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WAVEFORM DESIGN FOR ACTIVE SENSING SYSTEMS – A COMPUTATIONAL APPROACH

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Active sensing applications such as radar, sonar and medical imaging, demand proper designs of the probing waveform. A well-synthesized waveform can significantly increase the system performance in terms of signal-to-interference ratio, spectrum containment, beampattern matching, target parameter estimation and so on. The focus of this work is on designing probing waveforms using computational algorithms.

We first investigate designing waveforms with good correlation properties, which are widely useful in applications including range compression, channel estimation and spread spectrum. We consider both the design of a single sequence and that of a set of sequences, the former with only auto-correlations and the latter with auto- and cross-correlations. The proposed algorithms leverage FFT (fast Fourier transform) operations and can efficiently generate long sequences that were previously difficult to synthesize. We present a new derivation of the lower bound for sequence correlations that arises from the proposed algorithm framework. We show that such a lower bound can be closely approached by the newly designed sequences.

A two-dimensional extension of the time-delay correlation function is the ambiguity function (AF) that involves a Doppler frequency shift. We give an overview of AF properties and discuss how to minimize AF sidelobes in a discrete formation.

Besides good correlation properties, we also consider the stopband constraint that is required in the scenario of avoiding reserved frequency bands or strong electronic
jammer. We present an algorithm that accounts for both correlation and stopband constraints.

We finally consider transmit beampattern synthesis, particularly in the wideband case. We establish the relationship between a desired beampattern and underlying waveforms by using the Fourier transform. We highlight the increased design freedom resulting from the waveform diversity of a MIMO (multi-input multi-output) system.
CHAPTER 1
INTRODUCTION

The goal of an active sensing system, such as radar or sonar, is to determine useful properties of the targets or of the propagation medium by transmitting certain waveforms toward an area of interest and analyzing the received signals. For example, a land-based surveillance radar sends electromagnetic waves in the direction of sky, where objects such as airplanes can reflect a (usually a very tiny) fraction of the transmitted signal back to the radar. By measuring the round-trip time delay, the distance (called range) between the radar and the target can be estimated since the speed of propagation for radio waves is known \((3 \times 10^8 \text{ m/s})\). Additional target properties can be obtained by performing further processing at the receiver side; e.g., the speed of a target can be estimated by measuring the Doppler frequency shift of the received signal.

Animals like dolphins and bats have used sound waves to communicate and detect objects for millions of years, while humans did not start to do so until the early twentieth century. In 1904, a German engineer named Christian Hülsmeyer carried out the first radar experiment using his “telemobiloscope” to detect ships in dense fog by means of radio waves. As to sonar, Reginald Fessenden, a Canadian engineer, demonstrated in 1914 using a sound echo device, though not successfully, for iceberg detection off the east coast of Canada. It was amongst several other experiments and patents said to be prompted by the 1912 Titanic disaster.

Radar and sonar underwent great development during the two world wars and later on spread into diverse fields including weather monitoring, flight control and underwater sensing. There are two factors that are critical to the system performance, namely the receive filter and the transmit waveform. The receive filter is used to extract from the received signals the information of interest, e.g., the target locations in radar/sonar applications [1] or the channel conditions in communications [2]. The transmit waveform,
not surprisingly, interplays with the receive filter. A good design of the transmit waveform lends itself to accurate parameter estimation and a reduced computational burden at the receiver.

Arguably the most commonly used receive filter is the matched filter, which maximizes the signal-to-noise ratio (SNR) in the presence of stochastic additive white noise [3]. Examples of other well-known receive filters include the mismatched filter which is also called the instrumental-variable (IV) method [4–6], the Capon estimator [7], the amplitude and phase estimation (APES) algorithm [8–10] and more advanced data-adaptive techniques such as the iterative adaptive approach (IAA) [11].

We concentrate our attention on transmit waveform design in this work. Particularly we are interested in synthesizing waveforms that have good correlation properties (Chapters 2, 3 and 4). In radar range compression, low auto-correlation sidelobes improve the detection performance of weak targets [12, 13]; in code division multiple access (CDMA) systems, low auto-correlation sidelobes are desired for synchronization purposes and low cross-correlations reduce interferences from other users [14, 15]; and the situation is similar in many other active sensing applications such as ultra-sonic imaging [16]. An emitted probing waveform with low auto-correlation sidelobes maximizes the signal-to-noise ratio when complemented by a matched filter at the receiver side while significantly weakening signals from adjacent range bins. In Chapter 5, the correlation function is extended to the two-dimensional ambiguity function (AF) by incorporating the impact of Doppler delays.

In addition to correlation properties, good spectrum containment is desired for transmitted signals (Chapter 6). In practice, many frequency bands have already been reserved for particular uses such as navigation or military communications; or there could exist strong emitters (e.g., electronic jamming) whose operating frequencies should be avoided. Therefore it is desired that the transmit waveform deliver as little energy as possible in those frequency bands.
Chapter 7 discusses the problem of transmit beampattern synthesis in a MIMO (multi-input multi-output) system. A classical phased-array steers a narrow beam toward different angles by adjusting only the waveform phases across antenna elements. In a modern MIMO system, however, waveforms can be chosen freely and this waveform diversity allows for more flexibility in beampattern synthesis.

_notation:_ We will use boldface lowercase and uppercase letters to denote vectors and matrices, respectively. See Table 1-1 for other common notation used throughout this work.

### 1.1 Signal Model

Let $s(t)$ denote the transmitted signal with $t$ indicating time. Suppose that $s(t)$ consists of $N$ symbols

$$s(t) = \sum_{n=1}^{N} x(n)p_n(t)$$

(1–1)

where $p_n(t)$ is the shaping pulse and $\{x(n)\}_{n=1}^{N}$ are the $N$ symbols. The shaping pulse $p_n(t)$ (with duration $t_p$) can be an ideal rectangular pulse

$$p_n(t) = \frac{1}{\sqrt{t_p}} \text{rect} \left( \frac{t - (n - 1)t_p}{t_p} \right), \quad n = 1, \ldots, N,$$

(1–2)

where

$$\text{rect}(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

(1–3)

or other pulses, such as the raised-cosine pulse [2].

Note that the actual transmitted waveform is composed of the in-phase and quadrature components of $s(t)e^{j2\pi f_c t}$ where $f_c$ is the carrier frequency. It is assumed that the signal demodulation has already been performed at the receiver side and thus the carrier term $e^{j2\pi f_c t}$ can be safely ignored in the analysis.
In practice, hardware components such as analog-to-digital converters and power amplifiers have a maximum signal amplitude clip. In order to maximize the transmitted power that is available in the system, it is desirable that the transmit sequences are unimodular or have low peak-to-average power ratios (PAR). In our design, we impose the following unit-modulus constraint whenever feasible:

$$x(n) = e^{j\phi(n)}, \quad n = 1, \ldots, N$$  \hspace{1cm} (1–4)$$

where \{\phi(n)\} are phases. Note that (1–1) combined with (1–4) provides a phase-coded signal representation. There are many other types of signals that are widely used or have been discussed in the literature, which include the well-known chirp waveform (Section 1.3), discrete frequency-coded waveforms [17, 18] and waveforms constructed from a particular set of functions such as the prolate spheroidal [19] or the Hermite wave functions [20]. In this work we chose to focus specifically on the phase-coded signal model, which serves as a practical and effective framework for designing waveforms with various desirable properties.

The waveform \(s(t)\) is transmitted in the direction of a scene of interest and is reflected by various targets at different range locations. The reflected signals, which are time-shifted and weighted versions of \(s(t)\), arrive linearly combined at the receiver side:

$$y(t) = \sum_k \alpha_k s(t - \tau_k) + e(t)$$  \hspace{1cm} (1–5)$$

where \(\tau_k\) is the round-trip time delay for the \(k\)th target, \(\alpha_k\) is the coefficient related to the target reflection such as the radar cross section (RCS) and \(e(t)\) is the noise.

Suppose that we aim to estimate the coefficient \(\alpha_{k'}\) by applying the filter \(w(t)\) at the receiver:

$$\hat{\alpha}_{k'} = \int_{-\infty}^{\infty} w^*(t)y(t)dt.$$  \hspace{1cm} (1–6)$$
More precisely speaking, according to a conventional convolution definition, (1–6) is the receiver output at time instant 0 when \( y(t) \) is the input and \( w(-t) \) is the filter. However, we simply refer to \( w(t) \) as the filter in the receiver processing indicated by (1–6), without introducing any ambiguity in discussions afterwards.

To determine a proper \( w(t) \), we decompose \( y(t) \) into three parts:

\[
y(t) = \alpha_{k'} s(t - \tau_{k'}) + \sum_{k \neq k'} \alpha_k s(t - \tau_k) + e(t). \tag{1–7}
\]

If there is no clutter and \( e(t) \) is zero-mean white noise, then the matched filter \( w(t) = s(t - \tau_{k'}) \) will give the largest signal-to-noise ratio (SNR). A direct proof goes as follows:

\[
\text{SNR} \triangleq \frac{\left| \int_{-\infty}^{\infty} w^*(t) \alpha_{k'} s(t - \tau_{k'}) dt \right|^2}{\text{E} \left\{ \left| \int_{-\infty}^{\infty} w^*(t) e(t) dt \right|^2 \right\}} = \frac{\left| \alpha_{k'} \right|^2 \left| \int_{-\infty}^{\infty} w^*(t) s(t - \tau_{k'}) dt \right|^2}{\sigma^2_e \int_{-\infty}^{\infty} |w(t)|^2 dt} \tag{1–8}
\]

\[
\leq \frac{\left| \alpha_{k'} \right|^2 \int_{-\infty}^{\infty} |s(t - \tau_{k'})|^2 dt}{\sigma^2_e^2} = \frac{\left| \alpha_{k'} \right|^2 \sigma^2_s}{\sigma^2_e^2} \tag{1–9}
\]

where \( \text{E} \) denotes the expectation, and \( \sigma^2_e \) and \( \sigma^2_s \) are the noise power and signal power, respectively. Note that (1–8) is due to the white noise assumption \( \text{E} \{ e(t_1) e^*(t_2) \} = \sigma^2_e \delta_{t_1,t_2} \) and that (1–9) results from the Cauchy-Schwartz inequality \( \left| \int w^*(t) s(t - \tau_{k'}) dt \right|^2 \leq \int |w(t)|^2 dt \int |s(t - \tau_{k'})|^2 dt \). The maximum value of SNR in (1–9) is achieved if and only if the filter \( w(t) \) is a scaled version of \( s(t - \tau_{k'}) \), which concludes the proof.

For the purpose of normalization, the matched filter \( w(t) \) is chosen to be \( s(t - \tau_{k'}) / \int |s(t)|^2 dt \) and the corresponding estimate of \( \alpha_{k'} \) in (1–6) is given by

\[
\hat{\alpha}_{k'} = \frac{\int_{-\infty}^{\infty} s^*(t - \tau_{k'}) y(t) dt}{\int_{-\infty}^{\infty} |s(t)|^2 dt}. \tag{1–10}
\]

Besides boosting the signal component and suppressing the noise, the matched filter can also eliminate the clutter component (as easily seen from (1–7) and (1–10)) if

\[
r(\tau) = \int_{-\infty}^{\infty} s(t) s^*(t - \tau) dt, \quad -\infty < \tau < \infty, \tag{1–11}
\]

16
is zero for all $\tau \neq 0$. The $r(\tau)$ as defined in (1–11) is called the auto-correlation of $s(t)$.

### 1.2 Design Metrics

The previous section has outlined the benefit of small auto-correlation sidelobes $r(\tau)$ (for $\tau \neq 0$). For most practical cases, we only need to focus on the delay $\tau$ that is an integer multiple of the symbol length $t_p$. One of the reasons is that in modern systems digital filtering is usually performed at the receiver side, that is, the integral in (1–10) is implemented as a summation of sampled signals. In addition, if the rectangular shaping pulse (Eq. (1–2)) is used, the values of $r(\tau)$ can be obtained exactly by a linear interpolation of two neighboring auto-correlation samples [e.g. 13, Chapter 6]:

$$r(\tau) = \frac{\tau - t_1}{t_p} r(t_2) + \frac{t_2 - \tau}{t_p} r(t_1),$$

where $t_1 = \left\lfloor \frac{\tau}{t_p} \right\rfloor t_p$ and $t_2 = t_1 + t_p$.

Such auto-correlations at integer multiple delays $\{kt_p\}_{k=-N+1}^{N-1}$ can be calculated for $k \geq 0$ as

$$r(kt_p) = \int_{-\infty}^{\infty} s(t) s^* (t - kt_p) dt = \int_{kt_p}^{Nt_p} \sum_{n=k+1}^{N} x(n)p_n(t)x^*(n-k)p_n^*(t) dt$$

$$= \sum_{n=k+1}^{N} x(n)x^*(n-k) \cdot (N-k) \int_{0}^{t_p} |p_n(t)|^2 dt$$

$$= (N-k) \sum_{n=k+1}^{N} x(n)x^*(n-k).$$

(1–13)

Correlations at negative delays can be obtained from $r(kt_p) = r^*(-kt_p)$. When shaping pulses other than the rectangular one are used, it can still be expected that $r(\tau)$ is well controlled as long as $r(k)$ is made sufficiently small.

It follows from the above discussions that the correlations of interest are given by

$$r(k) = \sum_{n=k+1}^{N} x(n)x^*(n-k) = r^*(-k), \quad k = 0, \ldots, N - 1.$$  

(1–14)
The above \( \{r(k)\} \) is called the auto-correlation of the discrete sequence \( \{x(n)\} \).

Note that the notation \( r \) is "slightly abused" in (1–11) and (1–14) to denote both continuous-time and discrete-time auto-correlations, yet distinction can be made easily by examining the two different time variables.

For \( \{r(k)\}_{k=-N+1}^{N-1} \) defined above, \( r(0) \) is called the in-phase correlation, which is always equal to the signal energy. All other auto-correlations, i.e., \( \{r(k), \ k = -N + 1, \ldots, -1, 1, \ldots, N - 1\} \), are collectively called the auto-correlation sidelobes. One of the main interests in Chapters 2 and 3 is to design phase-coded sequences \( \{x(n)\} \) whose auto-correlation sidelobes are as low as possible. Chapter 5 discusses the ambiguity function synthesis, which can be considered as a two-dimensional extension to the correlation design.

More precisely speaking, the \( \{r(k)\} \) defined in (1–14) is the aperiodic auto-correlation. The periodic auto-correlation of the sequence \( \{x(n)\} \) is defined as

\[
\tilde{r}(k) = \sum_{n=1}^{N} x(n)x^\ast((n-k) \mod N) = \tilde{r}^\ast(-k) \\
= \tilde{r}^\ast(N-k) \quad k = 0, \ldots, N - 1,
\]

where "mod" is the modulo operator

\[
p \mod N = \begin{cases} 
    p - \left\lfloor \frac{p}{N} \right\rfloor N, & p \neq 0 \text{ or } N, \\
    N, & p = 0 \text{ or } N.
\end{cases}
\]

The relationship between the aperiodic correlation (1–14) and the periodic correlation (1–15) can be easily obtained as follows:

\[
\tilde{r}(k) = \sum_{n=1}^{K} x(n)x^\ast(n - k + N) + \sum_{n=k+1}^{N} x(n)x^\ast(n - k) \\
= \sum_{m=(N-k)+1}^{N} x(m - (N-k))x^\ast(m) + \sum_{n=k+1}^{N} x(n)x^\ast(n - k) \\
= r^\ast(N - k) + r(k).
\]
As will be shown in Chapter 7, waveform correlations can serve as a bridge connecting the underlying waveform to the desired beampattern of, e.g., an antenna array. Particularly, the waveform diversity in a MIMO system leads to a flexible control of waveform correlations which further lead to an agile transmit beampattern synthesis.

In the next section, we will review several well-known waveforms that have good auto-correlation properties, especially those that are phase-coded (1–4).

1.3 Review of Existing Waveforms

We start with the well-known chirp waveform. A chirp waveform is a linear frequency-modulated (LFM) pulse, whose frequency is swept linearly over a bandwidth $B$ in a time duration $T$. Chirp signals have been widely used in radar applications since World War II, as they possess relatively low correlation sidelobes and are mostly tolerant to Doppler frequency shifts [13]. In addition, the power of a chirp signal is dispersed evenly throughout the frequency spectrum, which allows for high spectral efficiency.

A chirp signal $s(t)$ can be written as

$$s(t) = \frac{1}{\sqrt{T}} e^{j \frac{B}{T} t^2}, \quad 0 \leq t \leq T$$

where $\frac{B}{T}$ is the chirp rate. Figure 1-1A shows the real part of $s(t)$ with parameters $T = 100$ sec and $B = 1$ Hz. Figure 1-1B shows its auto-correlation function $r(\tau)$ (normalized by $r(0)$ and using a $20\lg$ scale), where the peak sidelobe is $-13.4$ dB.

Many phase codes can be derived from the chirp signal. We sample $s(t)$ at time intervals $t_s = \frac{1}{B} (N = BT)$ and obtain the following sequence:

$$x(n) \triangleq s(nt_s) = e^{j \pi \frac{B}{T} (\frac{n}{N})^2}$$

$$= e^{j \pi \frac{n^2}{N}} = e^{j \pi \frac{n^2}{N}}, \quad n = 1, \ldots, N.$$  

The sequence $\{x(n)\}$ shown above has perfect periodic auto-correlations if $N$ is even, meaning that all periodic auto-correlation sidelobes are zero: $\tilde{r}(k) = 0$ for $k \neq 0$. 
A sequence with perfect periodic correlations for any odd $N$ can be constructed by changing the sequence phases in (1–19) to

$$x(n) = e^{j\pi \frac{n(n-1)}{N}}, \quad n = 1, \ldots, N$$

which is the Golomb sequence [21]. The Chu sequence [22], interestingly, is a combination of the above two sequences:

$$x(n) = \begin{cases} e^{jQ\pi \frac{n^2}{N}}, & N \text{ is even} \\ e^{jQ\pi \frac{n(n-1)}{N}}, & N \text{ is odd} \end{cases}$$

(1–21)

where $Q$ is any integer that is prime to $N$. As expected, the Chu sequence has perfect periodic correlations for any (positive) integer $N$.

Besides the Golomb and the Chu sequences, there are many other phase-coded sequences whose phases are quadratic functions of $n$, such as the well-known Frank sequence and the P4 sequence. The Frank sequence is defined for $N = L^2$ as:

$$x((m-1)L + p) = e^{j2\pi \frac{(m-1)(p-1)}{L}}, \quad m, p = 1, \ldots, L.$$  

(1–22)

The P4 sequence is defined for any length $N$ as

$$x(n) = e^{j\frac{2\pi}{N}(n-1)(\frac{n-1-N}{2})}, \quad n = 1, \ldots, N.$$  

(1–23)

Both Frank and P4 sequences have perfect periodic correlations.

Figure 1-2 shows the auto-correlation $r(k)$ of the P4 sequence of length $N = 100$. Note that from the sequence $\{x(n)\}$, we can construct a continuous-time waveform $s(t)$ using (1–1). We choose rectangular pulse shaping and $t_p = 1$ sec so that the signal duration is 100 s and that the signal bandwidth is roughly 1 Hz (i.e., $1/t_p$), which are the same parameters as used in Figure 1-1. The real and imaginary parts of the so-constructed $s(t)$ are shown separately in Fig. 1-3A. The auto-correlation $r(\tau)$ of this
$s(t)$ is shown in Figure 1-3B. The peak sidelobe is $-26.3$ dB, which is much lower than that of the chirp waveform in Figure 1-1B.

The auto-correlation properties of the Golomb, Chu or Frank sequences are similar to those of the P4 sequence and are omitted for brevity.

Another widely used sequence is the maximum length sequence (m-sequence) [e.g. 2]. An m-sequence is a type of pseudo random binary sequence that is generated by a maximal linear feedback shift register (LFSR). Fig. 1-4 shows a length-3 LFSR where the plus operator indicates ‘exclusive or’. Each register block can store 0 or 1 so three blocks amount to eight different states. When fed with any initial binary sequence (not all zeros), such a shift register will cycle through all eight states except for the all-zero state. For instance, starting from ‘001’, the register in Fig. 1-4 will pass repeatedly through the following seven states ‘001’, ‘100’, ‘010’, ‘101’, ‘110’, ‘111’, ‘011’. By taking only the output from the third block and replacing ‘0’ with ‘-1’, we obtain a length-7 m-sequence \{1, -1, -1, 1, -1, 1, 1\}. Its aperiodic as well as periodic correlations are shown in Fig. 1-5.

One of the prominent features of an m-sequence is that its periodic correlation sidelobes are always equal to $-1$, as can be observed from Fig. 1-5B. Its aperiodic correlation sidelobes, though, do not have a regular pattern and can be relatively high. Also note that the LFSR can be efficiently implemented in hardware, which largely facilitates the use of m-sequences in practice.

The aforementioned sequences/waveforms all have closed-form construction methods. Researchers have also used gradient descent or stochastic optimization techniques to find sequences with lower auto-correlations [23–25]. These algorithms are usually computationally expensive and work well only for small values of $N$, such as $N \sim 10^2$. An example is the search for polyphase Barker sequences whose auto-correlation sidelobes are all less than or equal to 1. A length-45 polyphase Barker
sequence is given by [24]

\[ x(n) = \exp\{j\phi(n)\}, \quad n = 1, \ldots, 45 \]  \hspace{1cm} (1–24)

\[ \{\phi(n)\} = \frac{2\pi}{90} \{0 \ 0 \ 7 \ 1 \ 76 \ 71 \ 76 \ 63 \ 56 \ 73 \ 87 \ 9 \ 9 \ 14 \ 25 \ 53 \ 62 \ 5 \ 32 \ 35 \ 85 \]

69 \ 40 \ 76 \ 57 \ 26 \ 9 \ 83 \ 56 \ 57 \ 21 \ 52 \ 89 \ 48 \ 11 \ 68 \ 26 \ 62 \ 6 \ 37 \ 73

19 \ 58 \ 12 \}

and its auto-correlation is shown in Figure 1-6. Its peak sidelobe is \(-33.1\) dB. Note that

for a unit-modulus sequence, the lowest possible peak sidelobe is 1, since \(|r(N - 1)| =

|x(N)x^*(1)| = 1\). In the case of \(N = 45\), 1 corresponds to \(20\log(1/N) = -33.1\) dB. Thus

a polyphase Barker code has the best auto-correlation properties in terms of the lowest

peak sidelobe. We mention in passing on the fact that the Barker code was originally

defined as a \textit{binary} sequence with correlation sidelobes not larger than 1 [26]. The

longest known Barker code has length 13 and is shown below

\[ \{x(n)\} = \left\{1 \ 1 \ 1 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1 \right\}, \]  \hspace{1cm} (1–25)

whose auto-correlation is plotted in Figure 1-7.

We finally point out that the design of unit-modulus sequences with low \textit{aperiodic}

correlations is much more difficult than the design that is concerned with low \textit{periodic}

correlations. The aforementioned sequences such as Chu, P4 and m-sequences

have low or zero periodic correlation sidelobes; and there are many others with the

same property such as Gold sequences [27] and Kasami sequences [28]. Note the

symmetric property of periodic correlations in (1–15): \(\tilde{r}(k) = \tilde{r}^*(N - k)\). It leads to the

fact that in order to minimize all periodic correlation sidelobes, we need to consider only

\(\tilde{r}(1), \ldots, \tilde{r}(N/2)\) for an even \(N\) and only \(\tilde{r}(1), \ldots, \tilde{r}\left(\frac{N-1}{2}\right)\) for an odd \(N\). The aperiodic

correlations, on the other hand, do not have such symmetric property. In addition, the

absolute value of the maximum-lag aperiodic correlation \(r(N - 1)\) is always equal to 1

(thus cannot be minimized): \(|r(N - 1)| = |x(N)x^*(1)| = 1\) because each element of
the sequence has unit-modulus. We can also observe from (1–17) that all-zero periodic correlation sidelobes imply \( r(k) = -r^*(N - k) \) (for \( k = 1, \ldots, N - 1 \)) and vice versa.

In the following chapters we will focus on sequence design for the aperiodic correlation case that is relatively less touched in the literature. For the sake of brevity, the word “correlation” indicates the aperiodic correlation unless “periodic correlation” is explicitly pointed out.

Table 1-1. Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a^* )</td>
<td>complex conjugate of a scalar ( a )</td>
</tr>
<tr>
<td>( \text{Re}{a} )</td>
<td>real part of a scalar ( a )</td>
</tr>
<tr>
<td>( \text{Im}{a} )</td>
<td>imaginary part of a scalar ( a )</td>
</tr>
<tr>
<td>(</td>
<td></td>
</tr>
<tr>
<td>( A^* )</td>
<td>complex conjugate of a matrix ( A )</td>
</tr>
<tr>
<td>( A^T )</td>
<td>transpose of a matrix ( A )</td>
</tr>
<tr>
<td>( A^H )</td>
<td>conjugate transpose of a matrix ( A )</td>
</tr>
<tr>
<td>( \text{tr}(A) )</td>
<td>trace of a matrix ( A )</td>
</tr>
<tr>
<td>(</td>
<td></td>
</tr>
<tr>
<td>( A \geq 0 )</td>
<td>matrix ( A ) is positive semi-definite</td>
</tr>
<tr>
<td>( A \leq B )</td>
<td>matrix ( B - A ) is positive semi-definite</td>
</tr>
<tr>
<td>( \delta_n )</td>
<td>Kronecker delta: ( \delta_n = 1 ) if ( n = 0 ) and ( \delta_n = 0 ) otherwise</td>
</tr>
<tr>
<td>( \delta_{mn} )</td>
<td>an extension of ( \delta_n ): ( \delta_{mn} = 1 ) if ( m = n ) and ( \delta_{mn} = 0 ) otherwise</td>
</tr>
<tr>
<td>( I_M )</td>
<td>identity matrix of dimension ( M \times M )</td>
</tr>
<tr>
<td>( f(x) * g(x) )</td>
<td>convolution of two functions ( f(x) ) and ( g(x) )</td>
</tr>
<tr>
<td>( \lfloor x \rfloor )</td>
<td>biggest integer less than or equal to ( x ) (real-valued)</td>
</tr>
<tr>
<td>( \arg(x) )</td>
<td>phase angle (in radians) of ( x )</td>
</tr>
</tbody>
</table>
Figure 1-1. A chirp signal $s(t)$ and its auto-correlations. A) The real part of $s(t)$ in (1–18) with $T = 100$ s and $B = 1$ Hz. B) The auto-correlation function of $s(t)$. 
Figure 1-2. The auto-correlation function $r(k)$ of the P4 sequence, as defined in (1–23), of length $N = 100$. 

[Graph of the auto-correlation function $r(k)/N$ in dB vs. $k$ with peaks at $k = 0$, $k = 20$, $k = 40$, $k = 60$, and $k = 80$, and values ranging from approximately $-60$ dB to $-10$ dB.]
Figure 1-3. A P4 waveform and its auto-correlations. A) The real and imaginary parts of $s(t)$ in (1–1) when the P4 sequence of length 100 and rectangular shaping pulses are used. B) The auto-correlation function $r(\tau)$ of this $s(t)$. 
Figure 1-4. A linear feedback shift register of length 3.

Figure 1-5. Auto-correlation of an m-sequence. A) The aperiodic auto-correlation and B) the periodic correlation of a length-7 m-sequence \( \{1, -1, -1, 1, -1, 1, 1\} \).
Figure 1-6. The auto-correlation function $r(k)$ of a polyphase Barker sequence of length $N = 45$, as defined in (1–24).

Figure 1-7. The auto-correlation function $r(k)$ of a Barker sequence of length $N = 13$, as defined in (1–25).
CHAPTER 2
SINGLE SEQUENCE DESIGN

From the discussions given in Chapter 1, we have made it clear that the goal of aperiodic sequence design is to make \( \{r(k)\}_{k \neq 0} \) as small as possible. In this chapter, we focus on the integrated sidelobe level (ISL) metric which is defined as:

\[
\text{ISL} = \sum_{k=-(N-1)}^{N-1} |r(k)|^2 = 2 \sum_{k=1}^{N-1} |r(k)|^2.
\]  

(2–1)

Our goal is to present efficient computation algorithms to minimize the ISL metric or ISL-related metrics under the constraint of synthesizing unit-modulus sequences. Note that the minimization of the ISL metric is equivalent to the maximization of the merit factor (MF) defined as follows:

\[
\text{MF} = \frac{|r(0)|^2}{\sum_{k=-(N-1)}^{N-1} |r(k)|^2} = \frac{N^2}{\text{ISL}}.
\]  

(2–2)

Owing to the significant practical interest in the design of unit-modulus sequences with low ISL (or equivalently large MF) values, as pointed out in Chapter 1, it comes as no surprise that the literature on this topic is extensive [25, 29–32].

An extension to the ISL and MF metrics are the weighted ISL (WISL) and modified MF (MMF) metrics:

\[
\text{WISL} = 2 \sum_{k=1}^{N-1} w_k |r(k)|^2, \quad w_k \geq 0, 
\]  

(2–3)

\[
\text{MMF} = \frac{|r(0)|^2}{\text{WISL}}
\]  

(2–4)

where \( \{w_k\}_{k=1}^{N-1} \) is an arbitrary set of weights. Such weighted ISL metrics are important in applications where we want to reduce, as much as possible, the interference due to a known multipath or a known clutter discrete. For example, there are cases in which the maximum difference between the arrival times of the sequence of interest and of the
interference is (much) smaller than the duration of the emitted sequence \([30, 33–35]\). Consequently, in such cases the interest lies in making \(|r(k)|_{k=1}^{P-1}\) small for some \(P < N\), instead of trying to make all correlation sidelobes \(|r(k)|_{k=1}^{N-1}\) small. Here the value of \(P\) is selected based on the \textit{a priori} knowledge about the application at hand (for instance, in wireless communications it is usually known that significant channel tap coefficients can occur up to a certain maximum delay, and so we can choose \(P\) as the said delay).

Because the ISL metric is highly multimodal (i.e., it has many local minima), stochastic optimization algorithms have been suggested for its minimization \([24, 25, 36]\). However, the computational burden of these algorithms becomes prohibitive as \(N\) increases: such algorithms are hardly effective on the currently available computing machines for \(N \sim 10^3\) or larger. Optimization algorithms for locally minimizing the ISL metric have also been proposed. These algorithms can be used to provide quick solutions to the problem of reducing the ISL value of a given reasonably good sequence. They can also be used as local minimization blocks of a stochastic global optimization algorithm. However, most of the existing local minimization algorithms for the ISL metric are descent gradient methods whose convergence problems as well as computational burdens increase significantly as \(N\) increases.

In this chapter we introduce two cyclic algorithms \([37]\) for the local minimization of ISL-related metrics. The first algorithm is called CAN (cyclic algorithm-new) that can be used for the local minimization of the ISL metric. CAN is based on FFT (fast Fourier transform) operations and can be used virtually for any practically relevant values of \(N\) up to \(N \sim 10^6\) or even larger. We also modify CAN so that it can tackle the WISL metric. The resulting algorithm, which is called WeCAN (weighted CAN), requires \(N\) times more computations than CAN and it can be run on a PC for \(N\) up to \(N \sim 10^4\).
2.1 Cyclic Algorithm-New (CAN)

The derivation of CAN involves several steps, the first of which consists of expressing the ISL metric in the frequency domain. It is well-known that, for any \( \omega \in [0, 2\pi] \),

\[
\left| \sum_{n=1}^{N} x(n)e^{-j\omega n} \right|^2 = \sum_{k=-(N-1)}^{N-1} r(k)e^{-j\omega k} \triangleq \Phi(\omega) \tag{2–5}
\]

[38]. Furthermore, it can be shown that the ISL metric in (2–1) can be equivalently written as:

\[
\text{ISL} = \frac{1}{2N} \sum_{p=1}^{2N} [\Phi(\omega_p) - N]^2, \tag{2–6}
\]

where \( \{\omega_p\} \) are the following Fourier frequencies:

\[
\omega_p = \frac{2\pi}{2N} p, \quad p = 1, \ldots, 2N. \tag{2–7}
\]

(Note that (2–6) is a Parseval-type equality.) To prove (2–6), let \( \delta_k \) denote the Kronecker delta and use the correlogram-based expression for \( \Phi(\omega) \) in (2–5) to verify that:

\[
\sum_{p=1}^{2N} [\Phi(\omega_p) - N]^2 = \sum_{p=1}^{2N} \left[ \sum_{k=-(N-1)}^{N-1} (r(k) - N\delta_k)e^{-j\omega_p k} \right]^2 \tag{2–8}
\]

\[
= \sum_{k=-(N-1)}^{N-1} \sum_{\tilde{k}=-(N-1)}^{N-1} (r(k) - N\delta_k)(r(\tilde{k}) - N\delta_{\tilde{k}})^* \left[ \sum_{p=1}^{2N} e^{-j\omega_p (k-\tilde{k})} \right].
\]

Because, for \( |k - \tilde{k}| \leq 2N - 2 \),

\[
\sum_{p=1}^{2N} e^{-j\omega_p (k-\tilde{k})} = e^{-j\frac{2\pi}{2N} (k-\tilde{k})} \cdot \frac{e^{-j2\pi (k-\tilde{k})} - 1}{e^{-j\frac{2\pi}{2N} (k-\tilde{k})} - 1} = 2N\delta_{(k-\tilde{k})}, \tag{2–9}
\]

we obtain from (2–8) the following equation:

\[
\frac{1}{2N} \sum_{p=1}^{2N} [\Phi(\omega_p) - N]^2 = \sum_{k=-(N-1)}^{N-1} |r(k) - N\delta_k|^2 = 2 \sum_{k=1}^{N-1} |r(k)|^2 = \text{ISL}, \tag{2–10}
\]
which is (2–6). Using the periodogram-based expression for \( \Phi(\omega) \) (Eq. (2–5)) in (2–6) shows that the problem of minimizing the ISL is equivalent to the minimization of the following frequency-domain metric:

\[
\sum_{p=1}^{2N} \left[ \left( \sum_{n=1}^{N} x(n)e^{-j\omega_p n} \right)^2 - N \right]^2.
\] (2–11)

This equivalence result has an obvious intuitive interpretation: minimizing the ISL makes the sequence behave like white noise, and consequently its periodogram should be nearly constant in frequency.

The next point to note is that the criterion in (2–11) is a quartic function of \( \{x(n)\} \). However, it can be verified that the minimization of (2–11) with respect to \( \{x(n)\} \) is “almost equivalent” to the following simpler problem (whose criterion is a quadratic function of \( \{x(n)\} \)):

\[
\min_{\{x(n)\}_{n=1}^{N}, \{\psi_p\}_{p=1}^{2N}} \sum_{p=1}^{2N} \left[ \sum_{n=1}^{N} x(n)e^{-j\omega_p n} - \sqrt{N}e^{j\psi_p} \right]^2.
\] (2–12)

Briefly speaking, if the criterion in (2–12) takes on a small value, then so does (2–11), and vice versa. More specifically, (2–11) is equal to zero if and only if the criterion in (2–12) is equal to zero. Consequently, by continuity arguments, if the global minimum value of (2–11) is “sufficiently small”, then the sequences minimizing (2–11) and, respectively, the criterion in (2–12) can be expected to be close to one another. See Appendix A for a detailed discussion on this “almost equivalence”.

Let

\[
a_p^H = \begin{bmatrix} e^{-j\omega_p} & \ldots & e^{-j2N\omega_p} \end{bmatrix},
\] (2–13)
let $A^H$ be the following unitary $2N \times 2N$ FFT matrix:

$$A^H = \frac{1}{\sqrt{2N}} \begin{bmatrix} a_1^H \\ \vdots \\ a_{2N}^H \end{bmatrix},$$  \hspace{1cm} (2–14)

and let $z$ be the sequence $\{x(n)\}_{n=1}^N$ padded with $N$ zeros:

$$z = \begin{bmatrix} x(1) & \cdots & x(N) & 0 & \cdots & 0 \end{bmatrix}^T_{2N \times 1}. \hspace{1cm} (2–15)$$

Then the criterion in (2–12) can be rewritten in the following more compact form (to within a multiplicative constant):

$$\|A^H z - v\|^2,$$  \hspace{1cm} (2–16)

where

$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\psi_1} & \cdots & e^{j\psi_{2N}} \end{bmatrix}^T.$$  \hspace{1cm} (2–17)

For given $\{x(n)\}$, the minimization of (2–16) with respect to $\{\psi_p\}$ is immediate: let

$$f = A^H z$$  \hspace{1cm} (2–18)

denote the FFT of $z$; then

$$\psi_p = \arg(f_p), \quad p = 1, \ldots, 2N. \hspace{1cm} (2–19)$$

Similarly, for given $v$, let

$$g = Av$$  \hspace{1cm} (2–20)

denote the Inverse-FFT (IFFT) of $v$. Because $\|A^H z - v\|^2 = \|z - Av\|^2$, it follows that the minimizing sequence $\{x(n)\}$ is given by:

$$x(n) = e^{j\arg(g_n)}, \quad n = 1, \ldots, N. \hspace{1cm} (2–21)$$
The CAN for the cyclic local minimization of the ISL-related metric in (2–12) is summarized below. Owing to its simple (I)FFT operations, CAN can be used for very large values of $N$, such as $N \sim 10^6$.

The CAN algorithm:

- **Step 0**: Set $\{x(n)\}_{n=1}^{N}$ to random initial values (e.g., $\{x(n)\}$ can be set to $\{e^{j2\pi\theta(n)}\}$ where $\{\theta(n)\}$ are independent random variables uniformly distributed in $[0, 2\pi]$), or $\{x(n)\}_{n=1}^{N}$ can be initialized by a good existing sequence such as a Golomb sequence.

- **Step 1**: Compute the $\{p_p\}_{p=1}^{2N}$ that minimize the metric for $\{x(n)\}_{n=1}^{N}$ fixed at their most recent values (Eq. (2–19)).

- **Step 2**: Compute the sequence $\{x(n)\}_{n=1}^{N}$ that minimizes the metric, under the constraint $|x(n)| = 1$, for $\{p_p\}_{p=1}^{2N}$ fixed at their most recent values (Eq. (2–21)).

- **Iteration**: Repeat Steps 1 and 2 until a pre-specified stop criterion is satisfied e.g. $\|x^{(i)} - x^{(i+1)}\| < \epsilon$, where $x^{(i)}$ is the sequence obtained at the $i^{th}$ iteration, and $\epsilon$ is a predefined threshold, such as $10^{-3}$.

The CAN algorithm described above considers the constraint that the sequence $\{x(n)\}$ is unit-modulus, i.e., its peak-to-average power ratio (PAR) is equal to 1. Allowing the PAR value to be larger than 1 leads to better suppression of correlation sidelobes. See Chapter 4 for an extended CAN algorithm that deals with a relaxed PAR constraint.

In the next section, we present an extended version of CAN which can deal with the WISL metric (with arbitrarily chosen weights) as defined in (2–3). The extended algorithm is called WeCAN (weighted CAN). The price paid for WeCAN’s ability to deal with a general WISL metric is an increased computational burden compared to CAN. Specifically, as will be shown in the next section, each iteration of WeCAN requires $N$ computations of $2N$-point (I)FFT’s; thus the number of flops required by WeCAN is roughly $N$ times larger than that of CAN. Nonetheless, WeCAN can still be used for relatively large values of $N$, such as $N \sim 10^4$. 

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2.2 Weighted Cyclic Algorithm-New (WeCAN)

Similarly to the proof of (2–6) in Section 2.1, we can derive the following expression for the WISL metric ($\gamma_k$ below is related to the weight $w_k$ in (2–3) as $w_k = \gamma_k^2$):

\[
\text{WISL} = 2 \sum_{k=1}^{N-1} \gamma_k^2 |r(k)|^2
\]

\[
= \frac{1}{2N} \sum_{p=1}^{2N} [\hat{\Phi}(\omega_p) - \gamma_0 N]^2,
\]

where

\[
\hat{\Phi}(\omega_p) \triangleq \sum_{k=-(N-1)}^{N-1} \gamma_k r(k)e^{-j\omega_k k}, \quad \omega_p = \frac{2\pi}{2N} p, \quad p = 1, \ldots, 2N,
\]

and where \( \{\gamma_k\}_{k=1}^{N-1} \) are real-valued (with $\gamma_k = \gamma_{-k}$). Note that by choosing \( \{\gamma_k\}_{k=1}^{N-1} \) appropriately, we can weigh the correlation lags in (2–22) in any desired way. Regarding $\gamma_0$, which does not enter into (2–22), it will be chosen to ensure that the matrix

\[
\Gamma = \frac{1}{\gamma_0} \begin{bmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_{N-1} \\
\gamma_1 & \gamma_0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \gamma_1 \\
\gamma_{N-1} & \cdots & \gamma_1 & \gamma_0
\end{bmatrix}
\]

(2–25)

is positive semi-definite, which we denote by $\Gamma \succeq 0$. This can be done in the following simple way: let $\tilde{\Gamma}$ be the matrix $\gamma_0 \Gamma$ with all diagonal elements set to 0, and let $\lambda_{\min}$ denote the minimum eigenvalue of $\tilde{\Gamma}$; then $\Gamma \succeq 0$ if and only if $\gamma_0 + \lambda_{\min} \geq 0$, a condition that can always be satisfied by selecting $\gamma_0$.

Next we will derive a criterion that is “almost equivalent” to (2–23) and which depends quadratically on the unknowns \( \{x(n)\}_{n=1}^{N} \), similarly to what we have done in the previous section. To do so, we must apparently obtain a square root of $\hat{\Phi}(\omega_p)$ in (2–24) that is linear in \( \{x(n)\}_{n=1}^{N} \). Note the following DFT pairs:

\[
\{r(k)\} \leftrightarrow \hat{\Phi}(\omega) = |X(\omega)|^2
\]
\[
\{ \gamma_k r(k) \} \longleftrightarrow \tilde{\Phi}(\omega) = \Gamma(\omega) * |X(\omega)|^2, \tag{2–26}
\]

where

\[
X(\omega) = \sum_{n=1}^{N} x(n)e^{-jn\omega}, \quad \Gamma(\omega) = \sum_{k=-(N-1)}^{N-1} \gamma_k e^{-j\omega k}, \tag{2–27}
\]

and where \( * \) is the convolution operator. Thus \( \tilde{\Phi}(\omega_p) \) can be expressed as

\[
\tilde{\Phi}(\omega_p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma(\omega_p - \psi)|X(\psi)|^2 \, d\psi
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-(N-1)}^{N-1} \gamma_k e^{-jk(\omega_p - \psi)} \sum_{n=1}^{N} x(n)e^{-jn\psi} \sum_{n=1}^{N} x^*(n)e^{j\tilde{n}\psi} \, d\psi
= \sum_{k=-(N-1)}^{N-1} \sum_{n=1}^{N} \sum_{\tilde{n}=1}^{N} \gamma_k x(n)x^*(\tilde{n}) \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\psi(k-n+\tilde{n})} \, d\psi \right\} e^{-j\omega_p k}. \tag{2–28}
\]

It is easy to verify that

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\psi(k-n+\tilde{n})} \, d\psi = \delta_{k-(n-\tilde{n})}. \tag{2–29}
\]

Thus

\[
\tilde{\Phi}(\omega_p) = \sum_{n=1}^{N} \sum_{\tilde{n}=1}^{N} \gamma_{n-\tilde{n}} x(n)x^*(\tilde{n})e^{-j\omega_p(n-\tilde{n})} = \tilde{x}_p^H (\gamma_0 \Gamma) \tilde{x}_p, \tag{2–30}
\]

where

\[
\tilde{x}_p = \begin{bmatrix} x(1)e^{-j\omega_p} & x(2)e^{-j2\omega_p} & \cdots & x(N)e^{-jN\omega_p}\end{bmatrix}^T \tag{2–31}
\]

and \( \Gamma \) is defined in (2–25). Therefore the WISL metric in (2–23) can be written as

\[
\text{WISL} = \frac{\gamma_0^2}{2N} \sum_{p=1}^{2N} \left[ \tilde{x}_p^H \Gamma \tilde{x}_p - N \right]^2. \tag{2–32}
\]

This expression suggests that the following problem can be expected to be “almost equivalent” to the minimization of the WISL metric:

\[
\min_{\{x(n)\}_{n=1}^{N},\{\alpha_p\}_{p=1}^{2N}} \sum_{p=1}^{2N} \| C \tilde{x}_p - \alpha_p \|^2 \tag{2–33}
\]
\[
\text{s.t. } \|\alpha_p\|^2 = N, \quad p = 1, \ldots, 2N,
\]
\[
|x(n)| = 1, \quad n = 1, \ldots, N,
\]
where “s.t.” stands for “subject to” and the \(N \times N\) matrix \(C\) is a square root of \(\Gamma\), i.e., \(\Gamma = C^T C\).

A cyclic algorithm for (2–33), which we will call WeCAN, can be derived as follows. For given \(\{x(n)\}_{n=1}^{N}\), (2–33) decouples into \(2N\) independent problems each of which has the following form:

\[
\min_{\alpha_p} \|f_p - \alpha_p\|^2 \quad \text{s.t. } \|\alpha_p\|^2 = N
\]

where the \(N \times 1\) vector \(f_p = C\tilde{x}_p\) is given. Note that under the constraint \(\|\alpha_p\|^2 = N\) we have

\[
\|f_p - \alpha_p\|^2 = \text{const} - 2 \Re \{f_p^H \alpha_p\} \\
\geq \text{const} - 2\|f_p\|\|\alpha_p\| = \text{const} - 2N\|f_p\|,
\]

where the equality is achieved if and only if

\[
\alpha_p = \sqrt{N} \frac{f_p}{\|f_p\|}.
\]

This is therefore the solution to the minimization problem in (2–33) for given \(\{x(n)\}_{n=1}^{N}\). Note that the computation of \(\{f_p\}_{p=1}^{2N}\) can be done by means of the FFT. Indeed, let \(c_{kn}\) denote the \((k, n)\)th element of \(C\) and define

\[
z_k = \begin{bmatrix} c_{k1} \cdot x(1) & c_{k2} \cdot x(2) & \cdots & c_{kN} \cdot x(N) & 0 & \cdots & 0 \end{bmatrix}^T_{(2N \times 1)}
\]

and

\[
F = \sqrt{2N} A^H \cdot \begin{bmatrix} z_1 & z_2 & \cdots & z_N \end{bmatrix}_{2N \times N}
\]
where the unitary $2N \times 2N$ FFT matrix $A^H$ has been defined in (2–14). Then it is not difficult to see that the transpose of the vector $f_p$ is given by the $p^{th}$ row of $F$.

Next we show that, for given $\{\alpha_p\}_{p=1}^{2N}$, the minimization problem in (2–33) with respect to $\{x(n)\}_{n=1}^{N}$ also has a closed-form solution. Let $\alpha_{pk}$ denote the $k^{th}$ element of $\alpha_p$ and let $a_p^H$ be given by (2–13). Using this notation, the criterion in (2–33) can be written as

$$
\sum_{p=1}^{2N} \|C\vec{x}_p - \alpha_p\|^2 = \sum_{k=1}^{N} \sum_{p=1}^{2N} |a_p^H z_k - \alpha_{pk}|^2
$$

$$
= \sum_{k=1}^{N} \|A^H z_k - \beta_k\|^2 = \sum_{k=1}^{N} \|z_k - A\beta_k\|^2,
$$

where

$$
\beta_k = \frac{1}{\sqrt{2N}} \begin{bmatrix} \alpha_{1k} & \alpha_{2k} & \cdots & \alpha_{2N,k} \end{bmatrix}^T, \quad k = 1, \ldots, N.
$$

(2–40)

For a generic element of $\{x(n)\}_{n=1}^{N}$, denoted as $x$, (2–39) becomes

$$
\sum_{k=1}^{N} |\mu_k x - \nu_k|^2 = \text{const} - 2 \text{Re} \left[ \left( \sum_{k=1}^{N} \mu_k^* \nu_k \right) x^* \right],
$$

(2–41)

where $\mu_k$ and $\nu_k$ are given by the corresponding elements in $z_k$ and $A\beta_k$, respectively.

Under the unimodular constraint, the minimizer $x$ of the criterion in (2–41) is given by

$$
x = e^{j\phi}, \quad \phi = \arg \left( \sum_{k=1}^{N} \mu_k^* \nu_k \right).$$

(2–42)

This observation concludes the derivation of the main steps of the WeCAN algorithm, whose summary is as follows.

The WeCAN algorithm:

- **Step 0**: Set the $\{x(n)\}_{n=1}^{N}$ to some initial values and select the desired weights $\{\gamma_k\}_{k=1}^{N-1}$; also choose $\gamma_0$ such that the matrix $\Gamma$ in (2–25) is positive semidefinite.

- **Step 1**: Compute the $\{\alpha_p\}_{p=1}^{2N}$ that minimize the criterion in (2–33) for $\{x(n)\}_{n=1}^{N}$ fixed at their most recent values (Eq. (2–36)).
• Step 2: Compute the sequence \( \{x(n)\}_{n=1}^{N} \) that minimizes the criterion in (2–33) for \( \{\alpha_p\}_{p=1}^{2N} \) fixed at their most recent values (Eq. (2–42)).

• Iteration: Repeat Steps 1 and 2 until a pre-specified stop criterion is satisfied.

2.3 Numerical Examples

2.3.1 Integrated Sidelobe Level (ISL) Design

We compare the merit factors of the Golomb sequence [39], of the Frank sequence [40], and of the CAN sequence initialized by one of these two types of sequences (denoted as CAN(G) and CAN(F), respectively). The definitions of the Golomb and Frank sequences are given in (1–20) and (1–22), respectively. (Note that the Golomb or Frank sequences can be easily computed for any value of \( N \) of possible practical interest, with the only restriction that \( N \) must be a perfect square for the Frank sequence.) We compute the merit factors of the above four types of sequences (Golomb, Frank, CAN(G) and CAN(F)) for the following lengths: \( N = 3^2, 5^2, 10^2, 15^2, 20^2, 30^2, 70^2 \) and \( 100^2 \). The results are shown in Figure 2-1 using a log-log scale. For all sequence lengths we consider, the CAN(G) and CAN(F) sequences give nearly the same merit factors; both are much larger than the merit factors given by the Golomb or Frank sequence. When \( N = 10^4 \), the CAN(G) sequence provides the largest merit factor of 1839.76, which is more than ten times larger than that given by the Golomb sequence (which is 157.10). We also show the correlation levels of the Golomb and CAN(G) sequences of lengths \( N = 10^2 \) and \( 10^3 \) in Figures 2-2 and 2-3, respectively. The correlation level is defined as

\[
\text{correlation level} = 20 \log_{10} \left| \frac{r(k)}{r(0)} \right|, \quad k = 1, \ldots, N - 1.
\]

(2–43)

We note that the correlation sidelobes of the Golomb sequence are comparatively large for \( k \) close to 0 and \( N - 1 \) (the same is true for the Frank sequence), while the CAN(G) sequence has relatively more uniform correlation sidelobes as \( k \) increases from 0 to \( N - 1 \).
2.3.2 Weighted Integrated Sidelobe Level (WISL) Design

Consider the design of a sequence of length $N = 100$. Suppose that we are interested in suppressing the correlations $r_1, \ldots, r_{25}$ and $r_{70}, \ldots, r_{79}$. We apply the WeCAN algorithm with the following weights used in the matrix $\Gamma$ in (2–25):

$$\gamma_k = \begin{cases} 
1, & k \in [1, 25] \cup [70, 79] \\
0, & k \in [26, 69] \cup [80, 99]
\end{cases}.$$  

(2–44)

($\gamma_0$ is chosen to ensure the positive semi-definiteness of $\Gamma$; more exactly we choose $\gamma_0 = 12.05$ following the discussion right after Eq. (2–25).) In this scenario, the MMF is as defined in Eq. (2–4) with

$$w_k = \gamma_k^2 = \begin{cases} 
1, & k \in [1, 25] \cup [70, 79] \\
0, & k \in [26, 69] \cup [80, 99]
\end{cases}.$$  

(2–45)

A randomly generated sequence is used to initialize the WeCAN algorithm. The correlation level of the designed sequence is shown in Figure 2-4. The WeCAN sequence has correlation sidelobes that are practically zero at the required lags, and which are much smaller than the sidelobes of the Golomb or CAN(G) sequence in the last subsection (Figure 2-2A and 2-2B). Table 2-1 presents the corresponding MMF values. The MMF of the WeCAN sequence (which is practically infinite) is significantly larger than the other MMF values in the table.

2.3.3 Channel Estimation

Consider an FIR (finite impulse response) channel impulse response $\{h_p\}_{p=0}^{P-1}$ whose estimation is our main goal (the number of channel taps $P$ is assumed to be known). Suppose that we transmit a probing sequence $\{x(n)\}_{n=1}^{N}$ and obtain the received signal

$$y_n = \sum_{p=0}^{P-1} h_p x(n - p) + e_n, \quad n = 1, \ldots, N + P - 1,$$  

(2–46)
where \( \{e_n\}_{n=1}^{N+P-1} \) is an i.i.d. complex Gaussian white noise sequence with zero mean and variance \( \sigma^2 \). Eq. (2–46) can be written in the following more compact form:

\[
y = \bar{X}h + e
\]

where \( \bar{X} \) is defined as

\[
\bar{X} = \begin{bmatrix}
x(1) & 0 \\
\vdots & \ddots \\
\vdots & 0 & x(1) \\
x(N) & \vdots \\
0 & \cdots & x(N)
\end{bmatrix}_{(N+P-1) \times P}
\]

\[y = \begin{bmatrix} y_1 & \cdots & y_{N+P-1} \end{bmatrix}^T, \quad h = \begin{bmatrix} h_0 & \cdots & h_{P-1} \end{bmatrix}^T, \quad e = \begin{bmatrix} e_1 & \cdots & e_{N+P-1} \end{bmatrix}^T.
\]

Let \( \bar{x}_p \) denote the \( p^{th} \) column of the matrix \( \bar{X} \). We use \( \bar{x}_p \) as a “matched filter” to determine \( h_p \) from \( y \), which leads to the following estimate of \( h_p \):

\[
\hat{h}_p = \frac{1}{N} \bar{x}_p^* y.
\]

Let the number of channel taps be \( P = 40 \). Figure 2-5A shows the magnitude of the simulated channel impulse response \( \{|h_p|\}_{p=0}^{P-1} \). We perform two experiments to compare the Golomb sequence and the WeCAN sequence. The latter is generated with the following weights

\[
\gamma_k = \begin{cases} 
1, & k \in [1, 39] \\
0, & k \in [40, N - 1] 
\end{cases}
\]
\[ w_k = \gamma_k^2, \quad k = 1, \ldots, N - 1, \]  

(2–51)

(and as usual, \( \gamma_0 \) is selected such that \( \Gamma \geq 0 \)) and its auto-correlation level is shown in Fig. 2-5B for the case of \( N = 100 \). In one experiment the noise power \( \sigma^2 \) is fixed at \( 10^{-4} \) and the sequence length \( N \) is varied from 100 to 500; in the other experiment \( N \) is fixed at 200 and \( \sigma^2 \) is varied from \( 10^{-6} \) to 1. For each pair \( (N, \sigma^2) \), 500 Monte-Carlo trials are run (in which the noise sequence \( e \) is varied) and the mean-squared error (MSE) of \( \hat{h} \) is recorded. Figure 2-6 shows the MSE of \( \hat{h} \) in the two situations. Due to better autocorrelation properties, the WeCAN sequence generates consistently smaller MSE than the Golomb sequence. In particular, it is interesting to observe from Figure 2-6B that as \( \sigma^2 \) decreases, the MSE of \( \hat{h} \) corresponding to the WeCAN sequence is decreasing linearly (and it becomes 0 as \( \sigma^2 \) goes to 0), while the performance of the Golomb sequence is limited to a certain level because of its non-zero correlation sidelobes, which induce an estimation bias.

![Figure 2-1. The merit factors of the Golomb, Frank, CAN(G) and CAN(F) sequences of lengths from \( 3^2 \) up to \( 100^2 \).](image-url)
Figure 2-2. Correlation levels of the Golomb and CAN sequences of length $N = 10^2$ designed under the ISL metric. A) The Golomb sequence, $N = 10^2$ and B) the CAN(G) sequence, $N = 10^2$.

Table 2-1. MMF values for the weights in (2–45) and $N = 100$

<table>
<thead>
<tr>
<th></th>
<th>Golomb</th>
<th>CAN(G)</th>
<th>WeCAN</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMF</td>
<td>32.55</td>
<td>142.64</td>
<td>$1.06 \times 10^{21}$</td>
</tr>
</tbody>
</table>
Figure 2-3. Correlation levels of the Golomb and CAN sequences of length $N = 10^3$ designed under the ISL metric. A) the Golomb sequence, $N = 10^3$ and B) the CAN(G) sequence, $N = 10^3$. 
Figure 2-4. The correlation level of a WeCAN sequence designed under the WISL metric with weights in (2–45).
Figure 2-5. The simulated channel impulse response and the probing WeCAN sequence. A) The magnitude of the simulated channel impulse response $h$. B) The correlation level of the WeCAN sequence designed under the WISL metric with weights in (2–51).
Figure 2-6. The MSE of the estimated $\hat{h}$ using two training sequences: the Golomb sequence and the WeCAN sequence. A) The noise power $\sigma^2$ is fixed at $10^{-4}$ and the sequence length $N$ is varied from 100 to 500. B) $N$ is fixed at 200 and $\sigma^2$ is varied from $10^{-6}$ to 1.
CHAPTER 3  
SEQUENCE SET DESIGN

Chapter 2 deals with designing a single sequence with good auto-correlation properties. In a similar way, in many applications a set of sequences that have good correlation properties can be desired, such as in MIMO (multi-input and multi-output) radar and CDMA (code division multiple access) systems. For example, when transmitting orthogonal waveforms, a MIMO radar system can achieve a greatly increased virtual aperture compared to its phased-array counterpart. This increased virtual aperture enables many of the MIMO radar advantages, such as better detection performance [41], improved parameter identifiability [42], enhanced resolution [43] and direct applicability of adaptive array techniques [44].

In the case of waveform set design, which is also referred to as multi-waveform design, both auto- and cross-correlations are involved. Good auto-correlation means that a transmitted waveform is nearly uncorrelated with its own time-shifted versions, while good cross-correlation indicates that any one of the transmitted waveform is nearly uncorrelated with other time-shifted transmitted waveforms. Good correlation properties in the above sense reduce the risk that the received signal of interest is drawn in correlated multipath or clutter interferences.

There is an extensive literature about multi-waveform design. In [18] and [30], orthogonal waveforms are designed with good auto- and cross-correlation properties, a topic that is directly tied to this Chapter. More related to Chapter 7, [45] and [46] focus on optimizing the covariance matrix of the transmitted waveforms to achieve a given transmit beampattern and in [32] the waveforms are designed to approximate a given covariance matrix. Other works include [47][48] and [49], where some prior information is assumed (e.g., the target impulse response) and the waveforms are designed to optimize a statistical criterion (e.g., the mutual information between the target impulse response and the reflected signals). We also note that in the area of multiple access
wireless communications, the spreading sequence design basically addresses the same problem of synthesizing waveforms with good auto- and cross-correlation properties [50].

Let \( \{x_m(n)\} \) \( (m = 1, \ldots, M \) and \( n = 1, \ldots, N) \) denote a set of \( M \) sequences, each of which is of length \( N \). The (aperiodic) cross-correlation of \( \{x_m(k)\}_{k=1}^N \) and \( \{x_m(k)\}_{k=1}^N \) at lag \( n \) is defined as

\[
\begin{align*}
    r_{m_1m_2}(n) &= \sum_{k=n+1}^{N} x_{m_1}(k)x_{m_2}^*(k-n) = r_{m_2m_1}^*(-n), \\
    m_1, m_2 &= 1, \ldots, M, \quad n = 0, \ldots, N - 1.
\end{align*}
\]

When \( m_1 = m_2 \), \((3–1)\) becomes the auto-correlation of \( \{x_m(k)\}_{k=1}^N \). Extending the approaches in Chapter 2, we present in this chapter cyclic algorithms for multi-sequence design. The first algorithm is called Multi-CAN, which aims to minimize all correlation sidelobes. The second algorithm is called Multi-WeCAN, which focuses on minimizing the correlation sidelobes in a certain time lag interval.

### 3.1 The Multi-CAN Algorithm

The Multi-CAN algorithm aims to minimize the following metric

\[
E = \sum_{m=1}^{M} \sum_{n=-N+1, n\neq 0}^{N-1} |r_{mm}(n)|^2 + \sum_{m_1=1}^{M} \sum_{m_2=1, m_2\neq m_1}^{M} \sum_{n=-N+1}^{N-1} |r_{m_1m_2}(n)|^2. \tag{3–2}
\]

To facilitate the discussion, denote the matrix of the transmitted waveforms by

\[
X = \begin{bmatrix} x_1 & x_2 & \cdots & x_M \end{bmatrix}_{N \times M} \tag{3–3}
\]

where

\[
x_m = \begin{bmatrix} x_m(1) & x_m(2) & \cdots & x_m(N) \end{bmatrix}^T \tag{3–4}
\]
is the \( m \)-th waveform. The waveform covariance matrices for different time lags are given by

\[
R_n = \begin{bmatrix}
    r_{11}(n) & r_{12}(n) & \cdots & r_{1M}(n) \\
    r_{21}(n) & r_{22}(n) & \cdots & r_{2M}(n) \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{M1}(n) & \cdots & \cdots & r_{MM}(n)
\end{bmatrix}, \quad n = -N + 1, \ldots, 0, \ldots, N - 1. \tag{3–5}
\]

By using the following “shifting matrix”

\[
J_n = \begin{bmatrix}
    1 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 1 & 0
\end{bmatrix}_{N \times N}
\]

the \( R_n \) in (3–5) can be rewritten as:

\[
R_n = (X^H J_n X)^T = R_n^H, \quad n = 0, \ldots, N - 1. \tag{3–6}
\]

With the above notation, the criterion in (3–2) can be written more compactly as

\[
\mathcal{E} = \|R_0 - NI_M\|^2 + 2 \sum_{n=1}^{N-1} \|R_n\|^2 = \sum_{n=-N}^{N-1} \|R_n - NI_M \delta_n\|^2. \tag{3–8}
\]

The following Parseval-type equality holds true (the proof is similar to that for the case of \( M = 1 \) in Chapter 2):

\[
\sum_{n=-(N-1)}^{N-1} \|R_n - NI_M \delta_n\|^2 = \frac{1}{2N} \sum_{p=1}^{2N} \|\Phi(\omega_p) - NI_M\|^2, \tag{3–9}
\]

where

\[
\Phi(\omega) \triangleq \sum_{n=-N+1}^{N-1} R_n e^{-j\omega n} \tag{3–10}
\]
is the spectral density matrix of the vector sequence $[x_1(n) \ldots x_M(n)]^T$ and

$$\omega_p = \frac{2\pi}{2N} p, \quad p = 1, \ldots, 2N. \quad (3-11)$$

The $\Phi(\omega)$ defined in (3-10) can be written in the following “periodogram-like” form [38]:

$$\Phi(\omega) = \tilde{y}(\omega)\tilde{y}^H(\omega) \quad (3-12)$$

where

$$\tilde{y}(\omega) = \sum_{n=1}^{N} y(n)e^{-j\omega n}, \quad y(n) = [x_1(n) \ x_2(n) \ \cdots \ x_M(n)]^T. \quad (3-13)$$

It follows from (3–9) and (3–12) that (3–8) can be rewritten as

$$\mathcal{E} = \frac{1}{2N} \sum_{p=1}^{2N} \|\bar{y}_p\tilde{y}_p^H - NI_M\|^2. \quad \left(\bar{y}_p \triangleq \tilde{y}(\omega_p)\right) \quad (3-14)$$

**Remark:** The $\mathcal{E}$ in (3–14) cannot be made very small, even without the unit-modulus constraint on the elements of $X$, because the rank 1 matrix $\bar{y}_p\tilde{y}_p^H$ cannot approximate well a full rank matrix $NI$.

Eq. (3–14) is a quartic (i.e., fourth-order) function of the unknowns $\{x_m(n)\}_{m=1,n=1}^{M,N}$. To get a simpler quadratic criterion function of $\{x_m(n)\}$, note that

$$\mathcal{E} = \frac{1}{2N} \sum_{p=1}^{2N} \|\bar{y}_p\tilde{y}_p^H - NI\|^2 = \frac{1}{2N} \sum_{p=1}^{2N} \text{tr} \left[ (\bar{y}_p\tilde{y}_p^H - NI)(\bar{y}_p\tilde{y}_p^H - NI)^H \right]$$

$$= \frac{1}{2N} \sum_{p=1}^{2N} \left( \|\bar{y}_p\|^4 - 2N\|\bar{y}_p\|^2 + N^2M \right)$$

$$= 2N \sum_{p=1}^{2N} \left( \frac{\|\bar{y}_p\|}{\sqrt{2N}} \right)^2 - \frac{1}{2} + N^2(M - 1). \quad (3-15)$$

Instead of minimizing (3–15) with respect to $X$, we consider the following minimization problem:

$$\min_{x_p(\alpha_p)} \sum_{p=1}^{2N} \left\| \frac{1}{\sqrt{2N}}\bar{y}_p - \alpha_p \right\|^2 \quad (3-16)$$
\[ \text{s.t. } |x_m(n)| = 1, \quad m = 1, \ldots, M \text{ and } n = 1, \ldots, N \]
\[ \| \alpha_p \|^2 = \frac{1}{2}, \quad p = 1, \ldots, 2N \quad (\alpha_p \text{ is } M \times 1) \]

where “s.t.” stands for “subject to”, and \( \{ \alpha_p \} \) are auxiliary variables. Evidently, if (3–15) (without the constant term \( N^2(M - 1) \)) can be made equal to zero (or “small”) by choosing \( X \), so can (3–16), and vice versa. To solve the minimization problem in (3–16), define

\[
\begin{align*}
\mathbf{a}^H_p &= \left[ e^{-j\omega_p} \ldots e^{-j2N\omega_p} \right], \quad \mathbf{A} = \frac{1}{\sqrt{2N}} \begin{bmatrix} a_1 & \ldots & a_{2N} \end{bmatrix}, \\
\mathbf{X} &= \begin{bmatrix} X \\ 0 \end{bmatrix}_{2N \times M}, \quad \mathbf{V} = \begin{bmatrix} \alpha_1 & \ldots & \alpha_{2N} \end{bmatrix}^T.
\end{align*}
\]

Then it is not difficult to observe that

\[
\sum_{p=1}^{2N} \left\| \frac{1}{\sqrt{2N}} \mathbf{y}_p - \alpha_p \right\|^2 = \| \mathbf{A}^H \mathbf{X} - \mathbf{V} \|^2 = \| \mathbf{X} - \mathbf{AV} \|^2.
\] (3–18)

(The second equality in (3–18) follows from the fact that \( \mathbf{A} \) is unitary.) The criterion in (3–18) can be minimized by means of two iterative (cyclic) steps. For given \( \mathbf{X} \) (i.e., \( X \) is given), the minimizer \( \{ \alpha_p \}_{p=1}^{2N} \) of (3–18) is given by

\[
\alpha_p = \frac{1}{\sqrt{2} \| \mathbf{c}_p \|} \mathbf{c}_p, \quad p = 1, \ldots, 2N
\] (3–19)

where

\[
\mathbf{c}_p^T = \text{the } p^{th} \text{ row of } (\mathbf{A}^H \mathbf{X}).
\] (3–20)

For given \( \mathbf{V} \) (i.e., \( \{ \alpha_p \}_{p=1}^{2N} \) are given), the minimizer \( \{ x_m(n) \} \) of (3–18) is given by

\[
x_m(n) = \exp(j \arg(d_{nm})), \quad m = 1, \ldots, M \text{ and } n = 1, \ldots, N
\] (3–21)
where
\[ d_{nm} = \text{the } (n, m)\text{th element of } (AV). \] (3–22)

The Multi-CAN algorithm thus obtained is summarized in below:

- **Step 0**: Initialize \( X \) by a randomly generated \( N \times M \) matrix or by some good existing sequences.
- **Step 1**: Fix \( \tilde{X} \) and compute \( V \) according to (3–19).
- **Step 2**: Fix \( V \) and compute \( \tilde{X} \) according to (3–21).
- **Iteration**: Repeat Steps 1 and 2 until a pre-specified stop criterion is satisfied, e.g., \( \|X^{(i)} - X^{(i+1)}\| < 10^{-3} \), where \( X^{(i)} \) is the waveform matrix obtained at the \( i \text{th} \) iteration.

Note that the \( A^H \tilde{X} \) in (3–20) is the FFT of each column of \( \tilde{X} \) and that the \( AV \) in (3–22) is the IFFT of each column of \( V \). Because of these (I)FFT-based computations, the Multi-CAN algorithm is quite fast. Indeed, it can be used to design long sequences up to \( N \sim 10^3 \) and \( M \sim 10 \), which can hardly be handled by other algorithms suggested in the previous literature.

### 3.2 The Multi-WeCAN Algorithm

In some radar applications like synthetic aperture radar (SAR) imaging, the transmitted pulse is relatively long (i.e., \( N \) is large) and the signals backscattered from objects in the near and far range bins overlap significantly [37]. In this case, only the waveform correlation properties in a certain lag interval around \( n = 0 \) are relevant to range resolution and a minimization criterion different from (3–8) is given by
\[
\tilde{E} = \|R_0 - NI_M\|^2 + 2 \sum_{n=1}^{P-1} \|R_n\|^2, \tag{3–23}
\]
where \( P - 1 \) is the maximum lag that we are interested in. More specifically, \( (P - 1)t_p \) (\( t_p \) is as mentioned in (1–1)) should be chosen larger than the maximum difference of round trip delays of the signals backscattered from near and far range bins.
Remark: Recall that it is not possible to make the criterion $\mathcal{E}$ in (3–14) very small. One way to understand this problem is to examine the criterion $\tilde{\mathcal{E}}$ defined in (3–23) where only $R_0, R_1, \ldots, R_{P-1}$ (which are complex-valued $M \times M$ matrices) are considered. $R_0$ is Hermitian with all diagonal elements equal to $N$, so setting $R_0 = NI$ leads to $M(M-1)$ (real-valued) equations. $R_1, \ldots, R_{P-1}$ do not have any special structure and setting them to zero adds $2M^2$ equations for each of them. Thus the total number of equations is $K = M(M - 1) + (P - 1)2M^2$. Compared to this, the number of variables that we can manipulate is $M(N - 1)$ (for each of the $M$ waveforms there are $N - 1$ free phases, as the initial phase does not matter). Therefore, a basic requirement for good performance is that $K \leq M(N - 1)$, which can be simplified to: $P \leq \frac{N + M}{2M}$.

Put differently, only when $P \leq \frac{N + M}{2M}$ is it possible in principle to design unimodular waveforms $X$ that make $\tilde{\mathcal{E}}$ zero; in other cases $\tilde{\mathcal{E}}$ or $\mathcal{E}$ cannot be made equal to zero.

The Multi-WeCAN (multi-sequence weighted-CAN) algorithm aims at minimizing the following criterion:

$$\hat{\mathcal{E}} = \gamma_0^2 \| R_0 - NI_M \|^2 + 2 \sum_{n=1}^{N-1} \gamma_n^2 \| R_n \|^2,$$

where $\{\gamma_n\}_{n=0}^{N-1}$ are real-valued weights. For instance, if we choose $\gamma_n = 1$ for $n = 0, \ldots, P - 1$ and $\gamma_n = 0$ otherwise, $\hat{\mathcal{E}}$ becomes the $\tilde{\mathcal{E}}$ defined in (3–23).

Similarly to (3–9), we can show that

$$\hat{\mathcal{E}} = \frac{1}{2N} \sum_{p=1}^{2N} \| \Phi(\omega_p) - \gamma_0 NI_M \|^2,$$

where $\{\omega_p\}_{p=1}^{2N}$ is given by (3–11) and

$$\Phi(\omega) \triangleq \sum_{n=-(N-1)}^{N-1} \gamma_n R_n e^{-j\omega n},$$
and where $\gamma_n = \gamma_{-n}$ for $n = 1, \ldots, N - 1$. To facilitate later developments, $\gamma_0$ is chosen such that the matrix

$$\Gamma = \begin{bmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_{N-1} \\
\gamma_1 & \gamma_0 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \gamma_1 \\
\gamma_{N-1} & \cdots & \gamma_1 & \gamma_0
\end{bmatrix}$$ (3–27)

is positive semi-definite (denoted as $\Gamma \succeq 0$). See the discussions following (2–25) in Chapter 2 for a way to determine $\gamma_0$. The condition $\Gamma \succeq 0$ is necessary because the matrix square root of $\Gamma$ is needed later on (Eq. (3–31)).

Similarly to (3–12), it can be shown that:

$$\Phi(\omega) = Z^T(\omega) \Gamma Z^*(\omega), \quad (3–28)$$

where

$$Z^T(\omega) = \begin{bmatrix} y(1)e^{-j\omega} & y(2)e^{-j\omega^2} & \cdots & y(N)e^{-j\omega N} \end{bmatrix}_{M \times N}. \quad (3–29)$$

By combining (3–25) and (3–28), the criterion becomes

$$\hat{\mathcal{E}} = \frac{1}{2N} \sum_{p=1}^{2N} \|Z_p^H \Gamma Z_p - \gamma_0N I_M\|^2 \quad (Z_p \triangleq Z(\omega_p)). \quad (3–30)$$

Instead of minimizing (3–30) with respect to $X$, we consider the following minimization problem (the discussion following (3–16)):

$$\min_{X, U} \sum_{p=1}^{2N} \|CZ_p - \sqrt{\gamma_0N} U_p\|^2, \quad (3–31)$$

s.t. $|x_m(n)| = 1, \quad m = 1, \ldots, M$ and $n = 1, \ldots, N$,

$$U_p^H U_p = I, \quad p = 1, \ldots, 2N, \quad (U_p \text{ is } N \times M)$$

where the $N \times N$ matrix $C$ is a square root of $\Gamma$ (i.e., $C^H C = \Gamma$).
The minimization problem in \((3–31)\) can be solved in a cyclic way as follows. For given \(\{Z_p\}_{p=1}^{2N}\) (i.e., \(X\) is given), \((3–31)\) decouples into \(2N\) independent problems, each of which can be written as

\[
\left\| CZ_p - \sqrt{\gamma_0} N U_p \right\|^2 = \text{const} - 2 \text{Re} \left\{ \text{tr} \left[ \sqrt{\gamma_0} N U_p Z^H_p C^H \right] \right\}, \quad p = 1, \ldots, 2N, \tag{3–32}
\]

where “const” denotes a term that is independent of the variable \(U_p\). Let

\[
Z^H_p C^H = U_1 \Sigma U_2^H
\tag{3–33}
\]

denote the economic SVD (singular value decomposition) of \(Z^H_p C^H\), where \(U_1\) is \(M \times M\), \(\Sigma\) is \(M \times M\) and \(U_2\) is \(N \times M\). Then the minimizer \(U_p\) of \((3–32)\), for fixed \(Z_p\), is given by (Eq. \((3–46)-(3–47)\) for the same optimization problem and Appendix B for the proof):

\[
U_p = U_2 U_1^H. \tag{3–34}
\]

Note that the computation of \(\{CZ_p\}_{p=1}^{2N}\) can be done by means of the FFT. To see this, let

\[
\tilde{X}_m = C^T \odot \begin{bmatrix} x_m & x_m & \cdots & x_m \end{bmatrix}_{N \times N}, \quad m = 1, \ldots, M, \tag{3–35}
\]

and

\[
F = \sqrt{2N} A^H \cdot \tilde{F}, \quad \tilde{F} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 & \cdots & \tilde{X}_M \\ 0_{N \times N} & 0_{N \times N} & \cdots & 0_{N \times N} \end{bmatrix}_{2N \times NM}, \tag{3–36}
\]

where \(A\) has been defined in \((3–17)\). Let \(f_p^T\) denote the \(p^\text{th}\) row of \(F\). We divide the \(NM \times 1\) vector \(f_p\) equally into \(M\) pieces, which we can observe that correspond to the \(M\) columns (from left to right) of \(CZ_p\). Thus the matrix \(CZ_p\) can be obtained from \(F\), which is calculated from \(\sqrt{2N} A^H \tilde{F}\), an FFT operation.
For given \( \{ U_p \}_{p=1}^{2N} \), the minimization problem in (3–31) also has a closed-form solution with respect to \( X \). Let

\[
G_{2N \times NM} = \begin{bmatrix} g_1 & g_2 & \cdots & g_{2N} \end{bmatrix}^T,
\]

where \( g_p \) denotes the \( NM \times 1 \) vector given by the columns of \( \sqrt{\gamma_0 N} U_p \) stacked on top of each other. Then the criterion in (3–31) can be written as

\[
\sum_{p=1}^{2N} \left| CZ_p - \sqrt{\gamma_0 N} U_p \right|^2 = \left| \sqrt{2N} A^H \tilde{F} - G \right|^2
\]

\[
= 2N \left| \tilde{F} - \frac{1}{\sqrt{2N}} AG \right|^2.
\]

The above function can be minimized with respect to each element of \( \{ x_m(n) \}_{m=1,n=1}^{M,N} \) separately. Let \( x \) denote a generic element of \( \{ x_m(n) \} \). Then the corresponding problem is to minimize the following criterion with respect to \( x \):

\[
\sum_{k=1}^{N} \left| \mu_k x - \nu_k \right|^2 = \text{const} - 2 \Re \left[ \left( \sum_{k=1}^{N} \mu_k^* \nu_k \right) x^* \right],
\]

where \( \{ \mu_k \}_{k=1}^{N} \) are given by the elements of \( \tilde{F} \) which contain \( x \), and \( \{ \nu_k \}_{k=1}^{N} \) are given by the elements of \( \frac{1}{\sqrt{2N}} AG \) whose positions are the same as those of \( \{ \mu_k \}_{k=1}^{N} \) in \( \tilde{F} \). (More specifically, for \( k = 1, \ldots, N \), \( \mu_k \) is given by the \( (k,n)^{th} \) element of \( C \) and \( \nu_k \) is given by the \( (n,(m-1)N+k)^{th} \) element of \( \frac{1}{\sqrt{2N}} AG \).) Under the unit-modulus constraint, the minimizer \( x \) of the criterion in (3–39) is given by

\[
x = e^{j\phi}, \quad \phi = \arg \left( \sum_{k=1}^{N} \mu_k^* \nu_k \right).
\]

The Multi-WeCAN algorithm follows from the above discussions and it is summarized below:

- **Step 0:** Initialize \( X \) and select the desired weights \( \{ \gamma_n \}_{n=0}^{N-1} \) such that the matrix \( \Gamma \) in (3–27) is positive semi-definite.
- **Step 1:** Fix \( \{ Z_p \}_{p=1}^{2N} \) (i.e., \( X \) is given) and compute \( \{ U_p \}_{p=1}^{2N} \) according to (3–34).
Step 2: Fix \( \{ U_p \}_{p=1}^{2N} \) and compute \( X \) according to (3–40).

Iteration: Repeat Steps 1 and 2 until a pre-specified stop criterion is satisfied, e.g., \( \| X^{(i)} - X^{(i+1)} \| < \epsilon \), where \( X^{(i)} \) is the waveform matrix obtained at the \( i \)th iteration.

Multi-WeCAN is not so computationally efficient as Multi-CAN, but it can still be used for relatively large values of \( N \) and \( M \), up to \( N \sim 10^3 \) and \( M \sim 10 \).

3.3 The Multi-CA-Original (Multi-CAO) Algorithm

The original cyclic algorithm (CA) for waveform design proposed in [32, 37] aims at minimizing a particular form of the criterion \( \hat{\mathcal{E}} \) in (3–24):

\[
\hat{\mathcal{E}}_{CAO} = P \| R_0 - N I_M \|^2 + 2 \sum_{n=1}^{P-1} (P-n) \| R_n \|^2,
\]

which can be obtained from (3–24) by choosing the weights \( \gamma_n^2 = P - n \) for \( n = 0, \ldots, P - 1 \) and \( \gamma_n^2 = 0 \) otherwise. We refer to this original CA algorithm as Multi-CAO (multi-sequence CA original) to make the naming consistent with that of Multi-CAN and Multi-WeCAN.

The above choice of \( \{ \gamma_n \}_{n=0}^{N-1} \) results from the following problem formulation that is simple and direct. Consider the matrix

\[
\hat{X} = \begin{bmatrix}
X_1 & \cdots & X_M
\end{bmatrix}^{(N+P-1)\times MP}
\]

where

\[
\hat{X}_m = \begin{bmatrix}
x_m(1) \\
\vdots \\
x_m(N) \\
\vdots \\
\vdots \\
x_m(N)
\end{bmatrix}^{(N+P-1)\times P}, \quad m = 1, \ldots, M.
\]
Then it is easy to observe that the $\hat{\mathcal{E}}_{CAO}$ defined in (3–41) can be expressed as

$$\hat{\mathcal{E}}_{CAO} = \| \hat{X}^H \hat{X} - N I_{MP} \|^2. \tag{3–44}$$

The minimization of (3–44) can be tackled by solving the following problem

$$\min_{\hat{X}, U} \| \hat{X} - \sqrt{N} U \|^2; \tag{3–45}$$

s.t. $|x_m(n)| = 1, \quad m = 1, \ldots, M$ and $n = 1, \ldots, N, \quad U^H U = I, \quad (U \text{ is } (N + P - 1) \times MP).$

Note that the problem of minimizing (3–44) and that of (3–45) are “almost equivalent”, and so is the relationship between (3–15) and (3–16), as well as (3–30) and (3–31). See the discussion following (2–12) in Chapter 2 as well as Appendix A for more information.

Regarding (3–45), we note the following facts. For given $\hat{X}$, let

$$\hat{X}^H = U_1 S U_2^H \tag{3–46}$$

denote the economic SVD (singular value decomposition) of $\hat{X}$. Here $U_1$ is an $MP \times MP$ unitary matrix, $U_2$ is an $(N + P - 1) \times MP$ semi-unitary matrix and $S$ is an $MP \times MP$ diagonal matrix. Then the solution $U$ of (3–45), for fixed $\hat{X}$, is given by (Appendix B)

$$U = U_2 U_1^H. \tag{3–47}$$

Next note that for fixed $U$, the minimization of the criterion in (3–45) also has a simple closed-form solution. To see this, let $x$ denote an arbitrary element from $\{x_m(n)\}$. Then a generic form of the minimization problem in (3–45) with respect to $x$ is given by

$$\min_x \sum_{k=1}^{P} |x - \mu_k|^2, \tag{3–48}$$

where $\{\mu_k\}_{k=1}^{P}$ are the elements of the matrix $\sqrt{N} U$ whose positions are the same as the positions of $x$ in $\hat{X}$. More precisely, for $x = x_m(n)$ the corresponding sequence
\{\mu_k\}_{k=1}^P \text{ is given by the } [n - 1 + r, (m - 1)P + r]^\text{th} \text{ elements of } \sqrt{N}U, \text{ for } r = 1, \ldots, P.

Because \(|x| = 1\), the criterion in (3–48) can be rewritten as

\[
\sum_{k=1}^{P} |x - \mu_k|^2 = \text{const} - 2 \text{Re} \left[ x \sum_{k=1}^{P} \mu_k^* \right] = \text{const} - 2 \left| \sum_{k=1}^{P} \mu_k \right| \cdot \cos \left[ \arg(x) - \arg \left( \sum_{k=1}^{P} \mu_k \right) \right]
\]

where const denotes the term that does not depend on \(x\). Hence the minimizer \(x\) of the criterion in (3–45) is given by

\[
x = \exp \left\{ j \arg \left( \sum_{k=1}^{P} \mu_k \right) \right\}.
\]

The Multi-CAO algorithm for (3–45) follows from the above discussion and is summarized below:

- **Step 0**: Set the matrix \(\bar{X}\) to an initial value.
- **Step 1**: Fix \(\{x_m(n)\}\) (i.e., \(\bar{X}\) is given) and compute \(U\) according to (3–47).
- **Step 2**: Fix \(U\) and compute \(\{x_m(n)\}\) according to (3–50).
- **Iteration**: Repeat Steps 1 and 2 until a pre-specified stop criterion is satisfied.

The criterion minimized by Multi-CAO is a special case of that minimized by Multi-WeCAN. Multi-CAO and Multi-WeCAN provide similar performances in terms of correlation sidelobe suppression, though the derivation and iteration steps of Multi-CAO are relatively less involved.

### 3.4 Numerical Examples

#### 3.4.1 Multi-CAN

Consider minimizing the criterion \(\mathcal{E}\) in (3–8), i.e., minimizing all correlation sidelobes: \(r_{mm}(n)\) for all \(m\) and \(n \neq 0\), and \(r_{m_1m_2}(n)\) for all \(m_1 \neq m_2\) and \(n\). Suppose that the number of transmit sequences is \(M = 3\) and the length of each sequence is \(N = 40\). We compare the Multi-CAN sequence set with the CE (cross entropy) sequence set in [30]. We use randomly generated sequences to initialize Multi-CAN. 100 Monte-Carlo
trials are run (i.e., 100 random initializations) and the sequence set with the lowest
correlation sidelobe peak is kept. The 40-by-3 CE sequence set is given in Table 1 of
[30].

Figures 3-1, 3-2 and 3-3 show the correlations \((r_{11}, r_{12}, \ldots, r_{33},\) normalized by \(N)\)
of the Multi-CAN sequence set and CE sequence set. The CE sequence set is slightly
better than the Multi-CAN sequence set in terms of correlation sidelobe peaks. However,
our goal is to minimize \(E\) or equivalently the following normalized fitting error:

\[
E_{\text{norm}} = \frac{E}{MN^2} = \left( \| \mathbf{R}_0 - N \mathbf{I} \|^2 + 2 \sum_{n=1}^{N-1} \| \mathbf{R}_n \|^2 \right) / (MN^2).
\]

The Multi-CAN sequence set gives a fitting error of 2.00, whereas the CE sequence set
has a bigger fitting error equal to 2.23.

Note that although the Multi-CAN and CE sequence sets show comparable
performances (also comparable to the performance of other sequence sets like the
ones in [18]), the Multi-CAN algorithm works much faster than other existing algorithms,
because Multi-CAN is based on FFT computations. For the above parameter set
\((N = 40, M = 3)\), the Multi-CAN algorithm consumes less than one second on an
ordinary PC to complete one Monte-Carlo trial. The overall computation time is still short
if we run plenty of Monte-Carlo trials and pick up the best sequence set. Moreover, the
computation time of Multi-CAN grows roughly as \(O(MN \log N)\) so that Multi-CAN can
handle very large values of \(N\), up to \(N \sim 10^5\). In contrast, the Cross Entropy [30] or
Simulated Annealing based methods [18] are relatively involved and become impractical
for large values of \(N\). In fact, we were unable to find in the literature any code set that is
designed for good (aperiodic) correlations and at the same time is sufficiently long to be
comparable with what can be obtained using the Multi-CAN algorithm.

For relatively large values of \(N\), we decided to employ the Hadamard sequence set
for comparison [15], which is easy to generate (for virtually any length that is a power
of 2) and is frequently used in wireless communications. We scrambled the Hadamard
sequence set with a PN (pseudo-noise) sequence set to lower its correlation sidelobes. We compare the Multi-CAN sequence set (100 Monte-Carlo trials are run for each $N$ and the result with the lowest correlation sidelobe peak is shown) and the QPSK (quadrature phase-shift keying) Hadamard+PN sequence set for $M = 3$ and $N = 2^7, \ldots, 2^{13}$. Figures 3-4 and 3-5 compare the sequence sets in terms of three criteria: the auto-correlation sidelobe peak, the cross-correlation peak and the normalized fitting error (defined in (3–51)). The Multi-CAN sequence set outperforms the Hadamard+PN sequence set with respect to each criterion. In fact, the advantage of the Multi-CAN algorithm lies not only in the significant length and the low correlation sidelobes of the designed sequence sets, but also in the easy generation (using different initial conditions) of many sequence sets which are of the same $N$-by-$M$ dimension and all have reasonably low correlation sidelobes. These randomly distributed waveform sets are useful in some application areas, like for countering the coherent repeater jamming in radar systems [1, 18].

3.4.2 Multi-WeCAN

Consider minimizing the criterion $\tilde{\mathcal{E}}$ in (3–23), i.e., minimizing the correlation sidelobes for lags not larger than $P - 1$: $r_{mm}(n)$ for all $m$ and $1 \leq n \leq P - 1$, and $r_{m_1m_2}(n)$ for all $m_1 \neq m_2$ and $0 \leq n \leq P - 1$. Suppose that the number of transmit sequences is $M = 4$, the length of each sequence is $N = 256$ and the number of correlation lags we want to consider is $P = 50$. Similarly to (3–51), the normalized fitting error for this scenario is defined as

$$\tilde{\mathcal{E}}_{\text{norm}} = \frac{\tilde{\mathcal{E}}}{MN^2} = \left( \|R_0 - NI\|^2 + 2 \sum_{n=1}^{P-1} \|R_n\|^2 \right) / (MN^2). \quad (3–52)$$

We also define the correlation level as

$$\text{correlation level} = 20 \log \frac{\|R_n - NI\delta_n\|}{\sqrt{MN^2}}, \quad n = -N + 1, \ldots, 0, \ldots, N - 1 \quad (3–53)$$

which measures the correlation sidelobes for each time lag.
We compare the Multi-WeCAN algorithm and the Multi-CAO algorithm. We use a randomly generated unimodular sequence set to initialize both Multi-WeCAN and Multi-CAO. For Multi-WeCAN, we choose

\[
\gamma_n^2 = \begin{cases} 
1, & n \in [1, P - 1] \\
0, & n \in [P, N - 1] 
\end{cases} 
\]  

(3–54)

and \(\gamma_0\) is chosen to ensure that \(\Gamma > 0\) (more exactly we choose \(\gamma_0 = 25.5\)).

Table 3-1 compares the Multi-CAO sequence set and the Multi-WeCAN sequence set in terms of the auto-correlation sidelobe peak (in the considered lag interval), the cross-correlation peak (in the considered lag interval) and the \(\tilde{\mathcal{E}}_{\text{norm}}\) defined in (3–52). The 256 \times 4 Multi-CAN sequence set is also added in Table 3-1 for comparison. The Multi-WeCAN sequence set gives the lowest correlation sidelobe peak and fitting error. Figure 3-6 shows the correlation level of the Multi-CAO and Multi-WeCAN sequence sets. We observe from Figure 3-6 that the Multi-WeCAN sequence set provides a “uniformly low” correlation level in the required lag interval \([1, P - 1]\), while the correlation level of the Multi-CAO sequence set increases as the lag increases from 1 to \(P - 1\). This behavior is attributed to the fact that Multi-WeCAN makes use of uniform weights \(\{\gamma_n = 1\}_{n=1}^{P-1}\) in (3–54) whereas Multi-CAO implicitly assumes “uneven” weights \(\{\gamma_n = P - n\}_{n=1}^{P-1}\) (Eq. (3–41)), so the bigger the lag, the smaller the weight. We also note that the correlation level at \(n = 0\) for the Multi-WeCAN sequence set is very low (around \(-85\) dB). The reason is that we chose \(\gamma_0 = 25.5\), which is much larger than the other weights (Eq. (3–54)) and thus the “0-lag” correlation fitting error \(|R_0 - NI|\) is emphasized the most in the criterion \(\tilde{\mathcal{E}}\) in (3–24).
Consider using the Multi-WeCAN algorithm to minimize the criterion $\hat{E}$ in (3–24) with $N = 256$, $M = 4$ and the following weights:

$$
\gamma_n^2 = \begin{cases} 
1, & n \in [1, 19] \cup [236, 255] \\
0, & n \in [20, 235]
\end{cases}
$$

(3–55)

(as before, $\gamma_0$ is chosen to ensure the positive semi-definiteness of $\Gamma$ in (3–27)). We still use a randomly generated sequence to initialize Multi-WeCAN. In this scenario, the normalized fitting error is defined as $\hat{E}_{\text{norm}} = \hat{E}/(MN^2)$.

Table 3-2 compares the Multi-WeCAN sequence and the $256 \times 4$ Multi-CAN sequence. The Multi-WeCAN sequence provides much lower correlation sidelobe peaks and much smaller fitting error. Figure 3-7 shows the corresponding correlation levels of the Multi-CAN and Multi-WeCAN sequences, from which we see that Multi-WeCAN succeeds much better in suppressing the correlations at the required lags. Note that because $|r_{m_1 m_2}(N - 1)| = 1$ for all $m_1$ and $m_2$, the correlation level corresponding to the maximum lag $N - 1$ is always equal to $20 \log_{10}(\sqrt{M^2}/\sqrt{MN^2})$, which is $-42.14$ dB in this case (the end points in both Figures 3-7A and 3-7B).

### 3.4.4 Quantization Effects

We have assumed that the phases of the designed sequences can take on any values from 0 to $2\pi$. In practice it might be required that the phases are drawn from a discrete constellation. Thus we briefly examine here the performance of the designed sequences under quantization.

Let $\{x_m(n)\}_{m=1,n=1}^{M,N}$ denote the sequence set that is obtained from one of the algorithms discussed in this paper. Suppose that the quantization level is $2^q$ where $q \geq 1$ is an integer. Then the quantized sequence can be expressed as

$$
\hat{x}_m(n) = \exp \left\{ j \left[ \frac{\arg\{x_m(n)\}}{2\pi/2^q} \right] \frac{2\pi}{2^q} \right\}, \quad m = 1, \ldots, M \text{ and } n = 1, \ldots, N.
$$

(3–56)
We quantize the Multi-CAN sequence used in Figure 3-4 into 32 levels (i.e. \( q = 5 \)) and do the same comparisons with the Hadamard+PN sequence. The results are shown in Figure 3-8, from which we see that the curves representing the CAN sequence move up a little but they are still below the corresponding curves of the Hadamard+PN sequence (except for the point of \( N = 4096 \) in Figure 3-8B). We do not plot the fitting error here as was done in Figure 3-5, because the fitting error of the CAN sequence almost does not change after this 32-level quantization.

Similar situations occur if we quantize sequences generated from the other algorithms discussed in this chapter. In our test, the performance degradation (i.e. the correlation sidelobe increase) was quite limited provided that the quantization level was not very small (e.g. \( q \geq 6 \)).

### 3.4.5 Synthetic Aperture Radar (SAR) Imaging

Consider a MIMO radar angle-range imaging example (intra-pulse Doppler effects are assumed to be negligible) using uniform linear arrays with colocated \( M = 4 \) transmit and \( M_r = 4 \) receive antennas. The inter-element spacing of the transmit and receive antennas is equal to 2 and 0.5 wavelengths, respectively. Suppose that all possible targets are in a far field consisting of \( P = 30 \) range bins (which means that the maximum round trip delay difference within the illuminated scene is not larger than 29 subpulses) and a scanning angle area of \((-30, 30)\) degrees. The length of the probing waveform for each transmit antenna is \( N = 256 \).

Let \( \mathbf{X} \) denote the \( N \times M \) transmitted probing waveform matrix (Eq. (3–3)), and let

\[
\bar{\mathbf{X}} = \begin{bmatrix} \mathbf{X} \\ 0 \end{bmatrix}_{(N+P-1) \times M}
\]  

(3–57)
where $\mathbf{0}$ is a $(P - 1) \times M$ matrix of zeros. Then the $M_r \times (N + P - 1)$ received data matrix can be written as [44]

$$
\mathbf{D}^H = \sum_{p=0}^{P-1} \sum_{k=1}^{K} \alpha_{pk} a_k b_k^T \tilde{\mathbf{X}}^H \mathbf{J}_p + \mathbf{E}^H,
$$

(3–58)

where $\mathbf{J}_p$ is an $(N + P - 1) \times (N + P - 1)$ shifting matrix as defined in (3–6) (with the same structure but different dimension), $\mathbf{E}^H$ is the noise matrix whose columns are independent and identically distributed (i.i.d.) random vectors with mean zero and covariance matrix $\mathbf{Q}$, $\{\alpha_{pk}\}_{p=0, k=1}^{P-1, K}$ are complex amplitudes which are proportional to the radar-cross-sections (RCS) of the scatters, and $a_k$ and $b_k$ are the receive and transmit steering vectors, respectively, which are given by

$$
a_k = \begin{bmatrix} 1 & e^{-j\pi \sin(\theta_k)} & \ldots & e^{-j\pi (M_r - 1) \sin(\theta_k)} \end{bmatrix}^T,
$$

(3–59)

and

$$
b_k = \begin{bmatrix} 1 & e^{-j\pi M_r \sin(\theta_k)} & \ldots & e^{-j\pi (M-1) M_r \sin(\theta_k)} \end{bmatrix}^T,
$$

(3–60)

where $\{\theta_k\}_{k=1}^K$ are the scanning angles. Our goal is to estimate $\{\alpha_{pk}\}_{p=0, k=1}^{P-1, K}$ from the collected data $\mathbf{D}^H$.

First we apply the following matched filter to the data matrix $\mathbf{D}^H$:

$$
\tilde{\mathbf{X}}^\text{MF}_p \triangleq \mathbf{J}_p^H \tilde{\mathbf{X}}^H \tilde{\mathbf{X}} \mathbf{J}_p^{-1}, \quad (N + P - 1) \times M,
$$

(3–61)

(note that we assume $N + P - 1 \geq M$ and thus $\tilde{\mathbf{X}}^H \mathbf{J}_p \tilde{\mathbf{X}}^\text{MF}_p = \mathbf{I}_M$) to perform range compression for the $p^{th}$ range bin, i.e.,

$$
\tilde{\mathbf{D}}^H_p \triangleq \mathbf{D}^H \tilde{\mathbf{X}}^\text{MF}_p = \left( \sum_{q=0}^{P-1} \sum_{k=1}^{K} \alpha_{qk} a_k b_k^T \tilde{\mathbf{X}}^H \mathbf{J}_q \right) \tilde{\mathbf{X}}^\text{MF}_p + \mathbf{E}^H \tilde{\mathbf{X}}^\text{MF}_p
$$

(3–62)

$$
= \sum_{k=1}^{K} \alpha_{pk} a_k b_k^T \tilde{\mathbf{X}}^H \mathbf{J}_p \tilde{\mathbf{X}}^\text{MF}_p + \mathbf{Z}_p
$$

$$
= \sum_{k=1}^{K} \alpha_{pk} a_k \tilde{b}_k^H + \mathbf{Z}_p, \quad (\tilde{b}_k^H \triangleq b_k^T),
$$
where $Z_p = \left( \sum_{q=0}^{P-1} \sum_{k=1}^{K} \alpha_{pqk} a_k b_k^T \tilde{X}^H J_q \right) \tilde{X}_p^M + E^H \tilde{X}_p^M$. The filtered data in (3–62) leads naturally to the following least squares (LS) estimate of $\alpha_{pk}$:

$$\hat{\alpha}_{pk}^{\text{LS}} = \frac{a_k^H \hat{D}_p^H b_k}{\|a_k\|^2 \|b_k\|^2}, \quad k = 1, \ldots, K \text{ and } p = 0, \ldots, P - 1,$$

(3–63)

as well as to the following Capon estimate:

$$\hat{\alpha}_{pk}^{\text{Capon}} = \frac{a_k^H \hat{R}_p^{-1} \hat{D}_p^H b_k}{a_k^H \hat{R}_p^{-1} a_k \|b_k\|^2}, \quad k = 1, \ldots, K \text{ and } p = 0, \ldots, P - 1,$$

(3–64)

where $\hat{R}_p = \hat{D}_p^H \hat{D}_p$ denotes the covariance matrix of the “compressed” received data ([44] for more details about these estimates of $\alpha_{pk}$).

To obtain a larger synthetic aperture, we use the SAR principle and thus repeat the process of sending a probing waveform and collecting data at $\tilde{N} = 20$ different positions; the collected data matrices are denoted as $D_1^H, D_2^H, \ldots, D_{\tilde{N}}^H$ respectively. Suppose that two adjacent positions are spaced $\frac{MM}{2}$ wavelengths apart, which induces a phase shift of $\psi_k = -2\pi \frac{MM}{2} \sin(\theta_k)$ for both the transmit and receive steering vectors corresponding to the two adjacent positions. (As long as the “targets in the far-field” assumption holds, the distance between two adjacent positions can be chosen at will and can be different for different adjacent positions; we only need to change the phase shift $\psi_k$ accordingly.) In this case, we let

$$\hat{D}_p^H = \begin{bmatrix} D_1^H \tilde{X}_p^M & D_2^H \tilde{X}_p^M & \cdots & D_{\tilde{N}}^H \tilde{X}_p^M \end{bmatrix}_{M \times \tilde{N}M},$$

(3–65)

and

$$\hat{b}_k^H = \begin{bmatrix} b_k^T & b_k^T e^{j2\psi_k} & \cdots & b_k^T e^{j2(\tilde{N}-1)\psi_k} \end{bmatrix}_{1 \times \tilde{N}M}.$$ 

(3–66)

Using this notation, the expressions for the estimates of $\alpha_{pk}$ in (3–63) and (3–64) can be used mutatis mutandis.
Table 3-1. Comparison between Multi-CAN, Multi-CAO and Multi-WeCAN under $\hat{\epsilon}$ 
\[ (N = 256, M = 4, P = 50) \]
<table>
<thead>
<tr>
<th></th>
<th>Auto-corr sidelobe peak (dB)</th>
<th>Cross-corr peak (dB)</th>
<th>$\hat{\epsilon}_{\text{norm}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multi-CAN</td>
<td>-20.54</td>
<td>-18.19</td>
<td>0.91</td>
</tr>
<tr>
<td>Multi-CAO</td>
<td>-21.08</td>
<td>-20.77</td>
<td>0.088</td>
</tr>
<tr>
<td>Multi-WeCAN</td>
<td>-31.10</td>
<td>-29.09</td>
<td>0.072</td>
</tr>
</tbody>
</table>

Table 3-2. Comparison between Multi-CAN and Multi-WeCAN under $\hat{\epsilon}$ \((N = 256, M = 4)\)
<table>
<thead>
<tr>
<th></th>
<th>Auto-corr sidelobe peak (dB)</th>
<th>Cross-corr peak (dB)</th>
<th>$\hat{\epsilon}_{\text{norm}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multi-CAN</td>
<td>-20.53</td>
<td>-17.68</td>
<td>0.40</td>
</tr>
<tr>
<td>Multi-WeCAN</td>
<td>-45.17</td>
<td>-45.81</td>
<td>$9.54 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

In the numerical simulation, the noise covariance matrix $Q$ is chosen as $\sigma^2 I_{M_r}$, where $\sigma^2 = 0.001$. The targets are chosen to form a “UF” shape (Fig. 3-9) and the RCS-related parameters $\{\alpha_{pk}\}_{p=0,k=1}^{P-1,K}$ are simulated as i.i.d. complex symmetric Gaussian random variables with mean 0 and variance 1 at the target locations and zero elsewhere. The average (transmitted) signal-to-noise ratio (SNR) is given by
\[
\text{SNR} = \frac{\text{tr}(X^H X)/N}{\text{tr}(Q)} = \frac{M}{M_r \sigma^2} = 30 \text{ dB}. \tag{3–67}
\]

We use two different probing sequence sets: a QPSK Hadamard+PN sequence set and a Multi-WeCAN sequence set with $N = 256, M = 4$ and $P = 30$. The transmitted waveform is phase-modulated by the probing sequence set (one sequence element corresponds to one subpulse) and we assume proper sampling so that the considered discrete models are appropriate. The estimated $\{\alpha_{pk}\}_{p=0,k=1}^{P-1,K}$ using these two waveforms are shown in Figures 3-10 and 3-11. The Multi-WeCAN waveform gives much clearer angle-range images than the Hadamard+PN waveform. Note from Figure 3-11B that the Multi-WeCAN waveform leads to an almost perfect range compression via the matched filter (the false scatterers are due to the presence of noise) and that the Capon estimator provides a radar image with a high angle resolution.
Figure 3-1. The correlations of the 40-by-3 CE and Multi-CAN sequence sets. A) $r_{11}(k)$ and B) $r_{12}(k)$.
Figure 3-2. The correlations of the 40-by-3 CE and Multi-CAN sequence sets. A) \( r_{13}(k) \) and B) \( r_{22}(k) \).
Figure 3-3. The correlations of the 40-by-3 CE and Multi-CAN sequence sets. A) $r_{23}(k)$ and B) $r_{33}(k)$. 
Figure 3-4. Comparison between the Multi-CAN sequence set and the Hadamard+PN sequence set with $M = 3$ and $N = 2^7, \ldots, 2^{13}$ in terms of A) the auto-correlation sidelobe peak and B) the cross-correlation peak.
Figure 3-5. Comparison between the Multi-CAN sequence set and the Hadamard+PN sequence set with $M = 3$ and $N = 2^7, \ldots, 2^{13}$ in terms of the normalized fitting error as defined in (3–51).
Figure 3-6. Correlation levels of the Multi-CAO sequence set and the Multi-WeCAN sequence set with $N = 256$, $M = 4$ and $P = 50$. (The dotted vertical lines signify the boundary of the time lag interval under consideration.) A) The Multi-CAO sequence set and B) the Multi-WeCAN sequence set.
Figure 3-7. Correlation levels of the Multi-CAN sequence and the Multi-WeCAN sequence with $N = 256$, $M = 4$ and weights $\{\gamma_n\}_{n=0}^{N-1}$ as specified in (3–55). (The dotted vertical lines signify the boundaries of the time lag intervals under consideration.) A) The Multi-CAN sequence and B) the Multi-WeCAN sequence.
Figure 3-8. The same comparisons as shown in Figures 3-4, except that the phases of the CAN sequence used here are quantized into 32 levels.
Figure 3-9. The true target image (the absolute values of $\{\alpha_{pk}\}_{p=0, k=1}^{P-1, K}$ are shown).
Figure 3-10. The estimated target images in terms of the RCS-related parameters \( \{ |\alpha_{pk}| \}_{p=0,k=1}^{P-1,K} \) using the Hadamard+PN waveform. A) The LS estimate and B) the Capon estimate.
Figure 3-11. The estimated target images in terms of the RCS-related parameters \( \{ |\alpha_{pk}|^{p-1,K} \}_{p=0,k=1} \) using the Multi-WeCAN waveform. A) The LS estimate and B) the Capon estimate.
CHAPTER 4
CORRELATION LOWER BOUNDS

The previous chapters have shown the importance of designing sequences or sequence sets with low correlation sidelobes. Let \( \{x_k(n)\} \) \((k = 1, \ldots, M \text{ and } n = 1, \ldots, N)\) denote a set of \( M \) sequences of length \( N \), which are restricted to have the same energy:

\[
\sum_{n=1}^{N} |x_k(n)|^2 = N, \quad k = 1, \ldots, M.
\] (4–1)

Excluding the in-phase (i.e., the zero-delay) auto-correlations, all other correlations are categorized to be correlation sidelobes and correspondingly the peak sidelobe level (PSL) metric is defined as

\[
PSL = \max \left\{ |r_{ks}(l)| \right\}, \quad k, s = 1, \ldots, M \text{ and } l = 0, \ldots, N - 1 \ (l \neq 0 \text{ if } k = s).
\] (4–2)

The following PSL lower bound is due to Welch [51]:

\[
PSL \geq N \sqrt{\frac{M - 1}{2NM - M - 1}} \triangleq B_{PSL}. \tag{4–3}
\]

The ISL metric, used previously in the single sequence case (Eq. (2–1) in Chapter 2), is defined in the present multi-sequence case as (also (3–2) in Chapter 3):

\[
ISL = \sum_{k=1}^{M} \sum_{p=-N+1}^{N-1} |r_{kk}(p)|^2 + \sum_{k=1}^{M} \sum_{s=1}^{M} \sum_{p=-N+1}^{N-1} |r_{ks}(p)|^2. \tag{4–4}
\]

A lower bound on ISL was implicitly derived in [52]. In the next section we derive both the ISL and PSL lower bounds using the Multi-CAN framework.
4.1 Bound Derivation

It is shown in Chapter 3 that the ISL metric in (4–4) can be transformed to the frequency domain as:

\[ \text{ISL} = \frac{1}{2N} \sum_{p=1}^{2N} \left( \| y_p \|^2 - N \right)^2 + (M - 1)N^2; \]  

\[ (4–5) \]

where

\[ y_p = \begin{bmatrix} y_1(p) \\ \vdots \\ y_N(p) \\ y_{M}(p) \end{bmatrix}, \quad y_k(p) = \sum_{n=1}^{N} x_k(n)e^{-j\frac{2\pi}{2N}(p-1)(n-1)}, \quad k = 1, \ldots, M. \]  

\[ (4–6) \]

Note that \( \{y_k(p)\}_{p=1}^{2N} \) is the DFT (discrete Fourier transform) of the sequence \( \{x_k(n)\}_{n=1}^{N} \) padded with \( N \) zeros in the tail. In this section we start from this frequency-domain expression of ISL to derive a lower bound on it.

Let \( z_{kp} = |y_k(p)|^2 \). Then the energy constraint in (4–1) is related to \( \{z_{kp}\} \) via the Parseval equality:

\[ \sum_{p=1}^{2N} z_{kp} = 2N \sum_{n=1}^{N} |x_k(n)|^2 = 2N^2, \quad k = 1, \ldots, M. \]  

\[ (4–7) \]

Expanding (4–5) and plugging in (4–7), we obtain

\[ \text{ISL} = \frac{1}{2N} \sum_{p=1}^{2N} \left( \sum_{k=1}^{M} z_{kp} \right)^2 - MN^2. \]  

\[ (4–8) \]

Making use of the Cauchy-Schwartz inequality leads to the following result:

\[ \text{ISL} \geq \frac{1}{4N^2} \left( \sum_{p=1}^{2N} 1 \cdot \left( \sum_{k=1}^{M} z_{kp} \right) \right)^2 - MN^2 \]

\[ \geq M^2N^2 - MN^2, \]  

\[ (4–9) \]
where (4–7) was used to get (4–10) from (4–9).

The above result on the ISL lower bound is summarized as:

\[
\text{ISL} \geq N^2 M(M - 1) \triangleq B_{\text{ISL}}. \tag{4–11}
\]

The PSL lower bound in (4–3) can be easily obtained from \(B_{\text{ISL}}\) as a corollary. It follows from the definition of ISL in (4–4) that

\[
\text{ISL} \leq 2M(N - 1)\text{PSL}^2 + M(M - 1)(2N - 1)\text{PSL}^2. \tag{4–12}
\]

Substituting (4–11) in (4–12), we obtain \(B_{\text{PSL}}\) in (4–3).

Note that the equality in (4–9) holds if and only if \(\sum_{k=1}^{M} z_{kp} = c\) for all \(p = 1, \ldots, 2N\) where \(c\) is a constant. Because of the energy constraint in (4–7), it is easy to see that \(c = NM\). In other words, a set of energy-constrained sequences \(\{x_k(n)\}\) meet the ISL lower bound if and only if their 2N-point DFT values satisfy \(\|y_p\|^2 = NM\) for all \(p = 1, \ldots, 2N\) (Eq. (4–6) for the definition of \(\|y_p\|\)). An example of such a sequence set is given in (4–13) below.

\subsection*{4.2 Approaching the Bound}

A natural question arises as to whether we can generate sequence sets that achieve the correlation lower bound \(B_{\text{ISL}}\) or \(B_{\text{PSL}}\). Here we focus on trying to meet \(B_{\text{ISL}}\).

A trivial solution to meeting \(B_{\text{ISL}}\) is the following sequence set (recall that the energy constraint in (4–1) is always imposed):

\[
x_k(n) = \begin{cases} 
\sqrt{N}, & n = 1, \\
0, & n = 2, \ldots, N, \\
\end{cases} \quad k = 1, \ldots, M, \tag{4–13}
\]

whose correlation sidelobes are all zero except for the zero-lag cross-correlation which is \(N\). A set of \(M\) sequences leads to \(M(M - 1)\) pairs of cross-correlations and thus the ISL for the above sequence set is exactly equal to the lower bound \(N^2 M(M - 1)\). However, the sequence set in (4–13) is not practically useful because its PSL is as high
as the in-phase auto-correlation. Moreover, transmitting only at one time instant while keeping silent at all other times, as evidenced by the zeros for $n = 2, \ldots, N$ in (4–13), results in a high (in fact, the maximum possible) peak-to-average power ratio (PAR), which is once again undesirable in practice.

The Multi-CAN algorithm introduced in Chapter 3 aims to find unimodular sequence sets with low ISL. The unimodular constraint refers to the fact that every sequence element has unit modulus, i.e. $|x_k(n)| = 1$. In this case the energy constraint in (4–1) is automatically satisfied. Note that unimodular sequences are often preferred in practice due to hardware restrictions, such as using an economical non-linear amplifier which is essentially working well only when the PAR is 1 or close to 1.

Although the unimodular constraint is certainly more stringent than the energy constraint, the unimodular sequence sets generated by Multi-CAN have an ISL that is fairly close to $B_{\text{ISL}}$, provided that there are at least two sequences in the set (the $M = 1$ situation turns out to be special and is taken care of later on). To illustrate, we show the ISL of sequence sets generated by Multi-CAN and the corresponding $B_{\text{ISL}}$ in Table 4-1, for various combinations of $M$ and $N$. Note that the Multi-CAN algorithm is run from a random initialization, and that different random initializations lead to different sequence sets but with similarly low correlations. Regarding Table 4-1, only one such realization is presented for each pair of $(M, N)$.

The good performance of Multi-CAN synthesized unimodular sequence sets, compared to $B_{\text{ISL}}$, can no longer be guaranteed when $M = 1$ in which case $B_{\text{ISL}} = 0$. Note that the Multi-CAN algorithm becomes the CAN algorithm when $M = 1$. Hereafter in this section, only the auto-correlation of a single sequence is considered. For a sequence \( \{x(n)\}_{n=1}^{N} \) with $|x(n)| = 1$ for all $n$, it holds that $|r(n-1)| = |x(N)x^*(1)| = 1$ and thus ISL $\geq 1$. Hence, obviously $B_{\text{ISL}}$ cannot be reached by using unimodular sequences. Actually the ISL of a single sequence generated by CAN is much larger than 1 (e.g., on the order of $10^3$ when $N = 200$), although a CAN sequence can possess much lower
correlation sidelobes than many well-known unimodular sequences in the literature, such as the Golomb or Frank sequence (Chapter 2). We consider below relaxing the unimodular constraint in the CAN algorithm so as to obtain lower correlations. More precisely, define the PAR of the sequence $x = \begin{bmatrix} x(1) & \cdots & x(N) \end{bmatrix}^T$ as:

$$\text{PAR}(x) = \frac{\max_{n} |x(n)|^2}{\frac{1}{N} \sum_{n=1}^{N} |x(n)|^2} = \max_{n} |x(n)|^2,$$

(4–14)

where the second equality is due to the energy constraint. The CAN algorithm generates sequences with $\text{PAR} = 1$. Here we extend it to the more general case of $\text{PAR} \leq \rho$ where $\rho$ can be any number between 1 and $N$.

Following the discussions in Chapter 2, the ISL metric in (4–5), for the case of $M = 1$, can be made small by solving the following minimization problem:

$$\min_{\{x(n)\}_{n=1}^{N}, \{\psi(p)\}_{p=1}^{2N}} f = \|A^H z - v\|^2 = \|z - Av\|^2$$

(4–15)

s.t. $\|x\|^2 = N$, $\text{PAR}(x) \leq \rho$,

where

$$z = \begin{bmatrix} x(1) & \cdots & x(N) & 0 & \cdots & 0 \end{bmatrix}_{2N \times 1}^T,$$

(4–16)

$$v = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{j\psi(1)} & \cdots & e^{j\psi(2N)} \end{bmatrix}_{2N \times 1}^T,$$

$\{\psi(p)\}$ are auxiliary variables and $A^H$ is a unitary $2N \times 2N$ DFT matrix (i.e., $A^H x$ gives the $2N$-point DFT of any vector $x$ of length $2N$). Note that (4–15) would reduce to the problem discussed in Chapter 2 (Section 2.1) if its second constraint were replaced by $\text{PAR}(x) = 1$. 84
The problem in (4–15) can be solved in a cyclic way. We first fix \( z \) and compute the \( v \) that minimizes \( f \):

\[
\psi(p) = \arg\{\text{the } p\text{-th element of } A^H z\}, \quad p = 1, \ldots, 2N.
\]

(4–17)

Next we fix \( v \) and note that the minimization problem can be cast as

\[
\min_x \|x - s\|^2 \\
\text{s.t. } \|x\|^2 = N, \\
\text{PAR}(x) \leq \rho,
\]

where \( s \) is an \( N \times 1 \) vector made from the first \( N \) elements of \( Av \).

The “nearest-vector” problem in (4–18) has been considered in [53]; herein we briefly outline its solution. To begin with, note that the solution to (4–18) without the PAR constraint is given by \( \hat{x} = \sqrt{N}s/\|s\| \). Then note that the PAR constraint is equivalent to \( \max_n |x(n)| \leq \sqrt{\rho} \). Hence if the magnitudes of all elements in \( \hat{x} \) are below \( \sqrt{\rho} \), then \( \hat{x} \) is a solution; if not, we resort to a recursive procedure as follows. The element in \( x \) corresponding to the largest element (in terms of magnitude) in \( s \), say \( s_\alpha \), is given by \( \sqrt{\rho}\exp\{j \arg(s_\alpha)\} \). The other \( N - 1 \) elements in \( x \) are obtained by solving the same problem as in (4–18), except that now \( x \) and \( s \) are \( (N - 1) \times 1 \) and that the energy constraint is \( \|x\|^2 = N - \rho \). Since the scalar case of (4–18) is trivial, such a recursive procedure is guaranteed to yield a solution. We refer the readers to [53] for more details.

To summarize, we iterate between (4–17) and (4–18) until convergence (for instance, until the norm of the difference between the \( x \)'s obtained in two consecutive iterations is less than a predefined threshold, e.g. \( 10^{-3} \)). The criterion in (4–15) is decreased in every iteration step so local convergence is guaranteed (i.e. the so-obtained \( x \) is at least a local minimum solution to (4–15)). The iterative process can be started from a random phase initialization of \( x \), e.g. \( \{x(n) = e^{j\phi(n)}\}_{n=1}^{N} \), where each \( \phi(n) \) is drawn independently from a uniform distribution over \([0, 2\pi]\); such an
initialization is used whenever we consider random initialization below. Alternatively \( x \) can be initialized by any good existing sequence ("good" meaning that the sequence itself already has relatively low correlations), e.g., the P4 sequence. The resulting algorithm is still named CAN in view of the fact that the CAN algorithm proposed in Chapter 2 is just a special case of (4–15) (corresponding to \( \text{PAR} = 1 \)) and surely an important one; no ambiguity will be introduced by using this name since hereafter in this Chapter we will specify the PAR value whenever we apply CAN.

Consider next using CAN to generate a sequence of length \( N = 512 \) with energy \( N \). Figure 4-1A shows the auto-correlations (normalized by \( N \) and in dB) of two CAN sequences, one with \( \text{PAR} = 1 \) and the other with \( \text{PAR} = 4 \), both initialized by a randomly generated sequence. Figure 4-1B is for the same setting as 4-1A except that the P4 sequence was used to initialize the CAN algorithm. Clearly \( \rho \) plays an important role: a larger \( \rho \) leads to significantly lower correlation sidelobe levels. (Note that we do not plot, for comparison, the correlations of the P4 or other well-known sequences such as Golomb or Frank, because they have higher correlation sidelobes than the CAN sequence with \( \text{PAR} = 1 \); see Chapter 2 for examples.)

Figure 4-2 illustrates the ISL of a CAN sequence with length \( N = 512 \) and \( \rho \) ranging from 1 to 10. As before, we use either a randomly generated sequence or the P4 sequence to initialize CAN. The P4 initialization gives lower ISL than the random initialization. Interestingly, when \( \rho \) is relatively small, the decrease of ISL caused by even a small increase of \( \rho \) is significant. Note that in the case of P4 initialization, the ISL can be decreased by more than 2 orders of magnitude if \( \rho \) is increased just from 1 to 1.2. However, after reaching a certain point, the increase of \( \rho \) does not push ISL any lower. The ISL of the CAN sequence initialized by P4 when \( \rho = 4 \) is 5.38, a value relatively close to the ISL lower bound of \( B_{\text{ISL}} = 0 \). A full explanation is still lacking as to why the ISL of the CAN sequence does not go to zero when \( \rho \) is sufficiently large, though the possible trapping of the algorithm in local minima is a likely reason.
Figure 4-1. Auto-correlations of two CAN sequences with different PARs. A) The auto-correlations (normalized by $N$ and shown in dB) of two CAN sequences of length $N = 512$, one with PAR = 1 and the other with PAR = 4, both initialized by a randomly generated sequence. B) The same as A except that the P4 sequence is used to initialize the CAN algorithm.
Table 4-1. $B_{ISL}$ vs. ISL of Multi-CAN sequence sets

<table>
<thead>
<tr>
<th>M, N</th>
<th>ISL</th>
<th>$B_{ISL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 200</td>
<td>80013.7</td>
<td>80000</td>
</tr>
<tr>
<td>2, 512</td>
<td>524385.8</td>
<td>524288</td>
</tr>
<tr>
<td>4, 512</td>
<td>3145752.2</td>
<td>3145728</td>
</tr>
<tr>
<td>4, 1000</td>
<td>1200044.8</td>
<td>12000000</td>
</tr>
</tbody>
</table>

Figure 4-2. The ISL of CAN sequences (with length $N = 512$ and initialized either randomly or by P4) vs. $\rho$. 
CHAPTER 5
AMBIGUITY FUNCTION

The ambiguity function (AF) shows the response of a matched filter to a signal with various time delays and Doppler frequency shifts. Let \( u(t) \) denote a probing signal with time support \([0, T]\) (i.e., \( u(t) \) is assumed to be zero outside \([0, T]\)). The (continuous-time) AF of \( u(t) \) is defined as

\[
\chi(\tau, f) = \int_{-\infty}^{\infty} u(t) u^*(t - \tau) e^{-j2\pi f(t-\tau)} \, dt
\]  

(5–1)

where \( \tau \) is the time delay and \( \nu \) is the Doppler frequency shift. There exists an extensive literature on radar ambiguity functions, such as [13, 17, 20, 54–59].

5.1 AF Properties

Figure 5-1 shows the ambiguity function of a chirp signal (Eq. (1–18) in Chapter 1) with parameters \( T = 10 \) s and \( B = 5 \) Hz. In the figure note that the absolute value of \( \chi(\tau, f) \) is normalized so that the peak value at the origin is 1, that the delay \( \tau \) is normalized by \( T \) and that the Doppler shift \( f \) is normalized by \( 1/T \). Such normalizations will also be used in most other AF plots to provide a consistent scaling.

Two of the ambiguity function features can be easily observed from Figure 5-1. The first feature is that the maximum value of \( |\chi(\tau, f)| \) is achieved by \( |\chi(0, 0)| \), which in fact equals the energy of \( u(t) \). The other one is the symmetry with respect to the origin, i.e., \( |\chi(\tau, f)| = |\chi(-\tau, -f)| \), so that it suffices to show the ambiguity function only for half of the \((\tau, f)\) plane (as was done in Figure 5-1A).

Another prominent feature of AF, which is less obvious than the two mentioned above, is the constant volume property:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau, f)|^2 d\tau df = E^2
\]  

(5–2)

where

\[
E = \int_{-\infty}^{\infty} |u(t)|^2 dt
\]  

(5–3)
is the energy of \( u(t) \). See Appendix C for proofs of the three properties mentioned above.

The AF of the chirp signal illustrated in Fig. 5-1 has the Doppler-tolerance property, in the sense that a mismatch in the Doppler frequency can still lead to a peak in the matched-filtering process although at the cost of a time-delay estimation error. Therefore even if a bank of filters matched to different Doppler frequencies is not available at the receiver end, targets with unknown Doppler shifts may still be detected due to this delay-Doppler coupling property.

As discussed in Section 1.3 of Chapter 1, many sequences such as the Golomb and Frank sequences can be derived from the chirp signal. Not surprisingly, their ambiguity functions inherit the Doppler-tolerance property. Figure 5-2 shows the AF of the waveform in (1−1) when \( \{x(n)\} \) is a length-50 Golomb sequence and \( p_n(t) \) is the rectangular shaping pulse. (Later on when we refer to the AF of a sequence, we implicitly mean the AF of the underlying coded waveform.) Comparing Fig. 5-1 and Fig. 5-2, it is interesting to observe how part of the AF volume in the central ridge is moved to the edge in Fig. 5-2 (the total volume being constant, see (5–2)).

As discussed in Chapter 2, the CAN algorithm can be used to design sequences with low correlation sidelobes. The auto-correlation function is nothing but the zero-Doppler cut of AF. The AF of two length-50 CAN sequences are shown in Fig. 5-3. For Fig. 5-3A a randomly generated sequence is used to initialize the CAN algorithm while for Fig. 5-3B the Golomb sequence of the same length is used for initialization. The white stripe at zero-Doppler frequency indicates the low correlation sidelobes. We observe that the AF in Fig. 5-3A is thumbtack-shaped which leads to high resolution for both delay and Doppler estimation. We also observe that the AF in Fig. 5-3B exhibits Doppler tolerance due to the initialization by the Golomb sequence.

A fact worth noting is that the zero-Delay cut of AF is the Fourier transform of \( u(t)u^*(t) \). For the waveforms discussed above that have unit modulus, the zero-Delay cut
of AF can be easily calculated as

\[
\chi(0, f) = \int_{-\infty}^{\infty} u(t) u^*(t) e^{-j2\pi ft} dt = \int_{0}^{T} e^{-j2\pi ft} dt = \frac{1 - e^{-j2\pi fT}}{j2\pi f} = e^{-j\pi fT} \cdot T\text{sinc}(\pi fT)
\]

(5–4)

where \(\text{sinc}(x) = \sin(x)/x\). Therefore \(|\chi(0, f)| = |T\text{sinc}(\pi fT)|\) regardless of \(u(t)\) (provided that \(u(t)\) is unimodular). Since \(\text{sinc}(\pi fT)\) dies out quickly as \(f\) increases, the zero-Delay cut appears as a vertical stripe of small values in Fig. 5-3, as well as in Figs. 5-1 and 5-2 (though somewhat less obvious).

We have illustrated the ambiguity functions of several waveforms. How to realize a desired ambiguity function using practical signals has been a classical problem in the waveform design area and there exists a considerable literature on this topic [13, 17, 20, 55, 56, 58, 60, 61]. Despite this extensive literature, apparently there is no universal method that can synthesize an arbitrary ambiguity function. In fact, matching only the zero-Doppler cut of an ambiguity function or minimizing the sidelobes of the auto-correlation function is a difficult problem in itself, as discussed in Chapters 2 and 3.

In what follows, we introduce the discrete-AF concept and show that the sidelobes of a discrete-AF can be suppressed in a region close to the origin.

### 5.2 Discrete-AF

We restrain the attention to baseband waveforms modulated in the following way (e.g., (1–1) in Chapter 1):

\[
u(t) = \sum_{n=1}^{N} x(n) p_n(t), \quad 0 \leq t \leq T
\]

(5–5)

where \(\{x(n)\}_{n=1}^{N}\) is the modulating code sequence that is to be designed (assumed to be zero when \(n \notin [1, N]\)) and

\[
p_n(t) = \begin{cases} \frac{1}{\sqrt{T_p}}, & (n-1)T_p \leq t \leq nT_p \\ 0, & \text{elsewhere} \end{cases}
\]

(5–6)
is an ideal rectangular shaping pulse of time length \( t_p \) (and thus \( T = N t_p \)). The energy of \( \{x(n)\}_{n=1}^{N} \) is constrained to be \( N \), i.e.,

\[
\sum_{n=1}^{N} |x(n)|^2 = N. \tag{5–7}
\]

As pointed out several times before, it is usually desired to have a unimodular sequence:

\[
x(n) = e^{j\phi_n}, \quad n = 1, \ldots, N, \tag{5–8}
\]

where \( \{\phi_n\} \) are the phases.

For the waveform defined in (5–5), the ambiguity function (5–1) can be simplified as follows. Substitute (5–5) into (5–1) to obtain

\[
\chi(\tau, f) = \int_{0}^{T} \left( \sum_{n=1}^{N} x(n)p_n(t) \right) \left( \sum_{m=1}^{N} x^*(m)p_m(t-\tau) \right) e^{-j2\pi f(t-\tau)} dt
\]

\[
= \sum_{m=1}^{N} \sum_{n=1}^{N} x^*(m) \left( \int_{0}^{T} p_n(t)p_m(t-\tau)e^{-j2\pi f(t-\tau)} dt \right) x(n). \tag{5–9}
\]

Consider the time grid \( \{t = kt_p\} \) \( (k = -N + 1, \ldots, 0, \ldots, N - 1) \) whose points are integer multiples of the subpulse length \( t_p \). It is not difficult to calculate \( \chi(\tau, f) \) at \( \tau = kt_p \):

\[
\chi(kt_p, f) = \sum_{m=1}^{N} \sum_{n=1}^{N} x^*(m) \left( \int_{0}^{nt_p} |p_n(t)|^2 e^{-j2\pi f(t-kt_p)} dt \right) \delta_{m+k,n} x(n) \tag{5–10}
\]

\[
= \frac{e^{j\pi ft_p \sin(\pi ft_p)}}{\pi ft_p} \sum_{n=1}^{N} x(n) x^*(n-k)e^{-j2\pi ft_p(n-k)}. \tag{5–11}
\]

Note that \( \chi(kt_p, f) \) equals zero if \( k \) lies outside \([-N + 1, N - 1]\). Similarly to the time grid, we consider \( \{f = \frac{p}{Nt_p}\} \) for an integer \( p \). Then we obtain

\[
\chi(kt_p, \frac{p}{Nt_p}) = e^{j\pi \frac{p}{N}} \text{sinc}(\pi \frac{p}{N}) \tilde{r}(k, p) \tag{5–12}
\]

where \( \text{sinc}(x) = \sin(x)/x \) and \( \tilde{r}(k, p) \) is what we call the discrete-AF:

\[
\tilde{r}(k, p) = \sum_{n=1}^{N} x(n) x^*(n-k)e^{-j2\pi \frac{(n-k)p}{N}}, \tag{5–13}
\]
\[ k = -N + 1, \ldots, N - 1; \quad p = -\frac{N}{2}, \ldots, \frac{N}{2} - 1. \]

The range of \( p \) is chosen as \(-\frac{N}{2}, \ldots, \frac{N}{2} - 1\) because it corresponds to the largest Doppler frequency range that can be unambiguously identified (note that the bandwidth of \( u(t) \) is approximately equal to \( 1/t_p \)). Also note that without loss of generality, we have implicitly assumed and will assume hereafter an even \( N \) in this chapter. An odd \( N \) leads to the range \( p = -\frac{N-1}{2}, \ldots, \frac{N-1}{2} \), which will bring little difference in the discussions below.

In practice, the Doppler frequency \( f \) is usually much smaller than the bandwidth of the probing waveform. For example, assuming an X-band radar operating at the wavelength \( \lambda = 3 \text{ cm} \), a fighter jet moving at the speed of Mach 3 \((v = 1020 \text{ m/s})\) can only induce a Doppler frequency of

\[ f = \frac{2v}{\lambda} = \frac{2 \times 1020 \text{ m/s}}{0.03 \text{ m}} = 68 \text{ kHz}, \]

which is much smaller than the bandwidth that is on the order of many MHz. As another example, for a sonar operating at the frequency \( f = 20 \text{ kHz} \) (corresponding to a wavelength of \( 1500/20 = 75 \text{ mm} \)), a fast-moving submarine at the speed of 25 knots \((v = 13 \text{ m/s})\) induces a Doppler frequency of

\[ f = \frac{2v}{\lambda} = \frac{2 \times 13 \text{ m/s}}{0.075 \text{ m}} = 346 \text{ Hz}, \]

which can also be considered to be very small compared with the several-kHz bandwidth that is normally used. Therefore, it is safe to confine our attention to values of \(|p| \ll N\), in which case \( \text{sinc}(\pi \frac{p}{N}) \approx 1 \) and thus

\[ |\chi(kt_p, \frac{p}{Nt_p})| \approx |\hat{r}(k, p)|, \quad k = -N + 1, \ldots, N - 1; |p| \ll N. \]

Since almost always only the magnitude of AF matters in target detection applications, (5–16) shows that it suffices to consider the discrete-AF as defined in (5–13).
The well-known AF properties of symmetry and constant volume also hold for the
discretized version in (5–13):

\[ |\ddot{r}(k, p)| = |\ddot{r}(-k, -p)|, \quad (5–17) \]
\[ \sum_{k=-N+1}^{N+1} \sum_{p=-N/2}^{N/2-1} |\ddot{r}(k, p)|^2 = N^3. \quad (5–18) \]

In addition, \( \ddot{r}(0, 0) \) is always equal to \( N \) because of the energy constraint in (5–7); we
refer to \( \ddot{r}(k, p) \) for \( k \neq 0 \) or \( p \neq 0 \) as the sidelobes. Note that the above properties follow
directly from the discrete-AF definition in (5–13) in spite of the approximate relationship
with the original AF as shown in (5–16).

In the next section we will be mainly concerned with designing the sequence
\( \{x(n)\}_{n=1}^{N} \) so as to minimize the sidelobes of the discrete-AF in a certain region:

\[ \min_{\{x(n)\}} C_1 = \sum_{k \in K} \sum_{p \in P} |\ddot{r}(k, p)|^2 \quad (5–19) \]

where \( K \) and \( P \) are the index sets that specify the region of interest. Because the total
volume of \( \ddot{r}(k, p) \) is fixed, (5–19) is meaningful only when \( K \) and \( P \) are strictly subsets of
\( \{-N + 1, \ldots, N - 1\} \) and \( \{-N/2, \ldots, N/2 - 1\} \) respectively.

5.3 Minimizing the Discrete-AF Sidelobes

Following the notation specified in (5–19), assume that the time delay set of interest
is given by \( K = \{0, \pm 1, \ldots, \pm (K - 1)\} \) and that the Doppler frequency set of interest is
given by \( P = \{0, \pm 1, \ldots, \pm (P - 1)\} \). Define a set of \( P \) sequences \( \{x_m(n)\}_{m=1}^{P} \) as follows:

\[ \{x_1(n) = x(n)\}_{n=1}^{N} \quad (5–20) \]
\[ \{x_2(n) = x(n)e^{j2\pi \frac{n}{N}}\}_{n=1}^{N} \]
\[ \vdots \]
\[ \{x_P(n) = x(n)e^{j2\pi \frac{n(P-1)}{N}}\}_{n=1}^{N}. \]
Note that \( \{x_m(n)\}_{m=1}^{P} \) are zero when \( n \notin [1, N] \). Let \( \{r_{ml}(k)\} \) denote the correlation between \( \{x_m(n)\} \) and \( \{x_l(n)\} \):

\[
\begin{align*}
    r_{ml}(k) &= \sum_{n=1}^{N} x_m(n)x^*_l(n - k) \\
    &= e^{j2\pi \frac{(m-1)k}{N}} \sum_{n=1}^{N} x(n)x^*(n - k)e^{-j2\pi \frac{(n-k)(l-m)}{N}}, \quad k \in \mathcal{K} \\
    m, l &= 1, \ldots, P.
\end{align*}
\]  

(5–21)

It is straightforward to verify that all values of \( \{r(k,p)\} \) \( (k \in \mathcal{K} \) and \( p \in \mathcal{P} \) are contained in the set \( \{r_{ml}(k)\} \) \( (k \in \mathcal{K} \) and \( m, l = 1, \ldots, P \)\). Interestingly, \( m \) and \( l \) do not need to increase stepwise from 1 to \( P \). For example, \( \{r_{ml}(k)\} \) \( (m, l = 1, 2, 5, 7) \) already covers all values of \( \{r(k,p)\} \) \( (p = 0, \ldots, 6) \) (a fact which bears resemblance to the minimum redundancy linear array [62]). This observation saves computation but does not improve the algorithm performance, so (5–21) will be used for \( m, l = 1, \ldots, P \) to keep the notation simple.

The foregoing discussion implies that by minimizing the correlations of the sequence set in (5–20), we equivalently minimize the discrete-AF sidelobes, i.e., the criterion \( C_1 \) in (5–19). The Multi-CAO algorithm discussed in Chapter 3 which was used to design sets of waveforms with good correlations, can be adapted to the problem of minimizing \( C_1 \) as explained in what follows.

Define

\[
\mathbf{X} = \begin{bmatrix}
\mathbf{X}_1 & \cdots & \mathbf{X}_P
\end{bmatrix}_{(N+K-1) \times KP}
\]

(5–22)
where

\[ X_m = \begin{bmatrix}
    x_m(1) & 0 \\
    \vdots & \ddots \\
    \vdots & \ddots & x_m(1) \\
    x_m(N) & \vdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots \\
    0 & x_m(1) \\
  \end{bmatrix}_{(N+K-1)\times K}, \quad m = 1, \ldots, P \]  \tag{5–23}

and \( \{x_m(n)\} \) are defined in (5–20). It is not difficult to see that all \( \{r_{ml}(k)\} \) \( (k \in K \) and \( m, l = 1, \ldots, P \) appear in the matrix \( X^H X \). Also note that the diagonal elements of \( X^H X \) are equal to \( N \) because of the energy constraint in (5–7). Therefore, the correlations of the sequence set in (5–20) can be made small through minimizing the following criterion:

\[ \hat{C}_1 = \|X^H X - NI_{KP}\|^2. \]  \tag{5–24}

Note that the criterion \( \hat{C}_1 \) equals zero if the matrix \( X \) is a semi-unitary matrix scaled by \( \sqrt{N} \), an observation which leads to the following minimization problem that has a simpler form than (5–24):

\[ \min_{X,U} \|X - \sqrt{N}U\|^2, \]  \tag{5–25}

s.t. \( |x(n)| = 1, \quad n = 1, \ldots, N \)

\[ x_m(n) = x(n)e^{j2\pi\frac{n(m-1)}{N}}, \quad m = 1, \ldots, P; \quad n = 1, \ldots, N \]

\[ U^H U = I \quad (U \text{ is } (N + K - 1) \times KP). \]

The minimization problem in (5–25) can be solved by the cyclic algorithm described below:

- **Step 0:** Randomly initialize the sequence \( \{x(n)\} \).
- **Step 1:** For fixed \( X \), the minimizer \( U \) is given by (Section 3.3 of Chapter 3)

\[ U = U_2 U_1^H \]  \tag{5–26}
where the matrices $U_1 (KP \times KP)$ and $U_2 ((N + K - 1) \times KP)$ come from the economic SVD of $X^H$: $X^H = U_1 \Sigma U_2^H$.

- **Step 2:** For fixed $U$, the criterion in (5–25) can be written as (note the unit-modulus constraint in (5–8))

\[
\|X - \sqrt{N} U\|^2 = \sum_{n=1}^{N} \sum_{l=1}^{KP} |\mu_{nl} x(n) - f_{nl}|^2 = \text{const} - 2 \sum_{n=1}^{N} \text{Re} \left[ \left( \sum_{l=1}^{KP} \mu_{nl}^* f_{nl} \right) x^*(n) \right]
\]

(5–27)

where const is a constant that does not depend on $\{x(n)\}$, $\{\mu_{nl}\}$ are given by the elements of $X$ that contain $x(n)$:

\[
[\mu_{n1} \cdots \mu_{n,KP}] = \left[ \begin{array}{c} e^{j2\pi nKP} \cdots e^{j2\pi nKP} \cdots e^{j2\pi nKP} \cdots e^{j2\pi nKP} \end{array} \right]_{1 \times KP}
\]

(5–28)

and $\{f_{nl}\}$ are given by the elements of $\sqrt{N} U$ whose positions are the same as those of $\{\mu_{nl}\}$ in $X$. The minimizer $x(n)$ (more exactly its phase) is obtained immediately:

\[
\phi_n = \arg \left( \sum_{l=1}^{KP} \mu_{nl}^* f_{nl} \right), \quad n = 1, \ldots, N.
\]

(5–29)

- **Iteration:** Repeat Steps 1 and 2 until convergence.

As an example, consider a scenario with $N = 100$, $K = 10$ and $P = 3$. We use the cyclic algorithm outlined above to design a unimodular sequence $\{x(n)\}_{n=1}^{N}$. The so-obtained discrete-AF, $|\tilde{f}(k,p)|$, is shown in Fig. 5-4. The “white” area in the center indicates that the sidelobes near the origin were successfully suppressed.
Figure 5-1. The AF of a chirp signal with $T = 10$ s and $B = 5$ Hz. A) 3D plot of the positive Doppler plane and B) 2D plot of the whole plane.
Figure 5-2. The AF of a length-50 Golomb sequence. A) 3D plot of the positive Doppler plane and B) 2D plot of the whole plane.
Figure 5-3. The AF of a length-50 CAN sequence. A) A randomly generated sequence was used to initialize CAN and B) the Golomb sequence was used to initialize CAN.
Figure 5-4. The synthesized discrete-AF: $|\tilde{r}(k, p)|$. 
CHAPTER 6
STOPBAND CONSTRAINT

Among the tasks associated with cognitive radar [63], an important one is to adapt the spectrum of transmitted waveforms to the changing environment. In particular, the transmitted signal should not use certain frequency bands that have already been reserved, such as the bands for navigation and military communications; or there could exist strong emitters whose operating frequencies should be avoided. Therefore it is required that the spectral power of transmitted waveforms be small for certain frequency bands [64–67].

The main focus in this chapter is on designing a discrete sequence whose spectral power is small in certain specified frequency bands. The designed sequence can be used in active sensing systems like radar/sonar as a probing sequence. It can also be used as a spreading sequence in spread spectrum applications such as a CDMA (code division multiple access) system.

Besides frequency notching, we also take into account the correlation properties of the designed sequence. As pointed out several times in previous chapters (e.g., in Chapter 1), in radar/sonar applications low auto-correlation of the probing sequence improves target detection when range compression is applied in the receiver. Furthermore, practical hardware components such as analog-to-digital converters and power amplifiers have maximum signal amplitude clip. In order to maximize the transmitted power that is available in the system, unimodular sequences are desired.

In this chapter we propose an algorithm named SCAN (stopband CAN) for unimodular transmit sequence design. SCAN is an extension of the CAN algorithm introduced in Chapter 2. CAN aims at generating unimodular sequences with low correlation sidelobes. SCAN extends CAN in such a way that both frequency stopbands and correlation sidelobes are considered. The SCAN algorithm is computationally efficient (as is based on FFT operations) and thus it facilitates long sequence generation
and possibly real-time waveform update. Another advantage of SCAN is that the algorithm can start from random initializations and that different initializations lead to different sequences, but all with similarly good properties.

The problem formulation and the SCAN algorithm are presented in Section 6.1. A variation of the SCAN algorithm named WeSCAN, which has more flexibility in controlling the correlation levels (but at the cost of increased computation), is discussed in Section 6.2. Several simulation results are shown in Section 6.3.

6.1 Stopband CAN (SCAN)

We first formulate the two design criteria that are related, respectively, to spectral band suppression and correlation sidelobe suppression. Without loss of generality, only normalized frequencies (from 0 to 1 Hz) are considered for notational simplicity.

Suppose that the set of frequencies which \( \{x(n)\}_{n=1}^{N} \) should avoid can be expressed as

\[
\Omega = \bigcup_{k=1}^{N_s} (f_{k1}, f_{k2})
\]

(6–1)

where \((f_{k1}, f_{k2})\) identifies one stopband and \(N_s\) is the number of stopbands. Corresponding to \(\Omega\), we choose a number \(\tilde{N}\) that is large enough so that points of the DFT (discrete Fourier transform) frequency grid \(\{p/\tilde{N}\}_{p=0}^{\tilde{N}-1}\) cover \(\Omega\) densely. Let \(F_{\tilde{N}}\) denote the \(\tilde{N} \times \tilde{N}\) DFT matrix whose \((k, l)^{th}\) element is given by

\[
[F_{\tilde{N}}]_{kl} = \frac{1}{\sqrt{N}} \exp \left\{ j2\pi \frac{kl}{\tilde{N}} \right\}, \quad k, l = 0, \ldots, \tilde{N} - 1
\]

(6–2)

where the coefficient \(1/\sqrt{\tilde{N}}\) makes \(F_{\tilde{N}}\) unitary. We form a matrix \(S\) from the columns of \(F_{\tilde{N}}\) corresponding to the frequencies in \(\Omega\). For example, if \(\Omega = [0.2, 0.3]\) Hz and we choose \(\tilde{N} = 100\), \(S\) will be the \(100 \times 11\) submatrix of \(F_{\tilde{N}}\) comprising its 20\(^{th}\) to 30\(^{th}\) columns (indexed from 0). After constructing \(S\), let \(G\) denote the matrix comprising the remaining columns in \(F_{\tilde{N}}\).
It follows from the above discussion that we can suppress the spectral power of \( \{x(n)\} \) in \( \Omega \) by minimizing the following criterion:

\[
\| S^H \tilde{x} \|^2
\]  

(6–3)

where

\[
\tilde{x} = \begin{bmatrix}
x(1) & \cdots & x(N) & 0 & \cdots & 0 \\
\end{bmatrix}^T.
\]  

(6–4)

Observe that (6–3) would become zero if \( \tilde{x} \) lied in the null space of \( S^H \). Since the null space of \( S^H \) is spanned by the columns of \( G \), the problem of minimizing (6–3) can be equivalently formulated as

\[
\min_{x, \alpha} J_1(x, \alpha) = \| \tilde{x} - G\alpha \|^2
\]  

s.t. \(|x(n)| = 1, \quad n = 1, \ldots, N\)

(6–5)

where \( x = \begin{bmatrix} x(1) & \cdots & x(N) \end{bmatrix}^T \), \( \alpha \) is an auxiliary variable and “s.t.” is short for “subject to”.

The problem of suppressing the correlation sidelobes can be dealt with using the CAN algorithm formulation discussed in Chapter 2. As shown in Section 2.1 of Chapter 2, the correlation sidelobes can be suppressed by solving the following problem:

\[
\min_{x, \nu} J_2(x, \nu) = \left\| \mathbf{F}_{2N}^H \begin{bmatrix} x \\ 0_{N \times 1} \end{bmatrix} - \nu \right\|^2
\]  

s.t. \(|x(n)| = 1, \quad n = 1, \ldots, N\)

\(|\nu_n| = \frac{1}{\sqrt{2}}, \quad n = 1, \ldots, 2N\)

(6–6)

where \( \mathbf{F}_{2N} \) is the \( 2N \times 2N \) FFT matrix, \( x = [x(1) \ldots x(N)] \) and \( \nu \) is an auxiliary variable.
Combining (6–5) and (6–6), we obtain the following minimization problem that incorporates both frequency stopband and correlation sidelobe constraints:

$$\min_{x, \alpha, v} J(x, \alpha, v) = \lambda \| \tilde{x} - G\alpha \|^2 + (1 - \lambda) \left\| F_{2N}^H \begin{bmatrix} x \\ 0 \end{bmatrix} - v \right\|^2$$

s.t. \( |x(n)| = 1, \ n = 1, \ldots, N \)
\( |v_n| = \frac{1}{\sqrt{2}}, \ n = 1, \ldots, 2N \)

where \( 0 \leq \lambda \leq 1 \) controls the relative weight on the two penalty functions \( J_1 \) and \( J_2 \).

Note that there are three variables in the criterion \( J(x, \alpha, v) \). We minimize \( J(x, \alpha, v) \) with respect to only one variable at a time and then iterate. The iteration steps are summarized below:

- **Step 0:** Initialize \( \{x(n)\}_{n=1}^{N} \) with a randomly generated unimodular sequence.

- **Step 1:** For fixed \( x \) and \( v \), \( J(x, \alpha, v) \) is a convex quadratic function of \( \alpha \). By setting \( \frac{\partial J}{\partial \alpha} = 0 \) and using \( G^H G = I \), we get the minimizer \( \alpha \):

  $$\alpha = G^H \tilde{x}. \quad (6–8)$$

  $$v = \frac{1}{\sqrt{2}} \exp \left( j \arg \left\{ F_{2N}^H [x^T \ 0_{1\times N}]^T \right\} \right). \quad (6–9)$$

- **Step 3:** For fixed \( \alpha \) and \( v \), \( J \) can be written as

  $$J = \text{const.} - 2 \text{Re}\{x^H (\lambda c_1 + (1 - \lambda)c_2)\} \quad (6–10)$$

where

\( c_1 = \text{the first } N \text{ elements of } G\alpha \) \quad (6–11)

and

\( c_2 = \text{the first } N \text{ elements of } F_{2N}v. \) \quad (6–12)

Then the minimizer \( x \) is given by

$$x = \exp \left( j \arg \{ \lambda c_1 + (1 - \lambda)c_2 \} \right). \quad (6–13)$$
Iteration: Repeat (6–8), (6–9) and (6–13) until convergence (for instance, until the norm of the difference between the x’s obtained in two consecutive iterations is less than a predefined threshold, e.g. 10^{-3}).

The resulting algorithm is named SCAN (stopband CAN), as it is an extension of the CAN algorithm discussed in Chapter 2. Note that all matrix operations involved in the updating formulae of (6–8), (6–9) and (6–13) can be done via FFT. Therefore the SCAN algorithm is computationally efficient. Indeed, it can be used to generate sequences of length up to $N \sim 10^6$.

Before proceeding to the next section, we point out that a more general constraint than unimodularity is to constrain the peak-to-average power ratio (PAR) of the transmitted sequence (Eq. (4–14) in Chapter 4). If a PAR larger than 1 is allowed, the SCAN algorithm can be kept as is with the exception that the minimizer $x$ in (6–13) is now given by the solution to the following problem:

$$\min_x \| x - [\lambda c_1 + (1 - \lambda)c_2] \|^2$$
$$\text{s.t. } \text{PAR}(x) \leq \rho$$  \hspace{1cm} (6–14)

where $1 \leq \rho \leq N$ is the prescribed largest allowable PAR. The “nearest-vector” problem in (6–14) has already been discussed in Section 4.2 of Chapter 4 to which we refer for details. In the examples of Section 6.3, however, we will use mostly the unimodular constraint unless otherwise stated.

### 6.2 Weighted Stopband Cyclic Algorithm-New (WeSCAN)

In this section we present the WeSCAN (weighted SCAN) algorithm which can be viewed as an extension of both SCAN and WeCAN.

For SCAN, minimizing the function $J_2$ in (6–6) is a way of minimizing the integrated sidelobe level (ISL) metric (Eq. (2–1) in Chapter 2):

$$\text{ISL} = 2 \sum_{k=1}^{N-1} |r(k)|^2.$$  \hspace{1cm} (6–15)
The following more general WISL (weighted ISL) metric associates a weight \( \gamma_k^2 \) with each correlation term \( r(k) \) (Eq. (2–3) in Chapter 2):

\[
\text{WISL} = 2 \sum_{k=1}^{N-1} \gamma_k^2 |r(k)|^2.
\]  

(6–16)

The weights \( \{ \gamma_k \} \) can be chosen to satisfy our needs. For example, we can set \( \gamma_1 = 0, \gamma_2 = 0 \) and \( \gamma_k = 1 \) for \( k = 3, \ldots, N - 1 \) to trade the correlation mainlobe width for sidelobe suppression.

Define the weighting matrix

\[
\Gamma = \frac{1}{\gamma_0} \begin{bmatrix}
\gamma_0 & \gamma_1 & \cdots & \gamma_{N-1} \\
\gamma_1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
\gamma_{N-1} & \cdots & \gamma_0
\end{bmatrix}
\]  

(6–17)

where \( \gamma_0 > 0 \) is large enough to make \( \Gamma \) positive semi-definite. Let \( \mathbf{D} \) be a square root of \( \Gamma \) and let \( d_{kl} \) denote the \((k, l)^{th}\) element of \( \mathbf{D} \). Then it is shown in Section 2.2 of Chapter 2 that the WISL metric can be minimized by solving the following problem:

\[
\min_{x, \{ \mathbf{\alpha}_p \}_{p=1}^{2N}} \tilde{J}_2(x, \mathbf{V}) = \| \mathbf{F}^H_{2N} \mathbf{Z} - \mathbf{V} \|^2
\]  

(6–18)

\[
\text{s.t.} \quad |x(n)| = 1, \quad n = 1, \ldots, N \\
\| \mathbf{\alpha}_p \|^2 = 1, \quad p = 1, \ldots, 2N
\]

where

\[
\mathbf{Z} = \begin{bmatrix} z_1 & z_2 & \cdots & z_N \end{bmatrix}_{2N \times N};
\]

(6–19)

\[
z_k = \begin{bmatrix} d_{k1} x(1) & \cdots & d_{kN} x(N) \end{bmatrix}^T_{2N \times 1}, \quad k = 1, \ldots, N
\]

and

\[
\mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{\alpha}_1 & \mathbf{\alpha}_2 & \cdots & \mathbf{\alpha}_{2N} \end{bmatrix}^T_{(2N \times N)}.
\]  

(6–20)
We replace the penalty function $J_2$ in (6–6) by the $\tilde{J}_2$ in (6–18), follow all discussions after (6–7) in Section 6.1 and make the necessary changes that are straightforward. The resulting algorithm is named WeSCAN (weighted SCAN). We will show in Section 6.3 that compared to SCAN, WeSCAN is able to generate sequences with much better frequency stopband suppression, for instance at the cost of an increased correlation mainlobe width.

To conclude this section, we remark on the fact that the frequency stopband penalty function (e.g., in (6–3)) can be formulated using continuous frequencies\cite{64}. Let

$$X(f) = \sum_{n=1}^{N} x(n)e^{-j2\pi f(n-1)}, \quad f \in [0, 1]$$ \hfill (6–21)

be the discrete-time Fourier transform of \{x(n)\}_{n=1}^{N}. The frequency stopband is still given by (6–1). The spectral power in the $k^{th}$ band is calculated as

$$\int_{f_{k_1}}^{f_{k_2}} |X(f)|^2 \, df = \int_{f_{k_1}}^{f_{k_2}} \left| \sum_{n=1}^{N} x(n)e^{-j2\pi f(n-1)} \right|^2 \, df \hfill (6–22)$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} x^*(m) \left[ \int_{f_{k_1}}^{f_{k_2}} e^{j2\pi f(m-n)} \, df \right] x(n),$$

and the stopband criterion $\tilde{J}_1$ is defined to be the summation of (6–22) over the stopbands:

$$\tilde{J}_1 = \sum_{k_1}^{N_s} \int_{f_{k_1}}^{f_{k_2}} |X(f)|^2 \, df \hfill (6–23)$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} x^*(m) \left[ \sum_{k_1}^{N_s} \int_{f_{k_1}}^{f_{k_2}} e^{j2\pi f(m-n)} \, df \right] x(n).$$

Letting $R$ be an $N \times N$ matrix whose $(m,n)^{th}$ element is given by

$$R_{mn} = \sum_{k=1}^{N_s} \int_{f_{k_1}}^{f_{k_2}} e^{j2\pi f(m-n)} \, df \hfill (6–24)$$

$$= \sum_{k=1}^{N_s} \left\{ \frac{e^{j2\pi f_{k_2}(m-n)} - e^{j2\pi f_{k_1}(m-n)}}{j2\pi (m - n)} \right\}, \quad m \neq n$$

$$f_{k_2} - f_{k_1}, \quad m = n$$
leads to the following expression for $\tilde{J}_1$:

$$
\tilde{J}_1 = x^H R x.
$$

(6–25)

In general $R$ will be rank deficient [38]. Suppose that $\text{rank}(R) = \hat{N} < N$. Let $R = U \Sigma U^*$ denote the (ordered) eigenvalue decomposition of $R$, and let $B$ denote the $N \times (N - \hat{N})$ matrix made from the last $N - \hat{N}$ columns of $U$. Similarly to the argument that led from (6–3) to (6–5), the weighted problem of notching the power in $\Omega$ can be formulated as

$$
\begin{align*}
\min_{x, \alpha} \tilde{J}_1(x, \alpha) &= \|x - B\alpha\|^2 \\
\text{s.t. } |x(n)| &= 1, \quad n = 1, \ldots, N.
\end{align*}
$$

(6–26)

Compared to (6–5), the formulation in (6–26) precludes the use of FFT in the algorithm iterations and thus is not too appealing from a computational efficiency perspective.

Another problem with (6–26) is related to the need for selecting $\hat{N}$ (since the rank of a matrix is not always easy to determine). The penalty function $\tilde{J}_1$ in (6–26) will be used in the examples of the next section only when it is so specified.

6.3 Numerical Examples

6.3.1 SCAN

Suppose that we want to design a unimodular sequence of length $N = 100$ and that the frequency stopband is given by $\Omega = [0.2, 0.3]$ Hz. We use the SCAN algorithm to generate such a sequence with the parameters $\lambda = 0.7$ and $\tilde{N} = 1000$. Letting $x$ denote the generated sequence, we measure the properties of $x$ in terms of the peak stopband power ($P_{\text{stop}}$) and peak correlation sidelobe ($P_{\text{corr}}$). More specifically, we define

$$
P_{\text{corr}} = 20 \log_{10} \left( \max_{k=1,\ldots,N-1} \frac{|r(k)|}{N} \right).
$$

(6–27)

To calculate $P_{\text{stop}}$, letting $\{y(k)\}_{k=1}^{\tilde{N}}$ denote the $\tilde{N}$-point FFT of $\{x(n)\}_{n=1}^{N}$, we normalize $\{y(k)\}_{k=1}^{\tilde{N}}$ so that the average value of $|y(k)|^2$ in the passband is 1. Then $P_{\text{stop}}$ is
calculated as

\[ P_{\text{stop}} = 10 \log_{10} \left( \max_k |y(k)|^2 \right) \quad \text{for} \quad \frac{k - 1}{\tilde{N}} \in \Omega. \] (6–28)

For this example, the range of \( k \) in (6–28) is from 201 to 300 (note that \( k \) corresponds to the frequency \((k - 1)/\tilde{N}\)).

The spectral power (i.e. the normalized \(||y(k)||^2\)) and correlation level (\(||r(k)||/N||\)) of the generated sequence are shown in Figs. 6-1A and 6-1B, respectively. Corresponding to the two figures, we have \( P_{\text{stop}} = -8.3 \) dB and \( P_{\text{corr}} = -19.2 \) dB.

To illustrate how \( \lambda \) affects the performance, we increase \( \lambda \) from 0.1 to 1 and plot the values of \( P_{\text{stop}} \) and \( P_{\text{corr}} \) in Fig. 6-2 (other parameters are kept the same as before). We do not show the results for \( \lambda \) less than 0.1 because such a small \( \lambda \) leads to very little stopband suppression. From Fig. 6-2 it is easy to see that a larger \( \lambda \) gives more weight to the stopband penalty function and thus results in a smaller \( P_{\text{stop}} \) at the cost of an increased \( P_{\text{corr}} \).

Remark: The curve of \( P_{\text{corr}} \) in Fig. 6-2 does not increase strictly monotonically as \( \lambda \) increases, nor does \( P_{\text{stop}} \) decrease monotonically. The reason is that the SCAN algorithm is initialized by a random sequence. Different initializations lead to different final sequences whose \( P_{\text{stop}} \) or \( P_{\text{corr}} \) varies. While generally a larger \( \lambda \) favors the stopband suppression, it is not guaranteed that, e.g., \( \lambda = 0.75 \) gives a smaller \( P_{\text{stop}} \) and a larger \( P_{\text{corr}} \) than \( \lambda = 0.7 \).

Next we consider a situation where the allowed band is highly segmented by stopbands:

\[
\Omega = [0, 0.11) \cup [0.13, 0.19) \cup [0.25, 0.36) \\
\cup [0.40, 0.65) \cup [0.8, 0.87) \cup [0.94, 1) \text{ Hz.} \] (6–29)

The sequence length is \( N = 10^4 \) and we choose \( \tilde{N} = N \) since \( N \) is already large enough to ensure a dense DFT-frequency grid. We set \( \lambda = 0.9 \) to emphasize the
stopband suppression. Fig. 6-3 shows the spectral power and correlation level of the generated SCAN sequence. Here $P_{\text{stop}} = -15.1$ dB and $P_{\text{corr}} = -7.3$ dB. Note that the peak correlation sidelobe occurs close to the origin (i.e. the in-phase correlation point) and that correlation sidelobes far away from the origin are much lower than $P_{\text{corr}}$.

### 6.3.2 WeSCAN

In this subsection we show that by using the WeSCAN algorithm, we can trade-off correlation properties for an improvement of $P_{\text{stop}}$.

We use the same setting as in Fig. 6-1 except that we apply the WeSCAN algorithm. Assuming that a relatively wide correlation mainlobe is acceptable, we choose the correlation weights (Eq. (6–16)) as $\gamma_1 = 0$, $\gamma_2 = 0$ and $\gamma_k = 1$ for $k = 3, \ldots, N - 1$. We show the spectral power and correlation level of the so-generated WeSCAN sequence in Fig. 6-4. Compared to Fig. 6-1, the stopband power ($P_{\text{stop}} = -34.9$ dB) is smaller by more than 20 dB, at the cost of an enlarged correlation mainlobe.

### 6.3.3 Relaxed Amplitude Constraint

In this example we illustrate the effect of a relaxed peak-to-average ratio (i.e. PAR $> 1$). The same setting as in Fig. 6-1 is used except that (6–14) replaces (6–13) in the SCAN algorithm. We choose $\rho = 2$ which constrains the PAR of the designed sequence to be less than or equal to 2. The spectral power and correlation level of the so-generated SCAN sequence are shown in Fig. 6-5, where $P_{\text{stop}} = -9.0$ dB and $P_{\text{corr}} = -19.3$ dB. In terms of $P_{\text{stop}}$ and $P_{\text{corr}}$, this relaxed PAR does not lead to much better performances. However, compared to Fig. 6-1, the spectral power in Fig. 6-5A jitters less and the correlation level in Fig. 6-5B is smaller for large time lags. Thus such a design is worth considering if PAR $> 1$ is allowed in the real system.

### 6.3.4 Using a Different Frequency Formulation

Finally we show an example in which the discrete-frequency formulation in (6–4) is replaced by the continuous-frequency one in (6–26). We still use the parameter setting in Fig. 6-1. In this case the rank of $R$ equals 22. The spectral power and correlation level
of the generated SCAN sequence are shown in Fig. 6-6, where $P_{\text{stop}} = -9.0$ dB and $P_{\text{cor}} = -18.5$ dB. We can observe that there is no performance gain by using a relatively more involved continuous-frequency stopband formulation.

Figure 6-1. The spectral power and correlation level of a SCAN sequence generated with parameters $N = 100$, $\tilde{N} = 1000$, $\lambda = 0.7$ and $\Omega = [0.2, 0.3]$ Hz. A) The spectral power (the dashed vertical lines signify the stopband borders) and B) the correlation level.
Figure 6-2. $P_{\text{stop}}$ and $P_{\text{corr}}$ vs. $\lambda$ (other settings are the same as those in Fig. 6-1).
Figure 6-3. The spectral power and correlation level of a SCAN sequence generated with parameters $N = 10^5$, $\bar{N} = 10^5$, $\lambda = 0.9$ and $\Omega = [0, 0.11) \cup [0.13, 0.19) \cup [0.25, 0.36) \cup [0.40, 0.65) \cup [0.8, 0.87) \cup [0.94, 1)$ Hz. A) The spectral power and B) the correlation level.
Figure 6-4. The spectral power and correlation level of a WeSCAN sequence generated with the same parameters as in Fig. 6-1. The correlation weights are chosen as $\gamma_1 = 0, \gamma_2 = 0$ and $\gamma_k = 1$ for $k = 3, \ldots, N - 1$. A) The spectral power and B) the correlation level.
Figure 6-5. The spectral power and correlation level of a SCAN sequence generated with the same parameters as in Fig. 6-1, except that the constraint \( \text{PAR} \leq 2 \) is imposed instead of unimodularity. A) The spectral power and B) the correlation level.
Figure 6-6. The spectral power and correlation level of a SCAN sequence generated with the same parameters as in Fig. 6-1, except that (6–26) is used in lieu of (6–4) in the SCAN algorithm. A) The spectral power and B) the correlation level.
CHAPTER 7
TRANSMIT BEAMPATTERN SYNTHESIS

Antenna array beampattern design has been a well-studied topic and there is a considerable literature available from classic analytical design [62, 68–71] to more recent works that resort to numerical optimization [72–76]. The predominant problem considered in the literature refers to the receive beampattern design, which is concerned with designing weights for the received signal so that the signal component impinging from a particular direction is reinforced while those from other directions are attenuated, a way in which certain signal properties (e.g., the signal power or direction-of-arrival) can be estimated. Such a problem usually boils down to the design of an FIR (finite-impulse-response) filter in the narrowband case or a set of FIR filters in the wideband case.

The transmit beampattern design, on the other hand, refers to designing the probing signals to approximate a desired transmit beampattern (i.e., an energy distribution in space and frequency). It has been often stated that the receive and transmit beampattern designs are essentially equivalent, which is partly true in the sense that the two scenarios bear similar problem formulations and that the FIR filter taps obtained via receive pattern design can be used theoretically as the probing signal to achieve an identical transmit pattern. In practice, however, the transmit beampattern design problem appears to be much harder because of the energy and peak-to-average power ratio (PAR) constraints on the transmit waveforms. In particular, a digital-to-analog converter scales the signal by the maximum allowable amplitude and a saturated power amplifier works well only when the signal is constant-modulus [1, 77]. If the transmitted signals have wildly varying magnitudes, we risk energy-loss or even signal distortion. As a result, the transmit beampattern design must be subject to the constraint that the transmit waveforms have unit-modulus or low PARs. On the contrary in the receive beampattern design, the FIR taps can take on any values, although
certain easy-to-meet constraints (e.g., the symmetry of the filter coefficients) are usually imposed. Therefore, except in a few simple situations such as the phased array case, the transmit beampattern design should be treated differently from the more prevalent receive beampattern design.

The narrowband transmit beampattern design problem has been addressed in the MIMO radar area [32, 45, 46, 78], and in biomedical imaging [79]. Most of the proposed methods first relate the desired beampattern to the cross-correlations between the transmit signals, then aim to design the signal covariance matrix and finally synthesize the actual signals [32]. In the wideband case, similar approaches have been proposed to design the power spectral density matrix [75], but no signals have been synthesized due to the difficulty of imposing the unit-modulus or PAR constraints.

In this chapter we propose an algorithm named WB-CA (wideband beampattern CA) to design unimodular or low-PAR sequences for transmit beampattern synthesis in wideband active sensing systems. We do not formulate the problem in terms of the transmit spectral density matrix (as was done in [75]), but instead directly link the beampattern to the signals through their Fourier transform. The design criterion is formulated in Section 7.1, which is followed by the algorithm description in Section 7.2. Simulation examples are shown in Section 7.3.

### 7.1 Problem Formulation

We focus on far-field beampattern synthesis for uniform linear arrays (ULA) as illustrated in Fig. 7-1. Note that the proposed method can be easily extended to the non-ULA case by using a more general steering vector than the one in (7–7); see, e.g., the steering vectors used in [79].

Suppose that there are $M$ linearly spaced isotropic array elements and that the inter-element spacing is $d$. The signal transmitted by the $m$th element is denoted as $s_m(t)$. Consider the beampattern in the far field at angle $\theta$ ($0^\circ \leq \theta \leq 180^\circ$) measured with respect to (w.r.t.) the array line. It is easy to see that the time delay between
two neighboring elements is \( \frac{d \cos \theta}{c} \), where \( c \) is the speed of wave propagation. We let 
\[ s_m(t) = x_m(t)e^{j2\pi f_c t} \]
where \( f_c \) is the carrier frequency and \( x_m(t) \) is the baseband signal whose spectral support is assumed to be included in the interval \([-\frac{B}{2}, \frac{B}{2}]\).

By using the above notation, the resulting far-field signal at angle \( \theta \) can be written as
\[
z_\theta(t) = \sum_{m=1}^{M} s_m \left( t - \frac{(m-1)d \cos \theta}{c} \right)
= \sum_{m=1}^{M} x_m \left( t - \frac{(m-1)d \cos \theta}{c} \right) e^{j2\pi f_c (t - \frac{(m-1)d \cos \theta}{c})}.
\]  

Suppose that the time support of \( x_m(t) \) is \([0, \tau]\). Then the Fourier transform of \( x_m(t) \) is given by
\[
y_m(f) = \int_{0}^{\tau} x_m(t)e^{-j2\pi ft} dt, \quad f \in [-\frac{B}{2}, \frac{B}{2}]
\]  
and the inverse Fourier transform is accordingly
\[
x_m(t) = \int_{-\frac{B}{2}}^{\frac{B}{2}} y_m(f)e^{j2\pi ft} df.
\]

Substituting (7–3) into (7–1) yields:
\[
z_\theta(t) = \int_{-\frac{B}{2}}^{\frac{B}{2}} Y(\theta, f)e^{j2\pi (f+f_c)t} df
\]
where
\[
Y(\theta, f) = \sum_{m=1}^{M} y_m(f)e^{-j2\pi (f+f_c)(\frac{(m-1)d \cos \theta}{c})}.
\]

It follows from (7–4) that the beampattern at spatial angle \( \theta \) and frequency \( f + f_c \) can be defined as
\[
P(\theta, f + f_c) = |Y(\theta, f)|^2 = |a^H(\theta, f)y(f)|^2, \quad f \in [-\frac{B}{2}, \frac{B}{2}]
\]
where
\[
a(\theta, f) = \left[1, e^{j2\pi(f+f_c)\frac{d\cos \theta}{c}}, \ldots, e^{j2\pi(f+f_c)\frac{(M-1)d\cos \theta}{c}}\right]^T
\]  
(7–7)
and
\[
y(f) = \left[y_1(f), y_2(f), \ldots, y_M(f)\right]^T.
\]  
(7–8)

Our problem is to design the signals \(\{x_m(t)\}_{m=1}^M\) (band-limited to \([-\frac{B}{2}, \frac{B}{2}]\)) so that the beampattern \(P(\theta, f + f_c)\) in (7–6) matches a desired one. In the sequel, the baseband frequency range \([-\frac{B}{2}, \frac{B}{2}]\) is explicitly indicated when necessary.

Digital signal processing techniques deal with the sampled signal:
\[
x_m(n) \triangleq x_m(t = nT_s), \quad n = 1, \ldots, N
\]  
(7–9)
where \(T_s\) is the symbol period that satisfies \(T_s = \frac{1}{B}\) and \(N = \lfloor \tau/T_s \rfloor\). Then (7–2) becomes (by a slight abuse of notation)
\[
y_m(fT_s) = T_s \sum_{n=1}^{N} x_m(nT_s)e^{-j2\pi nfT_s}, \quad f \in \left[-\frac{B}{2}, \frac{B}{2}\right].
\]  
(7–10)
Since the interval for \(fT_s\) is \([-0.5, 0.5]\), it is enough to consider the DFT (discrete Fourier transform) of \(\{x_m(n)\}\):
\[
y_m(p) = \sum_{n=1}^{N} x_m(n)e^{-j2\pi \frac{(n-1)p}{N}}, \quad p = -\frac{N}{2}, \ldots, 0, \ldots, \frac{N}{2} - 1
\]  
(7–11)
where \(N\) was assumed to be even (\(p\) will run from \(-\frac{N-1}{2}\) to \(\frac{N-1}{2}\) if \(N\) is odd). Note that in (7–11) we have dropped the multiplicative constant \(T_s\) from (7–10) since the scaling of \(\{y_m(p)\}\) does not affect the proposed approach (the discussions following (7–26) in Section 7.2).
Similarly to the frequency grid considered above, we also use a grid with points denoted as \( \{\theta_k\}_{k=1}^K \) for the spatial angle interval \([0^\circ, 180^\circ]\). For notational simplicity, let

\[
a_{kp} = a(\theta_k, \frac{p}{NT_s})
\]  

and

\[
y_p = \begin{bmatrix} y_1(p) & y_2(p) & \cdots & y_M(p) \end{bmatrix}^T.
\]  

Then it follows from (7–6) that the beampattern can be expressed on the discrete angle-frequency grid as

\[
P_{kp} = |a_{kp}^H y_p|^2.
\]  

Letting \( d_{kp} \) denote the desired beampattern, we seek to solve the following beampattern matching problem:

\[
\min_{\{x_m(n)\}} \sum_{k=1}^{K} \sum_{p=-N/2}^{N/2-1} \left[ d_{kp} - |a_{kp}^H y_p| \right]^2
\]  

subject to \( \text{PAR}(x_m) \leq \rho, \ m = 1, \ldots, M \)

where \( \rho \geq 1 \) is a pre-defined threshold and \( \text{PAR}(x_m) \) denotes the PAR (Eq. (4–14) in Chapter 4) of the \( m \)th sequence. As usual, we impose the energy constraint on the designed sequence: \( \sum_{n=1}^{N} |x_m(n)|^2 = N \) for \( m = 1, \ldots, M \).

The optimization problem in (7–15) is non-convex (and thus difficult in general) because of the PAR constraint. This non-convexity can be easily seen in the case of \( \rho = 1 \): then each \( x_m(n) \) can only take values from the unit-circle, which is not a convex set. Global optimization algorithms, such as the simulated annealing method, are expected to be computationally too expensive for the problem in (7–15) due to its high dimensionality. In Section 7.2, an efficient cyclic algorithm will be proposed to search for the local minimum of (7–15).
Remark: Note that \( \{x_m(n)\} \) is typically related to \( \{x_m(t)\} \) through pulse shaping:

\[
x_m(t) = \sum_{n=1}^{N} x_m(n)p(t - (n - 1)T_s), \quad m = 1, \ldots, M
\]  

(7–16)

where \( p(t) \) is the pulse (Eq. (1–1) in Chapter 1). The spectrum of the baseband signal \( x_m(t) \) would be confined to \([-B/2, B/2]\) only if \( p(t) \) were a perfect Nyquist shaping pulse (i.e., a sinc function which is centered at 0 and has the first zero-crossing at \( T_s \)). The use of any practical shaping pulse such as a (truncated) raised cosine \([2]\) will result in a leakage of the spectrum outside the desired range \([-B/2, B/2]\); these facts make (7–6) and (7–14) only approximately equivalent. We will examine the effect this approximation has on the design via an example in Section 7.3.

In addition, note that: i) the narrowband transmit beampattern design is just a special case of the wideband problem considered here and that ii) the receive beampattern design can be given a similar formulation which, however, differs in important ways from the transmit problem; see Appendix D and Appendix E for more details on these aspects.

### 7.2 The Proposed Design Methodology

Minimization of the criterion in (7–15) directly w.r.t. \( \{x_m(n)\} \) appears to be a difficult task (unless the matrix \( [a_{1p} \ldots a_{Kp}] \) turns out to be semi-unitary for each value of \( p \), which is hardly true in general). For this reason we adopt a two-stage design approach:

Stage 1 (beampattern to spectrum): First we solve (7–15) w.r.t. \( \{y_p\} \) considered to be general vectors in \( \mathbb{C}^{M \times 1} \).

Stage 2 (spectrum to waveform): Then we fit the DFT of \( \{x(n)\} \) to the so-obtained \( \{y_p\} \), subject to (s.t.) the enforced PAR constraint on \( \{x(n)\} \).

#### 7.2.1 Beampattern to Spectrum

For a generic term \( [d - |a^H y|^2] \) of (7–15) it holds that \( (d \geq 0) \)

\[
\min_{\phi} |d e^{j\phi} - a^H y|^2
\]  

(7–17)
\[= \min_{\phi} \{ d^2 + |a^H y|^2 - 2 \Re \left[ d |a^H y| \cos(\phi - \arg(a^H y)) \right] \} \]
\[= [d - |a^H y|]^2 \quad \text{(for} \quad \phi = \arg(a^H y)). \]

Consequently we can get the minimizer of (7–15) from the \( \{y_p\} \) that minimize, along with the auxiliary variables \( \{\phi_{kp}\} \), the following criterion:

\[
\sum_k \sum_p \left| d_{kp} e^{j\phi_{kp}} - a_{kp}^H y_p \right|^2. \tag{7–18}
\]

The above criterion can be conveniently minimized (w.r.t \( \{y_p\} \) and \( \{\phi_{kp}\} \)) by the cyclic algorithm outlined below.

**Stage 1 of WB-CA (Beampattern to Spectrum):**

- **Step 0:** Initialize \( \{\phi_{kp}\} \), such as \( \{\phi_{kp}\} = 0 \) or \( \{\phi_{kp}\} \) = randomly generated variables uniformly distributed in \([0, 2\pi]\).

- **Step 1:** For \( \{\phi_{kp}\} \) set at their latest values (denoted as \( \{\hat{\phi}_{kp}\}\)), let

\[
A_p = \begin{bmatrix}
a_{1p}^H \\
\vdots \\
a_{Kp}^H
\end{bmatrix}, \quad b_p = \begin{bmatrix}
d_{1p} e^{j\hat{\phi}_{1p}} \\
\vdots \\
d_{Kp} e^{j\hat{\phi}_{Kp}}
\end{bmatrix} \tag{7–19}
\]

Then (7–18) can be written as \( \sum_p \|b_p - A_p y_p\|^2 \). Thus the minimizer \( \{y_p\} \) is given by the least-squares estimate:

\[
\hat{y}_p = (A_p^H A_p)^{-1} A_p^H b_p, \quad p = -\frac{N}{2}, \ldots, 0, \ldots, \frac{N}{2} - 1. \tag{7–20}
\]

- **Step 2:** For \( \{y_p\} \) set at their latest values, the minimizer \( \{\phi_{kp}\} \) is given by (Eq. (7–17))

\[
\hat{\phi}_{kp} = \arg(a_{kp}^H \hat{y}_p). \tag{7–21}
\]

- **Iteration:** Repeat Steps 1 and 2 until convergence, e.g., until the change of \( \{\phi_{kp}\} \) in two consecutive iterations is less than a predefined threshold.

The above algorithm monotonically decreases the criterion (7–18) at each iteration, and hence it monotonically decreases the original criterion in (7–15) as well. Thus it is bound to converge to at least a local minimum value of (7–15). The basic principle of the
algorithm is related to the operation of the Gerchberg-Saxton Algorithm as described in Appendix A.

Remark: It follows from the Parseval equality that the energy-constraint on \( f_{x_m(n)} \) imposes the following constraint on \( y_p \):

\[
\sum_{p=-N/2}^{N/2-1} |y_m(p)|^2 = N \sum_{n=1}^{N} |x_m(n)|^2 = N^2, \quad m = 1, \ldots, M
\]  

(7–22)

where \( y_m(p) \) is the \( m^{th} \) element of \( y_p \) (Eq. (7–13)). Note that the steps in Stage 1 of WB-CA omit the constraint in (7–22) for simplicity (observe that this constraint yields a coupling of \( \{y_p\} \), which therefore could no longer be determined separately as in (7–20)). Nonetheless, the proposed algorithm performs reasonably well likely because the energy constraint on \( \{x_m(n)\} \) is considered in Stage 2 anyway.

7.2.2 Spectrum to Waveform

In Stage 2 we aim to synthesize the waveform \( \{x(n)\}_{n=1}^{N} \), under the PAR constraint, so that its DFT approximates the \( \{\hat{y}_p\}_{p=-N/2}^{N/2-1} \) from Stage 1 as closely as possible.

We note first that the \( \{\hat{y}_p\}_{p=-N/2}^{N/2-1} \) obtained in Stage 1 have a phase ambiguity, which can be observed from the minimization criterion in (7–18): if \( \{y_p\}, \{\phi_{kp}\} \) are minimizers of (7–18), then \( \{y_p e^{j\psi_p}\}, \{\phi_{kp} + \psi_p\} \) are also minimizers of (7–18) for any \( \psi_p \). This phase ambiguity results in fact from the original matching problem in (7–15): \( y_p \) and \( y_p e^{j\psi_p} \) lead to the same value of (7–15).

To exploit this phase flexibility associated with \( \{\hat{y}_p\} \), we introduce auxiliary variables \( \{\psi_p\}_{p=-N/2}^{N/2-1} \) and minimize the following fitting criterion w.r.t. both \( \{x_m(n)\} \) and \( \{\psi_p\} \):

\[
\sum_{p=-N/2}^{N/2-1} \left\| \hat{y}_p^T e^{j\psi_p} - \left[ 1 \quad e^{-j2\pi \frac{p}{N}} \quad \cdots \quad e^{-j2\pi \frac{(N-1)p}{N}} \right] X \right\|^2
\]  

(7–23)

where

\[
X = \begin{bmatrix} x_1 & x_2 & \cdots & x_M \end{bmatrix}
\]  

(7–24)
We further define

\[ e_p^H = \begin{bmatrix} 1 & e^{-j2\pi \frac{N}{2}} & \cdots & e^{-j2\pi \frac{(N-1)p}{N}} \end{bmatrix}, \quad p = \frac{N}{2}, \ldots, \frac{N}{2} - 1, \] (7–25)

\[
F^H = \begin{bmatrix}
    e_{-N/2}^H \\
    \vdots \\
    e_{N/2-1}^H
\end{bmatrix}_{N \times N}, \quad S^T = \begin{bmatrix}
    \hat{y}_{-N/2}^T e^{j\psi_{-N/2}} \\
    \vdots \\
    \hat{y}_{N/2-1}^T e^{j\psi_{N/2-1}}
\end{bmatrix}_{N \times M}
\]

Then (7–23) can be written as

\[
\|S^T - F^HX\|^2 = N\|\frac{1}{N}FS^T - X\|^2
\] (7–26)

where the equality comes from the fact that \( \frac{1}{\sqrt{N}}F \) is a unitary matrix.

Once again we use a cyclic algorithm to minimize (7–26) (w.r.t. \( \{x_m(n)\} \) and \( \{\psi_p\} \)); see the steps below. Note that the required matrix calculations \( F^HX \) and \( FS^T \) therein can be done by the FFT, which reduces the computation time.

Stage 2 of WB-CA (Spectrum to Waveform):

- Step 0: Initialize \( \{\psi_p\} \), for instance as \( \{\psi_p\} = 0 \).

- Step 1: For \( \{\psi_p\} \) fixed at their latest values, the minimization of (7–26) w.r.t. \( \{x_m(n)\} \) depends on the considered PAR constraint. Under the unit-modulus constraint (i.e., \( |x_m(n)| = 1 \)), the minimization of (7–26) is immediate:

\[
\hat{x}_m(n) = \exp(j \arg \{\text{the } (n,m)\text{ element of } FS^T\}),
\] (7–27)

\[
m = 1, \ldots, M, \quad n = 1, \ldots, N.
\]

If the \( \text{PAR} \leq \rho \) (\( \rho > 1 \)) constraint is imposed, we need to solve \( M \) separate minimization problems:

\[
\min_{x_m} \|u_m - x_m\|^2
\] s.t. \( \text{PAR}(x_m) \leq \rho \) (7–28)
(for \( m = 1, \ldots, M \)) where \( u_m \) is the \( m \)th column of \( \frac{1}{M} F S^T \). This problem can be solved efficiently by the “nearest-vector” method outlined in Section 4.2 of Chapter 4.

- Step 2: For \( \{x_m(n)\} \) fixed at their most recent values, the minimizer \( \{\psi_p\} \) is given by (the derivation of (7–29) is similar to (7–17)):
  \[
  \hat{\psi}_p = \text{arg}\{y_p^H v_p\}, \quad p = -\frac{N}{2}, \ldots, \frac{N}{2} - 1
  \]  
  (7–29)
  where \( v_p^T \) is the \((p + N/2)^{th}\) row of \( F^H X \).

- Iteration: Repeat Steps 1 and 2 until convergence.

It is easy to see that a possible scaling of \( S \) has no effect on (7–27). The same is true for (7–28) (which follows from the operation of the method in [53] used to solve (7–28)). Therefore, we can choose the desired beampattern \( \{d_{kp}\} \) in (7–18) without any concern for a possible normalization, as \( \{y_p\} \) will automatically scale (Eq. (7–20)) to fit the chosen \( \{d_{kp}\} \) and the scaling of \( \{y_p\} \) does not affect the synthesis of \( \{x_m(n)\} \).

To summarize, the proposed two-stage design methodology, first determining \( \{y_p\} \) and then \( \{x_m(n)\} \), basically reduces the problem in (7–18) to the design of \( N \) beamforming vectors \( \{y_p\} \), one for each frequency bin and then to matching them by the selection of \( \{x_m(n)\} \). Note that there are \( 2MN \) real-valued elements in \( \{y_p\} \), and \( MN \) free variables in \( \{x_m(n)\} \) under the unit-modulus constraint, and more than \( MN \) degrees of freedom if the PAR is allowed to be larger than 1. In addition, the \( \{\psi_p\} \) provide \( N \) degrees of freedom. Hence we can expect a “reasonable” performance for the matching step of the proposed approach.

Although WB-CA relies on an iterative process, the updating formulas are relatively simple and the iteration turns out to converge very fast. For the numerical examples presented in the next section, the execution of the WB-CA algorithm coded in MATLAB takes only a few seconds on an ordinary PC.
7.3 Numerical Examples

Unless stated otherwise, the following setting is used in this section: a ULA with $M = 10$ elements, the carrier frequency of the transmitted signal is $f_c = 1$ GHz, the bandwidth is $B = 200$ MHz and the number of symbols is $N = 64$. The symbol period is $T_s = 1/B$. The inter-element spacing is given by $d = \frac{c}{2(f_c + B/2)}$, that is, half wavelength of the highest in-band frequency to avoid grating lobes. The spatial angle is divided into $K = 180$ grid points (i.e., one degree per grid step).

Remark: In practical applications the antenna elements of an array are typically mutually coupled. The inter-element spacing $d$ chosen above results in over-sampling (i.e. sampling interval less than half wavelength) for lower in-band frequencies and may render the mutual coupling effects non-negligible, which could lead to energy being coupled into transmitters. However, this issue lies outside the scope of this chapter (as it depends on the specific hardware implementation such as the system tolerance and antenna types), and we refer the interested reader to [80–82] for discussions on decoupling.

7.3.1 The Idealized Time-Delayed Case

It follows from (7–6) that we can steer the transmit beam towards the angle $\theta_0$ by choosing the following signal spectrum:

$$y(f) = \sqrt{N}a(\theta_0, f), \quad f \in \left[-\frac{B}{2}, \frac{B}{2}\right]$$

(7–30)

where $\sqrt{N}$ is due to the energy constraint. Eq. (7–30) leads to (Eq. (7–14))

$$P_{kp} = N \sum_{m=1}^{M} e^{2\pi j\left(\frac{m}{M} + f_c\left(m-1\right)\frac{c(\cos \theta_0 - \cos \theta_p)}{2}\right)}$$

(7–31)

where for a fixed value of $f$ (i.e., $p$), the beam is steered in the direction of $\theta_0$ as in the case of a narrowband phased array. The underlying signals, i.e., the inverse Fourier
transform of (7–30), are given (up to a multiplicative constant) by

\[ x_m(t) = \text{sinc} \left( \frac{\pi}{T_s} \left( t - \frac{(m-1)d\cos\theta_0}{c} \right) \right), \quad m = 1, \ldots, M \tag{7–32} \]

where \( \text{sinc}(t) = \sin(x)/x \). Note that such an idealized \( \{x_m(t)\} \) has a very high PAR which is undesirable. Moreover, because \( d/c = 1/(2f_c + B) \ll T_s \), the required time delay of \( \frac{md\cos\theta_0}{c} \) can be too small to be readily implemented in practice, especially when \( \theta_0 \) is close but not equal to \( 90^\circ \).

We show the beampattern \( 10 \log(P_{kp}/N) \) in (7–31) for \( \theta_0 = 120^\circ \) as a 2D plot in Figure 7-2A, as well as a 3D plot in Figure 7-2B. The beampattern exhibits a clean mainlobe at \( \theta_0 \) across the entire frequency range.

**Remark:** In the narrowband case, for a given ULA aperture, the transmit beampattern generated by a phased array has the smallest mainlobe width. In the above example, we used the impractical time-delayed \( \{x_m(t)\} \) in (7–32) to get the phased array-like beampattern in (7–31), which thus has the narrowest mainlobe for every fixed frequency. Therefore we call it the “idealized time-delayed” case.

It has been assumed that for each array element an energy equal to \( N \) is emitted in all directions; and the energy constraint \( \sum_p |y_m(p)|^2 = N^2 \) in (7–22) indicates that on average \( |y_m(p)|^2 \) equals \( N \). Therefore, were there only one array element, \( P_{kp} \) would equal \( N \) at every grid point in the angle-frequency plane (Eq. (7–31)). This is the reason why the normalization \( 10 \log(P_{kp}/N) \) is used in all plots. Now that there are \( M \) transmit waveforms, the coherent sum gives \( \max P_{kp} = \max |a_{kp}^H y_p|^2 = |M\sqrt{N}|^2 = M^2 N \), which leads to \( 10 \log(M^2 N/N) = 20 \) dB in the plot. In fact, in the above idealized time-delayed case, all \( M \) waveforms add coherently at \( \theta = \theta_0 \) and the energy is evenly distributed at \( \theta = \theta_0 \) for all frequencies, which produces a constant 20 dB mainlobe height (Figs. 7-2 and (7–31)). In other examples, however, the mainlobe height is not necessarily 20 dB and the upper limit of the colorbar always corresponds to the largest value in the plot.
7.3.2 A Narrow Mainbeam

We use the proposed WB-CA algorithm to synthesize the following desired transmit beampattern:

\[
d(\theta, f + f_c) = \begin{cases} 
1, & \theta = 120^\circ \quad \text{for all } f \in \left[-\frac{B}{2}, \frac{B}{2}\right], \\
0, & \text{elsewhere}
\end{cases} \tag{7–33}
\]

that is, a beampattern with the mainlobe (as narrow as possible) located at 120° across the frequency support.

Stage 1 of WB-CA generates the DFT vectors \(\{\hat{y}_p\}_{p=-N/2}^{N/2-1}\), which are further normalized to preserve the total energy (i.e., normalized such that \(\sum_p \|y_p\|^2 = MN^2\)). Fig. 7-3 shows the beampattern \(P_{kp}\) that is calculated directly from these \(\{\hat{y}_p\}\). The so-obtained beampattern is quite similar to the idealized one in Fig. 7-2. However, the underlying waveforms corresponding to Fig. 7-3, given by the inverse DFT (IDFT) of \(\{\hat{y}_p\}_{p=-N/2}^{N/2-1}\), do not satisfy the energy and PAR requirement. Indeed, the \(M\) sequences obtained from the IDFT of \(\{\hat{y}_p\}\) have energies varying from 55.4 to 71.2 and PARs varying from 1.3 to 1.8. Note that such transmit sequences need to be scaled in practice so that the maximum energy does not exceed the system specifications, which will inevitably result in an energy loss.

We then proceed to Stage 2 of WB-CA and synthesize the sequences \(\{\hat{x}_m(n)\}\) under the unit-modulus constraint. After that, we compute the DFT of \(\{\hat{x}_m(n)\}\) and obtain the beampattern using (7–14); see Fig. 7-4. It is clear that the strict unit-modulus constraint degrades the beampattern matching. Table 7-1 shows the minimum value of the fitting criterion (Eq. (7–15)) associated with Figs. 7-3 and 7-4, respectively.

Next we examine the beampattern of the continuous-time waveforms corresponding to the so-obtained \(\{\hat{x}_m(n)\}\) (the Remark at the end of Section 7.1). More specifically, we pass each \(\{\hat{x}_m(n)\}_{n=1}^N (m = 1, \ldots, M)\) through an FIR raised-cosine filter (with the roll-off factor equal to 0.5) to get the continuous-time signal \(\hat{x}_m(t)\). The spectral density
functions of \( \{\hat{x}_m(t)\} \) are shown in Fig. 7-5 in an overlapping manner, from which we observe that the spectrum is well contained within \([f_c - B/2, f_c + B/2]\) despite of a certain leakage outside the frequency range of interest. The beampattern of \( \{\hat{x}_m(t)\} \), as defined in (7–6), is shown in Fig. 7-6. Compared to Fig. 7-4, the beampattern in Fig. 7-6 is a poorer approximation of the desired one. As discussed in the Remark at the end of Section 7.1, a practical pulse shaping renders (7–6) and (7–14) not exactly equivalent. Since the WB-CA algorithm aims to match (7–14) to the desired beampattern, the nonequivalence between (7–6) and (7–14) explains the degradation from Fig. 7-4 to Fig. 7-6.

### 7.3.3 Two Mainbeams

In this example we consider the following desired beampattern:

\[
d(\theta, f + f_c) = \begin{cases} 
1, & f_c - B/2 \leq f \leq f_c \text{ and } \theta = 120^\circ \\
1, & f_c \leq f \leq f_c + B/2 \text{ and } \theta = 60^\circ \\
0, & \text{elsewhere}
\end{cases}
\]  

(7–34)

The WB-CA beampattern under the unit-modulus constraint is shown in Fig. 7-7 and that under the PAR \( \leq 2 \) constraint is shown in Fig. 7-8. While Fig. 7-7 already provides a reasonably good beampattern matching, relaxing the PAR from 1 to 2 leads to the visibly better result in Fig. 7-8 due to more degrees of freedom in the waveform design. Table 7-2 shows this performance improvement in terms of the corresponding fitting criterion values.

### 7.3.4 A Wide Mainbeam

In both examples above, we focused on achieving a beampattern where mainlobe(s) were as narrow as possible. Specifically the idealized phased array-like beampattern in Fig. 7-2, which has the narrowest possible mainlobe, was well approximated by using practical waveforms in Section 7.3.2. If we want to obtain a narrower mainlobe, we have to use a larger value of \( M \), i.e. more transmit antenna elements.
Here we consider instead the following beampattern with a wider mainlobe:

\[
\Psi(\theta, f + f_c) = \begin{cases} 
1, & 100^\circ \leq \theta \leq 140^\circ \\
0, & \text{elsewhere}
\end{cases} \quad \text{for all } f \in \left[ -\frac{B}{2}, \frac{B}{2} \right].
\] (7–35)

The corresponding WB-CA beampattern under the unit-modulus constraint is shown in Fig. 7-9 and that under the PAR < 2 constraint in Fig. 7-10. We observe that the mainlobe in Fig. 7-9 or 7-10 has an almost constant width for different frequencies, unlike the mainlobe in Fig. 7-2 whose width tends to (slightly) shrink as the frequency increases (the well-known beam squint phenomenon). Also note the “mainlobe splitting” in Figs. 7-9 or 7-10. Had we synthesized an even wider mainlobe than (7–35), the splitting would have been more severe (e.g., the mainlobe can split twice so that there are three local maxima in the mainlobe area).

In all above examples the bandwidth \( B \) was set to 200 MHz. A larger bandwidth means more constraints and thus the beampattern matching is expected to become more difficult. To illustrate this fact, we repeat the example corresponding to Fig. 7-4 except that the bandwidth is now equal to 350 MHz. The result is shown in Fig. 7-11, where the beampattern is more irregular than in Fig. 7-4.

Regarding choosing \( N \) (the number of transmitted symbols), we note that increasing \( N \) does not necessarily lead to a better beampattern matching. The reason is that while a larger \( N \) increases the number of degrees of freedom of the waveform \( \{x_m(n)\} \), it also increases proportionally the number of elements in \( \{y_p\} \) that are to be matched in Stage 2 of WB-CA (the discussion at the end of Section 7.2). At the same time, \( N \) cannot be chosen too small because the frequency grid should be dense enough to cover the frequency support finely.

We finally point out that the initialization of WB-CA (i.e. Step 0 in both Stages) does not play an important role in the algorithm performance. In all numerical examples presented, randomly generated phases were used for initialization; different initializations
led to different waveforms but all of these waveforms had similar beampatterns. This also signifies the fact that the beampattern matching problem is highly multi-modal.

![ULA array configuration](image)

**Figure 7-1.** The ULA array configuration.

<table>
<thead>
<tr>
<th></th>
<th>Fig. 7-3</th>
<th>Fig. 7-4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Criterion in (7–15)</td>
<td>5987914</td>
<td>6048430</td>
</tr>
</tbody>
</table>

**Table 7-1. Optimized criterion values for Fig. 7-3 and 7-4**

<table>
<thead>
<tr>
<th></th>
<th>Fig. 7-7</th>
<th>Fig. 7-8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Criterion in (7–15)</td>
<td>5698808</td>
<td>5619700</td>
</tr>
</tbody>
</table>

**Table 7-2. Optimized criterion values for Fig. 7-7 and 7-8**
Figure 7-2. The idealized time-delayed beampattern in (7–31). A) The 2D plot and B) the 3D plot.
Figure 7-3. The WB-CA beampattern under only the total energy constraint. The desired beampattern is given in (7–33). A) The 2D plot and B) the 3D plot.
Figure 7-4. The WB-CA beampattern under the unit-modulus constraint. The desired beampattern is given in (7–33). A) The 2D plot and B) the 3D plot.
Figure 7-5. The overlaid spectral densities of the continuous waveforms corresponding to the sequences used in Fig. 7-4. The two vertical dashed lines represent the boundaries of the frequency range of interest.
Figure 7-6. The WB-CA beampattern of the continuous waveforms corresponding to the sequences used in Fig. 7-4. The desired beampattern is given in (7–33). A) The 2D plot and B) the 3D plot.
Figure 7-7. The WB-CA beampattern under the unit-modulus constraint. The desired beampattern is given in (7–34). A) The 2D plot and B) the 3D plot.
Figure 7-8. The WB-CA beampattern under the PAR \leq 2 constraint. The desired beampattern is given in (7–34). A) The 2D plot and B) the 3D plot.
Figure 7-9. The WB-CA beampattern under the unit-modulus constraint. The desired beampattern is given in (7–35). A) The 2D plot and B) the 3D plot.
Figure 7-10. The WB-CA beampattern under the PAR ≤ 2 constraint. The desired beampattern is given in (7–35). A) The 2D plot and B) the 3D plot.
Figure 7-11. The WB-CA beampattern for the same settings as in Fig. 7-4, excepting that the bandwidth $B$ is changed from 200 to 350 MHz. A) The 2D plot and B) the 3D plot.
CHAPTER 8
CONCLUSIONS

Several computational algorithms have been presented for transmit waveform design in active sensing applications. Waveforms are designed with the goal of achieving certain properties that include low correlation sidelobes, stopband constraint and beampattern matching. Discussions in this work lean toward formulating practical problems mathematically and solving the mathematical problems using optimization techniques. Each algorithm is tested via extensive numerical examples to assess its effectiveness. Particular attention is paid to making the developed algorithms computationally efficient. Besides newly developed algorithms, a considerable portion of this work presents tutorial-like materials, e.g., reviewing existing waveforms, analyzing properties of ambiguity functions, describing application scenarios, etc.

It is worth pointing out that none of the discussed waveform design requirements (such as minimization of correlation sidelobes) has a closed-form solution. They yield non-convex optimization problems that cannot be solved globally. The proposed iterative algorithms start from an initialization, minimize the objective function in a cyclic way (i.e., with respect to variables alternately) and are guaranteed for a local convergence. While depending on the initialization is a downside of these algorithms, it is at the same time an advantage because different initializations lead to different final waveforms which usually all have desired properties.

We have focused mainly on phase-coded waveforms in this work. As mentioned in the chapter of introduction, there are many other types of signals that are widely used or have been discussed in the literature such as frequency-modulated waveforms. Different types of waveforms lead to different problem formulations, which can significantly complicate the discussion as we have many different requirements of waveform properties. The model of phase-coded waveform meets practical constraints such as the unit-modulus constraint, is comfortable to handle mathematically and most
importantly turns out to have good performances. By focusing on one waveform model, we were able to tell a consistent story throughout the discussions.

Among the discussed topics, minimizing the sidelobes of an ambiguity function (AF) is the most challenging one. The attained performance is rather limited due to the large number of constraints in a two-dimensional space. As a widely used tool in radar signal analysis (as well as in other fields of active sensing), AF deserves more attention in our future endeavors.
APPENDIX A
CONNECTIONS TO A PHASE-RETRIEVAL ALGORITHM

The CAN algorithm introduced in this chapter for designing code sequences with impulse-like correlations is related to the Gerchberg-Saxton Algorithm (GSA) introduced for phase retrieval in the optics literature [83] some 40 years ago. The technique used in GSA appeared in fact earlier in [55], so GSA might be better named the Sussman-Gerchberg-Saxton Algorithm. However, to be consistent with most other literature, we will still use the name GSA.

In this appendix, we attempt to describe and clarify the relationship between CAN and GSA. We also present some facts on GSA that appear to be useful in their own right.

Let \( x \) be an \( N \times 1 \) vector and consider the problem of minimizing w.r.t. (with respect to) \( x \) the criterion:

\[
C(x) = \sum_{k=1}^{K} \left| a_k^H x - d_k \right|^2
\]

where \( d_k \in \mathbb{R}^+ \) and \( a_k \in \mathbb{C}^{N \times 1} \) are given and \( K \) is an integer that typically satisfies \( K \geq N \). In some applications, the vector \( x \) is free to vary in \( \mathbb{C}^{N \times 1} \) [84]. In other applications \( x \) is constrained to a certain subset of \( \mathbb{C}^{N \times 1} \), such as to the set of vectors with unimodular elements (i.e. \( |x_k| = 1 \)). To take this fact into account, we let \( x \in S \subseteq \mathbb{C}^{N \times 1} \).

The GSA was introduced in [55, 83] for tackling recovery problems typically involving a sequence and its Fourier transform. When used for problems that can be formulated as in (A–1), GSA has the form described below.

The Gerchberg-Saxton algorithm:

- Step 0: Given initial values \( \{ \phi_k^0 \}_{k=1}^{K} \) (\( \{ \phi_k \} \) are auxiliary variables; see below for details), iterate Steps 1 and 2 below, for \( i = 0, 1, \ldots \) until convergence.
- Step 1: \( x^i = \arg \min_{x \in S} \sum_{k=1}^{K} |a_k^H x - d_k e^{j\phi_k}|^2 \).
- Step 2: \( \phi_k^{i+1} = \arg(a_k^H x^i) \) and \( i \leftarrow i + 1 \).
Note that [83] proposed the above algorithm on heuristic grounds, without any reference to the minimization of $C(x)$ in (A–1). However it was realized later on in [85] that GSA is a minimization algorithm for (A–1) which has the appealing property of monotonically decreasing the criterion as the iteration proceeds. A simple proof of this fact is as follows:

$$C(x^i) = \sum_{k=1}^{K} |a_k^H x - d_k|^2 = \sum_{k=1}^{K} |a_k^H x^i - d_k e^{j\phi_k^{i+1}}|^2$$ (A–2)

$$\geq \sum_{k=1}^{K} |a_k^H x^{i+1} - d_k e^{j\phi_k^{i+1}}|^2$$

$$\geq \sum_{k=1}^{K} |a_k^H x^{i+1} - d_k e^{j\phi_k^{i+2}}|^2 = C(x^{i+1})$$

where the first inequality is due to Step 1 and the second inequality is due to Step 2 (these inequalities are strict if the solutions computed in Steps 1 and 2 are unique, which is usually the case in applications).

The calculation in (A–2) provides a way to motivate GSA as a minimization algorithm for $C(x)$. In the following we outline a way to derive GSA as a minimizing procedure for $C(x)$.

Let $\phi$ denote a $K \times 1$ vector of auxiliary variables and let $D(x, \phi)$ be a function which has the property that:

$$\min_{\phi} D(x, \phi) = C(x).$$ (A–3)

Then, under rather general conditions, the $x$ that minimizes $C(x)$ is the same as the $x$ obtained from the minimization of $D(x, \phi)$ w.r.t. both $x$ and $\phi$. Evidently, for this approach to be useful the minimization of $D(x, \phi)$ should be easier to handle than that of $C(x)$. To use the above idea in the present case of (A–1), we let

$$D(x, \phi) = \sum_{k=1}^{K} |a_k^H x - d_k e^{j\phi_k}|^2$$ (A–4)
(where $\phi$ is the vector made from $\{\phi_k\}_{k=1}^{K}$) and note that the above function has the required property:

\[
\min_{\phi} D(x, \phi) = \min_{\phi} \sum_{k=1}^{K} \left[ |a_k^H x|^2 + d_k^2 - 2|a_k^H x|d_k \cos(\arg(a_k^H x) - \phi_k) \right]
\]

\[
= \sum_{k=1}^{K} [ |a_k^H x| - d_k ]^2 = C(x).
\]

Clearly the minimization of $D(x, \phi)$ w.r.t. $x$ (unconstrained as in [84] or constrained as in this chapter) for fixed $\phi$ and, respectively, w.r.t. $\phi$ for fixed $x$ has simple closed-form solutions. Consequently $D(x, \phi)$, and hence $C(x)$, can be minimized conveniently via a cyclic algorithm in which $\phi$ is fixed to its most recent value and $D(x, \phi)$ is minimized w.r.t. $x$, and vice versa. The so-obtained algorithm is nothing but GSA and its property in (A–2) follows immediately from (A–3) and the fact that the cyclic minimization of $D(x, \phi)$ yields a monotonically decreasing sequence of criterion values: $C(x^i) = D(x^i, \phi^{i+1}) \geq D(x^{i+1}, \phi^{i+2}) = C(x^{i+1})$.

The central problem dealt with in Chapter 2 was the design of a code sequence with impulse-like aperiodic and, respectively, periodic correlations. A main result proved is the fact that the said problem can be reduced to that of minimizing a criterion of the form:

\[
\tilde{C}(x) = \sum_{k=1}^{K} [ |a_k^H x|^2 - d_k^2 ]^2
\]

for a certain $K$ and certain $\{a_k\}$ and $\{d_k\}$ (whose exact definitions are not of importance to the present discussion).

The criterion in (A–6) might seem rather similar to the $C(x)$ in (A–1), but in fact there are important differences between these two criteria. A first difference is that (A–3)–(A–5) obviously do not hold for $\tilde{C}(x)$. Consequently one cannot derive a GS-type algorithm for (A–6) by following the approach based on (A–3) and (A–4). Of course, we
could use a $\tilde{D}(x, \phi)$ defined as

$$
\tilde{D}(x, \phi) = \sum_{k=1}^{K} |(a_k^H x)^2 - d_k^2 e^{j\phi_k}|^2
$$

(A–7)

for which it holds that $\min_{\phi} \tilde{D}(x, \phi) = \tilde{C}(x)$, as required. However, the minimization of $\tilde{D}(x, \phi)$ is not easier than that of $\tilde{C}(x)$.

To get around the above problem, a principal observation made in Chapter 2 was that, under certain conditions, the minimization of (A–6) is almost equivalent to that of $D(x, \phi)$ in (A–4). Using this observation and the minimization approach outlined in the paragraph following (A–5), the cyclic algorithm termed CAN was introduced for minimizing $D(x, \phi)$. CAN has the same form as GSA. However, note that now the minimization of $D(x, \phi)$ does not necessarily provide a solution to the problem of minimizing $\tilde{C}(x)$. In particular, a second difference between the criteria $C(x)$ and $\tilde{C}(x)$ is that the proposed algorithms do not guarantee that the criterion $\tilde{C}(x)$ monotonically decreases as the iteration proceeds (only $D(x, \phi)$ is monotonically decreased by each iteration).

Finally, we remark on the fact that the WeCAN (weighted CAN) and multi-variate CAN algorithms (Chapter 3), although related to GSA in their basic principles, have a weaker connection to GSA than CAN and PeCAN. These algorithms, which have been obtained by means of the “almost equivalent” minimization approach mentioned in the previous paragraph, can be viewed as extensions of GSA to problems that have more involved forms than (A–1).
APPENDIX B
DERIVATION OF A UNITARY MATRIX SOLUTION

We prove in the following that the solution of (3–45) for fixed $X$ is given by (3–47).

The criterion in (3–45) can be written as

$$\|\bar{X} - \sqrt{N}U\|^2 = \text{tr} \left\{ (\bar{X}^H - \sqrt{N}U^H)(\bar{X} - \sqrt{N}U) \right\}$$  \hspace{1cm} \text{(B–1)}

$$= \text{const} - 2\sqrt{N} \text{Re} \{ \text{tr}(\bar{X}^H U) \}$$

where const denotes the term that does not depend on $U$ (note that $X$ is assumed known and that $U^H U = I$).

Using (3–46) we obtain

$$\text{Re} \{ \text{tr}(\bar{X}^H U) \} = \text{Re} \{ \text{tr}(U_1 S U_2^H U) \} = \text{Re} \{ \text{tr}(U_2^H U U_1 S) \}$$

$$= \sum_{k=1}^{MP} \text{Re} \left\{ (U_2^H U U_1)_{k,k} \right\} S_{kk}$$  \hspace{1cm} \text{(B–2)}

where $[\cdot]_{kk}$ denotes the $(k, k)^{th}$ element of a matrix.

For notational simplicity, let $B = U_2^H U U_1$ and then we have

$$|\text{Re} \{ B_{kk} \}|^2 \leq |B_{kk}|^2 \leq [BB^H]_{kk} = [U_2^H U U_1 U_1^H U^H U_2]_{kk}$$

$$=[U_2^H U U^H U_2]_{kk}.$$  \hspace{1cm} \text{(B–3)}

Note that $U$ is a “tall” semi-unitary matrix, which leads to the fact that $UU^H \leq I$ and thus

$$|\text{Re} \{ B_{kk} \}|^2 \leq [U_2^H U_2]_{kk} = 1.$$  \hspace{1cm} \text{(B–4)}

It follows from (B–1) – (B–4) that

$$\|\bar{X} - \sqrt{N}U\|^2 = \text{const} - 2\sqrt{N} \sum_{k=1}^{MP} \text{Re} \left\{ (U_2^H U U_1)_{k,k} \right\} S_{kk} \geq \text{const} - 2\sqrt{N} \sum_{k=1}^{MP} S_{kk}$$  \hspace{1cm} \text{(B–5)}

which is another constant independent of $U$. It is not difficult to see that the equality in (B–5) holds if and only if $U = U_2 U_1^H$, which concludes the proof.
APPENDIX C
PROPERTIES OF AMBIGUITY FUNCTION

Here we prove the three AF properties mentioned in Section 5.1.

- **Maximum value at the origin:**
  By using the Cauchy-Schwartz inequality we can get
  \[ |\chi(\tau, f)|^2 \leq \int_{-\infty}^{\infty} |u(t)|^2 dt \cdot \int_{-\infty}^{\infty} |u^*(t-\tau)e^{-j2\pi f(t-\tau)}|^2 dt = E^2 \quad \text{(C–1)} \]
  where \( E \) denotes the energy of \( u(t) \) (Eq. (5–3)). Since \( |\chi(0, 0)| = E \), it follows that the maximum value of \( |\chi(\tau, f)| \) is achieved at the origin.

- **Symmetry**
  A simple variable change \((t \leftarrow t+\tau)\) shows that
  \[ \chi(-\tau, -f) = \int_{-\infty}^{\infty} u(t)u^*(t+\tau)e^{j2\pi f(t+\tau)} dt = \int_{-\infty}^{\infty} u(t-\tau)u^*(t)e^{j2\pi ft} dt. \quad \text{(C–2)} \]
  which implies the symmetry property: \( |\chi(\tau, f)| = |\chi(-\tau, -f)| \).

- **Constant volume**
  The volume of \( |\chi(\tau, f)|^2 \) is given by
  \[ V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau, f)|^2 d\tau df \]
  \[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} u(t)u^*(t-\tau)e^{-j2\pi ft} dt \right)^2 d\tau df. \quad \text{(C–3)} \]
  Let \( W_\tau(f) \) denote the Fourier transform of \( u(t)u^*(t-\tau) \). By using the Parseval equality we get
  \[ \int_{-\infty}^{\infty} |W_\tau(f)|^2 df = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} u(t)u^*(t-\tau)e^{-j2\pi ft} dt \right)^2 df \quad \text{(C–4)} \]
  \[ = \int_{-\infty}^{\infty} |u(t)u^*(t-\tau)|^2 dt. \]
  Therefore
  \[ V = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(t)u^*(t-\tau)|^2 dt d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |u(x)u^*(y)|^2 dxdy \]
  \[ = \int_{-\infty}^{\infty} |u(x)|^2 dx \int_{-\infty}^{\infty} |u(y)|^2 dy = E^2 \quad \text{(C–5)} \]
  where the change of variables \( \{x = t, y = t-\tau\} \) are used.
APPENDIX D
NARROWBAND TRANSMIT BEAMPATTERN

In the narrowband case, \( B \ll f_c \) and therefore the distribution of energy versus frequency \( f \) is of less interest. Instead, the total energy over \( f \) is the quality of interest, which is given by (Eq. (7–14))

\[
P(\theta_k) = \sum_{p=-N/2}^{N/2-1} P_{kp} = \sum_{p=-N/2}^{N/2-1} |a_{kp}^H y_p|^2. \quad (D–1)
\]

Since the inter-element spacing \( d \) is on the order of the carrier wavelength, the narrowband assumption \( B \ll f_c \) implies that \( \frac{fd\cos\theta}{c} \sim 0 \) \((-B/2 \leq f \leq B/2\)), which means that the steering vector \( a_{kp} \) is independent of frequency. Thus its subscript \( p \) can be dropped and (D–1) becomes

\[
P(\theta_k) = a_k^H \left[ \sum_{p=-N/2}^{N/2-1} y_p y_p^H \right] a_k
\]

\[
= a_k^H \left[ \sum_{u=1}^{N} \sum_{v=1}^{N} x(u) x^H(v) \left( \sum_{p=-N/2}^{N/2-1} e^{-j2\pi \frac{(u-v)p}{N}} \right) \right] a_k
\]

\[
= a_k^H \left[ N \sum_{n=1}^{N} x(n) x^H(n) \right] a_k \quad (D–2)
\]

where \( x(n) \) is defined as

\[
x(n) = \left[ x_1(n) \ x_2(n) \ \cdots \ x_M(n) \right]^T, \ n = 1, \ldots, N. \quad (D–3)
\]

The result in (D–2), up to a multiplicative constant, coincides with the narrowband beampattern expression used in [46].
APPENDIX E
RECEIVE BEAMPATTERN

Paralleling the discussion in Section 7.1, we briefly formulate here the receive
beampattern synthesis problem for wideband signals.

Suppose that a wideband signal \( g(t)e^{j2\pi f_c t} \) with frequency band \([f_c - B/2, f_c + B/2]\) is
impinging from angle \( \theta \) \((0^\circ \leq \theta \leq 180^\circ)\) on a ULA. Let \( G(f) \) denote the Fourier transform
of \( g(t) \). The signal received at the \( m \)th array element can be written as

\[
    r_m(t) = g\left( t - \frac{(m - 1)d\cos\theta}{c} \right) e^{j2\pi f_c (t - \frac{(m - 1)d\cos\theta}{c})}
    \]

\[
    = \int_{-B/2}^{B/2} G(f)e^{-j2\pi (f + f_c) \frac{(m - 1)d\cos\theta}{c}} e^{j2\pi (f + f_c) t} df.
    \tag{E-1}
\]

Let \( H_m(f) \) denote the frequency response of the FIR filter used to process the
demodulated signal \( r_m(t)e^{-j2\pi f_c t} \). Then the receive beampattern can be expressed
in the frequency domain as

\[
    A(\theta, f + f_c) = \left| \sum_{m=1}^{M} H_m(f)e^{-j2\pi (f + f_c) \frac{(m - 1)d\cos\theta}{c}} \right|^2,
    \tag{E-2}
\]

where \( G(f) \) is omitted because it is the same for all array elements. The receive
beampattern synthesis problem can be stated as designing a set of \( M \) filters \( \{h_m(t)\}_{m=1}^{M} \)
(the Fourier transform of \( h_m(t) \) is \( H_m(f) \)) such that \( A(\theta, f + f_c) \) matches a desired
pattern.

As pointed out in the beginning of Chapter 7, there is no essential constraint on
\( \{h_m(t)\}_{m=1}^{M} \), the design of which can therefore be done by a host of approaches, such
as classic filter design methods [71] or convex optimization [72]. On the other hand, the
transmit beampattern design, which has been the topic of this chapter, is much harder
because of the PAR constraint, despite the fact that (7–6) and (E–2) have the same
form.
REFERENCES


BIOGRAPHICAL SKETCH

Hao He received the degree of Bachelor of Science from the University of Science and Technology of China, Hefei, China, in 2007, and the degree of Master of Science from the University of Florida, Gainesville, FL, in 2009, both in electrical engineering. He graduated with the degree of Doctor of Philosophy from the Department of Electrical and Computer Engineering at University of Florida in the summer of 2011.

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