ON THE JOINT PRICE AND REPLENISHMENT DECISIONS FOR PERISHABLE PRODUCTS

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2009

To my parents and each of my family members for their tender love and caring support

ACKNOWLEDGMENTS

I would like to express my deepest appreciation to my committee chair, Dr. Amar Sapra for his persistent guidance over last four years. He has not only instilled me with comprehensive knowledge but also given me an insight into the problems that I have examined. He has challenged and motivated me to pursue quality research, and has made me a deep thinker.

I also want to thank my doctoral committee members, Dr. Joseph Geunes, Dr. Farid AitSahlia, and Dr. Aydin Alptekinoglu, who have provided valuable suggestions and constructive advice during the preparation of this work and have made my research more complete.

Last but not least, I can't thank my parents enough. They have always supported and encouraged me in the pursuit of higher education. Without them, I will not have been able to accomplish this.

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August 2009

Chair: Amar Sapra
Major: Industrial and Systems Engineering

A perishable item is characterized by its usefulness over a limited period of time, known as *"life."* Once the *"life"* is over, these items spoil, which obviously is a loss. The bottom line of a firm can improve significantly if some of this spoilage is prevented, i.e., if the perishable nature of products is managed properly. One mechanism by which this may be achieved is demand management using price. Through an appropriate selection of price, demand can be modulated to improve profit. The modulation of demand can not only increase revenue but also reduce shortage, holding, and spoilage costs. Potential spoilage due to limited life-time of the perishable products is the main reason demand management of perishable products is even more important than for non-perishable products. Whereas for non-perishable products the only cost of unsold inventory is the cost for holding inventory, for perishable products the unsold inventory not only incurs inventory holding cost but, in addition, with increasing age of the inventory the risk of it remaining unsold by the end of its lifetime increases. Therefore, this dissertation explores joint demand and replenishment decisions on the inventory control of perishable items with random demand.

The first part of the dissertation is primarily motivated by a dilemma routinely faced by food retailers: when to replace old inventory of perishable products with fresh units when economies of scale exist in order placement. On one hand, economies of scale make it more attractive to place orders for large quantities. On the other hand, the demand for perishable products declines as their age approaches their lifetime; the reduction occurs

since customers prefer fresh units and/or avoid units that are close to expiry. To answer the above question, we develop three models with increasing flexibility. In each model, we consider a finite horizon periodic review system for a single product at a single retailer with a fixed cost of order placement. The retailer faces price-dependent stochastic demand and loses excess demand. The goal of each model is to identify when to place an order, the quantity whenever an order is placed, and the price in each period, regardless of whether an order is placed or not.

The second part of the dissertation examines joint replenishment and price decisions when economies of scale in order placement do not exist and demand is age-independent. We consider a periodic review model over finite horizon for a perishable product with fixed lifetime equal to two review periods such that excess demand in a period is backlogged. The optimal replenishment and demand management decisions for such a product depend on the issuing rule. For both first-in, first-out (FIFO) and last-in, first-out (LIFO) issuing rules, we obtain insights on the nature of these decisions. We find that the insights on the optimal policies for non-perishable products do not extend to perishable products with multi-period lifetime. For the FIFO rule, we also obtain bounds on both the optimal replenishment quantity as well as expected demand. Taking a weighted average of these bounds, we propose an approximate policy; computational experiments indicate that the policy performs within 2% of the optimal profit for a wide range of system parameters. We also conduct experiments to understand whether demand management bridges the profit gap between the FIFO and LIFO rules as well as to explore whether the profit improvement due to demand management favors one issuing rule over another.

CHAPTER 1
INTRODUCTION

In this work, we examine the joint replenishment and demand management decisions for a perishable product with fixed lifetime. A perishable product is characterized by its usefulness over a limited period of time. Once its lifetime is over, the usefulness of the product declines rapidly. The cost impact of spoilage due to perishability is massive. For example, the $1.7 billion apple industry in the U.S. loses as much as $300 million every year to spoilage (Webb (2006)). Similarly, it is estimated that the top 40 retailers in the U.S. dump as much as 500 million pounds of food every year due to spoilage (Gallagher (2008)). However, spoilage is not limited to produce or consumer goods alone; several industrial products also have a limited lifetime. For example, Chen (2007) mentioned that adhesive materials used for plywood panels lose their strength within 7 days.

Obviously, spoilage is a loss, and the bottom line of a firm can improve significantly if some of this spoilage is prevented, i.e., if the perishable nature of products is managed properly. One mechanism by which this may be achieved is demand management using price. Through an appropriate selection of price, demand can be modulated to improve profit. The modulation of demand can not only increase revenue but also reduce shortage, holding, and spoilage costs. Potential spoilage due to limited life-time of the perishable products is the main reason demand management of perishable products is even more important than for non-perishable products. Whereas for non-perishable products the only cost of unsold inventory is the cost for holding inventory, for perishable products the unsold inventory not only incurs inventory holding cost but, in addition, with increasing age of the inventory the risk of it remaining unsold by the end of its lifetime increases.

This dissertation is composed of two parts. In the first part, we model a scenario in which demand for a perishable product is age-dependent and there exist economies of scale in order placement. We develop three models with increasing flexibility. In each model, we consider a finite horizon periodic review system for a single product at a single retailer with a fixed cost of order placement. The retailer faces price-dependent stochastic demand and loses excess demand. The goal of each model is to identify when to place an order, the quantity whenever an order is placed, and the price in each period, regardless of whether an order is placed or not.

In the second part, we consider an alternative scenario in which neither the demand is age-dependent nor the economies of scale exist. We develop a periodic review model over finite horizon such that excess demand is backlogged. The optimal replenishment and demand management decisions for a perishable product depend on the issuing rule. For both first-in, first-out (FIFO) and last-in, first-out (LIFO) issuing rules, we obtain insights on the nature of these decisions.

CHAPTER 2
LITERATURE REVIEW

Since this work looks at the joint replenishment and price decisions for a perishable product, it lies at the interface of operations and marketing decisions. In this chapter, we review two streams of relevant literature. The first stream corresponds to the inventory control decisions for perishable products, and the second stream consists of the papers that examine the joint replenishment and price decisions.

The literature on the inventory control of perishable products can be classified into two categories depending on how the perishability of a product is modeled. In the first approach, inventory is assumed to perish at a uniform rate. This approach is inspired by the perishability characteristics observed in meat and vegetable produce. In the second approach, it is assumed that the lifetime of the product is fixed and completely known. Once the product reaches the end of its usable lifetime, it becomes unfit for consumption and must be discarded (perhaps for a cost) or salvaged. This approach is motivated by items whose lifetime is predictable such as packaged and processed food products. Another example is blood and its components.

In the first approach, the lifetime of each unit is modeled as an exponentially distributed random variable. (This leads to the model being called an exponential decay process.) This approach has also been utilized in numerous EOQ-based models; see Dave (1991), Goyal and Giri (2001), and Raafat (1991) for a review of such models. The analysis of models that consider randomness in both demand and lifetime has not received much attention, with the notable exception of Nahmias (1977b).

On the other hand, most of the papers that adopt the second approach utilize periodic review models with random demand. Unlike non-perishable products, the optimal replenishment policy depends on the relative order of inventory arrival and consumption. It is obvious that an inventory manager should prefer a first-in, first-out (FIFO) issuing policy since it minimizes inventory wastage. (It is implicitly assumed that all units have the same age upon arrival. Otherwise, the optimal issuing policy is first-expiring, first-out.) This issuance policy could be implemented when the inventory manager determines the order in which the inventory is sold or consumed (as is the case in hospital blood banks). Since most of the early research in the 1970's was motivated by blood (Prastacos (1984)), the assumption of a FIFO issuance policy was standard. The opposite of FIFO is the last-in, first-out (LIFO) policy. If the user/customer determines the issuance policy and he derives greater utility from newer units (as is the case for food products such as milk), then the inventory is sold/issued according to the LIFO policy. (As before, the implicit assumption here is that all units have the same age upon arrival. Otherwise, the optimal issuance policy from a customer's perspective is the last-expiring, first-out policy.)

The analysis of inventory control using either issuance policy is difficult. The primary reason is the need to carry an inventory vector corresponding to units of different ages. In parallel efforts, Fries (1975) and Nahmias (1975c) characterize the form of the optimal policy for the lost-sales and backlogging case, respectively. Using the special characteristics of the optimal solution, many papers have developed myopic or near-myopic policies that ignore the age-distribution of the on-hand inventory (Brodheim et al. (1975), Nahmias (1975a), Nahmias (1975b), Nahmias (1976), Nahmias (1977a), and Nandakumar and Morton (1993)). The research related to the inventory policy for the LIFO rule is especially limited. As an example of research in this area, Cohen and Pekelman (1978) develop age-distributions in a periodic review inventory system with lost-sales to determine the order policy. Once again, Nahmias (1982) and Karaesmen et al. (2008) summarize many of these papers. Two review papers that exclusively focus on the inventory control of blood are Prastacos (1984) and Pierskalla (2004).

The major contribution of the first part of the dissertation is to capture another feature of perishable products, age-dependent demand, which has not been yet examined in the existing literature. The earlier mentioned research on periodic review models with random demand typically assumes that the units of different ages are identical. This assumption is justified when units are issued by the owner (as in hospital blood bank), but is difficult to defend when customers select the units themselves as is common in supermarkets. For such products whose demand declines with age, we examine the replenishment policy in the presence of economies of scale arising from the fixed cost of order placement. We also model demand management using price in our models. Unlike the inventory control literature on perishable products, we do not analyze the case in which inventories of two different ages may be sold together. (The detailed justification for this assumption is available in Section 3.1.) This assumption is a marked departure from several existing models in perishable inventory theory that assume that inventories of different ages are perfectly substitutable. The substitutability assumption often results in complicated analysis due to the presence of an inventory vector corresponding to stocks of different ages (Nahmias (1982)). Even though our analysis avoids some of these difficulties, the assumption that order placement results in salvage of the old inventory poses its own challenges.

Naturally, our work in the second part also contributes to this stream of literature given our consideration of a product with two-period lifetime. To the best of our knowledge, this is the first work that examines the scenario in which demand is endogenous for a perishable product with fixed and multi-period lifetime. We would like to point out that Chande et al. (2004), Chande et al. (2005), and Chandrashekar et al. (2003) also consider replenishment and demand management decisions for perishable products, but these papers model only a *single* promotion (the only demand-related decision) during the horizon. Relative to these papers, our contribution is to optimize demand dynamically. We also develop analytical results to derive insights, whereas these papers use a computational approach to obtain insights.

We would also like to mention the recent work in coordination of price (or demand) and production/ordering decisions for non-perishable products. Using a periodic review model with random demand, Chen and Simchi-Levi (2004) and Federgruen and Heching (1999) develop optimal policies with and without fixed cost of order placement and backlogging. Chen et al. (2006) consider the problem with fixed cost but assume that excess demand is lost. Huh and Janakiraman (2006) generalize these results by using alternative proof techniques. For short-life cycle products with a single replenishment opportunity, Petruzzi and Dada (1999) analyze a Newsvendor model with price-dependent demand. An excellent review of literature on the coordination of pricing and inventory decisions is Yano and Gilbert (2006).

CHAPTER 3
INVENTORY RENEWAL AND DEMAND MANAGEMENT FOR A PERISHABLE PRODUCT: ECONOMIES OF SCALE AND AGE-DEPENDENT DEMAND

In 2004 Interstate Bakeries Corp., then US's largest baker and the maker of Wonder Bread, Twinkies and Ding-Dongs, filed for bankruptcy. One of the reasons cited for its decline is the decision to increase the number of days its famous brands of bread is kept on shelf from 3 days to 7 days (Adamy (2004)). Recent innovations in the baking industry have increased the lifetime of several bakery products and Interstate sought to take full advantage of these innovations by increasing the shelf-duration of its products. The major advantage of increased shelf-duration is less spoilage; it is estimated that top 40 retailers in the US dump as much as 500 Million pounds of food every year due to spoilage (Gallagher (2008)).

Yet contrary to the company's expectation, the strategy did not help. The longer bread stayed on the shelf, the more frequently it was moved by store personnel and customers. This made the bread look shelf-worn even though it was still consumable. The net result was a reduction in demand of many popular products, which played a significant role in Interstate's filing for bankruptcy protection. A competitor of Interstate, Flower Foods, utilizing the same technology chose to increase the shelf-duration of bread from 3 to 4 days instead of full 7 days, yet its sales increased by 5% (Adamy (2004)). These contrasting examples illustrate the importance of the decision food companies routinely face: When to take-off the old inventory from the shelf and replace it with fresh inventory. In this work, we develop and analyze three models to derive insights on this problem.

The decision regarding when to renew inventory is driven by the total *value* of inventory, which is equal to the potential revenue earned through the sale of the inventory until it expires plus the savings in shortage cost less the holding cost. The inventory is renewed if fresh units provide more value than the existing inventory. As inventory ages, the potential revenue declines since the demand deteriorates as the lifetime approaches. There are three reasons for the deterioration of demand with age for perishable products.

Firstly, customers increasingly want fresh products. Secondly, products may look shelf-worn due to handling by the store employees and other customers. Finally, customers want to avoid products that are close to their expiry (or sell-by) date due to the risk of their expiring before consumption.

Taking the trade-offs regarding the renewal of inventory into consideration, we build and analyze three models that incorporate the effect of age on demand for a periodic review system. We consider a finite horizon model for a single retailer that faces stochastic demand that is price and age dependent. The retailer incurs a fixed cost whenever she places an order. Whenever demand exceeds available inventory in a period, the excess demand is lost. The retailer faces the following three decisions: When to place an order, for what quantity should an order be placed and what price to set. While an order may not be placed every period due to economies of scale associated with a fixed cost, the price could be adjusted every period.

The use of price to manage demand not only enhances revenue from inventory, it also improves the utilization of inventory. Given the limited life of perishable products, the demand management of such products is even more important than non-perishable products. Whereas for non-perishable products the only cost of unsold inventory is the holding cost, for perishable products the unsold inventory not only incurs the holding cost but also with increasing age the risk of its remaining unsold by the end of its lifetime increases.

With this motivation, a summary of the rest of the paper is as follows. In Section 3.1, we describe the notation and formulate the first model. In the first model, which we refer to as the fixed cycle model, we assume that the reorder interval is fixed; the reorder interval is at most the lifetime of the product. The price, however, may be adjusted every period. This model is suitable for products with a relatively less volatile demand. Using dynamic programming, we show that the optimal profit function is concave in the on-hand inventory every period and provide insights on the form of the optimal pricing policy. We also develop some insights on the optimal reorder interval.

In the second model that we describe in Section 3.2, we eliminate the restriction that the reorder interval is fixed, and we refer to this model as the flexible replenishment model. This model is thus more suitable for products with volatile demand. The analysis of the model demonstrates that the optimal profit function has a complicated structure. However, we are able to show the existence of two threshold levels of inventory such that an order is placed whenever the inventory is less than the smaller threshold level or greater than the larger threshold level.

In Section 3.3, we describe the third model, which we refer to as the partial salvage model. In this model we examine a partial salvage strategy in the presence of flexible replenishment intervals. The model allows for situations in which the retailer has an option to partially salvage the inventory when she has an excess amount of it. The partial salvage model, despite being more flexible, has simpler structure compared to the flexible replenishment model. We demonstrate the existence of a threshold inventory level such that an order is placed if and only if the inventory is less than the threshold level.

In Section 3.4, we conduct numerical experiments to identify the marginal profit improvements due to the demand management, flexible replenishment, and partial salvage strategies as a function of several model parameters such as the price sensitivity of demand and variance of demand. We find that while the demand management and flexible replenishment strategy yield profit improvements of 2-7 % and 3-8%, respectively, the partial salvage strategy adds little benefit. This implies that the retailer can garner most of the benefit by a combination of flexible replenishment and demand management strategies. We conclude and summarize our findings in Section 5.1.

3.1 Fixed Cycle Model

We construct a model in which the retailer places an order only once every R periods. The lifetime of the product S is greater than or equal to R. At the end of every R-th period, the retailer salvages any existing inventory at rate $w_R \geq 0$ and places a new order. Given that a replenishment opportunity is available only once every few periods, this model is applicable where demand volatility is relatively low.

We assume that the salvage value w_R decreases in R, that is, the salvage value decreases with the age of the inventory. Our private communication with grocery store chains indicates that food products are often salvaged upon removal from store shelves. For example, bread is salvaged to thrift stores at heavy discounts once removed from store shelves. Similarly, meat is salvaged to chemical manufacturers, who may extract useful compounds from it, upon removal from shelves.

The assumption that the order placement and the salvage of the old inventory occur simultaneously implies that inventories of two different ages are not sold together. This assumption is a good approximation to the reality if most of the customers buy a product only after inspecting a few units to make sure they are buying the freshest unit. (Our private communication with several retail stores indicates this to be the case.) In such a case, there will be no incentive for the retailer to try to sell units of different ages together (at the same price) since the old units will remain unsold. The retailer may, however, sell old units at a reduced price elsewhere in the store. Another alternative for the retailer is to just throw the old inventory or salvage it elsewhere for some additional revenue. Our analysis does not require a specification of how the old inventory is salvaged. However, if the retailer sells it within the store, we assume that it does not impact the demand of the new inventory due to substitution effects. We are currently working on a model that considers such substitution effects.

Some other assumptions for the model are as follows. Throughout the rest of this paper, for simplicity we will assume the cost parameters to be stationary over time. Whenever an order is placed, a fixed cost K and a variable cost c per unit are incurred. The lead-time for order placement is zero, that is, an order once placed arrives immediately. In periods in which no order is placed, the retailer may adjust the price depending upon the quantity and age of the inventory. In any period, if demand exceeds the available inventory, the excess demand is lost.

We model the demand in a generic period t as follows:

$$D_t(s,p) = (d_t(p)\xi_1 + \xi_2)f(s), \tag{3.1.1}$$

where p is the price in period t, s is the age of the inventory, and ξ_1 and ξ_2 are random variables such that $E[\xi_1] = 1$ and $E[\xi_2] = 0$. Further, ξ_1 and ξ_2 are assumed to be continuous random variables with densities and IID for different periods. Since $E(\xi_1) = 1$ and $E(\xi_2) = 0$, the expectation of $D_t(s,p)$ is equal to $d_t(p)f(s)$. When $\xi_1 \equiv 1$, we refer to the demand function as *additive*. We assume $f(s)$ is a non-increasing function of s and takes values in the interval $[0,1]$ such that $f(0) = 1$. We also take $d_t(p)$ to be a linear function of price p, that is, $d_t(p) = a - bp$. While most of our results only require that the expected revenue $p.d_t(p)$ be concave in p (as in Federgruen and Heching (1999) and Chen and Simchi-Levi (2004)), the linearity assumption provides a concrete interpretation of customer behavior. To see this, we think of a to be the expected number of customers who would purchase the product if the price were 0 and the product were fresh (age = 0). When price p is set, the number of interested customers in expectation dwindles to $a - bp$. Further, if the product is s periods old, another fraction $(1 - f(s))$ of customers lose interest. Thus, the expected number of customers that want to purchase a unit that is s periods old is $(a - bp)f(s)$.

We point out that it is possible to extend our results to more general demand functions: Most of our results only require that $D_t(s,p)$ be non-increasing in s. For example, the demand function may also be modeled as $D_t(s,p) = ((a - bp - f_1(s))\xi_1 + \xi_2)f(s)$. Given the above interpretation of a, the term $f_1(s)$ may represent the expected number of customers who walk away after observing the age of the inventory without even checking the price. The term $f(s)$ would then represent the fraction of customers that after observing the price and age decide to purchase the product.

Since the expected demand $d_t(p)$ has a one-to-one correspondence to price, given the expected demand we can compute the corresponding price. Given this observation, we will work with expected demand as a variable instead of price in the rest of this paper. This change not only simplifies notation but also, as we will see, results in a more intuitive interpretation of some results. For exposition, we will use demand and price interchangeably depending upon the context. We denote the inverse of expected demand function for the inventory of age s by $D_s^{-1}(\cdot)$. Thus, for a given expected demand d, $p = D_s^{-1}(d)$ when the age of the inventory is equal to s. Let $\mathcal{D}_s$ be the set of feasible values for d when the age is equal to s. We assume that this set is convex and is constructed to ensure that both demand $D_t(s, D_s^{-1}(d))$ and price $D_s^{-1}(d)$ remain non-negative. The convexity assumption implies that $\mathcal{D}_s$ is an interval. For demand function as defined in Equation 3.1.1, $\mathcal{D}_s$ is identical for all s. As a result, we will drop subscript s from $\mathcal{D}_s$ in the subsequent analysis.

Let $x > 0$ be the available inventory to satisfy demand in a period. If d is the expected demand and s is the age of the inventory, the expected one-period profit in that period is equal to

$$L_s(x, d) = d.D_s^{-1}(d) - hE[x - d\xi_1 - \xi_2]^+ - \pi E[d\xi_1 + \xi_2 - x]^+, \tag{3.1.2}$$

where h is the unit holding cost, π is the unit lost-sales cost and $[\cdot]^+ = \max[\cdot, 0]$. The lost-sales cost π not only includes lost margin for any unsatisfied demand but also the cost of not fulfilling demand. Since price is a variable in our model, the lost margin is also endogenous to the model. The assumption of lost-sales cost that is independent of the price is applicable when the lost-margin is much smaller compared to the cost of not fulfilling demand. This should hold true in the grocery industry where margins are often small. Further, the goodwill cost of not providing an essential food item could be significant since the customer may take her entire business elsewhere. We also note that a similar assumption was made by Petruzzi and Dada (1999) in the context of a Newsvendor model with price-dependent demand.

One crucial modeling difficulty with a lost-sales model, unlike a backlogging model, is to approximate the lost-sales cost in a period when there is no on-hand inventory at the beginning of a period. Clearly, the retailer cannot set a high price to drive down the demand to save on her lost-sales cost as she can when there is positive on-hand inventory. The reason is that the demand management using price cannot work when there is nothing on hand. We handle this difficult by assuming that expected one-period profit corresponding to $x = 0$ is strictly less than $\lim_{x \to 0^+} L_s(x, d)$. We denote the expected one-period profit when $x = 0$ by A. This assumption makes $L_s(\cdot, d)$ function discontinuous at $x = 0$.

We are now ready to formulate the problem. Let $v_t(s, x)$ be the optimal profit from period t through the end of horizon when x units of age s are on hand. If no order is placed in period t,

$$\begin{aligned} v_t(s,x) &= \max_{d \in \mathcal{D}} L_s(x,d) + \beta E[v_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)], \quad x > 0, \\ &= -(R-s)A, \quad x = 0, \end{aligned}$$

where β is the discounting factor. The only decision variable here is the expected demand. On the other hand, if an order is placed in period t, the optimal profit function is equal to

$$v_t(s,x) = w_R x + \max_{d \in \mathcal{D}, y \geq 0} \{-K\delta(y) + L_0(y,d) - cy + \beta E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+)]\},$$

such that $v_{T+1}(s, x) = w_s x$. In the above formulation, $\delta(y) = 1$ if $y > 0$ and 0 otherwise. Observe that the above model becomes a Newsvendor model with price-dependent demand when $S = 1$.

We assume that there exists a $y > 0$ such that $-K + L_0(y, d^*(y)) + w_1 E(y - d^*(y)\xi_1 - \xi_2)^+ - cy > A$, where $d^*(y)$ is the optimal demand corresponding to y for a one-period model (Newsvendor model with endogenous demand), so that it is sub-optimal to not place an order even though there is nothing on hand. We set the discount factor β equal to 1 through the rest of this paper to avoid notational overload. For simplicity, we assume that all parameters are time-stationary.

In the following theorem, we state some structural properties of the above formulation.

Theorem 3.1.1. *1. The optimal profit function $v_t(s, x)$ is concave in x for any s.*

2. There exists a unique $d_t^*(x)$ for each x, which is non-decreasing in x.

3. When an order is placed, there exists a unique order quantity y^*. Further, when the demand function is additive, that is, $\xi_1 \equiv 1$, the optimal demand is equal to $\arg\max_d d \cdot (D_0^{-1}(d) - c)$.

The proof of the above result as well as other results are available in the Appendix.

In multi-period dynamic inventory systems, the link that connects decisions (such as the price) in different periods is the inventory that is carried. In the above model the inventory is renewed every R periods, so the order placement decision taken in a period affects the profit of only the following $R - 1$ periods. We refer to this group of R periods as a cycle. Let $V(R)$ be the total optimal profit over a cycle and $y^*(R)$ be the optimal order quantity in a cycle of length R. In the following proposition, we show how $V(R)$ and $y^*(R)$ change with the cycle length R when the demand is deterministic.

Proposition 3.1.2. *Let $\xi_1 \equiv 1, \xi_2 \equiv 0$ and all demand is satisfied. The optimal profit over a cycle of length R, $V(R)$, is concave in R. Furthermore, the cycle order quantity $y^*(R)$ is concave in R.*

The above result implies that when demand is deterministic, the marginal improvement in the cycle profit declines as the cycle length increases. It also shows that the marginal increase in the order quantity decreases as the number of periods in a cycle increases. This result is intuitive. Suppose k units are being carried for the last period in both n and $(n + 1)$ period cycles. The marginal benefit due to one additional unit being carried for the last period in a n-period cycle is strictly greater than the corresponding unit for the last period in a $(n + 1)$-period cycle. The reason is that the marginal revenue earned by the additional unit in period n is greater than or equal to the marginal revenue due to the corresponding unit in period $n+1$ since $\frac{\partial}{\partial d}(dD_{n-1}^{-1}(d)) \geq \frac{\partial}{\partial d}(dD_{n}^{-1}(d))$. Further, it costs h more to carry a unit into period $n+1$ compared to a unit carried into period n. Since each unit for the last period in $n+1$-period cycle improves the cycle profit by a smaller amount, fewer units are kept for the last period in $n+1$-period cycle compared to the last period in n-period cycle. By the same logic, the marginal profit increase due to the $(n+1)$-th period is smaller compared to the n-th period. In general, we believe that this argument should hold for the random demand case as well. This is why we think that a result analogous to Proposition 3.1.2 should exist when demand is uncertain although we are unable to establish it.

The result in Proposition 3.1.2 is useful since it leads to an approach to compute a desirable cycle length. In this approach, we can choose a cycle length that maximizes the optimal profit per period, $\frac{V(R)-K}{R}$. The function $\frac{V(R)-K}{R}$ is quasi-concave if $V(R)$ is concave (Boyd and Vandenberghe (2004)). We formally state this result in the following corollary.

Corollary 3.1.3. *If $V(R)$ is concave in R, then there exists a unique value of cycle length R^* that maximizes average profit per period $\frac{V(R)-K}{R}$ in a cycle.*

3.2 Flexible Replenishment Model

In this section, we consider a flexible replenishment strategy. In this strategy, the retailer has the option to place an order in any period. As a result, the time between any two successive orders is not necessarily identical.

As before, the order placement decision results in the salvage of the old inventory. Therefore, in each period the retailer may pursue one of the following two options: Either retain the old inventory and set a new price or salvage the old inventory, place an order and set a price. Once the age of the inventory becomes equal to the lifetime of the product, an order is necessarily placed. We refer to the option in which no order is placed as Strategy 1. The other option in which an order is placed is referred to as Strategy 2.

With this, the optimal profit from period t through the end of horizon is

$$v_t(s,x) \;=\; \begin{cases} \max\{v_t^1(s,x), v_t^2(s,x)\}, & \text{for } s < S \\ v_t^2(s,x), & \text{for } s = S \\ w_s x, & \text{for } t = T+1 \end{cases}$$

where

$$v_t^1(s,x) = \max_{d\in\mathcal{D}} \left\{ L_s(x,d) + Ev_{t+1}(s+1,(x-d\xi_1-\xi_2)^+) \right\} \tag{3.2.3}$$

is the optimal profit from period t through the end of horizon when Strategy 1 is used in period t and

$$v_t^2(s,x) = -w_s x + \max_{d\in\mathcal{D}, y\geq 0} \left\{ -K\delta(y) + L_0(x,d) + Ev_{t+1}(1,(y-d\xi_1-\xi_2)^+) \right\} \tag{3.2.4}$$

is the optimal profit from period t through the end of horizon when Strategy 2 is followed in period t. The term $L_s(x,d)$ in Equations 3.2.3 and 3.2.4 is as defined in 3.1.2. To make analysis more interesting, we assume that $v_t^2(s,0) - K > v_t^1(s,0)$. This assumption ensures that an order is necessarily placed when there is nothing on-hand. The remaining assumptions remain the same as in Section 3.1.

Observe that similar to the model in Section 3.2, the above model becomes a Newsvendor model with price-dependent demand when $S = 1$. Further, when the demand is deterministic (that is, $\xi_1 = 1$ and $\xi_2 = 0$), the fixed cycle and flexible replenishment models produce identical results. Since the flexibility to place orders is more useful when demand is more volatile, the flexible replenishment is likely to be more useful in volatile demand environments.

For the flexible replenishment model, the optimal profit function $v_t(s,x)$ is not necessarily concave in x. To see this, observe that the optimal profit for either strategy is concave for $t = T$ (assuming $s < S$). However, the maximum of two concave function is not necessarily concave, so the optimal profit function for $t = T$ or any other period is not necessarily concave. Figure 3-1 shows a sample plot of the optimal profit function.

As can be observed, this function appears to lack properties, such as quasi-concavity or monotonicity, that are useful in establishing the form of the optimal policy. This makes the analysis inherently difficult. However, we are still able to derive several results that shed light on the form of the optimal policy.

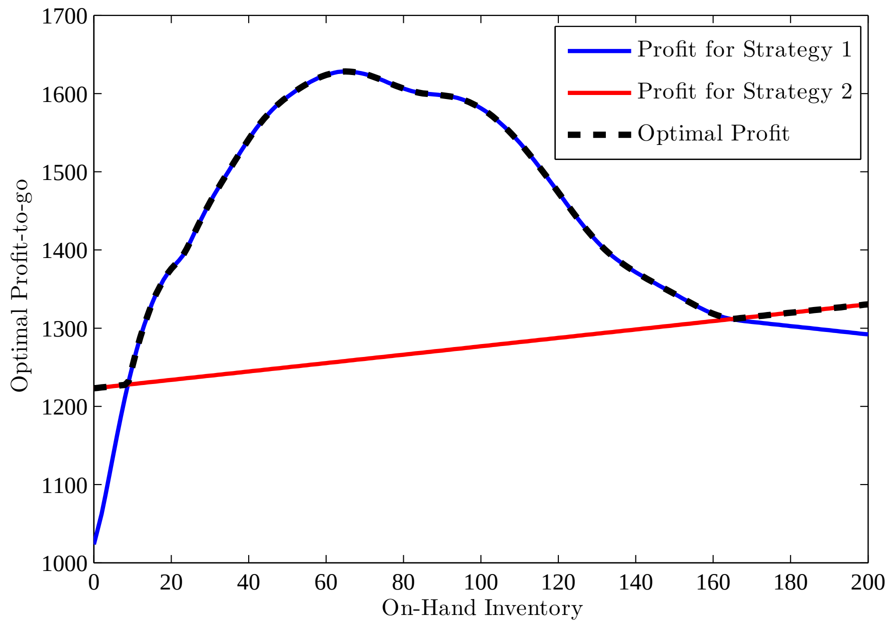


Figure 3-1. The profits for strategies 1 and 2 and the optimal profit as a function of on-hand inventory when $K = 200$, $h = 1$, $c = 5$, $\pi = 50$, $D_t(s,p) = (40 - p + \xi_2)f(1)$, $f(s) = \max(0, 2 - exp(0.1s))$, $t = 4$, $T = 8$, $\xi_2 \sim N(0, 100)$ truncated at ± 10.

We begin by stating in the following proposition the form of the optimal policy when the lifetime is equal to 2 periods.

Proposition 3.2.1. *Let $S = 2$.*

1. *For Strategy 1, $v_t^1(s, \cdot)$ is concave. Further, there exists a unique value of the optimal demand, whenever Strategy 1 is used.*
2. *Either $v_t^2(s, x) > v_t^1(s, x) \forall x$, in which case Strategy 2 is followed for every x, or there exist two thresholds $x_t^l(s)$ (possibly 0) and $x_t^u(s)$ such that Strategy 1 is followed for all values of $x \in (x_t^l(s), x_t^u(s))$ and Strategy 2 otherwise.*

The above result shows that when the inventory is too low or too high, an order is placed; otherwise, no order is placed. When the inventory is too low, it is insufficient to satisfy the demand in that period. As a consequence, the retailer places an order. On the other hand, when the inventory is too high, the cost of holding inventory outweighs the possible future gain (through sale of that inventory) that could be obtained by retaining that inventory. This means that salvaging the inventory results in higher profit than maintaining it.

To derive results for a general lifetime, we begin by examining the effect of the age of the inventory on the optimal profit function. In the following proposition we first show that the profit in a period for a given level of on-hand inventory and demand is non-increasing in the age of the inventory. Subsequently, we use this result to show that given the inventory x at the beginning of period t, the optimal profit from period t through the end of horizon is non-increasing in the age of the inventory. Finally, we show that the expected profit from period t through the end of horizon is non-decreasing in the lifetime of the product for any age s and inventory x.

Proposition 3.2.2. *1.* $L_s(x,d) \leq L_{s-1}(x,d)$*.*

2. $v_t(s,x) \leq v_t(s-1,x)$*.*

3. $v_t(s,x)$ *is a non-decreasing function of the lifetime* S*.*

In the following theorem, we characterize the form of the optimal policy for general lifetime values. Similar to the case when $S = 2$, there exist two thresholds such that an order is placed when the inventory is less than the lower threshold or greater than the upper threshold. However, unlike the case in which lifetime is equal to 2 periods, we are unable to show that no order is placed between the two thresholds. The second part of the theorem illustrates how the two thresholds change with the age of the inventory when the salvage value is independent of s.

Theorem 3.2.3. *1. If $v_t^1(s,x) > v_t^2(s,x)$ for some x, then there exists a lower threshold $x_t^l(s)$ and upper threshold $x_t^u(s)$ such that Strategy 2 is followed when $x \leq x_t^l(s)$ and $x \geq x_t^u(s)$.*

2. If $x_t^l(s)$ and $x_t^u(s)$ exist and the salvage value w_s is independent of s, then $x_t^l(s)$ increases in s and $x_t^u(s)$ decreases in s.

As we stated above, we are unable to prove that Strategy 1 is necessarily followed between $x_t^l(s)$ and $x_t^u(s)$, though our extensive numerical experiments show this to be the case usually. Our experiments indicate that only when demand is deterministic or has very low volatility that an order might be placed when the inventory falls between the two thresholds. In other words, when demand is sufficiently volatile, only Strategy 1 is likely to be used between $x_t^l(s)$ and $x_t^u(s)$.

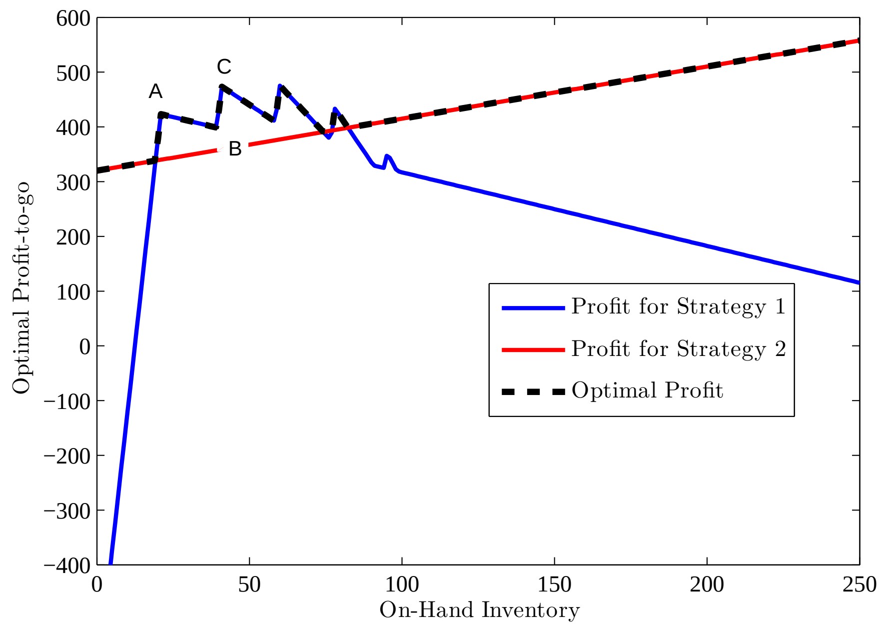


Figure 3-2. The profits for strategies 1 and 2 and the optimal profit as a function of on-hand inventory when $K = 200$, $h = 1$, $c = 5$, $\pi = 50$, $D_t(s,p) = (40 - p)f(1)$, $f(s) = \max(0, 2 - exp(0.1s))$, $p \in [1, 10]$, $t = 4$, $T = 8$.

To explain this observation, consider Figure 3-2. In this figure, we have plotted the optimal profits corresponding to Strategy 1, Strategy 2, and their maximum as a function of the inventory. To construct this example, we have assumed the demand to be deterministic. Note that since the demand is deterministic, it is possible to optimize over demand for all the future periods for any given level of inventory in period t.

Observe that the plot corresponding to Strategy 1 has several peaks and troughs. Consider the peak point denoted by A. This point is equal to the optimal inventory to satisfy exactly one period's demand. If the inventory is less than A, the retailer will have to either charge a high price or incur lost-sales. There is less need to charge a high price or incur lost-sales as the inventory increases, so the profit increases with the inventory. When the inventory increases beyond A, there is more than enough inventory to satisfy one period's demand but not enough to satisfy two periods' demands. Thus, the retailer will have to increase demand by giving discounts. Any remaining inventory will have to be salvaged next period when an order will be placed. This is true for all the inventory values between points A and B. Since the magnitude of the discount and the amount of the salvaged inventory increases with more inventory, the profit decreases with the inventory between points A and B. When the inventory increases beyond point B, it is now more beneficial to satisfy next period's demand than salvaging it. For the inventory values between points B and C, the retailer will have to charge a premium or incur lost-sales; the optimal amount of the inventory to satisfy demands of this period and next period corresponds to point C. As the inventory increases beyond C, the retailer once again has to give discounts as well as salvage some of it until it reaches the next trough.

As can be seen from Figure 3-2, multiple points of intersection between $v_t^1(s, \cdot)$ and $v_t^2(s, \cdot)$ may arise due to the zigzag nature of the plot for $v_t^1(s, \cdot)$. Sharp peaks and troughs arise since the optimal demand for next few periods can be computed exactly. This ceases to be true when demand is not deterministic. As a result, the curve becomes more "smooth" and peaks and troughs gradually vanish. Our experiments indicate that relatively mild values of demand uncertainty result in a smooth curve with mild or no peaks and troughs. In such a case, it is very likely that only Strategy 1 is followed between $x^l(s)$ and $x^u(s)$.

We now characterize a set of values of the inventory where Strategy 1 is certain to be used. We accomplish this objective by developing two lower bounds on $v_t^1(s,x)$. These bounds are concave functions of the inventory x and thus intersect $v_t^2(s,x)$ at none or two points since $v_t^2(s,x)$ is a linear function of x for any given s. When a bound intersects with $v_t^2(s,x)$ at two points, the values of x that lie between the two points follow Strategy 1. We also note that the left intersection point of $v_t^2(s,x)$ with either bound provides an upper bound on $x_t^l(s)$. Similarly, the right intersection point of $v_t^2(s,x)$ with either bound provides a lower bound on $x_t^u(s,x)$.

We obtain the first lower bound on $v_t^1(s,x)$ by imposing the restriction that Strategy 1 will be followed in period $t+1$ and that the optimal demand for period $t+1$ is computed in period t. Similarly, the second lower bound is obtained by assuming that Strategy 2 will be followed in period $t+1$. For small values of $x > x_t^l(s)$, it is more likely that Strategy 2 will be followed in period $t+1$. In that case, the second lower bound on $v_t^1(s,x)$ when intersected with $v_t^2(s,x)$ gives a relative tight upper bound on $x_t^l(s)$. Similarly, for larger values of x between $x_t^l(s)$ and $x_t^u(s)$, Strategy 1 is likely to be followed in period $t+1$. In that case, the first lower bound on $v_t^1(s,x)$ when intersected with $v_t^2(s,x)$ gives a relatively tight lower bound on $x_t^u(s)$.

In the following proposition, we state the result formally.

Proposition 3.2.4. *Let $t < T$. Let $B_t^1(s,x)$ and $B_t^2(s,x)$ be defined as follows:*

$$
\begin{aligned}
B_t^2(s,x) &= \max_{d_t, d_{t+1} \in \mathcal{D}} \left\{ L_s(x,d_t) + EL_{s+1}((x - d_t\xi_1 - \xi_2)^+, d_{t+1}) \right. \\
&\quad \left. + Ev_{t+2}(s+1, (x - d_t\xi_1^t - \xi_2^t)^+ - d_{t+1}\xi_1^{t+1} - \xi_2^{t+1}) \right\}, \\
B_t^1(s,x) &= \max_{d \in \mathcal{D}} \left\{ L_s(x,d) + Ev_{t+1}^2(s+1, (x - d\xi_1 - \xi_2)^+) \right\}
\end{aligned}
$$

$B_t^1(s,\cdot)$ and $B_t^2(s,\cdot)$ are concave. Thus, these functions intersect with $v_t^2(s,x)$ at none or two points. When $B_t^i(s,x)$ intersects with $v_t^2(s,x)$ at two points, let the left and right

intersection points be denoted by ℓ^i and u^i, respectively. If $B_t^i(s,x)$ does not intersect with $v_t^2(s,x)$, set $\ell^i = u^i = \infty$. Then $x_t^l(s)$ is bounded from above by $\min(\ell^1, \ell^2)$ and $x_t^u(s)$ is bounded from below by $\max(u^1, u^2)$.

We now discuss the optimal demand policy. The computation of the optimal demand for Strategy 2 in a period is relatively straightforward when demand is additive. We state the result formally in the following corollary.

Corollary 3.2.5. *When demand is additive, that is, $\xi_1 \equiv 1$, the optimal demand for Strategy 2 is obtained by maximizing the following concave function: $\arg\max_d d(D_0^{-1}(d) - c)$.*

For Strategy 1, however, the optimal demand (price) policy appears to lack a simple structure; see Figure 3-3 for a sample plot of the optimal price as a function of the inventory. As one would expect, the optimal price is chosen to maximize the profit from the available inventory. Our numerical experiments indicate that this may happen in two ways. Firstly, when there is sufficient inventory to satisfy demands of k future periods, including the current period, though it is less than ideal, the price may be increased to reduce the cost of lost-sales. For example, in Figure 3-3, for $x = x_t^l(s) = 10$ to $x = 24$ the price is used to reduce lost-sales since the amount of the inventory is less than ideal to satisfy one period's demand. The price stabilizes between $x = 24$ and $x = 35$ once the inventory becomes sufficiently large. Between $x = 35$ and $x = 59$, the variation in price is mainly to ensure that the inventory lasts 2 and 3 periods depending upon the value of x. Secondly, price is used when the inventory is high so that the inventory is salvaged (with a high probability) next period. In Figure 3-3, for $x \geq 72$ the inventory is excessive and is likely to be salvaged next period. The price is now chosen so that the combined profit from the sale of the inventory this period as well as salvage next period is maximized.

3.3 Partial Salvage Model

In the previous section, we showed that when the inventory is large, the retailer will have to salvage all of it. This is an extreme solution. In many practical settings, it is possible to salvage (or write-off) some of the units when the inventory is excessive. This

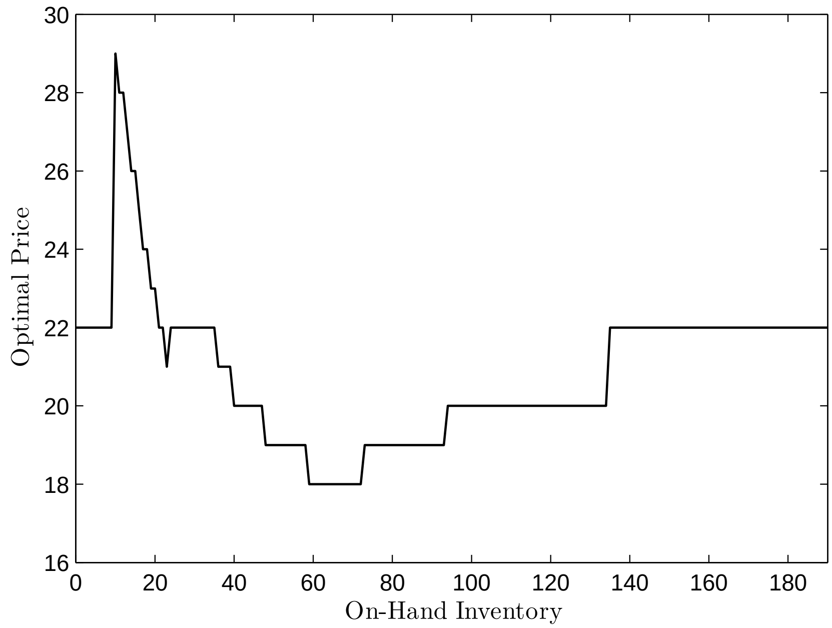


Figure 3-3. Optimal price as a function of on-hand inventory when $t = 4$, $T = 8$, $K = 200$, $h = 1$, $c = 5$, $\pi = 50$, $D_t(s, p) = (40 - p + \xi_2)f(s)$, $f(s) = \max(2 - \exp(0.1s), 0.75 \exp(-0.1s))$, $\xi_2 \sim N(0, 100)$ truncated at ± 10.

solution of partial salvage has several benefits. If some units are unlikely to be sold before the end of their lifetime, salvaging them in advance may fetch higher salvage value. The retailer also saves costs related to carrying inventory. In this section, we analyze a model in which some inventory may be salvaged when Strategy 1 is followed, that is, when no order is placed.

Inventory write-offs or disposals for both perishable or non-perishable products are quite frequent in the real-world. Yet few supply chain papers have examined such a decision in their models. We have come across only two papers that have considered this phenomenon for random demand, Rosenfield (1989) and Rosenfield (1992). (A number of papers have examined the salvage of excess inventory for deterministic demand. See Rosenfield (1989) for details.) It is optimal to dispose some of the inventory if the current salvage value of the inventory is greater than its future sale, which includes the revenue as well as savings in lost-sales cost due to it before the inventory expires, less the holding costs before the sale or end of the lifetime. Rosenfield (1989) develops an approach to identify when and in what quantity to salvage some of the inventory. He also looks at the case when the product is perishable. Compared to him, we differ in the modeling approach in that we use a periodic review model, whereas he utilizes a continuous review model with Poisson demand. We also include the possibility of demand management and consider age-dependent demand while identifying the amount to be salvaged.

To model partial salvage, we include a new decision variable z into the formulation which represents the partial salvage quantity. The modified expression for one-period expected profit is as follows:

$$L_s(x,z,d) = dD_s^{-1}(d) + w_s z - hE[x - z - d\xi_1 - \xi_2]^+ - \pi E[d\xi_1 + \xi_2 - x + z]^+,$$

where we have included z as an argument in the definition of L_s. We assume that the unit salvage value w_s for partial salvage is the same as for complete salvage. This assumption is not critical for the analysis; our analysis can be easily extended to the case in which the unit salvage values for the partial and complete salvages are different. From a practical perspective, for obvious reasons the units that are partially salvaged cannot be sold within the same store at a discount unlike the case when complete salvage occurs. In other words, the salvage cannot occur in the form of a price discount within the same store.

Given the definition of $L_s(x,z,d)$, we now state the formulations of the two strategies as follows:

$$\begin{aligned} v_t^1(s,x) &= \max_{0\le z\le x, d\in\mathcal{D}} \{L_s(x,z,d) + E[v_{t+1}(s+1,(x-z-d\xi_1-\xi_2)^+)]\} \qquad (3.3.5) \\ v_t^2(s,x) &= w_s x + \max_{d\in\mathcal{D}, y\ge 0} \{-K\delta(y) - cy + L_0(y,0,d) + E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+)]\} \end{aligned}$$

Observe that the formulation for Strategy 2 remains unchanged. The analysis is simplified if we substitute $x - z = q$ into the formulation. The variable q may be interpreted as the quantity not salvaged or the quantity remaining after the partial salvage. With this substitution, the new formulation is

$$
\begin{aligned}
v_t^1(s,x) &= \max_{0\le q\le x, d\in\mathcal{D}} \{Q_s(x,q,d) + E[v_{t+1}(s+1,(q-d\xi_1-\xi_2)^+)]\} \\
v_t^2(s,x) &= w_s x + \max_{d\in\mathcal{D}, y\ge 0} \{-K\delta(y) - cy + Q_0(y,y,d) + E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+)]\}
\end{aligned}
$$

where

$$Q_s(x,q,d) = dD_s^{-1}(d) + w_s(x-q) - hE[q-d\xi_1-\xi_2]^+ - \pi E[d\xi_1+\xi_2-q]^+.$$

Observe that $Q_s(x,q,d) = L_s(x,z,d)$ where $q = x - z$. The optimal profit function $v_t(s,x)$ remains the same as in Section 3.2, that is,

$$
v_t(s,x) = \begin{cases} \max\{v_t^1(s,x), v_t^2(s,x)\}, & \text{for } s < S, \\ v_t^2(s,x), & \text{for } s = S \text{ or } x = 0, \\ w_s x, & \text{for } t = T+1. \end{cases}
$$

Similar to the flexible replenishment model, we assume that $v_t^2(s,0) - K > v_t^1(s,0)$ to ensure that an order is necessarily placed when there is nothing on-hand. The remaining assumptions remain the same as for the fixed cycle model.

Compared to the flexible replenishment model, the optimal profit function has simpler structure. It still lacks concavity for the same reasons as in Section 3.2, but it now is non-decreasing in x. The reason it is non-decreasing is that the profit function corresponding to Strategy 1 is now non-decreasing in x. Since the profit function corresponding to Strategy 2, which is same as in Section 3.2, is also non-decreasing, the optimal profit function is also non-decreasing in x. The monotonicity of v makes analysis considerably easier. As a consequence, we are able to characterize the optimal policy for this model almost completely even though it has more decision variables compared to the model in Section 3.2.

We begin by showing in the following corollary that the expected profit in a period is non-increasing in the age of the inventory for any given levels of the inventory, the quantity not salvaged and the demand. We also show that the function Q_s is non-decreasing in x for any given levels of quantity not salvaged q and demand.

Corollary 3.3.1. *1. $Q_s(x,q,d) \geq Q_{s+1}(x,q,d)$.*

2. $Q_s(x_2,q,d) \geq Q_s(x_1,q,d)$ for $x_2 \geq x_1$.

The corollary can be easily proved using the non-increasing nature of $D_s^{-1}(d)$ and salvage value with respect to s. The details are omitted.

In the following proposition, we establish some properties of the profit functions v_t^1, v_t^2 and v_t that are useful in proving the form of the optimal policy. We first show that the profit function for Strategy 1 is non-decreasing in the inventory. Subsequently, we show that the optimal profit function v_t is non-increasing in the age for any given level of the inventory.

Proposition 3.3.2. *1. $v_t^1(s,x)$ is a non-decreasing function of x.*

2. $v_t(s,x) \leq v_t(s-1,x)$.

We now turn our attention to establishing the form of the optimal decision policy. To accomplish this objective, we transform the optimal profit function as follows:

$$g_t^i(s,x) = v_t^i(s,x) - w_s x \text{ for } i = 1,2.$$

Thus,

$$\begin{aligned}
g_t^1(s,x) &= \max_{d\in\mathcal{D},0\leq q\leq x} \{dD_s^{-1}(d) - w_s q - hE[q-d\xi_1-\xi_2]^+ - \pi E[d\xi_1+\xi_2-q]^+ \\
&\quad +E[g_{t+1}(s+1,(q-d\xi_1-\xi_2)^+) + w_s(q-d\xi_1-\xi_2)^+]\} \\
&= \max_{d\in\mathcal{D},0\leq q\leq x} \{dD_s^{-1}(d) - w_s q - (h-w_s)E[q-d\xi_1-\xi_2]^+ \\
&\quad -\pi E[d\xi_1+\xi_2-q]^+ + E[g_{t+1}(s+1,(q-d\xi_1-\xi_2)^+)]\} \\
&=: \max_{d\in\mathcal{D},0\leq q\leq x} J(d,q),
\end{aligned}$$

and

$$
\begin{aligned}
g_t^2(s,x) &= v_t^2(s,x) - w_s x \\
&= \max_{d\in\mathcal{D}, y\geq 0}\{-K\delta(y) - cy + L_0(y,0,d) + E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+)]\}.
\end{aligned}
$$

Note that $g_t^2(s,x)$ is independent of both x and s, so we define it as a constant M. If $M > g_t^1(s,x)$ for all x, it is optimal to always follow Strategy 2. The more interesting case occurs when $M \leq g_t^1(s,x)$ for some x. We state the optimal policy in that case in the following theorem.

Theorem 3.3.3. *Suppose (d_x^*, q_x^*) be the constrained maximizers of $J(d,q)$ over $d, q \in \mathcal{D} \times [0,x]$ and let (d^*, q^*) be the global maximizers of $J(d,q)$. Further, suppose $M \leq g_t^1(s, q^*)$.*

1. *There exists a threshold $x_t^l(s)$ such that an order is placed if and only if $x \leq x_t^l(s)$.*
2. *Suppose Strategy 1 is followed. Then, for any $x \geq q_x^*$, the quantity $x - q_x^*$ is salvaged. Further, for all $x \geq q^*$, the quantity $x - q^*$ is salvaged.*

The above theorem shows that there exists a threshold $(x_t^l(s))$ such that an order is placed if and only if the inventory is less than the threshold. Otherwise, no order is placed. When no order is placed and the inventory is large, the excess inventory is salvaged. In fact, whenever the system exceeds the inventory beyond a certain threshold (q^*), it will salvage every unit in excess of the threshold. Even when the inventory lies between $x_t^l(s)$ and q^*, salvage may occur though it is not necessary. In such a case, salvage will occur to bring the inventory level to a local maximum that lies within the interval $[x_t^l(s), x]$ and that has the largest profit value.

In the following proposition, we show that for the additive demand model, that is, when $\xi_1 \equiv 1$, the optimal demand has a simple structure. Further, the computation merely involves optimizing a single dimensional function over demand d. Note, however, that the form of this function depends on the strategy.

Proposition 3.3.4. *If $\xi_1 \equiv 1$, then*

1. *The optimal demand is $\arg\max_d d(D_0^{-1}(d) - c)$ when an order is placed.*

2. *When no order is placed and a positive amount of inventory is salvaged, the optimal demand is* $\arg\max_d d(D_s^{-1}(d) - w_s)$ *when the age of the inventory is* s*.*

The optimal price does not have a simple structure when there is no salvage. Figure 3-4 shows two sample plots of the optimal price as a function of the inventory. In this model, price is used only when there is sufficient inventory to satisfy demands of k future periods though the amount is less than ideal. In that case, price may be raised to save on the lost-sales cost. Unlike the flexible replenishment model, price is not utilized when there is excessive inventory since it can be salvaged. Figure 3-4 suggests that the optimal price may have a simpler structure when demand is more volatile.

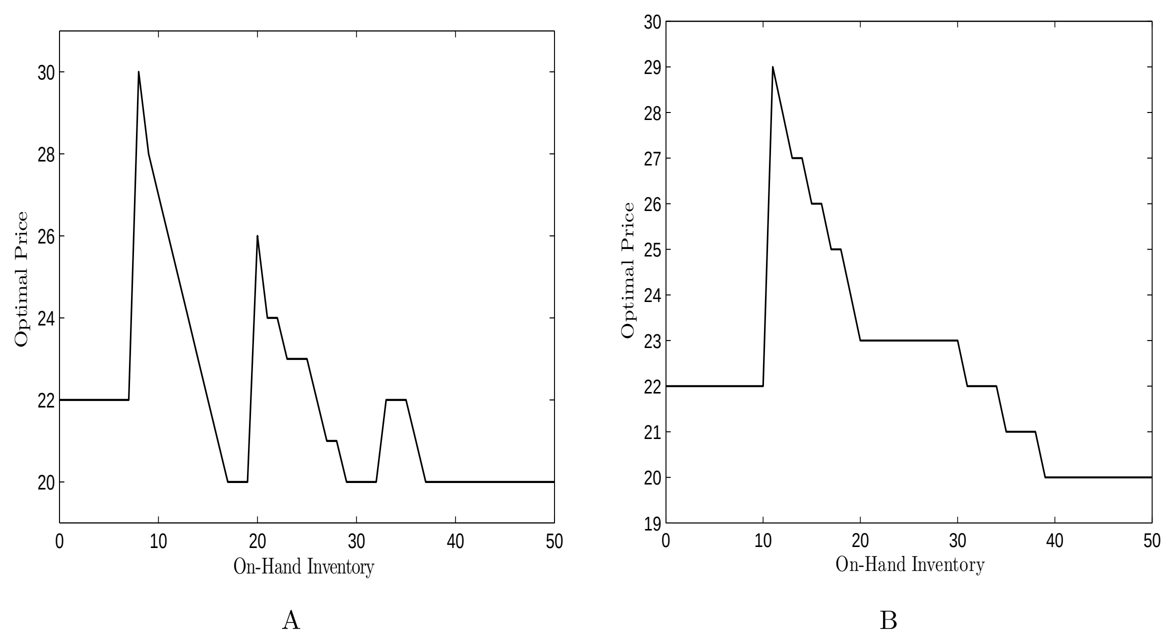


Figure 3-4. Optimal price as a function of on-hand inventory when $t = 5$, $T = 20$, $K = 200$, $h = 1$, $c = 5$, $\pi = 50$, $D_t(s, p) = (40 - p + \xi_2)f(s)$, $f(s) = \max(2 - \exp(0.1s), 0.75\exp(-0.1s))$. A) $\xi_2 \sim N(0, 1)$ truncated at ± 10. B) $\xi_2 \sim N(0, 100)$ truncated at ± 10.

3.4 Numerical Experiments

In this section, we numerically examine the benefits of the demand management, flexible replenishment, and partial salvage strategies. We also conduct a sensitivity analysis to understand how these benefits are affected by the unit lost-sales cost π, the

Table 3-1. Parameter values.

T	a	γ	K	c	h	U	w
20	40	0.75	200	5	1	10	0.6

price sensitivity of demand b, the volatility of demand and the rate of demand reduction with respect to the age of the inventory.

We consider an additive demand model for our experiments, and choose $f(s) = \max(2 - \exp(\alpha s), \gamma \exp(-\alpha s))$. For small values of s, this function is concave. This means that the reduction in the fraction of customers who do not want to purchase the product due to its age accelerates as the age increases. When s becomes large enough, this function is convex, which means that the rate of decline stabilizes as s becomes large implying that there will always be some customers who would buy the product. In the above function, the parameter α determines the rate at which demand declines with respect to the age. For small values of α, the demand reduction as the age increases is relatively small. This reduction occurs more and more rapidly as α increases. This is why we will refer to α as the rate of demand reduction with respect to the age in the rest of this section. Note that in the experiments, we do not impose an explicit upper limit on s.

Putting everything together, the demand model used in our experiments is as follows:

$$D_t(s, p) = (a - bp + \xi_2) \max(2 - \exp(\alpha s), \gamma \exp(-\alpha s)).$$

We take ξ_2 to have a truncated normal distribution with mean 0 and variance σ^2; the truncation occurs at $\pm U$. We set the salvage value in period s to be $wf(s)$. Thus, the reduction in salvage value with respect to the age mimics the reduction in demand with respect to the age.

The values of the parameters used in our experiments are reported in the following table.

In Figures 3-5, we plot the percentage improvement in the optimal profit due to the demand management, flexible replenishment and partial salvage strategies as a function of the lost-sales cost π, price sensitivity of demand b, demand volatility σ, and the rate of demand reduction with respect to age α. We compute the percentage improvement due to demand management as follows:

$$\frac{\text{Profit of fixed cycle model - Profit of fixed cycle model without demand management}}{\text{Profit of fixed cycle model without demand management}} \times 100\%.$$

To compute the profit of the fixed cycle model, we compute the optimal cycle profit for different cycle lengths. Subsequently, we choose the optimal cycle length R^* as the one that produces the highest average profit per period. Since there may not be an integer number of cycles corresponding to the optimal cycle length within the planning horizon, we approximate the total profit over the horizon by $\frac{T(\text{Optimal Cycle Profit Corresponding to } R^*)}{R^*}$. The computation of the fixed cycle model without demand management is similar; the only difference is that the same price is used every period. This price is optimized at the same time as the order quantity at the beginning of a cycle.

The percentage improvement due to flexible replenishment is defined as follows:

$$\frac{\text{Profit of flexible replenishment model - Profit of fixed cycle model}}{\text{Profit of fixed cycle model}} \times 100\%.$$

Similarly, the percentage improvement due to partial salvage is computed as follows:

$$\frac{\text{Profit of partial salvage model - Profit of flexible replenishment model}}{\text{Profit of flexible replenishment model}} \times 100\%.$$

Figures 3-5A-3-5D demonstrate that the benefit of both demand management and flexible replenishment strategies increases as any of the price sensitivity of demand, lost-sales cost, and variance of ξ_2; the variation with respect to α is not monotonic though it occurs in a narrow range. Broadly speaking, these strategies provide benefit by helping manage the demand risk. (We note that while the demand management strategy is beneficial even when demand is deterministic, the flexible replenishment strategy improves profit only when there is demand uncertainty.) This key idea will form the core of our explanation for each of these observations. As an example, the main advantage of the flexible replenishment strategy is that the retailer can select her reorder interval based on the inventory level. If demand has been slack for a few periods, the retailer does not have to wait until the end of the order cycle (as in the fixed cycle model) to place the new order; she can salvage the existing inventory and place an order immediately. Similarly, in the event of demand spikes, the retailer can place the next order early in the flexible replenishment model; she does not have to wait until the end of the order cycle.

On the other hand, the partial salvage strategy is useful only when there is a sub-optimal level (on the higher side) of the inventory. In that case, the retailer may partially salvage the inventory instead of renewing it or giving a discount. The root cause of any excess inventory is the uncertainty in demand. Combining these two factors, we conclude that the partial salvage strategy is likely to add value when demand risk increases or when the use of demand management is difficult.

Figure 3-5 shows that the demand management, flexible replenishment and partial salvage strategies provide a profit improvement of 2-7 % (without including the effect of b), 3-8%, and less than 0.1%, respectively. (The profit improvement due to the partial salvage strategy in the figure is multiplied by 10 so that it can be plotted on the same figure.) More details of the insights from the figure are as follows.

Figure 3-5A shows that the profit improvement due to the demand management strategy accelerates as the price sensitivity of demand b increases. As b increases, the demand-loss due to a sub-optimal price increases. As a result, demand management yields greater benefit compared to the static pricing strategy as b increases. The profit improvement due to the flexible replenishment strategy also increases as b increases. With the increase in the price sensitivity of demand, the mean demand decreases for a given value of price for either of the fixed order and flexible replenishment models, which results in lower prices and lower demand in optimality for any given combination of x, s and t. Due to lower demand mean, the optimal number of periods in a cycle increases in the fixed cycle model because of the fixed cost of order placement. The flexible replenishment model also orders for more periods. The increased number of periods in a cycle increases demand risk. Since the flexible replenishment model copes better with demand uncertainty, it performs better. On the other hand, the advantage of the partial salvage strategy declines with b. A higher value of b increases the ability of the retailer to modulate demand when faced with excessive inventory. This additional capability erodes the advantage of the partial salvage strategy to eliminate excess inventory, resulting in a reduced profit advantage as b increases.

The profit advantage of all the strategies increases with the lost-sales cost. As the lost-sales cost increases, the required customer service level increases. This results in higher order quantities for each of the three models. Higher order quantities increase the risk of excessive inventory. Since all the strategies reduce inventory risk, their profit advantage improves as the lost-sales cost increases.

As the uncertainty in demand increases, the profit improvement due to all the three strategies first increases rapidly before stabilizing for higher values of the variance of ξ_2. Greater demand variance increases safety stock requirements for all the three strategies. However, as in Part 2 above, the three strategies help the retailer cope with the inventory risk better. This results in more profit improvement as the demand uncertainty increases.

The benefits of the demand management and flexible replenishment strategies vary in a narrow range as the rate of demand reduction due to age α increases. This is contrary to our own initial intuition. Our own intuition is that with the increase of α the demand variance decreases in second period and beyond in a replenishment cycle. Since the benefit of either strategy increases with greater demand uncertainty, their profit advantage should decline as α increases. On the other hand, profit improvement due to the partial salvage strategy increases with α. We believe the reason lies in the reduced capability of the flexible replenishment strategy to use price to eliminate excess inventory.

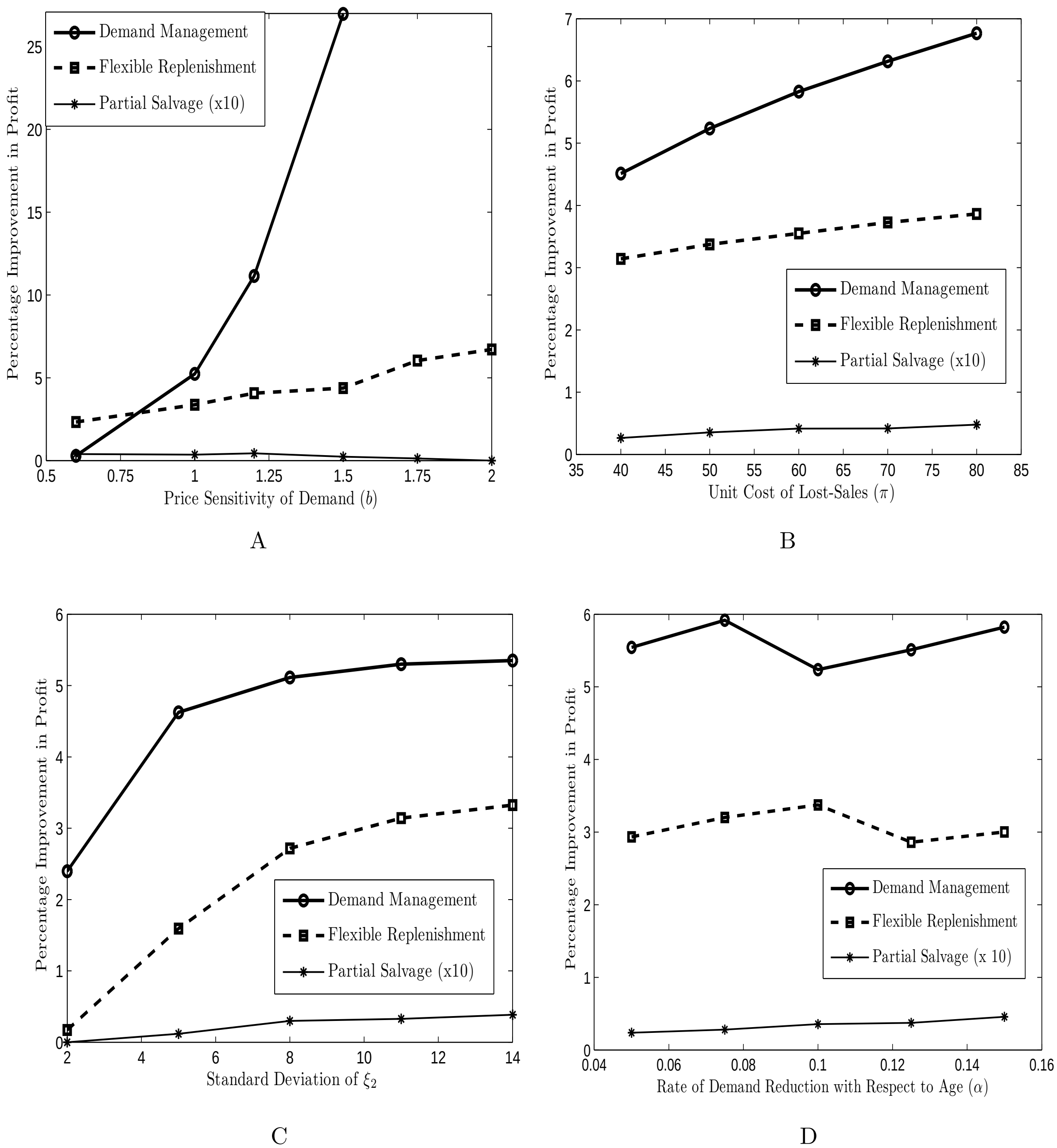


Figure 3-5. Profit improvement due to demand management, flexible replenishment, and partial salvage strategies as a function of model parameters. A) Different price sensitivity of demand. B) Different unit cost of lost-sales. C) Standard deviation of ξ_2. D) Rate of demand reduction with respect to age.

CHAPTER 4
JOINT INVENTORY AND DEMAND MANAGEMENT FOR PERISHABLE PRODUCTS: FIFO VERSUS LIFO

The second part of this work considers joint replenishment and price decisions when the economies of scale in order placement do not exist and demand is age-independent.

Much of the academic effort on the joint management of replenishment and demand exists only for non-perishable products or for products with a single replenishment opportunity during the life-time using Newsvendor-type models. The papers that utilize Newsvendor-type models ignore that the decisions regarding replenishment (by ordering fresh inventory) and demand management (by varying price) decisions can be made more than once during the lifetime of the product (Karaesmen et al. (2008)). This work aims to fulfill this important gap by considering a perishable product whose lifetime is equal to two review periods.

For such a product, it is well-understood that the profit during the planning horizon (and hence the optimal replenishment policy) depends on the issuing rule, that is, the relative order in which units arrive and are consumed (Nahmias (1982)). Naturally, this observation continues to hold even when demand is endogenous. In the existing literature, two commonly modeled issuing rules are first-in, first-out (FIFO) and last-in, first-out (LIFO). Under the FIFO rule, as the name suggests, when units of different ages are present in the system, they are consumed in the order of oldest first. The LIFO rule, on the other hand, is exactly opposite; the units are consumed in the order of newest first when units of different ages are present in the system. Clearly, the FIFO rule minimizes inventory wastage and is thus the favorite of an inventory manager. (It is implicitly assumed that all units have the same age upon arrival. Otherwise, the optimal issuing rule is first-expiring, first-out.) The FIFO rule can be easily implemented when the inventory manager determines the order in which inventory is sold or consumed (as is the case in hospital blood banks). However, if the user/customer determines the issuing rule and she derives greater utility from newer units (as is the case for food products such as milk), then inventory is sold/issued according to the LIFO policy. (As before, the implicit assumption here is that all units have the same age upon arrival. Otherwise, the optimal issuing rule from a customer's perspective is last-expiring, first-out.)

For both the FIFO and LIFO rules, we develop and analyze a periodic review model over finite horizon with backlogging to obtain insights on the nature of the optimal replenishment and demand management decisions. Our analysis reveals several interesting insights. As an example, we find that when demand is additive (i.e. the deterministic and random components are additive), the basestock list-price policy, which is optimal for non-perishable, is not necessarily optimal for either issuance rule. (In the basestock list-price policy, the optimal order-up-to level and price are constant whenever an order is placed.) In fact, even a list-price policy is not optimal for the FIFO rule. Since the structure of the optimal policy may not be simple, we propose an approximate policy using bounds on the optimal replenishment quantity and expected demand. For a wide-range of system parameters, the policy performs within 2% of the optimal profit.

Since the FIFO rule results in less spoilage in general, an inventory system operating under this rule generates greater profit than another system operating under the LIFO rule. This raises the question of whether demand management helps a LIFO system to bridge the profit gap with a FIFO system and if so, to what extent. To answer this question, we conduct computational experiments under two scenarios: Time-varying demand and capacity constraint on order quantity. The experiments reveal that while the LIFO system catches up with the FIFO system when demand is time-varying, the reverse happens when there is a capacity constraint. We also utilize the experiments to answer another question: Which of the two systems benefit more from the flexibility allowed by demand management? Once again, we find that the LIFO system benefits more when demand is time-varying but not when capacity is limited.

The rest of this paper is organized as follows. In Section 4.1, we discuss the basic assumptions and notation. In Sections 4.2 and 4.3, we develop and analyze periodic review models for the FIFO and LIFO rules, respectively. For the FIFO rule, we also develop upper and lower bounds on the optimal order quantity and expected demand in Subsection 4.2.2. We numerically examine the performance of an approximate policy obtained by taking a weighted average of these bounds in Section 4.4. In the same section, we also discuss other computational experiments. Finally, we conclude in Section 5.2. We begin by positioning our work in the basic notation and common assumptions.

4.1 Notation and Common Assumptions

We consider a finite-horizon, periodic-review model for a perishable product with a fixed lifetime equal to two periods at a single manufacturer. At the beginning of each period, the manufacturer inspects his net inventory, x_t, which is one-period old, and places an order for quantity, q_t. For simplicity, we assume that the lead-time is equal to zero. The assumption of zero lead-times is a standard convention in both perishable inventory theory as well as in the literature on joint replenishment and demand management decisions.

At the same time, the manufacturer determines price p_t for that period. Let $d_t(p_t)$ be the expected demand corresponding to p_t. We assume that the function $d_t(\cdot)$ is strictly decreasing. As a result, there is a one-to-one correspondence between price and expected demand. This also means that we can use price and expected demand interchangeably in analysis. In fact, the exposition is considerably simplified when expected demand is used as a variable instead of price. Accordingly, throughout the paper we use expected demand as a variable to present results. In doing so, we omit the argument p_t for simplicity and use only d_t to denote the expected demand. The interchangeability of expected demand and price also implies that the terms demand management and dynamic pricing have identical meanings in the context of our model.

We assume that the expected revenue $p_t \cdot d_t(p_t)$ is strictly concave in p_t. This assumption is standard in the literature on joint inventory-price decisions. One example of $d_t(p_t)$ for which the expected revenue is strictly concave is $d_t(p_t) = a_0 - b_0 p_t$, where

$a_0, b_0 > 0$. Another example of such a function is $d_t(p_t) = a_0 \exp(-b_0 p_t)$, where $a_0, b_0 > 0$, for $p_t \in [0, 2/b_0]$.

Once the order is delivered, customer demand arrives through the rest of the period. We assume that given expected demand d_t, the realized demand in period t is equal to $D_t = d_t + \xi_t$, where ξ_t is a random variable with support $[-a, \infty)$, where $a > 0$, such that $E(\xi_t) = 0$. Bounding the support of ξ_t at $-a$ is necessary to ensure that demand D_t remains non-negative. We assume that ξ_t is independently and identically distributed over time. Let F and f be the CDF and PDF of ξ_t, respectively.

The above demand model is referred to as the *additive* demand model in the existing literature. The word "additive" arises from the additive nature of the randomness (ξ_t). A more general model is the *multiplicative* model, which has the following form: $D_t = d_t\xi_t^1 + \xi_t^2$. Thus, the randomness is present in both additive and multiplicative forms. Although we will utilize the multiplicative model in computational experiments, our methodology is not useful in developing insights for this model. The extension of our results for this model thus remains a topic for future work.

We let $\mathcal{D}_t$ to be the set of all feasible values of expected demand in period t. Two requirements for any d to be contained in $\mathcal{D}_t$ are that (a) the realized demand D_t remains non-negative for all $\xi_t \in [-a, \infty)$ and (b) the corresponding price be non-negative. Since $\xi_t \geq -a$, requirement (a) ensures that any d in $\mathcal{D}_t$ is greater than or equal to a. Further, we assume $\mathcal{D}_t$ to be convex, which implies that the set is an interval.

At the end of the period, once all the demand is realized, holding cost is charged on any remaining inventory at rate h per unit. On the other hand, if demand exceeds inventory, the excess demand is backlogged, and backlogging cost is charged at π per unit backlogged.

Given values of x_t, q_t and d_t, the revenue and holding and shortage costs incurred in period t are equal to

$$L(x_t, q_t, d_t) = R(d_t) + hE[x_t + q_t - D_t]^+ + \pi E[D_t - x_t - q_t]^+,$$

where $R(d_t)$ is the expected revenue, which is equal to the product of expected demand and the corresponding price, and $[\cdot]^+$ stands for $\max(\cdot, 0)$.

Also, at the end of the period, the inventory that is two periods old is discarded. The amount of inventory discarded depends on the allocation rule. Under the FIFO allocation rule, the amount of inventory discarded is equal to $(x_t - D_t)^+$. On the other hand, the amount of inventory discarded under the LIFO rule is $(x_t - (D_t - q_t)^+)^+$. We let θ (possibly negative) be the unit cost of discarding old inventory. The parameter θ can be both positive or negative depending upon whether old inventory incurs a cost while being discarded or it is salvaged.

4.2 First-In, First-Out

In this section we derive insights on the joint replenishment and demand decisions for the case in which inventory is consumed according to the first-in, first-out (FIFO) rule. The setting considered in this section is similar to Nahmias and Pierskalla (1973); the major difference is our consideration of endogenous demand.

4.2.1 Analysis

It is well-understood in both academia and practice that if inventory is consumed in the order in which it arrives (that is, FIFO), the amount of inventory disposal is minimized (Nahmias (1982)). Naturally, an inventory manager would prefer to sell inventory according to this allocation rule. Therefore, if an inventory manager controls the order in which inventory is sold, then the inventory is likely to be sold in the FIFO order. This condition is satisfied in a business-to-business setting if the manufacturer selects the units to ship to a customer. For instance, in a vendor managed inventory system, the manufacturer selects which units to deliver to a customer. However, an inventory manager also controls the order of inventory consumption in online grocery stores, such as NetGrocer.com, where the grocer picks inventory and delivers it to the consumers' homes.

Even though excess demand is usually lost in a retail setting, in an online grocery store excess demand may be backlogged for food items. Unlike a consumer who visits a brick-and-mortar grocery store and is likely to obtain any unavailable product from another store, which results in a lost-sale, a consumer who purchases online is more likely to backorder an unavailable product for two reasons. One, it is clear the consumer prefers not to visit a brick-and-mortar store, so she is more likely to wait than to go to a brick-and-mortar store to buy the unavailable product. Two, the regular and personal nature of grocery shopping ensures that consumers have a close relationship with the grocer, so the consumer is unlikely to order the unavailable product with another e-grocer.

We next formulate the model. For simplicity, we omit the subscript t from all the variables through the rest of this paper unless necessary for exposition. Given the order of inventory consumption to be FIFO, the optimal profit from period t through the end of horizon v_t is equal to

$$v_t(x) = \max_{d \in \mathcal{D}, q \geq 0} L(x,q,d) - cq - \theta E(x-D)^+ + \alpha E v_{t+1}(q-(D-x)^+), \tag{4.2.1}$$

where α is the discount factor. We take the end of horizon profit $v_{T+1}(x)$ to be equal to $sx^+ - cx^-$, where s is the salvage value. Observe that the argument of v_{t+1}, $(q-(D-x)^+)^+$, is the amount of inventory that is one-period old at the beginning of period $t+1$. Also, define

$$G_t(x,q,d) = L(x,q,d) - cq - \theta E(x-D)^+ + \alpha E v_{t+1}(q-(D-x)^+), \tag{4.2.2}$$

so that $v_t(x) = \max_{d \in \mathcal{D}, q \geq 0} G_t(x,q,d)$.

We next discuss an observation regarding the form of the optimal policy. For non-perishable products, it is well-known that when the demand function is additive and excess demand is backlogged, the optimal value of d (or p) whenever an order is placed can be obtained by maximizing a single-dimensional, concave function of expected demand, $R(d) - cd$ (or, equivalently, price). One consequence of this result is that the replenishment and demand decisions become separable. That is, given the optimal value of d, the optimal replenishment quantity can be obtained by optimizing a one-dimensional function in the replenishment quantity.

We find that these results may not hold when the products are perishable, as in our model. That is, given that an order is placed in a period, it is not necessary that the optimal expected demand maximizes the function $R(d) - cd$. In fact, the optimal expected demand may not even be a constant; it may depend on the value of x. Figures 4-1A and 4-1B, in which we plot the optimal order quantity and expected demand as a function of net inventory, clearly illustrate that the optimal order quantity and expected demand may depend on net inventory. Further, the lack of separability between the replenishment and demand decisions can be easily deduced using Equation 4.2.1.

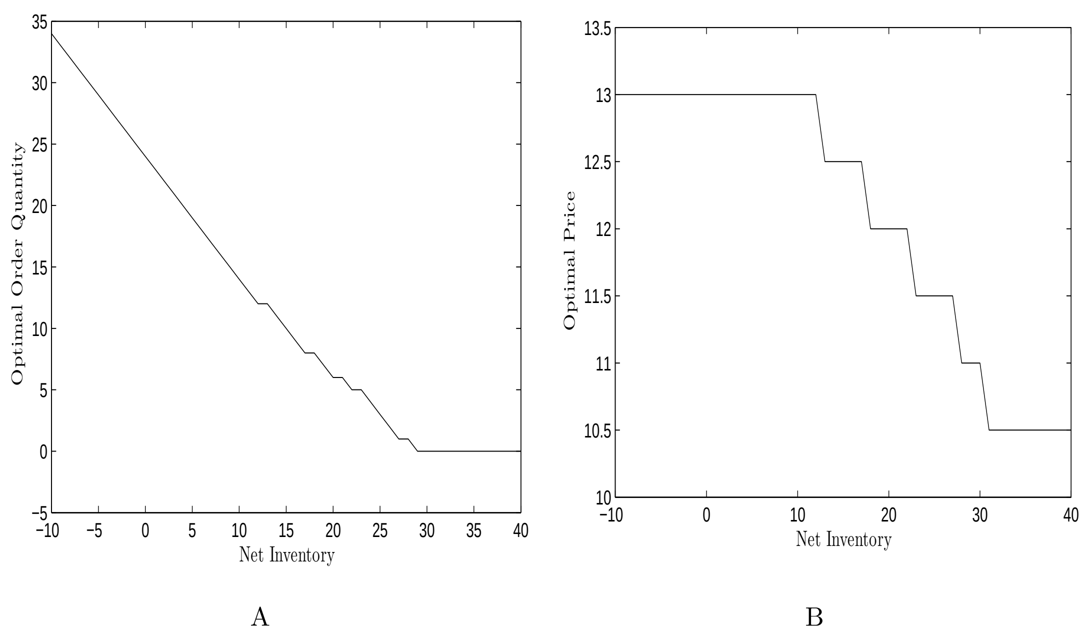


Figure 4-1. Trends of optimal order quantity and price when $\theta = -1$, $D = (42 - 2p + \xi)$, $\xi \sim$ truncated $N(10, 6)$ over $[-10, 10]$, $t = 1$, $T = 4$. A) Net inventory vs optimal order quantity. B) Net inventory vs optimal price.

We state these observations formally as follows.

Observation 4.2.1. *For the model defined in Equation 4.2.1, the replenishment and demand decisions are not separable. Further, the optimal expected demand in a period in which an order is placed may not maximize the function* $R(d) - cd$.

We formally state the potential dependence of the optimal expected demand and order quantity on net inventory in the following theorem in which we also shed light on the structure of the optimal profit function. We note that the structure of the optimal policy is similar to the case in which demand is not a decision variable (Nahmias and Pierskalla (1973)). Naturally, the presence of another variable (expected demand) complicates analysis, so the proofs require additional arguments compared to that paper. Another dimension, albeit minor, along which we contribute relative to Nahmias and Pierskalla is that we allow the end of horizon salvage value s to be less than c unlike them, who assume that the two are equal.

Let $d^*(x)$ and $q^*(x)$ be the optimal expected demand and order quantity when net inventory is equal to x. The theorem is as follows.

Theorem 4.2.2. *Let* $R''(d) \leq -h$ *and* $\pi > (1 - \alpha c)$. *Also, let* $\arg\max_d \{R(d) - cd\} \in \mathcal{D}$, *and* $f(w) \leq 1$ *for all* $w \in [-a, \infty)$.

1. *For each* t, $G_t(x, q, d)$ *is a jointly concave function of* q *and* d.
2. *For each* t, *there exists a unique* $\bar{x}_t > 0$ *such that* $\bar{x}_1 = \bar{x}_2 = \ldots = \bar{x}_{T-1} \geq \bar{x}_T$. *For* $x < \bar{x}_t$,
   (a) *The optimal order quantity* $q^*(x) > 0$.
   (b) $0 \leq d^{*\prime}(x) \leq 1$ *and* $-1 \leq q^{*\prime}(x) \leq 0$.
   (c) $c(1 - \alpha) - \theta \leq v_t'(x) \leq c$ *and* $v_t'(x) = c$ *for* $x \leq 0$.
   (d) $v_t(x)$ *is a concave function of* x.
3. *On the other hand, when* $x \geq \bar{x}_t$,
   (a) $q^*(x) = 0$ *and* $0 \leq d^{*\prime}(x) \leq 1$.
   (b) $-h - \theta \leq v_t'(x) \leq c$.
   (c) $v_t(x)$ *is a concave function of* x.

A brief discussion of the four technical assumptions required for the theorem is as follows. The first assumption requires that the second derivative of the expected revenue be less than $-h$. One consequence of this assumption is that the set of feasible values for expected demand $\mathcal{D}$ has bounded support. To see this, note that the left end-point of $\mathcal{D}$, a, is finite. Since the slope of $R(d)$ strictly decreases, there must exist some finite d_0 such that $R(d) \leq 0$ for all $d \geq d_0$. A negative value of $R(d)$ is possible only if price is negative. Clearly, any such values of d cannot be feasible, so the feasible interval for expected demand will be $[a, d_0]$. The second assumption, which imposes a lower bound on π, simplifies analysis by ensuring that backlogs are always satisfied the next period.

The next assumption that the maximizer of $R(d) - cd$ be feasible makes analysis convenient. This assumption is not critical for most of the proofs to hold. Finally, we require that the magnitude of the density at each point be no more than 1. This assumption holds for a wide range of parameter values for several common distributions. For instance, for the exponential distribution, the assumption is satisfied if the rate $\lambda \leq 1$, which means that $E(\xi) \geq 1$. Similarly, for the uniform distribution, the assumption holds if the length of the support is greater than 1.

The above result indicates that as net inventory increases, the optimal order quantity decreases though the rate of reduction is less than 1. This means that the order-up-to level $x + q^*(x)$ increases with x. Once the level of inventory becomes high enough, the order quantity becomes equal to 0. The threshold at which the order quantity becomes 0, interestingly, is the same for all $t \leq T - 1$, but has a lower value for period T due to the end-of-horizon effect. On the other hand, the optimal expected demand increases with net inventory, but the rate of increase is less than 1.

The result that $v_t'(x) = c$ for $x \leq 0$ is intuitive. This result means that a unit reduction in backlogged quantity increases profit by c. Since lead-time is 0 and π is large enough, all the backlogged demand is satisfied in the following period. To satisfy each unit of backlog costs c. Now, if the amount of backlog decreases by one unit, this cost c will be saved, resulting in a profit increase of c. On the other hand, we note that the slope of v_t is always at least $-\theta - h$. The lower bound is realized if a unit that is one period old is certain not to be consumed. In that case, the unit does not contribute to the revenue, but incurs both holding and salvage costs, which add up to $-\theta - h$. These bounds on the slope of v_t are useful not only in establishing other parts of the theorem, but also in developing bounds on the optimal order quantity and expected demand, as we show in Subsection 4.2.2.

4.2.2 Bounds on Optimal Replenishment Quantity and Demand

In this subsection, we develop upper and lower bounds on the order quantity and expected demand for the model developed in the last subsection. We obtain these bounds by exploiting the structural properties of the profit function as well as the optimal order quantity and expected demand stated in Theorem 4.2.3. For instance, in identifying an upper bound on the order quantity, we utilized the result that v_t' is bounded from above by c. The result is formally stated as follows.

Theorem 4.2.3. *1. When $x \geq \bar{x}$, the expected demand is bounded from above by $\max\{0, x - y\}$ and when $x \leq \bar{x}$, it is bounded from above by $\max\{0, \bar{x} - y\}$ where $y = F^{-1}[\frac{\pi - c + \alpha c}{\pi + h + \theta + \alpha c}]$.*

2. For any value of x, the expected demand is bounded from below by d_c such that $R'(d_c) = c$.

3. When $x \leq \bar{x}$, the order quantity is bounded from above by $\max\{0, \bar{r} + \bar{x} - y - x\}$ such that $\bar{r} = F^{-1}[\frac{\pi - c + \alpha c}{\pi + h}]$ and bounded from below by $\max\{0, \underline{r} + d_c - x\}$ such that $\underline{r} = F^{-1}[\frac{\pi - c - \alpha(h + \theta)]}{\pi + h}]$.

In Subsection 4.4.2, we numerically evaluate the effectiveness of approximate policies that are derived by taking a weighted average of the above bounds.

4.2.3 Relationship with One-Period and Infinite Lifetime Systems

In this subsection, we compare the optimal policy for the model defined in Subsection 4.2.1 to that of a system in which the product has a single-period lifetime as well as a system in which the product has an infinite-period lifetime (or is non-perishable). When there is no demand management, it can be easily shown that the order quantity for a two-period lifetime system in any period should be greater than the order quantity for a one-period lifetime system. The reason lies in the reduced risk of spoilage for a two-period lifetime system. The same reason also results in a larger order quantity for a non-perishable product as compared to a two-period lifetime system. Our objective is to explore whether the same intuition continues to hold in the presence of demand management.

We begin by formulating the model for a system in which the lifetime is equal to one period. All the modeling assumptions remain the same as in Subsection 4.2.1; the sole difference is that any unsold units at the end of each period are discarded. The optimal profit from period t through the end of horizon is

$$v_t^1(x) \quad = \quad \max_{q^1 \geq 0, d^1 \in \mathcal{D}} L(x, q^1, d^1) - cq^1 - \theta E(x + q^1 - d^1)^+ + \alpha E v_{t+1}((d^1 + \xi - x - q^1)^+), \tag{4.2.3}$$

where $v_{T+1}(x) = cx$. Observe that only backlogs are carried from one period to the next period. As a result, x can take only non-positive values.

Similarly, we can define a model for non-perishable items. The formulation is as follows:

$$v_t^\infty(x) = \max_{q^\infty \geq 0, d^\infty \in \mathcal{D}} L(x, q^\infty, d^\infty) - cq^\infty + \alpha E v_{t+1}(x + q^\infty - d^\infty - \xi), \tag{4.2.4}$$

where $v_{T+1}(x) = sx^+ - cx^-$. Since nothing ever perishes, there is no term involving θ in the above formulation. As noted before, it is well-known that the optimal demand when the product is non-perishable maximizes $R(d^\infty) - cd^\infty$ whenever an order is placed. The same can be also be easily shown for a system in which the product lifetime is equal to one period; we omit the details. Coupling these observations with Theorem 4.2.3, in which we shows that the optimal demand for a two-period lifetime system is bounded from below by the maximizer of $R(d) - cd$, we obtain the following corollary.

Corollary 4.2.4. *Whenever an order is placed, the optimal demand for a two-period lifetime system is bounded from below by the optimal demands for one-period lifetime as well as infinite lifetime systems.*

Next, we compare the order quantities. Since the net inventory at the beginning of a period in a one-period lifetime system is always non-positive, the comparison across the two systems can only be carried out for such values of x. It can be easily shown that for a one-period lifetime system, the ordering policy is a basestock policy. In the following proposition, we show that the optimal order quantity for a two-period lifetime system, $q^{2*}(x)$ is bounded from below by that for a one-period lifetime system, $q^{1*}(x)$ for all x non-positive. We also show that the threshold for order placement in a two-period lifetime system, $\bar{x}_t$, is bounded from below by the optimal basestock level, which is equal to $q^{1*}(0)$, of the one-period system.

Proposition 4.2.5.
1. *$q^{2*}(x) \geq q^{1*}(x)$ for all $x \leq 0$.*
2. *$\bar{x}_t \geq q^{1*}(0)$.*

The comparison of a two-period lifetime system with that of an infinite lifetime system is muddier. For instance, we find that the order quantity for the two-period lifetime system is larger than the infinite lifetime system for period T for any x. However, when $t << T$, so that the end-of-horizon effect is insignificant, the infinite lifetime system orders more compared to a two-period lifetime system. We are unable to determine a relationship between the two systems when t is relatively close to T. Observe that the end of horizon effect arises since the salvage cost at the end of horizon $s < c$. If $s = c$, then the order quantity in an infinite lifetime system will always dominate the order quantity in a two-period lifetime system.

On the other hand, the threshold for order-placement is always greater in the two-period lifetime system. The result is formally stated below.

Proposition 4.2.6. *1. $q^{2*}(x) \geq q^{\infty *}(x)$ for all x when $t = T$.*

2. When $t << T$, $q^{2*}(x) \leq q^{\infty *}(x)$.

3. If $s = c$, $q^{2*}(x) \leq q^{\infty *}(x)$.

4. For any t, $\bar{x}_t \geq q^{\infty *}(0)$.

We note that Proposition 4.2.5 can also be used to obtain a lower bound on the order quantity in a two-period lifetime system. (Proposition 4.2.6 can also be used to obtain an upper bound, but the bound turns out to be identical to Theorem 4.2.3, Part 3.) We state the lower bound in the following corollary.

Corollary 4.2.7. *The optimal order quantity in a two-period lifetime system is bounded from below by the optimal order quantity one-period lifetime as well as infinite lifetime systems.*

4.3 Last-In, First-Out

In this section, we discuss the case in which the consumption of inventory occurs according to the last-in, first-out (LIFO) rule. The LIFO rule is the opposite of the FIFO rule, so when new and old inventory is simultaneously present in the system, the new inventory is consumed before the old inventory. In general, the inventory of a perishable product is likely to be consumed according to this rule if every customer prefers and is able to obtain the freshest unit available. Customers may prefer to obtain the freshest unit available since such a unit has the most time-to-expiry, which reduces the risk of spoilage. The risk of spoilage, however, increases for the manufacturer under the LIFO rule.

In a business-to-business setting, inventory may be consumed according to the LIFO rule if a customer picks up inventory from the manufacturer's warehouse herself. The LIFO consumption order may be realized even when the manufacturer delivers inventory if the manufacturer is contractually required to ship the freshest units available to the customers.

We next develop a dynamic programming formulation corresponding to this rule. The optimal profit from period t to the end of horizon is equal to

$$\begin{aligned} v_t(x) &= \max_{q\geq 0, d\in\mathcal{D}} L(x,q,d) - cq - \theta E(x-(D-q)^+)^+ \\ &\quad +\alpha E v_{t+1}(x+q-D-(x-(D-q)^+)^+), \end{aligned} \tag{4.3.5}$$

such that $v_{T+1}(x) = sx^+ - cx^-$, where s is the salvage value of any inventory left at the end of horizon. The argument to v_{t+1} appears complex, but it can be derived in a simple manner. It is equal to net inventory at the end of period t $(x+q-D)$ less the amount of inventory spoiled $((x-(D-q)^+)^+)$. Similar to the model in Section 4.2, we consider an additive demand model, that is, $D = d + \xi$, where d takes values in $\mathcal{D}$.

Recall that for the FIFO rule, the optimal demand policy cannot be obtained by maximizing a single-dimensional function of d, $R(d) - cd$, whenever an order is placed, unlike for non-perishable products. In fact, the optimal value of d depends on the value of net inventory. Fortunately, the LIFO rule behaves differently, and the optimal expected demand can, once again, by obtained by maximizing $R(d) - cd$ whenever an order is placed. This also ensures that the objective function is separable in q and d. We state the result formally in the following proposition.

Proposition 4.3.1. *If the unconstrained optimal order quantity is non-negative in the optimal solution, the optimal expected demand is equal to* $d^*(x) = \max_{d\in\mathcal{D}} R(d) - cd$.

It is well-known that the optimal profit function for the LIFO rule lacks concavity (or any other type of simple structure) even when demand is exogenous (Nahmias (1982)). Naturally, the addition of another variable, expected demand, can only complicate the analysis, so the profit function continues to lack a simple structure. For instance, the optimal profit function defined in Equation 4.3.5 is not necessarily concave in netinventory. We state this observation formally as follows.

Observation 4.3.2. *The optimal profit function defined in Equation 4.3.5 is not necessarily concave in x.*

It is sufficient to consider the dynamic program for period T to establish this observation. We provide the complete argument in the Appendix.

The lack of concavity of the profit function makes it difficult to establish other properties of the optimal solution such as bounds on the slope of the optimal order quantity. In spite of this hardship, we are able to characterize a number of properties of the optimal profit function and optimal decisions. The result is stated as follows.

Theorem 4.3.3. *Let* $\pi > c(1-\alpha)$ *and* $\arg\max_d \{R(d) - cd\} \in \mathcal{D}$.

1. *There exists a unique* $\bar{x}$*, which is independent of* t*, such that an order is placed only if* $x < \bar{x}$*. For all* $x \geq \bar{x}$*, no order is placed.*
2. $d^{*\prime}(x) = 0$ *for* $x < \bar{x}$ *and* $0 \leq d^{*\prime}(x) \leq 1$ *for* $x \geq \bar{x}$.
3. *For* $x \leq \bar{x}$ *and* $t < T$, $-\theta - h \leq v_t'(x) \leq c + \alpha h - (1-\alpha)\theta$*. For* $x \leq \bar{x}$ *and* $t = T$, $-\theta - h \leq v_t'(x) \leq c$*. Further, for* $x \leq 0$, $v_t'(x) = c$.
4. *For* $x \geq \bar{x}$, $v_t'(x) \geq -\theta - h$.
5. *For* $x \geq \bar{x}$, $v_t(x)$ *is concave in* x.

Two remarks on the above theorem are as follows. Firstly, similar to the FIFO rule, there exists a unique threshold of the net inventory beyond which the order quantity is zero. Unlike the FIFO rule, however, this threshold is identical for all the periods, including period T. The expected demand beyond this threshold behaves in the same manner as for the FIFO case. Since the optimal demand whenever an order is placed is constant, naturally its slope is equal to 0. Because of a lack of concavity of v_t, we are unable to say anything about how the order quantity changes as the amount of net inventory increases. Extensive computational experiments, however, indicate that the optimal order quantity usually decreases with net inventory.

Secondly, similar to the FIFO rule, $v_t'(x) = c$ for $x \leq 0$. The reason is also the same, and the details are omitted. Finally, the theorem shows that the optimal profit function is concave for all net inventory values greater than $\bar{x}$.

4.4 Computational Experiments

The computational experiments, which we describe in this section, have three major objectives. The first objective is to explore whether demand management reduces the profit advantage of the FIFO rule over the LIFO rule and if so, to what extent. The same set of experiments also shed light on how benefits of demand management differ for the FIFO and LIFO issuing rules, which forms the second objective. The final objective is to measure the performance of the approximate policy for the FIFO rule obtained by taking a weighted average of the bounds developed in Theorem 4.2.3.

4.4.1 Benefits of Demand Management

To explore whether the flexibility allowed by demand management diminishes or even eliminates the profit advantage of the FIFO rule over the LIFO rule, we consider two scenarios. In the first scenario, we consider non-homogeneous demand over time, and in the second scenario we consider a capacity constraint on the order quantity. Specifically, we compare the percent profit difference between the FIFO and LIFO rules with and without demand management in these scenarios. We note that Chan et al. (2006) also conduct computational experiments under these scenarios, albeit their focus is on non-perishable products with lost-sales.

4.4.1.1 Non-homogeneous demand

We model the non-homogeneity of demand over time by letting expected demand vary over time. Specifically, we consider a multiplicative demand model in which the expected demand is a linear function of price ($D = (a-bp)\xi_1+\xi_2$), and we vary the parameter a over time to simulate non-homogeneity of demand. For simplicity, we assume that the demand function is identical every two periods. That is, periods $t, t+2, t+4, \ldots$ have identical demand functions. Let a_h and a_l be the values of a in odd- and even-numbered periods, respectively. Such a demand function may be observed by an e-grocer that makes ordering decisions twice every week, say every Thursday and Monday. The demand faced by the grocer is likely to be non-homogeneous across review periods since the review periods that

Table 4-1. Parameter values with non-homogeneous demand.

T	c	h	π	s	μ	γ	β	a_l
4	5	1	40	1.5	0	1.2	1.2	20

consist of weekends would see far more demand than the other periods. We note that the non-homogeneity of demand in our model may also be construed as seasonality with a periodicity of two periods.

The computation of the optimal profit over the planning horizon corresponding to the FIFO and LIFO rules when demand is a decision variable is accomplished using Equations 4.2.1 and 4.3.5, respectively. To identify the influence of demand management, we compare these profits with the profit corresponding to a fixed demand (or price) strategy in which the expected demand (or price) is optimized at the beginning of planning horizon but is not changed afterwards. For the FIFO rule, the corresponding dynamic program is as follows:

$$V_t(x,d) = \max_{q\geq 0} L(x,q,d) - cq - \theta(x-D)^+ + \alpha E V_{t+1}(q-(D-x)^+, d), \quad 1 \leq t \leq T, \tag{4.4.6}$$

where $V_{T+1}(x,d) = sx^+ - cx^-$. The optimal profit over the planning horizon is equal to $\max\limits_{d\in\mathcal{D}} V_1(0,d)$. Similarly, for the LIFO rule, the dynamic program corresponding to the fixed price strategy is as follows:

$$\begin{aligned} V_t(x,d) = & \max_{q\geq 0, d\in\mathcal{D}} L(x,q,d) - cq - \theta E(x-(D-q)^+)^+ \\ & + \alpha E V_{t+1}(x+q-D-(x-(D-q)^+)^+, d), \end{aligned} \tag{4.4.7}$$

such that $V_{t+1}(x,d) = sx^+ - cx^-$. Similar to the FIFO rule, the optimal profit over the planning horizon is equal to $\max\limits_{d\in\mathcal{D}} V_1(0,d)$.

To run the experiments, we take ξ_1 to be Beta distributed with parameters γ and β, and ξ_2 to be truncated Normal distributed with parameters μ and σ. The values of the model parameters are summarized in the following table.

Finally, we define our metric, *Percent Profit Improvement*, as follows:

$$\text{Percent Profit Improvement} \;=\; \frac{\text{Optimal Profit of FIFO-Optimal Profit of LIFO}}{\text{Optimal Profit of LIFO}} \times 100\%. \tag{4.4.8}$$

In Figure 4-2, we plot the profit improvement due to the FIFO rule both in the presence and absence of demand management when demand is time-heterogeneous. The main insight from the figure is that the profit advantage of the FIFO rule compared to the LIFO rule may diminish in the presence of demand management, although the FIFO rule continues to dominate the LIFO rule. It, however, is possible that for some parameter values that the profit advantage of the FIFO rule may increase even further in the presence of demand management. Our experiments indicate that for high values of a_h (relative to a_l), the profit advantage of the FIFO rule increases even further when b, θ, and σ take relatively small values. On the other hand, when a_h is close to a_l ($a_h/a_l \leq 1.8$), the profit advantage of the FIFO rule diminishes compared to the LIFO rule for all the parameter values that we considered.

To understand why the profit advantage of the FIFO rule diminishes, we breakdown the profit for all the four cases into revenue and different types of costs. We find that demand management allows LIFO to better utilize inventory and reduce shortage costs more compared to the FIFO. This results in lower profit difference between the FIFO and LIFO when demand is managed. Our computations also show that the profit advantage of the FIFO rule always reduces when demand is managed as b, θ, or σ increases. (See Figures 4-2B, 4-4C, and 4-4B as examples.) Once again, this is mainly caused by better utilization of inventory by the LIFO, which reduces shortage and wastage costs.

The same set of computations also allow us to examine if demand management favors one issuing rule over another. To further explore this question, we compute values of the following metric

$$\text{Percent Profit Improvement} = \frac{\text{Optimal Profit-Profit of Fixed Demand Strategy}}{\text{Profit of Fixed Demand Strategy}} \times 100\%. \tag{4.4.9}$$

for both FIFO and LIFO rules. We find that demand management favors the LIFO rule more than the FIFO rule when a_h/a_l is low and b, θ, and σ are relatively high. On the other hand, when a_h/a_l is high and b, θ, and σ are relatively low, the FIFO rule benefits more from demand management. (We omit the plots.) In particular, we find that the parameter values for which the profit advantage of the FIFO rule diminishes (enhances) compared to the LIFO rule are the same for which demand management favors the LIFO rule more (less).

4.4.1.2 Capacity constraint

We carryover almost all of the setup from our experiments on demand heterogeneity except for two differences. One, there is now a capacity constraint on the order quantity q. The production quantity remains constrained by the capacity even after the planning horizon comes to an end. This means that if there is backlog at the end of period T, it may take several more periods before production halts completely. Two, the expected demand is homogeneous over time, that is, the value of a is now identical in all the periods. With these changes the dynamic programs in Equations 4.2.1, 4.3.5, 4.4.6, and 4.4.7 can be easily modified, and the details are omitted.

In Figure 4-3, we plot the metric defined in Equation 4.4.8, *Percent Profit Improvement*, when the order quantity is capacity constrained. Contrary to Figure 4-2, demand management increases the profit advantage of the FIFO rule even further in the presence of a capacity constraint. A breakdown of profit into revenue and costs indicates that the main driver of the improvement in performance for the FIFO rule is revenue; all the costs change by roughly the same amount for the two rules when demand is managed. The reason for this observation lies in inventory availability. While the inventory availability is

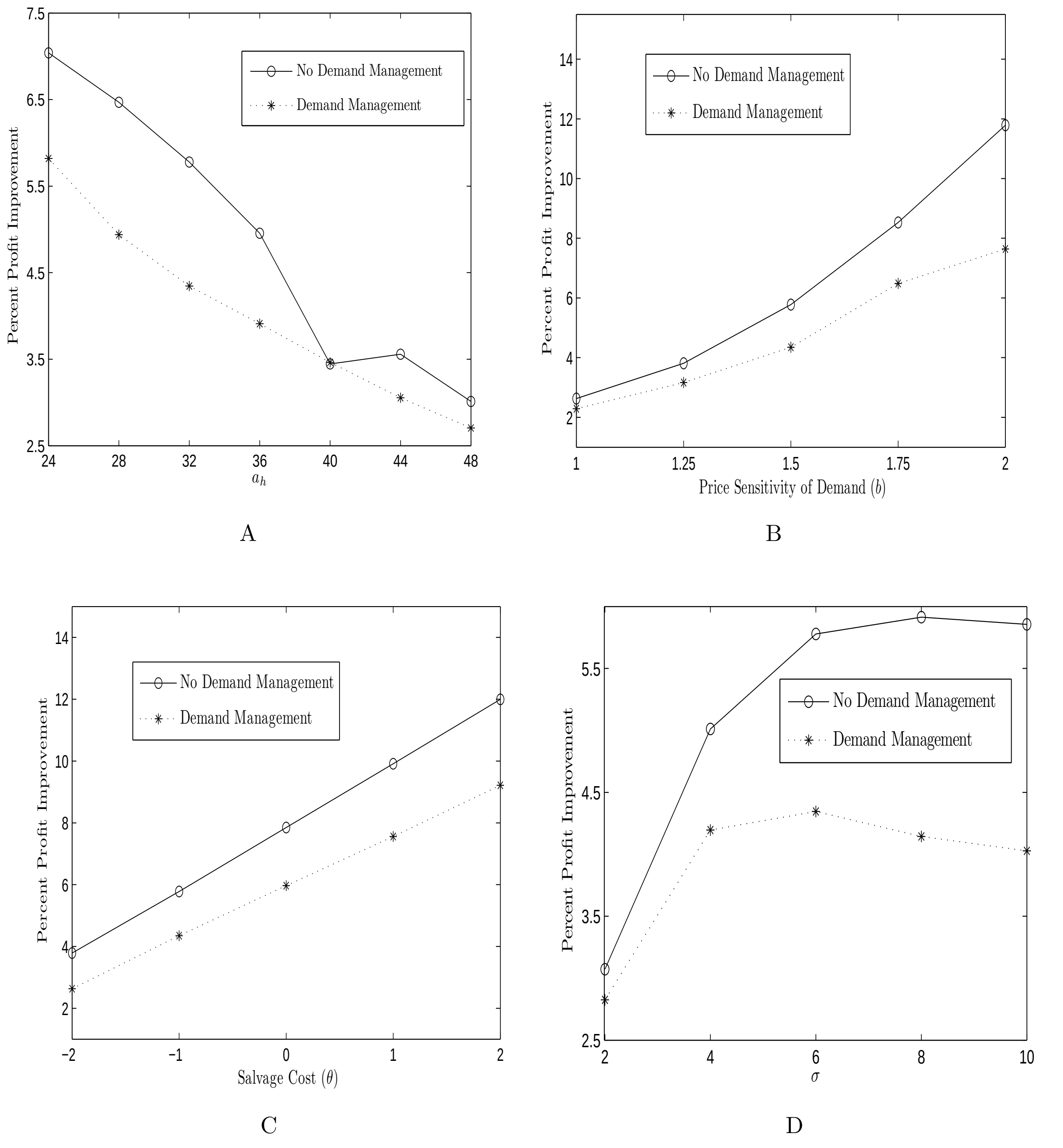


Figure 4-2. Percent profit advantage of FIFO with respect to LIFO in the presence of time-varying demand. A) Different a_h with $b = 1.5, \theta = -1, \sigma = 6$. B) Different price sensitivity of demand with $\theta = -1, \sigma = 6, a_h = 32$. C) Different salvage cost with $b = 1.5, \sigma = 6, a_h = 32$. D) Different standard deviation with $b = 1.5, \theta = -1, a_h = 32$.

limited under both the FIFO and LIFO rules due to the capacity constraint, the system under the LIFO rule suffers more due to greater spoilage. Less spoilage of inventory gives a FIFO system access to more inventory compared to a LIFO system, which gives it more leeway in modulating the price thus resulting in greater profit improvement. Note that the figure also shows that the difference in profit improvement in percent between demand management and no demand management usually increases as b, θ, or σ increases.

Finally, similar to the heterogeneous demand case, the same computational experiments can be used to determine if demand management favors one issuing rule over another, using Equation 4.4.9. Upon computation of this metric, we find that the profit improvement for the FIFO rule is greater than the profit improvement for the LIFO rule for all the model parameters that we considered. Once again, this result is not surprising given that the profit difference between the FIFO and LIFO rules increases when demand is endogenous in the presence of a capacity constraint.

4.4.2 Performance of Approximate Policy

Our objective in this section is to measure the performance of an approximate policy that is obtained by taking a weighted average of the bounds developed in Theorem 4.2.3 with respect to the optimal policy. We will perform this measurement for a wide range of values of three parameters, b, θ, and σ. The experiments also illustrate how the optimal weights change with these parameters.

The set up for this set of experiments is slightly different from the previous subsection, and the assumptions for this subsection are as follows. Firstly, we consider an additive demand model since Theorem 4.2.3 assumes that model; thus, $D = d + \xi$, where $d = a - bp$. The random variable ξ is normally distributed with mean μ and standard deviation σ. Secondly, demand is time-homogeneous and there is no capacity constraint. The values of model parameters are as follows:

In all the experiments that we conducted, we found that the optimal weights for demand are equal to 0 for the upper bound and 1 for the lower bound. That is, when the

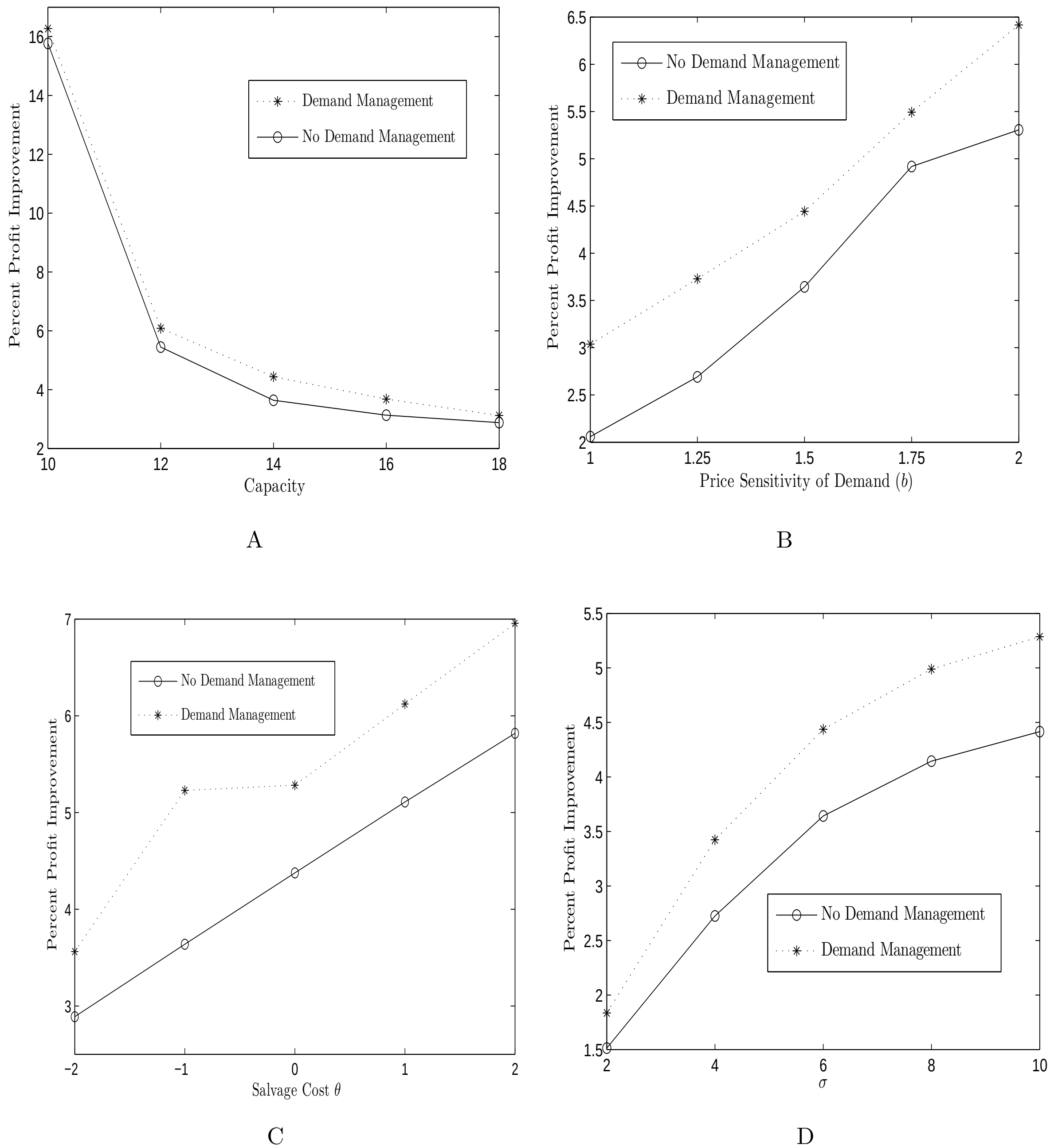


Figure 4-3. Percent profit advantage of FIFO with respect to LIFO in the presence of capacity constraint. A) Different capacity with $b = 1.5, \theta = -1, \sigma = 6$. B) Different price sensitivity of demand with $\theta = -1, \sigma = 6$, capacity=14. C) Different salvage cost with $b = 1.5, \sigma = 6$, capacity=14. D) Different standard deviation with $b = 1.5, \theta = -1$, capacity=14.

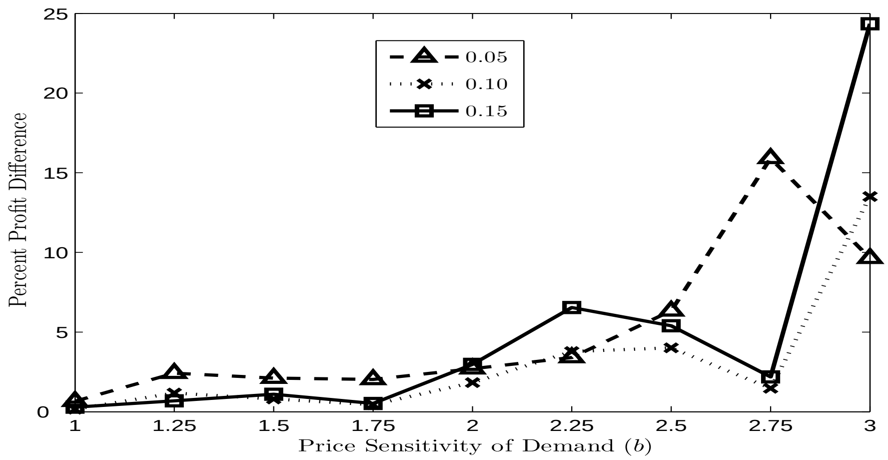
Table 4-2. Parameter values for the approximate policy.

T	c	h	π	s	μ	a
6	5	1	40	1.5	0	40

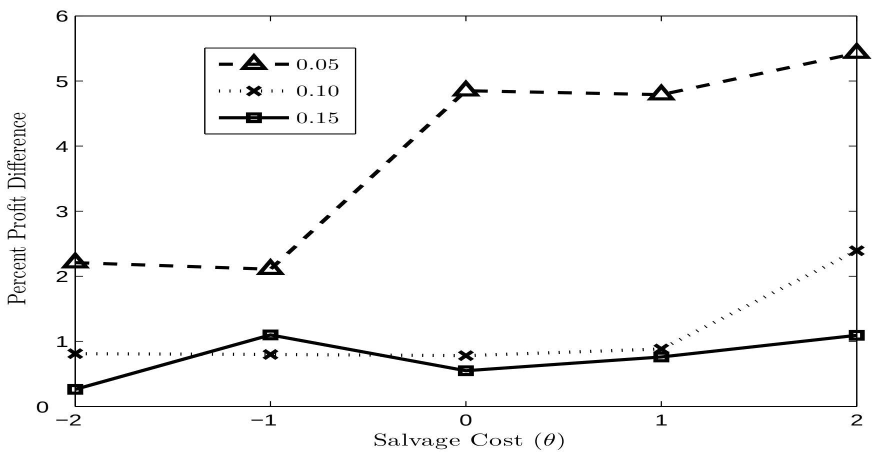
approximate policy is used, the best profit is obtained when expected demand is set equal to the lower bound, which is equal to the maximizer of $R(d) - cd$. On the other hand, the optimal weight for the order quantity is in the range of 0.1-0.15 for the upper bound. In fact, setting it to 0.15 usually gives very good results. In Figure 4-4, we plot the *Percent Profit Difference*, which is defined below, as a function of the weight on the upper bound.

$$\text{Percent Profit Difference} = \frac{\text{Optimal Profit-Profit of Approximate Policy}}{\text{Optimal Profit}} \times 100\%.$$

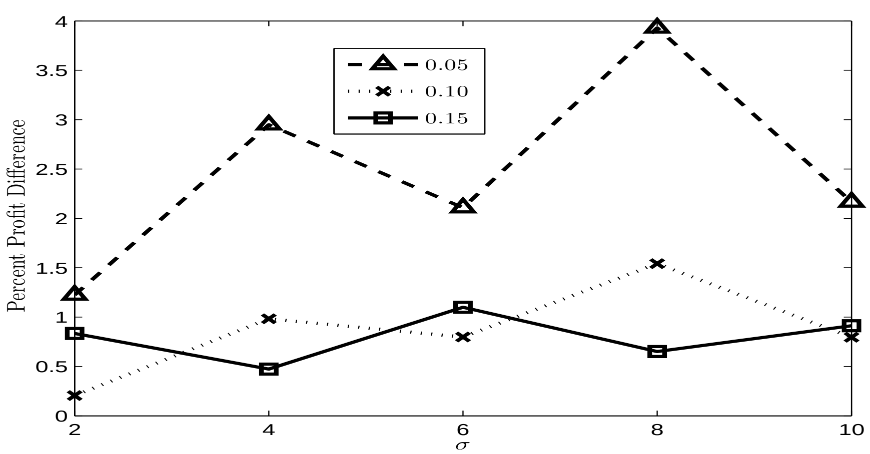
The figure shows that for low values of b, the approximate policy performs quite well. For instance, for $b \leq 2$, the profit of the approximate policy is within 2% of the optimal policy. However, as the value of b increases, the performance of the approximate policy worsens. For instance, for $b = 3$, the *Profit Difference* becomes equal to 10 %. The figure also illustrates that the performance of the policy appears to be relatively insensitive to the other two parameters, θ and σ.



A



B



C

Figure 4-4. Percent profit difference between approximate policy and optimal profit. A) Different price sensitivity of demand with $\theta = 0, \sigma = 6$. B) Different salvage cost with $b = 1.5, \sigma = 6$. C) Different standard deviation with $b = 1.5, \theta = 0$.

CHAPTER 5
CONCLUSIONS AND FUTURE RESEARCH

5.1 In the Presence of Economies of Scale and Age-dependent Demand

We develop three models to identify the optimal inventory renewal and pricing decisions for a perishable product by taking into account the reduction of demand with age. In the first model, the replenishment interval is static, and the price may be varied more frequently. We find that the optimal profit function in this model possesses nice properties, and our analysis sheds light on the properties of the optimal replenishment interval, order quantity and price. Compared to a model in which the same price is used every period, the use of dynamic pricing improves the profit by 2-7% for different combinations of model parameters.

The use of fixed replenishment intervals, although simple and convenient, is rigid. If demand uncertainty is significant, it may be more profitable for the retailer to have flexible replenishment intervals. Our second model captures this situation by deciding about whether to renew inventory at the beginning of each period. Price may still be varied every period. The optimal profit function for this model does not possess several properties that are useful in identifying the characteristics of the optimal decisions. Still, we are able to show that when lifetime is two periods an order is placed if and only if the inventory is too high or too low; otherwise, no order is placed. Our experiments show that this strategy of flexible replenishment may further improve profit by 3-8 %.

We also examine another strategy, the partial salvage strategy, in which some inventory may be salvaged if it is unlikely to be sold before the end of the lifetime or placement of the next order. We find that the optimal profit function is increasing in the inventory. We find that there exists a threshold point such that an order is placed if and only if the inventory is less than the point. The partial salvage strategy, however, contributes little to the profit on top of the flexible replenishment and demand management strategies; the profit improvement is less than 0.1 %.

One open research question is to develop simple price policies for the flexible replenishment and partial salvage models when no order is placed. Our analysis indicates that the relationship between the optimal price and the inventory could be complicated. More research is also needed to better understand the nature of the profit function for the flexible replenishment and partial salvage strategies as well as for simple extensions such as non-zero lead-time.

Our analysis makes one crucial assumption that order placement results in a salvage of the inventory, most likely away from the retail store. Thus, we ignore the likelihood that the retailer may sell old inventory at a reduced price within the store, which may influence the demand of the fresh inventory. The analysis of such a model is currently a work-in-progress.

5.2 Without the Economies of Scale and Age-dependent Demand

When we examine a scenario in which the demand is age-independent and the economies of scale do not exist, we develop a periodic review model to develop insights on the optimal replenishment and demand policies for a perishable product with a fixed lifetime of two periods. We consider two common issuing rules, FIFO and LIFO. For the FIFO rule, our analysis reveals that there exists a fixed threshold, which is the same for all but the last period, of inventory beyond which no order is placed even though demand can be varied. The optimal order quantity, whenever an order is placed, as well as the optimal demand depend on the value of inventory. Since the optimal policy may not have a simple structure, we propose an approximate policy that has a simple structure. Computational experiments indicate that this policy performs within 2% of the optimal profit for a wide range of system parameters.

While the profit function is concave for the FIFO rule, this property is lacking for the LIFO rule. However, the optimal replenishment policy continues to have a similar structure though the lack of a simple structure means that some of the properties that hold for the FIFO rule may not hold for the LIFO rule. One notable result is that when demand is additive, the optimal demand policy has a simpler structure compared to the FIFO; the optimal expected demand is constant whenever an order is placed unlike the FIFO rule, for which the optimal expected demand may depend on the level of inventory.

It is well-understood that the FIFO rule results in less inventory wastage than the LIFO rule and hence generates greater profit. Accordingly, many managers contrive of ways to ensure that inventory is consumed according to the FIFO rule. We conduct computational experiments to explore the extent to which the profit advantage of the FIFO rule is affected in the presence of demand management. In a surprising result, we find that the demand management may widen the profit gap between a FIFO and a LIFO system even more when there is a capacity constraint on the order quantity. However, the gap reduces when demand is time-varying and there is no capacity constraint.

One potential future research direction is the extension of our results to general lifetime. Given that the analysis of the problem in which demand is exogenous is fairly complex, we believe that the extension will not be easy. Another interesting future direction could arise by relaxing the assumption that units of different ages are priced equally. That is, the manager prices older units differently from newer units to maximize profit. Naturally, this research will be particularly useful in a system in which customers select the units, such as in a grocery store. We are currently working on this problem.

APPENDIX A
PROOFS IN CHAPTER 3

Proof of Theorem 3.1.1

Proof of Part 1

We first define $g_t(s,x) = v_t(s,x) - \pi x$ for $x > 0$ and $g_t(s,0) = \lim_{x \to 0^+} v_t(s,x)$. Note that unlike $v_t(s,x)$, $g_t(s,x)$ is a continuous function. With this definition,

$$g_t(s,x) = \begin{cases} \max_{d \in \mathcal{D}} \{L_s(x,d) - \pi x + E[g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+) \\ +\pi(x-d\xi_1-\xi_2)^+]\}, s < R, x > 0, \\ \lim_{x \to 0^+} v_t(s,x), \quad x = 0, \\ (w_R - \pi)x + \max_{d \in \mathcal{D}_0, y \geq 0} \{L_0(y,d) - K\delta(y) - cy + E[g_{t+1}(1,(y-d\xi_1-\xi_2)^+)] \\ +E[\pi(y-d\xi_1-\xi_2)]^+\}, \quad s = R. \end{cases}$$

Let

$$\begin{aligned} G_t(s,x,d) &= L_s(x,d) - \pi x + E[g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+) + \pi(x-d\xi_1-\xi_2)^+], \\ & \quad x > 0, s < R. \end{aligned} \tag{A.0.1}$$

By this definition, $g_t(s,x) = \max_{d \in \mathcal{D}} G_t(s,x,d)$ for $s < R$ and $x > 0$.

We will prove the result by induction on t. Since all the order cycles are probabilistically identical, it is enough to establish the result for one cycle. Accordingly, we prove this result for the first cycle which consists of periods 1 through R. We first show that $g_R(R-1,x)$ is a concave function of x. To accomplish this objective, it is enough to show that $G_R(R-1, x, d)$ is jointly concave in x and d (Boyd and Vandenberghe (2004)).

$$\begin{aligned} G_R(R-1,x,d) &= L_s(x,d) - \pi x + E[g_{R+1}(R,(x-d\xi_1-\xi_2)^+)] + \pi E(x-d\xi_1-\xi_2)^+ \\ &= dD_{R-1}^{-1}(d) - hE[x-d\xi_1-\xi_2]^+ - \pi E[d\xi_1+\xi_2-x]^+ - \pi x \\ &\quad +(w_R-\pi)E[x-d\xi_1-\xi_2]^+ + \pi E[x-d\xi_1-\xi_2]^+ \\ &= dD_{R-1}^{-1}(d) - (h-w_R+\pi)E[x-d\xi_1-\xi_2]^+ + \pi E[x-d\xi_1-\xi_2] - \pi x \\ &= dD_s^{-1}(d) - (h-w_R+\pi)E[x-d\xi_1-\xi_2]^+ - \pi E[d\xi_1+\xi_2]. \qquad \text{(A.0.2)} \end{aligned}$$

Clearly, the above expression is jointly concave in x and d for $\pi > w_R$ for any R.

Next, suppose that $g_k(s,x)$ is non-increasing and concave in x for $k = t+1, t+2, \ldots, R$ and consider $k = t$. Once again, consider

$$\begin{aligned} G_t(s,x,d) &= L_s(x,d) + \pi E[(x-d\xi_1-\xi_2)^+] - \pi x + E[g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)] \\ &= dD_s^{-1}(d) - hE[x-d\xi_1-\xi_2]^+ - \pi E[d\xi_1+\xi_2] \\ &\quad +E[g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)]. \qquad \text{(A.0.3)} \end{aligned}$$

As above, it is enough to show that $G_t(s,x,d)$ is jointly concave in x and d. To prove this, it is sufficient to show that $E[g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)]$ is jointly concave in x and d. By induction assumption, $g_{t+1}(s+1,x)$ is a non-increasing concave function of x. Consider any two feasible points (x_1,d_1) and (x_2,d_2). For any given realizations of ξ_1 and ξ_2 and any $\lambda \in (0,1)$,

$$\begin{aligned} & g_{t+1}(s+1,[(\lambda x_1+(1-\lambda)x_2)-(\lambda d_1+(1-\lambda)d_2)\xi_1-\xi_2]^+) \\ \geq\ & g_{t+1}(s+1,\lambda[x_1-d_1\xi_1-\xi_2]^+ + (1-\lambda)[x_2-d_2\xi_1-\xi_2]^+) \\ \geq\ & \lambda g_{t+1}(s+1,[x_1-d_1\xi_1-\xi_2]^+) + (1-\lambda)g_{t+1}(s+1,[x_2-d_2\xi_1-\xi_2]^+). \end{aligned}$$

The first inequality holds due to the non-increasing nature of g_{t+1}. The second inequality holds since $g_{t+1}(s,\cdot)$ is concave. Thus, $g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)$ is jointly concave in x and d.

Next, we establish that $g_t(s,x)$ is non-increasing in x. Let $x_1 \geq x_2$. Since $g_{t+1}(s+1,x)$ is a non-increasing function of x and $-E[x_2 - d\xi_1 - \xi_2]^+ \geq -E[x_1 - d\xi_1 - \xi_2]^+$, $G_t(s,x_1,d) \leq G_t(s,x_2,d)$. Now,

$$\begin{aligned} g_t(s,x_2) = G_t(s,x_2,d^*(x_2)) &\leq G_t(s,x_1,d^*(x_2)) \\ &\leq G_t(s,x_1,d^*(x_1)) = g_t(s,x_1), \end{aligned}$$

where $d^*(x_i) = \arg\max_{d \in \mathcal{D}} G_t(s,x_i,d)$ for $i = 1,2$.

Finally, we discuss the case when $s = R$, which is same as the first period of a new cycle. For any cycle, since $g_{R+1}(R,x)$ is equal to a constant plus $(w_R - \pi)x$, it is a non-increasing concave function of x.

Proof of Part 2

Consider $G_t(s,x,d)$ as defined in Part (a) (Equations A.0.2 and A.0.3). We first prove that $G_t(s,x,d)$ is supermodular for any given s. A function $F(u,v)$ is supermodular if $F(u_1,v) - F(u_2,v)$ is non-decreasing in v for $u_1 > u_2$ (Heyman and Sobel (1984)).

We only show the result for $s < R-1$; the result for $s = R-1$ can be proved similarly and the details are omitted. It is sufficient to establish the supermodularity in each term on the RHS of Equation A.0.3 since the sum of supermodular functions is also a supermodular function. Now, suppose $x_1 > x_2$ and $d_1 > d_2$. The first and third terms in Equation A.0.3 are supermodular since they only depend on d. Consider now the second term $-hE[x - d\xi_1 - \xi_2]^+$. For any realization of ξ_1 and ξ_2,

$$\begin{aligned} [x_1 - d_1\xi_1 - \xi_2]^+ - [x_2 - d_1\xi_1 - \xi_2]^+ &= [x_2 - d_1\xi_1 - \xi_2 + (x_1 - x_2)]^+ - [x_2 - d_1\xi_1 - \xi_2]^+ \\ &\leq [x_2 - d_2\xi_1 - \xi_2 + (x_1 - x_2)]^+ - [x_2 - d_2\xi_1 - \xi_2]^+ \\ &= [x_1 - d_2\xi_1 - \xi_2]^+ - [x_2 - d_2\xi_1 - \xi_2]^+, \end{aligned}$$

which means that $-h[x - d\xi_1 - \xi_2]^+$ is a supermodular function for any given ξ_1 and ξ_2, implying that $-E[x - d_1\xi_1 - \xi_2]^+$ is also a supermodular function since supermodularity holds under expectation.

We use the same logic for the remaining term, $E[g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)]$. We showed in Part (a) that $g_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)$ is jointly concave in x and d for any realizations of ξ_1 and ξ_2. Therefore,

$$\begin{aligned} & g_{t+1}(s+1,(x_1-d_1\xi_1-\xi_2)^+)-g_{t+1}(s+1,(x_2-d_1\xi_1-\xi_2)^+) \\ = \; & g_{t+1}(s+1,(x_2-d_1\xi_1-\xi_2+(x_1-x_2))^+)-g_{t+1}(s+1,(x_2-d_1\xi_1-\xi_2)^+) \\ \geq \; & g_{t+1}(s+1,(x_1-d_2\xi_1-\xi_2)^+)-g_{t+1}(s+1,(x_2-d_2\xi_1-\xi_2)^+). \end{aligned}$$

where the inequality follows from the concavity of g_{t+1}. Therefore, we conclude the supermodularity of G_t, which implies that $G_t(s,x_1,d_1)-G_t(s,x_2,d_1) \geq G_t(s,x_1,d_2)-G_t(s,x_2,d_2)$.

Now, suppose by way of contradiction that for $x_1 > x_2$, $d^*(x_2) > d^*(x_1)$ where $d^*(x_i) = \arg\max_{d\in\mathcal{D}} G_t(s,x_i,d)$. Then,

$$G_t(s,x_1,d^*(x_2))-G_t(s,x_2,d^*(x_2)) \geq G_t(s,x_1,d^*(x_1))-G_t(s,x_2,d^*(x_1)).$$

Rearranging terms, we get

$$G_t(s,x_2,d^*(x_1))-G_t(s,x_2,d^*(x_2)) \geq G_t(s,x_1,d^*(x_1))-G_t(s,x_1,d^*(x_2)).$$

But this is a contradiction since $G_t(s,x_2,d^*(x_1)) - G_t(s,x_2,d^*(x_2)) < 0$ and $G_t(s,x_1,d^*(x_1)) - G_t(s,x_1,d^*(x_2)) > 0$. (The inequalities are strict due to the strict concavity of G_t in d due to which $G_t(s,x,d_1) \neq G_t(s,x,d_2)$ for $d_1 \neq d_2$.) As a result, $d^*(x_1) \geq d^*(x_2)$ when $x_1 > x_2$.

Proof of Part 3

By assumption, y^* will automatically be greater than zero, so the constraint $y \geq 0$ may be eliminated from the formulation since.

Let $r = y - d$.

$$
\begin{aligned}
v_t(s,x) &= w_R x + \max_{d\in\mathcal{D},r}\{-K - c(r+d) + dD_0^{-1}(d) - hE[r-\xi_2]^+ - \pi E[\xi_2 - r]^+ \\
&\quad + E[v_{t+1}(1,(r-\xi_2)^+)]\} \\
&= w_R x - K + \max_{d\in\mathcal{D}}\{-cd + dD_0^{-1}(d)\} + \max_{r}\{-cr - hE[r-\xi_2]^+ - \pi E[\xi_2 - r]^+ \\
&\quad + E[v_{t+1}(1,(r-\xi_2)^+)]\}
\end{aligned}
$$

Thus, the optimal value of d is obtained by maximizing $d(D_0^{-1}(d) - c)$ over $d \in \mathcal{D}$.

Proof of Proposition 3.1.2

When $\xi_1 \equiv 1$, $\xi_2 \equiv 0$ and all demand is satisfied, the optimal profit over a cycle of length R is given by

$$
\begin{aligned}
V(R) &= \max_{y\geq 0, d_0, d_1, \ldots, d_{R-1}\in\mathcal{D}} -cy + \sum_{k=0}^{R-1} d_k D_k^{-1}(d_k) - h(y-d_0) - \ldots \\
&\quad -h(y - d_0 - d_1 - \ldots - d_{R-1}) + w_R(y - d_0 - d_1 - \ldots - d_{R-1}),
\end{aligned}
$$

where d_k is the demand satisfied in period $k+1$. Observe that since the demand is deterministic, the optimal demand for each period may be computed at the beginning of each cycle.

Firstly, observe that the optimal value of y is equal to the sum of the optimal demands over the cycle. Since all demand must be satisfied, y cannot be less than the sum of the optimal demands. To see the other side, suppose the optimal value of y, y^*, is greater than the sum of the optimal demands. If we now reduce y^* by $\delta > 0$ such that $y^* - \delta$ is still at least the sum of the optimal demands, the profit will increase by $(c + Rh - w_R)\delta > 0$, which is a contradiction.

Taking this observation into consideration, we now revise the above formulation as follows:

$$\begin{aligned} V(R) &= \max_{d_0,d_1,\ldots,d_{R-1}\in\mathcal{D}} \sum_{k=0}^{R-1} d_k(D_k^{-1}(d_k) - c) - hd_1 - 2hd_2 - \ldots - (R-1)hd_{R-1} \\ &= \max_{d_0,d_1,\ldots,d_{R-1}\in\mathcal{D}} \sum_{k=0}^{R-1} d_k(D_k^{-1}(d_k) - c - kh). \end{aligned}$$

The above problem is separable in $d_0, d_1, \ldots, d_{R-1}$, and the optimal value of d_k is found by solving $\max_{d_k\in\mathcal{D}} d_k(D_k^{-1}(d_k) - c - kh)$. Since $\frac{\partial}{\partial d} dD_k^{-1}(d) \le \frac{\partial}{\partial d} dD_{k-1}^{-1}(d)$, $\frac{\partial}{\partial d} d(D_k^{-1}(d) - c - kh) < \frac{\partial}{\partial d} d(D_{k-1}^{-1}(d) - c - (k-1)h)$. This implies that $d_k^* < d_{k-1}^*$ for $k = 1, 2, \ldots, R-1$. As a result, the increase in the cycle order quantity decreases as R increases.

Further,

$$\begin{aligned} V(R) - V(R-1) &= \int_0^{d_{R-1}^*} \left(\frac{\partial}{\partial d} d(D_{R-1}^{-1}(d) - c - (R-1)h) \right) dd \\ &\le \int_0^{d_{R-2}^*} \left(\frac{\partial}{\partial d} d(D_{R-2}^{-1}(d) - c - (R-2)h) \right) dd = V(R-1) - V(R-2), \end{aligned}$$

which establishes the concavity of $V(\cdot)$.

Proof of Proposition 3.2.1

Proof of Part 1

If Strategy 1 is followed in period t, then Strategy 2 is certain to be followed in period 2 given the length of the lifetime to be 2 periods. Also, Strategy 1 can only be used when the age is equal to 1. Therefore,

$$\begin{aligned} v_t^1(1,x) &= \max_{d\in\mathcal{D}} L_1(x,d) + Ev_{t+1}^2(2, (x - d\xi_1 - \xi_2)+) \\ &= \max_{d\in\mathcal{D}} L_1(x,d) + w_2 E(x - d\xi_1 - \xi_2) + . \end{aligned} \tag{A.0.4}$$

As in the proof of Theorem 3.1.1, the maximand in the above expression can be easily shown to be jointly concave in x and d. As a consequence, there exists a unique value of the optimal d for any given x.

Proof of Part 2

Since the maximand is jointly concave in x and d in A.0.4, $v_t^1(1,x)$ is a concave function of x. Further, $v_t^2(1,x)$ is a linear and non-decreasing function of x. Therefore, either $v_t^2(1,x)$ is greater than $v_t^1(s,x)$ for all x or the two functions intersect at exactly two points, x_t^l and x_t^u. In this case, it is optimal to employ Strategy 2 for all x less than x_t^l as well as for all x greater than x_t^u.

Proof of Proposition 3.2.2

Proof of Part 1

Since $D_s^{-1}(d) \leq D_{s-1}^{-1}(d)$,

$$\begin{aligned} L_s(x,d) &= dD_s^{-1}(d) - hE[x - d\xi_1 - \xi_2]^+ - \pi E[d\xi_1 + \xi_2 - x]^+ \\ &\leq dD_{s-1}^{-1}(d) - hE[x - d\xi_1 - \xi_2]^+ - \pi E[d\xi_1 + \xi_2 - x]^+ \\ &= L_{s-1}(x,d) \end{aligned}$$

Proof of Part 2

The result is established by induction. When $t = T+1$,

$$v_{T+1}(s,x) = w_s x \leq w_{s-1}x = v_{T+1}(s-1,x)$$

Suppose now that the result holds for periods $t+1, t+2, \ldots, T$. This implies that $v_{t+1}(s,x) \leq v_{t+1}(s-1,x)$. Consider period t such that $s < S$. From Part (1) and the induction assumption, for any given d,

$$L_s(x,d) + E[v_{t+1}(s+1,(x - d\xi_1 - \xi_2)^+)] \leq L_{s-1}(x,d) + E[v_{t+1}(s,(x - d\xi_1 - \xi_2)^+)].$$

If $d_s^*(x)$ maximizes $L_s(x, d_s^*(x)) + E[v_{t+1}(s+1,(x - d_s^*(x)\xi_1 - \xi_2)^+)]$, then

$$\begin{aligned} v_t^1(s,x) &= L_s(x,d_s^*(x)) + E[v_{t+1}(s+1,(x - d_s^*(x)\xi_1 - \xi_2)^+)] \\ &\leq L_{s-1}(x,d_s^*(x)) + E[v_{t+1}(s,(x - d_s^*(x)\xi_1 - \xi_2)^+)] \\ &\leq \max_{d\in\mathcal{D}}\{L_{s-1}(x,d) + E[v_{t+1}(s,(x - d\xi_1 - \xi_2)^+)]\} \\ &= v_t^1(s-1,x). \end{aligned}$$

On the other hand,

$$\begin{aligned}
v_t^2(s,x) &= w_s x + \max_{d\in\mathcal{D},y\geq 0}\{-K\delta(y) + L_0(y,d) - cy + E[v_t(1,(y-d\xi_1-\xi_2)^+)]\} \\
&\leq w_{s-1}x + \max_{d\in\mathcal{D},y\geq 0}\{-K\delta(y) + L_0(y,d) - cy + E[v_t(1,(y-d\xi_1-\xi_2)^+)]\} \\
&= v_t^2(s-1,x),
\end{aligned}$$

where the inequality follows since $w_s \leq w_{s-1}$. Therefore,

$$\begin{aligned}
v_t(s,x) &= \max\{v_t^1(s,x), v_t^2(s,x)\} \\
&\leq \max\{v_t^1(s-1,x), v_t^2(s-1,x)\} \\
&= v_t(s-1,x).
\end{aligned}$$

Finally, when $s = S, v_t(s,x) = v_t^2(s,x)$. This can be shown in the same manner as the case above in which we showed $v_t^2(s,x) \leq v_t^2(s-1,x)$.

Proof of Part 3

We prove by induction. Consider first $t = T$. For notational clarity, we add another argument, S, to v_t, v_t^1 and v_t^2 to indicate that the lifetime of the product is S periods.

Case 1: If $s = S$, then

$$\begin{aligned}
v_T(s,x,S) &= v_T^2(s,x,S) \\
&= w_s x + \max_{d\in\mathcal{D},y\geq 0}\{-K\delta(y) + L_0(y,d) - cy + w_1 E[y-d\xi_1-\xi_2]^+\} \\
&= v_T^2(s,x,S+1) \\
&\leq v_T(s,x,S+1)
\end{aligned}$$

Case 2: If $s < S$, $v_T^i(s,x,S) = v_T^i(s,x,S+1)$ for $i = 1,2$. To see this,

$$\begin{aligned}
v_T(s,x,S) &= \max\{v_T^1(s,x,S), v_T^2(s,x,S)\} \\
&= \max\{v_T^1(s,x,S+1), v_T^2(s,x,S+1)\} \\
&= v_T(s,x,S+1)
\end{aligned}$$

Thus, $v_T(s,x,S) \leq v_T(s,x,S+1)$ for $s \leq S$.

Suppose now that the result is true for periods $t+1$ and beyond. This implies that $v_{t+1}(s,x,S) \leq v_{t+1}(s,x,S+1)$ for $s \leq S$. Consider period t.

Case 1: If $s < S$, then

$$\begin{aligned}
v_t^1(s,x,S) &= \max_{d\in\mathcal{D}}\{L_s(x,d) + E[v_{t+1}(s+1,(x-d\xi_1-\xi_2)^+,S)]\} \\
&\leq \max_{d\in\mathcal{D}}\{L_s(x,d) + E[v_{t+1}(s+1,(x-d\xi_1-\xi_2)^+,S+1)]\} \\
&= v_t^1(s,x,S+1).
\end{aligned}$$

Similarly,

$$\begin{aligned}
v_t^2(s,x,S) &= w_s x + \max_{d\in\mathcal{D},y\geq 0}\{-K\delta(y) + L_0(y,d) - cy + E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+,S)]\} \\
&\leq w_s x + \max_{d\in\mathcal{D},y\geq 0}\{-K\delta(y) + L_0(y,d) - cy + E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+,S+1)]\} \\
&= v_t^2(s,x,S+1).
\end{aligned}$$

Now,

$$\begin{aligned}
v_t(s,x,S) &= \max\{v_t^1(s,x,S), v_t^2(s,x,S)\} \\
&\leq \max\{v_t^1(s,x,S+1), v_t^2(s,x,S+1)\} \\
&= v_t(s,x,S+1).
\end{aligned}$$

Case 2: If $s = S$, then

$$v_t(s,x,S) = v_t^2(s,x,S) = v_t^2(s,x,S+1) \leq v_t(s,x,S+1).$$

Proof of Theorem 3.2.3

Proof of Part 1

Since $v_t^2(s,0) - K > v_t^1(s,0)$, $v_t^2(s,0) > v_t^1(s,0)$.

Next, we prove that $v_t^1(s,x) < v_t^2(s,x)$ for x large enough using induction. Suppose this is true for period $t+1, t < T$, that is,

$$v_{t+1}^1(s,x) \le v_{t+1}^2(s,x) \text{ for } x \ge x_{t+1}^u,$$

where $x_{t+1}^u < \infty$. Since $d\xi_1 + \xi_2 \le M$ for some M,

$$\begin{aligned}
v_t^1(s,x) &= \max_{d\in\mathcal{D}}\{L_s(x,d) + E[v_{t+1}(s+1,(x-d\xi_1-\xi_2)^+)]\} \\
&= \max_{d\in\mathcal{D}}\{dD_s^{-1}(d) - hE[x-d\xi_1-\xi_2] + E[v_{t+1}(s+1,(x-d\xi_1-\xi_2))]\}, \text{ for } x \ge M \\
&= \max_{d\in\mathcal{D}}\{dD_s^{-1}(d) - hE[x-d\xi_1-\xi_2] + E[v_{t+1}^2(s+1,(x-d\xi_1-\xi_2))]\}, \\
&\quad \text{for } x \ge x_{t+1}^u + M
\end{aligned}$$

This implies that for large values of x,

$$\begin{aligned}
\frac{\partial}{\partial x}[v_t^1(s,x)] &= -h + w_{s+1} \\
&< w_s \\
&= \frac{\partial}{\partial x}[v_t^2(s,x)].
\end{aligned}$$

Since $v_t^2(s,0) > v_t^1(s,0)$ and $v_t^1(s,x) \le v_t^2(s,x)$ for large values of x, $v_t^1(s,\cdot)$ and $v_t^2(s,\cdot)$ must cross at least twice provided they cross at least once. The smallest and largest such crossing points are $x_t^l(s)$ and $x_t^u(s)$.

Proof of Part 2

Since the salvage value is independent of s,

$$v_t^2(s,x) = v_t^2(s-1,x).$$

From Proposition 3.2.2, we also know that

$$v_t^1(s,x) \le v_t^1(s-1,x).$$

This shows that $x_t^l(s-1) \leq x_t^l(s)$ and $x_t^u(s-1) \geq x_t^u(s)$ provided x_t^l and x_t^u exist for $s-1$ and s.

Proof of Proposition 3.2.4

To obtain the first bound, we consider the case in which Strategy 2 is followed in period $t+1$. Thus,

$$B_t^1(s,x) \;=\; \max_{d \in \mathcal{D}} \{L_s(x,d) + E[v_{t+1}^2(s+1,(x-d\xi_1-\xi_2)^+)]\},$$

which can be represented as

$$B_t^1(s,x) \;=\; \max_{d \in \mathcal{D}} \{L_s(x,d) + E[w_{s+1}(x-d\xi_1-\xi_2)^+]\} + \text{Constant}.$$

Clearly, the maximand on the RHS is jointly concave in x and d. As a result, $B_t^1(s,\cdot)$ is concave. Thus, it intersects $v_t^2(s,x)$ at either none or two points. When it intersects at two points, the two points, (l_t^1 and u_t^1), will be an upper bound on $x_t^l(s)$ and a lower bound on $x_t^u(s)$, respectively.

To obtain another bound, suppose that Strategy 1 is followed in period $t+1$ and Strategy 2 is followed in period $t+2$. Further, the expected demand for period $t+1$ is determined in period t. For ease of notation, we add superscripts t and $t+1$ to ξ_1 and ξ_2 to indicate which period they correspond to. Thus,

$$\begin{aligned}
B_t^2(s,x) &= \max_{d_t, d_{t+1} \in \mathcal{D}} \{L_s(x,d_t) + E[L_{s+1}((x-d_t\xi_1^t-\xi_2^t)^+, d_{t+1})] \\
&\quad + E[v_{t+2}^2(s+2,(x-d_t\xi_1^t-\xi_2^t-d_{t+1}\xi_1^{t+1}-\xi_2^{t+1})^+)]\} \\
&= \max_{d_t, d_{t+1} \in \mathcal{D}} \{L_s(x,d_t) + E[L_{s+1}((x-d_t\xi_1^t-\xi_2^t)^+, d_{t+1})] \\
&\quad + w_{s+2}E[(x-d_t\xi_1^t-\xi_2^t-d_{t+1}\xi_1^{t+1}-\xi_2^{t+1})^+] + \text{Constant}\}.
\end{aligned}$$

The maximand on the RHS of the last expression can be easily shown to be jointly concave in d_t, d_{t+1} and x. Thus, $B_t^2(s,x)$ is concave in x. As above, $B_t^2(s,x)$ intersects $v_t^2(s,x)$ at either none or two points. When it intersects at two points, the two points, (l_t^2 and u_t^2), will be an upper bound on $x_t^l(s)$ and a lower bound on $x_t^u(s)$, respectively.

Proof of Proposition 3.3.2

Proof of Part 1

Let $x_1 < x_2$.

$$\begin{aligned} v_t^1(s, x_1) &= \max_{0 \le q \le x_1, d \in \mathcal{D}} \{Q_s(x_1, q, d) + E[v_{t+1}(s+1, (q - d\xi_1 - \xi_2)^+)]\} \\ &\le \max_{0 \le q \le x_1, d \in \mathcal{D}} \{Q_s(x_2, q, d) + E[v_{t+1}(s+1, (q - d\xi_1 - \xi_2)^+)]\} \\ &\le \max_{0 \le q \le x_2, d \in \mathcal{D}} \{Q_s(x_2, q, d) + E[v_{t+1}(s+1, (q - d\xi_1 - \xi_2)^+)]\} \\ &= v_t^1(s, x_2), \end{aligned}$$

where the first inequality is valid since $Q_s(x_1, q, d) \le Q_s(x_2, q, d)$ using Corollary 3.3.1. The second inequality is valid since the feasible space for q is now larger.

Proof of Part 2

Suppose the result holds for period $t+1$. This means that $v_{t+1}(s, x) \le v_{t+1}(s-1, x)$. Then in period t for $s < S$,

$$\begin{aligned} v_t^1(s, x) &= \max_{0 \le q \le x, d \in \mathcal{D}} \{Q_s(x, q, d) + E[v_{t+1}(s+1, (q - d\xi_1 - \xi_2)^+)]\} \\ &\le \max_{0 \le q \le x, d \in \mathcal{D}} \{Q_{s-1}(x, q, d) + E[v_{t+1}(s, (q - d\xi_1 - \xi_2)^+)]\} \\ &= v_t^1(s-1, x) \end{aligned}$$

where the inequality follows since $Q_s \le Q_{s-1}$. Now,

$$\begin{aligned} v_t^2(s, x) &= w_s x + \max_{d \in \mathcal{D}, y \ge 0} \{-K\delta(y) - cy + Q_0(y, y, d) + E[v_{t+1}(1, (y - d\xi_1 - \xi_2)^+)]\} \\ &\le w_{s-1} x + \max_{d \in \mathcal{D}, y \ge 0} \{-K\delta(y) - cy + Q_0(y, y, d) + E[v_{t+1}(1, (y - d\xi_1 - \xi_2)^+)]\} \\ &= v_t^2(s-1, x). \end{aligned}$$

When $s = S$,

$$
\begin{aligned}
v_t(s,x) &= v_t^2(s,x) \\
&= w_s x + \max_{d\in\mathcal{D}, y\geq 0}\{-K\delta(y) - cy + Q_0(y,y,d) + E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+)]\} \\
&\leq w_{s-1} x + \max_{d\in\mathcal{D}, y\geq 0}\{-K\delta(y) - cy + Q_0(y,y,d) + E[v_{t+1}(1,(y-d\xi_1-\xi_2)^+)]\} \\
&= v_t^2(s-1,x) \\
&\leq v_t(s-1,x).
\end{aligned}
$$

Therefore, $v_t(s,x) \leq v_t(s-1,x)$ for $s \leq S$.

Proof of Theorem 3.3.3

If $q^* \leq x$, then

$$g_t^1(s,x) = g_t^1(s,q^*).$$

Since the profit is maximized by maintaining an on-hand inventory equal to q^*, it is optimal to salvage a quantity equal to $x - q^*$.

Now, since $M \leq g_t^1(s,q^*)$, there exists $x_t^l(s) \leq q^*$ such that $g_t^2(s,x_t^l(s)) = g_t^1(s,x_t^l(s))$. When $x \leq x_t^l(s)$, using Strategy 2 is optimal. Otherwise, Strategy 1 is chosen.

We prove the uniqueness of $x_t^l(s)$ by contradiction. Suppose no order is placed when the on-hand inventory is x_1 but an order is placed when the on-hand inventory level is x_2, where $x_t^l(s) \leq x_1 < x_2$. Then

$$
\begin{aligned}
g_t^1(s,x_1) &\geq g_t^2(s,x_1) \\
&= g_t^2(s,x_2) > g_t^1(s,x_2)
\end{aligned}
$$

We also know that

$$g_t^1(s,x_1) = \max_{d\in\mathcal{D}, 0\leq q\leq x_1} J(d,q) \leq \max_{d\in\mathcal{D}, 0\leq q\leq x_2} J(d,q) = g_t^1(s,x_2)$$

which is a contradiction. Thus, the uniqueness of $x_t^l(s)$ is established.

Proof of Proposition 3.3.4

When $\xi_1 \equiv 1$, then

$$
\begin{aligned}
v_t^1(s,x) &= \max_{0\le z\le x, d\in\mathcal{D}} \{L_s(x,z,d) + E[v_{t-1}(s+1,(x-z-d-\xi_2)^+)]\} \\
v_t^2(s,x) &= w_s x + \max_{d\in\mathcal{D}, y\ge 0} \{-K\delta(y) - cy + L_0(y,0,d) + E[v_{t+1}(1,(y-d-\xi_2)^+)]\}
\end{aligned}
$$

Proof of Part 1

The proof is identical to the proof of Theorem 3.1.1, Part 3.

Proof of Part 2

For Strategy 1, let $q = x - z$. Thus,

$$
\begin{aligned}
v_t^1(s,x) &= \max_{0\le q\le x, d\in\mathcal{D}} \{dD_s^{-1}(d) + w_s(x-q) - hE[q-d-\xi_2]^+ - \pi E[d+\xi_2-q]^+ \\
&\quad +E[v_{t+1}(s+1,(q-d-\xi_2)^+)]\}. \qquad \text{(A.0.5)}
\end{aligned}
$$

Using the same logic as in Part 1, the optimal q will be greater than or equal to zero.

We want to show that whenever the optimal q over $[0,x]$, $\bar{q}$, is strictly less than x (and so the difference of x and $\bar{q}$ is salvaged), the optimal demand is determined by solving $\max_{d\in\mathcal{D}}\{dD_s^{-1}(d) - w_s d\}$. Let the optimal solution to $\max_{d\in\mathcal{D}}\{dD_s^{-1}(d) - w_s d\}$ be equal to d^*.

Suppose by way of contradiction, the optimal demand $\bar{d}$ is not equal to d^* even though $\bar{q} \in (0,x)$. Without loss of generality, let $\bar{d} < d^*$. Increase the value of both q and d by $\delta > 0$ to $\bar{q}+\delta$ and $\bar{d}+\delta$ such that $\delta \le \min(x-\bar{q}, d^*-\bar{d})$. With this change, the maximand in Equation A.0.5 changes by $(\bar{d}+\delta)D_s^{-1}(\bar{d}+\delta) - (\bar{d})D_s^{-1}(\bar{d}) - w_s\delta$, which is identical to the change in the value of $dD_s^{-1}(d) - w_s d$ when d is increased from $\bar{d}$ to $\bar{d}+\delta$. Since $dD_s^{-1}(d) - w_s d$ is concave in d and $\bar{d} < d^*$, the expression $(\bar{d}+\delta)D_s^{-1}(\bar{d}+\delta) - (\bar{d})D_s^{-1}(\bar{d}) - w_s\delta$ is strictly positive, which contradicts our initial assertion that $\bar{q}$ and $\bar{d}$ are optimal.

APPENDIX B
PROOFS IN CHAPTER 4

Proof of Theorem 4.2.2

In the proof, for brevity we omit the dependence of $q^*(x)$ and $d^*(x)$ on x unless necessary for exposition. Note also that since $v'_{T+1}(0)$ is not defined, we take it to be equal to its right derivative, which is equal to s.

Proof of Part 1

We prove all the results by induction. We first establish the result for period T. Note that

$$\begin{aligned} G_T(x,q,d) &= R(d) - cq - h\int_{-a}^{x+q-d}(x+q-d-\xi)f(\xi)d\xi - \pi\int_{x+q-d}^{\infty}(\xi+d-x-q)f(\xi)d\xi \\ &\quad -\theta\int_{-a}^{x-d}(x-d-\xi)f(\xi)d\xi + \alpha s\int_{-a}^{x-d}qf(\xi)d\xi \\ &\quad +\alpha s\int_{x-d}^{x+q-d}(x+q-d-\xi)f(\xi)d\xi - \alpha c\int_{x+q-d}^{\infty}(d+\xi-x-q)f(\xi)d\xi, \end{aligned}$$

where we use the definition of $v_{T+1}(x) = sx^+ - cx^-$. We have represented partial expectations in the integral form for ease of computation, and we will pursue this approach throughout the proof. Now,

$$\frac{\partial G_T(x,q,d)}{\partial q} = -c(1-\alpha) + \pi - (h+\pi-\alpha s+\alpha c)F(x+q-d), \tag{B.0.1}$$

$$\begin{aligned} \frac{\partial G_T(x,q,d)}{\partial d} &= R'(d) - \alpha c + (h+\pi-\alpha s+\alpha c)F(x+q-d) - \pi \\ &\quad +(\theta+\alpha s)F(x-d), \end{aligned} \tag{B.0.2}$$

and so

$$\begin{aligned} \frac{\partial^2 G_T(x,q,d)}{\partial q^2} &= -(h+\pi-\alpha s+\alpha c)f(x+q-d) \le 0. \\ \frac{\partial^2 G_T(x,q,d)}{\partial d^2} &= R''(d) - (h+\pi-\alpha s+\alpha c)f(x+q-d) - (\theta+\alpha s)f(x-d) \le 0. \\ \frac{\partial^2 G_T(x,q,d)}{\partial d\partial q} &= (h+\pi-\alpha s+\alpha c)f(x+q-d). \end{aligned}$$

It can be easily shown that the determinant of the Hessian matrix of $G_T(x,q,d)$ is non-negative; the details are omitted. Further, since $\frac{\partial^2 G_T(x,q,d)}{\partial d^2} \leq 0$ and $\frac{\partial^2 G_T(x,q,d)}{\partial q^2} \leq 0$, $G_T(x,q,d)$ is jointly concave in q and d for $q \geq 0$ and $d \in \mathcal{D}$.

Suppose now that the result is true for periods $t+1, \cdots, T$, and consider period t. Then

$$\begin{aligned}
\frac{\partial G_t(x,q,d)}{\partial q} &= -c - h[F(x+q-d)] + \pi[1-F(x+q-d)] \\
&\quad +\alpha \int_{x-d}^{\infty} v'_{t+1}(x+q-d-\xi) f(\xi) d\xi + \alpha \int_{-a}^{x-d} v'_{t+1}(q) f(\xi) d\xi \quad \text{(B.0.3)} \\
\frac{\partial^2 G_t(x,q,d)}{\partial q^2} &= -(h+\pi) f(x+q-d) + \alpha \int_{x-d}^{\infty} v''_{t+1}(x+q-d-\xi) f(\xi) d\xi \\
&\quad +\alpha v''_{t+1}(q) F(x-d),
\end{aligned}$$

which is non-negative since $v''_{t+1} \leq 0$ by induction hypothesis. Also,

$$\begin{aligned}
\frac{\partial G_t(x,q,d)}{\partial d} &= R'(d) + (h+\pi) F(x+q-d) - \pi + \theta F(x-d) \\
&\quad -\alpha \int_{x-d}^{\infty} v'_{t+1}(x+q-d-\xi) f(\xi) d\xi \qquad \text{(B.0.4)} \\
\frac{\partial^2 G_t(x,q,d)}{\partial d^2} &= R''(d) - (h+\pi) f(x+q-d) - \theta f(x-d) \\
&\quad +\alpha \int_{x-d}^{\infty} v''_{t+1}(x+q-d-\xi) f(\xi) d\xi - \alpha v'_{t+1}(q) f(x-d) \\
&\leq R''(d) - \theta f(x-d) + \alpha(\theta + h) f(x-d), \qquad \text{(B.0.5)}
\end{aligned}$$

where the inequality follows since by induction hypothesis $v''_{t+1}(\cdot) \leq 0$ and $v'_{t+1}(\cdot) \geq -\theta - h$. The expression in B.0.5 is clearly non-positive if $\theta(1-\alpha) \geq \alpha h$. When $\theta(1-\alpha) < h\alpha$, the non-positivity of the expression in B.0.5 can be easily established using $f(\cdot) \leq 1$ and $R''(d) \leq -h$. Now,

$$\frac{\partial^2 G_t(x,q,d)}{\partial d \partial q} = (h+\pi) f(x+q-d) - \alpha \int_{x-d}^{\infty} v''_{t+1}(x+q-d-\xi) f(\xi) d\xi.$$

Finally, we compute the determinant of the Hessian matrix as follows:

$$
\begin{aligned}
= \quad & [-(h+\pi)f(x+q-d)+\alpha\int_{x-d}^{\infty} v''_{t+1}(x+q-d-\xi)f(\xi)d\xi] \\
& [R''(d)-(\theta+\alpha v'_{t+1}(q))f(x-d)] \\
& +[R''(d)-(h+\pi)f(x+q-d)+\alpha\int_{x-d^*(x)}^{\infty} v''_{t+1}(x+q-d-\xi)f(\xi)d\xi \\
& -(\theta+\alpha v'_{t+1}(q))f(x-d)][\alpha v''_{t+1}(q)F(x-d)].
\end{aligned}
$$

Since v_{t+1} is concave, to show that the above expression is non-negative it is sufficient to prove that $R''(d)-[\theta+\alpha v'_{t+1}(q)]f(x-d)\le 0$. Since $v'_{t+1}(q)\ge -\theta-h$,

$$
R''(d)-[\theta+\alpha v'_{t+1}(q)]f(x-d)\le R''(d)-[\theta+\alpha(-h-\theta)]f(x-d).
$$

The RHS can be show to be non-positive using the same argument that was used for proving that the RHS of B.0.5 is non-positive. Therefore, $G_t(x,q,d)$ is a jointly concave function of d and q for $q\ge 0$ and $d\in\mathcal{D}$.

Proof of Part 2

We first consider periods $t\in\{1,2,\ldots,T-1\}$. Given any x, suppose that the order quantity is 0. To determine the optimal expected demand in this scenario, note that

$$
\begin{aligned}
\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0} \quad = \quad & R'(d)+hF(x-d)-\pi[1-F(x-d)]+\theta F(x-d) \\
& -\alpha c[1-F(x-d)], \qquad \text{(B.0.6)}
\end{aligned}
$$

where we use $v'_{t+1}(x)=c$ for $x\le 0$ by induction hypothesis. If there exists a $d_0\in\mathcal{D}$ for which the RHS is equal to 0, then d_0 is the optimal demand. By setting Equation B.0.6 equal to 0, we fine that d_0 satisfies the following relationship: $F(x-d)=\frac{\pi+\alpha c-R'(d)}{\pi+h+\theta+\alpha c}$. However, if the RHS of Equation B.0.6 is negative (positive) for all $d\in\mathcal{D}$, then the optimal demand is equal to $\min\mathcal{D}$ ($\max\mathcal{D}$).

When the order quantity is 0, the values of net inventory x can be segmented into three mutually disjoint (one or more of which could be possibly empty) sets depending upon the value of corresponding optimal expected demand. Let the three sets be denoted by A, B and C. The sets A and C consist of all the inventory values for which $\frac{\partial G_t}{\partial d}|_{q=0} < 0$ for all $d \in \mathcal{D}$ (so $d^* = \min \mathcal{D}$) and $\frac{\partial G_t}{\partial d}|_{q=0} > 0$ for all $d \in \mathcal{D}$ (so $d^* = \max \mathcal{D}$), respectively. On the other hand, the set B consists of all x such that the unconstrained optimal value of d lies in $\mathcal{D}$.

For each $x \in B$, $d^{*\prime}(x) \in (0,1)$. To see that, we use Implicit Function Theorem to obtain $d^{*\prime}(x) = -\frac{\partial^2 G_t/\partial x \partial d}{\partial^2 G_t/\partial d^2} = \frac{(\pi+h+\theta+\alpha c)f(x-d^*)}{(\pi+h+\theta+\alpha c)f(x-d^*)-R''(d^*)}$. As a result, $x - d^*$ (and hence $F(x-d^*)$) is increasing in x. On the other hand, for $x \in A, C$, $d^{*\prime} = 0$. Once again, $x - d^*$ (and hence $F(x-d^*)$) is increasing in x for all $x \in A, C$.

Now, observe that any value of x for which the unconstrained optimal order quantity is 0 must satisfy the following equation for given optimal demand d^*:

$$\begin{aligned}
\frac{\partial G_t(x,q,d^*)}{\partial q}|_{q=0} = 0 \quad &= \quad -c - h[F(x-d^*)] + \pi[1-F(x-d^*)] + \alpha \int_{-a}^{x-d^*} v'_{t+1}(0) f(\xi) d\xi \\
&\quad +\alpha \int_{x-d}^{\infty} v'_{t+1}(x-d-\xi) f(\xi) d\xi \\
&= \quad \pi - c(1-\alpha) - (h+\pi)[F(x-d^*)], \qquad \text{(B.0.7)}
\end{aligned}$$

where we use $v'_{t+1}(x) = c$ for $x \le 0$ by induction hypothesis. Since $F(x-d^*)$ increases with x (as we proved above), there exists a unique value of x for which the RHS is equal to 0. This value is denoted by $\bar{x}_t$. Further, since $d^*(\bar{x}_t)$ is independent of t, it is obvious that $\bar{x}_t$ is also independent of t.

For period T, the analysis is same as above except that

$$\frac{\partial G_T(x,q,d^*)}{\partial q}|_{q=0} = \pi - c(1-\alpha) - (h+\pi-\alpha s+\alpha c)F(x-d^*). \qquad \text{(B.0.8)}$$

As a consequence, $\frac{\partial G_T(x,q,d)}{\partial q}|_{q=0} \le \frac{\partial G_t(x,q,d)}{\partial q}|_{q=0}$, $t < T$. Further, $\frac{\partial G_T(x,q,d)}{\partial d}|_{q=0}$ is identical to $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0}, t < T$. As a consequence, the optimal expected demand when $q = 0$ is the same for all t, including period T. Therefore, $\bar{x}_T \le \bar{x}_t$.

Proof of Part 2(a)

Consider some $x < \bar{x}_t$ and by way of contradiction, let $q^*(x) = 0$. We argued in the proof of Part 2 that $F(x - d^*(x))$ is increasing in x if the order quantity is zero. As a consequence, $F(x - d^*(x)) < F(\bar{x}_t - d^*(\bar{x}_t))$. Using Equation B.0.7, this means that $\frac{\partial G_t(x,q,d^*)}{\partial q}|_{q=0} > 0$ implying that the optimal profit will increase if q is increased, which contradicts the assumption that $q^*(x) = 0$.

Proof of Part 2(b)

First consider period T. Given x, suppose that q^* and d^* satisfy $\frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = \frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$. As a consequence, we can add Equations B.0.1 and B.0.2 to obtain

$$\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = R'(d^*) - c + (\theta + \alpha s)F(x - d^*) = 0.$$

Using the Implicit Function Theorem,

$$d^{*\prime}(x) = -\frac{\partial^2 G_T(x,q,d)/\partial x \partial d}{\partial^2 G_T(x,q,d)/\partial d^2}|_{q=q^*,d=d^*} = \frac{-(\theta + \alpha s)f(x - d^*)}{R''(d^*) - (\theta + \alpha s)f(x - d^*)} \in (0,1).$$

Similarly,

$$\frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} \;\; = \;\; -c(1-\alpha) + \pi - (h + \pi - \alpha s + \alpha c)F(x + q^* - d^*) = 0.$$

Once again, we use the Implicit Function Theorem to obtain

$$q^{*\prime}(x) = -\frac{\partial^2 G_T(x,q,d)/\partial x \partial q}{\partial^2 G_T(x,q,d)/\partial q^2}|_{q=q^*,d=d^*} = d^{*\prime}(x) - 1. \tag{B.0.9}$$

Since $d^{*\prime}(x) \in (0,1)$, $q^{*\prime}(x) \in (-1,0)$.

Consider now the case in which $\frac{\partial G_T}{\partial d}|_{d=d^*,q=q^*} \neq 0$. Since $q^*(x) > 0$, we still have $\frac{\partial G_T}{\partial d}|_{d=d^*,q=q^*} = 0$. If we substitute Equation B.0.1, which is equal to 0, in Equation B.0.2 for any given d, we get

$$\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*} = R'(d) - c + (\theta + \alpha s)F(x - d).$$

Given the concavity of G_T in d for any x, if no d satisfies $\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*} = 0$ implies that the optimal value of d lies at one of the boundary points of $\mathcal{D}$. Therefore, $d^{*\prime}(x) = 0$. Further, using Equation B.0.9, $q^{*\prime}(x) = -1$.

Next, we consider a generic period $t < T$. Suppose that $q^*(x)$ and $d^*(x)$ are such that $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = \frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$. Consequently, adding Equations B.0.3 and Equation B.0.4 results in

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = R'(d^*) - c + [\theta + \alpha v'_{t+1}(q^*)]F(x-d^*) = 0.$$

As before, using the Implicit Function Theorem we get

$$\begin{aligned} d^{*\prime}(x) &= -\frac{\partial^2 G_T(x,q,d)/\partial x \partial d}{\partial^2 G_T(x,q,d)/\partial d^2}|_{q=q^*,d=d^*} \\ &= \frac{[\theta + \alpha v'_{t+1}(q^*)]f(x-d^*) + \alpha v''_{t+1}(q^*)q^{*\prime}(x)F(x-d^*)}{-R''(d^*) + [\theta + \alpha v'_{t+1}(q^*)]f(x-d^*)}. \end{aligned} \tag{B.0.10}$$

On the other hand, we differentiate Equation B.0.3 with respect to x and get

$$\begin{aligned} & \frac{\partial}{\partial x}[\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*}] = 0 = -(h+\pi)f(x+q^*-d^*)[1+q^{*\prime}(x)-d^{*\prime}(x)] \\ + & \ \alpha \int_{x-d^*}^{\infty} v''_{t+1}(x+q^*-d^*-\xi)f(\xi)d\xi[1+q^{*\prime}(x)-d^{*\prime}(x)] \\ + & \ \alpha q^{*\prime}(x) \int_{-a}^{x-d^*} v''_{t+1}(q^*)f(\xi)d\xi. \end{aligned} \tag{B.0.11}$$

To establish the result by contradiction, suppose $q^{*\prime}(x) > 0$. Three cases arise.

Case 1: $q^*(x) \leq \bar{x}_{t+1}$**.** In this case, $\theta + \alpha v'_{t+1}(q^*(x)) \geq 0$. We prove the result in the following lemma.

Lemma B.0.1. *If* $z \leq \bar{x}_{t+1}$*, then* $\theta + \alpha v'_{t+1}(z) \geq 0$.

Proof. Note that using the induction hypothesis, $v'_k(\cdot) \leq c, k \geq t+1$. Let $t+1 \leq T-1$. Then,

$$\begin{aligned} \theta + \alpha v'_{t+1}(z) &= \theta + \alpha c - \alpha[\theta + \alpha v'_{t+2}(q^*(z))]F(z - d^*(z)) \\ &\geq \theta + \alpha c - \alpha(\theta + \alpha c)F(z-d^*(z)) \\ &= (\theta + \alpha c)[1 - \alpha F(z - d^*(z))] \quad \geq 0. \end{aligned}$$

where we have used the following expression for $v'_{t+1}(z)$:

$$v'_{t+1}(z) = c - [\theta + \alpha v'_{t+2}(q^*(z))F(z - d^*(z))], \quad z \leq \bar{x}_{t+1},$$

which we formally derive later in the proof of Part 2(c). (See Equation B.0.16.) A similar argument can be developed when $t + 1 = T$ using Equation B.0.15; the details are omitted. $\square$

Now, since $q^{*\prime}(x) > 0$, Equation B.0.11 can only be true if $1 + q^{*\prime}(x) - d^{*\prime}(x) \leq 0$. That is, $d^{*\prime}(x) \geq 1 + q^{*\prime}(x) > 1$. On the other hand, the denominator in Equation B.0.10 is not only positive and but also larger than the numerator, which means that $d^{*\prime}(x) \leq 1$. But this is a contradiction and hence $q^{*\prime}(x) \leq 0$.

Given $q^{*\prime}(x) \leq 0$, we obtain $d^{*\prime}(x) \geq 0$ using Equation B.0.10 since both the numerator and denominator are non-negative. The non-positivity of $q^{*\prime}$ also requires that $1 + q^{*\prime}(x) - d^{*\prime}(x) \geq 0$ for Equation B.0.11 to be true. That is, $d^{*\prime}(x) \leq 1 + q^{*\prime}(x) \leq 1$. The same inequality also implies that $q^{*\prime}(x) \geq d^{*\prime}(x) - 1 \geq -1$.

Case 2: $q^*(x) \geq \bar{x}_{t+1}$ and $\theta + \alpha v'_{t+1}(q^*(x)) \geq 0$. In this case, the argument in Case 1 can be repeated to obtain $0 \leq d^{*\prime}(x) \leq 1$ and $-1 \leq q^{*\prime}(x) \leq 0$.

Case 3: $q^*(x) \geq \bar{x}_{t+1}$ and $\theta + \alpha v'_{t+1}(q^*) < 0$. In this case, the denominator in Equation B.0.10 is equal to

$$-R''(d^*) + [\theta + \alpha v'_{t+1}(q^*)]f(x - d^*) \geq -R''(d^*) + [\theta - \alpha(h + \theta)]f(x - d^*)$$

where the inequality follows since $v'_{t+1}(\cdot) \geq -\theta - h$ by induction hypothesis. Since $R''(d^*) \leq -h$ and $f(\cdot) \leq 1$, the above expression is positive. Since the numerator in B.0.10 is negative as well, $d^{*\prime}(x) \leq 0$. Since $q^{*\prime}(x) > 0$ and $d^{*\prime}(x) \leq 0$, $1 + q^{*\prime}(x) - d^{*\prime}(x) > 0$. However, if $1 + q^{*\prime}(x) - d^{*\prime}(x) > 0$, Equation B.0.11 cannot be true. Thus, we have a contradiction here. Hence, $q^{*\prime}(x) \leq 0$.

For $t < T-1$, since $q^{*\prime}(x) \leq 0$ for all $x \leq \bar{x}_t$ and $\bar{x}_t = \bar{x}_{t+1}$, $q^*(x) \leq \bar{x}_{t+1}$ if $q^*(0) \leq \bar{x}_t$. In such a scenario, the only case possible for $t < T-1$ will be Case 1 above, in which $q^*(x) \leq \bar{x}_{t+1}$. Hence, $q^{*\prime}(x) \in [-1, 0]$ and $d^{*\prime}(x) \in [0, 1]$ for $x \in [0, \bar{x}_t]$.

Next, we show that $q^*(0) \leq \bar{x}_t$. If $q^*(0) = 0$, then we have nothing to prove since $\bar{x}_t \geq 0$ by Part 2(a). So assume that $q^*(0) > 0$. Then it must satisfy the following equation obtained by setting $\frac{\partial G_t(0,q,d^*)}{\partial q}|_{q=q^*(0)} = 0$:

$$F(q^*(0) - d^*(0)) = \frac{\pi - c + \alpha \int_{-a}^{\infty} v'_{t+1}(q^*(0) - d^*(0) - \xi) f(\xi) d\xi}{\pi + h}. \tag{B.0.12}$$

Since $v'_{t+1}(\cdot) \leq c$, $F(q^*(0) - d^*(0)) \leq \frac{\pi - c + \alpha c}{\pi + h}$. On the other hand, from the proof of Part 2, we know that $F(\bar{x}_t - d^*(\bar{x}_t)) = \frac{\pi - c + \alpha c}{\pi + h}$. Clearly, $q^*(0) - d^*(0) \leq \bar{x}_t - d^*(\bar{x}_t)$. Thus, it suffices to show that $d^*(0) \leq d^*(\bar{x}_t)$ to show that $q^*(0) \leq \bar{x}_t$.

By substituting $\frac{\partial G_t(0,q,d^*(0))}{\partial q}|_{q=q^*(0)}$, which is equal to 0, into $\frac{\partial G_t(0,q^*(0),d)}{\partial d}$, we get $\frac{\partial G_t(0,q^*(0),d)}{\partial d} = R'(d) - c$. By assumption, the value of d at which $R'(d) - c = 0$ is feasible. Hence $d^*(0)$ is equal to the solution of $R'(d) = c$.

Now, at $x = \bar{x}_t$, if we substitute $\frac{\partial G_t}{\partial q}$, which is equal to 0 by definition of $\bar{x}_t$, into $\frac{\partial G_t}{\partial d}$, we get

$$\frac{\partial G_t}{\partial d} = R'(d) - c + [\theta + \alpha c] F(\bar{x}_t - d).$$

If there exists a $d_0 \in \mathcal{D}$ at which the above equation is equal to 0, then $R'(d_0) \leq c$, implying that $d_0 \geq d^*(0)$. If however d_0 is not feasible, then the optimal solution should be $\max \mathcal{D}$. (Note that the optimal solution cannot be at $\min \mathcal{D}$ since that would imply $R'(d) < c$ for all $d \in \mathcal{D}$, which violates the assumption that the value of that d satisfies $R'(d) = c$ is feasible.) Since $\arg\max_d (R(d) - cd) \in \mathcal{D}$, $R'(\max \mathcal{D}) \leq c$, implying that $\max \mathcal{D} \geq d^*(0)$. Consequently, $d^*(0) \leq d^* \bar{x}_t$. Hence, $q^*(0) \leq \bar{x}_t$.

The above argument requires that $\bar{x}_t$ be equal to $\bar{x}_{t+1}$. As a result, it does not hold for $t = T-1$ since $\bar{x}_T$ may be less than $\bar{x}_{T-1}$. Fortunately, it is sufficient to show that $\theta + \alpha v'_T(z) \geq 0$ for $z \leq \bar{x}_{T-1}$. Once this inequality is established, the analysis for Cases

1 and 2 presented above can be replicated depending upon whether $q^*(x)$ is less than or greater than $\bar{x}_T$.

Since $v_T'(\cdot)$ decreases due to the hypothesized concavity of v_T, it is enough to establish that $\theta + \alpha v_T'(z) \geq 0$ for $z = \bar{x}_{T-1}$. From the proof of Part 2, we note that $\bar{x}_{T-1}$ is characterized by the following equation:

$$F(\bar{x}_{T-1} - d^*(\bar{x}_{T-1})) = \frac{\pi - c + \alpha c}{\pi + h}.$$

Now, since $\bar{x}_{T-1} \geq \bar{x}_T$,

$$v_T'(\bar{x}_{T-1}) = -(h + \pi + \theta + \alpha c)F(\bar{x}_{T-1} - d^*(\bar{x}_{T-1})) + \pi + \alpha c,$$

where we use Equation B.0.17, which is derived in the proof of Part 3(b). Substituting for $F(\bar{x}_{T-1} - d^*(\bar{x}_{T-1}))$, we get $v_T'(\bar{x}_{T-1}) = -(h + \pi + \theta + \alpha c)\left(\frac{\pi - c + \alpha c}{\pi + h}\right) + \pi + \alpha c$. Therefore, $\theta + \alpha v_T'(\bar{x}_{T-1}) > 0$.

Finally, we consider the case in which $\frac{\partial G_t(x,q,d}{\partial d}|_{q=q^*,d=d^*}$ is not necessarily equal to 0 for all $x \leq \bar{x}_t$. Recall from our argument above that $d^*(0)$ is in the interior of $\mathcal{D}$. Thus, $\frac{\partial G_t(x,q,d}{\partial d}|_{q=q^*,d=d^*}$ is equal to 0 at $x = 0$. As x increases, $d^*(x)$ also increases since its slope is non-negative. Suppose now that there exists a $\underline{x}_t < \bar{x}_t$ such that the unconstrained optimal value of d at $x = \underline{x}_t$ is equal to $\max(\mathcal{D})$. We claim that for all $x \in (\underline{x}_t, \bar{x}_t]$, $d^*(x) = \max(\mathcal{D})$. To see this, observe that for $x > \underline{x}_t$,

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} < \frac{\partial G_t(\underline{x}_t,q,d)}{\partial q}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} = 0,$$

and

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} > \frac{\partial G_t(\underline{x}_t,q,d)}{\partial d}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} = 0.$$

Both of these inequalities can be proved by showing that the derivatives of $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*(\underline{x}_t,d=\max\mathcal{D}}$ and $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q^*(\underline{x}_t,d=\max\mathcal{D}}$ with respect to x increase and decrease, respectively. As a consequence of above inequalities, profit can be improved in only one way: decrease q. (Increasing d will also increase profit but that is not possible.) Suppose now that we gradually increase the value of q to q_0 such that $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q_0,d=\max\mathcal{D}}$ becomes equal to 0. If for $q = q_0$, $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q_0,d=\max\mathcal{D}}$ is still non-negative, then that would imply that $(q_0, \max\mathcal{D})$ is optimal at x since profit can only be further increased by increasing d, but that is not possible. Now, adding Equation B.0.3, which is equal to 0, to Equation B.0.4, we get

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q_0,d=\max\mathcal{D}} = R'(\max\mathcal{D}) - c + [\theta + \alpha v'_{t+1}(q_0)]F(x - \max\mathcal{D}).$$

Recall that $\theta + \alpha v'_{t+1}(q^*(\underline{x}_t)) \geq 0$ from above since $q^*(\underline{x}_t) \leq x_{t+1}$. This combined with the concavity of v_{t+1} and $q_0 \leq q^*(\underline{x}_t)$ implies that $\theta + \alpha v'_{t+1}(q_0) \geq \theta + \alpha v'_{t+1}(q^*(\underline{x}_t))$. Finally, $F(x - \max\mathcal{D}) > F(\underline{x}_t - \max\mathcal{D})$. As a result,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=q_0,d=\max\mathcal{D}} > \frac{\partial G_t(\underline{x}_t,q,d)}{\partial d}|_{q=q^*(\underline{x}_t),d=\max\mathcal{D}} = 0.$$

As a consequence, $d^{*\prime}(x) = 0$. Substituting $d^{*\prime}(x) = 0$ in Equation B.0.11, we get

$$\begin{aligned} q^{*\prime}(x) \quad = \quad & \frac{(h+\pi)f(x+q^*-d^*) - \alpha\int_{x-d^*}^{\infty} v''_{t+1}(x+q^*-d^*-\xi)f(\xi)d\xi}{-(h+\pi)f(x+q^*-d^*) + \alpha\int_{x-d^*}^{\infty} v''_{t+1}(x+q^*-d^*-\xi)f(\xi)d\xi + \alpha v''_{t+1}(q^*)F(x-d^*)} \\ & \in (-1,0). \end{aligned}$$

Proof of Part 2(c)

Using the implicit differentiation rule,

$$\begin{aligned} \frac{\partial v_T(x)}{\partial x} \quad = \quad & \frac{\partial G_T(x,q,d)}{\partial x}|_{q=q^*,d=d^*} + \frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} \cdot q^{*\prime}(x) \\ & + \frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \cdot d^{*\prime}(x), \end{aligned} \tag{B.0.13}$$

where the second term is equal to zero at $q = q^*$ since $\frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0$ for $x \leq \bar{x}_t$. The third term is also 0 since either d^* satisfies $\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$ or d^* is equal to $\max\mathcal{D}$. Recall that we showed in the proof of Part 2(b) that if $\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \neq 0$ for all $x \leq \bar{x}_t$, then there exists $\underline{x}_t$ such that $d^*(x) = \max\mathcal{D}$ for $x \in (\underline{x}_t, \bar{x}_t]$ and $d^*(x)$ satisfies

$\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} = 0$ for $x \leq \underline{x}_t$. Clearly, $d^{*\prime}(x) = 0$ for $x > \underline{x}_t$. Therefore,

$$\begin{aligned} v_T'(x) &= \frac{\partial G_T(x,q,d)}{\partial x}|_{q=q^*,d=d^*} && \text{(B.0.14)} \\ &= \alpha c - (h + \pi - \alpha s + \alpha c)F(x + q^* - d^*) + \pi - (\theta + \alpha s)F(x - d^*) \\ &= c - (\theta + \alpha s)F(x - d^*) + \frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} && \text{(B.0.15)} \end{aligned}$$

where $\frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0$. In a similar manner,

$$\begin{aligned} v_t'(x) &= -(h+\pi)F(x + q^* - d^*) + \pi - \theta F(x - d^*) + \alpha \int_{x-d^*(x)}^{\infty} v_{t+1}'(x + q^* - d^* - \xi) f(\xi) d\xi \\ &= c - \alpha \int_{-a}^{x-d^*} v_{t+1}'(q^*) f(\xi) d\xi - \theta F(x - d^*) + \frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*} \\ &= c - [\theta + \alpha v_{t+1}'(q^*)]F(x - d^*). && \text{(B.0.16)} \end{aligned}$$

Now, from Equation B.0.15,

$$c \geq v_T'(x) \geq c - (\theta + \alpha s).$$

On the other hand, using Equation B.0.16,

$$v_t'(x) \geq c - (\theta + \alpha c)F(x - d^*) \geq c(1 - \alpha) - \theta,$$

where the inequality holds by using the induction hypothesis, $v_{t+1}'(\cdot) \leq c$. Further, since $\theta + \alpha v_{t+1}(q^*(x)) \geq 0$ for $x \leq \bar{x}_t$, as we proved in the proof of Part 2(b), $v_t'(x) \leq c$.

Now, when $x \leq 0$ $v_T'(x) = c$ since $F(x - d^*) = 0$ for $x \leq 0$. Using the same argument, $v_t'(x) = c$. for $x \leq 0$.

Proof of Part 2(d)

Using Equation B.0.16,

$$v_t''(x) = -\alpha v_{t+1}''(q^*) q^{*\prime}(x) F(x - d^*) - (\theta + \alpha v_{t+1}'(q^*)) f(x - d^*)(1 - d^{*\prime}(x)) \leq 0,$$

where we use the concavity of v_{t+1}, the non-positivity of $q^{*\prime}(x)$, the non-negativity of $(\theta + \alpha v_{t+1}'(q^*))$, as proved in the Proof of Part 2(b), and $d^{*\prime}(x) \in [0, 1]$. The proof for period T is similar, and the details are omitted.

Proof of Part 3(a)

Consider the following for $x > \bar{x}_t$:

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} = -c - (h+\pi)F(x - d^*(\bar{x}_t)) + \pi + \alpha c,$$

and

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} = R'(d^*(\bar{x}_t)) + (h+\pi+\theta+\alpha c)F(x - d^*(\bar{x}_t)) - \pi - \alpha c.$$

Clearly, $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} < \frac{\partial G_t(\bar{x}_t,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} = 0$ and $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} < \frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)}$. At x, we can increase profit in the following three ways: (i) Increase q by $\delta > 0$, (ii) Increase or decrease d from $d^*(\bar{x}_t)$ by $\delta > 0$, and (iii) A combination of (i) and (ii). Implementing (i) will reduce profit since $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} < 0$, so we ignore it. To consider (ii), assume first that $d^*(\bar{x}_t) < \max \mathcal{D}$. Thus, $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} = 0$. In this case, increasing d by δ increases profit at x. On the other hand, if $d^*(\bar{x}_t) = \max \mathcal{D}$, then $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} > 0$, implying that $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x}_t)} > 0$. While the profit will improve if we increase d, but that is not possible since d is already at the upper bound. Decreasing the value of d will definitely not increase the profit. Finally, using a similar argument as above, we can show that the profit improvement by (iii) cannot be more than that by (ii). Hence, we can ignore it.

Suppose now that we are able to increase the value of d by δ. While this will increase the value of $\frac{\partial G_t(x,q,d)}{\partial q}$ and decrease the value of $\frac{\partial G_t(x,q,d)}{\partial d}$, their signs, however, remain unchanged. As a result, once again, the profit can only be improved by increasing the value of d from $d^*(\bar{x}_t) + \delta$ to $d^*(\bar{x}_t) + 2\delta$, provided the new value of d remains feasible. We can keep repeating this procedure until one of $\frac{\partial G_t(x,q,d)}{\partial q}$ and $\frac{\partial G_t(x,q,d)}{\partial d}$ hits zero or d reaches the upper bound. If d reaches the upper bound before either of the two partial derivatives hits zero, the optimal solution will be $q^* = 0$ and $d^* = \max D$.

Suppose now that the either of the two derivatives hits zero before d reaches the upper bound. We claim that the partial derivative of G_t with respect to d will hit zero before the partial derivative of G_t with respect to q does. To see this, suppose on the contrary that as we are increasing the value of d, there exists a d_0 such that $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d_0} = 0 < \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d_0}$. This implies that $x - d_0 = \bar{x}_t - d^*(\bar{x}_t)$. Therefore,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d_0} = R'(d_0) + (h+\pi+\theta+\alpha c)F(\bar{x}_t - d^*(\bar{x}_t)) - \pi - \alpha c.$$

Since $\frac{\partial G_t(\bar{x}_t,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x}_t)} = 0$, $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d_0} = R'(d_0) - R'(d^*(\bar{x}_t)) < 0$ as $d_0 < d^*(\bar{x}_t)$. This is a contradiction.

Thus, $\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(x)} < 0$, and it is optimal to not order. Further, in this case, using the Implicit Function Theorem,

$$d^{*\prime}(x) = -\frac{\partial^2 G_t(x,q,d)/\partial x \partial d}{\partial^2 G_t(x,q,d)/\partial d^2}|_{q=0,d=d^*(x)} = \frac{(h+\pi+\theta+\alpha c)f(x-d^*)}{R''(d^*) + (h+\pi+\theta+\alpha c)f(x-d^*)} < 1,$$

if $\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(x)} = 0$. Otherwise, $d^{*\prime}(x) = 0$.

Proof of Part 3(b)

First, we consider period T.

$$\begin{aligned}
v_T(x) &= G_T(x,0,d^*) \\
&= R(d^*) - (h+\theta)\int_{-a}^{x-d^*}(x-d^*-\xi)f(\xi)d\xi - (\pi+\alpha c)\int_{x-d^*}^{\infty}(d^*+\xi-x)f(\xi)d\xi \\
v_T'(x) &= R'(d^*)d^{*\prime}(x) - (h+\theta)(1-d^{*\prime}(x))F(x-d^*) + (\pi+\alpha c)(1-d^{*\prime}(x))[1-F(x-d^*)] \\
&= d^{*\prime}(x)\{R'(d^*) + (h+\theta)F(x-d^*) - (\pi+\alpha c)[1-F(x-d^*)]\} \\
&\quad -(h+\theta)F(x-d^*) + (\pi+\alpha c)[1-F(x-d^*)] \\
&= d^{*\prime}(x)[\frac{\partial G_T(x,q,d)}{\partial d}|_{d=d^*(x),q=0}] - (h+\theta)F(x-d^*) + (\pi+\alpha c)[1-F(x-d^*(x))] \\
&= -(h+\theta+\pi+\alpha c)F(x-d^*) + \pi + \alpha c \qquad \text{(B.0.17)}
\end{aligned}$$

where either $\frac{\partial G_T(x,q,d)}{\partial d}|_{d=d^*(x),q=0}$ or $d^{*\prime} = 0$; the reason is similar to the one given in the Proof of Part 2(c) following Equation B.0.13, and the details are omitted. Next, we consider a period $t < T$.

$$\begin{aligned} v_t(x) &= G_t(x, 0, d^*) \\ &= R(d^*) - (h+\theta)\int_{-a}^{x-d^*} (x - d^* - \xi) f(\xi) d\xi - \pi \int_{x-d^*}^{\infty} (d^* + \xi - x) f(\xi) d\xi \\ &\quad + \alpha \int_{x-d^*}^{\infty} v_{t+1}(x - d^* - \xi) f(\xi) d\xi + \alpha v_{t+1}(0) F(x - d^*) \end{aligned}$$

$$\begin{aligned} v_t'(x) &= R'(d^*) d^{*\prime}(x) - (h+\theta)(1 - d^{*\prime}(x)) F(x - d^*) + \pi(1 - d^{*\prime}(x))[1 - F(x - d^*)] \\ &\quad + \alpha \int_{x-d^*}^{\infty} v_{t+1}'(x - d^* - \xi) f(\xi) d\xi (1 - d^{*\prime}(x)) \\ &= d^{*\prime}(x)[\frac{\partial G_t(x,q,d)}{\partial d}|_{d=d^*,q=0}] - (h+\theta) F(x - d^*) + (\pi + \alpha c)[1 - F(x - d^*)] \end{aligned}$$

where we use $v_{t+1}'(x) = c$ for $x \leq 0$. Since either $d^{*\prime}(x) = 0$ or $\frac{\partial G_t(x,q,d)}{\partial d}|_{d=d^*,q=0} = 0$, the rationale for which can be explained in the same manner as in the proof of Part 2(c) following Equation B.0.13, we get

$$v_t'(x) = -(h + \theta + \pi + \alpha c) F(x - d^*) + \pi + \alpha c. \tag{B.0.18}$$

From equations B.0.17 and B.0.18 for any given period t,

$$v_t'(x) \geq -(h + \theta + \pi + \alpha c) + \pi + \alpha c \geq -(h + \theta)$$

which provides a lower bound. Also,

$$\frac{\partial G_T(x,q,d)}{\partial q}|_{q=0,d=d^*} = -c(1-\alpha) + \pi - (h + \pi - \alpha s + \alpha c)[F(x - d^*)] \leq 0, \tag{B.0.19}$$

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*} = -c - h[F(x - d^*)] + \pi[1 - F(x - d^*) + \alpha c \leq 0, \quad t < T, \tag{B.0.20}$$

where the inequalities follow from the Proof of Part 3(a). Using B.0.19 and B.0.20, $F(x - d^*) \geq \frac{\pi - c + \alpha c}{\pi + h - \alpha s + \alpha c}$ for $t = T$, and $F(x - d^*) \geq \frac{\pi - c + \alpha c}{\pi + h}$ for $t < T$. Substituting these lower bound on $F(x-d^*)$ in Equations B.0.17 and B.0.18, we get

$$\begin{aligned} v'_T(x) - c &\leq \frac{(\pi - c + \alpha c)(-\theta - \alpha s)}{\pi + h - \alpha s + \alpha c} \leq 0, \\ &\leq \frac{(\pi - c + \alpha c)(-\theta - \alpha c)}{\pi + h} \leq 0, \quad t < T, \end{aligned}$$

implying that $v'_t(x) \leq c$.

Proof of Part 3(c)

Using equations B.0.17 and B.0.18,

$$v''_t(x) = -(h + \theta + \pi + \alpha c) f(x - d^*(x))[1 - d^{*\prime}(x)]$$

in any given period t. Since $d^{*\prime}(x) \in [0,1]$, $v''_t(x) \leq 0$.

Proof of Theorem 4.2.3

Proof of Part 1

We first derive the upper bound on the optimal expected demand. When $x > \bar{x}$ and $t \leq T$, using Equations B.0.17 and B.0.18,

$$v'_t(x) = -(h + \theta + \pi + \alpha c)F(x - d^*(x)) + \pi + \alpha c \leq c,$$

where the inequality follows from Theorem 4.2.2 Part 3(b). Therefore,

$$d^*(x) \leq x - F^{-1}[\frac{\pi - c + \alpha c}{h + \theta + \pi + \alpha c}] = x - y,$$

where we define $y = F^{-1}[\frac{\pi - c + \alpha c}{h + \theta + \pi + \alpha c}]$. Thus, the expected demand corresponding to net inventory x is bounded from above by $x - y$.

On the other hand, when $x \leq \bar{x}_t$, $d^*(x) \leq d^*(\bar{x}_t) \leq \bar{x}_t - y$, since $d^*(x)$ is increasing in x.

Proof of Part 2

Let d_c be defined such that $\frac{\partial[R(d)]}{\partial d}|_{d=d_c} = c$. Recall from the proof of Theorem 1 Part 2(a), $d^*(x) = d_c$, $x \leq 0$. Further, Theorem 1 shows that $d^{*\prime}(x) \geq 0$ for all x. As a consequence, $d^*(x) \geq d_c$ for all x.

Proof of Part 3

When $x \leq \bar{x}_t$,

$$\begin{aligned}\frac{\partial G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0 &= -c-(h+\pi)F(x+q^*-d^*)+\pi+\alpha[v'_{t+1}(q^*)F(x-d^*)\\ &\quad +\alpha\int_{x-d^*}^{\infty} v'_{t+1}(x+q^*-d^*-\xi)f(\xi)d\xi]\\ &\leq (-c+\pi+\alpha c)-(h+\pi)F(x+q^*-d^*),\end{aligned}$$

where the inequality follows since $v'_{t+1}(x) \leq c$ using Theorem 4.2.2. Therefore,

$$x+q^*-d^* \leq F^{-1}[\frac{-c+\pi+\alpha c}{h+\pi}] =: \bar{r}.$$

Consequently,

$$x+q^* \leq F^{-1}[\frac{-c+\pi+\alpha c}{h+\pi}]+d^* \leq \bar{r}+\bar{x}_t-y,$$

where the second inequality follows since $d^*(x) \leq \bar{x}_t - y$ from Part 1 above. Thus, the order quantity is bounded from above by $\bar{r}+\bar{x}_t-y-x$ when $x \leq \bar{x}_t$.

The lower bound on order quantity is obtained by using $v'_{t+1}(x) \geq -h-\theta$. As a consequence,

$$x+q^*-d^* \geq F^{-1}[\frac{\pi-c-\alpha(h+\theta)]}{h+\pi}] = \underline{r},$$

which implies $q^* \geq \underline{r}+(d^*-x) \geq \underline{r}+d_c-x$, where d_c is the maximizer of $(R(d)-cd)$.

Proof of Proposition 4.2.5

Recall from Equation B.0.12 that

$$F(q^2(0)-d^2(0)) = \frac{\pi-c+\alpha\int_{-a}^{\infty} v'_{t+1}(q^2(0)-d^2(0)-\xi)f(\xi)d\xi}{h+\pi}.$$

Further, since $\bar{x}_t \geq 0$, using Lemma B.0.1, $\theta+\alpha v'_{t+1}(q^2(0)-d^2(0)-\xi) \geq 0$. By substituting $v'_{t+1}(q^2(0)-d^2(0)-\xi) \geq -\frac{\theta}{\alpha}$ and $v'_{t+1}(z)=c$ for $z \leq 0$ in the above equation, we get

$$F(q^2(0)-d^2(0)) \geq \frac{\pi+\alpha c-c}{h+\theta+\pi+\alpha c}. \tag{B.0.21}$$

On the other hand, let $G_t^1(x,q,d)$ be the maximand in Equation 4.2.3. Now,

$$\begin{aligned}
\frac{\partial G_t^1(x,q^1,d^1)}{\partial q^1} &= -c-(h+\theta+\pi)F(x+q^1-d^1)+\pi] \\
&\quad +\alpha\int_{x+q^1-d^1}^{\infty} v'_{t+1}(x+q^1-d^1-\xi)f(\xi)d\xi \\
&= c(-1+\alpha)+\pi-(h+\theta+\pi+\alpha c)F(x+q^1-d^1) \qquad \text{(B.0.22)}
\end{aligned}$$

where we use $v'_{t+1}(z)=c$ (without proof) for $z\leq 0$. Thus, the optimal value of q^1 satisfies

$$F(x+q^{1*}-d^{1*})=\frac{\pi-c+\alpha c}{\pi+h+\theta+\alpha c}. \qquad \text{(B.0.23)}$$

Comparing Equations B.0.23 and B.0.21, $q^{1*}(0)\leq q^{2*}(0)$ where we use $d^{1*}(0)=d^{2*}(0)$.

Next, using Equation B.0.6,

$$F^2(\bar{x}_t^2-d^2(\bar{x}_t^2))=\frac{\pi+\alpha c-c}{h+\pi}.$$

Comparing the above equation with Equation B.0.23,

$$q_t^1(0)-d_t^1(0)\leq \bar{x}_t^2-d^2(\bar{x}_t^2).$$

Since the optimal value of $d^2(x)$ is increasing in x, $d^{2*}(\bar{x}_t^2)\geq d^{2*}(0)=d^{1*}(0)$. Therefore, $q_t^1(0)\leq \bar{x}_t^2$.

Proof of Proposition 4.2.6

Proof of Part 1

Let $G^\infty(x,q^\infty,d^\infty)$ be used to denote the maximand in Equation 4.2.4. For $t=T$,

$$\frac{\partial G_T^\infty(x,q^\infty,d^\infty)}{\partial q^\infty} = -c(1-\alpha)+\pi-(h+\pi-\alpha s+\alpha c)F(x+q^\infty-d^\infty),$$

which is identical to Equation B.0.1. Hence for any x, $q^{\infty*}(x)-d^{\infty*}(x)=q^{2*}(x)-d^{2*}(x)$. Since $d^{2*}(x)\geq d^{2*}(0)=d^{\infty*}(x)$, $q^{\infty*}(x)\leq q^{2*}(x)$.

Proof of Part 2

When $t << T$, $v_t\prime(x) = c$ for x less than the optimal base-stock level. As a result,

$$\begin{aligned}\frac{\partial G_t^{\infty}(x, q^{\infty}, d^{\infty})}{\partial q^{\infty}} &= -c + \pi - (h+\pi)F(x + q^{\infty} - d^{\infty}) + \alpha E v'_{t+1}(x + q^{\infty} - d^{\infty} - \xi) \\ &= -c + \pi - (h+\pi)F(x + q^{\infty} - d^{\infty}) + \alpha c.\end{aligned}$$

By setting the above equation to 0, we find that the optimal value of q^{∞} must satisfy the following equation:

$$F(x + q^{\infty *} - d^{\infty *}) = \frac{\pi - c + \alpha c}{\pi + h}. \tag{B.0.24}$$

On the other hand, using Equation B.0.12,

$$F(x + q^{2*}(0) - d^{2*}(0)) \leq \frac{\pi - c + \alpha c}{\pi + h},$$

where we use the upper bound on $v'_{t+1}(\cdot)$, which is equal to c, to obtain the upper bound. Comparing the above equation with Equation B.0.24,

$$q^{2*}(0) - d^{2*}(0) \leq q^{\infty *}(0) - d^{\infty *}(0).$$

Since $d^{2*}(0) = d^{\infty *}(0)$, $q^{2*}(0) \leq q^{\infty *}(0)$.

Proof of Part 3

Follows directly from Parts 1 and 2.

Proof of Part 4

Recall from the proof of Proposition 4.2.5 that $\bar{x}_t$ satisfies

$$F^2(\bar{x}_t^2 - d^2(\bar{x}_t^2)) = \frac{\pi + \alpha c - c}{h + \pi}.$$

Comparing the above equation with Equation B.0.24, we see that

$$\bar{x}_t^2 - d^2(\bar{x}_t^2)) = q^{\infty *}(0) - d^{\infty *}(0).$$

Since $d^2(\bar{x}_t^2)) \geq d^2(0) = d^{\infty *}(0)$, $\bar{x}_t^2 \geq q^{\infty *}(0)$.

Proof of Proposition 4.3.1

$$v_t(x) = \max_{q\geq 0, d\in\mathcal{D}} L(x,q,d) - cq - \theta E(x-(D-q)^+)^+ + \alpha E v_{t+1}(x+q-D-(x-(D-q)^+)^+).$$

Let $G_t(x,q,d)$ be the maximand on the RHS of Equation 4.3.5. That is,

$$\begin{aligned} G_t(x,q,d) &= R(d) - cq - hE[x+q-d-\xi]^+ - \pi E[d+\xi-x-q]^+ \\ &\quad -\theta E[x-[d+\xi-q]^+]^+ + \alpha E v_{t+1}(x+q-d-\xi-(x-(d+\xi-q)^+)^+). \end{aligned}$$

Let $z = q-d$. Then the optimization problem in any period t becomes

$$\begin{aligned} G_t(x,q,d) &= \max_{d\in\mathcal{D}, z\geq -d} R(d) - c(z+d) - hE[x+z-\xi]^+ - \pi E[\xi-x-z]^+ \\ &\quad -\theta E[x-[\xi-z]^+]^+ + \alpha E v_{t+1}(x+z-\xi-(x-(\xi-z)^+)^+). \end{aligned}$$

Consider some x such that $q^*(x) > 0$. In that case, the constraint $z \geq -d$ becomes redundant. Thus, the above optimization problem becomes

$$\begin{aligned} G_t(x,q,d) &= \max_{d\in\mathcal{D}}\{R(d) - cd\} + \max_{z}\{-cz - hE[x+z-\xi]^+ - \pi E[\xi-x-z]^+ \\ &\quad -\theta E[x-[\xi-z]^+]^+ + \alpha E v_{t+1}(x+z-\xi-(x-(\xi-z)^+)^+)\}. \end{aligned}$$

Clearly, the optimal value of d is the maximizer of $R(d) - cd$.

Justification Behind Observation 4.3.2

As noted in the main body of the paper, it is sufficient to demonstrate this observation for period T. Define

$$\begin{aligned} v_T(x) &= \max_{d\in\mathcal{D}, q\geq 0} R(d) - cq - hE[x+q-d-\xi]^+ - \pi E[d+\xi-x-q]^+ \\ &\quad -\theta E[x-(d+\xi-q)^+]^+ + \alpha s E[q-d-\xi]^+ - \alpha c[d+\xi-x-q]^+. \end{aligned}$$

Consider some x such that $q^* > 0$ so that $\frac{\partial G_T(x,q,d)}{\partial q} = 0 = \frac{\partial G_T(x,q,d)}{\partial d}$. (From Proposition 4.3.1, d^* maximizes $R(d) - cd$ and thus lies in the interior of $\mathcal{D}$.) Now,

$$\begin{aligned}
\frac{\partial v_T(x)}{\partial x} &= \frac{\partial G_T(x,q^*,d^*)}{\partial x} + \underbrace{\frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*}}_{=0} \cdot \frac{\partial q^*(x)}{\partial x} + \underbrace{\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*}}_{=0} \cdot \frac{\partial d^*(x)}{\partial x} \\
&= -(\pi + h + \theta + \alpha c)F(x + q^* - d^*) + \pi + \alpha c.
\end{aligned}$$

Therefore,

$$\frac{\partial^2 v_T(x)}{\partial x^2} = -(\pi + h + \theta + \alpha c)f(x + q^* - d^*)(1 + q^{*\prime} - d^{*\prime}), \qquad \text{(B.0.25)}$$

where $d^{*\prime} = 0$ since d^* is a constant. To show that v_T is not necessarily concave in x, it suffices to show that $1 + q^{*\prime}$ is not necessarily non-negative. Using the Implicit Function Theorem,

$$\begin{aligned}
q^{*\prime}(x) &= -\frac{\partial^2 G_T(x,q,d)/\partial x \partial q}{\partial^2 G_T(x,q,d)/\partial q^2}|_{q=q^*,d=d^*} \\
&= \frac{(\pi + \alpha c + h + \theta)f(x + q^* - d^*)}{(\alpha s + \theta)f(q^* - d^*) - (\pi + \alpha c + h + \theta)f(x + q^* - d^*)},
\end{aligned}$$

which is not necessarily greater than -1 if s or θ are strictly positive.

Proof of Theorem 4.3.3

Proof of Part 1

In period T,

$$\begin{aligned}
\frac{\partial G_T(x,q,d)}{\partial q} &= -c + \pi + \alpha c - (h + \pi + \alpha c + \theta)F(x + q - d) \\
&\quad +(\theta + \alpha s)F(q - d), \qquad \text{(B.0.26)} \\
\frac{\partial G_T(x,q,d)}{\partial d} &= R'(d) - \pi - \alpha c + (h + \pi + \alpha c + \theta)F(x + q - d) \\
&\quad -(\theta + \alpha s)F(q - d). \qquad \text{(B.0.27)}
\end{aligned}$$

Define $x = \bar{x}$ such that the constrained optimal value of q, $q^*(\bar{x}) = 0$. If there exist multiple such values of x, then we take the maximum of those values. It can be easily shown, however, that there exists at least one such value. Therefore,

$$\begin{aligned}
\frac{\partial G_T(x,q,d)}{\partial q}|_{q=0} &= -c+\pi+\alpha c-(h+\pi+\alpha c+\theta)F(x-d), \\
\frac{\partial G_T(x,q,d)}{\partial d}|_{q=0} &= R'(d)-\pi-\alpha c+(h+\pi+\alpha c+\theta)F(x-d),
\end{aligned}$$

where we take $F(-d)=0$ since $d+\xi\geq 0$. Since $\frac{\partial G_T(\bar{x},q,d)}{\partial q}|_{q=0}=0$, $\frac{\partial G_T(\bar{x},q,d)}{\partial d}|_{q=0}=R'(d)-c$. Thus, $d^*(\bar{x})$ satisfies the equation $R'(d)=c$.

Now, for $x<\bar{x}$,

$$\frac{\partial G_T(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} > \frac{\partial G_T(\bar{x},q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})}=0, \text{ and}$$

$$\frac{\partial G_T(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_T(\bar{x},q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}=0.$$

Given the signs of the slopes of G_T, there are three ways to improve the profit at x: (i) increase q by δ or (ii) decrease d by δ or (iii) increase q by $\frac{\delta}{2}$ and decrease d by $\frac{\delta}{2}$ where $\delta>0$ and sufficiently small. In Case (i), the profit increases by

$$G_T(x,\delta,d^*(\bar{x}))-G_T(x,0,d^*(\bar{x}))=-c\delta+A,$$

where

$$\begin{aligned}
A &= -hE[x+\delta-d^*(\bar{x})-\xi]^+ + hE[x-d^*(\bar{x})-\xi]^+ - \pi E[d^*(\bar{x})+\xi-x-\delta]^+ \\
&\quad +\pi E[d^*(\bar{x})+\xi-x]^+ - \theta E[x-(d^*(\bar{x})+\xi-\delta)^+]^+ + \theta E[x-(d^*(\bar{x})+\xi)^+]^+ \\
&\quad +\alpha sE[\delta-d^*(\bar{x})-\xi]^+ - \alpha sE[-d^*(\bar{x})-\xi]^+.
\end{aligned}$$

In Case (ii), the profit improves by

$$G_T(x,0,d^*(\bar{x})-\delta)-G_T(x,0,d^*(\bar{x})) = [R(d^*(\bar{x})-\delta)-R(d^*(\bar{x}))]+A < -c\delta+A,$$

where the inequality follows since $R(d^*(\bar{x}))=c$ and $R(d)$ is a strictly concave function of d. Consequently, $[R(d^*(\bar{x})-\delta)-R(d^*(\bar{x}))]<c\delta$. Finally, in case (iii), the profit improves

by

$$
\begin{aligned}
G_T(x, \frac{\delta}{2}, d^*(\bar{x}) - \frac{\delta}{2}) - G_T(x, 0, d^*(\bar{x})) &= [R(d^*(\bar{x}) - \frac{\delta}{2}) - R(d^*(\bar{x}))] - c[\frac{\delta}{2}] + A \\
&< -c\frac{\delta}{2} - c\frac{\delta}{2} + A.
\end{aligned}
$$

The reason for the inequality remains the same as for Case (ii). From the above analysis, Case (i) produces the best profit improvement.

Suppose now that $q = \delta$, $d = d^*(\bar{x})$, $\frac{\partial G_T(x,q,d)}{\partial q}|_{q=\delta, d=d^*(\bar{x})} > 0$, and $\frac{\partial G_T(x,q,d)}{\partial d}|_{q=\delta, d=d^*(\bar{x})} < 0$. Once again, we repeat the same argument as above, and find that increasing q by δ will improve profit the most. We keep increasing q by δ until $\frac{\partial G_T(x,q,d)}{\partial q}|_{q, d=d^*(\bar{x})} = 0$. Observe that the value of q that results in $\frac{\partial G_T(x,q,d)}{\partial q}|_{q, d=d^*(\bar{x})} = 0$ will also result in $\frac{\partial G_T(x,q,d)}{\partial d}|_{q, d=d^*(\bar{x})} = 0$. Therefore, $q^*(x) > 0$ and $d^*(\bar{x})(x) = d^*(\bar{x})$ for $x < \bar{x}$.

On the other hand, for $x > \bar{x}$,

$$
\begin{aligned}
& \frac{\partial G_T(x,q,d)}{\partial q}|_{q=0, d=d^*(\bar{x})} < \frac{\partial G_T(\bar{x},q,d)}{\partial q}|_{q=0, d=d^*(\bar{x})} = 0 \\
= \quad & \frac{\partial G_T(\bar{x},q,d)}{\partial d}|_{q=0, d=d^*(\bar{x})} < \frac{\partial G_T(x,q,d)}{\partial d}|_{q=0, d=d^*(\bar{x})}.
\end{aligned}
$$

Given the signs of the slopes of G_T, to increase profit, we must either decrease q or increase d (or do both). Given that q must remain non-negative, we cannot decrease q. Therefore, our only option is to increase d. Suppose we increase d by δ where $\delta > 0$ and sufficiently small. Then

$$
\frac{\partial G_T(x,q,d)}{\partial q}|_{q=0, d=d^*(\bar{x})+\delta} - \frac{\partial G_T(x,q,d)}{\partial q}|_{q=0, d=d^*(\bar{x})} = B,
$$

where $B = -(h + \pi + \theta + \alpha c)(F(x - d^*(\bar{x}) - \delta) - F(x - d^*(\bar{x})))$. Furthermore,

$$
\frac{\partial G_T(x,q,d)}{\partial d}|_{q=0, d=d^*(\bar{x})+\delta} - \frac{\partial G_T(x,q,d)}{\partial d}|_{q=0, d=d^*(\bar{x})} = [R'(d^*(\bar{x}) + \delta) - R'(d^*(\bar{x}))] - B.
$$

Since $R(\cdot)$ is strictly concave, $R'(d^*(\bar{x})+\delta)-R'(d^*(\bar{x}))<0$ and so

$$\begin{aligned} & \left|\frac{\partial G_T(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})+\delta}-\frac{\partial G_T(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}\right| \\ > \quad & \left|\frac{\partial G_T(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})+\delta}-\frac{\partial G_T(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})}\right|. \end{aligned}$$

This means that the slope of G_T with respect to q increases less in magnitude than the amount by which the slope of G_T with respect to d decreases. Now, if we repeat the same procedure of increasing expected demand by a small amount $\delta>0$ over and over again, the above inequality continues to hold. As a result, either we will reach the upper bound of $\mathcal{D}$ or the slope of G_T with respect to d will become 0 before the slope of G_T with respect to q does. Thus, the optimal value of q will remain equal to 0.

Now, consider period $t<T$.

$$\begin{aligned} \frac{\partial G_t(x,q,d)}{\partial q} &= -c+\pi-(h+\pi+\theta)F(x+q-d)+\theta F(q-d) \\ &\quad +\alpha\int_{-a}^{q-d} v'_{t+1}(q-d-\xi)f(\xi)d\xi+\alpha\int_{x+q-d}^{\infty} v'_{t+1}(x+q-d-\xi)f(\xi)d\xi \end{aligned} \tag{B.0.28}$$

$$\begin{aligned} \frac{\partial G_t(x,q,d)}{\partial d} &= R'(d)-\pi+(h+\pi+\theta)F(x+q-d)-\theta F(q-d) \\ &\quad -\alpha\int_{-a}^{q-d} v'_{t+1}(q-d-\xi)f(\xi)d\xi-\alpha\int_{x+q-d}^{\infty} v'_{t+1}(x+q-d-\xi)f(\xi)d\xi \end{aligned} \tag{B.0.29}$$

Once again, set $x=\bar{x}$ such that the unconstrained optimal value of $q^*(\bar{x})=0$. In case of multiple such values, choose the highest one. Using the induction hypothesis, $v'_{t+1}(x)=c$ for $x\leq 0$ and for any given d,

$$\frac{\partial G_t(\bar{x},q,d)}{\partial q}|_{q=0} = -c+\pi+\alpha c-(h+\pi+\theta+\alpha c)F(\bar{x}-d) \tag{B.0.30}$$

$$\frac{\partial G_t(\bar{x},y,d)}{\partial d}|_{q=0} = R'(d)-\pi-\alpha c+(h+\pi+\theta+\alpha c)F(\bar{x}-d) \tag{B.0.31}$$

Following the same argument as for period T, we can show that $R'(d^*(\bar{x})) = c$. Now, for $x < \bar{x}$,

$$\begin{aligned} &\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} > \frac{\partial G_t(\bar{x},q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = 0 \\ &= \frac{\partial G_t(\bar{x},q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} > \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}. \end{aligned}$$

Using the same argument as for period T, we can show that q^* is strictly positive and the optimal demand satisfies $R'(d^*(\bar{x})) = c$. For $x > \bar{x}$,

$$\begin{aligned} &\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_t(\bar{x},q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = 0 \\ &= \frac{\partial G_t(\bar{x},q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} < \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})}. \end{aligned}$$

Given the signs of the slopes of G_t, we must either decrease q or increase d to improve profit. However, since q is constrained to be non-negative, we cannot decrease q. Our only option is to increase d. Suppose we increase d by δ where $\delta > 0$ and is sufficiently small. Then

$$\frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_t(x,q,d)}{\partial q}|_{q=0,d=d^*(\bar{x})} = B,$$

where $B = -(h+\pi+\theta+\alpha c)(F(x-d^*(\bar{x})-\delta) - F(x-d^*(\bar{x})))$. Similarly,

$$\frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})+\delta} - \frac{\partial G_t(x,q,d)}{\partial d}|_{q=0,d=d^*(\bar{x})} = [R'(d^*(\bar{x})+\delta) - R'(d^*(\bar{x}))] - B.$$

Observe that the value of B is the same as for $t = T$. Hence, we can easily replicate the argument for $t = T$; the details are omitted.

Proof of Part 2

When $x < \bar{x}$, using the implicit differentiation rule,

$$v_T'(x) \quad = \quad \frac{\partial G_T(x,q,d)}{\partial x}|_{q=q^*,d=d^*} + \frac{\partial G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} \cdot q^{*\prime} + \frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \cdot d^{*\prime}$$

Since we know that d^* is a constant when $x \leq \bar{x}$, the third term is equal to zero. Also, $\frac{G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*} = 0$. Therefore,

$$\begin{aligned} v_T'(x) &= \frac{\partial G_T(x,q,d)}{\partial x}|_{q=q^*,d=d^*} = \pi + \alpha c - (h + \pi + \alpha c + \theta)F(x + q^* - d^*) \\ &= c - (\theta + \alpha s)F(q^* - d^*) + \underbrace{\frac{G_T(x,q,d)}{\partial q}|_{q=q^*,d=d^*}}_{=0}. \end{aligned}$$

While the first equation shows that $v_T'(x) \geq -h - \theta$, the second equation establishes that $v_T'(x)$ is bounded from above by c. Similarly, for period t,

$$\begin{aligned} v_t'(x) &= \frac{\partial G_t(x,q,d)}{\partial x}|_{q=q^*,d=d^*} = \pi - (h + \pi + \theta)F(x + q^* - d^*) \\ &\quad +\alpha \int_{x+q^*-d^*}^{\infty} v_{t+1}'(x + q^* - d^* - \xi)f(\xi)d\xi \\ &= \pi + \alpha c - (h + \pi + \theta + \alpha c)F(x + q^* - d^*), \end{aligned} \tag{B.0.32}$$

where we use $v_{t+1}'(x) = c$ for $x \leq 0$. It can be easily seen that $v_t'(x) \geq -\theta - h$. Now, using Equation B.0.28, Equation B.0.32 can also be written as

$$v_t'(x) = c - \theta F(q^* - d^*) - \alpha \int_{-a}^{q^*-d^*} v_{t+1}'(q^* - d^* - \xi)f(\xi)d\xi + \underbrace{\frac{G_t(x,q,d)}{\partial q}|_{q=q^*,d=d^*}}_{=0}.$$

Using the induction hypothesis, $v_{t+1}'(x) \geq -\theta - h$. Therefore, $v_t'(x) \leq c + \alpha h - (1 - \alpha)\theta$.

Proof of Part 3

When $x \geq \bar{x}$, $q^*(x) = 0$. As a result,

$$\frac{\partial G_T(x,0,d)}{\partial d} = R'(d) - \pi - \alpha c + (h + \pi + \alpha c + \theta)F(x - d).$$

Now, when there exists a feasible d at which the above equation is 0, then we can use the Implicit Function Theorem to obtain

$$d^{*\prime}(x) = -\frac{\partial^2 G_t(x,0,d)/\partial x \partial d}{\partial^2 G_t(x,0,d)/\partial d^2}|_{d=d^*} = \frac{(h + \pi + \theta + \alpha c)f(x - d^*)}{-R''(d^*) + (h + \pi + \theta + \alpha c)f(x - d^*)} \in (0,1).$$

Otherwise, if no d feasible exists at which $\frac{\partial G_T(x,0,d)}{\partial d} = 0$, then as we argued in the proof of Part 1, $d^* = \max \mathcal{D}$ and so $d^{*\prime}(x) = 0$.

The argument for any other period $t < T$ is identical since the expression for $\frac{\partial G_t}{\partial d}$ is same as for $\frac{\partial G_T}{\partial d}$, and the details are omitted. For $x < \bar{x}$, $d^{*\prime}(x) = 0$ since the optimal order quantity is a constant.

Next, consider $x \leq 0$. For such x,

$$v_t(x) = \max_{d \in \mathcal{D}, q \geq 0} R(d) - cq - hE[x + q - d - \xi]^+ - \pi E[d + \xi - x - q]^+ + \alpha E v_{t+1}(x + q - d - \xi).$$

Let $y = x + q$. Then, the above formulation becomes

$$\begin{aligned} v_t(x) &= cx + \max_{d \in \mathcal{D}, y \geq x} R(d) - cy - hE[y - d - \xi]^+ - \pi E[d + \xi - y]^+ \\ &\quad + \alpha E v_{t+1}(y - d - \xi). \end{aligned} \tag{B.0.33}$$

Similar to Proposition 4.3.1, it can be easily shown that the optimal value of d satisfies $R'(d) = c$ for all such x. Now, we claim that there exists a value of $y > x$ that produces strictly greater profit than $y = x$. To see this, we compute the derivative of the maximand with respect to y at $y = x$ as follows:

$$-c - (h + \pi)F(x - d^*) + \pi + \alpha c,$$

where we use $v'_{t+1}(x) = c$ for $x \leq 0$ by induction hypothesis. Since $x - d^* \leq -a$, $F(x - d^*) = 0$. Thus, the derivative becomes $\pi - c + \alpha c > 0$. Hence, increasing y from x to $x + \delta$ will increase profit, so $y^* > x$. With this observation, we can now drop the constraint $y \geq x$ from the RHS in Equation B.0.33. It can now be easily seen that $v'_t(x) = c$.

Proof of Parts 4 and 5

For $x \geq \bar{x}$, $q^*(x) = 0$.

$$\begin{aligned} \frac{\partial v_T(x)}{\partial x} &= \frac{\partial G_T(x, q^*, d^*)}{\partial x} + \underbrace{\frac{\partial G_T(x, q, d)}{\partial d}\Big|_{q=q^*, d=d^*} \cdot d^{*\prime}(x)}_{=0} \\ &= -(\pi + h + \theta + \alpha c)F(x - d^*) + \pi + \alpha c. \end{aligned}$$

where $\frac{\partial G_T(x,q,d)}{\partial d}|_{q=q^*,d=d^*} \cdot d^{*\prime}(x) = 0$ since either $d^{*\prime}(x)$ or $\frac{\partial G_T(x,q,d)}{\partial d}|_{q=0,d=d^*(x)} = 0$. Clearly, $v_T'(x) \geq -h - \theta$, and

$$v_T''(x) = -(h + \pi + \theta + \alpha c) f(x - d^*(x))[1 - d^{*\prime}(x)] \leq 0$$

since $d^{*\prime}(x) < 1$, as shown in Part 4 above. Similarly, in period t,

$$\begin{aligned} v_t'(x) &= \frac{\partial G_t(x,q,d)}{\partial x}|_{q=0,d=d^*} \\ &= \pi - (h + \pi + \theta)F(x - d^*) + \alpha \int_{x-d^*}^{\infty} v_{t+1}'(x - d^* - \xi) f(\xi) d\xi \\ &= \pi + \alpha c - (h + \pi + \theta + \alpha c)F(x - d^*), \end{aligned}$$

where we use $v_{t+1}'(x) = c$ for $x \leq 0$. It can be easily seen that $v_t'(x) \geq -h - \theta$. Further, we take the derivative for the above equation, then

$$v_t''(x) = -(h + \pi + \theta + \alpha c) F(x - d^*(x))(1 - d^{*\prime}(x)) \leq 0$$

since $d^{*\prime}(x) < 1$.

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BIOGRAPHICAL SKETCH

Li-Ming Chen was born in Taipei Taiwan on November 5, 1979. Upon completion of high school, he attended National Chiao Tung University in HsinChu, earning a degree in industrial engineering and management. After he obtained his bachelor's degree in June 2001, he served as a second lieutenant in the military for 2 years.

In June 2003, Li-Ming finished his military career in Taiwan. Realizing the desire for knowledge, he went to the United States for the pursuit of higher education. He began his graduate studies at the University of Michigan in the fall of 2003, and gained his master's degree in industrial and operations engineering in January 2005.

Li-Ming began to find enjoyment in exploring new ideas, resolving complex mathematical problems, and seeking efficient solutions. Therefore, he decided to pursue his doctorial degree in industrial and systems engineering at the University of Florida, which offered programs aligned with his interests in Operations Research and Supply Chain Management. While at the University of Florida, he received department funding as a full-time research assistant, and during which time, he also served as a teaching assistant for several courses in support of completing course objectives. He earned the Doctor of Philosophy degree in August, 2009

Dr. Chen's recent work discussed the replenishment decision from multiple sourcing when Supply Chain Disruption happens.