MATHEMATICAL MODELS OF COMPETITION WITH EXPLICIT COST CONSIDERATIONS IN SUPPLY CHAINS

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To my mom, Refika Konur and dad, Cengiz Konur

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TABLE OF CONTENTS

APPENDIX

LIST OF TABLES

LIST OF FIGURES

Figure page

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

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In this dissertation, we first consider a competitive facility location problem when the competitive firms are identical. Competing firms simultaneously determine their facility location and distribution quantity decisions on a congested distribution network. Each firm may locate more than one facility simultaneously. We adopt a two-stage solution approach to analyze the resulting symmetric competitive multi-facility location problem. First, the firms' supply quantity decisions are solved, given that the firms have chosen identical facility locations. Then, we focus on the firms' facility location decisions, and explain why the firms choose identical facility locations. A heuristic solution method is proposed for determining high-quality solution for the first stage. Numerical studies are conducted to illustrate the efficiency of our heuristic method.

We then use the aforementioned setting and analyses of the symmetric competitive multi-facility location game to analyze the effects of traffic congestion on supply chain activities. We utilize the model to provide analytical characterization of the effects of traffic congestion costs on equilibrium distribution flows. We present the results of extensive numerical studies to further illustrate the effects of traffic congestion costs on location, market supply quantity, and distribution decisions.

Next, we study the competitive multi-facility location game with traffic congestion costs when the firms are non-identical. Similar to the symmetric case, we utilize a two-stage solution approach. However, heterogeneity of the competing firms requires

distinct solution approaches in each stage. Particularly, firms' market-supply decisions for given facility locations are characterized using a variational inequality formulation. Then, a heuristic search method is provided for finding equilibrium locations, and its computational efficiency is compared to a random search method.

Finally, we consider pricing decisions for a supplier who sells his/her product via a distributor who, in turn, serves small, localized retailers. The retailers' orders may be horizontally decentralized or centrally managed by the distributor, depending on the distributor's procurement strategy. We model this problem as a Stackelberg game and determine its solution using backward induction: we first analyze the distributor's procurement strategy and the retailers' order quantity decisions, then, the results of the retailers' quantity decisions are used to determine the supplier's wholesale price.

CHAPTER 1
INTRODUCTION

Efficient supply chain management requires effective decision making in three main stages: Supply Chain Design, Supply Chain Planning, and Supply Chain Operations. In the Supply Chain Design phase, firms decide on the basic structure of their supply chains and set strategic goals that will be in effect over a relatively long time. The Supply Chain Planning phase focuses on utilizing the given configurations to increase profitability while Supply Chain Operations ensure that the processes required to function in the chain, given the design and plans, are performed effectively. In today's competitive industry, it is crucial for firms to design, plan, and operate their supply chains as efficiently as possible. An ideal supply chain management policy would take into account every factor that can affect the success of the supply chain.

Competition is a fact of the business world, and it is a key factor that should be regarded in supply chain decisions. Modeling a competitive firm as a monopoly is often unrealistic and can be fatal if used to make decisions. In this dissertation, we consider competitive supply chain management problems, which explicitly account for the inherent competition within the business world. Competition among firms mainly prevails in situations when there exists one or more resources that must be shared by a set of noncooperative firms. An end customer market for a product or service, for instance, is a resource shared by firms providing the same (or perfectly substitutable) product or service to this market. The limited consumption in end customer markets is a common driver of competition; however, it is not the only source of competition. Transportation networks and limited supplies are often shared by noncooperative firms and, hence, motivate additional competition. Firms should thus design, plan, and operate their supply chains while explicitly considering any possible competitive factors present in the system.

In addition to weighing competition in their decisions, it is also important for firms to account for explicit cost terms in their decision making processes for truly optimizing their supply chains via more accurate reflection of real-life scenarios. In Chapters 2-4, we model explicit transportation costs, which are compounded by traffic congestion costs. Recent studies at the intersection of traffic congestion and supply chain management empirically demonstrate the negative effects traffic congestion has on supply chain performance. Specifically, given that an underlying distribution network is shared by multiple firms, recognizing traffic congestion, which hinders logistical efficiency, is essential. Furthermore, traffic congestion cost modeling enables representation of the aforementioned competition on the transportation network. Therefore, we explicitly model traffic congestion costs in the problems of interest in Chapters 2-4.

Facility location and supply quantity (or equivalently production level) decisions are important in the design and planning of a flourishing supply chain. Moreover, as noted above, competition and explicit cost modeling should not be disregarded in any decision phase of supply chains. In Chapters 2-4, we analyze a set of competitive facility location problems with traffic congestion costs. Classical facility location problems focus on analyses of a single decision maker's facility location choices in the absence of competitive factors. Competitive facility location problems extend the classical facility location problems by directly accounting for the effects of competition present in the system. In particular, these problems formulate facility location and associated supply quantity or price decisions of a set of firms, who compete in order to serve a set of end customer markets. Chapters 2-4 model competitive facility location problems (also referred to as a *facility location game* or simply *location game*), in which a set of firms is engaged in Cournot type quantity competition within multiple end customer markets. The potential facility location and customer markets are represented by a finite number of nodes of a connected network; that is, we consider location games on a discrete network. There is a limited number of studies that examine location games on discrete networks (most of the competitive location problems assume spatial competition). Moreover, these studies have the restrictive assumption that the competing firms locate at most one facility. From a practical point of view, economies or diseconomies of scale may force firms to locate more than one facility even to supply a single customer market. Thus, in our analyses, the firms are allowed to locate multiple facilities simultaneously. Additionally, we consider nonlinear traffic congestion costs in the class of location games analyzed in this dissertation, which motivates firms to locate multiple facilities.

In particular, we first analyze a symmetric competitive multi-facility location game with traffic congestion costs in Chapter 2. The competing firms are recognized as identical (or homogeneous), and that is why we define this problem class as *symmetric*. Each firm's objective is to maximize their profits by determining their facility locations and supply quantities from each facility to each market while taking into account the competition in the markets as well as the competition on the common distribution network due to traffic congestion. We use a two-stage solution approach in our analyses. A two-stage solution approach first determines the equilibrium supply quantities (second stage decisions) given the firms' facility location choices. Then, the solution of the second stage is utilized in finding the equilibrium facility locations (first stage decisions). In applying the two-stage solution approach to the symmetric multi-facility location game under consideration in Chapter 2, we first solve for firms' equilibrium supply quantity decisions under the assumption that the firms have chosen identical facility locations. It is shown that the equilibrium solution of the second stage is symmetric, i.e., firms will supply the quantities to the markets in equilibrium, given that they have located facilities at the same locations. We then turn our attention to the first stage decisions, and investigate the rationale behind identical of firm location choices. Following this, a heuristic method is proposed to determine a set of locations where all firms will locate a facility. This heuristic method ranks locations with respect to specific parameters and is intended to mimic an individual firm's approach to the facility location decision problem. We demonstrate the efficiency of the heuristic method via our numerical studies and discuss a counter-intuitive observation.

Including traffic congestion costs in the symmetric competitive multi-facility location problem of Chapter 2 not only allows substantiation of competition on the distribution network but also enables analyses of the effects of traffic congestion on supply chain performance. There are past studies combining traffic congestion and supply chain management in the literature. Nevertheless, these studies have the following three major drawbacks: (i) the analyses are mostly based on empirical data and, hence, lack theoretical results, (ii) congestion costs are considered exogenously, and (iii) the competitive nature of traffic congestion is ignored. In Chapter 3, our aim is to utilize the model described in Chapter 2 to provide a complete analysis on the effects of traffic congestion on supply chain activities. Our approach overcomes the previously mentioned drawbacks as we model traffic congestion costs endogenously in such a way that the competitive nature of traffic congestion is captured. Furthermore, we manage to analytically characterize how changes in traffic congestion costs affect equilibrium flow decisions. These analytical results further grant qualitative characterization of the effects of traffic congestion levels on firms' location and distribution decisions. Chapter 3 also summarizes extensive numerical studies conducted to further illustrate the effects of traffic congestion costs on location, market supply quantity, and distribution decisions.

While our work on the symmetric multi-facility location problem with traffic congestion costs contributes to the theory of location games by studying a discrete competitive multi-facility location game (and to the supply chain management and transportation literatures by enabling detailed analyses of the effects of traffic congestion costs on supply chain management) we are naturally interested in the nuances of asymmetric competitive multi-facility location problems. Specifically, it may be the case that competing firms utilize different technologies and value congestion differently, which results in heterogeneity of the firms. Chapter 4 studies a class of asymmetric competitive multi-facility location problems with traffic congestion costs. Similar to the symmetric case, we adopt a two-stage solution approach; nevertheless, the asymmetry of the firms necessitates diverse analyses in each stage. In particular, we utilize the well known result that states that noncooperative games can be formulated as variational inequality problems under certain concavity conditions. Firms' supply quantity decisions can be formulated as an asymmetric linear variational inequality problem, and we discuss a self-adaptive projection method as a solution tool for the second stage of the asymmetric competitive multi-facility location problem. The challenging part lies in the analysis of the first stage game, i.e., searching for equilibrium facility location choices. We first focus on defining properties of equilibrium facility location choices and propose routines that are intended to ease the search process. Then, these routines are embedded in a heuristic search method. Finally, we compare the efficiency of the heuristic method to a random search algorithm.

Analyses of the symmetric and asymmetric competitive multi-facility location problems complete our study on multi-facility location games on discrete networks. These problems deal with competition within a single echelon of a supply chain. In Chapter 5, we turn our attention to a multi-echelon competitive supply chain. In particular, an agent in an upper level echelon of a supply chain must acknowledge the competition among the parties in a lower level echelon. For instance, it is a common practice that a supplier sells his/her product to a set of retailers, who then sell the product in the end customer market. These retailers compete within the end customer market, and it is important for the supplier to acknowledge this competition among the retailers in determining his/her wholesale price, i.e., selling price to the retailers.

Chapter 5 investigates a two-echelon competitive supply chain problem. Specifically, we are interested in the wholesale price setting problem of a supplier who sells his/her product via a distributor who, in turn, supplies a set of retailers. The retailers are competitive in the end customer market, however, they can be centralized by the distributor depending on the distributor's procurement strategy. In Chapter 5, we model a Stackelberg game to analyze the supplier's wholesale price setting problem, the distributor's procurement strategy, and the retailers' order quantity decisions. The supplier is the leader of the Stackelberg game and the distributor and the retailers are the followers. We use backward induction to solve for a Stackelberg equilibrium: first the distributor's procurement strategy and the retailers' order quantity decisions are analyzed; then, these analyses are used to solve the supplier's wholesale price setting problem. The supplier does not have control over the distributor's procurement strategy, which affects the total order quantity demanded from the supplier. Hence, the supplier can achieve substantial savings if s/he can control the distributor's procurement strategy. We define the supplier's potential savings when s/he has control over the distributor's procurement strategy as the value of the control for the supplier. We conduct numerical studies to quantify the value of control.

Finally, in Chapter 6, we conclude this dissertation by summarizing our contributions. We further discuss future research directions related to Chapters 2-5. In this dissertation, we introduce competitive multi-facility location problems with traffic congestion costs and provide detailed analyses of these problems. Mathematical formulation of traffic congestion problems with explicit consideration of competition helps us analytically characterize the effects of traffic congestion on firms' facility location and supply flow decisions under realistic settings. Furthermore, we study a two-echelon competitive supply chain and discuss the implications of centralized vs. decentralized retailing on the channel performance. Distinct managerial insights are gained through our analyses.

CHAPTER 2
A SYMMETRIC COMPETITIVE MULTI-FACILITY LOCATION GAME WITH CONVEX CONGESTION COSTS

2.1 Motivation and Literature Review

Facility location problems have been extensively studied in the literature. Most of the past operations research studies on facility location theory focus on formulating a single decision maker's problem in the absence of competitive factors. This stream of research is discussed in the facility location books by Drezner (1995) and Drezner and Hamacher (2002), and the review papers by Hale and Moberg (2004), Owen and Daskin (1998), and Tansel et al. (1983), as well as the references contained therein. As noted by Plastria (2001), an assumption of no competition is often impractical. Rhim et al. (2003) also observe that location competition is an important factor in competitive supply chains. As a result, another stream of research focuses on facility location problems under competition. The problems studied within this research stream comprise the fundamentals of competitive location theory. In this problem class, firms' location decisions (along with other strategic decisions, such as pricing decisions, supply quantity decisions, or capacity decisions) are studied by applying competitive equilibrium tools and concepts.

The classical study of Hotelling (1929) introduces the first competitive location problem. In this study, two firms compete in a market and each wishes to maximize its market share under a demand inelasticity assumption. Smithies (1941) considers the same problem with demand elasticity. Teitz (1968) extends Hotelling's problem by allowing firms to locate more than one facility. Following these basic studies, competitive location problems have been studied under different settings in the literature. These settings differ in their assumptions on the number of competing firms (two firms versus more general multiple firm problems), the number of strategic decisions (facility locations, product pricing, supply quantities and facility capacities), and the nature of the competition and strategic game (sequential facility location decisions, simultaneous location decisions, and decisions when facilities already exist at some locations). The reader may refer to Eiselt and Laporte (1996), Eiselt et al. (1993), and Plastria (2001) for reviews of competitive facility location problems under different assumptions.

In this chapter, we address a competitive facility location problem that considers facility location and market-supply quantity decisions for a set of homogeneous firms selling a product. In particular, the problem we study is a symmetric competitive facility location game. Multiple competitive firms selling a common product type are noncooperative and homogeneous, i.e., they incur identical marginal delivery costs in serving a market from the same location (i.e., from supply facilities located within the same city or town). Firms are subject to transportation and congestion costs as a result of deliveries to markets, as well as fixed facility location costs for maintaining supply facilities. We assume that the markets and the potential facility locations are represented as vertices of a connected network. Firms must simultaneously determine their supply facility locations (stage-one decisions) and the quantities they will ship from these facilities to each market (stage-two decisions). We thus adopt a two-stage solution: we first study the stage-two decisions, given that the firms have identical facility locations and, then, we focus on the stage-one decisions and characterize the firms' facility location decisions. Similar two-stage solution approaches are used in the analysis of competitive facility location problems by Lederer and Thisse (1990), Labbé and Hakimi (1991), Sarkar et al. (1997), Pal and Sarkar (2002), Rhim et al. (2003), and Sáiz and Hendrix (2008).

During the stage-one decisions, firms are allowed to locate more than one facility, and each facility is assumed uncapacitated. In the stage-two decisions, we assume that the firms are engaged in Cournot competition in multiple markets. Cournot competition and Bertrand competition are the most common concepts used in modeling competitive markets. In Cournot models, the competition is quantity based, whereas Bertrand models are based on price competition. Quantity competition is justified for industries where the production decisions are made before actual sales begin. As pointed out by Hamilton et al. (1994), technology is one reason why firms may experience a lag between production and sales. In such cases (e.g., when firms choose production capacities before actual production), when output levels are not adjustable in the short run, the current quantity in the market will determine the price. Hamilton et al. (1994) note that, given production decisions, market-clearing prices can be set. On the other hand, in Bertrand models, competitive firms should have robustness in their production systems in order to adjust output levels in response to price-sensitive demand. Furthermore, when firms compete for sales of identical products, Cournot competition may provide a better representation of the market, since firms will focus on determining production output rather than setting prices for homogeneous products (perfect substitutes). One may refer to Kreps and Scheinkman (1983), Anderson and Neven (1991), and Hamilton et al. (1994) for further discussions on the justification of Cournot competition. In this chapter, firms are assumed to be homogeneous and they sell homogeneous products; hence, we use Cournot competition to model the markets. In particular, the Cournot model defined in Hamilton et al. (1994) assumes that firms set mill prices, i.e., spatially separated markets have the same price, and customers pay for transportation costs. On the other hand, we assume that the transportation costs are paid by competing firms and different markets may observe different prices, i.e., the Cournot model of interest in this chapter is the so-called *delivered* Cournot competition, defined similar to the Cournot model of Hamilton et al. (1989). We use this competition model in Chapters 3, 4, and 5 as well.

We note that competitive location problems are commonly modeled using Cournot competition in the literature. Spatial competition of two firms under Cournot competition is studied by Labbé and Hakimi (1991). This study was extended to multiple firms by Sarkar et al. (1997). Both of these studies assume that firms locate a single facility. Conversely, Pal and Sarkar (2002) consider spatial competition in a Cournot duopoly where the competing firms may locate more than one facility. The distinguishing assumption of these studies is that competing firms enter each market by supplying a positive quantity to each market. The work in Rhim et al. (2003) and Sáiz and Hendrix (2008) relaxes this assumption and considers the case of free entry. In both of these studies, Cournot competition exists and firms choose the location of their single facility and the quantity they will supply from this facility to each market, if they choose to enter any market. Markets and potential facility locations are located on the vertices of a network. It should be noted that defining potential facility locations as vertices of a network is more practical and parallels the results of Labbé and Hakimi (1991), Lederer and Thisse (1990), and Sarkar et al. (1997), which state that equilibrium facility locations tend to be on the vertices of an underlying network under spatial competition. While a homogeneous cost structure is assumed by Rhim et al. (2003), Sáiz and Hendrix (2008) study a heterogeneous cost structure. The problem considered in this chapter applies similar assumptions as those of Rhim et al. (2003) and Sáiz and Hendrix (2008); however, firms are allowed to locate more than one facility, and firms are subject to nonlinear traffic congestion costs (for a discussion on the effects of traffic congestion on supply chain activities see, e.g., McKinnon, 1999, Rao et al., 1991, Sankaran et al., 2005, Weisbrod et al., 2001, and Konur and Geunes, 2011).

We first focus on the stage-two decisions, given that the firms choose identical facility locations. In this stage, we first show that the firms' supply quantity decisions correspond to a symmetric pure strategy Nash equilibrium, or simply Pure Nash Equilibrium (PNE) solution, given that the firms choose identical facility locations. Using this result, we show that the game among the firms actually reduces to a game of the locations. In particular, the game among the firms is similar to the game defined in Rhim et al. (2003), which assumes homogeneous firms; however, in our model, firms may locate more than one facility. Our results then imply that this game can also be reduced to a game of locations, resulting in an asymmetric game similar to the one studied in Sáiz and Hendrix (2008), which does not consider nonlinear traffic congestion costs as we do in this chapter. Therefore, we characterize the properties of this reduced game and provide an exact algorithm to find the PNE solution. Then, we study the firms' stage-one decisions and explain the rationale behind the assumption that firms choose identical facility locations. We note that firms choose identical facility locations in the case of a unique PNE location decision. However, when a unique PNE solution does not exist, since the equilibrium concept does not characterize what firms will actually do, we use the maximization of expected profits as an objective, assuming that any location decision is equally likely for each firm. We show that a mixed strategy Nash Equilibrium (MSNE) implies that it is equally likely for any firm to choose any given location decision. Thus, when firms are homogeneous, they will end up with identical facility locations, and therefore, we study the optimal location decision set for the individual firm.

Particularly, a heuristic solution method, based on how an individual firm might approach the location decision problem, is proposed for determining high-quality location decisions for the firms. The method is a two phase method and is intended to mimic how an individual firm might plan its facility location strategy in practice. We discuss results of extensive numerical studies to show the efficiency of the heuristic method. Moreover, in our numerical studies, we observe a counter-intuitive example, where the firms may be better off when they ignore congestion costs in their decisions.

Our work contributes to the literature by extending the work of Rhim et al. (2003) and Sáiz and Hendrix (2008) to allow competitive firms to locate more than one supply facility, as well as by explicitly considering nonlinear traffic congestion costs, as firms share a common distribution network. We establish important results for characterizing the stage-two (market supply) decisions that will be used in analyzing traffic congestion effects in a competitive supply chain in Chapter 3.

The rest of this chapter is organized as follows. Section 2.2 describes the detailed problem setting and solution approach. In Section 2.3, we characterize the properties of the equilibrium supply quantities and provide a solution algorithm, given that firms make identical facility location decisions. Section 2.4 focuses on facility location decisions. We explain the rationale behind the assumption that firms choose identical facility locations, and provide a total enumeration scheme and a heuristic method to characterize facility location decisions. In Section 2.5, the results of extensive numerical studies are discussed to characterize the efficiency of the heuristic method, and the impacts of accounting for congestion in the decision making process.

2.2 Problem Formulation and Solution Approach

Consider a set of k competitive firms, indexed by $r \in R = \{1, 2, \cdots, k\}$, who wish to determine the locations of facilities at m possible locations, indexed by $i \in I = \{1, 2, \cdots, m\}$, as well as the supply quantities from these facilities to each of n customer markets, indexed by $j \in J = \{1, 2, \cdots, n\}$. The supply firms incur linear transportation and convex congestion costs as a result of their supply quantity decisions. Additionally, firms are subject to fixed facility location costs that depend on their location decisions. We assume that firms may locate more than one facility; however, any facility ultimately located is assumed to be uncapacitated. Therefore, none of the firms will locate more than one facility at a specific location.

Firms are engaged in Cournot competition in multiple markets. The unit price in any market is determined by the total quantity supplied to that market. In particular, the unit price in market j is defined as a linear and decreasing function of the total supply quantity into market j, $q_{\bullet j \bullet}$. Let p_j denote the price in market j. Then

$$p_j(q_{\bullet j \bullet}) = a_j - b_j q_{\bullet j \bullet}, \tag{2–1}$$

where $a_j \geq 0$ and $b_j > 0$. The parameter a_j can be interpreted as the maximum demand or the consumption capacity in market j. The parameter b_j can be interpreted as the price sensitivity of market j. (In particular, all linear demand curves can be represented as $p = 1 - q$, where p denotes the delivered price, q denotes the demand (or quantity supplied), and b is a scaling parameter.) We note that this type of functional form is commonly used to model the behavior of market price under quantity competition (see, e.g., Rhim et al., 2003, Sáiz and Hendrix, 2008).

We assume that the transportation cost incurred by a firm on the link connecting location i to market j is a linear function of the quantity flow on that link, and we let $c_{ij} \geq 0$ denote the per unit transportation cost on link (i, j) for any firm. It should be noted that c_{ij} can be assumed to include per unit production costs as well. That is, a parameter $v_i > 0$ specific to location i can be included within c_{ij} to account for per unit production cost at location i. In addition to transportation costs, we assume that firms are subject to traffic congestion costs as a result of their distribution volume decisions. We note that traffic congestion affects supply chain activities, and recent studies document the negative effects traffic congestion has on supply chain performance (see, e.g., McKinnon, 1999, Rao et al., 1991, Sankaran et al., 2005, Weisbrod et al., 2001, and Konur and Geunes, 2011). Therefore, we consider congestion costs explicitly in our model. In particular, the congestion cost a firm incurs on link (i, j) amounts to $q_{ijr} g_{ij}$, where q_{ijr} denotes the quantity of flow on link (i, j) by firm r and g_{ij} is defined to be a function of the total quantity of flow on link (i, j), $q_{ij\bullet}$ ($q_{ij\bullet} = \sum_{r \in R} q_{ijr}$), and reads

$$g_{ij}(q_{ij\bullet}) = \alpha_{ij} q_{ij\bullet}. \tag{2–2}$$

In Equation (2–2), α_{ij} defines traffic congestion cost factor specific to link (i, j) and we assume that $\alpha_{ij} > 0$. Thus, any firm's congestion cost is a nondecreasing convex function of the quantity sent by the firm on a link. In Chapter 3, we provide justification for why this functional form is chosen to model traffic congestion costs. Finally, a firm incurs a fixed facility location cost, f_i, if it locates a facility at location i.

The profit function of a firm consists of four terms: total revenues gained from supplying markets, less transportation, congestion, and facility location costs. Explicitly, the profit function of firm r reads as

$$\Pi_r(\mathbf{Q},\mathbf{X}) = \sum_{j\in J} p_j \left(\sum_{i\in I}\sum_{r\in R} q_{ijr}\right)\sum_{i\in I} q_{ijr} - \sum_{j\in J}\sum_{i\in I} c_{ij} q_{ijr} - \sum_{j\in J}\sum_{i\in I} q_{ijr} g_{ij}\left(\sum_{r\in R} q_{ijr}\right) - f_r(\mathbf{x}_r), \tag{2–3}$$

where $\mathbf{Q}$ is $k \times m \times n$ matrix of q_{ijr} values and $\mathbf{X}$ is $m \times k$ binary matrix representing firms' location decisions. In particular, $\mathbf{x}_r$ is an $m-$vector representing location decisions of firm r with entries x_{ir} such that $x_{ir} = 1$ if firm r locates a facility at location i, $x_{ir} = 0$ otherwise. Hence, $f_r(\mathbf{x}_r) = \sum_{i\in I} x_{ir} f_i$ denotes the total facility location cost for firm r. For notational simplicity, we further define $q_{\bullet jr}$ as the total quantity shipped to market j by firm r ($q_{\bullet jr} = \sum_{i\in I} q_{ijr}$).

As noted in the previous section, we adopt a two-stage solution approach considering the following sequence of decisions: first, firms must determine their facility locations and, then, they must decide on their supply quantities at each market. As a result of competition, any firm's profit depends on the decisions of the other firms. Note that, unlike the previous studies by Rhim et al. (2003) and Sáiz and Hendrix (2008), firms not only compete based on market price, but also as a result of the congestion cost functions on supply links. Our two-stage solution approach first solves stage-two decisions given that the firms have identical facility locations. We employ the equilibrium concept of Nash (1951) to determine the firms' supply quantity decisions and propose a method that solves the associated Cournot oligopoly. Then, we focus on the solution of stage-one decisions and characterize the firms' facility location decisions.

2.3 Stage-Two Decisions: Market-Supply Game

In this section, we study the second stage of the game, which determines the firms' supply quantity decisions for a given location decision. This restricted game of determining equilibrium quantity decisions is referred to as the *Market-Supply Game*. The *Market-Supply Game* is a non-cooperative game in which the supply firms are the players. Firms simultaneously determine how much to send from facilities to markets.

To determine the firms' flows, we use the PNE concept, i.e., no firm will be better off by altering its supply quantity decisions under the given location decisions.

Now let us assume that the location decision for each firm, i.e., the vector $\mathbf{x}_r$ for each $r = 1, 2, \cdots, k$, is pre-determined. That is, $\mathbf{X}$ is fixed. Since $f_r(\mathbf{x}_r)$ is fixed for the given $\mathbf{X} = \mathbf{X}^0$, it can be omitted from Equation (2–3) for the analysis of the *Market-Supply Game*. Using the notation introduced in the previous section and Equations (2–1), (2–2), and (2–3), the profit function of firm r for the given $\mathbf{X} = \mathbf{X}^0$ can be rewritten as

$$\Pi_r(\mathbf{Q}|\mathbf{X} = \mathbf{X}^0) = \sum_{j \in J} \left[(a_j - b_j q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I} c_{ij} q_{ijr} - \sum_{i \in I} \alpha_{ij} q_{ijr} q_{ij\bullet} \right]. \qquad (2–4)$$

The function in Equation (2–4) is strictly concave in each $q_{ijr} \geq 0$, as $b_j > 0$ and $\alpha_{ij} > 0$. Note that $q_{ijr} = 0$ for all $j \in J, i \notin I^0$, where I^0 denotes the locations where firms have facilities for the given $\mathbf{X} = \mathbf{X}^0$. It further follows from Equation (2–4) that any firm's stage-two decisions can be analyzed separately for each market. Therefore, we focus on the *Market-Supply Game* for market j in the rest of this section. The discussion that follows on the firms' *Market-Supply Game* at market j is valid for the *Market-Supply Game* across all markets.

Given the firms' location decisions $\mathbf{X} = \mathbf{X}^0$, the profit function of firm r at market j, $\Pi_r^j(\mathbf{Q}_j|\mathbf{X} = \mathbf{X}^0)$, is

$$\Pi_r^j(\mathbf{Q}_j|\mathbf{X} = \mathbf{X}^0) = p_j(q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I} c_{ij} q_{ijr} - \sum_{i \in I} \alpha_{ij} q_{ijr} q_{ij\bullet}, \qquad (2–5)$$

where $\mathbf{Q}_j$ is the vector of firms' supply quantities at market j. Let q^*_{ijr} denote the equilibrium quantities. To determine the equilibrium solution for the *Market-Supply Game* at market j given that $\mathbf{X} = \mathbf{X}^0$, one should find the set of locations and the set of firms such that $q^*_{ijr} > 0$, and solve the first order conditions, $\partial \Pi_r^j(\mathbf{Q}|\mathbf{X} = \mathbf{X}^0)/\partial q_{ijr} = 0$, for each $q^*_{ijr} > 0$ simultaneously due to concavity of Equation (2–5). Explicitly, the first order condition when $q_{ijr} > 0$ reads

$$a_j - b_j[q_{\bullet j \bullet} + q_{\bullet jr}] - c_{ij} - \alpha_{ij}[q_{ijr} + q_{ij\bullet}] = 0. \tag{2–6}$$

At this point, we assume that $\mathbf{X}^0$ consists of identical columns, where the r^{th} column corresponds to the location decision of firm r. That is, $\mathbf{x}_r = \mathbf{x}^0 \; \forall r \in R$, where $\mathbf{x}^0$ denotes any column of $\mathbf{X}^0$. Thus, the number of facilities at any candidate location is either k or 0 for some positive k. Note that we do not need to consider locations where no firm has located a facility. Therefore, we only study quantity decisions at supply locations with k facilities. That is, I^0 denotes the set of locations with k facilities associated with $\mathbf{X}^0$. In the next proposition, we show that the quantity supplied from location i to market j is the same for each firm.

Proposition 2.1. *Given that $\mathbf{x}_r = \mathbf{x}^0 \; \forall r \in R$, $q^*_{ijr} = Q^*_{ij}/k \; \forall r \in R$, where Q^*_{ij} is the total equilibrium flow on link (i, j).*

Proof: Please see Appendix A.

It follows from Proposition 2.1 that, when the total equilibrium quantity supplied from location i to market j, i.e., Q^*_{ij} is known, we can readily obtain the associated q^*_{ijr} values. For the given $\mathbf{X}^0$, with $\mathbf{x}_r = \mathbf{x}^0 \; \forall r \in R$, it follows from Proposition 2.1 that (i) $q^*_{ijr} = Q^*_{ij}/k$, (ii) $q^*_{ij\bullet} = Q^*_{ij}$, (iii) $q^*_{\bullet jr} = \sum_{i \in I^0} Q^*_{ij}/k$ and (iv) $q^*_{\bullet j \bullet} = \sum_{i \in I^0} Q^*_{ij}$. Recall that Equation (2–6) states the first order equilibrium condition for quantities such that $q_{ijr} > 0$; that is, it gives the first order condition when $Q^*_{ij} > 0$. Substituting (i)-(iv) into Equation (2–6), we get

$$\delta_{ij} - b_j \left(\sum_{i \in I^0} Q^*_{ij} + \sum_{i \in I^0} \frac{Q^*_{ij}}{k} \right) - \alpha_{ij} \left(Q^*_{ij} + \frac{Q^*_{ij}}{k} \right) = \delta_{ij} - \gamma b_j \sum_{i \in I^0} Q^*_{ij} - \gamma \alpha_{ij} Q^*_{ij} = 0, \tag{2–7}$$

where $\delta_{ij} = a_j - c_{ij}$ and $\gamma = (k + 1)/k$. Note that we may have at most $k \times m$ such first order conditions defined for market j. Nevertheless, the first order conditions associated with a location use the same equation for each firm, given by Equation (2–7). This actually reduces the game of the firms to a game of locations by reducing the number of decision variables from $k \times m$ to m. Therefore, we focus on simultaneous solution of at most m first order conditions, one for each location, defined by Equation (2–7). In the next proposition, we state conditions that must be satisfied by Q^*_{ij} values.

Proposition 2.2. *The equilibrium quantities must satisfy the following conditions:*

$$(a) \quad Q^*_{ij} > 0 \quad \textit{if and only if} \quad \delta_{ij} > \gamma b_j \textstyle\sum_{i \in I^0} Q^*_{ij},$$

$$(b) \quad Q^*_{ij} = 0 \quad \textit{if and only if} \quad \delta_{ij} \leq \gamma b_j \textstyle\sum_{i \in I^0} Q^*_{ij}.$$

Proof: Considering Proposition 2.1 and Equation(2–7), the KKT conditions for any firm at location i, $i \in I^0$, can be written as follows:

$$\begin{aligned} \delta_{ij} - \gamma b_j \textstyle\sum_{i \in I^0} Q^*_{ij} - \alpha_{ij} Q^*_{ij} + u_i &= 0, \\ u_i Q^*_{ij} &= 0, \\ u_i &\geq 0. \end{aligned}$$

We first prove Statement (a). Suppose $Q^*_{ij} > 0$, then it implies that $u_i = 0$, hence, we have $\delta_{ij} = \gamma b_j \sum_{i \in I^0} Q^*_{ij} + \alpha_{ij} Q^*_{ij}$. Since $\alpha_{ij} Q^*_{ij} > 0$ as $\alpha_{ij} > 0$ and $Q^*_{ij} > 0$, it follows that $\delta_{ij} > \gamma b_j \sum_{i \in I^0} Q^*_{ij}$. Now suppose that $\delta_{ij} > \gamma b_j \sum_{i \in I^0} Q^*_{ij}$ and $Q^*_{ij} = 0$, then it implies that $u_i = \gamma b_j \sum_{i \in I^0} Q^*_{ij} - \delta_{ij} < 0$ which is a contradiction since $u_i \geq 0$. Hence, $Q^*_{ij} > 0$. We now prove Statement (b). Suppose $Q^*_{ij} = 0$ and $\delta_{ij} > \gamma b_j \sum_{i \in I^0} Q^*_{ij}$, then it implies that $u_i = \gamma b_j \sum_{i \in I^0} Q^*_{ij} - \delta_{ij} < 0$ which is a contradiction since $u_i \geq 0$. Hence, $\delta_{ij} \leq \gamma b_j \sum_{i \in I^0} Q^*_{ij}$. Now suppose that $\delta_{ij} \leq \gamma b_j \sum_{i \in I^0} Q^*_{ij}$ and $Q^*_{ij} > 0$, then it implies that $u_i = 0$, hence, we have $\delta_{ij} - \gamma b_j \sum_{i \in I^0} Q^*_{ij} = \alpha_{ij} Q^*_{ij} > 0$, which is a contradiction since $\alpha_{ij} > 0$. □

It follows from Proposition 2.2 that the set of active locations at market j will be determined by δ_{ij} values. We refer to any location i as *active* at market j if $Q^*_{ij} > 0$. Similarly, we refer to any firm r as *active* at market j whenever $q^*_{ijr} > 0$. The next proposition is a direct result of Proposition 2.2 and states the *activeness* relations between two locations.

Proposition 2.3. *If $\delta_{i_1 j} \geq \delta_{i_2 j}$ for locations $i_1, i_2 \in I^0$, then in an equilibrium solution*

$$(a) \quad \textit{if} \quad Q^*_{i_2 j} > 0, \quad \textit{then} \quad Q^*_{i_1 j} > 0,$$

$$(b) \quad \textit{if} \quad Q^*_{i_1 j} = 0, \quad \textit{then} \quad Q^*_{i_2 j} = 0.$$

Proof: Suppose that $\delta_{i_1 j} \geq \delta_{i_2 j}$ for locations $i_1, i_2 \in I^0$. Now consider that $Q^*_{i_2 j} > 0$. Then it follows from Proposition 2.2 that $\delta_{i_2 j} > \gamma b_j \sum_{i \in I^0} Q^*_{ij}$. This implies that $\delta_{i_1 j} > \gamma b_j \sum_{i \in I^0} Q^*_{ij}$.

Thus, it follows from Proposition 2.2 that $Q^*_{i_1j} > 0$, which proves Statement (a). Now consider that $Q^*_{i_1j} = 0$. Then it follows from Proposition 2.2 that $\delta_{i_1j} \leq \gamma b_j \sum_{i \in I^0} Q^*_{ij}$. This implies that $\delta_{i_2j} \leq \gamma b_j \sum_{i \in I^0} Q^*_{ij}$. Thus, it follows from Proposition 2.2 that $Q^*_{i_2j} = 0$, which proves Statement (b). □

Proposition 2.3 highlights the importance of sorting locations according to their δ_{ij} values which, for a given market j, is equivalent to sorting locations based on c_{ij} values. We note that in case of tied δ_{ij} values, Proposition 2.3 implies activeness of either all or none of the locations with tied δ_{ij} values. Nevertheless, the equilibrium solution of the *Market-Supply Game* at market j does not change with the sorting order of tied values of δ_{ij}, as the same set of first order conditions defined in Equation (2–7) will be solved. This holds true when $\alpha_{ij} > 0$. We note that it is possible to have multiple PNE quantity decisions when $\alpha_{ij} = 0$ for more than one facility. In such a case, some of the locations with tied values can be active while the others are inactive. Furthermore, the set of active locations can be defined by any combination of the locations with tied δ_{ij} values, which results in multiple equilibria.

Now let us sort locations according to their δ_{ij} values, and without loss of generality, let us assume that $\delta_{ij} \geq \delta_{(i+1)j}$. Therefore; if ℓ locations are active at market j, these locations are $1, 2, \ldots, \ell$ with $\ell \leq |I^0|$, where $|I^0|$ denotes the cardinality of the set I^0. Then for any firm at any location i, $i \leq \ell$ (since $q^*_{ijr} > 0$ as $Q^*_{ij} > 0$) the following first order condition must be satisfied:

$$\delta_{ij} - \gamma b_j (Q^*_{1j} + Q^*_{2j} + \cdots + Q^*_{\ell j}) - \gamma \alpha_{ij} Q^*_{ij} = 0 \; \forall i \leq \ell.$$

In matrix notation, the first order conditions can be represented as

$$\begin{bmatrix} \delta_{1j} \\ \delta_{2j} \\ \vdots \\ \delta_{\ell j} \end{bmatrix} = \gamma \begin{bmatrix} \alpha_{1j} + b_j & b_j & \cdots & b_j \\ b_j & \alpha_{2j} + b_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & b_j \\ b_j & \cdots & b_j & \alpha_{\ell j} + b_j \end{bmatrix} \begin{bmatrix} Q^*_{1j} \\ Q^*_{2j} \\ \vdots \\ Q^*_{\ell j} \end{bmatrix}.$$

It follows from the above representation that we can find the Q_{ij}^* values easily by inverting the $\ell \times \ell$ matrix for a given set of active locations. Note that inverting this matrix basically involves solving the first order conditions for locations $1, 2, \ldots, \ell$ together. However, our aim is to determine the set of active locations and then find the equilibrium quantities to solve the *Market-Supply Game* at market j. In the next proposition, we provide an algorithm that determines the solution of the *Market-Supply Game* at market j. The algorithm is based on Propositions 2.2 and 2.3.

Proposition 2.4. *Suppose that* $\boldsymbol{x}_r = \boldsymbol{x}^0\ \forall r \in R$*. Then Algorithm 1, stated below, determines the number of the active locations and the corresponding equilibrium flow quantities.*

Algorithm 1. *Given* $\boldsymbol{x}_r = \boldsymbol{x}^0\ \forall r \in R$*, the number of firms,* b_j*,* δ_{ij} *and* α_{ij} *values for market* j*:*

Step 0. Set $Q_{ij}^* = 0\ \forall i \notin I^0$*. Sort the remaining locations such that location 1 has the greatest* δ_{ij} *value. If* $\delta_{1j} > 0$*, set* $\ell = 2$ *and go to Step 1; otherwise* $Q_{ij}^* = 0\ \forall i \in I^0$*.*

Step 1. For location ℓ*, find* $Q_{\ell j}^{(\ell)}$ *by solving the following set of equations represented in matrix form*

$$\begin{bmatrix} \delta_{1j} \\ \delta_{2j} \\ \vdots \\ \delta_{\ell j} \end{bmatrix} = \gamma \begin{bmatrix} \alpha_{1j} + b_j & b_j & \cdots & b_j \\ b_j & \alpha_{2j} + b_j & \cdots & \vdots \\ \vdots & \vdots & \ddots & b_j \\ b_j & \cdots & b_j & \alpha_{\ell j} + b_j \end{bmatrix} \begin{bmatrix} Q_{1j}^{(\ell)} \\ Q_{2j}^{(\ell)} \\ \vdots \\ Q_{\ell j}^{(\ell)} \end{bmatrix}. \tag{2–8}$$

Step 2. If $Q_{\ell j}^{(\ell)} > 0$ *and* $\ell < |I^0|$*, set* $\ell = \ell + 1$ *and go to Step 1. If* $Q_{\ell j}^{(\ell)} > 0$ *and* $\ell = |I^0|$*, stop, locations* $1, 2, \ldots, \ell$ *are active and* $Q_{ij}^* = Q_{ij}^{(\ell)}\ \forall i \in I^0$*. Else if,* $Q_{\ell j}^{(\ell)} \leq 0$*, stop; locations* $1, 2, \ldots, \ell - 1$ *are active at market* j*.* $Q_{ij}^* = Q_{ij}^{(\ell-1)}$ *for* $i \leq \ell - 1$ *and* $Q_{ij}^* = 0$ *for* $i \geq \ell$*.*

Proof: When Algorithm 1 stops at $(\ell-1)^{th}$ iteration (an iteration refers to an execution of Step 2), it means that $Q_{\ell j}^{(\ell)} < 0$. We now show that if we continue the algorithm one step further, that is, if we assume that first $\ell+1$ locations are active, we should have $Q_{\ell+1}^{(\ell+1)} < 0$. Hence, this means that $\ell+1$ locations cannot be active. Moreover, since $Q_{\ell j}^{(\ell)} < 0$, ℓ locations cannot be active. Now suppose that $Q_{\ell j}^{(\ell)} < 0$. Note that $Q_{\ell j}^{(\ell)}$ is determined by the solution of the Equation (2–8), of which solution should satisfy (i) $\delta_{ij} - \gamma\alpha_{ij}Q_{ij}^{(\ell)} = \gamma b_j(Q_{1j}^{(\ell)} + Q_{2j}^{(\ell)} + \cdots + Q_{\ell j}^{(\ell)})\ \forall i \leq \ell$ and (ii) $\delta_{1j} - \gamma\alpha_{1j}Q_{1j}^{(\ell)} = \delta_{2j} - \gamma\alpha_{2j}Q_{2j}^{(\ell)} = \cdots = \delta_{\ell j} - \gamma\alpha_{\ell j}Q_{\ell j}^{(\ell)}$. It follows from (i) and (ii) that

$$Q_{\ell j}^{(\ell)} = \frac{\delta_{\ell j} + b_j \displaystyle\sum_{i=1}^{\ell} \frac{(\delta_{\ell j} - \delta_{ij})}{\alpha_{ij}}}{\gamma\left(\alpha_{\ell j} + b_j \displaystyle\sum_{i=1}^{\ell} \frac{\alpha_{\ell j}}{\alpha_{ij}}\right)},$$

which means, if $Q_{\ell j}^{(\ell)} < 0$, then $\delta_{\ell j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{\ell j} - \delta_{ij})}{\alpha_{ij}} < 0$. Similarly,

$$Q_{(\ell+1)j}^{(\ell+1)} = \frac{\delta_{(\ell+1)j} + b_j \displaystyle\sum_{i=1}^{(\ell+1)} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}}}{\gamma\left(\alpha_{(\ell+1)j} + b_j \displaystyle\sum_{i=1}^{\ell+1} \frac{\alpha_{(\ell+1)j}}{\alpha_{ij}}\right)}.$$

Now suppose that $Q_{(\ell+1)j}^{(\ell+1)} > 0$, then $\delta_{(\ell+1)j} + b_j \sum_{i=1}^{(\ell+1)} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} = \delta_{(\ell+1)j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} > 0$. On the other hand, $\delta_{(\ell+1)j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} < \delta_{\ell j} + b_j \sum_{i=1}^{\ell} \frac{(\delta_{\ell j} - \delta_{ij})}{\alpha_{ij}}$ as $\delta_{(\ell+1)} < \delta_{\ell}$, which implies $\delta_{(\ell+1)j} + b_j \sum_{i=1}^{(\ell+1)} \frac{(\delta_{(\ell+1)j} - \delta_{ij})}{\alpha_{ij}} < 0$. This is a contradiction, thus, $Q_{(\ell+1)j}^{(\ell+1)} < 0$. $\square$

We note that similar algorithms were proposed for the problems studied by Rhim et al. (2003) and Sáiz and Hendrix (2008), in which each firm may open at most one facility. The algorithm proposed by Sáiz and Hendrix (2008) follows a similar sorting procedure as in Algorithm 1. However, Algorithm 1 is substantively different because we directly consider the impacts of congestion costs (and, therefore, the associated first-order conditions are different). Moreover, although we sort supply locations based on c_{ij} values, as do Sáiz and Hendrix (2008), the interpretation of c_{ij} values is different in our case. That is, in Sáiz and Hendrix (2008), c_{ij} is a constant marginal delivery cost from firm i to market j. In our model, c_{ij} only corresponds to the marginal cost from location i to market j when flow from i to j equals zero (that is, c_{ij} is the intercept of the marginal cost as a function of flow on link (i, j)). Under nonlinear congestion costs, the marginal cost for delivery of an additional unit of flow depends on the existing flow level because of congestion costs, and is therefore a nondecreasing function of the flow on a link.

Algorithm 1 captures some interesting properties of equilibrium solutions when nonlinear congestion costs are considered. In particular, the condition $Q_{\ell j}^{(\ell)} > 0$ in Step 2 holds if only if $\delta_{ij} > \gamma b_j \left(Q_{1j}^{(\ell)} + Q_{2j}^{(\ell)} + \ldots + Q_{(\ell-1)j}^{(\ell)}\right)$, which follows from Equation (2–8). This implies that the parameter $\alpha_{\ell j}$ has no influence on whether or not a location ℓ is active in serving market j (although this parameter does affect the quantity of flow from supply location ℓ to market j when supply point ℓ is active in serving market j). However, $\alpha_{\ell j}$ does influence whether or not location $\ell + 1$ is active in serving market j, through the dependence of $Q_{ij}^{(\ell)}$ on $\alpha_{\ell j}$ for $i < \ell + 1$.

Then, applying Algorithm 1 separately for each market, one can solve for the equilibrium supply quantities. Next, we study the stage-one decisions.

2.4 Stage-One Decisions: Facility Locations

This section studies the firms' supply facility location decisions. We first discuss the rationale behind our prior assumption that all firms make identical location decisions. Then, we seek the best location decision for a single firm, assuming that it will also be the best location decision for the other firms.

2.4.1 Identical Location Decisions

Suppose that we are able to determine the optimal supply quantity decisions for any given location decision matrix $\mathbf{X}^0$, which implies that we can determine the total profit, including the facility location costs, for any given $\mathbf{X}^0$. In the next proposition, we show that if there exists a unique PNE location decision, then each firm chooses the same facility locations in equilibrium.

Proposition 2.5. *Suppose that there exists a unique PNE location matrix, $\boldsymbol{X}^*$. Then, $\boldsymbol{x}_r = \boldsymbol{x}^* \ \forall r \in R$, where $\boldsymbol{x}^*$ denotes the column vector decision for each firm in $\boldsymbol{X}^*$.*

Proof: Suppose that $\mathbf{X}^*$ is the unique PNE location decision such that $\mathbf{x}_{r_1} \neq \mathbf{x}_{r_2}$ for any two firms r_1 and r_2. Then there exists at least one location, say location i, such that firm r_1 does not have a facility while firm r_2 has a facility at location i, that is, $x^*_{ir_1} = 1$ and $x^*_{ir_2} = 0$. Now, if we make $x^*_{ir_1} = 0$ and $x^*_{ir_2} = 1$ in $\mathbf{X}^*$ and construct $\mathbf{X}^{**}$, then $\mathbf{X}^{**}$ is also a PNE location decision since firms are homogeneous with respect to transportation, traffic congestion and facility location costs. This contradicts that $\mathbf{X}^*$ is the unique PNE location decision. □

Proposition 2.5 also follows from the fact that the firms' location decisions form a multi-player symmetric (strongly symmetric; Brant et al., 2009) game with a finite number of strategies (Nash, 1951). For symmetric games, it is well known that a symmetric equilibrium exists, either under pure strategies or mixed strategies (Nash, 1951). Therefore, when there exists a unique PNE location matrix, it will be a symmetric PNE, i.e., each firm makes the same location decisions.

It is important to note that Proposition 2.5 parallels the agglomeration result discussed in Anderson and Neven (1991). In particular, Anderson and Neven (1991) show that the unique equilibrium within spatial competition occurs when two competing firms spatially agglomerate by locating their single facility at the center of a linear market (i.e., the customers are uniformly located on a line segment). A similar result is given in Hamilton et al. (1989). Proposition 2.5 can be considered as a further generalization of the agglomeration result for the case of the competition model studied in this chapter on a discrete network given that a unique PNE location decision exists.

Recall that the method described in the previous section characterizes the equilibrium quantity decisions given that the firms have identical facility locations.

Thus, Proposition 2.5 implies that we can use this method to determine the profits associated with any location matrix, which consists of identical columns, and choose the best among all such solutions to determine the unique PNE. Nevertheless, the uniqueness of a PNE location decision is not guaranteed.

While uniqueness of PNE location decisions implies existence of a symmetric PNE (which is the unique PNE location matrix itself as implied by Proposition 2.5), in the case of multiple PNE location decisions, it is possible that none of the equilibrium points under pure strategies is symmetric. Cheng et al. (2004) discuss the existence of at least one PNE for symmetric games with multiple players such that each player has two strategies. That is, if firms only have one option for locating their facilities, the game corresponding to the location decisions of the firms has at least one PNE solution. We note that the single location case can be solved by considering 2^k solutions with each firm either locating or not locating a facility at the single location. It easily follows from the discussion in the previous section that for any such configuration, the quantity decisions of the firms with a facility will be identical. Moreover, Rhim et al. (2003) prove the existence of a PNE in a competitive facility location game in which firms are allowed to locate at most one facility, by noting that the game can be modeled as a congestion game under the assumption that each market will be supplied from a single location. Then the existence result follows, as congestion games have PNE solutions (Rosenthal, 1973). Nevertheless, the game we study cannot be modeled as a congestion game due to the fact that firms may locate more than one facility. Furthermore, for a given set of facility locations, it is possible that a firm chooses to supply a given market from multiple facilities due to nonlinear traffic congestion costs. Cheng et al. (2004) note that even for symmetric games with two strategies, the existence or uniqueness of a symmetric PNE (i.e., when each player chooses the same strategy) is not guaranteed. Amir et al. (2008) show that a Pareto dominant symmetric PNE exists for supermodular, doubly symmetric games. However, the location decisions for our problem do not constitute a doubly symmetric game.

When a symmetric PNE solution does not exist, this implies that either multiple PNE solutions exist or no PNE location exists. For both of these cases, as previously noted, the corresponding mixed strategy Nash equilibrium (MSNE) will be symmetric. Next, we study MSNE for such cases under the following assumptions:

Assumption 2.1. *Given the location decisions of other firms, a firm will never locate an additional facility if locating this facility reduces profit.*

Assumption 2.2. *Given the location decisions of other firms, if locating an additional facility does not change the firm's total profit, the firm will add this facility.*

Assumption 2.3. *Given the location decisions of other firms, there do not exist multiple distinct location decisions containing an identical number of facilities that result in the same profit level for any firm.*

Note that Assumptions 2.1-2.3 imply that, given the location decisions of other firms, a firm will have a unique choice of location vector. In the next proposition, we show that, under Assumptions 2.1-2.3, a MSNE exists such that the probability of a firm choosing any particular location vector $\mathbf{x}$ is either 0 or equal to some value ρ such that $1 \geq \rho > 0$.

Proposition 2.6. *Suppose that Assumptions 2.1-2.3 hold and that no firm will choose a location decision that is weakly or strictly dominated. Then, there exists a mixed strategy Nash equilibrium with $\rho_r(\boldsymbol{x}) = \rho$ or $\rho_r(\boldsymbol{x}) = 0$ for any location vector $\boldsymbol{x}$, for all $r \in R$, where $\rho_r(\boldsymbol{x})$ denotes the probability that any firm r will choose location vector $\boldsymbol{x}$ and $1 \geq \rho > 0$.*

Proof: Please see Appendix B.

It follows from the proof of Proposition 2.6 that when a unique symmetric PNE location decision does not exist, firms will assign the same probabilities to location vectors that are not dominated in a mixed strategy, and dominated location vectors will be assigned probability 0. Moreover, due to the symmetry of the mixed strategy equilibrium, firms will assign the same probability to each particular location vector.

The problem with using the equilibrium concept as a solution tool for the stage-one decisions is that it fails to determine how firms will choose their facility locations in cases where multiple PNE solutions exist or no PNE location decision exists. We already know from Proposition 2.5 that when the PNE is unique, all firms will choose the same locations and, hence, we can search over one firm's decisions to find an equilibrium solution, as the profits of the firms will be the same when the location decisions are the same. Nevertheless, when multiple PNE solutions exist or when no PNE solution exists, we cannot characterize the firms' actions using the PNE concept. As noted in Proposition 2.6, in the case of multiple or no PNE solutions, when firms determine the probability of choosing a location vector, they will assign the same probabilities, and the probabilities associated with location vectors that are not weakly or strictly dominated are the same for each firm. Therefore, when firms' objectives are to maximize expected profits, they will choose the same facility locations due the symmetry assumption. Thus, firms' location decisions will correspond to a location matrix with identical columns, which produces the maximum profits. Therefore, from this point on, we focus on determining the best location decision of a single firm, assuming that other firms will choose the same locations. We note that the corresponding solution is a PNE when a unique PNE decision exists, and it is the best symmetric PNE when multiple symmetric PNE points exists. In both of these cases, the resulting solution will be a Subgame Perfect Nash equilibrium (Selten, 1975).

Now suppose that either $\mathbf{x}^{1*}$ or $\mathbf{x}^{2*}$ is the best location decision for firm r. To determine which of these is better for firm r, we need to compare the profits of firm r given $\mathbf{X}^0 = \mathbf{X}^1$ and $\mathbf{X}^0 = \mathbf{X}^2$, where each column of $\mathbf{X}^1$ equals $\mathbf{x}^{1*}$ and each column of $\mathbf{X}^2$ equals $\mathbf{x}^{2*}$. Note that we can find the total profit for firm r associated with $\mathbf{X}^1$ and $\mathbf{X}^2$ by determining the profit from supplying markets using the method described in the previous section, and then subtracting the facility location costs associated with $\mathbf{x}^{1*}$ and $\mathbf{x}^{2*}$. A total enumeration scheme would determine the profit for each $\mathbf{X}^0$ such that $\mathbf{X}^0$ has identical columns, and pick the one with maximum profit. In case of alternative optimal solutions, Assumptions 2.1-2.2 can be used as a selection tool.

The resulting matrix $\mathbf{X}^*$ will give the best location decision for firm r as well as for all other firms. However, total enumeration requires evaluating exponentially many location decisions for a firm. In particular, a firm must determine the profit for 2^m location decisions, and choose the one with the maximum profit. As total enumeration is computationally burdensome, we next provide a heuristic method intended to be representative of how individual firms may approach simultaneous location decisions in practice. Our heuristic method first chooses the number of facilities to be located based on a ranking of locations derived from the problem parameters and then, chooses the best locations for these facilities. The comparison of the heuristic method with total enumeration that we later provide in Section 2.5 will characterize conditions under which the method of analyzing location decisions in two steps leads to optimal or near-optimal performance.

2.4.2 Heuristic Method for Identifying Location Matrix

Because we consider a simultaneous game in which a player may not possess all relevant information associated with the other players, it is impossible to provide a general characterization of how an individual firm will approach the decision problem (and to, therefore, characterize the solution that will result). In an attempt to emulate a reasonable approach that might be taken by an individual firm under such conditions, we have constructed a ranking-based heuristic approach in which potential locations are ranked in a preference order based on problem data. The heuristic method we provide is thus based on assigning weights to candidate locations. In particular, the weight of a location is determined by considering the distance from the location to markets, the congestion cost factors on the links connecting the location to markets and the facility location cost at the location. Regarding the profit function of a firm given in Equation (2–3), a firm has transportation, congestion and facility location costs. Furthermore, a firm gains revenue by supplying the markets. Hence, when assigning weights to locations, we consider cost terms related to the location as well as price information from the markets. For instance, a location that is close to a market with a low initial price or high price sensitivity may be less appealing than a location that is further from a market with high initial price or low price sensitivity. To account for market price information, we consider the ratio a_j/b_j for each market as the market potential; that is, a greater a_j/b_j value means that a firm will get more revenue by supplying the market than supplying the same amount to a market with a lower a_j/b_j value.

Specifically, the weight of location i is determined by the expression

$$\omega_i = \frac{\sum_{j\in J} \frac{c_{ij}}{(a_j/b_j)}}{\sum_{i\in I}\sum_{j\in J} \frac{c_{ij}}{(a_j/b_j)}} + \frac{\sum_{j\in J} \frac{\alpha_{ij}}{(a_j/b_j)}}{\sum_{i\in I}\sum_{j\in J} \frac{\alpha_{ij}}{(a_j/b_j)}} + \frac{f_i}{\sum_{i\in I} f_i} \tag{2–9}$$

The weight of location i, ω_i, contains three normalized terms. The first term accounts for the transportation costs from the location to markets, as well as the market potentials. That is, a location close to markets with higher potentials will have a smaller weight than a location close to markets with lower potentials. The second term accounts for congestion costs associated with the location and follows the same logic as the first term. The last term is the normalized facility location cost at the location. Based on Equation (2–3), a location with lower weight is more favorable to firms.

The heuristic method has two phases. In the first phase, a firm decides on the number of facilities to locate as follows. Suppose that a firm is planning to locate ℓ facilities, $\ell \leq m$. We assume that the locations of these ℓ facilities will be the ℓ locations with the lowest weights, and we compute the profit associated with such a location decision. We repeat this process for each $0 \leq \ell \leq m$, and assume that the firm chooses the number of facilities that provides the maximum profit. In the second phase of the heuristic method, a firm determines the best locations for the number of facilities determined in the first phase. Below, we provide a step-by-step description of the algorithm.

Algorithm 2. *2-Phase Heuristic method:*

Phase I: Determining the number of facilities to be located

Step 0. Calculate the location weights using Equation (2–9). Sort locations in non-decreasing order of weight, i.e., $w_i < w_{i+1}$. *Set* $\ell = 0$, $\ell^* = 0$, $\pi^* = 0$, *and go to Step 1.*

Step 1. Construct $\boldsymbol{x}^\ell$ *by locating facilities at locations 1 to* ℓ *and determine the profit of any firm,* π^ℓ, *using Algorithm 1 with* $\boldsymbol{X}^0 = \boldsymbol{X}^\ell$, *where each column of* $\boldsymbol{X}^\ell$ *is* $\boldsymbol{x}^\ell$. *Go to Step 2.*

Step 2. If $\pi^\ell \geq \pi^*$, *set* $\pi^* = \pi^\ell$ *and* $\ell^* = \ell$. *If* $\ell \leq m - 1$ *set* $\ell = \ell + 1$ *and Go to Step 1. If* $\ell = m$, *go to Step 3.*

Step 3. If $\pi^* \geq 0$, *set* $\pi^* = \pi^\ell$ *and* $\ell^* = \ell$ *and, go to Step 4. Else, set* $\pi^* = 0$ *and* $\ell^* = 0$, *and stop.*

Phase II: Finding the best location decision with ℓ^* *facilities*

Step 4: Find the best location decision with ℓ^* *facilities by enumerating the location decisions containing* ℓ^* *ones (locations). Return the best solution.*

Algorithm 2 assumes that a firm determines facility locations in two phases; first, the number of facilities to be located is determined and then the locations for these facilities are determined. We note that Algorithm 2 provides the best location decision for a firm when the firm believes that all other firms will utilize the same weight ranking based approach in deciding the number of facilities to be located. In the next section, we provide numerical results on the efficiency of the heuristic method described in Algorithm 2. We further discuss a counter-intuitive result.

2.5 Numerical Studies

Our numerical studies focus on two kinds of analysis. We first consider the efficiency of the heuristic method provided in the previous section. Following this, we compare the firms' best decisions (i) when firms consider traffic congestion costs in decision making and (ii) when firms disregard traffic congestion costs in decision making.

2.5.1 Efficiency of The Heuristic Method

Our first analysis is aimed at characterizing the efficiency of the heuristic method provided in the previous section. We generate data for our computational tests in the following way. We consider eight problem classes, where each problem class differs in congestion cost factors, α_{ij}, transportation costs, c_{ij}, and facility location costs, f_i. By considering different problem classes, the goal is to provide a more conclusive analysis (rather than solving a specific class of problem for which the heuristic method is quite efficient). For each of the classes, we use all combinations of $k \in \{3, 5\}$, $n \in \{3, 5, 7\}$ and $m \in \{3, 5, 7, 10, 15\}$, resulting in 30 combinations of the values of k, n, and m. For each of these combinations, we generate 10 problem instances. For every problem, we let $a_j \sim U[50, 150]$ and $b_j \sim U[1, 2]$, where $U[l, u]$ denotes the uniform distribution on $[l, u]$. Table 2-1 gives the distribution range for α_{ij}, c_{ij} and f_i values in each problem class. In each problem class, we solve 300 problem instances and each problem instance is solved using total enumeration and the heuristic method stated in Algorithm 2.

Table 2-2 compares total enumeration with Algorithm 2 for a given firm's number of facilities (# of fac.), total quantity supplied to markets (Supply Quant.), total profit, and CPU time in seconds, and documents the optimality gap (Opt. gap). As can be seen from Table 2-2, the 2-Phase heuristic method is of course faster than total enumeration, and the average solution obtained by the 2-Phase heuristic method has an average optimality gap of 2.23% over the 2400 problem instances solved. Moreover, the 2-Phase

Table 2-1. Data ranges for problem classes 1-8 for analysis 1

	α_{ij}	c_{ij}	f_i
Class 1	(0,4]	(0,50]	[75,125]
Class 2	(0,4]	(0,50]	[100,150]
Class 3	(0,4]	[25,75]	[75,125]
Class 4	(0,4]	[25,75]	[100,150]
Class 5	[4,8]	(0,50]	[75,125]
Class 6	[4,8]	(0,50]	[100,150]
Class 7	[4,8]	[25,75]	[75,125]
Class 8	[4,8]	[25,75]	[100,150]

heuristic solution has parallel results with total enumeration for the average number of facilities located and the average of total quantities supplied to markets.

Table 2-2. Comparison of total enumeration and 2-phase heuristic method

	Total Enumeration				2-Phase Heuristic				
	# of fac.	Supply Quant.	Total Profit	CPU time	# of fac.	Supply Quant.	Total Profit	CPU time	Opt. gap
Class 1	4.24	50.81	2885.37	37.53	4.92	51.17	2854.41	4.22	1.47%
Class 2	3.86	50.12	2725.29	52.76	4.44	50.48	2687.25	5.45	1.51%
Class 3	3.32	35.36	1478.10	51.70	4.04	35.83	1441.29	5.09	2.96%
Class 4	2.95	34.68	1404.47	86.33	3.64	35.23	1356.59	6.94	3.86%
Class 5	5.81	33.37	1125.47	90.08	5.95	33.52	1121.38	7.95	1.64%
Class 6	5.18	32.38	1009.38	38.88	5.35	32.60	1005.28	4.09	0.63%
Class 7	3.95	20.69	445.14	54.39	4.08	20.74	439.42	6.10	3.23%
Class 8	3.16	18.41	336.58	41.34	3.25	18.47	330.20	4.02	2.57%
Avg.	4.06	34.48	1426.22	56.63	4.46	34.76	1404.48	5.48	2.23%

In Table 2-3, we compare the total enumeration and 2-Phase heuristic method solutions for problem instances with the same number of potential facility locations, i.e., for problems with $m = 3$, $m = 5$, $m = 7$, $m = 10$ and $m = 15$. We note that as the number of potential locations increases, the computation time advantage of the 2-Phase heuristic method increases as well. On the other hand, the optimality gap does not show a clear increasing or decreasing trend as the number of locations increases. Therefore, we can say that 2-Phase heuristic method is robust and the solution quality of Algorithm 2 is not clearly decreasing as the problem size increases, although Algorithm 2 becomes substantially more efficient computationally.

Table 2-3. Comparison of total enumeration and 2-phase heuristic method for each m

	Total Enumeration				2-Phase Heuristic				
m	# of fac.	Supply Quant.	Total Profit	CPU time	# of fac.	Supply Quant.	Total Profit	CPU time	Opt. gap
3	2.39	25.99	982.30	0.02	2.43	26.04	980.28	0.01	0.83%
5	3.40	30.42	1200.24	0.08	3.62	30.66	1189.57	0.02	2.08%
7	4.03	33.25	1366.70	0.37	4.41	33.57	1347.37	0.08	2.64%
10	4.66	35.74	1503.70	4.05	5.15	35.96	1473.26	0.52	2.85%
15	5.80	46.98	2078.18	278.62	6.70	47.54	2031.91	26.79	2.77%
Avg.	4.06	34.48	1426.22	56.63	4.46	34.76	1404.48	5.48	2.23%

From the analysis of Tables 2-2 and 2-3, we conclude that when a firm determines its facility locations using a two-phase approach (such that in the first phase, the number of facilities is determined by sorting potential facility locations with respect to weights; Equation (2–9) in our case), the resulting solution approach is computationally efficient, and the relative performance as measured by the optimality gap is relatively strong. Furthermore, the number of potential locations does not heavily influence the optimality gap. This suggests that the strategy of deciding locations in two phases makes sense. This also suggests a future research direction beyond the scope of this chapter, in which the game of the firms corresponds to a three-stage game. In the first stage, the number of facilities to be located is determined; then, in the second stage, facility locations are chosen and, finally, in the third stage, the supply quantities are determined.

Based on the analysis of the heuristic method, we can also argue that ranking potential facility locations based on a weight defined similar to Equation (2–9), considering the low optimality gap of the heuristic method, suggests that firms will prefer to locate facilities at locations that are connected to markets with high potentials, i.e., high a_j/b_j values, via shorter and less congested links and that have lower facility location costs.

2.5.2 Accounting for Congestion in Decision Making

This section compares the decisions of the firms (i) when all of the firms explicitly consider traffic congestion costs and (ii) when all firms disregard traffic congestion costs in their location and supply quantity decisions in the general case. In particular, we compare two cases: (i) when all of the firms are aware of congestion in the network and account for congestion costs in their decisions (i.e., they are indeed maximizing the profit function defined in Equation (2–3)) and (ii) when all of the firms disregard congestion costs in their decisions (i.e., they are maximizing $\Pi_r(\mathbf{Q}, \mathbf{X}) = \sum_{j \in J} p_j \left(\sum_{i \in I} \sum_{r \in R} q_{ijr} \right) \sum_{i \in I} q_{ijr} - \sum_{j \in J} \sum_{i \in I} c_{ij} q_{ijr} - f_r(\mathbf{x}_r)$ instead of Equation (2–3)), but still face congestion costs after they implement their decisions (i.e., they pay traffic congestion costs $\sum_{j \in J} \sum_{i \in I} q_{ijr} g_{ij} \left(\sum_{r \in R} q_{ijr} \right)$ as a result of their decisions after they make their decisions). Firms in Case (i) will determine their quantity decisions using Algorithm 1, and determine facility location decisions using total enumeration. Firms in Case (ii) do not consider traffic congestion costs in their decisions and, hence, we cannot use Algorithm 1 directly to determine equilibrium quantity decisions. On the other hand, using the next proposition, we show that when firms are not aware of congestion, they will supply a market from the closest facility to the market, and each firm will supply the same quantity.

Proposition 2.7. *Suppose that $\alpha_{ij} = 0 \ \forall i \in I, j \in J$. Given $\boldsymbol{X}^0$ such that $\boldsymbol{X}^0$ consists of identical columns, $q^*_{ijr} = b_j \delta_{ij}/(k+1)$ for $i = i^*$ and $q^*_{ijr} = 0$ for $i \neq i^* \ \forall r \in R$, where $i^* = \arg\max_{i \in I^0} \{\delta_{ij}\}$.*

Proposition 2.7 provides a solution method to find the equilibrium quantities for given location decisions $\mathbf{X}^0$ such that $\mathbf{X}^0$ consists of identical columns for Case (ii). Regarding the discussion in the previous section, total enumeration can still be used for Case (ii) to determine the location decisions.

We generate data for our computational tests in the following way. We consider two problem classes, where each problem class has 8 parameter distribution settings. That is, for each problem class, and for each of the three parameters of interest (c_{ij}, f_i, and α_{ij}), we have two uniform distributions from which parameter values are drawn (resulting in $2^3 = 8$ combinations of distribution settings). For each of these 8 combinations within a class, we use all combinations of $k \in \{3, 5\}$, $n \in \{3, 5, 7\}$ and $m \in \{3, 5, 7, 10\}$, resulting in 24 combinations of the values of k, n, and m. For each of these combinations, we generate 25 problem instances. For every problem, we let $a_j \sim U[50, 150]$ and $b_j \sim U[1, 2]$. Table 2-4 gives the distribution range for the α_{ij}, c_{ij} and f_i values in each data category, where B_i denotes data category i. We solve each problem instance for firms in Cases (i) and (ii). If the total profit of any single firm in Case (ii) is negative, we exclude this instance from our analysis, since we assume that firms will not participate when they have negative profits. In particular, this results in more than 15 problem instances in each of the 24 sets for each of the 8 categories for Problem Classes 1 and 2.

Table 2-4. Data categories for problem classes 1 and 2

	Class 1			Class 2		
	α_{ij}	c_{ij}	f_i	α_{ij}	c_{ij}	f_i
B1	(0,0.25]	[25,75]	[50,100]	[0.5,1]	[25,75]	[50,100]
B2	(0,0.25]	[25,75]	[75,125]	[0.5,1]	[25,75]	[75,125]
B3	(0,0.25]	[50,100]	[50,100]	[0.5,1]	[50,100]	[50,100]
B4	(0,0.25]	[50,100]	[75,125]	[0.5,1]	[50,100]	[75,125]
B5	[0.25,0.5]	[25,75]	[50,100]	[0.75,1.25]	[25,75]	[50,100]
B6	[0.25,0.5]	[25,75]	[75,125]	[0.75,1.25]	[25,75]	[75,125]
B7	[0.25,0.5]	[50,100]	[50,100]	[0.75,1.25]	[50,100]	[50,100]
B8	[0.25,0.5]	[50,100]	[75,125]	[0.75,1.25]	[50,100]	[75,125]

Intuitively, we would expect that firms in Case (i) have higher profits since they consider traffic congestion in their decisions, whereas firms in Case (ii) disregard traffic congestion in their decisions, but pay for congestion after their decisions are implemented. However, our numerical results imply that the opposite is also possible. Tables 2-5 and 2-6 compare Cases (i) and (ii) for each Problem Class. For Problem Class 1, we see that the average total profit for a single firm in Case (ii) is higher than the average total profit of a single firm in Case (i), whereas, we have the opposite for Problem Class 2. This result for Problem Class 1 implies that firms may actually increase their profits if they do not consider traffic congestion in their decisions. This phenomenon can be explained as follows. For our problem, firms are competing on two dimensions: the price in a market and the congestion on links connecting supply locations and markets. For Case (ii), since the congestion cost is disregarded in the decision making process, firms compete only on market price. So when the impact of congestion cost is relatively small and when firms compete only on market price, they may actually end up with higher profit.

Table 2-5. Statistics of cases (i) and (ii) for problem class 1

	# of fac.	Supply Quant.	Trans. Cost	Cong. Cost	Loc. Cost	Total Profit
Case (i)	2.47	34.38	1467.66	201.54	210.56	1628.94
Case (ii)	2.24	39.08	1642.61	415.57	191.85	1661.44

Table 2-6. Statistics of cases (i) and (ii) for problem class 2

	# of fac.	Supply Quant.	Trans. Cost	Cong. Cost	Loc. Cost	Total Profit
Case (i)	3.05	28.63	1275.48	365.74	257.76	1107.56
Case (ii)	2.26	39.30	1656.18	1427.80	193.18	662.54

Next, we provide a simple example to illustrate the phenomenon in which Case (ii) results in higher profit.

Example 2.1. *Consider two firms competing in a single market, market* 1. *There are two potential locations,* 1 *and* 2, *at which the firms may locate facilities. Suppose that facility location costs are 0 at both locations, i.e.,* $f_1 = f_2 = 0$. *Let* $c_{11} = 80$, $c_{21} = 90$, *and* $\alpha_{11} = 0.25$, $\alpha_{21} = 0.5$. *The market parameters are* $a_1 = 100$ *and* $b_1 = 1$. *The total quantity supplied to the market and the corresponding total profit for a single firm for Case (i) are* 5.33 *and* 64.00, *respectively. The total quantity supplied to the market and the corresponding total profit for a single firm for Case (ii) are* 6.67 *and* 66.67, *respectively. In both of the cases, only the facilities at location* 1 *supply market* 1.

As can be observed from the solution of Example 2.1, a firm is more profitable under Case (ii), i.e., disregarding congestion costs in decision making may result, in some cases, in higher profits, even under a PNE solution for both quantity and facility location decisions (note that when both firms locate facilities at both locations, this corresponds to a PNE location decision, since facility location costs are 0).

CHAPTER 3
ANALYSIS OF TRAFFIC CONGESTION COSTS IN A COMPETITIVE SUPPLY CHAIN

3.1 Motivation and Literature Review

Research on traffic network equilibrium problems, toll pricing (congestion pricing), and methods to mitigate traffic congestion have typically focused on the welfare of individual road users. However, recent studies identify the negative impacts traffic congestion has on supply chain operations. In particular, the performance of logistics operations is affected by traffic congestion, and these impacts are more drastic in Just-in-Time (JIT) production systems. Despite the fact that traffic congestion affects supply chain operations, most of the studies combining traffic congestion and supply chains are based on empirical data and lack theoretical results. Past literature also typically assumes that traffic congestion effects are exogenous, and these effects are analyzed indirectly by assuming that increased congestion either implies increased travel times or decreased travel time reliability. More importantly, traffic congestion effects are only studied insofar as they affect the distribution network of a single firm. In this chapter, we focus on the effects of traffic congestion on supply chain operations by modeling traffic congestion costs endogenously.

We study two primary supply chain decisions: facility location decisions and supply quantity decisions. McKinnon et al. (2008) note that companies may restructure their distribution systems due to increased traffic congestion. Moreover, Rao et al. (1991) note that changes in facility locations are often a long-term reaction to increased traffic congestion. For example, Lee (2004) points out that when 7-11 Japan (SEJ), a convenience-store company, located stores in key locations, SEJ was subject to more dramatic effects of traffic congestion. It is also worth noting that the effects of traffic congestion sometimes depend on the facility location choices of a company (see, e.g., Sankaran et al., 2005). Therefore, it is important to gain a better understanding of the effects of traffic congestion on facility location and distribution flow decisions.

McKinnon (1999) presents survey results on the negative effects of traffic congestion on the efficiency of logistics operations. In a similar study, McKinnon et al. (2008) note that, on average, traffic congestion accounts for 23% of the total delay times in shipments of the companies completing the survey. This rate can be higher (up to 34%) in some industries (McKinnon et al., 2008). For instance, Fernie et al. (2000) point out that traffic congestion is one of the most important factors affecting cost and service in grocery retailing in the UK. Sankaran et al. (2005) also document the results of a survey and discuss the effects of traffic congestion on supply chain operations. A systematic review of studies at the intersection of traffic congestion and supply chains can be found in Weisbrod et al. (2001), which also discusses how traffic congestion affects costs and productivity. Another stream of research studies traffic congestion in JIT systems. Because JIT systems require small lot sizes, Rao et al. (1991) note that this results in increased traffic congestion. Moreover, their survey results indicate that companies are aware of the associated congestion impacts and they propose short- and long-term methods to mitigate the effects of congestion. Moinzadeh et al. (1997) study the relationship between small lot sizes and traffic congestion for a company's distribution system, with multiple retailers using a common, congested road. Rao and Grenoble (1991) also study the effects of JIT replenishment and the resulting traffic congestion on distribution costs. One other field of research that combines traffic congestion and supply chains focuses on freight distribution. For example, Figliozzi (2009) studies the effects of traffic congestion on the costs associated with commercial vehicle tours, while Figliozzi (2006) and Figliozzi et al. (2007) analyze freight tours in congested urban areas. Similarly, Golob and Regan (2001, 2003) also study the impacts of traffic congestion on freight operations. We study the effects of traffic congestion on firms' facility location and distribution flow decisions in a competitive environment. This study is motivated by the observation that traffic congestion on a shared distribution network constitutes a form of competition for common distribution resources. Furthermore, we assume that the firms also compete within common markets in their sales volumes.

The model under consideration in this chapter considers facility locations and supply quantity decisions in a competitive environment on a congested distribution network. In particular, we utilize the competitive location game defined in Chapter 2. That is, the following setting is assumed. Competing firms are noncooperative and they must simultaneously determine their facility locations (first stage decisions); then, the supply quantities flowing out of these facilities into each market (second stage decisions) must be determined. Firms may locate more than one facility. In practice, firms may prefer to locate more than one facility rather than a single facility even to supply a single facility due to economies or diseconomies of scale. Specifically, Weisbrod et al. (2001) point out that higher level of congestion related costs are caused by higher shipping levels. Hence, in our formulation of traffic congestion costs, firms are subject to diseconomies of scale, which motivates firms to locate more than one facility to avoid severe traffic congestion related costs. Markets and possible facility locations are represented as vertices in a network. Firms are assumed to compete under a homogeneous cost structure; that is, they have identical cost parameters. For this reason, we assume that firms make the same facility location decisions when maximizing expected profits (when a unique Pure Nash Equilibrium does not exist). We assume oligopolistic Cournot competition in the second stage, i.e., the total supply to a market determines the price in that market.

As noted in Chapter 2, Cournot competition is one of the most common concepts used in modeling competitive markets. A Cournot oligopoly can be used to represent energy markets (Salant, 1982, Oren, 1997, Ventosa et al., 2005), agricultural markets (Klemperer and Meyer, 1986), grocery retailing markets (Mazzarotto, 2001, Ellickson, 2006, Arnade et al., 2007, Colangelo, 2008), and airline industries (Brander and Zhang, 1990, 1993, Oum et al., 1995, Park and Zhang, 1998, Pels and Verhoef, 2004).

Moreover, congestion is present in such markets (Oren, 1997, Fernie et al., 2000, Pels and Verhoef, 2004, Basso and Zhang, 2008). Therefore, studying a competitive location problem with Cournot competition and congestion costs is important for a more complete analysis of such markets.

This chapter contributes to the literature by providing an analytical characterization of the effects of traffic congestion on competitive firms' equilibrium facility location and supply quantity decisions. Considering the case with identical supply firms enables us to explicitly analyze and characterize how congestion costs affect the structure of equilibrium decisions, and to use this analysis to provide insights into how equilibrium solutions change in response to changes in congestion levels and costs. We first summarize the results of Chapter 2. Then, we provide analytical results on the effects of traffic congestion costs on the equilibrium quantities flowing from supply points to markets. We also discuss results for a special case of the problem when the potential facility locations are within market areas. We note that the findings of this study will have analogous results in the case without competition, i.e., the structural and qualitative results are analogous to the analysis of traffic congestion costs on a single firm's supply quantity and facility location decisions, as the firms will end up with the same decisions. Nevertheless, as aforementioned, we assume competition among the firms in the distribution network in order to highlight the way in which competitors' decisions affect a firm. We perform extensive numerical studies that illustrate the effects of traffic congestion on a firm's location and quantity decisions. Further numerical studies are conducted for the special case of the problem in which market areas serve as the only potential sites for facilities as well.

The rest of this chapter is organized as follows. In Section 3.2, we define the model used for the analysis of traffic congestion effects. Moreover, we briefly summarize the results of Chapter 2 that are required in this chapter. Section 3.3 analyzes the effects of increased traffic congestion on equilibrium supply quantities and discusses the implications of the results for a special case. In Section 3.4, we provide the results of extensive numerical studies that characterize the effects of traffic congestion on facility location and supply quantity decisions, and the effects of congestion for the special case.

3.2 Model and Analysis

In this section, we summarize the model used to analyze the effects of traffic congestion on facility location and equilibrium supply quantities along with its analysis. As noted, the symmetric competitive multi-facility location game with traffic congestion costs defined in Chapter 2 is utilized in this chapter. That is, we consider a set of k competitive firms considering the location of facilities at m possible locations in order to serve customer markets at n locations. Each firm incurs transportation, traffic congestion, and facility location costs as a result of their location and distribution volume decisions. More explicitly, firms are subject to linear transportation costs in the quantity shipped from a facility to a market and the traffic congestion cost incurred is convex in the total quantity shipped from a facility to a market. A fixed facility location cost exists for each location i. Moreover, we assume that any open facility is effectively uncapacitated and, hence, a firm will not open more than one facility at a location. The notation of Chapter 2 is used throughout this chapter as well. We define additional notation as needed.

We assume the unit price in each market is defined by Equation (2–1). We assume that the transportation cost is linear in the quantity of flow on link (i, j) and $c_{ij} \geq 0$ represents a per unit transportation cost. The function g_{ij} defined in Equation (2–2), which is a function of the total quantity of flow on link (i, j), determines the traffic congestion cost on link (i, j). That is, $g_{ij}(q_{ij\bullet}) = \alpha_{ij} q_{ij\bullet}$, where $\alpha_{ij} > 0$ denotes the traffic congestion cost factor. Hence, the congestion cost incurred by a firm using link (i, j) increases with the total quantity of flow on the link as well as with the quantity sent by the firm on that link. In particular, the congestion cost for firm r is $\alpha_{ij} q_{ijr} q_{ij\bullet} = 0$

when $q_{ijr} = 0$. On the other hand, when $q_{ijr} > 0$, the congestion cost of firm r equals $\alpha_{ij} q_{ijr} q_{ij\bullet} > 0$ and is convex and increasing in q_{ijr} when the quantities sent by other firms on the link are fixed. Thus, the firm's congestion cost is a nondecreasing convex function of the quantity sent by the firm on the link. It should be noted that the functional form of Equation (2–2) does not consider a fixed capacity limit on the distribution network flow. Nevertheless, this choice of functional form reflects the nature of traffic congestion, as congestion costs increase in volume at an increasing rate. This is compatible with the note in Weisbrod et al. (2001), which emphasizes that companies with higher shipping levels are subject to a higher level of congestion related costs. Furthermore, this function is different than the functional forms used to formulate congestion related costs in queueing models. Queueing models are studied to analyze traffic flow problems for individual road users (Heidemann, 1994, Heidemann and Wegmann, 1997, Vandaele et al., 2000, Woensel and Vandaele, 2006, Woensel and Vandaele, 2007). Generally, congestion costs are formulated as a function of the flow on a link, speed at the current flow, and the value of time for a traveler, along with the road characteristics such as traffic density and free flow speed. This type of congestion modeling also implies that traffic congestion costs increase at an increasing rate with the flow on a link, i.e., convexity of the congestion costs (Li, 2002, Woensel and Cruz, 2009). Therefore, the congestion modeling approach used in Chapters 2 and 3 is consistent with queueing models studied for traffic flow problems.

Recall that, the profit function of firm r reads as

$$\Pi_r(\mathbf{Q}, \mathbf{X}) = \sum_{j \in J} \left[(a_j - b_j q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I} c_{ij} q_{ijr} - \sum_{i \in I} \alpha_{ij} q_{ijr} q_{ij\bullet} \right] - f_r(\mathbf{x}_r),$$

where the first term is the revenue gained by serving markets, the second term is the total transportation cost, the third term is the total traffic congestion cost, and the last term is the total facility location cost.

In Chapter 2, we adopted a two-stage solution approach, which first employs the Nash Equilibrium concept of Nash (1951) to determine the firms' supply quantity decisions for a fixed set of location decisions, and then focuses on the solution for the Stage-one decisions. Next, we summarize the results associated with Stage-one and Stage-two decisions.

3.2.1 Stage-One Decisions

Recall from Chapter 2 that the firms' location decisions form a multi-player symmetric (strongly symmetric; Brant et al., 2009) game with a finite number of strategies. It thus follows that the uniqueness of the PNE location decision implies a symmetric PNE location matrix (Nash, 1951). Furthermore, when there exists a unique PNE location matrix, the search for an equilibrium location matrix can be restricted to location decisions such that each firm chooses the same facility locations. We can thus use the Algorithm 1, which determines the quantity decisions under identical facility location decisions to characterize the profit of each such location matrix. Thus, choosing the best among all solutions with identical columns determines the unique PNE. On the other hand, it is possible that multiple PNE location decisions exist, or that a PNE location decision does not exist. The problem with using the equilibrium concept as a decision mechanism for location decisions in this case is that it fails to explain and characterize firms' actual decisions in such cases (Harsanyi and Selten, 1988). Thus, if firms determine facility locations purely based on expected profits (assuming that any location vector is equally likely for any firm), then since firms are homogeneous, they will make the same decisions. We can therefore determine firms' location decisions by choosing the best among all location matrices with identical columns. Moreover, in the case of multiple or no PNE solutions, when firms determine the probability of choosing a location vector, it is shown in Chapter 2 that firms will assign the same probabilities, and the probability of choosing each location vector will be the same in a mixed-strategy Nash Equilibrium.

A total enumeration scheme would determine the profit for each location vector $\mathbf{X}^0$ such that $\mathbf{X}^0$ has identical columns, and pick the one with maximum profit. Although total enumeration is computationally burdensome, in our numerical studies we use total enumeration because our goal is to analyze the effects of congestion costs on the best decisions of the firms.

3.2.2 Stage-Two Decisions

The second-stage decisions constitute a non-cooperative game among the firms, who simultaneously determine how much to send from facilities to markets given the location decision for each firm. Note that unlike the previous studies by Rhim et al. (2003) and Sáiz and Hendrix (2008), the firms not only compete based on price, but also as a result of traffic congestion cost functions on supply links.

As is discussed in Section 2.3 in Chapter 2, given facility location decisions, the Stage-two decisions can be analyzed separately for each market. Algorithm 1 determines the number of the active locations and the corresponding equilibrium flow quantities, given that $\mathbf{x}_r = \mathbf{x}^0 \; \forall r \in R$, for a single market. Note that we study the multiple-market scenario in our analysis of traffic congestion as firms' problems are not separable for each market when the facility location costs are included (i.e., the first-stage decisions). This is because a facility that might not be profitable in the case of a single market may be profitable when it serves several markets. Furthermore, for a special case of our problem that we later discuss, we analyze the case in which firms' facility location choices are confined to a subset of the markets. In what follows, we provide a property of Algorithm 1 that will be utilized in the analysis of effects of traffic congestion on equilibrium supply quantities.

Suppose that there are ℓ active locations at market j. Consider the w^{th} iteration of Step 2 in Algorithm 1. Let $Q_{ij}^{(w)}$ be the tentative quantities calculated at the w^{th} iteration using Equation (2–8). (Note that $Q_{ij}^{(\ell)} = Q_{ij}^{*}$.) In the next proposition, we show that the quantity supplied from an active location decreases as the number of active locations increases at each iteration of Algorithm 1, whereas the total quantity supplied to the market increases.

Proposition 3.1. *(a)* $Q_{sj}^{(w)} > Q_{sj}^{(w+1)}$ *for location* s, $s \leq w$ *and* $w+1 \leq \ell$. *(b)* $\sum_{i=1}^{w} Q_{ij}^{(w)} < \sum_{i=1}^{w+1} Q_{ij}^{(w+1)}$, $w+1 \leq \ell$.

Proof: In the w^{th} iteration, we have tentative equilibrium quantities for locations 1 to w, which are the solutions to

$$\delta_{ij} - \gamma\alpha_{ij} Q_{ij}^{(w)} = \gamma b_j (Q_{1j}^{(w)} + Q_{2j}^{(w)} + \cdots + Q_{wj}^{(w)}) \ \forall i \leq w. \tag{3–1}$$

It follows from Equation (3–1) that

$$\delta_{1j} - \gamma\alpha_{1j} Q_{1j}^{(w)} = \delta_{2j} - \gamma\alpha_{2j} Q_{2j}^{(w)} = \cdots = \delta_{wj} - \gamma\alpha_{\ell j} Q_{wj}^{(w)}. \tag{3–2}$$

Equations (3–1) and (3–2) imply that, for location s, $s \leq w$, in the w^{th} iteration

$$Q_{sj}^{(w)} = \frac{\delta_{\ell j} + b_j \displaystyle\sum_{i=1}^{w} \frac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}}}{\gamma \left(\alpha_{sj} + b_j \displaystyle\sum_{i=1}^{w} \frac{\alpha_{sj}}{\alpha_{ij}} \right)}. \tag{3–3}$$

Equation (3–3) gives the equilibrium quantity for location s when $w = \ell$. Now suppose that $Q_{sj}^{(w)} \leq Q_{sj}^{(w+1)}$ for any $s \leq w < \ell$. By Equation (3–3) this means

$$\frac{\delta_{sj} + b_j \displaystyle\sum_{i=1}^{w} \frac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}}}{\alpha_{sj} + b_j \displaystyle\sum_{i=1}^{w} \frac{\alpha_{sj}}{\alpha_{ij}}} \leq \frac{\delta_{sj} + b_j \displaystyle\sum_{i=1}^{w} \frac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}} + b_j \frac{(\delta_{sj} - \delta_{(w+1)j})}{\alpha_{(m+1)j}}}{\alpha_{sj} + b_j \displaystyle\sum_{i=1}^{w} \frac{\alpha_{sj}}{\alpha_{ij}} + b_j \frac{\alpha_{sj}}{\alpha_{(w+1)}}}$$

It follows from the above inequality that

$$\delta_{(w+1)j} \leq \frac{-\alpha_{sj} A + \delta_{sj} B}{B + \alpha_{sj}} = \delta_{sj} - \alpha_{sj} \frac{\delta_{sj} + A}{\alpha_{sj} + B}$$

where $A = b_j \sum_{i=1}^{w} \frac{(\delta_{sj} - \delta_{ij})}{\alpha_{ij}}$ and $B = b_j \sum_{i=1}^{w} \frac{\alpha_{sj}}{\alpha_{ij}}$. Thus, considering Equation (3–3), the above inequality reads as

$$\delta_{(w+1)j} \leq \delta_{sj} - \gamma \alpha_{sj} Q_{sj}^{(w)}.$$

Equation (3–1) implies that the above inequality can be written as $\delta_{(w+1)j} \leq \gamma b_j (Q_{1j}^{(w)} + Q_{2j}^{(w)} + \cdots + Q_{wj}^{(w)})$. Furthermore, it follows from Equation (3–1), when $Q_{sj}^{(w)} \leq Q_{sj}^{(w+1)}$, we have $\gamma b_j \sum_{i=1}^{w} Q_{ij}^{(w)} \geq \gamma b_j \sum_{i=1}^{w} Q_{ij}^{(w+1)}$. This implies that $\delta_{(w+1)j} \leq \gamma b_j (Q_{1j}^{(w+1)} + Q_{2j}^{(w+1)} + \cdots + Q_{wj}^{(w+1)})$. We can write the last inequality as $\delta_{(w+1)j} \leq \gamma b_j (Q_{1j}^{(w+1)} + Q_{2j}^{(w+1)} + \cdots + Q_{wj}^{(w+1)} + Q_{(w+1)j}^{(w+1)}) - \gamma b_j Q_{(w+1)j}^{(w+1)}$. Moreover, from Equation (3–1), we have that $\gamma b_j (Q_{1j}^{(w+1)} + Q_{2j}^{(w+1)} + \cdots + Q_{wj}^{(w+1)} + Q_{(w+1)j}^{(w+1)}) = \delta_{(w+1)j} - \gamma \alpha_{(w+1)j} Q_{(w+1)j}^{(w+1)}$. Thus, we have $\delta_{(w+1)j} \leq \delta_{(w+1)j} - \gamma \alpha_{(w+1)j} Q_{(w+1)j}^{(w+1)} - \gamma b_j Q_{(w+1)j}^{(w+1)}$, which means $(\gamma \alpha_{(w+1)j} + \gamma b_j) Q_{(w+1)j}^{(w+1)} \leq 0$, which is a contradiction since at the $(w+1)^{th}$ iteration of the algorithm we check that $Q_{(w+1)j}^{(w+1)} > 0$ and then define $Q_{sj}^{(w+1)}$ values. This contradiction proves Statement (a). Statement (b) is a direct result of Statement (a) and Equation (3–1). □

The next section characterizes the effects of traffic congestion costs on the equilibrium quantity decisions, given that firms' facility location decisions are identical.

3.3 Effects of Traffic Congestion on Equilibrium Supply Quantities

In this section, we analyze the changes in the equilibrium supply quantities when the congestion cost factor on one of the links connecting a supply location to a market increases. Note that we assume the facility location decisions of the firms are fixed and identical. Suppose that locations are sorted such that location 1 has the greatest δ_{ij} value. Hence, when there are ℓ locations active in a market, these locations will be the first ℓ locations.

We first note that when there are ℓ locations active at a market initially, an increase in the traffic congestion cost factor for one of these active locations will not result in any of the initially active locations at that market becoming inactive at the market. The next proposition formalizes this result.

Proposition 3.2. *Consider α_{sj}^1 and α_{sj}^2 such that $\alpha_{sj}^1 < \alpha_{sj}^2$, and suppose that locations 1 to ℓ are active at market j under the α_{sj}^1 value, $1 \leq s \leq k$. Then locations 1 to ℓ are also active at market j under the α_{sj}^2 value.*

Proof: We first show that $Q_{zj}^{1(\ell)} < Q_{zj}^{2(\ell)}$ for location $z \leq \ell$, $z \neq s$, where $Q_{zj}^{1(\ell)}$ and $Q_{zj}^{2(\ell)}$ represents the quantities at the ℓ^{th} iteration of Algorithm 1 under α_{sj}^1 and α_{sj}^2 values, respectively. Suppose that $Q_{zj}^{1(\ell)} \geq Q_{zj}^{2(\ell)}$. By Equation (3–3) this means

$$\frac{\delta_{zj} + b_j \displaystyle\sum_{i=1, i\neq s}^{\ell} \frac{(\delta_{zj} - \delta_{ij})}{\alpha_{ij}} + b_j \frac{(\delta_{zj} - \delta_{sj})}{\alpha_{sj}^1}}{\alpha_{zj} + b_j \displaystyle\sum_{i=1, i\neq s}^{\ell} \frac{\alpha_{zj}}{\alpha_{ij}} + b_j \frac{\alpha_{zj}}{\alpha_{sj}^1}} \geq \frac{\delta_{zj} + b_j \displaystyle\sum_{i=1, i\neq s}^{\ell} \frac{(\delta_{zj} - \delta_{ij})}{\alpha_{ij}} + b_j \frac{(\delta_{zj} - \delta_{sj})}{\alpha_{sj}^2}}{\alpha_{zj} + b_j \displaystyle\sum_{i=1, i\neq s}^{\ell} \frac{\alpha_{zj}}{\alpha_{ij}} + b_j \frac{\alpha_{zj}}{\alpha_{sj}^2}}.$$

After simplifications, the above inequality implies that $\delta_{sj} \leq 0$, which is a contradiction since location s is assumed to be active at market j initially. This contradiction establishes that $Q_{zj}^{1(\ell)} < Q_{zj}^{2(\ell)}$. Hence, as $Q_{zj}^{1(\ell)} > 0$, we have $Q_{zj}^{2(\ell)} > 0$, i.e., location z is still active at market j. Moreover, considering Equation (3–1), $Q_{zj}^{1(\ell)} < Q_{zj}^{2(\ell)}$ implies that $\sum_{i=1}^{\ell} Q_{ij}^{1(\ell)} > \sum_{i=1}^{\ell} Q_{ij}^{2(\ell)}$. Since, $\delta_{sj} > \gamma b_j \sum_{i=1}^{\ell} Q_{ij}^{1(\ell)}$ we have $\delta_{sj} > \gamma b_j \sum_{i=1}^{\ell} Q_{ij}^{2(\ell)}$, i.e., we have $Q_{sj}^{2(\ell)} > 0$ and location s is still active at market j. □

Proposition 3.2 implies that when the traffic congestion cost factor for one of the initially active locations at a market increases, it is possible that the total number of active locations at that market may increase. Moreover, the initially active locations at the market will continue to be active at the market. Next we study the cases (i) when the number of active locations at the market remains the same and (ii) when the number of active locations at the market increases.

(i) When the number of active locations at a market remains the same, we know that all of the initially active locations will remain active at that market. That is, the set of active locations at the market remains the same. This case also captures the situation when all of the locations are initially active at the market. In this case, the quantity supplied to the market from the location for which the traffic congestion cost factor increased, will decrease. On the other hand, the quantity supplied to the market from the other locations will increase. Moreover, the *total quantity* supplied to the market decreases. We formalize this discussion in the next proposition.

Proposition 3.3. *Suppose that α^1_{sj} and α^2_{sj} are such that $\alpha^1_{sj} < \alpha^2_{sj}$, and that locations 1 to ℓ are active at market j under α^1_{sj} and α^2_{sj}, $1 \leq s \leq \ell$; that is, the number of active locations and the set of active locations at market j remain the same. Then (a) $Q^{1*}_{ij} > Q^{2*}_{ij}$ for $i = s$ and, $Q^{1*}_{ij} < Q^{2*}_{ij}$ $i \neq s$. Moreover, (b) $\sum_{i=1}^{\ell} Q^{1*}_{ij} > \sum_{i=1}^{\ell} Q^{2*}_{ij}$, where Q^{1*}_{ij} and Q^{2*}_{ij} denote the equilibrium quantities supplied from location i to market j under the α^1_{sj} and α^2_{sj} values, respectively.*

Proof: Since the number of active locations and the set of active locations at market j remain the same, $Q^{1*}_{ij} = Q^{1(\ell)}_{ij}$ and $Q^{2*}_{ij} = Q^{2(\ell)}_{ij}$. Now it directly follows from Equation (3–3) that $Q^{1*}_{ij} > Q^{2*}_{ij}$ for $i = s$ and, it follows from the proof of Proposition 3.2 that $Q^{1*}_{ij} < Q^{2*}_{ij}$ $i \neq s$. This completes the proof of Statement (a). Statement (b) is a direct result of Equation (3–1) and Statement (a). □

Statement (a) of Proposition 3.3 implies that each firm will reduce the quantity that it supplies to market j on link (i, j) if the set of active locations at market j does not change when the traffic congestion cost factor increases on the link. On the other hand, each firm will increase the quantity it supplies to market j on the other links in this case. Moreover, it follows from Statement (b) of Proposition 3.3 that the total quantity sent to market j by any firm will decrease. This discussion highlights the fact that firms will reduce their supply to market j and, hence, increase the price in market j, while decreasing their transportation costs by supplying less, to balance the increase in their traffic congestion costs. Next we study the case when the number of active locations at the market increases.

(ii) When the number of active locations at a market increases, the total quantity supplied to the market from the location for which the traffic congestion cost factor increases will decrease. On the other hand, the total quantity supplied to the market from the other locations that were initially active at the market may increase or decrease. However, if the total quantity supplied to the market from one of the initially active locations at the market (for which the traffic congestion cost factor remains the same) increases (decreases), the total quantity supplied to the market from the other initially active locations at the market (with unchanged traffic congestion cost factors) also increases (decreases). The next proposition formalizes this discussion.

Proposition 3.4. *Suppose that α^1_{sj} and α^2_{sj} are such that $\alpha^1_{sj} < \alpha^2_{sj}$, and suppose that locations 1 to ℓ are active at market j under α^1_{sj}, $s \leq \ell$, and locations 1 to $\ell + \wp$ are active at market j under α^2_{sj}. Then (a) $Q^{1*}_{ij} > Q^{2*}_{ij}$ for $i = s$. Moreover, (b) if $Q^{1*}_{ij} < Q^{2*}_{ij}$ for a location i, $i \neq s$, then $Q^{1*}_{ij} < Q^{2*}_{ij}$ $\forall i \leq \ell$, $i \neq s$ and $\sum_{i=1}^{\ell} Q^{1*}_{ij} > \sum_{i=1}^{\ell+\wp} Q^{2*}_{ij}$, where Q^{1*}_{ij} and Q^{2*}_{ij} denote the equilibrium quantities supplied from location i to market j under α^1_{sj} and α^2_{sj}, respectively.*

Proof: Note that, $Q^{1*}_{ij} = Q^{1(\ell)}_{ij}$ and $Q^{2*}_{ij} = Q^{2(\ell+\wp)}_{ij}$. We know from Proposition 3.3, that $Q^{1(\ell)}_{sj} > Q^{2(\ell)}_{sj}$. Moreover, we know from Proposition 3.1 that $Q^{2(\ell)}_{sj} > Q^{2(\ell+1)}_{sj}$. Thus it follows that $Q^{1(\ell)}_{sj} > Q^{2(\ell+\wp)}_{sj}$, which proves Statement (a). Statement (b) directly follows from Equation (3–1). In particular, suppose $Q^{1*}_{tj} < Q^{2*}_{tj}$ for location t, $t \leq k$, $t \neq s$. Then it follows from Equation (3–1) that $\sum_{i=1}^{\ell} Q^{1*}_{ij} > \sum_{i=1}^{\ell+\wp} Q^{2*}_{ij}$. Then it again follows from Equation (3–1) that $Q^{1*}_{ij} < Q^{2*}_{ij}$ for any location i, $i \leq k$, $i \neq s$. This completes the proof of Statement (b). $\square$

Proposition 3.4 implies that each firm will reduce the quantity it supplies to market j on link (s, j) if the number of active locations at market j increases when the traffic congestion cost factor increases on link (s, j). On the other hand, each firm may increase or decrease the quantity it supplies to market j on the other links in this case. However, the reaction of the firms will be the same for the quantity decisions on the other links, i.e., if firms increase (decrease) the flow on link (i, j), $i \neq s$, they will increase (decrease) the flow on any link (i, j), $i \neq s$. Moreover, when firms increase (decrease) the flow on link (i,j), $i \neq s$, the total quantity supplied to market j and the total quantity supplied to market j by any firm decreases (increases).

When the total quantity supplied to market j by a firm decreases, this implies that all of the firms decrease supply to market j, increasing the price in market j to balance the increase in the traffic congestion costs. Nevertheless, when the total quantity supplied to market j by a firm increases and the number of supply points increases, this illustrates how firms may choose to divert flow to market j using links that are not as close to market j but are less congested.

Our discussion of Propositions 3.3 and 3.4 implies that increased congestion hampers efficient planning of supply chain activities, because it pushes firms to supply markets using either more congested links or links that are not close to the market. In Section 3.4, we provide the results of extensive numerical studies to characterize the effects of increased congestion on firms' facility location decisions as well as supply quantity decisions. Next, we discuss the Stage-two decisions for a practical special case of the problem stated in Section 3.2, in which market locations may also serve as supply facility locations.

3.3.1 Implications for A Special Case: Facilities Located within Market Areas

A common case in practice occurs when potential facility locations are within the market areas. For instance, if the markets are considered to be a set of spatially separated cities, firms may locate their facilities within these cities. Therefore, we study the relevant case in which $I \subseteq J$. In this case, the transportation cost from a facility within a market area to that market will be very small and thus can be approximated by setting $c_{ij} = 0$ if $i = j$. In Chapter 2, we noted that c_{ij} may include a location-specific per unit production cost, v_i. Hence, assuming $c_{jj} = 0$ implies an assumption that $v_j = 0$ as well. Nevertheless, this assumption is not restrictive if we consider the following transformation when $v_j > 0$. For $v_j > 0$, we can define a parameter $a_j := a_j - v_j$, with $c_{ij} := c_{ij} - v_j$ for $i \neq j$. This transformation approximates transportation costs to market j from a facility within the market as equal to 0, while accounting for variable production costs. However, for simplicity, we assume that $c_{jj} = 0$. Any problem instance with $v_j > 0$ can be transformed to an equivalent problem instance with $c_{jj} = 0$ using the noted transformation. Therefore, for the special case of interest here, traffic congestion costs will be the main cost driver when supplying a market from a local facility that is active in the market. In this subsection, we document the implications of the results presented earlier for this special case and analyze the effects of traffic congestion costs on equilibrium supply quantities when the supply firms have fixed and identical facility locations.

Suppose that $I \subseteq J$ and firms have made identical location decisions; that is, if one of the firms has a facility in a market, then all firms have a facility in that market. Note that, as in the previous discussion of the the Stage-two decisions for the general case, we can analyze each market separately. Furthermore, the supply quantity decisions for any market without any supply facilities will correspond to the Stage-two decisions of the general case. Therefore, we only focus on the firms' supply quantity decisions for a market in which each firm has a facility. Suppose that each firm has a facility in market j and, thus, the unit transportation cost from these facilities to customers in market j is 0. Then it follows from Algorithm 1 that the facilities in market j will be active in market j if there is any positive supply to market j. Next, we show that there will be a positive supply to market j when firms have facilities in that market area.

Proposition 3.5. *Suppose that $I \subseteq J$ and firms make identical facility location decisions. When each firm locates a facility in market j and $a_j > 0$, then $q^*_{\bullet j \bullet} > 0$, where $q^*_{\bullet j \bullet}$ denotes the total quantity supplied to market j in equilibrium.*

Proof: We know from Algorithm 1 that the facilities at market j will supply a positive amount to market j if $q^*_{\bullet j \bullet} > 0$. Moreover, since $a_j > 0$, it follows from Step 1 of Algorithm 1 that the total quantity supplied from the facilities at market j is positive, i.e., $Q^*_{jj} > 0$. This implies that $q^*_{\bullet j \bullet} > 0$. □

Note that for the general case, it is possible that a market will not be supplied by any of the firms due to high transportation costs. On the other hand, for the case with $I \subseteq J$, if firms locate facilities in a market, then this market will necessarily receive some positive supply. It should be noted that when per unit production cost at market j is accounted for, the condition $a_j > 0$ reads as $a_j - v_j > 0$ considering the aforementioned transformation. Hence, market j will be supplied from the facilities within the market as long as the per unit production cost in market j is less than the market price at zero supply, i.e., $a_j > v_j$. However, it is not necessary that the market will be supplied strictly from the facilities within the market area; that is, firms may also supply the market from the facilities outside the market area, if congestion costs within the market are high. Next, we discuss the effects of variation in α_{jj}, the congestion cost factor in market j, on equilibrium supply quantities.

Propositions 3.2-3.4 remain valid for this special case. Furthermore, the discussion following Propositions 3.2-3.4 continues to hold. In particular, consider a scenario in which a market is only supplied from facilities located in that market. When congestion costs increase within the market area, firms will react to this increase by either continuing to supply the market from the same facilities or using additional facilities located outside the market area. If firms still supply the market only using facilities within the market area, the total supply to customers in the market will decrease. From a practical point of view, this implies that firms will choose to supply less within the market because they will not be able to deliver within a specified delivery time frame or with a high enough service level in the case of high congestion costs. Furthermore, in doing so, the decreased supply in the market will lead to a price increase (or, equivalently, firms will increase price, resulting in a reduced market demand). If suppliers decide to use additional facilities outside the market area to supply the market, they will still reduce their supply coming from facilities within the market. This will avoid high congestion costs; however, they now pay higher unit transportation costs by supplying the market from facilities outside the market area. From a practical perspective, this implies that firms will avoid congestion costs and maintain a specified delivery time or service level commitments by supplying a portion of the market demand from outside the market. In Section 3.4, we present the results of numerical studies that illustrate how congestion costs affect firms' facility location decisions for this special case.

3.4 Numerical Studies

Our numerical studies focus on characterizing the effects of traffic congestion costs on the firms' best decisions for the general case. We then present our numerical studies for the special case defined in Section 3.3.1.

We next discuss the results of our computational studies on the general case. We generate data for our computational studies on the general case in the following way. We consider four problem classes, where each problem class differs in transportation costs, c_{ij}, and facility location costs, f_i. For each of the classes, we use all combinations of $k \in \{3, 5\}$, $n \in \{3, 5, 7\}$ and $m \in \{3, 5, 7, 10\}$, resulting in 24 combinations of the values of k, n, and m. For each of these combinations, we generate 10 problem instances and each problem instance is solved for 16 different intervals of traffic congestion cost factor, α_{ij}, starting from 0 and increasing to 8 in increments of 0.5. This way we can analyze the effects of increasing congestion cost on the facility location and supply quantity decisions of the firms. For every problem, we let $a_j \sim U[50, 150]$ and $b_j \sim U[1, 2]$. Table 3-1 gives the distribution interval of c_{ij} and f_i values in each problem class.

Table 3-1. Data intervals for problem classes 1-4

	c_{ij}	f_i
Class 1	(0,50]	[75,125]
Class 2	(0,50]	[100,150]
Class 3	[25,75]	[75,125]
Class 4	[25,75]	[100,150]

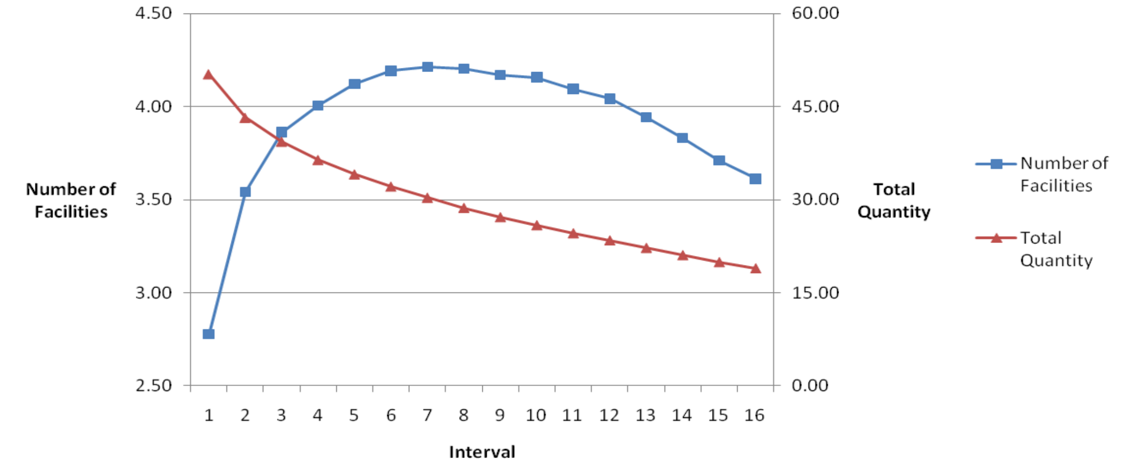
In each problem class, we solve 240 problem instances, and each instance is solved 16 times, once for each interval of α_{ij} values. For each problem instance, we determine the best location decision (using total enumeration) and the corresponding equilibrium quantity decisions for a single firm. We document the following average statistics over 960 problem instances (240 in each in each Problem Class) for each interval of α_{ij} values in Table 3-2: a given firm's number of facilities (# of fac.), total quantity supplied to markets (Supply Quant.), total transportation costs (Trans. Cost), total traffic congestion costs (Cong. Cost), total facility location costs (Loc. Cost) and total profit.

Table 3-2. Average statistics over problem classes 1-4 for each interval

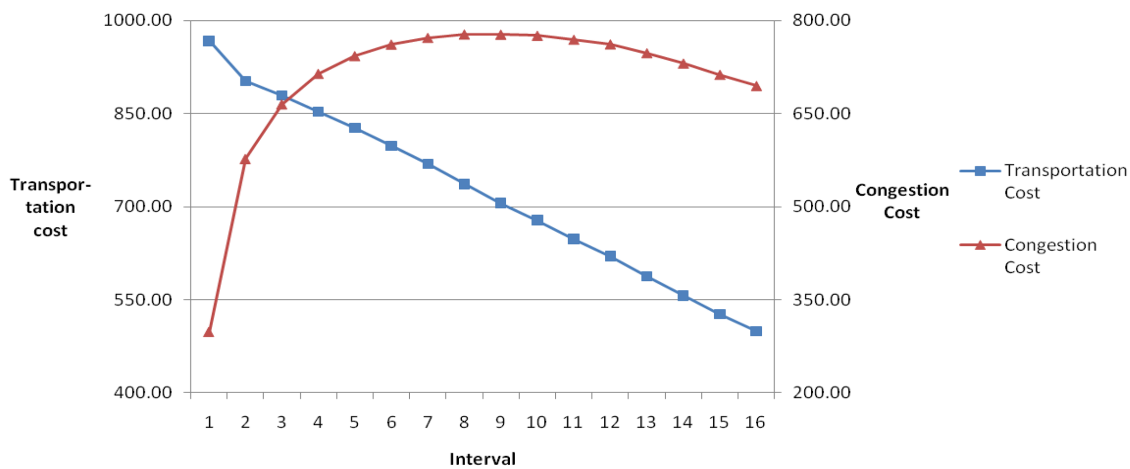
	Range of α_{ij}	# of fac.	Supply Quant.	Trans. Cost	Cong. Cost	Loc. Cost	Total Profit
Interval 1	$(0, 0.5]$	2.78	50.26	967.00	297.91	308.61	3089.29
Interval 2	$[0.5, 1]$	3.54	43.22	902.27	576.97	391.98	2261.97
Interval 3	$[1, 1.5]$	3.86	39.37	879.07	665.10	427.38	1855.16
Interval 4	$[1.5, 2]$	4.00	36.42	853.04	714.36	442.61	1574.12
Interval 5	$[2, 2.5]$	4.12	34.09	827.08	743.01	455.17	1361.77
Interval 6	$[2.5, 3]$	4.19	32.12	797.90	761.66	462.93	1194.12
Interval 7	$[3, 3.5]$	4.21	30.34	768.56	772.38	465.25	1054.87
Interval 8	$[3.5, 4]$	4.20	28.70	736.29	777.83	463.28	936.21
Interval 9	$[4, 4.5]$	4.17	27.19	705.62	777.58	458.74	835.39
Interval 10	$[4.5, 5]$	4.16	25.90	678.09	775.77	456.82	748.39
Interval 11	$[5, 5.5]$	4.09	24.59	647.55	769.48	449.28	671.05
Interval 12	$[5.5, 6]$	4.04	23.45	620.06	761.90	442.83	603.85
Interval 13	$[6, 6.5]$	3.94	22.23	587.91	747.70	431.71	544.33
Interval 14	$[6.5, 7]$	3.83	21.06	556.68	730.90	419.15	491.24
Interval 15	$[7, 7.5]$	3.71	19.94	526.62	712.79	405.20	444.18
Interval 16	$[7.5, 8]$	3.61	18.93	499.51	694.84	394.05	402.12

Figure 3-1 illustrates how different performance measures behave as the congestion cost factor increases (increasing interval on the horizontal axis corresponds to increasing congestion cost factor). The following conclusions can be drawn by analysis of Figure 3-1 (and the underlying data in Table 3-2).

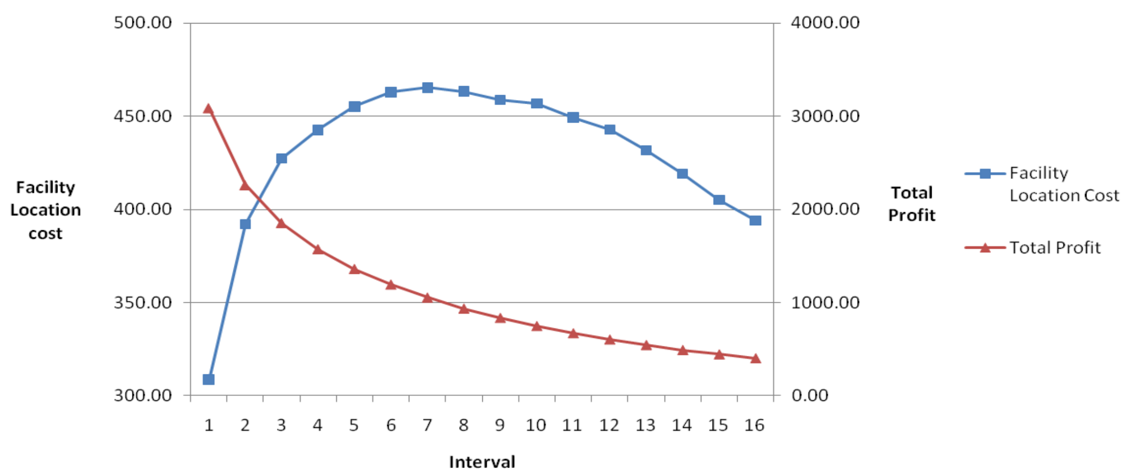
1. The number of facilities located increases with the congestion cost parameter values up to Interval 8. After Interval 8, it decreases. That is, firms will locate more facilities when congestion cost parameter values increase up to a point. However, after a point, firms will locate fewer facilities with the increase in congestion cost parameter values. Note that the facility location cost follows the same pattern.



A Number of facilities vs. interval & total quantity vs. interval



B Transportation cost vs. interval & congestion cost vs. interval



C Facility location cost vs. interval & total profit vs. interval

Figure 3-1. Patterns of each column in table 3-2

2. The total quantity supplied by a firm decreases as the congestion cost parameter values increase. This result is consistent with Propositions 3.3 and 3.4. Total transportation cost also follows the same pattern.

3. The total traffic congestion cost increases with the congestion cost parameter values up to Interval 8. After Interval 8, it decreases.

4. The total profit decreases as the congestion cost parameter values increase.

We note that the patterns observed in Table 3-2 were also observed within each problem class studied individually. Considering the points noted above, a firm's reaction to an increase in congestion cost can be explained as follows. Up to a point, a firm will locate more facilities and supply less to markets, in order to maximize profit by increasing the market price and decreasing transportation costs to compensate for the increase in congestion costs. However, when the congestion cost becomes significantly high, the firm will send less supply to markets from fewer supply points to avoid congestion costs in order to retain profitability.

Next, we discuss numerical studies for the case when the potential facility locations are within the market areas. We generate data for our computational studies in the following way. Similar to previous numerical studies, we consider four problem classes. Table 3-3 gives the distribution interval for c_{ij} and f_i values in each problem class. We note that the ranges of c_{ij} and f_i values are narrower when compared to the ranges used for the four problem classes studied previously. The reason behind this is that we also intend to analyze the firms' location decisions across markets with similar transportation and facility location cost characteristics but, with different market parameters. To account for different market characteristics, we define the market potential as the ratio a_j/b_j for market j. A greater a_j/b_j means that a firm will get more revenue by supplying the market than supplying the same amount to a market with lower a_j/b_j value. As a result, it is possible to observe how the market potentials, i.e., a_j/b_j values, affect firms' location decisions. For each of the problem classes, we use all combinations of $k \in \{3, 5\}$ and $n \in \{3, 5, 7, 10\}$, resulting in 8 combinations of k and n. For each combination, we let the maximum number of facilities that can be located equal the number of markets, i.e., $n = m$. Furthermore, we define $c_{ij} = 0$ when $i = j$, i.e., if firms locate facilities in market j, they will have no unit transportation cost from these facilities to market j. For each combination, we generate 10 problem instances and each problem instance is solved for 16 different intervals of traffic congestion cost factor, α_{ij}, starting from 0 and increasing to 8 in increments of 0.5 to analyze the effects of increasing congestion cost on firms' facility location and supply quantity decisions. Moreover, to analyze the firms' location choices as congestion increases, we consider different a_j and b_j values for each market. In particular, we assign the largest a_j and the lowest b_j values to the first market, and the lowest a_j and the highest b_j values to the last market, by letting $a_j = 50+(100/n)(j-1)+(100/n)u$ and $b_j = 2-(1/n)(j-1)-(1/n)u$, where u is uniformly distributed over $(0, 1]$. For instance, for problems with 5 markets, $a_5 \sim U[130, 150]$ and $b_5 \sim [1, 1.2]$ while $a_4 \sim U[110, 130]$ and $b_4 \sim [1.2, 1.4]$. In each problem class, we solve 80 problem instances, and each problem instance is solved 16 times, for each interval of α_{ij} values. For each problem instance, we determine the best location decision (using total enumeration) and the corresponding equilibrium quantity decisions for a single firm, as well as the markets in which firms locate facilities. Table 3-4 documents the following statistics for each interval of α_{ij} values: a given firm's number of facilities (# of fac.), total quantity supplied to markets (Supply Quant.), total transportation costs (Trans. Cost), total traffic congestion costs (Cong. Cost), total facility location costs (Loc. Cost), and total profit.

Table 3-3. Data intervals for problem classes 1-4

	c_{ij}	f_i
Class 1	[50,75]	[75,100]
Class 2	[50,75]	[100,125]
Class 3	[75,100]	[75,100]
Class 4	[75,100]	[100,125]

A graph of each statistic in Table 3-4 is shown in Figure 3-2. The following conclusions can be drawn by analysis of Table 3-4.

Table 3-4. Average statistics over problem classes 1-4 for each interval

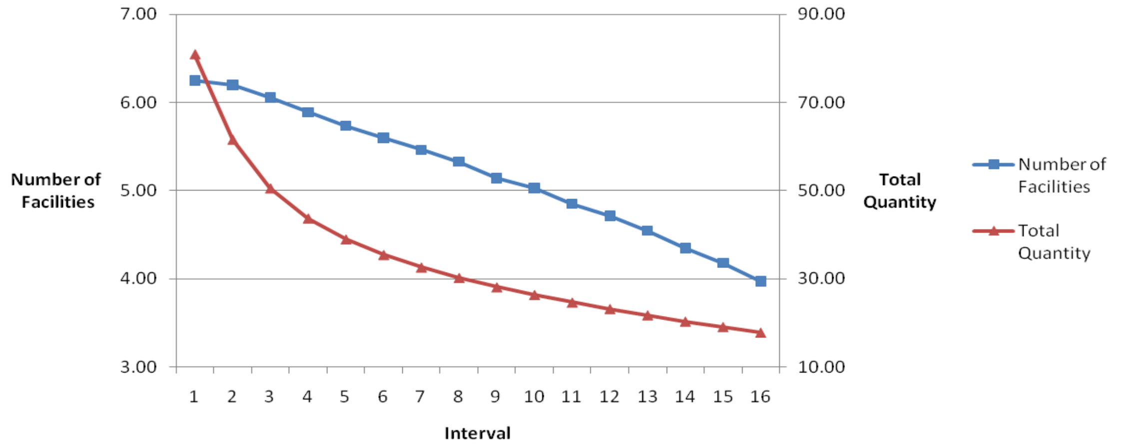
	Range of α_{ij}	# of fac.	Supply Quant.	Trans. Cost	Cong. Cost	Loc. Cost	Total Profit
Interval 1	$(0, 0.5]$	6.25	80.99	0.00	1060.42	624.40	5765.99
Interval 2	$[0.5, 1]$	6.20	61.63	33.98	1899.23	618.34	3390.28
Interval 3	$[1, 1.5]$	6.06	50.56	159.24	1960.67	603.25	2313.11
Interval 4	$[1.5, 2]$	5.89	43.75	293.48	1841.49	585.98	1745.82
Interval 5	$[2, 2.5]$	5.73	38.99	408.51	1694.78	569.51	1389.53
Interval 6	$[2.5, 3]$	5.60	35.46	488.72	1565.77	555.20	1144.15
Interval 7	$[3, 3.5]$	5.46	32.63	544.61	1453.36	541.46	960.31
Interval 8	$[3.5, 4]$	5.33	30.27	579.77	1358.20	527.44	818.32
Interval 9	$[4, 4.5]$	5.14	28.12	599.85	1270.58	508.43	704.50
Interval 10	$[4.5, 5]$	5.03	26.41	609.81	1200.56	496.85	610.32
Interval 11	$[5, 5.5]$	4.85	24.67	609.85	1129.61	478.24	529.01
Interval 12	$[5.5, 6]$	4.71	23.23	605.85	1071.15	464.04	461.50
Interval 13	$[6, 6.5]$	4.54	21.80	596.25	1012.03	447.07	402.49
Interval 14	$[6.5, 7]$	4.35	20.38	580.06	950.14	426.32	352.30
Interval 15	$[7, 7.5]$	4.18	19.12	563.77	894.02	408.74	307.95
Interval 16	$[7.5, 8]$	3.97	17.85	543.13	834.78	387.15	269.76

1. The number of facilities located decreases as the congestion cost parameter values increase for the special case. This is because when congestion costs are low, firms prefer to locate facilities in each market area; hence, they do not pay transportation costs. As congestion increases, firms prefer to locate fewer facilities so that they can avoid high congestion costs in each market area by paying transportation costs for shipments from facilities in market areas to market areas without facilities. Note that facility location cost follows the same pattern.

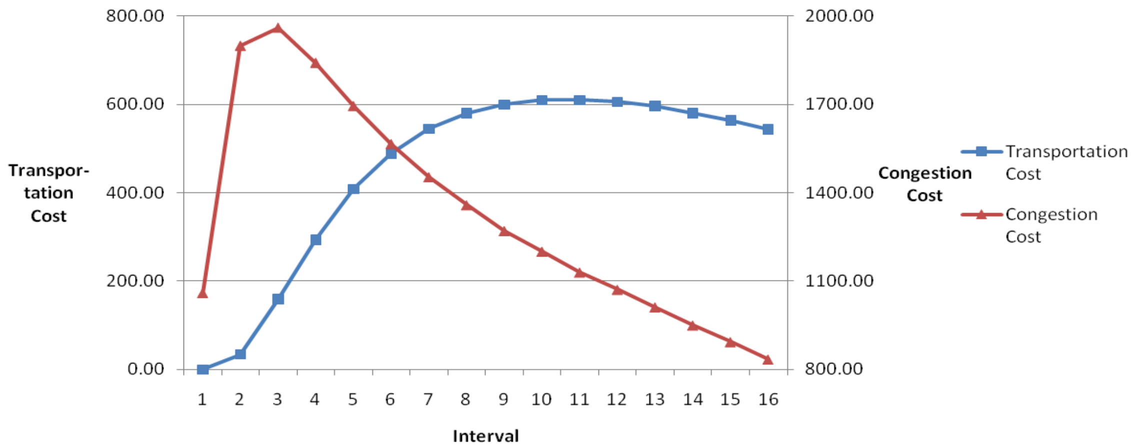
2. The total quantity supplied by a firm decreases as the congestion cost parameter values increase. On the other hand, total transportation cost increases up to a point and then decreases after that point. This is due the fact that, as congestion costs increase within markets, firms prefer to pay transportation costs from outside the markets, even if they decrease their total quantities supplied to the markets; when congestion costs are high enough, the suppliers reduce supplies and tend to supply markets from facilities within the market (and thus do not pay transportation costs).

3. The total traffic congestion cost increases with the congestion cost parameter values initially and then it decreases. This is because, up to an interval congestion cost parameter values, firms agree to pay congestion costs; but after this point, firms now trade transportation costs for higher congestion costs.

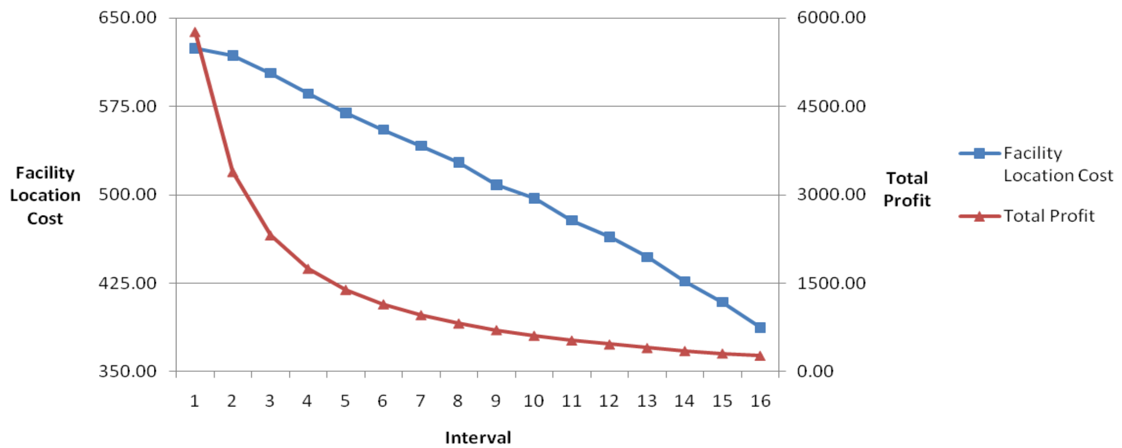
4. The total profit decreases as the congestion cost parameter values increase.



A Number of facilities vs. interval & total quantity vs. interval



B Transportation cost vs. interval & congestion cost vs. interval



C Facility location cost vs. interval & total profit vs. interval

Figure 3-2. Patterns of each column in table 3-4

Considering the points noted above, a firm's reaction to an increase in congestion cost when $I \subseteq J$ can be explained as follows. For low levels of congestion costs, firms locate facilities in more market areas so that they do not pay transportation costs. As congestion costs increase, however, firms prefer to locate facilities in fewer market areas, so that they can prevent paying high congestion costs; however, in this case, they pay higher transportation costs. Moreover, the total quantities supplied to markets decrease, which results in an increase in market prices. Next, we analyze the firms' choice of market areas to locate facilities for the special case.

To account for different market characteristics, we define the market potential as the ratio a_j/b_j for market j. A higher a_j/b_j value implies that a firm will obtain more revenue by supplying this market than supplying the same amount to a market with a lower a_j/b_j value. As a result, it is possible to observe how the market potentials, i.e., a_j/b_j values, affect firms' location decisions through our numerical studies. In Table 3-5, we show the average number of times that firms locate facilities in each market area over all of the problem instances. It follows from Table 3-5 that firms prefer to locate facilities in market areas with high market potentials, i.e., markets with greater a_j/b_j values.

Table 3-5. Average number of times firms located facilities in markets

	n=10	n=7	n=5	n=3
Market 1	1.00	0.99	0.98	0.85
Market 2	1.00	0.99	0.97	0.75
Market 3	1.00	0.96	0.92	0.49
Market 4	0.98	0.93	0.73	-
Market 5	0.95	0.83	0.52	-
Market 6	0.90	0.71	-	-
Market 7	0.82	0.47	-	-
Market 8	0.76	-	-	-
Market 9	0.61	-	-	-
Market 10	0.45	-	-	-

Furthermore, Tables 3-6 through 3-9 show the average number of times firms locate in each market area for each interval of congestion cost factor over all problem instances with the same number of markets. These results indicate that as congestion costs

Table 3-6. Average number of times firms located facilities in markets, $n = 3$ (M: market)

	M1	M2	M3
Interval 1	1.00	1.00	1.00
Interval 2	1.00	1.00	1.00
Interval 3	1.00	1.00	0.93
Interval 4	1.00	1.00	0.85
Interval 5	1.00	1.00	0.78
Interval 6	1.00	1.00	0.73
Interval 7	1.00	0.99	0.66
Interval 8	1.00	0.95	0.60
Interval 9	1.00	0.91	0.51
Interval 10	1.00	0.86	0.44
Interval 11	0.99	0.79	0.36
Interval 12	0.96	0.73	0.28
Interval 13	0.91	0.66	0.19
Interval 14	0.84	0.60	0.19
Interval 15	0.75	0.53	0.15
Interval 16	0.69	0.49	0.13
Average	0.85	0.75	0.49

Table 3-7. Average number of times firms located facilities in markets, $n = 5$ (M: market)

	M1	M2	M3	M4	M5
Interval 1	1.00	1.00	1.00	1.00	1.00
Interval 2	1.00	1.00	1.00	1.00	0.95
Interval 3	1.00	1.00	1.00	1.00	0.84
Interval 4	1.00	1.00	1.00	0.98	0.70
Interval 5	1.00	1.00	1.00	0.91	0.64
Interval 6	1.00	1.00	1.00	0.91	0.56
Interval 7	1.00	1.00	1.00	0.86	0.55
Interval 8	1.00	1.00	0.99	0.81	0.50
Interval 9	1.00	1.00	0.96	0.73	0.46
Interval 10	1.00	1.00	0.98	0.68	0.41
Interval 11	1.00	0.99	0.90	0.61	0.34
Interval 12	1.00	0.99	0.88	0.56	0.31
Interval 13	1.00	0.98	0.81	0.48	0.30
Interval 14	0.94	0.91	0.78	0.41	0.29
Interval 15	0.94	0.86	0.74	0.38	0.25
Interval 16	0.86	0.81	0.65	0.36	0.23
Average	0.98	0.97	0.92	0.73	0.52

Table 3-8. Average number of times firms located facilities in markets, $n = 7$ (M: market)

	M1	M2	M3	M4	M5	M6	M7
Interval 1	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Interval 2	1.00	1.00	1.00	1.00	1.00	1.00	0.95
Interval 3	1.00	1.00	1.00	1.00	1.00	1.00	0.84
Interval 4	1.00	1.00	1.00	1.00	1.00	0.99	0.63
Interval 5	1.00	1.00	1.00	1.00	1.00	0.94	0.50
Interval 6	1.00	1.00	1.00	1.00	0.95	0.85	0.43
Interval 7	1.00	1.00	1.00	1.00	0.94	0.78	0.40
Interval 8	1.00	1.00	1.00	0.99	0.90	0.71	0.39
Interval 9	1.00	1.00	0.99	0.98	0.80	0.63	0.38
Interval 10	1.00	1.00	0.99	0.95	0.76	0.60	0.36
Interval 11	1.00	1.00	0.96	0.91	0.76	0.51	0.34
Interval 12	1.00	1.00	0.93	0.91	0.74	0.53	0.30
Interval 13	1.00	1.00	0.93	0.85	0.71	0.49	0.28
Interval 14	1.00	1.00	0.89	0.81	0.61	0.45	0.26
Interval 15	0.98	0.96	0.86	0.78	0.56	0.44	0.25
Interval 16	0.91	0.90	0.81	0.75	0.56	0.41	0.23
Average	0.99	0.99	0.96	0.93	0.83	0.71	0.47

Table 3-9. Average number of times firms located facilities in markets, $n = 10$ (M: market)

	M1	M2	M3	M4	M5	M6	M7	M8	M9	M10
Interval 1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Interval 2	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.89
Interval 3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.94	0.69
Interval 4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.84	0.59
Interval 5	1.00	1.00	1.00	1.00	1.00	1.00	0.99	0.95	0.78	0.46
Interval 6	1.00	1.00	1.00	1.00	1.00	0.99	0.96	0.89	0.68	0.45
Interval 7	1.00	1.00	1.00	1.00	0.98	1.00	0.89	0.83	0.59	0.40
Interval 8	1.00	1.00	1.00	0.99	0.98	0.96	0.89	0.78	0.50	0.39
Interval 9	1.00	1.00	1.00	0.99	0.98	0.89	0.85	0.73	0.46	0.34
Interval 10	1.00	1.00	1.00	0.99	0.96	0.88	0.80	0.66	0.45	0.36
Interval 11	1.00	1.00	1.00	0.99	0.95	0.84	0.73	0.66	0.45	0.33
Interval 12	1.00	1.00	0.99	0.98	0.91	0.81	0.68	0.64	0.45	0.30
Interval 13	1.00	1.00	0.99	0.98	0.90	0.79	0.65	0.58	0.45	0.28
Interval 14	1.00	1.00	1.00	0.95	0.89	0.75	0.59	0.54	0.43	0.28
Interval 15	1.00	1.00	0.99	0.93	0.86	0.75	0.59	0.51	0.41	0.26
Interval 16	1.00	1.00	0.98	0.88	0.85	0.73	0.59	0.49	0.38	0.23
Average	1.00	1.00	1.00	0.98	0.95	0.90	0.82	0.76	0.61	0.45

increase, the average number of times that firms locate facilities in a specific market area decreases. This result is consistent with the first point above. Furthermore, it can be seen that within any interval of congestion cost parameter values, firms prefer to locate facilities in markets with higher market potential as congestion costs increase.

CHAPTER 4
COMPETITIVE MULTI-FACILITY LOCATION GAMES WITH NON-IDENTICAL FIRMS AND CONVEX TRAFFIC CONGESTION COSTS

4.1 Motivation and Literature Review

The problem of interest in this chapter assumes competition between multiple, non-identical firms supplying a single product to multiple markets. Each firm must determine its supply facility locations and the quantities it will supply from each facility to every market. Firms are noncooperative and must make simultaneous decisions. Potential facility locations and markets are located on a finite number of vertices of a network. The competition base is that of Cournot, i.e., the price in a market is determined by the total quantity supplied to the market. In particular, we extend the problems studied in Chapters 2 and 3 by considering heterogeneous firms. That is, an asymmetric competitive multi-facility location problem with convex traffic congestion costs is analyzed. One may refer to Chapters 2 and 3 for further discussion on competitive facility location problems, motivation to consider multi-facility location games with traffic congestion costs.

Similar to Chapter 2, a two-stage solution approach is adopted: first, Pure Nash Equilibrium (PNE) supply quantities for given facility locations (the Stage-two game) are determined and these are then used to search for equilibrium facility locations (the Stage-one game). The main contribution in this work lies in the analysis of the case of heterogenous suppliers, which requires substantially different techniques. In particular, a variational inequality approach is utilized for solving the Stage-two game and the resulting solution technique is embedded within a heuristic search method for the Stage-one game.

Gabay and Moulin (1980) suggest variational inequalities as a mechanism to determine equilibrium solutions in noncooperative games. One may refer to Facchinei and Pang (2003), Harker and Pang (1990) and Kinderlehrer and Stampacchia (1980) for an introduction to variational inequalities, solution approaches and the problems studied in variational inequality theory. Applications of variational inequalities on equilibrium problems can be seen in Konnov (2007) and Nagurney (1999). Dong et al. (2004) and Nagurney et al. (2002) provide representative examples of variational inequality formulations of equilibrium problems in competitive supply chains. In the literature, different solution approaches have been proposed for different types of variational inequality problems (VIP). Han and Lo (2002), He (1997), He and Liao (2002) and Wang et al. (2001) consider nonlinear VIPs, whereas Andreani et al. (1997), He and Zhou (2000) and Liao and Wang (2002) focus on linear VIPs.

In particular, spatial network equilibrium problems have been studied using associated variational inequality formulations in the literature. Spatial network equilibrium problems focus on price competition (or Cournot-type quantity competition) for a set of firms on a network. Friesz et al. (1983), Friesz et al. (1984), Chao and Friesz (1984), Harker (1984, 1986), Tobin (1987), Dafermos and Nagurney (1984, 1987, 1989), Nagurney (1987, 1988), and Miller et al. (1991) study spatial network equilibrium problems under different assumptions, and study variational inequality formulations for these equilibrium problems. Tobin and Friesz (1986) introduce facility location decisions within spatial network equilibrium problems, and they define spatially competitive network facility location problems. In spatially competitive network facility location problems, an entering firm's facility location and production decisions are analyzed by anticipating the reactions of competing firms who already have existing facilities on the given network. As noted by Friesz et al. (1988b), this problem corresponds to a Stackelberg game, where the entering firm is the leader and the firms with existing facilities are the followers. Tobin and Friesz (1986), Tobin et al. (1995), Friesz et al. (1988a), Friesz et al. (1988b, 1989), Miller et al. (1992a, 1996), and Miller et al. (1992b), also study spatially competitive network facility location problems under various assumptions. This chapter studies the simultaneous facility location decisions of a set of non-identical competing firms, i.e., the game defined in this study is not a

Stackelberg game. It is worth nothing that Stackelberg games are often modeled as mathematical programs with equilibrium constraints (MPECs). MPECs consider an optimization problem of a leader with constraints defining the followers' equilibrium conditions. One may refer to Luo et al. (1996) for detailed discussion on MPECs. The chapter does not use a leader-follower setting.

Finding a solution to the Stage-one game is important for understanding and characterizing the structural properties of equilibrium facility locations. Government agencies, land use planners, and suppliers to competing firms may benefit from understanding the locations private decision makers will choose in equilibrium. On the other hand, finding equilibrium facility locations can be computationally challenging. Heuristic methods are discussed for the problems studied by Rhim et al. (2003) and Sáiz and Hendrix (2008). Rhim (1997) proposes a genetic algorithm to find a PNE location decision for the model described in Rhim et al. (2003). On the other hand, Sáiz and Hendrix (2008) provide a multi-start search algorithm. In the case of identical firms, it is noted in Chapter 2 that firms will choose identical facility locations in equilibrium. Thus, when firms are homogeneous, they will ultimately choose identical facility locations in equilibrium, which substantially reduces the required search space. In the analysis of the heterogenous case, however, the search cannot be restricted to the case of identical facility location choices for each firm; hence, a heuristic method is discussed in the analyses of the Stage-one game for the more general case of heterogenous firms. In particular, conditions that must be satisfied by a PNE location decision are defined and a heuristic method is designed that enables a fast search for a solution that satisfies these conditions.

The rest of this chapter is organized as follows. In Section 4.2, the problem under consideration is formulated and details of the problem setting and solution approach are presented. Section 4.3 discusses the solution of equilibrium supply flows for given location decisions of the firms. In Section 4.4, the Stage-one game is studied. Sections

4.4.1–4.4.4 analyze the conditions that an equilibrium location decision must satisfy and provide a heuristic search method for the Stage-one game. The model is then extended to the multi-product case and implications for multi-echelon supply channels are discussed in Section 4.5. In Section 4.6, numerical studies on the efficiency of the heuristic method are documented.

4.2 Problem Formulation and Solution Approach

The model of interest in this chapter is a generalization of the one discussed in Chapter 2, which considers identical firms. That is, a set of k non-identical firms, indexed by $r \in R = \{1, 2, \cdots, k\}$, who wish to supply a set of n customer markets, indexed by $j \in J = \{1, 2, \cdots, n\}$, is considered. The firms compete with each other in the markets for the sales of a single product (this setting is extended to account for multiple products in Section 4.5). Firms may locate supply facilities at m potential locations, indexed by $i \in I = \{1, 2, \cdots, m\}$, in order to supply the markets. The costs incurred by supply firms include transportation (linear in the quantity shipped from facilities to markets), traffic congestion (convex and non-decreasing in the quantity supplied from a facility to a market), and fixed facility location costs. A market's price for the good is a linear, decreasing function of the total quantity supplied to the market from all firms, and each firm wishes to maximize its own profit. Moreover, it is assumed that a firm will not open more than one facility at a given location, implying that the firm will create sufficient capacity at the location to accommodate the quantity supplied by the facility to all markets in equilibrium.

In particular, let $q_{ijr} \geq 0$ denote the quantity shipped from the facility of firm r at location i to market j, $q_{\bullet jr}$ denote the total quantity shipped to market j by firm r (i.e., $q_{\bullet jr} = \sum_{i \in I} q_{ijr}$), $q_{i \bullet r}$ denote the total quantity shipped from location i by firm r (i.e., $q_{i \bullet r} = \sum_{j \in J} q_{ijr}$), $q_{ij \bullet}$ denote the total quantity shipped from location i to market j (i.e., $q_{ij \bullet} = \sum_{r \in R} q_{ijr}$), and $q_{\bullet j \bullet}$ denote the total quantity shipped to market j (i.e., $q_{i \bullet r} = \sum_{r \in R} \sum_{i \in I} q_{ijr}$). Furthermore, let us define $\mathbf{Q}$ as $k \times m \times n$ matrix of q_{ijr} values,

$\mathbf{X}$ as $m \times k$ binary matrix representing firms' location decisions, $\mathbf{x}_r$ as an $m-$vector representing location decisions of firm r, and x_{ir} such that $x_{ir} = 1$ if firm r locates a facility at location i, $x_{ir} = 0$ otherwise.

The price in market j, p_j, is defined identically with Equation (2–1), i.e.,

$$p_j(q_{\bullet j \bullet}) = a_j - b_j q_{\bullet j \bullet}, \tag{4–1}$$

where $a_j \geq 0$ and $b_j > 0$ denote the price at zero demand and the price sensitivity for market j (both parameters are assumed to be finite numbers). Note that Equation (4–1) is the inverse demand function associated with Cournot competition. As illustrated by the profit function, the transportation cost is linear in the quantity sent from facility i to market j for any firm r with marginal cost $c_{ijr} \geq 0$. It should be remarked that c_{ijr} can be easily adjusted to account for any per-unit production costs without loss of generality. That is, a location-specific parameter $v_{ir} \geq 0$ denoting the per-unit production cost at location i for firm r can be added to c_{ijr}. The traffic congestion cost coefficient for link (i, j) for firm r is defined as g_{ijr} (which is a function of the total quantity of flow on the link), where

$$g_{ijr}(q_{ij\bullet}) = \alpha_{ijr} q_{ij\bullet}. \tag{4–2}$$

The parameter $\alpha_{ijr} > 0$ is a traffic congestion cost multiplier for flow on link (i, j) for firm r. Hence, the congestion cost incurred by a firm using link (i, j) increases in the total flow on the link. Chapter 3 provides a justification of this functional form, which assumes that the congestion cost incurred by firm r on link (i, j) is nondecreasing and convex in q_{ijr}. Furthermore, Equation (4–2) introduces another type of competition, competition on the distribution network, in addition to the competition within the markets implied by Equation (4–1).

The profit function of a firm consists of the supply firm's total revenue, less variable transportation costs, traffic congestion costs, and facility location costs ($f_r(\mathbf{x}_r) = \sum_{i \in I} x_{ir} f_{ir}$ denotes the total facility location cost for firm r, where f_{ir} denotes the fixed cost of opening a facility at location i for firm r). Then, the profit function for each firm r can then be formulated as follows:

$$H_r(\mathbf{Q}, \mathbf{X}) = \sum_{j \in J} p_j \left(\sum_{i \in I} \sum_{r \in R} q_{ijr} \right) \sum_{i \in I} q_{ijr} - \sum_{j \in J} \sum_{i \in I} c_{ijr} q_{ijr} - \sum_{j \in J} \sum_{i \in I} q_{ijr} g_{ijr} \left(\sum_{r \in R} q_{ijr} \right) - f_r(\mathbf{x}_r). \tag{4–3}$$

As is typical in practice, a firm will determine the locations of facilities prior to determining the quantities to supply from facilities to markets; thus, it is assumed that firms simultaneously determine their facility locations (the first stage) and then their supply quantities (the second stage). The following two-stage solution approach is adopted. First, given facility location decisions, the Stage-two decisions are analyzed. Then, using the solution of the Stage-two game, the Stage-one decisions are analyzed.

In particular, the Stage-one game corresponds to a k-matrix game (each firm is a player), where each player has the same set of 2^m strategies (each strategy of firm r is a binary vector, $\mathbf{x}_r$, denoting firm r's facility location decisions at m locations). Therefore, checking whether a location matrix $\mathbf{X}$ is an equilibrium requires $k(2^m - 1)$ comparisons (as one must check whether $\mathbf{x}_r$ is the best option for firm r, which requires $2^m - 1$ comparisons, and this will be repeated for all players). Furthermore, there are $2^{m \times k}$ alternative outcomes of the Stage-one game. Nevertheless, as is discussed in Section 4.4, not all alternative location matrices are candidates for an equilibrium solution. Section 4.4 introduces properties of an equilibrium location matrix, denoted by $\mathbf{X}^*$, to narrow the search for $\mathbf{X}^*$, and to apply these properties in a search heuristic that returns $\mathbf{X}^*$, if one exists.

On the other hand, one needs to determine the equilibrium supply quantities for a given location matrix to check whether the given location matrix is an equilibrium solution. Section 4.3 formulates the problem of finding equilibrium supply quantities as a variational inequality problem (VIP) and describes a procedure to determine equilibrium supply quantities for any given location matrix $\mathbf{X}$, denoted by $\mathbf{Q}^*(\mathbf{X})$. This procedure is also utilized within the search heuristic described in Section 4.4; hence, an equilibrium solution $\mathbf{X}^*$, if one exists, and a corresponding $\mathbf{Q}^*(\mathbf{X}^*)$, are ultimately generated.

The solution generated by the two-stage solution approach, i.e., the $\mathbf{X}^*$, $\mathbf{Q}^*(\mathbf{X}^*)$ pair, is called a Subgame Perfect Nash Equilibrium (Selten, 1975), as the problem of finding equilibrium location and supply quantities is solved in two stages (each stage represents a subgame). However, it should be noted that the two-stage solution approach solves the integrated equilibrium problem, which seeks equilibrium facility locations and supply quantities simultaneously. This follows from the uniqueness of $\mathbf{Q}^*(\mathbf{X})$ for any $\mathbf{X}$ (the uniqueness result will be discussed in Section 4.3); therefore, the set of Subgame Perfect Nash Equilibrium solutions achieved by the two-stage solution approach is equal to the set of Nash Equilibrium solutions of the integrated equilibrium problem.

4.3 Stage-Two Decisions

This section studies the Stage-two game for a given $\mathbf{X}$, i.e., when $x_{ir} \in \{0, 1\}$ $\forall i = 1, 2, \dots, m$, $r = 1, 2, \dots, k$ have been pre-determined. This implies that $f_r(\mathbf{x}_r)$ is fixed, and can be ignored when analyzing the Stage-two game. Based on the profit function (4–3) and definitions of the price (4–1) and congestion (4–2) functions, firm r's optimization problem, given the facility locations, can then be written as

$$\begin{aligned}
\max \quad & \sum_{j \in J} \left[(a_j - b_j q_{\bullet j \bullet}) q_{\bullet jr} - \sum_{i \in I} c_{ijr} q_{ijr} - \sum_{i \in I} \alpha_{ijr} q_{ijr} q_{ij\bullet} \right] \\
\text{s.t.} \quad & q_{\bullet jr} \leq W x_{ir} \ \forall i \in I, \\
& q_{ijr} \geq 0 \ \forall i \in I, j \in J,
\end{aligned} \tag{4–4}$$

where W is a large number. The first set of constraints ensures that firm r can only supply a market from an open facility, and the second set of constraints imposes the nonnegativity on the supply quantities. Due to the market price and traffic congestion cost functions, any firm's quantity decisions will be affected by the other firms' quantity decisions; hence, a Nash Equilibrium solution is sought to characterize the firms' quantity decisions. It is straightforward to show that the objective function in Equation

(4–4) is a strictly concave function in each variable $q_{ijr} \geq 0$, because $b_j > 0$ and $\alpha_{ijr} > 0$. It further follows from Equation (4–4) that the objective function of any firm is separable in the markets, i.e., the equilibrium conditions for market j can be analyzed independently of the other markets (similar results are given in Rhim et al., 2003, Sáiz and Hendrix, 2008, and Chapter 2). Therefore, in what follows, the Stage-two game for an arbitrary market j is examined, given the location choices of the firms.

Let I_r denote the set of locations at which firm r has opened a facility and $|I_r|$ denote its cardinality (in the corresponding location matrix $\mathbf{X}$). Observe that $q_{ijr} = 0$ for all $j \in J, i \notin I_r$. Therefore, supply quantities for market j, given the location matrix $\mathbf{X}$, can be represented by a λ-vector, where $\lambda = \sum_{r \in R} |I_r|$. Let $\mathbf{Q}_j$ denote this vector such that $\mathbf{Q}_j \in R_+^\lambda$. Then, the profit function for firm r in market j under these supply quantities, denoted by $\Pi_r^j(\mathbf{Q}_j)$, can be written as

$$\Pi_r^j(\mathbf{Q}_j) = p_j(q_{\bullet j \bullet})q_{\bullet jr} - \sum_{i \in I_r} c_{ijr} q_{ijr} - \sum_{i \in I_r} \alpha_{ijr} q_{ijr} q_{ij\bullet}. \tag{4–5}$$

Due to the concavity of Equation (4–5), the first-order conditions ($\partial \Pi_r(\mathbf{Q}_j)/\partial q_{ijr} = 0$, for q_{ijr} values such that $q_{ijr} > 0$) must be satisfied at a Nash equilibrium solution for the Stage-two game (Nash, 1951). In particular, a Nash equilibrium solution must satisfy the conditions

$$a_j - b_j[q_{\bullet j \bullet} + q_{\bullet jr}] - c_{ijr} - \alpha_{ijr}[q_{ijr} + q_{ij\bullet}] = 0 \quad \textit{if} \quad q_{ijr} > 0, \tag{4–6}$$

$$a_j - b_j[q_{\bullet j \bullet} + q_{\bullet jr}] - c_{ijr} - \alpha_{ijr}[q_{ijr} + q_{ij\bullet}] \leq 0 \quad \textit{if} \quad q_{ijr} = 0. \tag{4–7}$$

Then, to find an equilibrium solution, one must determine q_{ijr} values that solve the associated first-order conditions simultaneously.

The solution approach for the case of homogenous firms, discussed in Chapter 2, is a relatively simple sorting based iterative approach similar to the ones proposed by Rhim et al. (2003) and Sáiz and Hendrix (2008). Specifically, the locations at which firms have open facilities are sorted based on cost parameters, and then the total quantities sent from these locations are determined. However, a sorting method will not work for the Stage-two game of the current model, as Equation (4–4) is not separable in i (the facility location choices) due to the convex traffic congestion costs (neither does this sorting method work for the symmetric case when firms choose distinct location decisions). Next, a variational inequality formulation for the Stage-two game for market j for the heterogenous case is presented. It should be noted that variational inequality formulations have been developed for spatial network equilibrium problems (see, e.g., Friesz et al., 1983, Friesz et al., 1984, Harker, 1984, 1986, Dafermos and Nagurney, 1984, 1987, 1989, Chao and Friesz, 1984, Tobin, 1987, Nagurney, 1987, 1988). The associated variational inequality formulation for the Stage-two game for market j is an asymmetric variational inequality problem (VIP) and, hence, one can use efficient solution algorithms designed for asymmetric VIPs for the Stage-two game for market j.

The Nash equilibrium quantities must be optimal for each firm, given the optimal decisions of all other firms. As the profit function of each firm is strictly concave in every q_{ijr}, the optimality conditions for each firm can be written in variational inequality form. It follows from Gabay and Moulin (1980), Nagurney (1999), and Nagurney et al. (2002) that $\mathbf{Q}_j^*$ is a Nash equilibrium if it satisfies the following variational inequality:

$$-\sum_{r\in R}\sum_{i\in I_r}\frac{\partial \Pi_r^j(\mathbf{Q}_j)}{\partial q_{ijr}}\times(q_{ijr}-q_{ijr}^*)\geq 0, \forall \mathbf{Q}_j\in R_+^{\lambda}. \tag{4–8}$$

The variational inequality in Equation (4–8) for market j then takes the following explicit form

$$\sum_{r\in R}\sum_{i\in I_r}[c_{ijr}-a_j+b_j(q_{\bullet j\bullet}+q_{\bullet jr})+\alpha_{ijr}(q_{ijr}+q_{ij\bullet})]\times(q_{ijr}-q_{ijr}^*)\geq 0, \forall \mathbf{Q}_j\in R_+^{\lambda}. \tag{4–9}$$

To study the properties of Equation (4–9), a classical variational inequality representation is given next. In particular, let $\mathbf{Q}_j^r\in\mathcal{S}_r$ denote the vector of supplies from firm r facilities to market j; that is, $\mathbf{Q}_j^r=(q_{1jr},\dots,q_{|I_r|jr})^T\ \forall r\in R$, where $\mathcal{S}_r$ denotes the strategy set of firm r. Then $\mathbf{Q}_j^*=((\mathbf{Q}_j^{1*})^T,\dots,(\mathbf{Q}_j^{k*})^T)^T\in\mathcal{S}$, where $\mathcal{S}=\mathcal{S}_1\times\dots\times\mathcal{S}_k$, is a Nash equilibrium solution if it satisfies

$$\Pi_r^j(\mathbf{Q}_j^{r*}, \mathbf{Q}_j^{[-r]*}) \geq \Pi_r^j(\mathbf{Q}_j^r, \mathbf{Q}_j^{[-r]*}) \; \forall \mathbf{Q}_j^r \in \mathcal{S}_r, \; \forall r \in R, \tag{4–10}$$

where $\mathbf{Q}_j^{[-r]*} = \left((\mathbf{Q}_j^{1*})^T, \ldots, (\mathbf{Q}_j^{(r-1)*})^T, (\mathbf{Q}_j^{(r+1)*})^T, \ldots, (\mathbf{Q}_j^{k*})^T\right)^T$. It can be shown that $\mathbf{Q}_j^* \in \mathcal{S}$ solves the Stage-two Game at market j if it solves the following VIP for the given location decision $\mathbf{X}$:

$$\langle F(\mathbf{Q}_j^*), \mathbf{Q}_j - \mathbf{Q}_j^* \rangle \geq 0, \forall \mathbf{Q}_j \in \mathcal{S}, \tag{4–11}$$

where $F(\mathbf{Q}_j) = (-\nabla_{\mathbf{Q}_j^1} \Pi_1^j(\mathbf{Q}_j), \ldots, -\nabla_{\mathbf{Q}_j^k} \Pi_k^j(\mathbf{Q}_j))$ is a λ-row vector function and $\nabla_{\mathbf{Q}_j^r} \Pi_r^j(\mathbf{Q}_j) = (\partial \Pi_r^j(\mathbf{Q}_j)/\partial q_{1jr}, \ldots, \partial \Pi_r^j(\mathbf{Q}_j)/\partial q_{|I_r|jr}) \; \forall r \in R$ (see, e.g., Nagurney, 1999). Note that $F(\mathbf{Q}_j)$ is a linear, continuous, and differentiable function.

Observe that as firms will not supply against a negative price, supply quantities can be considered to have upper bounds (in particular, one can set $W = a_j/b_j$ in Equation (4–4) as $p_j > 0$ and $q_{ijr} \leq (a_j - c_{ijr})/b_j$, as firms will not end up with negative profits). These linear upper bounds on the supply quantities, together with nonnegativity of supply quantities, imply a compact and convex subset of the strategy set, within which Equation (4–11) admits at least one solution. Moreover, Equation (4–9) indicates that $\nabla_{\mathbf{Q}_j^r} \Pi_r^j(\mathbf{Q}_j) = ([c_{1jr} - a_j + b_j(q_{\bullet j \bullet} + q_{\bullet jr}) + \alpha_{1jr}(q_{1jr} + q_{1j\bullet})], \ldots, [c_{|I_r|jr} - a_j + b_j(q_{\bullet j \bullet} + q_{\bullet jr}) + \alpha_{|I_r|jr}(q_{|I_r|jr} + q_{|I_r|j\bullet})])$. Then the Jacobian matrix of $F(\mathbf{Q}_j)$, $\nabla F(\mathbf{Q}_j)$, consists of the following values: $2b_j + 2\alpha_{ijr}, 2b_j, b_j + \alpha_{ijr}$ or b_j. Noting that $b_j > 0$ and $\alpha_{ijr} > 0$, it follows that each component of the Jacobian matrix is positive. Thus, for any $\mathbf{Q}_j \neq 0$, $\mathbf{Q}_j^T \nabla F(\mathbf{Q}_j) \mathbf{Q}_j > 0$. Then it follows from the mid-value theorem as noted in Nagurney (1999) that $F(\mathbf{Q}_j)$ is strictly monotone on $\mathcal{S}$. In particular, $F(\mathbf{Q}_j)$ is strictly monotone on the entire space R_+^λ. As a direct result of the strict monotonicity, Equation (4–11) has a unique solution. (One can refer to Nagurney (1999) for proofs of the discussion on the existence and uniqueness of the equilibrium solution of the Stage-two game.) Before presenting a solution method for Equation (4–11), it is worth noting that the equilibrium problem of the Stage-two game can equivalently be stated as the maximization of a quadratic concave function as well.

Next, an algorithm that solves the VIP stated in Equation (4–11) is discussed. Because Equation (4–11) implies an asymmetric linear VIP, an algorithm for asymmetric linear VIPs will be given. However, it should be noted that the algorithms studied for spatial network equilibrium problems can also be used to solve the Stage-two game (see, e.g., Friesz et al., 1983, Friesz et al., 1984, Harker, 1984, 1986, Dafermos and Nagurney, 1984, 1987, 1989, Chao and Friesz, 1984, Tobin, 1987, Nagurney, 1987, 1988). The algorithm stated below is the *self-adaptive projection method* proposed by Han (2006) for solving linear variational inequalities of the following form:

$$(M\mathbf{y}^* + z)^T(\mathbf{y} - \mathbf{y}^*) \geq 0, \forall \mathbf{y} \in K,$$

where $\mathbf{y}$ is the vector of decision variables (and $\mathbf{y}^*$ denotes a solution), K is a nonempty, convex, and closed subset of R^n, $M \in R^{n\times n}$ is a given matrix, and $z \in R^n$ is a given vector. As mentioned before, the resulting linear VIP of the Stage-two game for the given $\mathbf{X}$ is asymmetric, for which M is defined by the partial derivatives as stated in Equation (4–11), and the vector z consists of the $c_{ijr} - a_j$ values. Moreover, $K = \mathcal{S} = R^{\lambda}_{+}$. The algorithm can be formalized as follows.

Algorithm 3. *Self-adaptive Projection method for the VIP of the Stage-two game at market j.*

Step 0. Start with a $\boldsymbol{Q}_j^0 \in R^{\lambda}$. Set $\ell := 0$. Set $0 < \gamma < 2$, $\beta_0 > 0$, $\epsilon \geq 0$, and a sequence $\{\tau_\ell\} \subseteq [0, \infty)$ with $\sum_{\ell=0}^{\infty} \tau_\ell < \infty$. Set $\ell := 0$.

Step 1. Determine $e(\boldsymbol{Q}_j^\ell, \beta) = \boldsymbol{Q}_j^\ell - P_{\mathcal{S}}[\boldsymbol{Q}_j^\ell - \beta(M\boldsymbol{Q}_j^\ell + z)]$, $P_{\mathcal{S}}[\cdot]$ being the orthogonal projection from R^{λ} onto $\mathcal{S}$. If $\|e(\boldsymbol{Q}_j^\ell, \beta_\ell)\|_\infty \leq \epsilon$, stop.

Step 2. Compute the next iterate using $\boldsymbol{Q}_j^{\ell+1} = \boldsymbol{Q}_j^\ell - \gamma(I + \beta_\ell M)^{-1} e(\boldsymbol{Q}_j^\ell, \beta_\ell)$.

Step 3. Choose the next parameter $\beta_{\ell+1}$ from the interval $\frac{1}{1+\tau_\ell}\beta_\ell \leq \beta_{\ell+1} \leq (1 + \tau_\ell)\beta_\ell$. Set $\ell := \ell + 1$ and go to Step 1.

The algorithm requires calculating the inverse of a matrix and taking the projection of a point onto the set $\mathcal{S}$. In Equation (4–11), $\mathcal{S}$ is the nonnegative orthant and, hence, projection is easily carried out. In particular, as noted by Han (2006) as well, projection onto $\mathcal{S}$ using the Euclidean-norm is defined component-wise for each element of the vector to be projected. Explicitly, $P_{\mathcal{S}}[\mathbf{y}]_j = y_j$ if $y_j \geq 0$, and, $P_{\mathcal{S}}[\mathbf{y}]_j = 0$ if $y_j < 0$. The Self-adaptive Projection method, stated in Algorithm 1, converges to a solution of the variational inequality formulation in Equation (4–11) as M is positive definite for the variational inequality formulation in Equation (4–11). Moreover, there exists a solution for Equation (4–11) when $\mathcal{S}$ is the nonnegative orthant. Then, it follows from Han (2006) that the algorithm converges to a solution of Equation (4–11).

4.4 Stage-One Decisions

Before providing a method for finding an equilibrium location solution, it is first important to discuss existence results for PNE solutions of the Stage-one game. Rhim et al. (2003) demonstrate the existence of a PNE location decision under identical firms by showing that the associated Stage-one game can be modeled as a congestion game when each market is only supplied from facilities at a single location. It is a well-known result that congestion games have PNE points (Rosenthal, 1973). The condition that a market is supplied from a single location is possible when the marginal delivery cost to the market is the lowest from that supply location. As Rhim et al. (2003) model delivery cost as a linear function of the supply quantity, the marginal delivery cost to a market is constant for each location; thus, one can verify the stated existence condition. However, the problem under consideration in this study applies nonlinear costs, and because of the convex congestion costs, firms may prefer to supply a market from more than one location, and these locations may be distinct for each firm. Sáiz and Hendrix (2008) extend the model of Rhim et al. (2003) by relaxing the assumption of identical firms. The model of interest in Sáiz and Hendrix (2008) considers firm- and location-specific linear delivery costs, and the existence of a PNE location decision is not guaranteed. The current study is a further generalization of Sáiz and Hendrix (2008), and the existence of a PNE location decision is not guaranteed. However, the proposed search method discussed next finds an equilibrium location matrix, if one exists, or concludes that there is not an equilibrium solution.

4.4.1 Searching for An Equilibrium Location Matrix

To determine an equilibrium location decision matrix, if one exists, the initial focus lies in defining dominant strategies, which are candidates for a possible equilibrium location matrix. Later, a given location matrix in a dominant strategy should be checked to determine whether it is an equilibrium location matrix.

Recall that $\mathbf{Q}^*(\mathbf{X})$ defines the equilibrium quantities for a given location matrix $\mathbf{X}$, and the previous section showed how to find $\mathbf{Q}^*(\mathbf{X})$ for any given $\mathbf{X}$. Let $\Pi_r(\mathbf{X}) = H_r(\mathbf{Q}^*(\mathbf{X}), \mathbf{X})$ (i.e., $\Pi_r(\mathbf{X})$ includes facility location costs $f_r(\mathbf{X}) = f_r(\mathbf{x}_r) = \sum_{i \in I} f_{ir} x_{ir}$). Then, similar to Equation (4–10), the condition required for a location matrix $\mathbf{X}$ to correspond to an equilibrium decision reads

$$\Pi_r\left(\mathbf{x}_r^*, \mathbf{X}^{[-r]*}\right) \geq \Pi_r\left(\mathbf{x}_r, \mathbf{X}^{[-r]*}\right) \ \forall \mathbf{x}_r, \forall r \in R, \tag{4–12}$$

where $\mathbf{x}_r^*$ denotes the equilibrium location decision of firm r, $\mathbf{X}^{[-r]*}$ denotes the equilibrium decisions of all other firms, i.e., $\mathbf{X}^{[-r]*} = [\mathbf{x}_1^*, \ldots, \mathbf{x}_{r-1}^*, \mathbf{x}_{r+1}^*, \ldots, \mathbf{x}_k^*]$, and $\mathbf{X}^*$ denotes an equilibrium location matrix, if one exists. The following corollary is a direct implication of Equation (4–12) and characterizes a non-equilibrium location matrix.

Corollary 1. *If $\exists\ r \in R$ such that $\Pi_r(\boldsymbol{X}) < 0$, then $\boldsymbol{X}$ is not an equilibrium location matrix.*

Corollary 1 follows from the fact that if $\Pi_r(\mathbf{X}) < 0$, firm r will be better off by not locating any facility, hence, Equation (4–12) implies that $\mathbf{X}$ is not an equilibrium location matrix. That is, $\mathbf{X}$ can be an equilibrium decision if each one of its non-zero columns produces positive profit for the corresponding firm. This condition is similar to the *viability* condition used in Rhim et al. (2003). At this point, it is important to mention the *Stable Set* concept introduced in Dobson and Karmarkar (1987) and used by Rhim

(1997) and Rhim et al. (2003) to study location decisions in a similar competitive facility location setting. Dobson and Karmarkar (1987) define stability with respect to three different competitive factors: viability of locations, conditions of entry, and survival. In Rhim et al. (2003), each firm may open at most one facility, and a set of facility locations is defined to be *stable* as long as those firms with a facility make a positive profit (viability condition) and the firms without a facility can not make a positive profit by opening a facility (survival condition). A firm is referred to as an entrant whenever the firm has a facility. However, in the problem of interest in this study, a firm may open more than one facility. Thus, a firm is an entrant if the firm opens at least one facility. Corollary 1 implies that an entrant firm should make positive profit as a result of its location and corresponding quantity decisions, which can be referred to as the viability condition. It should be noted that while viability is necessary for a location decision to be an equilibrium, it is not a sufficient condition. That is, a viable location decision is not necessarily an equilibrium location decision. On the other hand, defining a survival condition can be ambiguous. It should be noted that the survival condition defined in Rhim et al. (2003) does not imply that an entrant firm must choose each location that is individually profitable. This follows from the fact that the facility location decisions of an entrant firm are not independent. In particular, suppose that an entrant firm may make positive profit by locating a facility at a location where the firm has no facility. It is possible that locating a facility at that location may decrease the overall profit of the entrant firm. However, a non-entrant firm, by definition, can not make a positive profit by locating a set of facilities at any subset of the locations and this discussion is already indicated by Corollary 1.

In what follows, a heuristic routine is stated to move to a viable location decision from a randomly given location decision $\mathbf{X}$. Prior to this, however, another routine is defined to ease the process of generating a viable location decision.

In particular, it should be noted that while checking whether $\mathbf{X}$ is an equilibrium location decision, one should set $x_{ir} = 0$ if $q^*_{i\bullet r}(\mathbf{X}) = 0$, and then check the conditions in Equation (4–12). This simply follows from the fact that no firm will locate a null facility, i.e., a facility that does not supply any market. Hence, the following routine is defined that will be used in the heuristic approach. A list of matrices denoted by $\mathcal{L}$ is introduced to keep track of the location matrices that are shown to be non-equilibrium.

Routine 0: Given $\mathbf{X}$ and $\mathcal{L}$, the following procedure generates a dominating location matrix:

1: Check whether $\mathbf{X} \in \mathcal{L}$

2: If $\mathbf{X} \in \mathcal{L}$, set $continue = 0$, stop and return $continue = 0$

3: Else, set $continue = 1$, $\mathcal{L} = \mathcal{L} \cup \{\mathbf{X}\}$, determine $\mathbf{Q}^*(\mathbf{X})$

4: If $q^*_{i\bullet r}(\mathbf{X}) = 0$, set $x_{ir} = 0$

5: Return $\mathbf{X}^0 = \mathbf{X}$ and $continue = 1$.

The set $\mathcal{L}$ consists of the location matrices which are not in equilibrium and, hence, if the given location matrix is in $\mathcal{L}$, checking whether it satisfies the equilibrium conditions is not required. However, when $\mathbf{X} \notin \mathcal{L}$, the output of Routine 0 is the location matrix $\mathbf{X}^0$, which dominates the original location matrix $\mathbf{X}$ for at least one firm (unless $\mathbf{X}^0 = \mathbf{X}$), which is why the term *dominating location matrix* is used. This routine is next used in generating a viable location matrix from a randomly given location decision $\mathbf{X}$.

4.4.2 Generating A Viable Location Decision

Let $\mathbf{X}^0$ be generated from a randomly given $\mathbf{X}$ by using Routine 0. If $\mathbf{X}^0$ is not viable, then there exists at least one firm with negative total profit. This further implies that there exists at least one facility of that firm with negative profit, i.e., for which the facility location cost exceeds the total profit of the firm gained by supplying markets from the facility. This discussion does not imply that each facility must be profitable in a viable location matrix. Instead, it implies that there must be a facility with negative profit in a location matrix that is not viable. For such matrices that are not viable, it thus makes sense to set $x_{ir} = 0$ when $\pi_{ir}(\mathbf{X}^0) < 0$ for a firm r such that $\Pi_r(\mathbf{X}^0) < 0$, where $\pi_{ir}(\mathbf{X}^0)$ denote the firm's profit at location i. Then, the equilibrium quantities for the modified matrix $\mathbf{X}$ can be determined and the corresponding modified matrix $\mathbf{X}^0$ can be generated. Repeating this process, a viable location matrix can be found. Specifically, this routine is defined as follows.

Routine 1: Given $\mathbf{X}$ and $\mathcal{L}$, the following procedure generates a viable location matrix:

1: Apply Routine 0

2: If $continue = 0$, stop and return $continue = 0$

3: Else, define R^- as the set of firms such that $\Pi_r(\mathbf{X}^0) < 0, \forall r \in R^-$

4: If $R^- = \emptyset$, $\mathbf{X}^0$ is viable; stop and return $\mathbf{X}^1 = \mathbf{X}^0$

5: Else, let $(\hat{\imath}, \hat{r}) = \arg\min\{\pi_{ir}(\mathbf{X}^0) : i \in I, r \in R^-\}$ and set $x_{\hat{\imath}\hat{r}} = 0$ in $\mathbf{X}$

6: Go to Step 1.

Note that the quantity decisions for $\mathbf{X}$ and $\mathbf{X}^0$ are the same, i.e., $q_{i\bullet r}(\mathbf{X}) = q_{i\bullet r}(\mathbf{X}^0)$. Hence, the $\pi_{ir}(\mathbf{X}^0)$ values can easily be calculated from the $\pi_{ir}(\mathbf{X})$ values by simply letting $\pi_{ir}(\mathbf{X}^0) = \pi_{ir}(\mathbf{X})$ when $x^0_{ir} = 1$ or $x_{ir} = 0$ and, $\pi_{ir}(\mathbf{X}^0) = 0$ when $x^0_{ir} = 0$ and $x_{ir} = 1$. In Step 5 of Routine 1, the initial location matrix $\mathbf{X}$ is modified; that is, Routine 1 does not close the facilities with negative profits under $\mathbf{X}^0$ and try to get a viable location matrix that has fewer facilities located than $\mathbf{X}^0$. The reason for such a modification is to capture the possibility that the null facilities closed with Routine 0 can have positive supply after the facilities with positive supply but with negative profits are closed. Thus, a viable location matrix with potentially more open facilities is generated. At this point, it should be noted that the matrix $\mathbf{X}$ entering Step 1 of Routine 1, and the modified matrix generated at the end of Step 5 of Routine 1, differ in only one entry. In particular, let $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$ are two consecutive location matrices entering Step 1 of Rule 1 and let $\mathbf{X}^0_{(i)}$ and $\mathbf{X}^0_{(i+1)}$ be the matrices generated by Routine 0 in Step 1 of Rule 1 corresponding to $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$, respectively. Then $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$ differ in only one entry. However,

$\mathbf{X}^0_{(i)}$ and $\mathbf{X}^0_{(i+1)}$ may differ in more than one entry. Observe that the facility with the most negative profit is closed in Step 5 of Routine 1, as such a facility is less likely to be open in an equilibrium solution. It should be remarked that in a viable location matrix, there may be some facilities with negative profit for some firms, although no firm will have negative total profit. Routine 1 will always find a viable location matrix after starting with a random location matrix. This follows because Step 5 of Routine 1 can be repeated at most $m \times k$ times, and after at most $m \times k$ repetitions, the return is $\mathbf{X} = \mathbf{0}$, which is viable, in the worst case. A viable location matrix generated at the end of Routine 1 corresponding to the given location matrix $\mathbf{X}$ is denoted by $\mathbf{X}^1$.

4.4.3 Equilibrium Check

Now suppose that $\mathbf{X}^1$ is a viable location matrix generated from $\mathbf{X}$ by using Routine 1. To determine whether $\mathbf{X}^1$ is an equilibrium location matrix, one needs to check if $\mathbf{x}^1_r$, the r^{th} column of $\mathbf{X}^1$, is the best response of firm r, $\forall r \in R$. To do so, it is required to check all possible location vectors for firm r while the location decisions of the other firms are fixed. Note that there are 2^m different location decisions for each firm and, hence, the profit of firm r should be evaluated for $2^m - 1$ location vectors, while keeping the other firms' location decisions unchanged. If $\mathbf{x}^1_r$ is shown not to be the best response of firm r for some $r \in R$, then $\mathbf{X}^1$ is not an equilibrium location matrix. As a result, another viable location matrix should be considered as a potential equilibrium location matrix. Note that it is sufficient to show that there exists a better location decision for at least one firm in $\mathbf{X}^1$ to conclude that $\mathbf{X}^1$ is not an equilibrium location matrix. To this end, two additional routines, referred to as Routine 2 and Routine 3, are defined to check whether a better location decision exists for firm r under $\mathbf{X}^1$. The intuition behind Routine 2 is as follows. If there exists a firm r facility with negative profit, it is determined whether closing this facility will increase firm r's total profit. As previously noted, there may exist facilities with negative profits for a viable location matrix. If the total profit increases by closing this facility, then the current location vector for the firm under $\mathbf{X}^1$ is not the best response of the firm and, hence, $\mathbf{X}^1$ is not an equilibrium location matrix. Therefore, another viable location matrix is considered as a candidate equilibrium matrix. This routine proceeds in the same way as Routine 1, which is used as a subroutine in Routine 2, but is applied only to one column of $\mathbf{X}^1$ each time.

Routine 2: Given $\mathbf{X}$ and $\mathcal{L}$, the following procedure searches for an improving location matrix:

1: Apply Routine 1

2: If $continue = 0$, stop and return $continue = 0$

3: Else, set $r = 1$ and $\mathbf{X}^2 = \mathbf{X}^1$

4: Define I_r^- such that $\pi_{ir}(\mathbf{X}^2) < 0, \forall i \in I_r^-$

5: If $I_r^- = \emptyset$, set $r = r + 1$

6: If $r > k$, stop and return $\mathbf{X}^2$

7: Else, go to Step 4

8: Else, let $(\hat{\imath}, r) = \arg\min\{\pi_{ir}(\mathbf{X}^1) : i \in I_r^-\}$ and $x_{\hat{\imath}r}^2 = 0$

9: If $\pi_r(\mathbf{X}^1) < \pi_r(\mathbf{X}^2)$, set $x_{\hat{\imath}r} = 0$ in $\mathbf{X}$ and go to Step 1

10: Else, set $I_r^- = I_r^- \backslash \{\hat{\imath}\}$ and go to Step 5

11: Return $\mathbf{X}^2$.

The purpose of Routine 2 is to determine whether the viable location matrix $\mathbf{X}^1$ generated from $\mathbf{X}$ is not an equilibrium without finding the best response of any firm. Routine 2 checks whether closing one of the facilities with negative profit will improve the total profit of a firm. If there is an improvement, $\mathbf{X}$ is updated and a new viable location matrix is generated. Now suppose that $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$ are two consecutive location matrices entering Step 1 of Routine 2 and let $\mathbf{X}_{(i)}^1$ and $\mathbf{X}_{(i+1)}^1$ be the viable matrices generated by Routine 1 in Step 1 of Routine 2 corresponding to $\mathbf{X}_{(i)}$ and $\mathbf{X}_{(i+1)}$, respectively. Note that $\mathbf{X}_{(i)}^1 \neq \mathbf{X}_{(i+1)}^1$, because an entry of $\mathbf{X}_{(i)}$ is changed to 0 whose value is 1 in $\mathbf{X}_{(i)}^1$; hence, $\mathbf{X}_{(i+1)}^1$ cannot have 1 in this entry. Note that Routine 2 terminates when either (i) there is no facility with negative profit or (ii) closing any facility with negative profit does not increase the profit of the corresponding firm. The output of Routine 2 is either $continue = 0$ or $\mathbf{X}^2$, which is a viable location matrix and satisfies (i) or (ii). When $continue = 0$, this means that if continued, one will end up with a matrix $\mathbf{X}^2$ that has already been analyzed and, hence, Routine 2 should be started with another location matrix $\mathbf{X}$. Now suppose that $\mathbf{X}^2$ is generated using Routine 2. It is still not known whether $\mathbf{X}^2$ is an equilibrium location matrix. The next step is to determine whether $\mathbf{X}^2$ contains the best responses for each firm. For this purpose, a full neighborhood search for each firm, as explained in Routine 3, is performed.

Routine 3: the following procedure determines whether $\mathbf{X}$ is equilibrium:

1: Set $r = 1$

2: If $r \leq k$, find the best response of firm r, denoted by $x_r^*(\mathbf{X})$, via total enumeration

3: If $x_r^*(\mathbf{X}) = \mathbf{x}_r$, set $r = r + 1$ and go to Step 2

4: Else, stop and return $equilibrium = 0$

5: Else, return $\mathbf{X}^* = \mathbf{X}$ and $equilibrium = 1$.

Total enumeration in Routine 3 generates all possible $\mathbf{x}_r$ vectors and then determines the best response of firm r when the other firms' location decisions are fixed by comparing the total profit of firm r for each matrix $\overline{\mathbf{X}}$ (which differs from $\mathbf{X}$ only in the r^{th} column). The purpose of Routines 2 and 3 is to determine if a given viable location matrix is not in equilibrium as quickly as possible. If Routine 2 cannot guarantee that the viable location matrix is not an equilibrium location matrix, then Routine 3 completes the check by considering all other options for each firm. Hence, at the conclusion of Routine 3, either an improved location decision for a firm is spotted, which implies that the location matrix is not in equilibrium, or an equilibrium location matrix is reached. In what follows, a heuristic method to search for an equilibrium location matrix is explained.

4.4.4 Heuristic Algorithm for Finding An Equilibrium Location Decision

The heuristic algorithm starts with a random location matrix and first moves to a viable location matrix. Then, it checks whether the equilibrium conditions are satisfied by this viable matrix. During the move from a random location matrix to a viable location matrix, Routines 0 and 1 are utilized. Routines 2 and 3 are used to check for equilibrium conditions. Routines 2 and 3 are mainly aimed at simplifying the process of checking equilibrium conditions by easily showing whether the equilibrium conditions are not satisfied, when the current viable matrix is not an equilibrium location matrix. However, a full search is needed to determine an equilibrium location matrix. It should be emphasized that the algorithm does not perform a full search for each non-equilibrium location matrix, which eases the computational burden, as a complete search is burdensome. In particular, a total enumeration scheme to find all of the equilibrium location decisions, or to find out that no equilibrium location decision exists, requires checking the equilibrium conditions for $2^{m \times k}$ location decisions. Moreover, checking the equilibrium conditions for any given location decision requires analyzing $k(2^m - 1)$ other options. Then it follows that a total enumeration scheme would require solving for equilibrium quantities $k2^{m \times k}(2^m - 1)$ times, which is exponential in both m and k. Hence, the following heuristic method that utilizes the previously defined routines is proposed.

Algorithm 4. *Heuristic method to find an equilibrium location matrix, if one exists.*

Step 0. Let $\mathcal{L} = \emptyset$.

Step 1. If $|\mathcal{L}| = 2^{m \times k}$, *stop; there does not exist an equilibrium location matrix. Else, generate a random location matrix,* $\boldsymbol{X}$, *such that* $\boldsymbol{X} \notin \mathcal{L}$.

Step 2. Apply Routine 2 with $\boldsymbol{X}$ *and* $\mathcal{L}$. *If* $\mathit{continue} = 0$, *go to Step 1. Else, generate* $\boldsymbol{X}^2$.

Step 3. Apply Rule 3 to $\boldsymbol{X}^2$. *If* $\mathit{equilibrium} = 0$, *go to Step 1. Else, equilibrium is found, stop and return* $\boldsymbol{X}^*$.

During the algorithm, the set $\mathcal{L}$ keeps track of the location matrices that have been processed. Note that if the algorithm does not stop at Step 3, then the locations in $\mathcal{L}$ are not equilibrium location decisions, and one cannot generate an equilibrium location decision from these location matrices using Routines 0, 1, 2 and 3. Hence, a new location matrix that is not in $\mathcal{L}$ is generated. Moreover, since there are $2^{m \times k}$ possible location decisions, it is concluded that there does not exist an equilibrium location decision when $|\mathcal{L}| = 2^{m \times k}$.

The efficiency of the heuristic method follows from the fact that it reduces the number of full equilibrium checks. Algorithm 4 does not perform a full equilibrium check for any non-viable location matrices, and even avoids this for some viable location matrices (those that are non-equilibrium). Algorithm 4 finds an equilibrium for any given problem if an equilibrium solution exists for the particular problem. If no equilibrium location decision exists for the given problem, then Algorithm 4 outputs the non-existence of an equilibrium. Furthermore, if multiple equilibria exist for the Stage-one game, then Algorithm 4 finds one of them, although it can be easily modified to return all of the equilibria. In Section 4.6, Algorithm 4 is compared with a random search algorithm and numerical results are presented that demonstrate the efficiency of Algorithm 4 in finding an equilibrium location matrix.

4.5 Extensions: Multi-Product and Multi-Echelon Channels

In this section, the model studied in the previous sections is first extended for multiple products, i.e., when firms supply a set of different items to the markets using the same distribution network. Then, implications of the previously documented methods are discussed for multiple echelon supply channels.

It is a common practice that firms supply a variety of products to end customer markets simultaneously. The shipment requirements of various items, of course, will result in higher levels of congestion on the underlying distribution network. The analysis thus far of the location-supply game with traffic congestion costs for a single product not only contribute to the current literature on competitive facility location games, but also permit generalization to more realistic multiple product cases. Therefore, in what follows, the setting defined in Section 4.2 is extended to a multiple product case and the implications on the results in Sections 4.3 and 4.4 are discussed for this case.

In particular, let Z be the set of l products that any firm r supplies to any market j. Furthermore, superscript z is used to index the previous notation for product type, $z = 1, 2, \ldots, l$. For instance, q_{ijr}^z denotes the quantity of product z shipped from the facility of firm r at location i to market j. It is assumed that the sales level of a product type is independent of other products, and that Cournot competition exists among firms for each product type in every market. Specifically, similar to Equation (4–1), the price for product z at market j is defined by $p_j^z(q_{\bullet j \bullet}^z) = a_j^z - b_j^z q_{\bullet j \bullet}^z$. The congestion cost function is as follows:

$$g_{ijr}^z \left(\sum_{z \in Z} q_{ij\bullet}^z \right) = \alpha_{ijr}^z \sum_{z \in Z} q_{ij\bullet}^z. \tag{4–13}$$

Equation (4–13) defines product-specific traffic congestion costs, that is, the total traffic congestion cost paid by firm r for shipping on link (i, j) is characterized as $\sum_{z \in Z} \left[\alpha_{ijr}^z q_{ijr}^z \sum_{z \in Z} q_{ij\bullet}^z \right]$. On the other hand, letting $\alpha_{ijr}^z = \alpha_{ijr} \; \forall z \in Z$, the total traffic congestion cost incurred by firm r for shipping products on link (i, j) amounts to $\alpha_{ijr} \sum_{z \in Z} \left[q_{ijr}^z \sum_{z \in Z} q_{ij\bullet}^z \right]$. Then the profit function for each firm r reads

$$\begin{aligned} H_r(\mathbf{Q}, \mathbf{X}) \;=\; & \sum_{j \in J} \sum_{z \in Z} p_j^z \left(\sum_{i \in I} \sum_{r \in R} q_{ijr}^z \right) \sum_{i \in I} q_{ijr}^z - \sum_{j \in J} \sum_{z \in Z} \sum_{i \in I} c_{ijr}^z q_{ijr}^z \\ & - \sum_{j \in J} \sum_{z \in Z} \sum_{i \in I} q_{ijr}^z g_{ijr}^z \left(\sum_{r \in R} \sum_{z \in Z} q_{ijr}^z \right) - f_r(\mathbf{x}_r), \end{aligned} \tag{4–14}$$

which equals revenue less transportation, traffic congestion, and facility location costs. The two-stage solution approach can be applied to the multiple product case as well. First, the second stage problem determines firms' equilibrium supply quantities at any market for each product type, and then the first stage problem solves for equilibrium location choices. Given firms' facility locations, the second stage game can be shown to be separable in markets; however, it is not separable by product type. Nevertheless, the concavity results and the following properties of the corresponding VIP for the equilibrium problem with multiple products still hold. Therefore, the associated solution method for the single product case discussed in Section 4.3 can be easily adjusted to handle multiple products, and it can be efficiently used to determine equilibrium supply quantities. (In the multiple product case, $\mathbf{Q}_j$ would correspond to a $l\lambda$-vector, which denotes the supply quantities of each product type for market j.) Analysis of the first stage game requires minor modifications (the only modification required is in Step 4 of Routine 0: x_{ir} should be set to 0 when $\sum_{z \in Z} q_{ijr}^{z(*)} = 0$, where $q_{ijr}^{z(*)}$ denotes q_{ijr}^{z} in equilibrium). Next, some possible generalizations to apply to multi-echelon supply chains are discussed.

A two-echelon supply chain is studied in Nagurney et al. (2002). In particular, they consider a supply chain network with manufacturers, retailers, and consumer markets; however, this study assumes that facility locations are predetermined. In an attempt to study competitive facility location problems in multi-echelon channels, first the setting of the game should be defined. Suppose that there are k_e parties at the e^{th} echelon of a t-echelon supply channel. If parties at different echelons decide on their control variables simultaneously, one game can be solved, assuming concavity conditions are satisfied, to determine equilibrium outcomes of the whole channel. A two-stage solution approach can be utilized when the parties' control variables are supply quantities and/or price and facility location choices: first, the supply quantity and price decisions can be solved via variational inequality formulations (see, e.g., Nagurney et al., 2002), and then equilibrium location solutions can be searched.

On the other hand, if there are priorities in the timing or sequence of decisions at different echelons, then distinct analyses would be required. This scenario corresponds to a sequential entry game, and decision makers with priorities in taking action would anticipate and consider the followers' reactions in their decisions. For instance, we might consider a two-echelon supply chain in which a set of competitive manufacturers, who simultaneously determine their plant locations and production levels (or wholesale prices), make decisions prior to a set of competitive retailers, who simultaneously determine store locations and order quantities from the manufacturers. The underlying game would correspond to a Stackelberg game with a set of competitive leaders (manufacturers) and a set of competitive followers (retailers). While Stackelberg games with a single leader and a set of competitive followers have been studied to determine the leader's wholesale price and the followers' order quantity decisions in the literature (see, e.g., Bernstein and Federgruen, 2003, 2005), to the best of our knowledge, there are no studies that consider the wholesale price decisions of multiple competitive leaders and the quantity decisions of multiple competitive followers in the literature. Furthermore, including location decisions, which introduces binary variables, would result in a challenging problem class. The models and analyses presented in this chapter may thus serve as a starting point for analyses of more general multi-echelon competitive facility location problems.

4.6 Numerical Study

In this section, the heuristic search method stated in Algorithm 4 is compared with a random search method. It should be noted that in order to solve for the equilibrium flow quantities for a given location matrix, Algorithm 3 is used with the parameter settings provided in Han (2006) (the efficiency of Algorithm 3 is discussed in Han, 2006). This section aims at demonstrating the efficiency and benefits of Algorithm 4. However, because the model under consideration is new to the literature, no benchmark algorithm exists for comparison. Thus, the numerical studies intend to demonstrate the potential benefits of the proposed heuristic algorithm when compared to a naïve or random search algorithm that might be applied in practice in the absence of an alternative approach. As a result, Algorithm 4 is compared with the following random search algorithm.

Algorithm 5. *Random search method to find an equilibrium location matrix, if one exists:*

Step 0. Let $\mathcal{L} = \emptyset$*. Go to Step 1.*

Step 1. If $|\mathcal{L}| = 2^{m \times k}$*, stop; there does not exist an equilibrium location matrix. Else, generate a random location matrix,* $\boldsymbol{X}$*, such that* $\boldsymbol{X} \notin \mathcal{L}$*. Go to Step 2.*

Step 2. Apply Routine 3 to $\boldsymbol{X}$*. If* $\text{equilibrium} = 0$*, go to Step 1. Else, equilibrium is found, stop and return* $\boldsymbol{X}^*$*.*

Note that the only difference between Algorithm 5 and Algorithm 4 is that Routine 2 is not used in Algorithm 5. This also means that Routine 1 (embedded in Routine 2) and, hence, Routine 0 (embedded in Routine 1) are not used in Algorithm 5 as well. Algorithm 5 applies a full equilibrium check to the given random location matrix and repeats Steps 1 and 2 until either an equilibrium location matrix is found or all of the location matrices are determined to be non-equilibrium. It is worth pointing out that the list of matrices in Algorithm 5 increases by 1 at each occurrence of Step 1. On the other hand, the list of matrices in Algorithm 4 may increase by more than 1 in each occurrence of Step 1. Furthermore, Algorithm 5 applies Routine 3 to each element of the list, whereas Algorithm 4 applies Routine 3 only to the location matrices generated by Routine 2.

For comparison purposes, the same sequence of random location matrices was used within Algorithm 5 and Algorithm 4 for each problem instance. A total of 27 different combinations of $k = \{2, 3, 4\}$, $m = \{2, 3, 4\}$, and $n = \{2, 3, 4\}$ were considered for eight different problem classes with varying per-unit transportation cost, congestion cost factor, and facility location cost distributions. Table 4-1 gives the distribution interval for c_{irj}, α_{ijr}, and f_{ir} values in each class. Ten randomly generated problem instances were solved within each problem class for each problem combination. For each class of problems, $a_j \sim U[50, 100]$ and $b_j \sim U[1, 2]$, where $U[l, u]$ denotes the uniform distribution on $[l, u]$.

Table 4-1. Data intervals for problem classes 1-8

	c_{ijr}	α_{ijr}	f_{ir}
Class 1	$[0, 50]$	$[0, 0.75]$	$[50, 125]$
Class 2	$[0, 50]$	$[0, 0.75]$	$[125, 250]$
Class 3	$[0, 50]$	$[0.75, 1.5]$	$[50, 125]$
Class 4	$[0, 50]$	$[0.75, 1.5]$	$[125, 250]$
Class 5	$[50, 100]$	$[0, 0.75]$	$[50, 125]$
Class 6	$[50, 100]$	$[0, 0.75]$	$[125, 250]$
Class 7	$[50, 100]$	$[0.75, 1.5]$	$[50, 125]$
Class 8	$[50, 100]$	$[0.75, 1.5]$	$[125, 250]$

An equilibrium location decision was found for every problem instance. Each row in Table 4-2 summarizes the average of 80 problem instances (10 from each problem class), for each combination of k, m and n (resulting in 2160 total instances) for the following data: length of the list at termination (list length), number of full equilibrium checks (# of checks) and CPU time in seconds. As can be seen in Table 4-2, the heuristic method is much faster than the random search method. This is due to the following two points: (i) the heuristic method does not perform a full equilibrium check (which is computationally burdensome) for each element within the list and, (ii) it moves to a viable location matrix from the given random location matrix and may determine that the viable matrix is not an equilibrium location matrix without a full equilibrium check. Notice that CPU times tend to be linear with n. It should also be noted that in the worst case, both algorithms would require $2^{k \times m}$ equilibrium checks (or executing Algorithm 4 $k2^{k \times m}(2^m - 1)$ times, which would be required for total enumeration of the location matrices). Nonetheless, when the list lengths at termination are compared, Algorithm 4 analyzes fewer matrices and performs full equilibrium checks for around 40% of these matrices when compared to Algorithm 5, which performs full equilibrium checks for all of the location matrices it analyzes. Table 4-3 compares the average values of 270 problem instances within each problem class.

Table 4-2. Comparison of heuristic method with random search method

			Algorithm 2			Algorithm 3		
			list	# of	CPU	list	# of	CPU
k	m	n	length	checks	time	length	checks	time
2	2	2	6.60	5.55	0.16	7.90	7.90	0.14
2	2	3	6.93	6.28	0.23	8.88	8.88	0.24
2	2	4	7.03	6.48	0.35	8.00	8.00	0.30
2	3	2	12.20	7.78	0.43	25.83	25.83	1.07
2	3	3	18.18	14.13	1.07	37.43	37.43	2.33
2	3	4	24.78	19.48	2.14	32.25	32.25	2.76
2	4	2	28.90	18.85	1.96	97.13	97.13	9.31
2	4	3	42.13	31.73	4.46	125.33	125.33	18.61
2	4	4	81.70	68.88	11.70	134.70	134.70	25.32
3	2	2	16.95	12.68	0.42	31.38	31.38	0.69
3	2	3	20.15	15.85	0.79	33.75	33.75	1.17
3	2	4	19.23	15.78	1.07	29.30	29.30	1.42
3	3	2	34.00	20.85	1.40	271.33	271.33	14.76
3	3	3	83.10	58.28	5.36	266.80	266.80	22.37
3	3	4	78.85	51.73	6.25	289.48	289.48	28.93
3	4	2	94.38	40.80	5.15	1764.38	1764.38	215.50
3	4	3	251.08	133.73	24.74	2257.53	2257.53	380.39
3	4	4	682.28	480.35	111.86	2023.58	2023.58	470.06
4	2	2	20.70	11.80	0.51	120.20	120.20	3.12
4	2	3	30.35	17.70	1.31	104.18	104.18	4.46
4	2	4	47.60	37.75	3.00	105.98	105.98	6.03
4	3	2	149.00	28.38	9.56	2169.65	2169.65	155.44
4	3	3	176.90	101.25	14.37	1646.43	1646.43	178.07
4	3	4	240.20	168.85	32.49	1854.90	1854.90	272.25
4	4	2	1767.73	146.85	803.84	29366.95	29366.95	10784.85
4	4	3	1797.20	789.25	600.56	27287.48	27287.48	12388.53
4	4	4	1663.70	682.85	312.42	26259.40	26259.40	10954.59
average			274.14	110.88	72.50	3568.89	3568.89	1331.21

It can be observed from Table 4-3 that for problem instances with higher values of c_{ijr}, Algorithm 4 is more efficient (i.e., shorter list length, fewer number of checks, and less time), while Algorithm 4 is slightly less efficient for problem instances with higher congestion cost factors. There is not any significant relationship between the efficiency of Algorithm 4 and facility location costs. Nevertheless, the results presented clearly illustrate that Algorithm 4 outperforms Algorithm 5 in average computational time, as

Table 4-3. Comparison of heuristic method with random search method for each problem class

	Algorithm 2			Algorithm 3		
	list length	# of checks	CPU time	list length	# of checks	CPU time
Class 1	242.14	151.65	31.37	3106.27	3106.27	684.64
Class 2	397.53	179.13	135.99	4039.97	4039.97	1577.19
Class 3	522.75	318.56	111.46	3061.44	3061.44	1009.38
Class 4	782.01	149.55	269.94	3722.40	3722.40	1208.88
Class 5	65.68	23.95	9.41	4294.39	4294.39	2206.29
Class 6	44.24	12.13	4.75	3114.65	3114.65	1369.39
Class 7	103.25	43.24	14.60	3506.11	3506.11	1253.74
Class 8	35.53	8.86	2.50	3705.90	3705.90	1340.18
average	274.14	110.88	72.50	3568.89	3568.89	1331.21

well as in average list size and the average number of full equilibrium checks, for each problem class.

CHAPTER 5
SUPPLIER WHOLESALE PRICING: IMPLICATIONS OF DECENTRALIZED VS. CENTRALIZED PROCUREMENT UNDER QUANTITY COMPETITION

5.1 Motivation

This work in this chapter is motivated by the structure and operation of an air-conditioning products supply channel in Florida. In particular, the channel operates as follows. A supplier manufactures air conditioning products, which it sells to local retailers via a distribution company in Florida. The retailers compete with each other and independently determine their equipment order quantities and inform the distributor; the distributor then transmits these orders to the supplier. The retailers' orders are shipped to the distributor and the distributor then ships the orders to retailers. The distributor's livelihood is, therefore, dependent on meeting the needs of both the supplier and the retailers. In its role as intermediary, the distributor directly affects the system's economic performance. Currently, the distributor does not engage in procurement quantity decisions, and the retailer orders are therefore individually transmitted to the supplier. This context raises several interesting research questions about the role a distributor can play in centralizing retailer procurement, and how this role influences the competition for channel profit between a supplier and its competitive retailers. In particular, this chapter is interested in how the distributor's procurement strategy can affect supplier wholesale pricing decisions, as well as the value to the supplier of being able to control this strategy.

A supplier's wholesale pricing strategy represents one of the most crucial decisions influencing the profitability and efficiency of a supply chain for a product or service. The supply chain management literature has, therefore, focused intensely on pricing problems in recent years. In this chapter, we study a supplier's pricing problem for a good sold to multiple retailers from a *vertically decentralized* perspective, and discuss the implications from a *vertically centralized* perspective. Under a vertically decentralized approach, parties at different stages of a distribution channel make decisions based on local objectives only. Conversely, a vertically centralized approach assumes existence of a central decision maker whose goal is to achieve a channel-wide optimal solution. The problem we study corresponds to a Stackelberg game in a vertically decentralized setting, where the supplier acts as the leader by setting a wholesale price, and the retailers act as followers by determining how much they will supply to the market, i.e., their order quantities. We consider the case when retailers' orders are transmitted via a single company (e.g., a distributor) and can be operated using one of three different strategies: decentralized, centralized, and partially centralized. It should be noted that centralization or decentralization of the retailers does not imply centralization or decentralization of the channel.

The channel we study, as noted previously, currently operates in a completely vertically decentralized manner; however, in principle, the decisions and actions at the retail stage may be horizontally centralized or decentralized. In particular, when the retailers are horizontally decentralized (as is currently the case), each retailer determines his/her order quantity from the supplier independently of other retailers. This case corresponds to a Cournot Oligopoly among the retailers, which assumes that the market price is determined by the total supply to the market. Under a horizontally centralized procurement approach, however, retailer order quantities would be determined by a central decision maker on behalf of the retail stage. In the setting we have described, this central decision maker would correspond to the distributor. In particular, because of the distributor's relative channel power, its relationship with the supplier, and the fragmented nature of the localized retailers, the distributor can potentially assume the procurement function for the retailers it serves with little or no risk. As the distributor is interested in the financial viability of both its supplier and the retailers it serves, this context raises interesting questions about how procurement centralization (or decentralization) impacts the performance of the distributor's upstream and downstream partners, as well as the entire channel. We will also consider the potential for applying a so-called partially centralized approach, under which we assume that individual retailers permit the distributor to procure on their behalf, provided that each retailer retains its current market share under independent ordering. Although the problem we discuss is motivated by a particular channel, the model and analysis we provide may apply more broadly, not only to other channels involving a distributor that serves multiple, separate retailers, but also to settings in which a chain of retail stores wishes to consider the potential benefits of centralized procurement.

When the retailers are horizontally decentralized, they compete in the end-customer market to sell the supplier's product. When competition over homogeneous products exists (i.e., under low product differentiation), as noted in previous chapters, the market can often be represented using Cournot competition, as quantity decisions will be of greater interest than pricing decisions. Hence, because we assume that the retailers sell the same product, a Cournot competition assumption readily applies to the problem class we study. Furthermore, Iyer (1998) notes that decentralized decision making is efficient for markets where product differentiation is low (low product differentiation implies that multiple products serve as demand substitutes). In this case, the supplier sets his/her wholesale price for a set of competitive retailers. For this case, we discuss the supplier's wholesale price setting problem as well as the competing retailers' order quantity decisions from the supplier.

On the other hand, retailer orders can be centrally managed by the distributor. Problems accounting for centralized control of retailers have been analyzed in the supply chain management literature (Dong and Rudi, 2004, Yang and Zhou, 2006, Shao et al., 2009). Through horizontal centralization, as noted by Shao et al. (2009), the distributor can benefit from reduced retailer competition. In our model, as we later show, a distributor would prefer to operate through a single retailer for a given product, i.e., the one with the lowest operating cost in the market for that product under horizontal centralization. In practice, this case corresponds to the scenario where the most efficient retailer takes over the distributor's role and interfaces directly with the supplier. We will discuss how the supplier sets it wholesale price for this case. Although horizontally centralized control of retailers is ultimately the most profitable solution for the retail stage for a single product, individual retailers may be more efficient at selling different products. As a result, individual retailers may wish to preserve the market share they gained under horizontal decentralization for a given product for a number of reasons (e.g., in cases where the product's availability may bring customers into the retail store and, therefore, affect the retailer's profit from sales of other products). Recent studies in assortment planning point out the fact that a product's demand depends not only on the price of the product, but also on the inventory levels of the other products in the retail shop (Bitran et al., 2010). Therefore, as an alternative to strict horizontal centralization of retailers, we consider the case in which retailers transition from decentralized control to centrally controlled ordering, but the market share gained under decentralized control must be maintained as control of ordering is transitioned to a central decision maker. We refer to this strategy as the *partial centralization* of retail procurement and discuss the distributor's decisions under this scenario. Furthermore, we illustrate how the supplier determines his/her optimal wholesale price under partially centralized retailer procurement.

In the Stackelberg game we consider, the supplier is the leader and the distributor and retailers are followers. The supplier determines its wholesale price and the retailers and distributor react by determining downstream order quantities (or, equivalently, market supply quantities) based on the distributor's procurement strategy. We use backward induction to determine the Stackelberg equilibrium of this game; that is, we first characterize the decisions at the downstream (distributor and retail) stages and use this solution to solve the supplier's problem. In particular, the sequence of events is as follows: (i) the supplier sets his/her wholesale price, (ii) the distributor determines whether retailer orders are decentralized, centralized, or partially centralized, (iii) the distributor announces retailer order quantities, and (iv) the supplier ships the retailers' order quantities, and a market clearing price is determined. It should be noted that the supplier does not control the distributor's procurement strategy. It is obvious that the supplier's wholesale price decision under an assumption of decentralized retailers will be suboptimal if the distributor horizontally centralizes retailer orders. However, we will show that centralized or partially centralized control of retail orders is beneficial to the retail stage for any given supplier wholesale price; hence, the distributor will choose either centralized or partially centralized control if retailers are willing to relinquish this control. It will then follow that the Stackelberg equilibrium solution is achieved when the supplier sets his/her wholesale price for centralized or partially centralized retailers. Nevertheless, we will show that this Stackelberg equilibrium is not necessarily an optimal solution for the system (e.g., when the supplier's production costs are linear in the production quantity). In particular, the supplier may be better off under decentralized retail ordering; that is, the supplier can achieve substantial savings if s/he is able to dictate the distributor's procurement strategy. This introduces an important concept which we refer to as the *value of control*. The value of control defines how much the supplier can save if s/he controls the distributor's procurement strategy, and we quantify the value of this control via our numerical studies.

This chapter contributes to the literature by modeling a supplier's pricing decision for a set of competitive retailers whose ordering processes can be controlled by a distributor under different procurement strategies. In our analysis, we allow an arbitrary number of non-identical retailers. Furthermore, we consider general cost functions in our models. To the best of our knowledge, the value of control for a supplier on the retail procurement strategy in a competitive market has not been studied in the literature. Thus, another contribution of this chapter lies in introducing and quantifying the value of this control for the supplier. We characterize properties of the supplier's profit function, and gain important managerial insights through our analytical and numerical studies of the channel. We argue how a supplier may encourage centralized or decentralized management of the retailers, depending on whether s/he faces economies or diseconomies of scale in production costs. Our numerical studies indicate that when the supplier's production cost is linear in production quantity, equilibrium is achieved when the retailers are centralized by the distributor and the supplier sets his/her wholesale price accordingly. However, system-wide profit is maximized when retailers are decentralized and the supplier sets the wholesale price for decentralized retailers. This observation indicates that an effective coordination mechanism should preclude horizontal centralization, i.e., vertical centralization may require horizontal decentralization. Based on these observations, we discuss why a coordination mechanism may require restrictions on retail procurement strategy.

The rest of this chapter is organized as follows. Section 5.2 discusses relevant work in the literature. Section 5.3 defines the problem setting and our solution approach. In Section 5.4, for a given wholesale price, we analyze the retailers' order quantity decisions under different procurement strategies imposed by the distributor and the distributor's best choice of procurement strategy. Then, in Section 5.5, we first derive an explicit expression for the retailers' total order quantity as a function of the wholesale price. Then, we formulate the supplier's problem in terms of the wholesale price and provide a solution algorithm for this problem. Section 5.6 extends our results to a multi-market (or multi-product) setting and discusses generalizations that permit the supplier to offer quantity discount pricing. Section 5.7 proceeds with our numerical studies that analyze the value of control. In addition, we determine the Stackelberg equilibrium and argue that channel-wide profits are not maximized at this equilibrium.

5.2 Literature Review

In the supply chain literature, a supplier's pricing strategy in a two-echelon channel is often considered within the broader context of *channel coordination*. Channel coordination focuses on mechanisms that ensure profit levels at or near the optimal centralized solution while using decentralized decision making. If a coordination mechanism exists that results in an optimal centralized solution under decentralized decisions, then *perfect coordination* is achieved. The application of pricing as a channel coordination mechanism appears in many inventory control problems. Monahan (1984) shows that an all-units quantity discount schedule can be used to increase a supplier's profit without reducing the buyer's profit with respect to that under decentralized decisions in a single-supplier, single-buyer system (under demand and cost assumptions similar to those in the economic order quantity, or EOQ, model). The studies by Lal and Staelin (1984), Banerjee (1986), Lee and Rosenblatt (1986), and Toptal et al. (2003) focus on coordinated pricing decisions for more general models under deterministic demand. When stochastic demand is considered, the coordination mechanisms proposed include revenue-sharing contracts (Cachon, 2003, Giannoccaro and Pontrandolfo, 2004, Cachon and Lariviere, 2005), buy-back policies (Pasternack, 1985, Emmons and Gilbert, 1998), returns policies (Taylor, 2001), and rebate policies (Taylor, 2002). Broader classes of pricing problems under different problem settings and coordination mechanisms have been studied in the operations research literature, see, e.g., the reviews by Tsay et al. (2000), Cachon (2003), and Sarmah et al. (2006).

The studies we have cited thus far, whether considering deterministic or stochastic demand, assume that the retailer's price is exogenously determined. In contrast, Jeuland and Shugan (1983) consider the pricing decision of a manufacturer in a channel coordination context when demand is a function of the retailer's price. Weng (1995a,b) considers a supplier's quantity discount pricing problem under price-sensitive demand, where the single supplier and a single retailer (or multiple, identical retailers) operate under the classical EOQ model. Additional work on pricing decisions of a supplier for coordinated channels with price-sensitive demand can be found in Moorthy (1987), Jeuland and Shugan (1988), Chen et al. (2001), Boyacı and Gallego (2002), Viswanathan and Wang (2003), and Qin et al. (2007). While these papers recognize a supplier's pricing decision as a Stackelberg game (involving the supplier and buyers), they do not consider competition among the buyers. We note that, beyond Stackelberg games, various cooperative, bargaining, and coalition games between a supplier and its buyers have been analyzed in the literature. For literature on game-theoretic analyses of these problems, one may refer to Kohli and Park (1989), Abad (1994), Iyer and Padmanabhan (2005), Nagarajan and Sošić (2008), Sarmah et al. (2006), Xie and Wei (2009), and the references therein. Competition at the retail-level of the supply chain is disregarded in the papers cited thus far.

This study considers a supplier who charges a fixed wholesale price to a set of retailers, who sell the supplier's product in an end-customer market. The channel is assumed to be vertically decentralized. Decentralized decision making often prevails in situations involving a dominant party (Toptal and Çetinkaya, 2006), when there is competition at the supplier level (Moorthy, 1988), and in other situations, such as supply chains with multiple inventory sites (Lee and Billington, 1993, Lee and Whang, 1999). Lee and Billington (1993) note that even when these multiple sites are controlled by a single firm, the firm may still prefer decentralized decision making. As noted previously, vertical decentralization is preferred for markets with low product differentiation (Iyer, 1998). Furthermore, the decentralized decision setting serves as a benchmark for comparing the effectiveness of coordination mechanisms that can increase overall channel profits. While the channel is vertically decentralized, we consider both horizontally decentralized and centralized decisions at the retail echelon. If the retail stage is horizontally decentralized, the retailers compete in the end-customer market. In particular, we model the retailers' competition using quantity or Cournot competition, as this type of competition often applies in cases with low product differentiation.

A significant amount of literature exists on coordination problems under downstream (e.g., retail-level) competition in a supply chain. Ingene and Parry (1995, 1998, 2000) study a coordination problem between a manufacturer and two retailers competing on price. Iyer (1998) studies channel coordination when retailers are engaged in price and non-price competition. Bernstein and Federgruen (2003) focus on coordinated inventory replenishment and pricing decisions in a two-echelon supply chain consisting of a single manufacturer and multiple competitive retailers (under both price- and quantity-based competition). They model the manufacturer's and the retailers' profits under deterministic demand. Following this, Bernstein and Federgruen (2005) analyze coordination mechanisms between a manufacturer and a set of competitive retailers who observe random demand in a single-period setting. This study is then extended by Bernstein and Federgruen (2004) to an infinite planning horizon. Other studies focusing on channel coordination with retailer competition consider returns policies (Padmanabhan and Png, 1997), two-part tariffs (Tsay and Agrawal, 2000), quantity discounts (Balachander and Srinivasan, 1998, Xiao and Qi, 2008), vendor managed inventory (Bernstein et al., 2006), and revenue sharing contracts (Yao et al., 2008). Additional references in this research stream can be found in Cachon (2003).

The underlying Stackelberg game we study is defined similarly to the games studied by Yang and Zhou (2006) and Keskinocak and Savaşaneril (2008). Yang and Zhou (2006) analyze a Stackelberg game where a manufacturer leads by determining a fixed wholesale price, and two non-identical competitive retailers follow by determining their order quantities. The manufacturer's optimal pricing decision is considered when the retailers are competitive (i.e., decentralized), united (i.e., centralized), and take actions sequentially. However, in this study, Yang and Zhou (2006) do not consider the market entry condition. (The competition base for the two retailers is price; hence, both retailers enter the market at any manufacturer wholesale price, and their selling prices differ in the market. However, in Cournot competition, a retailer may refuse to enter the market based on his/her operating costs as well as the supplier's wholesale price; that is, the supplier affects the number of retailers entering the market). In our study, we consider

Cournot-type quantity competition; hence, entering the market is a retailer's choice and the supplier can effectively control retailer entry decisions. As we discuss in Section 5.5, the supplier considers retailers' market entry decisions while setting his/her wholesale price. Keskinocak and Savaşaneril (2008) study a supplier's pricing problem when an arbitrary number of identical retailers compete on sales volumes. They analyze the decisions of competitive and cooperative retailers. While the competitive case corresponds to what we call horizontal decentralization, the cooperative case is different from the horizontal centralization we apply. The retailers cooperate via collaborative purchasing, given a supplier's linear discount pricing function. However, in their study, retailers' cooperation levels are determined by the supplier, i.e., the supplier decides whether the retailers are cooperative by setting a spillover factor, which defines the additional discount a retailer achieves from the supplier as a result of other retailers' order quantities. Our study extends the setting of Keskinocak and Savaşaneril (2008) to the case of non-identical retailers. Further, we consider horizontal centralization via an intermediary (the distributor) instead of cooperation.

In particular, we consider two different centralization scenarios for retailers: centralization and partial centralization. Centralization occurs when individual retail stores' decisions are controlled by a single firm, as would occur in the case of a chain store (Shao et al., 2009) or when a distributor orders on behalf of multiple retailers (Anupindi and Bassok, 1999). Centralized control of retailers has been analyzed in different forms in the supply chain management literature (Dong and Rudi, 2004, Yang and Zhou, 2006, Shao et al., 2009). Dong and Rudi (2004) consider a manufacturer and multiple, centralized, identical-cost retailers with a focus on transhipment and wholesale price decisions. As mentioned previously, Yang and Zhou (2006) consider the case when two retailers unite and behave in a centralized manner. Shao et al. (2009) consider a manufacturer's wholesale price decision for two identical retailers who can be centralized or decentralized. The centralization of the retail stage we apply in this chapter is similar to the aforementioned literature. When retailers are centralized, the objective is for the distributor, who acts as a central decision maker for the retail stage, to maximize the total profit at the retail stage. As we later demonstrate, while centralized retail procurement leads to the highest retail-stage profit, it may be difficult or impossible to implement because an optimal centralized policy requires that the end-customer market be supplied by a single retailer for a given product. When a retailer sells multiple products, however, the stock of a given product can affect a retailer's profit from other products. For instance, Bitran et al. (2010) note that along with the price of a particular product, the availability of other products in a store affects the demand of the product. Recent studies on retailer assortment planning analyze the effects of product availability on retail profits. For instance, a basket shopper (a consumer intending to buy multiple products from a retail shop) may prefer to purchase his/her entire basket from another shop when a given shop does not have one of the items in his/her basket (Bell and Lattin, 1998). Van Ryzin and Mahajan (1999) note that increasing the variety within an assortment increases the possibility a customer will buy something from a retailer. We refer the interested reader to Smith and Agrawal (2000), Cachon and Kök (2007), and Kök et al. (2009) for further discussions on assortment planning. Because of these assortment planning issues, we consider partial centralization as an alternative to complete centralization. The objective of partial centralization is the same as centralization, except that the distributor must maintain each retailer's market share in order to avoid the disadvantages centralization introduces with respect to assortment planning requirements.

5.3 Problem Formulation and Methodology

Consider a supplier (manufacturer) who sells a product to an end customer market via a set of n retailers, indexed by $i \in \{1, 2, \ldots, n\}$. We assume that the retailers' orders for the supplier's products are placed through a distributor. The distributor may choose decentralized, centralized, or partially centralized control of the retailer orders. The market price (i.e., the retailers' selling price) is determined by the total quantity supplied to the market. Thus, when the retailers are decentralized, they are engaged in Cournot (quantity- or sales-volume-based) competition. In particular, let q_i denote the quantity that retailer i, $i \in \{1, 2, \dots, n\}$, supplies to the end customer market. Then p, the market price, is determined by the function $p(Q)$, where Q denotes the total quantity supplied to the market, i.e., $Q = \sum_{i=1}^{n} q_i$. In particular, similarly to Chapters 2-4, we assume that $p(Q)$ is a linear decreasing function of Q such that

$$p(Q) = a - bQ = a - b\sum_{i=1}^{n} q_i, \tag{5–1}$$

where the parameters $a \geq 0$ and $b > 0$ denote the price at zero demand and the market price sensitivity, respectively (and both parameters are assumed to be real-valued finite numbers). The market magnitude and demand elasticity can also be represented by these parameters, respectively. We note that $p(Q)$, represents the inverse demand function associated with a Cournot setting, and is widely used to model situations in which market price is determined by the total supply to the market (see, e.g., Bernstein and Federgruen, 2003, Keskinocak and Savaşaneril, 2008).

The supplier must deliver the entire order for each retailer; thus, the supplier's production quantity equals the sum of the retailers' order quantities. As each retailer will order the quantity that s/he will supply to the market, the supplier's production quantity is $Q = \sum_{i=1}^{n} q_i$. The supplier incurs production costs, which are a function of Q and, hence, the supplier's profit equals total revenue gained from selling to retailers less production costs. Now let c denote the supplier's wholesale price (i.e., the supplier's selling price to retailers). We assume that the wholesale price set by the supplier is the same for each retailer. This is consistent with the Robinson-Patman Act, which prohibits price discrimination. Furthermore, a supplier may choose to utilize a simple wholesale pricing scheme (Ingene and Parry, 1995, 1998). Then the supplier's profit as a function of Q, $\Pi_S(Q)$, reads

$$\Pi_S(Q) = cQ - f(Q), \tag{5–2}$$

where cQ is the total revenue and $f(Q)$ represents the production-related cost function. We assume that $f(Q)$ is a continuous and differentiable function of Q. In particular, $f(Q)$ is a nondecreasing function, since production cost is expected to increase with production quantity. Further characteristics of $f(Q)$ will be discussed in more detail when we analyze the supplier's problem.

We assume that each retailer is subject to operating costs, and let $v_i(q_i)$ denote the operating cost function of any retailer i. In particular, let $w_i > 0$ denote the per-unit operating cost of retailer i. Observe that w_i can be interpreted as a per-unit shipping cost from retailer i to the market (and retailers located in different areas will therefore have different w_i values) or simply a per-unit operating (e.g., material handling) cost of retailer i (when retailers utilize different technologies, this may also lead to different w_i values). Considering the practice of retailers' orders being shipped to the distributor and then to the retailers, w_i can be interpreted as $w_i = w + r_i$, where w would denote shipping cost from the supplier to the distributor warehouse and r_i would denote the shipping cost from the distributor to retailer i plus the shipping cost from retailer i to the market. The total operating cost of retailer i, then, amounts to $v_i(q_i) = w_i q_i$. (Similar retailer cost functions are used in Keskinocak and Savaşaneril, 2008.) We assume that retailers are non-identical; thus, $w_{i_1} \neq w_{i_2}$ for $i_1 \neq i_2, i_1, i_2 \in \{1, 2, \dots, n\}$. The profit of retailer i equals the total revenue gained from supplying the market, less purchase and operating costs. Then, considering Equation (5–1), the profit function of retailer i reads

$$\Pi_i(\overrightarrow{\mathbf{Q}}) = p(Q)q_i - cq_i - v_i(q_i) = \left(a - b\sum_{i=1}^{n} q_i\right) q_i - cq_i - w_i q_i, \tag{5–3}$$

where $\overrightarrow{\mathbf{Q}}$ denotes the n-vector of retailer supply quantities. The first term of Equation (5–3) denotes the total revenue, the second term is the total purchase cost, and the last term represents the operating costs.

The supplier and the retailers, when horizontally decentralized, are profit maximizing agents; thus, the supplier's problem, $\mathcal{P}_S$, and the problem of retailer i, $\mathcal{P}_i$, can be formulated as follows:

$$(\mathcal{P}_S) \quad \max \quad \Pi_S(Q) \qquad\qquad (\mathcal{P}_i) \quad \max \quad \Pi_i(\overrightarrow{\mathbf{Q}})$$
$$\text{s.t.} \quad Q \geq 0, \qquad\qquad\qquad \text{s.t.} \quad q_i \geq 0.$$

The distributor can affect the procurement strategy under which the retailers operate. If the distributor centralizes retailer procurement (and determines the market supply), then the retailer order quantities can be collectively set to maximize the sum of the retailers' profits. On the other hand, if the distributor does not centralize the retailers, i.e., when retailers are horizontally decentralized, and the decision of any individual retailer is affected by the decisions of the other retailers. Therefore, we use the equilibrium concept of Nash (1951) to characterize the quantity decisions of the retailers in case of horizontal decentralization.

We first focus on solving for the retailers' order quantities given the distributor's procurement strategy, and, then the supplier's wholesale price-setting problem is solved. Here, given the supplier's wholesale price, the retailers' order quantity decisions are determined under three different distributor procurement strategies. Then, for each such strategy, the supplier's wholesale price setting problem is solved.

5.4 Retail Stage: Supply Quantities and Procurement Strategy

In this section, for a given supplier wholesale price, we provide a solution method for the retailers' order quantity decisions under three different distributor procurement strategies. Specifically, we characterize retailer order quantities when retailer procurement is decentralized, centralized, and partially centralized. At the end of this section, we compare these quantity decisions under different procurement strategies. Furthermore, we discuss the distributor's choice of procurement strategy and implications on the supplier's preference, if s/he can control or influence the distributor's procurement strategy.

5.4.1 Decentralized Retailing

When retailer procurement is decentralized, the distributor allows retailers to independently determine their order quantities. In this case, the distributor's role is to provide communication between the supplier and the retailers who want to sell the supplier's product. (The distributor can charge retailers on per-unit basis, in particular, when orders are shipped to the distributor's warehouse first, and then to retailers. This per-unit charge can be considered as part of the retailers' per-unit operating costs; hence, this will not change the analytical results. Nevertheless, as the distributor is the central decision maker for the retail stage, this per-unit charge is not necessary.) Horizontal decentralization at the retail stage has been analyzed in the literature (see, e.g., Dong and Rudi, 2004, Yang and Zhou, 2006, Shao et al., 2009). Under horizontal decentralization, each retailer's objective is to maximize his/her profit by determining its market supply quantity, which is affected by the supply quantity decisions of all other retailers. In particular, retailer decisions correspond to a Cournot oligopoly and our aim is to determine the Nash equilibrium solution of this game. Recall that retailer i's problem is defined by $\mathcal{P}_i$. It can be shown that $\Pi_i(\overrightarrow{\mathbf{Q}})$ is strictly concave, given the order quantities of the other retailers. (This follows, as $\partial^2\Pi_i(\overrightarrow{\mathbf{Q}})/\partial q_i^2 = -2b < 0$ and as $b > 0$.) This implies that the first-order conditions ($\partial\Pi_i(\overrightarrow{\mathbf{Q}})/\partial q_i = 0$, for $q_i > 0$) must be satisfied at the unique Nash equilibrium solution. The uniqueness follows from (i) strict concavity of the profit functions, (ii) the assumption of non-identical retailers, and (iii) bounded retailer quantity decisions, a and b are real-valued numbers, i.e., the retailers will not supply against a negative market price, and, thus, their supply quantities will be bounded from above, which, together with $q_i \geq 0$, $\forall i$, implies the compactness of the retailers' strategy sets. It follows from Equation (5–3) that if $q_i > 0$, then the Nash equilibrium solution must satisfy the condition

$$a - b\sum_{i=1}^{n} q_i - bq_i - c - w_i = 0. \tag{5–4}$$

Let q_i^* denote the quantity supplied by retailer i at equilibrium. Then retailer i is defined as *active* if $q_i^* > 0$. In what follows, we discuss important characteristics of the equilibrium supply quantities. The proof of the following proposition and the correctness of the algorithm that follows are provided in Appendix C under general market price and retailer operating cost functions (Appendix C shows that Proposition 5.1 and Algorithm 6 hold when $p(Q)$ is a decreasing concave function of Q and $v_i(q_i)$ is an increasing convex function of q_i. As these conditions are satisfied for linear market price and operating cost functions, we do not repeat the proofs for the case when $p(Q) = a - bQ$ and $v_i(q_i) = w_i q_i$). The following proposition provides important characteristics of the equilibrium supply quantities.

Proposition 5.1. *(i) $q_i^* > 0$ if and only if $c + w_i < a - b\sum_{i=1}^{n} q_i^*$, for all $i \in \{1, 2, \dots, n\}$. (ii) Suppose that $w_{i_1} < w_{i_2}$ for retailers i_1 and i_2 such that $i_1, i_2 \in \{1, 2, \dots, n\}$. Then, (a) if $q_{i_2}^* > 0$, then $q_{i_1}^* > 0$, and (b) if $q_{i_1}^* = 0$, then $q_{i_2}^* = 0$.*

Proof: Please see Appendix C.

Condition (i) of Proposition 5.1 simply states that a retailer is active if and only if its marginal cost ($c + w_i$) is less than the equilibrium market price. Condition (ii) indicates that sorting retailers with respect to their per-unit operating costs is important in characterizing equilibrium supply quantities. In particular, if we know that there are $\ell \leq n$ retailers active at equilibrium, it then ensues that these retailers are the first ℓ retailers with the lowest w_i values. Now, without loss of generality, suppose that retailers are sorted in increasing order of w_i values and let us assume that ℓ retailers are active for a given wholesale price c. That is, $q_i^* > 0$ for retailers $i = 1, 2, \dots, \ell$ and $q_i^* = 0$ for retailers $i = \ell + 1, \ell + 2, \dots, n$. Then the equilibrium supply quantities are determined by the simultaneous solution of Equation (5–4) for all $i \leq \ell$. However, we do not know the number of active retailers a priori. The next algorithm is a realization of the algorithm stated in Appendix C, which solves for the equilibrium quantities under generalized market price and operating cost functions, determining the number of active retailers as well as the associated equilibrium supply quantities when $p(Q) = a - bQ$ and $v_i(q_i) = w_i q_i$.

Algorithm 6. *Without loss of generality, suppose that retailers are sorted in increasing order of w_i values. Given a, b, c and w_i $\forall i \in \{1, 2, \dots, n\}$;*

Step 0. If $a - (c + w_1) \leq 0$, set $q_i^* = 0$ $\forall i \in \{1, 2, \dots, n\}$ and $\ell^* = 0$. Else, set $\ell = 1$ and go to Step 1.

Step 1. Determine $q_i^{(\ell)}$ for $i \leq \ell$ by solving the following system of equations. Go to Step 2.

$$a - b \sum_{i=1}^{\ell} q_i^{(\ell)} - b q_i^{(\ell)} - c - w_i = 0 \; \forall i \leq \ell.$$

Step 2. If $\ell = n$, stop. All of the retailers are active; $q_i^* = q_i^{(\ell)}$ for $i \leq n$ and $\ell^* = n$. Else, if $\ell < n$ and if $c + w_{\ell+1} \geq a - b \sum_{i=1}^{\ell} q_i^{(\ell)}$, stop. Retailers $i \leq \ell$ are active; $q_i^* = q_i^{(\ell)}$ for $i \leq \ell$ and $q_i^* = 0$ for $\ell < i \leq n$ and, $\ell^* = \ell$. Else, if $\ell < n$ and if $c + w_{\ell+1} < a - b \sum_{i=1}^{\ell} q_i^{(\ell)}$, set $\ell = \ell + 1$ and go to Step 1.

It follows from Algorithm 6 that the determination of retailer supply quantities consists of two main steps: determining the number of active retailers, which is defined as ℓ^*, and then solving for the equilibrium supply quantities of the active retailers, i.e., q_i^* values for $i \leq \ell^*$. Algorithm 6 terminates either when a set of retailers keeps the next retailer out of market or when all the retailers are active. Next, we analyze the details of centralized and partially centralized control of the retailers' procurement quantities.

5.4.2 Centralized Retailing

While decentralized procurement decisions can be implemented because of their practical simplicity and marketing benefits, it is possible that the distributor may achieve substantial savings under horizontally centralized control of retail procurement. As noted by Shao et al. (2009), the distributor can benefit from decreased competition. Therefore, we consider the case when the distributor prefers centralized control of the retailers. The distributor's objective under a centralized procurement strategy is to determine each retailer's supply quantity such that the total profit of the retailers is maximized.

Considering Equation (5–3), the solution under centralized retailing is obtained by solving the following problem:

$$\begin{aligned} \mathcal{P}_c \quad \max \quad & \textstyle\sum_{i=1}^{n} \Pi_i(\overrightarrow{\mathbf{Q}}) \\ \text{s.t.} \quad & q_i \geq 0, \quad i = 1, \ldots, n. \end{aligned}$$

One can easily see that the solution of the decentralized procurement strategy is feasible for $\mathcal{P}_c$, and centralized procurement, therefore, cannot reduce the total profit of the retailers. The following proposition characterizes the solution to $\mathcal{P}_c$. All of the proofs are presented in the Appendix.

Proposition 5.2. *$q_1^c = \frac{a-c-w_1}{2b} \geq 0$ and $q_i^c = 0$ for $i = 2, 3, \ldots, n$, where q_i^c denotes the supply quantity of retailer i under centralized retailing.*

Proof: We first show that the objective function of $\mathcal{P}_c$ is strictly concave by showing that the Hessian matrix of the objective function, $\mathbf{H}$, is positive definite for all $\overrightarrow{\mathbf{Q}} \in R^n$. Note that $\sum_{i=1}^{n} \Pi_i(\overrightarrow{\mathbf{Q}}) = \sum_{i=1}^{n} \left[(a - b\sum_{i=1}^{n} q_i)q_i - cq_i - w_i q_i\right] = \left(a - b\sum_{i=1}^{n} q_i\right) \sum_{i=1}^{n} q_i - \sum_{i=1}^{n} w_i q_i$ and one can show that $\mathbf{H} = -2b\mathbf{1}$ where $\mathbf{1}$ is an $n \times n$ matrix of 1's, i.e., $\mathbf{H}$ is symmetric. As $b > 0$, we have $\mathbf{H} < 0$, which implies strict concavity of $\sum_{i=1}^{n} \Pi_i(\overrightarrow{\mathbf{Q}})$. Then the KKT conditions are necessary and sufficient for problem $\mathcal{P}_c$ as we have linear constraints. The KKT conditions read

$$\begin{array}{lrcll} (\mathbf{a}) & a - 2b\sum_{i=1}^{n} q_i - c - w_i + u_i & = & 0 & i = 1, 2, \ldots, n, \\ (\mathbf{b}) & u_i q_i & = & 0 & i = 1, 2, \ldots, n, \\ (\mathbf{c}) & u_i & \geq & 0 & i = 1, 2, \ldots, n. \end{array}$$

We first argue that only one of the q_i values can be non-zero To establish a contradiction, suppose that $q_r > 0$ and $q_s > 0$, which implies $u_r = u_s = 0$ by ($\mathbf{b}$). Then it follows from ($\mathbf{a}$) for retailers r and s that $a - 2b\sum_{i=1}^{n} q_i = c + w_r$ and $a - 2b\sum_{i=1}^{n} q_i = c + w_s$, which means $w_r = w_s$. This establishes a contradiction as $w_r \neq w_s$; hence, only one of the q_i values can be non-zero. Furthermore, only $q_1 \geq 0$ as otherwise ($\mathbf{a}$) implies $u_1 < 0$ which contradicts ($\mathbf{c}$). In this case, it follows from ($\mathbf{a}$) that $q_1^c = \frac{a-c-w_1}{2b} \geq 0$. □

Proposition 5.2 implies that only the retailer with the lowest per-unit operating cost, i.e., only retailer 1 will supply the product to the end customer market. In practice, this would indicate that the retailer with the lowest per-unit operating cost acts as the distributor, or, equivalently, the distributor vertically integrates with the most efficient retailer. However, this degree of centralization based on a single product or product line can be impractical. Furthermore, when the distributor also carries other products sold via retailers, or retailers sell additional products, centralization of the retailers for the supplier's product may not be preferred. Therefore, we next study a procurement strategy that maintains the market share under decentralized decisions, while targeting increased profit levels.

5.4.3 Partially Centralized Retailing

The idea of partial centralization of retailers is to use the same objective as under centralization, but to apply constraints that preserve each retailer's market share. Let k_i denote retailer i's fraction of market supply at equilibrium under decentralized procurement for the given c (that is, $k_i = q_i^* / \sum_{i=1}^n q_i^*$ and can be determined using Algorithm 6 for any given c). For a given c, k_i is a constant. On the other hand, when we study the supplier's problem under partially centralized retailers, k_i will be a function of c. Note that $\sum_{i=1}^n k_i = 1$. Now, if we let q_i^p define the supply quantity of retailer i, $i \leq n$, under partial centralization, we should have $q_i^p/(\sum_{i=1}^n q_i^p) = k_i$. The solution to the partially centralized procurement problem will then be obtained by solving the following problem:

$$\begin{aligned} \mathcal{P}_p \quad \max \quad & \textstyle\sum_{i=1}^n \Pi_i(\overrightarrow{\mathbf{Q}}) \\ \text{s.t.} \quad & q_i = k_i \textstyle\sum_{i=1}^n q_i, \quad i = 1, \dots, n. \end{aligned}$$

Since the solution of the retailers' game under decentralization is feasible for $\mathcal{P}_p$, partial centralization leads to a solution that is at least as good for the retail stage. The solution of $\mathcal{P}_p$ is stated next.

Proposition 5.3. $q_i^p = k_i \left(\frac{a - c - \sum_{i=1}^n w_i k_i}{2b} \right)$.

Proof: Under partial centralization, the objective function of $\mathcal{P}_p$ reads as

$$\sum_{i=1}^{n} \Pi_i(\overrightarrow{\mathbf{Q}}) = \sum_{i=1}^{n} \left[(a - b\sum_{i=1}^{n} q_i)q_i - cq_i - w_i q_i \right] = Q \left(a - bQ - c - \sum_{i=1}^{n} w_i k_i \right).$$

Note that the objective function is concave in Q; hence, using the first order condition, one can conclude that $Q = \frac{a-c-\sum_{i=1}^{n} w_i k_i}{2b}$ in the optimal solution. Therefore, $q_i^p = k_i \left(\frac{a-c-\sum_{i=1}^{n} w_i k_i}{2b} \right)$. □

Unlike centralization, all of the retailers maintain the market share associated with decentralized procurement under partial centralization. Even though centralization is still more profitable for the retail stage (as the solution under partial centralization is feasible under centralization), the distributor may prefer partial centralization as a result of the practical concerns we have noted.

5.4.4 Comparison of Procurement Strategies

It is clear that the total profit of the retail stage is maximized under a centralized procurement strategy. In cases where complete centralization is not practical or possible and market share must be maintained, partial centralization maximizes the total profit at the retail stage. Decentralization as a procurement strategy results in the lowest level of profit at the retail stage. As a result we have the following relation:

$$\sum_{i=1}^{n} \Pi_i(\overrightarrow{\mathbf{Q}}^c) \geq \sum_{i=1}^{n} \Pi_i(\overrightarrow{\mathbf{Q}}^p) \geq \sum_{i=1}^{n} \Pi_i(\overrightarrow{\mathbf{Q}}^*), \tag{5–5}$$

where $\overrightarrow{\mathbf{Q}}^*$, $\overrightarrow{\mathbf{Q}}^c$, and $\overrightarrow{\mathbf{Q}}^p$ denote the quantity decision vectors of the retailers under decentralization, centralization, and partial centralization, respectively. Therefore, the distributor will prefer centralization of the retailers if possible, or partial centralization when horizontal centralization is not practical or possible.

As noted previously, the supplier does not have control over the distributor's procurement strategy. The supplier's profit levels for a given wholesale price will, however, be strongly affected by the distributor's procurement strategy. Specifically, the total quantity ordered from the supplier differs under each of the distributor's procurement strategies. In what follows, we compare the total quantity ordered from the supplier for different procurement strategies.

Proposition 5.4. *Let $Q^* = \sum_{i=1}^{n} q_i^*$, $Q^c = \sum_{i=1}^{n} q_i^c$, and $Q^p = \sum_{i=1}^{n} q_i^p$ denote the retailers' total order quantity under decentralization, centralization, and partial centralization, respectively. Then $Q^p \leq Q^c \leq Q^*$.*

Proof: Please see Appendix D.

Proposition 5.4 states that the retailers' aggregated order quantity is maximized under decentralized procurement. While centralized procurement results in a smaller total order quantity than decentralized procurement, it results in a greater total order quantity than partially centralized procurement. Hence, the distributor can increase total retail profit via centralization or partial centralization, while decreasing the total quantity ordered from the supplier. On the other hand, the supplier's preferences will differ according to the economies or diseconomies of scale the supplier faces. For a given wholesale price, we summarize the supplier's best interests with respect to the distributor's procurement strategy as follows.

- Given c, if the supplier is subject to economies of scale, it is in the supplier's best interest if retailer procurement is decentralized. If distributor centralizes retailer procurement, it is then in the supplier's best interest if retailer procurement is centralized rather than partially centralized.
- Given c, if the supplier is subject to diseconomies of scale, it is in the supplier's best interest if retailer procurement is centralized or partially centralized. If the distributor centralizes retailer procurement, it is then in the supplier's best interest if the retailers are partially centralized rather than centralized.

The above points lead to important insights on coordination and centralization of supply and retail stages. It should be noted that the channel-wide profits under different procurement strategies vary depending on the supplier's economies or diseconomies of scale. Therefore, since the maximum channel-wide profits can be achieved under a specific procurement strategy, a coordination policy or approach may need to constrain the retail procurement strategy. In Section 5.7, we discuss further insights on the coordination of the channel under different downstream characteristics. The next section analyzes the supplier's wholesale price-setting problem for each procurement strategy.

5.5 The Supplier's Problem: Optimal Wholesale Price

In this section, we first characterize the supplier's profit as a function of the wholesale price. Recall from Equation (5–2) that the supplier's profit function depends on the total order quantity of the retailers. Furthermore, it is clear from Algorithm 6 and Propositions 5.2 and 5.3 that the total retailer order quantity under any procurement strategy depends on the supplier's wholesale price. Hence, we derive closed-form expressions for total retailer order quantity as a function of the wholesale price. This enables stating the supplier's problem in terms of c only, the supplier's wholesale price. In what follows, we discuss the supplier's wholesale price-setting problem for decentralized, centralized, and partially centralized retailers using this approach.

5.5.1 Wholesale Pricing for Decentralized Retailing

Suppose that the distributor prefers decentralized retailing, i.e., when retailers are competitive in the end customer market. Next, we denote the sum of retailer supply quantities by $Q(c)$ (to stress its' dependence on the supplier's wholesale price), and derive a closed-form expression for this function. This enables stating $\mathcal{P}_S$ in terms of c only.

Let $\ell(c)$ denote the number of active retailers for a given value of c. Assuming without loss of generality that retailers are sorted in increasing order of w_i values, it then follows that if $\ell(c) = \ell$, then $q_i^* > 0$ for retailers $i \leq \ell$, and it can be shown that (see proof of Proposition 5.4 given in Appendix D)

$$Q(c|\ell(c) = \ell) = \frac{\ell(a-c)}{b(\ell+1)} - \frac{\sum_{i=1}^{\ell} w_i}{b(\ell+1)}, \tag{5–6}$$

where $Q(c|\ell(c) = \ell)$ denotes the total quantity ordered by the retailers as a function of c, given that $\ell(c) = \ell$. However, as is clear from Algorithm 6, $\ell(c)$ is itself a function of c. For this reason, we first characterize $\ell(c)$ and then use it to derive $Q(c)$.

Suppose that $c_1 < c_2$ and $\ell(c_1) = \ell(c_2) = \ell$. Then it follows from Equation (5–6) that $Q(c_1|\ell(c_1) = \ell) > Q(c_2|\ell(c_2) = \ell)$. That is, when the number of active retailers does not change with an increase in the wholesale price, the total order quantity decreases with the increase in the wholesale price. Let us define c_ℓ as the minimum wholesale price resulting in $\ell - 1$ active retailers, i.e., $c_\ell = \min\{c : \ell(c) = \ell - 1\}$ for $\ell \geq 1$ and observe from Equation (5–6) that $Q(c)$ is decreasing in c for $c \in [c_\ell, c_{\ell-1})$. Thus, if $c_\ell < c_{\ell-1}$, we can say that $Q(c)$ is decreasing for all $c \geq 0$. We next show that $c_\ell < c_{\ell-1}$ holds. The definition of $\ell(c)$ implies that $\ell(c) = \max\{i : q_i^* > 0\}$. Considering Step 2 of Algorithm 6, it follows that $\ell(c) = \max\{\ell : c + w_{\ell+1} \geq a - b\sum_{i=1}^{\ell} q_i^{(\ell)}\} = \max\left\{\ell : c + w_{\ell+1} \geq a - b\left(\frac{\ell(a-c)}{b(\ell+1)} - \frac{\sum_{i=1}^{\ell} w_i}{b(\ell+1)}\right)\right\}$; then

$$\ell(c) = \max\left\{\ell : w_{\ell+1} \geq \frac{a-c}{\ell+1} + \frac{\sum_{i=1}^{\ell} w_i}{\ell+1}\right\}. \tag{5–7}$$

Equation (5–7) implies that $\ell(c)$ is a decreasing step function of c, and $c_\ell < c_{\ell-1}$. In other words, $\ell(c) = \ell - 1$ for $c \in [c_\ell, c_{\ell-1})$ and $c_{\ell(0)} < c_{\ell(0)-1} < \ldots < c_2 < c_1$, where $\ell(0)$ denotes the number of active retailers when $c = 0$. We also define $c_{\ell(0)+1} = 0$. See Figure 5-1 for an illustration of $\ell(c)$. This figure can be explained as follows. As c increases, the per-unit delivery cost ($c + w_i$) to the market increases and, therefore, fewer retailers will enter the market. Furthermore, the retailers who do not enter will be the ones with higher unit operating costs. Now that we have established $c_\ell < c_{\ell-1}$, it easily follows from Equation (5–6) that $Q(c)$ is decreasing over $c \geq 0$. In particular, $Q(c)$ is determined by

$$Q(c) = \frac{\ell(c)(a-c)}{b(\ell(c)+1)} - \frac{\sum_{i=1}^{\ell(c)} w_i}{b(\ell(c)+1)}. \tag{5–8}$$

Note that $\ell(c)$ is a discontinuous function of c; hence, one may expect that Equation (5–8) represents a discontinuous function as well. However, in the next proposition, we show that $Q(c)$, unlike $\ell(c)$, is actually continuous in c, and the rate of decrease in slope for consecutive pieces is decreasing.

Proposition 5.5. *$Q(c)$ is a piecewise linear and continuous function of c. Furthermore, each piece is a decreasing linear function of c such that $\frac{Q(c_{r+2})-Q(c_{r+1})}{c_{r+1}-c_{r+2}} \geq \frac{Q(c_{r+1})-Q(c_r)}{c_r-c_{r+1}}$ for any r, $1 \leq r \leq \ell(0)-1$.*

Proof: We first note that only the c_ℓ values, $\ell = 1, 2, \ldots, n$, are possible values of discontinuity. Hence, to prove that $Q(c)$ is a piecewise continuous function of c, we show that $Q(c)$ is continuous at any such point, say c_{r+1}. To show that $Q(c)$ is continuous at c_{r+1}, we show that $\lim_{c\to c_{r+1}^+} Q(c) = \lim_{c\to c_{r+1}^-} Q(c)$. To do so, we first characterize c_{r+1}. By definition, $c_{r+1} = \min\{c : \ell(c) = r\} = \min\{c : \max\{\ell : w_{\ell+1} \geq \frac{a-c}{\ell+1} + \frac{\sum_{i=1}^{\ell} w_i}{\ell+1}\} = r\} = \min\{c : w_{r+1} \geq \frac{a-c}{r+1} + \frac{\sum_{i=1}^{r} w_i}{r+1}\} = \min\{c : c \geq a + \sum_{i=1}^{r} w_i - (r+1)w_{r+1}\} = a + \sum_{i=1}^{r} w_i - (r+1)w_{r+1}$. Now, note that $\lim_{c\to c_{r+1}^+} Q(c) = Q(c_{r+1}) = Q(c_{r+1}|\ell(c_{r+1}) = r)$. Then $\lim_{c\to c_{r+1}^+} Q(c) = \frac{r(a-c_{r+1})}{b(r+1)} - \frac{\sum_{i=1}^{r} w_i}{b(r+1)} = \frac{r(a-(a+\sum_{i=1}^{r} w_i-(r+1)w_{r+1}))}{b(r+1)} - \frac{\sum_{i=1}^{r} w_i}{b(r+1)} = \frac{rw_{r+1}-\sum_{i=1}^{r} w_i}{b}$. On the other hand, $\lim_{c\to c_{r+1}^-} Q(c) = Q(\lim_{c\to c_{r+1}^-} c) = Q(c_{r+1}, \ell(\lim_{c\to c_{r+1}^-})) = Q(c_{r+1}|\ell(c_{r+1}) = r+1)$. Then $\lim_{c\to c_{r+1}^-} Q(c) = \frac{(r+1)(a-c_{r+1})}{b(r+2)} - \frac{\sum_{i=1}^{r+1} w_i}{b(r+2)} = \frac{(r+1)(a-(a+\sum_{i=1}^{r} w_i-(r+1)w_{r+1}))}{b(r+2)} - \frac{\sum_{i=1}^{r+1} w_i}{b(r+2)} = \frac{((r+1)^2-1)w_{r+1}-(r+2)\sum_{i=1}^{r} w_i}{b(r+2)} = \frac{(r+2)rw_{r+1}-(r+2)\sum_{i=1}^{r} w_i}{b(r+2)} = \frac{rw_{r+1}-\sum_{i=1}^{r} w_i}{b}$. Thus $\lim_{c\to c_{r+1}^+} Q(c) = \lim_{c\to c_{r+1}^-} Q(c)$ for any r, $1 \leq r \leq \ell(0)-1$. Hence, $Q(c)$ is continuous in c. Recall that $Q(c_{r+1}) = \frac{rw_{r+1}-\sum_{i=1}^{r} w_i}{b}$. Then it follows that $Q(c_{r+2}) - Q(c_{r+1}) = (r+1)(w_{r+2} - w_{r+1})/b$. Similarly, $Q(c_{r+1}) - Q(c_r) = r(w_{r+1} - w_r)/b$. Furthermore, we know that $c_{r+1} = a + \sum_{i=1}^{r} w_i - (r+1)w_{r+1}$. This implies that $c_{r+1} - c_{r+2} = (r+2)(w_{r+2} - w_{r+1})$. Similarly, $c_r - c_{r+1} = (r+1)(w_{r+1} - w_r)$. Thus, $\frac{Q(c_{r+2})-Q(c_{r+1})}{c_{r+1}-c_{r+2}} = \frac{r+1}{b(r+2)} \geq \frac{Q(c_{r+1})-Q(c_r)}{c_r-c_{r+1}} = \frac{r}{b(r+1)}$. $\square$

Figure 5-1 illustrates $\ell(c)$ and $Q(c)$. Proposition 5.5 implies that $Q(c)$ is a continuous decreasing convex function of c.

Now consider problem $\mathcal{P}_S$. Let $c \in [c_\ell, c_{\ell-1})$, i.e., suppose there are ℓ active retailers. Within this segment, the supplier's problem is to maximize $\Pi_S(Q) = cQ - f(Q)$ subject to $c_\ell \leq c \leq c_{\ell-1}$ (note that we can write $c_\ell \leq c \leq c_{\ell-1}$ instead of $c_\ell < c \leq c_{\ell-1}$ as a result of Proposition 5.5). Explicitly accounting for the dependence of Q on c, the objective of $\mathcal{P}_S$ is to maximize $\Pi_S(Q(c))$. Recall that we assume $f(Q)$, the supplier's

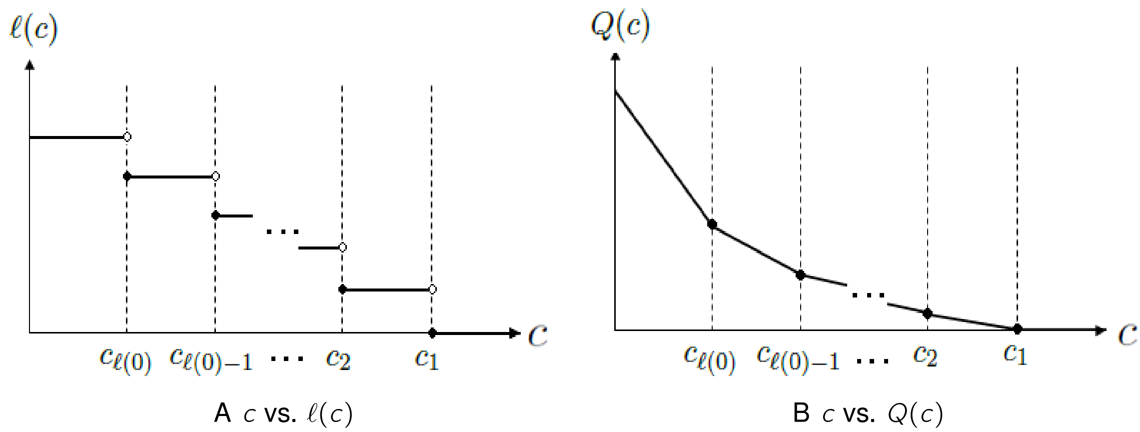


Figure 5-1. Illustrations of $\ell(c)$ and $Q(c)$

production cost function, to be a nondecreasing continuous and differentiable function of Q. Using the second derivative test, we can show that if $\frac{d^2f(Q)}{dQ^2} \geq -2b\left(\frac{\ell(0)+1}{\ell(0)}\right)$ then $\Pi_S(Q(c))$ is concave with respect to c. Note that this condition can be satisfied by concave and convex production cost functions. We next argue alternative conditions for concave production cost functions, when satisfied, imply the concavity of $\Pi_S(Q(c))$ with respect to c. Furthermore, we show that the convexity of $f(Q)$ implies strict concavity of $\Pi_S(Q(c))$ with respect to c.

Proposition 5.6. *Suppose that $f(Q)$ is concave in Q and consider $c_1, c_2 \in [c_\ell, c_{\ell-1})$ such that $c_1 > c_2$ for $1 \leq \ell \leq \ell(0)+1$. Define Q_1 and Q_2 as the total order quantity of the noncooperative buyers when $c = c_1$ and $c = c_2$, respectively. If $c_1 - c_2 \geq \frac{df(Q_1)/dQ - df(Q_2)/dQ}{2}$ then $\Pi_S(Q(c))$ is concave in c on $[c_\ell, c_{\ell-1})$. Furthermore, if $f(Q)$ is convex in Q for $Q \geq 0$ then $\Pi_S(Q(c))$ is strictly concave in c for $c_\ell \leq c \leq c_{\ell-1}$ such that $1 \leq \ell \leq \ell(0)+1$.*

Proof: Suppose that $f(Q)$ is concave in Q and let $c_1 > c_2$ such that $c_1, c_2 \in [c_\ell, c_{\ell-1})$ for $0 \leq \ell \leq \ell(0)$. Define $\nabla_x h(x)$ as the first derivative of $h(x)$ with respect to x. Let Q_1 and Q_2 be defined as in the proposition. Considering Equation (5–6), we can write $Q(c) = \theta - \mu c$, where $\theta = \frac{\ell a - \sum_{i=1}^{\ell} w_i}{b(\ell+1)}$ and $\mu = \frac{\ell}{b(\ell+1)}$. $\Pi_S(Q(c)) = \Pi_S(c)$ is concave if $[\nabla_c \Pi_S(c_2) - \nabla_c \Pi_S(c_1)]\,[c_2 - c_1] \leq 0$ for all $c_1, c_2 \geq 0$ (Bazaraa et al., 2006). First note that $\Pi_S(c) = cQ(c) - f(Q(c))$ and $\nabla_c \Pi_S(c) = -2c\mu + \theta + \mu \nabla_Q f(Q)$. Thus, $[\nabla_c \Pi_S(c_2) - \nabla_c \Pi_S(c_1)] [c_2 - c_1] = [-2\mu(c_2 - c_1) + \mu(\nabla_Q f(Q_2) - \nabla_Q f(Q_1))] [c_2 - c_1]$. Then if $[-2\mu(c_2 - c_1) + \mu(\nabla_Q f(Q_2) - \nabla_Q f(Q_1))] \leq 0$, we have $\Pi_S(c)$ concave over $[c_\ell, c_{\ell-1})$. That is, if $c_1 - c_2 \geq \frac{\nabla_Q f(Q_1) - \nabla_Q f(Q_2)}{2}$ then $\Pi_S(Q(c))$ is concave with respect to c in the interval $[c_\ell, c_{\ell-1})$. Now suppose that $f(Q)$ is convex in Q for $Q \geq 0$. Note that $\Pi_S(Q(c))$ consists of two parts: $cQ(c)$ and $f(Q(c))$. We first show that $cQ(c)$ is strictly concave for $c_\ell \leq c \leq c_{\ell-1}$. We know from Equation (5–6) that $Q(c) = \frac{\ell(a-c)}{b(\ell+1)} - \frac{\sum_{i=1}^{\ell} w_i}{b(\ell+1)}$ for $c_\ell \leq c \leq c_{\ell-1}$. Then it easily follows that $cQ(c)$ is strictly concave in c. Next we show that $f(Q(c))$ is convex. Note that we assume $f(Q)$ is nondecreasing and convex function. Furthermore, we know that $Q(c)$ is linear in c for $c_\ell \leq c \leq c_{\ell-1}$, i.e., it is convex. Then it follows that $f(Q(c))$ is convex (see, Bazaraa et al., 2006). Thus, $\Pi_S(Q(c))$ is strictly concave over $c_\ell \leq c \leq c_{\ell-1}$ such that $0 \leq \ell \leq \ell(0)$. $\square$

Proposition 5.6 states that if the difference in the wholesale price is greater than the half of the difference in the marginal production costs at the quantities corresponding to the wholesale prices c_1 and c_2, then the supplier's profit function will be piecewise concave. Proposition 5.6 further implies that when the supplier's production cost is convex in Q, the supplier's profit function is strictly concave in the wholesale price c for $c \in [c_\ell, c_{\ell-1}]$; hence, it is a piecewise concave function. Convex production costs are sometimes observed in practice and have been studied in the literature (see, e.g., Klein, 1961, Veinott, 1964, Eliashberg and Steinberg, 1987, Smith and Zhang, 1998). In particular, when the marginal production cost increases in the production quantity, the production cost function will be convex. This can be due to increased capacity requirements (Johnson and Montgomery, 1974) or expensive overtime requirements (Smith and Zhang, 1998, Ming-hui and Cheng-xiu, 2005).

We henceforth assume that $\Pi_S(Q(c)) = \Pi_S(c)$ is a piecewise concave function of c. The solution method we next propose for the supplier's problem is based on this piecewise concave structure; thus, this approach can also be used for any kind of concave production cost function by using a piecewise linear approximation of the production cost function (as linear pieces of $f(Q)$ will imply concavity of each piece of $\Pi_S(c)$). The supplier's problem in each interval involves the maximization of a concave function subject to linear boundary constraints. Let c_ℓ^* denote the wholesale price that maximizes $\Pi_S(c)$ over $c_{\ell+1} \leq c \leq c_\ell$ for $1 \leq \ell \leq \ell(0)$ (note that we do not consider $c > c_1$, because for $c > c_1$, $\ell(c) = 0$ and $Q(c) = 0$, hence, we can define $\Pi_S(c) = 0$). The next Corollary, which characterizes c_ℓ^*, is a direct result of the concavity of each piece of $\Pi_S(c)$.

Corollary 2. *Let $\Pi_S^{(\ell)}(c)$ denote the supplier's profit function on the interval $c_{\ell+1} \leq c \leq c_\ell$ and let c_ℓ^0 be defined such that $d\Pi_S^{(\ell)}(c)/dc = 0$ at c_ℓ^0. Then*

$$c_\ell^* = \begin{cases} c_{\ell+1} & \text{if } c_\ell^0 < c_{\ell+1}, \\ c_\ell^0 & \text{if } c_{\ell+1} \leq c_\ell^0 \leq c_\ell, \\ c_\ell & \text{if } c_\ell^0 > c_\ell, \end{cases}$$

where $c_\ell = a + \sum_{i=1}^{\ell-1} w_i - \ell w_\ell$ for $1 \leq \ell \leq \ell(0)$ and $c_{\ell(0)+1} = 0$ (please see the proof of Proposition 5.5 for the derivation of this equation for c_ℓ).

The following corollary characterizes the solution of the supplier's problem under decentralized procurement.

Corollary 3. *Given a, b, and w_i $\forall i \in \{1, 2, \ldots, n\}$, the supplier's optimal wholesale price when retailers are decentralized is*

$$c^* = \arg \max_{1 \leq \ell \leq \ell(0)} \{\Pi_S(c_\ell^*)\}. \tag{5–9}$$

Corollary 3 follows from Corollary 2 and the piecewise concave structure of $\Pi_S(c)$. Note that multiple optima may exist for the supplier's problem. In the case of alternative optima, if the supplier prefers to work with fewer (more) retailers, s/he may choose the maximum (minimum) of the alternative optimal wholesale prices. Next, we discuss the supplier's problem for centralized and partially centralized retailing.

5.5.2 Wholesale Pricing for Centralized and Partially Centralized Retailing

First, suppose that the distributor centralizes retailer procurement. In this case, we know from Proposition 5.2 that $Q(c) = \frac{a-c-w_1}{2b}$. Therefore, $Q(c)$ is independent of the number of active retailers, and is a linearly decreasing, continuous function of c. Then the supplier's profit function is not a piecewise function, i.e., it is differentiable over $c > 0$. We can easily generalize the discussion on the concavity of $\Pi_S(c)$ and conclude that the optimal wholesale price, c^*, is determined by the first-order optimality condition for the supplier's concave profit function when $f(Q)$ is convex in Q. Next, suppose that the distributor partially centralizes retailer procurement. Then Proposition 5.3 implies that $Q(c) = \frac{a-c-\sum_{i=1}^{\ell(c)} w_i k_i}{2b}$. Similarly to the case of decentralized retailing, under partial centralization, $Q(c)$ not only depends on c but also $\ell(c)$, which itself is a function of c. Thus, it follows that $Q(c)$ has a piecewise concave structure as in the case of decentralized retailers. We note that although k_i is a function of c, we can still show that $\Pi_S(c)$ is piecewise concave in c if $f(Q)$ is convex in Q. Thus, we can use Corollaries 2 and 3 to solve the supplier's problem when the distributor partially centralizes the retailers and the supplier's production cost function is convex. The next section extends our model to consider multiple markets and supplier discount pricing.

5.6 Extensions: Multiple Markets and Discount Pricing

We first extend the previous results to allow for multiple markets. Furthermore, we show how our results can be generalized to a setting in which the supplier offers a quantity discount pricing scheme.

Now suppose that the supplier sells his/her product to m end customer markets through n competitive retailers, whose orders may be controlled by a distributor (or are under joint ownership of a chain store). Let j index markets and suppose the price in market j is determined by the equation $p_j(Q_j) = a_j - b_j \sum_{i=1}^{n} q_{ij}$, where q_{ij} denotes the quantity supplied to market j by retailer i and $Q_j = \sum_{i=1}^{n} q_{ij}$. Then retailer i's problem under decentralized procurement is to maximize $\Pi_i(\widehat{\mathbf{Q}}) = \sum_{j=1}^{m} \left[\left(a_j - b_j \sum_{i=1}^{n} q_{ij} \right) q_{ij} \right] -$

$c\sum_{j=1}^{m} q_{ij} - \sum_{j=1}^{m} w_{ij}q_{ij}$, where we use $\widehat{\mathbf{Q}}$ to denote the $n \times m$ matrix of q_{ij} values, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, and w_{ij} denotes the operating cost of retailer i for serving market j. Note that the objective function of retailer i is strictly concave in its q_{ij} values for given supply quantities of the other retailers, as $b_j > 0$. Thus, the first-order condition, $a_j - b_j \sum_{j=1}^{n} q_{ij} - b_j q_{ij} - c - v_{ij} = 0$, must be satisfied at a Nash equilibrium solution if $q_{ij} > 0$. Note that the first-order condition associated with retailer i's supply quantity at market j does not depend on the supply quantities for other markets or on other market parameters. Hence, the retailers' problems are separable by market, which can be observed in $\Pi_i(\widehat{\mathbf{Q}})$ as well. Then, using Algorithm 6 for each market separately, one can solve for the equilibrium supply quantities at the markets. Furthermore, the total quantity supplied to market j as a function of c reads $Q_j(c) = \frac{\ell_j(c)(a_j - c)}{b_j(\ell_j(c)+1)} - \frac{\sum_{i=1}^{\ell_j(c)} w_{ij}}{b_j(\ell_j(c)+1)}$, where $\ell_j(c)$ denotes the number of active retailers in market j for a given value of c. It follows from Proposition 5.5 that $Q_j(c)$ is piecewise continuous and decreasing in c, which implies that $Q(c) = \sum_{j=1}^{m} Q_j(c)$ is a piecewise continuous and decreasing function of c. Using the breakpoints of the $Q_j(c)$ functions, we can determine the range of c values for which each piece of $Q(c)$ is defined. Then the solution method proposed in Corollaries 2 and 3 can be used to solve the supplier's problem. Furthermore, when the retailers are centralized or partially centralized by the distributor in all of the markets, it can again be shown that each market can be considered separately, and the solutions discussed in Sections 5.4.2 and 5.4.3 for a single market can be used for each market. We note that centralization of the retailers in the multiple-market case would imply that each market is occupied by a single retailer; in particular, the one with the lowest operating cost in that market. Finally, it should be noted that the multiple-market case can also be represented as a single-market, multiple-product scenario, such that the products are supplied by the same supplier and are differentiated (i.e., they are not demand substitutes; hence, they have independent inverse demand functions).

We next return to the single-market setting, but suppose that the supplier offers a quantity discount pricing scheme, such that the wholesale price is a linear decreasing function of the total quantity ordered from the supplier. Specifically, the supplier's wholesale price is given by $c(Q) = \lambda - \beta \sum_{i=1}^{n} q_i$, where λ is the maximum wholesale price and β is a per-unit discount. We note that this kind of discount price scheme has been used in past literature (see, e.g., Ingene and Parry, 1995, 1998, Keskinocak and Savaşaneril, 2008). When the supplier offers such a quantity discounting schedule, retailer i's problem under decentralized procurement is to maximize $\Pi_i(\overrightarrow{\mathbf{Q}}) = (a - b\sum_{i=1}^{n} q_i)q_i - (\lambda - \beta\sum_{i=1}^{n} q_i)q_i - w_i q_i$. We can show that when $b > \beta$ the objective function of retailer i is strictly concave in q_i for given supply quantities of the other retailers. In a similar setting, Keskinocak and Savaşaneril (2008) assume that $b > \beta$, as marginal revenues decrease faster than marginal costs. Hence, the method of Section 5.4.1 can be used under this assumption. Furthermore, when the retailers are centralized or partially centralized by the distributor, the discussion in Sections 5.4.2 and 5.4.3 still applies, that is, the total order from the supplier decreases due to centralization. When the supplier offers discount pricing, the methods and all of the results of Section 5.4 can apply by letting b be $b - \beta$ and letting c be λ. It should also be noted that when $c = \lambda$ and $\beta > 0$, the supplier increases his/her sales to the retailers under any distributor procurement strategy by offering quantity discounts. If the supplier's decision variable is λ only, i.e., when the discount factor is a fixed constant, the method described in Section 5.5 to determine the optimal wholesale price can be used to determine the optimal value of λ. On the other hand, when β or the (λ, β) pair are supplier decision variables, the supplier's problem becomes challenging. We pose these cases as future research questions in Chapter 6. The following section documents our numerical studies.

5.7 Numerical Study

In this section, we discuss a set of numerical studies intended to analyze the value of control. The value of control in this context can be interpreted as the amount the supplier can save if s/he can control or influence the distributor's procurement strategy. Recall that the retailers may be decentralized, centralized or partially centralized. That is, the distributor may choose one of the three distinct procurement strategies: decentralized retailing (DP), centralized retailing (CP), or partially centralized retailing (PC). As discussed previously, the supplier can benefit from controlling the distributor's procurement strategy. To analyze the value of control and the effects of channel parameters on the value of control, we conduct two sets of numerical studies detailed next.

Throughout our numerical studies, we assume that the supplier's production cost function consists of a fixed set up cost and a linear term in the production quantity, that is, $f(Q) = \gamma + \alpha Q$, where γ is the production setup cost and α is the per-unit production cost. A given problem instance is solved for 9 different scenarios consisting of each pair of combinations of DP, CP, and PC. Any pair of values represents the supplier's wholesale price decision when assuming a specific procurement strategy along with the distributor's actual procurement strategy. For instance, (DP,CP) implies that the distributor prefers decentralized procurement and the supplier sets the wholesale price assuming centralized procurement. Therefore, for the (DP,CP) scenario, the supplier's decision is suboptimal as s/he sets the wholesale price assuming (incorrectly) centralized procurement. The supplier's action is optimal only in scenarios (DP,DP), (CP,CP), and (PC,PC). The first set of numerical studies studies the value of control. To do so, we solve 10 problem instances for each combination of the following problem parameters in the single-market case: $n \in \{3, 4, 5\}$, $a \in \{100, 110, 120\}$, $b \in \{1, 1.25, 1.5\}$, $\mathbf{W} = \{U[5, 20], U[20, 35], U[35, 50]\}$, $\gamma = \{50, 75, 100\}$, and $\alpha = \{25, 50, 75\}$, where $\mathbf{W}$ denotes the n-vector of w_i values generated from a uniform distribution over the specified range. Thus, we solve 729 different problem classes and 7290 problem instances in the first set of numerical studies. Table 5-1 provides the average supplier profit over all of the problem instances for each of the 9 scenarios.

Table 5-1. Supplier's profit

		Supplier		
		DP	CP	PC
	DP	222.74	220.97	220.05
Distributor	CP	149.20	150.08	149.84
	PC	137.05	137.99	138.16

Note that we consider the case when the distributor's procurement strategy is known to the supplier. In this case, the supplier will end up in cells on the diagonal of Table 5-1, i.e., the supplier's profit levels are defined in (DP, DP) or (CP, CP) or (PC, PC) if the distributor applies decentralized, centralized, or partially centralized procurement, respectively.

Observation 5.1. *The supplier's optimal profit in scenario* (DP, DP) *is greater than the supplier's optimal profit in scenario* (CP, CP)*; and, the supplier's optimal profit in scenario* (CP, CP) *is greater than its optimal profit in scenario* (PC, PC)*. That is,* $(c_D^* - \alpha)Q^*(c_D^*) - \gamma \geq (c_C^* - \alpha)Q^c(c_C^*) - \gamma \geq (c_P^* - \alpha)Q^p(c_P^*) - \gamma$*, where* c_D^**,* c_C^**, and* c_P^* *denote the supplier's optimal wholesale prices for decentralized, centralized, and partially centralized procurement, respectively.*

Observation 5.1 follows from Proposition 5.4. In particular, given any wholesale price c, the supplier's profit equals $(c-\alpha)Q^*(c)-\gamma$, $(c-\alpha)Q^c(c)-\gamma$, and $(c-\alpha)Q^c(p)-\gamma$, respectively when retailers are decentralized, centralized, and partially centralized. As $Q^*(c) \geq Q^c(c) \geq Q^p(c)$, it then readily follows that $(c-\alpha)Q^*(c)-\gamma \geq (c-\alpha)Q^c(c)-\gamma \geq (c - \alpha)Q^c(p) - \gamma$ for any $c \geq \alpha$. The above relation implies that the value of control is always positive for the supplier when the supplier is subject to linear production costs. For the problem instances solved, we can use Table 5-1 to quantify the value of control for the supplier.

In particular, suppose that the retailers are partially centralized. The supplier can increase his/her profit by 8.63% ($100 \times (150.08 - 138.16)/138.16 = 8.63$) by persuading the distributor to apply centralized retailing and by 61.22% ($100 \times (222.74 - 138.16)/138.16 = 61.22$) by persuading the distributor to apply decentralized retailing.

Now, suppose that the retailers are centralized. In this case, the supplier can increase his/her profit by 48.41% ($100 \times (222.74 - 150.08)/150.08 = 48.41$) by persuading the distributor to apply decentralized retailing. Next, we discuss the distributor's procurement strategy choice.

Table 5-2 provides the average retail profit over all of the problem instances for each of the 9 scenarios. As can be observed from Table 5-2, the distributor will always prefer centralized retailing in the absence of any incentive. This observation was presented earlier in Equation (5–5).

Table 5-2. Retailers' total profit

		Supplier		
		DP	CP	PC
	DP	76.09	67.84	67.55
Distributor	CP	110.25	98.14	97.54
	PC	96.36	86.18	86.23

Observation 5.2. *The channel will end up in scenario* (CP, CP)*. That is, the distributor will centralize the retailers and the supplier will set his/her wholesale price for centralized procurement.*

Observation 5.2 is readily implied as the distributor will prefer centralization for any supplier wholesale price and, therefore, the supplier will set his/her optimal wholesale price for centralized retailers. The (CP, CP) scenario is also the equilibrium solution of the following game. Suppose that the distributor (row player) has three possible strategies: decentralized procurement (DP), centralized procurement (CP), or partially centralized procurement (PC). Similarly, the supplier (column player) has three strategies in setting the wholesale price: assuming decentralized procurement (DP), centralized procurement (CP), or partially centralized procurement (PC). Combining Tables 5-1 and 5-2, one obtains the payoff matrix given in Table 5-3 below. It is clear from Table 5-3 that the (CP, CP) scenario is the unique equilibrium in pure strategies.

Table 5-4 provides the channel-wide profits associated with Table 5-3. It follows that the state (DP, DP) results in the highest average channel-wide profits over all problem

Table 5-3. Payoff matrix

		Supplier		
		DP	CP	PC
	DP	76.09,**222.74**	67.84,220.97	67.55,220.05
Distributor	CP	**110.25**,149.20	**98.14**,**150.08**	**97.54**,149.84
	PC	96.36,137.05	86.18,137.99	86.23,**138.16**

instances solved, i.e., when the distributor chooses decentralized procurement and the supplier sets the wholesale price for decentralized procurement. We note that this case is observed in all of the problem instances solved. However, we cannot provide a formal proof of this observation for any problem instance, as it is not possible to get a closed-form expression for the supplier's optimal wholesale price for decentralized procurement. (We note that to show that (DP, DP) is channel-wide optimal, we need to show that channel-wide profit in (DP, DP) is greater than channel-wide profit in both (CP, CP) and (PC, PC). However, we can show that channel-wide profit in (CP, CP) is greater than the channel-wide profit in (PC, PC). This follows from definition of centralized procurement and Proposition 5.4.) This implies that the system-wide profits can be increased by moving to state (DP, DP) from (CP, CP). Furthermore, the supplier's gain is greater than the distributor's loss when both parties choose strategy DP instead of CP. Thus, the channel can be coordinated by the supplier, and any coordination mechanism should forbid centralization of retailing procurement.

Table 5-4. Channel profit

		Supplier		
		DP	CP	PC
	DP	**298.83**	288.81	287.60
Distributor	CP	259.45	248.22	247.38
	PC	233.41	224.17	224.39

The next set of numerical studies is intended to analyze how the average percentage retail loss, value of control, and increase in channel-wide profit are affected by moving from state (CP, CP) to (DP, DP) under different parameter values. We define the percentage retail loss as $100\% \times \frac{\Pi^R(CP,CP)-\Pi^R(DP,DP)}{\Pi^R(CP,CP)}$, where $\Pi^R(CP, CP)$ denotes, for example, the total retail-level profit when the supplier plans for CP and the distributor uses CP. Similarly, the percentage value of control is $100\% \times \frac{\Pi^S(DP,DP)-\Pi^S(CP,CP)}{\Pi^S(CP,CP)}$, and percentage increase in channel-wide profit is $100\% \times \frac{\Pi^C(DP,DP)-\Pi^C(CP,CP)}{\Pi^C(CP,CP)}$, where $\Pi^S(\cdot)$ and $\Pi^C(\cdot)$ denote the supplier and channel-wide profit, respectively.

We consider the effects of the number of retailers (n), market potential (a), sensitivity of market price to total supply (b), retailers' per-unit operation costs (w_i's), and the supplier's production setup cost (γ) and per-unit production cost (α). The numerical studies are constructed as follows: we change one parameter value at a time, leaving the remaining parameters at their previously defined values. For instance, when we vary n, for each $n \in \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$, we solve 10 problem instances for each combination of the following problem parameters in the single-market case: $a \in \{100, 110, 120\}$, $b \in \{1, 1.25, 1.5\}$, $\mathbf{W} = \{U[5, 20], U[20, 35], U[35, 50]\}$, $\gamma = \{50, 75, 100\}$, and $\alpha = \{25, 50, 75\}$. Similarly, when we analyze a, for each $a \in \{100, 105, 110, 115, 120, 125, 130, 135, 140\}$, we solve 10 problem instances for each combination of the following problem parameters in the single-market case: $n \in \{3, 4, 5\}$, $b \in \{1, 1.25, 1.5\}$, $\mathbf{W} = \{U[5, 20], U[20, 35], U[35, 50]\}$, $\gamma = \{50, 75, 100\}$, and $\alpha = \{25, 50, 75\}$. The combinations for other parameters are defined similarly. We note that all of the observations associated with the previous set of numerical studies are valid for this set of numerical studies as well. The horizontal axes of the graphs shown in Figures 5-2-5-4 define the range for each analyzed parameter. The vertical axis give the average percentage retail loss, value of control, and increase in channel-wide profit. In what follows, we interpret our results on the effects of different channel parameters.

Effects of Retail Parameters

The retail parameters we varied corresponded to the number of retailers, n, and retailer units costs, or w_i values. Based on Figure 5-2, we observe that:

- As n increases, the retail loss from moving from (CP, CP) to (DP, DP) is increasing. This result is expected because as the number of retailers increases, the competition increases under decentralized procurement; hence, the retail stage can achieve substantial savings when the distributor centralizes procurement. Furthermore, as n increases, the value of control due to moving from (CP, CP) to (DP, DP) is increasing. This result follows, as an increased number of retailers implies higher competition, and the supplier benefits from this. Finally, as n increases, the increase in channel-wide profits from moving from (CP, CP) to (DP, DP) is increasing. This follows because more competing retailers lead to greater market profitability.

- As the w_i's increase, the retail loss due to moving from (CP, CP) to (DP, DP) is decreasing. This result follows as fewer retailers supply the market under decentralized procurement when retailers are subject to higher operating costs. Therefore, the retail stage is not likely to achieve substantial savings when the distributor centralizes procurement. On the other hand, as w_i's increase, the value of control is increasing and the increase in channel-wide profits due to moving from (CP, CP) to (DP, DP) is increasing.

Effects of Market Parameters

The market potential, a, and elasticity, b, serve as the market parameters. Based on Figure 5-3, we observe that:

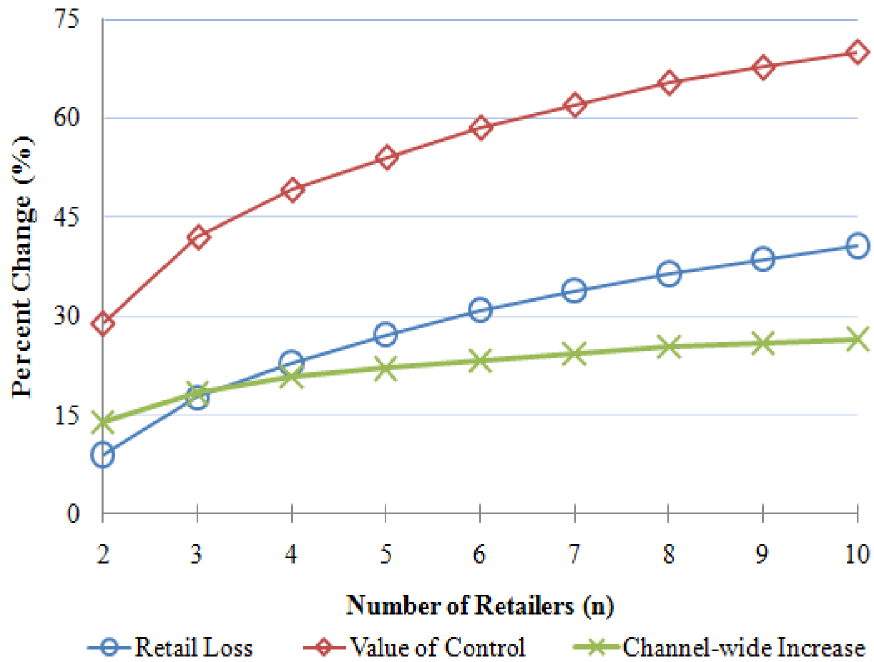
- As market potential (a) increases, the retail loss due to moving from (CP, CP) to (DP, DP) is increasing. This result is expected because as the market capacity increases, the distributor will achieve greater savings via centralization. However, the value of control due to moving from (CP, CP) to (DP, DP) decreases relatively slowly. This result follows as the influence of supplier's wholesale price on the retailers' quantity decisions diminishes as a increases. Finally, as a increases, the increase in the channel-wide profits due to moving from (CP, CP) to (DP, DP) follows relatively a stable pattern. That is, market capacity does not affect the percentage increase in channel-wide profits due to moving from (CP, CP) to (DP, DP).

- As elasticity (b) increases, the retail loss due to moving from (CP, CP) to (DP, DP) tends to decrease. However, a straight decreasing trend is not observed. On the other hand, as b increases, the value of control and the increase in channel-wide profits due to moving from (CP, CP) to (DP, DP) are increasing.

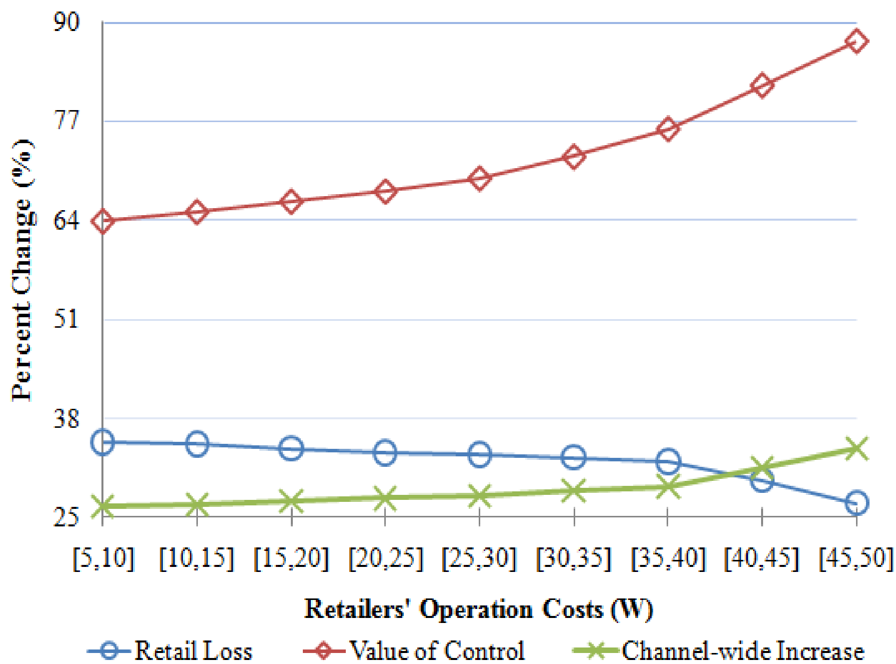
Effects of Supplier Parameters

The supplier's fixed cost (γ) and unit production cost (α) serve as the supplier's parameters of interest. Based on Figure 5-4, we observe that:

- As the fixed cost (γ) increases, the retail loss due to moving from (CP, CP) to (DP, DP) follows a stable pattern. This follows as γ is not a direct determinant of the supplier's wholesale price decision (excluding cases where the supplier's profit

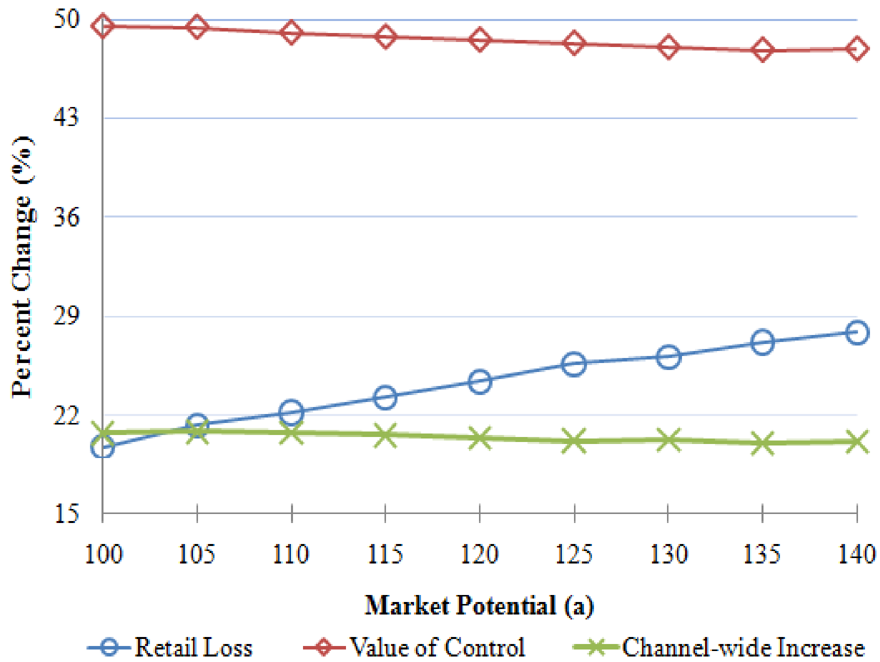


A Percent change in measures of interest vs. n

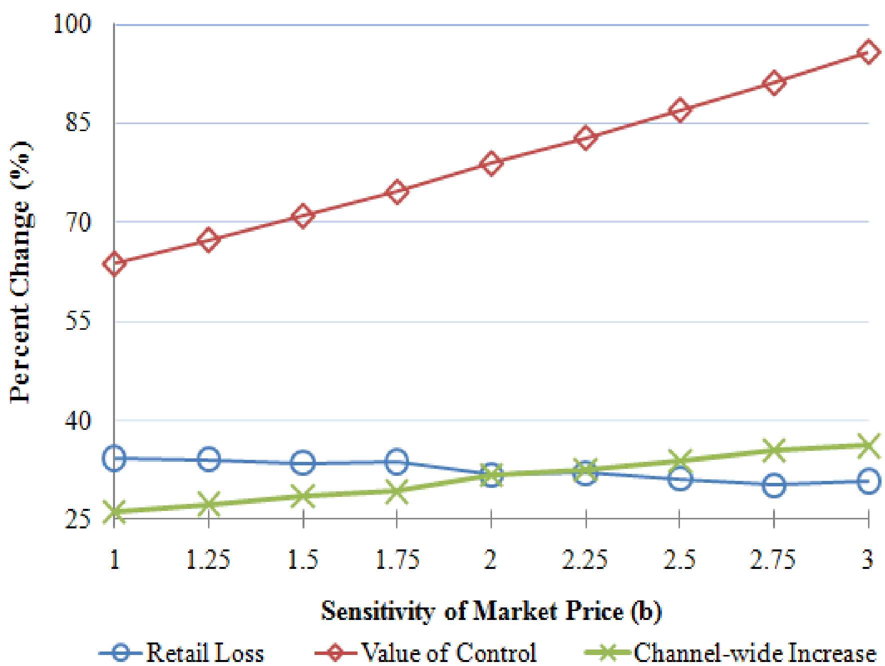


B Percent change in measures of interest vs. w_i's

Figure 5-2. Effects of retail parameters

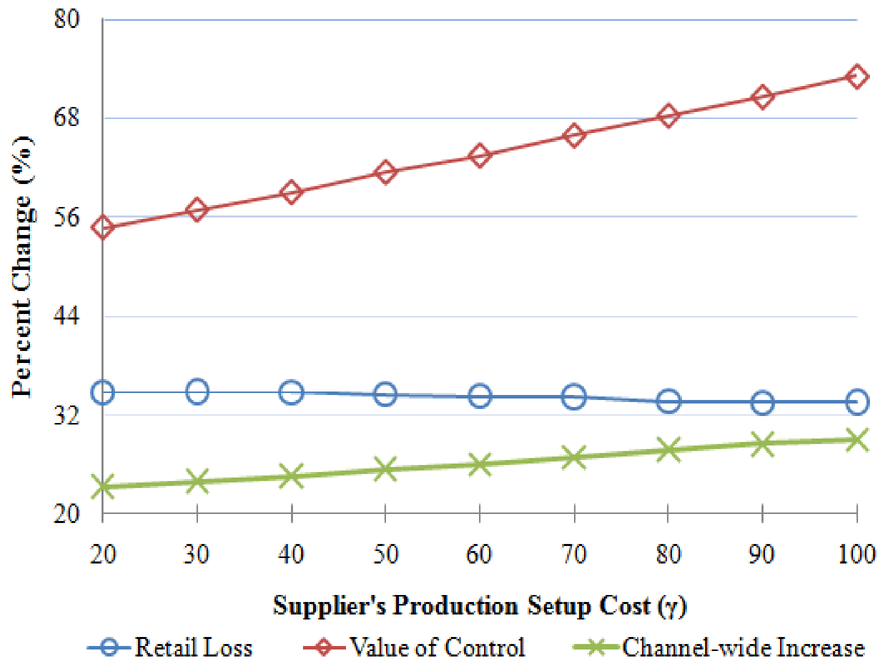


A Percent change in measures of interest vs. *a*

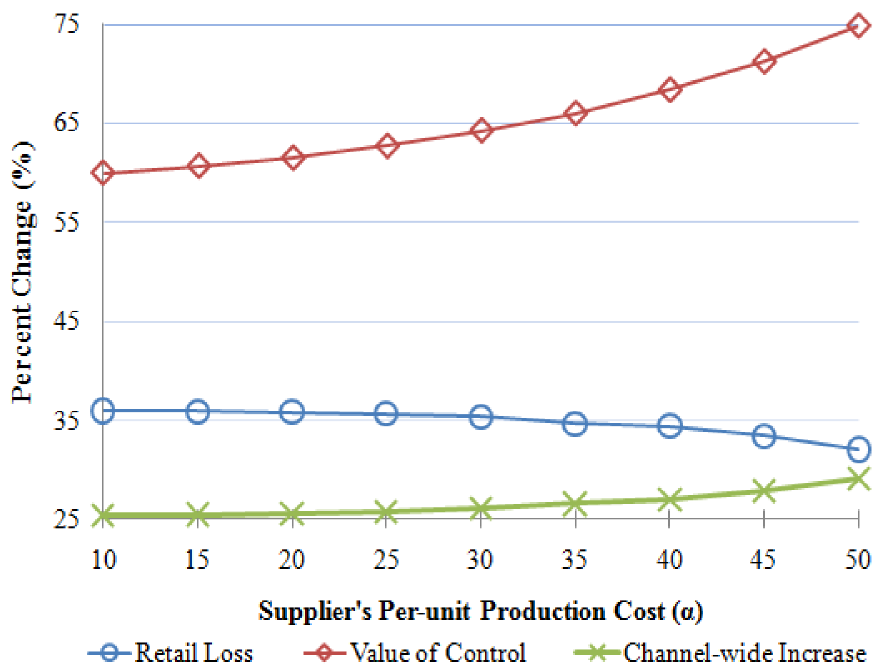


B Percent change in measures of interest vs. *b*

Figure 5-3. Effects of market parameters



A Percent change in measures of interest vs. γ



B Percent change in measures of interest vs. α

Figure 5-4. Effects of supplier parameters is negative due to high production setup costs). On the other hand, as γ increases, the value of control due to moving from (CP, CP) to (DP, DP) is increasing. This is due to the fact that, in the case of high production setup costs, the increase in the supplier's profit due to decentralization is higher. Finally, as γ increases, the increase in the channel-wide profits due to moving to moving from (CP, CP) to (DP, DP) is increasing. This follows from the discussions on the retail loss and value of control due to moving from (CP, CP) to (DP, DP).

- As the supplier's unit production cost (α) increases, the retail loss due to moving from (CP, CP) to (DP, DP) is decreasing. This result follows as fewer retailers supply the market under decentralized procurement for a larger α since the supplier's wholesale price will be higher. However, as α increases, the value of control and the increase in channel-wide profits due to moving from (CP, CP) to (DP, DP) are both increasing, which follow from a similar discussion as in the case of γ.

CHAPTER 6
CONCLUSION AND FUTURE RESEARCH DIRECTIONS

This chapter concludes the dissertation by summarizing our work in Chapters 2-5, providing concluding remarks, and highlighting our contributions to the existing literature. We further discuss future research directions related to Chapters 2-5.

6.1 Competitive Multi-Facility Location Problems with Congestion Costs

We studied a symmetric competitive facility location problem in Chapter 2 and an asymmetric competitive facility location problem in Chapter 4. Both of these chapters extend the current literature by allowing firms to simultaneously locate more than one facility and by accounting for nonlinear congestion costs.

In Chapter 2, we provided a solution method to determine PNE quantity decisions of the firms in Section 2.3. We first showed the symmetry of the equilibrium supply quantities given that the firms have identical facility location decisions. This result reduced the game of the firms to a so called game of the locations, for which we proposed an iterative solution method. Section 2.4 discussed the facility location decisions of the firms. We explained why it is safe to assume that firms will choose identical facility locations. Therefore, the location decisions of the firms can be determined by finding the most profitable location matrix containing identical columns. As a total enumeration scheme is computationally burdensome, we proposed a heuristic method that finds a good location decision, which may be adopted by all of the firms. As implied by our numerical studies, the heuristic is an efficient method that ranks locations based on certain problem parameters in the first phase.

The results presented in Chapter 2 are used in the analysis of the impacts of traffic congestion costs on equilibrium supply flows by Konur and Geunes (2011), which constitutes Chapter 3. The analysis of the heuristic method proposed in Chapter 2 suggests a future research direction: firms' decisions can be modeled as a three-stage game. In this game, firms first determine the number of facilities (stage-one), then the locations of these facilities (stage-two), and, finally, the supply quantities (stage-three). Furthermore, we observed a counter-intuitive result in our numerical studies. It is possible that firms may be better off when they ignore congestion costs in their decisions. In particular, the problem we formulate assumes two kinds of competition among the firms: competition in the markets and competition on the distribution network. Hence, our counter-intuitive result indicates that firms may have substantial savings when they ignore one of the types of competition (competition over the distribution network) in our numerical studies. We note that the analysis of firms' decisions when they compete over more than one resource serves as an interesting future research area.

In Chapter 4, we formulated a competitive location game with nonlinear costs for multiple, non-identical firms in a multiple-market setting under Cournot competition. Each firm incurs firm-specific linear transportation costs, convex traffic congestion costs, and fixed facility location costs. Chapter 4 determines equilibrium flows for any given facility locations via formulating the equilibrium problem as a variational inequality problem. The resulting formulation is an asymmetric linear variational inequality problem defined over the nonnegative orthant. Projection methods can be used as a solution method and a self-adaptive projection method proposed in Han (2006) was utilized. Second, an equilibrium facility location decision was sought. To find such a solution, routines were defined based on properties of equilibrium solutions in order to ease the search for an equilibrium location matrix. Utilizing these routines, a heuristic search method that finds an equilibrium location decision, if one exists, was discussed. The results of numerical studies imply that the heuristic method for finding an equilibrium location decisions is quite efficient when compared to a random search method. Furthermore, potential generalizations for multi-product and multi-echelon supply chains were discussed.

Chapter 4 provides tools to determine equilibrium facility locations and supply flows from these locations to multiple markets on a congested distribution network. The main contributions of this chapter lie in analyzing the case when heterogeneous firms are allowed to simultaneously locate more than one facility, as well as the consideration of nonlinear traffic congestion costs. Noting the recent studies on the negative effects of traffic congestion costs on supply chains (see, e.g., Rao et al., 1991, McKinnon, 1999, Weisbrod et al., 2001, Sankaran et al., 2005, Konur and Geunes, 2011), this study can be used to analyze the effects of traffic congestion costs on heterogeneous firms' distribution and facility location decisions in competitive supply chains.

The problems of interest in Chapters 2 and 4 were analyzed assuming that the firms are non-cooperative and they take simultaneous actions. On the other hand, studying competitive location games with nonlinear cost terms when cooperation is allowed among the firms, or under sequential actions by the firms, remain as future research directions. It should be noted that studying cooperative competitive location games on congested networks is important in the analysis of methods to mitigate inherent traffic congestion. The results in Chapters 2 and 4 provide tools that may be useful in such analysis.

One additional future research direction would include studying competitive facility location games in multi-echelon supply chains (an equilibrium problem for a two-echelon supply chain is studied in Nagurney et al. (2002); however, this study assumes that facility locations are predetermined). Analysis of two-echelon competitive facility location problems with traffic congestion is important for government agencies. These organizations have a substantial interest in developing congestion mitigation policies and may act as the upper level decision makers in such a setting. Again, the analysis in Chapters 2 and 4 will provide a foundation for studying such problems.

6.2 Traffic Congestion and Supply Chain Management

In Chapter 3, we studied facility location and supply quantity decisions for multiple firms in a competitive environment on a congested network. The main contribution of this chapter to the literature lies in characterizing the effects of traffic congestion on performance of a competitive supply chain by including traffic congestion costs directly. We used the symmetric competitive facility location problem defined in Chapter 2 in our analysis because firms typically share a common distribution network and, hence, they in essence compete for traffic capacity on the common distribution network. Furthermore, to capture a broader picture of practical realities, we considered firms' competition in the end customer markets. This modeling approach is useful in providing a more complete analysis of different markets such as grocery retailing, energy, airline and agricultural markets, as noted in Section 3.1.

As noted in Chapter 2, since the resulting game of the firms is symmetric, the firms will choose identical location decisions in equilibrium. Therefore, our analysis in Chapter 3 also explains the effects of traffic congestion on a single firm's supply quantity and facility location decisions on a congested distribution network with price sensitive markets. Studying more general problem formulations with explicit traffic congestion costs remains as a future research area. One direction in this area would be to analyze congestion effects by relaxing the homogeneity assumption of the firms. The results of Chapter 4 will help in studying such settings. We modeled traffic congestion costs endogenously and provided analytical results on how traffic congestion cost affects equilibrium supply quantity decisions. Section 3.3 of Chapter 3 gives a detailed discussion of these effects. Increased traffic congestion hinders efficient use of the distribution network, as firms may choose to supply a market from multiple, distant, and decentralized facilities. Similar results are observed for a highly relevant special case: when the potential facility locations are within the market areas. Moreover, our numerical studies characterize the effects of traffic congestion on facility location decisions as well. For the general case of the problem studied, we observed that firms tend to locate more facilities as congestion increases, up to a certain level. This is due to the fact that, with increased congestion, firms are not able to cover a market within a specific delivery time or at a desired service level and, thus, more facilities are located. However, beyond a level of congestion, firms locate fewer facilities, as their supply flows decrease in order to avoid high congestion costs. In addition to the general case, we conducted numerical studies for the special case. Numerical studies for the special case indicated that firms prefer to locate facilities in market areas with higher potentials, i.e., higher initial market price and/or lower sensitivity to the supply quantities. Also, firms will locate fewer facilities as congestion increases within market areas. The reason for this is that, as congestion increases, firms tend to locate facilities in market areas with higher market potentials and pay transportation costs to supply other markets, instead of paying congestion costs within each market area.

The results of Chapter 3 document the negative effects of traffic congestion on competitive firms. As a result, it is possible that firms may be willing to cooperate with government agencies to reduce traffic congestion. It is even possible that firms may cooperate with each other to mitigate traffic congestion, and, thereby reduce the negative effects of traffic congestion, as noted by Hensher and Puckett (2005). Studying such traffic congestion mitigation policies, using mathematical modeling techniques, remains as a future research area.

6.3 Pricing for Competitive Retailers

Chapter 5 models a supplier's pricing decisions for sales of a good to an end customer market via a set of retailers, who are jointly served by a distributor. In particular, we studied a Stackelberg game for which the supplier is the leader and the distributor and retailers act as followers. We considered different procurement strategies at the retail stage: decentralized, centralized, and partially centralized procurement. Under decentralized procurement, retailers engage in Cournot quantity competition in the end customer market. For this case, to determine retailers' quantity decisions (the equilibrium solution of the Cournot game), we proposed an iterative solution method based on sorting retailers with respect to certain parameters (this method is generalized to the cases in which the market price function is decreasing concave and the retailer operating cost function is increasing convex; and, this generalization provides an alternative approach to variational inequality based algorithms for broad classes of equilibrium problems, referred to as market equilibrium problems).

We contribute to the literature on supplier pricing problems for competitive retailers (i.e., when the retail stage is decentralized) by considering an arbitrary number of non-identical retailers, and by considering more general forms of the supplier's production cost function. We further contribute by studying two different centralization levels of retail procurement: centralization and partial centralization. Our findings imply that the supplier may encourage or discourage centralization at the retail stage, depending on whether s/he observes economies or diseconomies of scale in production.

Considering different strategies at the retail stage permitted studying the important concept of the value of control. To the best of our knowledge, the value of control in the context of Chapter 5 has not been previously analyzed. We showed that if the supplier controls the procurement strategy of the distributor, s/he may extract substantial savings. In particular, when the supplier's production costs are linear in the production quantity, we showed that it is in the supplier's best interest if the retail stage is decentralized. However, the equilibrium state indicates centralized procurement and the supplier, therefore, sets his/her wholesale price for centralized retailers. This state, nevertheless, does not maximize channel-wide profit in our numerical studies. The channel-wide profit is maximized under decentralized procurement for all of the problem instances solved. Therefore, a coordination mechanism should restrict centralization at the retailing stage; that is, vertical centralization is likely to require horizontal decentralization.

The setting of Chapter 5 serves as a benchmark for important future research directions. Studying the supplier's problem with competing retailers under more general market price and retailer cost functions remains as a future research direction. While generalized Stackelberg games have been analyzed in the literature (see, e.g., Sherali et al., 1983, Miller et al., 1991, Tobin, 1992), these Stackelberg games assume that retailers compete with each other and one of the retailers acts as a leader. Another future research direction would involve analyzing coordination mechanisms under different procurement strategies. For decentralized, competitive retailers, pricing as a coordination mechanism has been studied in the literature (see, e.g., Ingene and Parry, 1995, 1998, 2000, Bernstein and Federgruen, 2003, Keskinocak and Savaşaneril, 2008). Our study will be helpful in the analysis of coordination mechanisms for the generalized settings considered in Chapter 5. Furthermore, analysis of such coordination mechanisms for centralized and partially centralized retailers remains as a future research area. Studying the case in which the supplier plays the role of the distributor provides another future research direction. This case would correspond to analyzing the effects of Vendor-Managed-Inventory for multiple competitive retailers, where the supplier may decide on retailer order quantities as well as the wholesale price.

Another important future research direction consists of analyzing the supplier's problem when the competitive retailers can be cooperative. In the absence of a chain store or distributor, it is still possible that the competing retailers act together via coalitions/cooperations. For instance, Keskinocak and Savaşaneril (2008) study a supplier's discount pricing for retailers who purchase from the supplier collaboratively. Furthermore, retailers or buyers can form group purchasing organizations. As noted by Nagarajan et al. (2010), Group Purchasing Organizations are observed in various industries including health care, education, and retailing; we refer the interested reader to Nagarajan et al. (2010) for a broader discussion on modeling, formation, and issues related to Group Purchasing Organizations. Nagarajan et al. (2010) study stability of Group Purchasing Organization formation. In their setting, however, the supplier's discount pricing is fixed. An interesting research question, therefore, would be to analyze a supplier's discount pricing decisions when retailers can form coalitions to take advantage of the supplier's discount pricing. In particular, Group Purchasing Organizations are formed to take advantage of discounts offered by the supplier in many cases (Nagarajan et al., 2010, Chen and Roma, 2011). Thus, when the supplier sets a discount policy assuming non-cooperative retailers, discounts can be a detriment to a supplier in cases with retailer cooperation. The question of interest, then, would be how a supplier should set his/her wholesale price schedule to avoid *abuse* of discounts by retailer cooperatives.

APPENDIX A
SYMMETRY OF EQUILIBRIUM SUPPLY QUANTITIES GIVEN IDENTICAL FACILITY LOCATIONS

First note that for the locations where there is no facility, $q^*_{ijr} = 0 \; \forall r \in R$. Hence, we only focus on the locations where there are k facilities. We prove the statement using the KKT conditions defined for the optimal q_{ijr} values. Together with $q_{ijr} \geq 0$, the KKT conditions read

$$\begin{aligned} \delta_{ij} - b_j q_{\bullet j \bullet} - b_j q_{\bullet jr} - \alpha_{ij} q_{ij\bullet} - \alpha_{ij} q_{ijr} + u_{ijr} &= 0, \\ u_{ijr} q_{ijr} &= 0, \\ u_{ijr} &\geq 0, \end{aligned}$$

where $\delta_{ij} = a_j - c_{ij}$. Now consider any two firms r_1 and r_2. We show that $q^*_{ijr_1} = q^*_{ijr_2}$ $\forall i \in I^0$, where I^0 denotes the locations with k facilities for the given $\mathbf{X}^0$, by considering Cases 1 and 2, defined below. Then it follows that $q^*_{ijr_1} = q^*_{ijr_2} \; \forall i \in I^0$ for any two firms r_1 and r_2. Thus, we show that q^*_{ijr} is the same for all of the firms. Hence, letting Q^*_{ij} denote the total equilibrium quantity flow on the link (i, j), since there exist k firms at any location $i \in I^0$, it follows that $q^*_{ijr} = Q^*_{ij}/k$.

Case I: $q^*_{ijr_1} > 0$ **and** $q^*_{ijr_2} > 0 \; \forall i \in I^0$

Suppose that $q^*_{ijr_1} > 0$ and $q^*_{ijr_2} > 0 \; \forall i \in I^0$. Then $q^*_{ijr_1}$ and $q^*_{ijr_2}$ must satisfy the first order conditions, i.e., $u_{ijr_1} = u_{ijr_2} = 0 \; \forall i \in I^0$. Without loss of generality, we assume that $I^0 = \{1, 2, 3, \ldots, s\}$ such that $s \leq m$. The first order conditions, then, read

$$\delta_{ij} - b_j q^*_{\bullet j \bullet} - b_j q^*_{\bullet jr_1} - \alpha_{ij} q^*_{ij\bullet} - \alpha_{ij} q^*_{ijr_1} = 0 \; \forall i \in I^0, \tag{A–1}$$

$$\delta_{ij} - b_j q^*_{\bullet j \bullet} - b_j q^*_{\bullet jr_2} - \alpha_{ij} q^*_{ij\bullet} - \alpha_{ij} q^*_{ijr_2} = 0 \; \forall i \in I^0. \tag{A–2}$$

It follows from (A–1) and (A–2) that

$$b_j q^*_{\bullet jr_1} + \alpha_{ij} q^*_{ijr_1} = b_j q^*_{\bullet jr_2} + \alpha_{ij} q^*_{ijr_2} \; \forall i \in I^0, \tag{A–3}$$

$$\alpha_{zj}(q^*_{zjr_1} - q^*_{zjr_2}) = -b_j(q^*_{\bullet jr_2} - q^*_{\bullet jr_2}) = -b_j \sum_{i \in I^0} (q^*_{ijr_1} - q^*_{ijr_2}) \; \forall z \in I^0. \tag{A–4}$$

Subtracting (A–3) for location 2 from (A–3) for location 1, we get

$$\alpha_{1j} q^*_{1jr_1} - \alpha_{2j} q^*_{2jr_1} = \alpha_{1j} q^*_{1jr_2} - \alpha_{2j} q^*_{2jr_2}. \tag{A–5}$$

It follows from (A–5) that $\alpha_{1j}(q^*_{1jr_1} - q^*_{1jr_2}) = \alpha_{2j}(q^*_{2jr_1} - q^*_{2jr_2})$. Following similar argument, subtracting (A–3) for location $i+1$ from expression (A–3) for location i, $i \le s-1$, we get

$$\alpha_{1j}(q^*_{1jr_1} - q^*_{1jr_2}) = \alpha_{2j}(q^*_{2jr_1} - q^*_{2jr_2}) = \alpha_{3j}(q^*_{3jr_1} - q^*_{3jr_2}) = \ldots = \alpha_{sj}(q^*_{sjr_1} - q^*_{sjr_2}). \tag{A–6}$$

Considering (A–6), (A–4) can be written as

$$\alpha^2_{zj}(q^*_{zjr_1} - q^*_{zjr_2}) = -b_j \sum_{i \in I^0} \alpha_{ij}(q^*_{zjr_1} - q^*_{zjr_2}). \tag{A–7}$$

Since $\alpha_{ij} > 0$, (A–7) is only satisfied when $q^*_{zjr_1} = q^*_{zjr_2}$ for any location $z \in I^0$. Thus, it follows that $q^*_{ijr_1} = q^*_{jr_2}$ $\forall i \in I^0$.

Case II: $q^*_{ijr_1} = 0$ for $i \in I^0_{r_1} \subset I^0$ and $q^*_{ijr_2} = 0$ for $i \in I^0_{r_2} \subset I^0$

Suppose that $q^*_{ijr_1} = 0$ for locations $i \in I^0_{r_1} \subset I^0$ and $q^*_{ijr_2} = 0$ for locations $i \in I^0_{r_2} \subset I^0$. We consider the following three subcases of Case II, which capture all of the possibilities of Case II, and prove the statement for these three subcases.

Subcase I: $I^0_{r_1} = I^0_{r_2} = I^0_r \subset I^0$

Suppose that $I^0_{r_1} = I^0_{r_2} = I^0_r \subset I^0$, i.e., $q^*_{ijr_1} = q^*_{ijr_2} = 0$ for locations $i \in I^0_r \subset I^0$. For locations $i \notin I^0_r$, that is, for locations $i \in I^0 \backslash I^0_r$, we have $q^*_{ijr_1} > 0$ and $q^*_{ijr_2} > 0$. Thus, Subcase I reduces to Case I with $I^0 \backslash I^0_r$ instead of I^0, which means we have $q^*_{ijr_1} = q^*_{ijr_2}$ for locations $i \in I^0 \backslash I^0_r$. Thus, for Subcase I, we have $q^*_{ijr_1} = q^*_{ijr_2}$ $\forall i \in I^0$.

Subcase II: $I^0_{r_1} \neq I^0_{r_2}$ and either $I^0_{r_1} = \emptyset$ or $I^0_{r_2} = \emptyset$

Suppose that $I^0_{r_1} \neq I^0_{r_2}$ and either $I^0_{r_1} = \emptyset$ or $I^0_{r_2} = \emptyset$. Without loss of generality, suppose that $I^0_{r_2} = \emptyset$.

Situation (i): Consider $I^0_{r_1} = I^0$. Situation (i) implies that $q^*_{ijr_1} = 0$ $\forall i \in I^0$, thus, $q^*_{\bullet jr_1} = 0$. Now consider any location $z \in I^0$ and suppose that $q^*_{zjr_2} > 0$. It follows from the KKT conditions that $u_{zjr_2} = 0$. Moreover, from the KKT conditions for $q^*_{zjr_1}$ and $q^*_{zjr_2}$, we have

$$\delta_{zj} - b_j q^*_{\bullet j \bullet} - \alpha_{zj} q^*_{zj\bullet} + u_{zjr_1} = 0, \tag{A–8}$$

$$\delta_{zj} - b_j q^*_{\bullet j \bullet} - b_j q^*_{\bullet jr_2} - \alpha_{zj} q^*_{zj\bullet} - \alpha_{zj} q^*_{ijr_2} = 0. \tag{A–9}$$

It follows from (A–8) and (A–9) that $-u_{zjr_1} = b_j q^*_{\bullet jr_2} + \alpha_{zj} q^*_{zj\bullet} + \alpha_{zj} q^*_{ijr_2}$, which implies

$$b_j q^*_{\bullet jr_2} + \alpha_{zj} q^*_{zj\bullet} + \alpha_{zj} q^*_{zjr_2} = 0 \tag{A–10}$$

since $u_{zjr_1} \geq 0$, $\alpha_{zj} > 0$ and $b_j > 0$. Moreover, since $q^*_{ijr_2} \geq 0$, (A–10) is only satisfied when $q^*_{\bullet jr_2} = q^*_{zj\bullet} = q^*_{zjr_2} = 0$. Therefore, we have a contradiction with $q^*_{zjr_2} > 0$, thus, $q^*_{zjr_1} = q^*_{zjr_2} = 0$ for any location $z \in I^0$ for Situation (i), i.e., $q^*_{ijr_1} = q^*_{jr_2}\ \forall i \in I^0$.

Situation (ii): Consider $I^0_{r_1} \subset I^0$. Situation (ii) implies that there is at least one location, say location t, $t \in I^0 \backslash I^0_{r_1}$ such that $q^*_{tjr_1} > 0$ and $q^*_{tjr_2} > 0$. We show by contradiction that $q^*_{ijr_2} = 0\ \forall i \in I^0_{r_1}$. Suppose that $q^*_{zjr_2} > 0$ for any location $z \in I^0_{r_1}$. It follows from the KKT conditions that $u_{zjr_2} = 0$. Moreover, from the KKT conditions for $q^*_{zjr_1}$ and $q^*_{zjr_2}$, we have

$$\delta_{zj} - b_j q^*_{\bullet j \bullet} - b_j q^*_{\bullet jr_1} - \alpha_{zj} q^*_{zj\bullet} + u_{zjr_1} = 0, \tag{A–11}$$

$$\delta_{zj} - b_j q^*_{\bullet j \bullet} - b_j q^*_{\bullet jr_2} - \alpha_{zj} q^*_{zj\bullet} - \alpha_{zj} q^*_{ijr_2} = 0. \tag{A–12}$$

It follows from (A–11) and (A–11) that $b_j q^*_{\bullet jr_1} - u_{zjr_1} = b_j q^*_{\bullet jr_2} + \alpha_{zj} q^*_{ijr_2}$. Since $u_{zjr_1} \geq 0$, $\alpha_{zj} > 0$ and $q^*_{zjr_2} > 0$, it implies that

$$b_j q^*_{\bullet jr_1} > b_j q^*_{\bullet jr_2}. \tag{A–13}$$

Now consider any location $t \in I^0 \backslash I^0_{r_1}$ such that $q^*_{tjr_1} > 0$ and $q^*_{tjr_2} > 0$. Then it follows from KKT conditions that $u_{tjr_1} = u_{tjr_2} = 0$ and $b_j q^*_{\bullet jr_1} + \alpha_{tj} q^*_{tjr_1} = b_j q^*_{\bullet jr_2} + \alpha_{tj} q^*_{tjr_2}$. Since $b_j q^*_{\bullet jr_1} > b_j q^*_{\bullet jr_2}$, we have $q^*_{tjr_1} < q^*_{tjr_2}$ for any location

$t \in I^0 \backslash I^0_{r_1}$. Thus, it follows that

$$\sum_{t \in I^0 \backslash I^0_{r_1}} q^*_{tjr_1} < \sum_{t \in I^0 \backslash I^0_{r_1}} q^*_{tjr_2}. \tag{A–14}$$

Moreover, since $q^*_{zjr_1} = 0$ and $q^*_{zjr_2} > 0$ for any location $z \in I^0_{r_1}$, we have

$$\sum_{t \in I^0_{r_1}} q^*_{tjr_1} < \sum_{t \in I^0_{r_1}} q^*_{tjr_2}. \tag{A–15}$$

Inequalities (A–14) and (A–15) together imply that $b_j q^*_{\bullet jr_1} < b_j q^*_{\bullet jr_2}$, which is contradiction with inequality (A–13). Thus, we should have $q^*_{zjr_1} = q^*_{zjr_2} = 0$ for any location $z \in I^0_{r_1}$. For other locations, i.e., any location $i \in I^0 \backslash I^0_{r_1}$, we have $q^*_{ijr_1} > 0$ and $q^*_{ijr_2} > 0$. Since, we know $q^*_{zjr_1} = q^*_{zjr_2} = 0$ for any location $z \in I^0_{r_1}$, we can ignore such locations. Then, Situation (ii) reduces to Case 1 with $I^0 \backslash I^0_r$ instead of I^0, which means we have $q^*_{ijr_1} = q^*_{ijr_2}$ for locations $i \in I^0 \backslash I^0_r$.

Thus, $q^*_{ijr_1} = q^*_{ijr_2}$ $\forall i \in I^0$ for Situation (ii).

Situations (i) and (ii) together imply that $q^*_{ijr_1} = q^*_{ijr_2}$ $\forall i \in I^0$ for Subcase II.

Subcase III: $I^0_{r_1} \neq \emptyset$, $I^0_{r_2} \neq \emptyset$ and $I^0_{r_1} \neq I^0_{r_2}$

Suppose that $I^0_{r_1} \neq \emptyset$, $I^0_{r_2} \neq \emptyset$ and $I^0_{r_1} \neq I^0_{r_2}$. First note that for any location $i \in I^0_{r_1} \cap I^0_{r_2}$, we have $q^*_{ijr_1} = q^*_{ijr_2} = 0$, thus, we can disregard such locations and only study the situation when $I^0_{r_1} \neq \emptyset$, $I^0_{r_2} \neq \emptyset$ and $I^0_{r_1} \cap I^0_{r_2} = \emptyset$. This situation implies the following conditions:

$$q^*_{ijr_1} = 0 \textit{ and } q^*_{ijr_2} > 0 \; \forall i \in I^0_{r_1}, \tag{A–16}$$

$$q^*_{ijr_1} > 0 \textit{ and } q^*_{ijr_2} = 0 \; \forall i \in I^0_{r_2}. \tag{A–17}$$

We now show by contradiction that conditions (A–16) and (A–17) cannot be satisfied at the same time. Consider any location $z \in I^0_{r_1}$, that is, $q^*_{zjr_1} = 0$ and $q^*_{zjr_2} > 0$. It follows from the KKT conditions that $u_{zjr_2} = 0$ and (i) $b_j q^*_{\bullet jr_1} - u_{zjr_1} = b_j q^*_{\bullet jr_2} + \alpha_{zj} q^*_{zjr_2}$, which means that $b_j q^*_{\bullet jr_1} > b_j q^*_{\bullet jr_2}$ as $u_{zjr_2} \geq 0$, $\alpha_{zj} > 0$ and $q^*_{zjr_2} > 0$. Now consider any location $t \in I^0_{r_2}$, that is, $q^*_{tjr_1} > 0$ and $q^*_{tjr_2} = 0$. It follows from the KKT conditions that $u_{tjr_2} = 0$ and (ii) $b_j q^*_{\bullet jr_1} + \alpha_{tj} q^*_{tjr_1} = b_j q^*_{\bullet jr_2} - u_{zjr_2}$, which means that $b_j q^*_{\bullet jr_1} < b_j q^*_{\bullet jr_2}$ as $u_{tjr_1} \geq 0$, $\alpha_{tj} > 0$ and $q^*_{tjr_1} > 0$. (i) and (ii) establishes a contradiction. That is, we cannot satisfy the conditions (A–16) and (A–17) at the same time. Without loss of generality, suppose that we do not have condition (A–16), i.e., $q^*_{ijr_1} = 0$ and $q^*_{ijr_2} = 0 \; \forall i \in I^0_{r_1}$. Hence, we can disregard any location $i \in I^0_{r_1}$. Then, Subcase III reduces to Subcase II. Thus, $q^*_{ijr_1} = q^*_{ijr_2} \; \forall i \in I^0$ for Subcase III.

APPENDIX B
MIXED STRATEGY NASH EQUILIBRIUM FOR SYMMETRIC LOCATION GAME

We first note that, the location decisions of the firms corresponds to a symmetric game. It is well known that for symmetric games, there exist a symmetric Mixed Strategy Nash Equilibrium (MSNE) in cases of multiple equilibria or there does not exist a Pure Nash Equilibrium (PNE). Symmetry of MSNE means that the probability of choosing a specific location decision is the same for each firm, hence, if we know the probability assigned to location vector $\mathbf{x}$ by a firm, we know the probabilities assigned by each firm at equilibrium. Now, let us focus on a single firm and consider any location vector $\mathbf{x}$. Suppose Assumptions 2.1-2.3 hold. To capture the preferences in Assumptions 2.1 and 2.2, we formulate utility function of a firm and use this function as the firm's objective. We characterize the utility function of firm r, given the location decisions of all other firms as a function of $\mathbf{x}_r$ as follows

$$\mu_r(\mathbf{x}_r) = \begin{cases} -M & \text{if } \exists\, \mathbf{x} \text{ such that } \Pi_r(\mathbf{Q}^*(\mathbf{x}), \mathbf{x}) > \Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r), \\ -M & \text{if } \exists\, \mathbf{x} \text{ such that } \Pi_r(\mathbf{Q}^*(\mathbf{x}), \mathbf{x}) = \Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r) \text{ and } |\mathbf{x}| > |\mathbf{x}_r|, \\ \Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r) & \text{otherwise,} \end{cases} \tag{B–1}$$

where $M \to \infty$, $\Pi_r(\mathbf{Q}^*(\mathbf{x}_r), \mathbf{x}_r)$ denotes the total profit, including facility location costs, of firm r when $\mathbf{x}_r$ is the location vector, and $|\mathbf{x}|$ denotes the number of facilities located under location vector $\mathbf{x}$. Note that, the purpose of formulating a utility function as in Equation B–1 and letting $M \to \infty$ is just to reflect Assumptions 2.1 and 2.2 mathematically. Now given that any firm uses Equation (B–1) as an objective, we focus on determining the probability assigned to location vector $\mathbf{x}$ by any firm, say firm r_1, using the utilities of any other firm, say firm r_2, i.e., we compare two firms. Suppose there are T possible location vectors and firm r_1 assigns probability $\rho_{r_1 t}$ to location vector $t \leq T$. Now let us focus on utility matrix of firm r_2, say $\mathbf{A}$. Due to Equation (B–1), each row of **A** consists of 1 nonnegative and $t-1$ of $-M$ values. We consider the following two cases.

Case I: Each column on A has 1 nonnegative value

In this case, no strategy is weakly or strictly dominated, hence, $\rho_{r_1 t} > 0\ \forall 0 \leq t \leq T$. Then, we should have $a\rho_{r_1 t} - M(1-\rho_{r_1 t}) = b\rho_{r_1 z} - M(1-\rho_{r_1 z})$, where $a > 0$ and $b > 0$ for any t and z, $1 \leq t \leq T$ and $1 \leq z \leq T$. Then it follows that $\frac{\rho_{r_1 t}}{\rho_{r_1 z}} = \frac{a+M}{b+M}$. Then $\lim_{M\to\infty} \frac{\rho_{r_1 t}}{\rho_{r_1 z}} = 1$, i.e., $\rho_{r_1 t} = \rho_{r_1 z}$ for any $1 \leq t \leq T$ and $1 \leq z \leq T$. Moreover, since there are finite number of strategies for any firm, $\rho_{r_1 t} = \rho_{r_1 z} > 0$. Letting ρ denote this probability, we have $\rho_{r_1}(\mathbf{x}) = \rho$ for any location vector **x**, as $M \to \infty$. Then it easily follows from the symmetry of the MSNE, $\rho_r(\mathbf{x}) = \rho$ for any firm $r \in R$.

Case II: There are columns with no nonnegative values

In this case, the location vectors corresponding to the columns with nonnegative values weakly or strictly dominates the location vectors corresponding to the columns without nonnegative values. Hence, we can assign probability 0 to the columns without nonnegative values. For the remaining columns, then, Case II reduces to Case I.

We note that a weakly or strictly dominated strategy, i.e., a location vector, when utility function is used as an objective, is also weakly or strictly dominated when the profit function is used as an objective by the firms. It follows from Cases I and II that any firm will assign probability 0 to weakly or strictly dominated location vectors and any firm will assign probability ρ to any other location vector as $M \to \infty$.

APPENDIX C
SOLUTION OF DECENTRALIZED RETAILING UNDER GENERALIZED MARKET PRICE AND OPERATING COST FUNCTIONS

Here, we provide a solution method for decentralized, i.e., competitive retailers' quantity decisions for a given supplier wholesale price. We assume that the market price function, $p(Q)$, satisfies the following conditions.

Assumption C.1. *(i)* $p(Q)$ *is a continuously, twice differentiable function of* Q. *(ii)* $p(Q)$ *is a decreasing concave function of* Q*, i.e.,* $dp(Q)/dQ < 0$ *and* $d^2p(Q)/dQ^2 \leq 0$*, for* $Q \geq 0$*, and* $\partial p(Q)/\partial q_i < 0$ *and* $\partial^2 p(Q)/\partial q_i^2 \leq 0$ *for* $q_i \geq 0$ *and for all* $i = 1, \ldots, n$*.*

Note that when $p(Q) = a - bQ$, Assumption C.1 is satisfied. Furthermore, we assume that $v_i(q_i)$ satisfies the following conditions.

Assumption C.2. *(i)* $v_i(q_i)$ *is a continuously, twice differentiable function of* q_i. *(ii)* $v_i(q_i)$ *is an increasing convex function of* q_i*, i.e.,* $dv_i(q_i)/dq_i > 0$ *and* $d^2v_i(q_i)/dq_i^2 \geq 0$ *for* $q_i \geq 0$ *and for all* $i = 1, \ldots, n$*.*

Again note that when $v_i(q_i) = w_i q_i$, Assumption C.2 is satisfied. Under Assumptions C.1 and C.2, the profit function of retailer i reads

$$\Pi_i(\overrightarrow{\mathbf{Q}}) = p(Q)q_i - cq_i - v_i(q_i), \tag{C–1}$$

and it can be shown that $\Pi_i(\overrightarrow{\mathbf{Q}})$ is concave, given the order quantities of the other retailers. (This follows, as $\partial^2\Pi_i(\overrightarrow{\mathbf{Q}})/\partial q_i^2 = 2\partial p(Q)/\partial q_i + q_i \partial^2 p(Q)/\partial q_i^2 - d^2 v_i(q_i)/dq_i^2 \leq 0$ for $q_i \geq 0$, under Assumptions C.1 and C.2.) This implies that the first-order conditions ($\partial \Pi_i(\overrightarrow{\mathbf{Q}})/\partial q_i = 0$, for $q_i > 0$) must be satisfied at a Nash equilibrium solution. In particular, if $q_i > 0$ then a Nash equilibrium solution must satisfy the condition

$$p(Q) + q_i \frac{\partial p(Q)}{\partial q_i} - c - \frac{dv_i(q_i)}{dq_i} = 0. \tag{C–2}$$

It can be shown that $\overrightarrow{\mathbf{Q}}^* \in R^n_+$ solves the retailers' game if it solves the following variational inequality problem:

$$\langle F(\overrightarrow{\mathbf{Q}}^*), \overrightarrow{\mathbf{Q}} - \overrightarrow{\mathbf{Q}}^* \rangle \geq 0, \forall \overrightarrow{\mathbf{Q}} \in R^n_+, \tag{C–3}$$

where $F(\overrightarrow{\mathbf{Q}}) = (-\partial \Pi_1(\overrightarrow{\mathbf{Q}})/\partial q_1, \ldots, -\partial \Pi_n(\overrightarrow{\mathbf{Q}})/\partial q_n)$ is an n-row vector function (see, e.g., Gabay and Moulin, 1980, Friesz et al., 1983, Harker, 1984, Nagurney, 1999, Miller et al., 1996). Hence, one can use existing variational inequality methods to solve Equation (C–3) and thus the retailers' game. We refer the interested reader to the books by Miller et al. (1996) and Nagurney (1999), and the references therein, for algorithmic solutions for variational inequality formulations of equilibrium problems similar to the retailers' game. The method we discuss next is an iterative method that requires solving a system of equations of the form $H^\ell(\overrightarrow{\mathbf{Q}}^\ell) = 0$, where $H^\ell(\overrightarrow{\mathbf{Q}}^\ell)$ is an ℓ-vector function and $\overrightarrow{\mathbf{Q}}^\ell$ is an ℓ-vector, with $\ell \leq n$.

The following lemma is generalization of Proposition 5.1.

Lemma 1. *(i) $q_i^* > 0$ if and only if $c + dv_i(q_i^*)/dq_i < p(Q^*)$, for all $i \in \{1, 2, \ldots, n\}$. (ii) Suppose that $dv_{i_1}(0)/dq_{i_1} < dv_{i_2}(0)/dq_{i_2}$ for retailers i_1 and i_2 such that $i_1, i_2 \in \{1, 2, \ldots, n\}$. Then, (a) if $q_{i_2}^* > 0$, then $q_{i_1}^* > 0$, and (b) if $q_{i_1}^* = 0$, then $q_{i_2}^* = 0$.*

Proof: Suppose that the equilibrium supply quantities are known and fixed at q_i^* for all of the retailers except retailer i. Then the optimal supply quantity of retailer i is given by the KKT conditions, since retailer i's problem is to maximize his/her profit, which is a concave function of q_i, with the constraint $q_i \geq 0$, which is a linear constraint. The KKT conditions for retailer i, then, are

$$p(Q^*) + q_i^* \frac{\partial p(Q^*)}{\partial q_i} - c - \frac{dv_i(q_i^*)}{dq_i} + u_i = 0, \tag{C–4}$$

$$u_i q_i^* = 0, \tag{C–5}$$

$$u_i \geq 0. \tag{C–6}$$

Suppose that $q_i^* > 0$, which means $u_i = 0$. Then, condition (C–4) implies that $p(Q^*) + q_i^* \partial p(Q^*)/\partial q_i = c + dv_i(q_i^*)/dq_i$. Furthermore, as $\partial p(Q^*)/\partial q_i < 0$ from Assumption C.1 and $q_i^* > 0$, it follows that $c + dv_i(q_i^*)/dq_i < p(Q^*)$. Now suppose that $c + dv_i(q_i^*)/dq_i < p(Q^*)$. Let us assume that $q_i^* = 0$ to establish a contradiction. Then, condition (C–4) implies that $u_i + q_i^* \partial p(Q^*)/\partial q_i < 0$ as $c + dv_i(q_i^*)/dq_i < p(Q^*)$. This further implies that $u_i < 0$ as $q_i^* = 0$, which contradicts condition (C–6). Therefore, when $c + dv_i(q_i^*)/dq_i < p(Q^*)$, we have $q_i^* > 0$. This proves the condition (i) of Lemma 1.

Now let $dv_{i_1}(0)/dq_{i_1} < dv_{i_2}(0)/dq_{i_2}$. Suppose that $q_{i_2}^* > 0$. Then it follows from condition (i) that

$$c + dv_{i_2}(q_{i_2}^*)/dq_{i_2} < p(Q^*). \tag{C–7}$$

Let us assume that $q_{i_1}^* = 0$ to establish a contradiction. Then it follows from condition (i) that $c + dv_{i_1}(0)/dq_{i_1} \geq p(Q^*)$. This implies that

$$c + dv_{i_2}(0)/dq_{i_2} \geq p(Q^*) \tag{C–8}$$

as $dv_{i_1}(0)/dq_{i_1} < dv_{i_2}(0)/dq_{i_2}$. Combining (C–7) and (C–8), one concludes that

$$c + dv_{i_2}(0)/dq_{i_2} \geq p(Q^*). \tag{C–9}$$

However, (C–9) establishes a contradiction with Assumption C.2 as $v_{i_2}(q_{i_2})$ is convex, i.e., $d^2 v_{i_2}(q_{i_2}^*)/dq_{i_2}^2 \geq 0$, which means $dv_{i_2}(q_{i_2})/dq_{i_2}$ is an increasing function and, hence, one should have $dv_{i_2}(0)/dq_{i_2} \leq dv_{i_2}(q_{i_2}^*)/dq_{i_2}$ as $q_{i_2}^* > 0$. This contradiction proves statement (a) of condition (ii) of Lemma 1. Now suppose that $q_{i_1}^* = 0$. Let us assume that $q_{i_2}^* > 0$. Then statement (a) of condition (ii) implies that $q_{i_1}^* > 0$, which contradicts that $q_{i_1}^* = 0$, hence, $q_{i_2}^* = 0$. This proves statement (b) of condition (ii) of Lemma 1 and completes the proof of condition (ii). □

The following algorithm is generalization of Algorithm 6.

Algorithm 7. *Without loss of generality, suppose that retailers are sorted in increasing order of* $dv_i(0)/dq_i$ *values. Given* a*,* b*,* c *and* $v_i(q_i)$ $\forall i \in \{1, 2, \dots, n\}$*;*

Step 0. If $c + dv_1(0)/dq_1 \geq p(0)$*, set* $q_i^* = 0$ $\forall i \in \{1, 2, \dots, n\}$ *and* $\ell^* = 0$*. Else, set* $\ell = 1$ *and go to Step 1.*

Step 1. Determine $q_i^{(\ell)}$ *for* $i \leq \ell$ *by solving the following system of equations:*

$$p(Q) + q_i \partial p(Q)/\partial q_i - c - dv_i(q_i)/dq_i = 0, \;\; \forall i \leq \ell. \qquad \text{(C–10)}$$

Define $Q^{(\ell)} = \sum_{i=1}^{\ell} q_i^{(\ell)}$*, and go to Step 2.*

Step 2. If $\ell = n$*, stop. All of the retailers are active;* $q_i^* = q_i^{(\ell)}$ *for* $i \leq n$ *and* $\ell^* = n$*. Else, if* $\ell < n$ *and if* $c + dv_{\ell+1}(0)/dq_{\ell+1} \geq p(Q^{(\ell)})$*, stop. Retailers* $i \leq \ell$ *are active;* $q_i^* = q_i^{(\ell)}$ *for* $i \leq \ell$ *and* $q_i^* = 0$ *for* $\ell < i \leq n$ *and,* $\ell^* = \ell$*. Else, if* $\ell < n$ *and if* $c + dv_{\ell+1}(0)/dq_{\ell+1} < p(Q^{(\ell)})$*, set* $\ell = \ell + 1$ *and go to Step 1.*

Next, we prove correctness of Algorithm 7.

Lemma 2. *Suppose that retailers are sorted in increasing order of* $dv_i(0)/dq_i$ *values. Given* c*, and that* $p(Q)$*,* $v_i(q_i)$ $\forall i \in \{1, 2, \dots, n\}$*, satisfy Assumptions C.1 and C.2, Algorithm 7 gives the equilibrium solution of the retailers' game.*

Proof: To prove the correctness of Algorithm 7, we first show that when $q_i^{(\ell)} > 0$ $\forall i \leq \ell$ and $c + dv_{\ell+1}(0)/dq_{\ell+1} \geq p(Q^{(\ell)})$, we should have $q_{\ell+1}^* = 0$. Suppose that $q_i^{(\ell)} > 0$ $\forall i \leq \ell$ and $c + dv_{\ell+1}(0)/dq_{\ell+1} \geq p(Q^{(\ell)})$. To establish a contradiction, let us assume that $q_{\ell+1}^* > 0$, which means $q_{(\ell+1)}^{(\ell+1)} > 0$. Since $q_{\ell+1}^* > 0$, it follows from condition (ii) of Lemma 1 that $q_i^* > 0$ $\forall i \leq \ell$, which means $q_i^{(\ell+1)} > 0$ $\forall i \leq \ell$. Furthermore, as $q_{\ell+1}^{(\ell+1)} > 0$, it follows from condition (i) of Lemma 1 that $c + dv_{\ell+1}(q_{\ell+1}^{(\ell+1)})/dq_{\ell+1} < p(Q^{(\ell+1)})$. It then follows that $c + dv_{\ell+1}(0)/dq_{\ell+1} < p(Q^{(\ell+1)})$ as $v_{\ell+1}(q_{\ell+1})$ is assumed to be an increasing convex function in Assumption C.2, i.e., $c + dv_{\ell+1}(0)/dq_{\ell+1} < c + dv_{\ell+1}(q_{\ell+1}^{(\ell+1)})/dq_{\ell+1}$ for $0 < q_{\ell+1}^{(\ell+1)}$. Now, since $c + dv_{\ell+1}(0)/dq_{\ell+1} \geq p(Q^{(\ell)})$, one concludes that

$$p(Q^{(\ell)}) < p(Q^{(\ell+1)}). \qquad \text{(C–11)}$$

(C–11) implies that

$$Q^{(\ell)} > Q^{(\ell+1)}, \tag{C–12}$$

$$\partial p(Q^{(\ell)})/\partial q_i \leq \partial p(Q^{(\ell+1)})/\partial q_i < 0 \tag{C–13}$$

$\forall i \leq \ell + 1$ as $p(Q)$ is assumed to be a decreasing concave function of Q in Assumption C.1. Now consider any retailer i, $i \leq \ell$. Then considering Equation (C–10) at iterations ℓ and $\ell + 1$ for retailer i, we have

$$p(Q^{(\ell)}) + q_i^{(\ell)} \frac{\partial p(Q^{(\ell)})}{\partial q_i} - c - \frac{dv_i(q_i^{(\ell)})}{dq_i} = 0, \tag{C–14}$$

$$p(Q^{(\ell+1)}) + q_i^{(\ell+1)} \frac{\partial p(Q^{(\ell+1)})}{\partial q_i} - c - \frac{dv_i(q_i^{(\ell+1)})}{dq_i} = 0. \tag{C–15}$$

Combining (C–11), (C–14), and (C–15), one can conclude that

$$q_i^{(\ell+1)} \frac{\partial p(Q^{(\ell+1)})}{\partial q_i} - q_i^{(\ell)} \frac{\partial p(Q^{(\ell)})}{\partial q_i} < \frac{dv_i(q_i^{(\ell+1)})}{dq_i} - \frac{dv_i(q_i^{(\ell)})}{dq_i}. \tag{C–16}$$

Now suppose that $q_i^{(\ell)} \geq q_i^{(\ell+1)}$. It then follows from (C–13) that $q_i^{(\ell+1)}\partial p(Q^{(\ell+1)})/\partial q_i - q_i^{(\ell)}\partial p(Q^{(\ell)})/\partial q_i \geq 0$ for $q_i^{(\ell)} \geq q_i^{(\ell+1)}$. This further implies from (C–16) that $dv_i(q_i^{(\ell+1)})/dq_i - dv_i(q_i^{(\ell)})/dq_i > 0$. However, Assumption C.2 implies that $dv_i(q_i^{(\ell+1)})/dq_i - dv_i(q_i^{(\ell)})/dq_i \leq 0$ for $q_i^{(\ell)} \geq q_i^{(\ell+1)}$, which contradicts $dv_i(q_i^{(\ell+1)})/dq_i - dv_i(q_i^{(\ell)})/dq_i > 0$. Therefore,

$$q_i^{(\ell)} < q_i^{(\ell+1)}, \tag{C–17}$$

which implies that $\sum_{i=1}^{\ell}(q_i^{(\ell+1)} - q_i^{(\ell)}) > 0$. We know from (C–12) that $Q^{(\ell+1)} - Q^{(\ell)} < 0$, i.e., $q_{\ell+1}^{(\ell+1)} + \sum_{i=1}^{\ell}(q_i^{(\ell+1)} - q_i^{(\ell)}) < 0$. It then follows that $q_{\ell+1}^{(\ell+1)} < 0$ as $\sum_{i=1}^{\ell}(q_i^{(\ell+1)} - q_i^{(\ell)}) > 0$. This contradicts $q_{\ell+1}^{(\ell+1)} > 0$. The other cases considered in Step 2 of Algorithm 7 are obvious. Furthermore, Step 1 of Algorithm 7 is the simultaneous solution of the first order conditions for each active retailer and Step 0 is a direct application of Proposition 1. Therefore, Algorithm 7 gives the number of active retailers as well as the corresponding equilibrium supply quantities at the market. □

APPENDIX D
COMPARISON OF TOTAL ORDER QUANTITIES UNDER DIFFERENT RETAILING STRATEGIES

We first show that $Q^p \leq Q^c$. We know from Propositions 5.2 and 5.3 that $Q^c = \frac{a-c-w_1}{2b}$ and $Q^p = \frac{a-c-\sum_{i=1}^n k_i\left(\sum_{j=1}^n w_j k_j\right)}{2b}$. Now let $w^p = \sum_{j=1}^n w_j k_j$, i.e., $Q^p = \frac{a-c-w^p}{2b}$. From definition of w_1, we have $w_1 \leq w^p$. Thus, it follows that $Q^p \leq Q^c$.

Next, we show that $Q^c \leq Q^*$. The following lemma states a property of Algorithm 6, and, it indicates that $Q^c \leq Q^*$.

Lemma 3. *Suppose that the retailers are decentralized, and Algorithm 6 is used to determine the equilibrium number of active retailers and their order quantities. Let $\ell^* = \ell$ in the output of Algorithm 6. Then (i) $\sum_{i=1}^r q_i^{(r)} < \sum_{i=1}^{r+1} q_i^{(r+1)}$ for $r < \ell$, (ii) $q_i^{(r)} > q_i^{(r+1)}$ for $r < \ell$ and $i \leq r$.*

Proof: Suppose that $\ell^* = \ell$. We first prove statement (i). For $r < \ell$, we know from Algorithm 6 that $bq_1^{(r)} + c + w_1 = bq_2^{(r)} + c + w_2 = \ldots = bq_l^{(r)} + c + w_r$ and, using these equalities, one can derive $q_s^{(r)} = q_i^{(r)} + (w_i - w_s)/b$ for $i, s \leq l$. This expression and the linear equation given in Algorithm 6 for retailer i together imply that $q_i^{(r)} = \frac{a-c-(r+1)w_i+\sum_{i=1}^r w_i}{b(r+1)}$, which implies that $\sum_{i=1}^r q_i^{(r)} = \frac{r(a-c)}{b(r+1)} - \frac{\sum_{i=1}^r w_i}{b(r+1)}$. Similarly, it can be shown that $\sum_{i=1}^{r+1} q_i^{(r+1)} = \frac{(r+1)(a-c)}{b(r+2)} - \frac{\sum_{i=1}^{r+1} w_i}{b(r+2)}$. To establish a contradiction, let us assume that $\sum_{i=1}^r q_i^{(r)} \geq \sum_{i=1}^{r+1} q_i^{(r+1)}$, that is, $\frac{r(a-c)}{b(r+1)} - \frac{\sum_{i=1}^r w_i}{b(r+1)} \geq \frac{(r+1)(a-c)}{b(r+2)} - \frac{\sum_{i=1}^{r+1} w_i}{b(r+2)}$. It then follows that

$$(r+1)w_{(r+1)} \geq a - c + b\sum_{i=1}^r w_i. \tag{D–1}$$

Nevertheless, since ℓ retailers are active, $q_{(r+1)}^{(r+1)} > 0$ for $r < \ell$ thus, it follows from Step 2 of Algorithm 6 that $c + w_{(r+1)} < a - b\sum_{i=1}^r q_i^{(r)}$. Considering that $\sum_{i=1}^r q_i^{(r)} = \frac{r(a-c)}{b(r+1)} - \frac{\sum_{i=1}^r w_i}{b(r+1)}$, this further implies that

$$(r+1)w_{(r+1)} < a - c + b\sum_{i=1}^r w_i. \tag{D–2}$$

(D–1) contradicts (D–1) and this contradiction completes the proof of statement (i). Statement (ii) is a direct implication of Statement (i) with respect to the following relations: $a - b\sum_{i=1}^{r} q_i^{(r)} - bq_i^{(r)} - c - w_i = 0 \ \forall i \leq r$ and $a - b\sum_{i=1}^{r+1} q_i^{(r+1)} - bq_i^{(r+1)} - c - w_i = 0 \ \forall i \leq r+1$. Since $\sum_{i=1}^{r} q_i^{(r)} < \sum_{i=1}^{r+1} q_i^{(r+1)}$, it then follows that $q_i^{(r)} > q_i^{(r+1)}$ for $r < \ell$ and $i \leq r$. $\square$

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BIOGRAPHICAL SKETCH

Dinçer Konur was born in Istanbul, Turkey in 1984. He has completed his primary, secondary, and high school education in Istanbul. He has earned his B.S. degree in industrial engineering from Bilkent University in Ankara, Turkey, in 2007. Upon graduation, he started his doctoral study in the Department of Industrial and Systems Engineering at the University of Florida. He has received his M.S. degree in December 2009 during his doctoral studies. Dinçer Konur will complete his Ph.D. in August 2011. Upon graduation, he will join Intermodal Freight Transportation Institute at the University of Memphis as a post-doctoral researcher.