# COORDINATING REPLENISHMENT AND DISPOSAL DECISIONS IN CLOSED-LOOP SUPPLY CHAINS

By

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To my family

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We consider the characterization of the optimal inventory and production planning policies for multi-product closed-loop supply chain systems for direct reuse and value-added recovery by considering a class of deterministic discrete time dynamic demand and return models. In particular, we consider three problems. First, motivated by the an original equipment manufacturer (OEM) in the power generation equipment industry that provides spare part kits for power turbine maintenance services, we study a replenishment and disposal planning problem that arises in settings where customer returns are in as-good-as-new condition. These returns can be placed into inventory to satisfy future demand or can be disposed of, in case they lead to excess inventory. We propose a Lagrangian Relaxation approach that relies on the relaxation of the capacity constraints and develop a smoothing heuristic that uses the solution of the Lagrangian problem to obtain near-optimal solutions. Our computational results demonstrate that the proposed approach is very effective in obtaining high-quality solutions with a reasonable computational effort. Next, to take the uncertainty associated with directly reusable customer returns into account explicitly, we consider a procurement planning problem encountered by a catalogue retailer that also receives customer returns. To this end, we develop a discrete time dynamic deterministic demand and stochastic return model for the characterization of inventory and production planning policy. We develop solution algorithms using continuous optimization techniques, and identify the optimal solution

structures. Finally, we consider more general models for value-added recovery systems. Specifically, motivated by an OEM in the automotive industry that provides vehicle maintenance and repair services for vehicles for which replacement parts are often needed, we study manufacturing, remanufacturing, and disposal planning problem with shared manufacturing and product-specific remanufacturing and disposal capacities. We develop solution approaches based on Lagrangian Decomposition and Relaxation techniques used together with a constructive heuristic and Genetic Algorithms to obtain lower and upper bound solutions of the underlying problem. We conduct extensive computational experiments to investigate efficiency and effectiveness of proposed solution algorithms. Our computational results shows that although the Lagrangian decomposition approach is computationally more efficient, the Lagrangian relaxation approach is effective in finding high-quality solutions.

# CHAPTER 1
# INTRODUCTION

## 1.1 Background and Motivation

According to the United States (US) Environmental Protection Agency, more than 12 billion tons of industrial and 480 million tons of residential waste are produced in the US each year. Although some of the solid waste generated is recovered via recycling, the majority is directed to landfills and incineration facilities. While the rate of solid waste generation is expected to increase significantly, landfill and incineration capacities are anticipated to drop at a considerable rate (Pohlen and M.T. Farris, 1992; Rogers and Tibben-Lembke, 1999). To reduce the economical and environmental burden of solid waste, product recovery practices are becoming increasingly popular.

Product recovery management (PRM) encompasses the planning and management of a broad set of activities to reclaim the economical and environmental value residing in end-of-use or end-of-life products. The economical value that can be reclaimed from the end-of-use and end-of-life products is related to the cost savings realized by recovering products, components, and materials for reuse. Similarly, the environmental value involves the environmental impact reduction realized by diminishing the reliance on landfilling and incineration for solid waste disposal.

Historically, both the theory and practice of supply chain management has placed emphasis on the *forward channel*, which consists of suppliers, manufacturers, distributors, retailers, and customers. Recent interest in product recovery, however, has extended the scope of traditional supply chain management by drawing attention to the *reverse channel*, which consists of final-users, retailers, collectors, and remanufacturers. Since product recovery processes returned and used products in the reverse channel, the efficiency with which these products are processed has a direct impact on the profitability of product recovery practices. Moreover, as returns are processed or remanufactured to satisfy the customer demand, this impacts the forward channel flows and introduces a strong

interdependence between forward and reverse flows in the underlying closed-loop supply chain (CLSC). Consequently, one of the fundamental issues in the management of CLSC systems for product recovery is concerned with the coordination of activities of the agents in the forward and reverse channels via optimal inventory and production planning.

A closer examination of existing product recovery practices reveal that it is possible to distinguish between two forms of product recovery:

- **Direct reuse:** This type of product recovery is closely related to commercial returns, where returns are typically in as-good-as-new condition and can be reused without any processing. In direct reuse, the returns can be placed in finished goods inventory to satisfy demand in future periods, and hence, the returns need to be taken into account while making replenishment decisions. Also, if inventory accumulates excessively, then some items can be disposed of to reduce the finished goods inventory level. Hence, the inventory and production planning in the context of direct reuse considers a single stock point and focuses on the coordination of replenishment and disposal activities. Numerous examples of direct reuse can be observed in catalogue, internet, and brick-and-mortar retailing.

- **Value-added recovery:** For this type of product recovery, returns are used products that must be restored into reusable condition by remanufacturing. In value-added recovery, the returns can be kept in used item inventory. If inventory accumulates excessively, then some items can be disposed of to reduce the used item inventory level. Moreover, used items can be remanufactured to replenish finished goods inventory. Hence, used items inventory that can be remanufactured need to be taken into account while making replenishment decisions. Hence, the inventory and production planning in context of value-added recovery considers two stock points and focuses on the coordination of replenishment and disposal activities. Some specific examples of value-added recovery apply to single-use cameras, automotive parts, photocopy equipment, computers, and cellular phones.

## 1.2 Overview

In our research, we propose to focus on the characterization of the optimal inventory and production planning policies for multi-product CLSC systems for direct reuse and value-added recovery by considering a class of deterministic discrete time dynamic demand and return models. We develop efficient solution approaches that are based on dynamic programming, relaxation, decomposition, and meta-heuristic approaches to address this class of models. Moreover, we consider a discrete time dynamic deterministic demand and

stochastic return model for the characterization of inventory and production planning policy a direct reuse setting. The solution approach for this model relies on dynamic programming and continuous optimization techniques.

## 1.3 Impact

Upon a closer examination of the current literature on inventory and production planning for product recovery reveals that the existing work mainly focuses on uncapacitated models for single-product environments. There has been little work on capacitated models for multi-item environments. In our research, by focusing primarily on capacitated models for multi-item environments, we will attempt to fill the gap in the current literature. Our objective is to develop a basic set of models and effective solution algorithms to coordinate the underlying manufacturing, remanufacturing, and disposal decisions for multi-product capacitated models for CLSC systems for direct reuse and value-added recovery. By concentrating on these models and solution algorithms, we will develop decision making tools that ensure the efficiency of underlying CLSC systems for product recovery.

# CHAPTER 2
# LITERATURE REVIEW

## 2.1 Introduction

The existing work on inventory and production planning can broadly be classified according to the modeling of the demand and return processes by making a distinction between *deterministic* models and *stochastic* models. Deterministic models assume that the demands and returns are known throughout the planning horizon, whereas stochastic models take into account the uncertainty associated with (i) the quantity and timing of demands and (ii) the quantity, quality, and timing of returns. Deterministic models can further be classified according to the modeling of the demand and return processes by making a distinction between *continuous* and *discrete* time models. Deterministic continuous time models assume that that the demands and returns arrive continuously over time, whereas deterministic discrete time models are based on the assumption that the quantity of demands and returns in each time period are known throughout the planning horizon. In this context, the problems of interest in our research are related to three streams of research on *deterministic discrete time models for inventory and production planning for product recovery*, *stochastic models for inventory and production planning for product recovery*, and *multi-item capacitated lot sizing*.

## 2.2 Deterministic Discrete Time Models for Inventory and Production Planning for Product Recovery

Due to the increased need in practice (Guide, 2000), the general topic of inventory and production planning for product recovery has received considerable attention in recent years. For a comprehensive review of the literature, we refer the reader to (Dekker and van der Laan, 2003; Inderfurth et al., 2004; Inderfurth and Teunter, 2003; van der Laan et al., 2004; Richter and Dobos, 2004). A careful examination of the inventory and production planning for product recovery, however, reveals that this line of work mainly concentrates on value added recovery (Golany et al., 2001, 2005; Li et al., 2006b,a; Richter

and Gobsch, 2003; Richter and Sombrutzki, 2000; Richter and Weber, 2001; Yang et al., 2005) with one exception that focuses on direct reuse (Beltrán and Krass, 2002).

Motivated by practical applications in handling returns in catalogue retailing and managing aircraft engine service component kits, Beltrán and Krass (Beltrán and Krass, 2002) consider a single item uncapacitated lot sizing for a system where returns received from customers are placed in finished goods inventory without any processing. Their model is aimed to determine the optimal quantity of items to procure, quantity of items to dispose, and inventory level for items throughout the planning horizon with the objective of minimizing the sum of concave and piecewise differentiable procurement, disposal, and inventory holding costs. To obtain the solution to this model, they develop a DP algorithm that runs in polynomial time.

Richter and Sombrutzki (2000) consider the single item uncapacitated lot sizing problem for a pure remanufacturing system with fixed-charge remanufacturing and linear inventory holding costs. Their model seeks the optimal quantity of items to remanufacture in each period and the inventory levels for remanufacturable and remanufactured items throughout the planning horizon with the objective of minimizing the sum of remanufacturing setup and inventory holding costs. They show that a modification of the classical lot sizing algorithm can be used to obtain the solution (Wagner and Whitin, 1958). They also consider the problem in the context of a hybrid remanufacturing and manufacturing system, considering fixed-charge manufacturing costs where the model seeks also the optimal quantity of items manufactured in each period. However, they do not provide a solution approach for this general problem, but consider a special case of the problem where the quantity of remanufacturable items available at the beginning of the planning horizon is sufficiently large to cover the entire demand throughout the planning horizon. They develop a dynamic programming (DP) approach to obtain the solution. Richter and Weber (2001) extend the model formulations given by Richter and Sombrutzki (2000) by including linear remanufacturing and manufacturing costs and

develop DP approaches to obtain solutions. They also consider extend their models to take into account disposal option with linear disposal cost, which is allowed only in the first period. Richter and Gobsch (2003) concentrate on the single item uncapacitated lot sizing problem for a hybrid manufacturing and remanufacturing system considering fixed-charge manufacturing and remanufacturing as well as linear new material purchase, remanufacturable item acquisition, and inventory holding costs. Their model seeks the optimal quantity of remanufactured items and material to purchase, quantity of new items to manufacture, quantity of remanufacturable items to remanufacture, and inventory levels for new materials, remanufacturable items, and remanufactured/new items throughout the planning horizon with the objective of minimizing the total costs. They provide a nested DP approach to solve this general problem. They also analyze two special cases of the problem where (i) no material or remanufacturable item inventory is held and (ii) no remanufactured or new item inventory is held.

Golany et al. (2001, 2005) consider a single item uncapacitated lot sizing problem for a hybrid manufacturing and remanufacturing system with manufacturing, remanufacturing, disposal and inventory holding costs. They show that the problem (i) is NP-hard for general concave costs and (ii) can be transformed into a minimum cost network flow problem if all costs are linear. Yang et al. (2005) show that the problem remains NP-hard for stationary concave costs and develop a DP approach for the problem. By exploring the characteristics of the extreme point solutions, they develop a DP based heuristic approach that runs in polynomial time.

Li et al. (2006b) consider a multi-item uncapacitated lot sizing problem for a hybrid manufacturing and remanufacturing system, where downward substitution of items is allowed, i.e., a higher grade item can be substituted for a lower grade item to satisfy the demand. We note that the grade does not refer to whether the item is new or remanufactured, but to the characteristics of the item, e.g., a faster processor can be substituted for a processor with a lower speed, or a memory card with more storage space

can be substituted for a memory card with less storage space. Their model seeks the optimal quantity to manufacture and remanufacture for each item, remanufacturable and new/remanufactured inventory levels for each item as well as the quantity of a higher grade item used to satisfy the demand for a lower grade item with the objective of minimizing the sum of fixed-charge manufacturing and remanufacturing as well as linear inventory holding and substitution costs. For a special case of the problem where return flows do not impose capacity restrictions on remanufacturing quantities, i.e., the quantity of returns for an item in each period is larger than the demand for the item in that period, they develop a DP algorithm that exploits the structural properties of an optimal solution. For the general problem, they propose a two-stage heuristic approach, where a modification of the DP algorithm is used to find an initial feasible solution and an improvement procedure to obtain a better one. Li et al. (2006a) develop a two-item model in a capacitated setting where emergency procurements are allowed in case of a potential shortage. Their model seeks the optimal quantity to manufacture, remanufacture, and procure for each item, remanufacturable and new/remanufactured inventory levels for each item as well as the quantity of a lower grade item used to satisfy the demand for a higher grade item with the objective of minimizing the sum of fixed-charge manufacturing and remanufacturing as well as linear manufacturing, remanufacturing, emergency procurement, and inventory holding costs. Although downward substitution is allowed to satisfy the demand, the cost of substitution is assumed to be zero. The authors develop a hybrid DP and genetic algorithm (GA) approach to obtain a solution for the problem, where GA approach is used to specify the timing of the manufacturing and remanufacturing setups, whereas the DP algorithm is used to determine the manufacturing, remanufacturing, substitution, and emergency procurement quantities.

## 2.3 Stochastic Models for Inventory and Production Planning for Product Recovery

There is a rich body of literature on stochastic models for inventory and production planning for product recovery. Stochastic inventory and production planning models can be classified further according to the review policy in effect by making a distinction between *continuous* and *periodic* review. Under continuous review, the inventory level is monitored continuously over time, whereas under periodic review, the inventory level is observed at discrete, equally spaced points in time.

The existing continuous review models attempt to compute inventory and production control policy parameters mainly for value-added recovery systems. Assuming that all the returns are remanufactured one at a time upon receipt, a line of work focuses on systems with a single stock point Bayındır et al. (2003, 2005); Fleischmann et al. (2002); Heyman (1977); van der Laan (2003); Toktay et al. (2000); Yuan and Cheung (1998). Another line of work focuses on settings where returns are remanufactured in batches Aras et al. (2004); Muckstadt and Isaac (1981); Teunter et al. (2000); Teunter (2002); Teunter et al. (2004); van der Laan (2003); van der Laan et al. (1996a,b); van der Laan and Salomon (1997); van der Laan et al. (1999a,b); van der Laan and Teunter (2006). The emphasis is mainly on the analysis of single-item settings Aras et al. (2004); Bayındır et al. (2003, 2005); Fleischmann et al. (2002); Heyman (1977); Muckstadt and Isaac (1981); Teunter et al. (2000); Teunter (2002); Teunter et al. (2004); Toktay et al. (2000); van der Laan (2003); van der Laan et al. (1996a,b); van der Laan and Salomon (1997); van der Laan et al. (1999a,b); van der Laan and Teunter (2006); Yuan and Cheung (1998), considering both finite Teunter et al. (2000); Teunter (2002); Teunter et al. (2004) and infinite Aras et al. (2004); Bayındır et al. (2003, 2005); Fleischmann et al. (2002); Heyman (1977); Muckstadt and Isaac (1981); Toktay et al. (2000); van der Laan (2003); van der Laan et al. (1996a,b); van der Laan and Salomon (1997); van der Laan et al. (1999a,b); van der Laan and Teunter (2006); Yuan and Cheung (1998) planning horizons. Only a limited amount of

research on continuous review models aims to characterize the optimal policy Fleischmann et al. (2002); Heyman (1977), and the main emphasis of these models is on analyzing system performance under a particular inventory control policy. The underlying system is typically modeled as a continuous time Markov chain Aras et al. (2004); Bayındır et al. (2005); Muckstadt and Isaac (1981); van der Laan (2003); van der Laan et al. (1996a,b); van der Laan and Salomon (1997); van der Laan et al. (1999a,b); van der Laan and Teunter (2006); Yuan and Cheung (1998) or a queueing network Bayındır et al. (2003); Toktay et al. (2000). To compute the inventory control policy parameters, numerical optimization techniques Aras et al. (2004); Bayındır et al. (2003, 2005); Muckstadt and Isaac (1981); van der Laan (2003); van der Laan et al. (1996a,b); van der Laan and Salomon (1997); van der Laan et al. (1999a,b); van der Laan and Teunter (2006); Yuan and Cheung (1998) or analytical approximations Toktay et al. (2000); van der Laan (2003); van der Laan and Teunter (2006) are used. A more recent line of work also uses simulation modeling and analysis to compute effective values for policy parameters or to study system performance Teunter et al. (2000); Teunter (2002); Teunter et al. (2004); Toktay et al. (2000).

The majority of existing periodic review models aim to compute inventory and production control policy parameters for value-added recovery systems. While some models are developed considering a single stock point Buchanan and Abad (1998); Cohen et al. (1980); Fleischmann and Kuik (2003); Inderfurth (1997); Kelle and Silver (1989); Kiesmüller and Scherer (2003); Kiesmüller and van der Laan (2001); Mostard and Teunter (2006); Nakashima et al. (2002, 2004); Vlachos and Dekker (2003); Whisler (1967), others focus on settings with two stock points Bayındır et al. (2007); Inderfurth (1997, 2004); Kiesmüller (2003); Kiesmüller and Minner (2003); Kiesmüller and Scherer (2003); Mahadevan et al. (1999); Simpson (1978); Teunter and Vlachos (2002),and the general emphasis of the existing models has been on single-item settings considering single period Bayındır et al. (2007); Inderfurth (2004); Mostard and Teunter (2006); Vlachos and

Dekker (2003), finite horizon Buchanan and Abad (1998); Cohen et al. (1980); Heisig and Fleischmann (2001); Inderfurth (1997); Kelle and Silver (1989); Kiesmüller and Minner (2003); Kiesmüller and Scherer (2003); Kiesmüller and van der Laan (2001); Simpson (1978); Teunter and Vlachos (2002); Whisler (1967), and infinite horizon Fleischmann and Kuik (2003); Heisig and Fleischmann (2001); Kiesmüller (2003); Mahadevan et al. (1999); Nakashima et al. (2002, 2004); Whisler (1967) models.

A limited stream of research on periodic review models characterizes the optimal policy using differential calculus techniques in single period settings Bayındır et al. (2007); Inderfurth (2004); Mostard and Teunter (2006); Vlachos and Dekker (2003) and dynamic-programming-based optimization techniques in finite horizon settings Cohen et al. (1980); Fleischmann and Kuik (2003); Inderfurth (1997); Simpson (1978); Whisler (1967). Also, Markov decision process models are developed to compute the optimal inventory control parametersNakashima et al. (2002, 2004). Other existing work in the area focuses on analyzing system performance under a particular inventory control policy and uses dynamic programming Buchanan and Abad (1998); Cohen et al. (1980); Kelle and Silver (1989); Kiesmüller and Scherer (2003) or simulation based Kiesmüller (2003); Kiesmüller and Minner (2003); Kiesmüller and van der Laan (2001); Mahadevan et al. (1999); Teunter and Vlachos (2002) approaches to compute the control parameters.

### 2.4 Multi-Item Capacitated Lot Sizing

Production planning problems with setups between consecutive production batch releases are known as lot sizing problems (Brahimi et al., 2006). It is possible to classify the existing work on lot sizing problems according to the (i) number of products (i.e., single- vs. multi-item) considered, (ii) number of levels in the bill-of-materials (i.e., single- vs. multi-level) considered, and (iii) restriction on the capacity availability of production resources (i.e., uncapacitated vs. capacitated) taken into account (Brahimi et al., 2006; Karimi et al., 2003). In this context, the class of problems we consider in the proposed

research is most closely related to single-level, capacitated single- and multi-item lot sizing problems.

There are comprehensive reviews of single-level single-item (Brahimi et al., 2006) and multi-item (Karimi et al., 2003) capacitated lot sizing models in the literature. More recently, Jans and Degraeve (Jans and Degraeve, 2007) review the meta-heuristic approaches developed for difficult lot sizing problems. The general single-item capacitated lot sizing problem (Florian et al., 1980) as well as some special cases (Bitran and Yanasse, 1982) are NP-hard. The general multi-item capacitated lot sizing problem is strongly NP-hard (Chen and Thizy, 1990). In what follows, we review some of the existing work that develops exact and heuristic solution approaches for the multi-item capacitated lot sizing problem.

A line of existing work focuses on the development of exact solution approaches for the single-level multi-item capacitated lot sizing problem (Barany et al., 1984; Eppen and Martin, 1987; Leung et al., 1989). While Barany et al. (1984) propose cut generation technique which are the facets of the single-item uncapacitated lot sizing problem, Leung et al. (1989) explore the use of the facets of the single-item capacitated lot sizing problem. Eppen and Martin (1987) propose alternative formulations that yield much tighter linear relaxation models for both the single- and multi-item capacitated lot sizing problem. Belvaux and Wolsey (2000, 2001) develop a framework for the solution of the multi-level, multi-resource, multi-item capacitated lot sizing problem with backlogging that performs some preprocessing, generates cutting planes, and utilizes a primal heuristic for the lot sizing problem.

There are a number of papers that develops specialized constructive and/or improvement heuristics for the multi-item capacitated lot sizing problem (Dogramaci et al., 1981; Dixon and Silver, 1981; Eisenhut, 1975; Günther, 1987; Karni and Roll, 1982; Kırca and Kökten, 1994; Lambrecht and Vanderveken, 1979; Maes and Wassenhove, 1986; Selen and Heuts, 1987; Trigeiro, 1989). Eisenhut (1975) proposes a single forward-pass

approach that uses a marginal cost coefficient. On each iteration, all the items are ordered in their non-decreasing order of cost coefficients, and the requirements for the items with highest positive coefficients are added to the lot until capacity is reached on each iteration. As Eisenhut's heuristic can end up with an infeasible solution, both Lambrecht and Vanderveken (1979) and Dixon and Silver (1981) propose procedures to remove capacity infeasibilities. Dogramaci et al. (1981) and Karni and Roll (1982) propose multi-phase heuristic approaches that obtain improved solutions from initial feasible solutions. Maes and Wassenhove (1986) propose a flexible approach that can use different rules for the apriori ordering of items, criteria used to decide whether or not to include the demand for an item in the current period, and search strategy. Their approach includes procedures to ensure feasibility and to improve a given feasible solution. All these approaches build a plan using a rolling horizon approach. Kırca and Kökten (1994) focus on items rather than periods, and on each iteration of the approach, a feasible production plan for an item is developed. That is, once the plan for an item is generated, the available capacity in each period is updated. Then, a DP based procedure is used to generate the plan for the next item. Günther (1987) propose to start from a lot-for-lot solution and build a capacity feasible solution using a priority index to combine lots, a look-ahead procedure to ensure cumulative capacity feasibility, and a fix-up procedure to remove period capacity infeasibilities. Selen and Heuts (1987) propose the use of a modified priority index for Günther's heuristic. Trigeiro (1989) proposes a multi-pass approach for the version of the problem with setup times. In the first forward pass an initial plan is built using a modified version of the well-known Silver-Meal priority index. In the second backward pass, capacity infeasibilities are removed by shifting excess quantities. The final third pass ensures that the resulting solution satisfies the properties of an extreme point solution.

Another line of work focuses on the development of relaxation-based heuristics for the multi-item capacitated lot sizing problem (Chen and Thizy, 1990; Diaby et al., 1992b,a; Millar and Yang, 1994; Thizy, 1991; Thizy and Wassenhove, 1985; Trigeiro, 1987; Trigeiro

et al., 1989). Thizy and Wassenhove (1985) and Trigeiro (1987) propose Lagrangian relaxation approaches that focus on the relaxation of the capacity constraints. While Thizy and Wassenhove (1985) develop a transportation problem based upper bounding heuristic to construct a feasible schedule from the dual solution to the original problem, Trigeiro (1987) implements a smoothing heuristic for the same purpose. Chen and Thizy (1990) provide a comprehensive overview of relaxation approaches and show that the Lagrangian relaxation of the capacity constraints yield the tightest lower bound on the optimal solution. Thizy (1991) investigates the quality of bounds generated by alternative Lagrangian decomposition approaches. Trigeiro et al. (1989) develop a Lagrangian relaxation based solution approach for the version of the problem with setup times. In addition, Diaby et al. (1992a) and Diaby et al. (1992b) consider single- and multi-resource, respectively, version of the problem and develop Lagrangian relaxation based solution approaches. Millar and Yang (1994) also concentrate on the version of the problem with setup times and study Lagrangian relaxation and decomposition techniques.

# CHAPTER 3
# REPLENISHMENT AND DISPOSAL PLANNING FOR MULTIPLE PRODUCTS WITH REUSABLE RETURNS

## 3.1 Introduction

A typical spare part supplier for industrial equipment (e.g., power generation turbines, airplane engines, and radiology equipment) offers multiple types of kits each of which is tailored for a different type of maintenance operation.A kit includes all of the parts required for a particular type of maintenance operation. To reduce the downtime associated with the maintenance of expensive industrial equipment, kits are usually prepared in advance and shipped to customers. Upon the completion of on-site diagnosis and maintenance operations, the customers may return unused kits to the spare part supplier. As these kits are in as-good-as-new condition, they can be placed into inventory to satisfy future demand. If the inventory of a particular type of kit accumulates excessively, extra kits can be disassembled, and although some costs are incurred the parts can be placed in part inventories. Clearly, for optimal inventory planning and control of the kits, returns must be taken into account explicitly when making both assembly (i.e., replenishment) and disassembly (i.e., disposal) decisions. Our focus in this chapter is on the corresponding *Replenishment and Disposal Planning Problem* (RDPP) with multiple product types (i.e., kits).

More specifically, the problem is concerned with determining when, and how much, to replenish or dispose of each type of product. We consider the problem in a deterministic demand and return modeling framework over a finite planning horizon under the presence of processing capacities for replenishment and disposal activities. We develop an integer programming formulation to seek an optimal solution that characterizes the

- timing of replenishment and disposal setups,
- lot size decisions for replenishment and disposal activities, and
- stocking decisions for finished goods,

while considering the replenishment and disposal capacity restrictions explicitly. It can be argued that the quantity and timing of returns as well as the associated processing times are generally unpredictable in various practical situations such as the fashion industry in particular and B2C settings in general. In the context of spare part kitting for industrial equipment maintenance, however, a typical supplier has service contracts with a stable customer base which allows the supplier to use historical data to estimate the future demand and return quantities accurately for different types of products. We consider two variants of the RDPP both of which are motivated by the spare part kitting application of interest. In the first variant, the replenishment capacity is shared among multiple product types while the disposal capacity is product specific. In the second variant, both the replenishment and disposal capacities are shared among multiple product types. The practical motivations of these two variants are discussed in detail in Sections 3.2 and 3.5, respectively.

The remainder of this paper is organized as follows. We proceed with considering the first variant of the RDPP. Section 3.2 presents a mathematical formulation for this problem, and the solution approach is described in Section 3.3. Results from computational experiments are presented and analyzed in Section 3.4. Section 3.5 details how the proposed mathematical formulation and the solution approach can be modified for addressing the second variant of RDPP where both replenishment and disposal capacities are shared among multiple product types. Finally, conclusions and future research directions are summarized in Section 3.6.

### 3.2 Shared Replenishment and Product-Specific Disposal Capacities

The problem with shared replenishment and product-specific disposal capacities is motivated by an original equipment manufacturer (OEM) in the power generation equipment industry that provides spare part kits for power turbine maintenance services. The OEM installs power turbines at customer sites and provides service parts for

maintenance repairs. There are multiple types of turbine blades, and each type requires a different type of maintenance repair. Each type of maintenance repair is associated with a set of replacement parts to be used during maintenance. OEM purchases parts from part suppliers and assembles replacement part kits for the customers. These kits are assembled using an assembly line that is configured for the type of kit to be assembled, and the capacity of the line is shared among different types of kits. Therefore, there is an assembly capacity restriction, which is shared among different types of kits, that limits the quantity of kits that can be assembled.

The customers may order several kits but use fewer kits than ordered. The kits that are not used by the customer can be returned to the OEM. In our work, we consider full kits only. That is, partial kits are not accepted by the OEM. Hence, the OEM has two sources for replenishing the inventory of assembled kits: returns received from the customers and new kits assembled by using the parts in the part inventories. If too much inventory of a particular type of assembled kit accumulates, then the OEM can 'dispose' of the excess inventory by disassembling the kits and restocking parts in part inventories. The disassembly of kits can be performed on work benches which do not need to be configured; however, the labor required imposes a disassembly capacity restriction that limits the quantity of kits that can be disassembled. Therefore, there is a disassembly capacity restriction that limits the quantity of a particular type of kit that can be disassembled, but the disassembly capacity is not shared among different types of kits.

To ensure uninterrupted spare part kit availability, the OEM has to coordinate replenishment (i.e., assembly) and disposal (i.e., disassembly) decisions for the kits, which yields the multi-item capacitated lot sizing problem of interest in this chapter. The problem setting is shown in Figure 3-1. Specifically, given the forecasted demands and returns for multiple types of products (kits) as well as the shared replenishment and product-specific disposal capacity restrictions in each time period, our objective is

to satisfy the demand for the products and process all the returns, minimizing the sum of processing costs associated with replenishment and disposal activities as well as the inventory holding costs while obeying the capacity restrictions throughout the planning horizon.

We note that without loss of generality the lead times associated with replenishment and disposal activities can assumed to be zero. Moreover, there are no speculative motives that encourage replenishment with the purpose of disposal in the future. The problem parameters are as follows:

$T$ length of the planning horizon indexed by $t$ for $t = 1, \ldots, T$;

$P$ number of products indexed by $p$ for $p = 1, \ldots, P$;

$\alpha_p$ the replenishment capacity required to produce a unit of product $p$;

$C_t^m$ the shared replenishment capacity available in period $t$;

$C_{pt}^d$ the product-specific disposal capacity available for product $p$ in period $t$;

$D_{pt}$ the demand for product $p$ in period $t$;

$\overline{D}_{pt}$ the cumulative demand for product $p$ from period $t$ until period $T$ (i.e., $\overline{D}_{pt} = \sum_{\tau=t}^{T} D_{p\tau}$);

$R_{pt}$ the quantity of returns of product $p$ received in period $t$;

$\overline{R}_{pt}$ the cumulative returns for product $p$ from period one until period $t$ (i.e., $\overline{R}_{pt} = \sum_{\tau=1}^{t} R_{p\tau}$);

$D'_{pt}$ the *net demand* for product $p$ in period $t$ (i.e., $D'_{pt} = D_{pt} - R_{pt}$);

$\mu_{pt}$ the unit cost of replenishing product $p$ in period $t$;

$\delta_{pt}$ the unit cost/revenue of disposing product $p$ in period $t$;

$\varepsilon_{pt}$ the unit inventory holding cost associated with carrying product $p$ in inventory in period $t$;

$F^m_{pt}$ the fixed cost of a replenishment setup for product $p$ in period $t$; and

$F^d_{pt}$ the fixed cost of a disposal setup for product $p$ in period $t$.

We note that without loss of generality the lead times associated with replenishment and disposal activities can assumed to be zero. Moreover, there are no speculative motives that encourage replenishment with the purpose of disposal in the future. The decision variables are as follows:

$x^m_{pt}$ the replenishment quantity associated with product $p$ in period $t$;

$x^d_{pt}$ the replenishment quantity associated with product $p$ in period $t$;

$y^m_{pt}$ takes the value of one if replenishment activity takes place for product $p$ in period $t$, and zero otherwise;

$y^d_{pt}$ takes the value of one if disposal activity takes place for product $p$ in period $t$, and zero otherwise; and

$i_{pt}$ the quantity of units of product $p$ in inventory at the end of period $t$.

The RDPP with shared replenishment and product-specific disposal capacities is formulated as

$$\min \quad \sum_{p=1}^{P}\sum_{t=1}^{T}(F^m_{pt}y^m_{pt} + \mu_{pt}x^m_{pt}) + \sum_{p=1}^{P}\sum_{t=1}^{T}(F^d_{pt}y^d_{pt} + \delta_{pt}x^d_{pt}) + \sum_{p=1}^{P}\sum_{t=1}^{T}\varepsilon_{pt}i_{pt} \tag{3–1}$$

subject to

$$\sum_{p=1}^{P}\alpha_p x^m_{pt} \le C^m_t \qquad t = 1,\ldots,T; \tag{3–2}$$

$$x^d_{pt} \le C^d_{pt}y^d_{pt} \qquad p = 1,\ldots,P; t = 1,\ldots,T; \tag{3–3}$$

$$i_{p,t-1} + x^m_{pt} = D'_{pt} + x^d_{pt} + i_{pt} \qquad p = 1,\ldots,P; t = 1,\ldots,T; \tag{3–4}$$

$$x^m_{pt} \le \overline{D}_{pt}y^m_{pt} \qquad p = 1,\ldots,P; t = 1,\ldots,T; \tag{3–5}$$

$$x^d_{pt} \le \overline{R}_{pt}y^d_{pt} \qquad p = 1,\ldots,P; t = 1,\ldots,T; \tag{3–6}$$

$$i_{p0},\ i_{pT} = 0 \qquad p = 1, \ldots, P; \tag{3–7}$$

$$x^m_{pt},\ x^d_{pt},\ i_{pt}, \geq 0 \text{ and integer} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{3–8}$$

$$y^m_{pt},\ y^d_{pt} \in \{0, 1\} \qquad p = 1, \ldots, P; t = 1, \ldots, T. \tag{3–9}$$

Objective function (3–1) minimizes the sum of the fixed cost of performing replenishment and disposal setups as well as the linear replenishment, disposal, and inventory holding costs throughout the planning horizon. Constraint set (3–2) ensures that the total replenishment capacity required for all the products does not exceed the available replenishment capacity in period $t$. Constraint set (3–3) ensures that the disposal capacity required for each product $p$ does not exceed the available disposal capacity for product $p$ in period $t$. Constraint set (3–4) is the inventory balance equation for each product $p$ in each period $t$, which ensures that for each product $p$, the sum of the quantity in inventory carried into period $t$ and the quantity produced in period $t$ is equal to the sum of the net demand in period $t$, the quantity disposed in period $t$, and the quantity in inventory carried to period $t + 1$. Constraint set (3–5) ensures that a replenishment setup is performed for product $p$ in period $t$ if the quantity of product $p$ produced in period $t$ is positive; this quantity should not exceed the cumulative demand until the end of the planning horizon. Similarly, constraint set (3–6) ensures that a disposal setup is performed for product $p$ in period $t$ if the quantity of product $p$ disposed in period $t$ is positive; this quantity should not exceed the cumulative returns received since the beginning of the planning horizon. Constraint set (3–7) initializes the beginning and ending inventory levels for each product $p$. Constraint sets (3–8) and (3–9) ensure the integrality of the decision variables.

### 3.3 Solution Approach

Our solution approach is based on the Lagrangian relaxation of constraint sets (3–2) and (3–3). Let $\kappa_t \geq 0$ be the Lagrangian multiplier associated with the replenishment capacity constraint in period $t$ and $\kappa$ be the $T$ dimensional Lagrangian vector associated

with constraint set (3–2). Similarly, let $\lambda_{pt} \geq 0$ be the Lagrangian multiplier associated with the disposal capacity constraint for product $p$ in period $t$ and $\lambda$ be the $P \times T$ dimensional Lagrangian matrix associated with constraint set (3–3). Given $\kappa$ and $\lambda$, the Lagrangian problem obtained by relaxing constraint sets (3–2) and (3–3) can be stated as follows:

$$\min \sum_{p=1}^{P}\sum_{t=1}^{T}(F_{pt}^{m}y_{pt}^{m} + (\mu_{pt} + \alpha_p\kappa_t)x_{pt}^{m}) + \sum_{p=1}^{P}\sum_{t=1}^{T}((F_{pt}^{d} - C_{pt}^{d}\lambda_{pt})y_{pt}^{d} + (\delta_{pt} + \lambda_{pt})x_{pt}^{d})$$
$$+ \sum_{p=1}^{P}\sum_{t=1}^{T}\varepsilon_{pt}i_{pt} - \sum_{t=1}^{T}\kappa_t C_t^m \tag{3–10}$$

subject to (3–4)–(3–9).

For a given $\kappa$ and $\lambda$, the Lagrangian problem (3–10) can be solved efficiently to obtain a lower bound. Moreover, using the solution to the Lagrangian problem, a smoothing heuristic can be utilized to identify a solution that does not violate replenishment and disposal capacity restrictions when finding an upper bound. Using these lower and upper bounds, the values of the multipliers $\kappa$ and $\lambda$ can be updated accordingly using subgradient optimization.

Let $k$ be the iteration counter. Suppose that the termination criteria are defined by $\epsilon$, the tolerance gap between the best lower and upper bound, and $k_{\max}$, the maximum number of iterations. Also, $UB^k$ and $LB^k$ denote the upper and lower bounds, respectively, on the objective function value of the Lagrangian problem at iteration $k$. Similarly, $UB^*$ and $LB^*$ denote the incumbent upper and lower bounds, respectively, on the objective function value of the Lagrangian problem. Then, the general LR procedure we use can be stated as follows:

1. Initialize $\epsilon$, $k_{\max}$ and the Lagrangian multipliers. Set $k = 1$; $UB^* = UB^k = \infty$; and $LB^* = LB^k = -\infty$.
2. While $((LB^* \leq (1-\epsilon)UB^*)$ and $(k \leq k_{\max}))$
   (a) Formulate and solve the Lagrangian dual problem to obtain $LB^k$.

(b) If ($LB^k > LB^*$) then set $LB^* = LB^k$.

(c) Apply the smoothing heuristic to find a feasible solution to obtain $UB^k$.

(d) If ($UB^k < UB^*$) then set $UB^* = UB^k$ and update the best solution.

(e) Revise Lagrangian multipliers using the subgradient optimization method. Update step length if necessary.

(f) Set $k = k + 1$.

3. Report $LB^*$, $UB^*$, and the best solution.

We describe how we solve the Lagrangian problem to obtain a lower bound (in Section 3.3.1), find an upper bound using the smoothing heuristic (in Section 3.3.2), and update the Lagrangian multipliers (in Section 3.3.3), on each iteration of the LR approach.

### 3.3.1 Obtaining Lower Bounds

For a given $\kappa$ and $\lambda$, the Lagrangian problem (3–10) can be solved efficiently. First, we note that the last term in the objective function of this problem is constant. Moreover, the problem (3–10) is separable into $P$ subproblems, each of which corresponds to a product $p$. The single-product Lagrangian subproblem (for product $p$) can be stated as follows:

$$\min \quad \sum_{t=1}^{T}(F_{pt}^m y_{pt}^m + (\mu_{pt} + \alpha_p \kappa_t)x_{pt}^m) + \sum_{t=1}^{T}((F_{pt}^d - C_{pt}^d \lambda_{pt})y_{pt}^d + (\delta_{pt} + \lambda_{pt})x_{pt}^d) + \sum_{t=1}^{T} \varepsilon_{pt} i_{pt} \tag{3–11}$$

subject to

$$i_{p,t-1} + x_{pt}^m = D'_{pt} + x_{pt}^d + i_{pt} \qquad t = 1, \ldots, T; \tag{3–12}$$

$$x_{pt}^m \leq \overline{D}_{pt} y_{pt}^m \qquad t = 1, \ldots, T; \tag{3–13}$$

$$x_{pt}^d \leq \overline{R}_{pt} y_{pt}^d \qquad t = 1, \ldots, T; \tag{3–14}$$

$$i_{p0},\ i_{pT} = 0 \tag{3–15}$$

$$x_{pt}^m,\ x_{pt}^d,\ i_{pt}, \geq 0 \text{ and integer} \qquad t = 1, \ldots, T; \tag{3–16}$$

$$y_{pt}^m,\ y_{pt}^d \in \{0, 1\} \qquad t = 1, \ldots, T. \tag{3–17}$$

Note that the coefficient $(F^d_{pt} - C^d_{pt}\lambda_{pt})$ of binary variable $y^d_{pt}$ in the objective function (3–11) can be negative since $\lambda_{pt} \geq 0$. We observe that in the optimal solution to this problem, if $(F^d_{pt} - C^d_{pt}\lambda_{pt}) < 0$, then $y^d_{pt} = 1$. Then, we can modify the objective function (3–11) by (i) replacing $F^d_{pt}$ with $F'^d_{pt}$ where $F'^d_{pt} = \max\{(F^d_{pt} - C^d_{pt}\lambda_{pt}), 0\}$ for $t = 1, \ldots, T$; and (ii) adding a constant term $\sum_{t=1}^{T} \min\{F^d_{pt} - C^d_{pt}\lambda_{pt}, 0\}$. Consequently, for a given $\kappa$ and $\lambda$, the modified single-product Lagrangian subproblem is given by:

$$\min \sum_{t=1}^{T}(F^m_{pt}y^m_{pt} + (\mu_{pt} + \alpha_p\kappa_t)x^m_{pt}) + \sum_{t=1}^{T}(F'^d_{pt}y^d_{pt} + (\delta_{pt} + \lambda_{pt})x^d_{pt}) + \sum_{t=1}^{T}\varepsilon_{pt}i_{pt} + \sum_{t=1}^{T}\min\{F^d_{pt} - C^d_{pt}\lambda_{pt}, 0\} \tag{3–18}$$

subject to (3–12)-(3–17).

This modified single-product Lagrangian subproblem resembles the single-item uncapacitated lot-sizing problem with returns (SULSPwR) (in Beltrán and Krass (2002)) with one difference: there is a finite disposal cost associated with any excess inventory at the end of the planning horizon in the subproblem above, while any excess inventory at the end of the planning horizon has no value in Beltrán and Krass (2002). This difference does not influence the structural properties of the optimal solution presented in Beltrán and Krass (2002) but it does require a modified DP algorithm. Next, we proceed with a review of the properties of the optimal solution to the SULSPwR presented in Beltrán and Krass (2002), and then we extend the DP algorithm in Beltrán and Krass (2002) to obtain an optimal solution to our modified single-product Lagrangian subproblem. As in Beltrán and Krass (2002), in the remainder of this paper, a period in the planning horizon is called a decision period, if a replenishment or disposal setup has to be performed.

**Property 3.1.** *A feasible policy for SULSPwR is called a modified zero-inventory policy if it satisfies the following conditions:*

- *Replenishment and disposal setups are not performed simultaneously in any decision period.*

- *There exists a period with zero inventory between any two successive decision periods and after the last decision period.*

*As a result, the planning horizon can be divided into regeneration intervals each of which has*

- *zero beginning and ending inventory levels and*
- *either a single replenishment or a disposal setup.*

*Moreover, under a modified zero-inventory policy, it can be shown that each replenishment or disposal setup should satisfy the net demand of an integer number of periods, i.e., a modified zero-inventory policy satisfies the exact requirements property.*

**Property 3.2.** *To determine the optimal policy for SULSPwR, it suffices to consider the modified zero-inventory policies described in Property 1.*

Now, let us consider the modified single-product Lagrangian subproblem for product $p$. We identify the possible decision periods for the product that does not violate the modified zero-inventory policy as follows. Let $S_{pt}^{t\prime} = \sum_{j=t}^{j=t\prime} D\prime_{pj}$ denote the cumulative net demand for product $p$ between periods $t$ and $t\prime$ for $t < t\prime < T$. Also, let $\bar{I}_{pt}$ be the set of possible periods up to which the *positive* net demand for product $p$ is satisfied by a replenishment setup in period $t$. That is, set $\bar{I}_{pt}$ is comprised of time periods with *increasing* cumulative net demands for product $p$, where the cumulative net demand corresponding to the first element in the set is bounded from below by the current on-hand inventory for the product (i.e., $i_{pt-1}$). Similarly, let $\underline{I}_{pt}$ denotes the possible periods up to which the *negative* net demand for product $p$ is eliminated by a disposal setup in period $t$. That is set $\underline{I}_{pt}$ includes the time periods with *increasing* cumulative net demands for product $p$, for which the cumulative net demands corresponding to all of the elements in the set are bounded from above by the current on-hand inventory (i.e., $i_{pt-1}$) and from below by the current period's net demand (i.e., $D'_{pt}$). We present the procedure used to identify $\bar{I}_{pt}$ and $\underline{I}_{pt}$ for a product $p$ in period $t$ in Figure 3-2 and Figure 3-3, respectively. Let $\delta$ is a parameter.

In our DP algorithm for the modified single-product Lagrangian subproblem, the state is defined as the current period ($t$) for $t = 1, \ldots, T$. We employ a backward recursion where the initial state is represented by $(T + 1)$ and the ending state by (1). As we noted earlier, the decision in each state is whether to replenish or dispose, as well as when, and how much, to replenish or dispose. Moreover, the timing and quantity of both replenishment and disposal decisions for product $p$ in period $t$ are limited by the sets $\bar{I}_{pt}$ and $\underline{I}_{pt}$ (due to Property 1). Since there exists an optimal policy that satisfies the modified zero-inventory policy (due to Property 2), we only consider periods with zero incoming inventory.

Let $\tau_{pt}$ denote the maximum number of periods by which a decision period (either with a replenishment or disposal setup) for product $p$ can be postponed beyond period $t$. That is, we have $\tau_{pt} = \max\{w \colon t \leq w \leq T \text{ and } S_{pt}^{t\prime} \leq 0, \forall t \leq t\prime \leq w\}$. Let $g_p(t)$ denote the optimal value function for product $p$ in period $t$, which is defined as the minimum cost of satisfying demand and processing return from period $t$ until the end of the planning horizon with zero starting inventory for product $p$. The recurrence relation of the DP algorithm can be stated as

$$g_p(t) = \min \left\{ \begin{array}{l} \min\limits_{v \in \{t, \ldots, \tau_{pt}\}} \left\{ \sum\limits_{\ell=t}^{v-1} (\varepsilon_{pl}(-S_{pt}^{\ell})) + \psi(\bar{I}_{pv}) \right\} \\ \\ \min\limits_{v \in \{t, \ldots, \tau_{pt}\}} \left\{ \sum\limits_{\ell=t}^{v-1} (\varepsilon_{pl}(-S_{pt}^{\ell})) + \left\{ \begin{array}{ll} 0 & \tau_{pt} = T \text{ and } S_{pt}^{T} = 0 \\ \psi(\underline{I}_{pv}) & \text{otherwise} \end{array} \right\} \right\} \end{array} \right\},$$

where

$$\psi(\bar{I}_{pv}) = \min_{w \in \bar{I}_{pv}} \left\{ \begin{array}{ll} F_{pv}^{m} + (\mu_{pv} + \alpha_p \kappa_v) S_{pt}^{w} + \sum\limits_{\ell=v}^{w-1} (\varepsilon_{p\ell} S_{p\ell+1}^{w}) + g_p(w+1) & \bar{I}_{pv} \neq \emptyset \\ \infty & \bar{I}_{pv} = \emptyset \end{array} \right\}$$

$$\psi(\underline{I}_{pv}) = \min_{w \in \underline{I}_{pv}} \left\{ \begin{array}{ll} F\prime_{pv}^{d} + (\delta_{pv} + \lambda_{pv})(-S_{pt}^{w}) + \sum\limits_{\ell=v}^{w-1} (\varepsilon_{p\ell} S_{p\ell+1}^{w}) + g_p(w+1) & \underline{I}_{pv} \neq \emptyset \\ \infty & \underline{I}_{pv} = \emptyset \end{array} \right\}$$

In this recursive relation, (i) the first minimization characterizes whether the replenishment or disposal option yields the minimum value; (ii) the second minimization identifies the decision period, i.e., timing of the replenishment or disposal setup; and (iii) the third minimization specifies the quantity associated with the option, i.e., replenishment or disposal quantity, and the next regeneration period. There are $T$ major iterations of the DP algorithm, and for each iteration, the cost calculations and comparisons require $\mathrm{O}(T^2)$ operations. Therefore, the proposed DP algorithm runs in $\mathrm{O}(T^3)$ for the modified single-product Lagrangian subproblem for each product.

Let $LB_p^{k*}$ denote the optimal solution to the single-product Lagrangian subproblem for product $p$ at iteration $k$ obtained using the DP algorithm described above. Then, the lower bound for the Lagrangian problem at iteration $k$, $LB^k$, is given by

$$LB^k = \sum_{p=1}^{P} LB_p^{k*} - \sum_{t=1}^{T} \kappa_t C_t^m$$

### 3.3.2 Obtaining Upper Bounds

For a given $\kappa$ and $\lambda$, an optimal solution to the Lagrangian problem (3–10) can be identified efficiently with the DP approach described in Section 3.3.1. As this optimal solution may violate the replenishment and disposal capacity restrictions, we develop a smoothing heuristic to remove such infeasibilities. This approach is similar to, yet, extends, the one presented in Trigeiro et al. (1989). Our smoothing heuristic has three phases and is guaranteed to remove all replenishment and disposal capacity violations for a feasible instance of the problem. The first phase of the approach is a forward pass (i.e., starts from the beginning of the planning horizon and proceeds towards the last period) that adjusts disposal quantities to eliminate disposal capacity violations. The second phase adjusts replenishment quantities to eliminate replenishment capacity violations using a backward pass (i.e., starts from the end of the planning horizon and proceeds towards the first period) and an additional forward pass if needed. The third, and last, phase eliminates any unnecessary setups and/or inventory for the products in three steps.

We note that the heuristic maintains the feasibility of the inventory balance equations in all phases, while ensuring that the demand cannot backlogged and returns cannot be disposed of before they are received, and it proceeds as follows. The smoothing heuristic proceeds as follows:

1. **Eliminate disposal capacity violations:** Given a product that violates any of the disposal capacity restrictions, we identify the earliest period $t$ with disposal capacity violation. To eliminate/reduce the capacity violation in this period, we consider four options to shift the maximum amount possible to a (i) previous period with a disposal setup; (ii) future period with a disposal setup; (iii) previous period scheduling an additional disposal setup; and (iv) future period scheduling an additional disposal setup. For each option, we determine the marginal cost of eliminating/reducing the disposal capacity violation, which is obtained by dividing the total cost associated with implementing the option by the amount that can be shifted by the option. The option that yields the minimum marginal cost is executed, and the marginal costs of options are reevaluated. Execution of the minimum marginal cost option and reevaluation of the marginal costs is repeated until there is no more disposal capacity violation in period $t$. Then, the procedure is repeated for the next period with disposal capacity violations. When the current period is $t = T$, or no other periods with disposal capacity violation are left for the product, the procedure is guaranteed to resolve all disposal capacity violations for the product. The procedure is repeated for all the products with disposal capacity violations.

2. **Eliminate replenishment capacity violations:** This phase is executed in two steps:

   (a) Eliminate replenishment capacity violations for periods $t > 1$: We begin from the latest period $t$ where the replenishment capacity is violated and consider all the products that are produced in this period. If the replenishment quantity associated with a product is smaller than the amount of capacity violation, then we consider two options to shift the entire replenishment quantity to (i) the most recent period with a replenishment setup or (ii) the previous period. If the replenishment quantity associated with a product is larger than the amount of capacity violation, then we consider three options to shift (i) the minimum amount possible to reduce/eliminate the capacity violation to the previous period; (ii) the entire replenishment quantity to the previous period or (iii) the entire replenishment quantity to the most recent period with a replenishment setup. For each option, we determine the marginal cost of eliminating the replenishment capacity violation, which is obtained by dividing the total cost associated with implementing the option by the total amount that can be shifted by the option. These options are executed until there is no more replenishment capacity violation in the current period $t$. Then, the procedure

is repeated for period $(t-1)$. At the end of this step when the current period is $t = 1$, all replenishment capacity violations are guaranteed to be eliminated with possibly the exception of the first period.

(b) Eliminate replenishment capacity violation in period $t = 1$: We consider all the products that are produced in the first period. For all the products, we consider the option of shifting the minimum amount possible to eliminate the product capacity violation or the entire replenishment quantity to the next period. We determine the marginal cost of eliminating the replenishment capacity violation for each product. The violation is eliminated by shifting as many products as necessary to the next period $(t+1)$, starting with the product that has the lowest marginal cost. Then, the procedure is repeated for period $(t+1)$. At the end of this step when the current period has no replenishment capacity violation, all replenishment capacity violations are guaranteed to be eliminated.

3. **Eliminate unnecessary setups and inventory:** This phase is executed in three steps:

(a) Eliminate any unnecessary setups: Given a product, we start from period $t = T$. If both replenishment and disposal setups are performed for the product in the current period, then the replenishment and disposal quantities for the product can be reduced by the minimum of these two quantities, i.e., effectively eliminating the the setup that is associated with the smaller batch size. At the end of this step when the current period is $t = 1$, unnecessary setups have been eliminated for the product. The procedure is repeated for all the products.

(b) Adjust replenishment quantities to eliminate any unnecessary inventory: Given a product, we start from period $t = T$. If there is any unused replenishment capacity, then the products with positive incoming inventory are considered. For each such product, the cost reduction associated with increasing the replenishment quantity in $t$ and decreasing the replenishment quantity in $t'$ (where $t'$ is the most recent replenishment setup period, and ending inventory is positive for all periods from $t'$ up to and including $(t-1)$ for the product) is identified. If the cost reduction is positive, then the shift is performed, i.e., the maximum amount of replenishment for product $p$ that can be moved from period $t'$ to period $t$ is moved. The procedure continues until either there are no other products with positive savings or all the replenishment capacity is used in period $t$. Then, the procedure is repeated for period $(t-1)$. At the end of this step when the current period is $t = 1$, unnecessary inventory due to replenishment has been eliminated for the product. This step is repeated for all the products.

(c) Adjust disposal quantities to eliminate any unnecessary inventory: For a given product, we start from period $t = 1$. If there is any unused disposal capacity and the inventory is positive in all periods up to the next disposal setup period, $t'$, then the cost reduction associated with decreasing the disposal

quantity in $t'$ and increasing the disposal quantity in $t$ is identified. If the cost reduction is positive, then the shift is performed, i.e., the maximum amount of disposal for product $p$ that can be moved from period $t$ to period $t'$ is moved. The procedure is repeated for period $(t+1)$. At the end of this step when the current period is $t = 1$, unnecessary inventory due to disposal has been eliminated for the product. The step is repeated for all the products.

### 3.3.3 Subgradient Optimization

For a given $\kappa$ and $\lambda$, an optimal solution to the Lagrangian problem gives a lower bound on the optimal objective function value of the RDPP. To find the best possible lower bound, we need to determine the values of $\kappa$ and $\lambda$ that maximize this lower bound. We solve this problem approximately using the subgradient optimization method. Specifically, we initialize the Lagrangian multipliers to zero in the beginning of the procedure. Then, at each iteration $k$, using the lower and upper bound values, we update the multipliers using

$$\kappa_t^{k+1} = \max\left\{0, \kappa_t^k + a^k \frac{(UB^* - LB^k)(\sum_{p=1}^{P}(\alpha_p x_{pt}^m) - C_t^m)}{\sum_{t=1}^{T}((\sum_{p=1}^{P}\alpha_p x_{pt}^m - C_t^m)^2 + \sum_{p=1}^{P}(x_{pt}^d - C_{pt}^d y_{pt}^d)^2)}\right\}; \quad (3\text{–}19)$$

$$\lambda_{pt}^{k+1} = \max\left\{0, \lambda_{pt}^k + a^k \frac{(UB^* - LB^k)(x_{pt}^d - C_{pt}^d y_{pt}^d)}{\sum_{t=1}^{T}((\sum_{p=1}^{P}\alpha_p x_{pt}^m - C_t^m)^2 + \sum_{p=1}^{P}(x_{pt}^d - C_{pt}^d y_{pt}^d)^2)}\right\}. \quad (3\text{–}20)$$

where $a^k$ is the step size at iteration $k$. We update the step size by halving its value when there is no improvement in the lower bound in a fixed number of iterations.

## 3.4 Computational Experiments

We test the performance of the proposed LR approach on a set of randomly generated test instances. For benchmarking purposes, we solve the test instances using the branch-and-cut (B&C) approach as well. To this end, we employ CPLEX 10.1 and Concert Technology with default settings for cut generation, preprocessing, branching, and upper bounding heuristics. We implement the proposed LR approach using Visual Studio C++ programming language and use a computer with a 3.4Ghz Intel Pentium 4 with 2GB installed memory.

### 3.4.1 Random Test Instance Generation

In our preliminary computational experiments, we observed that the correlations between (i) the demand and returns, (ii) the fixed setup and inventory holding costs, (iii) the replenishment capacity and positive net demand, and (iv) the disposal capacity and negative net demand influence the computational difficulty of the problem instances. Therefore, in generating random test instances, we pay particular attention to these correlations. In particular, for a given $(P, T)$ pair, we begin by generating demand, return, replenishment capacities, and disposal capacities by inducing correlations among them. Then, we generate the cost parameters. We focus on instances with constant unit replenishment and disposal costs as well as inventory holding costs. We consider the time between replenishment (TBR) setups and the time between disposal (TBD) setups to generate the fixed replenishment and disposal setup costs based on the inventory holding cost. In our detailed description of random test instance generation below, $\bar{Z}$ and $\hat{Z}$ denote the population and sample mean, respectively, of a random variable $Z$. Also $\lfloor x \rfloor$ is the largest integer that is smaller than $x$.

1. **Demand:** Let $\varphi_D > 0$ be a parameter that controls the variability in demand. Let $\bar{D}$ be the expected demand in each period for each product.

   (a) Generate integer $D_{pt}$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$ uniformly from $[(1 - \varphi_D)\bar{D}, (1 + \varphi_D)\bar{D}]$.

   (b) Obtain the realized value of expected demand for $p = 1, \ldots, P$ using $\hat{D}_p = \sum_{t=1}^{T} D_{pt}/T$.

2. **Returns:** Let $\rho_{RD}$ be a parameter that induces correlation between returns and demand. Let $\varphi_R > 0$ be a parameter that controls the variability in returns.

   (a) Set the expected value of the returns per period for $p = 1, \ldots, P$ using $\bar{R}_p = \rho_{RD}\hat{D}_p$.

   (b) Generate integer $R_{pt}$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$ uniformly from $[(1 - \varphi_R)\bar{R}_p, (1 + \varphi_R)\bar{R}_p]$.

   (c) Let $D'^{+}_{pt}$ and $D'^{-}_{pt}$ denote the positive and negative net demand respectively, i.e., $D'^{+}_{pt} = \max\{0, D'_{pt}\}$, and $D'^{-}_{pt} = \max\{0, -D'_{pt}\}$, for $p = 1, \ldots, P$ and $t = 1, \ldots, T$. Similarly, let $T^{+}$ ($T^{-}$) denote the number of periods where net demand is

positive (negative). Obtain the realized value of the expected positive net demand for $p = 1, \ldots, P$, $\hat{D'^{+}_p}$, i.e., $\hat{D'^{+}_p} = \sum_{t=1}^{T} D'^{+}_{pt}/T^{+}$. Similarly, obtain the realized value of the expected negative net demand for $p = 1, \ldots, P$, $\hat{D'^{-}_p}$, i.e., $\hat{D'^{-}_p} = \sum_{t=1}^{T} D'^{-}_{pt}/T^{-}$.

3. **Shared Replenishment and Product-Specific Disposal Capacities:** Let $\rho_{D'^{+}}$ be a parameter that induces correlation between the positive net demand and the replenishment capacity. Let $\rho_{D'^{-}}$ be a parameter that induces correlation between the negative net demand and the disposal capacity.

   (a) Set the replenishment capacity for $t = 1, \ldots, T$ using $C^m_t = \lfloor \rho_{D'^{+}} \sum_{p=1}^{P} \alpha_p \min\{\hat{D'^{+}_p}, \hat{D}_p\} \rfloor$.

   (b) Set the disposal capacity for $p = 1, \ldots, P$ and $t = 1, \ldots, T$ using $C^d_{pt} = \lfloor \rho_{D'^{-}} \min\{R_{pt}, D'^{-}_{pt}\} \rfloor$.

4. **Replenishment Setup Costs:** Let $TBR$ be a parameter that induces the correlation between the fixed replenishment setup cost, positive net demand, and inventory holding cost. Let $\bar{F^m_p}$ be the expected replenishment setup cost for $p = 1, \ldots, P$. Let $\varphi_{F^m} > 0$ be a parameter that controls the variability in replenishment setup costs.

   (a) Set the expected value of replenishment setup cost for $p = 1, \ldots, P$ using $\bar{F^m_p} = \varepsilon_{pt}(\min\{\hat{D'^{+}_p}, \hat{D}_p\})TBR^2/2$.

   (b) Generate $F^m_{pt}$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$ uniformly from $[(1 - \varphi_{F^m})\bar{F^m_p}, (1 + \varphi_{F^m})\bar{F^m_p}]$.

5. **Disposal Setup Costs:** Let $TBD$ be the parameter that induces the relationship between the fixed disposal setup cost, negative net demand, and inventory holding cost. Let $\bar{F^d_p}$ be the expected disposal setup cost for $p = 1, \ldots, P$. Let $\varphi_{F^d} > 0$ be a parameter that controls the variability in disposal setup costs.

   (a) Set the expected value of disposal setup cost for $p = 1, \ldots, P$ using $\bar{F^d_p} = \varepsilon_{pt}(\min\{\hat{D'^{-}_p}, \hat{R}_p\})TBD^2/2$.

   (b) Generate $F^d_{pt}$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$ uniformly from $[(1 - \varphi_{F^d})\bar{F^d_p}, (1 + \varphi_{F^d})\bar{F^d_p}]$.

To develop a set of random instances of realistic size, we vary the number of products $P$ and the length of the planning horizon, $T$. Specifically, we consider four levels for $P$ (20, 30, 40, and 50) and three levels for $T$ (12, 24, and 36). Consequently, we have 12 classes of test instances. Moreover, we consider five levels for the correlation between total demand

to total returns throughout the planning horizon (from 0.15 for low to 0.75 for high in increments of 0.15), two levels for each of TBR and TBD ([1, 3] for low or [3, 5] for high), and two levels for each replenishment and disposal capacity tightness (1.5 for low or 2.5 for high). We consider the parameter values summarized in Table 3-1. For each $(P, T)$ we consider 80 different problem settings, and for each setting we generate 10 random test instances. Therefore, we generate and solve 9,600 instances in total.

### 3.4.2 Discussion of Computational Results

To investigate the performance of the LR approach, we solve the test instances with the proposed approach, and examine the gap between the resulting $LB^*$ and $UB^*$. In particular, we use

$$200 \times \frac{UB^* - LB^*}{UB^* + LB^*}$$

to evaluate the quality of the gap (as in Millar and Yang (1994)). In our experiments, the LR approach terminates when the gap between the lower and upper bound is less than 0.5 % or a maximum of 500 iterations are executed. The step size is set initially to 2 and halved if the lower bound value is not improved in five consecutive iterations.

We summarize the optimality gaps obtained by the LR approach and its CPU requirements in Table 3-2. As we solve a total of 9,600 test instances, each cell in this table corresponds to the average or maximum value representing 800 instances of the same size. We note that the observed average optimality gap over all instances is 0.25 percent, whereas the maximum is slightly above 2 percent. We also observe that for a given number of products, the average and maximum values of the optimality gaps become smaller as the length of the planning horizon increases. For a given planning horizon length, the average and maximum values of the optimality gaps are not significantly influenced by an increase in the number of products. When the number of products is large and/or the planning horizon is relatively long, (i) the influence of a 'misplaced' setup on the overall objective function value decreases and (ii) the smoothing heuristic

becomes more effective in removing capacity violations. Consequently, the quality of the gap identified by the LR approach is better for instances with a larger number of products and/or a longer planning horizon.

Moreover, we observe that the LR approach is computationally efficient as the observed average CPU requirement over all instances is less than one second, whereas the maximum is less than 22 seconds.

In Tables 4-3, 4-4, and 3-5, we provide a detailed analysis of the influence of the data characteristics on the performance of the LR approach.

Not surprisingly, we observe that the quality of the optimality gaps deteriorates as (i) the TBR and TBD values increase and/or (ii) the replenishment and/or disposal capacities become more restraining. It is worthwhile to note that the influence of an increase in TBR (and/or a decrease in replenishment capacity) appears to be more influential than an increase in TBD (and/or a decrease in disposal capacity), which can be attributed to problem characteristics. That is, as the replenishment capacity is shared among multiple product types, an increase in TBR (or a decrease in replenishment capacity) influences replenishment decisions for multiple products simultaneously, and a 'mislocated' replenishment setup decision influences the objective function value more than a 'mislocated' disposal setup decision. In addition, we observe that the correlation between returns and demand has a slight influence on the quality of the gaps obtained by the LR approach.

Although the results in Table 3-2 provide empirical evidence for the effectiveness of the proposed LR approach to address the RDPP, an interesting question that remains unanswered is the effectiveness of the B&C approach for obtaining solutions to the problem. To this end, we conducted a preliminary experiment where we solved an instance using the LR approach and recorded the CPU time required. Then, setting the time limit to this value, we attempted to solve the same instance with CPLEX. For numerous test instances, we observed the optimality gaps to be quite large. Therefore, we conducted an

experiment where we set the time limit to 30 seconds (which is more than the maximum time required by the LR approach across all instances as shown in Table 3-2) and the stopping gap to 0.1 percent. We summarize the results of this experiment in Table 3-6.

Again, each cell in this table represents the average or maximum value of 800 observations for test instances of the same size. We note that the average optimality gap over all instances is around 8 percent whereas the maximum is around 34 percent. In Tables 3-7, 3-8, and 3-9, we provide a detailed analysis of the influence of data characteristics on the performance of CPLEX. Again, we observe that the quality of optimality gaps deteriorates as (i) the TBR and TBD values increase and/or (ii) the replenishment and/or disposal capacities become more restraining. We also observe that the quality of the optimality gaps is more sensitive to changes in the TBR and TBD values than to those in the tightness of the replenishment and/or disposal capacities.

To test whether the performance of CPLEX would improve if the time limit were increased and the optimality gap were reduced, we conducted a follow-up experiment to consider instances with $T = 12$ only. In particular, we increased the time limit to 300 seconds (i.e., a ten-fold increase) and increased the stopping gap to 0.5 percent (i.e., a five-fold increase). In Table 3-10 below, we summarize the results from this experiment; we report the number of test instances that cannot be solved by CPLEX (i.e., the optimality gap cannot be reduced to 0.5 percent in 300 seconds) as well as the average and maximum optimality gaps upon termination for the unsolved instances. For example, we observe that only 183 (out of 800) test instances with size (20,12) were solved by CPLEX within 30 seconds with a stopping gap of 0.1 percent. Increasing the time limit to 300 seconds and reducing the optimality gap to 0.5 percent allowed CPLEX to solve 9 additional instances (i.e., a one percent increase). Keeping the number of products constant and increasing the number of time periods, we observed that increasing the time limit and the stopping gap did not help CPLEX obtain solutions for additional test instances.

Therefore, we conclude that CPLEX is far less effective than the proposed LR approach to address realistically-sized instances of the RDPP.

### 3.5 Shared Replenishment and Disposal Capacities

We now consider the case where both the replenishment and disposal capacities are shared among multiple types of products. This case is applicable when the return volumes are relatively small and it is not economically feasibly to setup separate disassembly lines for different product types (i.e., kits). Consequently, different types of products share the disposal capacity. To address the RDPP in this context, we let $\beta_p$ denote the disposal capacity required to dispose a unit of product $p$ and $C_t^d$ denote the shared disposal capacity available in period $t$.

The mathematical formulation in Section 3.2 can be modified for this variant of the RDPP by replacing constraint set (3–3) with

$$\sum_{p=1}^{P} \beta_p x_{pt}^d \leq C_t^d \qquad t = 1, \ldots, T \tag{3–21}$$

which ensures that the total disposal capacity required for all products does not exceed the available disposal capacity in period $t$.

The proposed LR approach can easily be modified as well. In particular, the approach is now based on the relaxation of constraint sets (3–2) and (3–21). Let $\nu = \{\nu_t\}$ denote the vector of Lagrange multipliers $\nu_t \geq 0, t = 1, \ldots, T$, each of which is associated with the disposal capacity constraint in period $t = 1, \ldots, T$. The Lagrangian problem obtained by relaxing constraint sets (3–2) and (3–21) for given $\kappa$ and $\nu$ can be stated as

$$\begin{aligned} \min \; & \sum_{p=1}^{P}\sum_{t=1}^{T}(F_{pt}^m y_{pt}^m + (\mu_{pt} + \alpha_p \kappa_t) x_{pt}^m) + \sum_{p=1}^{P}\sum_{t=1}^{T}(F_{pt}^d y_{pt}^d + (\delta_{pt} + \beta_p \nu_t) x_{pt}^d) + \sum_{p=1}^{P}\sum_{t=1}^{T} \varepsilon_{pt} i_{pt} \\ & - \sum_{t=1}^{T}(\kappa_t C_t^m + \nu_t C_t^d) \end{aligned} \tag{3–22}$$

subject to (3–4)–(3–9).

As before, the last term in the objective function of the Lagrangian problem (3–22) is constant, and the problem (3–22) is separable into $P$ subproblems, each of which corresponds to a product $p$. The single-product Lagrangian subproblem (for product $p$) now can be stated as

$$\min \quad \sum_{t=1}^{T}(F_{pt}^{m}y_{pt}^{m} + (\mu_{pt} + \alpha_p\kappa_t)x_{pt}^{m}) + \sum_{t=1}^{T}(F_{pt}^{d}y_{pt}^{d} + (\delta_{pt} + \beta_p\nu_t)x_{pt}^{d}) + \sum_{t=1}^{T}\varepsilon_{pt}i_{pt} \qquad (3\text{–}23)$$

subject to (3–12)–(3–17).

Each of these subproblems can also be solved efficiently with the DP algorithm presented in Section 3.3.1, and a lower bound can be obtained easily. To obtain the upper bounds, we modify Phase 1 of the smoothing heuristic presented in Section 3.3.2 as discussed next.

**Eliminate disposal capacity violations:**

This phase is executed in the following two steps.

(a) **Eliminate disposal capacity violations for periods $t < T$:** We begin with the earliest period $t$ where there is a disposal setup and the disposal capacity is violated. We classify the products that are disposed of in this period into two sets. The products for which the disposal quantity is *smaller than or equal to* the amount of disposal capacity violation are included in the first set. The second set comprises of the products for which the disposal quantity is *larger* that the amount of disposal capacity violation. For the products in the first set, we consider to shift the entire disposal quantity for the product (i) to the next future period with an existing disposal setup, or (ii) to the next period $(t + 1)$. In evaluating the total cost change associated with these options, we account for changes in the total unit disposal and inventory holding costs. Moreover, for the second option, we also take into account the fixed cost if the shift requires the scheduling of an additional disposal setup in period $(t + 1)$. Similarly, for the products in the second set, we consider to shift (i) the amount of capacity violation to the next period $(t + 1)$, or (ii) the entire disposal quantity to the next period $(t + 1)$, or (iii) the entire disposal quantity to the next future period with an existing disposal setup. Again, in evaluating these options, we account for the changes in the total unit disposal and inventory holding costs. Moreover, for the first and second options, we take into account the fixed disposal setup cost if the shift requires the scheduling of an additional disposal setup in period $(t + 1)$.
For each product, we pick the option that yields the minimum marginal cost. Then we pick the product (and, hence, the option) leading to the best marginal cost and shift the amount implied by the option for the product. We repeat the procedure

until there is no disposal capacity violation in the current period $t$. Then, the procedure is repeated for the next period with a disposal capacity violation. When the current period is $t = T$, all disposal capacity violations are guaranteed to be eliminated with, possibly, the exception of the last period.

(b) **Eliminate the disposal capacity violation in period $t = T$:** We use the following routine to eliminate the any remaining disposal capacity violation for $t = T$.
We consider all the products that are disposed of in the current period and classify them into two sets as before. For each product in the first set (whose disposal quantity is *smaller than or equal to* the amount of disposal capacity violation), we consider to shift the entire disposal quantity of the product to the previous period. For each product in the second set (whose disposal quantity is *larger than* the amount of disposal capacity violation) we consider to shift (i) the amount of capacity violation to the previous period, or (ii) the entire replenishment quantity to the previous period. In evaluating these options, we account for the changes in the total unit disposal and inventory holding costs as well as the additional disposal setup cost, if needed. Moreover, in identifying the amount that can be shifted by these options, we bear in mind that returns that are not received cannot be disposed of. The marginal cost of each option is obtained by dividing the total cost change associated with implementing the option by the amount that can be shifted by the option.
For each product, we pick the option that yields the minimum marginal cost which is obtained by dividing the total cost change associated with implementing the option by the amount that can be shifted by the option. Then, we pick the product (and, hence, the associated option) leading to the best marginal cost and shift the amount implied by the option for the product. We repeat the procedure (starting with classifying the products into two sets) until there is no disposal capacity violation in the current period $t$.
If the previous period has no disposal capacity violation, all disposal capacity violations are guaranteed to be eliminated. Otherwise, the routine is repeated for the previous period.

Considering the lower and upper bound values at each iteration $k$, the new set of multipliers can be updated using

$$\nu_t^{k+1} = \max\left\{0, \nu_t^k + a^k \frac{(UB^* - LB^k)(\sum_{p=1}^{P}(\beta_p x_{pt}^d) - C_t^d)}{\sum_{t=1}^{T}((\sum_{p=1}^{P}(\alpha_p x_{pt}^m) - C_t^m)^2 + (\sum_{p=1}^{P}(\beta_p x_{pt}^d) - C_t^d)^2)}\right\}. \quad (3\text{–}24)$$

To test the computational efficiency of our proposed approach for this variant of the RDPP, we obtained set of test instances using the random instance generation scheme we describe in Section 4.4.1 by modifying the disposal capacity for each period.

We summarize the optimality gaps obtained by proposed LR approach for the RDPP with shared replenishment and disposal capacities, as well as the CPU requirement, in Table 3-11.

Again, we solve a total of 9,600 instances. As in Table 3-2, each cell in this table represents the average or maximum value of 800 observations for test instances of the same size. We observe that the average optimality gap over all instances is 0.22 percent, whereas the maximum is less than 2 percent. Therefore, with the proposed modifications, the LR approach can easily be revised to obtain high-quality solutions for this variant of the RDPP as well. Upon a detailed investigation of the computational results, it is possible to make the same observations discussed in Section 4.4.2 in regards to the influence of the data characteristics on the performance of the LR approach for this variant of the RDPP.

To test the effectiveness of the B&C approach in obtaining solutions for this problem variant, we solved these test instances using CPLEX with a time limit of 30 seconds and a stopping gap of 0.1 percent as before. We summarize the results of this experiment in Table 3-12. We note that the average optimality gap over all instances is around 8 percent whereas the maximum is around 37 percent. In Table 3-13 below, we summarize the results from an experiment where we increased the time limit to 300 seconds and increased the stopping gap to 0.5 percent.

We observe that CPLEX is able to obtain the optimal solution for more test instances with size (20,12) for this variant of the RDPP. However, the number of instances for which CPLEX can determine and verify the optimal solution decreases as the number of products increases. Therefore, we conclude that CPLEX is not effective for addressing this variant of the RDPP either.

## 3.6 Concluding Remarks

We considered a spare part kitting application where customer returns are generally in as-good-as new condition and they need to be taken into account explicitly for the purpose of replenishment and disposal planning. The problem of interest, called the

RDPP, was to determine the timing of the replenishment and disposal setups, along with the associated replenishment and disposal quantities for multiple product types throughout a finite planning horizon so as to minimize the total costs of replenishment, disposal, and inventory holding. We examined two variants of the problem. In the first variant, the replenishment capacity was shared among multiple product types while the disposal capacity is product specific. In the second variant, both the replenishment and disposal capacities were shared among different product types. We developed a LR approach that relied on the relaxation of the replenishment and disposal capacity constraints to obtain lower bounds on the optimal solution values as well as near-optimal solutions with a reasonable computational requirement. For both variants of the problem, we conducted extensive computational experiments to investigate the computational efficiency of the LR approach and examined the impact of problem data characteristics on the quality of the gap obtained by the proposed approach. Our computational results showed that the proposed LR approach is effective in obtaining high-quality solutions for realistically-sized instances of the RDPP with a reasonable computational requirement.

An immediate extension of our work that would be of practical relevance is to consider a generalization of the RDPP where (i) the replenishment and/or disposal setup costs depend on the specific products included in the setup and/or (ii) setup times are explicitly modeled. Moreover, our models can be extended by considering demand backlogging. Also, the LR approach presented in this paper can be extended to settings where certain capacitated resources are shared by the replenishment and disposal activities. Another possible extension is to develop rolling-horizon heuristics (possibly with provable bounds) for the RDPP.

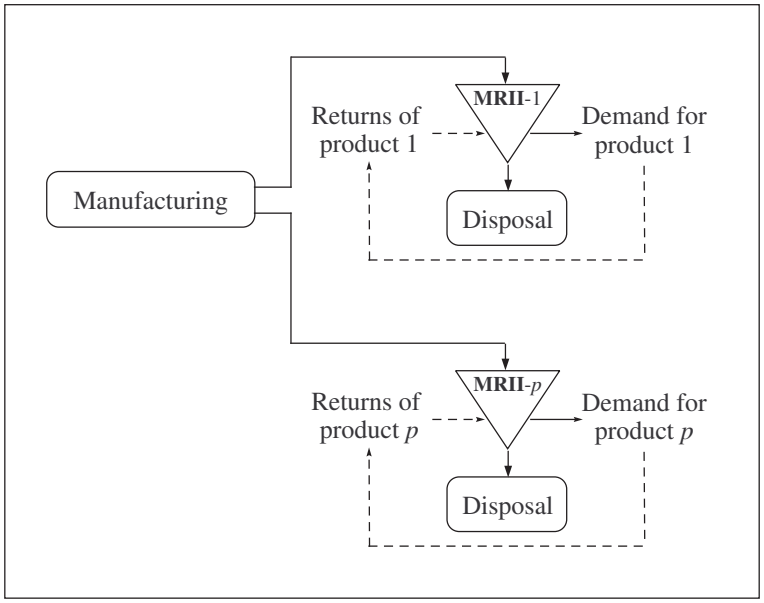


Figure 3-1. Problem setting.

```
Set Ī_pt = ∅, t′ = t, and δ = 0.
while (t′ < T) do
    if ((S^{t′}_pt > i_{pt−1}) and (S^{t′}_pt > δ)) then
        Set Ī_pt = Ī_pt ∪ {t′} and δ = S^{t′}_pt.

        Set t′ = t′ + 1.
    end if
end while
```

Figure 3-2. Pseudo-code of the Identification of Set $\bar{I}_{pt}$.

```
Set I_pt = ∅, t′ = t, and δ = D′_pt .
while (t′ < T) do
    if ((S^{t′}_pt < i_{pt−1}) and (S^{t′}_pt > δ)) then
        Set I_pt = I_pt ∪ {t′}.
        Set t′ = t′ + 1.
    end if
end while
```

Figure 3-3. Pseudo-code of the Identification of Set $\underline{I}_{pt}$.

Table 3-1. Parameter values used in computational experiments.

| **Parameter** | **Level(s)** | **Value(s)** |
|---|---|---|
| Average demand ($\bar{D}$) | Constant | 100 |
| Variability of demand ($\varphi_D$) | Constant | 0.5 |
| Variability of returns ($\varphi_R$) | Constant | 0.5 |
| Variability of fixed replenishment setup cost ($\varphi_{F^m}$) | Constant | 0.2 |
| Variability of fixed disposal setup cost ($\varphi_{F^d}$) | Constant | 0.2 |
| Unit inventory holding cost ($\varepsilon_{pt}$) | Constant | 1 |
| Unit replenishment cost ($\pi_{pt}$) | Constant | 3 |
| Unit disposal cost ($\delta_{pt}$) | Cost | 1 |
| Unit capacity consumption ($\alpha_p$) | Constant | 1 |
| Time between replenishment setups ($TBR$) | Low<br>High | [1, 3]<br>[3, 5] |
| Time between disposal setups ($TBD$) | Low<br>High | [1, 3]<br>[3, 5] |
| Replenishment capacity tightness ($\rho_{D'^+}$) | Low<br>High | 1.5<br>2.5 |
| Disposal capacity tightness ($\rho_{D'^-}$) | Low<br>High | 1.5<br>2.5 |
| Return/Demand ratio ($\rho_{RD}$) | Low<br>High | 0.15<br>0.75 |

Table 3-2. Performance of the LR Approach: Average and Maximum Optimality Gaps (%) and CPU Requirement (sec.).

| | Optimality Gap | | CPU Requirement | |
|---|---|---|---|---|
| Instance Size ($P, T$) | Avr. Gap (%) | Max. Gap (%) | Avr. Time (sec.) | Max. Time (sec.) |
| (20,12) | 0.30 | 1.74 | 0.08 | 0.55 |
| (30,12) | 0.28 | 2.34 | 0.07 | 0.78 |
| (40,12) | 0.28 | 1.41 | 0.07 | 1.05 |
| (50,12) | 0.28 | 2.02 | 0.06 | 1.25 |
| (20,24) | 0.25 | 1.43 | 0.15 | 2.94 |
| (30,24) | 0.24 | 0.91 | 0.11 | 4.30 |
| (40,24) | 0.24 | 0.81 | 0.16 | 5.83 |
| (50,24) | 0.24 | 0.83 | 0.17 | 7.36 |
| (20,36) | 0.22 | 0.68 | 0.22 | 8.73 |
| (30,36) | 0.22 | 0.91 | 0.28 | 13.19 |
| (40,36) | 0.22 | 0.76 | 0.27 | 17.50 |
| (50,36) | 0.21 | 0.62 | 0.19 | 21.80 |
| Overall | 0.25 | 2.34 | 0.15 | 21.80 |

Table 3-3. Effect of Time Between Replenishment (TBR) and Time Between Disposal (TBD) Setups on the Average and Maximum Optimality Gaps (%) of the LR Approach.

| (TBR, TBD) | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| (U[1,3],U[1,3]) | 0.08 | 0.86 |
| (U[1,3],U[3,5]) | 0.13 | 2.34 |
| (U[3,5],U[1,3]) | 0.37 | 1.34 |
| (U[3,5],U[3,5]) | 0.40 | 2.02 |

Table 3-4. Effect of Replenishment and Disposal Capacity Tightness on the Average and Maximum Optimality Gaps (%) of the LR Approach.

| $(\varphi_R, \varphi_D)$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| (1.5,1.5) | 0.33 | 2.34 |
| (1.5,2.5) | 0.30 | 1.74 |
| (2.5,1.5) | 0.19 | 2.02 |
| (2.5,2.5) | 0.17 | 1.38 |

Table 3-5. Effect of Correlation Between Returns and Demand on the Average and Maximum Optimality Gaps (%) of the LR Approach.

| $\rho_{RD}$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| 0.15 | 0.28 | 1.64 |
| 0.30 | 0.28 | 1.74 |
| 0.45 | 0.26 | 1.51 |
| 0.60 | 0.20 | 0.94 |
| 0.75 | 0.22 | 2.34 |

Table 3-6. Performance of CPLEX (with a Time Limit of the 30 Seconds and a Stopping Gap of 0.1 Percent): Average and Maximum Optimality Gaps (%).

| Instance Size $(P, T)$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| (20,12) | 3.85 | 14.88 |
| (30,12) | 4.74 | 15.85 |
| (40,12) | 5.67 | 18.50 |
| (50,12) | 6.20 | 18.20 |
| (20,24) | 7.47 | 23.47 |
| (30,24) | 8.33 | 27.06 |
| (40,24) | 9.13 | 28.44 |
| (50,24) | 10.03 | 34.80 |
| (20,36) | 9.00 | 28.29 |
| (30,36) | 10.61 | 34.13 |
| (40,36) | 11.00 | 32.01 |
| (50,36) | 11.48 | 32.87 |
| Overall | 8.13 | 34.80 |

Table 3-7. Effect of Time Between Replenishment (TBR) and Time Between Disposal (TBD) Setups on the Average and Maximum Optimality Gaps (%) of CPLEX.

| (TBR, TBD) | Avr. Gap (%) | Max. Gap(%) |
|---|---|---|
| (U[1,3],U[1,3]) | 4.53 | 21.48 |
| (U[1,3],U[3,5]) | 4.62 | 22.56 |
| (U[3,5],U[1,3]) | 11.68 | 34.13 |
| (U[3,5],U[3,5]) | 11.69 | 34.80 |

Table 3-8. Effect of Replenishment and Disposal Capacity Tightness on the Average and Maximum Optimality Gaps (%) of CPLEX.

| $(\tau_R, \tau_D)$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| (1.5,1.5) | 8.31 | 32.87 |
| (1.5,2.5) | 8.30 | 34.13 |
| (2.5,1.5) | 7.96 | 34.13 |
| (2.5,2.5) | 7.93 | 34.80 |

Table 3-9. Effect of Correlation Between Returns and Demand on the Average and Maximum Optimality Gaps (%) of CPLEX.

| $\rho_{RD}$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| 0.15 | 3.04 | 12.16 |
| 0.30 | 5.62 | 34.13 |
| 0.45 | 8.45 | 23.74 |
| 0.60 | 10.95 | 34.80 |
| 0.75 | 12.57 | 32.87 |

Table 3-10. Influence of Increasing the Time Limit and the Stopping Gap on the Performance of CPLEX: Number of Unsolved Instances, Average and Maximum Optimality Gaps (%).

| | 30 Seconds and 0.1 % Stopping Gap | | | 300 Seconds and 0.5 % Stopping Gap | | |
|---|---|---|---|---|---|---|
| | No. of Instances Unsolved | Gap Upon Termination | | No. of Instances Unsolved | Gap Upon Termination | |
| $(P, T)$ | (out of 800) | Avr. Gap (%) | Max. Gap (%) | (out of 800) | Avr. Gap (%) | Max. Gap (%) |
| (20,12) | 617 | 4.33 | 14.88 | 608 | 3.85 | 13.16 |
| (20,24) | 774 | 7.57 | 23.47 | 738 | 7.22 | 21.34 |
| (20,36) | 800 | 9.00 | 28.29 | 800 | 8.40 | 26.06 |

Table 3-11. Performance of the LR Approach for the RDPP with Shared Replenishment and Disposal Capacities: Average and Maximum Optimality Gaps (%) and CPU Requirement (sec.).

| | Optimality Gap | | CPU Requirement | |
|---|---|---|---|---|
| Instance Size $(P,T)$ | Avr. Gap (%) | Max. Gap (%) | Avr. Time (sec.) | Max. Time (sec.) |
| (20,12) | 0.26 | 1.74 | 0.07 | 0.70 |
| (30,12) | 0.24 | 1.31 | 0.07 | 0.84 |
| (40,12) | 0.24 | 0.91 | 0.07 | 1.08 |
| (50,12) | 0.24 | 0.80 | 0.08 | 1.31 |
| (20,24) | 0.23 | 1.21 | 0.21 | 3.17 |
| (30,24) | 0.22 | 0.66 | 0.20 | 4.97 |
| (40,24) | 0.22 | 0.60 | 0.24 | 6.50 |
| (50,24) | 0.22 | 0.50 | 0.22 | 4.34 |
| (20,36) | 0.28 | 0.81 | 0.40 | 9.48 |
| (30,36) | 0.20 | 0.64 | 0.35 | 14.77 |
| (40,36) | 0.20 | 0.50 | 0.38 | 10.83 |
| (50,36) | 0.19 | 0.50 | 0.34 | 9.84 |
| Overall | 0.22 | 1.74 | 0.22 | 14.77 |

Table 3-12. Performance of CPLEX for the RDPP with Shared Replenishment and Disposal Capacities (with a Time Limit of the 30 seconds and a Stopping Gap of 0.1 Percent): Average and Maximum Optimality Gaps (%).

| Instance Size $(P,T)$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| (20,12) | 3.89 | 14.62 |
| (30,12) | 4.78 | 16.37 |
| (40,12) | 5.72 | 18.39 |
| (50,12) | 6.25 | 18.84 |
| (20,24) | 7.51 | 23.68 |
| (30,24) | 8.38 | 26.92 |
| (40,24) | 9.20 | 28.01 |
| (50,24) | 10.13 | 36.97 |
| (20,36) | 8.99 | 28.45 |
| (30,36) | 10.55 | 32.54 |
| (40,36) | 11.13 | 32.20 |
| (50,36) | 11.50 | 28.91 |
| Overall | 8.17 | 36.97 |

Table 3-13. Influence of Increasing the Time Limit and the Stopping Gap on the Performance of CPLEX for the RDPP with Shared Replenishment and Disposal Capacities: Number of Unsolved Instances, Average and Maximum Optimality Gaps (%).

| | 30 Seconds and 0.1 % Stopping Gap | | | 300 Seconds and 0.5 % Stopping Gap | | |
|---|---|---|---|---|---|---|
| | No. of Instances Unsolved | Gap Upon Termination | | No. of Instances Unsolved | Gap Upon Termination | |
| $(P,T)$ | (out of 800) | Avr. Gap (%) | Max. Gap (%) | (out of 800) | Avr. Gap (%) | Max. Gap (%) |
| (20,12) | 616 | 4.38 | 14.62 | 579 | 4.04 | 13.76 |
| (20,24) | 774 | 7.61 | 23.68 | 738 | 7.26 | 21.56 |
| (20,36) | 800 | 8.99 | 28.45 | 800 | 8.42 | 26.34 |

# CHAPTER 4
# MANUFACTURING, REMANUFACTURING, DISPOSAL PLANNING FOR MULTIPLE PRODUCTS WITH REMANUFACTURABLE RETURNS

## 4.1 Introduction

An original equipment manufacturer (OEM) in the automotive industry provides vehicle maintenance and repair services for vehicles for which replacement parts are often needed. An OEM typically provides service parts for about 7 years for passenger vehicles and for about 12 years for light and heavy duty trucks beyond vehicle launch. The OEM may manufacture these replacements parts or procure them from an outside manufactured parts supplier. However, as an automobile and its parts are durable goods, failed parts that are removed from vehicles during maintenance and repair services typically have residual economic value that can be recovered via remanufacturing. Therefore, the OEM collects these failed yet remanufacturable parts from retail locations (service shops at the dealerships and warehousing distributors) and sends them to remanufacturing suppliers. Since the remanufactured parts are upheld to high quality standards and are typically offered with the same warranty coverage as the new parts, the customers in the automotive industry do not distinguish between new and remanufactured parts. Hence, the OEM has two sources to satisfy the replacement part demand: new part manufacturing (manufactured by the OEM or procured from new part suppliers) and used part remanufacturing (obtained from remanufacturing suppliers).

Since automotive parts remanufacturing has been a popular practice since the inception of the automotive industry, remanufacturable automotive parts are purchased and sold as commodities. Therefore, in some cases, the retailers may sell some of the parts to independent remanufacturers and return only a fraction of the parts to the OEM, which limits the supply of remanufacturable parts. In some other cases, the quantity of remanufacturable parts received may be more than the demand for remanufactured parts, in which case the OEM disposes of the remanufacturable parts by selling them to bulk metal recyclers for material recovery. In order to ensure uninterrupted replacement

part availability throughout the duration that the vehicles are in use, the OEM has to coordinate manufacturing, remanufacturing, and disposal decisions, which gives rise to the inventory and production planning problems addressed in this chapter.

Upon a closer examination of automotive parts remanufacturing practices, we observe that there are several factors that influence the complexity of the inventory and production planning problems encountered in this context. A factor that influences the complexity of the underlying problem pertains to the manufacturing and remanufacturing capacities. There are usually processing capacity restrictions that limit the quantity of new and remanufactured items that can be acquired from new part and remanufacturing suppliers. Some automotive parts (e.g., engines and transmissions) are manufactured by the OEM and different parts share the same manufacturing capacity. Some other parts (e.g., alternators, distributors) are purchased from part suppliers and hence they do not share the same manufacturing capacity. Remanufacturing of used parts are usually outsourced to remanufacturing suppliers. Typically, remanufacturing of some types of parts (e.g., transmissions or torque converters) are outsourced to a single remanufacturing supplier to take advantage of learning-by-doing effects and to be able to negotiate a lower unit cost due to consolidated volume. Therefore, some parts share the available remanufacturing capacity. For other parts (e.g., electronic control units, distributors), however, remanufacturing can be outsourced to different remanufacturing suppliers, each of which has his own processing capacity. An additional related factor concerns the available disposal capacity. There may be transportation and/or recycling capacity considerations that limit the quantity of used parts that can be disposed of.

In this chapter, we concentrate on the *Manufacturing, Remanufacturing and Disposal Planning Problem* (MRDPP) with multiple product types to address the inventory and production planning problem for remanufacturable automotive parts. More specifically, MRDPP is concerned with determining when and how much to manufacture, remanufacture, and dispose of each type of product. We consider the problem in a

deterministic demand and return modelling framework over a finite planning horizon under the presence of processing capacities for manufacturing, remanufacturing and disposal activities. In our work, we consider the problem environment where the manufacturing capacity is shared among multiple product types, while remanufacturing and disposal capacities are product specific. We develop an integer programming formulation to seek an optimal solution that characterizes the

- timing of manufacturing, remanufacturing, and disposal setups,
- lot sizing decisions for manufacturing, remanufacturing, and disposal activities, and
- stocking decisions for returns and finished goods,

while considering the manufacturing, remanufacturing, and disposal capacities explicitly.

The remainder of this chapter is organized as follows. Section 4.2 presents the modelling assumptions and a mathematical formulation for the MRDPP. Solution approaches are described in Section 4.3. Results from computational experiments are presented and analyzed in Section 4.4. Finally, conclusions and future research directions are summarized in Section 4.5.

## 4.2 Modelling Assumptions and Mathematical Formulation of MRDPP

MRDPP with shared manufacturing and product-specific remanufacturing and disposal capacities is motivated by an OEM in the automotive industry that provides vehicle maintenance and repair services for vehicles for which replacement parts (products) are often needed. A pictorial representation of the MRDPP setting is given in Figure 4-1. There are *two* stock points for each product type, one for the used item inventory (UII), and one for the serviceable (manufactured or remanufactured) items inventory (MRII). UII accumulates as returns of products are received. Excess returns may be disposed of from the UII. Manufacturing and remanufacturing activities replenish the MRII, and the demand for the product type is satisfied from the MRII. As we noted earlier, we focus on discrete time dynamic demand and return models to address the inventory and production planning problem encountered in this setting. Our objective is to minimize the sum of the

setup and processing costs associated with manufacturing, remanufacturing, and disposal activities and inventory holding costs associated with UII and MRII.

In our work, we make the following modeling assumptions:

1. Demand for all product types throughout the planning horizon is known. Demand for each product type can be satisfied by using remanufacturing used items or by manufacturing new items.

2. Returns for all product types throughout the planning horizon is also known. Returns for each product type can be remanufactured to replenish the MRII (i.e.,satisfy the demand) or be disposed of.

3. Manufacturing activity incurs a fixed setup and linear variable costs. The available manufacturing capacity is shared among multiple product types.

4. Remanufacturing activity incurs a fixed setup and linear variable costs. There is a processing capacity associated with the remanufacturing activity for each product type.

5. Disposal activity incurs a fixed setup and linear variable costs. There is a processing capacity associated with the disposal activity for each product type.

6. Items in MRII-$p$ of product type $p$ incur a linear inventory holding cost.

7. Items in UII-$p$ of product type $p$ incur a linear inventory holding cost.

8. Backlogging of demand is not allowed.

The problem parameters are defined as follows:

$T$ length of the planning horizon, indexed by $t$;

$P$ number of products, indexed by $p$;

$\alpha_p$ the manufacturing capacity required to manufacture a unit of product $p$ for $p = 1, \ldots, P$;

$K_t^m$ the manufacturing capacity available in period $t$ for $t = 1, \ldots, T$;

$K_{pt}^r$ the remanufacturing capacity available for product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$K_{pt}^d$ the disposal capacity available for product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$D_{pt}$ the demand for product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$\overline{D}_{pt}$ the cumulative demand for product $p$ from period $t$ thought period $T$ (i.e., $\overline{D}_{pt} = \sum_{\tau=t}^{T} D_{p\tau}$) for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$R_{pt}$ the quantity of returns of product $p$ received in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$\overline{R}_{pt}$ the cumulative returns for product $p$ from period one through period $t$ (i.e., $\overline{R}_{pt} = \sum_{\tau=1}^{t} R_{p\tau}$) for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$c^m_{pt}$ the unit cost of manufacturing product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$c^r_{pt}$ the unit cost of remanufacturing product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$c^d_{pt}$ the unit cost of disposing product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$h_{pt}$ the unit inventory holding cost associated with carrying product $p$ in the finished goods inventory in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$k_{pt}$ the unit inventory holding cost associated with carrying product $p$ in UII-$p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$F^m_{pt}$ the fixed cost of performing a manufacturing setup for product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$F^r_{pt}$ the fixed cost of performing a remanufacturing setup for product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$; and

$F^d_{pt}$ the fixed cost of performing a disposal setup for product $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$.

We have the following decision variables:

$x^m_{pt}$ the quantity of units of product $p$ manufactured in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$x^r_{pt}$ the quantity of units of product $p$ remanufactured in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$x^d_{pt}$ the quantity of returns of product $p$ disposed of in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$y^m_{pt}$ takes the value of one if manufacturing activity takes place for product $p$ in period $t$, and zero otherwise for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$y^r_{pt}$ takes the value of one if remanufacturing activity takes place for product $p$ in period $t$, and zero otherwise for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$y^d_{pt}$ takes the value of one if disposal activity takes place for product $p$ in period $t$, and zero otherwise for $p = 1, \ldots, P$ and $t = 1, \ldots, T$;

$i_{pt}$ the quantity of items in MRII-$p$ for product $p$ at the end of period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$; and

$j_{pt}$ the quantity of items in UII for product $p$ at the end of period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$.

A mathematical programming formulation for MRDPP is as follows:

$$\min \quad \sum_{p=1}^{P}\sum_{t=1}^{T}(F^m_{pt}y^m_{pt} + c^m_{pt}x^m_{pt} + F^r_{pt}y^r_{pt} + c^r_{pt}x^r_{pt} + F^d_{pt}y^d_{pt} + c^d_{pt}x^d_{pt} + k_{pt}j_{pt} + h_{pt}i_{pt}) \tag{4–1}$$

subject to

$$i_{p,t-1} + x^m_{pt} + x^r_{pt} - D_{pt} = i_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–2}$$

$$j_{p,t-1} + R_{pt} - x^r_{pt} - x^d_{pt} = j_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–3}$$

$$\sum_{p=1}^{P} \alpha_p x^m_{pt} \leq K^m_t \qquad t = 1, \ldots, T; \tag{4–4}$$

$$x^r_{pt} \leq K^r_{pt} y^r_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–5}$$

$$x^d_{pt} \leq K^d_{pt} y^d_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–6}$$

$$x^m_{pt} \leq \overline{D}_{pt} y^m_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–7}$$

$$x^r_{pt} \leq \overline{D}_{pt} y^r_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–8}$$

$$x^d_{pt} \leq \overline{R}_{pt} y^d_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–9}$$

$$j_{p0},\, j_{pT}, i_{p0},\, i_{pT} = 0 \qquad p = 1, \ldots, P; \tag{4–10}$$

$$x^m_{pt},\, x^r_{pt},\, x^d_{pt},\, j_{pt}\, i_{pt} \geq 0 \text{ and integer} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–11}$$

$$y^m_{pt},\, y^r_{pt},\, y^d_{pt} \in \{0,1\} \qquad\qquad p = 1,\ldots,P; t = 1,\ldots,T. \tag{4–12}$$

Objective function (4–1) minimizes the sum of fixed cost of performing manufacturing and disposal setups as well as linear manufacturing, disposal, and inventory holding (in UII and MRII) costs for all products throughout the planning horizon. Constraint set (4–2) gives the inventory balance equations for MRII for product $p$ in each period $t$, which ensures that the sum of the quantity in inventory carried from period $t-1$ to period $t-1$, the quantity manufactured in period $t$, and the quantity remanufactured in period $t$ is equal to the sum of the demand in period $t$ and the quantity in inventory carried from period $t$ to period $t+1$. Constraint set (4–3) is the inventory balance equation for UII for product $p$ in each period $t$, which ensures that the sum of the quantity in inventory carried from period $t-1$ to period $t$ and the quantity of returns received in period $t$ is equal to the sum of the quantity remanufactured in period $t$, the quantity disposed in period $t$ and the quantity in inventory carried to from period $t$ to period $t+1$. Constraint set (4–4) ensures that the manufacturing capacity required for all products in period $t$ does not exceed the available manufacturing capacity in period $t$. Constraint set (4–5) ensures that the remanufacturing capacity required for product $p$ in period $t$ does not exceed the available remanufacturing capacity for product $p$ in period $t$. Constraint set (4–6) ensures that the disposal capacity required for product $p$ in period $t$ does not exceed the available disposal capacity for product $p$ in period $t$. Constraint set (4–7) ensures that a manufacturing setup for product $p$ is performed in period $t$ if the quantity manufactured for product $p$ in period $t$ is positive and this quantity should not exceed the cumulative demand until the end of the planning horizon. Constraint set (4–8) ensures that a remanufacturing setup is performed for product $p$ in period $t$ if the quantity remanufactured for product $p$ in period $t$ is positive and this quantity should not exceed the cumulative demand until the end of the planning horizon. Similarly, constraint set (4–9) ensures that a disposal setup for product $p$ is performed in period $t$ if the quantity of product $p$ disposed of in

period $t$ is positive and this quantity should not exceed the cumulative returns received since the beginning of the planning horizon. Constraint set (4–10) initializes the beginning and ending inventory levels for all products in UII and MRII. Constraint sets (4–11) and (4–12) ensure the integrality of the decision variables.

Capacitated multi-item lot sizing problem is a special case of MRDPP problem. More specifically, when the return quantity is zero in every period for every product, the problem is equivalent the classical capacitated multi-item lot sizing problem. Since classical capacitated multi-item lot sizing problem is NP-Hard, so is MRDPP. Therefore, we focus on obtaining near-optimal solutions for MRDPP. To this end we develop a class of solution algorithms based on Lagrangian Decomposition and Relaxation techniques used in conjunction with Genetic Algorithms.

## 4.3 Solution Approaches

Our solution approach is based on Lagrangian Decomposition and Relaxation techniques used together with a constructive heuristic and Genetic Algorithms to obtain lower and upper bound solutions of MRDPP. In our work, we consider two solution approaches. First one relies on the decomposition and relaxation of MRDPP, while the second one relies on the relaxation of MRDPP.

In the Lagrangian Decomposition (LD) approach, we use variable redefinition and relaxation of some constraint sets to decompose the problem into polynomially solvable subproblems. We observe that the remanufacturing quantity decision variables connect the UII and MRII inventory balance constraint sets. Hence, we redefine the remanufacturing quantity and setup variables and replace the remanufacturing variable in the UII inventory balance constraint set. We keep the original remanufacturing variables in the MRII inventory balance constraint set, whereas we replace the remanufacturing variables in the UII inventory balance constraint set with the new remanufacturing variables. We add an equality (linkage) constraint set to define the equivalence of the existing and redefined decision variables to the original problem formulation. We then relax this set of

equality constraints together with the manufacturing, remanufacturing, disposal capacity constraint sets. The resulting relaxed problem becomes separable for each product type as well as forward and reverse channels. In particular, for each product type $p$, we have two subproblems. The first subproblem includes the forward channel decisions, i.e., manufacturing and remanufacturing, and the second one the reverse channel decisions, i.e, remanufacturing and disposal. The first subproblem deals with satisfying demand with no limitations on the return quantity (i.e., remanufacturing quantity), whereas the second subproblem deals with using returns with no limitations on the demand quantity (i.e., remanufacturing quantity). Each such subproblem is an uncapacitated single-item lot sizing problem that is solvable in polynomial time using a DP algorithm. For a given set of Lagrangian multipliers, we solve these subproblems to obtain a lower bound solution of the MRDPP. We obtain upper bound solutions utilizing a constructive heuristic, which is a smoothing heuristic that makes use of the lower bound solutions obtained from DP algorithms. Moreover, to further improve the upper bound, we use a GA as an improvement heuristic. Using lower and upper bound solutions, we update the Lagrangian multipliers by the subgradient optimization approach.

In the Lagrangian Relaxation approach, we only relax the manufacturing capacity constraint set (4–4). The resulting Lagrangian subproblems are single item MRDPP problems, which are NP-Hard in their own right. For a given set of Lagrangian multipliers, we solve these subproblems using CPLEX to obtain a lower bound solution of the MRDPP. We obtain upper bound solutions utilizing a constructive heuristic, which is a smoothing heuristic that makes use of the lower bound solutions obtained from CPLEX. Again, we update the Lagrangian multipliers by the subgradient optimization approach using lower and upper bound solutions.

Next, we analyze these two approaches in detail by giving the corresponding Lagrangian problem formulations, algorithms developed to obtain lower bounds and upper bounds, and the subgradient optimization approach.

### 4.3.1 A Lagrangian Decomposition Approach for MRDPP

In this approach, we employ variable redefinition to remanufacturing quantity and setup variables. The new decision variables are:

$x^u_{pt}$ the quantity of returns of product type $p$ remanufactured in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$; and

$y^u_{pt}$ takes the value of one if remanufacturing activity takes place for product type $p$ in period $t$, and zero otherwise for $p = 1, \ldots, P$ and $t = 1, \ldots, T$; [0.25cm]

We also define new cost parameters for the new decision variables:

$c^u_{pt}$ the unit cost of remanufacturing product type $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$; and

$F^u_{pt}$ the fixed cost of performing a remanufacturing setup for product type $p$ in period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$. [0.25cm]

Let $c^u_{pt} = F^u_{pt} = 0$ for $p = 1, \ldots, P$, and $t = 1, \ldots, T$. The new formulation can be stated as follows:

$$\min \sum_{p=1}^{P}\sum_{t=1}^{T}(F^m_{pt}y^m_{pt} + c^m_{pt}x^m_{pt} + F^r_{pt}y^r_{pt} + c^r_{pt}x^r_{pt} + F^d_{pt}y^d_{pt} + c^d_{pt}x^d_{pt} + k_{pt}j_{pt} + h_{pt}i_{pt} + F^u_{pt}y^u_{pt} + c^u_{pt}x^u_{pt}) \tag{4–13}$$

subject to

$$i_{p,t-1} + x^m_{pt} + x^r_{pt} - D_{pt} = i_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–14}$$

$$j_{p,t-1} + R_{pt} - x^u_{pt} - x^d_{pt} = j_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–15}$$

$$\sum_{p=1}^{P} \alpha_p x^m_{pt} \leq K^m_t \qquad t = 1, \ldots, T; \tag{4–16}$$

$$x^r_{pt} \leq K^r_{pt} y^r_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–17}$$

$$x^d_{pt} \leq K^d_{pt} y^d_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–18}$$

$$x^m_{pt} \leq \overline{D}_{pt} y^m_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–19}$$

$$x^r_{pt} \leq \overline{D}_{pt} y^r_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–20}$$

$$x^d_{pt} \leq \overline{R}_{pt} y^d_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–21}$$

$$x^u_{pt} \leq \overline{R}_{pt} y^u_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–22}$$

$$x^r_{pt} = x^u_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–23}$$

$$y^r_{pt} = y^u_{pt} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–24}$$

$$j_{p0},\, j_{pT}, i_{p0},\, i_{pT} = 0 \qquad p = 1, \ldots, P; \tag{4–25}$$

$$x^m_{pt},\, x^r_{pt},\, x^u_{pt},\, x^d_{pt},\, j_{pt}\, i_{pt} \geq 0 \text{ and integer} \qquad p = 1, \ldots, P; t = 1, \ldots, T; \tag{4–26}$$

$$y^m_{pt},\, y^r_{pt},\, y^u_{pt},\, y^d_{pt} \in \{0, 1\} \qquad p = 1, \ldots, P; t = 1, \ldots, T. \tag{4–27}$$

Note that we exchange the $x^r_{pt}$ decision variables with $x^u_{pt}$ in UII inventory balance constraint set. Constraint set (4–23) ensures that $x^r_{pt}$ and $x^u_{pt}$ take the same values. Similarly, constraint set (4–24) ensures that $y^r_{pt}$ and $y^u_{pt}$ take the same values. We also add auxiliary constraint set (4–22) to define bounds for $x^u_{pt}$ and $y^u_{pt}$. Since, the unit cost of remanufacturing for $x^u_{pt}$, and the fixed cost for $y^u_{pt}$ are zero, the formulation, i.e., (4–13)-(4–27), is equivalent to the original formulation , i.e, (4–1)-(4–12).

We then relax the constraint sets associated with manufacturing, remanufacturing and disposal capacities, i.e., (4–16)-(4–18) as well as the linkage constraint sets between remanufacturing decision variables, i.e. (4–23)-(4–24). When constraint sets (4–16), (4–17) and (4–18) are relaxed, the resulting relaxed problem becomes separable for each product type $p$ for $p = 1, \ldots, P$. Moreover, when constraint sets (4–23) and (4–24) are relaxed, the resulting relaxed problem becomes separable for forward and reverse channel subproblems. Let $\kappa = \{\kappa_t\}$ denote the vector of Lagrangian multipliers $\kappa_t \geq 0, t = 1, \ldots, T$, associated with the manufacturing capacity constraint set (4–16). Let $\gamma = \{\gamma_{pt}\}$, $\lambda = \{\lambda_{pt}\}$ denote the vector of Lagrangian multipliers $\gamma_{pt}, \lambda_{pt} \geq 0, p = 1, \ldots, P, t = 1, \ldots, T$, associated with the product-specific remanufacturing and disposal capacity constraint sets (4–17) and(4–18), respectively. Similarly let $\theta = \{\theta_{pt}\}, \omega = \{\omega_{pt}\}$ denote the vectors of Lagrangian multipliers $\theta_{pt}, \omega_{pt} \in (-\infty, \infty), p = 1, \ldots, P, t = 1, \ldots, T$, associated with the constraints sets (4–23) and (4–24). Given $\kappa, \gamma, \lambda$, $\theta$ and $\omega$, the Lagrangian problem obtained upon

relaxing the constraint sets (4–16)-(4–18), and (4–23)-(4–24) can be stated as follows:

$$\min \quad \sum_{p=1}^{P}\sum_{t=1}^{T}(F_{pt}^{m}y_{pt}^{m} + (c_{pt}^{m}+\kappa_t)x_{pt}^{m} + (F_{pt}^{r} - K_{pt}^{r}\gamma_{pt})y_{t}^{r} + (c_{pt}^{r}+\gamma_{pt}+\theta_{pt})x_{pt}^{r} + (F_{pt}^{d} - K_{pt}^{d}\lambda_{pt})y_{pt}^{d} + (c_{pt}^{d}+\lambda_{pt})x_{pt}^{d}(F_{pt}^{u}-\omega_{pt})y_{pt}^{u} + (c_{pt}^{u}-\omega_{pt})x_{pt}^{u} + k_{pt}j_{pt} + h_{pt}i_{pt}) - \sum_{t=1}^{T}(\kappa_t K_t^{m}) \tag{4–28}$$

subject to (4–14)-(4–15), (4–19)-(4–22), (4–25)-(4–27).

For a given set of Lagrangian multipliers, the solution of Lagrangian problem yields a lower bound solution.

We provide the specific outline of our LD implementation in Figure 4-2 where $\ell$, $\phi$ and $\tau$ denote the iteration counters. The termination criteria are defined by $\epsilon$, the tolerance gap between the best lower and upper bounds, and $\ell_{\max}$, the maximum number of iterations of LD. Upon the completion of $\phi_{\max}$ LD iterations, new individuals added to GA population. Upon the completion of $\tau_{\max}$ LD iterations, GA is executed. Moreover, $UB^{\ell}$ and $LB^{\ell}$ denote the upper and lower bounds, respectively, at iteration $\ell$. Similarly, $UB^{*}$ and $LB^{*}$ denote the incumbent upper and lower bounds, respectively. We proceed with a detailed discussion of how we solve the Lagrangian problem to obtain a lower bound (Section 4.3.1.1), identify an upper bound using the smoothing heuristic (Section 4.3.1.2) and GA, and update the Lagrangian multipliers (Section 4.3.1.3), on each iteration of the LD approach.

#### 4.3.1.1 Obtaining lower bounds

For given $\kappa$, $\gamma$, $\lambda$, $\theta$, and $\omega$ the Lagrangian problem (4–28) can be solved efficiently. First, we note that the last term in the objective function of this problem is constant. Furthermore, the problem (4–28) is separable into $P$ subproblems. Since the link between the return and the demand is broken by relaxing the constraint sets (4–23)-(4–24), each subproblem corresponding to a product type $p$ is separable into two subproblems that correspond to the forward and reverse channel activities for the product. Let $LR1SP1^{p}$

and $LR1SP2^p$ denote the subproblems for the forward and the reverse channel for product type $p$ for $p = 1, \ldots, P$.

The single-product forward-channel Lagrangian subproblem for product type $p$ ($LR1SP1^p$) can be stated as follows:

$$\min \quad \sum_{t=1}^{T} (F_{pt}^m y_{pt}^m + (c_{pt}^m + \kappa_t) x_{pt}^m + (F_{pt}^r - K_{pt}^r \gamma_{pt}) y_{pt}^r + (c_{pt}^r + \gamma_{pt} + \theta_{pt}) x_{pt}^r + h_{pt} i_{pt}) \quad (4\text{–}29)$$

subject to

$$i_{p,t-1} + x_{pt}^m + x_{pt}^r - D_{pt} = i_{pt} \qquad t = 1, \ldots, T; \quad (4\text{–}30)$$

$$x_{pt}^m \leq \overline{D}_{pt} y_{pt}^m \qquad t = 1, \ldots, T; \quad (4\text{–}31)$$

$$x_{pt}^r \leq \overline{D}_{pt} y_{pt}^r \qquad t = 1, \ldots, T; \quad (4\text{–}32)$$

$$i_{p0},\, i_{pT} = 0 \quad (4\text{–}33)$$

$$x_{pt}^m,\, x_{pt}^r,\, i_{pt} \geq 0 \text{ and integer} \qquad t = 1, \ldots, T; \quad (4\text{–}34)$$

$$y_{pt}^m,\, y_{pt}^r \in \{0, 1\} \qquad t = 1, \ldots, T. \quad (4\text{–}35)$$

Note that the coefficient $(F_{pt}^r - K_{pt}^r \gamma_{pt})$ of binary variable $y_{pt}^r$ in the objective function (4–29) can be negative since $\gamma_{pt} \geq 0$. We observe that in the optimal solution to this problem, if $(F_{pt}^r - K_{pt}^r \gamma_{pt}) < 0$, then $y_{pt}^r = 1$. As a result, we can modify the objective function (4–29) by (i) replacing $F_{pt}^r$ with $F_{pt}^{\prime r}$ where $F_{pt}^{\prime r} = \max\{(F_{pt}^r - K_{pt}^r \gamma_{pt}), 0\}$ for $t = 1, \ldots, T$; and (ii) adding a constant term $\sum_{t=1}^{T} \min\{F_{pt}^r - K_{pt}^r \gamma_{pt}, 0\}$. Consequently, for a given $\kappa$, $\gamma$, $\lambda$, and $\theta$ the modified $LR1SP1^p$ for product $p$ is given by:

$$\min \quad \sum_{t=1}^{T} (F_{pt}^m y_{pt}^m + (c_{pt}^m + \kappa_t) x_{pt}^m + (F_{pt}^{\prime r}) y_{pt}^r + (c_{pt}^r + \gamma_{pt} - \theta_{pt}) x_{pt}^r + h_{pt} i_{pt}) \quad (4\text{–}36)$$

subject to (4–30)-(4–35)

The modified $LR1SP1^p$ for product type $p$ is same as the single-item uncapacitated lot-sizing with two supply sources. Therefore, the zero-inventory property is satisfied for this problem. We develop an $O(T^2)$ DP algorithm similar to Wagner-Whitin algorithm to

solve the modified $LR1SP1^p$ problem. The state is defined as the current period ($t$) for $t = 1, \ldots, T$. We propose a backward recursion where the initial state is represented by $(T + 1)$ and the ending state by is denoted by (1). The decisions in each state include whether to manufacture or remanufacture and the associated manufacturing or remanufacturing quantity. Since there exists an optimal policy that satisfies the zero-inventory policy, we only consider periods with zero incoming inventory. Moreover, the quantity decisions for both manufacturing and remanufacturing activities in each period are limited by the cumulative demand of the consecutive future periods.

Let $f_p(t)$ denote the optimal value function for product $p$ in period $t$, which is defined as the minimum cost of satisfying demand from period $t$ through the end of the planning horizon with zero starting inventory for product $p$. The recurrence relation of the DP algorithm can be stated as follows:

$$f_p(t) = \min \begin{cases} \min\limits_{v=t,\ldots,T} \left\{ F^m_{pt} + (c^m_{pt} + \alpha_p \kappa_t) S^v_{pt} + \sum\limits_{l=t}^{v-1} h_{pl} S^v_{pl+1} + f_p(v+1) \right\} \\ \\ \min\limits_{v=t,\ldots,T} \left\{ F'^r_{pt} + (c^r_{pt} + K^r_{pt} \gamma_{pt} + \omega_{pt}) S^v_{pt} + \sum\limits_{l=t}^{v-1} h_{pl} S^v_{pl+1} + f_p(v+1) \right\} \end{cases}$$

In this recursive relation, the first minimization characterizes whether the manufacturing or remanufacturing option yields the minimum value, and the second minimization specifies the quantity associated with the option, i.e., manufacturing or remanufacturing, and the next regeneration period. There are $T$ major iterations of the DP algorithm, and for each iteration, the cost calculations and comparisons require $\mathrm{O}(T)$ operations. Therefore, the proposed DP algorithm runs in $\mathrm{O}(T^2)$ for the $LR1SP1^p$. Let $LB1^*_p$ denote the optimal solution to the $LR1SP1^p$ obtained using the DP algorithm described above.

The single-product reverse-channel Lagrangian subproblem for product type $p$ ($LR1SP2^p$) can be stated as follows:

$$\min \quad \sum_{t=1}^{T}((F^d_{pt} - K^d_{pt}\lambda_{pt})y^d_{pt} + (c^d_{pt} + \lambda_{pt})x^d_{pt} + (F^u_{pt} - \omega_{pt})y^u_{pt} + (c^u_{pt} - \omega_{pt})x^u_{pt} + k_{pt}j_{pt}) \tag{4–37}$$

subject to

$$j_{p,t-1} + R_{pt} - x^u_{pt} - x^d_{pt} = j_{pt} \qquad t = 1, \ldots, T; \tag{4–38}$$

$$x^d_{pt} \leq \overline{R}_{pt} y^d_{pt} \qquad t = 1, \ldots, T; \tag{4–39}$$

$$x^u_{pt} \leq \overline{R}_{pt} y^u_{pt} \qquad t = 1, \ldots, T; \tag{4–40}$$

$$j_{p0},\, j_{pT} = 0 \tag{4–41}$$

$$x^d_{pt},\, x^u_{pt},\, j_{pt} \geq 0 \text{ and integer} \qquad t = 1, \ldots, T; \tag{4–42}$$

$$y^u_{pt},\, y^d_{pt} \in \{0, 1\} \qquad t = 1, \ldots, T. \tag{4–43}$$

Note that the coefficient $(F^d_{pt} - K^d_{pt}\lambda_{pt})$ of binary variable $y^d_{pt}$, and $(F^u_{pt} - \omega_{pt})$ of binary variable $y^u_{pt}$ in the objective function (4–37) can be negative since $\lambda_{pt} \geq 0$ and $\omega > -\infty$. We observe that in the optimal solution to this problem, if $(F^d_{pt} - K^d_{pt}\lambda_{pt}) < 0$, then $y^d_{pt} = 1$. Similarly, if $(F^u_{pt} - \omega_{pt}) < 0$, then $y^u_{pt} = 1$. Accordingly, we can modify the objective function (4–37) by (i) replacing $F^d_{pt}$ with $F'^d_{pt}$, and $F^u_{pt}$ with $F'^u_{pt}$ where $F'^d_{pt} = \max\{(F^d_{pt} - K^d_{pt}\lambda_{pt}), 0\}$ and $F'^u_{pt} = \max\{(F^u_{pt} - \omega_{pt}), 0\}$ for $t = 1, \ldots, T$; and (ii) adding a constant term $\sum_{t=1}^{T} \min(\{F^d_{pt} - K^d_{pt}\lambda_{pt}, 0\} + \{F^u_{pt} - K^u_{pt}\omega_{pt}, 0\})$. Consequently, for given $\kappa$, $\gamma$, $\lambda$, $\theta$, $\omega$ the modified $LR1SP2^p$ is given by:

$$\min \quad \sum_{t=1}^{T}(F'^d_{pt}y^d_{pt} + (c^d_{pt} + \lambda_{pt})x^d_{pt} + F'^u_{pt}y^u_{pt} + (c^u_{pt} - \omega_{pt})x^u_{pt} + k_{pt}j_{pt}) \tag{4–44}$$

subject to (4–38)-(4–43)

Note that modified $LR1SP2^p$ is same as the uncapacitated lot sizing with two supply sources, where returns are processed by remanufacturing or disposal activities. We develop

an $O(T^2)$ DP algorithm similar to the one developed for $LR1SP1^p$ to solve the modified $LR1SP2^p$ problem. The state is defined as the current period $(t)$ for $t = 1, \ldots, T$. We propose a forward recursion where the initial state is represented by (1) and the ending state by $(T)$. The decisions in each state are whether to remanufacture or dispose, as well as how much. Since there exists an optimal policy that satisfies the zero-inventory policy, we only consider periods with zero ending inventory. Moreover, the quantity of both remanufacturing and disposal decisions in each period are limited by the cumulative return of the consecutive prior periods.

Let $f_p(t)$ denote the optimal value function for product $p$ in period $t$, which is defined as the minimum cost of processing return from period 1 through period $t$ with zero ending inventory for product $p$. The recurrence relation of the DP algorithm can be stated as follows:

$$f_p(t) = \min \begin{cases} \min\limits_{v=1,\ldots,t} \left\{ F'^{u}_{pt} + (c^u_{pt} - \omega_{pt}) R^t_{pv} + \sum\limits_{l=v}^{t-1} k_{pl} R^l_{pv} + f_p(v-1) \right\} \\ \\ \min\limits_{v=t,\ldots,T} \left\{ F'^{d}_{pt} + (c^d_{pt} + K^d_{pt}\lambda_{pt}) R^t_{pt} + \sum\limits_{l=v}^{t-1} (k_{pl} R^l_{pv}) + f_p(v-1) \right\} \end{cases}$$

In this recursive relation, (i) the first minimization characterizes whether the remanufacturing or disposal option yields the minimum value; (ii) the second minimization specifies the quantity associated with the option, i.e., remanufacturing or disposal quantity, and the previous regeneration period. There are $T$ major iterations of the DP algorithm, and for each iteration, the cost calculations and comparisons require $O(T)$ operations. Therefore, the proposed DP algorithm runs in $O(T^2)$ for $LR1SP2^p$. Let $LB2_p$ denote the optimal solution to $LR1SP2^p$ obtained using the DP algorithm described above. Then, the lower bound for the Lagrangian problem, $LB$, is given by:

$$LB = \sum_{p=1}^{P} (LB1_p + LB2_p) + \sum_{t=1}^{T} (\min\{F^r_{pt} - K^r_{pt}\gamma_{pt}, 0\} + \min\{F^d_{pt} - K^d_{pt}\lambda_{pt}, 0\} + \min\{F^u_{pt} - \omega_{pt}, 0\} - \kappa_t K^m_t)$$

In the next section we explain how we use this lower bound solution to obtain upper bound solution for MRDPP.

#### 4.3.1.2 Obtaining upper bounds

For given $\kappa$, $\gamma$, $\lambda$, $\theta$, and $\omega$; an optimal solution to the Lagrangian problem (4–13) can be identified efficiently with the DP approaches described in Section 4.3.1.1. However, this optimal solution may violate the manufacturing, remanufacturing, and disposal capacity restrictions, as well as linkage constraint sets for remanufacturing quantity and setup variables, i.e., (4–23) and (4–24). Moreover, the quantity of remanufactured items may exceed the available quantity of returns. We employ a smoothing heuristic to obtain upper bound on each iteration. We use both the lower bound solutions and solutions obtained after smoothing heuristic at every fix number of iterations to generate an initial population for the GA. Between every fixed number of iterations, we use the genetic algorithm to improve the current best upper bound. Moreover, at the end of the Lagrangian iterations, we use GA again. Below we explain the details of the smoothing heuristic and the GA.

**Smoothing Heuristic**. Recall that the lower bound solution is obtained in Section 4.3.1.1 may violate the manufacturing, remanufacturing, and disposal capacity restrictions, as well as the linkage constraints for the remanufacturing quantity and the setup variables. Note that the remanufacturing quantities from the solution of $LR1SP1^p$ may violate the available return quantity, similarly the remanufacturing quantities from the solution of $LR1SP2^p$ may violate the demand quantity. Hence, we apply the smoothing heuristic to these two solutions differ by only the remanufacturing quantities, where the first one $(\mathbf{y}^{\mathbf{r}*}, \mathbf{x}^{\mathbf{r}*})$ is obtained by the solution of $LR1SP1^p$ and the second one $(\mathbf{y}^{\mathbf{u}*}, \mathbf{x}^{\mathbf{u}*})$ is obtained by the solution of $LR1SP2^p$. We first fix the remanufacturing quantity and the setup variables for both solutions, then apply the same procedure to obtain two upper bounds to MRDPP as well as two individuals for the population of the GA.

Our smoothing heuristic has five phases and stops when a feasible solution is found or an infeasible solution with the least amount of capacity violation. The first phase of the approach is a forward pass (i.e., starts from the beginning of the planning horizon and proceeds towards the end of planning horizon) that adjusts the remanufacturing quantities, to eliminate the remanufacturing capacity violations and available return and demand violations. Once the remanufacturing quantities are adjusted, the current solution may no longer satisfy the inventory balance constraints for both UII and MRII. Therefore, in the second phase, we remodel the $LR1SP1^p$ and $LR2SP2^p$ by fixing the values of the remanufacturing variables to obtain manufacturing and disposal quantities, which would satisfy the inventory balance constraints. The third phase of the approach is a forward pass that adjusts the disposal quantities to eliminate disposal capacity violations. The fourth phase is a backward and forward pass that eliminates the manufacturing capacity violations. The fifth, and the last, pass eliminates the unnecessary inventory for the products. Note that the third through fifth phases of the smoothing heuristic maintain the feasibility of the inventory balance equations for UII and MRII while ensuring demand cannot be backlogged and returns cannot be remanufactured or disposed of before they are received.

**1. Adjusting remanufacturing quantities:**

Given a product that violates any of the remanufacturing capacity restrictions, and/or available return, and/or required demand, we identify the earliest period $t$ with a remanufacturing setup where any of these violations occur. If the violation is related to the available return, then the remanufacturing quantity is decreased to match the available return level. If there is violation to required demand, then the remanufacturing quantity is decreased to match the required demand. If the adjusted remanufacturing quantity violates the remanufacturing capacity, then we consider four options to eliminate/reduce the violation in this period to shift the *maximum amount possible* to a (i) previous period with an existing remanufacturing setup, or (ii) future period

with an existing remanufacturing setup, or (iii) previous period scheduling an additional remanufacturing setup, or (iv) future period scheduling an additional remanufacturing setup. Here, the *maximum amount possible* is dictated by the minimum of the amount of capacity violation in the period considered for the shift, available return, and required demand. For each option, we determine the marginal cost of eliminating/reducing the remanufacturing capacity violation, which is obtained by dividing the total cost change associated with implementing the option by the amount that can be shifted by the option. In evaluating the total cost change for each option, we consider changes in the total unit remanufacturing, and UII and MRII inventory holding costs for options (i) and (ii). For options (iii) and (iv), we take into account the changes in total unit remanufacturing, and UII and MRII inventory holding as well as fixed remanufacturing costs. Moreover, in identifying the amount that can be shifted for options (i) and (iii), we ensure that returns are not remanufactured before they are received. Similarly, in identifying the amount that can be shifted for options (ii) and (iv), we ensure that demand is not backlogged and the remanufacturing quantity is not higher than the required demand in future periods. The option that yields the minimum marginal cost is executed, and the marginal costs of options are reevaluated until there is either no more remanufacturing capacity violation in period $t$, or no more possible shift option left. If there is still capacity violation that cannot be shifted, the remanufacturing quantity is decreased so that the capacity violation is removed. Then, the procedure is repeated for the subsequent periods with violations. The procedure is stopped when the current period is $t = T$ or there are no other periods with a any kind of violation for the remanufacturing.

The above procedure is repeated for all the products with remanufacturing capacity and/or available return or required demand violations.

**2. Eliminating UII and MRII inventory balance violations:**

Since linkage constraints are relaxed and remanufacturing quantities are adjusted, the inventory balance equations may not be satisfied. In order to get a solution that satisfies

the inventory balance equations for UII and MRII, we fix the remanufacturing quantities and resolve the subproblems $LR1SP1^p$ and $LR1SP2^p$ for each product type $p$. Since we fix the remanufacturing variables (i.e.,$\mathbf{y^r}, \mathbf{x^r}, \mathbf{y^u}, \mathbf{x^u}$), the subproblems yield solutions that satisfy the inventory balance equations. This solution, however, may violate the manufacturing or disposal capacity restrictions.

**3. Eliminate disposal capacity violations:**

Given a product that violates any of the disposal capacity restrictions, we identify the earliest period $t$ with a disposal setup where the disposal capacity is violated. To eliminate/reduce the capacity violation in this period, we consider four options to shift the *maximum amount possible* to a (i) previous period with an existing disposal setup, or (ii) future period with an existing disposal setup, or (iii) previous period scheduling an additional disposal setup, or (iv) future period scheduling an additional disposal setup. Here, the *maximum amount possible* is dictated by the minimum of the amount of capacity violation and the amount of available disposal capacity in the period considered for the shift. For each option, we determine the marginal cost of eliminating/reducing the disposal capacity violation, which is obtained by dividing the total cost change associated with implementing the option by the amount that can be shifted by the option. In evaluating the total cost change for each option, we consider changes in the total unit disposal and UII inventory holding costs for options (i) and (ii). For options (iii) and (iv), we take into account the changes in total disposal, UII nventory holding and fixed disposal costs. Moreover, in identifying the amount that can be shifted for options (i) and (iii), we ensure that returns are not disposed of before they are received. The option that yields the minimum marginal cost is executed, and the marginal costs of options are reevaluated until there is no more disposal capacity violation in period $t$. Then, the procedure is repeated for the subsequent periods with disposal capacity violations. When the current period is $t = T$, or there are no other periods with a disposal capacity violation for the product, all disposal capacity violations for the product are guaranteed to be eliminated.

The above procedure is repeated for all the products with disposal capacity violations.

**4. Eliminate manufacturing capacity violations:**

This phase is executed in two steps.

(a) **Eliminate manufacturing capacity violations for periods $t > 1$:** We begin with the latest period $t$ where there is a manufacturing setup and the manufacturing capacity is violated.

We classify the products that are manufactured in this period into two sets. The products for which the manufacturing quantity is *smaller than or equal to* the amount of manufacturing capacity violation are included in the first set. The second set comprises of the products for which the manufacturing quantity is *larger* that the amount of manufacturing capacity violation. For each product in the first set, we consider to shift the entire manufacturing quantity of the product (i) to the most recent period with an existing manufacturing setup, or (ii) to the previous period $(t-1)$. In evaluating the total cost change associated with the above two options, we account for changes in the total unit manufacturing and MRII inventory holding costs. Moreover, for the second option, we also take into account the fixed cost if the shift requires the scheduling of an additional manufacturing setup in period $(t-1)$. Similarly, for each product in the second set, we consider to shift (i) the amount of capacity violation to the previous period $(t-1)$, or (ii) the entire manufacturing quantity to the previous period $(t-1)$, or (iii) the entire manufacturing quantity to the most recent period with an existing manufacturing setup. Again, in evaluating these options, we account for the changes in the total unit manufacturing and MRII inventory holding costs. Moreover, for the first and second options, we take into account the fixed manufacturing setup cost if the shift requires the scheduling of an additional manufacturing in period $(t-1)$.

For each product, we pick the option that yields the minimum marginal cost which is obtained by dividing the total cost change associated with implementing the option by the amount that can be shifted by the option. Then, we pick the product (and, hence, the associated option) leading to the best marginal cost and shift the amount implied by the option. We repeat the procedure (starting with classifying the products into two sets) until there is no manufacturing capacity violation in the current period $t$.

Then, the routine is applied for the previous period which has a capacity violation, and this is continued until the first period is reached. When the current period is $t = 1$, all manufacturing capacity violations are guaranteed to be eliminated with, possibly, the exception of the first period.

(b) **Eliminate the manufacturing capacity violation in period $t = 1$:** We use the following routine to eliminate the manufacturing capacity violation in period $t = 1$. We consider all the products that are manufactured in the period and classify them into two sets as before. For each product in the first set (whose manufacturing quantity is *smaller than or equal to* the amount of manufacturing capacity violation), we consider to shift the entire manufacturing quantity of the product to the next period. For each product in the second set (whose manufacturing quantity is *larger*

*than* the amount of manufacturing capacity violation) we consider to shift (i) the amount of capacity violation to the next period, or (ii) the entire manufacturing quantity to the next period. To evaluate the total cost change associated with the above options, we account for the changes in the total unit manufacturing and MRII inventory holding costs as well as the additional manufacturing setup cost, if needed. Moreover, in identifying the amount that can be shifted by these options, we bear in mind that backlogging is not allowed.

For each product, we pick the option that yields the minimum marginal cost which is obtained by dividing the total cost change associated with implementing the option by the amount that can be shifted by the option. Then, we pick the product (and, hence, the associated option) leading to the best marginal cost and shift the amount implied by the option for the product. We repeat the procedure (starting with classifying the products into two sets) until there is no manufacturing capacity violation in the current period $t$.

If the next period has no manufacturing capacity violation, all manufacturing capacity violations are guaranteed to be eliminated. Otherwise, the routine is repeated for the next period.

**5. Eliminate unnecessary inventory:**

This phase is executed in the following three steps.

(a) **Adjust manufacturing quantities to eliminate any unnecessary inventory:**
We use the following routine to eliminate any unnecessary inventory carried into a period adjusting the manufacturing quantities of the products.

We start from the latest period $t$ where there is a manufacturing setup and unused manufacturing capacity. We consider all the products that are manufactured in this period and have positive incoming MRII inventory to the period.

For each such product, we compute the potential cost reduction associated with increasing the corresponding manufacturing quantity in $t$ and decreasing it in $t'$ (where $t'$ is the product's most recent manufacturing setup period, and the product's ending inventory is positive for all periods from $t'$ up to and including $(t-1)$). Then, we pick the product with the largest cost reduction and shift the *maximum amount possible* for this product from $t'$ to $t$ bearing in mind the manufacturing capacities in both periods as well as the inventory balance constraints from $t'$ to $t$. Here, the *maximum amount possible* is dictated by the minimum of (i) the amount of incoming inventory of the product to period $t$, (ii) the manufacturing quantity in period $t'$, and (iii) the available manufacturing capacity in period $t$.

The procedure is continued until either there is no other product leading to a cost reduction or all the manufacturing capacity is used. Then, the procedure is repeated for the previous period with a manufacturing setup and unused manufacturing capacity.

When the current period is $t = 1$, or there are no other periods with a manufacturing setup and unused manufacturing capacity, unnecessary inventory due to early manufacturing has been eliminated for all the products.

(b) **Adjust remanufacturing quantities to eliminate unnecessary MRII inventory:** We use the following routine to eliminate any unnecessary MRII inventory for a product by adjusting its remanufacturing quantities.
Given a product, we start from the latest period $t$ where there is a remanufacturing setup for the product. If there is any unused capacity in period $t$ and the incoming MRII inventory is positive in all periods from the previous remanufacturing setup period $t'$ for the product, then we compute the potential cost reduction associated with decreasing the remanufacturing quantity in $t'$ and increasing the remanufacturing quantity in $t$. If the cost reduction is positive, then we shift the *maximum amount possible* that can be moved for the product from $t'$ to $t$ bearing in mind the remanufacturing capacities in both periods as well as the UII and MRII inventory balance constraints from $t'$ to $t$.
Here, the *maximum amount possible* is dictated by the minimum of (i) the minimum amount of MRII inventory of the product from period $t'$ to period $t$, (ii) the remanufacturing quantity in period $t'$, and (iii) the available remanufacturing capacity in period $t$. The procedure is repeated for the previous period with a remanufacturing setup and unused remanufacturing capacity. When the current period is $t = 1$ or there are no other periods with a remanufacturing setup and unused remanufacturing capacity for the product, unnecessary MRII inventory due to earlier remanufacturing has been eliminated for the product.
Above routine is repeated for all the products.

(c) **Adjust disposal quantities to eliminate any unnecessary inventory:** We use the following routine to eliminate any unnecessary UII inventory for a product by adjusting its disposal quantities.
Given a product, we start from the earliest period $t$ where there is a disposal setup for the product. If there is any unused disposal capacity in period $t$ and the UII inventory is positive in all periods up to the next disposal setup period $t'$ for the product, then we compute the potential cost reduction associated with decreasing the disposal quantity in $t'$ and increasing the disposal quantity in $t$. If the cost reduction is positive, then we shift the *maximum amount possible* that can be moved for the product from $t'$ to $t$ bearing in mind the disposal capacities in both periods as well as the UII inventory balance constraints from $t$ to $t'$. Here, the *maximum amount possible* is dictated by the minimum of (i) the minimum amount of UII inventory of the product from period $t$ to period $t'$, (ii) the disposal quantity in period $t'$, and (iii) the available disposal capacity in period $t$. The procedure is repeated for the next period with a disposal setup and unused disposal capacity. When the current period is $t = T$ or there are no other periods with a disposal setup and unused disposal capacity for the product, unnecessary UII inventory due to delayed disposal has been eliminated for the product.
Above routine is repeated for all the products.

Recall that we employ the above heuristic using remanufacturing quantities $\mathbf{x^r}$ or $\mathbf{x^u}$) separately. Therefore, we obtain two upper bound solutions. If any of these upper

bounds is better than the current best upper bound, then the current upper bound is updated. Moreover, we will use both of these upper bound solutions in the generation of the population of the GA.

**Genetic Algorithm**. We represent each individual (chromosome) of the population with setup variables for manufacturing, remanufacturing, and disposal for all products. Hence, each individual is represented by a matrix of size $3 \times P \times T$ with $\{0, 1\}$ values. We note that if the values of the setup variables are known, the resulting problem reduces to a minimum cost network flow problem. The LP relaxation of the problem for given setup variables yields integer solutions for manufacturing, remanufacturing, and disposal variables as well as the inventory variables for UII and MRII. Hence, the complete solution for each chromosome can be obtained by solving the LP relaxation of the problem. In order to ensure the feasibility, two types of slack variables are defined and added to the inventory balance constraints, i.e., (4–2) and (4–3), with some penalty added to the objective function.

To generate the population for an iteration of GA, we use the lower bound solution and smoothing heuristic. We obtain four chromosomes at every fixed number of iterations. The first chromosome is obtained using the setup variables' values corresponding to the manufacturing and the remanufacturing from the solution to the first subproblem, and the setup variables' values corresponding to the disposal from the solution of the second subproblem, i.e., $[\mathbf{y^{m*}}, \mathbf{y^{r*}}, \mathbf{y^{d*}}]$. The second chromosome is obtained by using the setup variables' values of manufacturing from the solution of the first subproblem, and the setup variables' values of remanufacturing and disposal from the solution of the second subproblem, i.e., $[\mathbf{y^{m*}}, \mathbf{y^{u*}}, \mathbf{y^{d*}}]$. The third and fourth chromosomes are obtained from the modification of the solutions obtained from lower bouding using the smoothing heuristics.

After a fixed number of Lagrangian iterations are executed, GA is employed using the generated population up to that iteration. We use the objective function value as a fitness value for each individual in the population. From the current population we first choose

the best (fittest) individual (i.e., chromosome or solution) as the first parent. We then randomly choose another individual from the rest of the population as the second parent. We apply random multi-point crossover (a random break point for each product and each type of activity) to these parents to generate two new solutions (children or offsprings). The objective function (fitness) values and quantity decision variables are found by solving the LP relaxation problem for both children. The fittest child randomly replaces either the weak parent or weakest individual in the population. This procedure is repeated for fixed number of times (preset number of generations). Below we give the details of the procedure.

We provide the specific outline of our GA implementation in Figure 4-3 where $\mathcal{M}$ denotes the given population size, $\Delta$ denotes the iteration counter, and $\Delta_m ax$ denotes the number of generations. Let $P1$ and $P2$ denote two parent chromosomes. Similarly, let $C1$ and $C2$ denote two children chromosomes. We let $\mathcal{L}(j)$ denote the objective function (fitness) value of individual $j$.

After Lagrangian iterations stops either with predefined gap has been reached or predefined number of iterations executed. We use the GA by using the most recent population to find a better upper bound solution. We report the solution with best objective function value.

#### 4.3.1.3 Subgradient optimization

For given $\kappa$, $\gamma$, $\lambda$, $\theta$, and $\omega$ an optimal solution to the Lagrangian problem gives a lower bound on the optimal objective function value of the MRDPP. To find the best possible lower bound, we need to determine the values of $\kappa$, $\gamma$, $\lambda$, $\theta$, and $\omega$ that maximize this lower bound. We solve this problem approximately using the subgradient optimization method. Specifically, we initialize the Lagrangian multipliers to zero at the beginning of the procedure. Then, at each iteration $\ell$, using the lower and upper bound values, we

update the multipliers using :

$$\kappa_t^{\ell+1} = \max\left\{0, \kappa_t^\ell + a^\ell \frac{(UB^* - LB^\ell)(\sum_{p=1}^P(\alpha_p x_{pt}^m) - K_t^m)}{\Omega^\ell}\right\},$$

$$\gamma_{pt}^{\ell+1} = \max\left\{0, \gamma_{pt}^\ell + a^\ell \frac{(UB^* - LB^\ell)(x_{pt}^r - K_{pt}^r y_{pt}^r)}{\Omega^\ell}\right\},$$

$$\lambda_{pt}^{\ell+1} = \max\left\{0, \lambda_{pt}^\ell + a^\ell \frac{(UB^* - LB^\ell)(x_{pt}^d - K_{pt}^d y_{pt}^d)}{\Omega^\ell}\right\},$$

$$\theta_{pt}^{\ell+1} = \theta_{pt}^\ell + a^\ell \frac{(UB^* - LB^\ell)(x_{pt}^r - x_{pt}^u)}{\Omega^\ell},$$

$$\omega_{pt}^{\ell+1} = \omega_{pt}^\ell + a^\ell \frac{(UB^* - LB^\ell)(y_{pt}^r - y_{pt}^u)}{\Omega^\ell}.$$

where $\Omega^\ell = \sum_{t=1}^T((\sum_{p=1}^P(\alpha_p x_{pt}^m) - K_t^m)^2 + \sum_{p=1}^P(x_{pt}^r - K_{pt}^r y_{pt}^r)^2 + (x_{pt}^d - K_{pt}^d y_{pt}^d)^2 + (x_{pt}^r - x_{pt}^u)^2 + (y_{pt}^r - y_{pt}^u)^2)^2$ and $a^\ell$ is the step size at iteration $\ell$.

We update the step size by halving its value when there is no improvement in the lower bound in a fixed number of iterations.

### 4.3.2 A Lagrangian Relaxation Approach for MRDPP

In this approach, we only relax the manufacturing capacity constraint set, i.e, (4–4). When constraint set (4–4) is relaxed, the resulting Lagrangian problem becomes separable for each product $p, p = 1, \ldots, P$. Each subproblem is a single product MRDPP, and it is NP-Hard. Hence, we use CPLEX to solve the Lagrangian subproblems. Instead of solving the subproblems to optimality, a 1% optimality gap is used to obtain solutions faster.

Let $\kappa = \{\kappa_t\}$ denote the vector of Lagrangian multipliers $\kappa_t \geq 0, t = 1, \ldots, T$, associated with the manufacturing capacity constraints in set (4–16). Then the Lagrangian problem can be stated as follows:

$$\min \sum_{p=1}^P \sum_{t=1}^T (F_{pt}^m y_{pt}^m + (c_{pt}^m + \kappa_t) x_{pt}^m + F_{pt}^r y_t^r + c_{pt}^r x_{pt}^r + F_{pt}^d y_{pt}^d + c_{pt}^d x_{pt}^d + k_{pt} j_{pt} + h_{pt} i_{pt}) - \sum_{t=1}^T (\kappa_t K_t^m) \tag{4–45}$$

subject to (4–2)-(4–3), (4–5)-(4–12).

For a given set of Lagrangian multipliers $\kappa$, the solution of the Lagrangian problem yields a lower bound.

We provide the specific outline of our LR implementation in Figure 4-4 where $\ell$ and $\tau$ denote the iteration counters. The termination criteria are defined by $\epsilon$, the tolerance gap between the best lower and upper bounds, and $\ell_{\max}$, the maximum number of iterations of LR. When there are $\tau_{\max}$ LR iterations, then GA is applied. Moreover, $UB^{\ell}$ and $LB^{\ell}$ denote the upper and lower bounds, respectively, at iteration $\ell$. Similarly, $UB^{*}$ and $LB^{*}$ denote the incumbent upper and lower bounds, respectively. We proceed with how we solve the Lagrangian problem to obtain a lower bound (Section 4.3.2.1), find an upper bound using the smoothing heuristic (Section 4.3.2.2) and GA, and update the Lagrangian multipliers (Section 4.3.2.3), on each iteration of the LD approach.

#### 4.3.2.1 Obtaining lower bounds

For a given $\kappa$ the Lagrangian problem (4–45) is separable into $P$ subproblems. Let $LR2SP^{p}$ denotes the subproblem for product $p$ for $p = 1 \ldots, P$. We note that the last term in the objective function of this problem is constant and does not effect the solution of the Lagrangian problem. The single product Lagrangian subproblem (for product $p$), i.e., $LR2SP^{p}$ can be stated as follows:

$$\min \quad \sum_{t=1}^{T} (F^{m}_{pt} y^{m}_{pt} + (c^{m}_{pt} + \kappa_t) x^{m}_{pt} + F^{r}_{pt} y^{r}_{pt} + c^{r}_{pt} x^{r}_{pt} + F^{d}_{pt} y^{d}_{pt} + c^{d}_{pt} x^{d}_{pt} + k_{pt} j_{pt} + h_{pt} i_{pt}) \quad (4\text{–}46)$$

subject to

$$i_{p,t-1} + x^{m}_{pt} + x^{r}_{pt} - D_{pt} = i_{pt} \qquad t = 1, \ldots, T; \qquad (4\text{–}47)$$

$$j_{p,t-1} + R_{pt} - x^{r}_{pt} - x^{d}_{pt} = j_{pt} \qquad t = 1, \ldots, T; \qquad (4\text{–}48)$$

$$x^{r}_{pt} \leq K^{r}_{pt} y^{r}_{pt} \qquad t = 1, \ldots, T; \qquad (4\text{–}49)$$

$$x^{d}_{pt} \leq K^{d}_{pt} y^{d}_{pt} \qquad t = 1, \ldots, T; \qquad (4\text{–}50)$$

$$x^{m}_{pt} \leq \overline{D}_{pt} y^{m}_{pt} \qquad t = 1, \ldots, T; \qquad (4\text{–}51)$$

$$x^r_{pt} \leq \overline{D}_{pt} y^r_{pt} \qquad t = 1, \ldots, T; \tag{4–52}$$

$$x^d_{pt} \leq \overline{R}_{pt} y^d_{pt} \qquad t = 1, \ldots, T; \tag{4–53}$$

$$j_{p0},\, j_{pT}, i_{p0},\, i_{pT} = 0 \tag{4–54}$$

$$x^m_{pt},\, x^r_{pt},\, x^d_{pt},\, j_{pt}\, i_{pt} \geq 0 \text{ and integer} \qquad t = 1, \ldots, T; \tag{4–55}$$

$$y^m_{pt},\, y^r_{pt},\, y^d_{pt} \in \{0, 1\} \qquad t = 1, \ldots, T. \tag{4–56}$$

We solve $LR2SP^p$ for product $p$ by CPLEX with 1% optimality gap for $p = 1, \ldots, P$. Let $LB_p$ denote best objective function value obtained from CPLEX, then the lower bound for the MRDPP, $LB$, is given by:

$$LB = \sum_{p=1}^{P} (LB_p) - \sum_{t=1}^{T} (\kappa_t K^m_t)$$

#### 4.3.2.2 Obtaining upper bounds

For a given $\kappa$, an optimal solution to the Lagrangian problem (4–45) is obtained by solving $LR2SP^p$ for $p = 1, \ldots, P$ by CPLEX. This optimal solution may only violate the shared manufacturing capacity restrictions. We employ smoothing heuristics to obtain an upper bound solution. Moreover, we use GA to improve the upper bound solution. We use the lower bound solution and smoothing heuristic to generate the population for GA for fixed number of Lagrangian iterations. We obtain four chromosomes at each iteration. We then use GA to find a better solution from the current population.

The first chromosome is obtained using the setup variables' values of the manufacturing, remanufacturing and disposal activities obtained from the lower bound solution. The second chromosome is obtained by applying mutation to the first chromosome. We randomly choose a fixed set of products, and for each chosen product, we randomly choose fixed set of periods. For each of these products and periods, we randomly choose manufacturing, remanufacturing, and disposal activity and change the current associated setup variable value, from 1 to 0 or 0 to 1. The third chromosome is obtained by applying

smoothing heuristic to the lower bound solution. The fourth chromosome is also chosen by applying mutation to the third chromosome.

Recall that the lower bound solution is obtained in Section 4.3.2.1 may only violate the manufacturing capacity restrictions. We fix the manufacturing capacity violations by the routine explained in Section 4.3.1.2. If the obtained upper bound is better than the current best upper bound, then the current best upper bound is updated. This upper bound solution is used as a third chromosome for the current Lagrangian iteration. The fourth chromosome is generated by applying mutation to this upper upper bound.

After a fixed number of Lagrangian iterations, GA is employed using the generated population up to that iteration. We use the GA given in Section 4.3.1.2. The only difference comes from the size and quality of the population.

Lagrangian iterations stops either with predefined gap has been reached or predefined number of iterations hit. We conduct the genetic algorithm once more using the most recent population to find a better upper bound solution. We report the solution with best objective function value.

#### 4.3.2.3 Subgradient optimization

For a given $\kappa$, an optimal solution to the Lagrangian problem gives a lower bound on the optimal objective function value of the MRDPP. To find the best possible lower bound, we need to determine the values of $\kappa$ that maximize this lower bound. We solve this problem approximately using the subgradient optimization method. Specifically, we initialize the Lagrangian multipliers to zero at the beginning of the procedure. Then, at each iteration $\ell$, using the lower and upper bound values, we update the multipliers using

$$\kappa_t^{\ell+1} = \max\left\{0, \kappa_t^{\ell} + a^{\ell}\frac{(UB^* - LB^{\ell})(\sum_{p=1}^{P}(\alpha_p x_{pt}^m) - K_t^m)}{\sum_{t=1}^{T}((\sum_{p=1}^{P}(\alpha_p x_{pt}^m) - K_t^m)^2}\right\}$$

where $a^{\ell}$ is the step size at iteration $\ell$.

We update the step size by halving its value when there is no improvement in the lower bound in a fixed number of iterations.

## 4.4 Computational Experiments

We test the performance of the proposed LD and LR approaches on a set of randomly generated test instances. For benchmarking purposes, we solve the test instances using the branch-and-cut (B&C) approach as well. To this end, we employ CPLEX 10.1 and Concert Technology with default settings for cut generation, preprocessing, branching, and upper bounding heuristics. We implement the proposed LD and LR approach using Visual Studio C++ programming language and use a computer with a 3.4Ghz Intel Pentium 4 with 2GB installed memory.

### 4.4.1 Random Test Instance Generation

In our preliminary computational experiments, we observed that the correlations between (i) the demand and returns, (ii) the fixed setup and inventory holding costs, (iii) the manufacturing, remanufacturing and disposal capacity levels influence the computational difficulty of the problem instances. Therefore, in generating random test instances, we pay particular attention to these correlations.

In our experiments, we begin by generating the demands, returns, and manufacturing, remanufacturing and disposal capacities by inducing correlations among them for each $(P, T)$ pair. We first generate the demands, then by using the correction between demand and return, and realized demand value, we generate returns. We then generate the return parameters inducing correlation between demands and returns. With the given capacity tightness, we generate the manufacturing capacities by using average realized demand in each period. Similarly, we generate the remanufacturing and disposal capacities for each product and each period by using the capacity tightness level and the realized return and demand values.

In our experiments, we consider instances with constant unit manufacturing, unit remanufacturing, and unit disposal, and inventory holding costs for UII and MRII. We generate the fixed manufacturing, remanufacturing, and disposal setup costs based on the inventory holding costs. In particular, we begin by generating the time between

manufacturing (TBM) setups and the product-specific time between remanufacturing and disposal (TBR, and TBD) setups. Then, we generate the fixed manufacturing setup cost for each period in view of the average demand, TBM value, MRII inventory holding cost and variability of fixed manufacturing setup costs. We generate the fixed remanufacturing setup cost for each product in each period considering the average demand and return, TBR value, average UII and MRII inventory holding cost, and variability of fixed remanufacturing setup costs.

Similarly, we generate the fixed disposal setup cost for each product in each period considering the average return, TBD value, inventory holding cost, and variability of fixed disposal setup costs.

To develop a set of random instances, we vary the number of products $P$ and the length of the planning horizon, $T$. Specifically, we consider two levels for $P$ (10, 20) and two levels for $T$ (12, 24). Consequently, we have 4 classes of test instances. Moreover, we consider five levels for the correlation between total demand to total returns throughout the planning horizon (from 0.15 for low to 0.75 for high in increments of 0.15), two levels for each of TBM, TBR, and TBD (U[1, 3] for low or U[3, 5] for high), and two levels for each manufacturing, remanufacturing and disposal capacity tightness (1.5 for low or 2.5 for high). We consider the parameter values summarized in Table 4-1.

For each $(P, T)$ we consider 20 different problem settings, and for each setting we generate 10 random test instances. Therefore, we generate and solve 800 instances in total.

### 4.4.2 Discussion of Computational Results

To investigate the performance of the proposed LD and LR approaches, we solve the problem instances and examine the gap between the resulting $LB^*$ and $UB^*$. As a performance criteria we use

$$\frac{UB^* - LB^*}{UB^* + LB^*} \times 200$$

In our experiments, the LD approach terminates when the gap between the lower and upper bound is less than 0.5 percent or a maximum of 500 iterations are executed. The step size is set initially to 2 and halved if the lower bound value is not improved in five consecutive iterations. We add solutions to GA population at every 10 iterations, and use the GA every 50 iterations. We use 20 generations in each of the GA applications. When the LD approach terminates we use 100 generations on final GA application.

We summarize the optimality gaps obtained by the LD approach and its CPU requirements in Table 4-2. As we solve a total of 800 test instances, each cell in this table corresponds to the average or maximum value representing 200 instances of the same size. We note that the observed average optimality gap over all instances is 6.86 percent, whereas the maximum is 15.34 percent. We also observe that for a given number of products, the average and maximum values of the optimality gaps become smaller as the length of the planning horizon increases. For a given planning horizon length, the average and maximum values of the optimality gaps become smaller by an increase in the number of products.

Moreover, we observe that the LD approach is computationally efficient as the observed average CPU requirement over all instances is less than 17 seconds, whereas the maximum is less than 37 seconds.

In Tables 4-3 and 4-4, we provide a detailed analysis of the influence of the data characteristics on the performance of the LD approach.

We observe that the quality of the optimality gaps deteriorates as the time between activity setups increase and/or (ii) the activity capacities become more restraining. It is worthwhile to note that the influence of an increase in capacity tightness is to be more influential than an increase in time between activity setups. In addition, we observe that the optimality gaps obtained by the LD approach deteriorates as the correlation between returns and demand increases in both the average and maximum optimality gap.

We also solve the same problem instances with LR approach. In our experiments, the LR approach terminates when the gap between the lower and upper bound is less than 0.5 percent or a maximum of 40 iterations are executed. The step size is set initially to 2 and halved if the lower bound value is not improved in five consecutive iterations. We add 4 solutions to GA population at the end of every iteration, and use the GA at the end of every 10 iterations. We use 20 generations in each of the GA applications. When the LR approach terminates we use 50 generations on final GA application.

We summarize the optimality gaps obtained by the LR approach and its CPU requirements in Table 4-5. We note that the observed average optimality gap over all instances is 0.95 percent, whereas the maximum is 3.69 percent. We observe the same influence of number of products and length of planning horizon on the optimality gaps of the LR approach as in the LD approach. Consequently, the quality of the gap identified by the LR approach is better for instances with a larger number of products and/or a longer planning horizon. Our computational results showed that the proposed LD approach is more efficient than LR approach, but LR approach is much more effective than LD approach.

In Tables 4-6 and 4-7, we provide a detailed analysis of the influence of the data characteristics on the performance of the LR approach.

Surprisingly, we observe that the quality of the optimality gaps of LR approach does not deteriorate as much as LD approach when the time between activity setups increase and/or (ii) the activity capacities become more restraining. More interestingly, average optimality gaps almost remain same independent from the capacity tightness and/or time between activity setups. In addition, the correlation between returns and demand does not have much effect on the quality of LR approach.

Although the results in Table 4-2 provide empirical evidence for the effectiveness of the proposed LR approach to address the MRDPP, an interesting question that remains unanswered is the effectiveness of the B&C approach for obtaining solutions to the

problem. We attempted to solve same instances with CPLEX by setting the time limit to maximum time required for LR approach for each $(P, T)$ pair. For example we set the stopping time for problem instances with 10 product types and 12 periods to 65 seconds. We summarize the optimality gaps obtained by CPLEX in Table 4-8

Again, each cell in this table represents the average or maximum value of 800 observations for test instances of the same size. We note that the average optimality gap over all instances is around 6.22 percent whereas the maximum is around 32 percent. This clearly shows that our LR approach outperforms the CPLEX.

## 4.5 Concluding Remarks

We considered an original equipment manufacturer (OEM) in the automotive industry that provides vehicle maintenance and repair services for vehicles for which replacement parts (products) are often needed. To ensure uninterrupted replacement product availability, OEM has to coordinate manufacturing, remanufacturing, and disposal decisions for the replacement products. The problem of interest, called MRDPP, was to determine the timing of the manufacturing, remanufacturing and disposal setups, along with the associated manufacturing, remanufacturing and disposal quantities for multiple product types throughout a finite planning horizon so as to minimize the total costs of manufacturing, remanufacturing, disposal, and inventory holding. We considered a variant of the problem where manufacturing capacity is shared among multiple product types while the remanufacturing and disposal capacities are product specific. We developed a LD and a LR approach. The LD approach relied on the decomposition of the problem along with the relaxation of the manufacturing, remanufacturing, and disposal capacity constraints. To obtain lower bounds on the optimal solution values, we developed DP approaches to solve subproblems in polynomial time. To obtain upper bounds, we developed a smoothing heuristic and utilized a GA approach. The LR approach relied on the relaxation of the shared manufacturing capacity constraints. We used CPLEX

to solve the resulting subproblems. Again, to obtain upper bounds, we developed a smoothing heuristic.

We conducted extensive computational experiments to investigate the computational efficiency of the LD and LR approaches. Our computational results showed that the proposed LD approach is computationally more efficient than the LR approach, but the LR approach is more effective than the LD approach.

An immediate extension of our work that would be of practical relevance is to consider a generalization of the MRDPP where (i) the manufacturing and/or remanufacturing and/or disposal setup costs depend on the specific products included in the setup and/or (ii) setup times are explicitly modeled. Moreover, MRDPP with shared remanufacturing capacity and/or linear disposal costs can be possible extensions. Our solution algorithms can be applied to these problems with slight modifications. Also, the LR approach presented in this paper can be extended to settings where certain capacitated resources are shared by the manufacturing and remanufacturing activities.

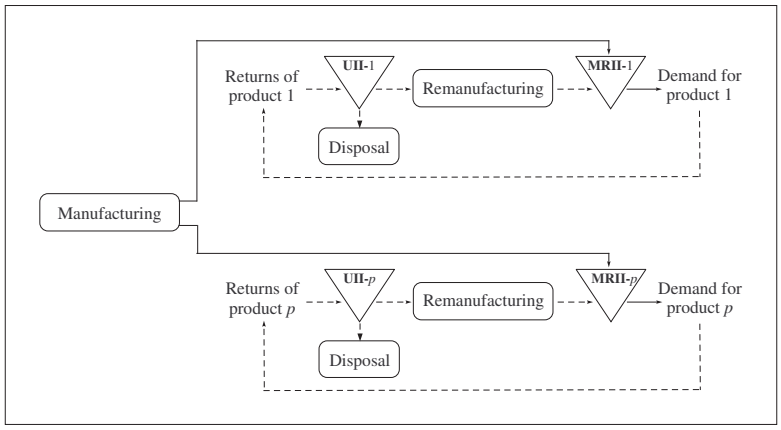


Figure 4-1. Problem Setting of MRDPP

```
Initialize ϵ, ℓ_max, τ_max, and the Lagrangian multipliers. Set ℓ = 1; τ = 1; UB* =
UB^ℓ = ∞; and LB* = LB^ℓ = −∞.
while ((LB* ≤ (1 − ϵ)UB*) and (ℓ ≤ ℓ_max)) do
   Formulate and solve the Lagrangian dual problem by DP algorithms to
   obtain LB^ℓ.
   if (LB^ℓ > LB*) then
      Set LB* = LB^ℓ.
   end if
   if ϕ = ϕ_max then
      Add two individuals obtained by the lower bounding approach to GA
      population.
   end if
   Find a feasible solution using the smoothing heuristic to obtain UB^ℓ.
   if (UB^ℓ < UB*) then
      Set UB* = UB^ℓ. Update the best solution.
   end if
   if ϕ = ϕ_max then
      Add two individuals obtained by the smoothing heuristic to GA
      population.
      Set ϕ = 0
   end if
   if (τ = τ_max) then
      Apply GA to current population to improve UB*. Update UB* if upper
      bound is improved.
      Set τ = 0
   end if
   Revise Lagrangian multipliers using the subgradient optimization method.
   Update step length if necessary.
   Set k = k + 1, ϕ = ϕ + 1, τ = τ + 1
end while
Apply GA to current population to improve UB*. Update UB* if upper bound
is improved.
Report LB*, UB*, and the best solution.
```

Figure 4-2. Pseudo-code of the LD Approach for the MRDPP.

```
Initialize Δ_max. Set Δ = 1. Choose best individual from population as P1,
  L(P1) < L(j), j = 1, ..., M.
while (Δ ≤ Δ_max)) do
   Randomly choose P2 from the population.
   Generate 2 children by applying multi-point crossover for P1 and P2:
   Evaluate objective function values of C1 and C2, L(C1), L(C2).
   if (min{L(C1), L(C2)} < UB*) then
      Set UB* = min{L(C1), L(C2)}. Update the best solution.
   end if
   Randomly replace either the weakest individual of the population or P2 by
     the fittest child.
   Set Δ = Δ + 1
end while
```

Figure 4-3. Pseudo-code of the GA Approach for the LD.

```
Initialize ϵ, ℓ_max, τ_max, and the Lagrangian multipliers. Set ℓ = 1; τ = 1; UB* =
  UB^ℓ = ∞; and LB* = LB^ℓ = −∞.
while ((LB* ≤ (1 − ϵ)UB*) and (ℓ ≤ ℓ_max)) do
   Formulate and solve the Lagrangian dual problem by CPLEX to obtain LB^ℓ.

   if (LB^ℓ > LB*) then
      Set LB* = LB^ℓ.
   end if
   Apply mutation to individuals obtained from lower bound two times, and add
     them to GA population.
   Find a feasible solution using the smoothing heuristic to obtain UB^ℓ.
   if (UB^ℓ < UB*) then
      Set UB* = UB^ℓ. Update the best solution.
   end if
   Add the individual obtained from smoothing heuristic to GA population.
   Apply mutation to upper bound solution and add that to GA population.
   if (τ = τ_max) then
      Apply GA to current population to improve UB*. Update UB* if upper
        bound is improved.
      Set τ = 0
   end if
   Revise Lagrangian multipliers using the subgradient optimization method.
     Update step length if necessary.
   Set k = k + 1, τ = tau + 1
end while
Apply GA to current population to improve UB*. Update UB* if upper bound
  is improved.
Report LB*, UB*, and the best solution.
```

Figure 4-4. Pseudo-code of the LR Approach for the MRDPP.

Table 4-1. Parameter Values Used in Computational Experiments of MRDPP

| **Parameter** | **Level(s)** | **Value(s)** |
|---|---|---|
| Average demand | Constant | 100 |
| Variability of demand | Constant | 0.5 |
| Variability of returns | Constant | 0.5 |
| Variability of fixed manufacturing setup cost | Constant | 0.2 |
| Variability of fixed remanufacturing setup cost | Constant | 0.2 |
| Variability of fixed disposal setup cost | Constant | 0.2 |
| Unit MRII inventory holding cost | Constant | 1 |
| Unit UII inventory holding cost | Constant | 0.5 |
| Unit manufacturing cost | Constant | 3 |
| Unit remanufacturing cost | Constant | 2 |
| Unit disposal cost | Constant | 1 |
| Unit capacity consumption | Constant | 1 |
| Time between manufacturing, remanufacturing, and disposal setups ($TBM, TBR, TBD$) | Low<br>High | U[1, 3]<br>U[3, 5] |
| Manufacturing, remanufacturing and disposal capacity tightness ($\varphi_M, \varphi_R, \varphi_D$) | Low<br>High | 1.5<br>2.5 |
| Return/Demand ratio ($\rho_{RD}$) | Low<br>High | 0.15<br>0.75 |

Table 4-2. Performance of the LD Approach: Average and Maximum Optimality Gaps (%) and CPU Requirement (sec.).

| | Optimality Gap | | CPU Requirement | |
|---|---|---|---|---|
| Instance Size ($P, T$) | Avr. Gap (%) | Max. Gap (%) | Avr. Time (sec.) | Max. Time (sec.) |
| (10,12) | 7.68 | 15.34 | 11.65 | 19.25 |
| (20,12) | 6.43 | 12.63 | 16.72 | 28.56 |
| (10,24) | 7.21 | 14.68 | 15.21 | 27.14 |
| (20,24) | 6.12 | 11.24 | 22.43 | 36.79 |
| Overall | 6.86 | 15.34 | 16.50 | 36.79 |

Table 4-3. Effect of Time Between Activity Setups and Capacity Tightness on the Average and Maximum Optimality Gaps (%) of the LD Approach.

| (TBM, TBR, TBD) | Avr. ($\varphi_M, \varphi_R, \varphi_D$) | Max. Gap (%) | Gap (%) |
|---|---|---|---|
| (U[1,3],U[1,3],U[1,3]) | (1.5,1.5,1.5) | 5.95 | 13.23 |
| (U[3,5],U[3,5],U[3,5]) | (1.5,1.5,1.5) | 8.95 | 17.34 |
| (U[1,3],U[1,3],U[1,3]) | (2.5,2.5,2.5) | 4.92 | 10.72 |
| (U[3,5],U[3,5],U[3,5]) | (2.5,2.5,2.5) | 7.62 | 16.34 |

Table 4-4. Effect of Correlation Between Returns and Demand on the Average and Maximum Optimality Gaps (%) of the LD Approach.

| $\rho_{RD}$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| 0.15 | 4.79 | 9.45 |
| 0.30 | 5.57 | 10.79 |
| 0.45 | 6.67 | 14.84 |
| 0.60 | 7.86 | 16.92 |
| 0.75 | 9.42 | 17.34 |

Table 4-5. Performance of the LR Approach: Average and Maximum Optimality Gaps (%) and CPU Requirement (sec.).

| | Optimality Gap | | CPU Requirement | |
|---|---|---|---|---|
| Instance Size $(P, T)$ | Avr. Gap (%) | Max. Gap (%) | Avr. Time (sec.) | Max. Time (sec.) |
| (10,12) | 0.98 | 3.69 | 14.85 | 52.25 |
| (20,12) | 0.94 | 2.43 | 27.25 | 98.68 |
| (10,24) | 0.97 | 3.21 | 44.52 | 138.53 |
| (20,24) | 0.92 | 2.98 | 72.91 | 356.12 |
| Overall | 0.95 | 3.69 | 39.88 | 356.12 |

Table 4-6. Effect of Time Between Activity Setups and Capacity Tightness on the Average and Maximum Optimality Gaps (%) of the LD Approach.

| (TBM, TBR, TBD) | Avr. $(\varphi_M, \varphi_R, \varphi_D)$ | Max. Gap (%) | Gap (%) |
|---|---|---|---|
| (U[1,3],U[1,3],U[1,3]) | (1.5,1.5,1.5) | 0.94 | 1.59 |
| (U[3,5],U[3,5],U[3,5]) | (1.5,1.5,1.5) | 1.03 | 3.69 |
| (U[1,3],U[1,3],U[1,3]) | (2.5,2.5,2.5) | 0.91 | 1.31 |
| (U[3,5],U[3,5],U[3,5]) | (2.5,2.5,2.5) | 0.92 | 1.44 |

Table 4-7. Effect of Correlation Between Returns and Demand on the Average and Maximum Optimality Gaps (%) of the LD Approach.

| $\rho_{RD}$ | Avr. Gap (%) | Max. Gap (%) |
|---|---|---|
| 0.15 | 0.89 | 3.69 |
| 0.30 | 0.94 | 2.72 |
| 0.45 | 0.93 | 2.43 |
| 0.60 | 0.98 | 1.91 |
| 0.75 | 1.01 | 2.34 |

Table 4-8. Performance of CPLEX (with a Time Limit of 400 Seconds): Average and Maximum Optimality Gaps (%).

| | Optimality Gap | |
|---|---|---|
| Instance Size $(P, T)$ | Avr. Gap (%) | Max. Gap (%) |
| (10,12) | 3.98 | 8.98 |
| (20,12) | 6.23 | 17.42 |
| (10,24) | 5.42 | 14.61 |
| (20,24) | 9.25 | 32.05 |
| Overall | 6.22 | 32.05 |

# CHAPTER 5
# PROCUREMENT PLANNING FOR MULTIPLE PRODUCTS WITH REUSABLE RANDOM RETURNS

## 5.1 Introduction

In catalogue retailing, usually 35-40 percent of the sold products are returned (Mostard and Teunter, 2006). In some cases, the return rate can be as high as 75 percent (Mostard and Teunter, 2006). Since a large proportion of these returned products are received in very good condition, these returns can be placed into inventory to satisfy future demand with little processing, such as inspection, cleaning, testing, repackaging etc. A typical catalogue retailer offers multiple products through its catalogues and internet sites and the (finished goods) inventory for a product can be replenished by either procuring products from the suppliers or processing and restocking the returns received from consumers. In this setting, for effective inventory planning and control, the returns received from consumers must be taken into account while making procurement decisions. An important issue that needs to be taken into account is the uncertainty associated with the return stream, which can be significant in the context of catalogue retailing.

In the context of spare part kitting for industrial equipment maintenance, a typical supplier has service contracts with a stable customer base. This allows the supplier to use historical data to estimate the future demand and return quantities accurately for different types of products. In the context of catalogue retailing, however, retailer's ability to forecast returns accurately is limited, as the quantity and/or timing of consumer returns are highly variable; influenced by several factors such as buyer's remorse or mismatch between customer's expectations about the product. Typically, a retailer procures the products from a supplier at the beginning of a finite planning horizon. Reusable returns received from customers throughout the planning horizon can be placed into inventory as they arrive to satisfy the demand in future periods. Clearly, for optimal inventory planning and control for this setting, returns must be taken into account explicitly when deciding on the procurement quantity at the beginning of the planning horizon. Our focus

in this chapter is on the corresponding *Procurement Planning Problem* (RDPP) with multiple product types.

More specifically, the problem is concerned with determining the procurement quantities for multiple product types at the beginning of the planning horizon. We consider the problem in deterministic demand and stochastic return framework over a finite planning horizon under finite procurement capacity. In our work, we consider four variants of the PPP. We begin by considering the problem in a single-period setting for a single product type (PPP1) and extend this model to include multiple product types (PPP2). Then, we study the problem in multiple-period setting for a single product type (PPP3) as well as for multiple product types (PPP4).

The remainder of this chapter is organized as follows. We proceed with the modelling assumptions in 5.2 We develop mathematical models and solution approaches for a single product type in Sections 5.3 and 5.4 for PPP1 and PP2, respectively. We analyze these models when the procurement costs are linear and when there is a fixed component in addition to the linear component. Similarly, we investigate models and solution approaches for multiple product types in Sections 5.5 and 5.6 for PPP3 and PPP4, respectively. Finally, conclusions and future research directions are summarized in Section 5.7.

## 5.2 Modelling Assumptions

In our work, we make the following modeling assumptions:

- There is a finite planning horizon with $T$ periods, indexed by $t = 1, \ldots, T$.
- There are $P$ different types of products, indexed by $p = 1, \ldots, P$.
- Demand forecast for product type $p$ in the planning period $t$ is known and denoted by $d_{pt}$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$.
- Unsatisfied demand for product type $p$ is backordered and shortage cost is incurred. Let $\pi_{pt}$ denote the shortage cost incurred for product type $p$ in period $t$.
- All of the returns are processed upon arrival and placed in inventory, i.e., no direct disposal. Let $c^r_{pt}$ denote the processing cost of a return for product type $p$ in period $t$.

- There is a single procurement opportunity at to the beginning of the planning horizon. Procurement items are received prior to the beginning of the planning horizon in advance of receiving and processing the returns. Let $x_p$ denote the quantity of product type $p$ procured. Also, let $c^n_{pt}$ denote the unit procurement cost for product type $p$ in period $t$.

- There is a finite available procurement capacity, $K$, that is shared among multiple product types. Let $\alpha_p$ denote the procurement capacity required to procure a unit of product type $p$.

- Inventory holding cost is incurred for items in inventory at the end of period $t$. Let $h_{pt}$ denote the inventory holding cost per unit of product type $p$.

The sequence of events are as follows:

1. Manufactured items for product $p$ are procured prior to the beginning of the planning horizon, denoted by $x_p$, for $p = 1, \ldots, P$.

2. Returns for product type $p$ in period $t$ are received for $p = 1, \ldots, P$ and $t = 1, \ldots, T$.

3. Demand for product type $p$ in period $t$ is satisfied for $p = 1, \ldots, P$ and $t = 1, \ldots, T$.

4. Excess stock for product type $p$ at the end of period $t$ is carried to the next period and inventory holding cost incurred for $p = 1, \ldots, P$ and $t = 1, \ldots, T$.

5. Excess demand is backordered or shortage cost is incurred for product type $p$ in period $t$, for $p = 1, \ldots, P$ and $t = 1, \ldots, T$.

The objective is to minimize the sum of expected procurement, return processing, backordering, and inventory holding costs.

### 5.3 PPP1: Single Period and Single Product

We first consider the case with a single product type in a single-period setting. We drop the indices $t$ and $p$. Return quantity for the product has a known distribution function. Let $f(r)$ denote the probability distribution function, and $F(r)$ the cumulative distribution function, of returns for the product. In this setting, if the realization of the return quantity, $r$, is less than the net demand, $(d - x)$, we have a shortage of $((d - x) - r)$ units. Similarly, if the realization of the return quantity, $r$, is more than the net demand net, $(d - x)$, we have to salvage $(r - (d - x))$ units. Hence, the expected cost as a function

of the procurement quantity is given by:

$$E[\mathcal{C}(x)] = \begin{cases} c^n x + c^r E[r] + \pi \int_0^{d-x} (d-x-r) f(r) dr + h \int_{d-x}^{\infty} (r-d+x) f(r) dr & x < d \\ c^n x + c^r E[r] + h \int_0^{\infty} (r-d+x) f(r) dr & x \geq d \end{cases} \tag{5–1}$$

Rearranging the terms, the total expected cost can be written as follows:

$$E[\mathcal{C}(x)] = \begin{cases} (c^n + h)x + (c^r + h)E[r] - hd + (\pi + h) \int_0^{d-x} (d-x-r) f(r) dr & x < d \\ (c^n + h)x + (c^r + h)E[r] - hd & x \geq d \end{cases} \tag{5–2}$$

Then, we can formulate the PPP1 as follows:

$$\min \quad E[\mathcal{C}(x)] \tag{5–3}$$

subject to

$$x \geq d. \tag{5–4}$$

First order condition for (5–2) is given by:

$$\frac{dE[\mathcal{C}(x)]}{dx} = E'[\mathcal{C}(x)] = \begin{cases} c^n + h - (\pi + h) \int_0^{d-x} f(r) dr & x < d \\ c^n + h & x \geq d \end{cases} \tag{5–5}$$

Second order condition for (5–2) is given by:

$$\frac{d^2 E[\mathcal{C}(x)]}{dx^2} = E''[\mathcal{C}(x)] = \begin{cases} (\pi + h) f(d-x) & x < d \\ & x \geq 0 \end{cases} \tag{5–6}$$

We have $E''[\mathcal{C}(x)] \geq 0$. Hence, (5–2) is convex, and the first order condition can be used to characterize the optimal procurement quantity, $x$. Note that when $x \geq d$ first derivative is positive, so the first order conditions can only be satisfied when $x < d$. Note also that if $c^n \geq (\pi + h)F(d) - h$ then the first derivative is nonnegative so the minimum of (5–2) occurs when $x$ is at its minimum, $x^* = 0$. Otherwise, if $c^n < (\pi + h)F(d) - h$, then the

optimal procurement quantity can be specified using:

$$F(d - x^*) = \frac{c^n + h}{\pi + h}$$
$$x^* = d - F^{-1}\left(\frac{c^n + h}{\pi + h}\right) \tag{5–7}$$

Note that we have $c^n < (\pi + h)F(d) - h$. Hence, $x^*$ always satisfies (5–4).

In the classical newsboy (newsvendor) problem, the critical fractile plays an important role to characterize the optimal order quantity. Recall that the critical fractile is the ratio of the cost of underage to the sum of the costs of underage and overage. In PPP1, the demand is deterministic, but returns are random. We can interpret (5–7) as follows: The cost of underage in our problem is $(c^n + h)$. Also, the cost of overage is $(\pi - c^n)$. That is, if we *underestimate* the quantity of returns by one unit, and, as a result, we procure one more manufactured item, then the cost of this extra unit is its procurement cost plus the inventory holding cost, i.e., $(c^n + h)$. Similarly, if we *overestimate* the return by one unit, and, as a result procure one unit manufactured item less, then the cost of not procuring this unit is the penalty cost of unsatisfied demand minus the procurement cost, i.e., $(\pi - c^n)$. If we use the critical fractile formula using these underage and overage costs, we obtain (5–7). Therefore, if we know the cost of underage, $c_u$, and cost of overage, $c_o$, then we can determine the optimal procurement quantity using:

$$x^* = d - F\left(\frac{c_u}{c_u + c_o}\right)$$

Next we consider PPP1 when there is a fixed cost associated with procurement in addition to the linear unit cost, $c^n$. Let, $C^n$ is the fixed cost of procurement. Also, let $z$ be a decision variable that takes the value of one when the procurement quantity is positive, i.e., $x > 0$, and zero otherwise. The total expected cost, (5–2), can be rewritten as follows:

$$E[\mathcal{C}(x, z)] = C^n z +$$

$$\begin{cases} (c^n+h)x+(c^r+h)E[r]-hd+(\pi+h)\int_0^{d-x}(d-x-r)f(r)dr & x<d \\ (c^n+h)x+(c^r+h)E[r]-hd & x\geq d \end{cases} \tag{5–8}$$

Let $M$ denote a sufficiently large number. Then, the problem formulation (5–3)-(5–4) becomes:

$$\min \quad E[\mathcal{C}(x,z)] \tag{5–9}$$

subject to

$$x \leq Mz \tag{5–10}$$

$$z \in \{0,1\} \tag{5–11}$$

$$x \geq 0 \tag{5–12}$$

The optimal solution to (5–9)-(5–12) can be characterized as follows. Let $x'$ be the procurement quantity identified by solving (5–3)-(5–4). If $x' > 0$, then the total expected cost is $E[\mathcal{C}(x',1)]$. Moreover, $E[\mathcal{C}(0,0)]$ denotes the total expected cost when the procurement quantity is zero. We can compare these two values, and if the former is smaller, then the optimal procurement quantity is $x'$ (i.e., $x^*=x'$); otherwise the optimal procurement quantity is 0 (i.e., $x^*=0$).

Recall that when $c^n<(\pi+h)F(d)-h$, $x'=d-F^{-1}(\frac{c^n+h}{\pi+h})$, then

$$\begin{aligned} E[\mathcal{C}(x',1)] =& C^n+(c^n+h)x'+(c^r+h)E[r]-hd+(\pi+h)\int_0^{d-x'}(d-x'-r)f(r)dr \\ =& C^n+c^nd-(c^n+h)F^{-1}(\frac{c^n+h}{\pi+h}))+(c^r+h)E[r]+ \\ & (\pi+h)\int_0^{F^{-1}(\frac{c^n+h}{\pi+h})}(F^{-1}(\frac{c^n+h}{\pi+h})-r)f(r)dr \end{aligned} \tag{5–13}$$

We also have

$$E[\mathcal{C}(0,0)]=(c^r+h)E[r]-hd+(\pi+h)\int_0^d(d-r)f(r)dr \tag{5–14}$$

If $E[\mathcal{C}(0,0)]$ is less than $E[\mathcal{C}(x',1)]$, then $x^*=0, z^*=0$, otherwise, $x^*=d-F^{-1}(\frac{c^n+h}{\pi+h}), z^*=1$. More specifically, we can find a condition on the fixed procurement cost such that the optimal procurement quantity is either $x^*=d-F^{-1}(\frac{c^n+h}{\pi+h})$ or zero. Let $\delta$ denote the break-even value of the fixed cost, which can be identified as follows:

$$\begin{aligned}\delta =&(c^r+h)E[r]-hd+(\pi+h)\int_0^d(d-r)f(r)dr-(c^nd-(c^n+h)F^{-1}(\frac{c^n+h}{\pi+h}))+\\ &(c^r+h)E[r]+(\pi+h)\int_0^{F^{-1}(\frac{c^n+h}{\pi+h})}(F^{-1}(\frac{c^n+h}{\pi+h})-r)f(r)dr)\\ =&(\pi+h)\int_0^d(d-r)f(r)dr-hd\\ &-\left(c^nd-(c^n+h)F^{-1}(\frac{c^n+h}{\pi+h}))+(\pi+h)\int_0^{F^{-1}(\frac{c^n+h}{\pi+h})}(F^{-1}(\frac{c^n+h}{\pi+h})-r)f(r)dr\right)\end{aligned}$$

If $\delta > C^n$, then it is optimal to procure $x^*=x'=d-F^{-1}(\frac{c^n+h}{\pi+h})$. Otherwise, if $\delta \leq C^n$, then it is optimal not to procure, i.e., $x^*=0$.

### 5.4 PPP2: Single Period and Multiple Products

We now consider the case with multiple-product types in a single-period setting. We drop the index $t$. Return quantity for product type $p$ has a known distribution function. Let $f_p(r_p)$ denote the probability distribution function, and $F_p(r_p)$ the cumulative distribution function, of returns for product type $p$. In this setting, retailer chooses the procurement quantities for each of the product types considering the finite procurement capacity shared among multiple product types. For ease of exposition, let $\overrightarrow{x}=[x_1,\ldots,x_P]$. The total expected cost as a function of the procurement quantities for the products is given by:

$$\begin{aligned}\min\ E[\mathcal{C}(\overrightarrow{x})]=\sum_{p=1}^{P}\Bigg\{&\int_0^{d_p-x_p}\left(c_p^nx_p+c_p^rr_p+\pi_p((d_p-x_p)-r_p)\right)f_p(r_p)dr_p\\ &+\int_{d_p-x_p}^{\infty}\left(c_p^nx_p+c_p^rr_p+h_p(r_p-(d_p-x_p))\right)f_p(r_p)dr_p\Bigg\}\end{aligned}\tag{5–15}$$

subject to

$$x_p \geq 0 \qquad\qquad p = 1, \ldots, P; \tag{5–16}$$

$$\sum_{p=1}^{P} \alpha_p x_p \leq K \tag{5–17}$$

Objective function (5–15) is separable for each product $p$, but constraint (5–17), shared procurement capacity constraint, links the products. We consider relaxing constraint (5–17), using a Lagrangian multiplier, $\theta > 0$. When (5–17) is relaxed, (5–15)-(5–17) becomes separable for each product type $p$ for $p = 1, \ldots, P$. For a given Lagrangian multiplier $\theta$ we have the following subproblem for each product type $p$:

$$\begin{aligned} \min\ E[\mathcal{C}(x_p|\theta)] = \Bigg\{ &\int_0^{d_p - x_p} \big((c_p^n + \alpha_p\theta)x_p + c_p^r r_p + \pi_p((d_p - x_p) - r_p)\big)\, f_p(r_p) dr_p \\ &+ \int_{d_p - x_p}^{\infty} \big((c_p^n + \alpha_p\theta)x_p + c_p^r r_p + h_p(r_p - (d_p - x_p))\big)\, f_p(r_p) dr_p \Bigg\} \end{aligned} \tag{5–18}$$

subject to

$$x_p \geq 0 \tag{5–19}$$

After some algebraic manipulation, (5–18) can be rewritten as follows:

$$\begin{aligned} E[\mathcal{C}(x_p|\theta)] =& (c_p^n + \alpha_p\theta + h_p)x_p + (c_p^r + h_p)E[r_p] - h_p d_p + \\ & (\pi_p + h_p) \int_0^{d_p - x_p} (d_p - x_p - r_p) f_p(r_p) dr_p \end{aligned} \tag{5–20}$$

First order condition for (5–20) is:

$$E'[\mathcal{C}(x_p|\theta)] = (c_p^n + \alpha_p\theta + h_p) - (\pi_p + h_p) \int_0^{d_p - x_p} f_p(r_p) dr_p \tag{5–21}$$

Second order condition for (5–20) is:

$$E''[\mathcal{C}(x_p|\theta)] = \begin{cases} (\pi_p + h_p) f_p(d_p - x_p) & x_p < d_p \\ & x_p \geq 0 \end{cases} \tag{5–22}$$

Since (5–22) is non-negative, (5–20) is convex. Hence, the optimal procurement quantity for a given Lagrangian multiplier $\theta$ can be identified using the first order conditions for the product types. For a given Lagrangian multiplier, $\theta$, the optimal procurement quantity $x_p^*$ is given by:

$$x_p^* = \begin{cases} d_p - F_p\left(\frac{c_p^n+\alpha_p\theta+h_p}{\pi_p+h_p}\right) & \theta < \frac{(\pi_p+h_p)F_p(d_p)-h_p-c_p^n}{\alpha_p} \\ \\ 0 & \theta \geq \frac{(\pi_p+h_p)F_p(d_p)-h_p-c_p^n}{\alpha_p} \end{cases}$$

Note that $x_p$ is decreasing in $\theta$. Also, if $\theta \geq \frac{(\pi_p+h_p)F_p(d_p)-h_p-c_p^n}{\alpha_p}$, then $E'[\mathcal{C}(x_p|\theta)]$ non-negative for any value of $x_p$. Therefore, if $\theta \geq \frac{(\pi_p+h_p)F_p(d_p)-h_p-c_p^n}{\alpha_p}$, then the optimal value of $x_p$ occurs at its minimum, i.e., 0.

**Solution Strategy.**. The optimal value of $\theta$ can be identified by the bisection method. We first find the optimal procurement quantities when the value of the Lagrangian multiplier is set to zero, i.e., $\theta = 0$. Note that this solution is, in fact, the solution with no capacity constraint. If this solution satisfies (5–17), then it is optimal to (5–15)-(5–17). However, if this solution does not satisfy (5–17), then we can search for the optimal value of $\theta$ that yields the best solution that satisfies (5–17). First, we identify the largest possible value that $\theta$ can take, which is given by the maximum $((\pi_p + h_p)F_p(d_p) - h_p - c_p^n)/\alpha_p$ value for $p = 1, \ldots, P$.

We develop the following algorithm which seeks for the optimal solution to (5–15)-(5–17) by performing a search over the Lagrangian multiplier $\theta$. Let $\epsilon > 0$ be an error bound on the capacity constraint such that if a solution that has a capacity usage in $[K - \epsilon, K]$ is identified, the algorithm stops.

**Algorithm** `APPP2`

*Step 0. Initialization:* Initialize $\epsilon$. Set $k \leftarrow 0$; $\theta^k \leftarrow 0$; $\theta_{\min} \leftarrow 0$; $\theta_{\max} \leftarrow \max_{p=1,\ldots,P}\left\{\frac{(\pi_p+h_p)F_p(d_p)-h_p-c_p^n}{\alpha_p}\right\}$;

*Step 1. Find an Initial Solution:* For $p = 1, \ldots, P$ find $x_p^k$:

$$x_p^k = \begin{cases} d_p - F_p(\frac{c_p^n + h_p}{\pi_p + h_p}) & c_p^n < (\pi_p + h_p)F_p(d_p) - h_p \\ 0 & c_p^n \geq (\pi_p + h_p)F_p(d_p) - h_p \end{cases}$$

If $\sum_{p=1}^{P} \alpha_p x_p^k \leq K$, then $x_p^* = x_p^k$ for $p = 1, \ldots, P$, and $\theta^* = 0$, go to Step 3. Otherwise $k \leftarrow k + 1$ and go to Step 2.

*Step 2. Find a Solution:* Set $\theta^k = \frac{\theta_{\min} + \theta_{\max}}{2}$.

$$x_p^k = \begin{cases} d_p - F_p(\frac{c_p^n + \alpha_p \theta^k + h_p}{\pi_p + h_p}) & \theta < \frac{(\pi_p + h_p)F_p(d_p) - h_p - c_p^n}{\alpha_p} \\ 0 & \theta \geq \frac{(\pi_p + h_p)F_p(d_p) - h_p - c_p^n}{\alpha_p} \end{cases}$$

If $\sum_{p=1}^{P} \alpha_p x_p^k \leq K$, and $\sum_{p=1}^{P} \alpha_p x_p^k \geq K - \epsilon$, then stop and go Step 3. Otherwise, if $\sum_{p=1}^{P} \alpha_p x_p^k > K$, then set $\theta_{\min} \leftarrow \theta^k$, else set $\theta_{\max} \leftarrow \theta^k$. Set $k \leftarrow k + 1$ and go to Step 2.

*Step 3. Report Solution:* Report $x_p^* = x_p^k$ for $p = 1, \ldots, P$, and $\theta^* = \theta^k$.

In Algorithm `APPP2`, we first initialize the iteration counter, $\epsilon$, and minimum and maximum values that $\theta$ can take. We then find an initial solution considering no capacity constraint, i.e., when $\theta = 0$, in Step 1. If this solution is feasible, then it is also optimal to (5–15)-(5–17). Otherwise, we search for the optimal $\theta$ value iteratively by halving the interval of $\theta$ at each iteration. When we find a solution that satisfies (5–17) with an error bound of $\epsilon$, we stop the search and report the current solution as an optimal solution.

We also analyze the problem if the procurement cost has a fixed cost component in addition to the linear component. Assume that there is a fixed cost, $C_p^n$ for product $p$ for $p = 1, \ldots, P$. Let $z_p$ be a decision variable for product $p$ that takes the value of one when the procurement quantity is positive, i.e., $x_p > 0$, and zero otherwise. (5–15)-(5–17) can be rewritten when there is a fixed cost as follows:

$$\min \; E[\mathcal{C}(\overrightarrow{x}, \overrightarrow{z})] = \sum_{p=1}^{P} C_p^n z_p \left\{ \int_0^{d_p - x_p} \left( c_p^n x_p + c_p^r r_p + \pi_p((d_p - x_p) - r_p) \right) f_p(r_p) dr_p \right.$$
$$\left. + \int_{d_p - x_p}^{\infty} \left( c_p^n x_p + c_p^r r_p + h_p(r_p - (d_p - x_p)) \right) f_p(r_p) dr_p \right\} \tag{5–23}$$

subject to

$$\sum_{p=1}^{P} \alpha_p x_p \leq K \tag{5–24}$$

$$x_p \leq M z_p \qquad p = 1, \ldots, P \tag{5–25}$$

$$x_p \geq 0 \qquad p = 1, \ldots, P; \tag{5–26}$$

We can still use the Algorithm `APPP2`. However, the solution of the subproblem for product $p$ may change depending on the fixed cost. More specifically, it may be less costly not to procure product type $p$ when there is a fixed cost. Recall that in Section (5.3), we identify two solutions, i.e., when $z = 1$ and $z = 0$. Then, we evaluate the objective function values of these solutions. The optimal solution is the solution that yields the lower objective function value. Subproblem for product type $p$ can be solved utilizing a similar approach. We next present the subproblem formulation for product $p$ when the capacity constraint (5–24) is relaxed:

$$\begin{aligned} \min \; E[\mathcal{C}(x_p, z_p|\theta)] = & C_p^n z_p + (c_p^n + \alpha_p \theta + h_p) x_p + (c_p^r + h_p) E[r_p] - h_p d_p + \\ & (\pi_p + h_p) \int_0^{d_p - x_p} (d_p - x_p - r_p) f_p(r_p) dr_p \end{aligned} \tag{5–27}$$

subject to

$$x_p \leq M z_p \tag{5–28}$$

$$x_p \geq 0 \tag{5–29}$$

For any given $\theta$, $z_p$ can be either zero or one. Hence, the solutions for both cases can be evaluated to identify the optimal solution. For a given $\theta$, when $z_p = 1$ the procurement quantity, $x'_p$ is given by:

$$x'_p = \begin{cases} d_p - F_p\left(\frac{c_p^n + \alpha_p \theta + h_p}{\pi_p + h_p}\right) & \theta < \frac{(\pi_p + h_p) F_p(d_p) - h_p - c_p^n}{\alpha_p} \\ \\ 0 & \theta \geq \frac{(\pi_p + h_p) F_p(d_p) - h_p - c_p^n}{\alpha_p} \end{cases}$$

We then find the total expected cost for product $p$, $E[\mathcal{C}(x'_p, 1)]$ when $x_p = x'_p$, $z_p = 1$. We also find the total expected cost for product $p$, $E[\mathcal{C}(0,0)]$ when $x_p = 0, z_p = 0$. If $E[\mathcal{C}(x'_p, 1)] \leq E[\mathcal{C}(0,0)]$ then $x^*_p = x'_p$, and $z^*_p = 1$. Otherwise, $x^*_p = 0$, and $z^*_p = 0$.

The second and third steps of Algorithm `APPP2` can be modified to solve the subproblems. That is, for a given Lagrangian multiplier, two solutions for each product type $p$ have to be compared, i.e., $(x'_p, 1)$ and (0,0), and the one that yields the lower objective function value is the optimal solution for the product. Hence, we can use Algorithm `APPP2`, when there is a fixed cost of procurement.

## 5.5 PPP3: Multiple Period and Single Product

We now consider the case with a single product type in a multiple-period setting. Retailer has to decide procurement quantity of the product at the beginning of the planning horizon. Therefore, the demand throughout the planning horizon can be satisfied by products procured at the beginning of the planning horizon, and the processed returns throughout the planning horizon. We drop the index $p$. Let $D_t$ denote cumulative demand through period $t$ for $t = 1, \ldots, T$, i.e., $D_t = \sum_{j=1}^{t} d_j$. Also, let $r_t(w)$ and $R_t(w)$ denote the return in period $t$ and cumulative return through period $t$, for $t = 1, \ldots, T$. Let $f_t(v)$ and $F_t(v)$ denote the pdf and c.d.f. of $R_t$, respectively Finally, let $I_t(w)$ denote the net ending inventory at period $t$ for $t = 1, \ldots, T$. The total expected cost as a function of the procurement quantity for the product is given by:

$$\min \; E[\mathcal{C}(x)] = c^n x + \sum_{t=1}^{T} \left( c^r_t E[r_t] + h_t E[\max\{I_t(w), 0\}] + \pi_t E[\max\{-I_t(w), 0\}] \right) \quad (5\text{–}30)$$

subject to

$$x + r_1(w) - d_1 = I_1(w) \quad (5\text{–}31)$$

$$I_{t-1}(w) + r_t(w) - d_t = I_t(w) \qquad t = 2, \ldots, T \quad (5\text{–}32)$$

$$x \geq 0 \quad (5\text{–}33)$$

Objective function (5–30) minimizes the total expected cost. The first term is the total procurement cost, second term is the expected processing cost of returns, and the third and fourth terms are total expected inventory holding and backordering costs, respectively. Constraint (5–31) ensures the inventory balance constraint for the first period. Constraint set (5–32) ensures the inventory balance for periods 2 through end of the planning horizon. Constraint (5–33) ensures the nonnegativity of procurement quantity.

By using (5–31) and (5–32), we can find the net ending inventory in any period $t$ in terms of procurement quantity, $x$, cumulative demand, $D_t$, and cumulative return, $R_t(w)$ as follows:

$$I_t(w) = x + R_t(w) - D_t \tag{5–34}$$

Note that for any period $t$ if $x < D_t$ then the ending net inventory at the end of period $t$, $I_t(w)$, can be positive or negative, whereas if $x \geq D_t$ then the ending net inventory at the end of period $t$, $I_t(w)$, can only be positive. Therefore, the total expected inventory holding and backordering costs incurred at the end of period $t$ can be written as follows:

$$\mathcal{L}_t(x) = \begin{cases} h_t \int_{D_t - x}^{\infty} (x + v - D_t)\, f_t(v) dv + \pi_t \int_0^{D_t - x} (D_t - x - v)\, f_t(v) dv & x < D_t \\ h_t \int_0^{\infty} (x + v - D_t)\, f_t(v) dv & x \geq D_t \end{cases} \tag{5–35}$$

More specifically, we can rewrite the $\mathcal{L}_t(x)$ as follows:

$$\mathcal{L}_t(x) = h_t\,(x + E[R_t(w)] - D_t) + \begin{cases} (h_t + \pi_t) \int_0^{D_t - x} (D_t - x - v)\, f_t(v) dv & x < D_t \\ 0 & x \geq D_t \end{cases} \tag{5–36}$$

The first order condition of $\mathcal{L}_t(x)$ is given by:

$$\frac{d\mathcal{L}_t(x)}{dx} = h_t - \begin{cases} (h_t + \pi_t)\, F_t(D_t - x) & x < D_t \\ 0 & x \geq D_t \end{cases} \tag{5–37}$$

Similarly, the second order condition of $\mathcal{L}_t(x)$ is given by:

$$\frac{d^2\mathcal{L}_t(x)}{dx^2} = \begin{cases} (h_t + \pi_t)\, f_t(D_t - x) & x < D_t \\ 0 & x \geq D_t \end{cases} \tag{5–38}$$

Since (5–38) is non-negative, $\mathcal{L}_t(x)$ is convex.

We now reformulate the problem without inventory variables by using (5–34) and (5–36) as follows:

$$\min\ E[\mathcal{C}(x)] = c^n x + \sum_{t=1}^{T} \left(c_t^r E[r_t] + L_t(x)\right) \tag{5–39}$$

subject to

$$x \geq 0 \tag{5–40}$$

Recall that the inventory holding and backordering costs at each period depends on the procurement quantity, $x$. By using different intervals of $x$, the objective function, (5–39), can be rewritten as follows:

$$E[\mathcal{C}(x)] = \begin{array}{l} c^n x + \sum_{t=1}^{T} \left(c_t^r E[r_t] + h_t \left(x + E[R_t(w)] - D_t\right)\right) + \\ \begin{cases} \sum_{j=1}^{T} (h_j + \pi_j) \int_0^{D_j - x} (D_j - x - v)\, f_j(v) dv & 0 \leq x < D_1 \\ \sum_{j=2}^{T} (h_j + \pi_j) \int_0^{D_j - x} (D_j - x - v)\, f_j(v) dv & D_1 \leq x < D_2 \\ \vdots & \vdots \\ \sum_{j=t}^{T} (h_j + \pi_j) \int_0^{D_j - x} (D_j - x - v)\, f_j(v) dv & D_{t-1} \leq x < D_t \\ \vdots & \vdots \\ \sum_{j=T}^{T} (h_j + \pi_j) \int_0^{D_j - x} (D_j - x - v)\, f_j(v) dv & D_{T-1} \leq x < D_T \\ 0 & x \geq D_T \end{cases} \end{array}$$

The objective function $E[\mathcal{C}(x)]$ is continuous and convex. Hence, the optimal procurement quantity can be characterized using the first order conditions.

The first order condition of $E[\mathcal{C}(x)]$ is as follows:

$$E'[\mathcal{C}(x)] = \quad c^n + \sum_{t=1}^{T} h_t - \begin{cases} \sum_{j=1}^{T} (h_j + \pi_j) F_j(D_j - x) & 0 \leq x < D_1 \\ \sum_{j=2}^{T} (h_j + \pi_j) F_j(D_j - x) & D_1 \leq x < D_2 \\ \vdots & \vdots \\ \sum_{j=t}^{T} (h_j + \pi_j) F_j(D_j - x) & D_{t-1} \leq x < D_t \\ \vdots & \vdots \\ \sum_{j=T}^{T} (h_j + \pi_j) F_j(D_j - x) & D_{T-1} \leq x < D_T \\ 0 & x \geq D_T \end{cases} \tag{5–41}$$

Note that the derivative, $E'[\mathcal{C}(x)]$, is nondecreasing in $x$ for two reasons, (i) $F_j(D_j - x)$ is nonincreasing in $x$, and (ii) $t$ increases in the summation $\sum_{j=t}^{T}(h_j + \pi_j)F_j(D_j - x)$. We formally present this property in the following Proposition.

**Proposition 5.1.** *The first derivative of objective function, $E'[\mathcal{C}(x)]$, is continuous and nondecreasing in procurement quantity, $x$ for $x \geq 0$.*

*Proof.* Note that $x$ can take values from 0 through $\infty$. As can be seen in (5–41), $x$ can be in any of the $T + 1$ intervals. Let $D_{T+1}$ be defined as $D_{T+1} = \infty$. Interval for period $t$ can be defined as $[D_{t-1}, D_t)$ for $t = 1, \ldots, T + 1$.It is enough to show that the derivative of $E[\mathcal{C}(x)]$ is (i) nondecreasing in any interval $t$, i.e., $[D_{t-1}, D_t)$, and (ii) nondecreasing through increasing intervals, i.e., $t'$ and $t''$ with $1 \leq t' < t'' \leq T + 1$.

(i) Let $x_1$ and $x_2$ be two procurement quantities in any interval $t$ with $D_{t-1} \leq x_1 < x_2 < D_t$ for $t = 1, \ldots, T + 1$. The first derivative $E'[\mathcal{C}(x)]$ when $x = x_1$ and $x = x_2$ are:

$$E'[\mathcal{C}(x)]|_{x_1} = c^n + \sum_{t=1}^{T} h_t - \sum_{j=t}^{T} (h_j + \pi_j) F_j(D_j - x_1)$$

$$E'[\mathcal{C}(x)]|_{x_2} = c^n + \sum_{t=1}^{T} h_t - \sum_{j=t}^{T} (h_j + \pi_j)\, F_j(D_j - x_2)$$

The difference between the first derivative values when procurement quantities are $x_2$ and $x_1$ is:

$$E'[\mathcal{C}(x)]|_{x=x_2} - E'[\mathcal{C}(x)]|_{x=x_1} = \sum_{j=t}^{T} (h_j + \pi_j)\,(F_j(D_j - x_1) - F_j(D_j - x_2)) \geq 0$$

Since $F_j(D_j - x_1) \geq F_j(D_j - x_2)$ for $j = t, \ldots, T$.

(ii) Let $x_1$ and $x_2$ be two procurement quantities with $x_1 \in [D_{t'-1}, D_{t'})$ and $x_2 \in [D_{t''-1}, D_{t''})$, where $1 \leq t' < t'' \leq T + 1$. Hence, $x_1 < x_2$. Similar to Part (i) we can compare the first derivative $E'[\mathcal{C}(x)]$ when $x = x_1$ and $x = x_2$.

$$E'[\mathcal{C}(x)]|_{x_1} = c^n + \sum_{t=1}^{T} h_t - \sum_{j=t'}^{T} (h_j + \pi_j)\, F_j(D_j - x_1)$$

$$E'[\mathcal{C}(x)]|_{x_2} = c^n + \sum_{t=1}^{T} h_t - \sum_{j=t''}^{T} (h_j + \pi_j)\, F_j(D_j - x_2)$$

The difference between the first derivative values when procurement quantities are $x_2$ and $x_1$ is:

$$\begin{aligned} E'[\mathcal{C}(x)]|_{x=x_2} - E'[\mathcal{C}(x)]|_{x=x_1} = & \sum_{j=t'}^{t''-1} (h_j + \pi_j)\, F_j(D_j - x_1) + \\ & \sum_{j=t''}^{T} (h_j + \pi_j)\,(F_j(D_j - x_1) - F_j(D_j - x_2)) \geq 0 \end{aligned}$$

Since $F_j(D_j - x_1) \geq F_j(D_j - x_2)$ for $j = t'', \ldots, T$, and $\sum_{j=t'}^{t''-1} F(D_j - x_1) \geq 0$. Moreover, $F_j(D_j - x)$ is continuous for $j = 1, \ldots, T$, and the derivative is continuous at the end points of the intervals. Hence, $E'[\mathcal{C}(x)]$ is continuous and nondecreasing in $x$ for $x \geq 0$. $\square$

To search for the optimal procurement quantity, $x^*$, we can use Proposition 5.1. In the following lemma we state whether the optimal procurement quantity is zero or positive.

**Lemma 1.** *If the first derivative of $E[\mathcal{C}(x)]$ is nonnegative when $x = 0$, i.e., $E'[\mathcal{C}(0)] \geq 0$, then the optimal procurement quantity is zero, $x^* = 0$, otherwise if $E'[\mathcal{C}(0)] < 0$, then the optimal procurement quantity is positive, i.e., $x^* > 0$.*

*Proof.* (i) If $E'[\mathcal{C}(0)] \geq 0$, then $E'[\mathcal{C}(x)] \geq 0$ for any $x > 0$, so that $x^* = 0$. (ii) If $E'[\mathcal{C}(0)] < 0$. Since $E'[\mathcal{C}(x)]|_{x=D_T} = c^n + \sum_{t=1}^{T}(h_t) > 0$, there should be some $0 < x < D_T$ satisfies $E'[\mathcal{C}(x)] = 0$, by intermediate value theorem. Hence $x^* > 0$. □

**Lemma 2.** *If the first derivative of $E[\mathcal{C}(x)]$ when $x = D_t$, i.e., $E'[\mathcal{C}(D_t)] = c^n + \sum_{j=1}^{T} h_t - \sum_{j=t+1}^{T}(F_j(D_j - D_t))$, is*

*(i)* $< 0$*, then there exists an optimal solution with $x^* > D_t$;*

*(ii)* $= 0$*, then optimal procurement quantity is $x^* = D_t$;*

*(iii)* $> 0$*, then there exists an optimal solution with $x^* < D_t$.*

*Proof.* The proof of each statement is as follows:

(i) Since $E'[\mathcal{C}(D_t)] < 0$, $E'[\mathcal{C}(D_T)] > 0$ and $E'[\mathcal{C}(x)]$ continuous and nondecreasing, there exists some $D_t < x < D_T$ that satisfies $E'[\mathcal{C}(x)] = 0$, by intermadiate value theorem.

(ii) Since, $E[\mathcal{C}(x)]$ is convex, the optimal solution is the one that satisfies the first order conditions.

(iii) Since $E'[\mathcal{C}(x)]$ is continuous and nondecreasing, there cannot be any $x \geq D_t$ that satisfy $E'[\mathcal{C}(x)] = 0$ Hence, the optimal solution should be less than $D_t$. Moreover, if $E'[\mathcal{C}(0)] > 0$, then $x^* = 0$, otherwise, $0 < x^* < D_t$.

□

As it is stated in Proposition 5.1 and Lemmas 1 and 2 we can search for the optimal procurement quantity by simply checking the derivative signs at the initial and end points of each interval, $[D_{t-1}, D_t)$ for $t = 1, \ldots, T$, i.e., when $x = D_{t-1}$ and $x = D_t$. If $E'[\mathcal{C}(0)] < 0$, then we can find an interval $t$ where first derivative at the end points have different signs, then the optimal solution should lie in that interval by Lemma 2. Otherwise the optimal procurement quantity is $x^* = 0$ by Lemma 1. To this end, we develop an iterative algorithm, which first tries to find the interval $t$, for which the

procurement quantity with different signs of first derivatives at its end points, $D_{t-1}$ and $D_t$. After the interval $t$ is identified, the optimal solution can be determined by using line search. We present the algorithm next.

**Algorithm** `APPP3`

*Step 0. Initialization:* Initialize $k \leftarrow 1$; $x_{\min} \leftarrow 0$; $x_{\max} \leftarrow D_1$

*Step 1. Calculate first derivative values at $x_{\min}$ and $x_{\max}$:*

$$E'[\mathcal{C}(x)]|_{x_{\min}} = c^n + \sum_{t=1}^{T} h_t - \sum_{j=k}^{T} (h_j + \pi_j) F_j(D_j - x_{min})$$

$$E'[\mathcal{C}(x)]|_{x_{\max}} = c^n + \sum_{t=1}^{T} h_t - \sum_{j=k}^{T} (h_j + \pi_j) F_j(D_j - x_{max})$$

*Step 2. Check conditions:*

- If $E'[\mathcal{C}(x)]|_{x_{\min}} < 0$ and
  - if $E'[\mathcal{C}(x)]|_{x_{\max}} < 0$, then $k \leftarrow k+1$; $x_{\min} \leftarrow D_{k-1}$; $x_{\max} \leftarrow D_k$, and go to Step 1.
  - if $E'[\mathcal{C}(x)]|_{x_{\max}} = 0$, then $x^* = x_{\max}$, and go to Step 4.
  - if $E'[\mathcal{C}(x)]|_{x_{\max}} > 0$, then $k^* \leftarrow k$, and go to Step 3.
- If $E'[\mathcal{C}(x)]|_{x_{\min}} = 0$, then $x^* = x_{\min}$, and go to Step 4.
- If $E'[\mathcal{C}(x)]|_{x_{\min}} > 0$, then $x^* = x_{\min}$, and go to Step 4.

*Step 3. Find the optimal solution via line search:* Note that$x^* \in (D_{k^*}, D_{k^*+1})$. Solve the following equation, and identify $x^*$.
$c^n + \sum_{t=1}^{T} h_t - \sum_{j=k^*}^{T} (h_j + \pi_j) F_j(D_j - x) = 0$. Go to Step 4.

*Step 4. Report Solution:* Report $x^*$.

We note that when $k = T$, the first derivative at $x = D_T$ is always positive, hence, the solution cannot be unbounded. Algorithm `APPP3` identifies a solution after a finite number of iterations, because same interval will not be revisited. The performance of the algorithm also depends on how efficiently the line search identifies the optimal solution at Step 3.

Finally, we discuss how the optimal procurement quantity can be obtained when there is a fixed cost component for procurement in addition to the linear component. Assume that there is a fixed cost, $C^n$ when the procurement quantity is positive. Let $z$ be a decision variable, that takes the value of one when the procurement quantity is positive, i.e., $x > 0$, and zero otherwise. Since there is single product and a single procurement opportunity, we have two options, to procure or not to procure, i.e., $z = 1$, or $z = 0$. When $z = 1$, the optimal procurement quantity, $x^* |_{z=1}$ can be identified by Algorithm `APPP3`. When $z = 0$, the optimal procurement quantity is zero. We can evaluate the objective function values of these two solutions, and the optimal solution is the one that yields lower objective function value. We can modify Algorithm `APPP3`, so that can solve the PPP3 when there is a fixed cost of procurement. At the end of Step 3, the objective function values of the solution identified by the line search and the solution with zero order quantity are evaluated. The solution that yields the lower objective function value is the optimal solution.

## 5.6 PPP4: Multiple Period and Multiple Product

Finally, we consider the case with a multiple product types in a multiple-period setting. Retailer has to decide procurement quantities for the product types at the beginning of the planning horizon by considering the finite procurement capacity that is shared among multiple products. Therefore, the demand throughout the planning horizon can be satisfied by products procured at the beginning of the planning horizon, and the processed returns throughout the planning horizon. We define the following additional notation. Let $D_{pt}$ denote cumulative demand for product type $p$ through period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$, i.e., $D_{pt} = \sum_{j=1}^{t} d_{pj}$. Also, let $r_{pt}(w)$ denote the returns for product $p$ in period $t$ and $R_{pt}(w)$ the cumulative returns for product type $p$ through period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$. Let $f_{pt}(v)$ and $F_{pt}(v)$ denote the p.d.ff and c.d.f. of $R_{pt}$, respectively Finally, let $I_{pt}(w)$ denote the net ending inventory for product type $p$ at period $t$ for $p = 1, \ldots, P$ and $t = 1, \ldots, T$. For ease of exposition, let $\overrightarrow{x}$ denote the

vector of procurement quantities. Similarly, let $\overrightarrow{I_p} = [I_{p1}, \ldots, I_{pT}]$ and $\overrightarrow{I} = [\overrightarrow{I_1}, \ldots, \overrightarrow{I_P}]$. Let $E[\mathcal{C}(\overrightarrow{x}, \overrightarrow{I})]$ denote the total expected cost as a function of the procurement quantity, $\overrightarrow{x}$ and ending net inventory, $\overrightarrow{I}$. Then, the total expected cost as a function of the procurement quantities and ending inventory levels for the product types is given by:

$$\min E[\mathcal{C}(\overrightarrow{x}, \overrightarrow{I})] = \sum_{p=1}^{P} \left( c_p^n x_p + \sum_{t=1}^{T} \left( c_{pt}^r E[r_{pt}] + h_{pt} E[\max\{I_{pt}(w), 0\}] + \pi_{pt} E[\max\{-I_{pt}(w), 0\}] \right) \right) \quad (5\text{–}42)$$

subject to

$$x_p + r_{p1}(w) - d_{p1} = I_{p1}(w) \qquad p = 1, \ldots, P \quad (5\text{–}43)$$

$$I_{pt-1}(w) + r_{pt}(w) - d_{pt} = I_{pt}(w) \qquad p = 1, \ldots, P, \;\; t = 2, \ldots, T \quad (5\text{–}44)$$

$$\sum_{p=1}^{P} \alpha_p x_p \leq K \quad (5\text{–}45)$$

$$x_p \geq 0 \qquad p = 1, \ldots, P \quad (5\text{–}46)$$

Objective function (5–42) minimizes the sum of expected procurement, processing, inventory holding and backordering costs throughout the planning horizon. Constraint set (5–43) captures the inventory balance equations for product $p$ in period 1. Constraint set (5–32) represents the inventory balance equations for product $p$ in each period $t$ for $t = 2, \ldots, T$, which ensures that for each product $p$ the sum of quantity in inventory carried into period $t$ and the quantity of returns processed is equal to the sum of the demand in period $t$ and the quantity of inventory carried to period $t + 1$. Constraint (5–45) ensures that the total procurement capacity required for all the products does not exceed the available procurement capacity. Constraint set (5–46) ensures the nonnegativity of the decision variables.

By using (5–43) and (5–44) for each product type $p$, we can find the net ending inventory in period $t$ in terms of procurement quantity $x_p$ for the product type $p$

cumulative demand and cumulative return as follows:

$$I_{pt}(w) = x_p + R_{pt}(w) - D_{pt} \tag{5–47}$$

Note that for any product $p$ for any period $t$ if $x_p < D_{pt}$ then the ending net inventory at period $t$, $I_{pt}(w)$, can be positive or negative, whereas if $x_p \geq D_{pt}$ then $I_{pt}(w)$ can only be positive. Therefore, the total expected holding and backordering costs incurred for product $p$ at the end of the period $t$ can be written as follows:

$$\mathcal{L}_{pt}(x_p) = \begin{cases} h_{pt} \int_{D_{pt}-x_p}^{\infty} (x_p + v - D_{pt})\, f_{pt}(v)dv + \pi_{pt} \int_0^{D_{pt}-x_p} (D_{pt} - x_p - v)\, f_{pt}(v)dv & x_p < D_{pt} \\ h_{pt} \int_0^{\infty} (x_p + v - D_{pt})\, f_{pt}(v)dv & x_p \geq D_{pt} \end{cases} \tag{5–48}$$

We now reformulate the problem without inventory variables by using (5–47) and (5–48) as follows:

$$\min\ E[\mathcal{C}(\overrightarrow{x})] = \sum_{p=1}^{P} \left( c_p^n x_p + \sum_{t=1}^{T} \left( c_{pt}^r E[r_{pt}] + L_{pt}(x_p) \right) \right) \tag{5–49}$$

subject to

$$\sum_{p=1}^{P} \alpha_p x_p \leq K \tag{5–50}$$

$$x_p \geq 0 \qquad p = 1, \ldots, P \tag{5–51}$$

Objective function (5–49) is separable for each product type $p$, but constraint (5–50), capacity constraint, links multiple product types. Hence, we consider relaxing constraint (5–50). Let $\theta > 0$ denote the associated Lagrangian multiplier. When (5–50) is relaxed, (5–49)-(5–51) becomes separable for each product type. The corresponding Lagrangian

problem is as follows:

$$\max_{\theta} \min_{\overrightarrow{x}} E[\mathcal{C}(\overrightarrow{x})] = \sum_{p=1}^{P} \left( (c_p^n + \alpha_p \theta) x_p + \sum_{t=1}^{T} \left( c_{pt}^r E[r_{pt}] + \mathcal{L}_{pt}(x_p) \right) \right) - \theta K$$

subject to

$$x_p \geq 0 \qquad p = 1, \ldots, P$$

$$\theta \geq 0$$

For a given Lagrangian multiplier $\theta$ we have:

$$\min \; E[\mathcal{C}(\overrightarrow{x}|\theta)] = \sum_{p=1}^{P} \left( (c_p^n + \alpha_p \theta) x_p + \sum_{t=1}^{T} \left( c_{pt}^r E[r_{pt}] + \mathcal{L}_{pt}(x_p) \right) \right) - \theta K \tag{5–52}$$

subject to

$$x_p \geq 0 \qquad p = 1, \ldots, P \tag{5–53}$$

Note that the last term of (5–52) is constant and the problem (5–52)-(5–53) is separable for each product type $p$. Hence, for a given Lagrangian multiplier $\theta$ we have $P$ subproblems. Subproblem for product type $p$ is as follows:

$$\min \; E[\mathcal{C}(x_p|\theta)] = (c_p^n + \alpha_p \theta) x_p + \sum_{t=1}^{T} \left( c_{pt}^r E[r_{pt}] + L_{pt}(x_p) \right) \tag{5–54}$$

subject to

$$x_p \geq 0 \tag{5–55}$$

Subproblem for product type $p$ is same as PPP3 problem. Therefore, Algorithm `APPP3` can be used to solve each subproblem. We develop an algorithm similar to Algorithm `APPP2` that seeks for the optimal $\theta$. We first solve each subproblem when the Lagrangian multiplier is equal to zero, i.e., $\theta = 0$. If the solution obtained satisfies the capacity constraint, then this solution is also optimal to original problem, (5–49)-(5–51). Otherwise,

we find an upper bound value for $\theta$ as follows:

$$\theta_{\max} = \max_{p=1,\ldots,P} \left\{ \frac{\sum_{j=1}^{T}((\pi_{pj} + h_{pj})F_j(D_j) - h_{pj}) - c_p^n}{\alpha_p} \right\}$$

Note that when Lagrangian multiplier is set to $\theta_{\max}$, the procurement quantities become zero for all products, and the capacity constraint is satisfied. Hence, $\theta_{\max}$ defines an upper bound on the Lagrangian multiplier. Let $\epsilon > 0$ be an error bound on the capacity constraint, such that a solution that has a capacity usage in $[K - \epsilon, K]$ is obtained, the algorithm stops. We present the algorithm next.

**Algorithm** `APPP4`

*Step 0 Initialization:* Initialize $k \leftarrow 0$; $\theta^k \leftarrow 0$; $\theta_{\min} \leftarrow 0$; $\theta_{\max} = \max_{p=1,\ldots,P} \left\{ \frac{\sum_{j=1}^{T}((\pi_{pj}+h_{pj})F_j(D_j)-h_{pj})-c_p^n}{\alpha_p} \right\}$;

*Step 1 Find an Initial Solution:* Find $x_p^k$ by solving subproblem (5–54)-(5–55) for product $p$ by using Algorithm `ASPMS`, for $p = 1, \ldots, P$
If $\sum_{p=1}^{P} \alpha_p x_p^k \leq K$, then $x_p^* = x_p^k$ for $p = 1, \ldots, P$, and $\theta^* = 0$, go to Step 3. Otherwise $k \leftarrow k + 1$ and go to Step 2.

*Step 2 Find a Solution:* Set $\theta^k = \frac{\theta_{\min}+\theta_{\max}}{2}$. Find $x_p^k$ by solving subproblem (5–54)-(5–55) for product $p$ by using Algorithm `ASPMS`, for $p = 1, \ldots, P$
If $\sum_{p=1}^{P} \alpha_p x_p^k \leq K$, and $\sum_{p=1}^{P} \alpha_p x_p^k \geq K - \epsilon$, then stop and go Step 3. Otherwise, if $\sum_{p=1}^{P} \alpha_p x_p^k > K$, then set $\theta_{\min} \leftarrow \theta^k$, else set $\theta_{\max} \leftarrow \theta^k$. Set $k \leftarrow k + 1$ and go to Step 2.

*Step 3 Report Solution:* Report $x_p^* = x_p^k$ for $p = 1, \ldots, P$, and $\theta^* = \theta^k$.

In Algorithm `APPP4` we first initialize the iteration counter, and bounds of $\theta$. We then find an initial solution considering no capacity constraint, i.e., when $\theta = 0$ in Step 1 by using the Algorithm `APPP3`. If this solution is feasible then it is also optimal to (5–49)-(5–51). Otherwise, we search for optimal $\theta$ value iteratively, by halving the interval of $\theta$ at each iteration. When we find a solution that satisfies (5–51) with an error bound of $\epsilon$, we stop the search and report the current solution as an optimal solution.

Assume that there is a fixed cost component for procurement for each product $p$, $C_p^n$, in addition to linear cost component. We can use Algorithm `PPP4` to solve this problem.

In Steps 2 and 3 of Algorithm `APPP4`, we can use modified Algorithm `APPP3` to solve the subproblems.

## 5.7 Concluding Remarks

We considered a catalogue retailing application where customer returns are in as-good-as new condition and they need to be taken into account explicitly for the purpose of procurement planning. The problem of interest, called the PPP, was to determine the the procurement quantity prior to the beginning of a finite planning horizon so as to minimize the total costs of procurement, return processing, backordering and inventory holding. We examined four variants of the problem. In the first variant, we considered the procurement of a single product type in a single period setting. In the second variant, we considered the procurement of multiple product types that share a finite procurement capacity in a single period setting. In the third variant, we considered the procurement of a single product type in a multiple period setting. Finally, we considered the procurement of multiple product types that share a finite procurement capacity in a multiple period setting. Our analysis of the first variant of the problem formed a basis for the development of a Lagrangian relaxation based solution approach for the second variant of the problem. Similarly, the iterative solution algorithm we developed for the third variant was instrumental for the Lagrangian relaxation based solution approach for the fourth variant of the problem.

An immediate extension of our work that would be of practical relevance is to consider a generalization of the PPP where (i) demand can be modeled as a random variable to take demand uncertainty into account and/or (ii) multiple procurement opportunities are considered throughout the planning horizon.

# CHAPTER 6
# CONCLUSION

In our study, we focused on the characterization of the optimal inventory and production planning policies for multi-product CLSC systems for direct reuse and value-added recovery by considering a class of deterministic discrete time dynamic demand and return models. We also consider a procurement planning problem for direct reuse systems with deterministic discrete time dynamic demand and stochastic returns. Most of the current literature on inventory and production planning for product recovery models for uncapacitated single-product environments. Hence, our contribution to literature is related to considering multi-product environments with capacity restrictions. Moreover, we develop efficient solution approaches that are based on dynamic programming, relaxation, decomposition, and meta-heuristic approaches to address this class of models.

In the first part of our study, we focused on replenishment and disposal planning problem for multi-product direct reuse systems. The problem of interest was motivated by OEM in the power generation equipment industry that provides spare part kits for power turbine maintenance services. To ensure uninterrupted spare part kit availability, the OEM has to coordinate replenishment and disposal decisions for the kits. We determine the timing of the replenishment and disposal setups, along with the associated replenishment and disposal quantities for multiple product types throughout a finite planning horizon so as to minimize variable replenishment, disposal, and inventory holding costs as well as fixed replenishment and disposal costs. In particular, we considered two variants of the problem. In the first variant, the replenishment capacity is shared among multiple product types while the disposal capacity is product specific. In the second variant, both the replenishment and disposal capacities are shared among multiple product types. We developed efficient solution approach for both variants of the problems. Our computational results show that the proposed approaches is effective in obtaining high-quality solutions for realistically-sized instances of the RDPP with a reasonable

computational requirement. A relevant extension of our work would be considering demand backlogging Another possible extension is to develop practically implementable rolling-horizon heuristics for the RDPP.

In the second part of our study, we studied a manufacturing, remanufacturing and disposal planning problem with multiple product types for value-added recovery systems. Problem of interest is motivated by the original equipment manufacturer in the automotive industry that provides vehicle maintenance and repair services for vehicles for which replacement parts (products) are often needed. We developed two solution approaches: Lagrangian Decomposition (LD) and Lagrangian Relaxation (LR). In LD approach, we decompose the problem into subproblems that are solved by polynomial time DP approaches. In LR approach, however, we use CPLEX to solve the resulting single product problems. Our computational results showed that although the LD approach is computationally more efficient, the LR approach is effective in finding high-quality solutions. Considering a generalization of the MRDPP where (i) the manufacturing and/or remanufacturing and/or disposal setup costs depend on the specific products included in the setup and/or (ii) setup times are explicitly modeled would be practically relevant extension of our work. Moreover, MRDPP with shared remanufacturing capacity and/or linear disposal costs can be possible extensions. Our solution algorithms can still be applied to these problems with slight modifications. Also, the LR approach presented in this paper can be extended to settings where certain capacitated resources are shared by the manufacturing and remanufacturing activities.

In the last part of our study, we focused on procurement planning problem for multi-product direct reuse systems. The problem of interest was motivated by catalogue retailing applications where returns are in as-good-as new condition and can be used to satisfy demand with little processing. To this end, we considered a discrete time dynamic deterministic demand and stochastic return model for the characterization of optimal . We study the procurement planning problems in catalogue retailing applications

where returns are in as-good-as new condition and can be used to satisfy demand with little processing. We examined four variants of the problem. We first variant focused on single-period setting for a single product type (PPP1) and extend this model to include multiple product types (PPP2). Then, we generalized our work to multiple-period settings for single product type (PPP3) as well as for multiple product types (PPP4). Our analysis of the first variant of the problem formed a basis for the development of a Lagrangian relaxation based solution approach for the second variant of the problem. Similarly, the iterative solution algorithm we developed for the third variant was instrumental for the Lagrangian relaxation based solution approach for the fourth variant of the problem.

An immediate extension of our work that would be of practical relevance is to consider a generalization of the PPP where (i) demand can be modeled as a random variable to take demand uncertainty into account and/or (ii) multiple procurement opportunities are considered throughout the planning horizon.

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