THE NON-COMMUTATIVE CARATHÉODORY-FEJÉR PROBLEM

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2010

I dedicate this dissertation to my family and friends.

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my advisor Dr. Scott McCullough, for his ever-present guidance, encouragement and help, without which this dissertation would never have been completed. I am forever grateful to him for having me as his student and for inspiring me with his tremendous knowledge and expertise in Mathematics.

I would also like to thank Dr. James Brooks, Dr. James Hobert, Dr. Michael Jury and Dr. Li-chien Shen for serving on my doctoral committee. My special thanks to Dr. Michael Jury for his many valuable suggestions.

Finally, I would like to thank my family and my friends especially Balaji, Krishna, Subbu and Vijay, for their constant encouragement and support.

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of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

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August 2010

Chair: Scott McCullough
Major: Mathematics

We pose the Carathéodory-Fejér interpolation problem for open, circled and bounded matrix-convex sets in $\mathbb{C}^d$. By using the Blecher-Ruan-Sinclair characterization of an abstract unital operator algebra, we obtain a necessary and sufficient condition for the existence of a minimum-norm solution to the problem.

CHAPTER 1
INTRODUCTION

A classical interpolation problem in function theory is the Carathéodory-Fejér interpolation problem (CFP): Given $n+1$ complex numbers $c_0, c_1, \ldots, c_n$ does there exist a complex valued analytic function $f(z) = \sum_{j=0}^{\infty} f_j z^j$ defined on the open unit disc $\mathbb{D} \subset \mathbb{C}$ such that $f_j = c_j$ for all $0 \leq j \leq n$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$?

The problem and some of its variants were studied by Carathéodory, Fejér and Schur during the early 20th century in [30], [31] and [8] . A necessary and sufficient condition for the solvability of the CFP, which is commonly referred to as the Schur Criterion, is that the Toeplitz matrix

$$\begin{pmatrix} c_0 & 0 & \cdots & 0 \\ c_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ c_n & \cdots & c_1 & c_0 \end{pmatrix} \tag{1.1}$$

is a contraction.

An operator theoretic view of the CFP was first presented by Sarason in his pioneering work [29]. His formulation has had a major impact not only on the CFP and the related Pick interpolation problem but the development of operator theory and the study of non-self adjoint operator algebras generally. We now present a proof of the equivalence between the solvability of the CFP and the Schur Criterion that uses Sarason's ideas. We will begin with some definitions and state some well-known facts (without proofs).

Let $H^2(\mathbb{D})$ denote the Hardy Hilbert Space defined by

$$H^2(\mathbb{D}) = \left\{ f : \mathbb{D} \to \mathbb{C} \ : \ f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

The inner product on $H^2(\mathbb{D})$ is given by

$$\langle f, g\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Let L^2 denote the Hilbert space of square-integrable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure. The inner product is defined as:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta$$

where $d\theta$ denotes the Lebesgue measure on $[0, 2\pi]$.

As is often done, we will view L^2 as a space of functions rather than as a space of equivalence classes of functions, by identifying two functions to be equal if they are equal a.e. with respect to the normalized Lebesgue measure. i.e.

$$L^2 = \left\{ f : \mathbb{T} \to \mathbb{C} \,:\, \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta < \infty \right\}.$$

The Hilbert space L^2 is separable with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$, where $e_n : \mathbb{T} \to \mathbb{C}$ is defined by $e_n(e^{i\theta}) = e^{in\theta}$. For a proof of this fact see [7]. The expansion of the function $f \in L^2$ with respect to this orthonormal basis is called the *Fourier Series* expansion of f. We write

$$f(e^{i\theta}) \approx \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}$$

where the *Fourier coefficients* $\widehat{f}(n)$ are given by

$$\widehat{f}(n) = \langle f, e_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta$$

and they satisfy

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 < \infty.$$

The Fourier Transform produces a canonical unitary equivalence between $H^2(\mathbb{D})$ and a subspace of L^2 as described below. It is a consequence of Fatou's Theorem

(see pp 15 - 20, [22]) that for $f \in H^2(\mathbb{D})$, the radial limit $\tilde{f}(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ exists for almost all θ. We will call $\tilde{f}$ the boundary function of f. An important property of $\tilde{f}$ is that $\tilde{f} \in L^2$. The linear mapping that takes $f \in H^2(\mathbb{D})$ to $\tilde{f} \in L^2$ is an isometry from $H^2(\mathbb{D})$ onto a closed subspace of L^2 which we will denote by H^2. If $g \in L^2$ is given by $g(e^{i\theta}) \approx \sum_{n=0}^{\infty} \widehat{g}(n)e^{in\theta}$, then it is well known that the function f defined on the unit disc by $f(z) = \sum_{n=0}^{\infty} \widehat{g}(n)z^n$ is in $H^2(\mathbb{D})$ and in addition, $\tilde{f} = g$. Conversely, if $f \in H^2(\mathbb{D})$ is given by $f(z) = \sum_{n=0}^{\infty} f_n z^n$ then, the Fourier series expansion of $\tilde{f}$ is $\tilde{f}(e^{i\theta}) \approx \sum_{n=0}^{\infty} f_n e^{in\theta}$. This equivalence gives us the following way to view the space H^2 in terms of Fourier coefficients, namely,

$$H^2 = \{g \in L^2 : \widehat{g}(n) = 0 \text{ for } n < 0\}.$$

Let f be a Lebesgue measurable function defined on $\mathbb{T}$. The essential supremum of f is defined by $\|f\|_\infty = \inf\{M \geq 0 : |f(z)| \leq M \ a.e.\}$. Let L^∞ denote the Banach space

$$L^\infty = \{f : \mathbb{T} \to \mathbb{C} : f \text{ is measurable and } \|f\|_\infty < \infty\}$$

and let H^∞ denote $H^2 \bigcap L^\infty$.

Remark 1.0.1. *If f is a bounded analytic function defined on the unit disc, then $\tilde{f} \in H^\infty$ and $\|\tilde{f}\|_\infty \leq \sup\{|f(z)| : z \in \mathbb{D}\}$. Conversely, if $\tilde{f} \in H^\infty$, then the function $f \in H^2(\mathbb{D})$ is bounded by $\|\tilde{f}\|_\infty$.*

For $\phi \in L^\infty$, the *Toeplitz Operator* with symbol ϕ is defined by

$$T_\phi f = P(\phi f)$$

for each $f \in H^2$, where P is the orthogonal projection of L^2 onto H^2. If $\phi \in H^\infty$, then

$$T_\phi f = P(\phi f) = \phi f.$$

In this case the Toeplitz operator is said to be *analytic* and its matrix with respect to the basis $\{e^{in\theta}\}_{n=0}^{\infty}$ is given by

$$T_\phi = \begin{pmatrix} \phi_0 & 0 & 0 & 0 & 0 & \\ \phi_1 & \phi_0 & 0 & 0 & 0 & \ddots \\ \phi_2 & \phi_1 & \phi_0 & 0 & 0 & \ddots \\ \phi_3 & \phi_2 & \phi_1 & \phi_0 & 0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

An important property of the Toeplitz operator T_ϕ is that

$$\| T_\phi \| = \|\phi\|_\infty \tag{1.2}$$

Lemma 1.0.1 (Sarason)**.** *Let*

(i) U *represent the Unilateral shift on* H^2*, i.e.* $(Uf)(z) = zf(z)$ *for all* $f \in H^2$

(ii) $K = H^2 \ominus \psi H^2$ *where* ψ *is an inner function, i.e.* $\psi \in H^\infty$ *and* $|\psi(z)| = 1$ *a.e.*

(iii) $S = PU|_K$ *where* P *denotes the orthogonal projection of* H^2 *onto* K*.*

(iv) T *be a (bounded) operator on* K*.*

If $TS = ST$*, then there exists a function* $\phi \in H^\infty$ *such that* $T = PT_\phi|_K$ *where* T_ϕ *is the analytic Toeplitz operator with symbol* ϕ *and* $\|\phi\|_\infty = \| T \|$*.*

A proof of the above lemma can be found in [29].

Theorem 1.0.1. *The CFP has a solution if and only if the matrix in equation* (1.1) *has norm at most one.*

Proof. Suppose that there exists a solution f to the CFP. It follows from remark 1.0.1 and equation (1.2) that $\tilde{f} \in H^\infty$ and,

$$1 \geq \sup\{|f(z)| : z \in \mathbb{D}\} \geq \|\tilde{f}\|_\infty = \| T_{\tilde{f}} \|.$$

Since the matrix in equation (1.1) is the compression of the analytic Toeplitz operator $T_{\tilde{f}}$ to the subspace spanned by the orthonormal set $\langle 1, z, z^2, \ldots, z^n \rangle$ of H^2 (and computed with respect to this basis), it follows that the norm of that matrix is at most 1.

For the converse, we apply Lemma 1.0.1 with $\psi(z) = z^{n+1}$, $K = \langle 1, z, z^2, \ldots, z^n \rangle$ and $T : K \to K$ being the matrix in (1.1). Suppose that $\|T\| \leq 1$. Let S and P be the operators that appear in the hypothesis of Lemma 1.0.1. The matrix of S with respect to the basis $\{1, z, z^2, \ldots, z^n\}$ is given by

$$S = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}. \tag{1.3}$$

Thus $TS = ST$ and Lemma 1.0.1 implies that there exists $\phi \in H^\infty$ such that $T = PT_\phi|_K$ and

$$\|\phi\|_\infty = \|T\|.$$

Since $\phi \in H^\infty$, $\phi = \tilde{f}$ for some $f \in H^2(\mathbb{D})$. It follows from Remark 1.0.1 that

$$\sup\{|f(z)| : z \in \mathbb{D}\} \leq \|\phi\|_\infty.$$

Hence this f is a solution to the CFP. □

From the operator theory/algebra point of view, the CFP is essentially unchanged if the coefficients $c_0, c_1, \ldots, c_n$ are taken to be elements of $B(\mathcal{U})$ for some separable Hilbert space $\mathcal{U}$. Indeed, even in this case, a necessary and sufficient condition for the solvability of the CFP is the same as before, only now the entries of the Toeplitz matrix in (1.1) are bounded operators in $B(\mathcal{U})$ and the matrix itself is an element of $B(\oplus_1^{n+1}\mathcal{U})$. An alternate way of viewing the Schur Criterion which is more convenient for our purposes is the following.

Lemma 1.0.2. *Let $p(z) = \sum_{j=0}^{n} c_j z^j \in B(\mathcal{U})$ and $\mathcal{H}$ be an arbitrary Hilbert space. The Schur Criterion is equivalent to the contractivity of the operator $p(T) = \sum_{j=0}^{n} c_j \otimes T^j \in B(\mathcal{U} \otimes \mathcal{H})$ for every contraction $T \in B(\mathcal{H})$ which is nilpotent of order $n+1$; i.e., $T^{n+1} = 0$.*

Proof. ($\Leftarrow$) Let S be the matrix in equation (1.3). Since S is a contraction and $S^{n+1} = 0$ we have $\|p(S)\| \leq 1$. The fact that the norm of the Toeplitz matrix in equation (1.1) is equal to the norm of the operator $p(S)$ completes the argument.

($\Rightarrow$) Fix a Hilbert space $\mathcal{H}$ and a contraction $T \in B(\mathcal{H})$ which satisfies $T^{n+1} = 0$. Let $D_{T^*} = (I_{\mathcal{H}} - TT^*)^{\frac{1}{2}}$. D_{T^*} is called the *defect operator* of T^*. Let $\{e_j\}_{j=0}^{n}$ denote the standard orthonormal basis of $\mathbb{C}^{n+1}$. Define the operator $V : \mathcal{H} \to \mathbb{C}^{n+1} \otimes \mathcal{H}$ by

$$Vh = \sum_{j=0}^{n} e_j \otimes D_{T^*}(T^*)^j h.$$

Then

$$\begin{aligned}
\langle Vh, Vh \rangle &= \langle \sum_{j=0}^{n} e_j \otimes D_{T^*}(T^*)^j h, \sum_{k=0}^{n} e_k \otimes D_{T^*}(T^*)^k h \rangle \\
&= \sum_{j=0}^{n} \langle D_{T^*}(T^*)^j h, D_{T^*}(T^*)^j h \rangle \\
&= \sum_{j=0}^{n} \langle T^j (I_{\mathcal{H}} - TT^*)(T^*)^j h, h \rangle \\
&= \langle h, h \rangle.
\end{aligned}$$

Thus V is an isometry. Moreover, For each $k = 0, 1, 2, \ldots$ we have,

$$\begin{aligned}
V(T^*)^k h &= \sum_{j=0}^{n} e_j \otimes D_{T^*}(T^*)^{j+k} h \\
&= \sum_{j=0}^{n-k} e_j \otimes D_{T^*}(T^*)^{j+k} h \\
&= ((S^*)^k \otimes I_{\mathcal{H}})(\sum_{j=0}^{n} e_j \otimes D_{T^*}(T^*)^j h) \\
&= ((S^*)^k \otimes I_{\mathcal{H}}) Vh.
\end{aligned}$$

This implies that for each $k = 0, 1, 2, \dots,$

$$T^k = V^*(S^k \otimes I_{\mathcal{H}})V. \tag{1.4}$$

Using equation (1.4) we get,

$$\begin{aligned}
\|p(T)\| &= \|\sum_{j=0}^{n} c_j \otimes V^*(S^j \otimes I_{\mathcal{H}})V\| \\
&= \|(I_{\mathcal{U}} \otimes V^*)(\sum_{j=0}^{n} c_j \otimes (S^j \otimes I_{\mathcal{H}}))(I_{\mathcal{U}} \otimes V)\| \\
&\leq \|\sum_{j=0}^{n} c_j \otimes (S^j \otimes I_{\mathcal{H}})\| \\
&= \|p(S)\| \\
&\leq 1.
\end{aligned}$$

□

1.1 Summary of Results

Several commutative multi-variable generalizations of the CFP have been obtained for different domains - the polydisc $\mathbb{D}^d \subset \mathbb{C}^d$ for example - and for different interpolating classes of functions, for example the Schur-Agler class of analytic functions that take contractive operator values on any d-tuple of commuting strict contractions in a manner discussed in [1]. For more details see [14], [6]. Some results on the problem for bounded circular domains in $\mathbb{C}^d$ can also be found in [11]. Some non-commutative generalizations of the CFP have also been studied in [25], [26], [9], [20], [4].

In this thesis, some of the existing results on the CFP have been extended to the non-commutative setting of the free algebra on a finite number of generators. An example of a domain we consider here is the $d \times \tilde{d}$ *non-commutative matrix mixed ball* defined by

$$\mathcal{D}^{d\tilde{d}} = \bigcup_{n \in \mathbb{N}} \{X = (X_{11}, X_{12}, \dots, X_{d\tilde{d}}) : X_{ij} \text{ are } n \times n \text{ matrices and } \|X\|_{op} < 1\}.$$

where $\|X\|_{op}$ is the norm of the operator $X = (X_{ij})_{i,j=1}^{d,\tilde{d}} : (\mathbb{C}^n)^{\tilde{d}} \to (\mathbb{C}^n)^d$.

What follows are some definitions which will lead us to the statement of the CFP for this particular domain.

- Let $\mathcal{F}_{d\tilde{d}}$ denote the semigroup of words generated by the symbols $\{g_{ij}\}_{i,j=1}^{d,\tilde{d}}$
- A set $\Lambda \subset \mathcal{F}_{d\tilde{d}}$ is said to be an *initial segment* if wg_{ij}, $g_{ij}w \notin \Lambda$ for all $w \notin \Lambda$, $1 \leq i \leq d$, $1 \leq j \leq \tilde{d}$.
- For $X \in \mathcal{D}^{d\tilde{d}}$ and $w = g_{i_1j_1}g_{i_2j_2} \dots g_{i_kj_k} \in \mathcal{F}_{d\tilde{d}}$, *the evaluation of* X *at* w is defined by $X^w = X_{i_1j_1}X_{i_2j_2} \dots X_{i_kj_k}$.
- $X \in \mathcal{D}^{d\tilde{d}}$ is said to be Λ-*nilpotent* if $X^w = 0$ for all $w \notin \Lambda$.
- A *formal power series* f is an expression of the form $f = \sum_{w \in \mathcal{F}_{d\tilde{d}}} f_w w$ where the coefficients f_w are complex numbers
- For $X \in \mathcal{D}^{d\tilde{d}}$ we define $f(X) = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w X^w$ whenever the series converges converges (in the operator norm) in the indicated order.

The CFP for the $d \times \tilde{d}$ *non-commutative matrix mixed ball* is the following: Let Λ, a finite initial segment, and

$$p = \sum_{w \in \Lambda} p_w w$$

be given. Does there exist a formal power series f that $f_w = p_w$ for $w \in \Lambda$ and $\sup\{\|f(X)\| : X \in \mathcal{D}^{d\tilde{d}}\} \leq 1$?

(A special case of) Our main result is the following.

Theorem 1.1.1. *There exists a (minimum-norm) solution f to the above problem if and only if*

$$\sup\{\|p(X)\| : X \in \mathcal{D}^{d\tilde{d}}, \ \ X \textit{ is } \Lambda - \textit{nilpotent}\} \leq 1.$$

In the body of the thesis we actually pose the CFP for more general domains that are matrix convex sets in $\mathbb{C}^d$, and using the Blecher-Ruan-Sinclair characterization of abstract operator algebras, prove a generalization of Theorem 1.1.1 which allows for operator-valued coefficients p_w and f_w.

Throughout this thesis, we will be working with non-commutative analytic functions, i.e. formal power series with matrix or operator coefficients that converge on some non-commutative neighborhood of the origin (see [32], [33], [34], [24], [25], [26], [21], [19]). These functions are not only objects of great mathematical interest, but have applications in areas such as control theory and optimization. Non-commutative polynomials, in particular, are of special interest since non-commutative polynomial inequalities (matrix inequalities where the unknowns are matrices too), occur naturally in the context of dimension-free linear systems. Recent advances in the study of non-commutative linear matrix inequalities can be found in [12], [18].

1.2 Organization

This thesis is organized as follows: In Chapter 2, the definition of a matrix convex set in $\mathbb{C}^d$ is given along with some examples and properties and a proof of the Effros-Winkler matricial Hahn-Banach Separation Theorem. The chapter ends with a discussion on the Non-commutative Fock Space and the associated Creation Operators.

In Chapter 3, the definition an abstract operator algebra is introduced along with some examples. It is also shown that the quotient of an abstract operator algebra by a closed two-sided ideal is an abstract operator algebra. The chapter ends with the statement of the Blecher-Ruan-Sinclair Theorem for abstract unital operator algebras.

Chapter 4 is where the interpolating class $\mathcal{A}(\mathcal{K})^\infty$ and the ideal $\mathcal{I}(\mathcal{K})$ are introduced. It is shown that $\mathcal{A}(\mathcal{K})^\infty$ and the quotient $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ are abstract operator algebras. Several key properties including a weak-compactness type property of the algebra $\mathcal{A}(\mathcal{K})^\infty$ and the norm attainment property of the algebra $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ are established. The chapter ends with a discussion on completely contractive representations of the algebra $\mathcal{A}(\mathcal{K})^\infty$, where it is shown that tuples of finite-dimensional compressions of operators that give rise to completely contractive representations of $\mathcal{A}(\mathcal{K})^\infty$ lie on the boundary of the underlying matrix convex set.

In Chapter 5, the matrix and the operator versions of the CFP (with a finite initial segment Λ) for a class of matrix-convex sets in $\mathbb{C}^d$ are posed and using the results from Chapters 2 - 4, a necessary and sufficient condition for the existence of a minimum-norm solution is obtained.

In Chapter 6, a version of Theorem 1.1.1 where it is assumed that the initial segment Λ is an infinite set, is proved for two special non-commutative domains namely, the d-dimensional non-commutative (operator) polydisc and the $d \times \tilde{d}$ non-commutative (operator) mixed ball.

In Chapter 7, two important and very interesting questions that came up while this work was in progress, are posed.

CHAPTER 2
MATRIX CONVEXITY

A basic object of study in this thesis is a *quantized*, or *non-commutative*, version of a convex set. While the definitions easily extend to convex subsets of arbitrary vector spaces, here the focus is on subsets of $\mathbb{C}^d$, the complex d-dimensional space. In this chapter we present the definition of a matrix convex subset of $\mathbb{C}^d$ and introduce our standard assumptions regarding these sets. Some properties and examples of such sets and a matricial Hahn-Banach separation result are also presented. The chapter ends with a discussion of the Creation Operators on the Non-commutative Fock Space.

2.1 Matrix Convex Sets in $\mathbb{C}^d$

Let $M_{m,n} = M_{m,n}(\mathbb{C})$ denote the $m \times n$ matrices over $\mathbb{C}$. In the case that $m = n$, we write M_n instead of $M_{n,n}$. Let $M_n(\mathbb{C}^d)$ denote d-tuples with entries from M_n. Thus, an $X \in M_n(\mathbb{C}^d)$ has the form $X = (X_1, \dots, X_d)$ where each $X_j \in M_n$.

A *non-commutative set* $\mathcal{L}$ is a sequence $(\mathcal{L}(n))$ where, for $n \in \mathbb{N}$, $\mathcal{L}(n) \subset M_n(\mathbb{C}^d)$, which is *closed with respect to direct sums*; i.e., if $X \in \mathcal{L}(n)$ and $Y \in \mathcal{L}(m)$, then

$$X \oplus Y = (X_1 \oplus Y_1, \dots, X_d \oplus Y_d) \in \mathcal{L}(n+m) \tag{2.1}$$

where

$$X_j \oplus Y_j = \begin{pmatrix} X_j & 0 \\ 0 & Y_j \end{pmatrix}.$$

A non-commutative set $\mathcal{L} = (\mathcal{L}(n))$ is *open* (*closed*) if each $\mathcal{L}(n)$ is open (closed).

A *matrix convex set* $\mathcal{K} = (\mathcal{K}(n))$ is a non-commutative set which is *closed with respect to conjugation by an isometry*; i.e., if $\alpha \in M_{m,n}$ and $\alpha^*\alpha = I_n$, and if $X = (X_1, \dots, X_d) \in \mathcal{K}(m)$, then

$$\alpha^* X \alpha = (\alpha^* X_1 \alpha, \dots, \alpha^* X_d \alpha) \in \mathcal{K}(n). \tag{2.2}$$

A subset $\mathcal{U}$ of $M_n(\mathbb{C}^d)$ is *circled* if $e^{i\theta}\mathcal{U} \subseteq \mathcal{U}$ for all $\theta \in \mathbb{R}$. A matrix convex set $\mathcal{K}$ is circled if each $\mathcal{K}(n)$ is circled. As a canonical example of a circled matrix convex set,

suppose $\gamma > 0$ and consider the *non-commutative γ-neighborhood* $\mathcal{C}_\gamma = (\mathcal{C}_\gamma(n))$ of $0 \in \mathbb{C}^d$ defined by

$$\mathcal{C}_\gamma(n) = \{X \in M_n(\mathbb{C}^d) : \sum_{j=1}^{d} X_j X_j^* \prec \gamma^2\}.$$

A matrix convex set $\mathcal{K} = (\mathcal{K}(n))$, is said to be *bounded* if there exists $\gamma \leq 1$, $\Gamma \geq 1$ such that, for each $n \in \mathbb{N}$,

$$\mathcal{C}_\gamma(n) \subseteq \mathcal{K}(n) \subseteq \mathcal{C}_\Gamma(n), \tag{2.3}$$

Assumption 2.1.1. *Here, it is typically assumed that the matrix convex set $\mathcal{K}$*

(a) is open;

(b) is bounded; and

(c) is circled;

2.1.1 Examples

The following are some examples of matrix convex sets in $\mathbb{C}^d$ that satisfy the conditions of Assumption 1.

(i) Let $\mathcal{K}(n) = \{(X_1, \dots, X_d) \ : \ \|X_j\| < 1\}$ with $\gamma = 1$ and $\Gamma = \sqrt{d}$. $\mathcal{K} = (\mathcal{K}(n))$ is the d-dimensional non-commutative matrix polydisc.

(ii) Let $\mathcal{K}(n) = \{X = (X_{11}, X_{12}, \dots, X_{d\tilde{d}}) \ : \ \|X\|_{op} < 1\}$, where $\|X\|_{op}$ is the norm of the operator $X = (X_{ij})_{i,j=1}^{d,\tilde{d}} : (\mathbb{C}^n)^{\tilde{d}} \to (\mathbb{C}^n)^d$, with $\gamma = \frac{1}{\sqrt{d\tilde{d}}}$ and $\Gamma = \sqrt{d\tilde{d}}$. $\mathcal{K} = \mathcal{K}(n)$ is the $d \times \tilde{d}$ non-commutative matrix mixed ball.

(iii) Let $\mathcal{K}(n) = \{(X_1, \dots, X_d) \ : \ \sup\{\sum_{j=1}^{d} |\langle X_j y, y\rangle| \ : \ \|y\| = 1\} < 1\}$ with $\gamma = \frac{1}{d}$ and $\Gamma = 2\sqrt{d}$. If d = 1, then $\mathcal{K}$ is the collection of all strict numerical radius contractions.

2.1.2 Properties

Below we present three important properties of a matrix convex set $\mathcal{L} = (\mathcal{L}(n))$ in $\mathbb{C}^d$ which we will use in the forthcoming chapters.

(i) $\mathcal{L}(n)$ is convex for each $n \in \mathbb{N}$.

(ii) If $0 \in \mathcal{L}$, then $\mathcal{L}$ is closed with respect to conjugation by a contraction; i.e. α in equation (2.2) can be assumed to be a contraction.

(ii) The closure of $\mathcal{L}$ namely $\overline{\mathcal{L}} = (\overline{\mathcal{L}(n)})$ is matrix convex.

Proof. (i) Let $X, Y \in \mathcal{K}(n)$ and $0 \leq t \leq 1$. Consider the the block matrix $\alpha^* = (\sqrt{t}I_n \quad \sqrt{1-t}I_n) \in M_{n,2n}$. Since α is an isometry it follows that,

$$tX + (1-t)Y = \alpha^*(X \oplus Y)\alpha \in \mathcal{K}(n).$$

(ii) Let $\alpha \in M_{n,m}$ be such that $\|\alpha\| \leq 1$ and $X \in \mathcal{L}(n)$. We first observe that $\tilde{\alpha} = (\alpha \oplus 0) \in M_{m+n,m+n}$ is a contraction. Let

$$\tilde{X} = (X \oplus 0) \in \mathcal{K}(n+m)$$

and

$$\widehat{X} = (\tilde{X} \oplus 0) \in \mathcal{K}(2(n+m)).$$

Let $D_{\tilde{\alpha}}$ denote the defect operator of $\tilde{\alpha}$. i.e.,

$$D_{\tilde{\alpha}} = (I_{m+n} - \tilde{\alpha}^*\tilde{\alpha})^{\frac{1}{2}}.$$

Then the Julia matrix of $\tilde{\alpha}$ namely,

$$J(\tilde{\alpha}) = \begin{pmatrix} \tilde{\alpha} & D_{\tilde{\alpha}^*} \\ D_{\tilde{\alpha}} & -\tilde{\alpha}^* \end{pmatrix} \tag{2.4}$$

is unitary. Therefore,

$$\tilde{\alpha}^*\tilde{X}\tilde{\alpha} = \begin{pmatrix} I_{m+n} & 0_{m+n} \end{pmatrix} J(\tilde{\alpha})^*\widehat{X}J(\tilde{\alpha}) \begin{pmatrix} I_{m+n} \\ 0_{m+n} \end{pmatrix} \in \mathcal{K}(m+n).$$

It follows that

$$\alpha^*X\alpha = \begin{pmatrix} I_m & 0_{m,n} \end{pmatrix} \tilde{\alpha}^*\tilde{X}\tilde{\alpha} \begin{pmatrix} I_m \\ 0_{n,m} \end{pmatrix} \in \mathcal{K}(m).$$

(iii) Consider $M_n(\mathbb{C}^d)$ with the topology defined by the norm

$$|||(X_1, \dots X_d)||| = \sum_{j=1}^{d} \|X_j\|.$$

where the norm on the RHS is the usual operator norm.

Let $X = (X_1, \dots, X_d) \in \overline{\mathcal{L}(p)}$ and $Y = (Y_1, \dots, Y_d) \in \overline{\mathcal{L}(q)}$. Choose sequences $X_m = (X_{m1}, \dots, X_{md})$ and $Y_m = (Y_{m1}, \dots, Y_{md})$ from $\mathcal{L}(p)$ and $\mathcal{L}(q)$ respectively such that $X_m \to X$ and $Y_m \to Y$.

We have,

$$\begin{aligned}
|||(X_m \oplus Y_m) - (X \oplus Y)||| &= \sum_{j=1}^{d} \|(X_{mj} \oplus Y_{mj}) - (X_j \oplus Y_j)\| \\
&= \sum_{j=1}^{d} \|(X_{mj} - X_j) \oplus (Y_{mj} - Y_j)\| \\
&= \sum_{j=1}^{d} \max\{\|X_{mj} - X_j\|, \|Y_{mj} - Y_j\|\}
\end{aligned}$$

Since $X_m \to X$ and $Y_m \to Y$, the sum on the RHS above can be made arbitrarily small for all large values m. Thus $X \oplus Y \in \overline{\mathcal{L}(p+q)}$.

If $\alpha \in M_{p,\ell}$ is an isometry, then we have

$$\begin{aligned}
|||\alpha^* X_m \alpha - \alpha^* X \alpha||| &= \sum_{j=1}^{d} \|\alpha^*(X_{mj} - X_j)\alpha\| \\
&\leq \sum_{j=1}^{d} \|X_{mj} - X_j\|.
\end{aligned}$$

Since $X_m \to X$, it follows that $\alpha^* X \alpha \in \overline{\mathcal{L}(\ell)}$.

□

2.2 Matricial Hahn-Banach Separation

In this section we present a Hahn-Banach separation theorem for matrix convex sets in $\mathbb{C}^d$ due to Effros and Winkler (See [13]). The following contents, are minor variants of lemmas and theorems from [16].

Given a positive integer n, let $\mathcal{T}_n$ denote the collection of all positive semi-definite $n \times n$ complex matrices of trace one. A $T \in \mathcal{T}_n$ corresponds to a state on M_n, via the

trace,

$$M_n \ni A \mapsto \mathrm{tr}(AT).$$

An affine linear mapping $f : \mathcal{T}_n \to \mathbb{R}$ is a function of the form $f(x) = a_f + \lambda_f(x)$, where λ_f is linear and $a_f \in \mathbb{R}$.

Lemma 2.2.1. *Suppose $\mathcal{F}$ is a cone of affine linear mappings $f : \mathcal{T}_n \to \mathbb{R}$. If for each $f \in \mathcal{F}$ there is a $T \in \mathcal{T}_n$ such that $f(T) \geq 0$, then there is an $S \in \mathcal{T}_n$ such that $f(S) \geq 0$ for every $f \in \mathcal{F}$.*

Proof. For $f \in \mathcal{F}$, let

$$B_f = \{S \in \mathcal{T}_n : f(S) \geq 0\}.$$

By hypothesis each B_f is non-empty and it suffices to prove that

$$\cap_{f \in \mathcal{F}} B_f \neq \emptyset.$$

Since each B_f is compact, it suffices to prove that the collection $\{B_f : f \in \mathcal{F}\}$ has the finite intersection property. Accordingly, let $f_1, \dots, f_m \in \mathcal{F}$ be given. Arguing by contradiction, suppose

$$\cap_{j=1}^m B_{f_j} = \emptyset.$$

In this case, the range $F(\mathcal{T}_n)$ of the mapping $F : \mathcal{T}_n \to \mathbb{R}^m$ defined by

$$F(S) = (f_1(S), \dots, f_m(S))$$

is both convex and compact because $\mathcal{T}_n$ is both convex and compact. Moreover, it does not intersect

$$\mathbb{R}^m_+ = \{x = (x_1, \dots, x_m) : x_j \geq 0 \text{ for each } j\}.$$

Hence there is a linear functional $\lambda : \mathbb{R}^m \to \mathbb{R}$ such that $\lambda(F(\mathcal{T}_n)) < 0$ and $\lambda(\mathbb{R}^m_+) \geq 0$. There exists λ_j such that

$$\lambda(x) = \sum_{j=1}^m \lambda_j x_j.$$

Since $\lambda(\mathbb{R}^m_+) \geq 0$ it follows that each $\lambda_j \geq 0$ and since $\lambda \neq 0$, for at least one k, $\lambda_k > 0$. Let

$$f = \sum_{j=1}^{m} \lambda_j f_j.$$

Since $\mathcal{F}$ is a cone and $\lambda_j \geq 0$, we have $f \in \mathcal{F}$. On the other hand, if $T \in \mathcal{T}_n$, then $f(T) < 0$. Hence for this f there does not exist a $T \in \mathcal{T}_n$ such that $f(T) \geq 0$, a contradiction which completes the proof. □

Lemma 2.2.2. *Let $\mathcal{C} = (\mathcal{C}(n))$ denote a matrix convex set in $\mathbb{C}^d$ which contains $0 \in \mathbb{C}^d$. Let $n \in \mathbb{N}$ and a linear functional $F : M_n(\mathbb{C}^d) \to \mathbb{C}$ be given. If*

$$\textit{Re } F(\mathcal{C}(n)) \leq 1,$$

then there exists $S \in \mathcal{T}_n$ such that for each $m \in \mathbb{N}$, $Y \in \mathcal{C}(m)$ and $C \in M_{m,n}$

$$\textit{Re } F(C^*YC) \leq \textit{tr}(CSC^*).$$

Proof. For each $m \in \mathbb{N}$, $Y \in \mathcal{C}(m)$ and $C \in M_{m,n}$, we define the affine linear map $f_{Y,C} : \mathcal{T}_n \to \mathbb{R}$ by

$$f_{Y,C}(T) = \text{tr}(CTC^*) - \text{Re } F(C^*YC).$$

Let $\mathcal{F}_n = \{f_{Y,C} : Y \in \mathcal{C}(m), C \in M_{m,n}, m \in \mathbb{N}\}$. The set $\mathcal{F}_n$ is a cone since

$$\beta f_{Y,C} = f_{Y,\sqrt{\beta}C}$$

and

$$f_{Y_1,C_1} + f_{Y_2,C_2} = f_{Z,D}$$

where $Z = Y_1 \oplus Y_2$, D^* is the block matrix $(C_1^* \;\; C_2^*)$ and $\beta \geq 0$ is arbitrary.

Choosing $\tilde{T} = \alpha\alpha^*$ where α is a unit vector such that

$$\|C\alpha\| = \|C\|,$$

it follows that

$$f_{Y,C}(\tilde{T}) = \|C\|^2 - \text{Re } F(C^*YC).$$

If $\|C\| = 1$, then by property (iii) from Subsection 2.1.2, $C^*YC \in \mathcal{C}(n)$ and so by the hypothesis of the lemma, the right hand side of the above equation is non-negative. If C does not have norm 1, but is not zero, then a simple scaling argument shows that $f_{Y,C}(\tilde{T}) \geq 0$.

Hence by Lemma 2.2.1, there exists an $S \in \mathcal{T}_n$ such that $f_{Y,C}(S) \geq 0$ for every m, $Y \in \mathcal{C}(m)$ and $C \in M_{m,n}$. □

A *linear pencil* L of size n is a formal expression of the form $\sum_{\ell=1}^{d} L_\ell g_\ell$ where $L_\ell \in M_n$. For a d-tuple $T = (T_1, \dots, T_d)$ of bounded operators on a Hilbert Space $\mathcal{H}$, the *evaluation of* L *at* T is defined as the operator $L(T) = \sum_{\ell=1}^{d} L_\ell \otimes T_\ell$.

Theorem 2.2.1 (**Matricial Hahn-Banach Separation**). *Let $\mathcal{C} = (\mathcal{C}(n))$ denote a closed matrix convex set in $\mathbb{C}^d$ which contains a non-commutative neighborhood of $0 \in \mathbb{C}^d$. If $X \notin \mathcal{C}(n)$, then there is a linear pencil L (of size n) that satisfies the following conditions.*

(i) $2 - L(Y) - L(Y)^* \succeq 0$ *for all* $m \in \mathbb{N}$ *and* $Y \in \mathcal{C}(m)$

(ii) $2 - L(X) - L(X)^* \not\succeq 0$.

Proof. By applying the usual Hahn-Banach separation theorem on the closed convex subset $\mathcal{C}(n)$ of $M_n(\mathbb{C}^d)$ and using the assumption that $\mathcal{C}(n)$ contains a non-commutative neighborhood of 0, we obtain a linear functional $F : M_n(\mathbb{C}^d) \to \mathbb{C}$ such that

$$\text{Re } F(\mathcal{C}(n)) \leq 1 < \text{Re } F(X).$$

Choose $0 < \epsilon < 1$ sufficiently small such that $G = (1-\epsilon)F$ satisfies

$$\text{Re } G(\mathcal{C}(n)) \leq 1 < \text{Re } G(X). \tag{2.5}$$

From Lemma 2.2.2 there exists $S \in \mathcal{T}_n$ such that

$$\text{Re } F(C^*YC) \leq \text{tr}(CSC^*)$$

for each $m \in \mathbb{N}$, $Y \in \mathcal{C}(m)$ and $C \in M_{m,n}$.

If we let $R = (1-\epsilon)S + \frac{\epsilon}{n} I_n$, then $R \in \mathcal{T}_n$, it is invertible and

$$\text{Re } G(C^* Y C) \leq \text{tr}(CRC^*). \tag{2.6}$$

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ denote the standard orthonormal basis for $\mathbb{C}^d$. Given $1 \leq \ell \leq d$, and column vectors $c, d \in \mathbb{C}^n$, define a bounded sesquilinear form on $\mathbb{C}^n$ by

$$\mathcal{B}_\ell(c, d) = G(R^{-\frac{1}{2}} \overline{d} c^T R^{-\frac{1}{2}} \otimes \mathbf{e}_\ell)$$

where c^T denotes the transpose of c.

There exists a unique matrix $B_\ell \in M_n$ such that

$$\mathcal{B}_\ell(c, d) = \langle B_\ell c, d \rangle.$$

Define the linear pencil L by $\sum_{\ell=1}^d B_\ell g_\ell$.

Fix a positive integer m. Let $Y = (Y_1, \dots, Y_d) \in \mathcal{C}(m)$ be given and consider $L(Y)$, the evaluation of L at Y. Let $\{e_1, \dots, e_m\}$ denote the standard orthonormal basis of $\mathbb{C}^m$. For $\delta = \sum_{j=1}^m \delta_j \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^m$, we have

$$\begin{aligned}
\langle L(Y)\delta, \delta \rangle &= \sum_{i,j=1}^m \sum_{\ell=1}^d \langle B_\ell \delta_j, \delta_i \rangle \langle Y_\ell e_j, e_i \rangle \\
&= \sum_{i,j=1}^m \sum_{\ell=1}^d G(R^{-\frac{1}{2}} \overline{\delta_i} \delta_j^T R^{-\frac{1}{2}} \otimes \mathbf{e}_\ell) \langle Y_\ell e_j, e_i \rangle \\
&= \sum_{i,j=1}^m \sum_{\ell=1}^d G((R^{-\frac{1}{2}} \overline{\delta_i})\, (R^{-\frac{1}{2}} \overline{\delta_j})^* \otimes \mathbf{e}_\ell) \langle Y_\ell e_j, e_i \rangle \\
&= G(\sum_{\ell=1}^d (\sum_{i,j=1}^m (R^{-\frac{1}{2}} \overline{\delta_i}) \langle Y_\ell e_j, e_i \rangle (R^{-\frac{1}{2}} \overline{\delta_j})^*) \otimes \mathbf{e}_\ell) \\
&= G((R^{-\frac{1}{2}} \overline{\delta}) Y (R^{-\frac{1}{2}} \overline{\delta})^*)
\end{aligned}$$

where $\overline{\delta_j}$ is the column vector whose entries are complex conjugates of the column vector δ_j and $\overline{\delta}$ is the $n \times m$ matrix with j-th column $\overline{\delta_j}$. Using equation (2.6) we get,

$$\begin{aligned}
\text{Re } \langle L(Y)\delta, \delta\rangle &= \text{Re } G((R^{-\frac{1}{2}}\overline{\delta})Y(R^{-\frac{1}{2}}\overline{\delta})^*) \\
&\leq \text{tr}((R^{-\frac{1}{2}}\overline{\delta})^* R(R^{-\frac{1}{2}}\overline{\delta})) \\
&= \sum_{j=1}^{m} \langle R(R^{-\frac{1}{2}}\overline{\delta_j}), (R^{-\frac{1}{2}}\overline{\delta_j})\rangle \\
&= \langle \sum_{i=1}^{m} \overline{\delta_i} \otimes e_i, \sum_{j=1}^{m} \overline{\delta_j} \otimes e_j\rangle \\
&= \|\delta\|^2.
\end{aligned}$$

On the other hand, computing as above and using equation (2.5) we get,

$$\begin{aligned}
\text{Re } \langle L(X) \sum_{i=1}^{n} \overline{R^{\frac{1}{2}}} e_i \otimes e_i, \sum_{j=1}^{n} \overline{R^{\frac{1}{2}}} e_j \otimes e_j\rangle &= \text{Re } G(I_n X I_n) \\
&> 1 \\
&= \| \sum_{i=1}^{n} \overline{R^{\frac{1}{2}}} e_i \otimes e_i\|^2.
\end{aligned}$$

□

2.3 The Non-commutative Fock Space and Creation Operators

The Fock space and the Creation Operators that act on it play a central role in the analysis to follow in the forthcoming chapters. One of the key properties is that tuples of finite-dimensional compressions of the creation operators lie in the underlying (open, circled and bounded) matrix convex set $\mathcal{K}$. We provide a proof of this fact in this section.

2.3.1 The Free Semi-group on d Letters and Intial Segments

The Fock space is defined in terms of the free semi-group on d letters.

Let $\mathcal{F}_d$ denote the set of all words generated by d symbols $\{g_1, ..., g_d\}$. Define the product on $\mathcal{F}_d$ by concatenation. i.e., if $w = g_{i_1}...g_{i_m}$ and $v = g_{j_1}...g_{j_n}$, then the product wv is given by $g_{i_1}...g_{i_m}g_{j_1}...g_{j_n}$. $\mathcal{F}_d$ is a semi-group with respect to this product, with the empty word $\emptyset$ acting as the identity element, i.e. $w\emptyset = w = \emptyset w$ for all $w \in \mathcal{F}_d$.

The length of the word $w = g_{i_1} \dots g_{i_m}$ is declared to be m (where it is assumed that $g_{i_j} \neq \emptyset$) and is denoted $|w|$. The length of $\emptyset$ is zero.

A set $\Lambda \subset \mathcal{F}_d$ is an *initial segment* if its complement is an ideal in the semi-group $\mathcal{F}_d$; i.e., if both $g_j w, w g_j \in \mathcal{F}_d \setminus \Lambda$ $(1 \leq j \leq d)$, whenever $w \in \mathcal{F}_d \setminus \Lambda$. In the case that $d = 1$ an initial segment is thus a set of the form $\{\emptyset, g_1, g_1^2, \dots, g_1^m\}$ for some m.

2.3.2 The Non-commutative Fock Space

Let $\mathbb{C}\langle g \rangle = \mathbb{C}\langle g_1, \dots, g_d \rangle$ denote the algebra of non-commuting polynomials in the variables $\{g_1, \dots, g_d\}$. Thus elements of $\mathbb{C}\langle g \rangle$ are linear combinations of elements of $\mathcal{F}_d$; i.e., an element of $\mathbb{C}\langle g \rangle$ of degree (at most) k has the form

$$\sum_{j=0}^{k} \sum_{|w|=j} p_w w,$$

where the p_w are complex numbers.

To construct the Fock space, $\mathbb{F}^2$, define an inner product on $\mathbb{C}\langle g \rangle$ by defining

$$\langle w, v \rangle = \begin{cases} 0 & \text{if } w \neq v \\ 1 & \text{if } w = v \end{cases} \tag{2.7}$$

for $w, v \in \mathcal{F}_d$ and extending it by linearity to all of $\mathbb{C}\langle g \rangle$. The completion of $\mathbb{C}\langle g \rangle$ in this inner product is then the Hilbert space $\mathbb{F}^2$.

2.3.3 The Creation Operators

There are natural isometric operators on $\mathbb{F}^2$ called the *creation operators* which have been studied intensely in part because of their connection to the Cuntz algebra [7]. Given $1 \leq j \leq d$, define $S_j : \mathbb{F}^2 \to \mathbb{F}^2$ by $S_j v = g_j v$ for a word $v \in \mathcal{F}_d$ and extend S_j by linearity to all of $\mathbb{C}\langle g \rangle$. It is readily verified that S_j is an isometric mapping of $\mathbb{C}\langle g \rangle$ into itself and it thus follows that S_j extends to an isometry on all of $\mathbb{F}^2$. In particular $S_j^* S_j = I$, the identity on $\mathbb{F}^2$. Also of note is the identity,

$$\sum_{j=1}^{d} S_j S_j^* = P, \tag{2.8}$$

where P is the projection onto the orthogonal complement of the one-dimensional subspace of $\mathbb{F}^2$ spanned by $\emptyset$, which follows by observing, for a word $w \in \mathcal{F}_d$ and $1 \leq j \leq d$, that

$$S_j^*(w) = \begin{cases} v & \text{if } w = g_j v \\ 0 & \text{otherwise.} \end{cases}$$

Of course, as it stands the tuple $S = (S_1, \dots, S_d)$ acts on the infinite dimensional Hilbert space $\mathbb{F}^2$. There are however, finite dimensional subspaces which are essentially determined by ideals in $\mathcal{F}_d$ and which are invariant for each S_j^*.

The subset $\Lambda(\ell) = \{w : |w| \leq \ell\}$ of $\mathcal{F}_d$ is a canonical example of a finite initial segment. And the subspace $\mathbb{F}(\ell)^2$ of $\mathbb{F}^2$ spanned by $\Lambda(\ell)$ is invariant for S_j^*, $1 \leq j \leq d$. Let $V(\ell)$ denote the inclusion of $\mathbb{F}(\ell)^2$ into $\mathbb{F}^2$ and let $S(\ell)$ denote the operator $V(\ell)^* S V(\ell)$. Thus, $S(\ell) = ((S(\ell))_1, \dots, (S(\ell))_d)$ where $(S(\ell))_j = V(\ell)^* S_j V(\ell)$.

Recall γ from the definition of the (open, bounded and circled) matrix-convex set $\mathcal{K}$.

Lemma 2.3.1. *If $t < \gamma$, then $tS(\ell) \in \mathcal{K}(n)$ for some $n \in \mathbb{N}$.*

Proof. Let P denote both the projection of $\mathbb{F}^2$ and $\mathbb{F}(\ell)^2$ onto the orthogonal complement of the span of $\emptyset$ in $\mathbb{F}^2$ and $\mathbb{F}(\ell)^2$ respectively. It follows from equation (2.8) that

$$\begin{aligned} P &= V(\ell)^* P V(\ell) \\ &= V(\ell)^* \left(\sum_{j=1}^{d} S_j S_j^* \right) V(\ell) \\ &= \sum_{j=1}^{d} (S(\ell))_j (S(\ell))_j^*. \end{aligned}$$

Thus for $t < \gamma$, that $tS(\ell) \in \mathcal{C}_\gamma(n) \subseteq \mathcal{K}(n)$, where $n = \sum_{j=0}^{\ell} d^j$ is the dimension of $\mathbb{F}(\ell)^2$. $\square$

Remark 2.3.1. $S(\ell)^w = 0$ *for all* $w \notin \Lambda(\ell)$.

CHAPTER 3
ABSTRACT OPERATOR ALGEBRAS

We begin this chapter with the definition of an abstract operator algebra. Following that we present some examples and the proof of the fact that the quotient of an abstract operator algebra by a closed two-sided ideal is an abstract operator algebra. Furthermore, we present a characterization of an abstract unital operator algebra due to Blecher, Ruan and Sinclair.

3.1 Abstract Operator Algebra

Let V be a complex vector space and $M_{p,q}(V)$ denote the set of all $p \times q$ matrices with entries from V. V is said to be a *matrix normed space* provided that there exist norms $\|\cdot\|_{p,q}$ on $M_{p,q}(V)$ that satisfy

$$\|A \cdot X \cdot B\|_{\ell,r} \leq \|A\|\|X\|_{p,q}\|B\|$$

for all $A \in M_{\ell,p}, X \in M_{p,q}(V), B \in M_{q,r}$.

A matrix normed space V is said to be an *abstract operator space* if

$$\|X \oplus Y\|_{p+\ell,q+r} = \max\{\|X\|_{p,q}, \|Y\|_{\ell,r}\}$$

where $X \in M_{p,q}(V)$ and $Y \in M_{\ell,r}(V)$ and $X \oplus Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$.

It is important to note that, without loss of generality, we can replace the rectangular matrices in the above definitions with square matrices.

V is an *abstract operator algebra* if V is an algebra, an abstract operator space and if the product on V is completely contractive i.e. $\|XY\|_p \leq 1$ whenever $\|X\|_p \leq 1$ and $\|Y\|_p \leq 1$ for all $X, Y \in M_p(V)$ and for all p. We say V is *unital* if V contains a multiplicative unit.

Before we look at examples of abstract operator algebras, we present, as a remark, an interesting fact about the abstract operator space $\mathbb{C}^d$.

Remark 3.1.1. *The closed unit balls of the abstract operator space $\mathbb{C}^d$ form a matrix convex set.*

A partial converse to the above remark is the following.

Lemma 3.1.1. *Let $\mathcal{L} = (\mathcal{L}(n))$ be a closed, bounded, absorbing and circled matrix convex set in $\mathbb{C}^d$. If $\mathcal{L}$ is strongly circled, i.e. $U\mathcal{L}(n) \subseteq \mathcal{L}(n)$ for all $n \in \mathbb{N}$ and unitary matrices $U \in M_n$, then there exists a sequence of norms $\|\cdot\|_n$ such that $\mathcal{L}(n)$ is the closed unit ball of $M_n(\mathbb{C}^d)$ with respect to $\|\cdot\|_n$, and $\mathbb{C}^d$ together with the sequence of norms $\|\cdot\|_n$ is an abstract operator space.*

Proof. It follows from the hypothesis and the definition of matrix convexity that for any unitary matrix $U \in M_n$,

$$\mathcal{L}(n)U = U^*(U\mathcal{L}(n))U \subseteq \mathcal{L}(n).$$

This implies that for unitaries $U, V \in M_n$,

$$U\mathcal{L}(n)V \subseteq \mathcal{L}(n). \tag{3.1}$$

Since $\mathcal{L}(n)$ is convex, closed, bounded, absorbing and circled, it is the closed unit ball of $M_n(\mathbb{C}^d)$ with respect to some norm, which we will denote $\|\cdot\|_n$. We need to show that $\mathbb{C}^d$ together with the sequence of norms $\|\cdot\|_n$ is an abstract operator space.

Let $X \in M_n(\mathbb{C}^d)$ be such that $\|X\|_n = 1$ and $A, B \in M_n$ be of unit norm. Consider the Julia matrices (see (2.4)) $J(A), J(B)$ of A and B. Since $J(A)$ and $J(B)$ are unitary, it follows from equation (3.1) that,

$$\|J(A)(X \oplus 0)J(B)\|_{2n} \leq 1.$$

Hence,

$$\|AXB\|_n = \| \begin{pmatrix} I_n & 0_n \end{pmatrix} J(A)(X \oplus 0)J(B) \begin{pmatrix} I_n \\ 0_n \end{pmatrix} \|_n \leq 1.$$

If any of $A, B \in M_n$ and $X \in M_n(\mathbb{C}^d)$ are not of unit norm, then a simple scaling argument shows that $\|AXB\|_n \leq \|A\|\|X\|_n\|B\|$. Thus $(\mathbb{C}^d, \|\cdot\|_n)$ is a matrix-normed

space. Next we show that if $X \in M_n(\mathbb{C}^d)$ and $Y \in M_m(\mathbb{C}^d)$ then, $\|X \oplus Y\|_{n+m} = \max\{\|X\|_n, \|Y\|_m\}$.

Since $(\mathbb{C}^d, \|\cdot\|_n)$ is a matrix normed space it follows that,

$$\|X \oplus Y\|_{n+m} \geq \max\{\|X\|_n, \|Y\|_m\}.$$

To prove the reverse inequality, observe that $\|\frac{X}{\max\{\|X\|_n,\|Y\|_m\}}\|_n$ and $\|\frac{Y}{\max\{\|X\|_n,\|Y\|_m\}}\|_m$ are at most one. Hence, by the matrix convexity of $\mathcal{L}$, it follows that,

$$\|(\frac{X}{\max\{\|X\|_n, \|Y\|_m\}} \oplus \frac{Y}{\max\{\|X\|_n, \|Y\|_m\}})\|_{n+m} \leq 1.$$

□

3.1.1 Examples

(i) Let $\mathcal{H}$ be a Hilbert space and $V = B(\mathcal{H})$. Define matrix norms on V by

$$\|(T_{ij})\|_{p,q} = \|T\|$$

where T is the operator $(T_{i,j})_{i,j=1}^{p,q} : \oplus_{\ell=1}^{q}\mathcal{H} \to \oplus_{\ell=1}^{p}\mathcal{H}$ and $\|T\|$ is its operator norm. Then V together with the sequence of norms $\|\cdot\|_{p,q}$ is an abstract unital operator algebra.

(ii) Let $\mathcal{H}$ be an arbitrary separable Hilbert space, and V denote the algebra of polynomials in d variables. Define matrix norms on V by

$$\|(x_{i,j})\|_{p,q} = \sup\{\|(x_{i,j}(T))\|\}$$

where the supremum is taken over all d-tuples $T = (T_1, \ldots T_d)$, where $\{T_k\}_{k=1}^{d} \subset B(\mathcal{H})$ is a set of commuting contractions. Then V together with the sequence of norms $\|\cdot\|_{p,q}$ is an abstract unital operator algebra.

3.1.2 The Quotient Operator Algebra

Let V be an abstract operator space with the sequence of norms $\|\cdot\|_{p,q}$, and let W be a closed subspace. Let $\eta : V \to V/W$ denote the quotient map $\eta(x) = x + W$. By identifying $M_{p,q}(V/W)$ with $M_{p,q}(V)/M_{p,q}(W)$ we get a sequence of norms $|||\cdot|||_{p,q}$ on $M_{p,q}(V/W)$ defined by

$$|||(\eta(x_{ij}))|||_{p,q} = \inf\{\|(x_{ij} + y_{ij})\|_{p,q} : y_{ij} \in W\}$$

Lemma 3.1.2. *V/W with the sequence of norms $|||\cdot|||_{p,q}$ defined as above is an abstract operator space.*

Proof. Let $A \in M_{\ell,p}$, $X = (\eta(x_{ij})) \in M_{p,q}(V/W)$ and $B \in M_{q,r}$. Choose $(y_{ij}) \in M_{p,q}(W)$ such that

$$|||(\eta(x_{ij}))|||_{p,q} + \epsilon > \|(x_{ij} + y_{ij})\|_{p,q}. \tag{3.2}$$

Observe that

$$AXB = A(\eta(x_{ij}))B = \eta(A(x_{ij})B) = \eta(A(x_{ij} + y_{ij})B). \tag{3.3}$$

Using equations (3.2) and (3.3) gives,

$$\begin{aligned}
|||AXB|||_{p,q} &= |||\eta(A(x_{ij} + y_{ij})B)|||_{p,q} \\
&\leq \|A(x_{ij} + y_{ij})B\|_{p,q} \\
&\leq \|A\|\|(x_{ij} + y_{ij})\|_{p,q}\|B\| \\
&< \|A\|(|||(\eta(x_{ij}))|||_{p,q} + \epsilon)\|B\| \\
&= \|A\|(|||X|||_{p,q} + \epsilon)\|B\|.
\end{aligned}$$

By letting $\epsilon \to 0$, it follows that V/W is a matrix-normed space.

Let $X = (\eta(x_{ij})) \in M_{p,q}(V/W)$ and $Y = (\eta(y_{mn})) \in M_{\ell,r}(V/W)$. Since V/W is a matrix-normed space, it can be seen that

$$|||X \oplus Y|||_{p+\ell,q+r} \geq \max\{\|X\|_{p,q}\,,\ \|Y\|_{\ell,r}\}.$$

To prove the reverse inequality, choose $(a_{ij}) \in M_{p,q}(W)$ and $(b_{mn}) \in M_{\ell,r}(W)$ such that

$$|||(\eta(x_{ij}))|||_{p,q} + \frac{\epsilon}{2} > \|(x_{ij} + a_{ij})\|_{p,q} \tag{3.4}$$

$$|||(\eta(y_{mn}))|||_{\ell,r} + \frac{\epsilon}{2} > \|(y_{mn} + b_{mn})\|_{\ell,r}, \tag{3.5}$$

and observe that

$$X \oplus Y = \eta((x_{ij} + a_{ij}) \oplus (y_{mn} + b_{mn})). \tag{3.6}$$

Using equations (3.4), (3.5) and (3.6) yields,

$$\begin{aligned}
|||X \oplus Y|||_{p+\ell,q+r} &= |||\eta((x_{ij} + a_{ij}) \oplus (y_{mn} + b_{mn}))|||_{p+\ell,q+r} \\
&\leq \|(x_{ij} + a_{ij}) \oplus (y_{mn} + b_{mn})\|_{p+\ell,q+r} \\
&= \max\{\|(x_{ij} + a_{ij})\|_{p,q}\,,\ \|(y_{mn} + b_{mn})\|_{\ell,r}\} \\
&< \max\{|||X|||_{p,q}\,,\ |||Y|||_{\ell,r}\} + \epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ completes the proof. □

Corollary 3.1.1. *Let V be an abstract unital operator algebra and W be a closed two-sided ideal in V. Then, V/W is an abstract unital operator algebra.*

Proof. From Lemma 3.1.2 we know that V/W is an abstract operator space. Moreover, $1 + W$ is the unit of V/W. It remains to show that the product on V/W is completely contractive. For that purpose, let $X = (\eta(x_{ij}))$ and $Y = (\eta(y_{ij})) \in M_p(V/W)$. Choose $(a_{ij}), (b_{ij}) \in M_p(W)$ such that

$$|||(\eta(x_{ij}))|||_p + \epsilon > \|(x_{ij} + a_{ij})\|_p \tag{3.7}$$

$$|||(\eta(y_{ij}))|||_p + \epsilon > \|(y_{ij} + b_{ij})\|_p \tag{3.8}$$

Since W is a two-sided ideal of V, it follows that

$$XY = (\eta(x_{ij}))(\eta(y_{ij})) = \eta((x_{ij})(y_{ij})) = \eta((x_{ij} + a_{ij})(y_{ij} + b_{ij})) \tag{3.9}$$

Using equations (3.7), (3.8), (3.9) and the fact that multiplication in V is completely contractive yields,

$$\begin{aligned}
|||XY|||_p &= |||\eta((x_{ij}+a_{ij})(y_{ij}+b_{ij}))|||_p \\
&\leq \|(x_{ij}+a_{ij})(y_{ij}+b_{ij})\|_p \\
&\leq \|(x_{ij}+a_{ij})\|_p \|(y_{ij}+b_{ij})\|_p \\
&< (|||(\eta(x_{ij}))|||_p + \epsilon)(|||(\eta(y_{ij}))|||_p + \epsilon) \\
&= (|||X|||_p + \epsilon)(|||Y|||_p + \epsilon).
\end{aligned}$$

The corollary follows by letting $\epsilon \to 0$. □

3.2 Representations of Abstract Unital Operator Algebras

The following is a characterization of abstract unital operator algebras due to Blecher, Ruan and Sinclair.

Let $\mathcal{V}$ and $\mathcal{W}$ be abstract operator spaces and $\phi : \mathcal{V} \to \mathcal{W}$ be a linear map. Define $\phi_q : M_q \otimes \mathcal{V} \to M_q \otimes \mathcal{W}$ by $\phi_q = I_q \otimes \phi$, where I_q is the $q \times q$ identity matrix.

The map ϕ is said to be *completely contractive (isometric)* if ϕ_q is a contraction (isometry) for each $q \in \mathbb{N}$.

A *completely contractive (isometric)* representation of an algebra $\mathcal{A}$ is a completely contractive (isometric) algebra homomorphism $\theta : \mathcal{A} \to B(\mathcal{M})$ for some Hilbert space $\mathcal{M}$.

Theorem 3.2.1. ***(Blecher-Ruan-Sinclair)*** *Every abstract unital operator algebra $\mathcal{A}$ admits a completely isometric representation. i.e. there exists a Hilbert space $\mathcal{M}$ and a unital completely isometric algebra homomorphism $\theta : \mathcal{A} \to B(\mathcal{M})$.*

CHAPTER 4
THE ABSTRACT OPERATOR ALGEBRAS $\mathcal{A}(\mathcal{K})^\infty$ & $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$

Recall $\mathcal{K}$, the matrix convex set satisfying the conditions of Assumption 2.1.1. In particular, $\mathcal{K}(1)$ is an circled open convex subset of $\mathbb{C}^d$. This chapter is divided into two parts. In the first part, which consists of four subsections, we construct an abstract unital operator algebra which is a natural non-commutative analog of the Banach algebra $H^\infty(\mathcal{K}(1))$. We also present a few lemmas on completely contractive representations of this algebra.

In the second part, Sections 4.5 and 4.6, we consider the ideal $\mathcal{I}(\mathcal{K})$ of the algebra $\mathcal{A}(\mathcal{K})^\infty$ determined by a finite inital segment Λ. We show that the quotient algebra $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ determined by the ideal is an abstract unital operator algebra. We also show that norms of classes in the quotient algebra are attained.

4.1 The Algebra $\mathcal{A}(\mathcal{K})^\infty$ of Scalar Formal Power Series

In this section we establish that the collection of scalar formal power series in non-commuting variables which converge uniformly on $\mathcal{K}$ is an algebra. We begin with the definition of a formal power series.

4.1.1 Formal Power Series

Let $\mathcal{U}$ and $\mathcal{U}'$ denote separable Hilbert spaces. A *formal power series* with coefficients from $B(\mathcal{U},\mathcal{U}')$ is an expression of the form

$$\sum_{w\in\mathcal{F}_d} f_w w \tag{4.1}$$

where $f_w \in B(\mathcal{U},\mathcal{U}')$. It is convenient to sum f according to its homogeneous of degree j terms; i.e.,

$$f = \sum_{j=0}^{\infty}\sum_{|w|=j} f_w w = \sum_{j=0}^{\infty} f_j. \tag{4.2}$$

Recall, for a d-tuple $T = (T_1, \dots, T_d)$ of operators on a common separable Hilbert space $\mathcal{H}$ and a word $w = g_{i_1}g_{i_2}\dots g_{i_n} \in \mathcal{F}_d$, $i_1, \dots, i_n \in \{1, 2, \dots, d\}$, the evaluation of w at T

is defined as

$$T^w = T_{i_1} T_{i_2} \dots T_{i_n}.$$

Given a formal power series f as above, define

$$f(T) = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w \otimes T^w \tag{4.3}$$

provided the sum converges in the operator norm in $B(\mathcal{U} \otimes \mathcal{H}, \mathcal{U}' \otimes \mathcal{H})$ in the indicated order. We note for clarity that if $\mathcal{U} = \mathcal{U}' = \mathbb{C}$, then $\otimes$ in (4.3) is the usual scalar product and if $f(T)$ converges, it is an element of $B(\mathcal{H})$.

Recall the matrix convex set $\mathcal{K} = (\mathcal{K}(n))$ which satisfies the conditions of Assumption 2.1.1. We will write $X \in \mathcal{K}$ to denote $X \in \bigcup_{n \in \mathbb{N}} \mathcal{K}(n)$. For the formal power series f as above, we define

$$\|f\| = \sup\{\|f(X)\| \,:\, X \in \mathcal{K}\}. \tag{4.4}$$

4.1.2 The Vector Space $\mathcal{A}(\mathcal{K})^\infty$

As it stands, the supremum in equation (4.4) can be infinite. We are only interested in those formal power series f for which this is not the case. Let

$$\mathcal{A}(\mathcal{K})^\infty = \left\{ f = \sum_{w \in F_d} f_w w : f_w \in \mathbb{C}, \|f\| < \infty \right\}.$$

Thus, elements of $\mathcal{A}(\mathcal{K})^\infty$ are in some sense analogous to elements of the classical (commutative) Hardy space, $H^\infty(\mathcal{K}(1))$ of bounded analytic functions on $\mathcal{K}(1)$. It is not hard to see that $\mathcal{A}(\mathcal{K})^\infty$ is a complex vector space with respect to term-wise addition and scalar multiplication.

Lemma 4.1.1. *$\|\cdot\|$ defines a norm on $\mathcal{A}(\mathcal{K})^\infty$.*

Proof. It follows from the definition that $\|\cdot\|$ is a seminorm. Thus it suffices to show that $\|f\| = 0$ implies $f = 0$. Let $\ell \in \{0, 1, 2, \dots\}$ and $S(\ell)$ be as in Subsection 2.3.3. For

$0 < t < \gamma$, using Lemma 2.3.1 and Remark 2.3.1, we get

$$0 = \|f\|^2 \tag{4.5}$$

$$\geq \|f(tS(\ell))(\emptyset)\|^2 \tag{4.6}$$

$$= \|\sum_{j=0}^{\ell} t^j \sum_{|w|=j} f_w w\|^2 \tag{4.7}$$

$$= \sum_{j=0}^{\ell} t^{2j} \sum_{|w|=j} |f_w|^2. \tag{4.8}$$

Thus $f_w = 0$ for all w such that $|w| \leq \ell$. Since ℓ is arbitrary, the lemma follows. □

4.1.3 Matrix Norms on $\mathcal{A}(\mathcal{K})^\infty$

Since it will be necessary to consider, in the sequel, matrices with entries from $\mathcal{A}(\mathcal{K})^\infty$, we define them here. Let

$$M_{p,q}(\mathcal{A}(\mathcal{K})^\infty) = \left\{ f = \sum_{w \in F_d} f_w w : f_w \in M_{p,q}, \|f\|_{p,q} < \infty \right\}$$

where the norm $\|\cdot\|_{p,q}$ is given by

$$\|f\|_{p,q} = \sup\{\|f(X)\| : X \in \mathcal{K}\}. \tag{4.9}$$

The following Lemmas plays an important role in the analysis to follow generally, and in proving that $\mathcal{A}(\mathcal{K})^\infty$ is an algebra, in particular.

Lemma 4.1.2. *If* $f = \sum_{w \in F_d} f_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$, *then* $\sum_{j=0}^{\infty} \gamma^{2j} \sum_{|w|=j} \|f_w\|^2$ *converges.*

Proof. Recall γ from equation (2.3) and $S(\ell)$ from Subsection 2.3.3. Fix $0 \leq t < \gamma$ and a unit vector $x \in \mathbb{C}^q$. Using Lemma 2.3.1 and Remark 2.3.1, we get

$$
\begin{aligned}
\|f\|^2 &\geq \|\sum_{j=0}^{\infty}\sum_{|w|=j} f_w \otimes (tS(\ell))^w (x\otimes\emptyset)\|^2 \\
&= \|\sum_{j=0}^{\ell} t^j \sum_{|w|=j} f_w \otimes S(\ell)^w\,(x\otimes\emptyset)\|^2 \\
&\geq \|\sum_{j=0}^{\ell} t^j \sum_{|w|=j} f_w x \otimes w\|^2 \\
&= \sum_{j=0}^{\ell} t^{2j} \sum_{|w|=j} \|f_w x\|^2.
\end{aligned}
$$

Since ℓ is arbitrary, allowing $t \uparrow \gamma$ yields

$$
\sum_{j=0}^{\infty} \gamma^{2j} \sum_{|w|=j} \|f_w x\|^2 \leq \|f\|^2. \tag{4.10}
$$

Let $\{e_1, e_2 \ldots, e_q\}$ denote the standard orthonormal basis for $\mathbb{C}^q$. We know that for each $w \in \mathcal{F}_d$,

$$
\|f_w\| \leq \left(\sum_{i=1}^{q} \|f_w e_i\|^2\right)^{\frac{1}{2}}. \tag{4.11}
$$

Moreover, for each $1 \leq i \leq q$, equation (4.10) implies that

$$
\sum_{j=0}^{\infty} \gamma^{2j} \sum_{|w|=j} \|f_w e_i\|^2 \leq \|f\|^2. \tag{4.12}
$$

Using equations (4.11) and (4.12), we get

$$
\begin{aligned}
\sum_{j=0}^{\infty} \gamma^{2j} \sum_{|w|=j} \|f_w\|^2 &\leq \sum_{j=0}^{\infty} \gamma^{2j} \sum_{|w|=j} (\sum_{i=1}^{q} \|f_w e_i\|^2) \\
&= \sum_{i=1}^{q} (\sum_{j=0}^{\infty} \gamma^{2j} \sum_{|w|=j} \|f_w e_i\|^2) \\
&\leq q\|f\|^2.
\end{aligned}
$$

□

Lemma 4.1.3. *Suppose that* $f = \sum_{w \in \mathcal{F}_d} f_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ *and* $X \in \mathcal{K}(n)$*. Let*

$$A_j = \sum_{|w|=j} f_w \otimes X^w.$$

If $0 < r < \sup\{s > 0 : sX \in \mathcal{K}(n)\}$*, then*

$$r^j \|A_j\| \leq \|f\|.$$

In particular, there is a $\rho < 1$ *such that* $\|A_j\| \leq \rho^j \|f\|$*.*

Proof. Because $\mathcal{K}(n)$ is open, convex, and circled, the function $F(z) = f(zX)$ is defined on a neighborhood of $\overline{\mathbb{D}}$; i.e. there exists a $\delta > 1$ for which the series,

$$F(z) = \sum_{j=0}^{\infty} A_j z^j$$

converges for all z such that $|z| < \delta$. Using the fact that the series $F(z)$ converges uniformly on the closed disc $\{z : |z| \leq \zeta\}$ for every $\zeta < \delta$, we get, choosing $\zeta = 1$, for each j that,

$$A_j = \frac{1}{2\pi} \int_0^{2\pi} F(e^{it}) e^{-ijt} \, dt.$$

It follows that

$$\|A_j\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|F(e^{it})\| \, dt.$$

Since $\|F(e^{it})\| = \|f(e^{it}X)\|$ and $e^{it}X \in \mathcal{K}(n)$, it follows that $\|F(e^{it})\| \leq \|f\|$ and the lemma follows. □

4.1.4 The Algebra $\mathcal{A}(\mathcal{K})^\infty$

There is a natural multiplication on $\mathcal{A}(\mathcal{K})^\infty$ which turns it into an algebra over $\mathbb{C}$.

Given $f = \sum_{w \in \mathcal{F}_d} f_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ and $g = \sum_{w \in \mathcal{F}_d} g_w w \in M_{q,r}(\mathcal{A}(\mathcal{K})^\infty)$, define the product fg of f and g as the convolution product; i.e.,

$$fg = \sum_{w \in \mathcal{F}_d} \left(\sum_{uv=w} f_u g_v \right) w.$$

The remainder of this subsection is devoted to demonstrating that this convolution product corresponds to pointwise product, extends the natural product of non-commutative polynomials (formal power series with only finitely many non-zero coefficients), and makes $\mathcal{A}(\mathcal{K})^\infty$ an algebra with unit $\emptyset$.

Lemma 4.1.4. *If $f \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ and $g \in M_{q,r}(\mathcal{A}(\mathcal{K})^\infty)$ and $X \in \mathcal{K}$, then*

(i) $fg(X)$ *converges;*

(ii) $fg(X) = f(X)g(X)$*;*

(iii) fg *is in* $M_{p,r}(\mathcal{A}(\mathcal{K})^\infty)$*; and*

(iv) $\|fg\| \leq \|f\|\|g\|$*.*

Proof. Fix $X \in \mathcal{K}(a) \subseteq \mathcal{K}$.

(i) As in the proof of Lemma 4.1.3, let

$$A_j = \sum_{|w|=j} f_w \otimes X^w,$$

$$B_j = \sum_{|w|=j} g_w \otimes X^w$$

$$C_j = \sum_{|w|=j} (\sum_{uv=w} f_u g_v) \otimes X^w.$$

Observe that $C_j = \sum_{k=0}^{j} A_k B_{j-k}$.

Let $F(z) = f(zX)$ and $G(z) = g(zX)$, both of which are defined in a neighborhood of $\overline{\mathbb{D}}$. From Lemma 4.1.3, there is a $\rho < 1$ such that $\|A_m\| \leq \rho^m \|f\|$ and $\|B_k\| \leq \rho^k \|g\|$. Hence

$$\|C_j\| \leq (j+1)\|f\|\,\|g\|\rho^j.$$

It follows that, for $|z| < \frac{1}{\rho}$, the series

$$\sum_{j=0}^{\infty} (\sum_{k=0}^{j} A_k B_{j-k}) z^j$$

converges. In particular $fg(X) = \sum_{j=0}^{\infty} C_j$ converges in norm.

(ii) Consider the function $FG(z) = fg(zX)$. From the proof of (i) we know that $FG(z)$ is defined whenever $|z| < \frac{1}{\rho}$. We will prove the more general fact that $FG(z) = F(z)G(z)$ whenever $|z| < \frac{1}{\rho}$, from which the claim will follow by setting $z = 1$. Recall γ, Γ from the definition of the matrix convex set $\mathcal{K}$. Let z be such that $|z| < \frac{\gamma^2}{d\Gamma}$. Observe that for any k,

$$\begin{aligned}\|\sum_{j=0}^{k} C_j z^j - \left(\sum_{m=0}^{\infty} A_m z^m\right)\left(\sum_{n=0}^{\infty} B_n z^n\right)\| \leq \|\sum_{j=0}^{k} C_j z^j - \left(\sum_{m=0}^{k} A_m z^m\right)\left(\sum_{n=0}^{k} B_n z^n\right)\| \\ + \|\sum_{m=0}^{k} A_m z^m\|\|\sum_{n=k+1}^{\infty} B_n z^n\| \\ + \|\sum_{m=k+1}^{\infty} A_m z^m\|\|\sum_{n=0}^{k} B_n z^n\| \\ + \|\sum_{m=k+1}^{\infty} A_m z^m\|\|\sum_{n=k+1}^{\infty} B_n z^n\|.\end{aligned} \tag{4.13}$$

We claim that the LHS of equation (4.13) converges to zero as $k \to \infty$. It suffices to show that the first term on the RHS of equation (4.13) converges to zero, in view of the convergence of the second, third and the fourth terms on the RHS to zero due to the following reasons:

(a) both $\|\sum_{m=0}^{k} A_m z^m\|$ and $\|\sum_{n=0}^{k} B_n z^n\|$ are finite.

(b) both $\sum_{m=k+1}^{\infty} A_m z^m$ and $\sum_{n=k+1}^{\infty} B_n z^n$, being tails of the convergent series $f(zX)$ and $g(zX)$ respectively, converge to zero as $k \to \infty$.

Consider the first term on the RHS of equation (4.13).

$$\| \sum_{j=0}^{k} C_j z^j - \left(\sum_{m=0}^{k} A_m z^m \right) \left(\sum_{n=0}^{k} B_n z^n \right) \|$$

$$\leq \| \sum_{|u|=1,|v|=k} (f_u g_v \otimes X^{uv}) + \sum_{|u|=2,|v|=k-1} (f_u g_v \otimes X^{uv}) + \ldots + \sum_{|u|=k,|v|=1} (f_u g_v \otimes X^{uv}) \| \ |z|^{k+1}$$
$$+ \| \sum_{|u|=2,|v|=k} (f_u g_v \otimes X^{uv}) + \sum_{|u|=3,|v|=k-1} (f_u g_v \otimes X^{uv}) + \ldots + \sum_{|u|=k,|v|=2} (f_u g_v \otimes X^{uv}) \| \ |z|^{k+2}$$
$$+ \ldots + \| \sum_{|u|=k,|v|=k} (f_u g_v \otimes X^{uv}) \| \ |z|^{2k}$$

$$\leq \left(\sum_{|u|=1,|v|=k} \|f_u g_v\| + \sum_{|u|=2,|v|=k-1} \|f_u g_v\| + \ldots + \sum_{|u|=k,|v|=1} \|f_u g_v\| \right) (\Gamma|z|)^{k+1}$$
$$+ \left(\sum_{|u|=2,|v|=k} \|f_u g_v\| + \sum_{|u|=3,|v|=k-1} \|f_u g_v\| + \ldots + \sum_{|u|=k,|v|=2} \|f_u g_v\| \right) (\Gamma|z|)^{k+2}$$
$$+ \ldots + \left(\sum_{|u|=k,|v|=k} \|f_u g_v\| \right) (\Gamma|z|)^{2k}$$

$$\leq \left(\sum_{|w|=k+1} \sum_{uv=w} \|f_u g_v\| \right) \left(\frac{\gamma^2}{d} \right)^{k+1} + \left(\sum_{|w|=k+2} \sum_{uv=w} \|f_u g_v\| \right) \left(\frac{\gamma^2}{d} \right)^{k+2}$$
$$+ \ldots + \left(\sum_{|w|=2k} \sum_{uv=w} \|f_u g_v\| \right) \left(\frac{\gamma^2}{d} \right)^{2k}$$

$$\leq \left(\sum_{|w|=k+1} \gamma^{4(k+1)} \sum_{uv=w} \|f_u g_v\|^2 \right)^{\frac{1}{2}} \frac{\sqrt{(k+2)d^{k+1}}}{d^{k+1}} + \left(\sum_{|w|=k+2} \gamma^{4(k+2)} \sum_{uv=w} \|f_u g_v\|^2 \right)^{\frac{1}{2}} \frac{\sqrt{(k+3)d^{k+2}}}{d^{k+2}}$$
$$+ \ldots + \left(\sum_{|w|=2k} \gamma^{4(2k)} \sum_{uv=w} \|f_u g_v\|^2 \right)^{\frac{1}{2}} \frac{\sqrt{(2k+1)d^{2k}}}{d^{2k}}$$

$$= \left(\sum_{|w|=k+1} \gamma^{4(k+1)} \sum_{uv=w} \|f_u\|^2 \|g_v\|^2 \right)^{\frac{1}{2}} \sqrt{\frac{k+2}{d^{k+1}}} + \left(\sum_{|w|=k+2} \gamma^{4(k+2)} \sum_{uv=w} \|f_u\|^2 \|g_v\|^2 \right)^{\frac{1}{2}} \sqrt{\frac{k+3}{d^{k+2}}}$$

$$+ \ldots + \left(\sum_{|w|=2k} \gamma^{4(2k)} \sum_{uv=w} \|f_u\|^2 \|g_v\|^2 \right)^{\frac{1}{2}} \sqrt{\frac{2k+1}{d^{2k}}}$$

$$\leq \left(\sum_{m=0}^{\infty} \gamma^{2m} \sum_{|u|=m} \|f_u\|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} \gamma^{2n} \sum_{|v|=n} \|g_v\|^2 \right)^{\frac{1}{2}} \left(\sqrt{\frac{k+2}{d^{k+1}}} + \sqrt{\frac{k+3}{d^{k+2}}} + \ldots + \sqrt{\frac{2k+1}{d^{2k}}} \right).$$

Note that to go from the 2nd to the 3rd inequality on the previous page, we use the Cauchy-Schwarz inequality on the Euclidean space $\mathbb{C}^{(\ell+1)d^\ell}$ and also the fact that the sum $\sum_{|w|=\ell} \gamma^{2\ell} \sum_{uv=w} \|f_u g_v\|^2$ contains $(\ell+1)d^\ell$ terms.

Since the series $\sum_{k=0}^{\infty} \sqrt{\frac{k+2}{d^{k+1}}}$, $\sum_{k=0}^{\infty} \sqrt{\frac{k+3}{d^{k+2}}}$, ..., $\sum_{k=0}^{\infty} \sqrt{\frac{2k+1}{d^{2k}}}$ are convergent, the corresponding k^{th} term sequences converge to zero, and so does the sum of the sequences namely $\left\{ \sqrt{\frac{k+2}{d^{k+1}}} + \sqrt{\frac{k+3}{d^{k+2}}} + \ldots + \sqrt{\frac{2k+1}{d^{2k}}} \right\}_{k=0}^{\infty}$. Moreover Lemma 4.1.2 implies that $\sum_{m=0}^{\infty} \gamma^{2m} \sum_{|u|=m} \|f_u\|^2$ and $\sum_{n=0}^{\infty} \gamma^{2n} \sum_{|v|=n} \|g_v\|^2$ are finite. These facts together imply the desired convergence. Thus $fg(zX) = f(zX)g(zX)$ whenever $|z| < \frac{\gamma^2}{d\Gamma}$.

Fix $x \in \mathbb{C}^r \otimes \mathbb{C}^a$ and $y \in \mathbb{C}^p \otimes \mathbb{C}^a$ and consider the complex valued functions $A(z) = \langle FG(z)x, y \rangle$ and $B(z) = \langle F(z)G(z)x, y \rangle$. Observe that $A(z)$ and $B(z)$ are analytic on the disc $\{z : |z| < \frac{1}{\rho}\}$ and $A(z) = B(z)$ on the subdisc $\{z : |z| < \frac{\gamma^2}{d\Gamma}\}$. Hence, $A(z) = B(z)$ on $\{z : |z| < \frac{1}{\rho}\}$. Since x and y are arbitrary, we have $FG(z) = F(z)G(z)$ whenever $|z| < \frac{1}{\rho}$. Choosing $z = 1$ gives $fg(X) = f(X)g(X)$.

(iii) & (iv) Since, for each $X \in \mathcal{K}$, $fg(X) = f(X)g(X)$ it follows that $\|fg(X)\| \leq \|f\| \, \|g\|$. Thus $\|fg\| \leq \|f\| \, \|g\|$ and $fg \in M_{p,r}(\mathcal{A}(\mathcal{K})^\infty)$. $\square$

Corollary 4.1.1. *$\mathcal{A}(\mathcal{K})^\infty$ is an algebra.*

Proof. Take $p = q = r = 1$ in the above lemma. $\square$

4.2 Weak Compactness and $\mathcal{A}(\mathcal{K})^\infty$

In this section it is shown that every bounded sequence in $\mathcal{A}(\mathcal{K})^\infty$ has a pointwise convergent subsequence. Indeed, $\mathcal{A}(\mathcal{K})^\infty$ has weak compactness properties with respect to bounded pointwise convergence mirroring those for $H^\infty(\mathbb{D})$, the usual space of bounded analytic functions on the unit disk $\mathbb{D}$.

Proposition 4.1. *Suppose that $f_m = \sum_{w\in\mathcal{F}_d}(f_m)_w w$ is a $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ sequence. If, for each $X \in \mathcal{K}$ the sequence $(f_m(X))$ converges or if for each $w \in \mathcal{F}_d$ the sequence $(f_m)_w$ converges and if (f_m) is a bounded sequence (so there is a constant c such that $\|f_m\| \leq c$ for all m), then there is an $f \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ such that $f_m(X)$ converges to $f(X)$ for each $X \in \mathcal{K}$ and moreover $\|f\| \leq c$.*

Proof. Recall $S(\ell)$ defined in Subsection 2.3.3. Fix $0 < t < \gamma$. If f_m converges pointwise, then the sequence $(f_m(tS(\ell)))$ is Cauchy. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_m(tS(\ell)) - f_n(tS(\ell))\| < \epsilon$$

for all $m, n \geq N$. Thus, if $x \in \mathbb{C}^q$ be a unit vector, then

$$\begin{aligned}
\epsilon^2 &> \|f_m(tS(\ell)) - f_n(tS(\ell))\|^2 \\
&\geq \|f_m(tS(\ell)) - f_n(tS(\ell))(x \otimes \emptyset)\|^2 \\
&= \sum_{j=0}^{\ell} t^{2j} \sum_{|w|=j} \|((f_m)_w - (f_n)_w)\ (x)\|^2 \\
&\geq t^{2j}\|((f_m)_w - (f_n)_w)\ (x)\|^2
\end{aligned}$$

for each word w of length at most ℓ. Since ℓ and the unit vector $x \in \mathbb{C}^q$ are arbitrary, it follows that, the sequence $((f_m)_w) \subset B(\mathbb{C}^q, \mathbb{C}^p)$ is Cauchy for each word $w \in \mathcal{F}_d$. Thus $(f_m)_w$ converges to some f_w for each w.

Hence, to prove the Proposition it suffices to prove that, if $(f_m)_w$ converges to f_w for each w and $\|f_m\| \leq c$ for each m, then for each $X \in \mathcal{K}$, the series

$$f(X) = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w \otimes X^w$$

converges and $(f_m(X))$ converges to $f(X)$.

For each j and $X \in \mathcal{K}$,

$$\sum_{|w|=j} (f_m)_w \otimes X^w \to \sum_{|w|=j} f_w \otimes X^w. \tag{4.14}$$

From Lemma 4.1.3, there is a $\rho < 1$ such that for each j,

$$\| \sum_{|w|=j} (f_m)_w \otimes X^w \| \leq \rho^j c. \tag{4.15}$$

From equations (4.14) and (4.15), it follows for each j that,

$$\| \sum_{|w|=j} f_w \otimes X^w \| \leq \rho^j c,$$

an estimate which implies that the series $f(X)$ converges.

Fix $J \in \mathbb{N}$ such that

$$\sum_{j=J+1}^{\infty} \rho^j < \frac{\epsilon}{4c}. \tag{4.16}$$

Recall Γ from the definition of the matrix convex set $\mathcal{K}$. Choose $K \in \mathbb{N}$ such that for all $m \geq K$,

$$\sum_{|w|=j} \|(f_m)_w - f_w\| < \frac{\epsilon}{2(J+1)\Gamma^J}. \tag{4.17}$$

for each $0 \leq j \leq J$.

Thus for all $m \geq K$, from equations (4.16) and (4.17), it follows that

$$
\begin{aligned}
&\|f_m(X) - f(X)\| \\
&\leq \sum_{j=0}^{J} \| \sum_{|w|=j} ((f_m)_w - f_w) \otimes X^w \| + \sum_{j=J+1}^{\infty} \| \sum_{|w|=j} ((f_m)_w - f_w) \otimes X^w \| \\
&\leq \sum_{j=0}^{J} \sum_{|w|=j} \|(f_m)_w - f_w\| \, \Gamma^j + \sum_{j=J+1}^{\infty} \| \sum_{|w|=j} (f_m)_w \otimes X^w \| + \sum_{j=J+1}^{\infty} \| \sum_{|w|=j} f_w \otimes X^w \| \\
&\leq \sum_{j=0}^{J} \frac{\epsilon}{2(J+1)} + 2c \sum_{j=J+1}^{\infty} \rho^j \\
&< \epsilon.
\end{aligned}
$$

Thus $f_m(X) \to f(X)$ for all $X \in \mathcal{K}$. Since $\|f_m(X)\| \leq c$, we have $\|f(X)\| \leq c$. This implies that $\|f\| \leq c$.

□

Lemma 4.2.1. *If $f_m = \sum_{w \in \mathcal{F}_d} (f_m)_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ satisfies $\|f_m\| \leq c$ for all $m \in \mathbb{N}$ then,*

(i) $\|(f_m)_w\| \leq \frac{c}{\gamma^{|w|}}$ *for all $w \in \mathcal{F}_d$ and for all $m \in \mathbb{N}$;*

(ii) *There exists a subsequence $\{f_{m_k}\}$ of $\{f_m\}$ and $f_w \in M_{p,q}$ such that $(f_{m_k})_w \to f_w$ for all w;*

(iii) *Let $f = \sum_{w \in \mathcal{F}_d} f_w w$. For each $X \in \mathcal{K}$ the sequence $(f_{m_k}(X))$ converges to $f(X)$ and moreover $\|f(X)\| \leq c$.*

Proof. To prove item (i), Recall γ from the definition of $\mathcal{K}$. Fix $0 < t < \gamma$ and a unit vector $x \in \mathbb{C}^q$. For $j = 0, 1, 2, \ldots, \ell$, the hypothesis $\|f_m\| \leq c$ together with the conclusion of Lemma 4.1.3 for $X = tS(\ell)$ imply that

$$\|t^j \sum_{|w|=j} (f_m)_w \otimes S(\ell)^w\| \leq c.$$

Hence

$$\begin{aligned} c^2 &\geq \|t^j \sum_{|w|=j} (f_m)_w x \otimes S(\ell)^w \emptyset\|^2 \\ &= t^{2j} \sum_{|w|=j} \|(f_m)_w x\|^2 \\ &\geq t^{2j} \|(f_m)_w x\|^2. \end{aligned}$$

Since x and ℓ are arbitrary, letting $t \uparrow \gamma$ it follows that $\|(f_m)_w\| \leq \frac{c}{\gamma^{|w|}}$ for all $m \in \mathbb{N}$.

The proof of item (ii) uses a standard diagonal argument. Let $\{w_1, w_2, ...\}$ be an enumeration of words in $\mathcal{F}_d$ which respects length (i.e., if $v \leq w$, then $|v| \leq |w|$). Since $\|(f_m)_{w_1}\| \leq \frac{c}{\gamma^{|w_1|}}$, there exists a subsequence say, $\{f_{1,m}\}$ of $\{f_m\}$ such that $(f_{1,m})_{w_1} \to f_{w_1}$. Since $\|(f_{1,m})_{w_2}\| \leq \frac{c}{\gamma^{|w_2|}}$, there exists a subsequence say, $\{f_{2,m}\}$ of $\{f_{1,m}\}$ and thereby of $\{f_m\}$, such that $(f_{2,m})_{w_2} \to f_{w_2}$. Continue this procedure to obtain a subsequence $\{f_{k,m}\}$ of $\{f_{k-1,m}\}$ and thereby of $\{f_m\}$ such that for all $k \in \mathbb{N}$,

$$(f_{k,m})_{w_k} \to f_{w_k}.$$

Now consider the diagonal sequence $\{f_{m,m}\}$. It follows that $\{f_{m,m}\}$ is a subsequence of $\{f_m\}$ and satisfies $(f_{m,m})_w \to f_w$ for all $w \in \mathcal{F}_d$.

In view of what has already been proved, an application of Proposition 4.1 proves item (iii). □

4.3 The Abstract Operator Algebra $\mathcal{A}(\mathcal{K})^\infty$

Consider $\mathcal{A}(\mathcal{K})^\infty$ with matrix norms $\|\cdot\|_{p,q}$ on $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$ as defined in (4.9).

Theorem 4.3.1. *$\mathcal{A}(\mathcal{K})^\infty$ with the family of norms $\|\cdot\|_{p,q}$, is an abstract unital operator algebra.*

Proof. Let $A \in M_{\ell,p}$, $F \in M_{p,q}(\mathcal{A}^\infty)$, $B \in M_{q,r}$. Interpret A and B as $A\emptyset \in M_{\ell,p}(\mathcal{A}(\mathcal{K})^\infty)$ and $B\emptyset \in M_{q,r}(\mathcal{A}(\mathcal{K})^\infty)$ respectively. As a notational convenience we will drop the

subscripts that go with the norms. It follows from Lemma 4.1.4 (ii) that for all $X \in \mathcal{K}(n)$,

$$\|AFB(X)\| = \|A(X)F(X)B(X)\| \leq \|A \otimes I_n\| \, \|F(X)\| \, \|B \otimes I_n\| \leq \|A\| \, \|F\| \, \|B\|.$$

Thus,

$$\|AFB\| \leq \|A\| \, \|F\| \, \|B\|. \tag{4.18}$$

Let $F \in M_{\ell,r}(\mathcal{A}(\mathcal{K})^\infty)$, $G \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$, $X \in \mathcal{K}(n)$. Observe that

$$\|F \oplus G\,(X)\| = \left\| \begin{pmatrix} F(X) & 0 \\ 0 & G(X) \end{pmatrix} \right\| \leq \max\{\|F(X)\|, \|G(X)\|\} \leq \max\{\|F\|, \|G\|\}.$$

Thus,

$$\|F \oplus G\| \leq \max\{\|F\|, \|G\|\} \tag{4.19}$$

Let $\epsilon > 0$ be given. Without loss of generality assume that $\|F\| \geq \|G\|$. Choose $m \in \mathbb{N}$ and $R \in \mathcal{K}(m)$ such that $\|F(R)\| > \|F\| - \epsilon$. Therefore

$$\|F \oplus G\| \geq \left\| \begin{pmatrix} F(R) & 0 \\ 0 & G(R) \end{pmatrix} \right\| \geq \|F(R)\| > \|F\| - \epsilon. \tag{4.20}$$

Letting $\epsilon \to 0$ in the inequality (4.20) and from the inequality (4.19) it follows that,

$$\|F \oplus G\| = \max\{\|F\|, \|G\|\}. \tag{4.21}$$

Lastly, complete contractivity of multiplication in $M_p(\mathcal{A}(\mathcal{K})^\infty)$ follows directly from Lemma 4.1.4 (iv). Thus $\mathcal{A}(\mathcal{K})^\infty$ is an abstract operator algebra. □

4.4 Completely Contractive Representations of $\mathcal{A}(\mathcal{K})^\infty$

Recall the definitions of a completely contractive and completely isometric representation from Section 3.2. Theorem 3.2.1 guarantees the existence of a completely isometric representation for the abstract unital operator algebra $\mathcal{A}(\mathcal{K})^\infty$.

Let $\pi : \mathcal{A}(\mathcal{K})^\infty \to \mathcal{B}(\mathcal{M})$ be a completely contractive unital representation and let $T = (T_1, \dots, T_d)$ where $T_j = \pi(g_j)$. As a notional device, we will write π_T for π. Further, we will also use π_T to denote the map $I_q \otimes \pi : M_q(\mathcal{A}(\mathcal{K})^\infty) \to M_q \otimes B(\mathcal{M})$.

In this section, we prove that for a completely contractive representation π_T of $\mathcal{A}(\mathcal{K})^\infty$, for any $n \in \mathbb{N}$ and finite dimensional subspace $\mathcal{W}$ of $\mathcal{M}$ of dimension n and $0 \le t < 1$ the tuple

$$tZ = tV^*TV = (tV^*T_1V, \dots, tV^*T_dV)$$

is in $\mathcal{K}(n)$. The proof begins with a couple of lemmas. Given $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$ and $0 \le r < 1$, define f_r as follows. If

$$f = \sum_{j=0}^{\infty} \sum_{|w|=j} f_w w = \sum_{j=0}^{\infty} f_j. \tag{4.22}$$

then

$$f_r = \sum_{j=0}^{\infty} r^j \sum_{|w|=j} f_w w = \sum_{j=0}^{\infty} r^j f_j.$$

Lemma 4.4.1. *If π_T is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$ and $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$, then $f_r(T)$ converges in operator norm. Moreover $\pi_T(f_r) = f_r(T)$ and $\|f_r(T)\| \le \|f_r\| \le \|f\|$. If in addition π_T is completely isometric, then $\lim_{r\to 1^-} \|f_r(T)\| = \|f\|$.*

Proof. Write f as in equation (4.22). Lemma 4.1.3 implies that $\|f_j\| \le \|f\|$. Because π_T is completely contractive $\|f_j(T)\| \le \|f\|$. It follows that $f_r(T)$ converges in norm. Since also the partial sums of f_r converge (to f_r) in the norm of $M_q(\mathcal{A}(\mathcal{K})^\infty)$, it follows that $\pi_T(f_r) = f_r(T)$ and so $\|f_r(T)\| \le \|f_r\|$.

The inequality $\|f_r\| \le \|f\|$ is straightforward because $r\mathcal{K} \subset \mathcal{K}$.

Now suppose that π_T is completely isometric. In this case $\|f_r(T)\| = \|f_r\|$. On the other hand $\lim_{r\to 1^-} \|f_r\| = \|f\|$. $\square$

Lemma 4.4.2. *Given $k \times k$ matrices $A_1, \dots, A_d$, let*

$$L = \sum_{j=1}^{d} A_j g_j.$$

Suppose

$$2 - L(X) - L(X)^* \succ 0$$

for all $X \in \mathcal{K}(\ell)$ and for all $\ell \in \mathbb{N}$. Let Φ_L denote the formal power series,

$$\Phi_L = L(2-L)^{-1} = \sum_{j=0}^{\infty} \frac{L^{j+1}}{2^{j+1}}.$$

(a) $2 - L(X) - L(X)^* \succ 0$ for all $X \in \mathcal{K}$ if and only if $2 - L(U) - L(U)^* \succeq 0$ for all $U \in \overline{\mathcal{K}}$.

(b) If $X \in \mathcal{K}(\ell)$, then $\Phi_L(X)$ converges in norm; i.e., the series

$$\sum_{j=0}^{\infty} \frac{L(X)^{j+1}}{2^{j+1}}$$

converges.

(c) $\|\Phi_L(X)\| < 1$ and hence Φ_L is in $M_k(\mathcal{A}(\mathcal{K})^\infty)$ and has norm at most one.

(d) If π_T is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$, then $2 - (L(T) + L(T)^*) \succeq 0$.

Proof. To prove part (a), suppose that $2 - L(X) - L(X)^* \succ 0$ for all $X \in \mathcal{K}$. Since $\overline{\mathcal{K}}$ is the closure of $\mathcal{K}$ it follows that $2 - L(U) - L(U)^* \succeq 0$ for all $U \in \overline{\mathcal{K}}$. To prove the converse, assume the contrary, i.e. suppose there exists $\tilde{X} \in \mathcal{K}$ and a unit vector v such that $\langle (2 - L(\tilde{X}) - L(\tilde{X})^*)v, v\rangle = 0$. The following argument has been adapted from [16]. Define the map $q : \mathbb{R} \to \mathbb{R}$ by

$$q(t) = \langle (2 - L(t\tilde{X}) - L(t\tilde{X})^*)v, v\rangle.$$

Observe that the q is an affine map that satisfies $q(0) = 2$ and $q(1) = 0$. Hence $q(t) < 0$ for all $t > 1$. Choose $s > 1$ such that $s\tilde{X} \in \mathcal{K}$. Such an s exists because $\mathcal{K}$ is open. For

this s, we get

$$q(s) = \langle (2 - L(s\tilde{X}) - L(s\tilde{X})^*)v, v\rangle < 0$$

which is a contradiction to the hypothesis.

To prove part (b) of the lemma, let $X \in \mathcal{K}(\ell)$ be given. Because $\mathcal{K}(\ell)$ is circled, it follows that $e^{i\theta}X \in \mathcal{K}(\ell)$ for each θ. Hence,

$$2 - e^{i\theta}L(X) - e^{-i\theta}L(X)^* \succ 0 \tag{4.23}$$

for each θ. For notational ease, let $Y = L(X)$. Thus Y is a $k\ell \times k\ell$ matrix and equation (4.23) implies that the spectrum of Y lies strictly within the disc; i.e., each eigenvalue of Y has absolute value less than one. Thus,

$$\frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{Y}{2}\right)^j = (2 - Y)^{-1}$$

converges in norm. It follows that

$$\Phi_L(X) = Y(2 - Y)^{-1} = \sum_{j=0}^{\infty}\frac{Y^{j+1}}{2^{j+1}}$$

converges.

To prove (c) observe that $\|Y(2 - Y)^{-1}\| < 1$ if and only if

$$(2 - Y)^*(2 - Y) \succ Y^*Y$$

which is equivalent to $2 - (Y + Y^*) \succ 0$. Thus $\|\Phi_L(X)\| < 1$ which implies that $\Phi_L \in M_k(\mathcal{A}(\mathcal{K})^\infty)$ with $\|\Phi_L\| \leq 1$. This completes the proof of (c).

To prove part (d), observe, Since π_T is completely contractive and $\Phi_L \in M_k(\mathcal{A}(\mathcal{K})^\infty)$ with norm at most one, an application of Lemma 4.4.1 yields, $\|\Phi_L(rT)\| \leq 1$. Arguing as in the proof of part (b), it follows that $2 - (L(rT) + L(rT)^*) \succeq 0$. This inequality holds for all $0 \leq r < 1$ and thus the conclusion of part (c) follows. $\square$

Proposition 4.2. *If $T = (T_1, \dots, T_d)$, and $T_j \in \mathcal{B}(\mathcal{M})$ for some Hilbert space $\mathcal{M}$, and $\pi(g_j) = T_j$ determines a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$, then, for each positive integer n and finite dimensional subspace $\mathcal{W}$ of $\mathcal{M}$ of dimension n and each $0 \leq t < 1$ the tuple*

$$tZ = tV^*TV = (tV^*T_1V, \dots, tV^*T_dV)$$

is in $\mathcal{K}(n)$, where $V : \mathcal{W} \to \mathcal{M}$ is the inclusion map.

Proof. Let n and $\mathcal{H}$ be given and define Z as in the statement of the proposition. Suppose that L is as in the statement of Lemma 4.4.2. From part (d) of the previous lemma, it follows that $2 - (L(T) + L(T)^*) \succeq 0$. Applying $I_k \otimes V^*$ on the left and $I_k \otimes V$ on the right of this inequality gives,

$$2 - (L(Z) + L(Z)^*) = (I_k \otimes V^*)(2 - (L(T) + L(T)^*)(I_k \otimes V) \succeq 0.$$

Part (a) of Lemma 4.4.2 and an application of Theorem 2.2.1 imply that $Z \in \overline{\mathcal{K}(n)}$. Hence $tZ \in \mathcal{K}(n)$ for all $0 \leq t < 1$. □

Lemma 4.4.3. *Let $\Lambda \subset \mathcal{F}_d$ be a finite initial segment, $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$ be as in equation (4.22) and suppose that π_T is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$ into $B(\mathcal{M})$ and T is $\Lambda -$ nilpotent. Then $\|f_r(T)\| \leq \sup\{\|f(X)\| : X \in \mathcal{K}, X$ is $\Lambda -$ nilpotent$\}$ for all $0 \leq r < 1$. Moreover if $f_w = 0$ for all $w \notin \Lambda$, then $\|f(T)\| \leq \sup\{\|f(X)\| : X \in \mathcal{K}, X$ is $\Lambda -$ nilpotent$\}$.*

Proof. Since Λ is finite and T is $\Lambda -$ nilpotent, $f_r(T) = \sum_{w \in \Lambda} f_w \otimes (rT)^w$. Let $\{e_j\}_{j=1}^q$ denote the standard basis of $\mathbb{C}^q$. Given $\epsilon > 0$, choose a unit vector $y = \sum_{j=1}^q e_j \otimes h_j \in \mathbb{C}^q \otimes \mathcal{M}$ such that

$$\|f_r(T)\| < \|f_r(T)y\| + \epsilon.$$

Let $\mathcal{H}$ denote the finite-dimensional subspace of $\mathcal{M}$ spanned by the vectors $\{T^w(h_j) : w \in \Lambda, 1 \leq j \leq q\}$ and $V : \mathcal{H} \to \mathcal{M}$ be the inclusion map. Then $Z = V^*TV$ is

Λ-nilpotent and

$$Z^w = \begin{cases} V^* T^w V & \text{if } w \in \Lambda \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.2 implies that $rZ \in \mathcal{K}$. Thus,

$$\begin{aligned} \|f_r(T)\| &< \| \left(\sum_{w \in \Lambda} f_w \otimes r^{|w|} T^w \right) y \| + \epsilon \\ &= \|f_r(Z) y\| + \epsilon \\ &\leq \|f_r(Z)\| + \epsilon \\ &\leq \sup\{\|f(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\} + \epsilon. \end{aligned}$$

Letting $\epsilon \to 0$ yields the desired inequality. If $f_w = 0$ for all $w \notin \Lambda$, then $f = \sum_{w \in \Lambda} f_w w$ is a non-commutative polynomial in which case we have $\lim_{r \to 1^-} \|f_r(T)\| = \|f(T)\|$ and this completes the proof. □

4.5 The Abstract Operator Algebra $\mathcal{A}(\mathcal{K})^\infty / \mathcal{I}(\mathcal{K})$

Recall the definition of an initial segment from Subsection 2.3.1. Fix a finite initial segment $\Lambda \subset \mathcal{F}_d$ and let

$$\mathcal{I}(\mathcal{K}) = \left\{ f = \sum_{w \notin \Lambda} f_w w \quad : \quad \|f\| < \infty \right\} \subset \mathcal{A}(\mathcal{K})^\infty$$

Observe that $\mathcal{I}(\mathcal{K})$ is a two-sided ideal in the operator algebra $\mathcal{A}(\mathcal{K})^\infty$.

Lemma 4.5.1. *$M_{p,q}(\mathcal{I}(\mathcal{K}))$ is closed in $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$.*

Proof. Let $f_m = \sum_{w \notin \Lambda} (f_m)_w w$ be a sequence in $M_{p,q}(\mathcal{I}(\mathcal{K}))$ be such that $f_m \to f = \sum_{w \in \mathcal{F}_d} f_w w \in M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)$. We need to show that $f_w = 0$ for all $w \in \Lambda$. Let ℓ be the least integer such that $\Lambda \subseteq \Lambda(\ell)$. Consider $tS(\ell)$ where $0 < t < \gamma$. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_m(tS(\ell)) - f(tS(\ell))\| < \epsilon$$

for all $m \geq N$. Let $x \in \mathbb{C}^q$ be an arbitrary unit vector. We have

$$\begin{aligned}
\epsilon^2 &\geq \|f_m(tS(\ell)) - f(tS(\ell))\|^2 \\
&\geq \|(f_m(tS(\ell)) - f(tS(\ell)))(x \otimes \emptyset)\|^2 \\
&= \|\sum_{w \in \Lambda(\ell)\setminus\Lambda} t^{|w|}((f_m)_w - f_w)x \otimes w + \sum_{w \in \Lambda} -t^{|w|} f_w x \otimes w\|^2 \\
&= \sum_{w \in \Lambda(\ell)\setminus\Lambda} t^{2|w|}\|((f_m)_w - f_w)x\|^2 + \sum_{w \in \Lambda} t^{2|w|}\|f_w x\|^2 \\
&\geq t^{2|w|}\|f_w x\|^2
\end{aligned}$$

for all $w \in \Lambda$ and $m \geq N$. Hence $f_w = 0$ for all $w \in \Lambda$. □

By identifying $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K}))$ with $M_{p,q}(\mathcal{A}(\mathcal{K})^\infty)/M_{p,q}(\mathcal{I}(\mathcal{K}))$, Corollary 3.1.1 implies that the quotient $\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ is an abstract operator algebra. We formally record this fact.

Theorem 4.5.1. *$\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ is an abstract unital operator algebra.*

4.6 Attainment of Norms of Classes in $M_q(\mathcal{A}(\mathcal{K})^\infty)/M_q(\mathcal{I}(\mathcal{K}))$

Let $p \in M_q(\mathcal{A}(\mathcal{K})^\infty)$. In this subsection it is shown that there exists $f \in M_q(\mathcal{I}(\mathcal{K}))$ such that

$$\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\| = \inf\{\|p + g\| : g \in M_q(\mathcal{I}(\mathcal{K}))\}.$$

Let $\{f_m\}$ be a sequence in $M_q(\mathcal{I}(\mathcal{K}))$ such that

$$\|p + M_q(\mathcal{I}(\mathcal{K}))\| \leq \|p + f_m\| \leq \|p + M_q(\mathcal{I}(\mathcal{K}))\| + \frac{1}{m}$$

It follows that the sequence $\{f_m\}$ is bounded and that $\|p + f_m\| \to \|p + M_q(\mathcal{I}(\mathcal{K}))\|$. An application of Lemma 4.2.1 yields a subsequence $\{f_{m_k}\}$ of $\{f_m\}$ and $f \in M_q(\mathcal{I}(\mathcal{K}))$ such that

$$(p + f_{m_k})(X) \to (p + f)(X)$$

for all $X \in \mathcal{K}$.

Proposition 4.3. *If $p, \{f_{mk}\}, f$ are as above, then $\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\|$.*

Proof. Let $\epsilon > 0$ be given. Choose $R \in \mathcal{K}$ such that

$$\|p + f\| < \|(p + f)(R)\| + \frac{\epsilon}{4}. \tag{4.24}$$

Since $\|(p + f_{m_k})(R)\| \to \|(p + f)(R)\|$, there exists $K_1 \in \mathbb{N}$ such that,

$$\|(p + f)(R)\| < \|(p + f_{m_k})(R)\| + \frac{\epsilon}{4} \tag{4.25}$$

for all $k \geq K_1$. Combining the inequalities from equations (4.24) and (4.25), implies that, for all $k \geq K_1$,

$$\|p + f\| < \|p + f_{m_k}\| + \frac{\epsilon}{2}. \tag{4.26}$$

Since $\|p + f_{m_k}\| \to \|p + M_q(\mathcal{I}(\mathcal{K}))\|$, there exists a Natural number K_2 such that for all $k \geq K_2$,

$$\|p + f_{m_k}\| < \|p + M_q(\mathcal{I}(\mathcal{K}))\| + \frac{\epsilon}{2}. \tag{4.27}$$

Setting $k = \max\{K_1, K_2\}$ in equations (4.26) and (4.27), and letting $\epsilon \to 0$ yields

$$\|p + f\| \leq \|p + M_q(\mathcal{I}(\mathcal{K}))\|.$$

On the other hand, since $f \in M_q(\mathcal{I}(\mathcal{K}))$,

$$\|p + f\| \geq \|p + M_q(\mathcal{I}(\mathcal{K}))\|.$$

□

CHAPTER 5
THE NON-COMMUTATIVE CARATHÉODORY-FEJÉR PROBLEM

In this chapter we pose the Carathéodory-Fejér Interpolation problem (CFP) for our open, bounded and circled matrix convex set $\mathcal{K}$. Using the results from Chapters 2 - 4, we prove a necessary and sufficient condition for the solvability of the problem.

5.1 The Carathéodory-Fejér Interpolation Problem (CFP)

The statement of the CFP is as follows: Fix a matrix convex set $\mathcal{K}$ satisfying the conditions of Assumption 2.1.1. Let $\Lambda \subset \mathcal{F}_d$ be a finite initial segment, and

$$p = \sum_{w \in \Lambda} p_w w \in \mathcal{A}(\mathcal{K})^\infty$$

be given. Does there exist $\tilde{x} \in \mathcal{A}(\mathcal{K})^\infty$ such that $\tilde{x}_w = p_w$ for $w \in \Lambda$ and $\|\tilde{x}\| \leq 1$?

Theorem 5.1.1. *There exists a (minimum-norm) solution $\tilde{x}$ to the above problem if and only if*

$$\sup\{\|p(X)\| : X \in \mathcal{K}, \;\; X \textit{ is } \Lambda - \textit{nilpotent}\} \leq 1.$$

The generalization of Theorem 5.1.1 allowing for operator coefficients is proved in this chapter.

The strategy is to first prove the result for matrix coefficients. This is done in Section 5.2 below. Passing from matrix to operator coefficients is then accomplished using well-known facts about the Weak Operator Topology (WOT) and the Strong Operator Topology (SOT) on the space of bounded operators on a separable Hilbert space. The details are in Section 5.3.

5.2 The Matrix Version

Fix $\Lambda \subset \mathcal{F}_d$, a finite initial segment, and a polynomial $p = \sum_{w \in \Lambda} p_w w \in M_q(\mathcal{A}(\mathcal{K})^\infty)$.

Proposition 5.1. *There exists $f \in M_q(\mathcal{A}(\mathcal{K})^\infty)$ such that $\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\| = \sup\{\|p(X)\| : X \in \mathcal{K}, X$ is $\Lambda -$ nilpotent$\}$.*

Proof. From Theorems 4.5.1 and 3.2.1 it follows that there exists a Hilbert space $\mathcal{M}$ and a completely isometric homomorphism $\theta : \mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K}) \to B(\mathcal{M})$. As

before, identify $M_q(\mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K}))$ with $M_q(\mathcal{A}(\mathcal{K})^\infty)/M_q(\mathcal{I}(\mathcal{K}))$. Let θ_q denote the map $I_q \otimes \theta : M_q(\mathcal{A}(\mathcal{K})^\infty)/M_q(\mathcal{I}(\mathcal{K})) \to M_q \otimes B(\mathcal{M})$. Let R be the d-tuple $(R_1, R_2, ..., R_d)$, where $R_j = \theta(g_j + \mathcal{I}(\mathcal{K})) \in B(\mathcal{M})$, for $1 \leq j \leq d$. Observe that R is Λ-nilpotent. Let $\eta : \mathcal{A}(\mathcal{K})^\infty \to \mathcal{A}(\mathcal{K})^\infty/\mathcal{I}(\mathcal{K})$ be the quotient map. The composition $\pi = \theta \circ \eta : \mathcal{A}(\mathcal{K})^\infty \to B(\mathcal{M})$ is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$. Since $\pi(g_j) = R_j$, consistent with the notation introduced in Section 4.4, we will use π_R to denote the map π.

It follows from Theorem 4.3 that there exists $f \in M_q(\mathcal{I}(\mathcal{K}))$ such that

$$\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{K}))\|. \tag{5.1}$$

The fact that θ is completely isometric implies that

$$\|p + M_q(\mathcal{I}(\mathcal{K}))\| = \|\theta_q(p + M_q(\mathcal{I}(\mathcal{K}))\| = \|p(R)\|. \tag{5.2}$$

Since π_R is a completely contractive representation of $\mathcal{A}(\mathcal{K})^\infty$, Lemma 4.4.3 implies that

$$\|p(R)\| \leq \sup\{\|p(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\}. \tag{5.3}$$

Combining the equations (5.1), (5.2) and (5.3), it follows that

$$\|p + f\| \leq \sup\{\|p(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\}.$$

But the definition of $\|p + f\|$ implies that

$$\|p + f\| \geq \sup\{\|p(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\}$$

and this completes the proof. □

(The matrix version of) Theorem 5.1.1 follows from the above proposition by setting $\tilde{x} = p + f$.

5.3 The Operator Version

As before, let $\Lambda \subset \mathcal{F}_d$ be a finite initial segment. Departing from the previous section, let $\mathcal{U}$ be an infinite dimensional separable Hilbert space and let the polynomial $p = \sum_{w\in\Lambda} p_w w$, where now $\{p_w\}_{w\in\Lambda} \subset B(\mathcal{U})$, be given.

Theorem 5.3.1. *There exists a formal power series $\tilde{x} = \sum_{w\in\mathcal{F}_d} \tilde{x}_w w$ such that $\tilde{x}_w = p_w$ for all $w \in \Lambda$ and $\|\tilde{x}\| = \sup\{\|p(X)\| : X \in \mathcal{K}, X$ is $\Lambda -$ nilpotent$\}$.*

Proof. Let $\{u_1, u_2, ...\}$ denote an orthonormal basis for the separable Hilbert space $\mathcal{U}$ and let $\mathcal{U}_m$ be the subspace of $\mathcal{U}$ spanned by the vectors $\{u_j\}_{j=1}^m$. For notation ease, let $C = \sup\{\|p(X)\| : X \in \mathcal{K}, X$ is $\Lambda -$ nilpotent$\}$. Observe that $C = 0$ if and only if $p = 0$.

For $w \in \Lambda$, define $M_m \ni (p_m)_w = V_m^* p_w V_m$ where $V_m : \mathcal{U}_m \to \mathcal{U}$ is the inclusion map. Let p_m denote the formal power series

$$p_m = \sum_{w\in\Lambda} (p_m)_w w$$

For each $X \in \mathcal{K}$, Observe that $\|p_m(X)\| \leq \|p(X)\| \leq \|p\|$. Thus $\|p_m\| \leq \|p\|$ and $p_m \in M_m(\mathcal{A}(\mathcal{K})^\infty)$ for all $m \in \mathbb{N}$. From Proposition 5.1, there exists $f_m \in M_m(\mathcal{I}(\mathcal{K}))$ such that $x_m = p_m + f_m \in M_m(\mathcal{A}(\mathcal{K})^\infty)$ and

$$\|x_m\| = \sup\{\|p_m(X)\| : X \in \mathcal{K}, X \text{ is } \Lambda - \text{nilpotent}\}.$$

For $w \in \mathcal{F}_d$, define $B(\mathcal{U}) \ni (\tilde{x}_m)_w = V_m (x_m)_w V_m^*$. Let $\tilde{x}_m$ denote the formal power series $\sum_{w\in\mathcal{F}_d} (\tilde{x}_m)_w w$. For $X \in \mathcal{K}$ and $j = 0, 1, 2, ...$, it follows from From Lemma 4.1.3 that there exists $0 \leq \rho < 1$ such that

$$\begin{aligned} \|\sum_{|w|=j} (\tilde{x}_m)_w \otimes X^w\| &\leq \|\sum_{|w|=j} (x_m)_w \otimes X^w\| \\ &\leq \rho^j \|x_m\| \\ &\leq C\rho^j \end{aligned} \tag{5.4}$$

This implies that the series for $\tilde{x}_m(X)$ converges for each $X \in \mathcal{K}$ and moreover we have

$$\|\tilde{x}_m\| \leq \|x_m\| \leq C. \tag{5.5}$$

Recall γ and $S(\ell)$ from Subsection 2.3.3. Fox $0 \leq t < \gamma$ and a unit vector $u \in \mathcal{U}$. For each $0 \leq j \leq \ell$ and $X \in \mathcal{K}$, it follows that

$$\begin{aligned} C^2 &\geq \|t^j \sum_{|w|=j} (\tilde{x}_m)_w \otimes S(\ell)^w (u \otimes \emptyset)\|^2 \\ &\geq \|t^j \sum_{|w|=j} (\tilde{x}_m)_w u \otimes w\|^2 \\ &\geq t^{2j} \sum_{|w|=j} \|(\tilde{x}_m)_w u\|^2 \end{aligned}$$

Since ℓ is arbitrary, letting $t \uparrow \gamma$ implies that $\|(\tilde{x}_m)_w\| \leq \frac{C}{\gamma^{|w|}}$ for all $w \in \mathcal{F}_d$ and $m \in \mathbb{N}$.

Since $\mathcal{U}$ is a separable Hilbert space and the sequence $\{(\tilde{x}_m)_w\}_{m=1}^{\infty}$ is bounded (by $\frac{C}{\gamma^{|w|}}$), for each $w \in \mathcal{F}_d$, there exists a subsequence of $\{(\tilde{x}_m)_w\}_{m=1}^{\infty}$ that converges with respect to the WOT on $B(\mathcal{U})$. By a diagonal argument similar to the one in Lemma 4.2.1, it follows that there exists a subsequence $\{\tilde{x}_{m_k}\}$ of $\{\tilde{x}_m\}$ and $\{\tilde{x}_w\}_{w \in \mathcal{F}_d} \subset B(\mathcal{U})$ such that for each $w \in \mathcal{F}_d$

$$(\tilde{x}_{m_k})_w \to \tilde{x}_w$$

with respect to the WOT on $B(\mathcal{U})$.

Let $X \in \mathcal{K}(n)$ for some n. Since $\sum_{|w|=j} (\tilde{x}_{m_k})_w \otimes X^w \to \sum_{|w|=j} \tilde{x}_w \otimes X^w$ with respect to the WOT on $B(\mathcal{U} \otimes \mathbb{C}^n)$, it follows from equation (5.4) that $\|\sum_{|w|=j} \tilde{x}_w \otimes X^w\| \leq C\rho^j$. Hence the series for $\tilde{x}(X)$ converges in norm.

Let $\epsilon > 0$ be given. Choose $L_X \in \mathbb{N}$ such that for all $k > L_X$,

$$\left\| \sum_{j=k}^{\infty} \sum_{|w|=j} \tilde{x}_w \otimes X^w \right\| < \epsilon. \tag{5.6}$$

Let $N_X \in \mathbb{N}$ be such that for all $k > N_X$,

$$\sum_{j=k}^{\infty} \rho^j < \frac{\epsilon}{C}. \tag{5.7}$$

Thus for all $k > N_X$ it follows from equations (5.4) and (5.7) that,

$$\left\| \sum_{j=k}^{\infty} \sum_{|w|=j} (\tilde{x}_{m_k})_w \otimes X^w \right\| < \epsilon. \tag{5.8}$$

Let $M_X = \max\{L_X, N_X\}$, $h \in \mathcal{U} \otimes \mathbb{C}^n$ be a unit vector and $y = \left(\sum_{j=0}^{M_X} \sum_{|w|=j} \tilde{x}_w \otimes X^w \right) h$.

From equations (5.5) and (5.8) it follows that

$$\left| \left\langle \left(\sum_{j=0}^{M_X} \sum_{|w|=j} (\tilde{x}_{m_k})_w \otimes X^w \right) h \, , \, y \right\rangle \right| \leq \|y\| \left(\|\tilde{x}_{m_k}(X)\| + \| \sum_{j=M_X+1}^{\infty} \sum_{|w|=j} (\tilde{x}_{m_k})_w \otimes X^w \| \right) \tag{5.9}$$

$$\leq \|y\|(C + \epsilon) \tag{5.10}$$

Since $\sum_{j=0}^{M_X} \sum_{|w|=j} (\tilde{x}_{m_k})_w \otimes X^w \to \sum_{j=0}^{M_X} \sum_{|w|=j} \tilde{x}_w \otimes X^w$ with respect to the WOT on $B(\mathcal{U} \otimes \mathbb{C}^n)$, it follows that

$$\left\langle \left(\sum_{j=0}^{M_X} \sum_{|w|=j} (\tilde{x}_{m_k})_w \otimes X^w \right) h \, , \, y \right\rangle \to \langle y \, , \, y \rangle. \tag{5.11}$$

Equations (5.10) and (5.11) together imply

$$\|y\| \leq C + \epsilon \tag{5.12}$$

Since h in the definition of y is arbitrary, it follows that

$$\| \sum_{j=0}^{M_X} \sum_{|w|=j} \tilde{x}_w \otimes X^w \| \leq C + \epsilon. \tag{5.13}$$

Thus from equations (5.6) and (5.13) it follows that

$$\|\tilde{x}(X)\| = \|\sum_{j=0}^{\infty}\sum_{|w|=j}\tilde{x}_w \otimes X^w\| \leq \|\sum_{j=0}^{M_X}\sum_{|w|=j}\tilde{x}_w \otimes X^w\| + \|\sum_{j=M_X+1}^{\infty}\sum_{|w|=j}\tilde{x}_w \otimes X^w\|$$
$$< C + 2\epsilon.$$

Letting $\epsilon \to 0$ implies that $\|\tilde{x}(X)\| \leq C$. Since $X \in \mathcal{K}$ was arbitrary, it follows that

$$\|\tilde{x}\| \leq C. \tag{5.14}$$

To prove the reverse inequality, observe that for $w \in \Lambda$, $(\tilde{x}_m)_w = V_m V_m^* p_w V_m V_m^*$ and $V_m V_m^* p_w V_m V_m^* \to p_w$ with respect to the WOT on $B(\mathcal{U})$. This implies that $\tilde{x}_w = p_w$ for all $w \in \Lambda$ and so by definition, we get $\|\tilde{x}\| \geq C$. Thus

$$\|\tilde{x}\| = C. \tag{5.15}$$

□

CHAPTER 6
INFINITE INITIAL SEGMENTS

In this chapter we present two examples of non-commutative operator domains, and consider the Carathéodory-Fejér Interpolation problem (CFP) for these domains under the assumption that the initial segment Λ is an infinite set. The examples we will consider here will be the operator (as opposed to matrix) versions of those presented in Subsection 2.1.1. So naturally, most of the definitions including non-commutative neighborhood of zero, circled etc. extend analogously. A slight modification in the notion of the $\gamma > 0$ neighborhood of 0 is necessary. Given an operator A on a Hilbert space $\mathcal{H}$, write $A \succ\succ 0$ if there is an $\epsilon > 0$ such that $A \succeq \epsilon I$; i.e., $A = A^*$ and for all vectors $h \in \mathcal{H}$, the inequality $\langle Ah, h\rangle \geq \epsilon\langle h, h\rangle$ holds. In the operator version, a non- commutative γ-neighborhood of zero, is the set of $T = (T_1, \dots, T_d)$ acting on $\mathcal{H}$ such that

$$\gamma^2 I \succ\succ \sum_{j=1}^{d} T_j T_j^*.$$

6.1 Examples of Non-commutative Operator Domains

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert Space.

The d-dimensional *Non-commutative Polydisc* is defined by

$$\mathcal{C}^d = \{(T_1, \dots, T_d) : T_j \in B(\mathcal{H}) \ \& \ \|T_j\| < 1\}$$

Just as for the non-commutative matrix polydisc, $\gamma = 1$ and $\Gamma = \sqrt{d}$.

The $d \times \tilde{d}$-dimensional *Non-commutative Mixed Ball* is defined by,

$$\mathcal{D}^{d\tilde{d}} = \{T = (T_{11}, T_{12}, \dots, T_{d\tilde{d}}) : T_{ij} \in B(\mathcal{H}) \ \& \ \|T\|_{op} < 1\}$$

where $\|T\|_{op}$ is the norm of the operator $(T_{ij})_{i,j=1}^{d,\tilde{d}} : B(\mathcal{H}^{\tilde{d}}) \to B(\mathcal{H}^d)$. As expected $\gamma = \frac{1}{\sqrt{d\tilde{d}}}$ and $\Gamma = \sqrt{d\tilde{d}}$.

We will demonstrate that the operator algebra approach that we used in Chapter 5 (for the finite Λ case) can also be applied here (to handle the infinite Λ case) and that it leads to a similar necessary and sufficient condition for the solvability of the CFP.

Fix an infinite initial segment $\Lambda \subset \mathcal{F}_d$. In order to make the proofs from the Chapters 2 - 5 work for this setting, some minor modifications need to be made.

Recall the Non-commutative Fock Space and the d-tuple of Creation Operators $S = (S_1, \dots, S_d)$ from Section 2.3. A more general property of S is the following.

Let $\mathbb{F}^2(\Lambda)$ denote the completion of the linear span of Λ with respect to the inner product defined in (2.7).

Lemma 6.1.1. *If $V : \mathbb{F}^2(\Lambda) \to \mathbb{F}^2$ denotes the inclusion map, then*

(i) $(V^*SV)^w = V^*S^wV$ *for all* $w \in \mathcal{F}_d$.

(ii) V^*SV *is* Λ*-nilpotent, i.e.* $(V^*SV)^w = 0$ *for all* $w \notin \Lambda$.

Here $V^*SV = (V^*S_1V, \dots, V^*S_dV)$.

Proof. We prove item (i) by induction. When $|w| = 0$ or 1, the statement is true. Assume the statement is true for all words of length at most n. Let w be a word of length $n+1$. Then $w = \tilde{w}g_j$ for some word $\tilde{w}$ of length n and some j such that $1 \leq j \leq d$. Let $u \in \Lambda$.

$$
\begin{aligned}
(V^*SV)^w(u) &= (V^*SV)^{\tilde{w}}(V^*SV)^{g_j}(u) \\
&= (V^*S^{\tilde{w}}V)(V^*S^{g_j}V)(u) \\
&= \begin{cases} V^*S^{\tilde{w}}(g_ju) & \text{if } g_ju \in \Lambda \\ 0 & \text{if } g_ju \notin \Lambda \end{cases} \\
&= \begin{cases} wu & \text{if } wu \in \Lambda \\ 0 & \text{if } wu \notin \Lambda \end{cases} \\
&= (V^*S^wV)(u).
\end{aligned}
$$

Thus we get $(V^*SV)^w = V^*S^wV$ for all w such that $|w| = n+1$, and this completes the proof.

To prove item (ii), fix $w \notin \Lambda$. In lieu of Part (i), it suffices to show that $V^*S^wV = 0$. Let $u \in \Lambda$. Since $w \notin \Lambda$ and Λ is an initial segment, we get $wu \notin \Lambda$. It follows that $V^*S^wV(u) = V^*(wu) = 0$.

□

6.2 The d-dimensional Non-commutative Polydisc

In this section we pose the CFP for $\mathcal{C}^d$, the d-dimensional Non-commutative Polydisc, with the infinite initial segment $\Lambda \subset \mathcal{F}_d$ and give our main result. We begin with the following remark. Let V be as in Lemma 6.1.1.

Remark 6.2.1. *If* $0 \leq t < 1$*, then*

(i) $tS = (tS_1, \dots, tS_d) \in \mathcal{C}^d$

(ii) $tV^*SV \in \mathcal{C}^d$.

We define $\mathcal{A}(\mathcal{C}^d)^\infty$ and $\mathcal{I}(\mathcal{C}^d)$ as before (See Sections 4.1 and 4.5). The proofs from Sections 4.1, 4.2 and 4.3, can be generalized to this current setting (Λ is infinite) by replacing $S(\ell)$ by S.

In order to generalize the proof of Lemma 4.5.1, to the current setting, we replace $S(\ell)$ with V^*SV. For clarity, we present the modified argument here.

Let $0 < t < 1$ be given. Since $f_m \in M_q(\mathcal{I}(\mathcal{C}^d))$, using Lemma 6.1.1 (ii), it follows that $f_m(tV^*SV) = 0$. Since $f_m \to f$ in norm, we have $f(tV^*SV) = 0$. If $x \in \mathbb{C}^q$ is a unit vector, then

$$\begin{aligned}
0 &= \|f(tV^*SV)\|^2 \\
&\geq \|f(tV^*SV)(x \otimes \emptyset)\|^2 \\
&= \sum_{j=0}^{\infty} t^{2j} \sum_{|w|=j, w\in\Lambda} \|f_w x\|^2 \\
&\geq t^{2|w|}\|f_w x\|^2
\end{aligned}$$

for all $w \in \Lambda$. Hence $f_w = 0$ for all $w \in \Lambda$.

We now state the CFP and our main result.

Let $\mathcal{U}$ be an infinite dimensional separable Hilbert space, $\Lambda \subset \mathcal{F}_d$ be an infinite initial segment and $p = \sum_{w \in \Lambda} p_w w$ be a formal power series such that $p_w \in B(\mathcal{U})$ and

$$\|p\| = \sup\{\|p(T)\| : T \in \mathcal{C}^d\} < \infty.$$

Does there exist $\tilde{x} = \sum_{w \in \mathcal{F}_d} \tilde{x}_w w$ such that $\tilde{x}_w = p_w$ for all $w \in \Lambda$ and $\|\tilde{x}\| \leq 1$?

Theorem 6.2.1. *There exists a solution $\tilde{x}$ to the above problem if and only if*

$$\sup\{\|p(T)\| : T \in \mathcal{C}^d, \quad T \textit{ is } \Lambda - \textit{nilpotent}\} \leq 1.$$

As before, we will prove Theorem 6.2.1 by first proving it for matrix coefficients, i.e. $p_w \in M_q$ and then by following that with a WOT approximation argument.

Let $\pi : \mathcal{A}(\mathcal{C}^d)^\infty / \mathcal{I}(\mathcal{C}^d) \to B(\mathcal{M})$ denote a completely isometric algebra homomorphism obtained by applying Theorem 3.2.1 to the abstract unital operator algebra $\mathcal{A}(\mathcal{C}^d)^\infty / \mathcal{I}(\mathcal{C}^d)$. For $1 \leq j \leq d$, let $R_j = \pi(g_j + \mathcal{I}(\mathcal{C}^d))$, and $R = (R_1, R_2, ..., R_d)$. Since $\|g_j + \mathcal{I}(\mathcal{C}^d)\| \leq \|g_j\| \leq 1$ and π is isometric, we have $\|R_j\| \leq 1$. Moreover R is Λ-nilpotent.

In Section 5.1, p was a polynomial, which automatically gave us the finiteness of $\|p(R)\|$. But here, since Λ is infinite, we will need an approximation argument to generalize Proposition 5.1. We present the generalization below.

Proposition 6.1. *There exists $f \in M_q(\mathcal{I}(\mathcal{C}^d))$ such that $\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{C}^d))\| = \sup\{\|p(T)\| : T \in \mathcal{C}^d, T$ is $\Lambda -$ nilpotent$\}$.*

Proof. We know from the generalization of Proposition 4.3 to the current setting of infinite Λ, that there exists an $f \in M_q(\mathcal{I}(\mathcal{C}^d))$ such that

$$\|p + f\| = \|p + M_q(\mathcal{I}(\mathcal{C}^d))\|.$$

Let $0 \leq t < 1$. Define the formal power series p_t by

$$p_t = \sum_{w \in \Lambda} t^{|w|} p_w w.$$

Since for each $T \in \mathcal{C}^d$ we have $p_t(T) = p(tT)$, it follows that $\|p_t\| \leq \|p\|$. It is also true that if $0 \leq t_1 < t_2 < 1$, then $\|p_{t_1}\| \leq \|p_{t_2}\|$.

For $0 \leq t < 1$, define $p_t^k = \sum_{w \in \Lambda, |w| \leq k} t^{|w|} p_w w$ and $p^k = \sum_{w \in \Lambda, |w| \leq k} t^{|w|} p_w w$. Let π_q denote the map $I_q \otimes \pi : M_q(\mathcal{A}(\mathcal{C}^d)^\infty)/M_q(\mathcal{I}(\mathcal{C}^d)) \to M_q \otimes B(\mathcal{M})$.

We will first prove that $\|\pi_q(p_t + M_q(\mathcal{I}(\mathcal{C}^d))\| = \|p_t(R)\| = \|p(tR)\|$.

We know that

$$\pi_q(p_t^k + M_q(\mathcal{I}(\mathcal{C}^d)) = p_t^k(R) = p^k(tR)$$

for each k. Moreover, for each $T \in \mathcal{C}^d$ and each j, we have

$$\| \sum_{|w|=j, w \in \Lambda} p_w \otimes T^w \| \leq \|p\|.$$

Let $\epsilon > 0$ be given. Choose a natural number N such that

$$\sum_{j=N+1}^{\infty} t^j < \frac{\epsilon}{\|p\|}.$$

Let $T \in \mathcal{C}^d$ be arbitrary. For $k \geq N$ we have,

$$\begin{aligned} \|(p_t^k - p_t)(T)\| &= \| \sum_{j=k+1}^{\infty} t^j \sum_{|w|=j, w \in \Lambda} p_w \otimes T^w \| \\ &\leq \sum_{j=k+1}^{\infty} t^j \| \sum_{|w|=j, w \in \Lambda} p_w \otimes T^w \| \\ &\leq \|p\| \sum_{j=k+1}^{\infty} t^j < \epsilon. \end{aligned}$$

This implies that $\|p_t^k - p_t\| \leq \epsilon$ for all $k \geq N$; i.e. the sequence of partial sums of p_t converges to p_t in norm.

Since $\|(p_t^k + M_q(\mathcal{I}(\mathcal{C}^d))) - (p_t + M_q(\mathcal{I}(\mathcal{C}^d)))\| \leq \|p_t^k - p_t\|$, we get that for all $k \geq N$,

$$\|(p_t^k + M_q(\mathcal{I}(\mathcal{C}^d))) - (p_t + M_q(\mathcal{I}(\mathcal{C}^d)))\| \leq \epsilon.$$

Since π_q is continuous (it is an isometry), we have

$$p_t^k(R) = \pi_q(p_t^k + M_q(\mathcal{I}(\mathcal{C}^d))) \to \pi_q(p_t + M_q(\mathcal{I}(\mathcal{C}^d))).$$

Thus $\pi_q(p_t + M_q(\mathcal{I}(\mathcal{C}^d))) = p_t(R)$ which in turn implies that

$$\|\pi_q(p_t + M_q(\mathcal{I}(\mathcal{C}^d)))\| = \|p_t(R)\|. \tag{6.1}$$

Let $\{t_m\}$ be an increasing positive sequence that converges to 1. Choose $h_m \in M_q(\mathcal{I}(\mathcal{C}^d))$ such that

$$\|p_{t_m} + h_m\| = \|p_{t_m} + M_q(\mathcal{I}(\mathcal{C}^d))\| \tag{6.2}$$

for each m.

We have

$$\|p_{t_m} + h_m\| = \|p_{t_m} + M_q(\mathcal{I}(\mathcal{C}^d))\| \leq \|p_{t_m}\| \leq \|p\|.$$

It follows that the sequence $\{h_m\}$ is bounded (by 2 $\|p\|$). Therefore, by the weak compactness property of the algebra $\mathcal{A}(\mathcal{C}^d)$ we get a subsequence of $\{h_m\}$ which we will again denote by $\{h_m\}$ and $g \in M_q(\mathcal{I}(\mathcal{C}^d))$ such that

$$h_m(T) \to g(T)$$

for each $T \in \mathcal{C}^d$.

Moreover, since $p_{t_m}(T) \to p(T)$ it follows that

$$\|(p_{t_m} + h_m)(T)\| \to \|(p + g)(T)\|$$

for each $T \in \mathcal{C}^d$.

Given $\epsilon > 0$, choose $H \in \mathcal{C}^d$ such that

$$\|p+g\| < \|(p+g)(H)\| + \frac{\epsilon}{2} \tag{6.3}$$

Choose a natural number N such that for all $m \geq N$ we have

$$\|(p+g)(H)\| < \|(p_{t_m} + h_m)(H)\| + \frac{\epsilon}{2} \tag{6.4}$$

From equations (6.3) and (6.4), we have,

$$\|p+g\| < \|(p_{t_m} + h_m)(H)\| + \epsilon < \|p_{t_m} + h_m\| + \epsilon \tag{6.5}$$

for all $m \geq N$. On the other hand,

$$\|p+g\| \geq \|p_{t_m} + g_{t_m}\| \geq \|p_{t_m} + h_m\| > \|p_{t_m} + h_m\| - \epsilon \tag{6.6}$$

for all $m \geq N$, Combining equations (6.5) and (6.6), we get

$$|\, \|p_{t_m} + h_m\| - \|p+g\| \,| < \epsilon$$

for all $m \geq N$. i.e. $\|p_{t_m} + h_m\| \to \|p+g\|$.

Combining equations (6.1) and (6.2), and using the fact that π_q is isometric and yields,

$$\|p_{t_m} + h_m\| = \|p(t_m R)\|.$$

Thus $\|p(t_m R)\| \to \|p+g\|$. Let $C = \sup\{\|p(T)\| : T \in \mathcal{C}^d, T \text{ is } \Lambda\text{-nilpotent}\}$. Since $\|p(t_m R)\| \leq C$, it follows that

$$\|p+g\| \leq C.$$

Hence,

$$\|p+f\| = \|p + M_q(\mathcal{I}(\mathcal{C}^d))\| \leq \|p+g\| \leq C.$$

On the other hand, by definition,

$$\|p+f\| \geq \sup\{\|(p+f)(T)\| : T \in \mathcal{C}^d, T \text{ is } \Lambda\text{-nilpotent}\} = C$$

and this completes the proof. $\square$

The matrix version of Theorem 6.2.1 follows as a consequence by setting $\tilde{x} = p + f$ in the above proposition.

To pass from the case of matrix coefficients to the case of operator coefficients, i.e. to prove Theorem 6.2.1, we can imitate the proof of Theorem 5.3.1 by using the d-tuple S in place of $S(\ell)$.

6.3 The $d \times \tilde{d}$ Non-commutative Mixed Ball

In this section we pose the CFP for $\mathcal{D}^{d\tilde{d}}$, the $d \times \tilde{d}$ Non-commutative Mixed Ball, with the infinite initial segment Λ and give our main result.

Let $\mathcal{F}_{d\tilde{d}}$ be the semi-group generated by the $d\tilde{d}$ symbols $\{g_{ij}\}_{i,j=1}^{d,\tilde{d}}$. Let $\mathbb{F}^2$ denote the corresponding Non-commutative Fock Space and $S = (S_{11}, S_{12}, \ldots, S_{d\tilde{d}})$, the $d\tilde{d}$-tuple of Creation Operators. Fix an infinite initial segment $\Lambda \subset \mathcal{F}_{d\tilde{d}}$.

As in Section 6.2, we begin with the following remark. Let $V : \mathbb{F}^2(\Lambda) \to \mathbb{F}^2$ be the inclusion map.

Remark 6.3.1. *If* $0 \leq t < \frac{1}{\sqrt{d\tilde{d}}}$, *then*

(i) $tS = (tS_{11}, S_{12}, \ldots, tS_{d\tilde{d}}) \in \mathcal{D}^{d\tilde{d}}$

(ii) $tV^*SV \in \mathcal{D}^{d\tilde{d}}$.

We define $\mathcal{A}(\mathcal{D}^{d\tilde{d}})^\infty$ and $\mathcal{I}(\mathcal{D}^{d\tilde{d}})$ as before (See Sections 4.1 and 4.5). The proofs from Sections 4.1, 4.2 and 4.3, can be generalized to this current setting (Λ is infinite) by replacing $S(\ell)$ by the $d\tilde{d}$-tuple S.

In order to generalize the proof of Lemma 4.5.1, to the current setting, we replace $S(\ell)$ with the $d\tilde{d}$-tuple V^*SV, and modify the proof as we did in Section 6.2.

We now state the CFP and give our main result.

Let $\mathcal{U}$ be an infinite dimensional separable Hilbert space, $\Lambda \subset \mathcal{F}_{d\tilde{d}}$ be an infinite initial segment and $p = \sum_{w \in \Lambda} p_w w$ be a formal power series such that $p_w \in B(\mathcal{U})$ and

$$\|p\| = \sup\{\|p(T)\| : T \in \mathcal{D}^{d\tilde{d}}\} < \infty.$$

Does there exist $\tilde{x} = \sum_{w \in \mathcal{F}_d} \tilde{x}_w w$ such that $\tilde{x}_w = p_w$ for all $w \in \Lambda$ and $\|\tilde{x}\| \leq 1$?

Theorem 6.3.1. *There exists a solution $\tilde{x}$ to the above problem if and only if*

$$\sup\{\|p(T)\| : T \in \mathcal{D}^{d\tilde{d}}, \quad T \textit{ is } \Lambda - \textit{nilpotent}\} \leq 1.$$

The same strategy that we used to prove Theorem 6.2.1 can be used to Theorem 6.3.1 as well.

As before, let 6.1, let $\pi : \mathcal{A}(\mathcal{D}^{d\tilde{d}})^\infty / \mathcal{I}(\mathcal{D}^{d\tilde{d}}) \to B(\mathcal{M})$ denote a completely isometric homomorphism obtained by applying Theorem 3.2.1 to the abstract unital operator algebra $A(\mathcal{D}^{d\tilde{d}})^\infty / \mathcal{I}(\mathcal{D}^{d\tilde{d}})$. And for $1 \leq i \leq d$ and $1 \leq j \leq \tilde{d}$, let $R_{ij} = \pi(g_{ij} + \mathcal{I}(\mathcal{D}^{d\tilde{d}}))$, and $R = (R_{11}, R_{12}, ..., R_{d\tilde{d}})$. It follows that R is Λ-nilpotent.

To prove the matrix version of Theorem 6.3.1, we can imitate the proof of Proposition 6.1. The only point that needs clarification is that the $d\tilde{d}$-tuple tR for $0 \leq t < 1$ lies in $\mathcal{D}^{d\tilde{d}}$, i.e. $\|tR\|_{op} < 1$.

To see this we first observe that the formal power series

$$\sum_{i,j=1}^{d,\tilde{d}} E_{ij} g_{ij} \in \mathcal{M}_\ell(A(\mathcal{D}^{d\tilde{d}})^\infty)$$

has norm at most one, where $\ell = \max\{d, \tilde{d}\}$, and E_{ij} is the $\ell \times \ell$ matrix whose (i,j)-th entry is 1 and other entries are 0; $1 \leq i \leq d$, $1 \leq j \leq \tilde{d}$.

The fact that the map $\pi_\ell : M_\ell(A(\mathcal{D}^{d\tilde{d}})^\infty) / M_\ell(\mathcal{I}(\mathcal{D}^{d\tilde{d}})) \to M_\ell \otimes B(\mathcal{M})$ is isometric implies that

$$\begin{aligned} \|R\|_{op} &= \|\sum_{i,j=1}^{d,\tilde{d}} E_{ij} \otimes R_{ij}\| \\ &= \|\pi_\ell(\sum_{i,j=1}^{d,\tilde{d}} E_{ij} g_{ij} + M_\ell(\mathcal{I}(\mathcal{D}^{d\tilde{d}}))\| \end{aligned}$$

$$
\begin{aligned}
&= \| \sum_{i,j=1}^{d,\tilde{d}} E_{ij} g_{ij} + M_\ell(\mathcal{I}(\mathcal{D}^{d\tilde{d}})) \| \\
&\leq \| \sum_{i,j=1}^{d,\tilde{d}} E_{ij} g_{ij} \| \\
&\leq 1.
\end{aligned}
$$

To pass from the matrix version to the operator version, i.e. to prove Theorem 6.3.1, we can imitate the techniques in the proof of Theorem 5.3.1 by using the $d\tilde{d}$-tuple S in place of $S(\ell)$.

CHAPTER 7
FUTURE RESEARCH

The following questions came up while this work was in progress: Could the operator algebras approach used in the examples discussed in Chapter 6 be generalized to handle non-commutative domains that are defined by a possibly infinite collection of Linear Matrix Inequalities? Is the algebra $\mathcal{A}(\mathcal{K})^{\infty}$ a dual algebra?

It would be interesting to know the answers to these questions.

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BIOGRAPHICAL SKETCH

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