called the Gauss pseudospectral method (GPM) was developed by Benson and Huntington in Ref. [48–50]. The GPM used Lagrange polynomials as basis functions and Legendre-Gauss (LG) points for collocation.

Despite the many advantages of direct methods, many of them do not give any information about the costate. The costate is important for verifying the optimality of
the solution, mesh refinement, sensitivity analysis, and real time optimization. Recently, costate estimates have been developed for pseudospectral methods. These estimates are derived by relating the Karush-Kuhn-Tucker (KKT) conditions of the NLP to the continuous costate dynamics as demonstrated by Seywald and Stryk in Ref. [60, 61]. A costate mapping principle has been derived by Fahroo et al. in Ref. [37] to estimate the costate from the KKT multipliers for the Lobatto pseudospectral method. However this principle does not hold at the boundary points. The resulting costate estimates at the boundaries do not satisfy the costate dynamics or boundary conditions, but only a linear combination of the two. It was shown by Benson in Ref. [48] that this is a result of the defects in the discretization when using LGL points.

As mentioned earlier, the Gauss pseudospectral method (GPM), which uses the LG points, was proposed by Benson in Ref. [48, 49]. The GPM differs from the Lobatto pseudospectral method (LPM) in the fact that the dynamic equations are not collocated at the boundary points. In this approach the KKT conditions of the NLP are found to be exactly equivalent to the discretized form of the first-order necessary conditions of the optimal control problem. This property allows for a costate estimate that is more accurate than the one obtained from the LPM. In the GPM, however, because the dynamics are not collocated at the initial and the final point, the control at either the initial or the final point is not obtained.

In this dissertation, a new method called the Radau pseudospectral method (RPM) is proposed. The RPM is a direct transcription method that uses parameterization of the state and the control by global polynomials (Lagrange polynomials) and collocation of differential constraints at the Legendre-Gauss-Radau (LGR) points [62]. The method developed in this dissertation differs from the Lobatto pseudospectral method in the fact that the dynamic equations are not collocated at the final point. It is shown that the KKT conditions from the resulting NLP are equivalent to the discretized form of the first-order necessary conditions of the optimal control problem. This method,
therefore, provides an approach to obtain accurate approximations of the costate for the continuous problem using the KKT multipliers of the NLP. Also, because the dynamics are not collocated at the final point, the costate at the final point is not obtained in the NLP solution. It is noted, however, that the costate at the final time can be estimated accurately using a Radau quadrature. The method of this dissertation differs from the Gauss pseudospectral method in the fact that the dynamic equations are collocated at the initial point. As a result, the method of this dissertation provides an approximation of the control at the initial point. It is noted that LGR points have previously been used for local collocation by Kameswaran et al. in Ref. [30] differing from the global collocation approach used in this research.

The Radau pseudospectral method derived in this dissertation has many advantages over other numerical methods for solving optimal control problems. First, the implementation of the method is easy and any change in constraints can be incorporated in the formulation without much work. Second, an accurate solution can be found using well-developed sparse NLP solvers with no need for an initial guess on the costate or derivation of the necessary conditions. Third, the costate can be estimated directly from the KKT multipliers of the NLP. The final advantage of the Radau pseudospectral method, is that they take advantage of the fast exponential convergence typical of spectral methods. This rapid convergence rate is shown empirically on a variety of example problems. The rapid convergence rate indicates that an accurate solution to the optimal control problem can be found using fewer collocation points and potentially less computational time, when compared with other methods. The rapid solution of the problem along with an accurate estimate for the costate and initial control could also enable real-time optimal control for nonlinear systems.

Next, in this dissertation, a unified framework is presented for the numerical solution of optimal control problems using the Gauss pseudospectral method (GPM) and the Radau pseudospectral method (RPM) [63]. The framework is based on
the common features and mathematical properties of the GPM and the RPM. The framework stresses the fact that, even though the GPM and the RPM appear to be only cosmetically different from the Lobatto pseudospectral method (LPM), the GPM and the RPM are, in fact, fundamentally different from the LPM. In the framework, the state is approximated by a basis of Lagrange polynomials, the system dynamics and the path constraints are enforced at the collocation points, and the boundary conditions are applied at the endpoints of the time interval. The GPM and the RPM employ a polynomial approximation that is the same degree as the number of collocation points while the LPM employs a state approximation that is one degree less than the number of collocation points. It is shown that the GPM and the RPM differentiation matrices are non-square and full rank, whereas the LPM differentiation matrix is square and singular. Consequently, the GPM and the RPM schemes can be expressed equivalently in either differential or integral form, while the LPM differential and integral forms are not equivalent. Furthermore, it is shown that the GPM and the RPM discrete costate systems are full rank while the LPM discrete costate system is rank-deficient. The LPM costate approximation is found to have an error that oscillates about the exact solution, and this error is shown by example to be due to the null space in the LPM discrete costate system. Finally, it is shown empirically that the discrete solutions for the state, control, and costate obtained from the GPM and the RPM converge exponentially as a function of the number of collocation points, whereas the LPM costate is potentially non-convergent. The framework presented in this dissertation provides the first rigorous analysis that identifies the key mathematical properties of pseudospectral methods using collocation at Gaussian quadrature points, enabling a researcher or end-user to see clearly the accuracy and convergence (or non-convergence) that can be expected when applying a particular pseudospectral method on a problem of interest.

Lastly, this dissertation presents two new direct pseudospectral methods that employ collocation at Legendre-Gauss (LG) and Legendre-Gauss-Radau (LGR)
points for solving infinite-horizon optimal control problems [64]. A smooth, strictly monotonic transformation is used to map the infinite time domain \( t \in [0, \infty) \) onto \( \tau \in [-1,1) \) interval. The resulting problem on the interval \( \tau \in [-1,1) \) is then transcribed to a nonlinear programming problem using collocation. By using the approach developed in this dissertation, the proposed methods provide approximations to the state and the costate on the entire horizon, including approximations at \( t = +\infty \).

In a manner similar to the finite-horizon Gauss and Radau pseudospectral methods, these infinite-horizon methods can be written equivalently in either a differential or an implicit integral form. In numerical experiments, the discrete solution is found to converge exponentially as a function of the number of collocation points. It is shown that the mapping \( \phi : [-1,+1) \rightarrow [0, +\infty) \) can be tuned to improve the quality of the discrete approximation. It is also shown that collocation at Legendre-Gauss-Lobatto (LGL) points cannot be used for solving infinite-horizon optimal control problems.

A pseudospectral method to solve infinite-horizon optimal control problems using LGR points has been previously proposed by Fahroo et al. in Ref. [65]. The methods proposed in this dissertation are significantly different from the method of Ref. [65] as the methods of this dissertation yield approximations to the state and costate on the entire horizon, including approximations at \( t = +\infty \) whereas the method of Ref. [65] does not provide the solution at \( t = \infty \). Furthermore, the method of Ref. [65] uses a transformation that grows rapidly as \( t \rightarrow \infty \), whereas in this dissertation a general change of variables \( t = \phi(\tau) \) of an infinite-horizon problem to a finite-horizon problem is considered. It is shown that better approximations to the continuous-time problem can be attained by using a function \( \phi(\tau) \) that grows slowly as \( t \rightarrow \infty \).

This dissertation is divided into the following chapters. Chapter 2 describes the mathematical background necessary to understand the pseudospectral methods used for solving optimal control problems. A general continuous-time optimal control problem is defined and the first-order necessary optimality conditions for that problem
are derived using the calculus of variations. The development and solution methods of finite-dimensional nonlinear programming problems are discussed next. Lastly, many mathematical concepts that are used in transcribing the continuous-time optimal control problem into a finite-dimensional NLP using the proposed pseudospectral method are reviewed. Chapter 3 provides motivation for the Radau pseudospectral method. It is demonstrated that the Lobatto pseudospectral method has an inherent defect in the costate dynamics at the boundaries and although the Gauss pseudospectral method does not suffer from this defect, it lacks the ability to give the initial control. The Radau pseudospectral method possesses the same accuracy as that of the GPM, but also provides the initial control from the solution of the NLP. Chapter 4 describes a direct transcription method, called the Radau pseudospectral method that transcribes a continuous-time optimal control problem into a discrete nonlinear programming problem. The method uses the Legendre-Gauss-Radau (LGR) points for collocation of the dynamic constraints, and for quadrature approximation of the integrated Lagrange cost term. The LGR discretization scheme results in a set of KKT conditions that are equivalent to the discretized form of the continuous first-order optimality conditions and, hence, provides a significantly more accurate costate estimate than that obtained using the Lobatto pseudospectral method. In addition, because collocation is performed at the LGR points, and the LGR points include the initial point, the control at the initial time is also obtained in the solution of the NLP. Lastly, the problem formulation for a flipped Radau pseudospectral method that uses the flipped LGR points is given, where the flipped LGR points are the negative of the LGR points.

Next, Chapter 5 presents a unified framework for two different pseudospectral methods based on collocation at the Legendre-Gauss (LG) and the Legendre-Gauss-Radau (LGR) points. Each of these schemes can be expressed in either a differential or an integral formulation. The LG and the LGR differentiation and integration matrices are invertible, and the differential and integral versions are equivalent. Each of these
schemes provide an accurate transformation between the Lagrange multipliers of the
discrete nonlinear programming problem and the costate of the continuous optimal
control problem. It is shown that both of these schemes form a determined system
of linear equations for costate dynamics. These schemes are different from the
pseudospectral method based on collocation at the Legendre-Gauss-Lobatto (LGL)
points. The LGL differentiation matrix is singular and hence the equivalence between the
differential and integral formulation is not established. For the LGL scheme, the linear
system of equations for costate dynamics is under-determined. The transformation
between the Lagrange multipliers of the discrete nonlinear programming problem and
the costate of the continuous optimal control problem for the LGL scheme is found
to be inaccurate. In Chapter 6, a variety of examples are solved using, the Gauss,
the Radau, and the Lobatto pseudospectral methods. Three main observations are
made in the examples. First, it is seen that the Gauss and the Radau pseudospectral
methods consistently generate accurate state, control and costate solutions, while
the solution obtained from the Lobatto pseudospectral method is inconsistent and
unpredictable. Second, it is seen for the examples that have exact analytical solutions
that the error in the solutions obtained from the Gauss pseudospectral method and
the Radau pseudospectral method goes to zero at an exponential rate as the number
of discretization points are increased. Third, it is shown that none of these methods
are well suited for solving problems that have discontinuities in the solution or where
the solutions lie on a singular arc. Chapter 7 describes two direct pseudospectral
methods for solving infinite-horizon optimal control problems numerically using the
Legendre-Gauss (LG) and the Legendre-Gauss-Radau (LGR) collocation. The proposed
methods yield approximations to the state and the costate on the entire horizon,
including approximations at $t = +\infty$. Finally, Chapter 8 summarizes the contributions of
this dissertation and suggests future research prospects.
CHAPTER 2
MATHEMATICAL BACKGROUND

In this chapter, first, a general continuous-time optimal control problem is defined and the first-order necessary optimality conditions for that problem are derived using the calculus of variations. Pontryagin’s principle, which is used to solve for the optimal control in some special cases, is also discussed. Next, this chapter describes the mathematical background necessary to understand the pseudospectral methods used for solving the optimal control problems. In a pseudospectral method, the continuous functions of time of an optimal control problem are approximated and the problem is transcribed into a finite-dimensional nonlinear programming problem (NLP). The NLP is then solved using well developed algorithms and software. The development and solution methods of finite-dimensional nonlinear programming problems are discussed in this chapter. Unconstrained, equality constrained, and inequality constrained problems are considered. Furthermore, the necessary conditions for optimality or the Karush-Kuhn-Tucker (KKT) conditions of the NLP are presented for each of the three cases.

Lastly, many mathematical concepts that are used in transcribing the continuous-time optimal control problem into a finite-dimensional NLP using the proposed pseudospectral method are reviewed in this chapter. The first and the most important is the idea of polynomial approximation using a basis of Lagrange polynomials. Polynomial approximation is used for approximating the continuous functions of time of the optimal control problem. Another important concept is the application of numerical methods for approximating the solution to differential equations. In a pseudospectral method, the differential equation constraints of an optimal control problem are transcribed to algebraic equality constraints. Two approaches are discussed to transcribe the differential equations to algebraic equations: time-marching methods and collocation. The last concept reviewed in this chapter is the numerical integration of functions.
Low-order numerical integrators and integration using Gaussian quadrature are discussed. The pseudospectral method of this research uses Legendre-Gauss-Radau quadrature for integration and orthogonal collocation at Legendre-Gauss-Radau points for approximating the solution to differential equations.

2.1 Optimal Control

The objective of an optimal control problem is to determine the state and the control that optimize a performance index while satisfying the physical and dynamic constraints of the system. Mathematically, an optimal control problem can be written in Bolza form as follows. Minimize the cost functional

\[
J = \Phi(y(t_0), t_0, y(t_f), t_f) + \int_{t_0}^{t_f} g(y(t), u(t), t) dt,
\]

subject to the dynamic constraints

\[
\dot{y}(t) = f(y(t), u(t), t),
\]

the boundary conditions

\[
\phi(y(t_0), t_0, y(t_f), t_f) = 0,
\]

and the inequality path constraints

\[
C(y(t), u(t), t) \leq 0,
\]

where \(y(t) \in \mathbb{R}^n\) is the state, \(u(t) \in \mathbb{R}^m\) is the control, and \(t \in [t_0, t_f]\) is the independent variable. The cost functional is composed of the Mayer cost, \(\Phi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}\), and the Lagrangian, \(g : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}\). Furthermore, \(f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^n\) defines the right-hand side of the dynamic constraints, \(C : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^s\) defines the path constraints, and \(\phi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^q\) defines the boundary conditions. The first-order necessary conditions for the optimal solution of the problem given in Eq. (2–1)-(2–4) are derived by using the calculus of variations as described in Section. 2.1.1 below.
2.1.1 Calculus of Variations and Necessary Conditions

Unconstrained optimization problems that depend on continuous functions of time require that the first variation, \( \delta J(y(t)) \), of the cost functional, \( J(y(t)) \), on an optimal path \( y^* \), vanish for all admissible variations \( \delta y \) [1]. In other words,

\[
\delta J(y^*, \delta y) = 0. \quad (2–5)
\]

For a constrained optimization problem, an extremal solution is generated from the continuous-time first-order necessary conditions by applying the calculus of variations to an augmented cost. The augmented cost is obtained by appending the constraints to the cost functional using the Lagrange multipliers. The augmented cost is given as

\[
J_a = \Phi(y(t_0), t_0, y(t_f), t_f) - \psi^T \phi(y(t_0), t_0, y(t_f), t_f) + \int_{t_0}^{t_f} \left[ g(y(t), u(t), t) - \lambda^T (t)(y(t) - f(y(t), u(t), t)) - \gamma^T (t)C(y(t), u(t), t) \right] dt,
\]

where \( \lambda(t) \in \mathbb{R}^n, \psi \in \mathbb{R}^q, \) and \( \gamma(t) \in \mathbb{R}^s \) are the Lagrange multipliers corresponding to Eqs. (2–2), (2–3), and (2–4), respectively. The quantity \( \lambda(t) \) is called the costate or the adjoint. The first-order variation with respect to all free variables is given as

\[
\delta J_a = \frac{\partial \Phi}{\partial y(t_0)} \delta y_0 + \frac{\partial \Phi}{\partial t_0} \delta t_0 + \frac{\partial \Phi}{\partial y(t_f)} \delta y_f + \frac{\partial \Phi}{\partial t_f} \delta t_f - \delta \psi^T \phi - \psi^T \frac{\partial \phi}{\partial y(t_0)} \delta y_0 - \psi^T \frac{\partial \phi}{\partial t_0} \delta t_0 - \psi^T \frac{\partial \phi}{\partial y(t_f)} \delta y_f - \psi^T \frac{\partial \phi}{\partial t_f} \delta t_f + ((g - \lambda^T (y - f) - \gamma^T C)|_{t=t_f}) \delta t_f - ((g - \lambda^T (y - f) - \gamma^T C)|_{t=t_0}) \delta t_0 + \int_{t_0}^{t_f} \left[ \frac{\partial g}{\partial y} \delta y + \frac{\partial g}{\partial u} \delta u - \delta \lambda^T (y - f) + \lambda^T \frac{\partial f}{\partial y} \delta y + \lambda^T \frac{\partial f}{\partial u} \delta u - \lambda^T \delta y - \delta \gamma^T C - \gamma^T \frac{\partial C}{\partial y} \delta y - \gamma^T \frac{\partial C}{\partial u} \delta u \right] dt.
\]

Fig. 2-1 shows the differences between \( \delta y_f \) and \( \delta y(t_f) \), and \( \delta y_0 \) and \( \delta y(t_0) \) where \( y^*(t) \) represents an extremal curve and \( y(t) \) represents a neighboring comparison curve. The quantities \( t_0 \) and \( t_f \) are the initial and the final times, \( \delta t_0 \) and \( \delta t_f \) are the variations in the initial and the final times, and \( y_0 \) and \( y_f \) are the state at the initial and the final times,
Figure 2-1. An extremal curve \( y^*(t) \) and a comparison curve \( y(t) \).

respectively. It is noted that \( \delta y(t_f) \) is the difference between \( y^*(t) \) and \( y(t) \) evaluated at \( t_f \) whereas \( \delta y_f \) is the difference between \( y^*(t) \) and \( y(t) \) evaluated at the end of each curve. Similar interpretations are true for \( \delta y(t_0) \) and \( \delta y_0 \). As can be seen in Fig. 2-1, the first-order approximations to \( \delta y_f \) and \( \delta y_0 \) are given as

\[
\delta y_0 = \delta y(t_0) + \dot{y}(t_0) \delta t_0, \quad (2-8)
\]

\[
\delta y_f = \delta y(t_f) + \dot{y}(t_f) \delta t_f. \quad (2-9)
\]

Furthermore, the term containing \( \delta \dot{y} \) in Eq. (2-7) is integrated by parts such that it can be expressed in terms of \( \delta y(t_0), \delta y(t_f), \) and \( \delta y \) as

\[
\int_{t_0}^{t_f} -\lambda^T \delta \dot{y} dt = -\lambda^T (t_f) \delta y(t_f) + \lambda^T (t_0) \delta y(t_0) + \int_{t_0}^{t_f} \lambda^T \delta y dt. \quad (2-10)
\]
Substituting Eqs. (2–8), (2–9), and (2–10) into Eq. (2–7), the variation of \( J_a \) is given as

\[
\delta J_a = \left( \frac{\partial \Phi}{\partial y(t_0)} - \psi^T \frac{\partial \phi}{\partial y(t_0)} + \lambda^T(t_0) \right) \delta y_0 \\
+ \left( \frac{\partial \Phi}{\partial y(t_f)} - \psi^T \frac{\partial \phi}{\partial y(t_f)} - \lambda^T(t_f) \right) \delta y_f - \delta \psi^T \phi \\
+ \left( \frac{\partial \Phi}{\partial t_0} - \psi^T \frac{\partial \phi}{\partial t_0} - (g + \lambda^T f - \gamma^T C)_{t=t_0} \right) \delta t_0 \\
+ \left( \frac{\partial \Phi}{\partial t_f} - \psi^T \frac{\partial \phi}{\partial t_f} + (g + \lambda^T f - \gamma^T C)_{t=t_f} \right) \delta t_f \\
+ \int_{t_0}^{t_f} \left[ \left( \frac{\partial g}{\partial y} + \lambda^T \frac{\partial f}{\partial y} - \gamma^T \frac{\partial C}{\partial y} + \lambda \right) \delta y + \left( \frac{\partial g}{\partial u} + \lambda^T \frac{\partial f}{\partial u} - \gamma^T \frac{\partial C}{\partial u} \right) \delta u \\
- \delta \lambda^T (y - f) - \delta \gamma^T C \right] dt.
\]

The first-order optimality conditions are formed by setting the variation of \( J_a \) equal to zero with respect to each free variable such that

\[
\dot{y} = f, \quad (2–12)
\]

\[
\frac{\partial g}{\partial y} + \lambda^T \frac{\partial f}{\partial y} - \gamma^T \frac{\partial C}{\partial y} = -\dot{\lambda}, \quad (2–13)
\]

\[
\frac{\partial g}{\partial u} + \lambda^T \frac{\partial f}{\partial u} - \gamma^T \frac{\partial C}{\partial u} = 0, \quad (2–14)
\]

\[
-\frac{\partial \Phi}{\partial y(t_0)} + \psi^T \frac{\partial \phi}{\partial y(t_0)} = \lambda^T(t_0), \quad (2–15)
\]

\[
\frac{\partial \Phi}{\partial y(t_f)} - \psi^T \frac{\partial \phi}{\partial y(t_f)} = \lambda^T(t_f), \quad (2–16)
\]

\[
\frac{\partial \Phi}{\partial t_0} - \psi^T \frac{\partial \phi}{\partial t_0} = (g + \lambda^T f - \gamma^T C)_{t=t_0}, \quad (2–17)
\]

\[
-\frac{\partial \Phi}{\partial t_f} + \psi^T \frac{\partial \phi}{\partial t_f} = (g + \lambda^T f - \gamma^T C)_{t=t_f}, \quad (2–18)
\]

\[
\phi = 0. \quad (2–19)
\]

Defining the augmented Hamiltonian as

\[
H(y(t), u(t), \lambda(t), \gamma(t), t) = g(y(t), u(t), t) + \lambda^T(t)f(y(t), u(t), t) - \gamma^T(t)C(y(t), u(t), t), \quad (2–20)
\]

\[
28
\]
the first-order optimality conditions are then conveniently expressed as

\[
\dot{y}(t) = \frac{\partial H}{\partial \lambda}, \quad (2-21)
\]

\[
\dot{\lambda}(t) = -\frac{\partial H}{\partial y}, \quad (2-22)
\]

\[
0 = \frac{\partial H}{\partial u}, \quad (2-23)
\]

\[
\lambda(t_0) = -\frac{\partial \phi}{\partial y(t_0)} + \psi^T \frac{\partial \phi}{\partial y(t_0)}, \quad (2-24)
\]

\[
\lambda(t_f) = \frac{\partial \phi}{\partial y(t_f)} - \psi^T \frac{\partial \phi}{\partial y(t_f)}, \quad (2-25)
\]

\[
H|_{t=t_0} = -\psi^T \frac{\partial \phi}{\partial t_0} + \frac{\partial \Phi}{\partial t_0}, \quad (2-26)
\]

\[
H|_{t=t_f} = \psi^T \frac{\partial \phi}{\partial t_f} - \frac{\partial \Phi}{\partial t_f}, \quad (2-27)
\]

\[
\phi = 0. \quad (2-28)
\]

Furthermore, using the complementary slackness condition, \(\gamma\) takes the value

\[
\gamma_i(t) = 0 \text{ when } C_i(y(t), u(t)) < 0, \quad 1 \leq i \leq s, \quad (2-29)
\]

\[
\gamma_i(t) < 0 \text{ when } C_i(y(t), u(t)) = 0, \quad 1 \leq i \leq s. \quad (2-30)
\]

When \(C_i < 0\) the path constraint in Eq. (2–4) is inactive. Therefore, by making \(\gamma_i(t) = 0\), the constraint is simply ignored in augmented cost. The negativity of \(\gamma_i\) when \(C_i = 0\) is interpreted such that improving the cost may only come from violating the constraint [5].

The continuous-time first-order optimality conditions of Eqs. (2–21)–(2–30) define a set of necessary conditions that must be satisfied for an extremal solution of an optimal control problem. This extremal solution can be a maxima, minima or saddle. The second-order sufficiency conditions must be inspected to determine which of the extremal solutions is a global minima. The derivation of the second-order sufficiency conditions, however, is beyond the scope of this dissertation. For a local minima, the particular extremal with the lowest cost is chosen.
2.1.2 Pontryagin’s Principle

The Pontryagin’s principle is used to determine the conditions for obtaining the optimal control. If the optimal control is interior to the feasible control set, the first-order necessary condition related to control given in Eq. (2–23) is used to obtain the optimal control. Eq. (2–23) is called the strong form of the Pontryagin’s principle. If, however, the solution lies on the boundary of the feasible control and state set, the strong form of the Pontryagin’s principle may not be used to compute the optimal control as the differential is one-sided. Such a control is called a “bang-bang” control. Furthermore, if the Hamiltonian defined in Eq. (2–20) is linear in control, i.e.,

\[ H(y(t), u(t), \lambda(t), \gamma(t), t) = g(y(t), t) + \lambda^T(t)f(y(t), t) + \lambda^T(t)u(t) - \gamma^T(t)C(y(t), t), \]

(2–31)

the derivative in Eq. (2–23) does not provide any information about the optimal control. For such problems, the weak form of the Pontryagin’s Principle must be used to determine the optimal control [66].

The control \( u^* \) that gives a local minimum of the cost \( J \) is by definition [1],

\[ J(u) - J(u^*) = \Delta J(u, u^*) \geq 0, \]

(2–32)

for all admissible control \( u \in U \) sufficiently close to \( u^* \). If \( u \) is defined as \( u = u^* + \delta u \), then the change in the cost can be expressed as

\[ \Delta J(u, u^*) = \delta J(u^*, \delta u) + \text{higher order terms}. \]

(2–33)

If \( \delta u \) is sufficiently small, then the higher order terms approach zero and the cost has a local minimum if

\[ \delta J(u^*, \delta u) \geq 0. \]

(2–34)

At the optimal solution, \( y^*, u^*, \lambda^*, \gamma^*, \psi^* \), the differential equations along with the boundary conditions and path constraints are satisfied. Therefore, all the coefficients
of the variation terms in Eq. (2–11) are zero, except the control term. This leaves the variation of the cost as

\[ \delta J_a(u^*, \delta u) = \int_{t_0}^{t_f} \left( \frac{\partial g}{\partial u} + \lambda^T \frac{\partial f}{\partial u} - \gamma^T \frac{\partial C}{\partial u} \right) \delta u \, dt \] (2–35)

The variation of the cost is the integral of the first-order approximation to the change in the Hamiltonian caused by a change in the control alone. The first-order approximation of the change in the Hamiltonian is by definition

\[ \left( \frac{\partial H}{\partial u} \right)_{y^*, u^*, \lambda^*, \gamma^*} \delta u = H(y^*, u^* + \delta u, \lambda^*, \gamma^*, t) - H(y^*, u^*, \lambda^*, \gamma^*, t). \] (2–36)

The variation of the cost for all admissible and sufficiently small \( \delta u \) becomes

\[ \delta J_a(u^*, \delta u) = \int_{t_0}^{t_f} (H(y^*, u^* + \delta u, \lambda^*, \gamma^*, t) - H(y^*, u^*, \lambda^*, \gamma^*, t)) \, dt. \] (2–37)

In order that \( \delta J(u^*, \delta u) \) is non-negative for any admissible variation in the control, the Hamiltonian must be greater than the optimal Hamiltonian for all time

\[ H(y^*, u^* + \delta u, \lambda^*, \gamma^*, t) \geq H(y^*, u^*, \lambda^*, \gamma^*, t). \] (2–38)

Therefore, the optimal control is the admissible control that minimizes the Hamiltonian.

The weak form of the Pontryagin’s principle is stated as [67]

\[ u^*(t) = \arg \min_{u \in U} [H(y^*(t), u(t), \lambda^*(t), \gamma^*(t), t)]. \] (2–39)

The Pontryagin’s principle in Eq. (2–39) is used to obtain optimal control when the control is bang-bang or when the Hamiltonian is linear in control.

### 2.2 Numerical Optimization

A nonlinear programming problem (NLP) arises in optimal control when a continuous-time optimal control problem is discretized. In this section, the development of solutions to the finite-dimensional optimization problems or nonlinear programming
problems (NLP) are discussed. The objective of a NLP is to find a set of parameters that minimizes some cost function that is subject to a set of algebraic equality or inequality constraints. The NLP is solved using well developed algorithms and software. Unconstrained minimization, equality constrained minimization, and inequality constrained minimization of a function are now discussed. The necessary conditions for optimality or the Karush-Kuhn-Tucker (KKT) conditions of the NLP are also presented for each of the three cases.

2.2.1 Unconstrained Optimization

The objective of an unconstrained optimization problem is to find the set of parameters that gives a minimum value of a scalar function. Consider the following problem of determining the minimum of a function defined as \[ J(y), \] where \( y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n \). If \( y^* \) is a locally minimizing set of parameters then the minimum value of the objective function is \( J(y^*) \). For \( y^* \) to be a locally minimizing point, the objective function evaluated at any neighboring point, \( \bar{y} \), must be greater than the optimal cost, i.e.,

\[ J(\bar{y}) > J(y^*). \] (2–41)

For \( y^* \) to be a locally minimizing point, the first-order necessary condition is stated as

\[ g(y^*) = 0, \] (2–42)

where \( g(y) \) is the gradient vector defined as

\[ g(y) \equiv \nabla_y J^T = \begin{bmatrix} \frac{\partial J}{\partial y_1} \\ \frac{\partial J}{\partial y_2} \\ \vdots \\ \frac{\partial J}{\partial y_n} \end{bmatrix}. \] (2–43)
The gradient condition is a set of \( n \) conditions to determine the \( n \) unknown variables of vector \( y \). The necessary condition by itself only defines an extremal point which can be a local minimum, maximum, or saddle point. In order to develop the sufficient condition defining a locally minimizing point \( y^* \), consider a three term Taylor series expansion of \( J(\bar{y}) \) about the extremal point, \( y^* \). The objective function at \( \bar{y} \) is approximated as

\[
J(\bar{y}) = J(y^*) + g^T(y^*)(\bar{y} - y^*) + \frac{1}{2}(\bar{y} - y^*)^T H(y^*)(\bar{y} - y^*) + \text{higher order terms}, \tag{2–44}
\]

where \( H(y) \) is the symmetric \( n \times n \) Hessian matrix defined as

\[
H(y) \equiv \nabla_{yy}J \equiv \frac{\partial^2 J}{\partial y^2} = \begin{bmatrix}
\frac{\partial^2 J}{\partial y_1 \partial y_1} & \frac{\partial^2 J}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 J}{\partial y_1 \partial y_n} \\
\frac{\partial^2 J}{\partial y_2 \partial y_1} & \frac{\partial^2 J}{\partial y_2 \partial y_2} & \cdots & \frac{\partial^2 J}{\partial y_2 \partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 J}{\partial y_n \partial y_1} & \frac{\partial^2 J}{\partial y_n \partial y_2} & \cdots & \frac{\partial^2 J}{\partial y_n \partial y_n}
\end{bmatrix}. \tag{2–45}
\]

If \( \bar{y} - y^* \) is sufficiently small, higher order terms can be ignored. Also, because of the first-order necessary condition given in Eq. (2–42),

\[
J(\bar{y}) = J(y^*) + \frac{1}{2}(\bar{y} - y^*)^T H(y^*)(\bar{y} - y^*). \tag{2–46}
\]

From the inequality given in (2–41),

\[
J(y^*) + \frac{1}{2}(\bar{y} - y^*)^T H(y^*)(\bar{y} - y^*) > J(y^*),
\]

\[
(\bar{y} - y^*)^T H(y^*)(\bar{y} - y^*) > 0. \tag{2–47}
\]

In order to ensure \( y^* \) is a local minimum, the additional condition in (2–47) must be satisfied. Eqs. (2–42) and (2–47) together define necessary and sufficient conditions for a local minimum.

### 2.2.2 Equality Constrained Optimization

Consider finding the minimum of the objective function

\[
J(y), \tag{2–48}
\]
subject to a set of $m \leq n$ constraints

$$f(y) = 0.$$  \hfill (2–49)

Similar to the calculus of variations approach for determining the extremal of functionals, in finding the minimum of an objective function subject to equality constraints, an augmented cost called the Lagrangian is used. Define the Lagrangian as

$$\mathcal{L}(y, \lambda) = J(y) - \lambda^T f(y),$$  \hfill (2–50)

where $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m$ are the Lagrange multipliers. Then, the necessary conditions for the minimum of the Lagrangian is that the point $(y^*, \lambda^*)$ satisfies

$$\nabla_y \mathcal{L}(y^*, \lambda^*) = 0,$$  \hfill (2–51)

$$\nabla_{\lambda} \mathcal{L}(y^*, \lambda^*) = 0,$$  \hfill (2–52)

where the gradient of $\mathcal{L}$ with respect to $y$ and $\lambda$ is

$$\nabla_y \mathcal{L} = g(y) - G^T(y) \lambda,$$  \hfill (2–53)

$$\nabla_{\lambda} \mathcal{L} = -f(y),$$  \hfill (2–54)

where $G(y)$ is the Jacobian matrix, defined as

$$G(y) \equiv \frac{\partial f}{\partial y} = \begin{bmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_n} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \cdots & \frac{\partial f_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial y_1} & \frac{\partial f_m}{\partial y_2} & \cdots & \frac{\partial f_m}{\partial y_n}
\end{bmatrix}.$$  \hfill (2–55)

It is noted that at a minimum of the Lagrangian, the equality constraint of Eq. (2–49) is satisfied. Next, this necessary condition alone does not specify a minimum, maximum or saddle point. In order to specify a minimum, first, define the Hessian of the Lagrangian
as
\[ H_L = \nabla_{yy} L = \nabla_{yy} J - \sum_{i=1}^{m} \lambda_i \nabla_{yy} f_i. \]  (2–56)

Then, a sufficient condition for a minimum is that
\[ v^T H_L v > 0, \]  (2–57)

for any vector \( v \) in the constraint tangent space.

2.2.3 Inequality Constrained Optimization

Consider the problem of minimizing the objective function
\[ J(y), \]  (2–58)

subject to the inequality constraints
\[ c(y) \leq 0. \]  (2–59)

Inequality constrained problems are solved by dividing the inequality constraints into a set of active constraints, and a set of inactive constraints. At the optimal solution \( y^* \), some of the constraints are satisfied as equalities, that is
\[ c_i(y^*) = 0, \quad i \in A, \]  (2–60)

where \( A \) is called the active set, and some constraints are strictly satisfied, that is
\[ c_i(y^*) < 0, \quad i \in A', \]  (2–61)

where \( A' \) is called the inactive set.

By separating the inequality constraints into an active set and an inactive set, the active set can be dealt with as equality constraints as described in the previous section, and the inactive set can be ignored. The added complexity of inequality constrained problems is in determining which set of constraints are active, and which are inactive. If the active and inactive sets are known, an inequality constrained problem becomes an
equality constrained problem stated as follows. Minimize the objective function

$$J(y),$$

subject to the constraints

$$c_i(y) = 0, \quad i \in A,$$  \hspace{1cm} (2–63)

and the same methodology is applied as in the previous section to determine a minimum.

Finally, consider the problem of finding the minimum of the objective function, $J(y)$, subject to the equality constraints

$$f(y) = 0,$$  \hspace{1cm} (2–64)

and the inequality constraints

$$c(y) \leq 0.$$  \hspace{1cm} (2–65)

The inequality constraints are separated into active and inactive constraints. Then, define the Lagrangian as

$$L(y, \lambda, \psi^{(A)}) = J(y) - \lambda^\top f(y) - \psi^{(A)^\top} c^{(A)}(y),$$  \hspace{1cm} (2–66)

where $\lambda$ are the Lagrange multipliers with respect to the equality constraints and $\psi^{(A)}$ are the Lagrange multipliers associated with the active set of inequality constraints. Furthermore, note that the inactive set of constraints are ignored by choosing $\psi^{(A')} = 0$.

Necessary condition for the minimum of the Lagrangian is that the point $(y^*, \lambda^*, \psi^{(A)^*})$ satisfies

$$\nabla_y L(y^*, \lambda^*, \psi^{(A)^*}) = 0,$$  \hspace{1cm} (2–67)

$$\nabla_\lambda L(y^*, \lambda^*, \psi^{(A)^*}) = 0,$$  \hspace{1cm} (2–68)

$$\nabla_{\psi^{(A)}} L(y^*, \lambda^*, \psi^{(A)^*}) = 0.$$  \hspace{1cm} (2–69)
Many numerical algorithms [69–71] exist for solving the nonlinear programming problems. Examples include the Newton’s method, conjugate direction methods, and gradient-based methods such as the sequential quadratic programming (SQP) and the interior-point methods. Various robust and versatile software programs have been developed for the numerical solution of the NLPs. Examples of well-known software that use the SQP methods include the dense NLP solver NPSOL [11] and the sparse NLP solver SNOPT [12]. Well known sparse interior point NLP solvers include KNITRO [13] and IPOPT [14].

2.3 Finite-Dimensional Approximation

In this section, the important mathematical concepts that are used to transcribe a continuous-time optimal control problem to a nonlinear programming problem (NLP) are reviewed. Three concepts are important in constructing a discretized finite-dimensional optimization problem from a continuous-time optimal control problem. These three concepts are polynomial approximation, numerical solution of differential equations, and numerical integration. In order to transcribe a continuous-time optimal control problem to a NLP, the infinite-dimensional continuous functions of the optimal control problem are approximated by a finite-dimensional Lagrange polynomial basis. Furthermore, the dynamic constraints are transcribed to algebraic constraints by setting the derivative of the state approximation (obtained using Lagrange polynomial approximation), equal to the right-hand side of dynamic constraints of Eq. (2–2) at a specified set of points called the collocation points. Lastly, the Lagrange cost is approximated by numerical integration using the Gaussian quadrature.

2.3.1 Polynomial Approximation

In this research, Lagrange polynomials are used to approximate continuous functions of time like state, control and costate. The Lagrange polynomial approximation [72] is based on the fact that given a set of \( N \) arbitrary support points, \( (t_1,\ldots,t_N) \), called the discretization points of a continuous function of time, \( y(t) \), on the interval \( t_i \in [t_0, t_f] \),
there exists a unique polynomial, $Y(t)$, of degree $N - 1$ such that

$$Y(t_i) = y(t_i), \quad 1 \leq i \leq N. \quad (2-70)$$

The unique polynomial approximation to the function, $y(t)$, is given by the Lagrange polynomial approximation formula

$$Y(t) = \sum_{i=1}^{N} y_i L_i(t), \quad (2-71)$$

where $y_i = y(t_i)$ and $L_i(t)$ are the Lagrange polynomials [73], defined as

$$L_i(t) = \prod_{k=1, k \neq i}^{N} \frac{t - t_k}{t_i - t_k}. \quad (2-72)$$

It is noted that these Lagrange polynomials satisfy the isolation property, i.e., they are one at the $i^{th}$ discretization point and zero at all others, so that

$$L_i(t_k) = \delta_{ik} = \begin{cases} 1 & : k = i \\ 0 & : k \neq i \end{cases} \quad (2-73)$$

This property is particularly advantageous in the finite-dimensional transcription of the optimal control problem. As noted before, an optimal control problem comprises of various functionals (functions of state and control). As will be seen further in this chapter, in order to transcribe the continuous-time problem to a finite-dimensional problem, these functionals are evaluated at some or all of the discretization points used in constructing the Lagrange polynomials. When these functionals are evaluated at a discretization point, isolated state and control at that particular discretization point appear in the expression and not a linear combination of all the support points. As a result of which the Jacobian matrix of the constraints of the NLP defined in Eq. (2–55) is a sparse matrix.
### 2.3.1.1 Approximation error

The error in the Lagrange approximation formula for functions in which \( N \) derivatives exist in \([t_0, t_f]\) is known to be [74]

\[
y(t) - Y(t) = \frac{(t - t_1) \cdots (t - t_N)}{N!} y^N(\zeta),
\]

(2–74)

where \( y^N(\zeta) \) is the \( N \)th derivative of the function \( y(t) \) evaluated at some \( \zeta \in [t_0, t_f] \). It is noted that at any support point, the error is zero. Furthermore, if \( y(t) \) is a polynomial of degree at most \( N - 1 \), the \( N \)th derivative in Eq. (2–74) vanishes. Thus, it is seen from Eq. (2–74) that the Lagrange interpolation approximation using \( N \) discretization points is exact for polynomials of degree at most \( N - 1 \). However, the behavior of the interpolation error as \( N \) approaches infinity for non-polynomial functions or analytic functions with singularities is rather interesting and is characterized as the Runge phenomenon [75].

The Runge phenomenon exists for Lagrange polynomials for a uniformly distributed set of discretization points. The Runge phenomenon is the occurrence of oscillations in the approximating function, \( Y(t) \), between discretization points. As the number of discretization points grows, the magnitude of the oscillations between support points also grows. The Runge phenomenon can be demonstrated on the following function defined in time interval \( \tau \in [-1,+1] \)

\[
y(\tau) = \frac{1}{1 + 50\tau^2}, \quad \tau \in [-1,+1].
\]

(2–75)

Fig. 2-2 shows the Lagrange polynomial approximation to the function \( y(\tau) = 1/(1 + 50\tau^2) \) utilizing \( N = 11 \) and \( N = 41 \) uniformly distributed discretization points, i.e., using a 10th-degree and a 40th-degree Lagrange polynomial basis. It is seen that as the number of points are increased, the approximation near the ends of the interval becomes increasingly worse and the error at the ends is much larger than the error in the middle of the interval. In order to avoid the Runge phenomenon in Lagrange polynomial approximation, the discretization points must be chosen so
A Approximation of \( y(\tau) = \frac{1}{1 + 50\tau^2} \) using 11 uniform discretization points.

B Approximation of \( y(\tau) = \frac{1}{1 + 50\tau^2} \) using 41 uniform discretization points.

Figure 2-2. Approximation of \( y(\tau) = \frac{1}{1 + 50\tau^2} \) using 11 and 41 uniformly spaced discretization points.
that the error is more equitably distributed and the maximum error on the interval is minimized. Non-uniform discretization points obtained from orthogonal polynomials like the Chebyshev polynomials and the Legendre polynomials are commonly used as discretization points to avoid the Runge phenomenon.

2.3.1.2 Family of Legendre-Gauss points

In pseudospectral methods, three sets of points are commonly used as discretization points in Lagrange polynomial approximation: Legendre-Gauss-Lobatto (LGL) points, Legendre-Gauss (LG) points, and Legendre-Gauss-Radau (LGR) points. All three sets of points are defined on the domain $[-1, +1]$, but differ significantly in that the LG points include neither of the endpoints, the LGR points include one of the endpoints, and the LGL points include both of the endpoints. In addition, the LGR points are asymmetric relative to the origin and are not unique in that they can be defined using either the initial point or the terminal point. The LGR points that include the terminal endpoint are often called the flipped LGR points. A schematic representation of these points is shown in Fig. 2-3. These sets of points are obtained from the roots of a Legendre polynomial and/or linear combinations of a Legendre polynomial and its derivatives. Denoting the $N$th degree Legendre polynomial by $P_N(\tau)$, then

LG: Roots obtained from $P_N(\tau)$
LGR: Roots obtained from $P_{N-1}(\tau) + P_N(\tau)$
LGL: Roots obtained from $\dot{P}_{N-1}(\tau)$ together with the points $-1$ and $1$

Because these points are defined on the domain $[-1, +1]$, the time domain $[t_0, t_f]$, is first mapped to $[-1, +1]$, by using the following affine transformation

$$t = \frac{t_f - t_0}{2} \tau + \frac{t_f + t_0}{2} \tag{2-76}$$

Let $(\tau_1, ..., \tau_N)$ be the $N$ LGL points such that $\tau_1 = -1$ and $\tau_N = 1$, then a function $y(\tau)$ is approximated on the interval $[-1, +1]$ using the Lagrange polynomials and the LGL
Figure 2-3. Schematic diagram of Legendre points.

points as

\[ Y(\tau) = \sum_{i=1}^{N} y_i L_i(\tau). \] \hspace{1cm} (2–77)

Next, let \((\tau_1, \ldots, \tau_N)\) be the \(N\) LGR points such that \(\tau_1 = -1\) and \(\tau_N < 1\), then the function \(y(\tau)\) is approximated on the interval \([-1, +1]\) using the Lagrange polynomials and the LGR points by defining \(\tau_{N+1} = 1\) and using the following approximation

\[ Y(\tau) = \sum_{i=1}^{N+1} y_i L_i(\tau). \] \hspace{1cm} (2–78)

Lastly, let \((\tau_1, \ldots, \tau_N)\) be the \(N\) LG points such that \(\tau_1 > -1\) and \(\tau_N < 1\), then the function \(y(\tau)\) is approximated on the interval \([-1, +1]\) using the Lagrange polynomials and the LG points by defining \(\tau_0 = -1\) and using the following approximation

\[ Y(\tau) = \sum_{i=0}^{N} y_i L_i(\tau). \] \hspace{1cm} (2–79)
A Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 11 LG discretization points.

B Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 41 LG discretization points.

Figure 2-4. Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 11 and 41 LG discretization points.
A Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 11 LGR discretization points.

B Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 41 LGR discretization points.

Figure 2-5. Approximation of $y(\tau) = 1/(1 + 50\tau^2)$ using 11 and 41 LGR discretization points.
Fig. 2-4 and Fig. 2-5 show the Lagrange polynomial approximation to the function 
\( y(\tau) = 1/(1 + 50\tau^2) \), utilizing \( N = 11 \) and \( N = 41 \) non-uniform LG and LGR discretization 
points, respectively. It is seen that by using the LG and the LGR discretization points, the 
Runge phenomenon is avoided, and the approximations become better as the number 
of discretization points is increased.

![Graph showing error vs. number of discretization points]

**Figure 2-6.** Base ten logarithm of infinity norm error vs. number of discretization points, 
\( N \), for approximating \( y(\tau) = 1/(1 + 50\tau^2) \).

Let the \( \log_{10} \) maximum infinity norm of error be defined as

\[
E_y = \max_k \log_{10} ||Y(\tau_k) - y(\tau_k)||_{\infty} .
\]  

(2–80)

Fig. 2-6 shows the error, \( E_y \), as a function of the number of discretization points used 
in constructing the Lagrange polynomials from the uniformly distributed discretization 
points, the LG, and the LGR discretization points for the function \( y(\tau) = 1/(1 + 50\tau^2) \). It 
is seen that as the number of discretization points increases, the Lagrange polynomial 
approximation using a uniformly distributed set of discretization points diverges. For 100
uniformly distributed discretization points, error is $O(10^{18})$. For a Lagrange polynomial defined by either the LG or the LGR support points, the approximation converges to the function. For 100 LG or LGR discretization points, the error in approximation is $O(10^{-6})$.

2.3.2 Numerical Solution of Differential Equations

Another important concept in transcribing the continuous-time optimal control problem to a finite-dimensional NLP is numerically approximating the solution to the differential equations by transcribing the dynamic constraints in Eq. (2–2), to algebraic equations. Consider the following differential equation whose solution is desired in the time interval $[t_0, t_f]$,\[ \dot{y}(t) = f(y(t), t), \quad y(t_0) = y_0. \quad (2–81) \]

If $(t_1, \ldots, t_N) \in [t_0, t_f]$, then solving the differential equation in Eq. (2–81) numerically involves approximating $(y(t_1), \ldots, y(t_N))$. Two approaches for solving such a differential equation are now considered: time-marching methods and collocation [4].

2.3.2.1 Time-marching methods

Suppose that the interval $[t_0, t_f]$ is divided into $N$ intervals $[t_i, t_{i+1}]$. Numerical methods for solving differential equations are sometimes implemented in multiple steps, i.e., the solution at time $t_{i+1}$ is obtained from a defined set of previous values $(t_{i-j}, \ldots, t_i)$ where $j$ is the number of steps. The simplest multiple-step method is a single-step method with $j = 1$. The Euler methods are the most common single-step methods. The Euler methods have the general form [3] \[
\begin{align*}
y_{i+1} &= y_i + h_i (\alpha f_i + (1 - \alpha) f_{i+1}),
\end{align*}
\]

where $f_i = f(y(t_i), t_i)$, $h_i = t_{i+1} - t_i$, and $\alpha \in [0, 1]$. The values $\alpha = (1, 1/2, 0)$ correspond respectively to the particular Euler methods called the Euler forward, the Crank-Nicolson, and the Euler backward method. More complex multiple-step methods involve the use of more than one previous time point, i.e., $j > 1$. The two most
commonly used multiple-step methods are the Adams-Bashforth and Adams-Moulton multiple-step methods.

The Euler backward and the Crank-Nicolson are examples of implicit methods because the value \( y(t_{i+1}) \) appears implicitly on the right-hand side of Eq. \((2–82)\) whereas the Euler forward is an example of an explicit method because the value \( y(t_{i+1}) \) does not appear on the right-hand side of Eq. \((2–82)\). When employing an implicit method, the solution at \( t_{i+1} \) is obtained using a predictor-corrector where the predictor is typically an explicit method (e.g., Euler-forward) while the corrector is the implicit formula. The implicit methods are more stable than the explicit methods, but an implicit method requires more computation at each step.

Next, suppose that we divide each of the \( N \) intervals \([t_i, t_{i+1}]\), into \( K \) subintervals \([\tau_k, \tau_{k+1}]\) where

\[
\tau_k = t_i + h_i \theta_k, \quad 1 \leq k \leq K, \tag{2–83}
\]

where \( h_i = t_{i+1} - t_i \) and \( \theta_k \in [0, 1] \). Then, the state at \( t_{i+1} \) is approximated as

\[
y(t_{i+1}) = y(t_i) + h_i \sum_{k=1}^{K} \beta_k f(y(\tau_k), \tau_k), \quad 1 \leq i \leq N. \tag{2–84}
\]

The value of state at \( \tau_k, 1 \leq k \leq K, \) in turn is approximated as

\[
y(\tau_k) = y(t_i) + h_i \sum_{j=1}^{K} \gamma_{kj} f(y(\tau_j), \tau_j), \quad 1 \leq k \leq K. \tag{2–85}
\]

Therefore, the state at \( t_{i+1} \) is obtained as

\[
y(t_{i+1}) = y(t_i) + h_i \sum_{k=1}^{K} \beta_k f\left(y(t_i) + h_i \sum_{j=1}^{K} \gamma_{kj} f(y(\tau_j), \tau_j), t_i + h_i \theta_k\right) \\
= y(t_i) + h_i \sum_{k=1}^{K} \beta_k f_{ik}, \quad \tag{2–86}
\]
where
\[ f_{ik} = f \left( y(t_i) + h_i \sum_{j=1}^{K} \gamma_{kj} f(y(\tau_j), \tau_j), t_i + h_i \theta_k \right) \]
\[ = f \left( y(t_i) + h_i \sum_{j=1}^{K} \gamma_{kj} f_j \right). \]  
(2–87)

The approximation obtained in Eq. (2–86) is called the \( K^{th} \)–order Runge-Kutta method \[72, 75] which is a multiple-stage method with \( K \) being the number of stages. A Runge-Kutta method is called explicit if \( \gamma_{kj} = 0 \) for all \( j \geq k \) and is called implicit otherwise. In an explicit Runge-Kutta method, the approximation at \( t_{i+1} \) is computed using information prior to \( t_{i+1} \) whereas in an implicit Runge-Kutta method \( y(t_{i+1}) \) is required in order to determine the solution at \( t_{i+1} \). In the implicit Runge-Kutta case, the solution is updated using a predictor-corrector approach.

A Runge-Kutta method is captured succinctly using the well-known Butcher array \[76]. The most well-known Runge-Kutta methods are the classical Runge-Kutta method which is a 4\(^{th}\)–order method, the Hermite-Simpson method which is a 3\(^{rd}\)–order method. The Euler methods are, in fact, 1\(^{st}\)–order Runge-Kutta methods.

2.3.2.2 Collocation

Another approach to transcribe continuous-time differential equations to finite-dimensional algebraic equations is by collocation. Suppose over an interval \([t_0, t_f]\) we choose to approximate the state using the following polynomial approximation \[57\] and \((t_1, \ldots, t_N) \in [t_0, t_f]\) as discretization points:
\[ y(t) \approx Y(t) = \sum_{i=1}^{N} a_i \phi_i(t), \]  
(2–88)
where \( a_i = a(t_i) \) is the test function evaluated at the \( i^{th} \) support point and \( \phi_i(t) \) is a trial function. The trial functions are used as a basis for the truncated series expansion of the solution, while the test functions are used to ensure that the differential equation is satisfied as closely as possible. Typical trial functions are orthogonal polynomials and
trigonometric functions. In spectral methods, the trial functions chosen are infinitely differentiable global functions.

In a pseudospectral method, the test functions used are the Lagrange polynomials. Furthermore, the test function is the state, \( y(t) \), itself, so that

\[
a_i = y(t_i), \quad 1 \leq i \leq N. \tag{2–89}
\]

The continuous-time approximation to state is then given as

\[
y(t) \approx Y(t) = \sum_{i=1}^{N} y(t_i) L_i(t). \tag{2–90}
\]

The derivative of this state approximation evaluated at a set of points, \( (\tau_1, \ldots, \tau_K) \in [t_0, t_f] \) is equated to the right-hand side of the state dynamics equation defined in Eq. (2–81) evaluated at the same set of points, i.e.,

\[
\dot{Y}(\tau_j) = \sum_{i=1}^{N} y(t_i) \dot{L}_i(\tau_j) = f(y(\tau_j), \tau_j), \quad 1 \leq j \leq K. \tag{2–91}
\]

Equation (2–91) is called a collocation condition because the approximation to the derivative is set equal to the right-hand side of the differential equation evaluated at each of the intermediate points, \( (\tau_1, \ldots, \tau_K) \), called the collocation points.

A subset of collocation methods that have seen extensive use in optimal control are the orthogonal collocation methods [77]. The key difference between an orthogonal collocation method and a standard collocation method is the manner in which the collocation points are chosen. Specifically, in an orthogonal collocation method the collocation points are the roots of a polynomial that is a member of a family of orthogonal polynomials. Common collocation points in orthogonal collocation are those obtained from the roots of the Chebyshev polynomials or the Legendre polynomials. Furthermore, the state in an orthogonal collocation method is typically approximated on the time interval \( \tau \in [-1, +1] \) as these points are defined on the domain \([-1, +1]\).
In pseudospectral methods, the LG, the LGR, and the LGL are commonly used set of collocation points.

2.3.3 Numerical Integration

The last step in finite-dimensional discretization of an optimal control problem is to approximate the cost function of Eq. (2–1) using numerical integration. Typically, the cost is approximated by using a numerical quadrature. Numerical quadrature can generally be described as sampling an integrand at a finite number of points \((t_1, \ldots, t_N) \in [t_0, t_f]\), and using a weighted sum of these points to approximate the integral as

\[
\int_{t_0}^{t_f} g(t) dt = \sum_{i=1}^{N} w_i g(t_i),
\]  

where \(w_i\) is the quadrature weight associated to the \(i^{th}\) sampling point.

The quadrature rule used for integration is consistent with the numerical method used for solving the differential equation. If one is using a Runge-Kutta method for solving the differential equation, the cost would also be approximated using Runge-Kutta integration. In the case of an orthogonal collocation method, the integration rule is an orthogonally collocated quadrature rule i.e., if one is using Legendre-Gauss points, then the Lagrange cost is approximated using a Gauss quadrature. The requirement for such a consistency in the approximation of the differential equations and the cost can be explained as follows.

Any Bolza cost can be converted to a Mayer cost by introducing a new state \(y_{n+1}\) and adding the differential equation

\[
\dot{y}_{n+1} = g(y(t), u(t)),
\]  

with the initial condition

\[
y_{n+1}(t_0) = 0.
\]
Then the cost functional of Eq. (2–1) is given in Mayer form as

\[ J = \Phi(y(t_0), t_0, y(t_f), t_f) + y_{n+1}(t_f). \] (2–95)

A common scheme would then be used to solve the augmented system of differential equations

\[ \dot{y}(t) = f(y(t), u(t)), \]
\[ \dot{y}_{n+1} = g(y(t), u(t)). \] (2–96)

Converting back to Bolza form, the quadrature approximation to the term

\[ \int_{t_0}^{t_f} g(y(t), u(t)) \, dt, \]

in Eq. (2–1) must be the same as the scheme used to solve the system of differential equations given in Eq. (2–96).

In this section, first, low-order numerical integrators are briefly discussed. Next, the Gaussian quadrature rules are presented and it is shown that the quadrature rules associated with the LG points are exact for the polynomials of degree at most \(2N - 1\), the LGR quadrature rule is exact for the polynomials of degree at most \(2N - 2\) and lastly, the LGL quadrature is exact for the polynomials of degree at most \(2N - 3\).

### 2.3.3.1 Low-order integrators

A common technique used to approximate the integral of a function is to use low-degree polynomial approximations that are easily integrated over many subintervals. The approximation to the integral of a function is then the sum of the integrals in each approximating subinterval. When these subintervals are uniformly distributed, these low-order integration rules are commonly known as the Newton-Cotes integration rules. A well-known low-order method is the composite trapezoidal rule [78]. The composite trapezoid rule divides a function into many uniformly distributed subintervals and each subinterval is approximated by a straight line approximation that passes through the
function at the ends of the subinterval. Fig. 2-7 shows \( g(\tau) = 1/(1 + 50\tau^2) \), \( \tau \in [-1, 1] \), and a four interval composite trapezoid rule approximation of the function.

![Graph of \( g(\tau) = 1/(1 + 50\tau^2) \)](image)

**Figure 2-7.** \( g(\tau) = 1/(1 + 50\tau^2) \) vs. \( \tau \) and a four interval trapezoid rule approximation.

For \( N \) approximating intervals, the composite trapezoid rule is given as [78]

\[
\int_{t_0}^{t_f} g(t) dt \approx \frac{t_f - t_0}{2N} \left( g(t_0) + 2g(t_1) + 2g(t_2) + \cdots + 2g(t_{N-1}) + g(t_N) \right),
\]

(2–97)

where \((t_0, \ldots, t_N)\) are the grid points such that the points \((t_{i-1}, t_i)\) define the beginning and the end of the \(i\)th interval. Consider using the composite trapezoid rule to approximate

\[
l = \int_{-1}^{1} \frac{1}{1 + 50\tau^2} d\tau.
\]

(2–98)

Fig. 2-8 shows the \( \log_{10} \) error from the trapezoid rule for approximating Eq. (2–98) as a function of the \( \log_{10} \) number of approximating intervals. As seen in Fig. 2-8, in order to achieve high levels of accuracy in the approximation, many approximating intervals are required. For one hundred thousand approximating intervals, the error is \( O(10^{-12}) \). The rate of convergence of approximation to the actual integral is linear in this case.
The rate of convergence for the Newton-Cotes rule is at best polynomial. The composite rectangle rule, the Romberg integration formula, the composite Simpson’s rule, and the Simpson’s $3/8^{th}$ rule are a few other low-order numerical integration methods. In the composite Simpson’s rule, for example, a second-degree polynomial is used to approximate the function in each subinterval.

![Graph showing the relationship between logarithm of error and logarithm of number of intervals](image)

Figure 2-8. Base ten logarithm of error vs. base ten logarithm of number of intervals for trapezoid rule approximation of Eq. (2–98).

### 2.3.3.2 Gaussian quadrature

The traditional Newton-Cotes formulae based on evenly spaced points do not converge for many well-behaved integrands as the number of support points is increased and, therefore, are not robust when being used as high degree approximations. In contrast to the composite Newton-Cotes rule, another approach is to use highly accurate Gaussian quadratures. In case of Gaussian quadrature, instability of the approximation does not exist as is the case with the Newton-Cotes rules. Furthermore, very high orders of accuracy are obtained for even a few quadrature points. In using a
N point Gaussian quadrature, two quantities must be specified for each point, namely, the location of the point, and a weighting factor multiplying the value of the function evaluated at this point.

The goal of a Gaussian quadrature is to construct an \( N \) point approximation

\[
\int_{-1}^{1} g(\tau) d\tau = \sum_{i=1}^{N} w_i g_i, \tag{2–99}
\]

where \( w_i \) is the quadrature weight associated with point \( \tau_i \) and \( g_i = g(\tau_i) \), such that the approximation in Eq. (2–99) is exact for the polynomial \( g(\tau) \) of as large a degree as possible [78]. Let \( E_N(g) \) be the error between the integral of \( g(\tau) \) and a \( N \) point quadrature approximation to the integral of \( g(\tau) \), that is,

\[
E_N(g) = \int_{-1}^{1} g(\tau) d\tau - \sum_{i=1}^{N} w_i g_i. \tag{2–100}
\]

It is now shown that if the points, \( \tau_i \), are chosen as the roots of a \( N^{th} \)–degree Legendre polynomial, then the error, \( E_N(g) \), is zero for a polynomial, \( g(t) \), of degree at most \( 2N - 1 \) [79]. Suppose \( g(\tau) \) is a polynomial of degree at most \( 2N - 1 \). Then, \( g(\tau) \) can be written as

\[
g(\tau) = P_N(\tau)f(\tau) + h(\tau), \tag{2–101}
\]

where \( f(\tau) \) and \( h(\tau) \) are polynomials of degree at most \( N - 1 \) and \( P_N(\tau) \) is the \( N^{th} \)–degree Legendre polynomial such that the roots of \( P_N(\tau) \) are \( (\tau_1, \ldots, \tau_N) \). Then the integral is,

\[
\int_{-1}^{1} g(\tau) d\tau = \int_{-1}^{1} (P_N(\tau)f(\tau) + h(\tau)) d\tau
\]

\[
= \int_{-1}^{1} P_N(\tau)f(\tau) d\tau + \int_{-1}^{1} h(\tau) d\tau \tag{2–102}
\]

Since, \( P_N(\tau) \) is orthogonal to all polynomials of degree at most \( N - 1 \) [80],

\[
\int_{-1}^{1} P_N(\tau)f(\tau) d\tau = 0. \tag{2–103}
\]
Therefore,
\[ \int_{-1}^{1} g(\tau) d\tau = \int_{-1}^{1} h(\tau) d\tau. \] (2–104)

Recall that \( h(\tau) \) is a polynomial of degree at most \( N - 1 \) and hence can be exactly approximated using \( N - 1 \) degree Lagrange polynomials as
\[ h(\tau) = \sum_{i=1}^{N} h(\tau_i) L_i(\tau). \] (2–105)

Substituting the Lagrange polynomial approximation into the integral given in Eq. (2–104)
\[ \int_{-1}^{1} g(\tau) d\tau = \int_{-1}^{1} \sum_{i=1}^{N} h(\tau_i) L_i(\tau) d\tau \]
\[ = \sum_{i=1}^{N} h(\tau_i) \int_{-1}^{1} L_i(\tau) d\tau. \] (2–106)

Evaluating \( g(\tau) \) at \( \tau_i \),
\[ g(\tau_i) = P_N(\tau_i) f(\tau_i) + h(\tau_i) \]
\[ = h(\tau_i), \] (2–107)

because, \( \tau_i \) is a root of \( P_N(\tau) \), that is,
\[ P_N(\tau_i) = 0, \quad 1 \leq i \leq N. \] (2–108)

Thus the integral of Eq. (2–104) is given as,
\[ \int_{-1}^{1} g(\tau) d\tau = \sum_{i=1}^{N} w_i g(\tau_i), \] (2–109)

where \( w_i \) are the quadrature weights given by
\[ w_i = \int_{-1}^{1} L_i(\tau) d\tau. \] (2–110)
Hence, it is shown that if the roots of the $N^{th}$ degree Legendre Polynomial are chosen as the quadrature points, then the quadrature approximation is exact for the polynomials of degree at most $2N - 1$. This set of points was previously defined as the Legendre-Gauss (LG) points and the quadrature rule obtained using LG points is called as the Legendre-Gauss quadrature. The LG quadrature weights are obtained using the relationship given in Eq. (2–110). Alternately, LG quadrature weights can also be obtained as \[ w_i = \frac{2}{1 - \tau_i^2 [P_N(\tau_i)]^2}, \quad 1 \leq i \leq N, \] (2–111)
where $P_N$ is the derivative of the $N^{th}$–degree Legendre polynomial.

The Legendre-Gauss-Radau (LGR) quadrature formula is similar to the Legendre-Gauss formula, except one of the boundary point is fixed at $-1$. The formula is created by choosing weights $w_i$ and $N - 1$ remaining points $\tau_i$, to integrate the highest degree polynomial possible with zero error, so that

$$E_N(g) = \int_{-1}^{1} g(\tau) d\tau - \sum_{i=1}^{N} w_i g(\tau_i) = 0, \quad \tau_1 = -1.$$ (2–112)

Because one degree of freedom has been removed, the LGR quadrature is exact for the polynomials of degree $2N - 2$. The LGR points are determined to be the zeros of the sum of the Legendre polynomials of degree $N$ and $N - 1$, $P_N(\tau) + P_{N-1}(\tau)$, where $\tau_1 = -1$. The weights are determined as \[ w_i = \frac{1}{(1 - \tau_i) [P_{N-1}(\tau_i)]^2}, \quad 2 \leq i \leq N. \] (2–113)

The weight at the boundary point is

$$w_1 = \frac{2}{N^2}.$$ (2–114)

In the Legendre-Gauss-Lobatto (LGL) quadrature formula the boundary points are fixed at $-1$ and $1$. The formula is created by choosing weights $w_i$ and $N - 2$ remaining
points $\tau_i$, to integrate the highest degree polynomial possible with zero error, so that

$$E_N(g) = \int_{-1}^{1} g(\tau) d\tau - \sum_{i=1}^{N} w_i g_i = 0, \quad \tau_1 = -1, \quad \tau_N = 1. \quad (2-115)$$

Because two degrees of freedom have been removed, the LGL quadrature is exact for the polynomials of degree $2N - 3$. The LGL points are determined to be the zeros of the derivative of the Legendre polynomial of degree $N - 1$, $\dot{P}_{N-1}(\tau)$, plus the two end points, $-1$ and $1$. The weights are determined as [81]

$$w_i = \frac{2}{N(N-1)[P_{N-1}(\tau_i)]^2}, \quad 2 \leq i \leq N-1. \quad (2-116)$$

The weights at the boundary points are

$$w_1 = w_N = \frac{2}{N(N-1)}. \quad (2-117)$$

The accuracy of the LG and the LGR points are now demonstrated for the integral given in Eq. (2–98). Fig. 2-9 shows the log$_{10}$ error for the approximation of Eq. (2–98) versus the number of LG and LGR points. It is seen that as the number of the LG or the LGR points is increased, the accuracy in approximation rapidly increases. Furthermore, the rate of convergence by the LG and LGR quadratures to the exact solution is exponential. For an accuracy of $O(10^{-12})$, 14 LGR points and 16 LG points are required as opposed to the one hundred thousand points required for composite trapezoidal rule. Therefore, for approximating the integral of Eq. (2–98), the LG or the LGR quadratures are significantly more accurate than a low-order method such as the composite trapezoid rule.
Figure 2-9. Base ten logarithm of error vs. number of Gaussian quadrature points, $N$, for approximation of Eq. (2–98).
CHAPTER 3
MOTIVATION FOR THE RADAU PSEUDOSPECTRAL METHOD

As described in Chapter 2, in a direct method for solving optimal control problems, the continuous-time optimal control problem is transcribed to a discrete nonlinear programming problem (NLP). The resulting NLP can then be solved by one of the many well developed nonlinear optimization algorithms. The direct methods are generally preferred over the indirect methods because in a direct method the optimality conditions do not need to be derived, a relatively low-quality initial guess may be used, a guess of the costate is not needed, and the problem can be modified relatively easily.

Extensive research has been done on various discretization schemes. Two of the early schemes, the Euler transcription and the Runge-Kutta transcription [20] are based on the time-marching methods for solving differential equations. The Euler transcription and the Runge-Kutta transcription exhibit polynomial rate of convergence. The collocation based pseudospectral methods are a significant improvement over the Euler transcription and the Runge-Kutta transcription. These methods have many advantages, the most important being the exponential convergence rate. The pseudospectral methods employ high accuracy quadrature rules, making it possible to use many fewer collocation points that may be necessary to achieve the same accuracy using another type of method (e.g., Euler or Runge-Kutta). In addition, because of their mathematical structure, the pseudospectral methods offer the opportunity to compute high accuracy costate.

The Lobatto pseudospectral method (LPM) [33–47] and the Gauss pseudospectral method (GPM) [48–50] are two of the most extensively researched pseudospectral methods. In the LPM, collocation is performed at the Legendre-Gauss-Lobatto (LGL) points, whereas in the GPM, collocation is performed at the Legendre-Gauss (LG) points. As described in Chapter 2, the LGL points include the boundary points $-1$ and $+1$. Because optimal control problems generally have boundary conditions at both ends,
using the LGL points appears to be the natural choice. The LPM, however, suffers from a defect in the optimality conditions at the boundary points. The optimality conditions of the NLP are not equivalent to the discretized form of the continuous-time optimality conditions. In particular, using the LPM leads to an inaccurate costate estimate. On the other hand, the LG points do not include either of the end points, making the GPM the least natural choice. It is shown, however, that in the GPM the optimality conditions of the NLP are equivalent to the discretized form of the continuous-time optimality conditions and an accurate costate estimate is obtained. However, because in the GPM collocation is not performed at the initial and the final point, the control at the initial point is not obtained from the NLP solution.

The motivation for the first part of this research is to find a method that possesses the same accuracy as that of the GPM, but also provides the initial control from the solution of the NLP. The need to find an accurate method that provides the initial control
is even more critical in the case where multiple-interval or \( hp \) pseudospectral methods are used [82, 83]. In a multiple-interval or \( hp \)-approach, the time interval is divided into several subintervals, where each subinterval is on \([-1, +1]\). The terminal point of one subinterval is the initial point of the next interval and is called a mesh point. Fig. 3-2 shows a time interval divided into three subintervals. Further, each of the subinterval uses collocation at 4 LGL, LG or LGR points. Because the LGL points include both the initial and final points, a redundancy in variables and information is observed at the mesh points. On the other hand, in case of the LG points, no information on control at the mesh point is obtained, because the LG points exclude both the initial and the terminal point. The pseudospectral method of this research, called the Radau pseudospectral method, uses Legendre-Gauss-Radau (LGR) points for collocation. The LGR points include the initial point. Therefore, by using the LGR points, control is obtained at each of the mesh points exactly once. It is noted, however, that the control at the final point is not obtained.

![Diagram of multiple-interval implementation of LGL, LG, and LGR points.](image)

**Figure 3-2.** Multiple-interval implementation of LGL, LG, and LGR points.
The second objective of this dissertation is to provide a researcher with a basis and justification to select a particular pseudospectral method while solving an optimal control problem. This objective is the motivation for presenting a unified framework which provides the first rigorous analysis that identifies the key mathematical properties of the pseudospectral methods using collocation at Gaussian quadrature points, enabling a researcher or end-user to see clearly the accuracy and convergence (or non-convergence) that can be expected when applying a particular pseudospectral method on a problem of interest. In this chapter, the Lobatto pseudospectral method and the Gauss pseudospectral method are explored.

3.1 Scaled Continuous-Time Optimal Control Problem

Consider the continuous-time optimal control problem of Section 2.1 again. Minimize the cost functional

$$J = \Phi(y(t_0), t_0, y(t_f), t_f) + \int_{t_0}^{t_f} g(y(t), u(t), t) dt,$$  \hspace{1cm} (3–1)

subject to the dynamic constraints

$$\dot{y}(t) = f(y(t), u(t), t),$$  \hspace{1cm} (3–2)

the boundary conditions

$$\phi(y(t_0), t_0, y(t_f), t_f) = 0,$$  \hspace{1cm} (3–3)

and the inequality path constraints

$$C(y(t), u(t), t) \leq 0,$$  \hspace{1cm} (3–4)

where it is noted that all vector functions of time are row vectors; that is, $y(t) = [y_1(t) \cdots y_n(t)] \in \mathbb{R}^n$. Define the augmented Hamiltonian as

$$H(y(t), u(t), \lambda(t), \gamma(t), t) = g(y(t), u(t), t) + \langle \lambda(t), f(y(t), u(t), t) \rangle$$
$$- \langle \gamma(t), C(y(t), u(t), t) \rangle.$$  \hspace{1cm} (3–5)
where $\langle a, b \rangle$ denotes the inner product between two vectors, $a$ and $b$, such that $\langle a, b \rangle = a^T b = b^T a$. Next, suppose $p : \mathbb{R}^n \to \mathbb{R}^m$, then $\nabla p$ is the $m$ by $n$ Jacobian matrix whose $i^{th}$ row is $\nabla p_i$. In particular, the gradient of a scalar-valued function is a row vector. Then first-order optimality conditions of the optimal control problem in Eqs. (3–1)-(3–4) are written in terms of the Hamiltonian as

$$\dot{y}(t) = \nabla_\lambda H,$$  
(3–6)  

$$\dot{\lambda}(t) = -\nabla_y H,$$  
(3–7)  

$$0 = \nabla_u H,$$  
(3–8)  

$$\lambda(t_0) = -\nabla_{y(t_0)}(\Phi - \langle \psi, \phi \rangle),$$  
(3–9)  

$$\lambda(t_f) = \nabla_{y(t_f)}(\Phi - \langle \psi, \phi \rangle),$$  
(3–10)  

$$H|_{t=t_0} = \nabla_{t_0}(\Phi - \langle \psi, \phi \rangle),$$  
(3–11)  

$$H|_{t=t_f} = -\nabla_{t_f}(\Phi - \langle \psi, \phi \rangle),$$  
(3–12)  

$$\gamma_i(t) = 0 \text{ when } C_i(y(t), u(t)) < 0, \quad 1 \leq i \leq s,$$  
(3–13)  

$$\gamma_i(t) < 0 \text{ when } C_i(y(t), u(t)) = 0, \quad 1 \leq i \leq s,$$  
(3–14)  

$$\phi = 0.$$  
(3–15)

The Hamiltonian at the initial time can be obtained using the following expression:

$$H|_{t=t_0} = \frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt - \int_{t_0}^{t_f} \frac{dH}{dt}(t_f - t) \, dt \right),$$  
(3–16)

since the second term on the right-hand side when integrated by part is written as

$$\frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt + (t_f - t_0)H|_{t=t_0} - \int_{t_0}^{t_f} H \, dt \right) = H|_{t=t_0}. \quad (3–17)$$

Similarly, the Hamiltonian at the final time is obtained from

$$H|_{t=t_f} = \frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt + \int_{t_0}^{t_f} \frac{dH}{dt}(t - t_0) \, dt \right), \quad (3–18)$$
because integrating the second term on the right-hand side by part gives
\[
\frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt + (t_f - t_0)H|_{t=t_f} - \int_{t_0}^{t_f} H \, dt \right) = H|_{t=t_f}.
\] (3–19)

The total time derivative of the Hamiltonian is written as
\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \dot{y}} \dot{y} + \frac{\partial H}{\partial \dot{\lambda}} \dot{\lambda} + \frac{\partial H}{\partial \dot{u}} \dot{u} + \frac{\partial H}{\partial \dot{\gamma}} \dot{\gamma},
\] (3–20)

Substituting the first-order optimality conditions of Eq. (3–6)-(3–8), the total derivative is reduced to
\[
\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial \dot{\gamma}} \dot{\gamma}.
\] (3–21)

Next, from the definition of the Hamiltonian,
\[
\frac{\partial H}{\partial \dot{\gamma}} = -C(y(t), u(t), t).
\] (3–22)

Using the complementary slackness conditions in Eq. (3–13)-(3–14),
\[
\frac{\partial H}{\partial \dot{\gamma}} = 0 \text{ when } C(y(t), u(t), t) = 0,
\] (3–23)
\[
\dot{\gamma}(t) = 0 \text{ when } C(y(t), u(t), t) < 0 \text{ because } \gamma(t) = 0.
\] (3–24)

Therefore, the total time derivative of the Hamiltonian is equal to the partial derivative of Hamiltonian with respect to time.
\[
\frac{dH}{dt} = \frac{\partial H}{\partial t}.
\] (3–25)

Hence Eqs. (3–16) and (3–18) can be rewritten as
\[
H|_{t=t_0} = \frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt - \int_{t_0}^{t_f} \frac{\partial H}{\partial t} (t_f - t) \, dt \right),
\] (3–26)
\[
H|_{t=t_f} = \frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt + \int_{t_0}^{t_f} \frac{\partial H}{\partial t} (t - t_0) \, dt \right).
\] (3–27)

The Legendre-Gauss-Lobatto (LGL) points used in the Lobatto pseudospectral method and the Legendre-Gauss (LG) points used in the Gauss pseudospectral method are defined on the domain \([-1, +1]\). Therefore, the time domain \([t_0, t_f]\), is mapped to
by using the following affine transformation 

\[ t = \frac{t_f - t_0}{2} \tau + \frac{t_f + t_0}{2}, \]  

(3–28)

such that 

\[ \frac{dt}{d\tau} = \frac{t_f - t_0}{2}. \]  

(3–29)

The mapping is used to convert the optimal control problem given in Eqs. (3–1)-(3–4) to the time domain \( \tau \in [-1, +1] \), such that the objective is to minimize the cost functional 

\[ J = \Phi(y(-1), t_0, y(+1), t_f) + \frac{t_f - t_0}{2} \int_{-1}^{1} g(y(\tau), u(\tau), \tau; t_0, t_f) d\tau, \]  

(3–30)

subject to the dynamic constraints 

\[ \frac{dy(\tau)}{d\tau} = y(\tau) = \frac{t_f - t_0}{2} f(y(\tau), u(\tau), \tau; t_0, t_f), \]  

(3–31)

the boundary conditions 

\[ \phi(y(-1), t_0, y(+1), t_f) = 0, \]  

(3–32)

and the inequality path constraints 

\[ \frac{t_f - t_0}{2} C(y(\tau), u(\tau), \tau; t_0, t_f) \leq 0. \]  

(3–33)

It is noted that, in order to write the optimality conditions more succinctly, Eq. (3–33) is multiplied by \((t_f - t_0)/2\) without actually affecting the constraint. The Hamiltonian for this problem is defined as 

\[
H(y(\tau), u(\tau), \lambda(\tau), \gamma(\tau), \tau; t_0, t_f) = g(y(\tau), u(\tau), \tau; t_0, t_f) + \langle \lambda(\tau), f(y(\tau), u(\tau), \tau; t_0, t_f) \rangle \\
- \langle \gamma(\tau), C(y(\tau), u(\tau), \tau; t_0, t_f) \rangle,
\]  

(3–34)

Consider the following derivation for obtaining the Hamiltonian at the initial time and the final time in \( \tau \) domain. From Eq. (3–26), the Hamiltonian at the initial time is

\[
H|_{\tau=t_0} = \frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt - \int_{t_0}^{t_f} \frac{\partial H}{\partial t} (t_f - t) \, dt \right).
\]  

(3–35)
Then using the time transformation in Eq. (3–28) the following expression is obtained:

\[ \frac{t_f - t}{t_f - t_0} = \frac{1 - \tau}{2}. \] (3–36)

Using Eq. (3–36) and the relationship in Eq. (3–29), the Hamiltonian at the initial time is given then in terms of \( \tau \) as

\[ H|_{t=t_0} = \frac{1}{2} \int_{-1}^{+1} H \, d\tau - \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t} \left( \frac{1 - \tau}{2} \right) d\tau. \] (3–37)

Again, from the time transformation in Eq. (3–28) the following expression is obtained:

\[ \frac{\partial t}{\partial t_0} = \frac{1 - \tau}{2}. \] (3–38)

Therefore, the Hamiltonian at the initial time is given as

\[ H|_{t=t_0} = \frac{1}{2} \int_{-1}^{+1} H \, d\tau - \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t_0} d\tau. \] (3–39)

Similarly, the Hamiltonian at the final time is obtained from Eq. (3–27) as

\[ H|_{t=t_f} = \frac{1}{t_f - t_0} \left( \int_{t_0}^{t_f} H \, dt + \int_{t_0}^{t_f} \frac{\partial H}{\partial t} (t - t_0) dt \right). \] (3–40)

From the time transformation in Eq. (3–28) the following expression is obtained:

\[ \frac{t - t_0}{t_f - t_0} = \frac{1 + \tau}{2}. \] (3–41)

Using Eq. (3–41) and the relationship in Eq. (3–29) the Hamiltonian at the final time is given in terms of \( \tau \) as

\[ H|_{t=t_f} = \frac{1}{2} \int_{-1}^{+1} H \, d\tau + \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t} \left( \frac{1 + \tau}{2} \right) d\tau. \] (3–42)

Again, from the time transformation in Eq. (3–28) the following expression is obtained:

\[ \frac{\partial t}{\partial t_f} = \frac{1 + \tau}{2}. \] (3–43)
Therefore, the Hamiltonian at the final time is given as

\[ H|_{t=t_f} = \frac{1}{2} \int_{-1}^{+1} H \, d\tau + \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t_f} \, d\tau. \]  

(3–44)

The first-order optimality conditions of the scaled continuous-time optimal control problem in Eqs. (3–30)-(3–33) are then given in terms of the Hamiltonian as

\[
\begin{align*}
\dot{y}(\tau) &= \frac{t_f - t_0}{2} \nabla_\lambda H, \\
\dot{\lambda}(\tau) &= -\frac{t_f - t_0}{2} \nabla_y H, \\
0 &= \nabla_u H, \\
\lambda(-1) &= -\nabla_{y(-1)}(\Phi - \langle \psi, \phi \rangle), \\
\lambda(+1) &= \nabla_{y(+1)}(\Phi - \langle \psi, \phi \rangle), \\
\nabla_{t_0}(\Phi - \langle \psi, \phi \rangle) &= \frac{1}{2} \int_{-1}^{+1} H \, d\tau - \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t_0} \, d\tau, \\
-\nabla_{t_f}(\Phi - \langle \psi, \phi \rangle) &= \frac{1}{2} \int_{-1}^{+1} H \, d\tau + \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t_f} \, d\tau, \\
\gamma_i(\tau) &= 0 \text{ when } C_i(y(\tau), u(\tau)) < 0, \quad 1 \leq i \leq s, \\
\gamma_i(\tau) &< 0 \text{ when } C_i(y(\tau), u(\tau)) = 0, \quad 1 \leq i \leq s, \\
\phi &= 0.
\end{align*}
\]

(3–45) – (3–54)

### 3.2 Lobatto Pseudospectral Method

The Lobatto pseudospectral method is a direct transcription method that converts a continuous-time optimal control problem into a discrete nonlinear programming problem. The method uses the Legendre-Gauss-Lobatto (LGL) points for the Lagrange polynomial approximation of continuous functions of time, for collocation of the differential dynamic constraints, and for quadrature approximation of the integrated Lagrange cost term.
3.2.1 NLP Formulation of the Lobatto Pseudospectral Method

Consider the \( N \) LGL points, \((\tau_1, \tau_2, \ldots, \tau_N)\), where \( \tau_1 = -1 \) and \( \tau_N = +1 \). Next, let \( L_i(\tau) \), \((i = 1, \ldots, N)\), be the Lagrange polynomials of degree \( N - 1 \) given as

\[
L_i(\tau) = \prod_{j=1 \atop j \neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 1, \ldots, N). \tag{3–55}
\]

The state, \( y(\tau) \), is approximated by a polynomial of degree at most \( N - 1 \) using the Lagrange polynomials as

\[
y(\tau) \approx Y(\tau) = \sum_{i=1}^{N} Y_i L_i(\tau), \tag{3–56}
\]

where \( Y_i = Y(\tau_i) \). The points, \((\tau_1, \ldots, \tau_N)\), that are used in state approximation are called the discretization points. Next, an approximation to the derivative of the state in \( \tau \) domain is given by differentiating the approximation of Eq. (3–56) with respect to \( \tau \),

\[
y(\tau) \approx \dot{Y}(\tau) = \sum_{i=1}^{N} Y_i \dot{L}_i(\tau). \tag{3–57}
\]

The following collocation conditions are then formed by equating the derivative of the state approximation in Eq. (3–57) to the right-hand side of the state dynamic constraints in Eq. (3–31) at the \( N \) LGL points:

\[
\sum_{i=1}^{N} Y_i \dot{L}_i(\tau_k) = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad (k = 1, \ldots, N), \tag{3–58}
\]

\[
\sum_{i=1}^{N} D_{ki} Y_i = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad D_{ki} = \dot{L}_i(\tau_k), \tag{3–59}
\]

where \( U_k = U(\tau_k) \) and \( D = [D_{ki}] \), \((1 \leq k \leq N)\), \((1 \leq i \leq N)\) is a \( N \times N \) square matrix and is called the \textit{Lobatto pseudospectral differentiation matrix}. The matrix \( D \) is square because the collocation points are the same as the discretization points used in
the Lagrange polynomial approximation of state. Let $Y^{LGL}$ be defined as

$$
Y^{LGL} = \begin{bmatrix}
Y_1 \\
\vdots \\
Y_N
\end{bmatrix}.
$$

Using the matrix $Y^{LGL}$, the collocated dynamics at the $N$ LGL collocation points in Eq. (3–59) are expressed as

$$
D_k Y^{LGL} = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad (k = 1, \ldots, N),
$$

where $D_k$ is the $k^{th}$ row of the differentiation matrix $D$. Next, the path constraints in Eq. (3–33) are enforced at the $N$ LGL collocation points as

$$
\frac{t_f - t_0}{2} C(Y_k, U_k, \tau_k; t_0, t_f) \leq 0, \quad (k = 1, \ldots, N).
$$

Lastly, the cost functional is approximated using the LGL quadrature as

$$
J = \Phi(Y(\tau_1), \tau_1, Y(\tau_N), \tau_N) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(Y_k, U_k, \tau_k; t_0, t_f),
$$

where $w_k$ is the quadrature weight associated with the $k^{th}$ LGL collocation point.

The finite-dimensional nonlinear programming problem (NLP) corresponding to the Lobatto pseudospectral method is then given as follows. Minimize the objective function

$$
J = \Phi(Y(\tau_1), \tau_1, Y(\tau_N), \tau_N) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(Y_k, U_k, \tau_k; t_0, t_f),
$$

subject to the following equality and inequality constraints:

$$
D_k Y^{LGL} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f) = 0, \quad (k = 1, \ldots, N),
$$

$$
\phi(Y(\tau_1), \tau_1, Y(\tau_N), \tau_N) = 0,
$$

$$
\frac{t_f - t_0}{2} C(Y_k, U_k, \tau_k; t_0, t_f) \leq 0, \quad (k = 1, \ldots, N),
$$
where the NLP variables are \((Y_1, \ldots, Y_N), (U_1, \ldots, U_N), t_0, \) and \(t_f\). The problem defined by Eqs. (3–63)-(3–66) is the discrete Lobatto pseudospectral approximation to the continuous-time optimal control problem defined by Eqs. (3–30)-(3–33).

A few key properties of the Lobatto pseudospectral method are now stated. The discretization points, at which the state is approximated are the \(N\) LGL points, \((\tau_1, \ldots, \tau_N)\). The state approximation uses the Lagrange polynomials of degree \(N - 1\), which is one less than the number of discretization points. Furthermore, the state dynamics are collocated at the same \(N\) LGL points, \((\tau_1, \ldots, \tau_N)\). Fig. 3-3 shows the discretization and collocation points for the Lobatto pseudospectral method. A consequence of approximating the state and collocating the dynamics at the same \(N\) LGL points is that the Lobatto pseudospectral differentiation matrix is a square, \(N \times N\) matrix.
3.2.2 Necessary Optimality Conditions

The necessary optimality conditions, also called the Karush-Kuhn-Tucker (KKT) conditions, of the NLP given in Eqs. (3–63)-(3–66) are now derived. The Lagrangian associated with the NLP is

\[
\mathcal{L} = \Phi(Y(\tau_1), \tau_1, Y(\tau_N), \tau_N) - \langle \Psi, \phi(Y(\tau_1), \tau_1, Y(\tau_N), \tau_N) \rangle \\
+ \frac{t_f - t_0}{2} \sum_{k=1}^{N} (w_k g(Y_k, U_k, \tau_k; t_0, t_f) - \langle \Gamma_k, C(Y_k, U_k, \tau_k; t_0, t_f) \rangle) \\
- \sum_{k=1}^{N} \langle \Lambda_k, D_k Y^{LGL} \rangle + \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f)), \tag{3–67}
\]

where \( \Lambda_k \) is the \( k^{th} \) row of the Lagrange multipliers matrix \( \Lambda \in \mathbb{R}^{N \times n} \) associated with the constraints in Eq. (3–64), \( \Psi \in \mathbb{R}^q \) are the Lagrange multipliers associated with the constraints in Eq. (3–65), and \( \Gamma_k \) is the \( k^{th} \) row of the Lagrange multipliers matrix \( \Gamma \in \mathbb{R}^{N \times s} \) associated with the constraints in Eq. (3–66). The KKT optimality conditions are then obtained by differentiating the Lagrangian with respect to each of the variables and equating these derivatives to zero such that

\[
\begin{align*}
\frac{t_f - t_0}{2} \nabla_{Y_1} (w_1 g_1 + \langle A_1, f_1 \rangle - \langle \Gamma_1, C_1 \rangle) - D_1^T \Lambda &= -\nabla_{Y_1} (\phi - \langle \Psi, \phi \rangle), \tag{3–68} \\
\frac{t_f - t_0}{2} \nabla_{Y_N} (w_N g_N + \langle A_N, f_N \rangle - \langle \Gamma_N, C_N \rangle) - D_N^T \Lambda &= -\nabla_{Y_N} (\phi - \langle \Psi, \phi \rangle), \tag{3–69} \\
\frac{t_f - t_0}{2} \sum_{k=1}^{N} \nabla_{y_k} (w_k g_k + \langle A_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) &= D_k^T \Lambda, \quad 2 \leq k \leq N - 1, \tag{3–70} \\
-\frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle A_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) &= -\nabla_{t_0} (\phi - \langle \Psi, \phi \rangle), \tag{3–71} \\
\frac{t_f - t_0}{2} \sum_{k=1}^{N} \nabla_{t_k} (w_k g_k + \langle A_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) \\
+ \frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle A_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) &= -\nabla_{t_f} (\phi - \langle \Psi, \phi \rangle), \tag{3–72} \\
\nabla_{u_k} (w_k g_k + \langle A_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) &= 0, \quad 1 \leq k \leq N. \tag{3–73}
\end{align*}
\]
\[ D_k Y^\text{LGL} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f) = 0, \quad 1 \leq k \leq N, \quad (3-74) \]

\[ \Gamma_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (3-75) \]

\[ \Gamma_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (3-76) \]

\[ \phi(Y(\tau_1), \tau_1, Y(\tau_N), \tau_N) = 0, \quad (3-77) \]

where \( D_i^T \) is the \( i^{th} \) row of \( D^T \), \( g_k = g(Y_k, U_k, \tau_k; t_0, t_f) \), \( f_k = f(Y_k, U_k, \tau_k; t_0, t_f) \) and \( C_k = C(Y_k, U_k, \tau_k; t_0, t_f) \).

The KKT conditions in Eqs. (3–68)-(3–77) are reformulated so that they become a discretization of the first-order optimality conditions given in Eqs. (3–45)-(3–54) for the continuous control problem of Eqs. (3–30)-(3–33). Let \( D^\dagger = [D_{ij}] \), \( 1 \leq i \leq N \), \( 1 \leq j \leq N \) be the \( N \times N \) matrix defined as follows:

\[ D_{ii}^\dagger = D_{ii} = 0, \quad 2 \leq i \leq N - 1, \quad (3-78) \]

\[ D_{11}^\dagger = -D_{11} - \frac{1}{w_1}, \quad (3-79) \]

\[ D_{NN}^\dagger = -D_{NN} + \frac{1}{w_N}, \quad (3-80) \]

\[ D_{ij}^\dagger = -\frac{w_j}{w_i} D_{ji}, \quad 1 \leq i, j \leq N, \quad i \neq j. \quad (3-81) \]

According to the definition of \( D^\dagger \),

\[ D_1^T = -w_1 D_1^\dagger W^{-1} - \frac{1}{w_1} e_1, \quad (3-82) \]

\[ D_N^T = -w_N D_N^\dagger W^{-1} + \frac{1}{w_N} e_N, \quad (3-83) \]

\[ D_k^T = -w_k D_k^\dagger W^{-1}, \quad 2 \leq k \leq N - 1, \quad (3-84) \]

where \( W \) is a diagonal matrix with weights \( w_k, 1 \leq k \leq N \), on the diagonal, \( e_i \) is the \( i^{th} \) row of \( N \times N \) identity matrix. Substituting Eqs. (3–82)-(3–84) in Eqs. (3–68)-(3–70), we
get the following relationships

\[- \nabla_{Y_1}(\Phi - \langle \Psi, \phi \rangle) = \frac{t_f - t_0}{2} \nabla_{Y_1}(w_1 g_1 + \langle \Lambda_1, f_1 \rangle - \langle \Gamma_1, C_1 \rangle) + w_1 D_1^\dagger W^{-1} \Lambda + \frac{1}{w_1} e_1 \Lambda, \]

(3–85)

\[- \nabla_{Y_N}(\Phi - \langle \Psi, \phi \rangle) = \frac{t_f - t_0}{2} \nabla_{Y_N}(w_N g_N + \langle \Lambda_N, f_N \rangle - \langle \Gamma_N, C_N \rangle) + w_N D_N^\dagger W^{-1} \Lambda - \frac{1}{w_N} e_N \Lambda, \]

(3–86)

\[- w_k D_k^\dagger W^{-1} \Lambda = \frac{t_f - t_0}{2} \nabla_{Y_k}(w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle), \]

(3–87)

\[2 \leq k \leq N - 1. \]

Next, defining the following change of variables:

\[\tilde{\lambda}_k = \frac{\Lambda_k}{w_k}, \quad 1 \leq k \leq N, \]

(3–88)

\[\tilde{\gamma}_k = \frac{\Gamma_k}{w_k}, \quad 1 \leq k \leq N, \]

(3–89)

\[\tilde{\psi} = \Psi. \]

(3–90)

Substituting Eqs. (3–88)-(3–90) in Eqs. (3–73)-(3–77) and in Eqs. (3–85)-(3–87), the transformed KKT conditions for the NLP are given as

\[0 = \nabla_{u_k}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 1 \leq k \leq N, \]

(3–91)

\[0 = D_k Y^{LGL} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad 1 \leq k \leq N, \]

(3–92)

\[\tilde{\gamma}_{ki} = 0 \quad \text{when} \quad C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \]

(3–93)

\[\tilde{\gamma}_{ki} < 0 \quad \text{when} \quad C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \]

(3–94)

\[0 = \phi(Y(\tau_1), \tau_1, Y(\tau_N), \tau_N), \]

(3–95)

\[\nabla_{t_0}(\Phi - \langle \tilde{\psi}, \phi \rangle) = \frac{1}{2} \sum_{k=1}^{N} w_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle) - \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k \nabla_{t_0}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \]

(3–96)
\[-\nabla_{\tau}(\Phi - \langle \tilde{\psi}, \phi \rangle) = \frac{1}{2} \sum_{k=1}^{N} w_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k \nabla_{\tau}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad (3-97)\]

\[D_i^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{Y_i}(g_1 + \langle \tilde{\lambda}_1, f_1 \rangle - \langle \tilde{\gamma}_1, C_1 \rangle) - \frac{1}{w_1} \left( \tilde{\lambda}_1 + \nabla_{Y_1}(\Phi - \langle \tilde{\psi}, \phi \rangle) \right), \quad (3-98)\]

\[D_N^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{Y_N}(g_N + \langle \tilde{\lambda}_N, f_N \rangle - \langle \tilde{\gamma}_N, C_N \rangle) + \frac{1}{w_N} \left( \tilde{\lambda}_N - \nabla_{Y_N}(\Phi - \langle \tilde{\psi}, \phi \rangle) \right), \quad (3-99)\]

\[D_k^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{Y_k}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 2 \leq k \leq N - 1. \quad (3-100)\]

Now, consider a comparison of the transformed KKT conditions in Eqs. (3–91)-(3–100) of the NLP to the first-order necessary optimality conditions in Eqs. (3–45)-(3–54) of the continuous-time optimal control problem. It is noted that the transformed KKT conditions in Eqs. (3–91)-(3–95) are the discretized forms of the continuous-time first-order optimality conditions in Eq. (3–47), Eq. (3–45), Eq. (3–52), Eq. (3–53), and Eq. (3–54), respectively. Next, the right-hand side of Eq. (3–96) and Eq. (3–97) is the quadrature approximation of the right-hand side of Eq. (3–50) and Eq. (3–51), respectively. Therefore, the transformed KKT conditions in Eq. (3–96) and Eq. (3–97) are the discretized version of continuous-time first-order optimality conditions in Eq. (3–50) and Eq. (3–51). Furthermore, it has been shown in Ref. [42] that $D^\dagger = D$ for LGL collocation, thus making $D^\dagger$ a differentiation matrix connected with the LGL points. Therefore, the left hand side of Eqs. (3–98) - (3–100) is an approximation of the costate dynamics at the $k^{th}$ collocation point, i.e.,

\[D_k^\dagger \tilde{\lambda} = \dot{\tilde{\lambda}}_k, \quad 1 \leq k \leq N. \quad (3–101)\]
As a result, Eq. \((3–100)\) represents the discretized version of the costate dynamics in Eq. \((3–46)\) at \(k = (2, \ldots, N – 1)\). It is noted, however, that at the boundary points, the discrete equivalents of continuous boundary conditions \((3–48)\) and \((3–49)\) are coupled in the discrete costate dynamics in \((3–98)\) and \((3–99)\), respectively. Hence, the system of the transformed KKT conditions of the NLP is \textit{not} equivalent to the first-order optimality conditions of the continuous-time optimal control problem.

The transformed KKT conditions for the Lobatto pseudospectral method define the set of conditions that approximate the continuous first-order optimality conditions. The costate can be estimated from the KKT multipliers using the relationship given in Eq. \((3–88)\). The costate estimate, however, does not satisfy the discrete form of the costate dynamics at the boundaries. This defect leads to significant errors in the costate estimate. Analysis and improvement of this defect of the Lobatto pseudospectral method is the one of the motivations for this dissertation.

\subsection*{3.3 Gauss Pseudospectral Method}

The Gauss pseudospectral method is also a direct transcription method that converts a continuous-time optimal control problem into a discrete nonlinear programming problem (NLP). The method uses the Legendre-Gauss (LG) points for collocation of the differential dynamic constraints and for quadrature approximation of the integrated Lagrange cost term. The Lagrange polynomial approximations of the continuous functions of time, however, utilize the LG points plus the initial point. This discretization scheme results in a set of KKT conditions that are exactly equivalent to the discretized form of the continuous first-order optimality conditions. Hence, a significantly better costate estimate than the Lobatto pseudospectral method is obtained. The method, however, does not provide any information about the initial control and the implementation of the method is more complex than that of the method of this research, the Radau pseudospectral method.
3.3.1 NLP Formulation of the Gauss Pseudospectral Method

Consider the \( N \) LG points, \((\tau_1, \tau_2, \ldots, \tau_N)\), where \( \tau_1 > -1 \) and \( \tau_N < +1 \). Define two new points such that \( \tau_0 = -1 \) and \( \tau_{N+1} = 1 \). Next, let \( L_i(\tau) \), \((i = 0, \ldots, N)\), be the Lagrange polynomials of degree \( N \) given by

\[
L_i(\tau) = \prod_{j=0, j \neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 0, \ldots, N). \tag{3–102}
\]

The state, \( y(\tau) \), is approximated by a polynomial of degree at most \( N \) using the Lagrange polynomials as

\[
y(\tau) \approx Y(\tau) = \sum_{i=0}^{N} Y_i L_i(\tau), \tag{3–103}
\]

where \( Y_i = Y(\tau_i) \). It is important to note that \( \tau_0 = -1 \) is not a LG point but is used in state approximation. Next, an approximation to the derivative of the state in \( \tau \) domain is given by differentiating the approximation of Eq. (3–103) with respect to \( \tau \),

\[
\dot{y}(\tau) \approx \dot{Y}(\tau) = \sum_{i=0}^{N} Y_i \dot{L}_i(\tau). \tag{3–104}
\]

The following collocation conditions are then formed by equating the derivative of the state approximation in Eq. (3–104) to the right-hand side of state dynamic constraints in Eq. (3–31) at the \( N \) LG points, \((\tau_1, \ldots, \tau_N)\),

\[
\sum_{i=0}^{N} Y_i \dot{L}_i(\tau_k) = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad (k = 1, \ldots, N), \tag{3–105}
\]

\[
\sum_{i=0}^{N} D_{ki} Y_i = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad D_{ki} = \dot{L}_i(\tau_k), \tag{3–106}
\]

where \( U_k = U(\tau_k) \). It is noted that \( \tau_0 \) is not a collocation point. The matrix \( D = [D_{ki}] \), \((1 \leq k \leq N), (0 \leq i \leq N)\) is a \( N \times (N + 1) \) non-square matrix and is called the Gauss pseudospectral differentiation matrix. The matrix \( D \) is non-square because the state approximation uses \( N + 1 \) points, \((\tau_0, \ldots, \tau_N)\), but the collocation is done at only the
\( N \) LG points, \( (\tau_1, \ldots, \tau_N) \). Let \( Y^{LG} \) be defined as
\[
Y^{LG} = \begin{bmatrix}
Y_0 \\
\vdots \\
Y_N
\end{bmatrix}.
\]

Using the matrix \( Y^{LG} \), the collocated dynamics at the \( N \) LG collocation points in Eq. (3–106) are expressed as
\[
D_k Y^{LG} = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad (k = 1, \ldots, N),
\]
(3–107)
where \( D_k \) is the \( k^{\text{th}} \) row of the differentiation matrix \( D \). It is noted here that the state at the final point is not present in the formulation. It is approximated as follows: First,
\[
y(+1) = y(-1) + \int_{-1}^{+1} \dot{y}(\tau) d\tau.
\]
(3–108)
The LG quadrature approximation of Eq. (3–108) is then given as
\[
Y(\tau_{N+1}) = Y(0) + \sum_{k=1}^{N} w_k Y(\tau_k)
\]
\[
= Y(0) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k f(Y_k, U_k, \tau_k; t_0, t_f),
\]
(3–109)
where \( w_k, 1 \leq k \leq N \), are the LG quadrature weights. Next, the path constraints in Eq. (3–33) are enforced at the \( N \) LG collocation points as
\[
\frac{t_f - t_0}{2} C(Y_k, U_k, \tau_k; t_0, t_f) \leq 0, \quad (k = 1, \ldots, N).
\]
(3–110)
Lastly, the cost functional is approximated using LG quadrature as
\[
J = \Phi(Y(0), \tau_0, Y(\tau_{N+1}), \tau_{N+1}) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(Y_k, U_k, \tau_k; t_0, t_f).
\]
(3–111)
The finite-dimensional nonlinear programming problem corresponding to the Gauss pseudospectral method is then given as follows. Minimize the cost function

$$J = \Phi(Y(\tau_0), \tau_0, Y(\tau_{N+1}), \tau_{N+1}) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(Y_k, U_k, \tau_k; t_0, t_f), \quad (3–112)$$

subject to the following equality and inequality constraints:

$$D_k Y_k^L - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f) = 0, \quad (k = 1, ..., N) \quad (3–113)$$

$$Y(\tau_{N+1}) - Y(\tau_0) - \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k f(Y_k, U_k, \tau_k; t_0, t_f) = 0, \quad (3–114)$$

$$\phi(Y(\tau_0), \tau_0, Y(\tau_{N+1}), \tau_{N+1}) = 0, \quad (3–115)$$

$$\frac{t_f - t_0}{2} C(Y_k, U_k, \tau_k; t_0, t_f) \leq 0, \quad (k = 1, ..., N) \quad (3–116)$$

where the NLP variables are \((Y_0, ..., Y_{N+1}), (U_1, ..., U_N), t_0\) and \(t_f\). It is noted that initial and terminal control, \(U_0\) and \(U_{N+1}\) are not obtained in the solution of the NLP. The problem defined by Eqs. (3–112)-(3–116) is the discrete Gauss pseudospectral

![Figure 3-4. Discretization and collocation points for Gauss pseudospectral method.](image)
approximation to the continuous-time optimal control problem defined by Eqs. (3–30)-(3–33).

A few key properties of the Gauss pseudospectral method are now stated. The discretization points, at which the state is approximated using Lagrange polynomials, are the \( N \) LG points plus the initial point, \((\tau_0, \ldots, \tau_N)\). The state approximation uses Lagrange polynomials of degree \( N \). The state dynamics are collocated at only the \( N \) LG points, \((\tau_1, \ldots, \tau_N)\). As a consequence, the Gauss pseudospectral differentiation matrix is a non-square, \( N \times (N + 1) \) matrix. Fig. 3-4 shows the discretization and the collocation points for the Gauss pseudospectral method.

### 3.3.2 Necessary Optimality Conditions

The necessary optimality conditions or the Karush-Kuhn-Tucker (KKT) conditions, of the NLP given in Eqs. (3–112)-(3–116) are now derived. The Lagrangian associated with the NLP is

\[
\mathcal{L} = \Phi(Y(\tau_0), \tau_0, Y(\tau_{N+1}), \tau_{N+1}) - \langle \Psi, \phi(Y(\tau_0), \tau_0, Y(\tau_{N+1}), \tau_{N+1}) \rangle \\
+ \frac{t_f - t_0}{2} \sum_{k=1}^{N} (w_k g(Y_k, U_k, \tau_k; t_0, t_f) - \langle \Gamma_k, C(Y_k, U_k, \tau_k; t_0, t_f) \rangle) \\
- \sum_{k=1}^{N} \langle \Lambda_k, D_k Y^{LG} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f) \rangle \\
- \langle \Lambda_{N+1}, Y(\tau_{N+1}) - Y(\tau_0) - \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k f(Y_k, U_k, \tau_k; t_0, t_f), \rangle \tag{3–117}
\]

where \( \Lambda_k \) is the \( k^{th} \) row of the Lagrange multipliers matrix \( \Lambda \in \mathbb{R}^{N \times n} \) associated with the constraints in Eq. (3–113), \( \Lambda_{N+1} \in \mathbb{R}^n \) are the Lagrange multipliers associated with the constraints in Eq. (3–114), \( \Psi \in \mathbb{R}^q \) are the Lagrange multipliers associated with the constraints in Eq. (3–115), and \( \Gamma_k \) is the \( k^{th} \) row of the Lagrange multipliers matrix \( \Gamma \in \mathbb{R}^{N \times s} \) associated with the constraints in Eq. (3–116) The KKT optimality conditions are then obtained by differentiating the Lagrangian with respect to each of the variables.
and equating these derivatives to zero such that

\[
\frac{t_f - t_0}{2} \nabla_{Y_k}(w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle) = D^T_k \Lambda, \quad 1 \leq k \leq N, \tag{3–118}
\]

\[
-\nabla_{Y_0}(\Phi - \langle \Psi, \phi \rangle) = \Lambda_{N+1} - D^T_0 \Lambda, \tag{3–119}
\]

\[
\nabla_{Y_{N+1}}(\Phi - \langle \Psi, \phi \rangle) = \Lambda_{N+1}, \tag{3–120}
\]

\[
\frac{t_f - t_0}{2} \sum_{k=1}^{N} \nabla_{t_0}(w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle)
\]

\[
- \frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle) = -\nabla_{t_0}(\Phi - \langle \Psi, \phi \rangle), \tag{3–121}
\]

\[
\frac{t_f - t_0}{2} \sum_{k=1}^{N} \nabla_{t_f}(w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle)
\]

\[
+ \frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle) = -\nabla_{t_f}(\Phi - \langle \Psi, \phi \rangle), \tag{3–122}
\]

\[
\nabla_{U_k}(w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle) = 0, \quad 1 \leq k \leq N, \tag{3–123}
\]

\[
D_k Y^L - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f) = 0, \quad 1 \leq k \leq N, \tag{3–124}
\]

\[
\Gamma_{ki} = 0 \quad \text{when} \quad C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \tag{3–125}
\]

\[
\Gamma_{ki} < 0 \quad \text{when} \quad C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \tag{3–126}
\]

\[
\phi(Y(\tau_0), \tau_0, Y(\tau_{N+1}), \tau_{N+1}) = 0, \tag{3–127}
\]

where \(D^T_i\) is the \(i^{th}\) row of \(D^T\), \(g_k = g(Y_k, U_k, \tau_k; t_0, t_f)\), \(f_k = f(Y_k, U_k, \tau_k; t_0, t_f)\) and \(C_k = C(Y_k, U_k, \tau_k; t_0, t_f)\).

Next, the KKT conditions given in Eqs. (3–118)-(3–127) are reformulated so that they become a discretization of the first-order optimality conditions given in Eqs. (3–45)-(3–54) for the continuous control problem given in Eqs. (3–30)-(3–33).

Let \(D^t = [D^t_{ij}]\), \((1 \leq i \leq N), \ (1 \leq i \leq N + 1)\) be the \(N \times (N + 1)\) matrix defined as follows:

\[
D^t_{ij} = -\frac{W_i}{W_j} D_{ji}, \quad (i, j) = 1, \ldots, N, \tag{3–128}
\]

\[
D^t_{i, N+1} = -\sum_{j=1}^{N} D^t_{ij}, \quad i = 1, \ldots, N. \tag{3–129}
\]
Theorem 1. The matrix $D^\dagger$ defined in (3–128) and (3–129) is a differentiation matrix for the space of polynomials of degree $N$. More precisely, if $p$ is a polynomial of degree at most $N$ and $p \in \mathbb{R}^{N+1}$ is the vector with $i^{th}$ component $p_i = p(\tau_i)$, $1 \leq i \leq N + 1$, then

$$(D^\dagger p)_i = \dot{p}(\tau_i), \quad 1 \leq i \leq N \quad (p \text{ of degree } \leq N).$$

Proof of Theorem 1. Let $E$ denote the differentiation matrix defined in the statement of the theorem. That is, $E$ is an $N \times (N + 1)$ matrix with the property that for all $p \in \mathbb{R}^{N+1}$, we have

$$(Ep)_i = \dot{p}(\tau_i), \quad 1 \leq i \leq N,$$

where $p$ is the polynomial of degree at most $N$ which satisfies $p_j = p(\tau_j)$, $1 \leq j \leq N + 1$.

If $p$ and $q$ are smooth, real-valued functions with $q(-1) = p(1) = 0$, then integration by parts gives

$$\int_{-1}^{1} \dot{p}(\tau)q(\tau)d\tau = -\int_{-1}^{1} p(\tau)\dot{q}(\tau)d\tau. \quad (3–130)$$

Suppose $p$ and $q$ are polynomials of degree at most $N$, with $N \geq 1$; in this case, $\dot{pq}$ and $pq$ are polynomials of degree at most $2N - 1$. Since Gauss quadrature is exact for polynomials of degree at most $2N - 1$, the integrals in (3–130) can be replaced by their quadrature equivalents to obtain

$$\sum_{j=1}^{N} w_j \dot{p}_j q_j = -\sum_{j=1}^{N} w_j p_j \dot{q}_j, \quad (3–131)$$

where $p_j = p(\tau_j)$ and $\dot{p}_j = \dot{p}(\tau_j)$, $1 \leq i \leq N$, $p$ is any polynomial of degree at most $N$ which vanishes at $-1$, and $q$ is any polynomial of degree at most $N$ which vanishes at $+1$. A polynomial of degree $N$ is uniquely defined by its value at $N + 1$ points. Let $p$ be the polynomial of degree at most $N$ which satisfies $p(-1) = 0$ and $p_j = p(\tau_j)$, $1 \leq j \leq N$. Let $q$ be the polynomial of degree at most $N$ which satisfies $q(+1) = 0$ and $q_j = q(\tau_j)$, $1 \leq j \leq N$. Substituting $\dot{p} = Dp$ and $\dot{q} = Eq$ in (3–131) gives

$$(WDp)^Tq_{1:N} = -(Wp_{1:N})^T Eq,$$
where $W$ is the diagonal matrix of quadrature weights. Since the first component of $p$ and the last component of $q$ vanish, this reduces to

$$p_{1:N}^T(D_{1:N}^T W + WE_{1:N})q_{1:N} = 0.$$  

Since $p_{1:N}$ and $q_{1:N}$ are arbitrary, we deduce that

$$D_{1:N}^T W + WE_{1:N} = 0,$$

which implies that

$$E_{ij} = -\frac{w_j}{w_i}D_{ji}, \quad (i, j) = 1, \ldots, N.$$  

(3–132)

Since $E$ is a differentiation matrix, $E1 = 0$, which yields

$$E_{i,N+1} = -\sum_{j=1}^N E_{ij}, \quad 1 \leq i \leq N.$$  

(3–133)

Comparing (3–128) with (3–132) and (3–129) with (3–133), we see that $D^\dagger = E$. The matrix $D^\dagger$ defined in (3–128) and (3–129) is a differentiation matrix for the space of polynomials of degree $N$.

According to the definition of $D^\dagger$,

$$D_k^\dagger = -w_k D_{k,1:N}^\dagger W^{-1}, \quad 1 \leq k \leq N,$$  

(3–134)

where $W$ is a diagonal matrix with weights $w_k, 1 \leq k \leq N$, on the diagonal. Substituting Eq. (3–134) in Eq. (3–118),

$$-w_k D_{k,1:N}^\dagger W^{-1} \Lambda = \frac{t_r - t_0}{2} \nabla_{\Lambda_k} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle), \quad 1 \leq k \leq N.$$  

(3–135)
Next, defining the following change of variables:

\[
\begin{align*}
\tilde{\lambda}_{N+1} &= \Lambda_{N+1}, \\
\tilde{\lambda}_0 &= \Lambda_{N+1} - D_0^T \Lambda, \tag{3–136} \\
\tilde{\lambda}_k &= \frac{\Lambda_k}{W_k} + \Lambda_{N+1}, \quad 1 \leq k \leq N, \tag{3–137} \\
\tilde{\gamma}_k &= \frac{\Gamma_k}{W_k}, \quad 1 \leq k \leq N, \tag{3–138} \\
\tilde{\psi} &= \psi. \tag{3–139}
\end{align*}
\]

Substituting Eqs. (3–136)-(3–140) in Eqs. (3–119)-(3–127) and in Eq. (3–135), the transformed KKT conditions for the NLP are given as

\[
\begin{align*}
0 &= \nabla U_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 1 \leq k \leq N, \tag{3–141} \\
0 &= D_k Y^L - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad 1 \leq k \leq N, \tag{3–142} \\
\tilde{\gamma}_{ki} &= 0 \quad \text{when} \quad C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \tag{3–143} \\
\tilde{\gamma}_{ki} &< 0 \quad \text{when} \quad C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \tag{3–144} \\
0 &= \phi(Y(\tau_0), \tau_0, Y(\tau_{N+1}), \tau_{N+1}), \tag{3–145} \\
\nabla_{t_0} (\Phi - \langle \tilde{\psi}, \phi \rangle) &= \frac{1}{2} \sum_{k=1}^{N} W_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle) \\
&- \frac{t_f - t_0}{2} \sum_{k=1}^{N} W_k \nabla_{t_0} (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \tag{3–146} \\
-\nabla_{t_f} (\Phi - \langle \tilde{\psi}, \phi \rangle) &= \frac{1}{2} \sum_{k=1}^{N} W_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle) \\
&+ \frac{t_f - t_0}{2} \sum_{k=1}^{N} W_k \nabla_{t_f} (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \tag{3–147} \\
\tilde{\lambda}_0 &= -\nabla_{\tau_0} (\Phi - \langle \tilde{\psi}, \phi \rangle), \tag{3–148} \\
\tilde{\lambda}_{N+1} &= \nabla_{Y_{N+1}} (\Phi - \langle \tilde{\psi}, \phi \rangle), \tag{3–149} \\
D_{k,1:N}^l \tilde{\lambda} + D_{k,N+1}^l \tilde{\lambda}_{N+1} &= -\frac{t_f - t_0}{2} \nabla_{Y_k} (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 1 \leq k \leq N. \tag{3–150}
\end{align*}
\]
Now, consider a comparison of the transformed KKT conditions in Eqs. (3–141)-(3–150) of the NLP to the first-order necessary optimality conditions in Eqs. (3–45)-(3–54) of the continuous-time optimal control problem. It is noted that the transformed KKT conditions in Eqs. (3–141)-(3–145) are the discretized forms of the continuous-time first-order optimality conditions in Eq. (3–47), Eq. (3–45), Eq. (3–52), Eq. (3–53), and Eq. (3–54), respectively. Next, the right-hand side of Eq. (3–146) and Eq. (3–147) is the quadrature approximation of the right-hand side of Eq. (3–50) and Eq. (3–51), respectively. Therefore, the set of transformed KKT conditions in Eq. (3–146) and Eq. (3–147) is the discretized version of the set of continuous-time first-order optimality conditions in Eq. (3–50) and Eq. (3–51). Furthermore, it is shown in Theorem 1 that the system (3–150) is a pseudospectral scheme for the costate dynamics, i.e.

\[ D^\dagger_{k,1:N} \dot{\lambda} + D^\dagger_{k,N+1} \dot{\lambda}_{N+1} = \dot{\lambda}_k, \quad 1 \leq k \leq N. \]  

(3–151)

Therefore, the left hand side of Eq. (3–150) is an approximation of the costate dynamics at the \( k^{th} \) collocation point. As a result, Eq. (3–150) represents the discretized version of the costate dynamics in Eq. (3–46) at \( k = (1, \ldots, N) \). Lastly, it is noted that, at the boundary points, the discrete equivalents of continuous boundary conditions (3–48) and (3–49) are the same as the discrete costate at the boundary points in (3–148) and (3–149), respectively. Hence, the system of transformed KKT conditions of the NLP is exactly equivalent to the first-order optimality conditions of the continuous-time optimal control problem. Therefore, an accurate costate estimate can be obtained from the KKT multipliers using the relationships given in Eqs. (3–136)-(3–138).

It is now shown that the initial costate can also be estimated from the Gauss quadrature approximation of costate dynamics. Let \( D = [D_0 \ D_{1:N}] \) where \( D_0 \) is the first column of \( D \) and \( D_{1:N} \) are the remaining columns. Then \( D \) is such that: \( D_0 = -D_{1:N} \mathbf{1}, \) where \( \mathbf{1} \) is a column vector of all ones.

**Proposition 1.** \( D_0 = -D_{1:N} \mathbf{1}; \text{ equivalently, } -D_{1:N}^{-1} D_0 = \mathbf{1}. \)
Proof of Proposition 1. The components of the vector $D_1$ are the derivatives at the collocation points of the constant polynomial $p(\tau) = 1$. Therefore, $D_1 = 0$, which implies that $D_1 = D_0 + D_{1:N}1 = 0$. Rearranging, we obtain

$$D_0 = -D_{1:N}1.$$  \hfill (3–152)

Returning to the definition of $\tilde{\lambda}_0$ in (3–137), we obtain

$$\tilde{\lambda}_0 = \Lambda_{N+1} - D_0^T \Lambda = \Lambda_{N+1} - \sum_{i=1}^{N} \Lambda_i D_{i,0} = \Lambda_{N+1} + \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_i D_{ij}$$  \hfill (3–153)

$$= \Lambda_{N+1} - \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_i D_{ij}^\dagger W_j = \Lambda_{N+1} - \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_i D_{ij}^\dagger \frac{W_j}{W_i}$$  \hfill (3–154)

$$= \tilde{\lambda}_{N+1} - \sum_{i=1}^{N} \sum_{j=1}^{N} W_i (\tilde{\lambda}_j - \tilde{\lambda}_{N+1}) D_{ij}^\dagger$$  \hfill (3–155)

$$= \tilde{\lambda}_{N+1} - \sum_{i=1}^{N} W_i D_{i,1:N} \tilde{\lambda} - \sum_{i=1}^{N} W_i D_{i,N+1} \tilde{\lambda}_{N+1}$$  \hfill (3–156)

$$= \tilde{\lambda}_{N+1} + \frac{t_f - t_0}{2} \sum_{i=1}^{N} W_i \nabla Y_i (g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle),$$  \hfill (3–157)

where (3–153) follows from the identity (3–152) given in Proposition 1, (3–154) is the definition (3–128) of $D^\dagger$, (3–155) is the definition (3–138) of $\tilde{\lambda}_j$, (3–156) is the definition (3–129) of $D^\dagger$ and (3–157) is the first-order optimality condition (3–150).

3.4 Summary

The LPM and the GPM are two collocation based direct transcription methods that transcribe the continuous-time optimal control problem into a discrete NLP. The most important advantage of these methods is their exponential convergence rate. The LPM, however, suffers from a defect in the optimality conditions at the boundary points. The optimality conditions of the NLP are not equivalent to the discretized form of the continuous-time optimality conditions which results in an inaccurate costate estimation. The GPM uses a collocation scheme such that the optimality conditions of the NLP are
exactly equivalent to the discretized form of the continuous-time optimality conditions. In this collocation scheme, however, control at the initial boundary is not obtained from the NLP solution. The Radau pseudospectral method of this research, discussed in Chapter 4, does not suffer from the defect in optimality conditions and has the ability to compute the initial control. The implementation of the Radau pseudospectral method is significantly less complex than that of the GPM.
CHAPTER 4
RADAU PSEUDOSPECTRAL METHOD

The Radau pseudospectral method is a direct transcription method that transcribes a continuous-time optimal control problem into a discrete nonlinear programming problem. The resulting NLP can then be solved by one of the many well developed nonlinear optimization algorithms. The method uses the Legendre-Gauss-Radau (LGR) points for collocation of the dynamic constraints, and for quadrature approximation of the integrated Lagrange cost term. The Lagrange polynomial approximation of the state, however, uses the LGR points plus the final point. The LGR discretization scheme developed here results in a set of KKT conditions that are equivalent to the discretized form of the continuous first-order optimality conditions and, hence, provides a significantly more accurate costate estimate than that obtained using the Lobatto pseudospectral method. In addition, because collocation is performed at the LGR points, and the LGR points include the initial point, the control at the initial time is also obtained in the solution of the NLP.

In this chapter, the NLP is given that arises from the discretization of a continuous-time optimal control problem using the Radau pseudospectral method. Next, the key properties of the Radau pseudospectral discretization are stated and the first-order necessary optimality conditions, called the Karush-Kuhn-Tucker (KKT) conditions, of the NLP are derived. It is then shown, that after a change of variables, these KKT conditions are equivalent to the discretized form of the continuous first-order necessary optimality conditions. The change of variables provides an accurate discrete approximation to the Lagrange multipliers of the continuous problem. Lastly, the problem formulation for a flipped Radau pseudospectral method that uses the flipped LGR points is given, where the flipped LGR points are the negative of the LGR points.
4.1 NLP Formulation of the Radau Pseudospectral Method

Consider again the continuous-time optimal control problem scaled to the time domain $\tau \in [-1, +1]$ of Section 3.1, such that the objective is to minimize the cost functional

$$J = \Phi(y(-1), t_0, y(+1), t_f) + \frac{t_f - t_0}{2} \int_{-1}^{1} g(y(\tau), u(\tau), \tau; t_0, t_f) d\tau,$$

subject to the dynamic constraints

$$\frac{dy(\tau)}{d\tau} = \dot{y}(\tau) = \frac{t_f - t_0}{2} f(y(\tau), u(\tau), \tau; t_0, t_f),$$

the boundary conditions

$$\phi(y(-1), t_0, y(+1), t_f) = 0,$$

and the inequality path constraints

$$\frac{t_f - t_0}{2} C(y(\tau), u(\tau), \tau; t_0, t_f) \leq 0,$$

where it is again noted that all vector functions of time are row vectors; that is, $y(\tau) = [y_1(\tau) \cdots y_n(\tau)] \in \mathbb{R}^n$. The first-order optimality conditions of the scaled continuous-time optimal control problem in Eqs. (4–1)-(4–4) are stated using the definition of Hamiltonian

$$H(y(\tau), u(\tau), \lambda(\tau), \gamma(\tau), \tau; t_0, t_f) = g(y(\tau), u(\tau), \tau; t_0, t_f) + \langle \lambda(\tau), f(y(\tau), u(\tau), \tau; t_0, t_f) \rangle$$

$$-\langle \gamma(\tau), C(y(\tau), u(\tau), \tau; t_0, t_f) \rangle.$$  

(4–5)

The first-order optimality conditions are

$$\dot{y}(\tau) = \frac{t_f - t_0}{2} \nabla_\lambda H,$$

$$\dot{\lambda}(\tau) = -\frac{t_f - t_0}{2} \nabla_y H,$$

$$0 = \nabla_u H,$$

$$\lambda(-1) = -\nabla_{y(-1)}(\Phi - \langle \psi, \phi \rangle),$$

$$\lambda(+1) = \nabla_{y(+1)}(\Phi - \langle \psi, \phi \rangle).$$

(4–6) (4–7) (4–8) (4–9) (4–10)
\[
\n\nabla_{t_0}(\Phi - \langle \psi, \phi \rangle) = \frac{1}{2} \int_{-1}^{+1} H \, d\tau - \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t_0} \, d\tau, \quad (4-11)
\]
\[
-\nabla_{t_f}(\Phi - \langle \psi, \phi \rangle) = \frac{1}{2} \int_{-1}^{+1} H \, d\tau + \frac{t_f - t_0}{2} \int_{-1}^{+1} \frac{\partial H}{\partial t_f} \, d\tau, \quad (4-12)
\]
\[
\gamma_i(\tau) = 0 \quad \text{when} \quad C_i(y(\tau), u(\tau)) < 0, \quad 1 \leq i \leq s, \quad (4-13)
\]
\[
\gamma_i(\tau) < 0 \quad \text{when} \quad C_i(y(\tau), u(\tau)) = 0, \quad 1 \leq i \leq s, \quad (4-14)
\]
\[
\phi = 0. \quad (4-15)
\]

Consider the \(N\) LGR points, \((\tau_1, \tau_2, \ldots, \tau_N)\), where \(\tau_1 = -1\) and \(\tau_N < +1\). Define a new point such that \(\tau_{N+1} = 1\). Next, let \(L_i(\tau), \,(i = 1, \ldots, N + 1)\), be the Lagrange polynomials of degree \(N\) given by
\[
L_i(\tau) = \prod_{j=1}^{N+1} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 1, \ldots, N + 1). \quad (4-16)
\]

The state, \(y(\tau)\), is approximated by a polynomial of degree at most \(N\) using the Lagrange polynomials as
\[
y(\tau) \approx Y(\tau) = \sum_{i=1}^{N+1} Y_i L_i(\tau), \quad (4-17)
\]
where \(Y_i = Y(\tau_i)\). It is important to note that \(\tau_{N+1} = 1\) is not a LGR point but is used in state approximation. Next, an approximation to the derivative of the state in \(\tau\) domain is given by differentiating the approximation of Eq. (4–17) with respect to \(\tau\),
\[
\dot{y}(\tau) \approx \dot{Y}(\tau) = \sum_{i=1}^{N+1} Y_i \dot{L}_i(\tau). \quad (4-18)
\]

The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (4–18) to the right-hand side of the state dynamic constraints in Eq. (4–2) at the \(N\) LGR points, \((\tau_1, \ldots, \tau_N)\) and are given as
\[
\sum_{i=1}^{N+1} Y_i \dot{L}_i(\tau_k) = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad (k = 1, \ldots, N), \quad (4-19)
\]
\[
\sum_{i=1}^{N+1} D_{ki} Y_i = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad D_{ki} = \dot{L}_i(\tau_k), \quad (4-20)
\]
where $\mathbf{U}_k = \mathbf{U}(\tau_k)$. It is noted that $\tau_{N+1}$ is not a collocation point. The matrix $\mathbf{D} = [D_{ki}]$, $(1 \leq k \leq N)$, $(1 \leq i \leq N + 1)$ is a $N \times (N + 1)$ non-square matrix and is called the Radau pseudospectral differentiation matrix. The matrix $\mathbf{D}$ is non-square because the state approximation uses $N + 1$ points, $(\tau_1, \ldots, \tau_{N+1})$, but the collocation is done at only the $N$ LGR points, $(\tau_1, \ldots, \tau_N)$. Let $\mathbf{Y}^{LGR}$ be defined as

$$
\mathbf{Y}^{LGR} = \begin{bmatrix}
Y_1 \\
\vdots \\
Y_{N+1}
\end{bmatrix}.
$$

Using the matrix $\mathbf{Y}^{LGR}$, the collocated dynamics at the $N$ LGR collocation points in Eq. (4–20) are expressed as

$$
\mathbf{D}_k \mathbf{Y}^{LGR} = \frac{t_f - t_0}{2} \mathbf{f}(\mathbf{Y}_k, \mathbf{U}_k, \tau; t_0, t_f), \quad (k = 1, \ldots, N),
$$

(4–21)

where $\mathbf{D}_k$ is the $k^{th}$ row of differentiation matrix $\mathbf{D}$. Next, the path constraints in Eq. (4–4) are enforced at the $N$ LGR collocation points as

$$
\frac{t_f - t_0}{2} \mathbf{c}(\mathbf{Y}_k, \mathbf{U}_k, \tau; t_0, t_f) \leq \mathbf{0}, \quad (k = 1, \ldots, N).
$$

(4–22)

Lastly, the cost functional is approximated using LGR quadrature as

$$
\mathcal{J} = \Phi(\mathbf{Y}(\tau_1), \tau_1, \mathbf{Y}(\tau_{N+1}), \tau_{N+1}) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(\mathbf{Y}_k, \mathbf{U}_k, \tau; t_0, t_f),
$$

(4–23)

where $w_k$ is the quadrature weight associated with the $k^{th}$ LGR collocation point.

The finite-dimensional nonlinear programming problem corresponding to the Radau pseudospectral method is then given as follows. Minimize the cost function

$$
\mathcal{J} = \Phi(\mathbf{Y}(\tau_1), \tau_1, \mathbf{Y}(\tau_{N+1}), \tau_{N+1}) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(\mathbf{Y}_k, \mathbf{U}_k, \tau; t_0, t_f),
$$

(4–24)
subject to the following equality and inequality constraints:

\[
D_k Y^{\text{LGR}} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f) = 0, \quad (k = 1, \ldots, N), \tag{4–25}
\]

\[
\phi(Y(\tau_1), \tau_1, Y(\tau_{N+1}), \tau_{N+1}) = 0, \quad (4–26)
\]

\[
\frac{t_f - t_0}{2} C(Y_k, U_k, \tau; t_0, t_f) \leq 0, \quad (k = 1, \ldots, N), \tag{4–27}
\]

where the NLP variables are \((Y_1, \ldots, Y_{N+1}), (U_1, \ldots, U_N), t_0, \) and \(t_f\). It is noted that the initial control, \(U_1\), is obtained in the solution of the NLP. The problem defined by Eqs. (4–24)-(4–27) is the discrete Radau pseudospectral approximation to the continuous-time optimal control problem defined by Eqs. (4–1)-(4–4).

A few key properties of the Radau pseudospectral method are now stated. The discretization points, at which the state is approximated using the Lagrange polynomials, are the \(N\) LGR points plus the final point, \((\tau_1, \ldots, \tau_{N+1})\). The state approximation uses the Lagrange polynomials of degree \(N\). The state dynamics are collocated at only the \(N\) LGR points, \((\tau_1, \ldots, \tau_N)\). As a consequence, the Radau pseudospectral differentiation
matrix is a non-square, \( N \times (N + 1) \) matrix. Fig. 4-1 shows the discretization and the collocation points for the Radau pseudospectral method.

### 4.2 Necessary Optimality Conditions

The necessary optimality conditions, or the Karush-Kuhn-Tucker (KKT) conditions, of the NLP given in Eqs. (4–24)-(4–27) are now derived. The Lagrangian associated with the NLP is

\[
\mathcal{L} = \Phi(Y(\tau_1), \tau_1, Y(\tau_{N+1}), \tau_{N+1}) - \langle \Psi, \phi(Y(\tau_1), \tau_1, Y(\tau_{N+1}), \tau_{N+1}) \rangle \\
+ \frac{t_f - t_0}{2} \sum_{k=1}^{N} (w_k g(Y_k, U_k, \tau; t_0, t_r) - \langle \Gamma_k, C(Y_k, U_k, \tau; t_0, t_r) \rangle) \\
- \sum_{k=1}^{N} \langle \Lambda_k, D_k Y_{LGR} \rangle - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_r)) ,
\]

(4–28)

where \( \Lambda_k \) is the \( k^{th} \) row of the Lagrange multipliers matrix \( \Lambda \in \mathbb{R}^{N \times n} \) associated with the constraints in Eq. (4–25), \( \Psi \in \mathbb{R}^q \) are the Lagrange multipliers associated with the constraints in Eq. (4–26), and \( \Gamma_k \) is the \( k^{th} \) row of the Lagrange multipliers matrix \( \Gamma \in \mathbb{R}^{N \times s} \) associated with the constraints in Eq. (4–27). The KKT optimality conditions are then obtained by differentiating the Lagrangian with respect to each of the variable and equating it to zero, such that

\[
\frac{t_f - t_0}{2} \nabla_{Y_k} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = D_k^T \Lambda, \quad 2 \leq k \leq N, \tag{4–29}
\]

\[
\frac{t_f - t_0}{2} \nabla_{Y_1} (w_1 g_1 + \langle \Lambda_1, f_1 \rangle - \langle \Gamma_1, C_1 \rangle) = D_1^T \Lambda - \nabla_{Y_1} (\Phi - \langle \Psi, \phi \rangle), \tag{4–30}
\]

\[
\nabla_{Y_{N+1}} (\Phi - \langle \Psi, \phi \rangle) = D_{N+1}^T \Lambda, \tag{4–31}
\]

\[
\frac{t_f - t_0}{2} \sum_{k=1}^{N} \nabla_{t_0} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) \\
- \frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = -\nabla_{t_0} (\Phi - \langle \Psi, \phi \rangle), \tag{4–32}
\]

\[
\frac{t_f - t_0}{2} \sum_{k=1}^{N} \nabla_{t_r} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) \\
+ \frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = -\nabla_{t_r} (\Phi - \langle \Psi, \phi \rangle). \tag{4–33}
\]
\[ \nabla u_k(w_k g_k + \langle A_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = 0, \quad 1 \leq k \leq N, \] (4–34)

\[ D_k Y_{LR}^{1} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f) = 0, \quad 1 \leq k \leq N, \] (4–35)

\[ \Gamma_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \] (4–36)

\[ \Gamma_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \] (4–37)

\[ \phi(Y(\tau_1), \tau_1, Y(\tau_{N+1}), \tau_{N+1}) = 0. \] (4–38)

where \( D_i^T \) is the \( i^{th} \) row of \( D^T \), \( g_k = g(Y_k, U_k, \tau; t_0, t_f) \), \( f_k = f(Y_k, U_k, \tau; t_0, t_f) \) and \( C_k = C(Y_k, U_k, \tau; t_0, t_f) \).

Next, the KKT conditions given in Eqs. (4–29)-(4–38) are reformulated so that they become a discretization of the first-order optimality conditions given in Eqs. (4–6)-(4–15) for the continuous control problem given in Eqs. (4–1)-(4–4). Let \( D^i = [D_{ij}^i] \), \( 1 \leq i \leq N \), \( 1 \leq i \leq N \) be the \( N \times N \) matrix defined as follows:

\[ D_{11}^i = -D_{11} - \frac{1}{w_i}, \] (4–39)

\[ D_{ij}^i = -\frac{w_j}{w_i} D_{ji}, \quad \text{otherwise.} \] (4–40)

**Theorem 2.** The matrix \( D^i \) defined in (4–39) and (4–40) is a differentiation matrix for the space of polynomials of degree \( N - 1 \). More precisely, if \( q \) is a polynomial of degree at most \( N - 1 \) and \( q \in \mathbb{R}^N \) is the vector with \( i^{th} \) component \( q_i = q(\tau_i), 1 \leq i \leq N \), then

\[ (D^i q)_i = \dot{q}(\tau_i), \quad 1 \leq i \leq N \quad (q \text{ of degree } \leq N - 1). \]

**Proof of Theorem 2.** Let \( E \) denote the differentiation matrix defined in the statement of the theorem. That is, \( E \) is an \( N \times N \) matrix with the property that for all \( q \in \mathbb{R}^N \), we have

\[ (Eq)_i = \dot{q}(\tau_i), \quad 1 \leq i \leq N, \]

where \( q \) is the polynomial of degree at most \( N - 1 \) which satisfies \( q_j = q(\tau_j), 1 \leq j \leq N \). If \( p \) and \( q \) are smooth, real-valued functions with \( p(1) = 0 \), then integration by parts
gives
\[ \int_{-1}^{1} \dot{p}(\tau) q(\tau) d\tau = -p(-1)q(-1) - \int_{-1}^{1} p(\tau) \dot{q}(\tau) d\tau. \] (4-41)

Suppose \( p \) is a polynomial of degree at most \( N \) and \( q \) is a polynomial of degree at most \( N - 1 \) with \( N \geq 1 \); in this case, \( pq \) and \( \dot{p}q \) are polynomials of degree at most \( 2N - 2 \).

Because Gauss-Radau quadrature is exact for polynomials of degree at most \( 2N - 2 \), the integrals in (4-41) can be replaced by their quadrature equivalents to obtain
\[ \sum_{j=1}^{N} w_j \dot{p_j} q_j = -p_1 q_1 - \sum_{j=1}^{N} w_j p_j \dot{q}_j, \] (4-42)

where \( p_j = p(\tau_j) \) and \( \dot{p}_j = \dot{p}(\tau_j) \), \( 1 \leq i \leq N \), \( p \) is any polynomial of degree at most \( N \) which vanishes at \( +1 \), and \( q \) is any polynomial of degree at most \( N - 1 \). A polynomial of degree \( N \) is uniquely defined by its value at \( N + 1 \) points. Let \( p \) be the polynomial of degree at most \( N \) which satisfies \( p(1) = 0 \) and \( p_j = p(\tau_j) \), \( 1 \leq j \leq N \). Let \( q \) be the polynomial of degree at most \( N - 1 \) such that \( q_j = q(\tau_j) \), \( 1 \leq j \leq N \). Substituting \( \dot{p} = D_{1:N} p \) and \( \dot{q} = Eq \) in (4-42) gives
\[ (WD_{1:N} p)^T q = -p_1 q_1 - (Wp)^T Eq, \]
\[ p^T D_{1:N}^T W q = -p_1 q_1 - p^T WEq, \]

where \( W \) is the diagonal matrix of Radau quadrature weights. This can be rearranged into the following form:
\[ p^T (D_{1:N}^T W + WE + e_1 e_1^T) q = 0, \]

where \( e_1 \) is the first column of the identity matrix. Since \( p \) and \( q \) are arbitrary, we deduce that
\[ D_{1:N}^T W + WE + e_1 e_1^T = 0, \]
which implies that

\[
E_{11} = -D_{11} - \frac{1}{w_1}, \quad (4-43)
\]

\[
E_{ij} = -\frac{w_j}{w_i} D_{ji}. \quad (4-44)
\]

Comparing (4–40) with (4–44) and (4–39) with (4–43), we see that \( D^\dagger = E \). The matrix \( D^\dagger \) defined in (4–40) and (4–39) is a differentiation matrix for the space of polynomials of degree \( N - 1 \).

\[\square\]

According to the definition of \( D^\dagger \),

\[
D_1^T = -w_1 D_1^\dagger W^{-1} - \frac{1}{w_1} e_1, \quad (4-45)
\]

\[
D_k^T = -w_k D_k^\dagger W^{-1}, \quad 2 \leq k \leq N, \quad (4-46)
\]

where \( W \) is a diagonal matrix with weights \( w_k \), \( 1 \leq k \leq N \), on the diagonal. The vector \( e_1 \) is the 1st row of \( N \times N \) identity matrix. Substituting Eqs. (4–45) and (4–46) in Eqs. (5–26) and (4–29),

\[
-\nabla Y_1 (\Phi - \langle \Psi, \phi \rangle) = \frac{t_r - t_0}{2} \nabla Y_1 (w_1 g_1 + \langle \Lambda_1, f_1 \rangle - \langle \Gamma_1, C_1 \rangle) + w_1 D_1^\dagger W^{-1} \Lambda + \frac{1}{w_1} e_1 \Lambda, \quad (4-47)
\]

\[
-w_k D_k^\dagger W^{-1} \Lambda = \frac{t_r - t_0}{2} \nabla Y_k (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle), \quad 2 \leq k \leq N \quad (4-48)
\]

Next, defining the following change of variables:

\[
\tilde{\lambda}_{N+1} = D_{N+1}^T \Lambda, \quad (4-49)
\]

\[
\tilde{\lambda}_k = \frac{\Lambda_k}{w_k}, \quad 1 \leq k \leq N, \quad (4-50)
\]

\[
\tilde{\gamma}_k = \frac{\Gamma_k}{w_k}, \quad 1 \leq k \leq N, \quad (4-51)
\]

\[
\tilde{\psi} = \Psi. \quad (4-52)
\]
Substituting Eqs. (4–49)-(4–52) in Eqs. (4–31)-(4–38) and in Eqs. (4–47)-(4–48), the transformed KKT conditions for the NLP are given as

\[ 0 = \nabla_{u_k} (g_k + \langle \hat{\lambda}_k, f_k \rangle - \langle \hat{\gamma}_k, c_k \rangle), \quad 1 \leq k \leq N, \quad (4–53) \]

\[ 0 = D_k Y^{LGR} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad 1 \leq k \leq N, \quad (4–54) \]

\[ \tilde{\gamma}_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (4–55) \]

\[ \tilde{\gamma}_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (4–56) \]

\[ 0 = \phi(Y(\tau_1), \tau_1, Y(\tau_{N+1}), \tau_{N+1}), \quad (4–57) \]

\[ \nabla_{t_0} (\Phi - \langle \tilde{\psi}, \phi \rangle) = \frac{1}{2} \sum_{k=1}^{N} w_k (g_k + \langle \hat{\lambda}_k, f_k \rangle - \langle \hat{\gamma}_k, c_k \rangle) \]

\[ - \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k \nabla_{t_0} (g_k + \langle \hat{\lambda}_k, f_k \rangle - \langle \hat{\gamma}_k, c_k \rangle), \quad (4–58) \]

\[ -\nabla_{t_f} (\Phi - \langle \tilde{\psi}, \phi \rangle) = \frac{1}{2} \sum_{k=1}^{N} w_k (g_k + \langle \hat{\lambda}_k, f_k \rangle - \langle \hat{\gamma}_k, c_k \rangle) \]

\[ + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k \nabla_{t_f} (g_k + \langle \hat{\lambda}_k, f_k \rangle - \langle \hat{\gamma}_k, c_k \rangle), \quad (4–59) \]

\[ D_1^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{Y_1} (g_1 + \langle \hat{\lambda}_1, f_1 \rangle - \langle \hat{\gamma}_1, c_1 \rangle) \]

\[ - \frac{1}{w_1} \left( \hat{\lambda}_1 + \nabla_{Y_1} (\Phi - \langle \tilde{\psi}, \phi \rangle) \right), \quad (4–60) \]

\[ \tilde{\lambda}_{N+1} = \nabla_{Y_{N+1}} (\Phi - \langle \tilde{\psi}, \phi \rangle), \quad (4–61) \]

\[ D_k^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{Y_k} (g_k + \langle \hat{\lambda}_k, f_k \rangle - \langle \hat{\gamma}_k, c_k \rangle), \quad (4–62) \]

\[ 2 \leq k \leq N. \]

Now, consider a comparison of the transformed KKT conditions in Eqs. (4–53)-(4–62) of the NLP to the first-order necessary optimality conditions in Eqs. (4–6)-(4–15) of the continuous-time optimal control problem. It is noted that the transformed KKT conditions in Eqs. (4–53)-(4–57) are the discretized forms of the continuous-time first-order optimality conditions in Eq. (4–8), Eq. (4–6), Eq. (4–13), Eq. (4–14), and Eq. (4–15), respectively. Next, the right-hand side of Eq. (4–58) and Eq. (4–59) is
the quadrature approximation of the right-hand side of Eq. (4–11) and Eq. (4–12), respectively. Therefore, the set of transformed KKT conditions in Eq. (4–58) and Eq. (4–59) is the discretized version of the set of continuous-time first-order optimality conditions in Eq. (4–11) and Eq. (4–12). Furthermore, it is shown in Theorem 2 that the system (4–62) is a pseudospectral scheme for the costate dynamics, i.e.

\[ D_k^\dagger \dot{\lambda}_k = \dot{\lambda}_k, \quad 1 \leq k \leq N. \]  

(4–63)

Therefore, the left hand side of Eq. (4–62) is an approximation of costate dynamics at the \( k^{th} \) collocation point, \( k = (2, \ldots, N) \). As a result, Eq. (4–62) represents the discretized version of the costate dynamics in Eq. (4–7) at \( k = (2, \ldots, N) \). Next, it is noted that at the final boundary point, the discrete equivalent of continuous boundary conditions (4–10) is the same as the discrete costate at the final boundary point in (4–61). However, at the initial boundary point, the discrete equivalent of the continuous boundary condition (4–9) is coupled in the discrete costate dynamics at the initial boundary point in (4–60). The equivalence of the transformed KKT condition at the initial boundary, (4–60), of the NLP to the discretized form of the continuous first-order optimality condition in (4–9) is now established by manipulating (4–49).

Let \( D = [D_{1:N} \quad D_{N+1}] \) where \( D_{1:N} \) is composed of first \( N \) columns and \( D_{N+1} \) is the last column of \( D \). Then \( D \) is such that: \( D_{N+1} = -D_{1:N}1 \), where \( 1 \) is a column vector of all ones.

**Proposition 2.** \( D_{N+1} = -D_{1:N}1 \)

**Proof of Proposition 2.** The components of the vector \( D1 \) are the derivatives at the collocation points of the constant polynomial \( p(\tau) = 1 \). Therefore, \( D1 = 0 \), which implies that \( D1 = D_{1:N}1 + D_{N+1} = 0 \). Rearranging, we obtain

\[ D_{N+1} = -D_{1:N}1. \]  

(4–64)
Returning to the definition of $\tilde{\lambda}_{N+1}$ in (4–49), we obtain

$$
\tilde{\lambda}_{N+1} = D_{N+1}^T \Lambda = \sum_{i=1}^{N} \Lambda_i D_{i,N+1} = -\sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_i D_{ij}
$$

(4–65)

$$
= \frac{\Lambda_1}{w_1} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\Lambda_i D_{ji}^* W_j}{W_j} = \frac{\Lambda_1}{w_1} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\Lambda_i D_{ij}^* W_j}{W_j}
$$

(4–66)

$$
= \tilde{\lambda}_1 + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i \tilde{\lambda}_j D_{ij}^* = \tilde{\lambda}_1 + \sum_{i=1}^{N} w_i D_{i}^* \tilde{\lambda}
$$

(4–67)

$$
= -\nabla_{\gamma_1}(\Phi - \langle \tilde{\psi}, \phi \rangle) - \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i \nabla_{\gamma_i}(g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle).
$$

(4–68)

where (4–65) follows from the identity (4–64) given in Proposition 2, (4–66) is the definition (4–39) and (4–40) of $D^\dagger$, (4–67) is the definition (4–50) of $\tilde{\lambda}_i$, and (4–68) is the first-order optimality condition (4–62) and (4–60). Rearranging (4–68) such that

$$
-\nabla_{\gamma_1}(\Phi - \langle \tilde{\psi}, \phi \rangle) = \tilde{\lambda}_{N+1} + \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i \nabla_{\gamma_i}(g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle).
$$

(4–69)

Next, the continuous costate dynamics in Eq. (4–7) are

$$
\dot{\lambda}(\tau) = -\frac{t_f - t_0}{2} \nabla_{\gamma} H = -\frac{t_f - t_0}{2} \nabla_{\gamma}(g + \langle \lambda, f \rangle - \langle \gamma, C \rangle).
$$

(4–70)

Integrating the continuous costate dynamics in (4–70) using Radau quadrature,

$$
\tilde{\lambda}_1 = \tilde{\lambda}_{N+1} + \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i \nabla_{\gamma_i}(g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle).
$$

(4–71)

Comparing (4–69) with (4–71) gives

$$
\tilde{\lambda}_1 = -\nabla_{\gamma_1}(\Phi - \langle \tilde{\psi}, \phi \rangle).
$$

(4–72)

Equation (4–72) is the missing boundary condition at the initial point that was coupled with the discrete costate dynamics at the initial point in (4–60). It is also implied by Eq. (4–72) that the extra term in (4–60) is in fact zero, thereby, making (4–60) consistent.
with discrete costate dynamics at the initial point. Hence, the system of transformed
KKT conditions of the NLP is equivalent to the first-order optimality conditions of the
continuous-time optimal control problem and accurate costate estimates are obtained
from the KKT multipliers using the relationship given in Eqs. (4–49)-(4–50).

4.3 Flipped Radau Pseudospectral Method

The Legendre-Gauss-Radau (LGR) points, like the Legendre-Gauss-Lobatto (LGL)
points and Legendre-Gauss (LG) points are defined on the interval $[-1, +1]$. However,
unlike LG and LGL points, LGR points are asymmetric about the origin. By flipping the
points about the origin, i.e., by taking the negative of LGR points, a new set of points,
called the flipped LGR points, is obtained. Radau pseudospectral method can be
implemented using this set of flipped LGR points. In this section, problem formulation
and KKT conditions for a flipped Radau pseudospectral method are presented. The
flipped Radau pseudospectral method differs from the standard Radau pseudospectral
method in that the final time is a collocation point in flipped Radau pseudospectral
method whereas in standard Radau pseudospectral method, initial time is a collocation
point. As a result, in flipped Radau pseudospectral method, control at the final time is
obtained. Furthermore, it is shown that the flipped Radau pseudospectral method also
results in a system of transformed KKT conditions that is equivalent to the discretized
form of continuous first-order optimality conditions.

4.3.1 NLP Formulation of the Flipped Radau Pseudospectral Method

Consider the $N$ flipped LGR points, $(\tau_1, \tau_2, \ldots, \tau_N)$, where $\tau_1 > -1$ and $\tau_N = +1$.
Define a new point such that $\tau_0 = -1$. Next, let $L_i(\tau), (i = 0, \ldots, N)$, be the Lagrange
polynomials of degree $N$ given by

$$L_i(\tau) = \prod_{j=0}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 0, \ldots, N).$$  (4–73)
The state, \( y(\tau) \), is approximated by a polynomial of degree at most \( N \) using the Lagrange polynomials as
\[
y(\tau) \approx Y(\tau) = \sum_{i=0}^{N} Y_i L_i(\tau),
\]  
(4-74)
where \( Y_i = Y(\tau_i) \). It is important to note that \( \tau_0 = -1 \) is not a flipped LGR point but is used in state approximation. Next, an approximation to the derivative of the state in \( \tau \) domain is given by differentiating the approximation of Eq. (4-74) with respect to \( \tau \),
\[
\dot{y}(\tau) \approx \dot{Y}(\tau) = \sum_{i=0}^{N} Y_i \dot{L}_i(\tau).
\]  
(4-75)
The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (4-75) to the right-hand side of state dynamic constraints in Eq. (4-2) at the \( N \) flipped LGR points, \( (\tau_1, \ldots, \tau_N) \) and are given as
\[
\sum_{i=0}^{N} Y_i \dot{L}_i(\tau_k) = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad (k = 1, \ldots, N),
\]  
(4-76)
\[
\sum_{i=0}^{N} D_{ki} Y_i = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad D_{ki} = \dot{L}_i(\tau_k),
\]  
(4-77)
where \( U_k = U(\tau_k) \). It is noted that \( \tau_0 \) is not a collocation point. The matrix \( D = [D_{ki}] \), \( (1 \leq k \leq N), \ (0 \leq i \leq N) \) is a \( N \times (N + 1) \) non-square matrix and is called the flipped Radau pseudospectral differentiation matrix. The matrix \( D \) is non-square because the state approximation uses \( N + 1 \) points, \( (\tau_0, \ldots, \tau_N) \), but the collocation is done at only \( N \) flipped LGR points, \( (\tau_1, \ldots, \tau_N) \). Let \( Y_{LGR} \) be defined as
\[
Y_{LGR} = \begin{bmatrix} Y_0 \\ \vdots \\ Y_N \end{bmatrix}.
\]
Using the matrix \( Y_{LGR} \), the collocated dynamics at the \( N \) LGR collocation points in Eq. (4-77) are expressed as
\[
D_k Y_{LGR} = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad (k = 1, \ldots, N),
\]  
(4-78)
where $D_k$ is the $k^{th}$ row of differentiation matrix $D$. Next, the path constraints in Eq. (4–4) are enforced at the $N$ flipped LGR collocation points as

$$\frac{t_f - t_0}{2} C(Y_k, U_k, \tau; t_0, t_f) \leq 0, \quad (k = 1, \ldots, N). \quad (4–79)$$

Lastly, the cost functional is approximated using LGR quadrature as

$$J = \Phi(Y(\tau_0), \tau_0, Y(\tau_N), \tau_N) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(Y_k, U_k, \tau; t_0, t_f), \quad (4–80)$$

where $w_k$ is the quadrature weight associated with the $k^{th}$ flipped LGR collocation point.

The finite-dimensional nonlinear programming problem corresponding to the flipped Radau pseudospectral method is then given as follows. Minimize the objective cost function

$$J = \Phi(Y(\tau_0), \tau_0, Y(\tau_N), \tau_N) + \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k g(Y_k, U_k, \tau; t_0, t_f), \quad (4–81)$$

subject to the following equality and inequality constraints:

$$D_k Y_{\text{LGR}}^1 - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f) = 0, \quad (k = 1, \ldots, N), \quad (4–82)$$

$$\phi(Y(\tau_0), \tau_0, Y(\tau_N), \tau_N) = 0, \quad (4–83)$$

$$\frac{t_f - t_0}{2} C(Y_k, U_k, \tau; t_0, t_f) \leq 0, \quad (k = 1, \ldots, N). \quad (4–84)$$

where the NLP variables are $(Y_0, \ldots, Y_N), (U_1, \ldots, U_N), t_0,$ and $t_f$. It is noted that the initial control, $U_0$, is not obtained in the solution of the NLP. However, the control at the final point, $U_N$, is obtained. The problem defined by Eqs. (4–81)-(4–84) is the discrete flipped Radau pseudospectral approximation to the continuous-time optimal control problem defined by Eqs. (4–1)-(4–4).

A few key properties of the flipped Radau pseudospectral method are now stated. The discretization points, at which the state is approximated using Lagrange polynomials, are the $N$ flipped LGR points plus the initial point, $(\tau_0, \ldots, \tau_N)$. The state approximation uses Lagrange polynomials of degree $N$. The state dynamics are
collocated at only the $N$ flipped LGR points, $(\tau_1, \ldots, \tau_N)$. As a consequence, the flipped Radau pseudospectral differentiation matrix is a non-square, $N \times (N + 1)$ matrix. Fig. 4-2 shows the discretization and collocation points for the flipped Radau pseudospectral method.

4.3.2 Necessary Optimality Conditions

The necessary optimality conditions, or the Karush-Kuhn-Tucker (KKT) conditions, of the NLP given in Eqs. (4–81)-(4–84) are now derived. The Lagrangian associated with the NLP is

\[ L = \Phi(Y(\tau_0), \tau_0, Y(\tau_N), \tau_N) - \langle \Psi, \phi(Y(\tau_0), \tau_0, Y(\tau_N), \tau_N) \rangle + \frac{t_f - t_0}{2} \sum_{k=1}^{N} \left( w_k g(Y_k, U_k, \tau; t_0, t_f) - \langle \Gamma_k, C(Y_k, U_k, \tau; t_0, t_f) \rangle \right) \]

\[ - \sum_{k=1}^{N} \langle \Lambda_k, D_k Y_{LGR}^{\tau} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f) \rangle, \]  

(4–85)
where $\Lambda_k$ is the $k^{th}$ row of the Lagrange multipliers matrix $\Lambda \in \mathbb{R}^{N \times n}$ associated with the constraints in Eq. (4–82), $\Psi \in \mathbb{R}^q$ are the Lagrange multipliers associated with the constraints in Eq. (4–83), and $\Gamma_k$ is the $k^{th}$ row of the Lagrange multipliers matrix $\Gamma \in \mathbb{R}^{N \times s}$ associated with the constraints in Eq. (4–84). The KKT optimality conditions are then obtained by differentiating the Lagrangian with respect to each of the variable and equating it to zero such that

$$\frac{t_f - t_0}{2} \nabla_{Y_k}(w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = D_k^T \Lambda, \quad 1 \leq k \leq N - 1,$$

$$\frac{t_f - t_0}{2} \nabla_{\tau_0}(w_N g_N + \langle \Lambda_N, f_N \rangle - \langle \Gamma_N, C_1 \rangle) = D_N^T \Lambda - \nabla_{\tau_0}(\Phi - \langle \Psi, \phi \rangle),$$

$$\nabla_{\tau_0}(\Phi - \langle \Psi, \phi \rangle) = D_0^T \Lambda,$$

$$\frac{t_f - t_0}{2} \sum_{k=1}^{N} \tau_0 (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle)$$

$$- \frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = -\nabla_{\tau_0}(\Phi - \langle \Psi, \phi \rangle),$$

$$\frac{t_f - t_0}{2} \sum_{k=1}^{N} \tau_0 (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle)$$

$$+ \frac{1}{2} \sum_{k=1}^{N} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = -\nabla_{\tau_0}(\Phi - \langle \Psi, \phi \rangle),$$

$$\nabla_{u_k}(w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle) = 0, \quad 1 \leq k \leq N,$$

$$D_k Y^{LGR} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f) = 0, \quad 1 \leq k \leq N,$$

$$\Gamma_{k} = 0 \text{ when } C_{k} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N,$$

$$\Gamma_{k} < 0 \text{ when } C_{k} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N,$$

$$\phi(Y(\tau_0), \tau_0, Y(\tau_N), \tau_N) = 0,$$

where $D_i^T$ is the $i^{th}$ row of $D^T$, $g_k = g(Y_k, U_k, \tau; t_0, t_f)$, $f_k = f(Y_k, U_k, \tau; t_0, t_f)$ and $C_k = C(Y_k, U_k, \tau; t_0, t_f)$.

Next, the KKT conditions given in Eqs. (4–86)-(4–95) are reformulated so that they become a discretization of the first-order optimality conditions given in Eqs. (4–6)-(4–15) for the continuous control problem given in Eqs. (4–1)-(4–4). Let $D_i^T = [D_{ij}^T], \quad 1 \leq i \leq
be the \(N \times N\) matrix defined as follows:

\[
D_{NN}^{\dagger} = -D_{NN} + \frac{1}{w_N}, \quad (4-96)
\]

\[
D_{ij}^{\dagger} = -\frac{w_j}{w_i} D_{ji}, \quad \text{otherwise.} \quad (4-97)
\]

**Theorem 3.** The matrix \(D^{\dagger}\) defined in (4–96) and (4–97) is a differentiation matrix for the space of polynomials of degree \(N - 1\). More precisely, if \(q\) is a polynomial of degree at most \(N - 1\) and \(q \in \mathbb{R}^N\) is the vector with \(i^{th}\) component \(q_i = q(\tau_i), 1 \leq i \leq N\), then

\[
(D^{\dagger}q)_i = \dot{q}(\tau_i), \quad 1 \leq i \leq N \quad (q \text{ of degree } \leq N - 1).
\]

**Proof of Theorem 3.** Let \(E\) denote the differentiation matrix defined in the statement of the theorem. That is, \(E\) is an \(N \times N\) matrix with the property that for all \(q \in \mathbb{R}^N\), we have

\[
(Eq)_i = \dot{q}(\tau_i), \quad 1 \leq i \leq N,
\]

where \(q\) is the polynomial of degree at most \(N - 1\) which satisfies \(q_j = q(\tau_j), 1 \leq j \leq N\). If \(p\) and \(q\) are smooth, real-valued functions with \(p(-1) = 0\), then integration by parts gives

\[
\int_{-1}^{1} \dot{p}(\tau)q(\tau)d\tau = p(1)q(1) - \int_{-1}^{1} p(\tau)\dot{q}(\tau)d\tau. \quad (4-98)
\]

Suppose \(p\) is a polynomial of degree at most \(N\) and \(q\) is a polynomial of degree at most \(N - 1\) with \(N \geq 1\); in this case, \(pq\) and \(p\dot{q}\) are polynomials of degree at most \(2N - 2\). Since Gauss-Radau quadrature is exact for polynomials of degree at most \(2N - 2\), the integrals in (4–98) can be replaced by their quadrature equivalents to obtain

\[
\sum_{j=1}^{N} w_j \dot{p}_j q_j = p_N q_N - \sum_{j=1}^{N} w_j p_j \dot{q}_j, \quad (4-99)
\]

where \(p_j = p(\tau_j)\) and \(\dot{p}_j = \dot{p}(\tau_j), 1 \leq i \leq N\), \(p\) is any polynomial of degree at most \(N\) which vanishes at +1, and \(q\) is any polynomial of degree at most \(N - 1\). A polynomial of degree \(N\) is uniquely defined by its value at \(N + 1\) points. Let \(p\) be the polynomial of degree at most \(N\) which satisfies \(p(-1) = 0\) and \(p_j = p(\tau_j), 1 \leq j \leq N\). Let \(q\) be
the polynomial of degree at most \( N - 1 \) such that \( q_j = q(\tau_j), \ 1 \leq j \leq N \). Substituting \( \dot{p} = D_{1:N} p \) and \( \dot{q} = Eq \) in (4–99) gives

\[
(WD_{1:N} p)^T q = p_N q_N - (Wp)^T Eq,
\]

\[
p^T D_{1:N}^T Wq = p_N q_N - p^T WEq,
\]

where \( W \) is the diagonal matrix of Radau quadrature weights. This can be rearranged into the following form:

\[
p^T (D_{1:N}^T W + WE - e_N e_N^T) q = 0,
\]

where \( e_N \) is the \( N^{th} \) column of the identity matrix. Since \( p \) and \( q \) are arbitrary, we deduce that

\[
D_{1:N}^T W + WE - e_N e_N^T = 0,
\]

which implies that

\[
E_{NN} = -D_{NN} + \frac{1}{w_N}, \quad (4–100)
\]

\[
E_{ij} = -\frac{w_i}{w_j} D_{ji}, \quad (4–101)
\]

Comparing (4–97) with (4–101) and (4–96) with (4–100), we see that \( D^\dagger = E \). The matrix \( D^\dagger \) defined in (4–97) and (4–96) is a differentiation matrix for the space of polynomials of degree \( N - 1 \).

According to the definition of \( D^\dagger \),

\[
D_{N}^T = -w_N D_{N}^\dagger W^{-1} + \frac{1}{w_N} e_N, \quad (4–102)
\]

\[
D_{k}^T = -w_k D_{k}^\dagger W^{-1}, \quad 1 \leq k \leq N - 1, \quad (4–103)
\]

where \( W \) is a diagonal matrix with weights \( w_k, \ 1 \leq k \leq N \), on the diagonal. Vector \( e_N \) is the \( N^{th} \) row of \( N \times N \) identity matrix. Substituting Eqs. (4–102) and (4–103) in
Eqs. (4–87) and (4–86),

\[- \nabla_{Y_N}(\Phi - \langle \Psi, \phi \rangle) = \frac{t_f - t_0}{2} \nabla_{Y_N}(w_N g_N + \langle \Lambda_N, f_N \rangle - \langle \Gamma_N, C_N \rangle) \]
\[+ w_N D_N^\top W^{-1} \Lambda - \frac{1}{w_N} e_N \Lambda, \quad (4–104)\]
\[- w_k D_k^\top W^{-1} \Lambda = \frac{t_f - t_0}{2} \nabla_{Y_k}(w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle), \quad 1 \leq k \leq N - 1 (4–105)\]

Next, defining the following change of variables:

\[\tilde{\lambda}_0 = -D_0^\top \Lambda, \quad (4–106)\]
\[\tilde{\lambda}_k = \frac{\Lambda_k}{w_k}, \quad 1 \leq k \leq N, \quad (4–107)\]
\[\tilde{\gamma}_k = \frac{\Gamma_k}{w_k}, \quad 1 \leq k \leq N, \quad (4–108)\]
\[\tilde{\psi} = \Psi. \quad (4–109)\]

Substituting Eqs. (4–106)-(4–109) in Eqs. (4–88)-(4–95) and in Eqs. (4–104)-(4–105),
the transformed KKT conditions for the NLP are given as

\[0 = \nabla_{u_k}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 1 \leq k \leq N, \quad (4–110)\]
\[0 = D_k Y^{\text{LGR}} - \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad 1 \leq k \leq N (4–111)\]
\[\tilde{\gamma}_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (4–112)\]
\[\tilde{\gamma}_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (4–113)\]
\[0 = \phi(Y(\tau_0), \tau_0, Y(\tau_N), \tau_N), \quad (4–114)\]
\[\nabla_{u_0}(\Phi - \langle \tilde{\psi}, \phi \rangle) = \frac{1}{2} \sum_{k=1}^{N} w_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle) \]
\[- \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k \nabla_{t_0}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad (4–115)\]
\[\nabla_{t_0}(\Phi - \langle \tilde{\psi}, \phi \rangle) = \frac{1}{2} \sum_{k=1}^{N} w_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle) \]
\[+ \frac{t_f - t_0}{2} \sum_{k=1}^{N} w_k \nabla_{t_0}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad (4–116)\]
\[
D_N^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{Y_N} (g_N + \langle \tilde{\lambda}_N, f_N \rangle - \langle \tilde{\gamma}_N, C_N \rangle) \\
+ \frac{1}{w_N} \left( \tilde{\lambda}_N - \nabla_{Y_N} (\Phi - \langle \tilde{\psi}, \phi \rangle) \right),
\]
(4–117)

\[
\tilde{\lambda}_0 = -\nabla_{Y_0} (\Phi - \langle \tilde{\psi}, \phi \rangle),
\]
(4–118)

\[
D_k^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{Y_k} (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle),
\]
(4–119)

\[
1 \leq k \leq N - 1.
\]

Now consider a comparison of the transformed KKT conditions in Eqs. (4–110)-(4–119) of the NLP to the first-order necessary optimality conditions in Eqs. (4–6)-(4–15) of the continuous-time optimal control problem. It is noted that the transformed KKT conditions in Eqs. (4–110)-(4–114) are the discretized forms of the continuous-time first-order optimality conditions in Eq. (4–8), Eq. (4–6), Eq. (4–13), Eq. (4–14), and Eq. (4–15), respectively. Next, the right-hand side of Eq. (4–115) and Eq. (4–116) is the quadrature approximation of the right-hand side of Eq. (4–11) and Eq. (4–12), respectively. Therefore, the transformed KKT conditions in Eq. (4–115) and Eq. (4–116) are the discretized version of continuous-time first-order optimality conditions in Eq. (4–11) and Eq. (4–12). Furthermore, it is shown in Theorem 3 that the system (4–119) is a pseudospectral scheme for the costate dynamics, i.e.

\[
D_k^\dagger \tilde{\lambda} = \dot{\tilde{\lambda}}_k \quad 1 \leq k \leq N.
\]
(4–120)

Therefore, the left hand side of Eq. (4–119) is an approximation of costate dynamics at the \(k^{th}\) collocation point, \(k = (1, \ldots, N - 1)\). As a result, Eq. (4–119) represents the discretized version of the costate dynamics in Eq. (4–7) at \(k = (1, \ldots, N - 1)\). Next, it is noted that at the initial boundary point, the discrete equivalent of the continuous boundary condition (4–9) is the same as the discrete costate at the initial boundary point in (4–118). However, at the final boundary point, the discrete equivalent of the continuous boundary condition (4–10) is coupled in the discrete costate dynamics at the final boundary point in (4–117).
The equivalence of the transformed KKT condition, at the final boundary (4–117), of the NLP to the discretized form of continuous first-order optimality condition in (4–10) is now established by manipulating (4–106).

Let $\mathbf{D} = [\mathbf{D}_0 \mathbf{D}_{1:N}]$ where $\mathbf{D}_0$ is the first column of $\mathbf{D}$ and $\mathbf{D}_{1:N}$ is the remaining columns. Then $\mathbf{D}$ is such that: $\mathbf{D}_0 = -\mathbf{D}_{1:N} \mathbf{1}$, where $\mathbf{1}$ is a column vector of all ones.

**Proposition 3.** $\mathbf{D}_0 = -\mathbf{D}_{1:N} \mathbf{1}$; equivalently, $-\mathbf{D}^{-1}_{1:N} \mathbf{D}_0 = \mathbf{1}$.

**Proof of Proposition 3.** The components of the vector $\mathbf{D}_1$ are the derivatives at the collocation points of the constant polynomial $p(\tau) = 1$. Therefore, $\mathbf{D}_1 = 0$, which implies that $\mathbf{D}_1 = \mathbf{D}_0 + \mathbf{D}_{1:N} \mathbf{1} = 0$. Rearranging, we obtain

$$\mathbf{D}_0 = -\mathbf{D}_{1:N} \mathbf{1}. \quad (4–121)$$

Returning to the definition of $\tilde{\lambda}_0$ in (4–106), we obtain

$$\tilde{\lambda}_0 = -\mathbf{D}_0^T \mathbf{\Lambda} = -\sum_{i=1}^{N} \mathbf{\Lambda}_i \mathbf{D}_{i,0} = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Lambda}_i \mathbf{D}_{ij} \quad (4–122)$$

$$= \frac{\mathbf{\Lambda}_N}{\mathbf{w}_N} - \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Lambda}_i \mathbf{D}_{ij}^T \mathbf{w}_j = \frac{\mathbf{\Lambda}_N}{\mathbf{w}_N} - \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Lambda}_j \mathbf{D}_{ij}^T \mathbf{w}_i = \tilde{\lambda}_N - \sum_{i=1}^{N} \mathbf{w}_i \mathbf{\Lambda}_i \mathbf{D}_{ij} \tilde{\lambda} \quad (4–123)$$

$$= \mathbf{\nabla}_{\mathbf{w}} \left( \Phi - \langle \tilde{\psi}, \mathbf{\phi} \rangle \right) + \frac{t_f - t_0}{2} \sum_{i=1}^{N} \mathbf{w}_i \mathbf{\nabla}_{\mathbf{w}} (g_i + \langle \tilde{\lambda}_i, \mathbf{f}_i \rangle - \langle \tilde{\gamma}_i, \mathbf{C}_i \rangle), \quad (4–124)$$

where (4–122) follows from the identity (4–121) given in Proposition 3, (4–123) is the definition (4–96) and (4–97) of $\mathbf{D}^\dagger$, (4–124) is the definition (4–107) of $\tilde{\lambda}_i$, and (4–125) is the first-order optimality condition (4–119) and (4–117). Rearranging (4–125) such that

$$\mathbf{\nabla}_{\mathbf{w}} (\Phi - \langle \tilde{\psi}, \mathbf{\phi} \rangle) = \tilde{\lambda}_0 - \frac{t_f - t_0}{2} \sum_{i=1}^{N} \mathbf{w}_i \mathbf{\nabla}_{\mathbf{w}} (g_i + \langle \tilde{\lambda}_i, \mathbf{f}_i \rangle - \langle \tilde{\gamma}_i, \mathbf{C}_i \rangle). \quad (4–126)$$
Next, the continuous costate dynamics in Eq. (4–7) are

\[ \dot{\lambda}(\tau) = -\frac{t_f - t_0}{2} \nabla_y H = -\frac{t_f - t_0}{2} \nabla_y (g + \langle \lambda, f \rangle - \langle \gamma, C \rangle). \]  

(4–127)

Integrating the continuous costate dynamics in (4–127) using Radau quadrature,

\[ \tilde{\lambda}_N = \tilde{\lambda}_0 - \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i \nabla_y (g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle). \]  

(4–128)

Comparing (4–126) with (4–128) gives

\[ \tilde{\lambda}_N = \nabla_y \Phi - \langle \tilde{\phi}, \phi \rangle. \]  

(4–129)

Equation (4–129) is the missing boundary condition at the final point that was coupled with the discrete costate dynamics at the final point in (4–117). It is also implied by Eq. (4–129) that the extra term in (4–117) is in fact zero, thereby, making (4–117) consistent with discrete costate dynamics at the final point. Hence, the system of transformed KKT conditions of the NLP is equivalent to the first-order optimality conditions of the continuous-time optimal control problem and an accurate costate estimate is obtained from the KKT multipliers using the relationship given in Eq. (4–106) and Eq. (4–107).

### 4.4 Summary

The Radau and the flipped Radau pseudospectral methods based on the collocation at the LGR and the flipped LGR points, respectively, have been presented. The continuous-time optimal control problem is transcribed into a discrete nonlinear programming problem (NLP) using either of the two methods. It has been shown that the transformed KKT conditions obtained in case of both the Radau pseudospectral method and the flipped Radau pseudospectral method are equivalent to the discretized version of the continuous-time first-order optimality conditions. While Radau pseudospectral method has the ability to compute the initial control, control at the final point is not
Continuous-Time Optimal Control Problem

Optimize

Continuous First-Order Optimality Conditions

Discretize

Discrete First-Order Optimality Conditions

Radau

Transformed KKT Conditions

flipped-Radau

Continuous-Time Optimal Control Problem

Discretize

Nonlinear Programming Problem

Optimize

Nonlinear Programming Problem

Figure 4-3. Relationship between the KKT conditions and the discrete first-order optimality conditions for Radau pseudospectral method and flipped Radau pseudospectral method.

obtained. In flipped Radau pseudospectral method, control at the final point is obtained but not the control at the initial point. The implementation of the Radau pseudospectral method is significantly less complex than the Gauss pseudospectral method because in the Radau pseudospectral method implementation, there is no need to compute the state at the final point using quadrature rule as is the case in the Gauss pseudospectral method.
CHAPTER 5
A UNIFIED FRAMEWORK FOR PSEUDOSPECTRAL METHODS

A unified framework is presented for two different pseudospectral methods based on collocation at the Legendre-Gauss (LG) and the Legendre-Gauss-Radau (LGR) points. For each of the schemes in this framework, (1) the state is approximated using collocation points and noncollocated points. (2) the state dynamics are evaluated at only the collocation points. As a result, schemes within this framework employ polynomials to approximate the state that are the same degree as the number of collocation points. Each of these schemes can be expressed in either a differential or an integral formulation. The LG and the LGR differentiation and integration matrices are invertible, and the differential and integral versions are equivalent. Each of these schemes provide an accurate transformation between the Lagrange multipliers of the discrete nonlinear programming problem and the costate of the continuous optimal control problem. It is shown that both of these schemes form a determined system of linear equations for costate dynamics. Lastly, it is shown empirically that the state, the control and the costate solutions obtained by these schemes converge. These schemes are different from the pseudospectral method based on collocation at the Legendre-Gauss-Lobatto (LGL) points. First, both state approximation and dynamics collocation use the same set of collocation points. As a result, in the LGL scheme the degree of polynomials used to approximate the state is one less than the number of collocation points. The LGL differentiation matrix is singular and hence the equivalence between the differential and integral formulation is not established. For the LGL scheme, the linear system of equations for costate dynamics is under-determined, the implications of which are shown by means of an example. The transformation between the Lagrange multipliers of the discrete nonlinear programming problem and the costate of the continuous optimal control problem for the LGL scheme is found to be inaccurate. Lastly, the solutions
obtained from the LGL scheme are typically non-convergent or have higher orders of magnitude error than the solution obtained from the LG or the LGR scheme.

5.1 Implicit Integration Scheme

In all the pseudospectral methods, the solutions to the dynamic constraints are approximated numerically using collocation. Consider the dynamic constraints of the form

$$\frac{d y(\tau)}{d \tau} = \dot{y}(\tau) = \frac{t_f - t_0}{2} f(y(\tau), u(\tau), \tau; t_0, t_f).$$

(5–1)

In collocation, first, the state, $y(\tau)$, is approximated using Lagrange polynomials. Let $Y(\tau)$ be the Lagrange polynomial state approximation. Then the derivative of the state approximation, $\dot{Y}(\tau)$, is set equal to the right-hand side of the differential equation in (5–1) evaluated at a set of intermediate points, $(\tau_1, \ldots, \tau_K)$, called the collocation points, resulting in a system of algebraic constraints. These algebraic constraints are the equality and constraints of the NLP.

An alternative approach to solve the differential equations in (5–1) is to use numerical integration. In this approach, the right hand side of the dynamic constraints in Eq. (5–1) is first approximated using Lagrange polynomials. Then the approximation is numerically integrated to obtain the solution of the state at the set of intermediate points.

In this section, it is shown that the differential and the integral schemes are equivalent for the methods that are a part of the framework, i.e., the Gauss pseudospectral method, the Radau pseudospectral method, and the flipped Radau pseudospectral method. It is also shown that for the Lobatto pseudospectral method, the differential and integral schemes are not equivalent.

5.1.1 Integral Formulation Using LG Collocation

Consider the $N$ LG points, $(\tau_1, \tau_2, \ldots, \tau_N)$, where $\tau_1 > -1$ and $\tau_N < +1$. Define two new points such that $\tau_0 = -1$ and $\tau_{N+1} = 1$. Next, let $L_i(\tau)$, $(i = 0, \ldots, N)$, be the
Lagrange polynomials of degree $N$ given by

$$L_i(\tau) = \prod_{\substack{j=0 \atop j \neq i}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 0, \ldots, N).$$  \hspace{1cm} (5–2)

The state, $y(\tau)$, is approximated by a polynomial of degree at most $N$ using the Lagrange polynomials as

$$y(\tau) \approx Y(\tau) = \sum_{i=0}^{N} Y_i L_i(\tau),$$  \hspace{1cm} (5–3)

where $Y_i = Y(\tau_i)$. It is important to note that $\tau_0 = -1$ is not a LG point but is used in the state approximation. Next, an approximation to the derivative of the state in $\tau$ domain is given by differentiating the approximation of Eq. (5–3) with respect to $\tau$,

$$\dot{y}(\tau) \approx \dot{Y}(\tau) = \sum_{i=0}^{N} Y_i \dot{L}_i(\tau).$$  \hspace{1cm} (5–4)

The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (5–4) to the right-hand side of state dynamic constraints in Eq. (5–1) at the $N$ LG points, $(\tau_1, \ldots, \tau_N)$.

$$\sum_{i=0}^{N} Y_i \dot{L}_i(\tau_k) = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad (k = 1, \ldots, N),$$  \hspace{1cm} (5–5)

$$\sum_{i=0}^{N} D_{ki} Y_i = \frac{t_f - t_0}{2} f_k, \quad D_{ki} = \dot{L}_i(\tau_k),$$  \hspace{1cm} (5–6)

where $U_k = U(\tau_k)$, $f_k = f(Y_k, U_k, \tau_k; t_0, t_f)$ and $D = [D_{ki}]$, $(1 \leq k \leq N)$, $(0 \leq i \leq N)$ is a $N \times (N + 1)$ non-square matrix and is called the Gauss pseudospectral differentiation matrix. The matrix $D$ is non-square because the state approximation uses $N + 1$ points, $(\tau_0, \ldots, \tau_N)$, but the collocation is done at only $N$ LG points, $(\tau_1, \ldots, \tau_N)$.

Let $D = [D_0 \ D_{1:N}]$ where $D_0$ is the first column of $D$ and $D_{1:N}$ are the remaining columns. Then $D$ is such that: (a) $D_{1:N}$ is nonsingular and (b) $D_0 = -D_{1:N}1$; equivalently, $-D_{1:N}^{-1}D_0 = 1$, where $1$ is a column vector of all ones.
**Proposition 4.** The matrix $D_{1:N}$ obtained by deleting the first column of the Gauss pseudospectral differentiation matrix $D$ is invertible.

**Proof of Proposition 4.** Suppose that for some nonzero $p \in \mathbb{R}^{N+1}$ with $p_0 = 0$, we have $Dp = 0$. Let $p$ be the unique polynomial of degree at most $N$ which satisfies $p(\tau_i) = p_i$, $0 \leq i \leq N$. Since the components of $Dp$ are the derivatives of $p$ evaluated at the collocation points, we have

$$0 = (Dp)_i = \dot{p}(\tau_i), \quad 1 \leq i \leq N.$$

Since $\dot{p}$ is a polynomial of degree at most $N - 1$, it must be identically zero since it vanishes at $N$ points. Hence, $p$ is constant. Since $p(-1) = 0$ and $p$ is constant, it follows that $p$ is identically 0. This shows that $p_i = p(\tau_i) = 0$ for each $i$. Since the equation $Dp = 0$ with $p_0 = 0$ has no nonzero solution, $D_{1:N}$ is nonsingular.

**Proposition 5.** $D_0 = -D_{1:N}1$; equivalently, $-D_{1:N}^{-1}D_0 = 1$.

**Proof of Proposition 5.** The components of the vector $D1$ are the derivatives at the collocation points of the constant polynomial $p(\tau) = 1$. Therefore, $D1 = 0$, which implies that $D1 = D_0 + D_{1:N}1 = 0$. Rearranging, we obtain

$$D_0 = -D_{1:N}1. \quad (5-7)$$

Multiplying by $D_{1:N}^{-1}$ gives $-D_{1:N}^{-1}D_0 = 1$.

Using the matrix $D = [D_0 \ D_{1:N}]$, the collocated dynamics at the $N$ LG collocation points in Eq. (5–6) are expressed as

$$D_0 Y_0 + D_{1:N} \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix} = \frac{t_f - t_0}{2} \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}. \quad (5-8)$$
Multiplying Eq. (5–8) by $B = D_{1:N}^{-1}$ and utilizing Proposition 5 to obtain

$$Y_k = Y_0 + \frac{t_f - t_0}{2} \sum_{i=1}^{N} B_{ki} f_i, \quad 1 \leq k \leq N.$$  \hspace{1cm} (5–9)

Next, the approach of numerical integration is applied to solve the differential equations in (5–1) by approximating the derivative on the right-hand side by Lagrange polynomials. Let $L_i^\dagger(\tau)$ be the Lagrange polynomials associated with the collocation points:

$$L_i^\dagger = \prod_{j=1, j \neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad 1 \leq i \leq N. \quad (5–10)$$

Notice that the Lagrange polynomials $L_i$ defined in (5–2) are degree $N$ while the Lagrange polynomials $L_i^\dagger$ are degree $N - 1$. Then the state dynamics on the right-hand side of Eq. (5–1) are approximated by the Lagrange polynomials as

$$F(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L_i^\dagger(\tau). \quad (5–11)$$

Equating the approximation in Eq. (5–11) to the time derivative of state such that

$$\dot{Y}(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L_i^\dagger(\tau). \quad (5–12)$$

Integrating Eq. (5–12) from $\tau_0$ to $\tau_k$, the following relationship is obtained

$$Y_k = Y_0 + \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i A_{ki}, \quad (5–13)$$

where $A = [A_{ki}]$, $(1 \leq k \leq N)$, $(1 \leq i \leq N)$ is a $N \times N$ matrix called the Gauss pseudospectral integration matrix.

It is now demonstrated that the differential scheme and the integral scheme are equivalent by showing that the matrix $B = D_{1:N}^{-1}$ in Eq. (5–9) is equal to the matrix $A$ in Eq. (5–13).
Let \( p \) be any polynomial of degree at most \( N \). By the construction of the \( N \times (N + 1) \) matrix \( D \), we have \( Dp = \dot{p} \) where

\[
p_k = p(\tau_k), \quad 0 \leq k \leq N, \quad (5-14)
\]
\[
\dot{p}_k = \dot{p}(\tau_k), \quad 1 \leq k \leq N. \quad (5-15)
\]

The identity \( \dot{p} = Dp \) can be written as

\[
\dot{p} = D_0 p_0 + D_{1:N} p_{1:N}. \quad (5-16)
\]

Multiplying by \( D_{1:N}^{-1} \) and utilizing Proposition 5 gives

\[
p_k = p_0 + (D_{1:N}^{-1} \dot{p})_k, \quad 1 \leq k \leq N. \quad (5-17)
\]

Next, a different expression for \( p_k - p_0 \) based on the Lagrange polynomial approximation of the derivative is obtained. Because \( \dot{p} \) is a polynomial of degree at most \( N - 1 \), it can be approximated exactly by the Lagrange polynomials \( L^\dagger_i \) as

\[
\dot{p} = \sum_{i=1}^{N} \dot{p}_i L^\dagger_i(\tau). \quad (5-18)
\]

Integrating \( \dot{p} \) from \(-1\) to \( \tau_k \), the following relationship is obtained

\[
p(\tau_k) = p(-1) + \sum_{i=1}^{N} \dot{p}_i A_{ki},
\]
\[
A_{ki} = \int_{-1}^{\tau_k} L^\dagger_i(\tau) d\tau, \quad 1 \leq k \leq N. \quad (5-19)
\]

Utilizing the notation (5–14) and (5–15), we have

\[
p_k = p_0 + (Ap)_k, \quad 1 \leq k \leq N. \quad (5-20)
\]

The relations (5–17) and (5–20) are satisfied for any polynomial of degree at most \( N \). Equating (5–17) and (5–20) to obtain

\[
Ap = D_{1:N}^{-1} \dot{p}.
\]
Choose \( p \) from the columns of the identity matrix to deduce that \( A = D_{1:N}^{-1} \). Thus, it is shown that the differential and integral forms of the state dynamics in the Gauss pseudospectral method are equivalent.

### 5.1.2 Integral Formulation Using Standard LGR Collocation

Consider the \( N \) LGR points, \((\tau_1, \tau_2, \ldots, \tau_N)\), where \( \tau_1 = -1 \) and \( \tau_N < +1 \). Define a new point such that \( \tau_{N+1} = 1 \). Next, let \( L_i(\tau) \), \((i = 1, \ldots, N + 1)\), be the Lagrange polynomials of degree \( N \) given by

\[
L_i(\tau) = \prod_{j=1}^{N+1} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 1, \ldots, N + 1). \tag{5–21}
\]

The state, \( y(\tau) \), is approximated by a polynomial of degree at most \( N \) using the Lagrange polynomials as

\[
y(\tau) \approx Y(\tau) = \sum_{i=1}^{N+1} Y_i L_i(\tau), \tag{5–22}
\]

where \( Y_i = Y(\tau_i) \). It is important to note that \( \tau_{N+1} = 1 \) is not a LGR point but is used in state approximation. Next, an approximation to the derivative of the state in \( \tau \) domain is given by differentiating the approximation of Eq. (5–22) with respect to \( \tau \),

\[
y'(\tau) \approx Y'(\tau) = \sum_{i=1}^{N+1} Y_i \dot{L}_i(\tau). \tag{5–23}
\]

The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (5–23) to the right-hand side of the state dynamic constraints in Eq. (5–1) at the \( N \) LGR points, \((\tau_1, \ldots, \tau_N)\).

\[
\sum_{i=1}^{N+1} Y_i \dot{L}_i(\tau_k) = \frac{t_r - t_0}{2} f(Y_k, U_k, \tau; t_0, t_r), \quad (k = 1, \ldots, N), \tag{5–24}
\]

\[
\sum_{i=1}^{N+1} D_{ki} Y_i = \frac{t_r - t_0}{2} f_k, \quad D_{ki} = \dot{L}_i(\tau_k), \tag{5–25}
\]

where \( U_k = U(\tau_k), f_k = f(Y_k, U_k, \tau_k; t_0, t_r) \) and \( D = [D_{ki}] \), \((1 \leq k \leq N), \quad (1 \leq i \leq N + 1)\) is a \( N \times (N + 1) \) non-square matrix and is called the Radau pseudospectral differentiation.
matrix. The matrix $D$ is non-square because the state approximation uses $N+1$ points, $(\tau_1, \ldots, \tau_{N+1})$, but the collocation is done at only $N$ LGR points, $(\tau_1, \ldots, \tau_N)$.

Let $D = [D_1 \ D_{2:N+1}]$ where $D_1$ is the first column of $D$ and $D_{2:N+1}$ are the remaining columns. Then $D$ is such that: (a) $D_{2:N+1}$ is nonsingular and (b) $D_1 = -D_{2:N+1} \mathbf{1}$; equivalently, $-D_{2:N+1}^{-1} D_1 = \mathbf{1}$, where $\mathbf{1}$ is a column vector of all ones.

**Proposition 6.** The matrix $D_{2:N+1}$ obtained by deleting the first column of the Radau pseudospectral differentiation matrix $D$ is invertible.

**Proof of Proposition 6.** Suppose that for some nonzero $p \in \mathbb{R}^{N+1}$ with $p_1 = 0$, we have $Dp = 0$. Let $p$ be the unique polynomial of degree at most $N$ which satisfies $p(\tau_i) = p_i$, $1 \leq i \leq N+1$. Since the components of $Dp$ are the derivatives of $p$ evaluated at the collocation points, we have

$$0 = (Dp)_i = \dot{p}(\tau_i), \quad 1 \leq i \leq N.$$ 

Since $p$ is a polynomial of degree at most $N - 1$, it must be identically zero since it vanishes at $N$ points. Hence, $p$ is constant. Since $p(-1) = 0$ and $p$ is constant, it follows that $p$ is identically 0. This shows that $p_i = p(\tau_i) = 0$ for each $i$. Since the equation $Dp = 0$ with $p_1 = 0$ has no nonzero solution, $D_{2:N+1}$ is nonsingular.

**Proposition 7.** $D_1 = -D_{2:N+1} \mathbf{1}$; equivalently, $-D_{2:N+1}^{-1} D_1 = \mathbf{1}$.

**Proof of Proposition 7.** The components of the vector $D1$ are the derivatives at the collocation points of the constant polynomial $p(\tau) = 1$. Therefore, $D1 = 0$, which implies that $D1 = D_1 + D_{2:N+1} \mathbf{1} = 0$. Rearranging, we obtain

$$D_1 = -D_{2:N+1} \mathbf{1}. \quad (5-26)$$

Multiplying by $D_{2:N+1}^{-1}$ gives $-D_{2:N+1}^{-1} D_1 = \mathbf{1}$.

\[\square\]
Using the matrix \( D = [D_1, D_{2:N+1}] \), the collocated dynamics at the \( N \) LGR collocation points in Eq. (5–25) are expressed as

\[
D_1 Y_1 + D_{2:N+1} \begin{bmatrix} Y_2 \\ \vdots \\ Y_{N+1} \end{bmatrix} = \frac{t_f - t_0}{2} \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}.
\] (5–27)

Multiplying Eq. (5–27) by \( B = D_{2:N+1}^{-1} \) and utilizing Proposition 7 gives

\[
Y_k = Y_1 + \frac{t_f - t_0}{2} \sum_{i=1}^{N} B_{ki} f_i, \quad 2 \leq k \leq N + 1.
\] (5–28)

Next, the approach of numerical integration is applied to solve the differential equations in (5–1) by approximating the derivative on the right-hand side by Lagrange polynomials. Let \( L_i^\dagger(\tau) \) be the Lagrange polynomials associated with the collocation points:

\[
L_i^\dagger = \prod_{j=1, j\neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad 1 \leq i \leq N.
\] (5–29)

Notice that the Lagrange polynomials \( L_i \) defined in (5–21) are degree \( N \) while the Lagrange polynomials \( L_i^\dagger \) are degree \( N - 1 \). Then the state dynamics on the right-hand side of Eq. (5–1) are approximated by the Lagrange polynomials as

\[
F(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L_i^\dagger(\tau).
\] (5–30)

Equating the approximation in Eq. (5–30) to the time derivative of state such that

\[
\dot{Y}(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L_i^\dagger(\tau).
\] (5–31)

Integrating Eq. (5–31) from \( \tau_1 \) to \( \tau_k \), the following relationship is obtained:

\[
Y_k = Y_1 + \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i A_{ki}, \quad 2 \leq k \leq N + 1,
\] (5–32)

\[
A_{ki} = \int_{-1}^{\tau_k} L_i^\dagger(\tau) d\tau, \quad 2 \leq k \leq N + 1,
\]
where \( \mathbf{A} = [A_{ki}], \) \((2 \leq k \leq N + 1), \) \((1 \leq i \leq N)\) is a \( N \times N \) matrix called the Radau pseudospectral integration matrix.

It is now demonstrated that the differential scheme and the integral scheme are equivalent by showing that the matrix \( \mathbf{B} = \mathbf{D}^{-1}_{2:N+1} \) in Eq. (5–28) is equal to the matrix \( \mathbf{A} \) in Eq. (5–32).

Let \( p \) be any polynomial of degree at most \( N \). By the construction of the \( N \times (N + 1) \) matrix \( \mathbf{D} \), we have \( \mathbf{D}p = \dot{p} \) where

\[
\begin{align*}
    p_k &= p(\tau_k), \quad 1 \leq k \leq N + 1, \\
    \dot{p}_k &= \dot{p}(\tau_k), \quad 1 \leq k \leq N.
\end{align*}
\]

The identity \( p = \mathbf{D}p \) can be written as

\[
\dot{p} = \mathbf{D}_1 p_1 + \mathbf{D}_{2:N+1} p_{2:N+1}.
\]

Multiplying by \( \mathbf{D}^{-1}_{2:N+1} \) and utilizing Proposition 7 gives

\[
\begin{align*}
    p_k &= p_1 + \left( \mathbf{D}^{-1}_{2:N+1} \dot{p} \right)_k, \quad 2 \leq k \leq N + 1.
\end{align*}
\]

Next, a different expression for \( p_k - p_1 \) based on the Lagrange polynomial approximation of the derivative is obtained. Because \( \dot{p} \) is a polynomial of degree at most \( N - 1 \), it can be approximated exactly by the Lagrange polynomials \( L^j_i \):

\[
\dot{p} = \sum_{i=1}^{N} \dot{p}_i L^j_i(\tau).
\]

Integrating \( \dot{p} \) from \(-1\) to \( \tau_k \), the following relationship is obtained

\[
\begin{align*}
    p(\tau_k) &= p(-1) + \sum_{i=1}^{N} \dot{p}_i A_{ki}, \\
    A_{ki} &= \int_{-1}^{\tau_k} L^j_i(\tau) \, d\tau, \quad 2 \leq k \leq N + 1.
\end{align*}
\]
Utilizing the notation (5–33) and (5–34), we have

\[ p_k = p_1 + (A\dot{p})_k, \quad 2 \leq k \leq N + 1. \quad (5–39) \]

The relations (5–36) and (5–39) are satisfied for any polynomial of degree at most \( N \).

Equating (5–36) and (5–39) to obtain

\[ A\dot{p} = D_{2:N+1}^{-1}\dot{p}. \]

Choose \( \dot{p} \) from the columns of the identity matrix to deduce that \( A = D_{2:N+1}^{-1} \). Thus, it is shown that the differential and integral forms of the state dynamics in the Radau pseudospectral method are equivalent.

**5.1.3 Integral Formulation Using Flipped LGR Collocation**

Consider the \( N \) flipped LGR points, \((\tau_1, \tau_2, \ldots, \tau_N)\), where \( \tau_1 > -1 \) and \( \tau_N = +1 \).

Define a new point such that \( \tau_0 = -1 \). Next, let \( L_i(\tau), (i = 0, \ldots, N) \), be the Lagrange polynomials of degree \( N \) given by

\[ L_i(\tau) = \prod_{j=0, j \neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 0, \ldots, N). \quad (5–40) \]

The state, \( y(\tau) \), is approximated by a polynomial of degree at most \( N \) using the Lagrange polynomials as

\[ y(\tau) \approx Y(\tau) = \sum_{i=0}^{N} Y_i L_i(\tau), \quad (5–41) \]

where \( Y_i = Y(\tau_i) \). It is important to note that \( \tau_0 = -1 \) is not a flipped LGR point but is used in state approximation. Next, an approximation to the derivative of the state in \( \tau \) domain is given by differentiating the approximation of Eq. (5–41) with respect to \( \tau \),

\[ y(\tau) \approx \dot{Y}(\tau) = \sum_{i=0}^{N} \dot{Y}_i L_i(\tau). \quad (5–42) \]

The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (5–42) to the right-hand side of the state dynamic constraints in
Eq. (5–1) at the $N$ flipped LGR points, $(\tau_1, \ldots, \tau_N)$.

\[
\sum_{i=0}^{N} Y_i \dot{L}_i(\tau_k) = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau; t_0, t_f), \quad (k = 1, \ldots, N),
\]

(5–43)

\[
\sum_{i=0}^{N} D_{ki} Y_i = \frac{t_f - t_0}{2} f_k, \quad D_{ki} = \dot{L}_i(\tau_k),
\]

(5–44)

where $U_k = U(\tau_k)$, $f_k = f(Y_k, U_k, \tau_k; t_0, t_f)$ and $D = [D_{ki}]$, $(1 \leq k \leq N)$, $(0 \leq i \leq N)$ is a $N \times (N + 1)$ non-square matrix and is called the flipped Radau pseudospectral differentiation matrix. The matrix $D$ is non-square because the state approximation uses $N + 1$ points, $(\tau_0, \ldots, \tau_N)$, but the collocation is done at only $N$ flipped LGR points, $(\tau_1, \ldots, \tau_N)$.

Let $D = [D_0 \ D_{1:N}]$ where $D_0$ is the first column of $D$ and $D_{1:N}$ are the remaining columns. Then $D$ is such that: (a) $D_{1:N}$ is nonsingular and (b) $-D_{1:N}^{-1} D_0 = 1$, where $1$ is a column vector of all ones.

**Proposition 8.** The matrix $D_{1:N}$ obtained by deleting the first column of the flipped Radau pseudospectral differentiation matrix $D$ is invertible.

**Proof of Proposition 8.** Suppose that for some nonzero $p \in \mathbb{R}^{N+1}$ with $p_0 = 0$, we have $Dp = 0$. Let $p$ be the unique polynomial of degree at most $N$ which satisfies $p(\tau_i) = \rho_i$, $0 \leq i \leq N$. Since the components of $Dp$ are the derivatives of $p$ evaluated at the collocation points, we have

\[
0 = (Dp)_i = \dot{p}(\tau_i), \quad 1 \leq i \leq N.
\]

Since $\dot{p}$ is a polynomial of degree at most $N - 1$, it must be identically zero since it vanishes at $N$ points. Hence, $p$ is constant. Since $p(-1) = 0$ and $p$ is constant, it follows that $p$ is identically 0. This shows that $\rho_i = p(\tau_i) = 0$ for each $i$. Since the equation $Dp = 0$ with $p_0 = 0$ has no nonzero solution, $D_{1:N}$ is nonsingular.

□
Proposition 9. \(-D_{1:N}^{-1}D_0 = 1\).

Proof of Proposition 9. The components of the vector \(D1\) are the derivatives at the collocation points of the constant polynomial \(p(\tau) = 1\). Therefore, \(D1 = 0\), which implies that \(D1 = D_0 + D_{1:N}1 = 0\). Rearranging, we obtain

\[D_0 = -D_{1:N}1.\]  \hspace{1cm} (5–45)

Multiplying by \(D_{1:N}^{-1}\) gives \(-D_{1:N}^{-1}D_0 = 1\).

Using the matrix \(D = [D_0\ D_{1:N}]\), the collocated dynamics at the \(N\) LG collocation points in Eq. (5–44) are expressed as

\[D_0 Y_0 + D_{1:N} \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix} = \frac{t_f - t_0}{2} \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}.\]  \hspace{1cm} (5–46)

Multiplying Eq. (5–46) by \(B = D_{1:N}^{-1}\) and utilizing Proposition 5–45 to obtain

\[Y_k = Y_0 + \frac{t_f - t_0}{2} \sum_{i=1}^{N} B_{ki}f_i, \quad 1 \leq k \leq N.\]  \hspace{1cm} (5–47)

Next, the approach of numerical integration is applied to solve the differential equations in (5–1) by approximating the derivative on the right-hand side by Lagrange polynomials. Let \(L_i^\dagger(\tau)\) be the Lagrange polynomials associated with the collocation points:

\[L_i^\dagger = \prod_{\begin{subarray}{c} j=1 \\ j \neq i \end{subarray}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad 1 \leq i \leq N.\]  \hspace{1cm} (5–48)

Notice that the Lagrange polynomials \(L_i\) defined in (5–40) are degree \(N\) while the Lagrange polynomials \(L_i^\dagger\) are degree \(N - 1\). Then the state dynamics on the right-hand side of Eq. (5–1) are approximated by the Lagrange polynomials as

\[F(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L_i^\dagger(\tau).\]  \hspace{1cm} (5–49)
Equating the approximation in Eq. (5–49) to the time derivative of state such that

\[ Y(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L^\dagger_i(\tau). \] (5–50)

Integrating Eq. (5–50) from \( \tau_0 \) to \( \tau_k \), following relationship is obtained

\[ Y_k = Y_0 + \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i A_{ki}, \] (5–51)

\[ A_{ki} = \int_{-1}^{\tau_k} L^\dagger_i(\tau) d\tau, \quad 1 \leq k \leq N, \]

where \( A = [A_{ki}], \ (1 \leq k \leq N), \ (1 \leq i \leq N) \) is a \( N \times N \) matrix called the flipped Radau pseudospectral integration matrix.

It is now demonstrated that the differential scheme and the integral scheme are equivalent by showing that the matrix \( B = D^{-1}_{1:N} \) in Eq. (5–47) is equal to the matrix \( A \) in Eq. (5–51).

Let \( p \) be any polynomial of degree at most \( N \). By the construction of the \( N \times (N + 1) \) matrix \( D \), we have \( Dp = \dot{p} \) where

\[ p_k = p(\tau_k), \quad 0 \leq k \leq N, \] (5–52)

\[ \dot{p}_k = \dot{p}(\tau_k), \quad 1 \leq k \leq N. \] (5–53)

The identity \( \dot{p} = Dp \) can be written as

\[ p = D_0p_0 + D_{1:N}p_{1:N}. \] (5–54)

Multiplying by \( D^{-1}_{1:N} \) and utilizing Proposition 5–45 gives

\[ p_k = p_0 + (D^{-1}_{1:N}\dot{p})_k, \quad 1 \leq k \leq N. \] (5–55)

Next, a different expression for \( p_k - p_0 \) based on the Lagrange polynomial approximation of the derivative is obtained. Because \( \dot{p} \) is a polynomial of degree at most \( N - 1 \), it can
be approximated exactly by the Lagrange polynomials $L_i^\dagger$ as

$$\dot{p} = \sum_{i=1}^{N} \dot{p}_i L_i^\dagger(\tau). \tag{5–56}$$

Integrating $\dot{p}$ from $-1$ to $\tau_k$, the following relationship is obtained

$$p(\tau_k) = p(-1) + \sum_{i=1}^{N} \dot{p}_i A_{ki}, \tag{5–57}$$

$$A_{ki} = \int_{-1}^{\tau_k} L_i^\dagger(\tau) \, d\tau, \quad 1 \leq k \leq N.$$

Utilizing the notation (5–52) and (5–53), we have

$$p_k = p_0 + (A \dot{p})_k, \quad 1 \leq k \leq N. \tag{5–58}$$

The relations (5–55) and (5–58) are satisfied for any polynomial of degree at most $N$. Equating (5–55) and (5–58) to obtain

$$A \dot{p} = D_{1:N}^{-1} \dot{p}.$$

Choose $\dot{p}$ from the columns of the identity matrix to deduce that $A = D_{1:N}^{-1}$. Thus, it is shown that the differential and integral forms of the state dynamics in the flipped Radau pseudospectral method are equivalent.

5.1.4 Integral Formulation Using LGL Collocation

Consider the $N$ LGL points, $(\tau_1, \tau_2, \ldots, \tau_N)$, where $\tau_1 = -1$ and $\tau_N = +1$. Next, let $L_i(\tau), (i = 1, \ldots, N)$, be the Lagrange polynomials of degree $N - 1$ given by

$$L_i(\tau) = \prod_{j=1}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 1, \ldots, N). \tag{5–59}$$

The state, $y(\tau)$, is approximated by a polynomial of degree at most $N - 1$ using the Lagrange polynomials as

$$y(\tau) \approx Y(\tau) = \sum_{i=1}^{N} Y_i L_i(\tau), \tag{5–60}$$
where \( Y_i = Y(\tau_i) \). The points, \((\tau_1, \ldots, \tau_N)\), that are used in state approximation are called the discretization points. Next, an approximation to the derivative of the state in \( \tau \) domain is given by differentiating the approximation of Eq. (5–60) with respect to \( \tau \),

\[
y(\tau) \approx \dot{Y}(\tau) = \sum_{i=1}^{N} Y_i \dot{L}_i(\tau). \tag{5–61}
\]

The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (5–61) to the right-hand side of state dynamic constraints in Eq. (5–1) at the \( N \) LGL points. The \( N \) LGL points at which the collocation is done are called the collocation points.

\[
\sum_{i=1}^{N} Y_i \dot{L}_i(\tau_k) = \frac{t_f - t_0}{2} f(Y_k, U_k, \tau_k; t_0, t_f), \quad (k = 1, \ldots, N), \tag{5–62}
\]

\[
\sum_{i=1}^{N} D_{ki} Y_i = \frac{t_f - t_0}{2} f_k, \quad D_{ki} = \dot{L}_i(\tau_k), \tag{5–63}
\]

where \( U_k = U(\tau_k), f_k = f(Y_k, U_k, \tau_k; t_0, t_f) \) and \( D = [D_{ki}], \ (1 \leq k \leq N), \ (1 \leq i \leq N) \) is a \( N \times N \) square matrix and is called the \textit{Lobatto pseudospectral differentiation matrix}. The matrix \( D \) is square because the collocation points are the same as the discretization points used in Lagrange polynomial approximation of state. Using the matrix \( D \), the collocated dynamics at the \( N \) LGL collocation points in Eq. (5–63) are expressed as

\[
D \begin{bmatrix} Y_1 \\
\vdots \\
Y_N \end{bmatrix} = \frac{t_f - t_0}{2} \begin{bmatrix} f_1 \\
\vdots \\
f_N \end{bmatrix}. \tag{5–64}
\]

It is now shown that the Lobatto pseudospectral differentiation matrix is a singular matrix.

**Proposition 10.** The \textit{Lobatto pseudospectral differentiation matrix} \( D \) is singular.

**Proof of Proposition 10.** Let \( p(\tau) = 1 \) be a constant polynomial. Next, let \( p \in \mathbb{R}^N \) be such that \( p_i = p(\tau_i) = 1, \ 1 \leq i \leq N \). The components of the vector \( Dp \) are the derivatives
at the collocation points of the constant polynomial \( p(\tau) = 1 \). Therefore, \( D1 = 0 \), which implies the equation \( Dp = 0 \) has a nonzero solution. Hence, \( D \) is singular.

Next, the approach of numerical integration is applied to solve the differential equations in (5–1) by approximating the derivative on the right-hand side by Lagrange polynomials. Let \( L^\dagger_i(\tau) \) be the Lagrange polynomials associated with the collocation points:

\[
L^\dagger_i = \prod_{j=1, j \neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad 1 \leq i \leq N.
\]

(5–65)

Notice that the Lagrange polynomials \( L_i \) defined in (5–59) and the Lagrange polynomials \( L^\dagger_i \) are degree \( N - 1 \). Then the state dynamics on the right-hand side of Eq. (5–1) are approximated by the Lagrange polynomials as

\[
F(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L^\dagger_i(\tau).
\]

(5–66)

Equating the approximation in Eq. (5–66) to the time derivative of state such that

\[
Y(\tau) = \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i L^\dagger_i(\tau).
\]

(5–67)

Integrating Eq. (5–67) from \( \tau_1 \) to \( \tau_k \), following relationship is obtained

\[
Y_k = Y_1 + \frac{t_f - t_0}{2} \sum_{i=1}^{N} f_i A_{ki},
\]

(5–68)

where \( A = [A_{ki}], \ (2 \leq k \leq N), \ (1 \leq i \leq N) \) is a \( N \times N \) matrix called the \textit{Lobatto pseudospectral integration matrix}. The Lobatto integration matrix is the same as that found in Ref. [84]

It is noted that the integrated scheme in Eq. (5–68) is not equivalent to the differential scheme in Eq. (5–64). Both the schemes are altogether different schemes.
The differential scheme consists of \( N \) equations while the integral scheme represents \( N - 1 \) equations.

### 5.2 Costate Dynamics for Initial-Value Problem

A very interesting feature of the costate dynamics obtained in the pseudospectral methods is observed in the case of initial-value problems. In an initial-value problem, the boundary conditions at only the initial endpoint are specified. This feature unifies the Gauss pseudospectral method, the Radau pseudospectral method and the flipped Radau pseudospectral method in the sense that the costate dynamics in all these three methods form a determined system of equations whereas, in case of the Lobatto pseudospectral method, the system of equations formed by the costate dynamics is under-determined. The discrete costate dynamics form a linear system of equations in the costate, \( \lambda \). It is shown that in case of Lobatto pseudospectral method, the matrix of the linear system has a null space and there exists an infinite number of solutions to the costate dynamics. It is shown empirically that in case of a one-dimensional optimal control problem the costate solution obtained from the Lobatto pseudospectral method oscillates about the actual costate and that the oscillations have the same pattern as the null space of the system.

#### 5.2.1 Gauss Pseudospectral Method

Consider the costate dynamics obtained in the transformed KKT conditions for the Gauss pseudospectral method

\[
D_{k,1:N}^\dagger \ddot{\lambda} + D_{k,N+1}^\dagger \ddot{\lambda}_{N+1} = -\frac{t_f - t_0}{2} \nabla_{\gamma_k}(g_k + \langle \ddot{\lambda}_k, f_k \rangle - \langle \ddot{\gamma}_k, C_k \rangle),
\]

(5–69)

\[1 \leq k \leq N.\]

It is observed that, given \( Y_k, U_k, \ddot{\gamma}_k, (k = 1, \ldots, N), t_0, \) and \( t_f \), (5–69) represents \( Nn \) equations in \((N + 1)n\) variables, \( \ddot{\lambda}_i \in \mathbb{R}^n, (i = 1, \ldots, N + 1) \). For an initial-value problem,
costate at the terminal endpoint, $\tilde{\lambda}_{N+1}$ is obtained as

$$\tilde{\lambda}_{N+1} = \nabla Y_{N+1} \Phi.$$  \hfill (5–70)

Substituting (5–70) in Eq. (5–69), the costate dynamics are obtained as

$$D_{1:N}^\dagger \tilde{\lambda} + D_{N+1}^\dagger \nabla y_{N+1} \Phi = -\frac{t_f - t_0}{2} \begin{bmatrix} \nabla y_1 (g_1 + \langle \tilde{\lambda}_1, f_1 \rangle - \langle \tilde{\gamma}_1, C_1 \rangle) \\ \vdots \\ \nabla y_N (g_N + \langle \tilde{\lambda}_N, f_N \rangle - \langle \tilde{\gamma}_N, C_N \rangle) \end{bmatrix},$$  \hfill (5–71)

where $D_{1:N}^\dagger$ are the first $N$ columns of $D^\dagger$ and $D_{N+1}^\dagger$ is the last column of $D^\dagger$. It is observed that (5–71) represents $Nn$ linear equations in $Nn$ variables, $\tilde{\lambda}_i \in \mathbb{R}^n$, $(i = 1, \ldots, N)$. Hence, it is a determined system of equations.

### 5.2.2 Radau Pseudospectral Method

Consider the costate dynamics obtained in the transformed KKT conditions for the Radau pseudospectral method

$$D_1^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla y_1 (g_1 + \langle \tilde{\lambda}_1, f_1 \rangle - \langle \tilde{\gamma}_1, C_1 \rangle)$$

$$-\frac{1}{w_1} \left( \tilde{\lambda}_1 + \nabla y_1 (\Phi - \langle \tilde{\psi}, \phi \rangle) \right),$$  \hfill (5–72)

$$D_k^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla y_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle),$$  \hfill (5–73)

$$2 \leq k \leq N.$$

Using the result of Eq. (4–69), we know that

$$-\nabla y_1 (\Phi - \langle \tilde{\psi}, \phi \rangle) = \tilde{\lambda}_{N+1} + \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i \nabla y_i (g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle).$$  \hfill (5–74)

For an initial-value problem, costate at the terminal endpoint, $\tilde{\lambda}_{N+1}$ is obtained as

$$\tilde{\lambda}_{N+1} = \nabla y_{N+1} \Phi.$$  \hfill (5–75)
Substituting (5–75) in Eq. (5–74), following result is obtained

\[- \nabla_{\gamma_1} \left( \Phi - \langle \tilde{\gamma}, \phi \rangle \right) = \nabla_{\gamma_{N+1}} \Phi + \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i \nabla_{\gamma_i} \left( g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle \right). \]  

(5–76)

Substituting (5–76) in Eq. (5–72), the costate dynamics are obtained as

\[
D^i \tilde{\lambda} + \frac{1}{w_1} e_i e_i^T \tilde{\lambda} = -\frac{t_f - t_0}{2} \begin{bmatrix} \nabla_{\gamma_1} (g_1 + \langle \tilde{\lambda}_1, f_1 \rangle - \langle \tilde{\gamma}_1, C_1 \rangle) \\ \vdots \\ \nabla_{\gamma_N} (g_N + \langle \tilde{\lambda}_N, f_N \rangle - \langle \tilde{\gamma}_N, C_N \rangle) \\ + \frac{1}{w_1} e_1 \left( \nabla_{\gamma_{N+1}} \Phi + \frac{t_f - t_0}{2} \sum_{i=1}^{N} w_i \nabla_{\gamma_i} \left( g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle \right) \right) \end{bmatrix},
\]  

(5–77)

where \( e_1 \) is the first column of \( N \times N \) Identity matrix. It is observed that, given \( Y_k, U_k, \tilde{\gamma}_k, \) \( (k = 1, ..., N), t_0, \) and \( t_f, \) (5–77) represents \( Nn \) linear equations in \( Nn \) variables, \( \tilde{\lambda}_i \in \mathbb{R}^n, \) \( (i = 1, ..., N). \) Hence, it is a determined system of equations.

### 5.2.3 Flipped Radau Pseudospectral Method

Consider the costate dynamics obtained in the transformed KKT conditions for the flipped Radau pseudospectral method

\[
D^i_N \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{\gamma_N} \left( g_N + \langle \tilde{\lambda}_N, f_N \rangle - \langle \tilde{\gamma}_N, C_N \rangle \right) + \frac{1}{w_N} \left( \tilde{\lambda}_N - \nabla_{\gamma_N} \Phi \right), \]  

(5–78)

\[
D^k_k \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla_{\gamma_k} \left( g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle \right), \]  

(5–79)

\( 1 \leq k \leq N - 1. \)

Rewriting the costate dynamics as

\[
D^i \tilde{\lambda} - \frac{1}{w_N} e_N e_N^T \tilde{\lambda} = -\frac{t_f - t_0}{2} \begin{bmatrix} \nabla_{\gamma_1} (g_1 + \langle \tilde{\lambda}_1, f_1 \rangle - \langle \tilde{\gamma}_1, C_1 \rangle) \\ \vdots \\ \nabla_{\gamma_N} (g_N + \langle \tilde{\lambda}_N, f_N \rangle - \langle \tilde{\gamma}_N, C_N \rangle) \end{bmatrix} - \frac{1}{w_N} e_N \nabla_{\gamma_N} \Phi, \]  

(5–80)

where \( e_N \) is the \( N^{th} \) column of \( N \times N \) Identity matrix. It is observed that, given \( Y_k, U_k, \) \( \tilde{\gamma}_k, \) \( (k = 1, ..., N), t_0, \) and \( t_f, \) Eq. (5–80) represents \( Nn \) linear equations in \( Nn \) variables,
\( \tilde{\lambda}_i \in \mathbb{R}^n, (i = 1, \ldots, N) \). Hence, the system of equations in \((5–80)\) is a determined system of equations.

### 5.2.4 Lobatto Pseudospectral Method

Consider the costate dynamics obtained in the transformed KKT conditions for the Lobatto pseudospectral method

\[
D_1^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla Y_1 (g_1 + \langle \tilde{\lambda}_1, f_1 \rangle - \langle \tilde{\gamma}_1, C_1 \rangle) \\
- \frac{1}{w_1} \left( \tilde{\lambda}_1 + \nabla Y_1 (\Phi - \langle \tilde{\psi}, \phi \rangle) \right), \quad (5–81)
\]

\[
D_N^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla Y_N (g_N + \langle \tilde{\lambda}_N, f_N \rangle - \langle \tilde{\gamma}_N, C_N \rangle) \\
+ \frac{1}{w_N} \left( \tilde{\lambda}_N - \nabla Y_N \phi \right), \quad (5–82)
\]

\[
D_k^\dagger \tilde{\lambda} = -\frac{t_f - t_0}{2} \nabla Y_k (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad (5–83)
\]

\[
2 \leq k \leq N - 1.
\]

Because for this discussion, the optimal control problem is being defined as an initial-value problem, consider the following form of initial boundary condition:

\[
\phi(x(-1)) \equiv x(-1) - x_0 = 0 \\
\Leftrightarrow x(\tau_1) - x_0 = 0. \quad (5–84)
\]

Therefore,

\[
\nabla Y_1 \langle \tilde{\psi}, \phi \rangle = \nabla Y_1 \langle \tilde{\psi}, x(\tau_1) - x_0 \rangle = \tilde{\psi}, \quad (5–85)
\]

where \( \tilde{\psi} \in \mathbb{R}^n \).
Rewriting the costate dynamics as

\[
D^T \ddot{\lambda} + \frac{1}{w_1} e_1 e_1^T \ddot{\lambda} - \frac{1}{w_N} e_N e_N^T \ddot{\lambda} - \frac{1}{w_1} e_1 \ddot{\psi} = -\frac{t_f - t_0}{2} \begin{bmatrix}
\nabla_{Y_1}(g_1 + \langle \dot{\lambda}_1, f_1 \rangle - \langle \dot{\gamma}_1, C_1 \rangle) \\
\vdots \\
\nabla_{Y_N}(g_N + \langle \dot{\lambda}_N, f_N \rangle - \langle \dot{\gamma}_N, C_N \rangle)
\end{bmatrix} - \frac{1}{w_1} e_1 \nabla_{Y_1} \Phi - \frac{1}{w_N} e_N \nabla_{Y_N} \Phi,
\]

(5–86)

\[
\begin{bmatrix}
D^T + \frac{1}{w_1} e_1 e_1^T - \frac{1}{w_N} e_N e_N^T & - \frac{1}{w_1} e_1 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{\lambda} \\
\ddot{\psi}
\end{bmatrix} = -\frac{t_f - t_0}{2} \begin{bmatrix}
\nabla_{Y_1}(g_1 + \langle \dot{\lambda}_1, f_1 \rangle - \langle \dot{\gamma}_1, C_1 \rangle) \\
\vdots \\
\nabla_{Y_N}(g_N + \langle \dot{\lambda}_N, f_N \rangle - \langle \dot{\gamma}_N, C_N \rangle)
\end{bmatrix} - \frac{1}{w_1} e_1 \nabla_{Y_1} \Phi - \frac{1}{w_N} e_N \nabla_{Y_N} \Phi,
\]

(5–87)

where \(e_1\) and \(e_N\) are the first and the \(N^{th}\) columns, respectively, of \(N \times N\) identity matrix.

It is observed that, given \(Y_k, U_k, \dot{\gamma}_k\), \((k = 1, \ldots, N)\), \(t_0\), and \(t_f\), Eq. (5–87) represents \(NNn\) linear equations in \((N + 1)n\) variables, \(\ddot{\lambda}_i \in \mathbb{R}^n\), \((i = 1, \ldots, N)\) and \(\ddot{\psi} \in \mathbb{R}^n\). Hence, the matrix of the linear system has a null space and there exists an infinite number of solutions to the LGL costate dynamics. The dimension of the null space is at least \(n\) since \(\ddot{\lambda}_i \in \mathbb{R}^n\).

Despite the null space in the LGL costate dynamics, a wealth of numerical examples published in the literature \([37, 40, 42–45, 65]\), along with the convergence theory of Kang \([85]\), all demonstrate that the Lobatto pseudospectral method leads to convergent approximations to the state and control variable. However, due to the null space in the discrete costate dynamics, the concept of convergence for the costate is open to interpretation \([63]\). In Ref. \([44]\), Gong et al. have shown in “Covector Mapping Theorem” that any solution to the first-order optimality conditions for the continuous control problem, approximately satisfies the transformed KKT conditions of the discrete LGL problem, and the error tends to zero as \(N \to \infty\). This provides evidence that among the infinite set of solutions associated with the discrete costate dynamics, there exists a good approximation to the continuous costate \([63]\). Moreover, a closure condition is
proposed for selecting a good approximation to the continuous costate from among the infinite set of solutions to the costate dynamics. In the context of a initial-value control problem, the closure condition amounts to choosing a solution to the discrete costate equation (5–87) which satisfies the conditions

\[ \| \tilde{\lambda}_1 - \tilde{\psi} \| \leq \delta, \]
\[ \| \tilde{\lambda}_N - \nabla_x \Phi(X_N) \| \leq \delta, \] (5–88)

where \( \delta \) is some given error tolerance. In [44, Thm. 4] it is shown that an asymptotically valid choice for \( \delta \) is of the form \( \delta = N^{(1.5 - m)} \), where \( m \geq 4 \), independent of \( N \), depends on the number of derivatives of smoothness of an optimal solution. In practice, the null space for the LGL costate dynamics is often observed to be highly oscillatory [63]. As a result, in Ref. [37] (page 276), Fahroo et al. suggested that the computed costate can be post-processed using a filter to obtain a good approximation to the continuous costate. Very recently, in Ref. [86], Gong et al. proposed a replacement to the closure condition (5–88) which entails relaxing the collocated dynamics at the beginning and end of the time interval.

### 5.3 Convergence

The equivalence between the transformed KKT conditions of the NLP and the discretized form of the continuous-time first-order optimality conditions provides a way to approximate the costate and the Lagrange multipliers of an optimal control problem. Furthermore, the equivalence between the differential and implicit integration scheme establishes the fact that in using the differential form of a pseudospectral method to solve an optimal control problem, the state dynamics are actually being integrated numerically. However, a formal mathematical proof that shows that the solution to the discrete NLP converges to the optimal solution of the original continuous-time optimal control problem is still missing.
It has been shown in literature that many numerical integration schemes (for e.g., Euler methods, trapezoidal method and Runge-Kutta methods) have certain convergence properties. However, convergence of an integration method does not necessarily mean that the same discretization scheme will converge when used to solve an optimal control problem. In Ref. [21], it is shown that a convergent Runge-Kutta method may not actually converge to the continuous optimal solution when used to solve an optimal control problem. With the inclusion of a cost function, the problem ceases to be merely a system of differential equations that can be solved using convergent numerical integration schemes. Thus, the convergence proof for pseudospectral methods is a non-trivial proof, and is currently being studied by several researchers [30, 44, 45]. Ref. [30] implements the Radau pseudospectral method locally, i.e., the time interval is broken into subintervals and solution in each subinterval is approximated using the Radau pseudospectral method. Convergence to the local approach is shown using the more traditional approach to convergence, namely the error tends to zero as the number of subintervals approaches infinity.

For global pseudospectral approaches discussed in this dissertation, an approximation to the solution is obtained over the time interval treated as a single interval, so some other approach to convergence must be used. One approach would be to show convergence as the number of discretization points increases to infinity. In Ref. [44, 45], it is demonstrated that the Lobatto pseudospectral method leads to convergent approximations to the state and control variable. In Ref. [44], it is shown in “Covector Mapping Theorem” that any solution to the first-order optimality conditions for the continuous control problem, approximately satisfies the transformed KKT conditions of the discrete LGL problem, and the error tends to zero as $N \rightarrow \infty$. However, due to the null space in the discrete costate dynamics, the concept of convergence for the costate is open to interpretation.
In the numerical experiments discussed in Chapter 6, it is seen for the examples that have exact analytical solutions that the error in the solutions obtained from the Gauss pseudospectral method and the Radau pseudospectral method goes to zero at an exponential rate as the number of discretization points are increased. Furthermore, the number of discretization points at which the error goes to zero is shown to be small, supporting the fact that a coarse discretization may be used to obtain an accurate solution using either the Gauss pseudospectral method or the Radau pseudospectral method. On the other hand, it is shown that the costate solution obtained from the Lobatto pseudospectral method is typically nonconvergent. In the case where the LPM solution does converge, the errors are found to be two to three orders of magnitude higher for the LPM than for the GPM or the RPM. The number of discretization points at which machine precision is achieved for the LPM is usually higher than that for the GPM or the RPM.

5.4 Summary

A unified framework has been presented based on the Gauss and the Radau pseudospectral methods. It was demonstrated that each of these schemes can be expressed in either a differential or an integral formulation that are equivalent to each other. It was also demonstrated that the Lobatto pseudospectral method does not exhibit such an equivalence. Furthermore, it was shown that the discrete costate systems in both the Gauss and the Radau pseudospectral methods are full rank while the discrete costate system in Lobatto pseudospectral method has a null space. A rigorous mathematical proof of convergence for the Gauss and the Radau pseudospectral methods is still missing. In the Chapter 6, however, it is shown empirically that the error in the solutions obtained from the Gauss pseudospectral method and the Radau pseudospectral method goes to zero at an exponential rate as the number of discretization points are increased.
CHAPTER 6
FINITE-HORIZON OPTIMAL CONTROL EXAMPLES

In this chapter, a variety of examples are solved using, the Gauss, the Radau, and the Lobatto pseudospectral methods. Three main observations are made in the examples. First, it is seen that the Gauss and the Radau pseudospectral methods consistently generate accurate state, control and costate solutions, while the solution obtained from the Lobatto pseudospectral method is inconsistent and unpredictable. Specifically, it is shown that the costate obtained from the Lobatto pseudospectral method typically oscillates about the exact costate. In one of the examples, it is shown that the oscillation has a pattern similar to the pattern of the null space of the transformed LPM costate dynamics. Furthermore, it is demonstrated that, when an optimal control problem has an incomplete set of boundary conditions, the costate from the LPM oscillates whereas these oscillations are are not present when a problem consists of a fully defined set of boundary conditions. The disadvantage of the LPM not being an implicit integration scheme is also demonstrated in one example. Second, it is seen for the examples that have exact analytical solutions that the error in the solutions obtained from the Gauss pseudospectral method and the Radau pseudospectral method goes to zero at an exponential rate as the number of discretization points are increased. Furthermore, the number of discretization points at which the error goes to zero is shown to be small, supporting the fact that a coarse discretization may be used to obtain an accurate solution using either the Gauss pseudospectral method or the Radau pseudospectral method. On the other hand, it is shown that the costate solution obtained from the Lobatto pseudospectral method is typically nonconvergent. In the case where the LPM solution does converge, the errors are found to be two to three orders of magnitude higher for the LPM than for the GPM or the RPM. The number of discretization points at which machine precision is achieved for the LPM is usually higher than that for the GPM or the RPM. Third, it is shown that none of these methods
are well suited for solving problems that have discontinuities in the solution or where the solutions lie on a singular arc.

6.1 Example 1: Nonlinear One-Dimensional Initial-Value Problem

The first example considered is a nonlinear one-dimensional Mayer cost initial-value problem \[ \text{[63]} \]. Determine the state, \( y(t) \), and the control, \( u(t) \), on \( t \in [0, t_f] \) that minimize the cost functional

\[
J = -y(t_f),
\]

subject to the dynamic constraint

\[
\dot{y} = \frac{5}{2}(-y + yu - u^2),
\]

and the boundary condition

\[
y(0) = 1,
\]

where \( t_f = 2 \). The optimal solution to this problem is

\[
y^*(t) = 4/(1 + 3 \exp(5t/2)),
\]

\[
u^*(t) = y^*(t)/2,
\]

\[
\lambda_n^*(t) = -\exp(2 \ln(1 + 3 \exp(5t/2)) - 5t/2)/(\exp(-5) + 6 + 9 \exp(5)).
\]

It is noted that the given optimal control problem is one-dimensional initial-value problem, the state dynamics are nonlinear, final time is fixed and the cost functional consists of only the Mayer cost.

The example was solved using the Gauss, Radau, flipped Radau, and the Lobatto pseudospectral methods for \( N = 30 \) collocation points using the NLP solver SNOPT with optimality and feasibility tolerances of \( 10^{-15} \) and \( 2 \times 10^{-15} \), respectively. For each method, the initial guess was zero. Figs. 6-1, 6-2, 6-3, 6-4 show the solution obtained from each of the methods along with the exact solution. It is seen that the Lobatto costate oscillates about the exact solution while the Gauss and Radau costate are indistinguishable from the optimal solution. The oscillation of the Lobatto costate is due
A State solution.

B Control solution.

C Costate solution.

Figure 6-1. Solution obtained from the Gauss pseudospectral method for Example 1.

to the null space in the Lobatto costate dynamics discussed in Chapter 5. Since $n = 1$ in this problem, the dimension of the null space is 1. In Fig. 6-5A, the difference between the exact costate and the costate obtained from the LPM is plotted. In Fig. 6-5B a vector in the null space is plotted. Comparing Fig. 6-5A to Fig. 6-5B, we see that the oscillations in the Lobatto costate around the correct costate are essentially due to the addition of a vector in the null space. For this particular example an improved Lobatto
costate can be obtained by adding 0.4 times the null space vector given in Fig. 6-5B to the LPM costate estimate obtained from the NLP solver.

Next, the optimal control problem was solved for \( N = (5, 10, 15, \ldots, 30) \) collocation points for the Gauss, the Radau, and the Lobatto pseudospectral methods with exact solution as the initial guess. Figs. 6-6A–6-6C show the base 10 logarithm of
the $L_{\infty}$-norm errors for the state, control, and costate, respectively, defined as follows:

\[
E_y = \max_k \log_{10} \| y(\tau_k) - y^*(\tau_k) \|_{\infty}, \\
E_u = \max_k \log_{10} \| u(\tau_k) - u^*(\tau_k) \|_{\infty}, \\
E_{\lambda_y} = \max_k \log_{10} \| \lambda_y(\tau_k) - \lambda_y^*(\tau_k) \|_{\infty}.
\]
Fig. 6-6A shows that the state error using either the Gauss or Radau pseudospectral methods is approximately two to four orders of magnitude smaller than the state error for the Lobatto pseudospectral method for $N \leq 15$. In Figure 6-6B, it is seen that the Gauss and Radau control is between two and seven orders of magnitude more accurate than the corresponding Lobatto controls for $N \leq 15$. For $N > 15$, the Gauss and Radau state and control errors drop to machine precision (approximately $10^{-16}$), while the Lobatto errors achieve machine precision at $N = 30$. In Fig. 6-6C it is seen that the Gauss and
Figure 6-5. LPM costate error and null space for Example 1.
the Radau costate errors decrease to near the optimizer tolerances (approximately $10^{-15}$) while the Lobatto costate error remains above $10^{-2}$. 

Figure 6-6. Solution error vs. number of collocation points, $N$, for Example 1
6.2 Example 2: Nonlinear One-Dimensional Boundary-Value Problem

The second example considered is a nonlinear one-dimensional Bolza cost boundary-value problem [62]. Minimize the cost functional

\[ J = \frac{1}{2} \int_0^{t_f} (y + u^2) dt, \]  
(6–6)

subject to the dynamic constraint

\[ \dot{y} = 2y + 2u\sqrt{y}, \]  
(6–7)

and the boundary conditions

\[
\begin{align*}
y(0) &= 2, \\
y(t_f) &= 1, \\
t_f &= 5.
\end{align*}
\] 
(6–8)

It is noted that the given optimal control problem is one-dimensional in the state and the control, the state dynamics are nonlinear and there are no path constraints in the problem. Furthermore, there is a final boundary condition, the cost functional consists of a Lagrange cost and the final time is fixed.

The exact solution to the optimal control problem of (6–6)–(6–8) is given as

\[
\begin{align*}
y^*(t) &= x^2(t), \\
\lambda_y^*(t) &= \frac{\lambda_x(t)}{2\sqrt{y}}, \\
u^*(t) &= -\lambda_x(t),
\end{align*}
\] 
(6–9)

where \(x(t)\) and \(\lambda_x(t)\) are given as

\[
\begin{bmatrix}
x(t) \\
\lambda_x(t)
\end{bmatrix} = \exp(At) \begin{bmatrix}
x_0 \\
\lambda_{x0}
\end{bmatrix},
\] 
(6–10)
where

\begin{align*}
A &= \begin{bmatrix}
1 & -1 \\
-1 & -1 \\
\end{bmatrix} \\
\begin{bmatrix}
\begin{bmatrix}
1 \\
-1 \\
\end{bmatrix} & 1 \\
-1 & -1 \\
\end{bmatrix}
\end{align*}
(6–11)

\begin{align*}
x_0 &= \sqrt{2} \\
x_f &= 1 \\
\lambda_{x_0} &= \frac{x_f - B_{11}x_0}{B_{12}}
\end{align*}

and

\begin{align*}
B &= \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22} \\
\end{bmatrix} = \exp(At_f)
(6–12)
\end{align*}

The example was solved using the Gauss, Radau, flipped Radau, and the Lobatto pseudospectral methods for $N = 40$ collocation points using the NLP solver SNOPT with optimality and feasibility tolerances of $10^{-10}$. For each method, the initial guess was zero. Figs. 6-7, 6-8, 6-9, 6-10 show the solution obtained from each of the method along with the exact solution. It is seen that the state, control, and the costate solution obtained from the NLP for each of the methods is indistinguishable from the exact solution. It is interesting to note that in this problem where both the boundary conditions at the initial and the final times are known, the LPM costate does not demonstrate oscillations about the exact solution as was seen in the previous example.

Next, consider the following formulation of the original Lagrange cost problem to a Mayer cost problem. Let $v(t)$ be a component of the state vector such that the dynamics are defined as

\begin{align*}
\dot{v} &= \frac{1}{2}(v + u^2).
(6–13)
\end{align*}

with the initial condition

\begin{align*}
v(0) &= -2.
(6–14)
\end{align*}
Figure 6-7. Solution obtained from the Gauss pseudospectral method for Example 2.

As a result, the original one-dimensional Lagrange cost problem can be written as the following two-dimensional Mayer cost problem. Minimize the cost functional

\[ J = v(t_f) - v(0), \]  

(6–15)
subject to the dynamic constraints

\[ \dot{y} = 2y + 2u\sqrt{y}, \]
\[ \dot{v} = \frac{1}{2}(y + u^2), \]  

(6–16)

Figure 6-8. Solution obtained from the Radau pseudospectral method for Example 2.
Figure 6-9. Solution obtained from the flipped Radau pseudospectral method for Example 2.

and the boundary conditions

\[
\begin{align*}
    y(0) &= 2, \\
    v(0) &= -2, \\
    y(t_f) &= 1, \\
    t_f &= 5.
\end{align*}
\] (6–17)
Figure 6-10. Solution obtained from the Lobatto pseudospectral method for Example 2.

This modified problem was solved using the GPM, the RPM, and the LPM for $N = 40$ collocation points using the NLP solver SNOPT with optimality and feasibility tolerances of $10^{-15}$. Figs. 6-11A-6-11E show the solution obtained for the problem. It is seen that the state, control, and the first component of the costate solution obtained from the NLP for each of the methods is indistinguishable from the exact solution. The second component of the costate obtained from the LPM, however, shows oscillations about the optimal costate. The magnitude of error is small ($10^{-7}$) as compared to the magnitude
of error obtained in the previous example. This error in the costate obtained from the LPM can be attributed to the fact that the modified problem has an incomplete set of boundary conditions and that the LPM scheme is not an implicit integrator. If the LPM had been an implicit integrator, the cost written as a Mayer cost or a Lagrange cost would not have made any difference in the solution. Unlike the result obtained using the LPM, the GPM or the RPM costate is indistinguishable from the exact costate.

Next, the optimal control problem was solved for \( N = (5, 10, 15, \ldots, 50) \) collocation points for the Gauss, the Radau, and the Lobatto pseudospectral methods with the exact solution as the initial guess. Figs. 6-12A–6-12E show the base 10 logarithm of the \( L_\infty \)-norm errors for the state, control, and costate, respectively, defined as follows:

\[
\begin{align*}
E_y &= \max_k \log_{10} \| y(\tau_k) - y^*(\tau_k) \|_\infty, \\
E_v &= \max_k \log_{10} \| v(\tau_k) - v^*(\tau_k) \|_\infty, \\
E_u &= \max_k \log_{10} \| u(\tau_k) - u^*(\tau_k) \|_\infty, \\
E_{\lambda_y} &= \max_k \log_{10} \| \lambda_y(\tau_k) - \lambda_y^*(\tau_k) \|_\infty, \\
E_{\lambda_v} &= \max_k \log_{10} \| \lambda_v(\tau_k) - \lambda_v^*(\tau_k) \|_\infty.
\end{align*}
\] (6–18)

It is seen that \( E_y, E_v, E_u, \) and \( E_{\lambda_y} \) decrease in a linear manner from \( N = 5 \) to 50 for all three methods. Moreover, for \( N > 40 \) \( E_v \) and \( E_u \) reached machine precision (approximately \( 10^{-16} \)). Again, the linear rate of decrease of the base 10 logarithm error, \( E \), for the lower number of nodes demonstrates an exponential convergence rate.

Fig. 6-12D shows the second component of costate error, \( E_{\lambda_v} \). For \( N \leq 30 \), the costate from the GPM and the RPM are up to 10 orders of magnitude more accurate than that from the LPM. For \( N > 30 \), the costate from the GPM and the RPM are up to 5 orders of magnitude more accurate than the costate from the LPM.
A First component of state.

B Second component of state.

C First component of costate.

D Second component of costate.

E Control solution.

Figure 6-11. Solution obtained from the Gauss, the Radau, and the Lobatto pseudospectral method for $N = 40$ for modified Example 2.
A First component of state errors for Example.

B Second component of state errors for Example.

C First component of costate errors for Example.

D Second component of costate errors for Example.

E Control errors for Example.

Figure 6-12. Solution error vs. number of collocation points, \( N \), for Example 2
6.3 Example 3: Orbit-Raising Problem

The next example considered is multi-dimensional in the state with an incomplete set of boundary conditions. Consider the following orbit-raising optimal control problem found in Ref. [5]. Minimize the cost functional

\[ J = -r(t_f) \]  

subject to the dynamic constraints

\[
\begin{align*}
\dot{r} &= v_r, \\
\dot{\theta} &= v_\theta/r, \\
\dot{v}_r &= v_\theta^2/r - \mu/r^2 + a \sin \beta, \\
\dot{v}_\theta &= -v_r v_\theta/r + a \cos \beta,
\end{align*}
\]  

and the boundary conditions

\[
\begin{align*}
(r(0), \theta(0), v_r(0), v_\theta(0)) &= (1, 0, 0, 1), \\
(v_r(t_f), v_\theta(t_f)) &= (0, \sqrt{\mu/r(t_f)}),
\end{align*}
\]

where

\[ a \equiv a(t) = \frac{T}{m_0 - |\dot{m}|t}. \]

It is noted for this example that \( \mu = 1, \ T = 0.1405, \ m_0 = 1, \ \dot{m} = 0.0749, \) and \( t_f = 3.32. \)

The orbit-raising problem was solved using the GPM, RPM, and LPM for \( N = 64. \) For this problem, the optimal solution is not known. Therefore, only a qualitative comparison between the solutions is performed. The state, control (after an unwraping of the angle), and costate solutions for all the methods are shown in Figs. 6-13-6-15. First, it is observed that each of the methods produce qualitatively similar values for the state and the control. On comparing the costate approximations for different methods, however, we observe an oscillation in the LPM costate, which is likely due to the contributions in the null space associated with the discrete costate dynamics. In particular, it is seen that the GPM and the RPM produce a very accurate result for
\( \theta(t) \) while LPM produces a value for \( \lambda_\theta(t) \) that oscillates around zero. In addition, it is seen that \( \lambda_r(t) \) for LPM also oscillates (unlike the smooth behavior shown by the costate obtained from the GPM and the RPM). Thus, the GPM and the RPM differ significantly from the LPM in costate accuracy, demonstrating a fundamental difference in the nature of the costate estimates obtained using either the GPM or the RPM as compared with the LPM.
Figure 6-14. Solution obtained from the Radau pseudospectral method for Example 3.
Figure 6-15. Solution obtained from the Lobatto pseudospectral method for Example 3.
6.4 Example 4: Bryson Maximum Range Problem

The next example considered is a variation of the Bryson maximum range problem taken from Ref. [5]. In this example, both the state and the control are multi-dimensional and the problem contains an equality control path constraint. The problem is stated as follows: Maximize

$$J = x(t_f),$$

subject to the dynamic constraints

$$\dot{x} = vu_1,$$
$$\dot{y} = vu_2,$$
$$\dot{v} = g/2 - gu_2,$$

the boundary conditions

$$(x(0), y(0), \nu(0)) = (0, 0, 0),$$
$$y(t_f) = 0.1,$$

and the path constraint

$$u_1^2 + u_2^2 = 1.$$

This example was solved for $g = 1$, using the GPM, RPM, and LPM for $N = 40$. The state, control, and costate for each method are shown in Figs. 6-16–6-19. Again, for this problem, the optimal solution is not known. Therefore, only a qualitative comparison between the solutions is performed. First, it is observed that each of the methods produce qualitatively similar values for the state. On comparing the control and the costate approximations for different methods, however, we observe that the LPM control is quite different from the GPM or the RPM control. Furthermore, oscillations in the LPM costate are also observed. As with the previous examples, this example demonstrates the fundamental difference in accuracy between the GPM or RPM and the LPM. In this example it was found that the state for all three methods matched, but both the LPM
costate and LPM control were significantly different from the control and the costate solutions from the GPM and the RPM. The errors in the LPM control and costate are attributed to the fact that the LPM transformed adjoint system has a coupling between the collocated costate dynamics and the transversality conditions, the LPM costate dynamics system has a null space associated with it, and that the LPM is not an implicit integrator.
Figure 6-17. Solution obtained from the Radau pseudospectral method for Example 4.
Figure 6-18. Solution obtained from the flipped Radau pseudospectral method for Example 4.
Figure 6-19. Solution obtained from the Lobatto pseudospectral method for Example 4.
6.5 Example 5: Bang-Bang Control Problem

The next example considered is taken from Ref. [48] and has a “Bang-Bang” optimal control. Minimize the final time,

\[ J = t_f, \]  

(subject to the dynamic constraints

\[ \dot{x} = y(t), \]  
\[ \dot{y} = u(t), \]  

the boundary conditions

\[ x(0) = x_0, \]  
\[ y(0) = y_0, \]  
\[ x(t_f) = 0, \]  
\[ y(t_f) = 0, \]  

and the control inequality constraint

\[ |u(t)| \leq u_{max} = 1. \]

It is noted that the exact solution to the optimal control problem of (6–27)–(6–30) is obtained using the weak form of Pontryagin’s principle. The exact solution is given as

\[ x = \begin{cases} -\frac{t^2}{2} + y_0 t + x_0 & : t \leq t_1 \\ \frac{t^2}{2} - t_r t + \frac{t_r^2}{2} & : t > t_1 \end{cases} \]  

\[ y = \begin{cases} -t + y_0 & : t \leq t_1 \\ t - t_r & : t > t_1 \end{cases} \]  

\[ u = \begin{cases} -1 & : t \leq t_1 \\ 1 & : t \geq t_1 \end{cases} \]  

\[ \lambda_x = c_1 \]  
\[ \lambda_y = -c_1 t + c_2 \]
where the switching time \( t_1 \), final time \( t_f \), constants \( c_1 \) and \( c_2 \) are given as

\[
\begin{align*}
t_1 & = y_0 + \sqrt{0.5y_0^2 + x_0} \\
t_f & = 2t_1 - y_0 \\
c_1 & = \frac{1}{t_1 - t_f} \\
c_2 & = c_1 t_1.
\end{align*}
\]

The example was solved using the Gauss, Radau, flipped Radau, and Lobatto pseudospectral methods for \( x_0 = 1 \), \( y_0 = 3 \), and \( N = 40 \) collocation points using the NLP solver SNOPT with default optimality and feasibility tolerances of \( 10^{-6} \) and \( 2 \times 10^{-6} \), respectively. For each method, the initial guess for the first component of state was 1, the second component of state was 3, the control was \(-1\) and that for final time was 7. Figs. 6-20, 6-21, 6-22, 6-23 show the solution obtained from each of the method along with the exact solution. It is seen that the state solution obtained from the NLP for each of the method is indistinguishable from the exact solution. The control and the costate, however, are not exactly the same as those of the optimal solution. The error in the costate obtained from the Gauss, the Radau, and the flipped Radau pseudospectral methods is smaller than the error in the costate obtained from the Lobatto pseudospectral method. The costate from the LPM oscillates about the optimal costate.
Figure 6-20. Solution obtained from the Gauss pseudospectral method for Example 5.
Figure 6-21. Solution obtained from the Radau pseudospectral method for Example 5.
Figure 6-22. Solution obtained from the flipped Radau pseudospectral method for Example 5.
Figure 6-23. Solution obtained from the Lobatto pseudospectral method for Example 5.
Next, the optimal control problem was solved for \( N = (5, 10, 15, \ldots, 50) \) collocation points for the Gauss, Radau, and Lobatto pseudospectral methods. Figs. 6-24A–6-24E show the base 10 logarithm of the \( L_\infty \)-norm errors for the state, control, and costate, respectively, defined as follows:

\[
\begin{align*}
E_x &= \max_k \log_{10} ||x(\tau_k) - x^*(\tau_k)||_\infty, \\
E_y &= \max_k \log_{10} ||y(\tau_k) - y^*(\tau_k)||_\infty, \\
E_u &= \max_k \log_{10} ||u(\tau_k) - u^*(\tau_k)||_\infty, \\
E_{\lambda x} &= \max_k \log_{10} ||\lambda_x(\tau_k) - \lambda_x^*(\tau_k)||_\infty, \\
E_{\lambda y} &= \max_k \log_{10} ||\lambda_y(\tau_k) - \lambda_y^*(\tau_k)||_\infty.
\end{align*}
\] (6–37)

It is seen that \( E_x, E_y, E_{\lambda x}, E_{\lambda y}, \) and \( E_u \) show significant errors in the solution for all three methods. Although, the Gauss and the Radau pseudospectral methods outperform the Lobatto pseudospectral method, neither one of the methods shows much improvement as the number of collocation points are increased. The divergence from the typical exponential behavior can be explained by examining the discontinuities in the solution. At the switching point, the control is discontinuous, the first state has a discontinuous second derivative, and the second state has a discontinuous first derivative. Pseudospectral methods use polynomials to approximate the state. The polynomial approximations are unable to approximate non-smooth functions accurately. As a consequence, global pseudospectral methods are not well suited for problems with discontinuities in the solution or in the derivatives of the solution.
Figure 6-24. Solution error vs. number of collocation points, \( N \), for Example 5
6.6 Example 5: Singular Arc Problem

The next example considered is a problem whose optimal solution lies on a singular arc and is taken from Ref. [48]. The problem is stated as follows. Minimize

\[ J = \frac{1}{2} \int_0^{t_f} (x^2 + y^2) \, dt, \tag{6–38} \]

subject to the dynamic constraints

\[ \dot{x} = y(t), \]
\[ \dot{y} = u(t), \tag{6–39} \]

the boundary conditions

\[ x(0) = 0.8, \quad y(0) = 0.8, \tag{6–40} \]
\[ x(t_f) = 0.01, \quad y(t_f) = -0.01, \tag{6–41} \]

and the control inequality constraint

\[ |u(t)| \leq u_{max} = 1. \tag{6–42} \]

The exact solution is given as

\[ u^*(t) = 0.8 \exp(-t), \]
\[ x^*(t) = 0.8 \exp(-t), \]
\[ y^*(t) = -0.8 \exp(-t), \tag{6–43} \]
\[ \lambda^*_x(t) = 0.8 \exp(-t), \]
\[ \lambda^*_y(t) = 0. \]

The optimal time is obtained as

\[ t^*_f = -\log \left( \frac{1}{80} \right) \tag{6–44} \]

170
The example was solved using the Gauss, Radau, flipped Radau, and Lobatto pseudospectral methods for $N = 30$ collocation points using the NLP solver SNOPT with default optimality and feasibility tolerances of $10^{-6}$ and $2 \times 10^{-6}$, respectively. For each method, the initial guess for the first component of state was 0.8, the second component of state was $-0.8$, the control was 1 and that for final time was 4. Figs. 6-25–6-28 show the solution obtained from each of the method along with the exact solution. It is seen
that the approximate solution for the state and costate is a relatively good approximation to the exact solution while the control obtained from each of the methods is inaccurate.

Next, the optimal control problem was solved for \( N = (5, 10, 15, \ldots, 50) \) collocation points for the Gauss, Radau, and Lobatto pseudospectral methods. Figs. 6-29A–6-29E show the base 10 logarithm of the \( L_\infty \)-norm errors for the state, control, and costate,
respectively, defined as follows:

\[
egin{align*}
E_x &= \max_k \log_{10} \|x(\tau_k) - x^*(\tau_k)\|_\infty, \\
E_y &= \max_k \log_{10} \|y(\tau_k) - y^*(\tau_k)\|_\infty, \\
E_u &= \max_k \log_{10} \|u(\tau_k) - u^*(\tau_k)\|_\infty, \\
E_{\lambda_x} &= \max_k \log_{10} \|\lambda_x(\tau_k) - \lambda^*_x(\tau_k)\|_\infty, \\
E_{\lambda_y} &= \max_k \log_{10} \|\lambda_y(\tau_k) - \lambda^*_y(\tau_k)\|_\infty.
\end{align*}
\]  

(6–45)
Figure 6-28. Solution obtained from the Lobatto pseudospectral method for Example 6.

It is seen that $E_x$, $E_y$, $E_{\lambda_x}$, $E_{\lambda_y}$, and $E_u$ are not converging as $N$ is increased demonstrating that these methods are not suitable to solve problems where optimal solution lies on a singular arc.
Figure 6-29. Solution error vs. number of collocation points, $N$, for Example 6.
6.7 Summary

It has been shown that typically the LPM costate approximation is found to have an error that oscillates about the exact solution, and this error was shown by example to be due to the null space in the LPM discrete costate dynamics. Empirical evidence has suggested that the Gauss and the Radau pseudospectral methods converge rapidly (exponentially) for a large class of problems and give a better costate estimate than the Lobatto pseudospectral method. However, it has been shown that these methods are not well suited for solving problems that have discontinuities in the solution or discontinuities in the derivatives of the solution, and problems that contain singular arcs.
CHAPTER 7
INFINITE-HORIZON OPTIMAL CONTROL PROBLEMS

Certain processes must be controlled indefinitely. Such processes are described on an infinite time domain, \([0, \infty)\). In this chapter, two direct pseudospectral methods are described for solving infinite-horizon optimal control problems numerically using the Legendre-Gauss (LG) and the Legendre-Gauss-Radau (LGR) collocation. An important aspect of numerically approximating the solution of an infinite-horizon optimal control problem is the manner in which the horizon is treated. Finding the state of the system at \(t = \infty\) is a challenge while solving infinite-horizon optimal control problems numerically.

To solve an infinite-horizon optimal control problem numerically, the infinite-horizon optimal control problem must be transformed to a finite-horizon problem. In this research, a smooth, strictly monotonic transformation \(t = \phi(\tau)\) is used to map the infinite time domain \(t \in [0, \infty)\) onto the domain \(\tau \in [-1, 1)\). The resulting problem on the finite interval is transcribed to a nonlinear programming problem using collocation. Two methods that employ collocation at the LG and the LGR points, called the infinite-horizon Gauss pseudospectral method and the infinite-horizon Radau pseudospectral method, are described in this chapter. Furthermore, it is shown that collocation based on the LGL points is not suitable for solving infinite-horizon optimal control problems.

It is noted that an LGR pseudospectral method for approximating the solution of nonlinear infinite-horizon optimal control problems has been previously developed in Ref. [65]. The methods presented in this dissertation are fundamentally different from the method of Ref. [65]. First, the proposed methods yield approximations to the state and the costate on the entire horizon, including approximations at \(t = +\infty\). Second, similar to the Gauss and the Radau pseudospectral methods for finite-horizon problems, these methods can also be written equivalently in either a differential or an implicit integral form. Third, in the proposed methods a general change of variables \(t = \phi(\tau)\) of an infinite-horizon problem to a finite-horizon problem is considered. It is
shown that the map $\phi : [-1, +1) \to [0, +\infty)$ can be tuned to improve the quality of the discrete approximation. In numerical examples, the discrete solution exhibits exponential convergence as a function of the number of collocation points.

### 7.1 Infinite-Horizon Optimal Control Problem

Consider the infinite-horizon optimal control problem: Minimize the cost functional

$$ J = \int_{0}^{\infty} g(y(t), u(t)) dt, \quad (7–1) $$

subject to the dynamic constraints

$$ \dot{y}(t) = f(y(t), u(t)), \quad (7–2) $$

the boundary conditions

$$ y(0) - y_0 = 0, \quad (7–3) $$

and the inequality path constraints

$$ C(y(t), u(t)) \leq 0, \quad (7–4) $$

where all vector functions of time are row vectors; that is, $y(t) = [y_1(t) \cdots y_n(t)] \in \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $C : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^s$, and $\dot{y}$ denotes the time derivative of $y$. It is noted that generally an infinite-horizon optimal control problem does not consist of a Mayer cost term and the independent variable, i.e., time is not present as an explicit variable in the problem. In this problem, the initial and the final times are treated as fixed at 0 and $\infty$, respectively.

The infinite time domain, $[0, \infty)$ is now mapped to $[-1, 1)$ using the change of variables $t = \phi(\tau)$ where $\phi$ is a differentiable, strictly monotonic function of $\tau$ that maps
the interval $[-1, 1)$ onto $[0, \infty)$. Three examples of such a function are

$$\phi_a(\tau) = \frac{1 + \tau}{1 - \tau}, \quad (7-5)$$

$$\phi_b(\tau) = \log_e \left( \frac{2}{1 - \tau} \right), \quad (7-6)$$

$$\phi_c(\tau) = \log_e \left( \frac{4}{(1 - \tau)^2} \right). \quad (7-7)$$

The change of variables $\phi_a(\tau)$ was originally proposed in Ref. [65], while the transformations $\phi_b(\tau)$ and $\phi_c(\tau)$ are introduced in this research. These latter changes of variables produce slower growth in $t = \phi(\tau)$ as $\tau$ approaches $+1$, than that of $\phi_a(\tau)$ as shown in Figure 7-1. Better discretization can be achieved by tuning the change of variables to the problem. For example, suppose the exact state of a problem is given as

$$y(t) = \exp \left( 1 - \frac{2}{e^t} \right). \quad (7-8)$$

Figure 7-2 shows the growth of $y(t)$. As is seen in the Figure, $y(t)$ changes slowly when $t$ is large. The logarithmic functions of $\phi_b(\tau)$ and $\phi_c(\tau)$ essentially move collocation points associated with large values of $t$ to the left. In the Figure, collocation points corresponding to $\phi_b(\tau)$ are shown. This leftward movement of the collocation points is beneficial since more collocation points are situated where the solution is changing most rapidly.

Next, define

$$T(\tau) = \frac{d\phi}{d\tau} \equiv \phi'(\tau). \quad (7-9)$$

After changing variables from $t$ to $\tau$, the infinite-horizon optimal control problem becomes

$$J = \int_{-1}^{+1} T(\tau) g(y(\tau), u(\tau)) d\tau, \quad (7-10)$$

subject to the dynamic constraints

$$\frac{dy(\tau)}{d\tau} = \dot{y}(\tau) = T(\tau)f(y(\tau), u(\tau)). \quad (7-11)$$
Figure 7-1. Growth in $\phi(\tau)$ at 40 LG points.
Figure 7-2. Growth of $y(t)$ and location of 40 collocation points using $\phi_b(\tau)$.

the boundary conditions

$$y(-1) - y_0 = 0.$$ (7–12)

and the inequality path constraints

$$T(\tau)C(y(\tau), u(\tau)) \leq 0.$$ (7–13)

It is noted that, $T(\tau)$, is multiplied to Eq. (7–13) without actually affecting the constraint so that the optimality conditions can be posed in a succinct manner. Here $y(\tau)$ and $u(\tau)$ denote the state and the control as a function of the new variable $\tau$. 
The first-order optimality conditions for the finite horizon control problem in Eqs. (7–10)-(7–13), are

\[ \dot{y}(\tau) = T(\tau)\nabla_\lambda H, \quad (7-14) \]

\[ \dot{\lambda}(\tau) = -T(\tau)\nabla_y H, \quad (7-15) \]

\[ 0 = \nabla_u H, \quad (7-16) \]

\[ \lambda(-1) = \psi, \quad (7-17) \]

\[ \lambda(+1) = 0, \quad (7-18) \]

\[ \gamma_i(\tau) = 0 \text{ when } C_i(y(\tau), u(\tau)) < 0, \quad 1 \leq i \leq s, \quad (7-19) \]

\[ \gamma_i(\tau) < 0 \text{ when } C_i(y(\tau), u(\tau)) = 0, \quad 1 \leq i \leq s, \quad (7-20) \]

\[ \phi = 0, \quad (7-21) \]

where the Hamiltonian \( H \) is defined as

\[ H(y(\tau), u(\tau), \lambda(\tau), \gamma(\tau)) = g(y(\tau), u(\tau)) + \langle \lambda(\tau), f(y(\tau), u(\tau)) \rangle \]

\[ -\langle \gamma(\tau), C(y(\tau), u(\tau)) \rangle. \quad (7-22) \]

### 7.2 Infinite-Horizon Gauss Pseudospectral Method

In this section, the discrete approximation to the nonlinear infinite-horizon optimal control problem is formulated using global collocation at Legendre-Gauss points. The state at the horizon is included by quadrature. Equivalence between the transformed KKT conditions and discrete version of the first-order optimality conditions is established. The equivalent implicit integration scheme is also derived.

#### 7.2.1 NLP Formulation

Consider the LG collocation points \((\tau_1, \ldots, \tau_N)\) on the interval \((-1, 1)\) and two additional noncollocated points \(\tau_0 = -1\) (the initial time) and \(\tau_{N+1} = 1\) (the terminal time, corresponding to \(t = +\infty\)). The state is approximated by a polynomial of degree at most
\[ y(\tau) \approx Y(\tau) = \sum_{i=0}^{N} Y_i L_i(\tau), \]  
(7–23)

where \( Y_i \in \mathbb{R}^n \) and \( L_i(\tau) \) is a Lagrange polynomial of degree \( N \) defined as

\[ L_i(\tau) = \prod_{j=0}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 0, \ldots, N). \]  
(7–24)

The basis includes the function \( L_0 \) corresponding to the initial time \( \tau_0 = -1 \), but not a function corresponding to \( \tau_{N+1} = +1 \). Differentiating the series of Eq. (7–23) with respect to \( \tau \), an approximation to the derivative of the state in \( \tau \) domain is obtained as

\[ \dot{y}(\tau) \approx \dot{Y}(\tau) = \sum_{i=0}^{N} Y_i \dot{L}_i(\tau). \]  
(7–25)

The following collocation conditions are then formed by equating the derivative of the state approximation in Eq. (7–25) to the right-hand side of the state dynamic constraints in Eq. (7–11) at the \( N \) LG points, \((\tau_1, \ldots, \tau_N)\):

\[ \sum_{i=0}^{N} Y_i \dot{L}_i(\tau_k) = T(\tau_k) f(Y_k, U_k), \quad (k = 1, \ldots, N), \]  
(7–26)

\[ \sum_{i=0}^{N} D_{ki} Y_i = T(\tau_k) f(Y_k, U_k), \quad D_{ki} = \dot{L}_i(\tau_k), \]  
(7–27)

where it is noted that \( \tau_0 \) is not a collocation point. Hence, the discrete approximation to the system dynamics is

\[ D_k Y^{LG} = T(\tau_k) f(Y_k, U_k), \quad 1 \leq k \leq N, \]  
(7–28)

where \( D_k \) is the \( k^{th} \) row of differentiation matrix \( D = [D_{ki}] \), \((1 \leq k \leq N), \ (0 \leq i \leq N)\), called the Gauss pseudospectral differentiation matrix and \( Y^{LG} \) is defined as

\[ Y^{LG} = \begin{bmatrix} Y_0 \\ \vdots \\ Y_N \end{bmatrix}. \]
Thus, the discrete approximation to the system dynamics $\dot{y}(\tau) = T(\tau)f(y(\tau), u(\tau))$ is obtained by evaluating the system dynamics at each collocation point and replacing $y(\tau_k)$ by its discrete approximation $D_k Y^{LG}$. It is important to observe that the left-hand side of Eq. (7–28) contains approximations for the state at the initial point plus the LG points while the right-hand side contains approximations for the state (and control) at only the LG points. Next, it is again noted that the Gauss differentiation matrix $D$ has the following properties: if $D = [D_0 \ D_{1:N}]$ where $D_0$ is the first column of $D$ and $D_{1:N}$ are the remaining columns, then $D$ is such that: (a) $D_{1:N}$ is nonsingular and (b) $D_0 = -D_{1:N}1$; equivalently, $-D_{1:N}^{-1}D_0 = 1$, where $1$ is a column vector of all ones.

The objective function in Eq. (7–10) is approximated by the Legendre-Gauss quadrature as

$$J = \int_{-1}^{+1} T(\tau)g(y(\tau), u(\tau))d\tau \approx \sum_{k=1}^{N} T(\tau_k)w_k g(Y_k, U_k),$$

(7–29)

where $w_k$ is the quadrature weight associated with $\tau_k$. Next, we have

$$y(+1) = y(-1) + \int_{-1}^{+1} T(\tau)f(y(\tau), u(\tau))d\tau.$$  

(7–30)

Eq. (7–30) can be approximated using the LG quadrature as

$$Y_{N+1} = Y_0 + \sum_{k=1}^{N} w_k T(\tau_k)f(Y_k, U_k),$$

(7–31)

where $Y_{N+1}$ is treated as an additional variable. Rearranging Eq. (7–31), the following equality constraint is then added in the discrete approximation:

$$Y_{N+1} - Y_0 - \sum_{k=1}^{N} w_k T(\tau_k)f(Y_k, U_k) = \mathbf{0}.$$  

(7–32)

Lastly, the path constraints in Eq. (7–13) are enforced at the $N$ LG collocation points as

$$T(\tau_k)c(Y_k, U_k) \leq 0, \quad (k = 1, \ldots, N).$$

(7–33)
The continuous-time nonlinear infinite-horizon optimal control problem of Eqs. (7–10)-(7–13) is then approximated by the following NLP: Minimize the cost

\[ J = \sum_{k=1}^{N} T(\tau_k)w_k g(Y_k, U_k), \quad (7–34) \]

subject to the following equality and inequality constraints

\[ D_k Y^{LG} - T(\tau_k)f(Y_k, U_k) = 0, \quad (k = 1, \ldots, N), \quad (7–35) \]

\[ Y_{N+1} - Y_0 - \sum_{k=1}^{N} w_k T(\tau_k)f(Y_k, U_k) = 0, \quad (7–36) \]

\[ Y_0 - y_0 = 0, \quad (7–37) \]

\[ T(\tau_k)C(Y_k, U_k) \leq 0, \quad (k = 1, \ldots, N), \quad (7–38) \]

where the NLP variables are \((Y_0, \ldots, Y_{N+1})\) and \((U_1, \ldots, U_N)\). It is noted that, the singularity in the change of variables \(t = \phi(\tau)\) and in \(T(\tau) = \phi'(\tau)\) at \(\tau = +1\) is avoided in the NLP formulation. \(T(\tau) = \phi'(\tau)\) is never evaluated at the singularity in Eqs. (7–34)-(7–38), rather \(T(\tau)\) is evaluated at the quadrature points only which are all strictly less than 1. Furthermore, because \(Y_{N+1}\) is a NLP variable, the solution of state at the final point, i.e., at \(t = \infty\) is also obtained.

### 7.2.2 Karush-Kuhn-Tucker Conditions

The necessary optimality conditions or the Karush-Kuhn-Tucker (KKT) conditions, of the NLP given in Eqs. (7–34)-(7–38) are now derived. The Lagrangian associated with the NLP is

\[ L = -\langle \Psi, Y_0 - y_0 \rangle \]

\[ + \sum_{k=1}^{N} T(\tau_k)(w_k g(Y_k, U_k) - \langle \Gamma_k, C(Y_k, U_k) \rangle) \]

\[ - \sum_{k=1}^{N} \langle \Lambda_k, D_k Y^{LG} - T(\tau_k)f(Y_k, U_k) \rangle \]

\[ - \langle \Lambda_{N+1}, Y_{N+1} - Y_0 - \sum_{k=1}^{N} T(\tau_k)w_k f(Y_k, U_k) \rangle, \quad (7–39) \]
where $\Lambda_k$ is the $k^{th}$ row of the Lagrange multipliers matrix $\Lambda \in \mathbb{R}^{N \times n}$ associated with the constraints in Eq. (7–35), $\Lambda_{N+1} \in \mathbb{R}^n$ are the Lagrange multipliers associated with the constraints in Eq. (7–36), $\Psi \in \mathbb{R}^q$ are the Lagrange multipliers associated with the constraints in Eq. (7–37), and $\Gamma_k$ is the $k^{th}$ row of the Lagrange multipliers matrix $\Gamma \in \mathbb{R}^{N \times s}$ associated with the constraints in Eq. (7–38). The KKT optimality conditions are then obtained by differentiating the Lagrangian with respect to each of the variable and equating the derivative to zero, such that

$$T(\tau_k) \nabla Y_k(w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle) = D_k^T \Lambda, \quad 1 \leq k \leq N, \quad (7–40)$$

$$\Psi = \Lambda_{N+1} - D_0^T \Lambda, \quad (7–41)$$

$$0 = \Lambda_{N+1}, \quad (7–42)$$

$$\nabla U_k(w_k g_k + \langle \Lambda_k + w_k \Lambda_{N+1}, f_k \rangle - \langle \Gamma_k, C_k \rangle) = 0, \quad 1 \leq k \leq N, \quad (7–43)$$

$$D_k Y^{LG} - T(\tau_k) f(Y_k, U_k) = 0, \quad 1 \leq k \leq N, \quad (7–44)$$

$$\Gamma_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, 1 \leq k \leq N, \quad (7–45)$$

$$\Gamma_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, 1 \leq k \leq N, \quad (7–46)$$

$$Y_0 - y_0 = 0, \quad (7–47)$$

where $D_i^T$ is the $i^{th}$ row of $D^T$, $g_k = g(Y_k, U_k)$, $f_k = f(Y_k, U_k)$ and $C_k = C(Y_k, U_k)$.

Next, the KKT conditions given in Eqs. (7–40)-(7–47) are reformulated so that they become a discretization of the first-order optimality conditions given in Eqs. (7–14)-(7–21) for the continuous control problem given in Eqs. (7–10)-(7–13).

Let $D^\dagger = [D^\dagger_{ij}]$, $(1 \leq i \leq N)$, $(1 \leq i \leq N+1)$ be the $N \times (N+1)$ matrix defined as follows:

$$D^\dagger_{ij} = -\frac{w_i}{w_j} D_{ij}, \quad (i, j) = 1, \ldots, N, \quad (7–48)$$

$$D^\dagger_{i,N+1} = -\sum_{j=1}^{N} D^\dagger_{ij}, \quad i = 1, \ldots, N. \quad (7–49)$$
It is noted again that the matrix $D^\dagger$ defined in (7–48) and (7–49) is a differentiation matrix for the space of polynomials of degree $N$. According to the definition of $D^\dagger$,

$$D_k^T = -w_k D_{k,1:N}^\dagger W^{-1}, \quad 1 \leq k \leq N, \quad (7–50)$$

where $W$ is a diagonal matrix with weights $w_k$, $1 \leq k \leq N$, on the diagonal. Substituting Eq. (7–50) in Eq. (7–40),

$$-w_k D_{k,1:N}^\dagger W^{-1} \Lambda = T(\tau_k) \nabla_{\chi_k}(w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle), \quad 1 \leq k \leq N. \quad (7–51)$$

Next, defining the following change of variables:

$$\tilde{\lambda}_{N+1} = \Lambda_{N+1}, \quad (7–52)$$

$$\tilde{\lambda}_0 = \Lambda_{N+1} - D_0^T \Lambda, \quad (7–53)$$

$$\tilde{\lambda}_k = \frac{\Lambda_k}{w_k} + \Lambda_{N+1}, \quad 1 \leq k \leq N, \quad (7–54)$$

$$\tilde{\gamma}_k = \frac{\Gamma_k}{w_k}, \quad 1 \leq k \leq N, \quad (7–55)$$

$$\tilde{\psi} = \Psi. \quad (7–56)$$

Substituting Eqs. (7–52)-(7–56) in Eqs. (7–41)-(7–47) and in Eq. (7–51), the transformed KKT conditions of the NLP are given as

$$0 = \nabla_{\chi_k}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 1 \leq k \leq N, \quad (7–57)$$

$$0 = D_k Y^\text{LG} - T(\tau_k) f(Y_k, U_k), \quad 1 \leq k \leq N, \quad (7–58)$$

$$\tilde{\gamma}_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (7–59)$$

$$\tilde{\gamma}_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (7–60)$$

$$0 = Y_0 - y_0, \quad (7–61)$$

$$\tilde{\lambda}_0 = \tilde{\psi}, \quad (7–62)$$

$$\tilde{\lambda}_{N+1} = 0, \quad (7–63)$$

$$D_{k,1:N}^\dagger \tilde{\lambda} + D_{k,N+1}^\dagger \tilde{\lambda}_{N+1} = -T(\tau_k) \nabla_{\chi_k}(g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 1 \leq k \leq N. \quad (7–64)$$
Now, consider a comparison of the transformed KKT conditions in Eqs. (7–57)-(7–64) of the NLP to the first-order necessary optimality conditions in Eqs. (7–14)-(7–21) of the continuous-time optimal control problem. It is noted that the transformed KKT conditions in Eqs. (7–57)-(7–61) are the discretized forms of the continuous-time first-order optimality conditions in Eq. (7–16), Eq. (7–14), Eq. (7–19), Eq. (7–20), and Eq. (7–21), respectively. Furthermore, it is shown in Theorem 1 that the system (7–64) is a pseudospectral scheme for the costate dynamics, i.e.

\[
D_{k,1:N}^\dagger \tilde{\lambda} + D_{k,N+1}^\dagger \tilde{\lambda}_{N+1} = \dot{\tilde{\lambda}}_k, \quad 1 \leq k \leq N.
\]  

(7–65)

Therefore, the left hand side of Eq. (7–64) is an approximation of costate dynamics at the \(k^{th}\) collocation point. As a result, Eq. (7–64) represents the discretized version of the costate dynamics in Eq. (7–15) at \(k = (1, \ldots, N)\). Lastly, it is noted that at the boundary points, the discrete equivalents of continuous boundary conditions (7–17) and (7–18) are the same as the discrete costate at the boundary points in (7–62) and (7–63), respectively. Hence, the system of transformed KKT conditions of the NLP is exactly equivalent to the first-order optimality conditions of the continuous-time optimal control problem. Therefore, accurate costate estimates can be obtained from the KKT multipliers using the relationship given in Eqs. (7–52)-(7–54) as

\[
\lambda_{N+1} \approx \tilde{\lambda}_{N+1} = \Lambda_{N+1},
\]

(7–66)

\[
\lambda_0 \approx \tilde{\lambda}_0 = \Lambda_{N+1} - D_0^T \Lambda,
\]

(7–67)

\[
\lambda_k \approx \tilde{\lambda}_k = \frac{\Lambda_k}{w_k} + \Lambda_{N+1}, \quad 1 \leq k \leq N,
\]

(7–68)

\[
\gamma_k \approx \tilde{\gamma}_k = \frac{\Gamma_k}{w_k}, \quad 1 \leq k \leq N,
\]

(7–69)

\[
\psi \approx \tilde{\psi} = \Psi.
\]

(7–70)
7.2.3 Equivalent Implicit Integration Scheme

It is now shown that the LG pseudospectral discretization of the state equation has an equivalent integrated formulation. Let \( p \) be any polynomial of degree at most \( N \). By the construction of the \( N \times (N + 1) \) matrix \( D \), we have \( Dp = \dot{p} \) where

\[
\begin{align*}
p_k &= p(\tau_k), \quad 0 \leq k \leq N, \quad (7-71) \\
\dot{p}_k &= \dot{p}(\tau_k), \quad 1 \leq k \leq N. \quad (7-72)
\end{align*}
\]

Let \( D = [D_0 \ D_{1:N}] \) where \( D_0 \) is the first column of \( D \) and \( D_{1:N} \) are the remaining columns. Then the identity \( p = Dp \) can be written as

\[
\dot{p} = D_0p_0 + D_{1:N}p_{1:N}. \quad (7-73)
\]

Multiplying by \( D_{1:N}^{-1} \) and utilizing Proposition 5 gives

\[
p_k = p_0 + (D_{1:N}^{-1}\dot{p})_k, \quad 1 \leq k \leq N. \quad (7-74)
\]

Next, a different expression for \( p_k - p_0 \) based on the Lagrange polynomial approximation of the derivative is obtained. Let \( L_i^\dagger(\tau) \) be the Lagrange interpolation polynomials associated with the collocation points:

\[
L_i^\dagger = \prod_{\substack{j=1 \atop j \neq i}}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad 1 \leq i \leq N. \quad (7-75)
\]

Notice that the Lagrange polynomials \( L_i \) defined in (7–24) are degree \( N \) while the Lagrange polynomials \( L_i^\dagger \) are degree \( N - 1 \). Then because \( \dot{p} \) is a polynomial of degree at most \( N - 1 \), it can be approximated exactly by the Lagrange polynomials \( L_i^\dagger \):

\[
\dot{p} = \sum_{i=1}^{N} \dot{p}_i L_i^\dagger(\tau). \quad (7-76)
\]
Integrating \( \dot{p} \) from \(-1\) to \( \tau_k \), following relationship is obtained

\[
p(\tau_k) = p(-1) + \sum_{i=1}^{N} \dot{p}_i A_{ki},
\]

\( A_{ki} = \int_{-1}^{\tau_k} L_i^1(\tau) d\tau, \quad 1 \leq k \leq N. \) \hspace{1cm} (7–77)

Utilizing the notation (7–71) and (7–72), we have

\[
p_k = p_0 + (A \dot{p})_k, \quad 1 \leq k \leq N. \]

(7–78)

The relations (7–74) and (7–78) are satisfied for any polynomial of degree at most \( N \).

Equating (7–74) and (7–78) to obtain

\[
A \dot{p} = D_{1:N}^{-1} p.
\]

Choose \( \dot{p} \) from the columns of the identity matrix to deduce that \( A = D_{1:N}^{-1} \).

Rewriting Eq. (7–28) such that

\[
D_0 Y_0 + D_{1:N} \begin{bmatrix} Y_1 \\
\vdots \\
Y_N \end{bmatrix} = \begin{bmatrix} T(\tau_1)f(Y_1, U_1) \\
\vdots \\
T(\tau_N)f(Y_N, U_N) \end{bmatrix}. \]

(7–79)

Multiplying by \( A = D_{1:N}^{-1} \) and utilizing Proposition 5 to obtain

\[
Y_k = Y_0 + \sum_{i=1}^{N} A_{ki} T(\tau_i)f(Y_i, U_i), \quad 1 \leq k \leq N. \]

(7–80)

Hence, the Eq. (7–80) is the equivalent implicit integral form of the state equation in (7–28). The elements of \( A \) are the integrals of the Lagrange basis \( L_i^1 \), while the elements of \( D \) are the derivatives of the Lagrange basis \( L_i \) defined in (7–24). In the implicit integration scheme, the state at the horizon is approximated using Gauss quadrature in Eq. (7–36) as in the differential scheme. Computationally, the differential formulation of Eq. (7–28) of the system dynamics is more convenient since any nonlinear terms in \( f \)
retain their sparsity in the discretization, while for the integrated version of Eq. (7–80), the nonlinear terms are nonsparse due to multiplication by the dense matrix \( A \).

7.3 Infinite-Horizon Radau Pseudospectral Method

In this section discrete approximation to the nonlinear infinite-horizon optimal control problem is formulated. This discrete scheme is based on global collocation using the Legendre-Gauss-Radau collocation points. The state at the horizon is included in the state approximation. Equivalence between the transformed KKT conditions and discrete version of the first-order optimality conditions is established. The equivalent implicit integration scheme is also derived.

7.3.1 NLP Formulation

Consider the LGR collocation points \((\tau_1, \ldots, \tau_N)\) on the interval \([-1, 1]\) where \(\tau_1 = -1\) and one additional noncollocated point \(\tau_{N+1} = 1\) (the terminal time, corresponding to \(t = +\infty\)). The state is approximated by a polynomial of degree at most \(N\) as

\[
y(\tau) \approx Y(\tau) = \sum_{i=1}^{N+1} Y_i L_i(\tau),
\]

(7–81)

where \(Y_i \in \mathbb{R}^n\) and \(L_i(\tau)\) is a Lagrange polynomial of degree \(N\) defined as

\[
L_i(\tau) = \prod_{j=1, j \neq i}^{N+1} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 1, \ldots, N+1).
\]

(7–82)

The basis includes the function \(L_{N+1}\) corresponding to the final time \(\tau_{N+1} = -1\). Differentiating the series of Eq. (7–81) with respect to \(\tau\), an approximation to the derivative of the state in \(\tau\) domain is obtained as

\[
y(\tau) \approx \dot{Y}(\tau) = \sum_{i=1}^{N+1} Y_i \dot{L}_i(\tau).
\]

(7–83)

The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (7–83) to the right-hand side of the state dynamic constraints in
Eq. (7–11) at the $N$ LGR points, $(\tau_1, \ldots, \tau_N)$.

\begin{align}
\sum_{i=1}^{N+1} Y_i \dot{L}_i(\tau_k) &= T(\tau_k)f(Y_k, U_k), \quad (k = 1, \ldots, N), \quad (7–84) \\
\sum_{i=1}^{N+1} D_{ki} Y_i &= T(\tau_k)f(Y_k, U_k), \quad D_{ki} = L_i(\tau_k). \quad (7–85)
\end{align}

It is noted that $\tau_{N+1}$ is not a collocation point. Hence, the discrete approximation to the system dynamics is

$$D_k Y^{\text{LGR}} = T(\tau_k)f(Y_k, U_k), \quad 1 \leq k \leq N,$$

where $D_k$ is the $k^{\text{th}}$ row of differentiation matrix $D = [D_{ki}], (1 \leq k \leq N), (1 \leq i \leq N + 1)$, called the *Radau pseudospectral differentiation matrix* and $Y^{\text{LGR}}$ is defined as

$$Y^{\text{LGR}} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_{N+1} \end{bmatrix}.$$

Thus, the discrete approximation to the system dynamics $\dot{y}(\tau) = T(\tau)f(y(\tau), u(\tau))$ is obtained by evaluating the system dynamics at each collocation point and replacing $\dot{y}(\tau_k)$ by its discrete approximation $D_k Y^{\text{LGR}}$. It is important to observe that the left-hand side of Eq. (7–86) contains approximations for the state at the LGR points plus the final point while the right-hand side contains approximations for the state (and control) at only the LGR points.

The objective function in Eq. (7–10) is approximated by the Legendre-Gauss-Radau quadrature as

$$J = \int_{-1}^{+1} T(\tau)g(y(\tau), u(\tau))d\tau \approx \sum_{k=1}^{N} T(\tau_k)w_k g(Y_k, U_k), \quad (7–87)$$

where $w_k$ is the quadrature weight associated with $\tau_k$. Lastly, the path constraints in Eq. (7–13) are enforced at $N$ LGR collocation points as

$$T(\tau_k)C(Y_k, U_k) \leq 0, \quad (k = 1, \ldots, N). \quad (7–88)$$
The continuous-time nonlinear infinite-horizon optimal control problem of Eqs. (7–10)-(7–13) is then approximated by the following NLP: Minimize the cost

$$J = \sum_{k=1}^{N} T(\tau_k)w_k g(Y_k, U_k),$$

subject to the following equality and inequality constraints

$$D_k Y^{LGR} - T(\tau_k)f(Y_k, U_k) = 0, \quad (k = 1, \ldots, N),$$

$$Y_1 - y_0 = 0,$$

$$T(\tau_k)C(Y_k, U_k) \leq 0, \quad (k = 1, \ldots, N),$$

where the NLP variables are \((Y_1, \ldots, Y_{N+1})\) and \((U_1, \ldots, U_N)\). It is noted that, the singularity in the change of variables \(t = \phi(\tau)\) and in \(T(\tau) = \phi'(\tau)\) at \(\tau = +1\) is avoided in the NLP formulation. \(T(\tau) = \phi'(\tau)\) is never evaluated at the singularity in Eqs. (7–89)-(7–92), rather \(T(\tau)\) is evaluated at the quadrature points only, which are all strictly less than 1. Furthermore, because \(Y_{N+1}\) is a NLP variable, the solution of state at the final point, i.e., at \(t = \infty\) is also obtained.

### 7.3.2 Karush-Kuhn-Tucker Conditions

The necessary optimality conditions or the Karush-Kuhn-Tucker (KKT) conditions, of the NLP given in Eqs. (7–89)-(7–92) are now derived. The Lagrangian associated with the NLP is

$$\mathcal{L} = -\langle \Psi, Y_1 - y_0 \rangle$$

$$+ \sum_{k=1}^{N} T(\tau_k)(w_k g(Y_k, U_k) - \langle \Gamma_k, C(Y_k, U_k) \rangle)$$

$$- \sum_{k=1}^{N} \langle \Lambda_k, D_k Y^{LGR} - T(\tau_k)f(Y_k, U_k) \rangle,$$

where \(\Lambda_k\) is the \(k^{th}\) row of the Lagrange multipliers matrix \(\Lambda \in \mathbb{R}^{N \times n}\) associated with the constraints in Eq. (7–90), \(\Psi \in \mathbb{R}^{q}\) are the Lagrange multipliers associated with the constraints in Eq. (7–91), and \(\Gamma_k\) is the \(k^{th}\) row of the Lagrange multipliers matrix.
\( \Gamma \in \mathbb{R}^{N \times s} \) associated with the constraints in Eq. (7–92) The KKT optimality conditions are then obtained by differentiating the Lagrangian with respect to each of the variable and equating the derivative to zero, such that

\[
T(\tau_k)\nabla_Y(\mathbf{g}_k + \langle \mathbf{f}_k, \mathbf{C}_k \rangle) = \mathbf{D}_k^T \mathbf{A}, \quad 2 \leq k \leq N, \quad (7–94)
\]
\[
T(\tau_k)\nabla_Y(\mathbf{g}_1 + \langle \mathbf{f}_1, \mathbf{C}_1 \rangle) = \mathbf{D}_1^T \mathbf{A} + \Psi, \quad (7–95)
\]
\[
0 = \mathbf{D}_{N+1}^T \mathbf{A}, \quad (7–96)
\]
\[
\nabla_Y(\mathbf{g}_k + \langle \mathbf{f}_k, \mathbf{C}_k \rangle) = \mathbf{0}, \quad 1 \leq k \leq N, \quad (7–97)
\]
\[
\mathbf{D}_k \mathbf{Y}^{LGR} - T(\tau_k)\mathbf{f}(\mathbf{Y}_k, \mathbf{U}_k) = \mathbf{0}, \quad 1 \leq k \leq N, \quad (7–98)
\]
\[
\Gamma_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (7–99)
\]
\[
\Gamma_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (7–100)
\]
\[
\mathbf{Y}_1 - \mathbf{y}_0 = \mathbf{0}, \quad (7–101)
\]

where \( \mathbf{D}_i^T \) is the \( i^{th} \) row of \( \mathbf{D}^T \), \( \mathbf{g}_k = g(\mathbf{Y}_k, \mathbf{U}_k) \), \( \mathbf{f}_k = f(\mathbf{Y}_k, \mathbf{U}_k) \) and \( \mathbf{C}_k = C(\mathbf{Y}_k, \mathbf{U}_k) \).

Next, the KKT conditions given in Eqs. (7–94)-(7–101) are reformulated so that they become a discretization of the first-order optimality conditions given in Eqs. (7–14)-(7–21) for the continuous control problem given in Eqs. (7–10)-(7–13).

Let \( \mathbf{D}_i^\dagger = [D_i^\dagger_{ij}], \quad (1 \leq i \leq N), \quad (1 \leq i \leq N) \) be the \( N \times N \) matrix defined as follows:

\[
D_1^\dagger_{11} = -D_{11} - \frac{1}{w_1}, \quad (7–102)
\]
\[
D_i^\dagger_{ij} = -\frac{w_j}{w_i}D_{ji}, \quad \text{otherwise.} \quad (7–103)
\]

The matrix \( \mathbf{D}_i^\dagger \) defined in (7–103) and (7–102) is a differentiation matrix for the space of polynomials of degree \( N - 1 \). According to the definition of \( \mathbf{D}_i^\dagger \),

\[
\mathbf{D}_1^T = -w_1 \mathbf{D}_1^\dagger \mathbf{W}^{-1} - \frac{1}{w_1} \mathbf{e}_1, \quad (7–104)
\]
\[
\mathbf{D}_k^T = -w_k \mathbf{D}_k^\dagger \mathbf{W}^{-1}, \quad 2 \leq k \leq N, \quad (7–105)
\]
where $W$ is a diagonal matrix with weights $w_k$, $1 \leq k \leq N$, on the diagonal. $e_1$ is the 1st row of $N \times N$ identity matrix. Substituting Eqs. (7–104) and (7–105) in Eqs. (7–95) and (7–94),

$$
\Psi = T(\tau_k) \nabla_{y_1} (w_1 g_1 + \langle \Lambda_1, f_1 \rangle - \langle \Gamma_1, C_1 \rangle) + w_1 D_{1}^T W^{-1} \Lambda + \frac{1}{w_1} e_1 \Lambda, \quad (7–106)
$$

$$
-w_k D_{k}^T W^{-1} \Lambda = T(\tau_k) \nabla_{y_k} (w_k g_k + \langle \Lambda_k, f_k \rangle - \langle \Gamma_k, C_k \rangle), \quad 2 \leq k \leq N. \quad (7–107)
$$

Next, defining the following change of variables:

$$
\tilde{\lambda}_{N+1} = D_{N+1}^T \Lambda, \quad (7–108)
$$

$$
\tilde{\lambda}_k = \frac{\Lambda_k}{w_k}, \quad 1 \leq k \leq N, \quad (7–109)
$$

$$
\tilde{\gamma}_k = \frac{\Gamma_k}{w_k}, \quad 1 \leq k \leq N, \quad (7–110)
$$

$$
\tilde{\psi} = \Psi. \quad (7–111)
$$

Substituting Eqs. (7–108)-(7–111) in Eqs. (7–96)-(7–101) and in Eqs. (7–106)-(7–107), the transformed KKT conditions for the NLP are given as

$$
0 = \nabla_{u_k} (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad 1 \leq k \leq N, \quad (7–112)
$$

$$
0 = D_k Y^L_{k} - T(\tau_k) f(Y_k, U_k), \quad 1 \leq k \leq N, \quad (7–113)
$$

$$
\tilde{\gamma}_{ki} = 0 \text{ when } C_{ki} < 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (7–114)
$$

$$
\tilde{\gamma}_{ki} < 0 \text{ when } C_{ki} = 0, \quad 1 \leq i \leq s, \quad 1 \leq k \leq N, \quad (7–115)
$$

$$
0 = Y_1 - y_0, \quad (7–116)
$$

$$
D_1^T \tilde{\lambda} = -T(\tau_k) \nabla_{y_1} (g_1 + \langle \tilde{\lambda}_1, f_1 \rangle - \langle \tilde{\gamma}_1, C_1 \rangle) - \frac{1}{w_1} (\tilde{\lambda}_1 - \tilde{\psi}), \quad (7–117)
$$

$$
\tilde{\lambda}_{N+1} = 0, \quad (7–118)
$$

$$
D_k^T \tilde{\lambda} = -T(\tau_k) \nabla_{y_k} (g_k + \langle \tilde{\lambda}_k, f_k \rangle - \langle \tilde{\gamma}_k, C_k \rangle), \quad (7–119)
$$

$$
2 \leq k \leq N. \quad (7–120)
$$
Now, consider a comparison of the transformed KKT conditions in Eqs. (7–112)-(7–119) of the NLP to the first-order necessary optimality conditions in Eqs. (7–14)-(7–21) of the continuous-time optimal control problem. It is noted that the transformed KKT conditions in Eqs. (7–112)-(7–116) are the discretized forms of the continuous-time first-order optimality conditions in Eq. (7–16), Eq. (7–14), Eq. (7–19), Eq. (7–20), and Eq. (7–21), respectively. Furthermore, it is shown in Theorem 2 that the system (7–119) is a pseudospectral scheme for the costate dynamics, i.e.

\[ D_k^\dagger \tilde{\lambda}_k = \tilde{\lambda}_k, \quad 1 \leq k \leq N. \]  

(7–120)

Therefore, the left hand side of Eq. (7–119) is an approximation of the costate dynamics at the \( k^{th} \) collocation point, \( k = (2, \ldots, N) \). As a result, Eq. (7–119) represents the discretized version of the costate dynamics in Eq. (7–15) at \( k = (2, \ldots, N) \). Next, it is noted that at the final point, the discrete equivalent of continuous boundary conditions (7–18) is the same as the discrete costate at the final point in (7–118). However, at the initial point, the discrete equivalent of continuous boundary condition (7–17) is coupled in the discrete costate dynamics at the initial point in (7–117).

The equivalence of the transformed KKT condition of the NLP at the initial boundary (7–117) to the discretized form of continuous first-order optimality condition in (7–17) is now established by manipulating (7–108).

Returning to the definition of \( \tilde{\lambda}_{N+1} \) in (7–108), we obtain

\[
\tilde{\lambda}_{N+1} = D_{N+1}^T \Lambda = \sum_{i=1}^{N} \Lambda_i D_i, N+1 = -\sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_i D_{ij} 
\]

(7–121)

\[
= \frac{\Lambda_1}{w_1} + \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_i D_{ij}^\dagger \frac{w_j}{w_i} = \frac{\Lambda_1}{w_1} + \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_i D_{ij}^\dagger \frac{w_j}{w_i} 
\]

(7–122)

\[
= \tilde{\lambda}_1 + \sum_{i=1}^{N} \sum_{j=1}^{N} w_i \tilde{\lambda}_j D_{ij}^\dagger = \tilde{\lambda}_1 + \sum_{i=1}^{N} w_i D_{ij}^\dagger \tilde{\lambda} 
\]

(7–123)

\[
= \tilde{\psi} - \sum_{i=1}^{N} T(\tau_i) w_i \nabla_{\gamma_i} (g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle), 
\]

(7–124)
where (7–121) follows from the identity (4–64) given in Proposition 2, (7–122) is the definition (7–102) and (7–103) of $D^1$, (7–123) is the definition (7–109) of $\tilde{\lambda}_i$, and (7–124) is the first-order optimality condition (7–119) and (7–117). Rearranging (7–124) such that
\[
\tilde{\psi} = \tilde{\lambda}_{N+1} + \sum_{i=1}^{N} T(\tau_i) w_i \nabla Y_i (g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle).
\] (7–125)

Next, the continuous costate dynamics in Eq. (7–15) are
\[
\dot{\lambda}(\tau) = - T(\tau) \nabla_y H = - T(\tau) \nabla_y (g + \langle \lambda, f \rangle - \langle \gamma, C \rangle).
\] (7–126)

Integrating the continuous costate dynamics in (7–126) using the Radau quadrature,
\[
\tilde{\lambda}_1 = \tilde{\lambda}_{N+1} + \sum_{i=1}^{N} T(\tau_i) w_i \nabla Y_i (g_i + \langle \tilde{\lambda}_i, f_i \rangle - \langle \tilde{\gamma}_i, C_i \rangle).
\] (7–127)

Comparing (7–125) with (7–127) gives
\[
\tilde{\lambda}_1 = \tilde{\psi}.
\] (7–128)

Eq. (7–128) is the missing boundary condition at the initial point that was coupled with the discrete costate dynamics at the initial point in (7–117). It is also implied by Eq. (7–128) that the extra term in (7–117) is in fact zero, thereby, making (7–117) consistent with discrete costate dynamics at the initial point. Hence, the system of transformed KKT conditions of the NLP is equivalent to the first-order optimality conditions of the continuous-time optimal control problem and accurate costate estimates are obtained from the KKT multipliers using the relationship given in
Eqs. (7–108)-(7–109) as
\[
\begin{align*}
\lambda_{N+1} & \approx \bar{\lambda}_{N+1} = D_{N+1}^T \Lambda, \\
\lambda_k & \approx \bar{\lambda}_k = \frac{\Lambda_k}{w_k}, \quad 1 \leq k \leq N, \\
\gamma_k & \approx \bar{\gamma}_k = \frac{\Gamma_k}{w_k}, \quad 1 \leq k \leq N, \\
\psi & \approx \bar{\psi} = \Psi.
\end{align*}
\] (7–129)

7.3.3 Equivalent Implicit Integration Scheme

It is now shown that the LGR pseudospectral discretization of the state equation has an equivalent integrated formulation. Let \( p \) be any polynomial of degree at most \( N \). By the construction of the \( N \times (N + 1) \) matrix \( D \), we have \( Dp = \dot{p} \) where
\[
\begin{align*}
p_k & = p(\tau_k), \quad 1 \leq k \leq N + 1, \\
\dot{p}_k & = \dot{p}(\tau_k), \quad 1 \leq k \leq N.
\end{align*}
\] (7–133)

Then the identity \( \dot{p} = Dp \) can be written as
\[
\dot{p} = D_1 p_1 + D_{2:N+1} p_{2:N+1}.
\] (7–135)

Multiplying by \( D_{2:N+1}^{-1} \) and utilizing Proposition 7 gives
\[
p_k = p_1 + (D_{2:N+1}^{-1} \dot{p})_k, \quad 2 \leq k \leq N + 1.
\] (7–136)

Next, a different expression for \( p_k - p_1 \) based on the Lagrange polynomial approximation of the derivative is obtained. Let \( L_i^\dagger(\tau) \) be the Lagrange interpolation polynomials associated with the collocation points:
\[
L_i^\dagger = \prod_{j=1}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad 1 \leq i \leq N.
\] (7–137)

Notice that the Lagrange polynomials \( L_i \) defined in (7–82) are degree \( N \) while the Lagrange polynomials \( L_i^\dagger \) are degree \( N - 1 \). Then because \( \dot{p} \) is a polynomial of degree at
most \( N - 1 \), it can be approximated exactly by the Lagrange polynomials \( L_i^\dagger \):

\[
p = \sum_{i=1}^{N} \dot{p}_i L_i^\dagger (\tau).
\]  

(7–138)

Integrating \( \dot{p} \) from \( -1 \) to \( \tau_k \), following relationship is obtained

\[
p(\tau_k) = p(-1) + \sum_{i=1}^{N} \dot{p}_i A_{ki},
\]  

(7–139)

\[A_{ki} = \int_{-1}^{\tau_k} L_i^\dagger (\tau) d\tau, \quad 2 \leq k \leq N + 1.\]

Utilizing the notation (7–133) and (7–134), we have

\[
p_k = p_1 + (A\dot{p})_k, \quad 2 \leq k \leq N + 1.
\]  

(7–140)

The relations (7–136) and (7–140) are satisfied for any polynomial of degree at most \( N \). Equating (7–136) and (7–140) to obtain

\[
A\dot{p} = D_{2:N+1}^{-1} p.
\]

Choose \( \dot{p} \) from the columns of the identity matrix to deduce that \( A = D_{2:N+1}^{-1} \). Rewriting Eq. (7–86) such that

\[
D_1 Y_1 + D_{2:N+1} \begin{bmatrix} Y_2 \\ \vdots \\ Y_{N+1} \end{bmatrix} = \begin{bmatrix} T(\tau_1) f(Y_1, U_1) \\ \vdots \\ T(\tau_N) f(Y_N, U_N) \end{bmatrix}.
\]  

(7–141)

Multiplying by \( A = D_{2:N+1}^{-1} \) and utilizing Proposition 7 to obtain

\[
Y_k = Y_1 + \sum_{i=1}^{N} A_{ki} T(\tau_i) f(Y_i, U_i), \quad 2 \leq k \leq N + 1.
\]  

(7–142)

Hence, the Eq. (7–142) is the equivalent implicit integral form of the state equation in (7–86). The elements of \( A \) are the integrals of the Lagrange basis \( L_i^\dagger \), while the elements of \( D \) are the derivatives of the Lagrange basis \( L_i \) defined in (7–82). It is noted that the state at the final point is also obtained in the implicit integration scheme.
7.4 Inapplicability of Lobatto Pseudospectral Method

Consider the LGL collocation points \((\tau_1, \ldots, \tau_N)\) on the interval \([-1, 1]\) where \(\tau_1 = -1\) and \(\tau_N = 1\) (the terminal time, corresponding to \(t = +\infty\)). The state is approximated by a polynomial of degree at most \(N - 1\) as

\[
y(\tau) \approx Y(\tau) = \sum_{i=1}^{N} Y_i L_i(\tau),
\]

(7–143)

where \(Y_i \in \mathbb{R}^n\) and \(L_i(\tau)\) is a Lagrange polynomial of degree \(N - 1\) defined as

\[
L_i(\tau) = \prod_{j=1, j \neq i}^{N} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad (i = 1, \ldots, N).
\]

(7–144)

Differentiating the series of Eq. (7–143) with respect to \(\tau\), an approximation to the derivative of the state in \(\tau\) domain is obtained as

\[
\dot{y}(\tau) \approx \dot{Y}(\tau) = \sum_{i=1}^{N} Y_i \dot{L}_i(\tau).
\]

(7–145)

The collocation conditions are then formed by equating the derivative of the state approximation in Eq. (7–145) to the right-hand side of the state dynamic constraints in Eq. (7–11) at the \(N\) LGL points, \((\tau_1, \ldots, \tau_N)\).

\[
\sum_{i=1}^{N} Y_i \dot{L}_i(\tau_k) = T(\tau_k)f(Y_k, U_k), \quad (k = 1, \ldots, N),
\]

(7–146)

\[
\sum_{i=1}^{N} D_{ki} Y_i = T(\tau_k)f(Y_k, U_k), \quad D_{ki} = \dot{L}_i(\tau_k).
\]

(7–147)

Hence, the discrete approximation to the system dynamics is

\[
D_k Y^{\text{LGL}} = T(\tau_k)f(Y_k, U_k), \quad 1 \leq k \leq N,
\]

(7–148)
where $D_k$ is the $k^{th}$ row of differentiation matrix $D = [D_{ki}]$, $(1 \leq k \leq N)$, $(1 \leq i \leq N)$, called the *Lobatto pseudospectral differentiation matrix* and $Y^{LGL}$ is defined as

$$Y^{LGL} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_N \end{bmatrix}.$$ 

Thus, the discrete approximation to the system dynamics $\dot{y}(\tau) = T(\tau)f(y(\tau), u(\tau))$ is obtained by evaluating the system dynamics at each collocation point and replacing $y(\tau_k)$ by its discrete approximation $D_k Y^{LGL}$.

The objective function in Eq. (7–10) is approximated by the Legendre-Gauss-Lobatto quadrature as

$$J = \int_{-1}^{1} T(\tau) g(y(\tau), u(\tau)) d\tau \approx \sum_{k=1}^{N} T(\tau_k) w_k g(Y_k, U_k), \quad (7–149)$$

where $w_k$ is the quadrature weight associated with $\tau_k$. Lastly, the path constraints in Eq. (7–13) are enforced at the $N$ LGL collocation points as

$$T(\tau_k) C(Y_k, U_k) \leq 0, \quad (k = 1, \ldots, N). \quad (7–150)$$

The continuous-time nonlinear infinite-horizon optimal control problem of Eqs. (7–10)-(7–13) is then approximated by the following NLP: Minimize the cost

$$J = \sum_{k=1}^{N} T(\tau_k) w_k g(Y_k, U_k), \quad (7–151)$$

subject to the following equality and inequality constraints

$$D_k Y^{LGL} - T(\tau_k) f(Y_k, U_k) = 0, \quad (k = 1, \ldots, N), \quad (7–152)$$

$$Y_1 - y_0 = 0, \quad (7–153)$$

$$T(\tau_k) C(Y_k, U_k) \leq 0, \quad (k = 1, \ldots, N). \quad (7–154)$$
It is noted that, the singularity in \( T(\tau) = \phi'(\tau) \) at \( \tau = +1 \) is present in the NLP formulation. Because \( \tau = +1 \) is a collocation point, \( T(\tau) = \phi'(\tau) \) is evaluated at the singularity in Eqs. (7–151)-(7–154). This causes a singularity to occur in the formulation of the NLP. Therefore, the Lobatto pseudospectral method in which collocation is performed at the terminal point of the interval cannot be used for solving an infinite-horizon optimal control problem.

### 7.5 Examples

In this section two examples of the infinite-horizon Gaussian and Radau pseudospectral methods developed in this chapter are considered. The first example is a one-dimensional nonlinear infinite-horizon optimal control problem. The first example was solved using both the infinite-horizon Gaussian and Radau pseudospectral methods developed in this chapter and the method of Ref. [65] using all three time transformations \( \phi(\tau) \). An error analysis is performed and error comparison between different time transformations is done. An error comparison between the errors from the methods of this dissertation and the method of Ref. [65] is also performed. The second example is the two-dimensional infinite-horizon LQR problem. The second example was solved using both the infinite-horizon Gaussian and Radau pseudospectral methods developed in this chapter using all three time transformations \( \phi(\tau) \). An error comparison for both the methods using all three transformations is performed.

#### 7.5.1 Example 1: Infinite-Horizon One-Dimensional Nonlinear Problem

Consider the following nonlinear infinite-horizon optimal control problem [64].

Minimize the cost functional

\[
J = \frac{1}{2} \int_{0}^{\infty} (\log^2 y(t) + u(t)^2) \, dt, \tag{7–155}
\]

subject to the dynamic constraint

\[
\dot{y}(t) = y(t) \log y(t) + y(t) u(t), \tag{7–156}
\]
with the initial condition
\[ y(0) = 2. \]  
(7–157)

The exact solution to this problem is
\[
\begin{align*}
y^*(t) &= \exp(x^*(t)), \\
u^*(t) &= -(1 + \sqrt{2})x^*(t), \\
\lambda^*(t) &= (1 + \sqrt{2})\exp(-x^*(t))x^*(t), \\
x^*(t) &= \log_2\exp(-t\sqrt{2}).
\end{align*}
\]  
(7–158)

The example of Eqs. (7–155)–(7–157) was solved for \( N = (5, 10, 15, 20, 25, 30) \) using both the infinite-horizon pseudospectral methods described in this chapter and the approach of Ref. [65] with the three strictly monotonic transformations of the domain \( \tau \in [-1, +1) \) given in Eqs. (7–5)–(7–7). All solutions were obtained using the NLP solver SNOPT with optimality and feasibility tolerances of \( 1 \times 10^{-10} \) and \( 2 \times 10^{-10} \), respectively.

Furthermore, the following initial guess of the solution was used:
\[
\begin{align*}
y(\tau) &= y_0, \\
u(\tau) &= \tau,
\end{align*}
\]  
(7–159)

where we recall that \( \tau \in [-1, 1] \). Figs. 7-3–7-8 show the solutions obtained from the infinite-horizon Gauss and Radau pseudospectral methods alongside the exact solution using the three transformations of the domain \( \tau \in [-1, +1) \) given in Eqs. (7–5)–(7–7) for \( N = 30 \) collocation points. It is seen that the GPM and the RPM solution are indistinguishable from the exact solution for all three quantities (state, control, and costate). In particular, it is seen that the infinite horizon GPM and RPM solve the problem on the entire infinite time domain.
Figure 7-3. Solution obtained from the Gauss pseudospectral method using $\phi_a(\tau)$.

Next, the maximum base ten logarithm of the state, control, and costate errors are defined as

$$E_y = \max_k \log_{10} \| y(\tau_k) - y^*(\tau_k) \|_\infty,$$

$$E_u = \max_k \log_{10} \| u(\tau_k) - u^*(\tau_k) \|_\infty,$$

$$E_\lambda = \max_k \log_{10} \| \lambda_y(\tau_k) - \lambda_y^*(\tau_k) \|_\infty.$$  \hfill (7–160)

In Eq. (7–160) the index $k$ spans the approximation points in the case of either the state and costate and spans only the collocation points in the case of the control. It is
noted that the state and costate are obtained on the entire horizon with the index \( N + 1 \) corresponding to the state and costate at \( \tau = +1 \), or equivalently, at \( t = +\infty \).

The state, control and costate errors obtained using the Gauss and Radau methods of this paper are shown, respectively, in Figs. 7-9, 7-10 and 7-11 alongside the error obtained using the method of Ref. [65] with the transformation given in Eqs. (7–5) and (7–7). It is seen for all three transformations and for both methods of this paper, the state, control, and costate errors decrease in essentially a linear manner until \( N = 30 \),
Figure 7-5. Solution obtained from the Gauss pseudospectral method using $\phi_6(\tau)$, demonstrating an approximately exponential convergence rate. Furthermore, it is observed that either the Gauss or Radau method of this paper yields approximately the same error for a particular value of $N$ and choice of transformation. Moreover, it is seen that the errors are largest and smallest, respectively, using the transformations of Eqs. (7–6) and (7–7). In fact, the transformation of Eq. (7–7) is at least one order of magnitude more accurate than either of the other two transformations. Finally, it is seen that the errors from the two methods of this paper using the transformation of Eq. (7–7)
Figure 7-6. Solution obtained from the Radau pseudospectral method using $\phi_b(\tau)$.

are significantly smaller than those obtained using the method of Ref. [65] (where the transformation of Eq. (7–5) are used). When the transformation of Eq. (7–7) is used, however, the state errors from the method of Ref. [65] are nearly the same as those obtained using the Gauss and Radau methods, while the control and costate errors are approximately one order of magnitude larger using the method of Ref. [65].

The different behavior of the functions given in Eqs. (7–5)–(7–7) is understood if we apply the change of variables to the continuous solution. The optimal state in the
Figure 7-7. Solution obtained from the Gauss pseudospectral method using $\phi_c(\tau)$.

transformed coordinates is as follows:

$$y_a^*(\tau) = \exp \left( \log 2 \exp \left( -\sqrt{2} \left( \frac{1 + \tau}{1 - \tau} \right) \right) \right)$$  \hspace{0.5cm} (7–161)

$$y_b^*(\tau) = \exp \left( \log 2 \left( \frac{1 - \tau}{2} \right)^{\sqrt{2}} \right)$$  \hspace{0.5cm} (7–162)

$$y_c^*(\tau) = \exp \left( \log 2 \left( \frac{1 - \tau}{2} \right)^{2\sqrt{2}} \right)$$  \hspace{0.5cm} (7–163)
Figure 7-8. Solution obtained from the Radau pseudospectral method using $\phi_c(\tau)$.

Here the subscripts $a$, $b$, and $c$ correspond to the three choices of $\phi$ given in Eqs. (7–5)–(7–7). An advantage of using a logarithmic change of variables given in Eqs. (7–6) or (7–7), as compared to the function given in Eq. (7–5), is that logarithmic functions essentially move collocation points associated with large values of $t$ to the left. Because the exact solution changes slowly when $t$ is large, this leftward movement of the collocation points is beneficial since more collocation points are situated where the solution is changing most rapidly. The disadvantage of a logarithmic change of variables is seen in the
Figure 7-9. Maximum state errors for example using infinite-horizon Gauss and Radau pseudospectral method alongside errors obtained using the Radau method of Ref. [65].
Figure 7-10. Maximum control errors for example using infinite-horizon Gauss and Radau pseudospectral method alongside errors obtained using the Radau method of Ref. [65].
Figure 7-11. Maximum costate errors for example using infinite-horizon Gauss and Radau pseudospectral method alongside errors obtained using the Radau method of Ref. [65].
function \( \log(1 - \tau) \) where the growth is so slow near \( \tau = +1 \) that the transformed solution possesses a singularity in a derivative at \( \tau = +1 \). In other words, the \( j^{th} \) derivative of a function of the form \( (1 - \tau)^\alpha \), where \( \alpha > 0 \) is not an integer, is singular at \( \tau = +1 \) for \( j > \alpha \). In particular, \( y^*_0(\tau) \) has one derivative at \( \tau = +1 \) but not two, while \( y^*_e(\tau) \) has two derivatives at \( \tau = +1 \) but not three. To achieve exponential convergence, \( y^*(\tau) \) should be infinitely smooth. For this particular problem, the choice of Eq. (7–7) has the following nice properties: \( y^*_e(\tau) \) is relatively smooth with two derivatives, although not infinitely smooth, and collocation points corresponding to large values of \( t \), where the solution changes slowly, are moved to the left [when compared to \( t = (1 + \tau)/(1 - \tau) \)] where the solution changes more rapidly. As a result, for \( 5 \leq N \leq 30 \), the function of Eq. (7–7) yields a solution that is often two or more orders of magnitude more accurate than the other choices for \( \phi \).

### 7.5.2 Example 2: Infinite-Horizon LQR Problem

Consider the following optimal control problem taken from Ref. [65]. Denoting \( x(t) = [x_1(t) \ x_2(t)]^T \in \mathbb{R}^2 \) as the state and \( u(t) \in \mathbb{R} \) as the control, minimize the cost functional

\[
J = \frac{1}{2} \int_0^\infty (x^TQx + u^TRu) \, dt, \tag{7–164}
\]

subject to the dynamic constraint

\[
\dot{x} = Ax + Bu, \tag{7–165}
\]

and the initial condition

\[
x(0) = \begin{bmatrix} -4 \\ 4 \end{bmatrix}. \tag{7–166}
\]

The matrices \( A, B, Q, \) and \( R \) for this problem are given as

\[
A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \frac{1}{2}. \tag{7–167}
\]
The exact solution to this problem is

\[ x^*(t) = \exp([A - BK]t)x(0) \]
\[ u^*(t) = -Kx^*(t) \]
\[ \lambda^*(t) = Sx^*(t) \]  \hspace{1cm} (7–168)

where \( K \) is the optimal feedback gain and \( S \) is the solution to the algebraic Riccati equation. In this case \( K \) and \( S \) are given, respectively, as

\[
K = \begin{bmatrix}
4.828427124746193 & 2.557647291327851 \\
6.031273049535752 & 2.414213562373097 \\
2.414213562373097 & 1.278823645663925
\end{bmatrix}
\]

\[
S = \begin{bmatrix}
6.031273049535752 & 2.414213562373097 \\
2.414213562373097 & 1.278823645663925
\end{bmatrix}
\]  \hspace{1cm} (7–169)

The optimal control problem of Eqs. (7–164)–(7–166) was solved using the infinite-horizon Gauss and Radau pseudospectral methods using the NLP solver SNOPT with default optimality and feasibility tolerances of \( 10^{-6} \) and \( 2 \times 10^{-6} \), respectively, for \( N = 5 \) to \( N = 40 \) by steps of 5. Figs. 7-12-7-17 show the solutions obtained from the infinite-horizon Gauss and Radau pseudospectral methods alongside the exact solution using the three transformations of the domain \( \tau \in [-1, +1) \) given in Eqs. (7–5)–(7–7) for \( N = 40 \) collocation points. It is seen that the GPM and the RPM solution are indistinguishable from the exact solution for all three quantities (state, control, and costate). In particular, it is seen that the infinite horizon GPM and RPM solve for the state and the costate on the entire infinite time domain whereas the control is obtained only at the collocation points, which do not include the final point.

Suppose now that we define the following maximum absolute errors between the solution obtained from the NLP solver and the exact solution:

\[
E_x = \max_k \log_{10} ||x(\tau_k) - x^*(\tau_k)||_{\infty},
\]
\[
E_u = \max_k \log_{10} ||u(\tau_k) - u^*(\tau_k)||_{\infty},
\]
\[
E_\lambda = \max_k \log_{10} ||\lambda(\tau_k) - \lambda^*(\tau_k)||_{\infty},
\]  \hspace{1cm} (7–170)
The values of $E_x$, $E_u$, and $E_\lambda$ are shown in Figs. 7-18–7-22. It is seen that all errors decrease linearly until approximately $N = 40$, again demonstrating an exponential convergence rate. Furthermore, it is observed that either the Gauss or the Radau method of this dissertation yields approximately the same error for a particular value of $N$ and choice of transformation. Moreover, it is seen that for low number of collocation points, i.e., $N \leq 30$ the errors are the largest and the smallest, respectively, using the transformations of Eqs. (7–6) and (7–7). In fact, the transformation of Eq. (7–7),
$\phi_c(\tau)$, is at least one order of magnitude more accurate than either of the other two transformations. For $N \geq 30$, however, the transformation of Eq. (7–5), $\phi_a(\tau)$, seems to be the most accurate of all the three transformations. More precisely, the rate of convergence for the transformation $\phi_b(\tau)$ is the highest and the rate of convergence for the transformation $\phi_c(\tau)$ is the least. For $N = 5$, the errors are the least and the greatest for the transformations $\phi_c(\tau)$ and $\phi_a(\tau)$, respectively. Due to the higher rate of convergence for transformation $\phi_a(\tau)$, at $N = 15$, the errors from the transformation...
Figure 7-14. Solution obtained from the Gauss pseudospectral method using $\phi_b(\tau)$.

$\phi_a(\tau)$ become smaller than the errors from the transformation $\phi_b(\tau)$ and finally, from the errors from the transformation $\phi_c(\tau)$ at $N = 35$. 
Figure 7-15. Solution obtained from the Radau pseudospectral method using $\phi_b(\tau)$. 
Figure 7-16. Solution obtained from the Gauss pseudospectral method using $\phi_c(\tau)$. 

A State solution. 

B Control solution. 

C Costate solution.
Figure 7-17. Solution obtained from the Radau pseudospectral method using $\phi_c(\tau)$. 
Figure 7-18. Maximum first component of state errors for example using infinite-horizon Gauss and Radau pseudospectral method
Figure 7-19. Maximum second component of state errors for example using infinite-horizon Gauss and Radau pseudospectral method
Figure 7-20. Control errors for example using infinite-horizon Gauss and Radau pseudospectral method
Figure 7-21. Maximum first component of costate errors for example using infinite-horizon Gauss and Radau pseudospectral method.
Figure 7-22. Maximum second component of costate errors for example using infinite-horizon Gauss and Radau pseudospectral method.
7.6 Summary

Two methods based on the LG and the LGR collocation, called respectively, the infinite-horizon Gauss pseudospectral method and the infinite-horizon Radau pseudospectral method have been presented in this chapter. The proposed methods yield approximations to the state and the costate on the entire horizon, including approximations at $t = +\infty$. Furthermore, it has been shown that the transformed KKT conditions for both the methods are equivalent to the discretized version of the continuous-time first-order optimality conditions. Both of the methods can be written equivalently in either a differential or an implicit integral form. Lastly, it has been shown that collocation based on the LGL points is not suitable for solving infinite-horizon optimal control problems. It has been shown that the map $\phi : [-1, 1) \rightarrow [0, +\infty)$ can be tuned to improve the quality of the discrete approximation. In numerical examples, the discrete solution exhibits exponential convergence as a function of the number of collocation points.
CHAPTER 8
CONCLUSION

8.1 Dissertation Summary

It is difficult, if not impossible, to find the solution to a continuous-time optimal control problem from the first-order necessary conditions derived from the calculus of variations for most of the engineering applications. Numerical approximations to the solution of the optimal control problem, therefore, must be used. Many numerical methods exist for approximating the solution of an optimal control problem. These methods generally fall into one of two categories: indirect methods and direct methods. Indirect methods attempt to find a solution to the optimal control problem by approximating the first-order necessary conditions derived from the calculus of variations and the Pontryagin’s principle. Direct methods convert the infinite-dimensional continuous control problem into a finite-dimensional discrete nonlinear programming problem (NLP). The resulting NLP can then be solved by well-developed NLP algorithms. Indirect methods generally are more accurate, while direct methods have simpler, more convenient, formulations and are more robust. Pseudospectral methods for solving optimal control problems are a class of direct transcription methods that use orthogonal collocation for approximating the solution of differential equations and the Gaussian quadrature to approximate integral cost. The Lobatto pseudospectral method (LPM) uses the Legendre-Gauss-Lobatto (LGL) points in the formulation where the costate estimates are derived directly from the Karush-Kuhn-Tucker (KKT) multipliers of the resulting NLP. The LPM, however, suffers from a defect in the costate estimates at the boundary points. At these points the costate estimates do not satisfy the costate boundary conditions or the discretized costate dynamics. This defect results in a relatively poor estimate of the costate. The Gauss pseudospectral method (GPM) uses the Legendre-Gauss (LG) points and does not suffer from a defect in costate estimates. In the GPM, the KKT conditions of the resulting NLP are a discrete form of
the continuous-time first-order necessary conditions. It is noted, however, that because
the differential equations are collocated at only the interior points, the control at the initial
and the final point is not obtained using the GPM.

In this dissertation, a method, called the Radau pseudospectral method (RPM),
has been presented for direct trajectory optimization and costate estimation using
global collocation at the Legendre-Gauss-Radau (LGR) points. A theoretical foundation
for the method has been provided. The primary property that distinguishes the RPM
from the LPM and the GPM is the fact that the dynamics are collocated at the interior
points plus the initial point. This leads to an elegant equivalence between the necessary
conditions of the NLP and the first-order optimality conditions of the continuous problem
while still providing the control at the initial point. The fact that the necessary conditions
of the NLP are a discrete form the continuous-time first-order necessary conditions
allows the RPM to take advantage of the convenient formulations and robustness of a
direct method, while preserving the accuracy of an indirect method. The results of this
research indicate that the RPM described here has the ability to determine accurate
solutions for the state, the control, and the costate of a general optimal control problem.

Next, a unified framework has been presented based on the GPM and the RPM. It
was demonstrated that each of these schemes can be expressed in either a differential
or an integral formulation that are equivalent to each other. It was also demonstrated
that the LPM does not exhibit such an equivalence. The fact that the differential forms
of the GPM and the RPM, as formulated in this research, can be expressed in an
equivalent integral form, shows that the dynamics are in fact being integrated when
implementing these methods. Either form can be used to solve the problem. The
differential form, however, results in a more sparse NLP that can be solved faster.

Furthermore, both the GPM and the RPM provide an accurate transformation
between the KKT multipliers of the discrete NLP and the costate of the continuous
optimal control problem. It has been shown that the discrete costate systems in
both these methods are full rank while the discrete costate system in the Lobatto pseudospectral method has a null space. The LPM costate approximation was found to have an error that oscillates about the exact solution, and this error was shown by example to be due to the null space in the Lobatto discrete costate system. By comparing the schemes of this unified framework with the Lobatto pseudospectral method, an effort has been made to understand the deficiencies in the Lobatto pseudospectral method that lead to an inaccurate solution.

Lastly, the infinite-horizon versions of the Gauss and the Radau pseudospectral methods have been derived. The proposed methods yield approximations to the state and costate on the entire horizon, including approximations at $t = +\infty$. These versions can also be written equivalently in either a differential or an implicit integral form. It was shown that the mapping $\phi : [-1, +1) \rightarrow [0, +\infty)$ can be tuned to improve the quality of the discrete approximation. It was also shown that the Lobatto pseudospectral method can not be used to solve the infinite-horizon optimal control problems. The dynamics are collocated at the final point in the Lobatto pseudospectral method and hence cannot be used for infinite-horizon problems as a singularity exists at the final point in the mapping $\phi : [-1, +1) \rightarrow [0, +\infty)$.

Empirical evidence has suggested that the Gauss and the Radau pseudospectral methods converge rapidly (exponentially) for a large class of problems and give a better costate estimate than the Lobatto pseudospectral method. These advantages have been shown on a variety of example problems. It has been shown that these methods, however, are not well suited for solving problems that have discontinuities in the solution or discontinuities in the derivatives of the solution, and problems that contain singular arcs.
8.2 Future Work

8.2.1 Convergence Proof for Gauss and Radau Pseudospectral Method

The equivalence between the transformed KKT conditions of the NLP and the discretized form of the continuous-time first-order optimality conditions provides a way to approximate the costate and the Lagrange multipliers of an optimal control problem. Furthermore, the equivalence between the differential and the implicit integration scheme establishes the fact that in using the differential form of a pseudospectral method to solve an optimal control problem, the state dynamics are actually being integrated numerically. The numerical examples suggest that the solution from the Gauss and the Radau pseudospectral method converge at an exponential rate. A formal mathematical proof, however, that shows that the solution to the discrete NLP converges to the optimal solution of the original continuous-time optimal control problem is still missing.

8.2.2 Costate Estimation Using Lobatto Pseudospectral Method

Both the Gauss and the Radau schemes provide an accurate transformation to obtain costate estimate from the KKT multipliers of the discrete nonlinear programming problem. The LPM costate approximation, however, is found to have an error that oscillates about the exact solution, and this error is shown by example to be due to the null space in the Lobatto discrete costate system. For the first example considered in this dissertation, it was possible to find the linear factor by which the null space can be combined to the solution obtained from the NLP to obtain an accurate costate estimate. For a multi-dimensional problem, this approach, however, is not trivial. Exploring the null space associated with the Lobatto discrete costate dynamics and obtaining accurate costate estimate using the Lobatto pseudospectral method is an interesting research problem.
REFERENCES


BIOGRAPHICAL SKETCH

Divya Garg was born in 1984 in Ghaziabad, Uttar Pradesh, India. She received her Bachelor of Technology degree in Mechanical Engineering from the Malaviya National Institute of Technology, Jaipur, India, in May 2007. She then received her Master of Science degree in Mechanical Engineering in December 2008, and her Doctor of Philosophy degree in Mechanical Engineering in August 2011, at the University of Florida. Her research interests include linear algebra, numerical optimization, optimal control theory, and numerical approximations to the solution of optimal control problems.