PATH-DEPENDENT OPTION PRICING: EFFICIENT METHODS FOR LÉVY MODELS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2010

To my three favorite guys: My husband, Árni; my dad, Gylfi; and my brother, Thröstur

ACKNOWLEDGMENTS

None of this would have been possible without my advisor and friend Farid. His enthusiasm and ambition inspired me and I am thankful for all the knowledge he shared with me during countless hours. Murali, of my committee, taught me a great deal of math, yet I am most appreciative for getting to know him as a friend. I would like to thank both Murali and Farid for all our inspirational conversations. Also, I would like to thank my committee members, Dr. Pardalos and Dr. Nimalendran for their support. Thanks go out to my family for their love, all the phone calls and for their lovely visits: My dad Gylfi, my brother Thröstur, my sister-in-law Una, my nephews Thorri and Fródi and my parents-in-law Erna and Jón. Also, thanks go out to my friends: Alex, Mireia, Ehsan, Kelly, Vera, Altannar, Ashwin, Shantih, Emily, May, Soheil, Behnam, Renee, Clay, Filip, Unnur, Helga Björk, Helga Björk, Svanhvít, Elín, Anna Gyda, Ragnheidur, Lara and Jacki. Florida's nature with all its magical wonders made being here an amazing experience. And lastly, my deepest gratitude goes to Árni, who is the most loving husband I could wish for and has been here for me all of this time. I am really grateful that we got to share this experience.

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August 2010

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This dissertation is concerned with the pricing of path-dependent options where the underlying asset is modeled as a continuous-time exponential Lévy process and is monitored at discrete dates. These options enable their users to tailor random payoff outcomes to their particular risk profiles and are widely used by hedgers such as large multinational corporations and speculators alike. The use of continuous–time models since the breakthrough paper of Black and Scholes has been greatly facilitated by advances in stochastic calculus and the mathematical elegance it provides. The recent financial crisis started in 2008 has highlighted the importance of models that incorporate the possibility of sudden, large jumps as well as the higher likelihood of adverse outcomes as compared with the classical Black-Scholes model. Increasingly, exponential Lévy processes have become preferred alternatives, thanks in particular to the explicit Lévy–Khinchin representation of their characteristic functions. On the other hand, the restriction of monitoring dates to a discrete set increases the mathematical and computational complexity for the pricing of path–dependent options even in the classical Black-Scholes model. This dissertation develops new techniques based on recent advances in the fast evaluation and inversion of Fourier and Hilbert transforms as well as classical results in fluctuation theory, particularly those involving random walk duality and ladder epochs.

CHAPTER 1
PATH-DEPENDENT OPTIONS

1.1 Introduction

Options are contracts in which the buyer of the option gets the right, but not the obligation, to buy or sell the underlying asset of the contract at some date in the future, for a predetermined strike price. If the date is pre-specified (labeled maturity or expiration date) then the option is of European exercise-style. Otherwise, it is of American exercise-style and can be exercised any time up to maturity. A call option is a contract that gives the right to buy the underlying asset, and a put option is a contract that gives the right to sell the underlying asset. The seller of the option collects a fee upfront in order to give this right to the option holder. The determination of the fair price of this fee for different kinds of options has been of interest for academics and practitioners alike. It has become an area of major intellectual and commercial development since 1973, when Black and Scholes published a breakthrough article that allowed for the pricing of so-called standard (or vanilla) options (Black & Scholes, 1973), by only using readily available parameters, namely the prevailing riskless rate in the market and the volatility (standard deviation of returns) of the underlying asset upon which the option is written. Vanilla options depend only on the price of the underlying security on the exercise date, whereas path-dependent options have an exercise payoff that depends on the price path of the underlying security from the beginning of the contract until the exercise date. An Asian option is an example of a path dependent option. The payoff of a European exercise-style Asian call option is $max(A_T - K, 0) = (A_T - K)^+$, where K is the strike price of the Asian option and A_T is the average of the security over the life of the contract. In contrast, the corresponding payoff of a standard (vanilla) call option is $max(S_T - K, 0) = (S_T - K)^+$, where S_T is the price of the underlying security at maturity.

The use of derivatives has become popular in recent years because investment banks have been able to hold them without having to put them on their balance sheets. Since options allow for leveraged transactions, this has allowed banks and investors to make highly leveraged transactions without them ever showing up on their balance sheets. (In simple terms, leverage refers to borrowing.) An argument made in support of this state of affair is described in the J.P. Morgan guide to credit derivatives (Morgan, 1999). After the crash of financial markets in late 2008, many became worried that unregulated use of derivatives was dangerous to financial markets. Path-dependent options are also called exotic options and are mostly traded between private parties, in so-called over-the-counter-trade (OTC), not in open markets. They have therefore been hard for the legislator to oversee. The U.S. House of Representatives and the U.S. Senate drafted a bill that was to limit OTC trading of exotic derivatives to respond to concerns that their opacity can be a source of instability (Gibson, 2010). In addition, the bill proposed that some uncovered (or 'naked') derivatives trading be banned. However, the bill came across hard opposition from a group of investors, politicians and academics and has had some alleviating amendments added to it, including the drop of the proposal to ban naked derivatives trading and the drop of most limits to OTC trading of options. Many suggested that even if the use of exotic options would be limited in the U.S. this would only spur life into foreign OTC trading since it would not be likely that people would stop using these investment vehicles since they have become so common.

At the end of the last decade, (Boyle & Boyle, 2001) noted that growth in option trading had increased significantly for the past 30 years and that in the first quarter of 2000 the estimated value underlying option contracts around the world was $102 trillion. In fact, this was only the estimated value underlying exchange traded contracts, the estimated value underlying over the counter (OTC) option contracts was estimated to be $88 trillion (BIS, 2000) so the total value underlying option contracts was $190 trillion in the beginning of 2000. For the last quarter of 2009, the estimated value

underlying exchange traded options around the world was $444 trillion (BIS, 2010), or roughly fourfold the value from 10 years earlier, even when it was down from $690 trillion in the beginning of 2008. However, OTC seems to have become the preferred method of trading options, with $605 trillion in underlying value for OTC contracts in June 2009.(BIS, 2009) In comparison, the GDP of the USA was $10 trillion in 2000 and $14 trillion in 2009 (BEA, 2010), so at the beginning of the decade, the total value underlying option contracts in the world was roughly 19 times the GDP, and at the end of the decade it was 75 times the GDP.

1.2 Asian Options

The first paper written on Asian options, by Boyle and Emanuel in 1980, was rejected by the Journal of Finance, since this kind of option was not traded at that time(Georgios Foufas and Mats G. Larson, 2008). The paper is still a working paper (Boyle & Emanuel, 1980). Boyle and Emanuel called this new option type, averaging options, but they were dubbed Asian options by Bankers Trust because the firms that bought the options from Bankers Trust, were Japanese. These firms' annual reports were based on average exchange rates over the year, so average rate options were appropriate for them to hedge their risk(Vorst, 1996). In practice today, Asian options are mostly traded on oil products, agricultural commodities such as corn and soybeans and on currencies. As far back as in 1998, Microsoft was already taking advantage of the elimination of downside risk that Asian options offer, along with the potential of an upside gain by hedging their foreign currency exposure by using Asian put options(William Falloon, 1998). Microsoft's treasurer at that time, Mr. Heitz, said in an interview with Risk magazine that Microsoft had 10-12 counterparties from which it could buy the put options. Today, Asian options are still most commonly traded over the counter. Asian options are particularly useful in thinly traded markets or to protect against large price variations. Investors who have an obligation due on a certain date will want an insurance against the counterparty being able to move prices against them.

Since it will be much harder to move the average price than the price on a specific date, Asian options have become common use in thinly traded stocks/currencies.

Nowadays, on the Chicago Mercantile Exchange (CME), average options are constructed in the following way: The option has a swap (i.e.; a contract to exchange an interest or currency rate for another) as the underlying security, and a fixed strike price. The final price on the swap is used to calculate the payoff of the average option. The final price on the swap is calculated by taking the arithmetic average of daily prices from each day for which a price for the underlying security for the swap is determined for the previous month. The daily price is found by taking the average of the high and low quotations on each day for the underlying security for the swap. The payoff for an average call option will be the final price on the swap minus the strike price, and the payoff for an average put option will be the strike price minus the final price on the swap. Even though this structure is intricate, the average price option payoff is simply the difference between the arithmetic average price of the security itself over the previous month minus a fixed strike price, so our pricing model for Asian options given in this paper is applicable to the average price options traded on the CME. On the CME, all the 19 average options available in early 2010, had an oil product as the underlying security, and they were all traded on CME's over the counter clearing service. The oil products included e.g. gasoline, jet kerosene, fuel oil, propane, butane, heating oil, gasoil, ethane and crude oil (CMEGroup, 2010). Through these examples, it is clear that Asian options are widely used.

Asian options are less likely than vanilla options to be manipulated because it is not possible to manipulate the price over such a long time as opposed to vanilla (or regular) options. A recent example from the drop in the Dow Jones by almost 10% within a few minutes (Mattich, 2010) shows that whether it is by mistake or manipulation, it is possible for the market to be affected severely from other factors than efficiency in just a matter of minutes. In the case of a vanilla call option, had the closing price of

the underlying asset been 10% lower than on the previous day, the option might have expired worthless on that day, but would have expired in the money on the previous day. In the case of an Asian option, this 10% lower price only affects the average by moving 1 out of n prices that are part of the average and therefore cannot affect the Asian option price as much. As a result, Asian options are perceived to be cheaper and therefore reduce the risk management costs of their bona fide users.

In the first published paper on Asian options, (Kemna & Vorst, 1990) used Monte Carlo methods to determine the price of the arithmetic Asian option. By using the geometric Asian option as a control variate, where the geometric average is given by

$$A_T = \left(\prod_{i=1}^{n} S_{t_i} \right)^{1/n}$$

they were able to price the Asian option faster than with plain Monte Carlo. Monte Carlo simulation works well but can be computationally expensive without the enhancement of variance reduction techniques. One must account for the inherent discretization bias resulting from the approximation of continuous-time processes through discrete sampling as shown by (Broadie et al., 1999). As previously noted, the arithmetic Asian option, where the arithmetic average is given by

$$A_T = \frac{1}{n} \sum_{i=1}^{n} S_{t_i}$$

is the one that is used in practice. However, it is not possible to find the exact analytical price for the arithmetic Asian option. The geometric Asian option on the other hand is lognormally distributed when the underlying price process is assumed to follow a geometric Brownian motion. So with that assumption it is possible to derive the exact analytical price for the geometric Asian option. (Turnbull & Wakeman, 1991) proposed using an approximation of the density function of the arithmetic Asian option by using an Edgeworth expansion. Among the first to derive analytic results,(Geman & Yor, 1993) computed the Laplace transform of the price of a continuously sampled

Asian option computed as $\frac{1}{T}\int_0^T S_t dt$. Its numerical inversion remains problematic for low volatility and/or short maturity as shown by (Fu et al., 1998). On the other hand, in practice, sampling is performed over a discrete set of dates (daily, weekly, etc.) In this case, no analytic results are available even in the Black-Scholes framework, where the main source of the problem stems from the lack of an explicit distribution for the sum of correlated log-normal random variables. As a result, a significant number of approximations that produce closed-form expressions have appeared. For example,(Thompson, 1998) provides tight analytical bounds and (Linetsky, 2004) derived a new integral formula for the price of a continuously sampled Asian option, which is again slowly convergent for low volatility cases. In general, the price of an Asian option can be found by solving a partial differential equation (PDE) in two-dimensional spaces (see (Ingersoll, 1987)), which is prone to oscillatory solutions. Ingersoll also observed that the two-dimensional PDE for a floating strike Asian option can be reduced to a one-dimensional PDE. (Rogers & Shi, 1995) simpler formulated a one-dimensional PDE that can model both floating and fixed strike Asian options. However this one-dimensional PDE is difficult to solve numerically since the diffusion term is very small for values of interest on the finite difference grid. Several articles contain attempts to improve the numerical performance of this PDE. (Andreasen, 1998) applies the reduction of Rogers and Shi to discretely sampled Asian option. Independent efforts in recent years have attempted to unify pricing techniques for different types of options and relate these methods to pricing Asian option. Using again Rogers and Shi's reduction, (Lipton, 1999) noticed similarities in pricing equations for the passport, lookback, and Asian options . (Shreve & Večeř, 2000) developed techniques for pricing options on a traded account, which include all options that could be replicated by self-financing trading in the underlying asset. They include European, passport, vacation, as well as Asian options. Numerical techniques for pricing contracts of this type are described in (Vecer, 2001). (Hoogland & Neumann, 2001) developed

an alternative framework for pricing various types of options using scale invariance methods and derived more general semi-analytic solutions for prices of continuously sampled Asian options. A major shortcoming of these approaches is their inability to help determine hedging parameters, which are crucial to the option writer. (Fusai & Meucci, 2008) derive pricing methods for both arithmetic and geometric Asian options under discrete monitoring and for a general Lévy process. They work on the Fourier space with a recursive pricing formula like we do. However, for their re-centering technique they require finite moments, which we don't. In addition, it is unclear how their method can lead to computing hedge parameters, while ours will be shown to produce them with minimal additional computations.

1.3 Lookback Options

The payoff for lookback options depends on the extremum price observed over the contract period. For floating lookback options, the holder of a call option gets the right to buy at the lowest price over the contract period and sell at the price on the expiration date, T and the holder of a put option gets the right to sell at the highest price over the contract period and buy at the price on the expiration date. For fixed lookback options, the holder of a call option gets the right to buy the security at a fixed strike price K, but the selling price is the highest price over the contract period. The holder of a fixed put lookback option gets the right to sell at the lowest price over the contract period and buy at a fixed strike price K. There are also other variations, where for example the holder of the option gets the right to buy or sell for a percentage of the extremum price observed.

Compared to other options, lookback options provide the biggest payoff potential because the investor can choose the exercise date in retrospect, that is by looking back over the life of the option. The reported uses for lookback options are mainly speculative. It is obvious that lookback options will be more expensive than vanilla or Asian options because the holder is getting the biggest potential payoff over the whole life of the option.

Some analytical solutions have been proposed when the monitoring of the price process is continuous and/or when the underlying price process follows geometric Brownian motion, see e.g. (Heynen & Kat, 1995), (Conze & Viswanathan, 1991) and (Goldman et al., 1979). In practice, monitoring occurs at discrete dates and the monitoring dates $t_1, \dots, t_N = T$ are predetermined. (Kou, 2008a) says that it is practical that monitoring is discrete, and that if monitoring were continuous, there would be arbitrage opportunities for barrier options, e.g. if a barrier is reached. Those could represent themselves while markets are open in only some parts of the world. Similarly, if the highest/lowest price for a lookback option during its contract period so far is reached at a time when not all exchanges are open simultaneously, it would be unfair to the traders in parts of the world where the markets are closed since they are unable to trade upon that information immediately while others in open markets would reap the profits. All traded lookback options have discrete monitoring, so even if a higher/lower price is observed outside of the monitoring dates, it is not taken into account for determining the extrema of prices over the contract period. As a consequence of the discrete monitoring, pricing is mathematically and computationally challenging. Substantial mis-pricing occurs when a discretely monitored contract is priced approximately by a continuous-monitoring formula (cf. (Broadie et al., 1999), (Heynen & Kat, 1995).) (Broadie et al., 1999) introduce correction terms so that the continuous-monitoring formulas can be used as approximations for the discretely monitored options. Their method also improves convergence by means of lattice methods.(Babbs, 2000) uses a binomial model to price continuously monitored floating–strike lookback options. Using discrete monitoring and pricing for both fixed strikes and floating strikes, (Cheuk & Vorst, 1997) also use a binomial model to price lookback options, improving upon Babbs. (Boyle & Tian, 1999) used a trinomial method to value the non-Gaussian CEV process and found the price for lookback and barrier options when the price follows the CEV process. Later they found that it was inaccurate

for lookback options and proposed a correction using Monte Carlo methods (Boyle et al., 1999). (Davydov & Linetsky, 2001) also found pricing formulas for the lookback option when the underlying follows the CEV process. Using Laplace transforms, their method is faster than that of Boyle and Tian. (AitSahlia & Lai, 1998) use the duality property of random walks to derive recursively the distribution of conditioned extrema of the geometric Brownian motion price process and use numerical integration methods to price lookback options. (Tse et al., 2001) use a tridiagonal procedure that takes advantage of the properties of the geometric Brownian motion price process and price the lookback options numerically and achieve more efficiency than previous methods. (Andricopoulos et al., 2003) develop a quadrature method that can be used to numerically price a wide range of options, including lookback options.(Broadie & Yamamoto, 2003) develop a fast Gauss transform for non-path-dependent option valuation under geometric Brownian motion and the Merton model.(Broadie & Yamamoto, 2005) extend their previous results and derive a double-exponential Gaussian model that can be applied to lookback options and other path-dependent options. (Petrella & Kou, 2004) find Laplace transforms of discrete lookback options using a recursion formula. These involve Spitzer's formula. They invert the Laplace transforms numerically to get the lookback option price and hedging parameters for several Lévy price models. For the geometric Brownian motion price process and discrete monitoring, (Atkinson & Fusai, 2007) find the distribution of the extrema of prices in closed form and are thus able to find the lookback price for fixed and floating options. The latest work on lookback options is by (Feng & Linetsky, 2009). They do a forward recursion on the prices of the lookback option, utilizing Hilbert transforms and Fourier transforms. Their method is efficient and accurate but is restricted by some conditions making it inapplicable to the important pure-jump processes. In contrast, our method, to be described in detail later, is more generally applicable and has the

same computational complexity as it also uses their fast algorithm for the evaluation and inversion of Hilbert and Fourier transforms.

1.4 Overview

Briefly, this dissertation is broadly organized as follows: Chapter 2 reviews Lévy processes and their use in finance as well as recent advances in Fourier-based techniques. In particular, we review those making use of Hilbert transforms due to (Feng & Linetsky, 2008) and (Feng & Linetsky, 2009), which enable us to make additional contributions to efficiently and accurately price lookback options as described in detail in Chapter 4. Chapter 3 deals with the pricing of Asian options and provides a detailed description of our approximation approach based on yet another type of path-dependent options, namely quantile options, that are not traded but which provide mathematical expediency. The contributions of this thesis consist of new techniques to price discretely monitored Asian and lookback options. Their distinguishing feature lies in working on the characteristic function of the option price distribution rather than on the characteristic function of the price itself, as is done in (Feng & Linetsky, 2008) and (Feng & Linetsky, 2009), the most competitive approach up-to-date. Ours has the significant advantage of enabling a direct computation of hedging parameters, the ”Greeks”, in contrast to the unstable numerical derivatives and the computationally complex Malliavian calculus required by all the other alternatives. In addition, the (Feng & Linetsky, 2009) pricing method for lookback options is slower than ours and excludes an important class of Lévy models in finance, the popular variance gamma specification (Madan & Seneta, 1990).

CHAPTER 2
LÉVY PROCESSES

2.1 Motivation for Lévy Pricing Models

Up until recently, most pricing models have assumed that the underlying process for any security follows a geometric Brownian motion a la Black Scholes, that is

$$dS_t = S_t \mu dt + S_t \sigma dW$$

where S_t is the price of the security at time t, μ is the drift rate of the security, σ is the volatility rate and W is a Wiener process. By modeling the price in this way, the assumption is that $ln(S_t)$ follows a Brownian motion, that is $ln(S_t)$ is continuous and has independent and normally distributed increments. There are several issues regarding modeling the underlying price process like this. First,(Merton, 1976) noted that far too many random jumps occur in the price process in practice to be justified by constant volatility or a continuous path of prices. He therefore suggested an addition of a jump term to the price process, so that

$$dS_t = S_t \mu dt + S_t \sigma dW + dq$$

where q is a Poisson process with normally distributed jumps, where both are independent of W. These random jumps lead to an empirical distribution that has fatter tails than the normal distribution. Other issues include the empirical observation of (log) price returns that are not symmetric, and with peaks higher than suggested by the normal distribution (a leptokurtotic curve). These issues are addressed in e.g. (Kou, 2002) and in (Carr et al., 2002). Kou and Carr et al. suggest models to remedy those issues, respectively, the Kou model which has both a diffusion component and a jump component and the CGMY model, which only has a jump component. All of the aforementioned models, including the Black and Scholes model, are specific cases of a general class of processes called Lévy processes. Lévy processes are fairly general and allow for a

wide range of models, including the Poisson process, Brownian motion, or the pure jump process of Carr et al.

2.2 Using Lévy Pricing Models

A process $(X_t)_{t>0}$ is called a Lévy process if it has

a) Independent increments: That is for all t_0, t_1, ..., t_n, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, ..., X_{t_n} - X_{t_{n-1}}$ are independent

b) Stationary increments: That is $X_t - X_s$ has the same distribution as $X_{t-s+u} - X_u$ and

c) Continuous paths a.e: That is $\lim_{h\to 0} P(|X_{t+h} - X_t| \geq \epsilon) = 0$ for any $\epsilon > 0$.

Every Lévy process can be fully described by three parameters. The first two parameters, a and σ^2, describe the continuous component of the Lévy process, and the third parameter is a function $\nu(x)$, called the Lévy density, which identifies the discrete component of the Lévy process. Furthermore, a is the constant drift of the continuous component and σ^2 is the constant variance of the continuous component.

Using only those three parameters, the Lévy-Khinchin formula:

$$ln\mathbb{E}[e^{i\theta X_t}] = ait\theta - \frac{1}{2}\sigma^2 t\theta^2 + t\int(e^{i\theta x} - 1 - i\theta x\mathbf{I}_{|x|<1})\nu(x)dx$$

where $a\epsilon\mathbb{R}, \sigma \geq 0$ and $\int_{\mathbb{R}/0} min\{1, x^2\}\nu(x)dx < \infty$, allows for an easy retrieval of the characteristic function,$\phi(\theta) = \mathbb{E}[e^{i\theta X_t}]$, of many Lévy processes, which makes them feasible for practical use.

Lévy processes can have either finite activity, which means that over any interval, there will be a finite amount of jumps, or they can have infinite activity, which means that any interval will have infinite amount of jumps (Wu et al., 2008). Pure jump processes with infinite activity, are often not distinguishable from pure diffusion processes, and when there is infinite activity it is not necessary to have a Brownian motion component as well. When σ is 0, we have a pure jump process and when $\nu(x)$ is zero we have a pure diffusion process. The arrival rate for jumps is determined by $\int_{\mathbb{R}/0}\nu(x)dx = \lambda$.

If $\lambda < \infty$, then the mean arrival rate of jumps is finite, and when $\lambda = \infty$ the number of jumps over any interval will be infinite. The simplest Lévy process as previously mentioned is the Black Scholes model, for which $\nu(x) = 0$ and therefore the characteristic function is simply

$$\mathbb{E}[e^{i\theta X_t}] = e^{ait\theta - \frac{1}{2}\sigma^2 t\theta^2}$$

and it is easy to derive the probability distribution function (pdf), which is simply the normal density with mean $a - \frac{1}{2}\sigma^2 t$ and variance $\sigma^2 t$. The Merton model has Lévy density:

$$\frac{\lambda}{\sqrt{2\pi\delta^2}} e^{-\frac{(x-\mu)^2}{2\delta^2}}$$

which describes in mathematical terms that the process will have jumps that are normally distributed with mean μ and variance δ^2, and that the jumps come with frequency λ. For the Merton model the characteristic function can be simplified to

$$\mathbb{E}[e^{i\theta X_t}] = e^{ait\theta - \frac{1}{2}\sigma^2 t\theta^2 + \lambda t\{e^{-\frac{\delta^2\theta^2}{2} + i\mu\theta} - 1\}},$$

see (Cont & Tankov, 2004), however the probability density can only be represented as an infinite series, and is thus not available in closed form. This provides an additional computational complexity in deriving an option value which has this price process as the underlying asset. On one hand, this is a better model for the price process, because it is more realistic that the price exhibit some jumps, just as it might when new information arrives to the market that immediately changes market participants' opinion on what the price should be, so that the price immediately adjusts. It is worth noting that a more realistic model (than the Black Scholes model) would only be useful in practice, if it enables us to price the derivatives of the price process. For the base case, a vanilla call option, the price of the option at initiation using the Merton model would be:

$$C_T(K) = e^{-rt}\mathbb{E}[S_0 e^{X_T} - K] = \int_{ln(K)}^{\infty} (S_0 e^{X_T} - K) dF(X_T)$$

$$= \int_{ln(K)}^{\infty} (S_0 e^{X_T} - K) e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{\frac{-(X_T - \gamma t - k\mu)^2}{2(\sigma^2 t + k\delta^2)}}}{k!\sqrt{2\pi(\sigma^2 t + k\delta^2)}} dX_T$$

This integral can not directly be evaluated, except in a few cases (Merton, 1976) where the infinite sum within the integral can be simplified.(Carr & Madan, 1999) derive a method in which it is not necessary to know the pdf of the price process to calculate the option price. Rather than working with the direct integral above, they work with its Fourier transform which they obtain in an explicit form, albeit not trivially because in order for the integral to be non-singular, they have to multiply the Fourier transform with a specific remedial function. The explicit formula involves the characteristic function of the price process, which as mentioned above, can always be retrieved from the Lévy-Khinchin formula for all Lévy processes. Once they have an explicit formula for the Fourier transform, they take the inverse Fourier transform, then multiply again with the inverse of the remedial function to retrieve the option price. When calculating the inverse Fourier transform, they use the discrete Fourier transform (DFT) on the integral, which means that they have to discretize the integral. To speed up the calculations they then transform the integral to conform to the setup for the Fast Fourier Transform (FFT) which is faster than calculating the DFT directly, $O(Nlog(N))$ vs $O(N^2)$ respectively. The FFT will give prices of several different strikes for each calculation of the FFT. When performing these calculations, there is a choice to be made for the FFT, if the grid for the DFT is chosen to be wide, the strike prices will be relatively close to each other, and if the grid for the DFT is chosen to be fine, the strike prices will be far apart. So the choice of the grid has to be made according to what strike price range is needed. Also, the choice of the remedial function has to be made carefully so that it ensures integrability. (Lee, 2004) discusses these choices of parameters in more detail, and shows how the

FFT method of pricing utilizing the characteristic function of the price process can be extended to other option classes.

(Kou, 2002) proposes a model that has jumps in addition to a diffusion process, but the jumps have double exponential distribution instead of normal distribution, like in the Merton model. Also, the distribution of jumps is different depending on whether it is an upward movement or a downward movement, reflecting the trend that stock price changes seem generally to be of different magnitude for good news and bad news (Chen et al., 2003). For the Kou model the Lévy density is:

$p\lambda_+ e^{-\lambda_+ x}$ if $x > 0$ and

$(1-p)\lambda_- e^{\lambda_- x}$ if $x < 0$

and although the probability density is not available in closed form, the characteristic function can be derived from the Lévy-Khinchin formula and the corresponding pricing of vanilla options can then be done by using the methods in Carr et al. and Lee. The Kou model achieves the high peak and the fat tails that are typical of stock returns and eliminates the phenomenon that is called volatility smile. A volatility smile or skewness is seen when options are priced using the Black and Scholes model (Hull, 2006). The standard deviation, or volatility as it is called in the finance literature, is assumed to be fixed in the Black and Scholes model. Yet, when vanilla option market prices are observed for different strike prices, and the Black Scholes model is solved to return the volatility, it is different for different strike prices, typically higher the further away from at the money the strike price is. It can also be skewed, referring to that the implied volatility is higher for strike prices under the at the money price, and lower for strike prices that are out of the money. Because of the jumps that the Kou model incorporates, this smile disappears and the implied volatility becomes constant.

There are two more prominent Lévy models that we will mention. First is the Variance Gamma model (Madan et al., 1998), that also makes implied volatility constant for vanilla options, so that no volatility smile is observed. This is done in a very different

way from the Kou model; in the Variance Gamma model, there is no diffusion part, but instead the number of jumps over any given interval is infinite, that is, it has infinite activity. There are three parameters in the Variance Gamma model that need to be calibrated. The CGMY model (Carr et al., 2002) is a generalization of both the Kou model and the Variance Gamma, and it has five parameters that need to be specified. The Lévy density of the CGMY model is

$C\frac{e^{-Mx}}{x^{1+Y}}$ if $x > 0$ and

$C\frac{e^{Gx}}{(-x)^{1+Y}}$ if $x < 0$

When Y is equal to -1, the CGMY model becomes the Kou model, and when Y equals zero, it is the same as the Variance Gamma model. The CGMY model exhibits infinite activity for Y between 0 and 2, and finite activity for Y less than 0. Y has to be less than 2 in all cases, so that the characteristic function may exist. It is not obvious how to specify the parameters of the CGMY model, so practitioners have found reasonable parameters for it by calibration with real world data, which is typically done by seeing which models fit historical data the best. For example, (Carr et al., 2002) specify the 5 parameters (in addition to the four parameters in the Lévy density, σ needs to be specified) that make the CGMY model fit the S&P 500 index the best, and also display how drastically the distribution function changes by just twisting even one of the parameters at a time, thereby showing how sensitive the model is to parameter changes. It should also be noted that even though Lévy pricing models solve a lot of the empirical issues that using the Black-Scholes model entails, model selection of a Lévy process is hard, mainly because there are so many parameters to estimate. The data needed to estimate the exact parameters and models would have to be enormous to justify using one good model rather than another.(Heyde & Kou, 2004)

2.3 The Fast Hilbert Transform

The Fourier-transform method of (Carr & Madan, 1999) can be utilized to price vanilla options for any Lévy process. (Feng & Linetsky, 2008) and (Feng & Linetsky,

2009) develop a Hilbert-transform based method to price barrier and lookback options when the underlying asset follows an exponential Lévy process. Their recourse to Hilbert transforms in the Fourier space stems from the presence of an indicator function multiplying the function of interest; the price. This indicator function captures the path-dependency of the option payoff such as the barrier crossing event prior to the option expiration, for example. Succinctly, they use the following property relating Fourier and Hilbert transforms for a given ϕ defined on $\Re$:

$$\mathcal{F}\left(\mathbf{1}_{(0,\infty)} \cdot \phi\right)(\xi) = \frac{1}{2}\hat{\phi} + \frac{i}{2}\mathcal{H}\left(\hat{\phi}\right)(\xi),$$

where the Fourier transform for $f \in \mathrm{L}^1(\mathbb{R})$ is

$$\hat{f}(\xi) \equiv \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} e^{i\xi x} f(x) dx.$$

and the Hilbert transform for $f \in \mathrm{L}^p(\mathbb{R})$, $1 \leq p < \infty$, is

$$\mathcal{H}(f)(x) = \frac{1}{\pi} P.V. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy.$$

The fast Fourier transform algorithm was available long before Carr et al's paper and enables us to find the discretization of the Fourier transform (DFT) with a computational complexity of $O\left(N \log_2 N\right)$, where N is the number of points in the discretization. This is an advantage over computing the DFT in the naive way, which results in a complexity of $O\left(N^2\right)$. Feng and Linetsky proceed to make their own fast Hilbert transform algorithm since none existed. They use Whittaker cardinal series (Sinc expansion) to approximate $\mathcal{H}$ with

$$\mathcal{H}(f)(\xi) \approx H_{h,M} f(\xi) = \sum_{m=-M}^{M} f(mh) \frac{1 - cos[\pi(\xi - mh)/h]}{\pi(\xi - mh)/h},$$

where h is the discretization step size and $M > 0$ is the truncating integer for the integral approximation. After this discretization step, they then use the FFT and Toeplitz matrix-vector multiplication to compute $H_{h,M} f(\xi)$. The overall computational complexity

to find the Hilbert transform is $O\left(M \log_2 M\right)$, or the same complexity as the FFT for the Fourier transform. Furthermore, the error in the approximation decays exponentially as h is taken smaller. The price of a down-and-out barrier option at time zero is given by

$$V(S) = e^{-rT}\mathbb{E}_S\left[(S_T - K)^+ 1_{(L,\infty)}(S_\Delta)\cdots 1_{(L,\infty)}(S_{N\Delta})\right],$$

where S_t is the price of the underlying at time t, Δ is the monitoring interval, $N\Delta = T$ and L is the barrier. All the indicator functions are within the expectation because if the price of the underlying drops below L on any monitoring date, the option becomes worthless. Feng and Linetsky do a backward recursion on the prices of the barrier option to find the time zero value of the option where $S_t = Ke^{X_t}$ for any Lévy process X_t, with $X_0 = \ln(S_0/K)$. They define the time-zero price of the option as

$$V(S_0) = e^{-rT} v^0\left(\ln(S_0/K)\right)$$

with v^0 obtained recursively through:

$$\begin{aligned}
v^N(x) &= K(e^x - 1)^+ \mathbf{1}_{(l,\infty)}(x),\\
v^{j-1}(x) &= \mathbf{1}_{(l,\infty)}(x)\cdot P_\Delta v^j(x), j = N, N-1, ..., 2,\\
v^0(x) &= P_\Delta v^1(x),
\end{aligned}$$

where $P_\Delta f(x) := E[f(X_{t+\Delta})|X_t = x]$ and $l := \ln(L/K)$. Then for j=N,N-1,...,2, they perform the recursion in Fourier space:

$$\begin{aligned}
\hat{v}^N(\xi) &= \frac{K(1-e^{i\xi l})}{i\xi} - \frac{K(1-e^{(1+i\xi)l})}{1+i\xi}\\
\hat{v}^{j-1}(\xi) &= \frac{1}{2}\hat{\psi}(-\xi)\hat{v}^j(\xi) + \frac{i}{2}e^{i\xi l}\mathcal{H}\left(e^{-i\eta l}\hat{\psi}(-\eta)\hat{v}^j(\eta)\right)(\xi),
\end{aligned}$$

where $\hat{\psi}$ is the characteristic function of X_Δ and for each recursion step they utilize the fast Hilbert transform to obtain the Fourier transform on the left. Then, finally they retrieve v^0 through a final Fourier transform

$$v^0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\xi}\hat{\psi}(-\xi)\hat{v}^1(\xi)d\xi$$

and accomplish an aggregate computational complexity of $O\left(NM\log_2 M\right)$ to find the barrier option price. In their extension to lookback options, (Feng & Linetsky, 2009) utilize the Fast Hilbert transform, by working forward in their recursive scheme, rather than backward. In our approach, based on evaluating the option price distribution instead, we still maintain the use of the fast Hilbert transform discretization algorithm of (Feng & Linetsky, 2008) in a backward recursive fashion. As mentioned in Chapter 1, the main advantage of our approach is its ability to generate hedge parameters much more seamlessly than any other alternative.

CHAPTER 3
QUANTILE APPROXIMATIONS FOR ASIAN OPTIONS

3.1 Introduction

Chapter 3 develops a new approximation approach to price and hedge discretely monitored Asian options when the underlying asset price follows a Lévy process. The option price is shown to be accurately approximated by a weighted sum of related quantile options. The latter are options on quantile values of the underlying asset process. Though they are currently not traded, our work in Chapter 3 shows how they can be used for efficient computation of Asian option prices. Furthermore, our method offers a way to directly approximate hedge parameters with practically negligible additional computational effort.

Chapter 3 is organized as follows. The first section summarizes the concept of a quantile option in both the original continuous-time setting and our discrete set-up for discrete monitoring of path-dependent options. The second section contains our quantile-based approximation in a general Lévy process framework. The last section presents a numerical illustration on the particular case of the Black-Scholes (Brownian) model.

3.2 Quantile Options

First introduced by (Miura, 1992), these options are path-dependent and are meant to generalize the concept of options on extrema (minimum or maximum). For a (μ, σ)–Brownian motion $\{X_t, t \geq 0\}$ and $\alpha \in (0, 1)$, define the α–quantile process $\{M(\alpha, t), t \geq 0\}$ by:

$$M(\alpha, t) = \inf \left\{ x : \int_0^t \mathbf{1}_{(X_s \leq x)} ds > \alpha t \right\}.$$

Then the α–quantile option payoff is defined as

$$\left(S_0 e^{M(\alpha, T)} - K\right)^+,$$

where S_0 is initial price of underlying asset (stock, currency, ...) and K is the strike price. The corresponding option price has been extensively studied by (Akahori, 1995) and (Dassios, 1995) who in the process generalize the arc-sine law for Brownian motion. More precisely, they obtain

$$Pr\left\{M(\alpha, t) \in dx\right\} = g(x; \alpha, t)dx, \tag{3–1}$$

where

$$g(x; \alpha, t) = \int_{-\infty}^{\infty} g_1(x-y; \alpha t) g_2\left(y; (1-\alpha)t\right) dy, \tag{3–2}$$

and g_1 and g_2 are the probability density functions associated with $\sup_{0\le s\le \alpha t} X_s$ and $\inf_{0\le s\le (1-\alpha)t} X_s$, respectively, i.e:

$$Pr\left(\sup_{0\le s\le \alpha t} X_s \in dx\right) = g_1(x; \alpha t)dx, \tag{3–3}$$

$$Pr\left(\inf_{0\le s\le (1-\alpha)t} X_s \in dx\right) = g_2(x; (1-\alpha)t)dx. \tag{3–4}$$

These functions are explicitly derived as

$$g_1(x;\tau) = \begin{cases} \frac{1}{\sigma}\left(\frac{2}{\pi\tau}\right)^{1/2} \exp\left\{-\frac{(x-\mu\tau}{2\sigma^2\tau}\right\} - \frac{2\mu}{\sigma^2}\exp\left(\frac{2\mu x}{2\sigma^2}\right)\left[1-\Phi\left(\frac{x+\mu\tau}{\sigma\sqrt{\tau}}\right)\right] & \text{for } x>0, \\ 0, & \text{for } x\le 0, \end{cases} \tag{3–5}$$

$$g_2(x;\tau) = \begin{cases} 0, & \text{for } x\ge 0, \\ \frac{1}{\sigma}\left(\frac{2}{\pi\tau}\right)^{1/2} \exp\left\{-\frac{(x-\mu\tau}{2\sigma^2\tau}\right\} + \frac{2\mu}{\sigma^2}\exp\left(\frac{2\mu x}{2\sigma^2}\right)\Phi\left(\frac{x+\mu\tau}{\sigma\sqrt{\tau}}\right) & \text{for } x<0. \end{cases} \tag{3–6}$$

The quantile option price at time 0 is then

$$E\left[e^{-rT}\left(S_0 e^{M(\alpha,T)} - K\right)^+\right], \tag{3–7}$$

which can be evaluated through numerical integration as the associated probability density function g is determined through Eq. 3–1 through Eq. 3–6.

The key to the derivation of the above results begins with the equivalence between the events $\{M(\alpha, t) > x\}$ and $\{\int_0^t \delta(X_s \le x)ds < \alpha t\}$, where $\delta(A)$ is the indicator of

whether event A has occurred, thus relating the quantile process to the occupation time. As a consequence, one can then show the following identity (cf. (Dassios, 1995)):

$$M(\alpha, t) \overset{law}{=} \sup_{0 \le s \le \alpha t} X^{(1)}(s) + \inf_{0 \le s \le 1-\alpha t} X^{(2)}(s), \tag{3–8}$$

where $X^{(1)}(t)$ and $X^{(2)}(t)$ are independent copies of the process $X(t) = \mu t + \sigma B(t)$, with $B(t)$ denoting a standard Brownian motion. Furthermore, (Dassios, 1995) also derives the joint distribution of $M(\alpha, t)$ and $X(t)$:

$$\begin{pmatrix} M(\alpha, t) \\ X(t) \end{pmatrix} \overset{law}{=} \begin{pmatrix} \sup_{0 \le s \le \alpha t} X^{(1)}(s) + \inf_{0 \le s \le 1-\alpha t} X^{(2)}(s) \\ X^{(1)}(\alpha t) + X^{(2)}((1-\alpha)t) \end{pmatrix} \tag{3–9}$$

In fact, both Eq. 3–8 and Eq. 3–9 hold when the reference X is a Lévy process as (Dassios, 1996) shows. While the derivation of the results for the Brownian case is based, respectively for Eq. 3–8 and Eq. 3–9, on the Feynman-Kac formula and the Girsanov theorem, the method of proof for the Lévy process relies in fact on an asymptotic discretization. The latter will turn out to be exactly what we need for the Asian option pricing with discrete monitoring. Specifically, (Dassios, 1996) develops the following:

Proposition. Let $\xi_1, \xi, \dots, \xi_n$ be i.i.d. random variables. Consider the random walk $\zeta_n = \sum_0^n \xi_k$, $0 \le n$, where $\zeta_0 = 0$ w.p. 1, and let $\zeta^{(1)}$ and $\zeta^{(2)}$ be two independent copies of ζ. Then

$$M_{j,n}(\zeta) \overset{law}{=} M_{j,j}(\zeta^{(1)}) + M_{0,n-j}(\zeta^{(2)}), \tag{3–10}$$

where, for integers $0 \le j \le n$ and a discrete process $X = (X_0, X_1, X_2, \dots)$, $M_{j,n}(X)$ is the $(j, n)^{th}$ quantile of X defined as

$$M_{j,n}(X) = \inf \left\{ x : \sum_{i=0}^{n} \delta\left(X_i \le x\right) > j \right\}.$$

We should note that in fact the joint distribution

$$\begin{pmatrix} M_{j,n}(\zeta) \\ \zeta_n \end{pmatrix} \overset{law}{=} \begin{pmatrix} M_{j,j}(\zeta^{(1)}) + M_{0,n-j}(\zeta^{(2)}) \\ \zeta_j^{(1)} + \zeta_{n-j}^{(2)}, \end{pmatrix} \tag{3–11}$$

has been known since (Wendel, 1960).

3.3 Distributions for Discrete Quantile Processes

Whereas the use of an order statistic to consistently estimate a single quantile implies its convergence in probability, our approach here via Eq. 3–10 deals with quantile *processes*. Thus we make use of corresponding collections of order statistics with the associated mode of weak convergence. For this purpose, we shall show that we can rely on either convergence of characteristic functions in the general Lévy case, or on random walk approximation in the case of Brownian motion. For the latter, we will show through a numerical illustration how Bernoulli random walks results due to (Takacs, 1996) can be exploited. For the former, we exploit the Lévy-Khinchine characterization theorem for the increment of a Lévy process and make use of results due to (Pollaczek, 1975) on order statistics as we show next.

Let $X_1, X_2, \ldots$ be a collection of i.i.d random variables. We are interested in determining the characteristic functions of the order statistics of the random walk samples $X_1, X_1 + X_2, \ldots, \sum_1^n X_i$. Thus, we define for $n \geq 1$ and $1 \leq \nu \leq n$,

$$X_{n,\nu} = \max^{(\nu)} \left(X_1, X_1 + X_2, \ldots, \sum_{i=1}^{n} X_i \right), \tag{3–12}$$

where, for real numbers $a_1, a_2, \ldots, a_n$, $\max^{(\nu)}(a_1, a_2, \ldots, a_n)$ represents the νth number taken in descending order in the collection. With this convention, we have $\max^{(1)}(a_1, a_2, \ldots, a_n) \equiv \max(a_1, a_2, \ldots, a_n)$. In other words, $X_{n,\nu}$, $1 \leq \nu \leq n$ represent (an) order statistics (process) for the random walks values $\left(X_1, X_1 + X_2, \ldots, \sum_{i=1}^n X_i\right)$. We now adopt the approach followed by (Pollaczek, 1975) in order to determine the moment generating functions of the characteristic functions for $X_{n,\nu}$. More specifically,

with q a complex number and ϕ the characteristic function of a Lévy increment with cdf F, namely,

$$\phi(-q) = E\exp(-qX_1) = \int_{-\infty}^{\infty} e^{-qs} dF(s), \tag{3–13}$$

let

$$G(q,x,y) = \sum_{n=1}^{\infty}\sum_{\nu=1}^{n} x^{n-1}y^{\nu-1}E\exp(-qX_{n,\nu}),$$

where $|x| < 1$, $|xy| < |\phi(-q)|^{n-1}$. (Pollaczek, 1975) then shows

$$G(i\tau,x,y) \;=\; \frac{\phi(i\tau)}{(1-xy)(1-xy\phi(-i\tau))}\exp\left[\sum_{n=1}^{\infty}\frac{x^n}{n}(1-y^n)\int_{-\infty}^{\infty} e^{-i\tau t^+} dF_n(t)\right], \tag{3–14}$$

for $|x| < 1$, $|xy| < 1$, where F_n is the n-fold convolution of F with itself, so that

$$\phi^n(-q) = \int_{-\infty}^{\infty} e^{-qt} dF_n(t)$$

and

$$\exp(-qa^+) = \frac{1}{2\pi}\int_C \frac{e^{-a\xi}q}{(q-\xi)\xi} d\xi$$

for a real, q such that $Re(q) > 0$, and where C is a parallel to the right of the imaginary axis such that $Re(q-\xi) > 0$ for $\xi \in C$.

For a Lévy process, $\phi(-q)$ is explicitly given and thus F_n can be obtained via Fast Fourier Transform. The characteristic function of any $X_{(n,\nu)}$ is then trivially retrievable through derivatives with respect to x and y evaluated at $x = 0$ and $y = 0$.

3.4 Quantile Approximations for Fixed Strike Asian Options

Under the risk-neutral measure, the time-0 price is

$$e^{-rT}E\left(A_T - K\right)^+, \tag{3–15}$$

which can be evaluated in closed-form with geometric averaging in the standard Black-Scholes model. In practice, averaging is arithmetic over discretely sampled prices of the underlying. In this case, there are no known closed-form expressions for the distribution of a sum of correlated log-normal random variables. As a result,

pricing approximations for fixed strike options (arithmetic average) have involved mostly Monte Carlo simulation, moment matching approaches, density perturbation, PDE, and convolution (FFT) techniques (cf. references in (Benhamou, 2002) and (Linetsky, 2004).) In Chapter 3, we propose using quantile options, for which analytic expressions are readily available, to approximate the price of a discretely sampled Asian option with a fixed strike.

In this section we detail our quantile approximation. It is based on three elements: (i) the payoff of an Asian option is a monotone transformation of the average price, (ii) the arithmetic average of a random sample is the same as that of the associated order statistics, and (iii) the latter are generally consistent estimators of quantiles. Ultimately, our task is to evaluate expectations of the form $E[Z]$, where $Z = \left(S_0 e^{M(\alpha,T)} - K\right)^+$ and $M(\alpha, T)$ is the α-quantile of the underlying process over the interval $[0, T]$. Note that for now we refer to a generic quantile. However, we will later define such processes using notation referring directly to the discrete sampling of the underlying.

With discrete monitoring, A_T in Eq. 3–15 is the arithmetic average taken over a set of prices monitored at times $t_1, t_2, \ldots, t_n := T$:

$$A_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i}$$

We now define discrete-time quantile and occupation-time processes, respectively $M(\alpha, T, n)$ and $\tau(x, T, n)$:

$$M(\alpha, T, n) = inf\left\{x : \frac{\sum_{i=1}^{n}\delta(X_{t_i} \leq x)}{n} > \alpha\right\}$$

$$\tau(x, T, n) = \frac{\sum_{i=1}^{n}\delta(X_{t_i} \leq x)}{n}$$

where X_{t_i} is the t_i-time value of the Lévy process X such that $X_0 \equiv 0$ and $\delta(A) = 1$ if A occurs and $\delta(A) = 0$ otherwise. Here, $S_{t_i} = S_0 e^{X_{t_i}}$ is the underlying asset price at time t_i.

Theorem 3.1. *For any positive integer β, there exist $\lambda_1, \lambda_2, \dots, \lambda_\beta$ and $\alpha_1, \alpha_2, \dots, \alpha_\beta \in (0,1)$ such that $\sum_{i=1}^{\beta} \alpha_i = 1$, and*

$$\sum_{i=1}^{\beta} \lambda_i E\left(S_0 e^{M(\alpha_i, T, n)} - K\right)^+ \to E(A_T - K)^+,$$

as $\beta \to \infty$.

Proof. Let $\left(S_{(1)} \leq S_{(2)} \leq \cdots \leq S_{(n)}\right)$ be the order statistic of the sample $(S_{t_1}, S_{t_2}, \cdots, S_{t_n})$. Then for any positive integer β, there trivially exist $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_\beta$ such that $\sum_{i=1}^{\beta} \alpha_i = 1$. Furthermore, we have

$$\begin{aligned}
\sum_{i=1}^{n} S_{t_i} &= \sum_{i=1}^{n} S_{(i)} \\
&= \sum_{i=1}^{\lceil n\alpha_1 \rceil} S_{(i)} + \sum_{j=1}^{\beta-1} \sum_{\lceil n\alpha_j \rceil + 1}^{\lceil n\alpha_{j+1} \rceil} S_{(i)} + \sum_{\lceil n\alpha_\beta \rceil}^{n} S_{(i)}
\end{aligned}$$

Note that the sequence $S_{(1)}, S_{(2)}, \dots, S_{(n)}$ is monotone, non-decreasing with probability 1. As such, it may be considered as deterministic and thus the sum $\sum_{i=1}^{b} S_{(i)}$ may be viewed as a Riemann sum (with probability 1) to the extent that one can write

$$\sum_{i=a}^{b} S_{(i)} \approx \left(S_{(b)} - S_{(a)}\right)(b-a)$$

almost surely. The quantities

$$\Gamma_j = \sum_{\lceil n\alpha_j \rceil + 1}^{\lceil n\alpha_{j+1} \rceil} S_{(i)}$$

have the same properties and thus one can also write

$$\sum_{j=1}^{\beta-1} \sum_{\lceil n\alpha_j \rceil + 1}^{\lceil n\alpha_{j+1} \rceil} S_{(i)} \approx \sum_{j=1}^{\beta-1} \left(S_{(\lceil n\alpha_{j+1} \rceil)} - S_{(\lceil n\alpha_j \rceil)}\right)\left(\lceil n\alpha_{j+1} \rceil - \lceil n\alpha_j \rceil\right)$$

almost surely. As a result, we can now write the following approximation

$$\sum_{i=1}^{n} S_{t_i} \approx (2\lceil n\alpha_1\rceil - \lceil n\alpha_2\rceil - 1)\, S_{(\lceil n\alpha_1\rceil)} + \sum_{j=2}^{\beta-1} (2\lceil n\alpha_j\rceil - \lceil n\alpha_{j+1}\rceil)\, S_{(\lceil n\alpha_j\rceil)}$$
$$+ (2\lceil n\alpha_\beta\rceil - n)\, S_{(\lceil n\alpha_\beta\rceil)} + (n - \lceil n\alpha_\beta\rceil)\, S_{(n)} - (\lceil n\alpha_1\rceil - 1)\, S_{(1)}.$$

Note that by choosing α_1 such that $\lceil n\alpha_1\rceil = 1$ (e.g. $\alpha \approx 1/n$) and α_β such that $\lceil n\alpha_\beta\rceil \approx n$ (e.g. $\alpha_\beta \geq 0.95$) we see that the extreme statistics $S_{(1)} = \min\{S_{t_i}\}$ and $S_{(n)} = \max\{S_{t_i}\}$ can be omitted from the approximation. Recall that $S_{(\lceil n\alpha\rceil)}$ is an estimator of the αth quantile of the price process $\{S\}$. Thus, with the monotonicity of the functions $x \mapsto e^x$ and $x \mapsto (x-K)^+$, we can write

$$\sum_{i=1}^{\beta} \lambda_i E\left(S_0 e^{M(\alpha_i, T, n)} - K\right)^+ \to E(A_T - K)^+$$

as $\beta \to \infty$. □

With this approximation and given the determination of the distributions of the variables $M(\alpha_i, T, n)$ as described in the previous section, we now have all the ingredients to proceed with the pricing of a discretely monitored Asian option.

3.5 Pricing in the Black-Scholes Model

In an earlier discussion we mentioned that the distributions of the discrete quantile process can also be determined through the random walk approximation route. We proceed to do so in this section, where we focus on the Black-Scholes model, with its underlying Brownian motion as a special Lévy process. As shown above, the core of our approximate pricing of an Asian option is now the determination of a set of expectations, namely $E\left(S_0 e^{M(\alpha_j, m, \sigma, T, n)} - K\right)^+$ for various values of α_j, where $m = r - \sigma^2/2$ is the drift of the Brownian motion followed by the natural logarithm of the underlying asset price, the volatility of which is σ in a market where the riskless rate of return is r. With this notation, $M(\alpha_j, m, \sigma, T, n)$ represents the α_j^{th} quantile of n equally spaced segments of this Brownian motion on the interval $[0, T]$. Correspondingly, we also use the notation $\tau(x, m, \sigma, T, n)$ for the occupation time as we soon shall exploit the space-time scaling

property of Brownian motion, thus justifying the explicit reference to the drift m and volatility σ. Using a basic property of expectation for non-negative random variables, we have

$$E\left(S_0 e^{M(\alpha,m,\sigma,T,n)} - K\right)^+ = S_0 \int_{ln(K/S_0)}^{\infty} P\{M(\alpha, m, \sigma, T, n) > x\} e^x \, dx.$$

Observe that

$$P\{M(\alpha, m, \sigma, T, n) > x\} \quad = \quad P\{\tau(x, m, \sigma, T, n) \leq \alpha\}.$$

Furthermore, by the space-time scaling property of Brownian motion, we can write

$$P\{\tau(x, m, \sigma, T, n) \leq \alpha\} = P\left\{\tau\left(\frac{x}{\sigma\sqrt{T}}, \frac{m\sqrt{T}}{\sigma}, 1, 1, n\right) \leq \alpha\right\},$$

where $\tau(x', m', 1, 1, n)$, for $x' = \frac{x}{\sigma\sqrt{T}}$ and $m' = \frac{m\sqrt{T}}{\sigma}$, is defined over the process $X_{t_i} = m' t_i + \xi(t_i)$ with $t_n = 1$ and standard Brownian motion $\xi(t)$. Assume that the number of monitoring dates n of the underlying asset process satisfies $n > m'^2$. This is generally easy to fulfill given that $m' = (r/\sigma - \sigma/2)\sqrt{T}$, where $0 < r/\sigma < 1$, with σ typically in the range of 0.2 to 0.60, and $T \leq 1$. Consider now a random walk $(\zeta_r, r \geq 0)$ with increments ξ such that $P\{\xi = 1\} = p$ and $P\{\xi = -1\} = q$, where

$$p = \frac{1}{2} + \frac{m'}{2\sqrt{n}} \text{ and } q = \frac{1}{2} - \frac{m'}{2\sqrt{n}}$$

for $n > m'^2$. For $j \in \{0, 1, 2, \dots n\}$, define $\Delta_n(j) = \sum_{i=1}^{n} \delta(\zeta_i > j)$, which counts the number of times the random walk is above j in the time interval $\{0, 1, \dots, n\}$. From (Takacs, 1996) we have for $x > 0$ the approximation

$$P\left\{\tau\left(\frac{x}{\sigma\sqrt{T}}, \frac{m\sqrt{T}}{\sigma}, 1, 1, n\right) \leq \alpha\right\} \sim P\{\Delta_n(k) \geq n - j\}$$

where $j = [n\alpha]$, $k = [\frac{x\sqrt{n}}{\sigma\sqrt{T}}]$, $k \geq 0$, and $0 < \alpha < 1$. Furthermore, for $n > \frac{mT^2}{\sigma^2}$

$$p = \frac{1}{2} + \frac{m\sqrt{T}}{2\sigma\sqrt{n}} \text{ and } q = \frac{1}{2} - \frac{m\sqrt{T}}{2\sigma\sqrt{n}}$$

Note that we can extend the definition of $\Delta_n(l)$ to $l < 0$ by observing that $\Delta_n(l)$ has the same the distribution as $n - \Delta_n^*(-l-1)$, where $\Delta_n^*(k)$, $k > 0$, is defined in the same manner as $\Delta_n(k)$ with the roles of p and q interchanged. Thus, for $x < 0$,

$$P\left\{\tau\left(\frac{x}{\sigma\sqrt{T}}, \frac{m\sqrt{T}}{\sigma}, 1, 1, n\right) \leq \alpha\right\} \sim P\{\Delta_n^*(-k-1) \leq j\}$$

where $j = [n\alpha]$, $k = \left[\frac{x\sqrt{n}}{\sigma\sqrt{T}}\right]$, $k \leq 0$ and $0 < \alpha < 1$. Furthermore, for $n > \frac{mT^2}{\sigma^2}$

$$p = \frac{1}{2} - \frac{m\sqrt{T}}{2\sigma\sqrt{n}} \text{ and } q = \frac{1}{2} + \frac{m\sqrt{T}}{2\sigma\sqrt{n}}.$$

Our expectation formula then becomes

$$E\left(S_0 e^{M(\alpha,m,\sigma,T,n)} - K\right)^+ \sim S_0 \int_{ln(K/S_0)}^{\infty} P\{\Delta_n(k) \geq n-j\} e^x \, dx,$$

where

$$P\{\Delta_n(k) \geq n-j\} = \begin{cases} \sum_{i=n-j}^{n-k} P\{\Delta_n(k) = i\} & \text{for } j \geq k > 0, \\ 0 & \text{for } j < k, \end{cases}$$

and

$$P\{\Delta_n(k) \geq n-j\} = \begin{cases} 1 - \sum_{i=j+1}^{n+k+1} P\{\Delta_n(-k-1) = i\} & \text{for } 0 > k \geq j-n, \\ 0 & \text{for } k < j-n\ . \end{cases}$$

For $1 \leq i \leq n-k$,

$$P\{\Delta_n(k) = i\} = P\{\Delta_i = i\}\left[P\{\rho(k+1) > n-i\} - P\{\rho(k) > n-i\}\right],$$

where

$$\rho(k) = \inf\{r : \zeta_r = k, r \geq 0\},$$

$$P\{\Delta_i = i\} = p - qP\{\rho(1) < i\},$$

$$P\{\rho(1) < i\} = 1 - (P\{\zeta_{i-1} = -1\} + P\{\zeta_{i-1} = 0\} + (1-p/q)P\{\zeta_{i-1} < -1\})$$

$$P\{\zeta_{i-1} < -1\} = \sum_{a=1-i}^{-2} P\{\zeta_{i-1} = a\}$$

and

$$P\{\zeta_{i-1} = a\} = \binom{i-1}{[(i-1+a)/2]} p^{(i-1+a)/2} q^{(i+1-a)/2}$$

and

$$P\{\rho(k+1) > n-i\} - P\{\rho(k) > n-i\} = P\{\zeta_{n-i} = k\} + (q/p)P\{\zeta_{n-i} = k+1\}$$

and

$$+(1-\frac{p}{q})\left(\frac{p}{q}\right)^k P\{\zeta_{n-i} < -k-1\}$$

and

$$P\{\zeta_{n-i} < -k-1\} = \sum_{a=i-n}^{-k-2} P\{\zeta_{n-i} = a\}$$

There are clearly several choices available for the weights and percentile levels for the approximation in Theorem 1. In fact, one may refer to a simple choice inspired from Tukey's tri-mean as a starting point. Through some numerical evidence, we show that this is amply adequate for practical purposes. In this case, we use the approximation

$$(A_T - K)^+ \sim \sum_{i=1}^{3} \lambda_i \left(S_0 e^{M(\alpha_i, T)} - K\right)^+$$

where $\alpha_1 = \lambda_1 = 0.25$, $\alpha_2 = \lambda_2 = 0.50$, $\alpha_3 = 1 - \lambda_3 = 0.75$.

3.6 Hedging Parameters

To obtain the Greeks, one needs the version below of the Leibniz integral formula

$$\frac{d}{d\xi}\int_{a(\xi)}^{\infty} g(x,\xi)dx = \int_{a(\xi)}^{\infty} \frac{\partial}{\partial\xi} g(x,\xi)dx - g(a(\xi),\xi)\frac{d}{d\xi}a(\xi).$$

Therefore, letting $g(x) = P\{M(\alpha, m, \sigma, T, n) > x\}e^x$, we have

$$\begin{aligned}
\text{Delta} &= \frac{d}{dS_0}\left[S_0 \int_{\ln(K/S_0)}^{\infty} g(x)dx\right] \\
&= \int_{\ln(K/S_0)}^{\infty} g(x)dx + S_0\left[\frac{d}{dS_0}\int_{\ln(K/S_0)}^{\infty} g(x)dx\right]
\end{aligned}$$

Now, by the Leibnitz rule, we therefore have

$$\text{Delta} = \int_{\ln(K/S_0)}^{\infty} g(x)dx + g\left(\ln(K/S_0)\right)$$

Another crucial hedging parameter, namely Gamma, can be computed as easily:

$$\begin{aligned}
\text{Gamma} &= \frac{d}{dS_0}\text{Delta} \\
&= \frac{d}{dS_0}\int_{\ln(K/S_0)}^{\infty} g(x)dx - \frac{1}{S_0}\frac{d}{dS_0}g\left(\ln(K/S_0)\right) \\
&= \frac{1}{S_0}g\left(\ln(K/S_0)\right) - \frac{1}{S_0}\frac{d}{dx}g\left(\ln(K/S_0)\right)
\end{aligned}$$

Recalling that $g(x) = P\{M(\alpha, m, \sigma, T, n) > x\}e^x$ and letting $f_M(x) = \frac{d}{dx}P\{M(\alpha, m, \sigma, T, n) < x\}$, we have

$$\frac{d}{dx}g(x) = \left(P\{M(\alpha, m, \sigma, T, n) > x\} + \frac{d}{dx}P\{M(\alpha, m, \sigma, T, n) > x\}\right)e^x$$

and

$$\text{Gamma} = \frac{1}{S_0}g\left(\ln(K/S_0)\right) - \frac{1}{S_0}\left[g\left(\ln(K/S_0)\right)\right]$$

Then we may use the following:

$$\begin{aligned}
\text{Delta} = \frac{dp}{dS_0} = \frac{d}{dS_0}\left[S_0\int_{ln(K/S_0)}^{\infty} P\{M(\alpha, m, \sigma, T, n) > x\}e^x\,dx\right] \\
= \int_{ln(K/S_0)}^{\infty} P\{M(\alpha, m, \sigma, T, n) > x\}e^x\,dx \\
-S_0\frac{S_0}{K}\left(\frac{-K}{S_0^2}\right)P\{M(\alpha, m, \sigma, T, n) > x\}e^x|_{x=ln(K/S_0)} \\
= \int_{ln(K/S_0)}^{\infty} P\{M(\alpha, m, \sigma, T, n) > x\}e^x\,dx + \frac{K}{S0}P\{M(\alpha, m, \sigma, T, n) > ln(K/S_0)\}
\end{aligned}$$

$$\begin{aligned}
\text{Gamma} &= \frac{d}{dS_0}\left(\frac{dp}{dS_0}\right) = \frac{K}{S_0}\frac{d}{dS_0}\left[P\{M(\alpha, m, \sigma, T, n) > ln(K/S_0)\}\right] \\
&= \frac{K}{S_0}\frac{d}{dx}\left[P\{M(\alpha, m, \sigma, T, n) < x\}\right]\frac{1}{S_0} \\
&= \frac{K}{S_0^2}f_M(ln(K/S_0)
\end{aligned}$$

where $f_M(x)$ is the probability distribution of $M(\alpha, m, \sigma, T, n)$ which can be approximated as

$$f_M(ln(K/S_0) \approx P\{\Delta_n(k) = j\}$$

$$\text{where } j = n - [n\alpha] \text{ and } k = \left[\frac{ln(K/S_0)\sqrt{n}}{\sigma\sqrt{T}}\right].$$

Additional hedging parameters, such as rho, vega and theta, can be approximated similarly.

3.7 Numerical Evaluation

In this section we compare the accuracy of our approximation against benchmark values computed via significantly much-slower Monte-Carlo simulation. Though our main theorem requires that $\beta \to \infty$, our results as displayed in Tables 3-1 and 3-2 indicate that the approximation is in fact very well behaved even when β is as small as 3. From Table 3-1, observe that the accuracy of the approximation deteriorates only in a small number of cases that have no practical interest. They are deep out-of-the money (thus unlikely to be exercised) options with negligible prices. In all the other cases, the deviations from the benchmark values are in fact well within the bid-ask spread for over-the-counter option contracts. Similar observations can be made regarding the results displayed in Table 3-2. In this case, we are able to obtain hedging parameters that are as important for the option writer, typically a bank as counterparty to a hedge fund, a manufacturer, or airline company. These hedging parameters have traditionally been omitted from the option pricing literature or relegated to numerical derivation via finite-differences, which are numerically unstable, or Monte Carlo simulation, which is very time-consuming.

3.8 Conclusion

Chapter 3 develops an approximation technique for Asian option pricing and hedging based on analytic expressions for quantile options when the underlying asset follows an exponential Lévy process. Our numerical results indicate that this

Table 3-1. Fixed Strike Asian call option with parameters $S_0 = 100$, $r = 0.1$, $n = 50$, and $T = 1$. Benchmark values result from Monte Carlo simulations with 100,000 paths (standard error in parentheses). Prices using quantile approximations (with $\beta = 3$) are given in the last column.

Volatility	K	From Benhamou's Paper (Monte Carlo Price and SE)		Benchmark Price (Expected Value and SE)		Option Price Using Quantile Options
0.1	80	22.78	(0.00)	22.78	(0.00)	22.71
0.1	90	13.73	(0.00)	13.73	(0.00)	13.68
0.1	100	5.24	(0.00)	5.25	(0.00)	5.29
0.1	110	0.72	(0.00)	0.73	(0.00)	1.07
0.1	120	0.03	(0.00)	0.03	(0.00)	0.13
0.3	80	23.07	(0.01)	23.09	(0.01)	22.94
0.3	90	15.22	(0.01)	15.20	(0.02)	15.23
0.3	100	9.01	(0.01)	9.00	(0.02)	9.07
0.3	110	4.83	(0.01)	4.86	(0.02)	5.15
0.3	120	2.35	(0.01)	2.39	(0.01)	2.83
0.5	80	24.83	(0.03)	24.86	(0.03)	24.56
0.5	90	18.32	(0.03)	18.29	(0.04)	18.13
0.5	100	13.18	(0.03)	13.13	(0.04)	12.99
0.5	110	9.23	(0.03)	9.24	(0.04)	9.33
0.5	120	6.36	(0.03)	6.32	(0.03)	6.69

Table 3-2. Fixed Strike Asian call option with parameters $S_0 = 100$, $r = 0.1$, $n = 50$, and $T = 1$. Approximation of option's delta with $\beta = 3$. Benchmark values result from Monte Carlo simulations with 100,000 paths (standard error in parentheses).

Sigma	K	Benchmark	Delta
0.1	80	0.95 (0.000)	0.95
0.1	90	0.95 (0.000)	0.94
0.1	100	0.78 (0.001)	0.72
0.1	110	0.22 (0.001)	0.20
0.1	120	0.01 (0.000)	0.03
0.3	80	0.91 (0.000)	0.86
0.3	90	0.79 (0.001)	0.72
0.3	100	0.61 (0.001)	0.55
0.3	110	0.41 (0.001)	0.38
0.3	120	0.24 (0.001)	0.23
0.5	80	0.82 (0.000)	0.75
0.5	90	0.71 (0.001)	0.61
0.5	100	0.58 (0.000)	0.52
0.5	110	0.46 (0.001)	0.43
0.5	120	0.35 (0.001)	0.28

approximation is very competitive with alternatives that are computationally more expensive.

CHAPTER 4
PRICING OF LOOKBACK OPTIONS USING LÉVY PROCESSES

4.1 Lookback Options

Chapter 4 presents an efficient method to price lookback options in the Lévy process context by extending the random walk duality results of (AitSahlia & Lai, 1998) originally developed in the Black-Scholes set-up and by exploiting the very fast numerical scheme recently developed by (Feng & Linetsky, 2008) and (Feng & Linetsky, 2009) to compute and invert Hilbert transforms. Though (Feng & Linetsky, 2009) also apply the Hilbert transform technology to price lookback options, their approach is significantly more complex than ours and is about twice as long. In addition, they need to determine the transition probability density of the Lévy process and impose conditions that exclude pure jumps processes, such as the popular Variance Gamma model (cf. (Madan & Seneta, 1990), (Milne & Madan, 1991), and (Madan et al., 1998).) In contrast, our approach is much simpler and makes use of only the characteristic function of the log-increment, which is central to Lévy processes. Furthermore, by focusing our approach on determining the distribution function of the maximum of the Lévy process we can also determine hedging parameters with minimal additional computational effort.

For ease of comparison we adopt the notation in (AitSahlia & Lai, 1998) originally developed for Brownian motion but now assume that the underlying price process $\{S_t\}$ follows an exponential Lévy process (i.e.; that which is followed by $\log S_t$.) Given N discrete monitoring dates $t_1, t_2, \ldots, t_N$, the maximum price $\tilde{M}_N = \max\{S_{t_1}, \ldots, S_{t_N}\}$ and minimum price $\tilde{\Lambda}_N = \min\{S_{t_1}, \ldots, S_{t_N}\}$ of the underlying asset lead to inception (time $t_0 = 0$) prices for both fixed strike and floating strike lookback options summarized in Table 4-1.

Table 4-1. Loookback option prices at time $t_0 = 0$

Fixed strike	Floating strike
$e^{-rT}E\left(\tilde{M}_N - K\right)^+$	$e^{-rT}E\left(\tilde{M}_N - S_{t_N}\right)^+$
$e^{-rT}E\left(K - \tilde{\Lambda}_N\right)^+$	$e^{-rT}E\left(S_{t_N} - \tilde{\Lambda}_N\right)^+$

The difficulty in pricing these options is essentially due to the fact that the distributions of $\tilde{M}_N$ and $\tilde{\Lambda}_N$ are not known in analytical form even for the standard geometric Brownian motion of the Black-Scholes model.

4.2 Duality and Extrema of Random Walks

Under the assumption that the underlying price $\{S_t\}$ follows an exponential Lévy process and given the discrete monitoring of the maximum and minimum at dates $t_1, t_2, \ldots, t_N$, we can write $S_{t_n} = S_0 e^{U_n}$, where $\{U_n : n \geq 1, U_0 = 0\}$ is a random walk with i.i.d. increments X_i such that their common characteristic function $\hat{\Psi}$ is explicitly known thanks to the Lévy-Khinchine formula.

Define now $\tau_- = \inf\{n : U_n \leq 0\}$ to be the first passage of the log-price process below zero, observed on a monitoring date, and $\tau_+ = \inf\{n : U_n > 0\}$ the corresponding first passage of the log-price process above zero. τ_- or τ_+ are called 'ladder epochs'. The duality property of this random walk will enable us, through τ_- and τ_+, to derive recursive expressions leading to the distributions of the extrema $\tilde{M}_N$ and $\tilde{\Lambda}_N$.

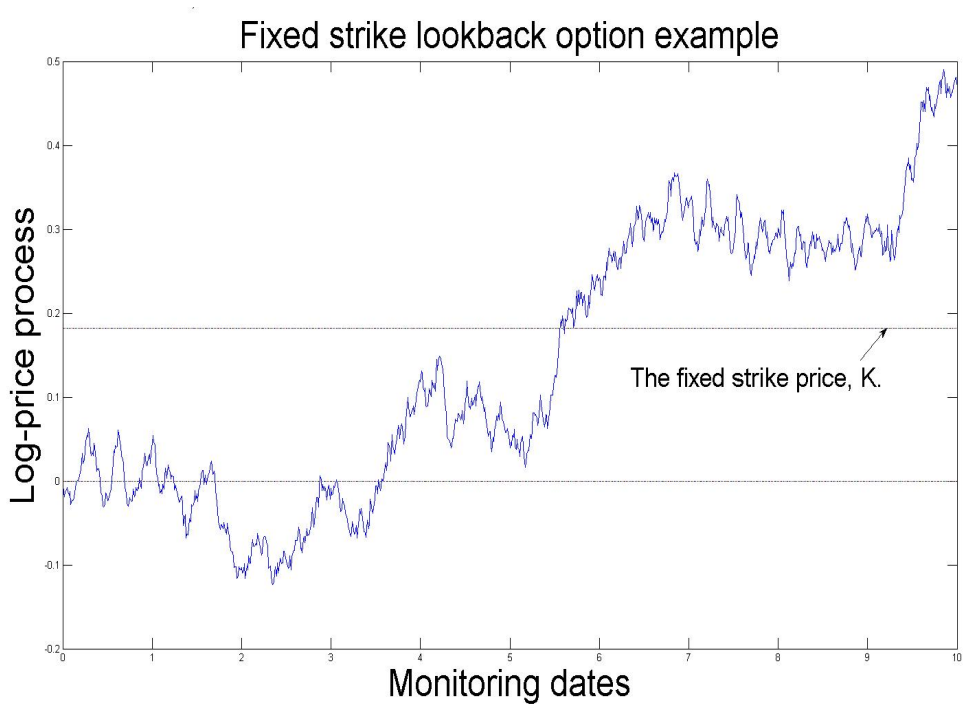

Figure 4-1. Sample path of a log-price process for a lookback option

Looking at Figure 4-1 we see that $\tau_- = 2$ even though the log-price has dropped below zero before time 1. Since we observe the prices only on the discrete monitoring dates, this does not affect τ_- as the price is back above zero at time 1. Also, $\tau_+ = 1$ and M_N is equal to the price on the 10th monitoring date, even though the continuous price process has a higher price since this higher price is not observed on a monitoring date. From (AitSahlia & Lai, 1998) we know that the distribution of the maximum log-price can be written as

$$
\begin{aligned}
&P\{M_N \in dx\} = \\
&\quad P\{U_1 \in dx\}\, P\{X_2 \le 0, X_2 + X_3 \le 0, \dots, X_2 + \cdots X_N \le 0\} \\
&\quad + \sum_{\nu=2}^{N} \Big[P\{U_\nu > U_i, i < \nu; U_\nu \in dx\} \times \\
&\qquad\qquad P\{X_{\nu+1} \le 0, X_{\nu+1} + X_{\nu+2} \le 0, \dots, X_{\nu+1} + \cdots + X_N \le 0\} \Big]
\end{aligned}
$$

for $x > 0$. Furthermore, the duality of random walks (Feller, 1971), lets us rewrite one of the above probabilities in terms of one of the ladder epochs

$$
\begin{aligned}
&P\{U_\nu > U_i, i < \nu; U_\nu \in dx\} \\
&\quad = P\{U_\nu - U_{\nu-1} > 0, \dots, U_\nu - U_1 > 0; U_\nu \in dx\} \\
&\quad = P\{U_1 > 0, \dots, U_{\nu-1} > 0; U_\nu \in dx\} \\
&\quad = P\{\tau_- > \nu; U_\nu \in dx\}
\end{aligned}
$$

And another of the above probabilities can also be written in terms of one of the ladder epochs

$$
\begin{aligned}
&P\{X_{\nu+1} \le 0, X_{\nu+1} + X_{\nu+2} \le 0, \dots, X_{\nu+1} + \cdots + X_N \le 0\} \\
&\quad = P\{U_1 \le 0, U_2 \le 0, \dots, U_{N-\nu} \le 0\} \\
&\quad = P\{\tau_+ > N - \nu\}
\end{aligned}
$$

Putting the simplified probabilities into the original equation yields, for $x > 0$,

$$P\{M_N \in dx\} = P\{U_1 \in dx\} P\{\tau_+ > N-1\} + \sum_{\nu=2}^{N} P\{\tau_- > \nu; U_\nu \in dx\} P\{\tau_+ > N-\nu\} \tag{4–1}$$

and for $x = 0$, it is clear that $P\{M_N = 0\} = P\{\tau_+ > N\}$. The advantage of writing the above probabilities in terms of the ladder epochs τ_- and τ_+ is that they can be determined recursively.

Define now the Fourier transform or characteristic function of a distribution function F of a real random variable X as (cf. (Chung, 1974)) as:

$$\mathcal{F}(F)(\xi) = E\left(e^{i\xi X}\right) = \int_{\mathbb{R}} e^{i\xi x} dF(x).$$

Alternatively, the notation $\widehat{F}$ will also be used. Furthermore, we define the Hilbert transform for such F by the Cauchy principal value integral

$$\mathcal{H}(F)(\xi) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{dF(x)}{\xi - x},$$

which reduces to the earlier definition of a Hilbert transform when F is absolutely continuous (with respect to the Lebesgue measure) with a density $f \in L^p(\mathbb{R})$. We can now state the following generalization to Proposition 1 in (AitSahlia & Lai, 1998).

Proposition 1. Let J be either $(0, \infty)$ or $(-\infty, 0]$ and $\tau = \inf\{n : U_n \notin J\}$. For $x \in J$, let $dF_n(x) = P\{\tau_- > n; U_n \in dx\}$ and let $\Psi(x)$ be the cumulative distribution function (cdf) of a log-increment X_i and $\hat{\Psi}$ its characteristic function. Then the characteristic functions $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_N$ can be determined recursively through the following relations:

$$\widehat{F}_1 = \widehat{\Psi} \tag{4–2}$$

$$\widehat{F}_n = \frac{1}{2}\widehat{F}_{n-1} \cdot \widehat{\Psi} + \frac{i}{2}\mathcal{H}\left(\widehat{F}_{n-1} \cdot \widehat{\Psi}\right) \quad \text{for } 2 \le n \le N \tag{4–3}$$

Proof. A straightforward generalization of the recursion on density functions in (AitSahlia & Lai, 1998), pg. 230, Eq. 10, can be expressed as

$$\begin{aligned} F_1(x) &= \Psi(x) \\ F_n(x) &= 1_J(x) \cdot (F_{n-1} * \Psi)(x), \quad \text{for } 2 \leq n \leq N \end{aligned}$$

We now recall the following property that relates Fourier and Hilbert transforms for a function ϕ on $\mathbb{R}$ (cf. (Stenger, 1993) and (Feng & Linetsky, 2008)):

$$\mathcal{F}\left(\mathbf{1}_{(0,\infty)} \cdot \phi\right)(\xi) = \frac{1}{2}\hat{\phi} + \frac{i}{2}\mathcal{H}\left(\hat{\phi}\right)(\xi),$$

which together with the independence of the Lévy increments leads, for $2 \leq n \leq N$, to:

$$\begin{aligned} \mathcal{F}(F_n) &= \mathcal{F}\left(1_J \cdot (F_{n-1} * \Psi)\right) \\ &= \frac{1}{2}\mathcal{F}\left(F_{n-1} * \Psi\right) + \frac{i}{2}\mathcal{H}\left(\mathcal{F}\left(F_{n-1} * \Psi\right)\right) \\ &= \frac{1}{2}\widehat{F}_{n-1} \cdot \widehat{\Psi} + \frac{i}{2}\mathcal{H}\left(\widehat{F}_{n-1} \cdot \widehat{\Psi}\right). \quad \square \end{aligned}$$

Remarks. First, note that the preceding applies to the distribution of the *minimum* of the random walk as well. Simply replace U_n by $-U_n$. Then

$$\Lambda_N = \min\{U_n : 0 \leq n \leq N\} = -\max\{-U_n : 0 \leq n \leq N\}$$

and for $x < 0$,

$$\begin{aligned} P\{\Lambda_N \in dx\} &= P\{U_1 \in dx\} P\{\tau_- > N-1\} \\ &+ \sum_{\nu=2}^{N} P\{\tau_+ > \nu; U_\nu \in dx\} P\{\tau_- > N-\nu\} \end{aligned}$$

Second, note that the recursions in Eq. 4–2 and Eq. 4–3 fit perfectly the set-up of (Feng & Linetsky, 2008) to apply their highly efficient algorithm to compute all the Fourier and Hilbert transforms and invert the last ($\widehat{F}_N$)for pricing purposes at a computational cost of $O(NM\log(M))$, where M is the number of quadrature points

in the integrals and N is the number of discrete observation dates, with a resulting error $O\left(M^{1/(1+\nu)}\exp(-cM^{\nu/(1+\nu)})\right)$, $c > 0$, which decays exponentially. The ultimate determination of $\widehat{F}_N$ (via its Fourier inversion) is at the root of the computation of the option price as we show next.

4.3 Fixed-Strike Lookback Options

We are now ready to apply the main result of the last section to price a fixed strike (a.k.a. hindsight) lookback option, which, upon exercise, grants the right to purchase the underlying asset at the minimum price and re-sell it at the strike K, for a put, or to buy it at the strike K and re-sell it at the maximum for a call. To enable comparisons with earlier results involving only Brownian motion, we shall focus on the call, whose payoff is $\left(S_0 e^{M_N} - K\right)^+$.

Proposition 2. The value of a hindsight (or fixed-strike) lookback call at inception is

$$e^{-rT} E\left(S_0 e^{M_N} - K\right)^+ = e^{-rT}\alpha_N \left(S_0 - K\right)^+ + e^{-rT}\sum_{\nu=1}^{N}\int_0^\infty \left(S_0 e^x - K\right)^+ dF_\nu(x), \qquad (4\text{–}4)$$

where $F_\nu(x)$ are obtained through the application of the numerical scheme of (Feng & Linetsky, 2008) to the recursions in Eq. 4–2 and Eq. 4–3 for $x > 0$, with $J = (-\infty, 0]$, and $\alpha_0, \alpha_1, \ldots, \alpha_N$ defined by

$$\alpha_0 = 1, \quad \alpha_n = G_n(0) - \lim_{x\to-\infty} G_n(x) \ \text{ for } n \geq 1,$$

where G_n defined for $x \leq 0$ by replacing $\widehat{F}_n$ by $\widehat{G}_n$ in Eq. 4–2 and Eq. 4–3 and using $J = (-\infty, 0]$.

Proof. By definition, we have

$$E\left(S_0 e^{M_N} - K\right)^+ = \int_0^\infty \left(S_0 e^x - K\right)^+ P\left\{M_N \in dx\right\},$$

the right hand side of which can be re-expressed as

$$\left(S_0 - K\right)^+ P\left\{M_N = 0\right\} + \int_{0+}^\infty \left(S_0 e^x - K\right)^+ P\left\{M_N \in dx\right\}.$$

Recall that $\tau_+ = \inf\{n : U_n > 0\}$ and $dG_n(x) = P\{\tau_+ > n; U_n \in dx\}$ for $x < 0$ and $n \geq 1$. Therefore

$$\alpha_n = \int_{-\infty}^{0} dG_n(x) = P\{\tau_+ > n\} = P\{U_1 \leq 0, \ldots, U_n \leq 0\}. \tag{4–5}$$

The latter, together with (4–1) and the decomposition above, yields

$$P\{M_N \in dx\} = \alpha_{N-1} P\{U_1 \in dx\} + \sum_{\nu=2}^{N} \alpha_{N-\nu} dF_\nu(x) \text{ for } x > 0,$$

which in turn concludes the proof by virtue of $P\{M_N = 0\} = P\{\tau_+ > N\}$.

4.4 Floating-Strike Lookback Options

We show in this section that the pricing via the recursions in Eq. 4–2 and Eq. 4–3 extends to floating-strike lookback options. These are contrasted to the fixed-strike by making the strike set to the price of the underlying upon exercise. Thus with a floating-strike put, its holder can purchase the underlying at its trading price upon exercise and sell it at the maximum it has achieved over the life of the contract, resulting in a payoff $\left(S_0 e^{M_N} - S_M\right)^+$. On the other hand, a floating-strike call allows its holder to purchase the asset at the minimum it achieved during its life and sell it at the price it trades upon exercise. Again, to allow for comparison with the classical Brownian process in the Black-Scholes model we illustrate the application of the approach on the put. Incidentally, floating-strike options are sometimes labeled standard.

Proposition 3. The value at inception of floating-strike lookback put is given by

$$e^{-rT} E\left(S_0 e^{M_N} - S_N\right)^+ = e^{-rT} S_0 \sum_{\nu=0}^{N-1} \beta_{N-\nu} I_\nu,$$

where

$$\begin{aligned} \beta_{N-\nu} &= \int_{-\infty}^{0} \left(1 - e^x\right) dG_{N-\nu}(x) \quad \text{for } 0 \leq \nu \leq N, \\ I_0 &= 1, \quad I_\nu = \int_0^\infty e^x dF_\nu(x) \quad \text{for } \nu \geq 1, \end{aligned}$$

with $\widehat{F}_\nu$ and $\widehat{G}_\nu$ obtained through the recursions in Eq. 4–2 and Eq. 4–3 as in Proposition 2.

Proof. Since $S_N = S_0 e^{U_N}$, we have $\left(S_0 e^{M_N} - S_N\right)^+ = S_0 \left(e^{M_N} - e^{U_N}\right)^+$, from which

$$\begin{aligned} E\left(e^{M_N} - e^{U_N}\right)^+ &= E\left(1 - e^{U_N}\right) 1_{\{U_1<0, U_2<0, \ldots, U_N<0\}} \\ &\quad + E\left(e^{U_1} - e^{U_N}\right) 1_{\{U_1>0, U_1>U_2, \ldots, U_1>U_N\}} \\ &\quad + \sum_{\nu=2}^{N-1} E\left(e^{U_\nu} - e^{U_N}\right) 1_{\{0<U_\nu, U_1<U_\nu, \ldots, U_{\nu-1}<U_\nu, U_\nu>U_{\nu+1}, \ldots, U_\nu>U_N\}}, \end{aligned}$$

where each of the above cases corresponds to the maximum being achieved at, respectively, $t_0 = 0$, t_1, or t_ν, $2 \le \nu \le N-1$. Observe that $P\{U_i = U_j\}$=0 for $i \neq j$. By definition, $\tau_+ = \inf\{n : U_n > 0\}$ and $\tau_- = \inf\{n : U_n \le 0\}$, but since $P\{U_n = 0\} = 0$ for all $n > 0$, we have $\tau_+ = \inf\{n : U_n \ge 0\}$ almost surely. Therefore

$$\begin{aligned} E\left(1 - e^{U_N}\right) 1_{\{U_1<0, U_2<0, \ldots, U_N<0\}} &= E\left(1 - e^{U_N}\right) 1_{\{\tau_+>0\}} \\ &= \int_{-\infty}^{0} (1 - e^x) dG_N(x). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} & E\left(e^{U_1} - e^{U_N}\right) 1_{\{U_1>0, U_1>U_2, \ldots, U_1>U_N\}} \\ = \; & E\left(e^{U_1} - e^{U_1 + \sum_{i=2}^{N} X_i}\right) 1_{\{U_1>0, U_2-U_1<0, \ldots, U_N-U_1<0\}} \\ = \; & \int_{x=0}^{\infty} \int_{y=-\infty}^{0} \left(e^x - e^{x+y}\right) \\ & \times P\left\{U_1 \in dx, X_2 < 0, X_2 + X_3 < 0, \ldots, X_2 + X_3 + \cdots + X_N < 0, \sum_{i=2}^{N} X_i \in dx\right\} \\ = \; & \int_0^{\infty} e^x P\{U_1 \in dx\} \\ & \times \left[\int_{-\infty}^{0} (1 - e^y) P\left\{X_2 < 0, X_2 + X_3 < 0, \ldots, X_2 + X_3 + \cdots + X_N < 0, \sum_{i=2}^{N} X_i \in dx\right\}\right] \\ = \; & \int_0^{\infty} e^x d\Psi(x) \left[\int_{-\infty}^{0} (1 - e^y) dG_{N-1}(y)\right], \end{aligned}$$

where we make use of the independence between U_1 and $(X_2, \dots, X_N)$ in the next to last step above.

Finally,

$$\begin{aligned}
&\textstyle\sum_{\nu=2}^{N-1} E\left(e^{U_\nu} - e^{U_N}\right) 1_{\{0<U_\nu, U_1<U_\nu, \dots, U_{\nu-1}<U_\nu, U_\nu>U_{\nu+1}, \dots, U_\nu>U_N\}} \\
&\quad = \sum_{\nu=2}^{N-1} \int_{x=0}^{\infty} \int_{y=-\infty}^{0} \left(e^x - e^{x+y}\right) P\{U_1 < U_\nu, \dots, U_{\nu-1} < U_\nu; U_n \in dx\} \\
&\qquad \times P\{X_{\nu+1} < 0, \dots, X_{\nu+1} + \cdots + X_N < 0; X_{\nu+1} + \cdots + X_N \in dy\} \\
&\quad = \sum_{\nu=2}^{N-1} \int_0^{\infty} e^x dF_\nu(x) \left[\int_{-\infty}^{0} (1 - e^y) dG_{N-\nu}(y)\right] \quad \square.
\end{aligned}$$

4.5 Extensions

Further applications of the technique presented above can be made with straightforward modifications to situations where the payoff depends on the minimum. In addition, all these options can be valued at other times than their inceptions by conditioning on the suprema up to the valuation time prior to expiration. Other variations on the pricing of these lookback include the situation, for example, where the suprema are observed over a predefined window within the life of the contract. In all these cases, the general relations provided by (AitSahlia & Lai, 1998) also apply here, with obvious modifications and will therefore not be repeated here.

Additionally, our approach is particularly well-suited for the computation of hedging parameters, which are especially crucial to the option writer's risk management practice. For example, the fixed-strike lookback price at time 0 of Proposition 2, Eq. 4–4, can be re-written as

$$\begin{aligned}
e^{-rT} E\left(S_0 e^{M_N} - K\right)^+ &= e^{-rT} \alpha_N (S_0 - K)^+ + e^{-rT} \sum_{\nu=1}^{N} \int_0^{\infty} (S_0 e^x - K)^+ dF_\nu(x) \\
&= \begin{cases} e^{-rT} \sum_{\nu=1}^{N} \int_{\log(K/S_0)}^{\infty} (S_0 e^x - K)) dF_\nu(x) & \text{if } S_0 \le K \\ e^{-rT} \alpha_N (S_0 - K) + e^{-rT} \sum_{\nu=1}^{N} \int_0^{\infty} (S_0 e^x - K) dF_\nu(x) & \text{if } S_0 > K \end{cases}
\end{aligned}$$

from which the delta and gamma parameters (first and second derivatives with respect to S_0, respectively) can easily be computed.

4.6 Summary

In Chapter 4 we extended a recursive algorithm that was originally developed for lookback option pricing when the underlying asset follows a geometric Brownian motion and is monitored at discrete dates within the life of the contract. Our extension to the geometric Lévy processes exploited the duality property of random walks through the use of ladder epochs resulting in recursion expressions for characteristic functions of the extrema that are perfectly tailored for a powerful algorithm for Hilbert transform akin to the Fast Fourier Transform. In addition, our approach yields hedging parameters with little additional computational effort. The ability to develop such results is inherently linked to the characterization of Lévy processes as consisting of continuous-time processes with independent and identically distributed increments. Thus their discrete monitoring is in fact very helpful as it enables us to use readily available results from fluctuation theory.

CHAPTER 5
CONCLUSION

Derivatives such as options are essential to the functioning of a modern economy. They provide opportunities for hedgers seeking to reduce their financial risks as well as speculators, whose hits and misses in the marketplace can provide additional liquidity. The pricing and hedging of these financial instruments has become increasingly challenging as ever more complex models have emerged to account for practical features that cannot be ignored. Over the past few years, continuous–time asset pricing models that rely on Lévy processes have gained significant prominence. Their widening adoption is due to their ability to capture salient features such as jumps and fat tails in asset return distributions that cannot be ignored. For example, if one were to maintain using the classical Black-Scholes-Merton model that gave mathematical finance its impetus in the early 1970's and which relies on the normality assumption of asset returns, one would seriously underestimate the actual probability of significant and unusual drops. For example, (Kou, 2008b) shows that over the period Jan 2, 1980 to December 31, 2005, the standardized (de-meaned and scaled by standard deviation) daily return of the critically watched S&P 500 index ranged from a minimum of -21.1550, to a maximum of 7.9967, which both occurred during the market crash year of 1987. Yet the probability of a standard normal distribution falling 21 units below its zero mean is approximately 1×10^{-107}. For comparison, it is estimated that the universe is about 15 billion years (or 5×10^{17} seconds) old. There is therefore clearly a need for alternative models, and those based on Lévy processes have many favorable features, including independence of increments and their infinite-divisibility, a variety of ways to capture large deviations, the possibility to incorporate jumps, particularly the popular pure-jump and jump-diffusion models. Finally, from a mathematical and computational tractability perspective, there is the remarkable Lévy-Khinchin representation which makes explicit the characteristic function of the process in terms of three parameters. In addition,

recent developments in the inversion of Fourier (otherwise known as characteristic functions in stochastic modeling) and related Hilbert transforms have spurred great interest in Lévy models.

The focus of this dissertation is on path-dependent options in the particular context of Lévy models. With payoffs depending on the entire path followed by the asset price of the underlying up until exercise, these options are especially useful when their holders wish to address a specific risk issue in a fashion that cannot be achieved by standard (or vanilla) options alone. For example, they could be concerned only if the underlying asset moves outside a certain range of values, say of interest or currency exchange rates, in which case they would be interested in barrier options, which come in the knock-in and knock-out flavors. The former entitle their holder the acquisition of a standard option only if the underlying asset price crosses a barrier. They however have to pay for the privilege upfront, with the possibility of never acquiring the option if the underlying does not cross the barrier before expiration. On the other hand, a knock-out option yields the same payoff as a standard option as long as the underlying asset price does not cross a barrier prior to expiration. Though barrier options were not explicitly addressed in this thesis, they are in fact intimately linked with lookback options, where the statistical distribution of the maximum (or minimum) is paramount as it is clear that a barrier above the initial asset price can only be breached if the maximum is above while, correspondingly, a barrier below would only be breached when the minimum is below it. Lookback options (or options on extrema) have the most flexible payoffs, and are thus the most expensive. They are used by either speculators or by very risk-averse operators. The other type of path-dependent options addressed in the present work concerns Asian (also known as average) options, which are widely used by multinational corporations to smooth their costs as well as their revenues in the face of highly variable raw material prices and large fluctuations in currency exchange rates.

Since the successful use of continuous-time modeling based on stochastic calculus to derive the celebrated Black-Scholes model, mathematical finance has developed mainly in this realm and has accomplished much. However, the monitoring of asset prices for path–dependent options is effected on a discrete set of dates. The resulting mathematical problem is significantly more complex than the operational use of stochastic calculus as it involves a mix of discrete and continuous methods. This dissertation contains new results regarding the efficient pricing of lookback options that exploit judiciously the random walk duality inherent to discretely observed Lévy processes together with recent algorithmic advances on Hilbert transforms that afford computational complexity comparable to the Fast Fourier Transform. The other topic in this dissertation concerns discretely monitored Asian options, the pricing and hedging of which we address through the use of conceptual quantile options. Though they may not be yet traded, history has proved that options initially started as concepts, such as Asian and lookback options, do eventually enjoy acceptance in practice. In our case, they enable a mathematical approach which when coupled with yet another set of results from fluctuation theory based on characteristic functions leads to efficient pricing and hedging computational advances.

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BIOGRAPHICAL SKETCH

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