# OPTIMIZATION METHODS IN FINANCIAL ENGINEERING

By

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To my parents.

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Our study developed novel approaches to solving and analyzing challenging problems of financial engineering including options pricing, market forecasting, and portfolio optimization. We also make connections of the portfolio theory with general deviation measures to classical portfolio and asset pricing theories.

We consider a problem faced by traders whose performance is evaluated using the VWAP benchmark. Efficient trading market orders include predicting future volume distributions. Several forecasting algorithms based on CVaR-regression were developed for this purpose.

Next, we consider assumption-free algorithm for pricing European Options in incomplete markets. A non-self-financing option replication strategy was modelled on a discrete grid in the space of time and the stock price. The algorithm was populated by historical sample paths adjusted to current volatility. Hedging error over the lifetime of the option was minimized subject to constraints on the hedging strategy. The output of the algorithm consists of the option price and the hedging strategy defined by the grid variables.

Another considered problem was optimization of the Omega function. Hedge funds often use the Omega function to rank portfolios. We show that maximizing Omega function of a portfolio under positively homogeneous constraints can be reduced to linear programming.

Finally, we look at the portfolio theory with general deviation measures from the perspective of the classical asset pricing theory. We derive pricing form of generalized CAPM relations and stochastic discount factors corresponding to deviation measures. We suggest methods for calibrating deviation measures using market data and discuss the possibility of restoring risk preferences from market data in the framework of the general portfolio theory.

# CHAPTER 1
# INTRODUCTION

Fast development of financial industry makes high demands of risk management techniques. Success of financial institutions operating in modern markets is largely affected by the ability to deal with multiple sources of uncertainty, formalize risk preferences, and develop appropriate optimization models. Recently, the synthesis of engineering intuition and mathematics led to the development of advanced risk management tools. The theory of risk and deviation measures has been created, with its applications to regression, portfolio optimization, and asset pricing; which encouraged the use of novel risk management methods in academia and industry and stimulated a lot or research in the area of modelling and formalizing risk preferences. Our study makes a connection between financial applications of the theory of general deviation measures and classical asset pricing theory. We also develop novel approaches to solving and analyzing challenging problems of financial engineering including options pricing, market forecasting, and portfolio optimization.

Chapter 2 considers a broker who is supposed to trade a specified number of shares over certain time interval (market order). Performance of the broker is evaluated by Volume Weighted Average Price (VWAP), which requires trading the order according to the market volume distribution during the trading period. A common approach to this task is to trade the order following the average historical volume distribution. We introduce a dynamic trading algorithm based on forecasting market volume distribution using techniques of generalized linear regression.

Chapter 3 presents an algorithm for pricing European Options in incomplete markets. The developed algorithm (a) is free from assumptions on the stock process; (b) achieves 0.5%-3% pricing error for European in- and at-the-money options on S&P500 Index; (c) closely matches the market volatility smile; (d) is able to price options using 20-50 sample paths. We use replication idea to find option price, however we allow the hedging strategy

to be non-self-financing and minimize cumulative hedging error over all sample paths. The constraints on the hedging strategy, incorporated into the optimization problem, reflect assumption-free properties of the option price, positions in the stock and in the risk-free bond. The algorithm synthesizes these properties with the stock price information contained in the historical sample paths to find the price of option from the point of view of a trader.

Chapter 4 proves two reduction theorems for the Omega function maximization problem. Omega function is a common criterion for ranking portfolios. It is equal to the ratio of expected overperformance of a portfolio with respect to a benchmark (hurdle rate) to expected underperformance of a portfolio with respect to the same benchmark. The Omega function is a non-linear function of a portfolio return; however, it is positively homogeneous with respect to instrument exposures in a portfolio. This property allows transformation of the Omega maximization problem with positively homogeneous constraints into a linear programming problem in the case when the Omega function is greater than one at optimality.

Chapter 5 looks at the portfolio theory with general deviation measures from the perspective of the classical asset pricing theory. In particular, we analyze the generalized CAPM relations, which come out as a necessary and sufficient conditions for optimality in the general portfolio theory. We derive pricing forms of the generalized CAPM relations and show how the stochastic discount factor emerges in the generalized portfolio theory. We develop methods of calibrating deviation measures from market data and discuss applicability of these methods to estimation of risk preferences of market participants.

# CHAPTER 2
# TRACKING VOLUME WEIGHTED AVERAGE PRICE

## 2.1 Introduction

Volume-weighed average price ($VWAP$) is one of the commonly used trade evaluation benchmarks in the stock market. $VWAP$ of a set of transactions is the sum of prices of these transactions weighted by their volumes. For example, in order to calculate the daily $VWAP$ of the market, one should sum up the amounts of money traded for each transaction and divide it by the total volume of stocks traded during the day. A trader's performance can then be evaluated by comparing the $VWAP$ of a trader's transactions with the market $VWAP$. Selling a block of stocks is well performed if the $VWAP$ of selling transactions is higher then the $VWAP$ of the market and vice versa.

There are several types of benchmarks similar to $VWAP$. $VWAP$, as it is defined above, is reasonable for evaluation of relatively small orders of liquid stocks. $VWAP$ *excluding own transactions* is appropriate when the total volume of transactions constitutes a significant portion of the market's daily volume. For highly volatile stocks, value-weighted average price is also used, where prices of transactions are weighted by dollar values of this transactions. $VWAP$ benchmarks are widespread mostly outside USA, for example, in Japan.

The purpose of the $VWAP$ *trading* is to obtain the volume-weighted price of transactions as close to the market $VWAP$ as possible. An investor may act differently when seeking for $VWAP$ execution of his order. He can make a contract with a broker who guarantees selling or buying orders at the daily $WVAP$. Since the broker assumes all the risk of failing to achieve the average price better than $VWAP$ and is usually risk averse, commissions are quite large.

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This chapter is based on joint work with Vladimir Bugera and Stan Uryasev.

An order may be sent to electronic systems where it is executed at the daily $VWAP$ price ($VWAP$ crosses). These orders are matched electronically before the beginning of a trading day and executed during or after trading hours. $VWAP$ crosses normally have low transaction costs; however, the price of execution is not known in advance and there may exist the possibility that the order will not be executed.

An investor with direct access to the market may trade his order directly. But since $VWAP$ evaluation motivates to distribute the order over the trading period and trade by small portions, this alternative is not preferable due to intensity of trading and the presence of transaction costs.

The most recent approach to $VWAP$ trading is participating in $VWAP$ automated trading, where a trading period is broken up into small intervals and the order is distributed as closely as possible to the market's daily volume distribution, that is traded with the minimal market impact. This strategy provides a good approximation to market's $VWAP$, although it generally fails to reach the benchmark. More intelligent systems perform careful projections of the market volume distribution and expected price movements and use this information in trading. A more detailed survey of $VWAP$ trading can be found in Madgavan (2002).

Although $VWAP$-benchmark has gained popularity, very few studies concerning $VWAP$ strategies are available. Several studies, Bertsimas and Lo (1998), Konishi and Makimoto (2001) have been done about block trading where optimal splitting of the order in order to optimize the expected execution cost is considered. In the setup of block trading, only prices are uncertain, whereas the purpose of $VWAP$ trading is to achieve a close match of the market $VWAP$, which implies dealing with stochastic volumes as well. Konishi (2002) develops a static $VWAP$ trading strategy that minimizes the expected execution error with respect to the market realization of $VWAP$. A static strategy is determined for the whole trading period and does not change as new information arrives.

It is especially suitable for trading low liquid stocks, due to statistical errors in historical data for such stocks that make them unsuitable for forecasting.

In this chapter we develop dynamic $VWAP$ strategies. We consider liquid stocks and small orders, that make negligible impact on prices and volumes of the market. The forecast of volume distribution is the target; the strategy consists in trading the order proportionally to projected market daily volume distribution. We split a trading day into small intervals and estimate the market volume consecutively for each interval using linear regression techniques.

## 2.2 Background and Preliminary Remarks

Consider the case when only one stock is available for trading. If at time $\tau$ a transaction of trading $\nu$ units of the stock at a price $p$ we denote this transaction by $\{\tau, \nu, p\}$. Let $\Omega = \{\{\tau_k, \nu_k, p_k\}, \quad k = 1, .., K\}$ be a set of all transactions in the market during a day. Then the $VWAP$ of the stock is

$$VWAP = \frac{\sum_k p_k \nu_k}{\sum_k \nu_k}. \tag{2–1}$$

If a trading day is split into $N$ equal intervals $\{(t_{n-1}, t_n] \mid n = 1, .., N\}$, $t_n = (n/N)T$ , where $T$ is the length of the day, then the corresponding expression for the daily $VWAP$ is given by

$$VWAP = \frac{\sum_{n=1}^{N} P_n V_n}{\sum_{n=1}^{N} V_n}. \tag{2–2}$$

where

$$V_n = \sum_{k:\ \tau_k \in (t_{n-1}, t_n]} \nu_k \tag{2–3}$$

is the volume traded during time period $(t_{n-1}, t_n]$,

$$P_n = \begin{cases} (\sum_{k:\tau_k \in (t_{n-1},..,t_n]} p_k \nu_k)/V_n, & \text{if } \ V_n > 0 \\ 0, & \text{if } \ V_n = 0 \end{cases} \tag{2–4}$$

can be thought of as an average market price during the $n^{th}$ interval.

Consider an order to sell $X$ units of stock during a trading day. We assume that an execution of this order does not affect the prices and the volume distribution of the market, which is reasonable for relatively small volumes $X$. We describe a trading strategy by the sequence

$$\{x_n \mid n = 1, .., N\}, \quad \sum_{n=1}^{N} x_n = 1,$$

where $x_n$ is the proportion of the order amount $X$ traded during the $n^{th}$ time interval. This strategy in terms of amount of stock would be

$$\{x_n X \mid n = 1, .., N\}.$$

Definition in terms of proportions is more appropriate because the value of $VWAP$ depends on proportions $V_i / \sum_{n=1}^{N} V_n$ rather than on volumes $V_i$:

$$VWAP = \sum_{n=1}^{N} P_n v_n, \quad v_n = \frac{V_n}{\sum_{j=1}^{N} V_j}. \tag{2–5}$$

Values of $x_n$ are assumed to be nonnegative (i.e. the trader is not allowed to buy stocks).

We construct the dynamic trading strategy by forecasting the volumes of stock traded in the market during each interval of a trading day. We assume that during a small interval (about 5 min) we can perform transactions at the average market price during this interval. Then, from (2–5) it follows that a possible way to meet the market $VWAP$ is to trade the order proportionally to the market volume during each interval, yielding the same daily distribution of the traded volume as the market's one. For each interval of a day we make a forecast of the market volume that will be traded during this interval and then trade according to this forecast. At the end of the day we obtain the forecast of the full daily volume distribution; the order is traded according to this distribution.

The way of dynamic computing of the distribution should be discussed first. Direct estimations of proportions of the market volume $v_1, ..., v_N$ does not guarantee that the

obtained proportions will sum up to one, since our procedure of finding each proportion $v_i$ does not take into account the previously found proportions $v_j,\ j \leq i$. To avoid this problem, we construct the distribution using the fractions of the remaining volume, that has not yet been traded at current time, rather than of the total daily volume. To make it more rigorous, suppose that $(V_1, V_2, ..., V_N)$ is the distribution of volume (in number of stocks) during a day. In terms of fractions of the daily volume this distribution can be represented as

$$(v_1, v_2, ..., v_N), \quad v_k = \frac{V_k}{\sum_{j=1}^{N} V_j}.$$

An alternative representation is

$$(w_1, w_2, ..., w_N), \quad w_k = \frac{V_k}{\sum_{j=k}^{N} V_j},$$

where $w_k$ is a fraction of the remaining volume after the $(k-1)^{th}$ interval, that is traded during the $k^{th}$ interval. Figure 2-1 demonstrates the two representations of the volume distribution. Note, that $w_N$ is always equal to 1. There is a one-to-one correspondence between representations $(v_1, ..., v_N)$ and $(w_1, ..., w_N)$; the transitions between them are given by formulas

$$w_1 = v_1, \quad w_k = \frac{v_k}{1 - \sum_{i=1}^{k-1} v_i}, \quad k = 2, ..., N \tag{2–6}$$

and

$$v_1 = w_1, \quad v_k = w_k \cdot \prod_{i=1}^{k-1} (1 - w_i), \quad k = 2, ..., N. \tag{2–7}$$

The last equations follow from the fact that

$$w_i(1 - w_{i-1}) \cdot ... \cdot (1 - w_{i-m}) = \frac{V_i}{V_{i-m} + ... + V_N}, \quad m = 1, ..., i-1.$$

Thus, for each interval $i$ we make a forecast of the fraction $w_i$ of the remaining volume. The fraction $w_i$ corresponds to the amount of the stock $V_i^{trade} = V_i^{rem} \cdot w_i$ to be

traded during $i^{th}$ interval, where $V_i^{rem}$ is the number of shares left to trade at the end of the $(i-1)^{th}$ interval. At the end of the day, $\sum_{i=1}^{N} V_i^{trade} = X$.

## 2.3 General Description of Regression Model

In the algorithm is described in detail in the next section we use the linear regression to make a forecast of the market volume distribution. For every interval $i$ the fraction $w_i$ is represented as a linear combination of several informative values obtained from the preceding time intervals. In this section we discuss some general questions regarding the types of deviation functions we use for the regression.

Consider the general regression setting where a random variable $Y$ is approximated by a linear combination

$$Y \sim c_1 X_1 + ... + c_n X_n + d \tag{2–8}$$

of indicator variables $X_1, ..., X_n$. In our study the variables are modelled by a set of scenarios

$$\{(Y^s;\ X_1^s, ..., X_n^s) \mid s = 1, ..., S\} \tag{2–9}$$

For a scenario $s$ the approximation error is

$$\epsilon^s = Y^s - c_1 X_1^s - ... - c_n X_n^s - d. \tag{2–10}$$

We consider our regression model as an optimization problem of minimizing the aggregated approximation error. Below we describe penalty functions we use as the objective.

### 2.3.1 Mean-Absolute Error

In the first regression model, the minimized objective is the mean-absolute error of the approximation (2–8)

$$D_{MAD}(\epsilon) = E|\epsilon|. \tag{2–11}$$

For the case of scenarios (2–9), the optimization problem is

$$\min D_{MAD} = \frac{1}{S}\sum_{s=1}^{S} \mid Y^s - c_1 X_1^s - ... - c_n X_n^s - d \mid . \tag{2–12}$$

### 2.3.2 CVaR-objective

The objective we used in the second regression model will be referred to as *CVaR-objective*. Mean-absolute deviation equally penalizes all outcomes of the approximation error (2–10), however our intention penalize the largest (by the absolute value) outcomes of the error. To give a more formal definition of the CVaR-objective and show the relevance of using it in regression problems, wee need to refer to the newly developed theory of deviation measures and generalized linear regression, see Rockafellar et al. (2002b).

CVaR-objective consists of two CVaR-deviations (Rockafellar et al. (2005a)) and penalizes the $\alpha$-highest and the $\alpha$-lowest outcomes of the estimation error (2–10) for a specified confidence level $\alpha$ ($\alpha$ is usually expressed in percentages). We will use a combination of CVaR-deviations as an objective:

$$\begin{aligned} D_{CVaR}(\epsilon) &= CVaR_\alpha^\triangle(\epsilon) + CVaR_\alpha^\triangle(-\epsilon) = \\ &= CVaR_\alpha(\epsilon) + CVaR_\alpha(-\epsilon). \end{aligned} \tag{2–13}$$

This expression is the difference between the average of $\alpha$ highest outcomes of random variable $X$ and the average of $\alpha$ lowest outcomes of $X$.

$D_{CVaR}(\epsilon)$ does not depend on the free term $d$ in (2–8) and the minimization (2–13) determines the optimal values of variables $c_1, ..., c_n$ only. The optimal value of the term $d$ can be found from different considerations; we use the condition that the estimator (2–8) is non-biased.

Thus, the regression problem takes the form:

$$\begin{aligned} \min_{c,d} \quad & CVaR_\alpha\left[Y-\bar{Y}\right]+CVaR_\alpha\left[\bar{Y}-Y\right] \\ s.t. \quad & E\left[\bar{Y}\right]=E[Y] \\ & \bar{Y}=\sum_{i=1}^n c_iX_i+d. \end{aligned} \tag{2–14}$$

Since

$$CVaR_{(1-\alpha)}\left[-X\right]=\frac{\alpha}{1-\alpha}CVaR_\alpha\left[X\right]-\frac{1}{1-\alpha}E[X], \tag{2–15}$$

optimization program (2–14) becomes:

$$\begin{aligned} \min_{c,d} \quad & \alpha CVaR_\alpha\left[Y-\bar{Y}\right]+(1-\alpha)CVaR_{1-\alpha}\left[Y-\bar{Y}\right] \\ s.t. \quad & E\left[\bar{Y}\right]=E[Y] \\ & \bar{Y}=\sum_{i=1}^n c_iX_i+d. \end{aligned} \tag{2–16}$$

The term $E[\bar{Y}-Y]$ is not included into the objective function since $E[\bar{Y}-Y]=0$ due to the first constraint.

For the case of scenarios (2–9) the optimization problem (2–16) can be reduced to the following linear programming problem.

$$\begin{aligned} \min \quad & \alpha\chi_+ + (1-\alpha)\chi_- \\ s.t. \quad & \textstyle\sum_{s=1}^S\left[\sum_{i=1}^n c_iX_i^s+d\right]=\sum_{s=1}^S Y^s \\ & \chi_\alpha \geq \xi_\alpha + \tfrac{1}{\alpha S}\textstyle\sum_{s=1}^S z_\alpha^s \\ & \chi_{1-\alpha} \geq \xi_{1-\alpha} + \tfrac{1}{(1-\alpha)S}\textstyle\sum_{s=1}^S z_{1-\alpha}^s \\ & z_\alpha^s \geq Y^s - (\textstyle\sum_{i=1}^n c_iX_i^s+d) - \xi_\alpha \\ & z_{1-\alpha}^s \geq Y^s - (\textstyle\sum_{i=1}^n c_iX_i^s+d) - \xi_{1-\alpha} \end{aligned} \tag{2–17}$$

Variables: $c_i, d \in \mathbb{R}$ for $i=1,...,n$; $\chi_\alpha, \chi_{1-\alpha} \in \mathbb{R}$; $z_\alpha^s, z_{1-\alpha}^s \geq 0$ for $s=1,...,S$.

### 2.3.3 Mixed Objective

Generally speaking, one can construct different penalizing functions using combinations the mean-absolute error function and $CVaR$-objectives with different confidence levels $\alpha$. Denote the objective in (2–14) by $D_{CVaR}^\alpha$, then the problem with the mixed objective is

stated as follows

$$\min \beta D_{MAD} + \sum_{i=1}^{I} \beta_i \cdot D_{CVaR}^{\alpha_i} \tag{2–18}$$
$$\textit{subject to constraints in (2–17)},$$

where $\beta_i \in [0,1]$, $i = 1, ..., I$, $\beta + \sum_{i=1}^{I} \beta_i = 1$.

In our experiments, we used convex combinations of two $CVaR$-objectives, one with the confidence level 50%:

$$\min \beta \cdot D_{CVaR}^{50\%} + (1-\beta) \cdot D_{CVaR}^{\alpha} \tag{2–19}$$
$$\textit{subject to constraints in (2–17)},$$

and of the mean-absolute error function and the $CVaR$-deviation:

$$\min \beta \cdot D_{MAD} + (1-\beta) \cdot D_{CVaR}^{\alpha} \tag{2–20}$$
$$\textit{subject to constraints in (2–17) without the first one},$$

where the balance coefficient $\beta \in [0,1]$. For comparison, different types of deviations are presented on Figure 2-2.

## 2.4 Experiments and Analysis

### 2.4.1 Model Design

Suppose that historical records for the last $S$ days are available, where each day is split into $N$ equal intervals. The purpose of our study is to estimate relative volumes for each interval of a day. Suppose we want to forecast the fraction of the remaining volume $w_k$ that will be traded in the market during the $k^{th}$ interval. In order to forecast $w_k$ we use the information about volumes and prices of the stock represented by variables $p^1_{(k-l),s}, ..., p^P_{(k-l),s}$, $l = 1, ..., L$, where $p^i_1, ..., p^i_P$ are variables taken from the $i^{th}$ interval and $L$ is the number of the preceding intervals.

We consider the following regression model

$$w_k \sim \sum_{l=1}^{L} \left( \gamma^1_{k-l} p^1_{k-l} + \cdots + \gamma^P_{k-l} p^P_{k-l} \right). \tag{2–21}$$

If from the beginning of the day up to the current time the number of intervals is less than $L$ then missing intervals are picked from the previous day. In order to approximate the $k^{th}$ ($k < L$) interval of the day parameters from intervals 1 through $k-1$ of the current day and intervals $N-(L-k)$ through $N$ of the previous day are used in linear combination (2–21).

Values of the corresponding parameters $p^j_{(k-l),s}$ and fractions of the remaining volume $w^k_s$, $s = 1, ..., S$, $i = 1, ..., L$, $j = 1, ..., P$, are collected from the preceding $S$ days of the history. Thus, we have the set of scenarios

$$\{(w^k_s,\ \{p^1_{(k-l),s}, ..., p^P_{(k-l),s}\}^L_{l=1}) \mid s = 1, ..., S\}. \tag{2–22}$$

Denote the linear combination

$$\sum_{l=1}^{L} \left(\gamma^1_{k-l} p^1_{(k-l),s} + \cdots + \gamma^P_{k-l} p^P_{(k-l),s}\right) \tag{2–23}$$

as $\hat{w}^k_s$, the collection of $\gamma^i_{k-j}$ as $\vec{\gamma}$.

In our study we consider the following optimization problems:

**P1**: MAD

$$\min_{\vec{\gamma}} \quad E|w^k - \hat{w}^k|; \tag{2–24}$$

**P2**: CVaR

$$\begin{aligned} \min_{\vec{\gamma}} \quad & CVaR_\alpha(w^k - \hat{w}^k) + CVaR_\alpha(\hat{w}^k - w^k) \\ & s.t.\ E[w^k] = E[\hat{w}^k] \end{aligned} \tag{2–25}$$

**P3**: MAD+CVaR

$$\min_{\vec{\gamma}} \quad \beta E|w^k - \hat{w}^k| + (1-\beta)\left(CVaR_\alpha(w^k - \hat{w}^k) + CVaR_\alpha(\hat{w}^k - w^k)\right) \tag{2–26}$$

**P4**: Mixed CVaR

$$\begin{aligned} \min_{\vec{\gamma}} \quad & \beta\left(CVaR_{50\%}(w^k - \hat{w}^k) + CVaR_{50\%}(\hat{w}^k - w^k)\right) + \\ & +(1-\beta)\left(CVaR_\alpha(w^k - \hat{w}^k) + CVaR_\alpha(\hat{w}^k - w^k)\right) \\ & s.t.\ \ E[w^k] = E[\hat{w}^k], \end{aligned} \tag{2–27}$$

where $\beta \in [0, 1]$. Each of these problems can be reduced to linear programming ones.

By solving these problems, the optimal value of $\vec{\gamma}^*$ is obtained. The forecast of $w_k^0$ is then made by the expression (2–21).

### 2.4.2 Nearest Sample

It is reasonable to choose for regression the "nearest" scenarios in the sense of similarity of historical days to the current day. Since for each day we are interested in the values of variables $p^1_{(k-l),s}, ..., p^P_{(k-l),s},\ l = 1, ..., L$, we define the "distance" between the current day and the scenario $s$ in the following way:

$$D_i = \sqrt{\frac{1}{L\,P}\sum_{i=1}^{P}\sum_{l=1}^{L}(p^i_{(k-l)} - p^i_{(k-l),s})^2}. \tag{2–28}$$

After calculating distances to all $S$ scenarios, we choose $S_{best}$ closest scenarios corresponding to lowest values of $D_i$ in (2–28). By doing so, we eliminate "outliers" with unusual, with respect to the current day, behavior of the market which favors the accuracy of forecasting.

### 2.4.3 Data Set

The model was verified with the historical prices of IBM stock for the period April 1997 - August 2002. Each day is split into 78 5-minute intervals (daily trading hours are 9:30 AM - 4:00 PM). For some experiments besides prices and volumes of the IBM stock we also used prices and volumes of index SPY.

### 2.4.4 Evaluation of Model Performance

We evaluated the performance of the model by applying it to the historical data set and forecasting the volume distribution of the IBM stock for the period of 100 (Feb. 2002 - Aug. 2002). In order to make the forecast for one day, a set of scenarios from the last $S$ admissible days was used. The day is "admissible" if this day and the previous day are full trading days starting and ending in usual hours, and there are no trading interruptions during these days. We compared the forecasted distributions with the actual ones and found the estimation error by averaging estimation errors for each interval over all output

days. Suppose that the algorithm was used to forecast volume distributions for $K$ days. If the historical volumes are

$$\{(v_1^k, ..., v_N^k) \mid k = 1, ..., K\}; \tag{2–29}$$

the forecasted volumes are

$$\{(\breve{v}_1^k, ..., \breve{v}_N^k) \mid k = 1, ..., K\}, \tag{2–30}$$

then the estimation error is

$$MAD = \frac{1}{K\,N} \sum_{k=1}^{K} \sum_{n=1}^{N} |v_n^k - \breve{v}_n^k|. \tag{2–31}$$

We also calculated another error

$$SD = \sqrt{\frac{1}{K\,N} \sum_{k=1}^{K} \sum_{n=1}^{N} (v_n^k - \breve{v}_n^k)^2}. \tag{2–32}$$

As a benchmark measuring the relative accuracy of the model "average daily volumes" ($ADV$) strategy was used. This very simple strategy provides a good approximation to $VWAP$. Suppose a set of historical volumes of the market:

$$\left\{ \ (V_s^1, ..., V_s^N), \ s = 1, .., S \right\}. \tag{2–33}$$

Denote

$$\bar{V}^n = \sum_{s=1}^{S} V_s^n, \ V^{total} = \sum_{n=1}^{N} \bar{V}^n. \tag{2–34}$$

Then the average volume distribution is

$$(\bar{v}_1, ..., \bar{v}_N), \ v^n = \frac{\bar{V}^n}{V^{total}}. \tag{2–35}$$

$ADV$ strategy is trading according to this distribution.

An example of average volume distribution versus the actual volume evolution is presented in Figure 2-3. It can be seen that daily volume exhibits the "$U$-shape" and that the average distribution provides a good approximation to the daily volume evolution.

In the case of the data set described above, we calculated the average volume distribution over $S$ admissible days. The estimation error of the $ADV$ strategy was calculated using (2–32). The relative gain in accuracy of the regression algorithm was judged by the value of

$$\mathrm{G_{MAD}} = \frac{MAD^{ADV} - MAD}{MAD^{ADV}} \cdot 100\%. \tag{2–36}$$

Relative gain in standard deviation is

$$\mathrm{G_{SD}} = \frac{SD^{ADV} - SD}{SD^{ADV}} \cdot 100\%. \tag{2–37}$$

## 2.5 Experiments and Results

In our experiments, we varied the type of the objective, coefficients in the objective, the "length" of the regression $L$, the number of admissible historical days $S$ and the number of (nearest) scenarios $S_{best}$ used in the regression.

With respect to the parameters (2–21) we took from each interval, the experiments were divided into two groups.

In the first group, the experiments were based on using only prices and volumes of the stock as useful information. Namely, from each interval we used the following information:

$$\ln V \text{ and } \ln \frac{P_{close}}{P_{open}} \tag{2–38}$$

where $V$ is market volume during the interval, $P_{open}$ and $P_{close}$ are open and close prices of the interval. $R = P_{close}/P_{open}$ is, therefore, the return during the interval. Logarithms were used to take into account the possibility that the ratios of returns and volumes, aside from returns and volumes themselves, contain some information about the future volume. A linear combination of logarithms of parameters can be represented as a linear combination of the parameters and their ratios.

In the second group, we added volumes and returns of INDEX SPY to the set of parameters taken from intervals. From each interval $P = 4$ parameters were used:

$$\ln V, \;\; \ln \frac{P_{close}}{P_{open}}, \;\; \ln V^{SPY} \text{ and } \ln \frac{P^{SPY}_{close}}{P^{SPY}_{open}}. \tag{2–39}$$

The idea of using index information comes from the fact that evolutions of index and stock are correlated and that the ratios of returns and prices of stock and index may also contain useful information.

Tables 2-1, 2-2 show the results for the mean-absolute deviation used as an objective and different values of $L, S$ and $S_{best}$. These tables show that including INDEX data does not improve the accuracy of prediction. Also, as one can notice, there is a balance between the number of terms $N_{term} = L\,P$ in the linear combination (2–21) and the number of scenarios ($S_{best}$) used in the regression model. As $N_{term}$ increases, the model becomes more flexible and more scenarios are needed to achieve the same level of accuracy. For example, the best two models that use stock data ($P = 2$), have values of $S_{best}$ and $N_{term}$ equal 450 and 4, 200 and 2, respectively. Also, when the index data is used, the number of parameters $P$ doubles, and the number of scenarios in the best models increases to $700 - 800$ for the same regression length $L$.

In the case of CVaR-objective and mixed objective (Tables 2-3, 2-4), different values of $L, S$ and $S_{best}$ yielded a similar order of superiority as in the case of the mean-absolute deviation.

Two more facts can be seen from the results. First, that the most successful models use information only from the last one or two intervals, which means that the information about the future volume is concentrated in the past few minutes. Second, the idea of choosing the closest scenarios from the preceding history does work, especially when a small portion of non-similar days (50 or 100 out of 500 or 800 potential scenarios) is excluded. This agrees with the observation that most of the days are "regular" enough to be used for the estimation of the future.

In Table 2-5 we changed the form of the mixed objective, that is, differed $\beta$ and $\alpha$ in (2–17). We found that the best models have all weight put on the CVaR objective and for a fixed balance $\beta$ the models with small values of $\alpha$ are superior.

The most accurate model turned out to be the one with CVaR- objective having the relative gain 4%.

## 2.6 Conclusions

In this study we designed several $VWAP$ trading strategies based on dynamic forecasting of market volume distribution. We made estimations of market volume during small time intervals as a linear combination of market prices and volumes and their ratios. We found that prices and volumes do not contain much information about the future volume. Linear regression techniques proved to be quite efficient and easily implementable for forecasting the volumes, although the considered sets of indicator parameters do not justify the use of regression instead of the simple average strategy.

Table 2-1. Performance of tracking models: stock vs. stock+index, full history regression

| S | $S_{best}$ | L | MAD,% | SD,% | $G_{MAD}$,% | $G_{SD}$,% |
|---|---|---|---|---|---|---|
| | | | STOCK | | | |
| 500 | 500 | 2 | 34.0 | 41.1 | 3.5 | 3.2 |
| 500 | 500 | 1 | 34.1 | 41.4 | 3.4 | 2.5 |
| 500 | 500 | 3 | 34.2 | 41.3 | 3.0 | 2.8 |
| 800 | 800 | 2 | 34.3 | 42.1 | 2.7 | 0.9 |
| 800 | 800 | 1 | 34.4 | 42.5 | 2.6 | -0.1 |
| 800 | 800 | 3 | 34.4 | 42.2 | 2.4 | 0.5 |
| | | | STOCK+INDEX | | | |
| 500 | 500 | 1 | 34.1 | 41.0 | 3.2 | 3.5 |
| 500 | 500 | 2 | 34.2 | 41.0 | 3.1 | 3.5 |
| 800 | 800 | 1 | 34.2 | 41.2 | 3.1 | 3.1 |
| 800 | 800 | 2 | 34.2 | 40.8 | 3.1 | 3.9 |
| 800 | 800 | 3 | 34.3 | 40.7 | 2.7 | 4.0 |
| 500 | 500 | 3 | 34.4 | 41.2 | 2.4 | 3.0 |

Table 2-2. Performance of tracking models: stock vs. stock+index, best sample regression

| S | $S_{best}$ | L | MAD,% | SD,% | $G_{MAD}$,% | $G_{SD}$,% |
|---|---|---|---|---|---|---|
| | | | STOCK | | | |
| 500 | 450 | 2 | 34.0 | 40.8 | 3.7 | 4.0 |
| 500 | 200 | 1 | 34.0 | 39.4 | 3.6 | 7.1 |
| 500 | 450 | 1 | 34.0 | 41.0 | 3.6 | 3.3 |
| 500 | 400 | 2 | 34.0 | 40.5 | 3.6 | 4.6 |
| 500 | 400 | 1 | 34.0 | 41.0 | 3.5 | 3.3 |
| 800 | 500 | 2 | 34.2 | 41.0 | 3.1 | 3.4 |
| 500 | 450 | 3 | 34.2 | 41.1 | 3.0 | 3.1 |
| 800 | 700 | 2 | 34.3 | 42.1 | 2.9 | 0.9 |
| | | | STOCK+INDEX | | | |
| 500 | 450 | 1 | 34.1 | 40.6 | 3.3 | 4.4 |
| 500 | 450 | 2 | 34.1 | 40.1 | 3.2 | 5.4 |
| 800 | 750 | 1 | 34.2 | 40.8 | 3.2 | 3.8 |
| 800 | 750 | 2 | 34.2 | 40.8 | 3.2 | 4.0 |
| 800 | 700 | 1 | 34.2 | 40.9 | 3.1 | 3.7 |
| 800 | 700 | 2 | 34.2 | 40.5 | 3.1 | 4.6 |
| 500 | 400 | 1 | 34.3 | 41.0 | 3.0 | 3.5 |
| 500 | 400 | 2 | 34.3 | 41.1 | 2.8 | 3.3 |

Table 2-3. Performance of tracking models: mixed objective, changing size of history and best sample

| S | $S_{best}$ | L | $\alpha$,% | $\beta$,% | MAD,% | SD,% | $G_{MAD}$,% | $G_{SD}$,% |
|---|---|---|---|---|---|---|---|---|
| | | | | | STOCK | | | |
| 500 | 450 | 2 | 30 | 50 | 34.0 | 40.8 | 3.7 | 4.0 |
| 500 | 200 | 1 | 30 | 50 | 34.0 | 39.4 | 3.6 | 7.2 |
| 500 | 450 | 1 | 30 | 50 | 34.0 | 41.0 | 3.6 | 3.4 |
| 500 | 400 | 2 | 30 | 50 | 34.0 | 40.5 | 3.6 | 4.6 |
| 500 | 400 | 1 | 30 | 50 | 34.0 | 41.0 | 3.5 | 3.3 |
| 500 | 500 | 2 | 30 | 50 | 34.0 | 41.1 | 3.5 | 3.2 |
| 500 | 500 | 1 | 30 | 50 | 34.1 | 41.4 | 3.4 | 2.5 |
| 800 | 500 | 2 | 30 | 50 | 34.2 | 41.0 | 3.0 | 3.4 |

Table 2-4. Performance of tracking models: CVaR deviation, changing size of history and best sample

| S | $S_{best}$ | L | $\alpha$,% | $\beta$,% | MAD,% | SD,% | $G_{MAD}$,% | $G_{SD}$,% |
|---|---|---|---|---|---|---|---|---|
| | | | | | STOCK | | | |
| 500 | 400 | 2 | 30 | 100 | 33.9 | 40.7 | 4.0 | 4.1 |
| 500 | 200 | 1 | 30 | 100 | 33.9 | 39.7 | 3.9 | 6.6 |
| 500 | 200 | 2 | 30 | 100 | 33.9 | 39.4 | 3.9 | 7.1 |
| 500 | 450 | 2 | 30 | 100 | 33.9 | 40.8 | 3.9 | 4.0 |
| 500 | 480 | 2 | 30 | 100 | 33.9 | 40.8 | 3.8 | 3.0 |
| 500 | 400 | 3 | 30 | 100 | 34.0 | 40.4 | 3.7 | 4.9 |
| 500 | 400 | 1 | 30 | 100 | 34.0 | 41.1 | 3.7 | 3.3 |
| 500 | 450 | 1 | 30 | 100 | 34.0 | 41.0 | 3.7 | 3.4 |

Table 2-5. Performance of tracking models: mixed objective

| S | $S_{best}$ | L | $\alpha$,% | $\beta$,% | MAD,% | SD,% | $G_{MAD}$,% | $G_{SD}$,% |
|---|---|---|---|---|---|---|---|---|
| | | | | | STOCK | | | |
| 500 | 450 | 2 | 20 | 100 | 33.9 | 40.7 | 4.0 | 4.2 |
| 500 | 450 | 2 | 30 | 100 | 33.9 | 39.6 | 3.9 | 4.4 |
| 500 | 450 | 2 | 10 | 100 | 33.9 | 39.8 | 3.9 | 4.0 |
| 500 | 450 | 2 | 5 | 30 | 33.9 | 40.6 | 3.9 | 4.5 |
| 500 | 450 | 2 | 10 | 30 | 33.9 | 40.7 | 3.9 | 4.1 |
| 500 | 450 | 2 | 20 | 30 | 34.0 | 40.7 | 3.8 | 4.1 |
| 500 | 450 | 2 | 5 | 100 | 34.0 | 41.6 | 3.8 | 4.4 |
| 500 | 450 | 2 | 30 | 30 | 34.0 | 41.7 | 3.8 | 4.2 |

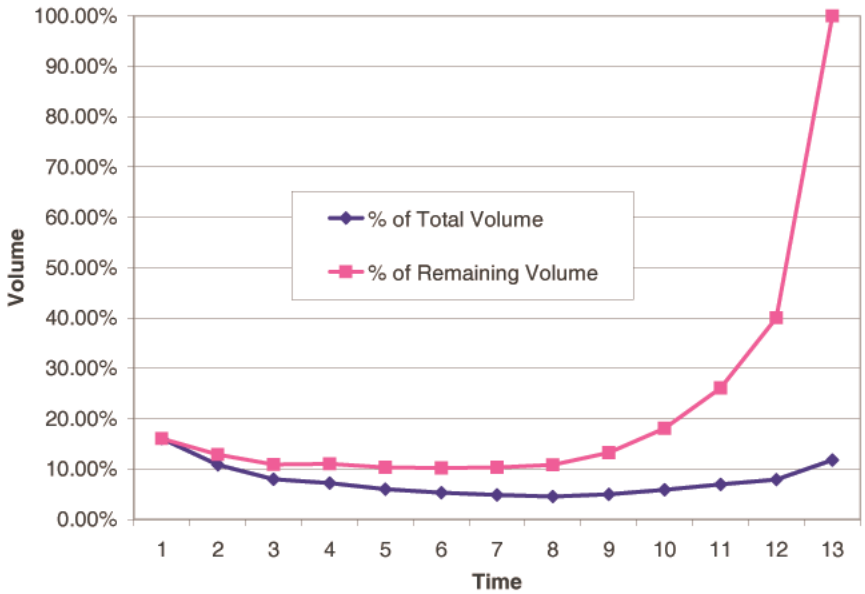


Figure 2-1. Percentages of remaining volume vs. percentages of total volume

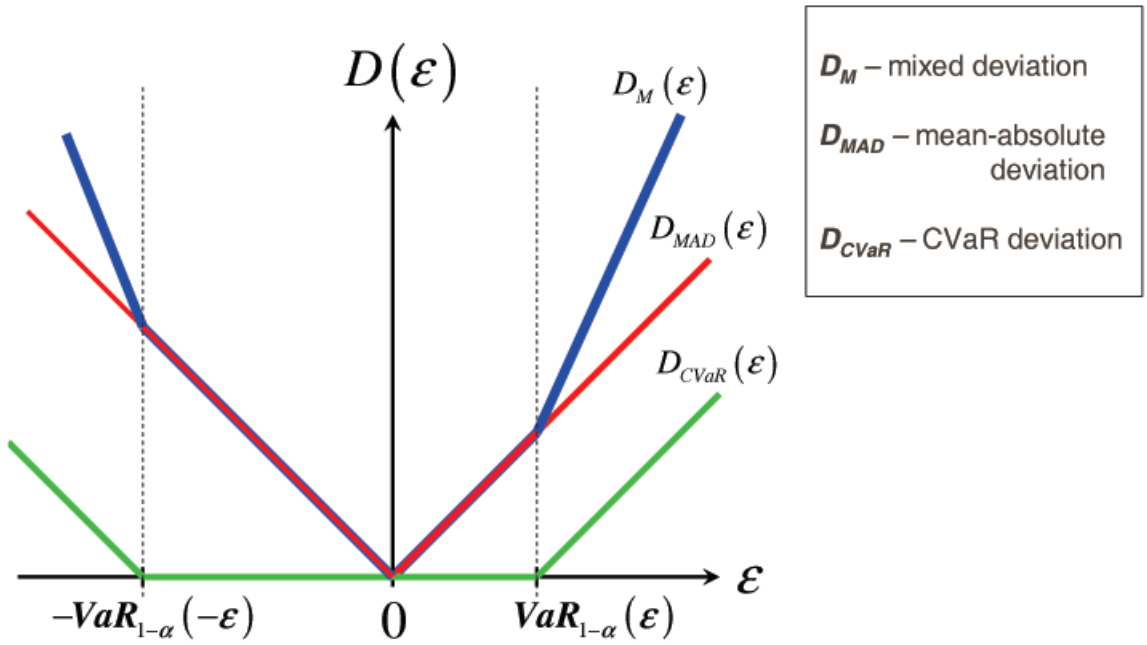


Figure 2-2. MAD, CVaR, and mixed deviations

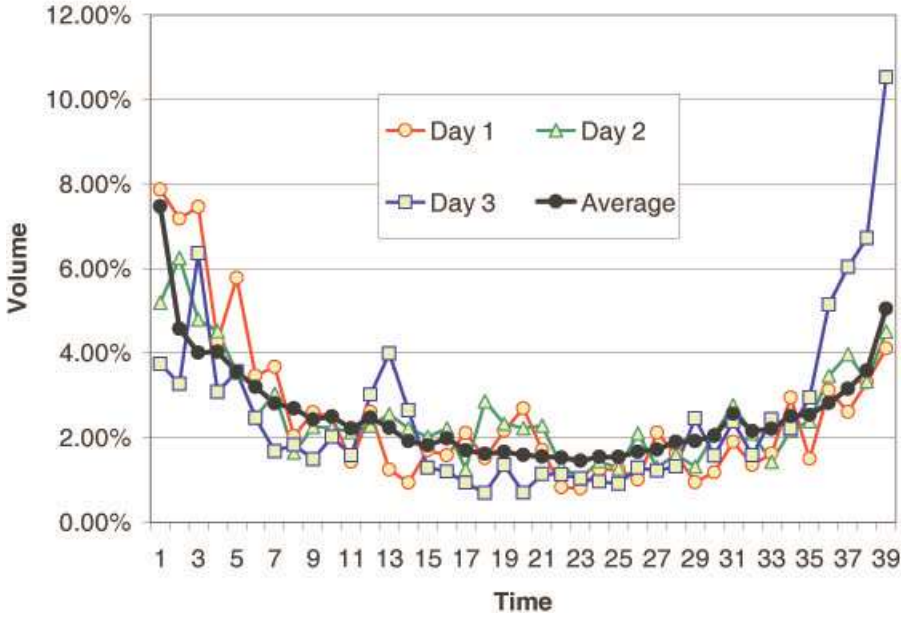


Figure 2-3. Daily volume distributions

# CHAPTER 3
# PRICING EUROPEAN OPTIONS BY NUMERICAL REPLICATION

## 3.1 Introduction

Options pricing is a central topic in financial literature. A reader can find an excellent overview of option pricing methods in Broadie and Detemple (2004). The algorithm for pricing European options in discrete time presented in this paper has common features with other existing approaches. We approximate an option value by a portfolio consisting of the underlying stock and a risk-free bond. The stock price is modelled by a set of sample-paths generated by a Monte-Carlo or historical bootstrap simulation. We consider a non-self-financing portfolio dynamics and minimize the sum of squared additions/subtractions of money to/from the hedging portfolio at every re-balancing point, averaged over a set of sample paths. This error minimization problem is reduced to quadratic programming. We also include constraints on the portfolio hedging strategy to the quadratic optimization problem. The constraints dramatically improve numerical efficiency of the algorithm.

Below, we refer to option pricing methods directly related to our algorithm. Although this paper considers European options, some related papers consider American options.

Replication of the option price by a portfolio of simpler assets, usually of the underlying stock and a risk-free bond, can incorporate various market frictions, such as transaction costs and trading restrictions. For incomplete markets, replication-based models are reduced to linear, quadratic, or stochastic programming problems, see, for instance, Bouchaud and Potters (2000), Bertsimas et al. (2001), Dembo and Rosen (1999), Coleman et al. (2004), Naik and Uppal (1994), Dennis (2001), Dempster and Thompson

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This chapter is based on the paper Ryabchenko, V., Sarykalin, S., and Uryasev, S. (2004) Pricing European Options by Numerical Replication: Quadratic Programming with Constraints. *Asia-Pacific Financial Markets*, **11**(3), 301-333.

(2001), Edirisinghe et al. (1993), Fedotov and Mikhailov (2001), King (2002), and Wu and Sen (2000).

Analytical approaches to minimization of quadratic risk are used to calculate an option price in an incomplete market, see Duffie and Richardson (1991), Föllmer and Schied (2002), Föllmer and Schweizer (1989), Schweizer (1991, 1995, 2001).

Another group of methods, which are based on a significantly different principle, incorporates known properties of the shape of the option price into the statistical analysis of market data. Ait-Sahalia and Duarte (2003) incorporate monotonic and convex properties of European option price with respect to the strike price into a polynomial regression of option prices. In our algorithm we limit the set of feasible hedging strategies, imposing constraints on the hedging portfolio value and the stock position. The properties of the option price and the stock position and bounds on the option price has been studied both theoretically and empirically by Merton (1973), Perrakis and Ryan (1984), Ritchken (1985), Bertsimas and Popescu (1999), Gotoh and Konno (2002), and Levi (85). In this paper, we model stock and bond positions on a two-dimensional grid and impose constraints on the grid variables. These constraints follow under some general assumptions from non-arbitrage considerations. Some of these constraints are taken from Merton (1973).

Monte-Carlo methods for pricing options are pioneered by Boyle (1977). They are widely used in options pricing: Joy et al. (1996), Broadie and Glasserman (2004), Longstaff and Schwartz (2001), Carriere (1996), Tsitsiklis and Van Roy (2001). For a survey of literature in this area see Boyle (1997) and Glasserman (2004). Regression-based approaches in the framework of Monte-Carlo simulation were considered for pricing American options by Carriere (1996), Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999, 2001). Broadie and Glasserman (2004) proposed stochastic mesh method which combined modelling on a discrete mesh with Monte-Carlo simulation. Glasserman (2004), showed that regression-based approaches are special cases of the stochastic mesh method.

The algorithm uses the hedging portfolio to approximate the price of the option. We aimed at making the hedging strategy close to real-life trading. The actual trading occurs at discrete times and is not self-financing at re-balancing points. The shortage of money should be covered at any discrete point. Large shortages are undesirable at any time moment, even if self-financing is present.

The pricing algorithm described in this paper combines the features of the above approaches in the following way. We construct a hedging portfolio consisting of the underlying stock and a risk-free bond and use its value as an approximation to the option price. We aimed at making the hedging strategy close to real-life trading. The actual trading occurs at discrete times and is not self-financing at re-balancing points. The shortage of money should be covered at any discrete point. Large shortages are undesirable at any time moment, even if self-financing is present. We consider non-self-financing hedging strategies. External financing of the portfolio or withdrawal is allowed at any re-balancing point. We use a set of sample paths to model the underlying stock behavior. The position in the stock and the amount of money invested in the bond (hedging variables) are modelled on nodes of a discrete grid in time and the stock price. Two matrices defining stock and bond positions on grid nodes completely determine the hedging portfolio on any price path of the underlying stock. Also, they determine amounts of money added to/taken from the portfolio at re-balancing points. The sum of squares of such additions/subtractions of money on a path is referred to as the squared error on a path.

The pricing problem is reduced to quadratic minimization with constraints. The objective is the averaged quadratic error over all sample paths; the free variables are stock and bond positions defined in every node of the grid. The constraints, limiting the feasible set of hedging strategies, restrict the portfolio values estimating the option price and stock positions. We required that the average of total external financing over all paths equals to zero. This makes the strategy "self-financing on average". We incorporated monotonic,

convex, and some other properties of option prices following from the definition of an option, a non-arbitrage assumption, and some other general assumptions about the market. We do not make assumptions about the stock process which makes the algorithm distribution-free. Monotonicity and convexity constraints on the stock position are imposed. Such constraints reduce transaction costs, which are not accounted for directly in the model. We aim to prohibit sharp changes in stock and bond positions in response to small changes in the stock price or in time to maturity.

We performed two numerical tests of the algorithm. First, we priced options on the stock following the geometric Brownian motion. Stock price is modelled by Monte-Carlo sample-paths. Calculated option prices are compared with the known prices given by the Black-Scholes formula. Second, we priced options on S&P 500 Index and compared the results with actual market prices. Both numerical tests demonstrated reasonable accuracy of the pricing algorithm with a relatively small number of sample-paths (considered cases include 100 and 20 sample-paths). We calculated option prices both in dollars and in the implied volatility format. The implied volatility matches reasonably well the constant volatility for options in the Black-Scholes setting. The implied volatility for S&P 500 index options (priced with 100 sample-paths) tracks the actual market volatility smile.

The advantage of using the squared error as an objective can be seen from the practical perspective. Although we allow some external financing of the portfolio along the path, the minimization of the squared error ensures that large shortages of money will not occur at any point of time if the obtained hedging strategy is practically implemented.

Another advantage of using the squared error is that the algorithm produces a hedging strategy such that the sum of money added to/taken from the hedging portfolio on any path is close to zero. Also, the obtained hedging strategy requires zero average external financing over all paths. This justifies considering the initial value of the hedging portfolio as a price of an option. We use the notion of "a price of an option in the practical setting" which is the price a trader agrees to buy/sell the option. In the example

of pricing options on the stock following the geometric Brownian motion the algorithm finds hedging strategy which delivers requested option payments at expiration with high precision on many considered sample paths. Therefore, we claim that the initial value of the portfolio can be considered as an estimate of the market price.

We assume an incomplete market in this paper. We use the portfolio of two instruments - the underlying stock and a bond - to approximate the option price and consider many sample paths to model the stock price process. As a consequence, the value of the hedging portfolio may not be equal to the option payoff at expiration on some sample paths. Also, the algorithm is distribution-free, which makes it applicable to a wide range of underlying stock processes. Therefore, the algorithm can be used in the framework of an incomplete market.

Usefulness of our algorithm should be viewed from the perspective of practical options pricing. Commonly used methods of options pricing are time-continuous models assuming specific type of the underlying stock process. If the process is known, these methods provide accurate pricing. If the stock process cannot be clearly identified, the choice of the stock process and calibration of the process to fit market data may entail significant modelling error. Our algorithm is superior in this case. It is distribution-free and is based on realistic assumptions, such as discrete trading and non-self-financing hedging strategy.

Another advantage of our algorithm is low back-testing errors. Time-continuous models do not account for errors of implementation on historical paths. The objective in our algorithm is to minimize the back-testing errors on historical paths. Therefore, the algorithm has a very attractive back-testing performance. This feature is not shared by any of time-continuous models.

## 3.2 Framework and Notations

### 3.2.1 Portfolio Dynamics and Squared Error

Consider a European option with time to maturity $T$ and strike price $X$. We suppose that trading occurs at discrete times $t_j$, $j = 0, 1, ..., N$, such that

$$0 = t_0 < t_1 < ... < t_N = T, \quad t_{j+1} - t_j = const, \; j = 0, 1, ..., N-1.$$

We denote the position in the stock at time $t_j$ by $u_j$, the amount of money invested in the bond by $v_j$, the risk-free rate by $r$, and the stock price at time $t_j$ by $S_j$.

The price of the option at time $t_j$ is approximated by the price $c_j$ of a hedging portfolio consisting of the underlying stock and a risk-free bond. The hedging portfolio is rebalanced at times $t_j$, $j = 1, ..., N-1$. Suppose that at the time $t_{j-1}$ the hedging portfolio consists of $u_{j-1}$ shares of the stock and $v_{k-1}$ dollars invested in the bond[1] . The value of the portfolio right before the time $t_j$ is $u_{j-1}S_j + (1+r)v_{j-1}$. At time $t_j$ the positions in the stock and in the bond are changed to $u_j$ and $v_j$, respectively, and the portfolio value changes to $u_jS_j + v_j$. We consider a non-self-financing portfolio dynamics by allowing the difference

$$a_j = u_jS_j + v_j - (u_{j-1}S_j + (1+r)v_{j-1}) \tag{3–1}$$

to be non-zero. The value $a_j$ is the excess/shortfall of the money in the hedging portfolio during the interval $[t_{j-1}, t_j]$. In other words, $a_j$ is the amount of money added to (if $a_j \geq 0$) or subtracted from (if $a_j < 0$) the portfolio during the interval $[t_{j-1}, t_j]$. Thus, the inflow/outflow of money to/from the hedging portfolio is allowed.

[1] Below, the number of shares of the stock and the amount of money invested in the bond are referred to as positions in the stock and in the bond.

We require that the value of the hedging portfolio at expiration be equal to the option payoff $h(S_N)$, $u_N S_N + v_N = h(S_N)$, where

$$h(S) = \begin{cases} \max\{0, S - X\} & \text{for call options;} \\ \max\{0, X - S\} & \text{for put options.} \end{cases}$$

The non-self-financing portfolio dynamics is given by

$$u_{j+1} S_{j+1} + v_{j+1} = u_j S_{j+1} + (1 + r) v_j + a_j, \ j = 0, ..., N - 1, \tag{3–2}$$

where the portfolio value at time $t_j$ is $c_j = u_j S_j + v_j, \ j = 0, ..., N$.

The degree to which a portfolio dynamics differs from a self-financing one is an important characteristic, essential to our approach. In this paper, we define a *squared error* on a path,

$$A = \sum_{j=1}^{N} \left( a_j e^{-rj} \right)^2, \tag{3–3}$$

to measure the degree of "non-self-financity". The reasons for choosing this particular measure will be described later on.

### 3.2.2 Hedging Strategy

We assume that the composition of the hedging portfolio depends on time and the stock price. We define a hedging strategy as a function determining the composition of a hedging portfolio for any given time and the stock price at that time. If the hedging strategy is defined, the corresponding portfolio management decisions for the stock price path $S_0, S_1, ..., S_N$ are given by the sequence $(u_0, v_0), (u_1, v_1), ..., (u_N, v_N)$.

A hedging strategy is modelled on a discrete grid with a set of approximation rules. Consider a grid consisting of nodes $\{(j, k); \ j = 0, ..., N, \ k = 1, ..., K\}$ in the time and the stock price. The index $j$ denotes time and corresponds to time $t_j$; the index $k$ denotes the stock price and corresponds to the stock price $\tilde{S}_k$ (we use the tilde sign for stock values on the grid to distinguish them from stock values on sample-paths). Stock prices $\tilde{S}_k$,

$k = 1, ..., K$ on the grid are equally distanced in the logarithmic scale, i.e.

$$\tilde{S}_1 < \tilde{S}_2 < ... < \tilde{S}_K, \;\; \ln(\tilde{S}_{k+1}) - \ln(\tilde{S}_k) = const.$$

Thus, the node $(j, k)$ of the grid corresponds to time $t_j$ and the stock price $\tilde{S}_k$. To every node $(j, k)$ we assigned two variables $U_j^k$ and $V_j^k$ representing the composition of the hedging portfolio at time $t_j$ with the stock price $\tilde{S}_k$. The pair of matrices

$$[U_j^k] = \begin{bmatrix} U_0^1 & U_1^1 & \dots & U_N^1 \\ U_0^2 & U_1^2 & \dots & U_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ U_0^K & U_1^K & \dots & U_N^K \end{bmatrix}, \quad [V_j^k] = \begin{bmatrix} V_0^1 & V_1^1 & \dots & V_N^1 \\ V_0^2 & V_1^2 & \dots & V_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ V_0^K & V_1^K & \dots & V_N^K \end{bmatrix} \tag{3–4}$$

are referred to as a *hedging strategy*. These matrices define portfolio management decisions on the discrete set of the grid nodes. In order to set those decisions on any path, not necessarily going through grid points, approximation rules are defined.

We model the stock price dynamics by a set of sample paths

$$\{(S_0, S_1^p, ..., S_N^p) |\; p = 1, ..., P\} \tag{3–5}$$

where $S_0$ is the initial price. Let variables $u_j^p$ and $v_j^p$ define the composition of the hedging portfolio on path $p$ at time $t_j$, where $p = 1, ..., P$, $j = 0, ..., N$. These variables are approximated by the grid variables $U_j^k$ and $V_j^k$ as follows. Suppose that $\{S_0, S_1^p, ..., S_N^p\}$ is a realization of the stock price, where $S_j^p$ denotes the price of the stock at time $t_j$ on path $p$, $j = 0, ..., N$, $p = 1, ..., P$. Let $u_j^p$ and $v_j^p$ denote the amounts of the stock and the bond, respectively, held in the hedging portfolio at time $t_j$ on path $p$. Variables $u_j^p$ and $v_j^p$ are linearly approximated by the grid variables $U_j^k$ and $V_j^k$ as follows

$$u_j^p = \alpha_j^p U_j^{k(j,p)+1} + (1 - \alpha_j^p) U_j^{k(j,p)}, \;\; v_j^p = \alpha_j^p V_j^{k(j,p)+1} + (1 - \alpha_j^p) V_j^{k(j,p)}, \tag{3–6}$$

where $\alpha_j^p = \dfrac{\ln S_j^p - \ln \tilde{S}_{k(j,p)}}{\ln \tilde{S}_{k(j,p)+1} - \ln \tilde{S}_{k(j,p)}}$, and $k(j,p)$ is such that $\tilde{S}_{k(j,p)} \leq S_j^p < \tilde{S}_{k(j,p)+1}$.

According to (3–1), we define the excess/shortage of money in the hedging portfolio on path $p$ at time $t_j$ by

$$a_j^p = u_{j+1}^p S_{j+1}^p + v_{j+1}^p - (u_j^p S_{j+1}^p + (1+r)v_j^p).$$

The squared error $\mathcal{E}_p$ on path $p$ equals

$$\mathcal{E}_p = \sum_{j=1}^{N} (a_j^p e^{-rj})^2. \tag{3–7}$$

We define the *average squared error* $\bar{\mathcal{E}}$ on the set of paths (3–5) as an average of squared errors $\mathcal{E}_p$ over all sample paths (3–5)

$$\bar{\mathcal{E}} = \frac{1}{P} \sum_{p=1}^{P} \sum_{j=1}^{N} \left(a_j^p e^{-rj}\right)^2. \tag{3–8}$$

The matrices $[U_j^k]$ and $[V_j^k]$ and the approximation rule (3–6) specify the composition of the hedging portfolio as a function of time and the stock price. For any given stock price path one can find the corresponding portfolio management decisions $\{(u_j, v_j) | j = 0, ..., N-1\}$, the value of the portfolio $c_j = S_j u_j + v_j$ at any time $t_j$, $j = 0, ..., N$, and the associated squared error.

The value of an option in question is assumed to be equal to the initial value of the hedging portfolio. First columns of matrices $[U_j^k]$ and $[V_j^k]$, namely the variables $U_0^k$ and $V_0^k$, $k = 1, ..., K$, determine the initial value of the portfolio. If one of the initial grid nodes, for example node $(0, \tilde{k})$, corresponds to the stock price $S_0$, then the price of the option is given by $U_0^{\tilde{k}} S_0 + V_0^{\tilde{k}}$. If the initial point $(t = 0, S = S_0)$ of the stock process falls between the initial grid nodes $(0, k)$, $k = 1, ..., K$, then approximation formula (3–6) with $j = 0$ and $S_0^p = S_0$ is used to find the initial composition $(u_0, v_0)$ of the portfolio. Then, the price of the option is found as $u_0 S_0 + v_0$.

## 3.3 Algorithm for Pricing Options

This section presents an algorithm for pricing European options in incomplete markets. Subsection 3.1, presents the formulation of the algorithm; subsections 3.2 - 3.4 discuss the choice of the objective and the constraints of the optimization problem.

### 3.3.1 Optimization Problem

The price of the option is found by solving the following minimization problem.

$$\min \ \bar{\mathcal{E}} = \frac{1}{P}\sum_{j=1}^{N}\sum_{p=1}^{P}\left(\left\{u_j^p S_j^p + v_j^p - u_{j-1}^p S_j^p - (1+r)v_{j-1}^p\right\} e^{-rj}\right)^2 \tag{3–9}$$

subject to

$$\frac{1}{P}\sum_{j=1}^{N}\sum_{p=1}^{P}\left\{u_j^p S_j^p + v_j^p - u_{j-1}^p S_j^p - (1+r)v_{j-1}^p\right\} e^{-rj} = 0$$

$$U_N^k \tilde{S}_k + V_N^k = h(\tilde{S}_k), \ \ k = 1, ..., K,$$

approximation rules (3–6),

constraints (3–10)-(3–18) (defined below) for call options,

or constraints (3–19)-(3–27) (defined below) for put options,

free variables: $U_j^k, \ V_j^k, \ j = 0, ..., N, \ k = 1, ..., K.$

The objective function in (3–9) is the average squared error on the set of paths (3–5). The first constraint requires that the average value of total external financing over all paths equals to zero. The second constraint equates the value of the portfolio and the option payoff at expiration. Free variables in this problem are the grid variables $U_j^k$ and $V_j^k$; the path variables $u_j^p$ and $v_j^p$ in the objective are expressed in terms of the grid variables using approximation (3–6). The total number of free variables in the problem is determined by the size of the grid and is independent of the number of sample-paths. After solving the optimization problem, the option value at time $t_j$ for the stock price $S_j$ is defined by $u_j S_j + v_j$, where $u_j$ and $v_j$ are found from matrices $[U_j^k]$ and $[V_j^k]$, respectively, using

approximation rules (3–6). The price of the option is the initial value of the hedging portfolio, calculated as $u_0 S_0 + v_0$.

The following constraints (3–10)-(3–18) for call options or (3–19)-(3–27) for put options impose restrictions on the shape of the option value function and on the position in the stock. These restrictions reduce the feasible set of hedging strategies. Subsection 3.3 discusses the benefits of inclusion of these constraints in the optimization problem.

Below, we consider the constraints for European call options. The constraints for put options are given in the next section, together with proofs of the constraints. Most of the constraints are justified in a quite general setting. We assume non-arbitrage and make 5 additional assumptions. Proofs of two constraints on the stock position (horizontal monotinicity and convexity) in the general setting will be addressed in subsequent papers. In this paper we validate these inequalities in the Black-Scholes case.

The notation $C_j^k$ stands for the option value in the node $(j, k)$ of the grid,

$$C_j^k = U_j^k \tilde{S}_j^k + V_j^k.$$

The strike price of the option is denoted by $X$, time to expiration by $T$, one period risk-free rate by $r$.

**Constraints on Call Option Value**

- Immediate exercise constraints. The value of an option is no less than the value of its immediate exercise[2] at the discounted strike price,

$$C_j^k \geq \left[\tilde{S}_j^k - Xe^{-r(T-t_j)}\right]^+. \tag{3–10}$$

[2] European options do not have the feature of immediate exercise. However, the right part of constraint (3–10) coincides with the immediate exercise value of an American option having the current stock price $\tilde{S}_j^k$ and the strike price $Xe^{-r(T-t_j)}$.

- Option price sensitivity constraints.

$$\begin{aligned} C_j^{k+1} \leq & \gamma_j^k C_j^k + X(\gamma_j^k - 1)e^{-r(T-t_j)},\ \gamma_j^k = \tilde{S}_j^{k+1}/\tilde{S}_j^k, \\ & j = 0, ..., N-1,\ k = 1, ..., K-1. \end{aligned} \tag{3–11}$$

This constraints bound sensitivity of an option price to changes in the stock price.

- Monotonicity constraints.

  0. Vertical monotonicity. For any fixed time, the price of an option is an increasing function of the stock price.

$$\frac{\tilde{S}_j^k}{\tilde{S}_j^{k+1}} C_j^{k+1} \geq C_j^k,\ j = 0, ..., N;\ k = 1, ..., K-1. \tag{3–12}$$

  0. Horizontal monotonicity. The price of an option is a decreasing function of time.

$$C_{j+1}^k \leq C_j^k,\ j = 0, ..., N-1;\ k = 1, ..., K. \tag{3–13}$$

- Convexity constraints. The option value is a convex function of the stock price.

$$\begin{aligned} & C_j^{k+1} \leq \beta_j^{k+1} C_j^k + (1 - \beta_j^{k+1}) C_j^{k+2}, \\ & \text{where } \beta_j^{k+1} \text{ is such that } \tilde{S}_j^{k+1} = \beta_j^{k+1} \tilde{S}_j^k + (1 - \beta_j^{k+1}) \tilde{S}_j^{k+2}, \\ & j = 0, ..., N;\ k = 1, ..., K-2. \end{aligned} \tag{3–14}$$

**Constraints on Stock Position for Call Options**

Let us define $\hat{k}$, such that $\tilde{S}^{\hat{k}} \leq X < \tilde{S}^{\hat{k}+1}$.

- Stock position bounds. The stock position value lies between 0 and 1

$$0 \leq U_j^k \leq 1,\ j = 0, ..., N,\ k = 1, ..., K. \tag{3–15}$$

- Vertical monotonicity. The position in the stock is an increasing function of the stock price,

$$U_j^{k+1} \geq U_j^k,\ j = 0, ..., N;\ k = 1, ..., K-1. \tag{3–16}$$

- Horizontal monotonicity. Above the strike price the position in the stock is an increasing function of time; below the strike price it is a decreasing function of time,

$$U_j^k \leq U_{j+1}^k, \text{ if } k > \hat{k}; \; U_j^k \geq U_{j+1}^k, \text{ if } k \leq \hat{k}. \tag{3–17}$$

- Convexity constraints. The position in the stock is a concave function in the stock price above the strike and a convex function in the stock price below the strike,

$$\begin{aligned} &(1-\beta_j^{k+1})U_j^{k+2} + \beta_j^{k+1}U_j^k \leq U_j^{k+1}, \text{ if } k > \hat{k}, \\ &(1-\beta_j^{k-1})U_j^{k-2} + \beta_j^{k-1}U_j^k \geq U_j^{k-1}, \text{ if } k \leq \hat{k}, \\ &\text{where } \beta_j^l \text{ is such that } \tilde{S}_j^l = \beta_j^l \tilde{S}_j^{l-1} + (1-\beta_j^l)\tilde{S}_j^{l+1}, \quad l = (k+1), (k-1). \end{aligned} \tag{3–18}$$

### 3.3.2 Financial Interpretation of the Objective

There are two reasons for considering the average squared error: financial interpretation and accounting for transaction costs. The financial interpretation is discussed here, while the accounting for transaction costs is considered in subsection (3.3.4).

The expected hedging error is an estimate of "non-self-financity" of the hedging strategy. The pricing algorithm seeks a strategy as close as possible to a self-financing one, satisfying the imposed constraints. On the other hand, from a trader's viewpoint, the shortage of money at any portfolio re-balancing point causes the risk associated with the hedging strategy. The average squared error can be viewed as an estimator of this risk on the set of paths considered in the problem.

There are other ways to measure the risk associated with a hedging strategy. For example, Bertsimas et al. (2001) considers a self-financing dynamics of a hedging portfolio and minimizes the squared replication error at expiration. In the context of our framework, different estimators of risk can be used as objective functions in the optimization problem (3–9) and, therefore, produce different results. However, considering other objectives is beyond the scope of this paper.

### 3.3.3 Constraints

We use the value of the hedging portfolio to approximate the value of the option. Therefore, the value of the portfolio is supposed to have the same properties as the value of the option. We incorporated these properties into the model using constraints in the optimization problem. The constraints (3–10)-(3–14) for call options and (3–19)-(3–23) for put options follow under quite general assumptions from the non-arbitrage considerations. The type of the underlying stock price process plays no role in the approach: the set of sample paths (3–5) specifies the behavior of the underlying stock. For this reason, the approach is distribution-free and can be applied to pricing any European option independently of the properties of the underlying stock price process. Also, as shown in section 5 presenting numerical results, the inclusion of constraints to problem (3–9) makes the algorithm quite robust to the size of input data.

The grid structure is convenient for imposing the constraints, since they can be stated as linear inequalities on the grid variables $U_j^k$ and $V_j^k$. An important property of the algorithm is that the number of the variables in problem (3–9) is determined by the size of the grid and is independent of the number of sample paths.

### 3.3.4 Transaction Costs

The explicit consideration of transaction costs is beyond the scope of this paper. We postpone this issue to following papers. However, we implicitly account for transaction costs by requiring the hedging strategy to be "smooth", i.e., by prohibiting significant rebalancing of the portfolio during short periods of time or in response to small changes in the stock price. For call options, we impose the set of constraints (3–16)-(3–18) requiring monotonicity and concavity of the stock position with respect to the stock price and monotonicity of the stock position with respect to time (constraints (3–25)-(3–27) for put options are presented in the next section). The goal is to limit the variability of the stock position with respect to time and stock price, which would lead to smaller transaction costs of implementing a hedging strategy. The minimization of the average squared error is

another source of improving "smoothness" of a hedging strategy with respect to time. The average squared error penalizes all shortages/excesses $a_j^p$ of money along the paths, which tends to flatten the values $a_j^p$ over time. This also improves the "smoothness" of the stock positions with respect to time.

## 3.4 Justification Of Constraints On Option Values And Stock Positions

### 3.4.1 Constraints for Put Options

This subsection presents constraints in optimization problem (3–9) for pricing European put options.

**Constraints on value of Put options.**

- Immediate exercise" constraints.

$$P_j^k \geq \left[Xe^{-r(T-t_j)} - \tilde{S}_j^k\right]^+. \tag{3–19}$$

- Option price sensitivity constraints.

$$\begin{aligned} P_j^k \leq & \gamma_j^k P_j^{k+1} + X(1-\gamma_j^k)e^{-r(T-t_j)},\ \gamma_j^k = \tilde{S}_j^k/\tilde{S}_j^{k+1}, \\ & j = 0, ..., N-1,\ k = 1, ..., K-1. \end{aligned} \tag{3–20}$$

- Monotonicity constraints.

  0. Vertical monotonicity.

$$P_j^k \geq P_j^{k+1},\ j = 0, ..., N;\ k = 1, ..., K-1. \tag{3–21}$$

  0. Horizontal monotonicity.

$$P_{j+1}^k \leq P_j^k + X(e^{-r(T-t_{j+1})} - e^{-r(T-t_j)}),\ j = 0, ..., N-1;\ k = 1, ..., K. \tag{3–22}$$

- Convexity constraints.

$$\begin{aligned} & P_j^{k+1} \leq \beta_j^{k+1} P_j^k + (1-\beta_j^{k+1}) P_j^{k+2} \\ & \text{where } \beta_j^{k+1} \text{ is such that } \tilde{S}_j^{k+1} = \beta_j^{k+1}\tilde{S}_j^k + (1-\beta_j^{k+1})\tilde{S}_j^{k+2} \\ & j = 0, ..., N;\ k = 1, ..., K-2. \end{aligned} \tag{3–23}$$

**Constraints on stock position for put options**

In the following constraints, $\hat{k}$ is such that $\tilde{S}^{\hat{k}} \leq X < \tilde{S}^{\hat{k}+1}$.

- **Stock Position Bounds**

$$0 \leq U_j^k \leq 1,\ j = 0, ..., N;\ k = 1, ..., K. \tag{3–24}$$

- **Vertical monotonicity**

$$U_j^{k+1} \geq U_j^k,\ j = 0, ..., N; k = 1, ..., K-1. \tag{3–25}$$

- **Horizontal monotonicity**

$$U_j^k \leq U_{j+1}^k,\ \text{if } k > \hat{k},\ U_j^k \geq U_{j+1}^k,\ \text{if } k \leq \hat{k} \tag{3–26}$$

- **Convexity constraints**

$$\begin{aligned}
&(1-\beta_j^{k+1})U_j^{k+2} + \beta_j^{k+1}U_j^k \leq U_j^{k+1},\ \text{if } k > \hat{k},\\
&(1-\beta_j^{k-1})U_j^{k-2} + \beta_j^{k-1}U_j^k \geq U_j^{k-1},\ \text{if } k \leq \hat{k},\\
&\text{where } \beta_j^l \text{ is such that } \tilde{S}_j^l = \beta_j^l \tilde{S}_j^{l-1} + (1-\beta_j^l)\tilde{S}_j^{l+1},\\
&l = (k+1), (k-1).
\end{aligned} \tag{3–27}$$

### 3.4.2 Justification of Constraints on Option Values

This subsection proves inequalities on put and call option values under certain assumptions. Properties of option values under various assumptions were thoroughly studied in financial literature. In optimization problem (3–9) we used the following constraints holding for options in quite a general case. We assume non-arbitrage and make technical assumptions 1-5 (used by Merton (1973) for deriving properties of call and put option values. Some of the considered properties of option values are proved by Merton (1973). Other inequalities are proved by the authors.

The rest of the section is organized as follows. First, we formulate and prove inequalities (3–10)-(3–14) for call options. Some of the considered properties of option

values are not included in the constraints of the optimization problem (3–9), they are used in proofs of some of constraints (3–10)-(3–14). In particular, weak and strong scaling properties and two inequalities preceding proofs of option price sensitivity constraints and convexity constraints are not included in the set of constraints.

Second, we consider inequalities (3–19)-(3–23) for put options. We provide proofs of vertical and horizontal option price monotonicity; proofs of other inequalities are similar to those for call options.

We use the following notations. $C(S_t, T, X)$ and $P(S_t, T, X)$ denote prices of call and put options, respectively, with strike $X$, expiration $T$, when the stock price at time $t$ is $S_t$. When appropriate, we use shorter notations $C_t$ and $P_t$ to refer to these options.

Similar to Merton (1973), we make the following assumptions to derive inequalities (3–10)-(3–14) and (3–19)-(3–23).

**Assumption 1.** Current and future interest rates are positive.

**Assumption 2.** No dividends are paid to a stock over the life of the option.

**Assumption 3.** Time homogeneity assumption.

**Assumption 4.** The distributions of the returns per dollar invested in a stock for any period of time is independent of the level of the stock price.

**Assumption 5.** If the returns per dollar on stocks $i$ and $j$ are identically distributed, then the following condition hold. If $S_i = S_j$, $T_i = T_j$, $X_i = X_j$; then $Claim_i(S_i, T_i, X_i) = Claim_j(S_j, T_j, X_j)$, where $Claim_i$ and $Claim_j$ are options (either call or put) on stocks $i$ and $j$ respectively.

Below are the proofs of inequalities (3–10)-(3–14).

1. "Immediate exercise" constraints. Merton (1973)

$$C_t \geq [S_t - X \cdot e^{-r\cdot(T-t)}]^+.$$

□ Put-Call parity, $C_t - P_t + X \cdot e^{-r\cdot(T-t)} = S_t$, and non-negativity of a put option price ($P_t \geq 0$) imply $C_t \geq S_t - Xe^{-r\cdot(T-t)}$. This inequality combined with $C_t \geq 0$ gives $C_t \geq Max(0, S_t - X \cdot e^{-r\cdot(T-t)}) = [S_t - X \cdot e^{-r\cdot(T-t)}]^+$. ■

2. Scaling property.

**a)** Weak scaling property. Merton (1973)

For any $k > 0$ consider two stock price processes $S(t)$ and $k \cdot S(t)$. For these processes, the following inequality is valid $C(k \cdot S_t, T, k \cdot X) = k \cdot C(S_t, T, X)$, where $S_t$ is the value of the process $S(t)$ at time $t$.

□ At expiration $T$, the price of the first stock is $S_T$, the value of the second stock is $k \cdot S_T$. By definition, the values of call options written on the first stock (with strike $X$) and on the second stock (with strike $k \cdot X$) are $C(S_t, T, X) = Max[0, S_T - X]$ and $C(k \cdot S_t, T, k \cdot X) = Max[0, k \cdot S_T - k \cdot X]$, respectively. From $Max[0, k \cdot S_T - k \cdot X] = k \cdot Max[0, S_T - X]$ and non-arbitrage considerations, it follows that $C(k \cdot S_t, T, k \cdot X) = k \cdot C(S_t, T, X)$. ■

**b)** Strong scaling property. Merton (1973)

Under assumptions 4 and 5, the call option price $C(S, T, X)$ is homogeneous of degree one in the stock price per share and exercise price. In other words, if $C(S, T, X)$ and $C(k \cdot S, T, k \cdot X)$ are option prices on stocks with initial prices $S$ and $k \cdot S$ and strikes $X$ and $k \cdot X$, respectively, then $C(k \cdot S, T, k \cdot X) = k \cdot C(S, T, X)$.

□ Consider two stocks with initial prices $S_1$ and $S_2$; define $k = S_2/S_1$. Let $z_i(t)$ be the return per dollar for stock $i$, $i = 1, 2$. Consider two call options, A and B, on stock 2. Option $A$ is written on $1/k$ shares of stock 2 and has strike price $X_1$; option $B$ is written on one share of stock 2 and has strike $X_2 = k \cdot X_1$. Prices $C_2(S_1, T, X_1)$ and $C_2(S_2, T, X_2)$ of these options are related as $C_2(S_2, T, X_2) = C_2(k \cdot S_1, T, k \cdot X_1) = k \cdot C_2(S_1, T, X_1)$, according to the weak scaling property.

Now consider an option $C$ with the strike $X_1$ written on one share of the stock 1. Denote its price by $C_1(S_1, T, X_1)$. Options $A$ and $C$ have equal initial prices $S_1 = \frac{1}{k}S_2$, time to expiration $T$, and $X_1$. Moreover, the distribution of returns per dollar $z_i(t)$ for

stocks $i = 1, 2$ are the same. Hence, from assumption 5, $C_1(S_1, T, X_1) = C_2(S_1, T, X_1)$, and, therefore, $C_2(S_2, T, X_2) = k \cdot C_1(S_1, T, X_1)$, which concludes the proof. ■

3. Option price sensitivity constraints.

**a)** First, we derive an inequality taken from Merton (1973). In part b) we apply it to obtain the sensitivity constraint on the call option price.

For any $X_1$, $X_2$ such that $0 \leq X_1 \leq X_2$, the following inequality holds

$$C(S_t, T, X_1) \leq C(S_t, T, X_2) + (X_2 - X_1) \cdot e^{-r \cdot (T-t)}.$$

□ Consider two portfolios. Portfolio $A$ contains one call option with strike $X_2$ and $(X_2 - X_1) \cdot e^{-r \cdot (T-t)}$ dollars invested in bonds. Portfolio $B$ consists of one call option with strike $X_1$. Both options are written on the stock following the process $S_t$. At expiration, the value of portfolio A is $\max\{0, S_T - X_2\} + X_2 - X_1$, the value of portfolio B is $\max\{0, S_T - X_1\}$. The value of portfolio $A$ is always greater than the value of portfolio $B$ at expiration. This statement with non-arbitrage considerations implies that $C(S_t, T, X_2) + (X_2 - X_1) \cdot e^{r \cdot (T-t)} \geq C(S_t, T, X_1)$. ■

**b)** Consider two options with strike $X$ and initial prices $S_2$ and $S_1$, $S_2 \geq S_1$. Denote $\gamma = S_2/S_1$. The following inequality takes place,

$$C(S_2, T, X) \leq \gamma C(S_1, T, X) + X(\gamma - 1)e^{-r(T-t)}.$$

□ Let $\alpha = \dfrac{1}{\gamma} = \dfrac{S_1}{S_2}$. Using inequality presented in a), we write $C(S_1, T, \alpha X) \leq C(S_1, T, X) + (X - \alpha X)e^{-r(T-t)}$. Applying scaling property to the left-hand side of this inequality yields $C(S_1, T, \alpha X) = C(S_1 \dfrac{S_2}{S_1} \dfrac{S_1}{S_2}, T, \alpha X) = C(S_2 \cdot \alpha, T, \alpha X) = \alpha C(S_2, T, X)$. Therefore,

$$\alpha C(S_2, T, X) \leq C(S_1, T, X) + X(1 - \alpha)e^{-r(T-t)}.$$

Dividing by $\alpha$ and substituting $1/\alpha = \gamma$ we get $C(S_2, T, X) \leq \gamma C(S_1, T, X) + X(\gamma - 1)e^{-r(T-t)}$. ■

4. Vertical option price monotonicity.

For two options with strike $X$ and initial prices $S_1$ and $S_2$, $S_2 \geq S_1$, there holds

$$C(S_1, T, X) \leq \frac{S_1}{S_2} \cdot C(S_2, T, X)\,.$$

□ For any strike $X_1 \leq X$, from non-arbitrage assumptions we have $C(S_1, T, X) \leq C(S_1, T, X_1)$. Applying scaling property to the right-hand side gives

$C(S_1, T, X) \leq \frac{X_1}{X}\, C(S_1 \dfrac{X}{X_1}, T, X)$. By setting $X_1 = \dfrac{S_1}{S_2} X \leq X$, we get $C(S_1, T, X) \leq \frac{S_1}{S_2} \cdot C(S_2, T, X)\,.$ ■

5. Horizontal option price monotonicity.

Let $C(t, S, T, X)$ denote the price of a European call option with initial time $t$, initial price at time $t$ equal to $S$, time to maturity $T$, and strike $X$. Under the assumptions 1, 2 and 3 for any $t$, $u$, $t < u$, the following inequality holds,

$$C(t, S, T, X) \geq C(u, S, T, X).$$

□ Similar to $C(t, S, T, X)$, define $A(t, S, T, X)$ to be the value of American call option with parameters $t$, $S$, $T$, and $X$ meaning the same as in $C(t, S, T, X)$. Time homogeneity assumption 2 implies that two options with different initial times, but equal initial and strike prices and times to maturity should have equal prices: $A(t, S, T, X) = A(u, S, T + u - t, X)$. On the other hand, non-arbitrage considerations imply $A(u, S, T + u - t, X) \geq A(u, S, T, X)$. Combining the two inequalities yields $A(t, S, T, X) \geq A(u, S, T, X)$. Since the value of an American call option is equal to the value of the European call option under assumption 1, the above inequality also holds for European options: $C(t, S, T, X) \geq C(u, S, T, X)$. ■

6. Convexity. Merton (1973).

**a)** $C$ is a convex function of its exercise price: for any $X_1 > 0$, $X_2 > 0$ and $\lambda \in [0, 1]$

$$C(S, T, \lambda \cdot X_1 + (1 - \lambda) \cdot X_2) \leq \lambda \cdot C(S, T, X_1) + (1 - \lambda) \cdot C(S, T, X_2).$$

□ Consider two portfolios. Portfolio $A$ consists of $\lambda$ options with strike $X_1$ and $(1-\lambda)$ options with strike $X_2$; portfolio $B$ consists of one option with strike $\lambda \cdot X_1 + (1-\lambda) \cdot X_2$. Convexity of function $\max\{0, x\}$ implies that the value of portfolio $A$ at expiration in no less than the value of portfolio $B$ at expiration. $\lambda \max\{0, S_T - X_1\} + (1-\lambda) \max\{0, S_T - X_2\} \geq \max\{0, S_T - (\lambda \cdot X_1 + (1-\lambda) \cdot X_2)\}$. Hence, from non-arbitrage assumptions, portfolio $A$ costs no less than portfolio $B$: $\lambda \cdot C(S, T, X_1) + (1-\lambda) \cdot C(S, T, X_2) \geq C(S, T, \lambda \cdot X_1 + (1-\lambda) \cdot X_2)$. ■

**b)** Under the assumption 4, option price $C(S, T, X)$ is a convex function of the stock price: for any $S_1 > 0$, $S_2 > 0$ and $\lambda \in [0, 1]$ there holds,

$$C(\lambda \cdot S_1 + (1-\lambda) \cdot S_2, T, X) \leq \lambda \cdot C(S_1, T, X) + (1-\lambda) \cdot C(S_2, T, X).$$

□ Denote $S_3 = \lambda S_1 + (1-\lambda) S_2$. Choose $X_1$, $X_2$ and $\alpha$ such that $X_1 = X/S_1$, $X_2 = X/S_2$, $\alpha = \lambda S_1 / S_3 \in [0, 1]$, and denote $X_3 = \alpha X_1 + (1-\alpha) X_2$.

Consider an inequality $C(1, T, X_3) \leq \alpha \cdot C(1, T, X_1) + (1-\alpha) \cdot C(1, T, X_2)$ following from convexity of option price with respect to the strike price (proved in a) ). Since

$$\alpha S_3 = \lambda S_1, \ (1-\alpha) S_3 = \left(1 - \frac{\lambda S_1}{S_3}\right) S_3 = S_3 - \lambda S_1 = (1-\lambda) S_2, \tag{3–28}$$

multiplying both sides of the previous inequality by $S_3$ gives $S_3 \cdot C(1, T, X_3) \leq \lambda \cdot S_1 \cdot C(1, T, X_1) + (1-\lambda) \cdot S_2 \cdot C(1, T, X_2)$. Further, using the weak scaling property, we get $C(S_3, T, S_3 \cdot X_3) \leq \lambda \cdot C(S_1, T, S_1 \cdot X_1) + (1-\lambda) \cdot C(S_2, T, S_2 \cdot X_2)$. Using definitions of $X_1$ and $X_2$ and expanding $S_3 X_3$ as

$$S_3(\alpha X_1 + (1-\alpha) X_2) = S_3 X \left( \frac{\alpha}{S_1} + \frac{1-\alpha}{S_2} \right) =$$

$$= S_3 X \left( \frac{\lambda S_1}{S_3} \frac{1}{S_1} + \frac{S_3 - \lambda S_1}{S_3} \frac{1}{S_2} \right) = S_3 X \left( \frac{\lambda}{S_3} + \frac{1-\lambda}{S_3} \right) = X,$$

we arrive at $C(S_3, T, X) \leq \lambda \cdot C(S_1, T, X) + (1 - \lambda) \cdot C(S_2, T, X)$, as needed. ■

Constraints on European put option values are presented below. We state them in the same order as the constraints for call options. Proofs are given for vertical option price monotonicity constraints; other inequalities can be proved using put-call parity and considerations similar to those in the proofs of corresponding inequalities for call options.

1. "Immediate exercise" constraints.

$$P_t \geq [X \cdot e^{-r \cdot (T-t)} - S_t]^+.$$

2. Scaling property.

**a)** Weak scaling property.

For any $k > 0$, consider two stock price processes $S(t)$ and $k \cdot S(t)$. For these processes the following inequality holds: $P_1(k \cdot S_t, T, k \cdot X) = k \cdot P_2(S_t, T, X)$, where $P_1$ and $P_2$ are options on the first and the second stocks respectively.

**b)** Strong scaling property.

Under the assumptions 4 and 5, put option value $P(S, T, X)$ is homogeneous of degree one in the stock price and the strike price, i.e., for any $k > 0$, $P(k \cdot S, T, k \cdot X) = k \cdot P(S, T, X)$.

3. Option price sensitivity constraints.

**a)** For any $X_1$, $X_2$, $0 \leq X_1 \leq X_2$, the following inequality is valid,

$$P(S_t, T, X_2) \leq P(S_t, T, X_1) + (X_2 - X_1) \cdot e^{r \cdot (T-t)}.$$

**b)** For initial stock prices $S_1$ and $S_2$, $S_1 \leq S_2$

$$P(S_1, T, X) \leq \gamma P(S_2, T, X) + X(1 - \gamma) e^{-r(T-t)},$$

where $\gamma = S_1/S_2$.

4. Vertical option price monotonicity.

**a)** For any $\alpha \in [0,1]$ the following inequality is valid:

$$P(S, X \cdot \alpha) \leq \alpha \cdot P(S, X).$$

$\square$ Consider portfolio $A$ consisting of one option with strike $\alpha \cdot X$, and portfolio $B$ consisting of $\alpha$ options with strike $X$. We need to show that portfolio $B$ always outperforms portfolio $A$. This follows from non-arbitrage consideration since at expiration the value of portfolio $B$ is greater or equal to the value of portfolio $A$: $[X \cdot \alpha - S_T]^+ \leq \alpha \cdot [X - S_T]^+,\ 0 < \alpha < 1$. $\blacksquare$

**b)** For any $S_1$, $S_2$, $S_1 \leq S_2$, there holds $P(S_2, T, X) \leq P(S_1, T, X)$.

$\square$ Consider an inequality $P(S_1, \alpha X) \leq \alpha P(S_1, X),\ 0 < \alpha < 1$, proved above. Set $\alpha = S_1/S_2 \in [0,1]$. Applying the weak scaling property, we get

$$P(S_1 \frac{1}{\alpha}\alpha, T, \alpha X) \leq \alpha P(S_1, T, X),$$

$$P(S_1 \frac{1}{\alpha}, T, X) \leq P(S_1, T, X),$$

$$P(S_2, T, X) \leq P(S_1, T, X). \blacksquare$$

5. Horizontal option price monotonicity.

Under assumptions 1, 2, and 3, for any initial times $t$ and $u$, $t < u$, the following inequality is valid:

$$P(t, S, T, X) \geq P(u, S, T, X) + X \cdot (e^{-r \cdot (T-t)} - e^{-r \cdot (T-u)}),$$

where $P(\tau, S, T, X)$ is the price of a European put option with initial price $\tau$, initial price at time $\tau$ equal to $S$, time to maturity $T$, and strike $X$.

6. Convexity.

**a)** $P(S, T, X)$ is a convex function of its exercise price $X$

**b)** Under assumption 4, $P(S, T, X)$ is a convex function of the stock price.

### 3.4.3 Justification of Constraints on Stock Position

This subsection proves/validates inequalities (3–15)-(3–18) and (3–24)-(3–27) on the stock position. Stock position bounds and vertical monotonicity are proven in the general case (i.e. under assumptions 1-5 and the non-arbitrage assumption); horizontal monotonicity and convexity are justified under the assumption that the stock process follows the geometric Brownian motion.

The notation $C(S,T,X)$ ($P(S,T,X)$) stands for the price of a call (put) option with the initial price $S$, time to expiration $T$, and the strike price $X$. The corresponding position in the stock (for both call and put options) is denoted by $U(S,T,X)$.

First, we present the proofs of inequalities (3–15)-(3–18) for call options.

1. Vertical monotonicity (Call options).

$U(S,t,X)$ is an increasing function of $S$.

□ This property immediately follows from convexity of the call option price with respect to the stock price, (property 6(b) for call options). ■

2. Stock position bounds (Call options).

$$0 \leq U(S,T,X) \leq 1$$

□ Since the option price $C(S,t,X)$ is an increasing function of the stock price $S$, it follows that $U(S,t,X) = C'_s(S,t,X) \geq 0$.

Now we need to prove that $U(S,t,X) \leq 1$. We will assume that there exists such $S^*$ that $C'_s(S^*) \geq \alpha$ for some $\alpha > 1$ and will show that this assumption contradicts the ineqiality[3] $C(S,t,X) \leq S$.

[3] This inequality can be proven by considering a portfolio consisting of one stock and one shorted call option on this stock. At expiration, the portfolio value is $S_T - \max\{0, S_T - X\} \geq 0$ for any $S_T$ and $X \geq 0$. Non-arbitrage assumption implies that $S \geq C(S,t,X)$.

Since $U(S,t,X)$ increases with $S$, for any $S \geq S^*$ we have $U(S,t,X) \geq \alpha$, $\int_{S^*}^{S} U(s,t,X)ds \geq \int_{S^*}^{S} \alpha ds$, $C(S,t,X) - C(S^*,t,X) \geq \alpha S - \alpha S^*$, $C(S,T,x) \geq C(S^*,t,X) - \alpha S^* + \alpha S$, $C(S,t,X) \geq S + (\alpha - 1)S + C(S^*,t,X) - \alpha S^*$.

Let $f(s) = (\alpha - 1)s + C(S^*) - \alpha S^*$. Since $(\alpha - 1) > 0$, there exists such $S_1 > S^*$ that $f(S_1) > 0$. This implies $C(S_1,t,X) > S_1$ which contradicts inequality $C(S,t,X) \leq S$. ■

The previous inequalities were justified in a quite general setting of assumptions 1-5 and a non-arbitrage assumption. We did not manage to prove the following two groups of inequalities (horizontal monotonicity and convexity) in this general setting. The proofs will be provided in further papers. However, here we present proofs of these inequalities in the Black-Scholes setting.

3) *Horizontal monotonicity (Call options)*

$U(S,t,X)$ is an increasing function of $t$ when $S \geq X$,

$U(S,t,X)$ is a decreasing function of $t$ when $S < X$.

□ We will validate these inequalities by analyzing the Black-Scholes formula and calculating the areas of horizontal monotonicities for the options used in the case study. The Black-Scholes formula for the price of a call option is

$$C(S,T,X) = S\,N(d_1) - Xe^{rT}N(-d_2),$$

where $S$ is the stock price, $T$ is time to maturity, $r$ is a risk-free rate, $\sigma$ is the volatility,

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}} dZ, \tag{3–29}$$

and $d_1$ and $d_2$ are given by expressions

$$d_1 = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{Se^{rT}}{X}\right) + \frac{1}{2}\sigma\sqrt{T},$$

$$d_2 = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{Se^{rT}}{X}\right) - \frac{1}{2}\sigma\sqrt{T}.$$

Taking partial derivatives of $C(S,T,X)$ with respect to $S$ and $t$, we obtain

$$C'_s(S,T,X) = U(S,T,X) = (S,T,X) = N(d_1),$$

$$C''_{st}(S,T,X) = U'_t(S,T,X) = \frac{\exp\left\{-\frac{\left(T(r+\frac{\sigma^2}{2})+\ln\left(\frac{S}{X}\right)\right)^2}{2T\sigma^2}\right\}\left(-T(2r+\sigma^2)+2\ln\left(\frac{S}{X}\right)\right)}{4\sqrt{2\pi}\,T^{\frac{3}{2}}\sigma}.$$

The sign of $U'_t(S,T,X)$ is determined by the sign of the expression $F(S) = -T(2r+\sigma^2) + 2\ln\left(\frac{S}{X}\right)$. $F(S) \geq 0$ (implying $U'_t(S,T,X) \geq 0$) when $S \geq L$ and $F(S) \leq 0$ (implying $U'_t(S,T,X) \leq 0$) when $S \leq L$, where $L = X \cdot e^{T(r+\sigma^2/2)}$.

For the values of $r = 10\%$, $\sigma = 31\%$, $T = 49$ days $L$ differs from $X$ less than 2.5%. For all options considered in the case study the value of implied volatility did not exceed 31% and the corresponding value of $L$ differs from the stike price less than 2.5%. Taking into account resolution of the grid, we consider the approximation of $L$ by $X$ in the horizonal monotonicity constraints to be reasonable. ■

4) Convexity (Call options).

$U(S,t,X)$ is a concave function of $S$ when $S \geq X$,

$U(S,t,X)$ is a convex function of $S$ when $S < X$.

□ We used MATHEMATICA to find the second derivative of the Black-Scholes option price with respect to the stock price ($U''_{SS}(S,t,X)$). The expression of the second derivative is quite involved and we do not present it here. It can be seen that $U''_{SS}(S,t,X)$ as a function of $S$ has an inflexion point. Above this point $U(S,t,X)$ is concave with respect to $S$ and below this point $U(S,t,X)$ is convex with respect to $S$. We calculated inflexion points for some options and presented the results in the Table (3-7).

The Error(%) column contains errors of approximating inflexion points by strike prices. These errors do not exceed 3% for a broad range of parameters. We conclude that inflexion points can be approximated by strike prices for options considered in the case study. ■

Next, we justify the constraints (3–24)-(3–27) for put options.

1. Vertical monotonicity (Put options).

$U(S,t,X)$ is an increasing function of $S$.

□ This property immediately follows from convexity of the put option price with respect to the stock price (property 6(b) for put options). ■

2. Stock position bounds (Put options).

$$-1 \leq U(S,T,X) \leq 0$$

□ Taking derivative of the put-call parity $C(S,T,X) - P(S,T,X) + X \cdot e^{-rT} = S$ with respect to the stock price $S$ yields $C'_s(S,T,X) - P'_s(S,T,X) = 1$. This equality together with $0 \leq C'_s(S,T,X) \leq 1$ implies $-1 \leq P'_s(S,T,X) \leq 0$, which concludes the proof. ■

3) Horizontal monotonicity (Put options).

$U(S,t,X)$ is an increasing function of $t$ when $S \geq X$,

$U(S,t,X)$ is a decreasing function of $t$ when $S < X$.

□ Taking the derivatives with respect to $S$ and $T$ of the put-call parity yields $C''_{st}(S,T,X) = P''_{st}(S,T,X)$. Therefore, the horizontal monotonic properties of $U(S,T,X)$ for put options are the same as the ones for call options. ■

4) Convexity (Put options).

$U(S,t,X)$ is a concave function of $S$ when $S \geq X$,

$U(S,t,X)$ is a convex function of $S$ when $S < X$.

□ Put-call parity implies that $C''_{SS}(S,T,X) = P''_{SS}(S,T,X)$. Therefore, the convexity of put options is the same as the convexity of call options. ■

### 3.5 Case Study

This section present the results of two numerical tests of the algorithm. First, we price European options on the stock following the geometric Brownian motion and compare the results with prices obtained with the Black-Scholes formula. Second, we price European options on S&P 500 index (ticker SPX) and compare the results with actual market prices.

Tables 3-1, 3-3, and 3-4 report "relative" values of strikes and option prices, i.e. strikes and prices divided by the initial stock price. Prices of options are also given in the implied volatility format, i.e., for actual and calculated prices we found the volatility implied by the Black-Scholes formula.

### 3.5.1 Pricing European options on the stock following the geometric brownian motion

We used a Monte-Carlo simulation to create 200 sample paths of the stock process following the geometric brownian motion with drift 10% and volatility 20%. The initial stock price is set to \$ 62; time to maturity is 69 days. Calculations are made for 10 values of the strike price, varying from \$ 54 to \$ 71. The calculated results and Black-Scholes prices for European call options are presented in Table 3-1.

Table 1 shows quite reasonable performance of the algorithm: the errors in the price (Err(%), Table 3-1) are less than 2% for most of calculated put and call options.

Also, it can be seen that the volatility is quite flat for both call and put options. The error of implied volatility does not exceed 2% for most call and put options (Vol.Err(%), Table 3-1). The volatility error slightly increases for out-of-the-money puts and in-the-money calls.

### 3.5.2 Pricing European options on S&P 500 Index

The set of options used to test the algorithm is given in Table 3-2. The actual market price of an option is assumed to be the average of its bid and ask prices. The price of the S&P 500 index was modelled by historical sample-paths. Non-overlapping paths of the index were taken from the historical data set and normalized such that all paths have the same initial price $S_0$. Then, the set of paths was "massaged" to change the spread of paths until the option with the closest to at-the-money strike is priced correctly. This set of paths with the adjusted volatility was used to price options with the remaining strikes.

Table 3-3 displays the results of pricing using 100 historical sample-paths. The pricing error (see Err(%), Table 3-3) is around 1.0% for all call and put options and increases

for out-of-the-money options. Errors of implied volatility follow similar patterns: errors are of the order of 1% for all options except for deep out-of-the-money options. For deep in-the-money options the volatility error also slightly increases.

### 3.5.3 Discussion of Results

Calculation results validate the algorithm. A very attractive feature of the algorithm is that it can be successfully applied to pricing options when a small number of sample-paths is available. (Table 3-4 shows that in-the-money S&P 500 index options can be priced quite accurately with 20 sample-paths.) At the same time, the method is flexible enough to take advantage of specific features of historical sample-paths. When applied to S&P 500 index options, the algorithm was able to match the volatility smile reasonably well (Figures 3.6, 3.6). At the same time, the implied volatility of options calculated in the Black-Scholes setting is reasonably flat (Figures 3.6, 3.6). Therefore, one can conclude that the information causing the volatility smile is contained in the historical sample-paths. This observation is in accordance with the prior known fact that the non-normality of asset price distribution is one of causes of the volatility smile.

Figures 3.6, 3.6, 3.6, and 3.6 present distributions of total external financing ($\sum_{j=1}^{N} a_j^p e^{-rj}$) on sample paths and distributions of discounted money inflows/outflows ($a_j^p e^{-rj}$) at re-balancing points for Black-Scholes and SPX call options. We summarize statistical properties of these distributions in Table (3-5).

Figures 3.6, 3.6, 3.6, and 3.6 also show that the obtained prices satisfy the non-arbitrage condition. With respect to pricing a single option, the non-arbitrage condition is understood in the following sense. If the initial value of the hedging portfolio is considered as a price of the option, then at expiration the corresponding hedging strategy should outperform the option payoff on some sample paths, and underperform the option payoff on some other sample paths. Otherwise, the free money can be obtained by shorting the option and buying the hedging portfolio or vise versa. The algorithm produces the price of the option satisfying the non-arbitrage condition in this sense. The value of external

financing on average is equal to zero over all paths. The construction of the squared error implies that the hedging strategy delivers less money than the option payoff on some paths and more money that the option payoff on other paths. This ensures that the obtained price satisfies the non-arbitrage condition.

The pricing problem is reduced to quadratic programming, which is quite efficient from the computational standpoint. For the grid consisting of $P$ rows (the stock price axis) and $N$ columns (the time axis), the number of variables in the problem (3–9) is $2PN$ and the number of constraints is $O(NK)$, regardless of the number of sample paths. Table 3-6 presents calculation times for different sizes of the grid with CPLEX 9.0 quadratic programming solver on Pentium 4, 1.7GHz, 1GB RAM computer.

In order to compare our algorithm with existing pricing methods, we need to consider options pricing from the practical perspective. Pricing of actually traded options includes three steps.

**Step 1: Choosing stock process and calibration.** The market data is analyzed and an appropriate stock process is selected to fit actually observed historical prices. The stock process is calibrated with currently observed market parameters (such as implied volatility) and historically observed parameters (such as historical volatility).

**Step 2: Options pricing.** The calibrated stock process is used to price options. Analytical methods, Monte-Carlo simulation, and other methods are usually used for pricing.

**Step 3: Back-testing.** The model performance is verified on historical data. The hedging strategy, implied by the model, is implemented on historical paths.

Most commonly used approach for practical pricing of options is time continuous methods with a specific underlying stock process (Black-Scholes model, stochastic volatility model, jump-diffusion model, etc). We will refer to these methods as process-specific methods. In order to judge the advantages of the proposed algorithm against the process-specific methods, we should compare them step by step.

**Comparison at step 1.** Choosing the model may entail modelling error. For example, stocks are approximately follow the geometric Brownian motion. However, the Black-Scholes prices of options would fail to reproduce the market volatility smile.

Our algorithm does not rely on some specific model and does not have errors related to the choice of the specific process. Also, we have realistic assumptions, such as discrete trading, non-self-financing hedging strategy, and possibility to introduce transaction costs (this feature is not directly presented in the paper).

Calibration of process-specific methods usually require a small amount of market data. Our algorithm competes well in this respect. We impose constraints reducing feasible set of hedging strategies, which allows pricing with very small number of sample paths.

**Comparison at step 2.** If the price process is identified correctly, the process-specific methods may provide an accurate pricing. Our algorithm may not have any advantages in such cases. However, the advantage of our algorithm may be significant if the price process cannot be clearly identified and the use of the process-specific methods would contain a significant modelling error.

**Comparison at step 3.** To perform back-testing, the hedging strategy, implied by a pricing method, is implemented on historical price paths. The back-testing hedging error is a measure of practical usefulness of the algorithm.

The major advantage of our algorithm is that the errors of back-testing in our case can be much lower than the errors of process-specific methods. The reason being, the minimization of the back-testing error on historical paths is the objective in our algorithm. Minimization of the squared error on historical paths ensures that the need of additional financing to practically hedge the option is the lowest possible. None of the process-specific methods possess this property.

### 3.6 Conclusions and Future Research

We presented an approach to pricing European options in incomplete markets. The pricing problem is reduced to minimization of the expected quadratic error subject to constraints. To price an option we solve the quadratic programming problem and find a hedging strategy minimizing the risk associated with it. The hedging strategy is modelled by two matrices representing the stock and the bond positions in the portfolio depending upon time and the stock price. The constraints on the option value impose the properties of the option value following from general non-arbitrage considerations. The constraints on the stock position incorporate requirements on "smoothness" of the hedging strategy. We tested the approach with options on the stock following the geometric Brownian motion and with actual market prices for S&P 500 index options.

This paper is the first in the series of papers devoted to implementation of the developed algorithm to various types of options. Our target is pricing American-style and exotic options and treatment actual market conditions such as transaction costs, slippage of hedging positions, hedging options with multiple instruments and other issues. In this paper we established basics of the method; the subsequent papers will concentrate on more complex cases.

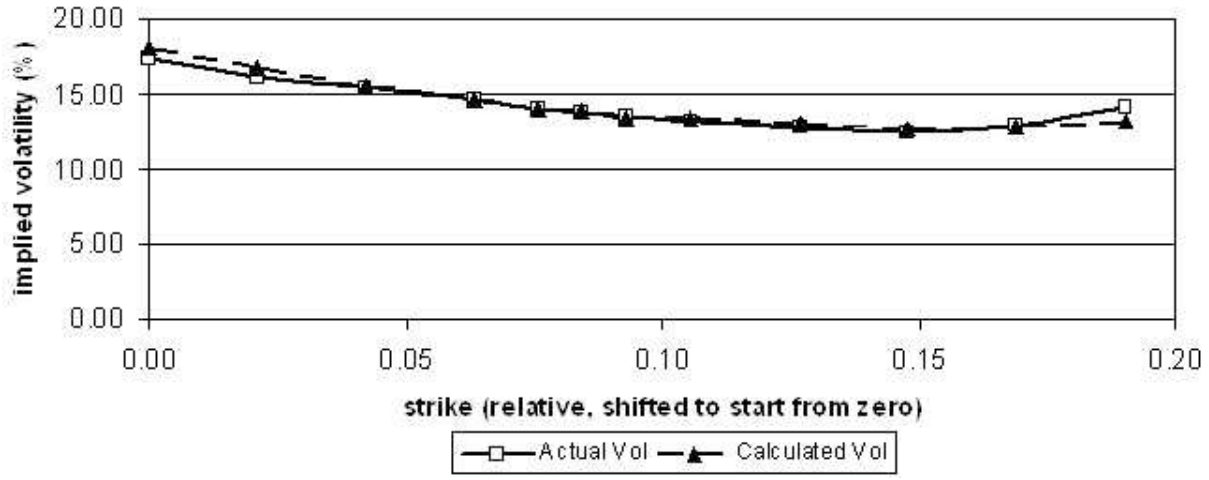


Figure 3-1. Implied volatility vs. strike: Call options on S&P 500 index priced using 100 sample paths. Based on prices in columns *Calc. Vol(%)* and *Act. Vol(%)* of Table 3-3.
Calculated Vol(%) = implied volatility of calculated options prices (100 sample-paths), Actual Vol(%) = implied volatility of market options prices, strike price is shifted left by the value of the lowest strike.

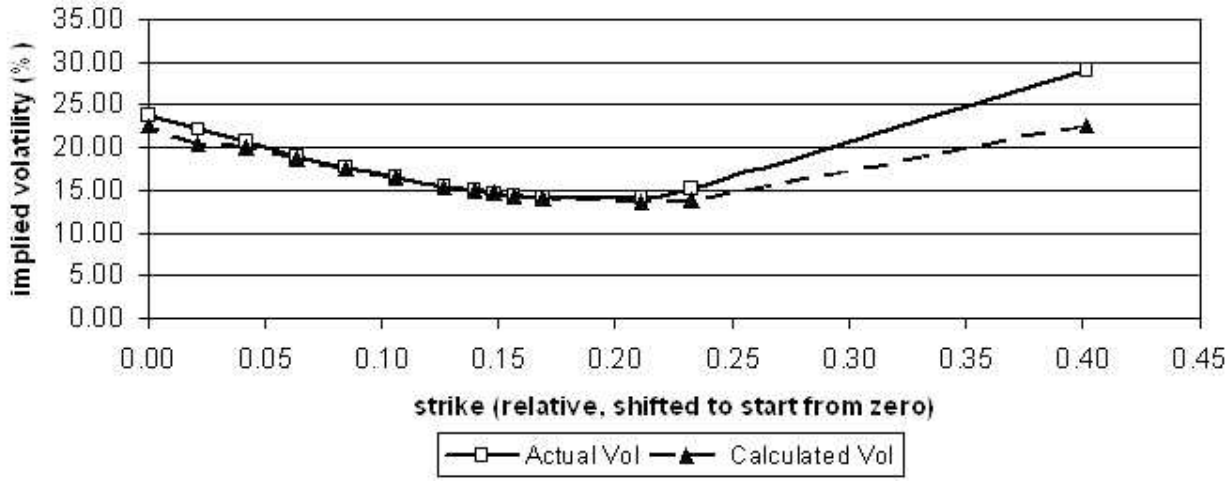


Figure 3-2. Implied volatility vs. strike: Put options on S&P 500 index priced using 100 sample paths. Based on prices in columns *Calc. Vol(%)* and *Act. Vol(%)* of Table 3-3.
Calculated Vol(%) = implied volatility of calculated options prices (100 sample-paths), Actual Vol(%) = implied volatility of market options prices, strike price is shifted left by the value of the lowest strike.

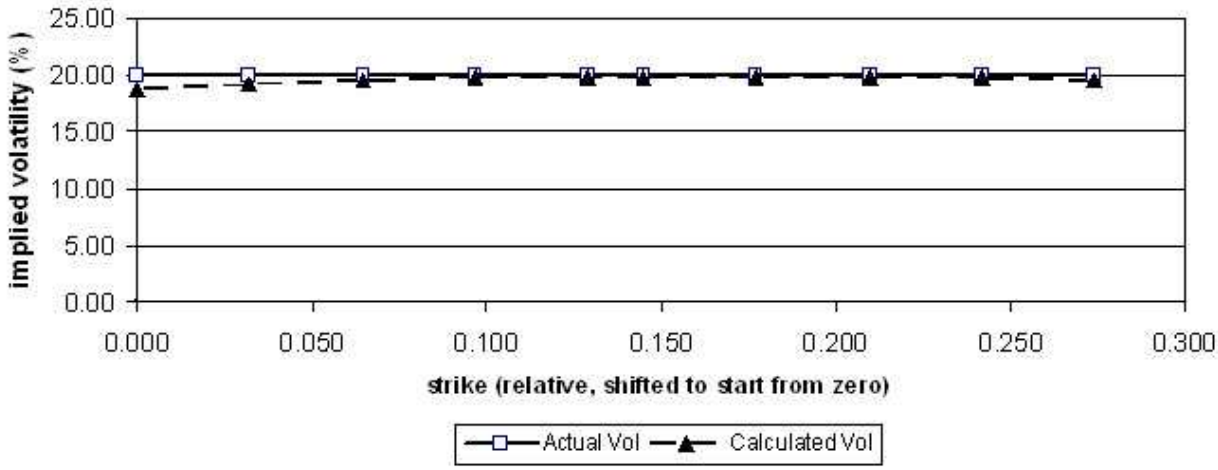


Figure 3-3. Implied volatility vs. strike: Call options in Black-Scholes setting priced using 200 sample paths. Based on prices in columns *Calc.Vol(%)* and *B-S.Vol(%)* of Table 3-1.
Calculated Vol(%) = implied volatility of calculated options prices (200 sample-paths), Actual Vol(%) = flat volatility implied by Black-Scholes formula, strike price is shifted left by the value of the lowest strike.

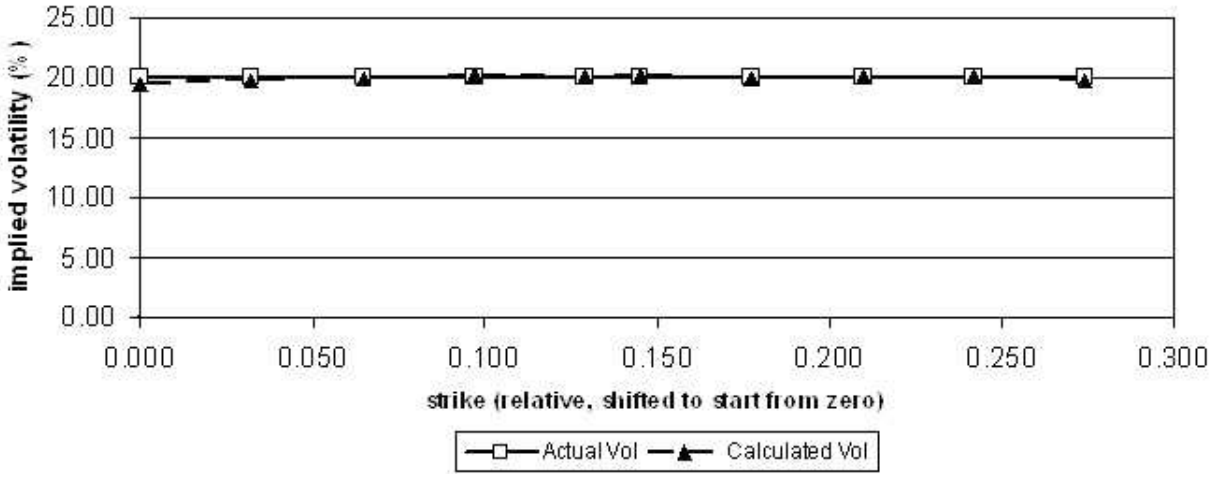


Figure 3-4. Implied volatility vs. strike: Put options in Black-Scholes setting priced using 200 sample paths. Based on prices in columns *Calc.Vol(%)* and *B-S.Vol(%)* of Table 3-1.
Calculated Vol(%) = implied volatility of calculated options prices (200 sample-paths), Actual Vol(%) = flat volatility implied by Black-Scholes formula, strike price is shifted left by the value of the lowest strike.

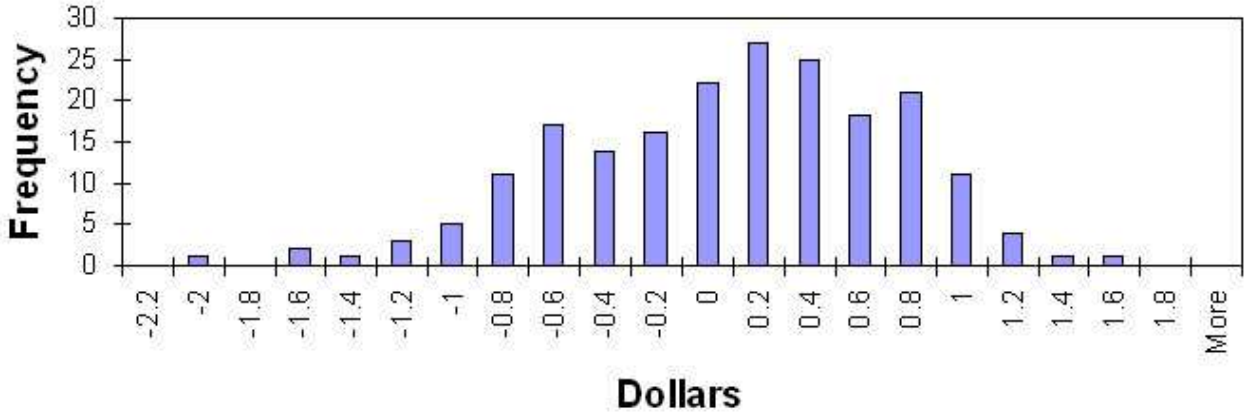


Figure 3-5. Black-Scholes call option: distribution of the total external financing on sample paths.
Initial price=$62, strike=$62 time to expiration=70, risk-free rate=10%, volatility=20%.
Stock price is modelled with 200 Monte-Carlo sample paths.

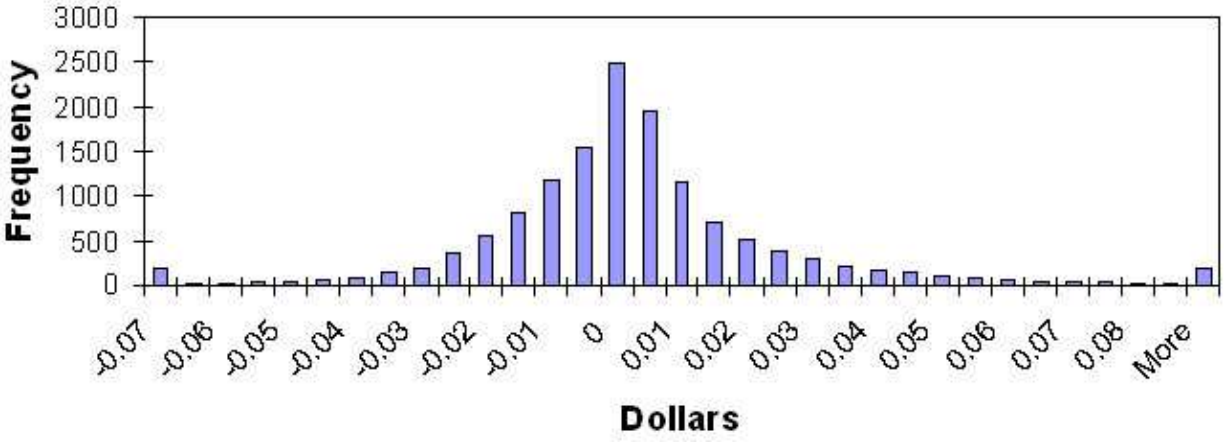


Figure 3-6. Black-Scholes call option: distribution of discounted inflows/outflows at re-balancing points.
Initial price=$62, strike=$62 time to expiration=70, risk-free rate=10%, volatility=20%.
Stock price is modelled with 200 Monte-Carlo sample paths.

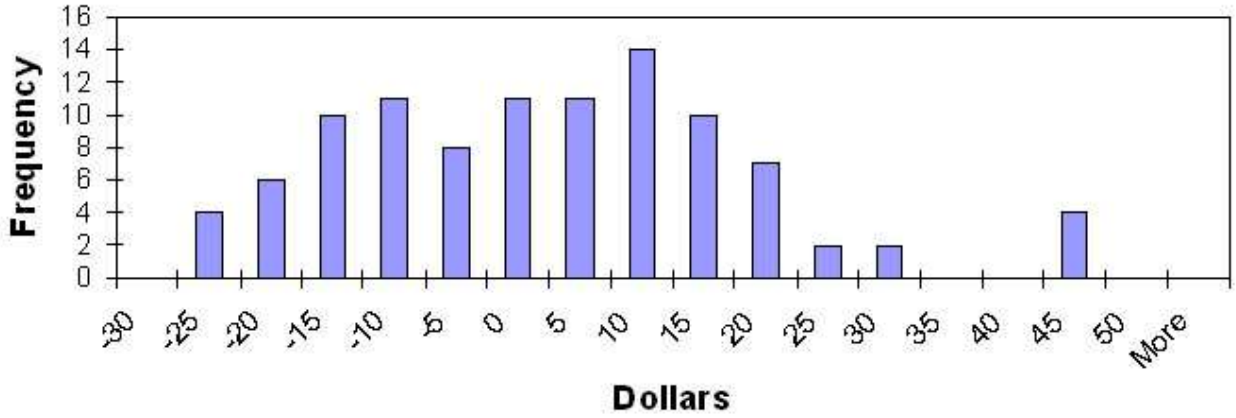


Figure 3-7. SPX call option: distribution of the total external financing on sample paths.
Initial price=$1183.77, strike price=$1190 time to expiration=49 days,
risk-free rate=2.3%.
Stock price is modelled with 100 sample paths.

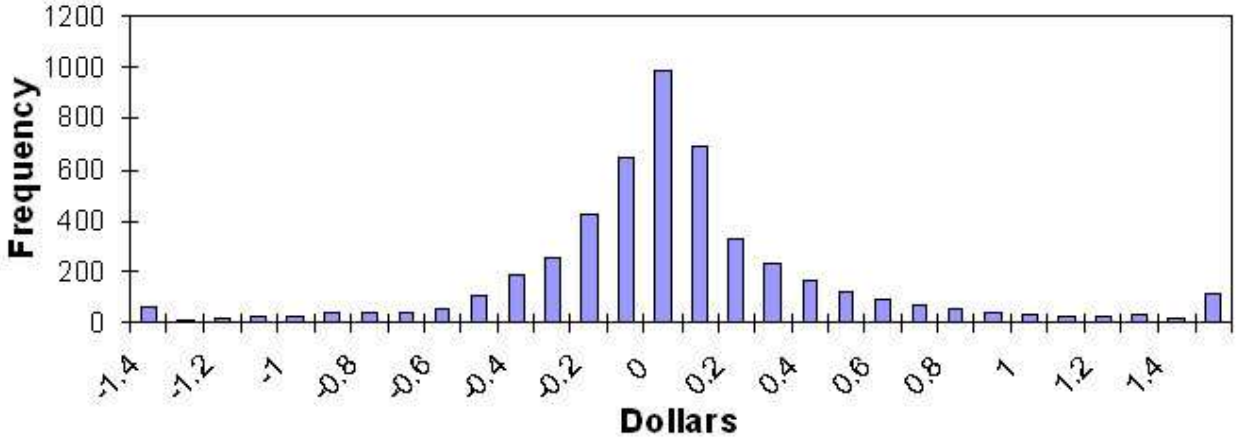


Figure 3-8. SPX call option: distribution of discounted inflows/outflows at re-balancing points.
Initial price=$1183.77, strike price=$1190 time to expiration=49 days,
risk-free rate=2.3%.
Stock price is modelled with 100 sample paths.

Table 3-1. Prices of options on the stock following the geometric Brownian motion: calculated versus Black-Scholes prices.

| Strike | Calc. | B-S | Err(%) | Calc.Vol.(%) | B-S.Vol.(%) | Vol.Err(%) |
|---|---|---|---|---|---|---|
| | | | Call options | | | |
| 1.145 | 0.0037 | 0.0038 | -3.78 | 19.63 | 20.00 | -1.86 |
| 1.113 | 0.0075 | 0.0074 | 1.35 | 19.91 | 20.00 | -0.46 |
| 1.081 | 0.0134 | 0.0133 | 0.65 | 19.87 | 20.00 | -0.65 |
| 1.048 | 0.0226 | 0.0227 | -0.04 | 19.79 | 20.00 | -1.04 |
| 1.016 | 0.0364 | 0.0361 | 0.80 | 19.94 | 20.00 | -0.28 |
| 1.000 | 0.0446 | 0.0445 | 0.19 | 19.82 | 20.00 | -0.92 |
| 0.968 | 0.0651 | 0.0648 | 0.47 | 19.94 | 20.00 | -0.31 |
| 0.935 | 0.0891 | 0.0892 | -0.08 | 19.59 | 20.00 | -2.07 |
| 0.903 | 0.1166 | 0.1168 | -0.11 | 19.29 | 20.00 | -3.56 |
| 0.871 | 0.1464 | 0.1465 | -0.07 | 18.71 | 20.00 | -6.44 |
| | | | Put options | | | |
| 1.145 | 0.1274 | 0.1276 | -0.16 | 19.73 | 20.00 | -1.36 |
| 1.113 | 0.0995 | 0.0994 | 0.04 | 20.03 | 20.00 | 0.17 |
| 1.081 | 0.0738 | 0.0738 | 0.05 | 20.02 | 20.00 | 0.12 |
| 1.048 | 0.0514 | 0.0514 | -0.10 | 19.97 | 20.00 | -0.16 |
| 1.016 | 0.0334 | 0.0332 | 0.71 | 20.14 | 20.00 | 0.68 |
| 1.000 | 0.0258 | 0.0258 | 0.15 | 20.02 | 20.00 | 0.11 |
| 0.968 | 0.0147 | 0.0144 | 1.82 | 20.19 | 20.00 | 0.93 |
| 0.935 | 0.0070 | 0.0071 | -1.60 | 19.89 | 20.00 | -0.56 |
| 0.903 | 0.0029 | 0.0031 | -5.77 | 19.71 | 20.00 | -1.45 |
| 0.871 | 0.0010 | 0.0011 | -12.88 | 19.52 | 20.00 | -2.41 |

Initial price=\$62, time to expiration=69 days, risk-free rate=10%, volatility=20%, 200 sample paths generated by Monte-Carlo simulation.
Strike(\$)=option strike price, Calc.=obtained option price (relative), BS=Black-Scholes option price (relative), Err=$(Found - BS)/BS$, Calc.Vol.=obtained option price in volatility form, BS.Vol.(%)=Black-Scholes volatility,
Vol.Err(%)=$(Calc.Vol. - BS.Vol.)/BS.Vol.$

Table 3-2. S&P 500 ptions data set.

| Strike | Bid | Ask | Price | Rel.Pr | Strike | Bid | Ask | Price | Rel.Pr |
|---|---|---|---|---|---|---|---|---|---|
| | | Call options | | | | | Put options | | |
| 1500 | N/A | 0.5 | N/A | N/A | 1500 | 311.3 | 313.3 | 312.3 | 0.2638 |
| 1325 | 0.3 | 0.5 | 0.4 | 0.0003 | 1300 | 112.7 | 114.7 | 113.7 | 0.0960 |
| 1300 | 0.45 | 0.8 | 0.625 | 0.0005 | 1275 | 88.8 | 90.8 | 89.8 | 0.0759 |
| 1275 | 1.15 | 1.65 | 1.4 | 0.0012 | 1225 | 46.9 | 48.9 | 47.9 | 0.0405 |
| 1250 | 3.7 | 4.2 | 3.95 | 0.0033 | 1210 | 36.9 | 38.9 | 37.9 | 0.0320 |
| 1225 | 8.6 | 9.6 | 9.1 | 0.0077 | 1200 | 31.0 | 33.0 | 32.0 | 0.0270 |
| 1210 | 13.2 | 14.8 | 14.0 | 0.0118 | 1190 | 26.1 | 28.1 | 27.1 | 0.0229 |
| 1200 | 17.5 | 18.9 | 18.2 | 0.0154 | 1175 | 19.8 | 21.4 | 20.6 | 0.0174 |
| 1190 | 22.1 | 24.1 | 23.1 | 0.0195 | 1150 | 12.5 | 14.0 | 13.25 | 0.0112 |
| 1175 | 30.8 | 32.8 | 31.8 | 0.0269 | 1125 | 8.0 | 9.0 | 8.5 | 0.0072 |
| 1150 | 48.0 | 50.0 | 49.0 | 0.0414 | 1100 | 5.1 | 5.9 | 5.5 | 0.0046 |
| 1125 | 68.3 | 69.5 | 68.9 | 0.0582 | 1075 | 3.3 | 4.1 | 3.7 | 0.0031 |
| 1100 | 90.2 | 92.2 | 91.2 | 0.0770 | 1050 | 2.2 | 3.0 | 2.6 | 0.0022 |
| 500 | 682.1 | 684.1 | 683.1 | 0.5771 | 1025 | 1.55 | 2.05 | 1.8 | 0.0015 |

Strike($)=option strike price, Bid($)=option bid price, Ask($)=option ask price, Price($)=option price (average of bid and ask prices), Rel.Pr=relative option price

Table 3-3. Pricing options on S&P 500 index: 100 paths

| Strike | Calc. | Actual | Err(%) | Calc.Vol.(%) | Act.Vol.(%) | Vol.Err(%) |
|---|---|---|---|---|---|---|
| | | | Call options | | | |
| 1.119 | 0.0002 | 0.0003 | -40.00 | 13.17 | 14.14 | -6.82 |
| 1.098 | 0.0005 | 0.0005 | -5.28 | 12.80 | 12.92 | -0.90 |
| 1.077 | 0.0013 | 0.0012 | 11.57 | 12.70 | 12.40 | 2.42 |
| 1.056 | 0.0035 | 0.0033 | 5.70 | 13.03 | 12.80 | 1.78 |
| 1.035 | 0.0079 | 0.0077 | 3.15 | 13.38 | 13.18 | 1.52 |
| 1.022 | 0.0117 | 0.0118 | -0.75 | 13.43 | 13.49 | -0.48 |
| 1.014 | 0.0156 | 0.0154 | 1.32 | 13.91 | 13.77 | 1.03 |
| 1.005 | 0.0195 | 0.0195 | 0.01 | 14.07 | 14.06 | 0.01 |
| 0.993 | 0.0269 | 0.0269 | 0.18 | 14.63 | 14.60 | 0.23 |
| 0.971 | 0.0416 | 0.0414 | 0.50 | 15.57 | 15.40 | 1.09 |
| 0.950 | 0.0589 | 0.0582 | 1.12 | 16.81 | 16.13 | 4.25 |
| 0.929 | 0.0775 | 0.0770 | 0.62 | 18.04 | 17.35 | 3.94 |
| 0.422 | 0.5789 | 0.5771 | 0.33 | 69.39 | N/A | N/A |
| | | | Put options | | | |
| 1.267 | 0.2633 | 0.2638 | -0.20 | 22.50 | 29.02 | -22.44 |
| 1.098 | 0.0956 | 0.0960 | -0.47 | 13.88 | 15.14 | -8.35 |
| 1.077 | 0.0756 | 0.0759 | -0.36 | 13.71 | 14.18 | -3.32 |
| 1.035 | 0.0406 | 0.0405 | 0.33 | 14.22 | 14.11 | 0.77 |
| 1.022 | 0.0319 | 0.0320 | -0.25 | 14.29 | 14.35 | -0.40 |
| 1.014 | 0.0274 | 0.0270 | 1.26 | 14.75 | 14.51 | 1.62 |
| 1.005 | 0.0229 | 0.0229 | -0.01 | 14.89 | 14.90 | -0.01 |
| 0.993 | 0.0176 | 0.0174 | 1.38 | 15.47 | 15.30 | 1.10 |
| 0.971 | 0.0111 | 0.0112 | -0.52 | 16.43 | 16.47 | -0.28 |
| 0.950 | 0.0070 | 0.0072 | -1.95 | 17.58 | 17.72 | -0.79 |
| 0.929 | 0.0045 | 0.0046 | -3.42 | 18.84 | 19.05 | -1.09 |
| 0.908 | 0.0028 | 0.0031 | -10.00 | 20.02 | 20.57 | -2.68 |
| 0.887 | 0.0015 | 0.0022 | -32.27 | 20.46 | 22.24 | -7.99 |
| 0.866 | 0.0011 | 0.0015 | -26.00 | 22.46 | 23.78 | -5.54 |

Initial price=\$1183.77, time to expiration=49 days, risk-free rate=2.3%. Stock price is modelled with 100 sample paths. Grid dimensions: $P = 15$, $N = 49$.
Strike=option strike price (relative), Calc.=calculated option price (relative), Actual=actual option price (relative), Err=$(Calc. - Actual)/Actual$, Calc.Vol.=calculated option price in volatility form, Act.Vol.(%)=actual option price in volatility terms, Vol.Err(%)=$(Calc.Vol. - Act.Vol.)/Act.Vol.$

Table 3-4. Pricing options on S&P 500 index: 20 paths

| Strike | Calc. | Actual | Err(%) | Calc.Vol.(%) | Act.Vol.(%) | Vol.Err(%) |
|---|---|---|---|---|---|---|
| | | | Call options | | | |
| 1.119 | 0.0005 | 0.0003 | 45.00 | 14.95 | 14.14 | 5.78 |
| 1.098 | 0.0010 | 0.0005 | 88.80 | 14.48 | 12.92 | 12.09 |
| 1.077 | 0.0020 | 0.0012 | 66.86 | 13.95 | 12.40 | 12.50 |
| 1.056 | 0.0047 | 0.0033 | 41.80 | 14.39 | 12.80 | 12.38 |
| 1.035 | 0.0092 | 0.0077 | 19.84 | 14.43 | 13.18 | 9.42 |
| 1.022 | 0.0132 | 0.0118 | 11.41 | 14.47 | 13.49 | 7.26 |
| 1.014 | 0.0160 | 0.0154 | 4.03 | 14.20 | 13.77 | 3.13 |
| 1.005 | 0.0195 | 0.0195 | 0.00 | 14.06 | 14.06 | 0.00 |
| 0.993 | 0.0264 | 0.0269 | -1.66 | 14.28 | 14.60 | -2.15 |
| 0.971 | 0.0393 | 0.0414 | -5.01 | 13.67 | 15.40 | -11.23 |
| 0.950 | 0.0548 | 0.0582 | -5.76 | 12.01 | 16.13 | -25.52 |
| 0.929 | 0.0737 | 0.0770 | -4.35 | 8.39 | 17.35 | -51.65 |
| 0.422 | 0.5790 | 0.5771 | 0.34 | N/A | N/A | N/A |
| | | | Put options | | | |
| 1.267 | 0.2633 | 0.2638 | -0.19 | 23.45 | 29.02 | -19.16 |
| 1.098 | 0.0959 | 0.0960 | -0.13 | 14.82 | 15.14 | -2.11 |
| 1.077 | 0.0762 | 0.0759 | 0.40 | 14.67 | 14.18 | 3.45 |
| 1.035 | 0.0415 | 0.0405 | 2.49 | 14.92 | 14.11 | 5.72 |
| 1.022 | 0.0332 | 0.0320 | 3.69 | 15.20 | 14.35 | 5.93 |
| 1.014 | 0.0278 | 0.0270 | 2.74 | 15.03 | 14.51 | 3.54 |
| 1.005 | 0.0229 | 0.0229 | 0.01 | 14.90 | 14.90 | 0.01 |
| 0.993 | 0.0168 | 0.0174 | -3.31 | 14.90 | 15.30 | -2.63 |
| 0.971 | 0.0089 | 0.0112 | -20.72 | 14.58 | 16.47 | -11.48 |
| 0.950 | 0.0030 | 0.0072 | -58.73 | 12.99 | 17.72 | -26.73 |
| 0.929 | 0.0000 | 0.0046 | -100.00 | 4.38 | 19.05 | -77.00 |
| 0.908 | 0.0000 | 0.0031 | -100.00 | 6.07 | 20.57 | -70.50 |
| 0.887 | 0.0000 | 0.0022 | -100.00 | 7.68 | 22.24 | -65.48 |
| 0.866 | 0.0000 | 0.0015 | -100.00 | 8.98 | 23.78 | -62.21 |

Initial price=\$1183.77, time to expiration=49 days, risk-free rate=2.3%. Stock price is modelled with 20 sample paths. Grid dimensions: $P = 15$, $N = 49$.
Strike=option strike price (relative), Calc.=calculated option price (relative), Actual=actual option price (relative), Err=$(Calc. - Actual)/Actual$, Calc.Vol.=calculated option price in volatility form, Act.Vol.(%)=actual option price in volatility terms, Vol.Err(%)=$(Calc.Vol. - Act.Vol.)/Act.Vol.$

Table 3-5. Summary of cashflow distributions for obitained hedging strategies presented on Figures 3.6, 3.6, 3.6, and 3.6.

| | Total financing | Re-bal. cashflow | Total financing | Re-bal. cashflow |
|---|---|---|---|---|
| | Black-Scholes Call | | SPX Call | |
| mean | 0.0 | 0.0 | 0.0 | 0.0 |
| st.dev. | 0.6274 | 0.0449 | 16.1549 | 1.2730 |
| median | 0.0770 | -0.0008 | 0.2695 | -0.0314 |

Total financing ($) = the sum of discounted inflows/outflows of money on a path; Re-bal. cashflow ($) = discounted inflow/outflow of money on re-balancing points.
Black-Scholes Call: Initial price=$62, strike=$62, time to expiration=70, risk-free rate=10%, volatility=20%. Stock price is modelled with 200 Monte-Carlo sample paths.
SPX Call: Initial price=$1183.77, strike price=$1190, time to expiration=49 days, risk-free rate=2.3%. Stock price is modelled with 100 sample paths.

Table 3-6. Calculation times of the pricing algorithm.

| # of paths | P | N | Building time(sec) | CPLEX time(sec) | Total time(sec) |
|---|---|---|---|---|---|
| 20 | 20 | 49 | 0.8 | 8.2 | 9.0 |
| 100 | 25 | 49 | 1.6 | 12.6 | 14.2 |
| 200 | 25 | 70 | 5.5 | 31.7 | 37.2 |

Calculations are done using CPLEX 9.0 on Pentium 4, 1.7GHz, 1GB RAM.
# of paths = number of sample-paths, P = vertical size of the grid, N = horizontal size of the grid, Building time = time of building the model (preprocessing time), CPLEX time = time of solving optimization problem, Total time = total time of pricing one option.

Table 3-7. Numerical values of inflexion points of the stock position as a function of the stock price for some options.

| Expir.(days) | Strike($) | Inflexion($) | Error(%) |
|---|---|---|---|
| 0 | 62 | 60.126 | 3.02 |
| 35 | 62 | 61.056 | 1.52 |
| 69 | 62 | 61.975 | 0.04 |
| 0 | 54 | 52.368 | 3.02 |
| 35 | 54 | 53.178 | 1.52 |
| 69 | 54 | 53.974 | 0.05 |
| 0 | 71 | 68.855 | 3.02 |
| 35 | 71 | 69.919 | 1.52 |
| 69 | 71 | 70.967 | 0.05 |

Expir.(days) = time to expiration, Strike($) = strike price of the option, Inflexion($) = inflexion point, Error(%)=(Strike-Inflexion)/Strike.

# CHAPTER 4
# METHODS OF REDUCING MAXIMIZATION OF OMEGA FUNCTION TO LINEAR PROGRAMMING

## 4.1 Introduction

The classical mean-variance portfolio theory is based on the assumption that the returns are normally distributed. One of important characteristics of a portfolio is the Sharpe ratio, the ratio of the excess return over the risk-free rate to the standard deviation of a portfolio. Maximization of the Sharpe ratio in portfolio management allows to pick a portfolio with the highest return or with the lowest risk. However, if the standard deviation does not adequately represent risk, Sharpe-optimal portfolios can produce highly non-optimal returns. Critique of the classical approach to the portfolio management is based on the fact that the mean and the variance of a non-normal random variable does not fully describe its distribution and, in particular, do not account for heavy tails of distributions, which are of particular interest for investors. Introduction of higher-order moments into portfolio analysis leads to more accurate solutions. One of the areas in which the mean-variance framework fails is the hedge fund analysis. Properties of tails of return distributions are the key characteristics of hedge funds. Portfolio measurement should incorporate the information about higher-order moments of return distributions in order to adequately represent hedge fund risk.

One of the alternatives to the mean-variance approach is the Omega function, recently introduced in Shadwick and Keating (2002). Omega function $\Omega_r(r_h)$ is the ratio of the upper and the lower partial moments of an asset rate of return $r$ against the benchmark rate of return $r_h$. The upper partial moment is the expected outperformance of an asset over a benchmark; lower partial moment is the expected underperformance of an asset with respect to the benchmark. The Omega function has several attractive features which made it a popular tool in risk measurement. First, it takes the whole distribution into account. A single value $\Omega_r(r_h)$ contains the impact of all moments of the distribution. A collection of $\Omega_r(r_h)$ for all possible $r_h$ fully describes the return distribution. Second,

Omega function has a simple and intuitive interpretation. For a fixed benchmark return $L$, the number $\Omega_r(r_h)$ is a ratio of the expected upside and the expected downside of an asset with respect to the benchmark. It also contains the investor's risk preferences by specifying the benchmark return. Third, given a benchmark $r_h$, comparison of two assets with returns $r_1$ and $r_2$ is done by comparing their Omega values $\Omega_{r_1}(r_h)$ and $\Omega_{r_2}(r_h)$. The asset with greater Omega is preferred to the asset with lower Omega.

The choice of the Omega-optimal portfolio with respect to a fixed benchmark with linear constraints on portfolio weights leads to a non-linear optimization problem. Several approaches to solving this problem has been proposed, among which are the global optimization approach in Avouyi-Govi et al. (2004) and parametric approach employing the family of Johnson distributions in Passow (2005). Mausser et al. (2006) proposes reduction of the Omega maximization problem to linear problem using change of variables. The suggested reduction is possible if the Omega function is greater than 1 at optimality, several non-linear methods are suggested otherwise.

This paper investigates reduction of the Omega-based portfolio optimization problem with fixed benchmark to linear programming. We consider a more general problem than Mausser et al. (2006) by allowing short positions in portfolio instruments and considering constraints of the type $h(x) \leq 0$ with the positively homogeneous function $h(\cdot)$, instead of linear constraints in Mausser et al. (2006). We prove that the Omega-maximizing problem can be reduced to two different problems. The first problem has the expected gain as an objective, and has a constraint on the low partial moment. Second problem has the low partial moment as an objective and a constraint on the expected gain. If the Omega function is greater than 1 at optimality, the Omega maximization problem can be reduced linear programming problem. If the Omega function is lower than 1 at optimality, the proposed reduction methods lead to the problem either of maximizing a convex function, or with linear objective and a non-convex constraint.

To illustrate the use of the reduction theorems, we consider a resource allocation problems frequently arising in the hedge fund management. We show how this problem can be reduced to linear programming and illustrate this with a case study based on the data set of a real hedge fund.

## 4.2 Omega Optimization

### 4.2.1 Definition of Omega Function

Let $t = 1, ..., T$ denote time periods. Each time period produces one scenario of asset returns. Consider a portfolio of $N$ instruments, with instrument $i$ having the rate of return $r_i^t$ at time $t$. The benchmark rate of return is called the hurdle rate of return and is denoted by $r_h$. The difference

$$\tilde{r}_i^t = r_i^t - r_h$$

is the excess rate of return of instrument $i$ over the hurdle rate at time $t$.

Let $x_i$ be the exposure in instrument $i$ in the portfolio; the corresponding weights are $w_i = x_i / \sum_{i=1}^N x_i$, $i = 1, ..., N$.

The loss function measuring underperformance of the portfolio with respect to the hurdle rate at time $t$ is defined by

$$L(t, x) = \sum_{i=1}^{N} (r_h - r_i^t) x_i.$$

The lower partial moment $\underline{\eta}(x)$ and the upper partial moment $\bar{\eta}(x)$ of the loss function $L(t, x)$ are defined as follows

$$\underline{\eta}(x) = \frac{1}{T} \sum_{t \in S_+(x)} L(t, x), \quad where \quad S_+(x) = \{t \mid L(t, x) \geq 0, t = 1, ..., T\};$$

$$\bar{\eta}(x) = -\frac{1}{T} \sum_{t \in S_-(x)} L(t, x), \quad where \quad S_-(x) = \{t \mid L(t, x) < 0, t = 1, ..., T\}.$$

The expected gain with respect to the hurdle rate $r_h$ is

$$q(x) = \frac{1}{T}\sum_{t=1}^{T} -L(t,x).$$

**Assumption (A1)**

We make the assumption that there are no $x \neq 0$ such that $L(t,x) = 0$ for all $t = 1, ..., T$. In other words, we assume that there are $N$ linear independent vectors among

$$\tilde{r}_t = [\tilde{r}_1^t, ..., \tilde{r}_N^t], \ t = 1, ..., T.$$

Since the number of scenarios $T$ is usually much bigger than the number of instruments $N$ in the portfolio, the assumption A1 is almost always satisfied. This assumption prohibits the case when both functions $q(x)$ and $\underline{\eta}(x)$ simultaneously equal to zero for some $x \neq 0$.

The Omega function is the ratio of the two partial moments

$$\Omega(x) = \frac{\bar{\eta}(x)}{\underline{\eta}(x)},$$

which can be expressed as

$$\Omega(x) = \frac{\bar{\eta}(x)}{\underline{\eta}(x)} = \frac{\bar{\eta}(x) - \underline{\eta}(x) + \bar{\eta}(x)}{\underline{\eta}(x)} = \frac{q(x) + \bar{\eta}(x)}{\underline{\eta}(x)} = 1 + \frac{q(x)}{\underline{\eta}(x)}.$$

Note that both functions $q(x)$ and $\underline{\eta}(x)$ are positively homogeneous[1] . This is trivial for $q(x)$ as it is linear with respect to $x$, and holds for $\underline{\eta}(x)$ since linearity of the loss function $L(t,x)$ with respect to $x$ implies

$$\underline{\eta}(\lambda x) = \sum_{t \in S_+(\lambda x)} L(t, \lambda x)/T = \sum_{t \in S_+(x)} L(t, \lambda x)/T = \sum_{t \in S_+(x)} \lambda L(t,x)/T = \lambda \underline{\eta}(x).$$

[1] A function $f(x)$ is called positively homogeneous if $f(\lambda x) = \lambda f(x)$ for all $\lambda > 0$.

### 4.2.2 General Problem

This paper deals with solving the following non-linear problem. We consider a fixed hurdle rate $r_h$ and form a portfolio of $N$ instruments subject to restrictions expressed by $K$ inequalities. The goal is to maximize the Omega function of the portfolio.

$$
\begin{aligned}
(\mathcal{P}_0) \qquad & max \quad \Omega(w) = 1 + \frac{q(w)}{\underline{\eta}(w)} \\
& \text{s.t.} \\
& \qquad h_k(w) \geq 0, \ k = 1, ..., K, \\
& \qquad \textstyle\sum_{i=1}^{I} w_i = 1, \\
& \qquad w_i \in \mathbb{R}, \ i = 1, ..., I,
\end{aligned}
$$

where functions $h_k(x)$ are positively homogeneous.

It is not necessary for variables in problem $\mathcal{P}_0$ to be weights. Note that the function $\Omega(x)$ is invariant to scaling its argument, since

$$
\Omega(\lambda x) = 1 + \frac{q(\lambda x)}{\underline{\eta}(\lambda x)} = 1 + \frac{\lambda q(x)}{\lambda \underline{\eta}(x)} = 1 + \frac{q(x)}{\underline{\eta}(x)} = \Omega(x)
$$

for any feasible $x$ and $\lambda > 0$. Moreover, if constraints $h_k(x) \geq 0$, $k = 1, ..., K$ hold for some $x$, they also hold for $\lambda x$, $\lambda > 0$.

Consider the following alternative to $\mathcal{P}_0$.

$$
\begin{aligned}
(\mathcal{P}_0') \qquad & max \quad \Omega(x) = 1 + \frac{q(x)}{\underline{\eta}(x)} \\
& \text{s.t.} \\
& \qquad h_k(x) \geq 0, \ k = 1, ..., K, \\
& \qquad \textstyle\sum_{i=1}^{I} x_i > 0, \\
& \qquad x_i \in \mathbb{R}, \ i = 1, ..., I.
\end{aligned}
$$

In order to simplify notations, we denote feasible sets in problems $\mathcal{P}_0$ and $\mathcal{P}_0'$ by $K_w$ and $K$, respectively,

$$K_w = \{x \mid h_k(x) \geq 0,\ \sum_{i=1}^{I} x_i = 1,\ x_i \in \mathbb{R},\ k = 1, ..., K,\ i = 1, ..., I\},$$

$$K = \{x \mid h_k(x) \geq 0,\ \sum_{i=1}^{I} x_i > 0,\ x_i \in \mathbb{R},\ k = 1, ..., K,\ i = 1, ..., I\}.$$

Problems $\mathcal{P}_0$ and $\mathcal{P}_0'$ can be written by $\max_{K_w} \Omega(x)$ and $\max_K \Omega(x)$, respectively. The relationship between problems $\mathcal{P}_0$ and $\mathcal{P}_0'$ is stated in the following lemma.

**Lemma 1.** *If the problem $\mathcal{P}_0$ has a solution, then the problem $\mathcal{P}_0'$ also has a solution, and vice versa. Moreover, if $x^*$ is the solution to $\mathcal{P}_0$ and $w^*$ is the solution to $\mathcal{P}_0'$, then $x^* = \lambda w^*$ for some $\lambda > 0$, and $\Omega(x^*) = \Omega(w^*)$.*

**Proof:**

Let $w^*$ be optimal solution to $\mathcal{P}_0$. If $\Omega(\hat{x}) > \Omega(w^*)$ for some $\hat{x} \in K$ in $\mathcal{P}_0'$, then for $\hat{\lambda} = (\sum_{i=1}^{I} \hat{x}_i)^{-1}$ we have $\Omega(\hat{\lambda}\hat{x}) = \Omega(\hat{x}) > \Omega(w^*)$. Since $\hat{\lambda}\hat{x} \in K_w$, we have a contradiction with $w^*$ being the optimal solution to $\mathcal{P}_0$. Therefore, the objective in $\mathcal{P}_0'$ is bounded from above by $\Omega(w^*)$. Since $w^* \in K$, it is the optimal solution to $\mathcal{P}_0'$: $x^* = w^*$. Any solution of the form $x^* = \lambda w^*$, $\lambda > 0$ will also be optimal.

Conversely, suppose $x^*$ is the optimal solution to $\mathcal{P}_0'$. Then the objective function in $\mathcal{P}_0$ are bounded from above by $\Omega(x^*)$. Take $\lambda^* = (\sum_{i=1}^{I} x_i^*)^{-1}$, then $\lambda^* x^*$ is feasible point in $\mathcal{P}_0$, and $\Omega(\lambda^* x^*) = \Omega(x^*)$, therefore $w^* = \lambda^* x^*$ is the optimal solution to $\mathcal{P}_0$. ■

### 4.2.3 Two Reduction Theorems

In the following reduction theorems, we use the notion of reduction of one problem to another, which is understood in the following sense. Problem $\mathcal{P}_1$ can be reduced to problem $\mathcal{P}_2$ (notation is $\mathcal{P}_1 \leftrightarrow \mathcal{P}_2$) if both problems have finite solutions or are unbounded. Furthermore, $x_1^*$ and $x_2^*$ are optimal solutions to $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, then $x_1^* = \lambda x_2^*$ for some $\lambda > 0$.

Equivalence of problems $\mathcal{P}_1$ and $\mathcal{P}_2$ is denoted by $\mathcal{P}_1 \Longleftrightarrow \mathcal{P}_2$.

In order to state the first theorem, we introduce the following sets

$$\begin{aligned} D_{q+} &= \{x \,|\, q(x) > 0\} \bigcap K, \\ D_{\underline{\eta}\leq 1} &= \{x \,|\, \underline{\eta}(x) \leq 1\}, \\ D_{\underline{\eta}\geq 1} &= \{x \,|\, \underline{\eta}(x) \geq 1\}, \end{aligned}$$

and define problems

$$(\mathcal{P}_{\underline{\eta}\leq 1}) \qquad \max_{K \bigcap D_{\underline{\eta}\leq 1}} q(x)$$

and

$$(\mathcal{P}_{\underline{\eta}\geq 1}) \qquad \max_{K \bigcap D_{\underline{\eta}\geq 1}} q(x).$$

**Theorem 1.** *Suppose that the feasible region in problem $\mathcal{P}_0$ is bounded. Then $\mathcal{P}_0$ either has a finite solution or is unbounded. If $D_{q+} \bigcap K_w \neq \varnothing$, then problem $\mathcal{P}_0$ can be reduced to problem $\mathcal{P}_{\underline{\eta}\leq 1}$. If $D_{q+} \bigcap K_w = \varnothing$, the problem $\mathcal{P}_0$ can be reduced to problem $\mathcal{P}_{\underline{\eta}\geq 1}$.*

For the second theorem, we introduce sets $D_{q=0} = \{x \,|\, q(x) = 0\}$, $D_{q\geq 1} = \{x \,|\, q(x) \geq 1\}$, and $D_{q\geq -1} = \{x \,|\, q(x) \geq -1\}$, and define problems

$$(\mathcal{P}_{q\geq 1}) \qquad \min_{K \bigcap D_{q\geq 1}} \underline{\eta}(x)$$

and

$$(\mathcal{P}_{q\geq -1}) \qquad \max_{K \bigcap D_{q\geq -1}} \underline{\eta}(x).$$

**Theorem 2.** *Suppose that the feasible region in problem $\mathcal{P}_0$ is bounded. If $D_{q+} \bigcap K_w \neq \varnothing$ and $D_{\underline{\eta}=0} \bigcap K_w \neq \varnothing$, then problem $\mathcal{P}_0$ is unbounded and the objective function in problem $\mathcal{P}_{q\geq 1}$ is equal to zero at optimality. If $D_{q+} \bigcap K_w \neq \varnothing$ and $D_{\underline{\eta}=0} \bigcap K_w = \varnothing$ then problem $\mathcal{P}_0$ can be reduced to $\mathcal{P}_{q\geq 1}$. If $D_{q+} \bigcap K_w = \varnothing$, and $D_{q=0} \bigcap K_w \neq \varnothing$, then the objective function in $\mathcal{P}_0$ is equal to zero at optimality, and the problem $\mathcal{P}_{q\geq -1}$ is unbounded. If $D_{q+} \bigcap K_w = \varnothing$ and $D_{q=0} \bigcap K_w = \varnothing$, the problem $\mathcal{P}_0$ can be reduced to problem $\mathcal{P}_{q\geq -1}$.*

Proofs of both theorems are given in the next section.

Theorems 1 and 2 require the knowledge if $D_{q+} \bigcap K_w = \varnothing$, i.e. if $q(x) > 0$ at least at one feasible point $x$ in problem $\mathcal{P}_0$. This information can be obtained by maximizing $q(x)$ over the feasible region in $\mathcal{P}_0$, i.e. by solving

$$\begin{aligned} \max \quad & q(w) \\ \text{s.t.} \quad & \\ & h_k(w) \geq 0, \ k = 1, ..., K, \\ & \textstyle\sum_{i=1}^{I} w_i = 1, \\ & w_i \in \mathbb{R}, \ i = 1, ..., I. \end{aligned} \tag{4–1}$$

If $q(x) > 0$ at optimality in (4–1), then problem $\mathcal{P}_0$ can be reduced to $\mathcal{P}_{\underline{\eta} \leq 1}$ (or $\mathcal{P}_{q \geq 1}$), otherwise it can be reduced to $\mathcal{P}_{\underline{\eta} \geq 1}$ (or $\mathcal{P}_{q \geq -1}$). The alternative to solving (4–1), one could solve

$$\begin{aligned} \max \quad & q(x) \\ \text{s.t.} \quad & \\ & h_k(x) \geq 0, \ k = 1, ..., K, \\ & \textstyle\sum_{i=1}^{I} x_i \geq 0, \\ & x_i \in \mathbb{R}, \ i = 1, ..., I, \end{aligned} \tag{4–2}$$

where the variables are not restricted to be weights. If $q(x^*) > 0$ at optimality in (4–2), then $q(\lambda^* x^*) > 0$ for $\lambda = 1/(\sum_{i=1}^{N} x_i^*)$, where $\lambda^* x^*$ is a feasible point in $\mathcal{P}_0$. If $q(x^*) \leq 0$ in (4–2), then $q(x) \leq 0$ for all feasible points in (4–1), since the feasible region in (4–2) contains the feasible region in (4–1).

Another prescription to determine if $D_{q+} \bigcap K_w = \varnothing$ is to solve $\mathcal{P}_{\underline{\eta} \leq 1}$ first. If $D_{q+} \bigcap K_w \neq 0$, then the reduction to $\mathcal{P}_{\underline{\eta} \leq 1}$ is correct, and $q(x) > 0$ ( or $\Omega(x) > 1$) at optimality. If $D_{q+} \bigcap K_w = 0$, then problem $\mathcal{P}_{\underline{\eta}} \leq 1$ has no solution or have the optimal objective value equal to zero. To see this, note that if $D_{q=0} \bigcap K_w \neq \varnothing$, then there exists a point $x^*$ such that $q(x^*) = 0$ and $\underline{\eta}(x) = 1$, therefore the objective in $\mathcal{P}_{\underline{\eta} \leq 1}$ is equal to zero at optimality. If $D_{q=0} \bigcap K_w = \varnothing$, then $q(x) < 0$ for any $x \in D_{q-}$. However,

any point of the form $\lambda x$, $x \in D_{q-}$, is feasible to $\mathcal{P}_{\underline{\eta}\leq 1}$ for significantly low $\lambda$, moreover, $q(\lambda x) = \lambda q(x) \longrightarrow 0$ as $\lambda \longrightarrow 0$, therefore problem $\mathcal{P}_{\underline{\eta}\leq 1}$ never attains its maximum.

Alternatively, the problem $\mathcal{P}_{q\geq 1}$ can be attempted. If $D_{q+} \bigcap K_w \neq \varnothing$, then the solution to $\mathcal{P}_{q\geq 1}$ after normalizing gives the solution to $\mathcal{P}_0$. If $D_{q+} \bigcap K_w = \varnothing$, the problem $\mathcal{P}_{q\geq 1}$ is infeasible, due to the constraint $q(x) \geq 1$.

### 4.3 Proofs Of Reduction Theorems For Omega Optimization Problem

We use the following notations.

$$D_{q+} = \{x \,|\, q(x) > 0\} \bigcap K, \qquad D_{q-} = \{x \,|\, q(x) \leq 0\} \bigcap K,$$

$$D_{q=0} = \{x \,|\, q(x) = 0\}, \qquad D_{q=1} = \{x \,|\, q(x) = 1\},$$

$$D_{q\geq 1} = \{x \,|\, q(x) \geq 1\}, \qquad D_{q\geq -1} = \{x \,|\, q(x) \geq -1\},$$

and

$$D_{\underline{\eta}+} = \{x \,|\, \underline{\eta}(x) > 0\}, \qquad D_{\underline{\eta}=0} = \{x \,|\, \underline{\eta}(x) = 0\},$$

$$D_{\underline{\eta}=1} = \{x \,|\, \underline{\eta}(x) = 1\}, \qquad D_{\underline{\eta}\leq 1} = \{x \,|\, \underline{\eta}(x) \leq 1\},$$

$$D_{\underline{\eta}\geq 1} = \{x \,|\, \underline{\eta}(x) \geq 1\}.$$

**Theorem 1**

*Suppose that the feasible region in problem $\mathcal{P}_0$ is bounded. Then $\mathcal{P}_0$ either has a finite solution or is unbounded. If $D_{q+} \bigcap K_w \neq \varnothing$, then problem $\mathcal{P}_0$ can be reduced to problem $\mathcal{P}_{\underline{\eta}\leq 1}$. If $D_{q+} \bigcap K_w = \varnothing$, the problem $\mathcal{P}_0$ can be reduced to problem $\mathcal{P}_{\underline{\eta}\geq 1}$.*

**Proof:** Consider the case when $D_{q+} \bigcap K_w \neq \varnothing$. If $K_w \bigcap D_{\underline{\eta}=0} \neq \varnothing$, both problems $\mathcal{P}_0$ and $\mathcal{P}_{\underline{\eta}\leq 1}$ are unbounded. Indeed, there exists $\hat{x} \in K_w$ such that $q(\hat{x}) > 0$ and $\underline{\eta}(\hat{x}) = 0$, therefore, $\Omega(\hat{x}) = +\infty$ and the problem $\mathcal{P}_0$ is unbounded. On the other hand, $\lambda\hat{x} \in K \bigcap D_{\underline{\eta}\leq 1}$ for any $\lambda > 0$ and $q(\lambda x) = \lambda q(x) \longrightarrow +\infty$ as $\lambda \longrightarrow +\infty$, therefore, the problem $P_{\underline{\eta}\leq 0}$ is also unbounded.

If $K_w \bigcap D_{\underline{\eta}=0} = \varnothing$, feasible sets in both problems $\mathcal{P}_0$ and $\mathcal{P}_{\underline{\eta}\leq 1}$ are bounded and closed, and objective function are continuous, therefore both problems have finite

solutions. By Lemma 1, the problem $\mathcal{P}_0$ can be reduced to $\mathcal{P}_0'$. The following sequence of reductions of the problem $\mathcal{P}_0'$ leads to the problem $\mathcal{P}_{\underline{\eta}\leq 1}$.

$$\mathcal{P}_0' = \max_{K} \Omega(x) \overset{(1')}{\Leftrightarrow} \max_{D_{q+}} \Omega(x) \overset{(2')}{\leftrightarrow} \max_{D_{q+}\bigcap D_{\underline{\eta}=1}} \Omega(x) \overset{(3')}{\Leftrightarrow} \max_{D_{q+}\bigcap D_{\underline{\eta}=1}} q(x) \overset{(4')}{\Leftrightarrow} \max_{D_{q+}\bigcap D_{\underline{\eta}\leq 1}} q(x) = \mathcal{P}_{\underline{\eta}\leq 1}.$$

$(1')$ Since $\Omega(x) > 1$ for any $x \in D_{q+}$ , and $\Omega(x) \leq 1$ otherwise. Therefore, the maximum in $\mathcal{P}_0'$ will never be attained in the set $D_{q-}$.

$(2')$ Let $x^*$ be solution to $\max_{D_{q+}} \Omega(x)$, $x^{**}$ be solution to $\max_{D_{q+}\bigcap D_{\underline{\eta}=1}} \Omega(x)$.

Then $\max_{D_{q+}\bigcap D_{\underline{\eta}=1}} \Omega(x) \leq \Omega(x^*)$. Take $\lambda^* = 1/\underline{\eta}(x^*)$, then $\underline{\eta}(\lambda^* x^*) = 1$ and $\Omega(\lambda^* x^*) = \Omega(x^*)$, so $x^{**} = \lambda^* x^*$.

$(3')$ $\max_{D_{q+}\bigcap D_{\underline{\eta}=1}} \left(1 + \dfrac{q(x)}{\underline{\eta}(x)}\right) = \max_{D_{q+}\bigcap D_{\underline{\eta}=1}} \left(1 + \dfrac{q(x)}{1}\right) = 1 + \max_{D_{q+}\bigcap D_{\underline{\eta}=1}} q(x)$.

$(4')$ Suppose that $x^*$ is the solution to $\max_{D_{q+}\bigcap D_{\underline{\eta}\leq 1}} q(x)$ and $\underline{\eta}(x^*) < 1$. Take $\lambda^* = 1/\underline{\eta}(x^*) > 1$. Then $\underline{\eta}(\lambda^* x^*) = 1$, $q(\lambda^* x^*) = \lambda^* q(x^*) > q(x^*)$, which is a contradiction. Therefore, $\underline{\eta}(x) = 1$ at optimality in problem $\mathcal{P}_{\underline{\eta}\leq 1}$, and the equivalence $(4')$ is justified.

Now consider the case $D_{q+}\bigcap K_w = \varnothing$. Definitions of functions $q(x)$ and $\underline{\eta}(x)$ imply that $D_{q-}\bigcap D_{\underline{\eta}=0} = \varnothing$, so $\underline{\eta}(x) > 0$ for any $x \in K_w$. By the same argument as above, both problems $\mathcal{P}_0$ and $\mathcal{P}_{\underline{\eta}\geq 1}$ have finite solutions, and $\mathcal{P}_0 \leftrightarrow \mathcal{P}_0'$.

First, consider the case when $D_{q=0}\bigcap K_w \neq \varnothing$. In this case, the optimal solution $x^*$ to $\mathcal{P}_0$ gives $\Omega(x^*) = 1$, and $q(x^*) = 0$, $\underline{\eta}(x^*) > 0$. Taking $\lambda^* = 1/\underline{\eta}(x^*)$, yields $q(\lambda^* x^*) = 0$, $\underline{\eta}(\lambda^* x^*) = 1$, so $x^{**}\lambda^* x^*$ is the optimal solution to $\mathcal{P}_{\underline{\eta}\geq 1}$, and $q(x^{**}) = 0$.

If $D_{q=0}\bigcap K_w = \varnothing$, then $q(x) < 0$ for all $x \in D_{q-}$. The following sequence of reductions leads to the problem $\mathcal{P}_{\underline{\eta}\geq 1}$.

$$\mathcal{P}_0' = \max_{D_{q-}} \Omega(x) \overset{(1'')}{\leftrightarrow} \max_{D_{q-}\bigcap D_{\underline{\eta}=1}} \Omega(x) \overset{(2'')}{\Leftrightarrow} \max_{D_{q-}\bigcap D_{\underline{\eta}=1}} q(x) \overset{(3'')}{\Leftrightarrow} \max_{D_{q-}\bigcap D_{\underline{\eta}\geq 1}} q(x) = \mathcal{P}_{\underline{\eta}\geq 1}.$$

$(1'')$ and $(2'')$ are proven similarly to $(1')$ and $(2')$, so here we consider $(3'')$. We need to show that if $x^*$ is the optimal solution to $\mathcal{P}_{\underline{\eta}\geq 1}$, then $\underline{\eta}(x^*) = 1$. Indeed, suppose that

$\underline{\eta}(x^*) > 1$. Take $\lambda^* = 1/\underline{\eta}(x^*) < 1$. Then $\underline{\eta}(\lambda^* x^*) = 1$ and $q(\lambda^* x^*) = \lambda^* q(x^*) > q(x^*)$, which is a contradiction. The equivalence $(3'')$ is justified. ■

**Theorem 2**

*Suppose that the feasible region in problem $\mathcal{P}_0$ is bounded. If $D_{q+} \bigcap K_w \neq \varnothing$ and $D_{\underline{\eta}=0} \bigcap K_w \neq \varnothing$, then problem $\mathcal{P}_0$ is unbounded and the objective function in problem $\mathcal{P}_{q\geq 1}$ is equal to zero at optimality. If $D_{q+} \bigcap K_w \neq \varnothing$ and $D_{\underline{\eta}=0} \bigcap K_w = \varnothing$ then problem $\mathcal{P}_0$ can be reduced to $\mathcal{P}_{q\geq 1}$. If $D_{q+} \bigcap K_w = \varnothing$, and $D_{q=0} \bigcap K_w \neq \varnothing$, then the objective function in $\mathcal{P}_0$ is equal to zero at optimality, and the problem $\mathcal{P}_{q\geq -1}$ is unbounded. If $D_{q+} \bigcap K_w = \varnothing$ and $D_{q=0} \bigcap K_w = \varnothing$, the problem $\mathcal{P}_0$ can be reduced to problem $\mathcal{P}_{q\geq -1}$.*

**Proof:**

If $D_{q+} \bigcap K_w \neq \varnothing$ and $D_{\underline{\eta}=0} \bigcap K_w \neq \varnothing$, then unboundedness of $\mathcal{P}_0$ is already shown in Theorem 1. Consider problem $\mathcal{P}_{q\geq 1}$. Take $\hat{x} \in D_{\underline{\eta}=0} \bigcap K_w$, then $q(\hat{x}) > 0$, $\underline{\eta}(\hat{x}) = 0$. Taking $\hat{\lambda} = 1/q(\hat{x})$, we have $q(\hat{\lambda}\hat{x}) = 1$, $\underline{\eta}(\hat{\lambda}\hat{x}) = 0$. Since $\underline{\eta}(x) \geq 0$ for all $x$, the optimal objective value in problem $\mathcal{P}_{q\geq 1}$ is zero.

If $D_{q+} \bigcap K_w \neq \varnothing$ and $D_{\underline{\eta}=0} \bigcap K_w = \varnothing$, then problem $\mathcal{P}_0$ can be reduced to problem $\mathcal{P}_0'$, as was shown in the proof of Theorem 1. The following sequence of reductions transforms the problem $\mathcal{P}_0'$ into $\mathcal{P}_{q\geq 1}$.

$$\mathcal{P}_0' = \max_{K} \Omega(x) \overset{(1')}{\Leftrightarrow} \max_{D_{q+}} \Omega(x) \overset{(2')}{\leftrightarrow} \max_{D_{q+} \bigcap D_{q=1}} \Omega(x) \overset{(3')}{\Leftrightarrow} \min_{D_{q+} \bigcap D_{q=1}} \underline{\eta}(x) \overset{(4')}{\Leftrightarrow} \min_{D_{q+} \bigcap D_{q\geq 1}} \underline{\eta}(x) = \mathcal{P}_{q\geq 1}.$$

$(1')$ is already shown in Theorem 1.

$(2')$ Let $x^*$ be solution to $\max_{D_{q+}} \Omega(x)$, $x^{**}$ be solution to $\max_{D_{q+} \bigcap D_{q=1}} \Omega(x)$.

Then $\max_{D_{q+} \bigcap D_{q=1}} \Omega(x) \leq \Omega(x^*)$. Take $\lambda^* = 1/q(x^*)$, then $q(\lambda^* x^*) = 1$ and $\Omega(\lambda^* x^*) = \Omega(x^*)$, so $x^{**} = \lambda^* x^*$.

$(3')$ $\max_{D_{q+} \bigcap D_{q=1}} \left(1 + \dfrac{q(x)}{\underline{\eta}(x)}\right) = \max_{D_{q+} \bigcap D_{q=1}} \left(1 + \dfrac{1}{\underline{\eta}(x)}\right) = 1 + \dfrac{1}{\min_{D_{q+} \bigcap D_{q=1}} \underline{\eta}(x)}.$

(4′) Suppose that $x^*$ is the solution to $\max_{D_{q+}\bigcap D_{q\geq 1}} \underline{\eta}(x)$ and $q(x^*) > 1$. Take $\lambda^* = 1/q(x^*) < 1$. Then $q(\lambda^* x^*) = 1$, $\underline{\eta}(\lambda^* x^*) = \lambda^* \underline{\eta}(x^*) < \underline{\eta}(x^*)$, which is a contradiction. Therefore, $q(x) = 1$ at optimality in problem $\mathcal{P}_{q\geq 1}$, and the equivalence (4′) is justified.

Now consider the case $D_{q+}\bigcap K_w = \varnothing$ and $D_{q=0}\bigcap K_w \neq \varnothing$. Take $\hat{x} \in D_{q=0}\bigcap K_w$. From the assumption A1, it follows that $\underline{\eta}(\hat{x}) > 0$. Since $\max_{K_w} \Omega(x) \leq 0$ and $\Omega(\hat{x}) = 0$, we conclude that the optimal objective value in problem $\mathcal{P}_0$ is zero. As for problem $\mathcal{P}_{q\leq 1}$, points of the form $\lambda\hat{x}$ for $\lambda > 0$ are all feasible, and $\underline{\eta}(\lambda x) \longrightarrow +\infty$ as $\lambda \longrightarrow +\infty$, so problem $\mathcal{P}_{q\leq 1}$ is unbounded.

If $D_{q+}\bigcap K_w = \varnothing$ and $D_{q=0}\bigcap K_w = \varnothing$, then the feasible region of problem $\mathcal{P}_0$ is closed and bounded, and the objective function is continuous, therefore, problem $\mathcal{P}_0$ has a solution. According to Lemma 1, $\mathcal{P}_0$ can be reduced to $\mathcal{P}_0'$. Consider the following sequence of reductions.

$$\mathcal{P}_0' = \max_{D_{q-}} \Omega(x) \overset{(1'')}{\leftrightarrow} \max_{D_{q-}\bigcap D_{q=-1}} \Omega(x) \overset{(2'')}{\Leftrightarrow} \max_{D_{q-}\bigcap D_{q=-1}} \underline{\eta}(x) \overset{(3'')}{\Leftrightarrow} \max_{D_{q-}\bigcap D_{q\geq -1}} q(x) = \mathcal{P}_{q\geq -1}.$$

(1″) Let $x^*$ be solution to $\max_{D_{q-}} \Omega(x)$, $x^{**}$ be solution to $\max_{D_{q-}\bigcap D_{q=-1}} \Omega(x)$.

Then $\max_{D_{q-}\bigcap D_{q=-1}} \Omega(x) \leq \Omega(x^*)$. Take $\lambda^* = -1/q(x^*) > 0$, then $q(\lambda^* x^*) = -1$ and $\Omega(\lambda^* x^*) = \Omega(x^*)$, so $x^{**} = \lambda^* x^*$.

(″) $\max_{D_{q-}\bigcap D_{q=-1}} \left(1 + \dfrac{q(x)}{\underline{\eta}(x)}\right) = \max_{D_{q+}\bigcap D_{q=1}} \left(1 - \dfrac{1}{\underline{\eta}(x)}\right) = 1 - \dfrac{1}{\max_{D_{q-}\bigcap D_{q=-1}} \underline{\eta}(x)}$.

(3″) Suppose that $x^*$ is the solution to $\max_{D_{q-}\bigcap D_{q\geq -1}} \underline{\eta}(x)$ and $q(x^*) > -1$. Take $\lambda^* = -1/q(x^*) > 1$. Then $q(\lambda^* x^*) = -1$, $\underline{\eta}(\lambda^* x^*) = \lambda^* \underline{\eta}(x^*) > \underline{\eta}(x^*)$, which is a contradiction. Therefore, $q(x) = -1$ at optimality in problem $\mathcal{P}_{q\geq -1}$, and the equivalence (3″) is justified. ■

### 4.4 Applications of Reduction Theorems to Problems with Linear Constraints

The set of constraints

$$h_k(x) \geq 0,\ k = 1, ..., K \tag{4–3}$$

with positively homogeneous functions $h_k(\cdot)$ in problem $\mathcal{P}_0$ is quite general. For example, any set of linear inequalities on portfolio weights

$$Ax \leq b, \ \sum_{i=1}^{N} x_i = 1$$

can be written in the form (4–3) by taking

$$h(x) = b \sum_{i=1}^{N} x_i - Ax$$

.

In this subsection, we discuss application of Theorems 1 and 2 to problems with linear constraints. In the case when $D_{q+} \bigcap K_w \neq \varnothing$ (alternatively, $\Omega(x) > 1$ at optimality), the problem $\mathcal{P}_0$ can be reduced to $\mathcal{P}_{\underline{\eta} \leq 1}$ or $\mathcal{P}_{q \geq 1}$. In problem $\mathcal{P}_{\underline{\eta} \leq 1}$, the constraint $\underline{\eta}(x) \leq 1$ can be reduced to linear programming. Recall that $\underline{\eta}(x) = \frac{1}{T} \sum_{i=1}^{T} [L(t,x)]^+$. Introduction of additional variables $z_t$, $t = 1, ..., T$, allows to enforce the constraint $\underline{\eta}(x) \leq 1$ by replacing it with

$$\sum_{t=1}^{T} z_t \leq 1, \ z_t \geq L(t,x), \ z_t \geq 0, \ \text{for } t = 1, ..., T.$$

The problem $\mathcal{P}_{q \geq 1}$ can similarly be reduced to linear programming. The minimization of the convex function $\underline{\eta}(x)$ can be reduced to maximization of $\sum_{t=1}^{T} z_t$ with additional constraints $z_t \geq L(t,x), \ z_t \geq 0$, $t = 1, ..., T$.

If $\Omega(x) \leq 1$ at optimality, the problem $\mathcal{P}_0$ is reduced to $\mathcal{P}_{\underline{\eta} \geq 1}$ or $\mathcal{P}_{q \geq -1}$. Both of these problems cannot be reduced to linear programming due to the presence of the constraint $\underline{\eta}(x) \geq 1$ in $\mathcal{P}_{\underline{\eta} \geq 1}$ or maximization of the convex objective $\underline{\eta}(x)$ in $\mathcal{P}_{q \geq -1}$.

### 4.5 Example: Resource Allocation Problem

As an example of applying Theorem 1, we solve the following problem arising in hedge fund management. Consider $N$ fund managers among which the resources should be allocated. Let $w_i$ be the fraction of resources allocated to manager $i$, $i = 1, ..., N$. Some managers have similar strategies; there are $M$ different strategies among all managers. Let $J_m$ be a set of managers pursuing strategy $m$, then $\sum_{j \in J_m} w_j$ is the fraction of resources

allocated to strategy $m$, $m = 1, ..., M$. The following optimization problem allocates money to groups of managers with similar strategies and to individual managers within each strategy.

$$
\begin{aligned}
&\max \quad \Omega(w) = 1 + \frac{q(w)}{\underline{\eta}(w)} \\
&\text{s.t.} \\
&\qquad \textstyle\sum_{i=1}^{I} w_i = 1 && \text{budget constraint,} \\
&\qquad b_m^l \le \textstyle\sum_{j \in J_m} w_j \le b_m^u, \ m = 1, ..., M && \text{constraints on allocation to strategies,} \\
&\qquad l_i \le w_i \le u_i, \ \text{where } l_i > -\infty, \ i = 1, ..., I && \text{box constraints for individual positions,} \\
&\qquad w_i \in \mathbb{R}, \ i = 1, ..., I.
\end{aligned}
\tag{4–4}
$$

The constraint $\sum_{i=1}^{I} x_i = 1$ allows to rewrite the set of constraints

$$b_m^l \le \sum_{j \in J_m} x_j \le b_m^u, \ m = 1, ..., M, \tag{4–5}$$

$$l_i \le x_i \le u_i, \ i = 1, ..., I, \tag{4–6}$$

in the following form

$$b_m^l \sum_{i=1}^{I} x_i \le \sum_{j \in J_m} x_j \le b_m^u \sum_{i=1}^{I} x_i, \ m = 1, ..., M, \tag{4–7}$$

$$l_i \sum_{i=1}^{I} x_i \le x_i \le u_i \sum_{i=1}^{I} x_i, \ i = 1, ..., I. \tag{4–8}$$

For any $x$ satisfying (4–7)-(4–8), $\lambda x$ for $\lambda > 0$ will also satisfy (4–7)-(4–8). Therefore, constraints (4–7)-(4–8) are special case of the constraints of type (4–3).

According to Theorem 1, the problem (4–4) can be reduced to the following problem.

$$
\begin{aligned}
\max \quad & q(x) \\
\text{s.t.} \quad & \\
& \underline{\eta}(x) \leq 1 \text{ (or } \underline{\eta}(x) \geq 1) \\
& b_m^l \sum_{i=1}^I x_i \leq \sum_{j \in J_m} x_j \leq b_m^u \sum_{i=1}^I x_i, \ m = 1, ..., M, \\
& l_i \sum_{i=1}^I x_i \leq x_i \leq u_i \sum_{i=1}^I x_i, \ i = 1, ..., I \\
& x_i \in \mathbb{R}, \ i = 1, ..., I.
\end{aligned} \tag{4–9}
$$

where the loss constraint is $\underline{\eta}(x) \leq 1$ if $q(x) > 0$ ($\Omega(x) > 1$) at least at one feasible point of the problem (4–4), and $\underline{\eta}(x) \geq 1$ otherwise.

We solve the above problem with the constraint $\underline{\eta}(x) \leq 1$ by reducing it to the following linear programming program. In the case study described below the solution gives $\Omega(x^*) > 1$ at optimality, implying that the reduction to LP is valid. The following LP formulation uses explicit expressions of functions $q(x)$ and $\underline{\eta}(x)$.

$$
\begin{aligned}
\max \quad & -\sum_{t=1}^T \sum_{i=1}^I (r_h - r_t^i) x_i \\
\text{s.t.} \quad & \\
& \sum_{t=1}^T z_t \leq 1, \\
& z_t \geq \sum_{i=1}^I (r_h - r_t^i) x_i, \ t = 1, ..., T, \\
& b_m^l \sum_{i=1}^I x_i \leq \sum_{j \in J_m} x_j \leq b_m^u \sum_{i=1}^I x_i, \ m = 1, ..., M \\
& l_i \sum_{i=1}^I x_i \leq x_i \leq u_i \sum_{i=1}^I x_i, \ \text{(where } l_i > -\infty), \ i = 1, ..., I \\
& z_t \geq 0, \ t = 1, ..., T. \\
& x_i \in \mathbb{R}, \ i = 1, ..., I.
\end{aligned} \tag{4–10}
$$

We solve the allocation problem 4–10 for a portfolio consisting of 10 strategies. We used historical daily rates of return for the funds from October 1, 2003, to March 17, 2006. The daily hurdle rate is set to $r_h = 0.00045$. The optimal solution $x^*$ to (4–10) gives $\Omega(x^*) = 1.164 > 1$, which indicates that the reduction of (4–4) to (4–10) is correct. The solution to (4–4) is obtained by normalizing the solution $x^*$. The optimal allocation is given in the Table 4-1.

Table 4-1. Optimal allocation

| Strategy | Allocation (%) |
|---|---|
| Manager 1 | 10.00 |
| Manager 2 | 20.00 |
| Manager 3 | 7.50 |
| Manager 4 | 2.22 |
| Manager 5 | 0.00 |
| Manager 6 | 7.50 |
| Manager 7 | 0.00 |
| Manager 8 | 12.78 |
| Manager 9 | 20.00 |
| Manager 10 | 20.00 |

## 4.6 Conclusions

We considered a problem of maximizing the Omega function of a portfolio with a fixed benchmark with positively homogeneous constraints. We proved that this problem can be reduced either to maximizing the expected gain with constraint of the low partial moment or to maximizing/minimizing the low partial moment with constraint on the expected gain. We showed that in case when the Omega function is greater than 1 at optimality, the proposed reductions lead to linear programming. We illustrate the use of the proposed methodology with the resource allocation problem from the real hedge fund practice.

CHAPTER 5
CALIBRATION OF GENERAL DEVIATION MEASURES FROM MARKET DATA

## 5.1 Introduction

General portfolio theory with general deviation measures, developed by Rockafellar et al. (2005a, 2006), was shown to have similar results to the classical portfolio theory Markowitz (1959). Replacement of the standard deviation in the classical portfolio optimization problem by some general deviation measure leads to generalization of concepts of masterfund, efficient froniter, and the CAPM formula. In particular, the necessary and sufficient conditions of optimality in the portfolio problem with general deviation measures were called CAPM-like relations in Rockafelar et al. (2006). In this chapter, we refer to them as generalized CAPM relations; and refer to the underlying theory as the generalized portfolio theory.

This paper makes a connection between the general portfolio theory and the classical asset pricing theory by examination of generalized CAPM relations. In particular, we derive discount factors, corresponding to the CAPM-like relations and consider pricing forms of generalized CAPM relations. We propose a method of calibrating deviation measures from market data and discuss ways of identifying risk preferences of investors in the market within the framework of the general portfolio theory.

### 5.1.1 Definitions and Notations

Following Rockafellar et al. (2005b), we define random variables as elements of $\mathcal{L}^2(\Omega) = \mathcal{L}^2(\Omega, \mathcal{M}, P)$, where $\Omega$ is a space of future states $\omega$, $\mathcal{M}$ is a $\sigma$-algebra on $\Omega$, and $P$ is a probability measure on $(\Omega, \mathcal{M})$. The inner product between elements $X$ and $Y$ in $\mathcal{L}^2(\Omega)$ is

$$\langle X, Y \rangle = E[XY] = \int_\Omega X(\omega)Y(\omega)dP(\omega).$$

In this paper we will use the notions of a deviation measure $\mathcal{D}$, its associated risk envelope $\mathcal{Q}$, and a risk identifier $\mathcal{Q}(X)$ for a random variable $X \in \mathcal{L}^2(\Omega)$ with respect to $\mathcal{D}$. The

reader is referred to Rockafellar and Uryasev (2002) and Rockafellar et al. (2005a, 2006) for details.

### 5.1.2 General Portfolio Theory

The general portfolio theory (Rockafellar et al. (2005a)) is derived in the following framework. The market consists of $n$ risky assets with rates of return modelled by r.v.'s $r_i$ for $i = 1, ..., n$ and a risk-free asset with the constant rate of return modelled by a constant r.v. $r_0$. Several modelling assumptions are made about these rates of returns.

Investors solve the following portfolio optimization problem.

$$\begin{aligned} \min \quad & \mathcal{D}(x_0 r_0 + x_1 r_1 + ... + x_n r_n) \\ \text{s.t.} \quad & E(x_0 r_0 + x_1 r_1 + ... + x_n r_n) \geq r_0 + \Delta \\ & x_0 + x_1 + ... + x_n = 1 \\ & x_i \in \mathrm{R}, \ i = 0, ..., n. \end{aligned} \tag{5–1}$$

In the case of a finite and continuous deviation measure $\mathcal{D}$, generalized CAPM relations come out as necessary and sufficient conditions for optimality in the above problem. It was shown in Rockafellar et al. (2005b) that problem (5–1) has three different types of solution depending on the magnitude of the risk-free rate, corresponding to cases of the master fund of positive type, the master fund of negative type, and the master fund of threshold type. Master fund of positive type is the one most commonly observed in the market, when return of the market portfolio is greater than the risk free rate, and investors would take long positions in the master fund when forming their portfolios.

In this paper, we consider the case of master funds of positive type and the corresponding CAPM-relations

$$Er_i - r_0 = \frac{\mathrm{cov}(-r_i, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})} [Er_M^{\mathcal{D}} - r_0], \ i = 1, ..., n, \tag{5–2}$$

where $r_M^D$ is the rate of return of the master fund, $Q_M^{\mathcal{D}}$ is the risk identifier for the master fund $r_M^D$ corresponding to the deviation measure $\mathcal{D}$.

### 5.1.3 Generalized CAPM relations and Pricing Equilibrium

Relationships (5–2) closely resemble the classical CAPM formula. However, generalized CAPM relations cannot play the same role in the general portfolio framework as CAPM formula plays in the classical theory, as discussed in Rockafellar et al. (2005b). The group of investors using the deviation measure $\mathcal{D}$ is viewed only as a subgroup of all the investors. generalized CAPM relations do not necessarily represent the market equilibrium, as the classical CAPM formula does, and therefore cannot be readily used as a tool for asset pricing. Another difficulty with using relations (5–2) for asset pricing is that neither the master fund nor the asset beta for a fixed master fund can be uniquely determined.

For the pricing using the generalized CAPM relations to make sense, we make the following assumptions.

(A1) All investors in the considered economy use the same deviation measure $\mathcal{D}$.

(A2) The master fund can be identified in the market (or some proxy for the master fund exists). If the set of risk identifiers for the master fund is not a singleton, the choice of a particular risk identifier from this set has negligible effect on asset prices obtained though the generalized CAPM relations. Therefore, we can fix a particular risk identifier for the purpose of asset pricing.

Assumption A2 makes sense because for most basic deviation measures members of the risk identifier set $\mathcal{Q}^{\mathcal{D}}(r_M^{\mathcal{D}})$ for a given master fund $r_M^{\mathcal{D}}$ differ on a set of the form $\{r_M^{\mathcal{D}} = C\}$, where $C$ is a constant. For deviation measures considered in Rockafellar et al. (2006), the risk identifier set for standard deviation and semideviations is a singleton; $C = -\mathrm{VaR}_\alpha(X)$ for CVaR-deviation with confidence level $\alpha$; $C = Er_M^{\mathcal{D}}$ for mean absolute deviation and semideviations. Since asset prices in generalized CAPM (5–2) depend on the risk identifier $Q_M^{\mathcal{D}}$ though $(-r_i, Q_M^{\mathcal{D}})$, assumption A2 suggests that $\mathrm{Prob}\{r_M^{\mathcal{D}} = C\} = 0$.

This is true for continuous distributions of $r_M^{\mathcal{D}}$, and usually holds in practice when the distribution of the master fund is modelled by scenarios.

Assumption A2 cannot be satisfied for worst-case deviation and semideviations, see Rockafellar et al. (2006)

Under assumptions A1 and A2, all quantities in generalized CAPM relations are fixed and well-defined, and the relations represent pricing equilibrium. In further chapters we will closely examine generalized CAPM relations under these assumptions.

## 5.2 Intuition Behind Generalized CAPM Relations

### 5.2.1 Two Ways to Account For Risk

Consider an asset with price $\pi$ and uncertain future payoff $\zeta$. In a risk-neutral world, the asset will be priced as follows.

$$\pi = \frac{1}{1+r_0} E[\zeta], \tag{5–3}$$

where $r_0$ is a risk-free rate of return. The price of an asset is the discounted expected value of its future payoff. The asset with random payoff $\zeta$ would have the same price as an asset with pays $\hat{\zeta} = E[\zeta]$ with probability 1 in the future.

If the risk is present, the price of an asset paying $\hat{\zeta}$ with certainty in future would, generally speaking, differ from the price of the asset having random payoff $\zeta$, such that $E[\zeta] = \hat{\zeta}$. The formula (5–3) needs to be corrected for risk. There are two ways to do it.

The first way is to modify the discounted quantity:

$$\pi = \frac{1}{1+r_0} \Phi(asset), \tag{5–4}$$

where $\Phi(asset)$ is called *the certainty equivalent*. It is a function of asset parameters and is equal to the payoff of a risk-free asset having the same price as the risky asset with payoff $\zeta$.

The second way is to modify the discounting coefficient:

$$\pi = \frac{1}{1 + r_{ra}(asset)} E[\zeta], \quad r_{ra}(asset) \neq -1. \tag{5–5}$$

where $r_{ra}(asset)$ is the *risk-adjusted rate of return.*

Pricing forms of the classical CAPM (see, for example, Luenberger (1998)) are as follows.

Certainty equivalent form of CAPM:

$$\pi = \frac{1}{1 + r_0} \left( E[\zeta] - \frac{cov(\zeta, r_M)(Er_M - r_0)}{\sigma_M^2} \right). \tag{5–6}$$

Risk-adjusted form of CAPM:

$$\pi = \frac{1}{1 + r_0 + \beta(Er_M - r_0)} E[\zeta]. \tag{5–7}$$

Here asset beta $\beta = \frac{cov(r, r_M)}{\sigma_M^2}$, $r_M$ is the rate of return of the master fund, and $r$ is the rate or return of the asset ($r = (\zeta - \pi)/\pi$).

Relevant to further discussion, there is a measure of asset quality known as the Shapre Ratio

$$S = \frac{E[r] - r_0}{\sigma(r)}.$$

It is a risk-return characteristic, measuring the increase in the access return of an asset if the asset volatility in increased by 1. The higher the Sharpe Ratio, the better the asset. Classical CAPM implies that master fund has the highest Shapre Ratio in the economy.

### 5.2.2 Pricing Forms of Generalized CAPM Relations

We now derive pricing forms of the generalized CAPM relations. Substituting $r_i = \zeta_i/\pi_i - 1$ into (5–2), we get

$$
\begin{aligned}
\frac{E\zeta_i}{\pi_i} - (r_0+1) &= \frac{-\operatorname{cov}(\zeta_i/\pi_i - 1, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})}[Er_M^{\mathcal{D}} - r_0] \\
E\zeta_i - \pi_i(r_0+1) &= -\frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}\operatorname{cov}(\zeta_i, Q_M^{\mathcal{D}}) \\
\pi &= \frac{1}{1+r_0}\left(E\zeta_i + \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}\operatorname{cov}(\zeta_i, Q_M^{\mathcal{D}})\right). \qquad (5\text{–}8)
\end{aligned}
$$

Pricing formula (5–8) the certainty equivalent pricing form of generalized CAPM relations (5–2) (compare it to (5–4)), where the certainty equivalent

$$
\Phi(\zeta_i) = E\zeta_i + \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} cov(\zeta_i, Q_M^{\mathcal{D}}) \qquad (5\text{–}9)
$$

is the payoff of a risk-free asset having the same price $\pi_i$.

We could rearrange the formula (5–2) in a different way, namely

$$
\begin{aligned}
Er_i - r_0 &= \frac{\operatorname{cov}(-r_i, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})}[Er_M^{\mathcal{D}} - r_0] \\
\frac{E\zeta_i}{\pi_i} - (r_0+1) &= \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}\operatorname{cov}(-r_i, Q_M^{\mathcal{D}}) \\
\frac{E\zeta_i}{\pi_i} &= (r_0+1) + \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}\operatorname{cov}(-r_i, Q_M^{\mathcal{D}}) \\
\pi_i &= \frac{E\zeta_i}{1 + r_0 + \dfrac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}\operatorname{cov}(-r_i, Q_M^{\mathcal{D}})}, \qquad (5\text{–}10) \\
&\quad \text{when } \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}\operatorname{cov}(-r_i, Q_M^{\mathcal{D}}) \neq 1 + r_0.
\end{aligned}
$$

Formula (5–10) is the risk-adjusted pricing form of generalized CAPM relations (5–2) (compare with (5–10)), where the risk-adjusted rate of return is

$$r_{ra}(r_i) = r_0 + \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} cov(-r_i, Q_M^{\mathcal{D}}). \tag{5–11}$$

The quantity $\dfrac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}$, which we denote by $S_M^{\mathcal{D}}$, in (5–8) and (5–10) is the generalized Sharpe Ratio for the master fund. It shows what increase in excess return can be obtained by increasing the deviation of the asset by 1. In the classical portfolio theory, master fund has the highest Sharpe Ratio among all assets. The same result holds in the generalized setting as we show next.

**Lemma 2.** *For the case of the master fund of positive type, the master fund has the highest generalized Shapre Ratio in the economy, i.e.*

$$\frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} \geq \frac{Er_i - r_0}{\mathcal{D}(r_i)}, \ i = 1, ..., n. \tag{5–12}$$

**Proof:** Consider generalized CAPM relations

$$Er_i - r_0 = \frac{\text{cov}(-r_i, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})} [Er_M^{\mathcal{D}} - r_0]$$

for some asset $i > 0$. The generalized Sharpe Ratio for the master fund is strictly positive $S_M^{\mathcal{D}} = \dfrac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} > 0$ since $Er_M^{\mathcal{D}} - r_0 > 0$ and $\mathcal{D}(r_M^{\mathcal{D}}) > 0$.

If $\text{cov}(-r_i, Q_M^{\mathcal{D}}) = 0$, then $Er_i = r_0$, therefore $\dfrac{Er_i - r_0}{\mathcal{D}(r_i)} = 0$, and (5–12) holds.

If $\text{cov}(-r_i, Q_M^{\mathcal{D}}) < 0$, then $Er_i < r_0$, therefore $\dfrac{Er_i - r_0}{\mathcal{D}(r_i)} < 0$, and (5–12) holds.

If $\text{cov}(-r_i, Q_M^{\mathcal{D}}) > 0$, then according to the dual representation of $\mathcal{D}(r_i)$, we have

$$\mathcal{D}(r_i) = \max_{Q \in \mathcal{Q}} \text{cov}(-r_i, Q) \geq \text{cov}(-r_i, Q_M^{\mathcal{D}}) > 0,$$

where $\mathcal{Q}$ is the risk envelope for the deviation measure $\mathcal{D}$, and $Q_M^{\mathcal{D}} \in \mathcal{Q}$. Dividing both sides of generalized CAPM relations by $\mathrm{cov}(-r_i, Q_M^{\mathcal{D}})$, we get

$$\frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} = \frac{Er_i - r_0}{\mathrm{cov}(-r_i, Q_M^{\mathcal{D}})} \geq \frac{Er_i - r_0}{\mathcal{D}(r_i)}.$$

■

Formulas (5–8) and (5–10) imply that the risk adjustment is determined by the correlation of the asset rate of return with the risk identifier of the master fund.

To gain a better intuition about the meaning of this form of risk adjustment, we compare the classical CAPM formula with the generalized CAPM relations for the CVaR-deviation $\mathcal{D}(X) = CVaR_\alpha^\Delta(X) \equiv CVaR_\alpha(X - EX)$.

First note, that more valuable assets are those with lower returns. When pricing two assets with the same expected return, investors will pay higher price for a more valuable asset, therefore its return will be lower than that of the less valuable asset.

We begin by analyzing the classical CAPM formula written in the form

$$Er_i = r_0 + \frac{\mathrm{cov}(r_i, r_M)}{\sigma_M^2}(Er_M - r_0), \tag{5–13}$$

where the left-hand side of the equation is the asset return. The return is governed by the correlation of the asset rate of return with the market portfolio rate of return, i.e. by the quantity $cov(r_i, r_M)$. Assets with higher return correlation with the market portfolio have higher expected returns, and vice versa. Formula (5–14) implies that assets with lower correlation with the market are more valuable. There is the following intuition behind this result. Investors hold the market portfolio and the risk-free asset; the proportions of holdings depend on the target expected portfolio return. The only source of risk of such investments is introduced by the performance of the market portfolio. The most undesirable states of future are those where market portfolio returns are low. The assets with higher payoff in such states would be more valued, since they serve as insurance against poor performance of the market portfolio. Therefore, the lower the correlation of

an asset return with the market portfolio return, the more protection against undesirable states of the world the asset offers, and the more valuable the asset is. The assets with higher correlation with the market would have higher returns, and vice versa.

Now consider the case of CVaR-deviation, $\mathcal{D}(X) = CVaR_\alpha(X - EX)$. Investors measuring uncertainty of the portfolio performance by this deviation measure are concerned about the value of the average of the $\alpha\%$ worst returns relative to the mean of the return distribution.

We consider generalized CAPM relations for the CVaR-deviation in the case of the master fund of positive type.

$$Er_i = r_0 + \beta_i(Er_M^{\mathcal{D}} - r_0), \tag{5–14}$$

where $\beta_i = \dfrac{\operatorname{cov}(-r_i, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})}$, and the risk identifier $Q_M^{\mathcal{D}}$ is given by

$$\begin{cases} 0 \le Q_M^{\mathcal{D}}(\omega) \le \alpha^{-1}, & EQ_M^{\mathcal{D}} = 1, \\ Q_M^{\mathcal{D}}(\omega) = 0 & \text{when } r_M^{\mathcal{D}}(\omega) > -VaR_\alpha(r_M^{\mathcal{D}}), \\ Q_M^{\mathcal{D}}(\omega) = \alpha^{-1} & \text{when } r_M^{\mathcal{D}}(\omega) < -VaR_\alpha(r_M^{\mathcal{D}}). \end{cases} \tag{5–15}$$

If $\operatorname{prob}\{r_M^{\mathcal{D}} = VaR_\alpha(r_M^{\mathcal{D}})\} = 0$, then

$$\beta_i = \frac{E[Er_i - r_i \,|\, r_M^{\mathcal{D}} \le -VaR_\alpha(r_M^{\mathcal{D}})]}{E[Er_M^{\mathcal{D}} - r_M^{\mathcal{D}} \,|\, r_M^{\mathcal{D}} \le -VaR_\alpha(r_M^{\mathcal{D}})]}. \tag{5–16}$$

For further discussion, assume $\alpha = 10\%$. Then the numerator of (5–16) is the expected underperformance of the asset rate of return with respect to its average rate of return, conditional on the master fund being in its 10% lowest values. The denominator of (5–16) is the the same quantity for the master fund. An investor holds the master fund and the risk-free asset in his portfolio. The portfolio risk is introduced by the performance of the master fund. Formula (5–16) suggests that assets are valued based on their relative performance versus the master fund performance in those future states where the master

fund is in its 10% lowest values. The most valued assets, i.e. assets with lowest returns, would have the lowest betas. Low betas correspond to relatively high asset returns (small values of $Er_i - r_i$) compared to the master fund returns (values of $Er_M^{\mathcal{D}} - r_M^{\mathcal{D}}$), when $r_M^{\mathcal{D}}$ is among 10% its lowest values.

From the general portfolio theory point of view, the value of the asset is, therefore, determined by the extent to which this asset provides protection against poor master fund performance. Depending on the specific form of the deviation measure, the need for this protection corresponds to different parts of the return distribution of the master fund. Most valuable assets drastically differ in performance from the master fund in those cases when protection is needed the most.

## 5.3 Stochastic Discount Factors in General Portfolio Theory

### 5.3.1 Basic Facts from Asset Pricing Theory.

The concept of a stochastic discount factor appears in the classical Asset Pricing Theory (see Cochrane (2001)). Under certain assumptions (stated below), there exists a random variable $m$, called the (stochastic) discount factor or the pricing kernel, which relates asset payoffs $\zeta_i$ to prices $\pi_i$ as follows.

$$\pi_i = E[m\,\zeta_i],\ \ i = 0, ..., n. \tag{5–17}$$

The discount factor is of fundamental importance to asset pricing. Below, we present two theorems due to Ross (1978), and Harrison and Kreps (1979) which emphasize connections between the discount factor and assumptions of absence of arbitrage and linearity of pricing. In the narration, we follow Cochrane (2001), Chapter 4.

Let $\underline{X}$ be the space of all payoffs an investor can form using all available instruments.

We will consider two assumptions, the *portfolio formation* assumption (A1) and the *law of one price* assumption (A2).

(A1) If $\zeta' \in \underline{X}$, $\zeta'' \in \underline{X}$, then $a\zeta' + b\zeta'' \in \underline{X}$ for any $a, b \in \mathbb{R}$.

Let Price($\zeta$) be the price of payoff $\zeta$.

(A2) If $\text{Price}(\zeta') = \pi'$ and $\text{Price}(\zeta'') = \pi''$, then $\text{Price}(a\zeta' + b\zeta'') = a\pi' + b\pi''$ for any $a, b \in \mathbb{R}$.

Under assumption (A1), the payoff space $\underline{X}$ is defined as follows.

$$\underline{X} = \{\zeta \,|\, \zeta = a_0 + a_1\zeta_1 + ... + a_n\zeta_n,\ a_i \in \mathbb{R},\ i = 0, ..., n\},$$

where $\zeta_i$ is the payoff of asset $i$, $i = 0, ..., 1$.

**Theorem 3.** *(1) The existence of a discount factor implies the law of one price A2. (2) Given portfolio formation A1 and the law of one price A2, there exists a unique payoff $\zeta^* \in \underline{X}$ such that the price $\pi$ of any payoff $\zeta \in \underline{X}$ is given by $\pi = E[\zeta^*\zeta]$.*

The second theorem has to do with absence of arbitrage, which is defined as follows.

*Absence of Arbitrage:* The payoff space $\underline{X}$ and the pricing function $\text{Price}(\cdot)$ leave no arbitrage opportunities if every payoff $\zeta$ that is always non-negative, $\zeta \geq 0$ (almost surely), and positive, $\zeta > 0$ with some positive probability, has positive price, $\text{Price}(\zeta) > 0$.

**Theorem 4.** *(1) Existence of a strictly positive discount factor implies absence of arbitrage opportunities. (2) No arbitrage implies the existence of a strictly positive discount factor, $m > 0$, $\pi = E[m\zeta]$ for any $\zeta \in \underline{X}$.*

These theorems for the case of assets with continuous payoffs are given in Hansen and Richard (1987).

From the perspective of discount factors, a complete market is characterized by a unique discount factor; in an incomplete market there exists an infinite number of discount factors and each discount factor produces the same prices of all assets with payoffs in $\underline{X}$ through (5–17). More details on pricing assets in compete and incomplete markets will be provided later on. Important implications of these theorems are as follows.

- There exists a strictly positive discount factor $m > 0$, and such factor might not be unique.
- In the space of payoffs $\underline{X}$ there exists only one discount factor $\zeta^* \in \underline{X}$, which may or may not be strictly positive.

- In a complete market with no arbitrage opportunities, the unique discount factor lies in the payoff space $\underline{X}$ and is strictly positive.
- In an incomplete market with no arbitrage opportunities, all discount factors can be generated as $m = \zeta^* + \varepsilon$, where $\zeta^*$ is the discount factor (unique) in the payoff space $\underline{X}$, and $\epsilon$ is a random variable, orthogonal to $\underline{X}$, $E[\varepsilon\zeta] = 0\ \forall\zeta \in \underline{X}$.
- The discount factor $\zeta^*$ is a projection of any discount factor $m$ on $\underline{X}$. For any asset, $\pi = E[m\zeta] = E[(proj(m|\underline{X}) + \varepsilon)\zeta] = E[proj(m|\underline{X})\,\zeta]$.

It should be mentioned that the existence of so-called "risk-neutral" measure is justified by the existence of a strictly positive discount factor. Indeed, we can rewrite (5–17) as follows.

$$\pi = E[m\zeta] = \int_\Omega m(\omega)\zeta(\omega)dP(\omega) = \frac{1}{1+r_0}\int_\Omega \zeta(\omega)dQ(\omega), \tag{5–18}$$

where $dQ(\omega) = (1 + r_0)m(\omega)dP(\omega)$. Since expectation of $(1 + r_0)m$ equals to one[1] and $m > 0$, $dQ(\omega)$ can be treated as a probability measure. It is usually called the "risk-neutral" probability measure; the risk-neutral pricing form of (5–17) is

$$\pi = \frac{1}{1+r_0}E^Q[\zeta],$$

where $E^Q[\cdot]$ denotes expectation with respect to the risk-neutral measure.

If one picks a discount factor $m$, which is not strictly positive, the transformation (5–18) will lead to the pricing equation $\pi = \int_\Omega \zeta(\omega)dQ(\omega)$ that correctly prices all assets with payoffs in $\underline{X}$. However, $dQ(\omega)$ will not be a probability measure.

[1] Application of (5–17) to the risk-free rate gives $1 = E[m(1 + r_0)]$.

Now consider application of the formula $\pi = E[m\zeta]$ for pricing new assets[2] in complete and incomplete markets.

In a complete market, the payoff of any new asset lies in $\underline{X}$, therefore any new asset will be uniquely priced by the law of one price (alternatively, since the discount factor is unique, there exists only one price, $\pi_{new} = E[m\zeta_{new}]$, for a new asset with payoff $\zeta_{new}$.

In an incomplete market, two cases are possible. (1) The payoff of a new asset belongs to $\underline{X}$; its price is uniquely determined by the law of one price (alternatively, the formula $\pi_{new} = E[m\zeta_{new}]$ will give the same price regardless of which discount factor $m$ is used). (2) The payoff of a new asset does not belong to $\underline{X}$, i.e. the new asset cannot be replicated by the existing ones. In this case, one cannot decide upon a single price of the asset. Let $\zeta_{new}$ be the payoff of a new asset. Upper $\overline{\pi}_{new}$ and lower $\underline{\pi}_{new}$ prices (forming the range of non-arbitrage prices $[\underline{\pi}_{new}, \overline{\pi}_{new}]$) of this asset can be defined as follows.

$$\overline{\pi}_{new} = \sup_{m \in \Phi} E[m\,\zeta_{new}], \quad \underline{\pi}_{new} = \inf_{m \in \Phi} E[m\,\zeta_{new}], \tag{5–19}$$

where $\Phi = \{m \,|\, m(\omega) > 0 \text{ with probability } 1\}$. Including only strictly positive discount factors to the set $\Phi$ leads to arbitrage-free prices given by formula $\pi = E[m\zeta]$.

[2] Originally, we assumed that the market consists of $n + 1$ assets with rates of returns $r_0, r_1, ..., r_n$. Any other asset is considered to be new to the market. A new asset may be replicable by the existing assets (in which case its payoff will belong to $\underline{X}$) or may not be (then its payoff will not belong to $\underline{X}$).

### 5.3.2 Derivation of Discount Factor for Generalized CAPM Relations

We begin by rewriting CAMP-like relations as follows.

$$
\begin{aligned}
Er_i - r_0 &= \frac{cov(-r_i, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})}[Er_M^{\mathcal{D}} - r_0], \\
\frac{E\zeta_i}{\pi_i} - r_0 - 1 &= \frac{1}{\pi_i}\frac{cov(-\zeta_i, Q_M^{\mathcal{D}})}{\mathcal{D}(_M^{\mathcal{D}})}[E_M^{\mathcal{D}} - r_0], \\
E\zeta_i - (r_0+1)\pi_i &= \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}(E\zeta_i E Q_M^{\mathcal{D}} - E[\zeta_i Q_M^{\mathcal{D}}]), \\
\pi_i &= \frac{1}{1+r_0}\left(E\zeta_i + \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}E[\zeta Q_M^{\mathcal{D}}] - \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}E\zeta_i E Q_M^{\mathcal{D}}\right) = \\
&= \frac{1}{1+r_0}E\left[\zeta_i\left((Q_M^{\mathcal{D}} - 1)\frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} + 1\right)\right], \qquad (5\text{–}20) \\
& i = 0, 1, ..., n.
\end{aligned}
$$

Letting

$$
m^{\mathcal{D}}(\omega) = \frac{1}{1+r_0}\left((Q_M^{\mathcal{D}}(\omega) - 1)\frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} + 1\right), \qquad (5\text{–}21)
$$

we arrive at the pricing formula in the form (5–17)

$$
\pi_i = E[m^{\mathcal{D}}\zeta_i], \ i = 0, 1, ..., n. \qquad (5\text{–}22)
$$

The discount factor corresponding to the deviation measure $\mathcal{D}$ is given by (5–21). Pricing formulas (5–22) corresponding to different deviation measures $\mathcal{D}$ will yield the same prices for assets $r_i$, $i = 0, ..., n$, and their combinations (defined by portfolio formation assumption A1), but will produce different prices of new assets, whose payoffs cannot be replicated by payoffs of existing $n+1$ assets. Each deviation measure $\mathcal{D}$ has the corresponding discount factor $m^{\mathcal{D}}$, which is used in (5–22) to determine a unique price of a new asset. An investor has risk related to imperfect replication of the payoff of a new asset, and specifies his risk preferences by choosing a deviation measure in pricing formula (5–22).

### 5.3.3 Geometry of Discount Factors for Generalized CAPM Relations

Consider two deviation measures, $D'$ and $D''$. Both measures provide the same pricing of assets $i = 0, ..., n$: $\pi_i = E[m_{\mathcal{D}'}\zeta_i]$ and $\pi_i = E[m_{\mathcal{D}''}\zeta_i]$. Subtracting these equations yields $E[(m_{\mathcal{D}'} - m_{\mathcal{D}''})\zeta_i] = 0$, $i = 0, ..., n$. The difference of discount factors for any two deviation measures is orthogonal to the payoff space $\underline{\mathrm{X}}$. It follows that discount factor $m^{\mathcal{D}}$ for any $\mathcal{D}$ can be represented as $m^{\mathcal{D}} = m^* + \varepsilon\mathcal{D}$, where $m^* \in \underline{\mathrm{X}}$ is the projection of all discount factors $m^{\mathcal{D}}$ on the payoff space $\underline{\mathrm{X}}$, and $\varepsilon\mathcal{D}$ is orthogonal to $\underline{\mathrm{X}}$. We call $m^*$ pricing generator for the general portfolio theory.

The pricing generator $m^*$ coincides with the discount factor for the standard deviation $\mathcal{D} = \sigma$, since

$$m_\sigma(\omega) = \frac{1}{1+r_0}\left(1 - \frac{r_M^\sigma(\omega) - Er_M^\sigma}{\sigma(r_M^\sigma)}\,\frac{Er_M^\sigma - r_0}{\sigma(r_M^\sigma)}\right) \tag{5–23}$$

together with $r_M^\sigma \in \underline{\mathrm{X}}$ imply $m_\sigma \in \underline{X}$.

For a given payoff space $\underline{\mathrm{X}}$, discount factors $m^{\mathcal{D}}$ for all $\mathcal{D}$ form a subset of all discount factors corresponding to $\underline{\mathrm{X}}$.

### 5.3.4 Strict Positivity of Discount Factors Corresponding to Deviation Measures

We now examine strict positivity of discount factors corresponding to general deviation measures.

The strict positivity condition $m^{\mathcal{D}}(\omega) > 0$ (a.s.) can be written as

$$\begin{gathered}\tfrac{1}{1+r_0}\left((Q_M^{\mathcal{D}}(\omega) - 1)\,\tfrac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})} + 1\right) > 0 \\ Q_M^{\mathcal{D}}(\omega) > 1 - \tfrac{\mathcal{D}(r_M^{\mathcal{D}})}{Er_M^{\mathcal{D}} - r_0}.\end{gathered} \tag{5–24}$$

Note that the left-hand side of condition (5–24) contains a random variable, while the right-hand side is a constant, and the inequality between them should be satisfied with probability one. Scaling the deviation measure $\mathcal{D}$ by some $\lambda > 0$ will change the value of the left-hand side. We show next that it does not change meaning of the condition (5–24).

**Lemma 3.** *Condition (5–24) is invariant with respect to re-scaling deviation measure $\mathcal{D}$.*

**Proof:** Condition (5–24) can be expressed as

$$Q_M^{\mathcal{D}}(\omega) \geq 1 - \frac{1}{S_M^{\mathcal{D}}}, \tag{5–25}$$

where $S_M^{\mathcal{D}} = \frac{\mathcal{D}(r_M^{\mathcal{D}})}{Er_M^{\mathcal{D}} - r_0}$. Consider a re-scaled deviation measure $\hat{\mathcal{D}} = \lambda\mathcal{D}$, $\lambda > 0$. Let $\hat{S}_M^{\hat{\mathcal{D}}}$ be the Sharpe Ratio corresponding to $\hat{\mathcal{D}}$. Since master funds for $\hat{\mathcal{D}}$ and $\mathcal{D}$ are the same, $\hat{S}_M^{\hat{\mathcal{D}}} = \frac{1}{\lambda} S_M^{\mathcal{D}}$.

Since the risk envelopes $\hat{\mathcal{Q}}$ and $\mathcal{Q}$ for deviation measures $\hat{\mathcal{D}}$ and $\mathcal{D}$ are related as

$$\hat{\mathcal{Q}} = (1-\lambda) + \lambda\mathcal{Q},$$

the risk identifiers $\hat{\mathcal{Q}}(r_M^{\mathcal{D}})$ and $\mathcal{Q}(r_M^{\mathcal{D}})$ will be related in the same way, as shown next.

$$\begin{aligned}
\hat{\mathcal{Q}}(r_M^{\mathcal{D}}) &= \operatorname*{argmin}_{\hat{Q}\in\hat{\mathcal{Q}}} \operatorname{cov}(-r_M^{\mathcal{D}}, \hat{Q}) \\
&= \operatorname*{argmin}_{\hat{Q}\in(1-\lambda)+\lambda\mathcal{Q}} \operatorname{cov}(-r_M^{\mathcal{D}}, \hat{Q}) \\
&= (1-\lambda) + \lambda \cdot \operatorname*{argmin}_{Q\in\mathcal{Q}} \operatorname{cov}(-r_M^{\mathcal{D}}, (1-\lambda) + \lambda Q) \\
&= (1-\lambda) + \lambda \cdot \operatorname*{argmin}_{Q\in\mathcal{Q}} \operatorname{cov}(-r_M^{\mathcal{D}}, Q) \\
&= (1-\lambda) + \lambda\mathcal{Q}(r_M^{\mathcal{D}}).
\end{aligned}$$

Finally, if (5–25) holds for $\mathcal{D}$, it holds for $\lambda\mathcal{D}$ as well, since

$$\begin{aligned}
\hat{Q}_M^{\hat{\mathcal{D}}}(\omega) &\geq 1 - \frac{1}{\hat{S}_M^{\hat{\mathcal{D}}}} \\
(1-\lambda) + \lambda Q_M^{\mathcal{D}}(\omega) &\geq 1 - \frac{\lambda}{S_M^{\mathcal{D}}} \\
\lambda Q_M^{\mathcal{D}}(\omega) &\geq \lambda - \frac{\lambda}{S_M^{\mathcal{D}}} \\
Q_M^{\mathcal{D}}(\omega) &\geq 1 - \frac{1}{S_M^{\mathcal{D}}}.
\end{aligned}$$

■

Next, we show that the pricing generator $m^*$ is not strictly positive. Indeed, the corresponding risk identifier is given by

$$Q_M^\sigma(\omega) = 1 - \frac{r_M^\sigma(\omega) - Er_M^\sigma}{\sigma(r_M^\sigma)}.$$

Condition $\mathcal{M}(\omega) > 0$ takes the form

$$\begin{aligned} 1 - \frac{r_M^\sigma(\omega) - Er_M^\sigma}{\sigma(r_M^\sigma)} &> 1 - \frac{\mathcal{D}(r_M^\sigma)}{Er_M^\sigma - r_0} \\ \frac{r_M^\sigma(\omega) - Er_M^\sigma}{\sigma(r_M^\sigma)} &< \frac{\mathcal{D}(r_M^\sigma)}{Er_M^\sigma - r_0} \\ r_M^\sigma(\omega) &< Er_M^\sigma + \frac{\sigma^2(r_M^\sigma)}{Er_M^\sigma - r_0}. \end{aligned}$$

The last inequality is violated with positive probability, for instance, for normally distributed random variables.

Consider an alternative representation of $m^{\mathcal{D}}(\omega)$ in (5–21). Letting $S_M^{\mathcal{D}} = \frac{Er_M^{\mathcal{D}} - r_0}{\mathcal{D}(r_M^{\mathcal{D}})}$, we get

$$m^{\mathcal{D}}(\omega) = \frac{1}{1+r_0}\left((Q_M^{\mathcal{D}}(\omega) - 1)S_M^{\mathcal{D}} + 1\right) = \frac{1}{1+r_0}\left((Q_M^{\mathcal{D}}(\omega)S_M^{\mathcal{D}} + (1 - S_M^{\mathcal{D}})\right). \tag{5–26}$$

In Lemma 1 we showed that risk identifiers $\hat{\mathcal{Q}}(r_M^{\mathcal{D}})$ and $\mathcal{Q}(r_M^{\mathcal{D}})$ for deviation measures $\hat{\mathcal{D}} = \lambda\mathcal{D}$ ($\lambda > 0$) and $\mathcal{D}$, respectively, are related as

$$\hat{\mathcal{Q}}(r_M^{\mathcal{D}}) = (1 - \lambda) + \lambda\mathcal{Q}(r_M^{\mathcal{D}}).$$

This allows to rewrite the expression for the discount factor as follows,

$$m^{\mathcal{D}}(\omega) = \frac{1}{1+r_0} Q_M^{\mathcal{D}_M}(\omega),$$

where $Q_M^{\mathcal{D}_M}(\omega)$ is a risk identifier for the deviation measure $\mathcal{D}_M = S_M^{\mathcal{D}} \cdot \mathcal{D}$. Strict positivity of a discount factor is then equivalent to strict positivity of the risk identifier $Q_M^{\mathcal{D}_M}(\omega)$.

## 5.4 Calibration of Deviation Measures Using Market Data

### 5.4.1 Identification of Risk Preferences of Market Participants

As was discussed earlier, if a general portfolio problem is posed for a set of basic assets $r_0, r_1, ..., r_n$, then deviation measures gives the same prices to these assets, as well as to any assets whose payoffs can be replicated by payoffs of the basic assets. In this section, we examine ways of estimation (calibration) of the deviation measure $\mathcal{D}$ in the general portfolio theory from market data. Numerical methods supporting the proposed algorithms are not considered in this work; we concentrate on the meaning of the calibration methods, their advantages and drawbacks, and their limitations in determination of risk preferences of market participants. Essentially, calibration of deviation measures is done by adjusting it until the generalized CAPM relations

$$Er_i = r_0 + \frac{\mathrm{cov}(-r_i, Q_M^{\mathcal{D}})}{\mathcal{D}(r_M^{\mathcal{D}})}[Er_M^{\mathcal{D}} - r_0],\ i = 1, ..., n. \tag{5–27}$$

provide the most accurate asset pricing. We could either take a set of asset returns $r_i$ from the market and estimate the master fund $r_M^{\mathcal{D}}$, or treat the master fund as given by the market and estimate expected returns $Er_i$. The obtained quantities, the master fund return or expected returns of the assets, will depend on the deviation measure $\mathcal{D}$, which can be calibrated by comparing estimated quantities to their market values.

We limit our consideration to the case of known master fund; the method based on estimation of a master fund given a set of assets is more computationally difficult, because the generalized portfolio problem should be solved for each choice of $\mathcal{D}$.

Assumption that a master fund can be obtained from the market is justified by the existence of indices, such as S&P 500, Dow Jones Industrial Average and Nasdaq 100, which represent the state of some large part of the market; moreover, investing in these indices can be thought of as investing in the market.

Any broad-based market index is associated with certain selection of assets; the index summarizes the behavior of the market of these assets. We could calibrate the generalized

CAPM relations by pricing assets from the index-associated pool, or by pricing foreign assets to this pool. These two ideas have different meaning as they refer to different ways risk preferences are manifested in the general portfolio theory.

The first calibration method is based on pricing assets from the index pool. The index serve as a master fund in a generalized portfolio problem posed for assets from the pool. Given a fixed selection of assets, different deviation measures would produce different master funds. The existence of a particular master fund for these assets in the market can, therefore, be used as a basis for estimation of a deviation measure. The "best" deviation measure is the one which yields the best match between the expected returns of assets from the pool and the index return through the generalized CAPM relations.

The second calibration method is based on pricing assets lying outside of the index pool. As we discussed earlier, when pricing a new asset whose payoff does not belong to the initially considered payoff space, the price investors would pay depends on their risk preferences, defined by the deviation measure. The second method, therefore, uses prices of "new" assets with respect to the index pool as the basis for estimation of risk preferences. It should be noted that in the setup of the general portfolio theory the selection of assets is fixed, and the master fund depends on the deviation measure. In the present method we assume that the master fund is fixed and change the deviation measure to obtain the best match between the master fund return and expected returns of new assets. By doing so, we imply that the choice of the index-associated pool of assets depends on the deviation measure.

We justify the assumption of a fixed master fund by the observation that master funds, expected returns of assets, and their generalized betas can be determined from the market data quite easily, while the selection of assets corresponding to an index can be determined much more approximately. An index usually represents behavior of a part of the market consisting of much more instruments that the index is comprised of. With much certainty, though, we could assume that assets constituting the index belong to the

pool of assets represented by the index. Therefore, the first calibration method can be based on matching the prices of assets the index consists of.

We also note that implementations of both methods are the same: selecting some index as a master fund, we adjust the deviation measure until the generalized CAPM relations provide most accurate pricing of a certain group of assets. We refer to this group of assets as the target group.

Finally, we discuss the question, should the two calibration methods give the same results. Generally speaking, for a fixed set of assets, the choice of risk preferences in terms of a deviation measure determines both the master fund and pricing of new assets with payoffs outside of the considered payoff space. When the generalized portfolio problem is posed for the whole market, risk preferences can be determined only through matching the master fund, since there are no "new" assets with respect to the whole market. The master fund coincides with the market portfolio, i.e. weight of an asset in the master fund equals the capitalization weight of this asset in the market.

If a certain index is assumed to represent the whole market, then calibration of the deviation measure based on different target groups of assets (for example, on a group of stocks and a group of derivatives on these stocks) should give the same result. If the obtained risk preferences do not agree, this may indicate that either the general portfolio theory with a single deviation measure is not applicable to the market or that the index does not adequately represent the market.

If indices track performance of some parts of the market, the two methods are not, generally speaking, expected to give the same results. Market prices of assets not belonging to an index group may not be directly influenced by risk preferences of investors holding the index in their portfolios. For example, it does not make sense to calibrate risk preferences by taking one index as a master fund and assets from another index as a target set of assets. However, it is reasonable to suppose that prices of derivatives (for example, options) on the assets belonging to an index group are formed by risk

preferences of investors holding this index. Derivatives on assets have non-linear payoffs and cannot be replicated by payoffs of these assets. The second calibration method applied to pricing derivatives on some stocks is expected to give similar risk preferences as the first calibration method applied to pricing the same stocks, where the master fund is taken to be the index representing these stocks. If the so-obtained risk preferences do not agree, either the general portfolio theory is does not adequately represent the chosen part of the market or option prices are significantly influenced by factors, not captured by the risk preferences of investors holding the corresponding index in their portfolios.

### 5.4.2 Notations

We consider two implementations of calibration methods. We assume that the index-associated group of assets consists of $n$ assets with rates of return $r_1, ..., r_n$, the master fund associated with the deviation measure $\mathcal{D}$ is a portfolio of these assets and the risk-free asset with the rate of return $r_0$; the rate of return of the master fund is $r_M^{\mathcal{D}}$. The target group of assets consists of $k$ assets with rates of return $r'_1, ..., r'_k$. The target assets may or may not belong to the index-associated group.

For the purposes of calibration, we assume a parametrization of a deviation measure $\mathcal{D} = \mathcal{D}_\alpha$, where $\alpha = (\alpha_1, ..., \alpha_l)$ is a vector of parameters.

### 5.4.3 Implementation I of Calibration Methods

The first implementation is based on direct estimation of expected returns of target assets and minimization of the estimation error with respect to parameters $\vec{\alpha}$. Let $E\hat{\mathbf{r}}'(\alpha) = (E\hat{r}'_1(\alpha), ..., E\hat{r}'_k(\alpha))$ be a vector of expected returns of target assets estimated using the deviation measure $\mathcal{D}_\alpha$, $E\mathbf{r}' = (Er'_1, ..., \mathrm{E}r'_k)$ be a vector of the true expected rates of return, and $\mathrm{Dist}(E\mathbf{r}', E\hat{\mathbf{r}}'(\alpha))$ be a measure of distance between the two vectors. The parameters $\alpha$ the deviation measure can be calibrated by solving the following

optimization problem.

$$\begin{aligned} &\min_{\vec{\alpha}} \quad \mathrm{Dist}(E\mathbf{r}', E\hat{\mathbf{r}}'(\alpha)) \\ &\text{s.t.} \qquad E\hat{r}_i'(\alpha) = r_0 + \frac{\mathrm{cov}(-r_i', Q_M^{\mathcal{D}_\alpha})}{\mathcal{D}_\alpha(r_M^{\mathcal{D}_\alpha})}[Er_M^{\mathcal{D}_\alpha} - r_0],\ i = 1, ..., k, \end{aligned}$$

where covariances $\mathrm{cov}(-r_i', Q_M^{\mathcal{D}_\alpha})$ are calculated from the market.

### 5.4.4 Implementation II of Calibration Methods

The second implementation is common in the literature of restoring risk preferences. It is based on estimating the ratio of risk-neutral and actual distributions of the master fund. We adapt the procedure to the setup of the general portfolio theory. The target group of assets now consists of European call options on the master fund. The implementation differs from the previous one as follows. Expected returns of target assets and the rate of return of the master fund are replaced by the risk-neutral density of the master fund and the actual density of the master fund, respectively; the match between these densities is optimized with respect to parameters $\alpha$. The mentioned risk-neutral density can actually be referred to as "density" only under certain assumptions, ensuring that this function is non-negative. However, methods of estimating this function from market data usually assume its non-negativity. The use of European options in this implementation is essential, therefore it can be an implementation of the second calibration method only. Options have non-linear payoffs with respect to the underlying assets and do not belong to the pool of assets associated with the chosen index.

Assume that the probability measure $P$ in the market has a density function $p(\omega)$. We consider the generalized CAPM relations in the form (5–22) and transform them as follows ($\Omega$ denotes the complete set of future events $\omega$).

$$\pi = E[m^{\mathcal{D}}\zeta] = \int_\Omega \zeta(\omega) m^{\mathcal{D}}(\omega) p(\omega) d\omega = \frac{1}{1+r_0}\int_\Omega \zeta(\omega)(1+r_0) m^{\mathcal{D}}(\omega) p(\omega) d\omega, \qquad (5\text{–}28)$$

where $\pi$ and $\zeta$ are the price and the payoff of an asset. Letting

$$q^{\mathcal{D}}(\omega) = (1 + r_0)m^{\mathcal{D}}(\omega)p(\omega), \tag{5–29}$$

we get

$$\pi = \frac{1}{1 + r_0} \int_{\Omega} \zeta(\omega) q^{\mathcal{D}}(\omega) d\omega. \tag{5–30}$$

As we discussed above, if the discount factor $m^{\mathcal{D}}(\omega)$ is strictly positive, the function $q^{\mathcal{D}}(\omega)$ could be called the "risk-neutral" density function.

The future event $\omega$ consists of future returns of all assets in the market and can be represented as $\omega = (r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r})$, where $\check{r}$ represents rates of returns of the rest of assets in the market.[3]

Now consider integrating relationship (5–29) with respect to $r_1', ..., r_k', \check{r}$.

$$\int_{\Omega} q^{\mathcal{D}}(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_1' ... dr_k' d\check{r} = (1+r_0) \int_{\Omega} m^{\mathcal{D}}(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) p(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_1' ... dr_k' d\check{r}. \tag{5–31}$$

Let

$$\tilde{q}^{\mathcal{D}}(r_M^{\mathcal{D}}) = \int_{\Omega} q^{\mathcal{D}}(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_1' ... dr_k' d\check{r}.$$

If $m^{\mathcal{D}}$ was strictly positive, $\tilde{q}^{\mathcal{D}}(r_M^{\mathcal{D}})$ would be a risk-neutral marginal distribution of the master fund. To simplify the right-hand side of (5–31), note that the discount factor $m^{\mathcal{D}}$ is a linear transformation of the risk identifier $Q_M^{\mathcal{D}}$ (both $m^{\mathcal{D}}$ and $Q_M^{\mathcal{D}}$ are random variables and are function of $\omega$). Due to the representation

$$Q_M^{\mathcal{D}} \in \mathcal{Q}_M^{\mathcal{D}} = \operatorname*{argmin}_{Q \in \mathcal{Q}} E[r_M^{\mathcal{D}} Q],$$

[3] The master fund is not an asset but a portfolio of assets with rates of return $r_1, ..., r_n$. The future state $\omega$ is initially represented as $\omega = (r_1, ..., r_n, r_1', ..., r_k', \check{r})$. Assuming that the asset $r_1$ is represented in the master fund with non-zero coefficient, we can represent $\omega$ as $\omega = (r_M^{\mathcal{D}}, r_2, ..., r_n, r_1', ..., r_k', \check{r})$. After including $r_2, ..., r_n$ to $\check{r}$, we get the representation $\omega = (r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r})$.

the risk identifier $Q_M^{\mathcal{D}}(\omega)$ depends on $\omega$ through $r_M^{\mathcal{D}}$, $Q_M^{\mathcal{D}}(\omega) = Q_M^{\mathcal{D}}(r_M^{\mathcal{D}}(\omega))$. For example, for the case of standard deviation $\mathcal{D} = \sigma$

$$Q_M^{\sigma}(\omega) = 1 - \frac{r_M^{\sigma}(\omega) - Er_M^{\sigma}}{\sigma(r_M^{\sigma})},$$

so $Q_M^{\sigma}(\omega) = Q_M^{\sigma}(r_M^{\sigma}(\omega))$.

Equation (5–31) becomes

$$\begin{aligned}
\tilde{q}^{\mathcal{D}}(r_M^{\mathcal{D}}) &= (1 + r_0) \textstyle\int_{\Omega} m^{\mathcal{D}}(r_M^{\mathcal{D}}) p(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_1' ... dr_k' d\check{r}, \\
\tilde{q}^{\mathcal{D}}(r_M^{\mathcal{D}}) &= (1 + r_0) m^{\mathcal{D}}(r_M^{\mathcal{D}}) \textstyle\int_{\Omega} p(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_1' ... dr_k' d\check{r}, \\
\tilde{q}^{\mathcal{D}}(r_M^{\mathcal{D}}) &= (1 + r_0) m(r_M^{\mathcal{D}}) \tilde{p}(r_M^{\mathcal{D}}),
\end{aligned} \tag{5–32}$$

where $\tilde{p}(r_M^{\mathcal{D}}) = \int_{\Omega} p(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_1' ... dr_k' d\check{r}$ is the actual marginal distribution of the master fund.

Relationship (5–29) is now transformed into

$$\tilde{q}^{\mathcal{D}}(r_M^{\mathcal{D}}) = (1 + r_0) m^{\mathcal{D}}(r_M^{\mathcal{D}}) \tilde{p}(r_M^{\mathcal{D}}), \tag{5–33}$$

where $r_M^{\mathcal{D}} = r_M^{\mathcal{D}}(\omega)$. This relationship provide the basis for calibration of $\mathcal{D}$. Let $\tilde{q}(r_M^{\mathcal{D}})$ denote the true risk-neutral distribution of the master fund. Both functions $\tilde{q}(r_M^{\mathcal{D}})$ and $\tilde{p}(r_M^{\mathcal{D}})$ can be estimated from market data; the error in estimation of $\tilde{q}(r_M^{\mathcal{D}})$ by $\tilde{q}^{\mathcal{D}} r_M^{\mathcal{D}})$ in (5–33) is minimized with respect to $\mathcal{D}$.

First, we consider estimation of $\tilde{q}(r_M^{\mathcal{D}})$. Let $q(\omega)$ be the true market risk-neutral distribution, $\tilde{q}(r_M^{\mathcal{D}}) = \int_{\Omega} q(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_1' ... dr_k' d\check{r}$. Applying formula (5–30) with $q(\omega)$ for pricing an option on the master fund, we get

$$\pi_c = \frac{1}{1 + r_0} \int_{\Omega} \zeta_c(r_M^{\mathcal{D}}) q(r_M^{\mathcal{D}}, r_1', ..., r_k', \check{r}) dr_M^{\mathcal{D}} dr_1' ... dr_k' d\check{r} = \frac{1}{1 + r_0} \int_{\Omega} \zeta_c(r_M^{\mathcal{D}}) \tilde{q}(r_M^{\mathcal{D}}) dr_M^{\mathcal{D}},$$

where $\pi_c$ and $\zeta_c$ are the price and the payoff of the option.

To simplify notations, we will consider applying the above formula for a call option on with price $C$, strike $K$, time to expiration $T$.[4] The option is written on a master fund with current price $S_0$ and price $S = S(\omega)$ at expiration of the option. The derivation below concerns estimation of the function $\tilde{q}(S)$, and $\tilde{q}(r_M^{\mathcal{D}}) = \tilde{q}(S/S_0)$.

$$\begin{aligned} e^{rT}C = \int_{-\infty}^{+\infty} [S-K]^+ \tilde{q}(S)dS = \int_K^{+\infty} (S-K)\tilde{q}(S)dS = \\ = \int_K^{+\infty} S\tilde{q}(S)dS - K\int_K^{+\infty} \tilde{q}(S)dS. \end{aligned} \tag{5–34}$$

Differentiating (5–34) with respect to $K$, we get

$$e^{rT}\frac{\partial C}{\partial K} = -K\tilde{q}(K) - \int_K^{+\infty} \tilde{q}(S)dS + K\tilde{q}(K) = -\int_K^{+\infty} \tilde{q}(S)dS.$$

Differentiating (5–34) twice with respect to $K$, we arrive at the formula for estimating risk-neutral density $q$ from cross section of option prices

$$e^{rT}\frac{\partial^2 C}{\partial K^2} = -\frac{\partial}{\partial K}\int_K^{+\infty} \tilde{q}(S)dS = \tilde{q}(K),$$

or in the most common form

$$\tilde{q}(S) = e^{rT}\left.\frac{\partial^2 C}{\partial K^2}\right|_{K=S}. \tag{5–35}$$

Formula (5–35) allows to estimate the function $\tilde{q}(r_M^{\mathcal{D}})$ when the cross-section of prices of options written on the master fund is available. It is worth mentioning that this method estimates $\tilde{q}(r_M^{\mathcal{D}})$ at a given point in time; it is based on options prices at this time.

Now consider estimation of the marginal probability density $\tilde{p}(r_M^{\mathcal{D}})$. The most common way to estimate this density is to use kernel density estimation based on certain period of historical data. However, this method assumes that the density does not change over time. When time dependence is taken into account, we are left with only one

[4] For options with time to maturity $T$, the discount coefficient is $e^{rT}$ rather than $1 + r_0$.

realization of this density at each point in time; direct estimation of the density is not possible.

However, the formula (5–32) provides a way of estimating $\tilde{p}(r_M^{\mathcal{D}})$ for a specific date, if the function $m^{\mathcal{D}}(r_M^{\mathcal{D}})$ is known. This idea is utilized in the utility estimation algorithm suggested in Bliss and Panigirtzoglou (2001). We develop a modification of this method to calibrate the deviation measure, as follows.

Assume the parametrization $\mathcal{D} = \mathcal{D}_\alpha$. Also assume that the master fund is known from the market and therefore is fixed, its rate of return is denoted by $r_M$. For each date $t = 1, ..., T$, we estimate the function $m_t^\alpha(r_M)$ using (5–21). Quantities $\mathcal{D}_\alpha(r_M)$, $Er_M^{\mathcal{D}_\alpha}$, and $Q_M^{\mathcal{D}_\alpha}$ in the definition of $m_t^\alpha(r_M)$ , are calculated based on a certain period of historical returns the index. Also, we estimate functions $\tilde{q}_t(r_M)$, $t = 1, ..., T$, using (5–35). Formula (5–33) allows to estimate function $\tilde{q}_t^\alpha(r_M)$ for each parametrization of $\mathcal{D}_\alpha$. The parameters $\alpha$ can be calibrated by hypothesizing that $\tilde{q}_t(r_M) = \tilde{q}_t^\alpha(r_M)$ for $t = 1, ..., T$ (which holds if $\mathcal{D}_\alpha$ is the correct deviation measure in the market) and maximizing the p-value of an appropriate statistic.

This hypothesis is further transformed as follows. Using the true risk-neutral distributions $\tilde{q}_t(r_M)$, the actual distributions $\tilde{p}_t(r_M)$ are estimated using (5–33),

$$\tilde{p}_t^\alpha(r_M) = \frac{\tilde{q}_t(r_M)}{(1 + r_0)m_t^\alpha(r_M)},$$

$t = 1, ..., T$. We then test the null hypothesis that risk-neutral distributions $\tilde{p}_t^\alpha(r_M)$, $t = 1, ..., T$, equal to the true risk-neutral distributions $\tilde{p}_t(r_M)$, $t = 1, ..., T$.

For each time $t = 1, ..., T$, only one realization $r_M(t)$ of the master fund is available; the value $r_M(t)$ is a single sample from the true density $\tilde{p}_t(r_M)$. Under the null hypothesis $\tilde{p}_t^\alpha(r_M) = \tilde{p}_t(r_M)$, therefore random variables $y_t^\alpha$ defined by

$$y_t^\alpha = \int_{-\infty}^{r_M(t)} \tilde{p}_t^\alpha(r)dr,$$

for $t = 1, ..., T$, are i.i.d. Uniform$[0, 1]$ random variables.

Joint uniformity and independence of $y_t^\alpha$, $t = 1, ..., T$, can be tested using Berkowitz $LR_3$ statistic (see Berkowitz (2001)), which has the chi-squared distribution with 3 degrees of freedom $\chi^2(3)$ under the null hypothesis. The deviation measure $\mathcal{D}_\alpha$ can be calibrated by maximizing the p-value of the $LR_3$ statistic with respect to parameters $\alpha$.

### 5.4.5 Discussion of Implementation Methods

Both considered implementations are based on the same idea (fitting the generalized CAPM relations to market data) but algorithmically are quite different.

Implementation I requires calculation expected returns of assets and estimation of actual distribution of the master fund. These quantities can be found from the market data quite easily and accurately. However, the results of this implementation depend on a particular choice of the objective function $\mathrm{Dist}(\cdot, \cdot)$. It can be argued that the choice of the objective should depend on a the parametrization $\mathcal{D}_\alpha$ of the deviation measure being calibrated. For example, if the deviation measure is calibrated in the form of the mixed CVaR-deviation, then $\mathrm{Dist}(\cdot, \cdot)$ should be based on the CVaR-deviation, rather than on the standard deviation. Another drawback of implementation I is that the financial literature did not use similar algorithms for calibration of utility functions. When risk preferences are estimated using this implementation for the general portfolio theory are compared with risk preferences estimated in financial literature for the utility theory, the results may differ just due to differences in numerical procedures, underlying the two estimations.

Implementation II is widely used in financial literature for estimation of risk-aversion coefficients of utility functions. However, this implementation suffers from some numerical challenges related to evaluation of the actual density in the formula

$$\tilde{p}_t^\alpha(r_M) = \frac{\tilde{q}_t(r_M)}{(1 + r_0) m_t^\alpha(r_M)}.$$

The first challenge is estimation of risk-neutral distributions $\tilde{q}_t(r_M)$. There are several methods of estimation the risk-neutral distribution from the cross-section of options prices used in the literature, but the results of estimations are sensitive to the data and may

significantly depend on a method used. Second challenge is estimation of the function $\tilde{p}_t^\alpha(r_M)$. Discount factors $m_t^\alpha(r_M)$ may be close to zero for some values of $r_M$, which makes accuracy of estimation of $\tilde{q}_t(r_M)$ crucial for calculation of densities $\tilde{p}_t^\alpha(r_M)$ and even more crucial for calculation of $y_t^\alpha$, $t = 1, ..., T$. It follows that risk preferences obtained using implementation II can only be trusted if the underlying numerical methods are very reliable.

There is one more drawback of this implementation when it is in the general portfolio theory. When using numerical estimation of risk-neutral densities, we have to assume that no arbitrage opportunities exist in the prices of options from the cross-section. This implies that only strictly positive discount factors $m^{\mathcal{D}}(r_M)$ should be used for calibration. Indeed, if $m^{\mathcal{D}}(r_M(\omega)) < 0$ with positive probability, then estimates of the risk-neutral density $\tilde{q}_t^\alpha(r_M) = (1 + r_0)m^\alpha(r_M)\tilde{p}(r_M)$ can be negative, and the hypothesis that the true risk-neutral densities $\tilde{q}_t(r_M(\omega)) > 0$ (estimated from options cross-section) are equal to the estimated densities $\tilde{q}_t^\alpha(r_M)$ does not make sense. However, it is not clear at this point, which deviation measures have the property that $m^{\mathcal{D}}(\omega) > 0$ with probability 1.

Finally, there is an issue relevant to estimation of return distributions of assets based on their historical returns. Historical data may contain outliers or effects of rare historical events. After such "cleaning", historical data may provide more reliable conclusions. However, filtering historical data from historical effects is an open question.

## 5.5 Coherence of Mixed CVaR-Deviation

One of the flexible parameterizations of a deviation measure is mixed CVaR-deviation of gains and losses. One of desirable properties of deviation measures is coherence. Coherent deviation measures express risk preferences which are more appealing from the point of view financial intuition and optimization than the deviation measures lacking coherence. In this section, we examine coherence of the mixed CVaR-deviation of gains and losses

The mixed-CVaR deviation of gains and losses is defined as follows.

$$\mathcal{D}(X) = \sum_{i=1}^{n} \gamma_i CVaR_{\alpha_i}(X - EX) + \sum_{j=1}^{n} \delta_j CVaR_{\beta_j}(-X + EX),$$
$$\sum_{i=1}^{n} \gamma_i + \sum_{j=1}^{n} \delta_j = 1,\ \alpha_i \geq 0, \beta_j \geq 0 \text{ for all } i,\ j. \tag{5–36}$$

We will refer to deviation measure $CVaR_\alpha(X - EX)$ as CVaR deviation of gains, to deviation measure $CVaR_\beta(-X + EX)$ as CVaR deviation of losses.

Risk identifier for a convex combination of deviation measures is a convex combination of their risk identifiers. Risk identifier for CVaR deviation of losses was derived in Rockafellar et al. (2006). Below, we derive the risk identifier for CVaR deviation of losses and mixed-CVaR deviation of losses and examine coherence of these deviation measures.

The following lemma will help to find risk-identifier for $\mathcal{D}(X) = CVaR_\alpha(-X + EX)$.

**Lemma 4.** *Consider the deviation measure $\mathcal{D}$ and let $\tilde{\mathcal{D}}$ be the reflection of $\mathcal{D}$, i.e. $\tilde{\mathcal{D}}(X) = \mathcal{D}(-X)$. Let $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ be risk envelopes and $\mathcal{Q}(X)$ and $\tilde{\mathcal{Q}}(X)$ be risk identifiers for the random variable $X$ for deviation measures $\mathcal{D}$ and $\tilde{\mathcal{D}}$, respectively. Then*

$$\tilde{\mathcal{Q}} = 2 - \mathcal{Q}, \tag{5–37}$$

*and*

$$\tilde{\mathcal{Q}}(X) = 2 - \mathcal{Q}(-X). \tag{5–38}$$

**Proof:**

To verify (5–37) we need to prove the dual representation $\tilde{\mathcal{D}}(X) = EX - \inf_{\tilde{Q}\in\tilde{\mathcal{Q}}} E[X\tilde{Q}]$ and also show that $\tilde{\mathcal{Q}}$ satisfies properties (Q1) - (Q3). The dual representation is correct since,

$$EX - \inf_{\tilde{Q}\in\tilde{\mathcal{Q}}} E[X\tilde{Q}] = EX - \inf_{Q\in\mathcal{Q}} E[X(2-Q)] = E[-X] - \inf_{Q\in\mathcal{Q}} E[-XQ] = \mathcal{D}(-X) = \tilde{\mathcal{D}}(X).$$

Properties (Q1) and (Q3) of $\tilde{\mathcal{Q}}$ follow immediately from properties of $\mathcal{Q}$. To prove property (Q2) we need to show that for each non-constant $X$ there exists $\tilde{Q} \in \tilde{\mathcal{Q}}$ so that $E[X\tilde{Q}] \leq EX$. Indeed, fix a non-constant $X$. According to property (Q2) of the risk envelope $\mathcal{Q}$ stated for the random variable $-X$ there exists $Q' \in \mathcal{Q}$ such that $E[-XQ'] < E[-X]$. The property (Q2) will hold with $\tilde{Q}' = 2 - Q'$, since

$$E[X\tilde{Q}'] = E[X(2 - Q')] = 2EX + E[-XQ'] < 2EX + E[-X] = EX.$$

To prove (5–38), we will use the formula $\partial\mathcal{D}(X) = 1 - \mathcal{Q}(X)$ and the fact that $\partial\tilde{\mathcal{D}}(X) = -\partial\mathcal{D}(-X)$.

$$1 - \tilde{\mathcal{Q}}(X) = \partial\tilde{\mathcal{D}}(X) = -\partial\mathcal{D}(-X) = -1 + \mathcal{Q}(-X)$$
$$\tilde{\mathcal{Q}}(X) = 2 - \mathcal{Q}(-X).$$

■

From (5–38), the risk envelope for the deviation measure $\tilde{\mathcal{D}}(X) = CVaR_\beta(-X + EX)$ is[5]

$$\tilde{\mathcal{Q}} = \{\, \tilde{Q} \,|\, 2 - \alpha^{-1} \leq \tilde{Q} \leq 2,\ E\tilde{Q} = 1\}.$$

To find the risk identifier $\tilde{\mathcal{Q}}(X)$, consider the risk identifier $\mathcal{Q}(X)$ for CVaR deviation of gains $\mathcal{D}(X) = CVaR_\alpha(X - EX)$, given by

$$Q \in \mathcal{Q}(X) \Longleftrightarrow \begin{cases} Q(\omega) = \alpha^{-1}, & \text{when } X(\omega) < -VaR_\alpha(X) \\ 0 \leq Q(\omega) \leq \alpha^{-1}, & \text{when } X(\omega) = -VaR_\alpha(X) \\ Q(\omega) = 0, & \text{when } X(\omega) > -VaR_\alpha(X). \end{cases}$$

[5] The risk envelope for $\mathcal{D}(X) = CVaR_\alpha(X - EX)$ is $\mathcal{Q} = \{\, Q \,|\, 0 \leq Q \leq \alpha^{-1},\ EQ = 1\}$.

Therefore $Q \in \mathcal{Q}(-X)$ is equivalent to having

$$\begin{cases} Q(\omega) = \alpha^{-1}, & \text{when } -X(\omega) < -VaR_\alpha(-X) \\ 0 \le Q(\omega) \le \alpha^{-1}, & \text{when } -X(\omega) = -VaR_\alpha(-X) \\ Q(\omega) = 0, & \text{when } -X(\omega) > -VaR_\alpha(-X), \end{cases}$$

or

$$\begin{cases} Q(\omega) = \alpha^{-1}, & \text{when } X(\omega) > VaR_\alpha(-X) \\ 0 \le Q(\omega) \le \alpha^{-1}, & \text{when } X(\omega) = VaR_\alpha(-X) \\ Q(\omega) = 0, & \text{when } X(\omega) < VaR_\alpha(-X), \end{cases}$$

and the risk identifier $\tilde{\mathcal{Q}}(X)$ is given by

$$\tilde{Q} \in \tilde{\mathcal{Q}}(X) \Longleftrightarrow \begin{cases} \tilde{Q}(\omega) = 2 - \beta^{-1}, & \text{when } X(\omega) > VaR_\beta(-X) \\ 2 - \beta^{-1} \le \tilde{Q}(\omega) \le 2, & \text{when } X(\omega) = VaR_\beta(-X) \\ \tilde{Q}(\omega) = 2, & \text{when } X(\omega) < VaR_\beta(-X). \end{cases} \tag{5–39}$$

Next, we examine coherence of CVaR and mixed-CVaR deviations of losses. Coherence of a deviation measure $\mathcal{D}$ is equivalent to having $Q \ge 0$ for all $Q \in \mathcal{Q}$, where $\mathcal{Q}$ is a risk envelope for the deviation measure $\mathcal{D}$. We will now show that it suffices to check the non-negativity of all risk identifiers $\mathcal{Q}(X)$ for all random variables $X$.

**Lemma 5.** *Let $\mathcal{D}$ be a deviation measure, $\mathcal{Q}$ be an associated risk envelope, $\mathcal{Q}(X)$ be the risk identifier for the r.v. $X$. Then $\mathcal{D}$ is coherent if and only if*

$$Q \ge 0 \textit{ for all } Q \in \mathcal{Q}(X) \textit{ for all } X. \tag{5–40}$$

**Proof:** If $\mathcal{D}$ is coherent, then (5–40) holds since $\mathcal{Q}(X) \in \mathcal{Q}$ for any X.

To prove the converse statement, we need to show that (5–40) implies $Q \ge 0$ for all $Q \in \mathcal{Q}$. Suppose this is not true, namely there exists $\tilde{Q} \in \mathcal{Q}$, such that $\tilde{Q}(\omega) < 0$ on some set $S \subset \Omega$. Since $\mathcal{Q}$ is convex, there exists a subset $\mathcal{Q}_S^- \subset \mathcal{Q}$ with the property $Q(\omega) < 0$ on $S$ for all $Q \in \mathcal{Q}_S^-$. Consider a random variable $\tilde{X}$ such that $\tilde{X}(\omega) = 1$ if $\omega \in S$, and

$\tilde{X}(\omega) = 0$ otherwise. Then,

$$\mathcal{Q}(\tilde{X}) = \underset{Q\in\mathcal{Q}}{\operatorname{argmin}}\, E[XQ] = \underset{Q\in\mathcal{Q}}{\operatorname{argmin}}\, E[\mathbf{1}_S \cdot Q] = \underset{Q\in\mathcal{Q}_S^-}{\operatorname{argmin}}\, E[\mathbf{1}_S \cdot Q] \subset \mathcal{Q}_S^-, \tag{5–41}$$

which contradicts with the condition $Q \geq 0$ for all $Q \in \mathcal{Q}(\tilde{X})$, as required by (5–40). This concludes the proof.

■

Risk identifiers (5–39) implies that the deviation measure $\tilde{\mathcal{Q}}_\beta(X) = CVaR_\beta(-X + EX)$ is coherent if $2 - \beta^{-1} \geq 0$, which is equivalent to having $\beta \geq 1/2$.

Now consider the mixed-CVaR measure

$$\mathcal{D}_{\beta_1,\ldots,\beta_n}(X) = \sum_{i=1}^{n} \gamma_i CVaR_{\beta_i}(-X + EX)$$

and examine its coherence. The risk identifier for this measure given by

$$\tilde{\mathcal{Q}}_{\beta_1,\ldots,\beta_n}(X) = \sum_{i=1}^{n} \tilde{\mathcal{Q}}_{\beta_i}(X),$$

where $\tilde{\mathcal{Q}}_{\beta_i}(X)$ are risk identifiers for measures $CVaR_{\beta_i}(-X + EX)$. Assume for further analysis that $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$, then $VaR_{\beta_1}(-X) \leq VaR_{\beta_2}(-X) \leq \ldots \leq VaR_{\beta_n}(-X)$. The graph of members of $\tilde{\mathcal{Q}}_{\beta_1,\ldots,\beta_n}(X)$ are step functions decreasing at the breakpoints $VaR_{\beta_i}(-X)$, so that having $\tilde{Q} \in \tilde{\mathcal{Q}}_{\beta_1,\ldots,\beta_n}(X)$ means that

$$\begin{cases} E\tilde{Q} = 1, & \\ \tilde{Q}(\omega) = 2, & \text{when } X(\omega) < VaR_{\beta_1}(-X), \\ \tilde{Q}(\omega) = 2 - \sum_{j=1}^{k}(\gamma_j/\beta_j), & \text{when } VaR_{\beta_k}(-X) < X(\omega) < VaR_{\beta_{k+1}}(-X), \\ \tilde{Q}(\omega) \in [2 - \sum_{j=1}^{k-1}(\gamma_j/\beta_j), 2 - \sum_{j=1}^{k}(\gamma_j/\beta_j)], & \text{when } X(\omega) = VaR_{\beta_k}(-X),\ k \geq 2, \\ \tilde{Q}(\omega) \in [2, 2 - \gamma_1/\beta_1], & \text{when } X(\omega) = VaR_{\beta_1}(-X). \end{cases} \tag{5–42}$$

The measure $\tilde{\mathcal{D}}_{\beta_1,\ldots,\beta_n}(X)$ is coherent if the lowest value of members of $\tilde{\mathcal{Q}}_{\beta_1,\ldots,\beta_n}(X)$ are greater than zero, i.e.

$$\sum_{i=1}^{n} \gamma_i/\beta_i \leq 2.$$

It is important to mention that a mixed measure $\tilde{\mathcal{D}}_{\beta_1,\ldots,\beta_n}(X)$ can be coherent even if the some of its components are not. For example, combining the non-coherent measure $CVaR_{45\%}(-X+EX)$ and a coherent one $CVaR_{\beta}(-X+EX)$, $\beta \geq 1/2$, with equal weights, we get a coherent mixed measure

$$\tilde{\mathcal{D}}_{45\%,\beta} = \frac{1}{2}CVaR_{45\%}(-X+EX) + \frac{1}{2}CVaR_{\beta}(-X+EX),$$

when $\beta \geq 9/16$.

## 5.6 Conclusions

Discount factors corresponding to generalized CAPM relations exist and depend on risk identifiers for master funds. The projection of these discount factors on the space of asset payoffs coincides with the discount factor corresponding to the standard deviation. It is possible to calibrate the deviation measure in the general portfolio theory from market data if a parametrization of the deviation measure is assumed. One of candidate parameterizations is mixed-CVaR deviation of gains and losses. The risk identifier of CVaR and mixed-CVaR deviations of losses are derived and coherence of these deviation measures is examined.

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