STUDIES OF MIXING AND RELAXATION IN TIME-DEPENDENT COLLISIONLESS DYNAMICAL SYSTEMS

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Publius Vergilius Maro (70 to 19 BC) said:
They are able because they think they are able
They can because they think they can
In memory of my father Mihail (1938 to 1997),
my advisor Henry (1955 to 2003),
my friend Beth (1973 to 2004),
For thinking I was able
For thinking I could
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STUDIES OF MIXING AND RELAXATION IN TIME-DEPENDENT COLLISIONLESS DYNAMICAL SYSTEMS

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The current distribution of matter in galaxies tells us about the Universe we live in: how it formed, what its building blocks are, and how it evolved. Computer simulations of structure formation in the Universe allow us to investigate the processes that drive the evolution of the early Universe and the formation of galaxies. To properly interpret how observations of nearby galaxies relate to the early state of the Universe, a detailed understanding of the processes that lead galaxies to equilibrium is needed. One such process is “mixing,” which controls how the observed distributions of galaxies relate to their initial state. This study has two main goals. The first is to investigate the processes that drive the evolution of self-gravitating systems to equilibrium and cause mixing in phase-space. The second objective is to understand mixing and other factors that result in an apparently universal power-law profile of phase-space density from cosmological N-body simulations, by performing a quantitative analysis of the evolution of phase-space density in binary N-body mergers and in cosmological simulations.
1.1 Structure Formation in the Universe

The standard Hot Big Bang model is formulated as a physical theory for the origin of the Universe in the context of Einstein’s theory of general relativity. The earliest physical models date back to Alexander Friedmann (1924), who formulated what today are called the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models.

These models lead to a picture in which the very fabric of space-time is expanding, and provide an explanation of the Hubble Law, \( v = Hr \), which is a relationship between the recession velocity \( (v) \) of a galaxy and its distance from us \( (r) \), first recognized by Edwin Hubble (1929). The discovery by the Cosmic Background Explorer (COBE) (Fixsen et al. 1996) satellite that the Cosmic Microwave Background Radiation (CMBR) was an excellent black-body supported the view that the Universe was expanding and had formed in a hot Big Bang. For nearly 70 years following Hubble’s discovery, the study of Cosmology was dominated by the search for two numbers, the Hubble Constant \( H_0 \) and the parameter determining the rate of deceleration of the expansion of the Universe. Due to large systematic uncertainties in determining distances to far away galaxies, the accurate determination of the Hubble constant was only achieved recently due to advances in instrumentation, the launch of the Hubble Space Telescope (HST), and the use of several complementary methods. Accurate distances to nearby galaxies obtained as part of an HST Key Project have allowed calibration of five different methods for determining the distances to galaxies out to 500 Mpc (Freedman et al. 2001).

Contrary to the expectation (based on standard FLRW cosmology) that the expansion of the Universe was slowing down, observations by two independent groups in the late 1990s found that supernovae at high redshifts are fainter than predicted for a slowing expansion and indicate that the expansion is actually speeding up (Riess et al. 1998; Perlmutter et al. 1999).
The WMAP (Wilkinson Microwave Anisotropy Probe) satellite observed the CMB with unprecedented spatial, temperature and frequency resolution and provided higher accuracy measurements of many cosmological parameters than had been available from the previous CMB instrument. The CMB measurements from WMAP are in agreement with the supernove measurements and according to the results of the analysis of the three-year data (Spergel et al. 2007), the Universe is composed of 3% ordinary baryonic matter, 23% dark matter (which does not emit or absorb light), and 74% dark energy, which acts to accelerate expansion.

More than 70 years ago, Fritz Zwicky (1933, 1937) realized that clusters of galaxies consisted predominantly of matter in some nonluminous form. Pioneering work on the rotation curves of spiral galaxies (Rubin & Ford 1970; Freeman 1970; Bosma & van der Kruit 1979) showed that the curves do not decrease outside the regions where most of the luminous mass ends, but are flat well beyond the radius of the stellar distribution. The rotation velocity rises to a maximum from the center outward, but outside of the central bulge, the rotation speed is nearly a constant function of radius. The most widely accepted explanation for this was that there is a substantial amount of matter far from the center of the galaxy, and its gravitational influence is best seen at large radii.

Several categories of dark matter (DM) have been postulated and are classified as either baryonic dark matter or non-baryonic dark matter. The latter is divided into three different types: hot dark matter (HDM), warm dark matter (WDM), and cold dark matter (CDM).

Candidates for baryonic DM are non-luminous cold gas (Gerhard & Silk 1996) and MACHOs (massive compact halo objects, Alcock et al. (1993)). Big Bang nucleosynthesis puts stringent constraints on the amount of baryons in the Universe. Current observations are consistent with these theoretical constraints, implying that all of the DM in the Universe cannot consist of baryons.
HDM consists of particles that travel with relativistic velocities. The best candidate for the hot DM is the neutrino. Neutrinos have very small masses, and do not interact through two of the four fundamental forces, the electromagnetic interaction and the strong interaction. They do interact by the weak interaction, and gravity, but due to the low strength of these forces, they are difficult to detect. The anisotropies in the microwave background radiation, as measured by the COBE and WMAP satellites, indicates that matter has clumped on very small scales and fast moving particles cannot clump together on such small scales. Also, if most DM consisted of neutrinos, matter in the Universe would be more uniformly distributed, which is not observed.

CDM is a type of DM which is cold (its velocity is non-relativistic, \( v \ll c \)), dissipationless (can not cool by radiating photons), and collisionless.

WDM particles have slightly lower velocities than neutrinos and are warmer than the CDM particles. Candidates for WDM are gravitinos and sterile neutrinos (a hypothetical neutrino that does not interact via any of the fundamental interactions). Because particle velocities are lower, WDM clumps more with its surroundings than neutrinos do, but not as much as CDM particles.

Within the ΛCDM model, DM collapses first into small halos, and these halos merge to form progressively larger halos over time. Baryons cool and condense to form galaxies in halo centers. Also known as the “bottom up” scenario, it implies that galaxies form first (from the merger of smaller units), then the clusters of galaxies, and then the superclusters.

A systematic study of DM halos density profiles for a wide range of halo masses and cosmologies was done by Navarro et al. (1996) (hereafter NFW) who argued that an analytical profile in the form of a double power-law provided a good description of halo profiles in their simulations. Other studies claim different values of the inner slope. However, all these theoretical models diverge as the radius approaches zero, producing a steep central density cusp. Density profiles with outer slopes larger than the NFW
\( (\rho(r) \propto r^{-3}) \) yield finite mass models, but for slopes with lesser values than the NFW, the total mass diverges as the radius increases. These models provide good fits to the simulated density distribution of DM halos up to about the virial radius.

Two of the outstanding puzzles of ΛCDM models of structure formation are the fact that DM halo profiles resulting from cosmological \( N \)-body simulations have a “nearly universal” form, and the finding that phase-space density profiles of these halos have power-law distributions over about 2.5 orders of magnitude in radial extent (Taylor & Navarro 2001; Arad et al. 2004; Ascasibar & Binney 2005). These facts point to a universality in the processes driving structure formation.

One of the reasons why both particle physicists and cosmologists are interested in knowing about the phase-space density (also known as the distribution function, Binney & Tremaine (1987)) of DM particles in the current Universe is that there are currently several particle physics experiments searching for candidate DM particles such as neutralinos and axions. If CDM candidates are the DM particles, then their interaction cross-sections for self-annihilation are highest at the centers of DM halos, where the density profile of the DM is in a cusp as predicted by NFW. Furthermore, if baryons fall into these steep cuspy DM potentials and dissipate they drag more DM in after them and this process (generally referred to as adiabatic compression) can produce even steeper cusps than predicted in DM-only simulations. In the centers of such cusps - either at the centers of galaxies like the MW or at the centers of the large numbers of predicted DM subhalos - WIMP particles should have a high cross-section of annihilation, and if they do annihilate, they will be detectable in gamma-rays in a particular frequency range with a particular spectrum. The phase-space density at the present time in the centers of galaxy cusps depends on both the initial phase-space density of DM particles in the early Universe, as well as on the processes that have lead to changes in this phase-space density over the course of galaxy and structure formation. In particular, processes such as mergers of collisionless DM halos result in continuous changes in the phase-space densities.
as structures evolve. One of the goals of this thesis is to understand how phase-space density of DM has evolved over time and to determine what regions in galaxies are likely to maintain the highest phase-space densities that are representative of the phase-space density in the early Universe. Such studies will influence both direct and indirect searches for DM particles.

**1.2 Evolution of Collisionless Systems to Dynamical Equilibrium**

One of the big puzzles in galactic astronomy is how galaxies achieve statistical equilibrium. Numerical simulations suggest that galaxy encounters, even for collisions which trigger very large deviations from statistical equilibrium, result in a perturbed galaxy which will, within a few dynamical times, readjust to a new dynamical equilibrium.

Several authors had noted that the time scale for two-body relaxation (Chandrasekhar 1943) was several orders of magnitude too long to be relevant for the evolution of galaxies (Zwicky 1939).

“Violent relaxation” (VR) was first proposed by Lynden-Bell (1967) as a mechanism to explain the regularity and smoothness of the light distributions in elliptical galaxies. VR, as considered by Lynden-Bell, consisted of “phase mixing” - a process by which an initially compact ensemble of non-interacting phase-space points in a time-independent potential can be stretched into a long ribbon in phase-space as a result of small differences in the initial energies (or angular momenta) of particles in the ensemble. Phase mixing conserves the fine grained distribution function but in a coarse grained sense, and it produces smooth/relaxed looking distributions on timescales that depend only on the range of values of the orbital integrals (or actions) in the ensemble (Binney & Tremaine 1987). The second aspect of VR studied by Lynden-Bell consists of exchange of energy between the particles and the global potential which is undergoing fluctuations. It has often been assumed that the fact that the potential is strongly time-dependent is crucial in making the mixing sufficiently efficient. Lynden-Bell derived the distribution function that would result from rapid fluctuations in the potential of the galaxy, such as
those that might occur during gravitational collapse or following a merger. The process of VR outlined by Lynden-Bell produces a smooth distribution function as a result of the gravitational scattering of particles by the time-dependent global gravitational potential. He argued that this process could result in a mass-independent and smooth final distribution of particles in phase-space.

In the mid-1990’s it was argued that it may not be time dependence by itself that is crucial. What seemed essential was that the flow be chaotic, exhibiting an exponentially sensitive dependence on initial conditions. Studies by (Mahon et al. 1995; Merritt & Valluri 1996; Habib et al. 1997; Kandrup et al. 2000; Kandrup & Siopis 2003) have shown that microscopic ensembles of iso-energetic test particles which occupy an infinitesimal volume of phase-space surrounding a chaotic orbit can evolve rapidly to a near-invariant distribution that uniformly covers the energy surface available to the ensemble. This process of diffusion to a near-invariant distribution is termed “chaotic mixing” and has been shown to occur in a variety of galaxy-type potentials. Thus, a localised ensemble of initial conditions corresponding to chaotic orbits will, when evolved in a time-independent Hamiltonian, begin by diverging exponentially from its localised state and then converge towards an equilibrium or near-equilibrium state (Kandrup 1998). Like phase mixing, chaotic mixing conserves the fine-grained phase-space density. Unlike phase mixing, its rate does not depend on the initial spread in the energy distribution of particles in the ensemble, but on how strongly chaotic the orbits in the ensemble are (Merritt & Valluri 1996).

Since all orbits in a time-dependent potential with long-lived oscillations can, in principle, exchange energy with the potential, a large fraction of the orbits will not conserve the energy integral. In such systems previous authors have found that a significant fraction of orbits can be chaotic (Kandrup et al. 2003). This raises the possibility that the underlying physical process driving relaxation in violently fluctuating potentials could be a consequence of chaotic mixing. The first step in making this
connection was studied by Kandrup et al. (2003) and Terzić & Kandrup (2004), who demonstrated that large fractions of chaotic orbits arise in potentials subjected to long-duration (damped) periodic oscillations. These authors found that the fraction of chaotic orbits depended on the frequency as well as the amplitude of the oscillations in the potential and attributed the chaos to a “broad resonance.”

Several processes affect the structure and evolution of collisionless systems. In particular, VR and the statistical distribution functions that result from this process were investigated (Arad & Lynden-Bell 2005; Arad & Johansson 2005; Dehnen 2005). Several studies of the remnant density profiles have shown that the remnants retain a surprisingly strong memory of their initial density profiles despite the strength of the relaxation process (Boylan-Kolchin & Ma 2004; Kazantzidis et al. 2006).

1.3 This Study

The first part of this work explores how orbits in a galactic potential can be impacted by large amplitude time-dependencies of the form that one might associate with galaxy or halo formation or strong encounters between pairs of galaxies. Parts of the work described in this chapter are published in Kandrup et al. (2003). A period of time-dependence with a strong oscillatory component can give rise to large amounts of transient chaos, and it is therefore investigated whether chaotic phase mixing associated with this transient chaos could play a major role in accounting for the speed of VR. Analysis of simple models involving time-dependent perturbations of an integrable Plummer potential indicates that this chaos results from a broad resonance between the frequencies of the orbits and the frequencies of the time-dependent perturbation. Numerical computations of orbits in potentials exhibiting damped oscillations suggest that, within a period of 10 dynamical times $t_D$ or so, one could achieve simultaneously both ‘near-complete’ chaotic phase mixing and a nearly time-independent, integrable end state.

As a continuation, N-body simulations of halo mergers were analyzed in order to investigate the mechanisms responsible for driving mixing in phase-space and the evolution
to dynamical equilibrium. In particular, the hypothesis that VR is the result of chaotic mixing in such systems is explored. The majority of the work in this chapter is published in Valluri et al. (2007). The focus is on mixing in energy and angular momentum and on determining how the rate of mixing of these quantities scales with local dynamical time. This allows a comparison between the rate and percent of mixing due to VR with other well-known mixing processes such as phase mixing and chaotic mixing.

It is important to mention that the terms “mixing” and “relaxation” are often used interchangeably in the astrophysics literature to mean the approach of a gravitating system to global as well as small-scale equilibrium. The most restrictive definition of “relaxation” is a process which leads to the complete “loss of memory of initial conditions” or loss of correlations between initially nearby particles in phase-space. The large changes in integrals of motion required for such relaxation only occur as a result of two-body interactions (Chandrasekhar 1943) and as a result of VR. Both processes cause changes in the fine-grained distribution function. In contrast, phase mixing and chaotic mixing in time-independent potentials, under the right conditions, can lead to large changes in correlations but conserve integrals of motions, and also conserve the fine-grained distribution function.

The merger remnants produced during the VR experiments mentioned above were investigated, with the goal of understanding how the mixing process redistributes particles in energy, angular momentum, and radius.

An interesting question regarding VR is, “What gives rise to the universal power-law phase-space density profiles seen in cosmological mergers?” In what follows the focus is on the evolution of the phase-space density of collisionless (DM) particles following mergers of spherical potentials with different density profiles.

The phase-space density at the present time in the centers of galaxy cusps is expected to depend on both the initial phase-space density of DM particles in the early Universe as well as on the processes that have lead to changes in this phase-space density over
the course of galaxy and structure formation. In particular, processes such as mergers of collisionless DM halos result in continuous changes in the phase-space densities as structures evolve. One of the goals of this study is to understand how phase-space density of DM has evolved over time and to determine what regions in galaxies are likely to maintain the highest phase-space densities that are representative of the phase-space density in the early Universe.

The last part of this work presents the investigation of the evolution of phase-space density in the formation and evolution of Milky-Way-sized halos in a $\Lambda$CDM cosmology. The main motivation for looking at evolution of the distribution function is because it is the primary physical quantity (its evolution determines evolution of collisionless DM component). Given the fact that the origin of CDM density profiles is still not fully understood, looking at the distribution function and its evolution is interesting as it may give clues as to what happens during nonlinear collapse and mass accretion history of CDM halos.
CHAPTER 2
PHASE MIXING AND CHAOS IN INTEGRABLE POTENTIALS

2.1 Physical Expectations

The basic idea is that an appropriate time-dependence added to a potential can trigger chaos via resonance: if orbits are subjected to a perturbation with power at appropriate frequencies, one might expect otherwise regular orbits to become chaotic. In particular, if unperturbed orbits have power concentrated at frequencies $\sim \Omega$ and the perturbation has significant power at frequencies $\omega$ sufficiently close to $\Omega$, there is the possibility of resonance overlap which, as is well known (Lichtenberg & Lieberman 1992), can lead to the onset of chaoticity.

As a particularly simple example, one knows that if a regular orbit with substantial power at some frequency $\Omega$ is perturbed by a sinusoidal perturbation with frequency $\omega = \Omega$, the orbit will typically exhibit a drastic response which can manifest sensitive dependence on initial conditions. The real issue is ‘how close’ must $\Omega$ and $\omega$ be to trigger strong exponential sensitivity?

Detailed analysis of the effects of resonance overlap suggests generically that the width of such resonances will be an increasing function of amplitude. Even if a slightly ‘detuned’ small amplitude perturbation has no apparent effect, a large perturbation with identical frequency may produce a huge response. Indeed, the numerical models described in Sections § 2.3 and § 2.4 of this chapter indicate that, for large amplitude perturbations – fractional amplitude of order $10 - 20\%$ or more – the resonance can be very broad. In particular, these models reveal that, if the power in the orbits is concentrated at frequencies $\sim \Omega$, one can get a nontrivial increase in chaos – in both the relative number of chaotic orbits and the value of the largest Lyapunov exponent – for sinusoidal perturbations with $0.1\Omega \leq \omega \leq 30\Omega$. In other words, the resonance can be more than two orders of magnitude wide!
The Lyapunov exponent (Lichtenberg & Lieberman 1992) is one of the most popular ways of measuring the degree of chaos of an orbit in a smooth potential. It measures the rate of divergence of two infinitesimally nearby orbits in phase-space. If the rate of divergence of phase-space points is linear, then the Lyapunov exponent $\chi = 0$ and the orbit is termed regular. In a smooth equilibrium potential with three-spatial dimensions such orbits will conserve at least three isolating integrals of motion. If, on the other hand, the rate of divergence of the phase-space points is exponential, the orbit is termed chaotic. Thus, the Lyapunov exponent defines the time scale on which infinitesimally nearby trajectories in phase-space undergo exponential separations in their phase-space coordinates.

The important point, then, is that if resonances of this sort are really so broad, one might expect transient chaos to be extremely common, if not frequently encountered, in time-dependent galactic potentials. Numerical simulations of galaxy encounters and mergers and most models of galaxy and halo formation imply that a system approaching equilibrium will exhibit damped oscillations. To the extent, however, that one is dealing with collisionless relaxation, there is dimensionally only one natural time scale in the problem, namely the dynamical $t_D \sim 1/\sqrt{G\rho}$, with $\rho$ a characteristic density. In particular, $t_D$ should set the oscillation time scale as well as the orbital time scale. The exact numerical values of these time scales will involve numerical coefficients which will in general be unequal and vary as a function of location within the galaxy. If, however, one only needs to assume that the oscillation and orbital time scale agree to within an order of magnitude or so, it would seem likely that the conditions appropriate for resonance-induced transient chaos could arise in much, if not all, of the galaxy.

The crucial recognition, then, is that even a relatively short period of exponential sensitivity can result in comparatively efficient chaotic phase mixing. Earlier analysis of flows in time-independent potentials (Mahon et al. 1995; Merritt & Valluri 1996; Kandrup 1998) have demonstrated that the presence of chaos can dramatically enhance
the degree to which a system becomes ‘shuffled’ and, hence, the rate at which a localised orbit ensemble evolves towards an equilibrium. By complete analogy, one might expect that transient chaos in a self-consistently determined time-dependent potential will again enhance the efficacy of ‘mixing’ and, by so doing, facilitate a more rapid approach towards an equilibrium.

It should be noted explicitly that the physical mechanism for the generation of chaos described here is very different from the mechanism whereby the amount of chaos can increase – or decrease – in more slowly varying potentials (Kandrup & Drury 1998). If the potential is slowly varying, one can visualise an orbit as moving in a phase-space which is slowly changing but which is nearly constant over a time scale $\sim t_D$. It is then natural to suppose that at some times the orbit finds itself in a phase-space region that is ‘chaotic’ in the sense that there is a sensitive dependence on initial conditions, but that at other times it finds itself in ‘regular’ regions where there is no such sensitive dependence. In the setting envisioned in this chapter, the phase-space is changing on a time scale comparable to – or even shorter than – the orbital time scale, so that this picture is necessarily lost. Moreover, in the setting described here one would expect generically a systematic increase in the amount and degree of chaos, not simply a change which could be either positive or negative.

It should be noted that, to a certain extent, the term transient chaos is necessarily ‘fuzzy.’ Chaos, like ordinary Lyapunov exponents, is only defined in an asymptotic $t \to \infty$ limit. The point, however, is that, even over finite time intervals, it is physically meaningful to ask whether orbits exhibit an exponentially sensitive dependence on initial conditions. This is, for example, the notion that motivates the definition of finite time Lyapunov exponents (Grassberger et al. 1988), which have become accepted tools in nonlinear dynamics. From a phenomenological point of view, it makes perfect sense to identify an orbit as exhibiting significant transient chaos if, for some finite time $> t_D$, the orbit exhibits significant exponential sensitivity. In many cases, it is clear by visual
inspection and/or through a computation of a finite time Lyapunov exponent whether such transient chaos is present. In other cases, however, things are not so completely clear. For example, for the models considered in this chapter it proves difficult to determine with complete reliability a minimum frequency at which transient chaos ‘turns on’ since this ‘turning on’ is comparatively gradual.

2.2 Numerical Experiments

The numerical computations described in this chapter involved perturbations of a Plummer sphere, which is characterised (in units with $G = m = 1$) by the potential

$$V_0(x, y, z) = -\frac{1}{(1 + x^2 + y^2 + z^2)^{1/2}}.$$  \hfill (2–1)

This form was selected especially because orbits in this potential are all strictly integrable. Any chaos that is observed can be associated unambiguously with the perturbations that were introduced. These were assumed to take the form

$$V_1(x, y, z, t) = -\frac{m}{(1 + x^2 + a^2 y^2 + z^2)^{1/2}} ,$$  \hfill (2–2)

where the quantities $m$ and $a$ are allowed to vary in time. A variety of different time-dependences were considered, including:

- Strictly sinusoidal oscillations in $m$ or $a$, i.e.,

  $$m = m_0 \sin \omega t \quad \text{and} \quad a = a_0 + \delta a \sin \omega t .$$  \hfill (2–3)

  This allowed to focus on the physical mechanism in its ‘purest’ form.

- Sinusoidal oscillations that damp exponentially or as a power law in time, i.e.,

  $$m = m_0 e^{-\alpha t} \sin \omega t \quad \text{or} \quad m = m_0 \frac{\sin \omega t}{(t_0 + t)^p} ,$$  \hfill (2–4)

with $p = 1$ or $2$, and analogous formulas for $a(t)$. This helps confirm that the physical effects associated with a purely oscillatory perturbation persist if, as is natural in the context of violent relaxation (VR), one allows for a galaxy that damps towards equilibrium.
• Nonoscillatory damping, i.e.,

\[ m = m_0 e^{-\alpha t} \quad \text{or} \quad m = \frac{m_0}{(t_0 + t)^p} \]  

and analogous formulas for \( a(t) \). This helps demonstrate that the presence of an oscillatory component is crucial in order to get a substantial amount of transient chaos. As will be described below, computations with perturbations given by Equations 2–6 or 2–7 typically exhibited comparatively minimal amounts of transient chaos.

• Variable driving frequency, assuming that

\[ \omega(t) = \omega_0 + \delta(t) , \]  

with \( \delta(t) \) treated as Gaussian coloured noise, so that its statistical properties are determined uniquely by its first two moments, which are assumed to satisfy

\[ \langle \delta(t) \rangle = 0 \quad \text{and} \quad \langle \delta(t_1)\delta(t_2) \rangle = \Delta^2 \exp(-|t_1 - t_2|/t_c) , \]  

where \( \Delta \) represents the typical ‘size’ of the random component and \( t_c \) the time scale over which it changes appreciably. Considering perturbations of this form helps confirm that, because the resonance is comparatively broad, permitting the driving frequency to drift (within limits) has a relatively minor – albeit potentially significant – effect. This again is important physically. Since the density distribution varies as a galaxy evolves towards an equilibrium, one would anticipate that the characteristic oscillation frequency (or frequencies) will change.

Computations were performed for ensembles of 1600 initial conditions which were generated in two different ways. (1) Broad ‘representative’ ensembles were generated by uniformly sampling a constant energy region. A study of orbits generated from such ensembles allows to derive systematic effects which might be expected to act in a galaxy as a whole. (2) Localized ensembles were generated by sampling small phase-space 6D cubes. Studying orbits generated from such ensembles allowed one to determine the extent to which the ‘smooth’ behaviour associated with the representative ensembles reflected the choice of ensemble as opposed to properties of the resonance.

Most experiments involved perturbations at the 50% level or less, i.e., \( m_0 \leq 0.5 \) and/or \( \delta a \leq 0.5 \). For the intermediate energies considered throughout, a ‘typical’ orbital frequency (derived from a Fourier transform) was \( \Omega \sim 0.3 - 0.35 \), corresponding
to $t_D = 2\pi/\Omega \sim 20$ in absolute units. On physical grounds one might expect that
the perturbations should damp on a time scale comparable to, but somewhat longer
than, $t_D$, so that most experiments assumed that (to within factors of a few) $t_0$ and
$\alpha^{-1} \sim 5t_D \sim 100$. Computations of chaotic phase mixing in time-independent potentials
(Kandrup & Novotny 2004) typically exhibit an exponential approach towards a
near-invariant distribution on a time scale $\sim 5t_D$!

### 2.3 Resonance and Transient Chaos

Computations involving representative orbit ensembles have led to several unambiguous
conclusions:

- Integrations which do not involve large scale time variations on a time scale $\sim t_D$
yield little, if any, transient chaos. In particular, as will be illustrated more carefully
in the following Section § 2.4, allowing for nonoscillatory damped perturbations of the
form given by Equation 2–6 do not result in large measures of chaos, even for large
amplitudes $m_0 \geq 0.5$. However, allowing for oscillations in $m$ and/or $a$ can trigger
substantial amounts of chaos.

- There are at least two reasons to believe that this chaos is triggered by a resonance.
More obvious, perhaps, is the fact that the largest increases in chaos – as reflected by
both the number of chaotic orbits and the size of a typical Lyapunov exponent – arise
for driving frequencies comparable to, or somewhat larger than, the frequencies for
which the unperturbed orbits have most of their power. Equally significant, however,
is the fact that the driving frequencies which result in the greatest amount of chaos
 correspond precisely to those frequencies for which the orbital energies are 'shuffled'
the most, i.e., where different particles with the same initial energy end up with the
largest spread in final energies. A priori there need be no direct connection between
increases in chaos and large-scale shuffling of energies. However, one would expect
resonant couplings to lead to significant changes in energies. A correlation between
changes in energy and the amount of chaos thus corroborates the interpretation that
this chaos is resonant in origin.

- At least for the case of large amplitude oscillations, $m_0 \geq 0.1$ or so, the resonance
is comparatively broad. One observes significant increases in both the relative
measure of chaotic orbits and the size of the largest Lyapunov exponent whenever
the driving frequency $\omega$ and the natural orbital frequencies $\Omega$ agree to within an
order of magnitude or so. If, for example, one considers an energy where the natural
frequencies peak for $\Omega_m \sim 0.3 - 0.35$ or so, noticeable increases in the degree of chaos
are observed for $0.035 \leq \omega \leq 10.0$, i.e., $0.1 \sim \omega/\Omega_m \sim 30$. 


• The width of the resonance, i.e., the range of values of \( \omega \) for which one sees an appreciable increase in the degree of chaos, appears to be an increasing function of amplitude.

• The resonance appears to be smooth in the sense that, for fixed amplitude, the relative number of chaotic orbits and the size of the largest Lyapunov exponent both vary smoothly as functions of the driving frequency \( \omega \). In particular, plots of the fraction \( f \) of the orbits that are chaotic or the mean size \( \langle \chi \rangle \) of the largest Lyapunov exponent as functions of driving frequency typically exhibit a single peak. There is no obvious evidence for complex structure within the resonance.

• For fixed \( \omega \), the relative number of chaotic orbits and the size of the largest Lyapunov exponent are both an increasing function of perturbation amplitude, although it appears that the fraction \( f \) of orbits that are chaotic can converge asymptotically towards a fixed value \( f_M \) for sufficiently large amplitude. For some values of \( \omega \), especially those near the center of the resonance, this \( f_M \) can approach unity; for other choices of frequency, \( f_M \) can be substantially smaller.

• Whether or not the perturbation breaks spherical symmetry appears to be relatively unimportant, for example, making the parameter \( a \) time-dependent or otherwise different from unity does not change things all that much. Provided that the system manifests large amplitude oscillations, one can apparently get just as much chaos for nonspherical systems as for spherical systems.

Evidence for several of these conclusions is provided in Figures 2-1–2-4, all generated from the same set of 1600 initial conditions, evolved for a total time \( t = 512 \) in the presence of a strictly sinusoidal perturbation of the form given by Equation 2-3 with \( a \equiv 1 \). The two panels of Figure 2-1 exhibit the relative measure \( f \) of chaotic orbits as a function of driving frequency \( \omega \) for five different choices of amplitude \( m_0 \). The top panel exhibits \( f(\omega) \) on a linear scale, thus allowing one to focus on the behaviour of the resonance at higher frequencies; the lower exhibits \( f \) as a function of \( \log_{10} \omega \), which allows one to see more clearly the behaviour at lower frequencies.

Most obvious from Figure 2-1 is the fact that the width of the resonance is an increasing function of \( m_0 \), although this increase appears to be comparatively small for \( m_0 > 0.2 \) or so. Also evident is the fact that, except perhaps for the lowest amplitude, \( m_0 = 0.1 \), the fraction \( f \) appears to vary smoothly with frequency. In each case, \( f \) peaks at a frequency \( \omega_{max} \) comparable to the typical natural frequencies \( \Omega \sim 0.3 - 0.35 \).
Figure 2-1. The relative fraction $f$ of chaotic orbits in a representative 1600 orbit ensemble which are subjected to strictly sinusoidal oscillations with variable frequency $\omega$. The different curves represent, from bottom to top, driving amplitudes $m_0 = 0.1, 0.2, 0.3, 0.4, \text{ and } 0.5$. The bottom panel shows the same data plotted as a function of $\log_{10}\omega$.

associated with the unperturbed orbits. However, $\omega_{\text{max}}$ does seem to vary somewhat as a function of $m_0$. For larger amplitudes, near $\omega_{\text{max}}$ essentially all the orbits are chaotic, i.e., $f(\omega_{\text{max}}) \approx 1$. For lower frequencies $f(\omega_{\text{max}})$ can be significantly less than unity. Overall, there is evidence for significant amounts of transient chaos for $0.35 \leq \omega \leq 10.0$.

The two panels of Figure 2-2 exhibit $\langle \chi \rangle$, the mean value of the largest finite time Lyapunov exponent for the same ensembles, again plotted on both linear and logarithmic scales. Here $\langle \chi \rangle$ was extracted by first identifying those orbits in the ensemble that were deemed chaotic and then computing the mean value of $\chi$ for those orbits.
Figure 2-2. The mean value $\langle \chi \rangle$ of the chaotic orbits in the ensembles used to generate Figure 2-1, again plotted as a function of $\omega$. The bottom panel shows the same data plotted as a function of $\log_{10} \omega$.

Figure 2-3 shows $\delta E$, the root mean spread in energies at time $t = 512$, for the chaotic orbits identified in Figure 2-1 and Figure 2-2. It is clear visually that, as one would expect if the transient chaos is associated with a resonant coupling, the frequencies which result in the largest measures of chaotic orbits and the largest $\langle \chi \rangle$ also result in the largest shifts in energy. However, it is also evident that, especially for the higher amplitude perturbations, $\delta E$ exhibits a more complex dependence on $\omega$ than do $f(\omega)$ or $\langle \chi(\omega) \rangle$. This does not contradict the assertion that transient chaos is induced by a resonant coupling, but it does suggest that the amount and degree of chaos in the broad resonance region is less sensitive to the pulsation frequency than is the shuffling in energies. This has
Figure 2-3. The root mean squared spread in energies, $\delta E_{\text{rms}}(t = 512)$, for the orbit ensembles used to generate Figures 2-1 and 2-2. The bottom panel shows same data plotted as a function of $\log_{10}\omega$.

potentially significant implications for VR, where one is interested in both (i) shuffling the energies of the individual masses and (ii) randomising their phase-space locations on a constant energy region.

One other point is also evident from Figure 2-3, namely that, at least for frequencies well into the resonance, the spread in energies can be very large. This reflects the fact that, because of the resonant coupling, a significant fraction of the orbits have acquired large energies that set them along trajectories involving excursions to very large radii, $r > 10$ or more. Allowing for a resonance that is this strong to last for as long as $t \approx 20t_D$
is completely unrealistic in the context of any model for VR. The computations described in Section § 2.4.2, which incorporate strong damping, avoid this artificial behaviour.

In part because of this effect, any uniform prescription used to distinguish between ‘regular’ and ‘chaotic’ orbits is necessarily somewhat improvised and, as such, fraught with difficulties. Figures 2-1 and 2-2 reflect an analysis in which ‘chaotic’ was defined as corresponding to a finite time Lyapunov exponent for the interval $0 < t < 512$ with a value $\chi \geq 0.012$, arguably a fairly large threshold value. Lowering the threshold value of 0.012 to a smaller value would of course increase the estimated fraction $f$ of chaotic orbits and decrease the mean value of $\langle \chi \rangle$ for those orbits deemed chaotic. In any event, the choice $\chi \geq 0.012$ implies that there is little if any chaos for $\omega \leq 0.01$ and $\omega \geq 10$, which is consistent with the fact that localised ensembles exhibit little if any evidence of chaotic phase mixing for $\omega < 0.01$ and $\omega > 10$ or so. Despite this, however, it should be recognised explicitly that this prescription really identifies orbits which are ‘obviously’ or ‘strongly’ chaotic, and that it may be ignoring a considerable number of orbits which are only very weakly chaotic.

This prescription, like any other which involves a fixed minimum value and a fixed integration time, is open to criticism. For those frequencies and amplitudes where the resonance is especially strong and many orbits quickly achieve large energies, it might seem more appropriate to consider a shorter time interval, during which the orbits are restricted to smaller radii. However, for frequencies and amplitudes for which the resonance is weaker, one might wish instead to integrate for much longer times so as (hopefully) to allow for clear distinctions to be made between regular and ‘extremely sticky’, nearly regular orbits. The choice of a time interval $t = 512$ represents a compromise necessitated by the idealised nature of the computations described in this section.

In any event, what is inevitable is that, for chaotic orbits in any potential, one must expect correlations between the energy of the orbit and the size of a typical finite time
Figure 2-4. The relative fraction $f$ of chaotic orbits in a representative 1600 orbit ensemble which are subjected to strictly sinusoidal oscillations of the form given by eq. (4) of variable amplitude $m_0$ with $\omega = 1.4$ (upper curve), $\omega = 3.5$ (middle curve), and $\omega = 7.0$ (lower curve).

Lyapunov exponent $\chi$. For the potential considered here, one finds typically that the orbits deemed regular are precisely those which exhibit the smallest changes in energy and hence, overall, the smallest excursions from the origin. However, for those orbits which are chaotic, those spending more time at larger radii and which, for this reason, have larger orbital time scales, tend to have somewhat smaller values of $\chi$.

Figure 2-4 exhibits the relative measure of chaotic orbits as a function of amplitude $m_0$ for three different frequencies, namely $\omega = 1.4$, 3.5, and 7.0. For $\omega = 1.4$, a frequency in the middle of the resonance, for amplitudes as large as $m_0 = 0.25$ almost all the orbits exhibit evidence of chaos. For $\omega = 3.5$, a value somewhat closer to the edge of the resonance, the relative abundance of chaotic orbits is somewhat smaller. However, for $m_0 = 0.25$ as many as 75% of the orbits are clearly chaotic, a fraction which does not increase significantly if $m_0$ is increased. For $\omega = 7.0$, a value near the edge of the
resonance, the fraction of the orbits which is chaotic is smaller and does not appear to level off for \( m_0 > 0.25 \) or so. Indeed, if the amplitude is increased to a value as large as \( m_0 = 1.0 \) as many as 50% of the orbits prove chaotic, i.e., \( f(m_0 = 1) \approx 0.5 \).

The uniform criterion used to identify chaotic orbits for Figures 2-1 and 2-2 fails for \( m_0 \leq 0.1 \). For that reason, Figure 2-4 implemented a different, somewhat more subjective criterion which has been used successfully in distinguishing between regular and ‘very sticky’ chaotic orbits in the triaxial generalisations of the Dehnen potentials (Siopis & Kandrup 2000). What this entailed was ordering the 1600 computed values of \( \chi \) at various times \( T_A \), plotting the ordered list of \( \chi(T_A) \)'s for different values of \( T_A \), and then in each case searching for a ‘kink’ in the curve. This prescription appears to permit an accurate determination of a fraction \( f \) of the orbits exhibiting chaotic behaviour which is insensitive to the precise choice of \( T_A \), although the estimated size of the finite time \( \chi(T_A) \) does depend on the sampling interval.

Additional insights into the nature of the resonance can be derived from Figure 2-5, which shows the \( x \)-components of the composite (normalised) Fourier transforms of coordinate and force per unit mass, \( |x(\Omega)| \) and \( |a_x(\Omega)| \), constructed by combining spectra for 1600 unperturbed \( \omega = 0 \) orbits generated from the same initial conditions used to generate Figures 2-1–2-3. (Because of spherical symmetry the \( y \)- and \( z \)-components are statistically identical.) As noted already, for this particular choice of energy, \( |x(\Omega)| \) peaks at \( \Omega_m \sim 0.3 - 0.35 \) and is comparatively negligible for much higher and lower frequencies. By contrast, \( |a_x(\Omega)| \) has multiple peaks associated with higher frequency harmonics which arise because the force per unit mass is a nonlinear function of position. It is natural to suppose that it is these harmonics that are responsible for the comparatively large responses at frequencies \( \Omega \) large compared with \( \Omega_m \) that are evident in Figures 2-1–2-3. It is particularly interesting that this response cuts off for driving frequencies \( \omega \) large compared with \( 2\Omega \), rather than \( \Omega \), a fact that would suggest that one is observing the effects of a broad 2 : 1 resonance.
Figure 2-5. The power spectrum $|x(\Omega)|$ associated with the $x$-component of the orbits generated from the 1600 initial conditions used to created Figures 2-1–2-3, evolved in the absence of any time-dependent perturbation and plotted as a function of $2\Omega$ (top panel). The bottom panel shows the power spectrum $|a_x(\Omega)|$ associated with the $x$-component of the force per unit mass associated with the same orbits.

Much of the smoothness in plots such as those given in Figure 2-2 reflects the fact that the data were generated from ‘representative’ ensembles of initial conditions. If the same analysis is repeated for different localised ensembles of initial conditions, even ensembles with the same energy, one observes significant variability in both the size of a typical finite time Lyapunov exponent and the value of the frequency $\omega$ for which $\langle \chi \rangle$ is maximised. And similarly, plots of quantities like $\langle \chi \rangle$ as a function of $\omega$ for such ensembles exhibit more structure than do plots for representative ensembles. This
presumably reflects the fact that the natural frequencies of the otherwise regular orbits vary with phase-space location, and that some frequencies are more susceptible to the resonance than others. This additional structure could have important implications for near-equilibria subjected to nearly periodic perturbations, which could impact certain orbits much more than others. However, one might expect that, when considering the ‘global’ properties of VR, such details would tend to wash out.

2.4 A Model for Violent Relaxation

2.4.1 Variable Pulsation Frequencies

Allowing for a randomly varying frequency of the form given by Equations 2–8 and 2–9 leads to several significant conclusions.

- Perhaps the most important is that allowing for a comparatively slow frequency drift can actually increase the relative measure of chaotic orbits. Suppose that the unperturbed frequency $\omega_0$ is well into the resonant region and that $\Delta$ is not sufficiently large to push $\omega$ outside. In this case, the introduction of a perturbation $\delta(t)$ with an autocorrelation time $t_c$ long compared with $t_D$ tends to convert many, if not all, the ‘regular’ orbits into orbits with appreciable exponential sensitivity. This behaviour can be understood by supposing that, in the absence of the frequency drift, there exist some regular orbits which, because of the particular values of their natural frequencies, barely manage to avoid resonating with the perturbation. Allowing for a drift facilitates an improved resonance coupling which can make (some of) these orbits chaotic for at least some of the time.

- Provided that $\omega_0$ and $\omega_0 \pm \Delta$ are well within the resonance region, it is also apparent that, as the autocorrelation time $t_c$ decreases, the degree of chaos, as probed by the mean $\langle \chi \rangle$, tends to decrease, even though the relative fraction of chaotic orbits may well remain equal to unity.

- For fixed autocorrelation time $t_c$, the mean $\langle \chi \rangle$ manifests only a relatively weak dependence on $\Delta$, at least for $0.1 \leq \Delta/\omega_0 \leq 0.8$. For some choices of $t_c$ increasing $\Delta$ causes an increase in the mean finite time $\chi$; in other cases increasing $\Delta$ decreases the mean. However, the observed variations are invariably small.

All three of these points are illustrated in Figures 2-6 and 2-7. The dashed and triple-dot-dashed curves in Figure 2-6 correspond to plots of ordered finite time Lyapunov exponents $\chi$ associated with 1600 initial conditions subjected to undamped pulsations with variable frequency $\omega = \omega_0 + \delta(t)$ for $m_0 = 0.1$, $\omega_0 = 3.5$ and different choices of
Figure 2-6. Finite time Lyapunov exponents for 1600 representative orbits evolved for a time $t = 512$ in an undamped Plummer potential pulsed with amplitude $m = 0.1$ and a frequency frequency $\omega(t) = \omega_0 + \delta(t)$, with $\omega_0 = 3.5$, autocorrelation time $t_c = 320 = 16t_D$, and $\Delta^2 = 0.35$ (dashed curve) and $\Delta^2 = 2.8$ (triple-dot-dashed). The solid curve contrasts exponents derived for an evolution with $\delta \equiv 0$. The same data is plotted for $t_c = 160, t_c = 80, t_c = 40, t_c = 20,$ and $t_c = 10$.

$\Delta$ and $t_c$. The solid curve in each panel contrasts the exponents generated for the same initial conditions evolved with $\delta = 0$. It is evident that, for the largest values of $t_c$, the differences between the curves with $\delta$ zero and nonzero are relatively small, at least for the largest values of $\chi$. However, it is also apparent that the time-dependent frequency drift has significantly reduced the relative number of orbits with very small finite time $\chi$’s. Also evident is the fact that decreasing the autocorrelation time $t_c$, i.e., making the frequency drift more quickly, tends generically to decrease the typical value of $\chi$, although the
Figure 2-7. The mean Lyapunov exponent $\langle \chi \rangle$ (upper two curves) and the associated dispersion $\sigma_\chi$ (lower two curves) computed for ensembles of 1600 orbits evolved in an undamped Plummer potential with amplitude $m = 0.5$, $\omega_0 = 3.5$, and variable autocorrelation time $t_c$. Diamonds correspond to $\Delta^2 = 0.35$, triangles to $\Delta^2 = 2.8$. (b) The same for data is plotted for $m_0 = 0.1$ and for $m_0 = 0.05$. 
relative measure of chaotic orbits may not change appreciably. For this particular choice
of $\omega_0$ and $m_0$, increasing $\Delta$ for fixed $t_c$ also tends to reduce the typical size of a finite
time $\chi$. This, however, is not generic. If, instead, the same computations are repeated for
$m_0 = 0.05$, one finds that the choice of $\Delta$ which leads to the largest $\chi$’s actually varies
with $t_c$.

The effects of varying $\Delta$ and $t_c$ on the mean $\langle \chi \rangle$ and the dispersion $\sigma_\chi$ associated with
these ensembles is exhibited in Figure 2-7.

2.4.2 Damped Oscillations

Allowing for damped oscillations of the form given by Equations 2–4 or 2–5 has a
comparatively minimal effect. Not surprisingly, the size of the mean Lyapunov exponent
$\langle \chi \rangle$ decreases as time elapses and there can be a gradual decrease in the relative fraction
$f$ of orbits exhibiting exponential sensitivity. However, the basic phenomenon of transient
chaos persists until the perturbation has damped almost completely away.

This is, for example, evident from Figure 2-8, which was again generated from
the same 1600 initial conditions used to generate Figures 2-1 – 2-5, now allowing for a
perturbation of the form given by Equation 2–5 with $m_0 = 0.5$, $t_0 = 100$, and $p = 2$, again
allowing for variable $\omega$. Here the four lower curves (from top to bottom) represent mean
values of the finite time Lyapunov exponent $\langle \chi \rangle$ generated separately for the intervals
$100 < t < 200$, $200 < t < 300$, $300 < t < 400$, and $400 < t < 500$. The top curve,
reproduced from Figure 2-2, represents $\langle \chi \rangle$ for $0 < t < 512$ for the same orbits evolved
in the presence of undamped ($p = 0$) oscillations with the same initial $m_0$. It is evident
that, as one would have expected, the degree of exponential sensitivity decreases with time
and that, by the end of the integration, the orbits are behaving in a nearly regular fashion.
The fact that the peak frequency $\omega$ drifts towards lower values at later times is consistent
with the fact, evident from Figure 2-2, that for undamped oscillations, lower amplitudes
$m_0$ correlate with lower values of $\omega$. 
Figure 2-8. The four lower curves exhibit the mean value \( \langle \chi \rangle \) as a function of \( \omega \) for chaotic orbits subjected to damped oscillations with \( m_0 = 0.5 \), \( p = 2 \), and \( t_0 \) for the intervals (from top to bottom) \( 100 < t < 200 \), \( 200 < t < 300 \), \( 300 < t < 400 \), and \( 400 < t < 500 \). The top curve exhibits \( \langle \chi \rangle \) for undamped \((p = 0)\) oscillations for the intervals \( 0 < t < 512 \). The bottom panel shows the same data plotted as a function of \( \log_{10} \omega \).

That it may be possible to achieve efficient chaotic phase mixing in an oscillating galactic potential while still relaxing towards a nearly integrable state within \( 10 - 20 t_D \) or so is illustrated visually in Figures 2-10 – 2-12, which exhibit the behaviour of the same localised ensemble of 1600 initial conditions, again subjected to damped oscillations of the form given by Equation 2–5 with \( m_0 = 0.5 \), \( p = 2 \), and \( t_0 = 100 \) for three different choices of frequency \( \omega \). Figure 2-9 exhibits the results of a corresponding evolution in the presence of a nonoscillatory perturbation of the form given by Equation 2–7. In each case, the six
The $x$ and $y$ coordinates of an initially localised ensemble of orbits evolved in a Plummer potential subjected to a nonoscillatory perturbation of the form given by Equation 2–7 with $m_0 = 0.5$ and $p = 2$, for $t = 0$, $t = 16$, $t = 32$, $t = 64$, $t = 128$, and $t = 256$. It is evident visually from Figure 2-9 that, in the absence of oscillations, mixing is comparatively inefficient. Indeed, the evolution in Figure 2-9 is qualitatively identical to examples of regular phase mixing in time-independent potentials (as in Figure 2 in Kandrup (1999)). As asserted in Section § 2.3, a non-oscillatory perturbation of the
integrable Plummer potential leads to little if any transient chaos and, as such, no evidence for chaotic phase mixing. Figures 2-10 – 2-12, which incorporate a systematic pulsation, all yield phase mixing that is substantially more efficient. Figure 2-10, which represents orbits that have been pulsed with $\omega = 0.035$, a frequency near the lower edge of the resonance, clearly exhibits more robust mixing, although the mixing is still considerably less efficient than what is observed for strongly chaotic flows in time-independent potentials (as in Figure 1 in Kandrup (1999)). Indeed, it is difficult
Figure 2-11. The same as in Figure 2-10, for $\omega = 0.70$.

to determine unambiguously whether the orbits used to generate Figure 2-10 really exhibit significant transient chaos.

By contrast, Figure 2-11, which was generated for $\omega = 1.4$, exhibits precisely the sort of behaviour which one has come to associate with chaotic phase mixing in time-independent potentials. For the first two dynamical times, the localised ensemble of orbits, which started with a phase-space size $< 10^{-2}$, still remains confined. By $t = 64$, however, corresponding to an interval $\approx 3.2t_D$, the orbits have begun to spread
Figure 2-12. The same as in Figure 2-11, for $\omega = 3.50$.

significantly; and, by $t = 128$, corresponding to roughly $6.4t_D$, the ensemble has dispersed to fill a large portion of the available phase-space, and hence relaxation might be occurring for this value of $\omega$. Nevertheless, despite this efficient mixing, the potential is damping rapidly towards a near integrable state. By a time $t = 256$, the overall amplitude of the perturbation $m_0/(1 + t/t_0)^2$, has decreased from $m_0 = 0.5$ to $m_0 \approx 0.04$, this corresponding to a much more nearly time-independent system.
Figure 2-12 exhibits an example of ‘incomplete’ chaotic phase mixing for the case of a frequency $\omega = 3.5$. Here again one sees clear evidence of the dispersive behaviour indicative of chaotic phase mixing; but, since one is closer to the upper edge of the resonance, the efficient mixing has in fact turned off before the ensemble could fill all the accessible phase-space. The lack of orbits near $x = y = 0$ is exactly what one would expect in a time-independent spherical potential for orbits with an appreciable angular momentum.

2.5 Summary

The numerical computations described here lead to several conclusions:

- Subjecting orbits to a possibly damped oscillatory time-dependence can give rise to substantial amounts of transient chaos, even if the initial and final form of the potential is completely integrable.

- This transient chaos appears to arise from a resonant coupling which, at least for large amplitudes, is very broad and, as such, might be expected to be present in many physical systems.

- This transient chaos can drive chaotic phase mixing which, in the context of a fully self-consistent evolution, might be expected to play an important role in VR, for example, during the formation of galaxies and galaxy halos and in mergers or close encounters of galaxies, if the oscillations are long lived as in the case considered here.

To the extent that the resonance described in this chapter is representative of the Plummer sphere, it would thus seem likely that all but the very highest energy orbits would in fact be able to resonate with a large scale ‘bulk’ oscillation. One might also be tempted to conclude that the possibility of resonances for $\omega \gg \Omega_m$ is largely unimportant physically. This, however, is not necessarily the case. One anticipates generically that, as the system ‘relaxes’, power will cascade from larger scales and lower frequencies to shorter scales and higher frequencies. To the extent, however, that the natural frequency of the perturbation is more important than its detailed form, one might expect that such higher frequency oscillations, once triggered, could play an important role in continuing the process of VR on shorter scales. This is consistent with the fact (Kandrup et al. 2003)
that shorter scale perturbations associated with supermassive black hole binaries orbiting near the center of a galaxy can trigger qualitatively similar effects in terms of resonant phase mixing and energy and mass transport.
CHAPTER 3
CHAOS AND VIOLENT RELAXATION IN N-BODY MERGERS

3.1 Numerical Methods

The mechanisms responsible for violent relaxation are investigated by analyzing
N-body simulations of binary mergers between dark matter (DM) halos. The halo models
follow density profiles that are described by the general \((\alpha, \beta, \gamma)\) spherical density law
(Equation 3–1) (for example, Hernquist 1990; Zhao 1996), where \(\gamma\) denotes the asymptotic
inner slope of the profile, \(\beta\) corresponds to the outer slope, and \(\alpha\) determines the sharpness
of the transition between the inner and outer profile.

\[
\rho(r) = \frac{\rho_s}{(r/r_s)^\gamma[1 + (r/r_s)^\alpha]^[(\beta-\gamma)/\alpha]} \quad (r \leq r_{\text{vir}})
\]  

(3–1)

Two main halo models specified by particular choices of the parameters \(\alpha, \beta,\) and \(\gamma\)
were considered. The first model follows the Navarro et al. (1996, hereafter NFW) profile
(with \((\alpha, \beta, \gamma) = (1, 3, 1)\)), while the second model (with \((\alpha, \beta, \gamma) = (2, 3, 0.2)\)) corresponds
to a profile with a shallower inner slope. N-body halo models are constructed using the
exact phase-space distribution function under the assumptions of spherical symmetry
and an isotropic velocity dispersion tensor (Kazantzidis et al. 2004). Each of the initial
dark matter (DM) halos has a virial mass of \(M_{\text{vir}} = 10^{12} M_\odot\) implying a virial radius of
\(r_{\text{vir}} \simeq 256.7 \text{kpc},\) and a concentration of \(c = 12,\) resulting in a scale radius of \(r_s \simeq 21.4 \text{kpc}.\)
It is worth emphasizing that the adopted value of \(M_{\text{vir}}\) serves merely practical purposes
and does not imply anything special about the particular choice of mass scale. Hence, the
conclusions can be readily extended to mergers between equal-mass systems of any mass
scale.

More recently, Navarro et al. (2004) proposed a profile in which the slope varies
continuously with radius, and Merritt et al. (2006) argued that an Einasto (1969) model
could be more widely used to fit to characterize the density profiles. One of the main
arguments of the latter’s group is that, unlike the NFW profile, Einasto’s model has a
finite central density and a finite mass. However, the adopted profiles constitute a realistic representation of the density distribution of DM halos from $N$-body simulations for the purpose of this study.

Four different merger experiments were analyzed. The simulations were conducted with the multi-stepping, parallel, tree $N$-body code PKDGRAV (Stadel 2001). PKDGRAV uses a spline softening length, such that the force is completely Keplerian at twice the quoted softening length, and multi-stepping based on the local acceleration of particles. The first experiment followed the encounter of two NFW halos (referred to hereafter as run Bp1), while in the second experiment an NFW halo merged with a halo having an inner slope of $\gamma = 0.2$ (referred to hereafter as run hBp1). The initial halo models in these two runs were sampled with $N = 2 \times 10^5$ particles and forces were softened with a spline gravitational softening length equal to $\epsilon = 1.5$ kpc. In order to minimize any concern that the results might be compromised by numerical resolution effects, we analyzed an additional merger experiment between two NFW halos increasing the mass resolution by a factor of 10 and scaling down the softening lengths according to $\epsilon \propto N^{-1/3}$ (hereafter referred to as run HRBp). Initial conditions for binary mergers were generated by building pairs of halo models and placing them at a distance equal to twice their virial radii. In this study only mergers of systems on parabolic orbits were discussed, owing to the fact that this particular orbital configuration is the most typical of merging halos in cosmological simulations (for example, Zentner et al. 2005). These merger simulations (labeled Bp1,hBp1, and HRBp) were analyzed and described in greater detail in Kazantzidis et al. (2006). Extensive convergence tests carried out by Kazantzidis et al. (2006) indicate that isolated halo models did not deviate from their original distribution functions on timescales as long as 100 crossing times even at small radii ($r \sim \epsilon$).

In order to distinguish the mixing that results from the exponential instability of the $N$-body problem from the effects of the mixing resulting during the merger, an $N$-body simulation of the evolution of a spherical isolated NFW halo was performed. All orbits in
an isolated spherical halo are expected (in the smooth potential or large \( N \) limit) to be “regular orbits” that conserve four isolating integrals of motion. This simulation therefore serves as an important control experiment with which to compare the evolution in the merger simulations.

### 3.2 Macroscopic Evolution of Merger Remnants

A self-gravitating distribution of particles that is out of equilibrium will experience potential fluctuations, and the potential and kinetic energies of particles will oscillate back and forth. This behavior is governed by the time-dependent virial theorem:

\[
\frac{1}{2} \ddot{I} = 2T + V
\]  

where \( I \) is the moment of inertia, \( T \) is the total kinetic energy, and \( V \) is the total potential energy of the system of particles, with all quantities defined with respect to the center of mass of the system. For a time-independent gravitating system, \( \ddot{I} = 0 = 2T + V \). A robust quantitative measure of how far a gravitating system is from equilibrium can be obtained from the virial ratio \( 2T/|V| \), which is unity for a system in global dynamical equilibrium.

In Figure 3-1, the top panel shows the separation of the most bound particle (MBP) of each halo in the merger, the middle panel shows the evolution of the virial ratio \( 2T/|V| \) and the lower panel shows the change in total potential energy of the system \( (V) \), as a function of time. We note that pericenter passages (seen as minima in the MBP separation) correspond to maxima in the virial ratio and minima in potential energy. For \( t \sim 7 \) Gyr the virial ratio \( 2T/|V| \) and potential \( V \) undergo strong fluctuations but remains close to unity thereafter. We refer to this time \( t \sim 7 \) Gyr to be the time when the system is globally relaxed.

### 3.3 Microscopic Chaos

In real galaxies, there are two components of the gravitational accelerations on particles, a component arising from the background potential which is smoothly varying with time, and a second component arising from the discrete nature of stars (and possibly
Figure 3-1. Top: The separation between the most bound particles of the two merging NFW halos (run Bp1) as a function of time; Middle: time evolution of the virial ratio $2T/|V|$; Bottom: time evolution of the total potential energy of the system $V$. 
DM particles). The importance of the discrete component is defined by the two-body relaxation timescale (Chandrasekhar 1943; Binney & Tremaine 1987) and is larger than the age of the Universe in galaxy-sized systems. Therefore, it is generally assumed that equilibrium and non-equilibrium evolution of the phase-space density distributions in galaxy-size objects can be studied in the mean-field limit, where the collisionless Boltzmann equation can be applied.

However, as was first noted by Miller (1964), the N-body problem is chaotic in the sense that the trajectory of the 6N-dimensional phase-space coordinate of the system exhibits exponential sensitivity toward small changes in initial conditions. This exponential instability has been investigated extensively in several studies over the past three decades (see Merritt 2005 for a review). These studies have shown that the largest $N$-body Lyapunov exponent $\chi$ does not converge toward zero as $N$ increases (Kandrup & Smith 1991), even for $N$-body realizations of integrable potentials (Kandrup & Sideris 2001). Hemsendorf & Merritt (2002) have shown that the direct $N$-body problem (with $N < 10^5$) is inherently chaotic and that the degree of chaos, as measured by the rate of divergence of nearby trajectories (with the Lyapunov exponent $\chi$), increases with increasing $N$ with characteristic $e$-folding time equal to $\sim 1/20$ of the system crossing time (in the absence of softening). It has also been shown that the mean $t_e$ ($e$-folding time) increases with increasing particle number in the case of softened potentials (El-Zant 2002) so that discreteness effects are reduced as the softening becomes comparable to interparticle separations (Kandrup & Smith 1991).

The exponential separation of particles in energy occurs on the $e$-folding timescale ($t_e$) which is much smaller than the two body relaxation time ($t_r$) (Kandrup et al. 1994; Hut & Heggie 2001). The divergence of particles in energy saturates after a few $t_e$ and then varies with time on the standard two-body relaxation timescale indicating that the microchaos is not an important process in changing the fine-grained distribution functions of gravitating systems.
Despite the fact that particles separate exponentially in phase-space, as $N$ increases, orbits in these systems become more and more regular, acquiring orbital characteristics closer and closer to those in smooth potentials, and the macroscopic separations of orbits saturate at smaller and smaller distances (Valluri & Merritt 2000; Kandrup & Sideris 2001; Sideris 2004). Additionally, decades of $N$-body integrations have demonstrated that, in many ways, the behavior of large-$N$ systems matches expectations derived from the collisionless Boltzmann equation (for example, Aarseth & Lecar 1975).

The fact that in an $N$-body system the Lyapunov exponent will primarily measure the microchaos and not the macroscopic chaos (arising from the background potential) means that it is has limited usefulness for measuring chaotic behavior in $N$-body systems. Other measures of chaos, such as those that measure changes in orbital characteristics (for instance, fundamental oscillation frequencies of orbits; for example, Laskar 1990, Laskar et al. 1992, Valluri & Merritt 1998), rely on the ability to identify such frequencies from oscillations that last between 30-100 orbital periods. Sideris (2006) developed a technique based on pattern recognition of orbital characteristics which is applicable in time-dependent systems whose oscillations last 10-30 crossing times. To our knowledge, none of the standard measures of chaos are applicable to $N$-body orbits in potentials with non-periodic potential fluctuations lasting only a few ($< 10$) crossing times. This makes it necessary to define new ways of quantifying chaos and mixing.

### 3.4 Definitions of Mixing in $N$-body Systems

In this chapter we estimate the degree of mixing at various stages of the evolution of the merging systems (a) by considering pairs of nearby particles in the $N$-body simulation and tracing their separation in four different phase-space quantities and (b) by carrying out mixing experiments that monitor the rate at which ensembles of 1000s of nearby particles evolve with time and reach a “near invariant” distribution in energy and angular momentum.
3.4.1 Separation of Pairs of Nearby Particles

Several previous studies of this type have focused on the configuration space separation $\Delta r$ of a pair of infinitesimally nearby test particles in a frozen $N$-body potential. These studies picked pairs of non-self-gravitating test particles that are arbitrarily nearby in phase-space (Valluri & Merritt 2000; Kandrup & Sideris 2003). Studies that used live $N$-body simulations (for example, Kandrup et al. 1994) have compared pairs of orbits in two different realizations of simulations where the initial conditions had been perturbed by an infinitesimal amount. Here we work with orbits in a given $N$-body simulation and are therefore limited (by the resolution of the simulations) in our ability to choose pairs of particles that are very close in phase-space.

We select a large number (1000) of nearest-neighbor pairs in one of the two merging $N$-body halos and follow their separations through the entire merger process. We have checked that when two identical halos merge, the results were independent of the halo chosen to perform the analysis. To pick particle pairs we adopt the following scheme:

- All the particles in a given halo are sorted in their separation $r$ from the MBP of that halo, and the particles are binned in ten radial shells of equal width, extending from the MBP to the initial virial radius of the halo ($r_{\text{vir}} \approx 256.7$ kpc). In the analysis that follows we compare behavior of particles in the 1st, 4th and 7th shell from the center of the potential. The outer radius of the first shell is just outside the scale radius, the 4th shell lies at the half mass radius and 7th shell lies at the 3/4 mass radius of the initial NFW halo.

- A particle $P^*$ with separation $r_P$ from the MBP is picked at random in a given shell, and the 5000 nearest particles in configuration space to the particle $P^*$ are selected.

- The velocity separations $v$ of each of the 5000 particles relative to particle $P^*$ are computed, and the 2500 particles with the smallest $v$ separation from $P^*$ are selected.

- The magnitudes of the separation of total angular momentum $J$ of each of the 2500 particles above relative to particle $P^*$ are computed, and the 1250 particles with the smallest $J$ separation from $P^*$ are selected.

- The angle between the angular momentum $J$ and the angular momentum $J_{P^*}$ is computed for each of the 1250 particles, and the 625 particles with the smallest angle are selected.
From the remaining 625 particles, the particle with the total energy $E = T + V$ closest to the particle $P^*$ is then picked as its nearest neighbor.

In general, after an initial phase of exponential growth, the separation of two nearby particles saturates and then oscillates about some median value, due to small but finite differences in the initial oscillation frequencies in the global potential (for example Kandrup & Sideris (2003)). To determine the behavior of a “typical pair” of particles in a given shell, we first pick 1000 pairs of particles in each shell. We then report the median value of the separation of that phase-space quantity as a function of time for the 1000 particle pairs. We find (as is well known) that the median is a more reliable estimator than the mean separation of particles because each particle has a small but finite probability of being ejected from the system as a result of the merger, and such ejections cause a small fraction of particles to form a long tail in the histogram of separations at each time step. Thus, for all quantities defined below it should be implicitly understood that we report the median value of the parameter for 1000 pairs of particles. Specifically, we monitor the following quantities:

- “$\ln(\Delta r)$” – the natural logarithm of the separation in the spatial coordinate $r$ of the median pair of particles.
- “$\ln(|\Delta v|)$” – the natural logarithm of the absolute value of the relative three dimensional velocity $v$ of the median pair of particles.
- “$\Delta E$” – the separation in total energy of the median pair of particles.
- “$\Delta J$” – the separation in the total amplitude of the angular momentum of the median pair of particles. (We pick pairs whose vector directions have angular separations of less than 90 deg. As the particles separate, we track the separation in the amplitude of $\mathbf{J}$ but not the direction.)

The quantities defined above differ in one important aspect from previous studies in which similar quantities have been used: all particles in the pairs are self-gravitating (and not test particles) and are picked from an initial distribution function that is self-consistent with the spherical NFW potential. This implies that the typical initial pair separation $\ln(\Delta r)$ of particles depends on their location in the potential such that
pairs in the central cusp will be much closer than the typical initial pair separation in the outer regions. We therefore often scale the quantities relative to their values at $t = 0$ Gyr. Moreover, especially when calculating the velocity separation $\ln(\Delta v)$, we observe the mutual gravitational attraction of nearby particles.

The quantities $\Delta E$ and $\Delta J$ provide additional insights because in the case of an isolated $N$-body halo, both quantities are expected to remain constant (to within numerical errors) for the entire duration of the simulation (following the initial increase in these quantities due to the Miller instability). Since $E$ and $J$ are integrals of motion in equilibrium potentials, any changes in the separation of these quantities in the case of the merger reflects the influence of the time-dependent potential. Evolution with time of these quantities for various simulations are described in Section § 3.5.1.

3.4.2 Mixing of Ensembles in Phase-Space

Another way to quantify the mixing rate is by monitoring the secular evolution of ensembles of infinitesimally nearby test particles due to the presence of chaotic orbits. We conducted a series of mixing experiments where we monitored the spread of an ensemble of 1000 nearby particles as a function of time in two phase-space coordinates.

Such experiments have been carried out previously to study the mixing of microscopic ensembles of chaotic orbits in static, non-linear potentials (Mahon et al. 1995; Merritt & Valluri 1996; Kandrup et al. 2000; Kandrup & Siopis 2003). These authors picked ensembles of orbits with initial conditions that uniformly sampled an infinitesimally small region of phase-space and measured the rate at which the ensemble spreads and fills a region of configuration space. Ensembles associated with strongly chaotic orbits were found to evolve to a near-invariant distribution within 5-10 orbital crossing times, while ensembles associated with weakly chaotic or “sticky” orbits (orbits that lie close to resonance in phase-space) evolved on much longer time scales and often did not reach an invariant distribution. Ensembles associated with regular orbits only spread slightly, due to phase mixing that resulted from the small differences in their initial integrals of motion.
Once again the resolution of the simulations determine how closely we can sample the phase-space. We pick ensembles of the nearest 1000 particles about a given particle ($P^*$) in the higher resolution merger simulation only (run HRBp), following a scheme similar to that for picking pairs of nearby particles (Section § 3.4.1) to obtain the 1000 nearest particles to $P^*$.

Configuration-space coordinates have traditionally been used to study chaotic mixing of microscopic ensembles in smooth potentials. In such experiments microscopic ensembles of regular orbits do not mix at all in configuration space so all the observed mixing can be attributed to chaos. However tests carried out with ensembles selected by the above scheme in the $N$-body isolated spherical NFW halo showed quite significant spreading in configuration-space. The evolution of the ensembles in configuration space coordinates followed the following pattern: ensembles spread rapidly in configuration space on a short timescale (due to the Miller instability, see Section § 3.3), and then continued to spread more slowly at a rate that scaled with local orbital period. At a given radius the mixing rate was found to be sensitive to the particular ensemble in question - the nature of the parent orbit (whether it is box-like or tube-like), the location of the orbit relative to the orientation of the merger etc. Since their mixing in configuration space is dominated by the Miller instability and phase mixing, our ensembles are too large to be usefully compared with the microscopic ensembles used to measure chaotic mixing in the works of previous authors. However, we found that ensembles selected by the above scheme in the isolated halo showed negligible mixing in the coordinates $E, J$ (in agreement with the previous findings of Hut & Heggie 2001). Consequently we focus our attention on mixing in these two variables. The main objective of carrying out these experiments is to observe mixing in $E, J$ that results from the merger.
Figure 3-2. Divergence in the quantities \( \ln(\Delta r)/\ln(\Delta r(t = 0)) \) as a function of time (in units of local crossing time) for particles in the 1\textsuperscript{st}, 4\textsuperscript{th} and 7\textsuperscript{th} radial shells. The solid lines are for the merger of two NFW halos (run Bp1). The dot-dashed lines are for the control simulation of an isolated NFW halo. The dashed straight lines are best-fits to both the solid lines and the dot-dashed lines in the region \( t/t_c = [0.05 - 0.35] \).

We define the quantities \((\mathcal{E}, \mathcal{J})\) for each particle in the ensemble as follows:

\[
\begin{align*}
\mathcal{E} & = \frac{E - E(P^*)}{E(P^*)} \\
\mathcal{J} & = \frac{J - J(P^*)}{J(P^*)}
\end{align*}
\]  

(3–3)  

(3–4)

where \(E(P^*), J(P^*)\) are the energy, magnitude of the total angular momentum of the particle \(P^*\) respectively. The quantities \(\mathcal{E}, \mathcal{J}\), are computed for each of the 1000 particles in the ensemble.

The results of these experiments are described in Section § 3.5.2.

3.5 Results

3.5.1 Separation of Nearby Particles in Phase-Space

We begin by demonstrating the ability of \(\ln(\Delta r)\) to identify chaotic behavior. This quantity (and all the others that we use) are compared to equivalent quantities measured in an isolated equilibrium NFW halo. The three panels of Figure 3-2 show the initial separation in spatial coordinate \(\ln(\Delta r)\) (relative to its value at \(t = 0\) Gyr) in three
different radial shells (1st, 4th and 7th). The horizontal axis gives time in units of the median crossing time $t_c$ in each shell of the corresponding isolated NFW halo. The local cross time are $t_c = 0.53, 2.59, 4.95$ Gyr for shells 1, 4, 7 respectively. We see that in both the merger (solid curve) and the isolated halo (dot-dash curve) the initial separation is nearly linear in the quantity $[\ln(\Delta r)/\ln(\Delta r(0))]$, indicating an exponential change in radial separation. The linear increase in $\ln(\Delta r)$ on this timescale constitutes a clear manifestation of the Miller instability (see Section § 3.3). This phase of exponential divergence is short lived (< 35% of the local crossing time). The $e$-folding time of the Miller instability $t_e = 0.39, 1.06, 2.73$ Gyr for shells 1, 4, 7 respectively (roughly half the local crossing time in each shell). In both the merger simulation and in the isolated halo, the initial rapid change in $\ln(\Delta r)$ saturates at the same value - a value determined by the length scale over which the smooth potential dominates over the graininess (Kandrup & Smith 1991). At smaller radii, the higher particle densities result in saturation values of $\ln(\Delta r)$ that are higher that at larger radii. Figure 3-2 is clear evidence that micro-chaos is detectable despite our inability (imposed by our use of a $N$-body distribution function) to choose particles arbitrarily close to each other in phase-space.

We now focus on the behavior of the four quantities ($\ln(\Delta r)$, $\ln(\Delta v)$, $\Delta E$ and $\Delta J$) over longer time-scales. As before all quantities are compared with the equivalent quantities measured in an isolated equilibrium NFW halo to control for exponential Miller instability. Figure 3-3 shows the evolution of $\ln(\Delta r)$ and $\ln(\Delta v)$ as a function of time in three different radial shells (1st, 4th and 7th from the center). The quantities plotted are scaled using the initial values of the quantities at $t = 0$ Gyr; i.e., the top panels show $[\ln(\Delta r)/\ln(\Delta r(0))]$ and the lower panels show $[\ln(\Delta v)/\ln(\Delta v(0))]$ (where $\Delta r(0), \Delta v(0)$ are the separations at $t = 0$). The solid lines show the evolution of the quantities in the low-resolution merger simulations (run Bp1: two NFW halos with $2 \times 10^5$ particles per halo), and the dot-dashed curves show the evolution of these quantities in the isolated
Figure 3-3. The top three panels show divergence in the quantities \( \ln(\Delta r)/\ln(\Delta r(t = 0)) \) as a function of time for three different shells in the halo (from left to right: the first, fourth, and seventh shells respectively). The lower three panels show the quantity \( \ln(\Delta v)/\ln(\Delta v(t = 0)) \), for the same three shells. The solid lines are for the merger of two NFW halos (run Bp1). The dot-dashed lines are for the control simulation of an isolated NFW halo.

In what follows we point out some characteristic features of Figure 3-3 that are seen elsewhere in this section.

The lower panels of Figure 3-3 show the separation in \( \ln(\Delta v) \) in both the isolated halo and the merging halos. There is a noticeable initial decrease in \( \ln(\Delta v) \) (which occurs simultaneously with the linear increase in \( [\ln(\Delta r)/\ln(\Delta r(0))] \)), which we attribute to the mutual gravitational attraction between the pair of particles that causes a deceleration of the particles while their radial separation increases. The initial decrease in \( \ln(\Delta v) \) is seen in both the merging halos and the isolated halo, confirming that this is not due to the merger but is a characteristic of the \( N \)-body simulation.
In the innermost (1st) shell, after the initial growth in \( \ln(\Delta r) \) and the initial decrease in \( \ln(\Delta v) \), there are periods of time during which these quantities are essentially constant, separated by several sharp decreases in \( \ln(\Delta r) \) and correspondingly sharp peaks in \( \ln(\Delta v) \) not seen in the isolated halo. The dips in \( \ln(\Delta r) \) and peaks in \( \ln(\Delta v) \) are coincident with each other and correspond to the pericenter passages of the MBPs of the two halos seen in Figure 3-1. It is the first and second pericenter passages that cause the most significant changes in the particle separations (\( \ln(\Delta r) \), \( \ln(\Delta v) \)).

It is quite remarkable how little systematic change in separation of either \( r \) or \( v \) is seen in between pericenter passages, suggesting that very little change in the macroscopic separations of particles occurring at any time during the merger except during the pericenter passages. This can be understood as follows: during pericenter passages of the MBPs of the two halos, there is significant overlap in the two potentials leading to a sudden deepening in the potential experienced by the particles leading to a transient compressive tidal field that simultaneously decreases the separations in particles (seen by the dips in \( \ln(\Delta r) \)) and increases their kinetic energies (seen in the sharp increases in \( \ln(\Delta v) \)). Dynamical friction between the two halos is also strongest at pericenter and consequently the majority of their orbital energy and angular momentum are lost impulsively at these times. When the two halos separate enough that their innermost shells no longer overlap they experience little or no change in their potentials, so there is little further change in separation of trajectories. Thus, the global potential fluctuations that occur between pericenter passages do not cause any further increase in particle separations. Behavior similar to that in shell 1 is seen in shells 2 and 3 but is not shown here.

The effect of the first pericenter passage is seen to occur simultaneously at all radii. In shells 4, 7 we observe a dip in \( \ln(\Delta r) \) and peak in \( \ln(\Delta v) \) at the first pericenter passage which is both broader and less intense than in shell 1. At larger radii the first pericenter passage is followed by a continuous change in \( \ln(\Delta r) \) and \( \ln(\Delta v) \) but subsequent pericenter
Figure 3-4. The panels show divergence in the quantities $\ln(\Delta r)/\ln(\Delta r(t = 0))$ as a function of time for $1^{st}$, $4^{th}$, $7^{th}$ shells. The solid curves are for the merger of two NFW halos (run Bp1) with pairs of particles selected at $t = 0$ Gyr. The dashed curves are for pairs of particles selected at $t = 3.0$ Gyr in the isolated halo and the dot-dashed curves are for pairs of particles selected at $t = 3.0$ Gyr. The curves for the pairs starting at $t = 3.0$ Gyr have been translated in time to $t = 0$ Gyr. The dot-dashed curves and the dashed curves indicate that the initial exponential instability of nearby orbits during the merger is not very different from that in the isolated halo, independent of the time at which the particle pairs are selected.

passages are less clearly visible. This indicates that while there is no propagation delay in the impulsive tidal shock with radius, the response is longer lived at larger radii. This is because, the centers of the two merging halos do not separate beyond the radius of the $4^{th}$ and $7^{th}$ shells after the first pericenter. Consequently the change in the external potential is not as sudden as for the inner shells.

As discussed in Section § 3.4 it is extremely difficult to make a reliable quantitative measurement of an exponential instability on a short timescale. However, both the qualitative and quantitative behavior of initially nearby particles seen in Figure 3-2 indicates that we are in fact detecting the Miller instability. If the oscillating external potential is also contributing to an exponential instability of orbits one might expect that if pairs of nearby particles were picked during the phase when the potential is changing very rapidly, the rate of change of separation in some quantity (say $r$) would be greater
than in either the isolated halo or at the start of the merger. Figure 3-4 compares the separations of particles $\ln(\Delta r)$ when pairs were picked at $t = 0$ Gyr with pairs picked at $t = 0$ Gyr in the isolated halo and separation of pairs of particles picked at $t = 3$ Gyr. For the pairs of particles picked at $t = 3$ Gyr in the merger, the time axis is $t - 3$ so that initial slopes of the three curves can be more readily compared. We chose a start time $t = 3$ Gyr to compare against since it corresponds to a time at which the virial ratio is changing very rapidly and corresponds to the first apocenter separation of the MBPs of the two halos (identical behavior was found when particles were picked at $t = 4$ Gyr).

Since the potential is changing rapidly at this time, we might expect to observe an enhanced rate of separation of nearby pairs of orbits due to chaos induced by the potential fluctuations provided the $e$-folding time for this instability is comparable to or greater than that due to the Miller instability. Figure 3-4 shows, on the contrary, that the initial exponential separation of particles in $r$ at $t = 3$ Gyr does not have a systematically greater slope than the rate of separation of particles at $t = 0$ Gyr. In shell 1 the slope of the dot-dash curve is smaller than that of the solid curve (which we attribute to the fact that heating from the first pericenter passage caused nearby particles to be further apart in phase-space); in shell 4 the slopes of all three curves are essentially identical; only in shell 7 is the slope of the dot-dash curve slightly steeper than that of the black curves indicating that chaoticity may be playing a bigger role at large radii.

While this is clearly not conclusive evidence, it is suggestive of the hypothesis that the degree of chaos due to the time-dependent potential, is not significant compared to the chaoticity due to the Miller instability. It may play a small role in the evolution of the system at larger radii. However there is little evidence to suggest that chaotic mixing from orbits in the time-dependent potential is driving the relaxation.

To understand better the mechanism driving the relaxation we now look at separation of particles in energy and angular momentum space. In integrable time-independent
Figure 3-5. The top three panels show divergence in the quantities $\Delta E/\Delta E(t = 0)$ as a function of time for three different shells in the halo (from left to right: the first, fourth and seventh shells respectively). The lower three panels show the quantity $\Delta J/\Delta J(t = 0)$, for the same three shells. The solid lines are for the merger of two NFW halos (run Bp1). The dot-dashed lines are for the isolated NFW halo.

potentials, energy and some quantity akin to angular momentum are generally integrals of motion. It is therefore of interest to quantify how these quantities change during a merger.

Figure 3-5 shows the separations in energies (top panels) and angular momentum\(^1\) (bottom panels) for the same pairs of particles as in Figure 3-3. As before, the dot-dashed lines are for pairs of particles in the isolated halo, while the solid lines are for pairs of particles in one of the merging halos. The quantities $\Delta E, \Delta J$ are scaled relative to their initial values at $t = 0$ Gyr. $\Delta E$ and $\Delta J$ (as expected) are very close to zero in the isolated

\(^{1}\) The angular momentum ($J$) is about an axis perpendicular to the plane containing the orbits of the center-of-masses of the two merging halos.
The sharp initial rise in $\Delta E$ and $\Delta J$ reflects the response of the particles to the presence of second halo. This a consequence of the fact that although each individual halo is in equilibrium separately, when the two halos are set up on a parabolic trajectory at $t = 0$ they are no longer in equilibrium, since particles now experience the external potential of the second halo.

After the initial increase in $\Delta E$ and $\Delta J$, there are sharp but transient increases in the separation of these quantities that also occurs primarily during pericenter passages. In fact, the peaks in $\Delta E$ and $\Delta J$ are strongly correlated with the pericenter passages in all radial shells and provide striking evidence that it is these events that are primarily responsible for scattering particles in $E$ and $J$. We also note that the step-like changes in these quantities and their correlation with pericenter passages is seen at all radii unlike in the case of $\Delta r$ and $\Delta v$ which changed more slowly and continuously in the outer regions. In the inner most shells there is little or no separation in phase-space quantities occurring between pericenter passages. The slight increase in $\Delta E$ and $\Delta J$ seen in shell 4 and shell 7 between $t = 2$ Gyr (first pericenter passage) and $t = 5$ Gyr (second pericenter passage) could be evidence for slower chaotic mixing. After the first pericenter passage the outer regions of the two halos never completely separate again and the continued overlap of the two potentials means that at larger radii the tidal compressive field and dynamical friction is able to operate over a longer duration and consequently more gradual changes occur as well. It is clear, however that the impulsive changes in $\Delta E$ and $\Delta J$ during multiple pericenter passages have the strongest effect on the mixing in $E$ and $J$.

Figure 3-6 compares separations in all four quantities $\ln(\Delta r)$, $\ln(\Delta v)$, $\Delta E$ and $\Delta J$ for two merger simulations with different numerical resolutions. We recall that the run HRBp

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2 We note that they are not precisely zero because, although the total energy of the system is conserved to within numerical error, the computation of $(\Delta E/\Delta E(t = 0))$, $E$ for a given particle is relative to the MBP - which can be a different particle at each time-step.
Figure 3-6. The four panels (top to bottom) show divergence in the four quantities $\ln(\Delta r)$, $\ln(\Delta v)$, $\Delta E$ and $\Delta J$ plotted as a function of time for pairs of particles in the innermost shell of one of the two merging NFW halos. The solid lines are for the merger of two halos with $2 \times 10^5$ particles in each halo (run Bp1); dot-dashed lines are for the merger of two halos with $2 \times 10^6$ particles in each halo (run HRBp). The inserts in the top two panels show differences in the initial separation due to the Miller instability. The dot-dash line starts at lower values in both $\ln(\Delta r)$, $\ln(\Delta v)$ which is a result of picking particles closer in phase-space in run HRBp. For run HRBp $\ln(\Delta r)$ has a shorter $e$-folding time, due to the higher particle number and smaller softening length. Apart from the behavior of initial separations in all four quantities, the low and high resolution mergers are almost identical.
was performed with a factor of 10 more particles and approximately a factor of 2 higher force resolution than the run Bp1. We present the comparison for only the innermost shell since this is expected to be the most susceptible to numerical effects, if any. There is clearly very little difference between the overall behavior of separations of particles in any of the 4 quantities plotted. The only noticeable difference is in the initial behavior of \( \ln(\Delta r), \ln(\Delta v) \).

The insert in the top most panel shows that the initial radial separation of particles in run HRBp is smaller at \( t = 0 \) Gyr than it is for run Bp1. This is purely a consequence of the fact that by increasing the mass resolution the phase-space coordinates become more densely sampled and this allows us to select pairs of particles that are closer in phase-space (see inserts in top two panels). The insert in the second panel shows that the initial velocity separation \( \ln(\Delta v) \) decreases to lower values in run HRBp compared to run Bp1. This also points to the increased gravitational deceleration due to the proximity of the nearest neighbor and the decrease in the softening parameter.

All curves are plotted without scaling them relative to their initial separations to illustrate that apart from differences in the initial separation (which we attribute to the Miller instability), the separations in all four quantities in the different resolution mergers saturate at the same values. This is confirmation that particle separations saturate at values that are determined by the global dynamics of the merger and not by the resolution of the simulations.

Figure 3-7 shows separations in \( \ln(\Delta r) \) and \( \ln(\Delta v) \) for particles in different radial shells in run Bp1 and in the merger of an NFW halo with a shallow cusp. The behavior is remarkably similar in both cases except in the innermost shell. This is expected, since the innermost shell is the only one in which the density profiles (and consequently the phase-space density distributions) differ significantly. It is well known from previous studies (Boylan-Kolchin & Ma 2004; Kazantzidis et al. 2006) that cusps in shallow potentials are less robust than cusps in NFW halos. Particles in the innermost shell of
Figure 3-7. Solid lines show separations in \( \ln(\Delta r) \) and \( \ln(\Delta v) \) for the merger of two NFW halos and the dot-dash lines show the separations of these quantities for the merger between an NFW halo and a halo with an inner shallow cusp of \( \gamma = 0.2 \). The behavior is similar in both cases except in the innermost shell where the differences in the density profiles are significant.

The results of our investigation of particle separations in \( \ln(\Delta r) \), \( \ln(\Delta v) \), \( \Delta E \) and \( \Delta J \) can be summarized as follows:

- The initial separation of pairs of nearby particles in \( N \)-body systems such as those studied here is a result of the exponential instability of the \( N \)-body problem, the so-called “Miller instability.” The qualitative and quantitative behavior of the separation
of particles: for example their dependence on local crossing time (Figure 3-3), their dependence on softening and force resolution (Figure 3-6) are completely consistent with previous studies of this instability. It is important to point out that the very fact that we are able to detect the exponential divergence of orbits due to the Miller instability indicates that despite the fact that pairs of nearby particles in the \( N \)-body simulations are separated by macroscopic distances (i.e., are not infinitesimally close by), our method for identifying an exponential instability can be applied successfully to an \( N \)-body system.

- Subsequent to the saturation of the separations due to the Miller instability, the most significant increase in separations of nearby particles (in \( r, v, E, \) or \( J \)) occur during pericenter passages of the MBP of the two halos and is a consequence of the compressive tidal shocking (due to the overlap of the two halos) and dynamical friction between the two halos that occurs during the pericenter passages. In the innermost shells there is little or no separation in phase-space quantities occurring between pericenter passages, despite the large global fluctuations in potential occurring during these times. At larger radii separation in \( r, v \) continue following pericenter passages. At larger radii, there is also a small (but short lived) increase in separation in \( E \) and \( J \) between the first and second pericenter passages.

- If we consider the increase in the macroscopic separation of particles in \( (r, v) \) to be a measure of the mixing of particles in these quantities, these plots would lead us to conclude that the majority of mixing occurs due to the Miller instability and several phases of mixing that occur during pericenter passages of the two halos. In all radial shells of merger simulation we observe that the macroscopic separation of initially nearby particles increases until it saturates. In the merging halos, saturation occurs at larger values of \( \ln(\Delta r), \ln(\Delta v) \) respectively, than in the isolated halo. This is a consequence of the fact that the accessible phase-space volume has increased during the merger process. We saw from Figure 3-5 that \( \Delta E \) and \( \Delta J \) (which represent increases in the available phase-space states accessible to particles) increased in step-wise fashion primarily during pericenter passages of the MPBs, as was predicted by Spergel & Hernquist (1992). By \( t = 7 \) Gyr the virial ratio \( 2T/|V| \approx 1 \), although the potential continues to experience long-lived but low-amplitude oscillations until 13 Gyr. The low-level potential fluctuations cause little change in the separation of either \( r \) or \( v \) beyond 7 Gyr.

In Lynden-Bell’s (1967) model for “violent relaxation”, particles experience changes in their phase-space coordinates due to their interaction with a time-varying background potential. It has been argued by various authors that a time-dependent potential alone is inadequate to cause relaxation, since nearby particles in phase-space will merely be relabeled in energy and will not separate (or mix). The results from this section indicate that during the merger of two halos, nearest neighbor particles in phase-space do mix in
Further, the primary causes of mixing are the compressive tidal shocks and dynamical friction which transfer energy and angular momentum from the orbital motion of the two merging halos to the particles during the pericenter passages (Gnedin et al. 1999; Valluri 1993). The particles respond to sudden increments in their orbital energy and angular momentum by jumping to new orbits, resulting in mixing in phase-space and evolution to a new distribution. This latter phase of evolution could indeed be the result of some weaker form of chaotic mixing, coupled with phase mixing, both of which occur on a longer timescale.

### 3.5.2 Mixing of Ensembles in Phase-Space

In this section we present results of the mixing experiments with an initially coherent ensemble of particles selected from the \( N \)-body simulation. As discussed in Section § 3.4.2 we confine our analysis to mixing in the scaled energy and angular-momentum variables \( \mathcal{E}, \mathcal{J} \).

Figure 3-8 shows the evolution of an ensemble of 1000 self-gravitating particles in the 4\(^{th}\) shell in run HRBp. The density contours are obtained using a kernel smoothing algorithm. At \( t = 2 \) and \( t = 5 \) Gyr which correspond to the first and second pericenter passage, respectively, the ensemble undergoes a sudden spreading in energy (\( \mathcal{E} \)) which is a consequence of the deepening of the potential during the overlap of the two halos. By \( t = 6 \) Gyr the particle distribution fills a triangular region in \( \mathcal{E}, \mathcal{J} \) space. A greater spreading in \( \mathcal{E} \) at the smallest values of angular momentum is a consequence of the fact that these orbits are on the most radial orbits and consequently experience greater changes in their potential energies. Particles with the smallest (most negative) values of \( \mathcal{E} \) experience the greatest spread in angular momentum largely as a consequence of their being ejected during the tidal shocks to larger radii.

We carried out similar mixing experiments for over 25 different ensembles in each shell. The behavior of the ensemble in Figure 3-8 was found to be quite representative of all the different ensembles.
The evolution of an ensemble of 1000 nearest neighbors in phase-space in shell 4 at $t = 0$ (run HRBp) plotted at one Gyr time intervals in the merger. The plot shows isodensity contours for the particle distribution. The red contours correspond to the highest density regions and black to the lowest density regions. The contours are spaced at logarithmic density intervals relative to the maximum density contour to enhance the visibility of low density regions. Pericenter passages (at $t = 2$ and $t = 5$ Gyr) are seen to cause a sudden spreading of the entire ensemble in $\mathcal{E}$ (a result of the tidal shocking at these times). Two additional pericenter passages occur between $t = 5$ and $t = 6$ (see Figure 3-1). By $t = 6$ Gyr the ensemble has experienced 4 pericenter passages and has settled to fill a triangular region in the $\mathcal{E}, \mathcal{J}$ space.
Figure 3-9. Isodensity contours for ensembles of particles plotted as a function of $\mathcal{E}, \mathcal{J}$ for shell 1 (left hand column), shell 4 (middle column) and shell 7 (right-hand column) plotted for different times in their evolution (see text for details).
Figure 3-9 shows contours of projected density of particles as a function of the quantities \((E, J)\) in run HRBp, for ensembles in 3 different radial shells at 4 different times in the evolution. The evolution of the ensembles in shells 1, 4 and 7 are very similar. At \(t = 2\) Gyr (first pericenter passage) the ensemble spreads in \(E\). Similar increases in spread and are seen during the 2nd pericenter passages (but are not shown here). After the first pericenter passage the majority of the particles in the ensemble at \(t = 4\) Gyr return to a more compact distribution in \((E, J)\) but one with a larger spread in \(J\) at small \(E\). The bottom panels correspond to \(t = 6\) Gyr, after the first three pericenter passages have occurred. At this time all the ensembles fill a roughly triangular region in \(E, J\) space. This characteristic triangular final distribution is seen in all shells and for essentially all of over a hundred different ensembles we examined. This final distribution is not uniform in density, but the density contrast across the ensemble is much smaller than in the initial distribution. We note that when these ensembles were re-observed at 15 Gyr there was slightly greater uniformity of density in the lower-density tails but otherwise they had changed very little from their distributions at 8 Gyr.

There are two important observations that can be made: (a) changes in \(E, J\) occur primarily during pericenter passages (Figure 3-8), and (b) the final distribution in \(E, J\) space of the ensembles at all radii are similar, and the rate at which this final distribution is reached is independent of radius (Figure 3-9).

The median crossing time in the innermost shell \((r \leq 25\) kpc) is \(t_c \sim 5 \times 10^8\) yrs, while in the 7th shell is \(t_c \sim 5\) Gyr. It is obvious from Figure 3-9 that although the crossing time in shell 7 is a factor of 10 longer than in shell 1, the diffusion of the ensembles in \((E, J)\)-space appears to occur primarily due to the pericenter passages of the MBPs and there is only a slightly increased rate of spreading at larger radii. Phase mixing and chaotic mixing in isolated potentials occur at rates that scale with the local dynamical time so one would expect that mixing effects that depend on the local dynamical time (such as chaotic mixing) would occur faster at small radii than at large radii. The
the evolution of the ensembles at different radii occurs episodically following
pericenter passages, rather than continuously at a rate that scales with local orbital
time points to the importance of the impulsive tidal events as the processes that drives the
transition to an equilibrium distribution in energy and angular momentum.

Indeed, the absence of a strong dependence on local crossing time in Figure 3-9
provides the strongest evidence so far that mixing in energy and angular momentum is
driven by compressive shocking and dynamical friction that occur at pericenter passages.
While chaotic mixing, as defined for static smooth potentials, might be occurring during
the merger of two halos, it plays a minor role in driving the remnant to equilibrium.

3.6 Summary

The principal goal of this study was to investigate the processes that drive the
evolution of self-gravitating systems to equilibrium and cause mixing in phase-space. The
goal was also to determine whether the possible presence of large fractions of chaotic
orbits arising from the time-dependent potential was responsible for driving the system to
equilibrium. We summarize the main results below.

• All orbits in the \( N \)-body simulation exhibit the well-documented “Miller instability”
of the \( N \)-body problem. This instability results in an initial nearly exponential
separation of nearby orbits in configuration space and energy, and a decrease
in velocity separation. The \( e \)-folding time (\( t_e \)) for the separation increases with
increasing softening length and decreasing particle number. The initial exponential
separation saturates on a short timescale (\( \sim 0.4t_e \)). The \( e \)-folding time and the
number of \( e \)-folds after which the instability saturates are identical in the merging
and isolated halos indicating that this is purely a consequence of the \( N \)-body nature
of the problem and not a result of the merger. To this end, we use control simulations
of isolated halos as a baseline for comparison with merging halos to identify the
mixing processes occurring in mergers.

• After the initial growth in \( \ln(\Delta r) \) and decrease in \( \ln(\Delta v) \), there are periods of time
for which these quantities are almost constant, separated by several sharp decreases
in \( \ln(\Delta r) \) and correspondingly sharp peaks in \( \ln(\Delta v) \) that are not seen in the isolated
halo. The transient changes in \( \ln(\Delta r) \) and \( \ln(\Delta v) \) are coincident with each other and
correspond to the pericenter passages of the centers of the merging halos. Pericenter
passages also cause sharp transient step-like changes in the energy and angular
momentum separation of pairs of particles. Between pericentric passages there is very
little evidence for mixing in the innermost shell, but at larger radii $\ln(\Delta r)$ and $\ln(\Delta v)$ continue to grow slowly with time due to the overlap of the merging halos.

- The changes in $\Delta E$ and $\Delta J$ are more strongly correlated with pericenter passages at all radii pointing to the importance of compressive tidal shocks. At larger radii there is also some growth in separation between pericentric passages, possibly pointing to the continued action of chaotic mixing. If we assume that separations of nearby particles in phase-space are linked to “mixing”, this implies that the principal drivers of strong mixing are the compressive tidal shock and increased dynamical friction that occurs during the pericentric passages.

- Mixing experiments with ensembles of 1000 nearest particles in phase-space showed that mixing in the phase-space variables $E$, $J$ also occurs primarily during pericenter passages. Mixing leads ensembles of nearby self-gravitating particles to spread and fill a triangular region in $E$, $J$ space. Mixing in these parameters occurs at the same rate at all radii, largely independent of the characteristic crossing time of the ensemble. This is further evidence that the mixing process is driven by pericentric passages and is unrelated to mixing processes that scale with the local crossing time.
CHAPTER 4
THE CONSEQUENCES OF MIXING DURING VIOLENT RELAXATION

4.1 Incomplete Relaxation and Redistribution of Particles in Mergers

Two interesting aspects of merger simulations of dark matter (DM) halos that have been pointed out recently are (a) that cusps are remarkably robust and (b) that about 40% of the total mass of the remnant lies beyond fiducial virial radius (Kazantzidis et al. 2006). Two questions that arise from these findings are: 1) how do the final radial distributions, energies and angular momenta of particles in the remnant depend on their original location in the merging halos and, 2) From where do the particles that end up outside the virial radius in the merger remnant originate? Addressing the first question is important since the merger causes the redistribution of particles to occur in a way that the density profiles and phase-space distribution functions preserve homology. This may be interpreted as an indication that particles at all radii are being heated uniformly at all radii. However, it is also reasonable to assume that less bound particles in the outer part of the halo are preferentially heated.

It is also of interest to understand how the relative change in energy or angular momentum of particles depends on their initial location in the halo. In this section we examine the redistribution of particles in radius, energy and angular momentum and the dependence of their final locations on their initial location in the merging halos.

We note in passing that while the isolated halos and the initial merging halos are spherical, justifying the use of spherical radial shells, the merger remnant is significantly prolate-triaxial with minor-to-major principal axis ratio $c/a$ varying from 0.5 at the center to 0.7 at the virial radius. Although the merger remnants exhibit significant departures from spherical symmetry, in what follows we ignore the triaxiality of the final distribution when binning particles in radius.

Each panel in Figure 4-1 shows how particles that were originally in a given shell (indicated by the caption) are redistributed in the final radial shells at $t = 9$ Gyr when
Figure 4-1. The fraction of particles ($N/N_{\text{max}}$) that lie in each of the radial shells at the final time step. The vertical dashed line and the caption in each panel indicate the shell in which particles were at $t = 0$ Gyr. Particles from one of the two NFW halos (in run Bp1) is shown as a solid line while particles in the same shells of the isolated NFW halo are shown by dot-dashed line.
the merger would conventionally be deemed complete. The results for isolated halo are an important baseline for comparison.

In the top panel (shell 1), the solid lines (merger) and dot-dash lines (isolated halo) are remarkably similar but indicate that while the cusp is quite robust, in fact as much as 20% of the particles in the isolated halo and over 40% of the particles in the merging halos end up outside the cusp at the end of the simulation. The primary reason is that 80% of the particles in the isolated halo have apocenters within the cusp. The merger only causes a small additional fraction (20%) of these bound particles to be spread to shells immediately outside the cusp (shells 2-4). In comparison with the outer shells, this level of redistribution for shell 1 is somewhat modest.

The middle panel (shell 4) shows that, for the isolated NFW halo (dot-dashed curve), ∼ 50% of the particles are in shells 3 and 4 at the end of the simulation, but the remainder are almost uniformly redistributed within shells 2, and 5-10. In shell 7 of the isolated halo ∼ 45% of the particles stay in shells 6 and 7 while the remainder are redistributed almost uniformly to all radii from shell 4 outward. Thus, there is significant radial redistribution of particles even in the non-evolving isolated NFW halo purely because the initial distribution function has an isotropic velocity distribution with roughly equal fractions of radial and tangential orbits. The peaks in shells (3,4) and (6,7) are due to the more tangentially biased orbits. A comparison of the dot-dash curves with the solid curves shows that the main effect of the merger is to flatten out the peaks due to tangential orbits thus making the final distributions more radially anisotropic. At all radii the merger is responsible for redistributing about ∼ 15 – 20% more particles from a given bin to other radial bins.

A similar study of the merger of a NFW halo with a halo having a shallow cusp showed that except in the innermost shell where the profiles of the two halos differ, the particles in the two halos with cusps of different slope are redistributed identically. A larger fraction (∼ 45%) of particles in the shallow cusp were redistributed by heating to
radial shells (2-4) directly outside the cusp (in comparison with 20% in NFW-NFW halo mergers). The differences in the distribution of particles at the end of the merger confirm that NFW cusps are robust and retain a much more significant fraction of their particles than do shallow cusps, confirming what has been previously found by other authors (Boylan-Kolchin & Ma 2004; Kazantzidis et al. 2006).

Figure 4-2 shows distributions of the changes in the kinetic, potential, and total energies of all particles in different shells at the end of the merger \((\Delta E = E_{\text{final}} - E_{\text{initial}})\). The \(y\)-axis gives the fractional change in number of particles (relative to the total number of particles in a given radial shell). Particles in the innermost shell experience a net decrease in total energy that is a result of the overall decrease in the potential energy of the particles pointing to an energy segregation phenomenon during mergers (Funato et al. 1992). This is expected, since there is an overall increase of \(\sim 60\%\) in the mass of the cusp compared to the mass in the initial cusp (Kazantzidis et al. 2006). At all other radii, there is a net increase in total energy that is most significant at larger radii. This is a result of significant fractions of particles gaining potential energy and being redistributed to larger radii as seen from the multiple peaks in the potential energy distributions arising from particles heated during each pericenter passage. It is striking to note that in all shells the kinetic energy distributions remain peaked about \(\Delta E = 0\) and are more peaked than Gaussian, indicating that the majority of particles experience only a small change in their kinetic energies despite experiencing large changes in their potential energies. This is additional confirmation that the redistribution in energy that we saw in the mixing experiments (Section § 3.5.2) is primarily due to changes in potential energies of particles due to interactions with the background potential during pericenter passages.

The innermost shell includes all particles within \(\sim 25\) kpc, which is slightly larger than the initial scale radius \((r_s = 21\) kpc) of the merging NFW halos. Figure 4-3 is similar to Figure 4-2 but shows how particles in three equal radial shells within \(r_s\) are
Figure 4-2. Histograms of the change in energy ($\Delta E = E_{\text{final}} - E_{\text{initial}}$) for all particles in shells 1 (top), 4 (middle), and 7 (bottom). The solid line shows the change in total energy, the dot-dashed shows the change in kinetic energy, and the dashed line shows the change in potential energy.
Figure 4-3. Histograms of the change in energy ($\Delta E = E_{\text{final}} - E_{\text{initial}}$) for all particles in three equal radius bins within the scale radius ($r_s = 21$ kpc). The solid line represents total energy, the dashed-dot represents kinetic energy and the dashed represents potential energy.
redistributed in energy. It is clear that the particles in the inner 33% of the scale radius experience the greatest deepening in the potential.

Figure 4-4 shows histograms of change in $J_z$ (the component of $\mathbf{J}$ perpendicular to the orbital plane of the merger) for all particles in radial shells 1, 4 and 7. The highly peaked distributions in the innermost shell indicates that there is virtually no net change in $J_z$ for the particles here. In the outer two shells, $\Delta J_z$ has an increasingly wider distribution. Although it is not plotted here, we note that in the outermost shells (shells 9, 10) the median of the distribution of $\Delta J_z$ shifts to slightly positive values, indicating a small gain in net angular momentum for the outermost particles. The observed increase is a result of transfer of the orbital angular momentum of the merging halos into the angular momentum of individual particles. Our results indicate that the angular momentum of the merger is absorbed primarily by particles at large radii.

As was noted previously, about 10% of the mass of the initial DM halos lies outside the fiducial virial radius, but it has been found that nearly 40% of the mass of the final remnant lies outside the virial radius of the remnant halo (Kazantzidis et al. 2006). We now examine the distribution of particles in the merger remnant of run Bp1 to determine where these particles originate from. Figure 4-5 shows the fraction ($F_{\text{vir}}$) of the total number of particles lying beyond the virial radius of the remnant that originated from each of the 10 shells of the original halos. All particles that lay outside the virial radius in the original halos are assigned to shell 11 (which extends from the virial radius to the outer edge of the simulation volume). The highest fraction (about 25%) of the particles outside the virial radius of the remnant were already outside the virial radius of the initial systems. The innermost shell is the most robust with fewer than (< 1%) being ejected beyond $r_{\text{vir}}$. Interestingly, all shells from the half mass radius onward (shell 3 and beyond) contribute roughly equally ($\sim 8 - 11\%$) to the particles that lie outside $r_{\text{vir}}$ in the final remnant. Figure 4-5 and Figure 4-1 together show that the radial redistribution
Figure 4-4. Histograms of change in $J_z$ (component of $\mathbf{J}$ perpendicular to the orbital plane of the merger) for all particles in the selected shells.
of particles occurs essentially independently of the original radius of the particle, provided that the particles lie outside the central 3 shells.

### 4.2 Summary

This chapter shows that a significant fraction of mass (≈ 40%) from the merger remnants studied in Chapter 3 lies outside its formal virial radius and that this matter is ejected roughly uniformly from all radii outside the inner regions. This highlights the fact that mass, in its standard virial definition, is not additive in mergers.

- We confirm the findings of several other authors that cusps of DM halos are remarkably robust. The robustness of cusps is a consequence of the fact that 80% of the particles in the first shell have apocenters that lie within the first shell. During the merger only a small fraction (20%) of particles with apocenters within the original cusp are ejected to larger radii. The majority of the ejected particles do not get beyond shell 4. A much larger fraction (∼ 45%) of particles in the central regions of core-like profiles is ejected to radial shells directly outside the cusp during a merger.
• Particles in a given radial shell outside the cusp are redistributed almost uniformly with radius primarily due to phase mixing. This result holds for both the isolated spherical halo and the merging halos. The merger helps to redistribute an additional $\sim 20\%$ of orbits that have predominantly tangential velocity distributions in the original halos.

• Particles within the scale radius of the merging NFW halos experience a net decrease in total energy following the merger. This is a result of the overall decrease in the potential energy of the particles, resulting from the ($\sim 60\%$) increase in the mass of the cusp in the final remnant.

• The majority of particles in the merging halos experience only a small change in their kinetic energies and moderate changes in their potential energies. The change in total energy of particles is driven predominantly by the changes in their potential energies during pericenter passages.

• At smaller radii the majority of particles experience only small change in angular momentum evidenced by highly peaked distributions of $\Delta J$. The angular momentum of the merger is absorbed primarily by particles in the outermost radial shells.

• The largest fraction of particles lying beyond the virial radius of the final remnant (25\%) were located beyond the virial radius of the original halos. The inner 2 shells are most robust to the ejection of particles beyond the virial radius. All shells between shell 4 and the virial radius (shell 10) contribute roughly equally to the particles ejected beyond $r_{\text{vir}}$ of the remnant.
CHAPTER 5
EVOLUTION OF DARK MATTER PHASE-SPACE DENSITY PROFILES IN DISSIPATIONLESS MERGERS

5.1 Fine and Coarse-Grained Phase-Space Density and its Measurement

The state of a collisionless stellar or DM system is completely described by its phase-space density (otherwise called the fine grained distribution function, \( f \)). At any time \( t \), the state of a collisionless system is given by specifying the mass of particles \( f(x, v, t)dx dv \) in the volume element \((dx dv)\) centered on the phase-space coordinate \((x, v)\). The evolution of the phase-space density \( f(x, v, t) \) is described by the collisionless Boltzmann equation (CBE),

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial v} = 0 ,
\]

(5–1)

where \( \Phi \) is the gravitational potential. The CBE states that \( df/dt = 0 \) in a collisionless system, which implies that \( f(x, v, t) \) is a constant.

In practice, it is impossible to measure the fine-grained distribution function \( f \), and what is measured is an average of \( f \) over some volume of phase-space. Thus, the coarse-grained distribution function \( \bar{f} \) at any phase-space point \((x, v)\) is the average value of \( f \) in some small volume element in phase-space centered on \((x, v)\). While \( \bar{f} \) does not obey the CBE, it is, in fact, the only quantity that can actually be measured for a dynamical system.

Since the coarse-grained distribution function \( \bar{f} \) is the only quantity that can be measured, it is interesting to determine how it varies with time. In this chapter, we will focus on the time evolution of the coarse-grained phase-space density during the merger and relaxation of two DM halos. The actual value of the computed function depends on the specific details of the coarse-graining employed. However, the Mixing Theorem (Binney & Tremaine 1987; Tremaine et al. 1986; Mathur 1988) states that processes that operate during the relaxation of collisionless systems (e.g. phase mixing, chaotic mixing, and the mixing of energy and angular momentum that accompanies violent relaxation) result in a
decrease in all convex functions of the coarse-grained phase-space density. In particular, the coarse-grained phase-space density in a particular region of phase-space \((x,v)\) is expected to decrease with time, because at any point in phase-space, both low and high phase-space density regions get highly mixed \(\text{Binney & Tremaine (1987); Tremaine et al. (1986)}\), especially following such events as mergers and violent relaxation.

One of the simplest methods for computing coarse-grained phase-space density of the DM halos that result from cosmological simulations was first used by \(\text{Taylor & Navarro (2001)}\), who defined a spherically-averaged space density

\[
Q(r) = \frac{\rho(r)}{\sigma(r)^3},
\]

where \(\rho(r)\) is the configuration-space density averaged in spherical shells, and \(\sigma(r)\) is the spherically averaged velocity dispersion. They found that \(Q(r)\) is well approximated by a single power law, \(Q \propto r^{-1.87}\), from \(10^{-2.5}\) of the virial radius to just beyond the virial radius (i.e. over more than 2.5 orders of magnitude in radius), fact that was predicted by \(\text{Bertschinger (1985)}\) for a secondary infall model.

In recent years, several groups have developed numerical techniques for computing the coarse-grained phase-space density in \(N\)-body simulations. All of these techniques rely upon ways to divide 6-dimensional phase-space into bins in which particles can be counted. These “binning” techniques are also referred to as tesselation schemes:

- \(\text{Arad et al. (2004)}\) uses a “Delaunay tesselation field estimator,” which computes the densities of a set of points from the volume of the Delaunay cells to which they belong. In a space of \(d\) dimensions, a “Delaunay cell” is defined as a \(d\)-dimensional polyhedron made by connecting every set of \(d+1\) points, such that a \((d-1)\)-dimensional sphere passes through all of them but does not encompass any other point from the sample. The code that they wrote to implement the scheme is referred to as “Sheshdel.” These authors use their code to show that the highest phase-space density regions in the Universe at \(z = 0\) are at the centers of DM halos and subhalos. The main problem with the Delaunay tesselation scheme is that the phase-space volumes, and thus, the density estimates derived are not metric-free.

- \(\text{Ascasibar & Binney (2005)}\) use a method (encoded in the algorithm “FiEstAS”) that is metric-free, and very fast compared with other tesselation schemes, e.g., Sheshdel.
The method is based on a repeated division of each dimension of phase-space into two regions that contain roughly equal numbers of particles. The process of sub-division continues until each volume element contains only one particle. The dimension to be divided is chosen either randomly or by providing an equal number of divisions in each dimension. The highest resolution is obtained for the dimension with the largest number of tesselations (or sub-divisions).

- “EnBiD” (Sharma & Steinmetz 2006) is an algorithm which employs a method which is very similar to the method used by Ascasibar & Binney (2005), but with the difference that, at each step, the space is tesselated along the dimension having minimum entropy (and maximum information). This scheme optimizes the number of divisions to be made in a particular dimension (as opposed to FiEstAS) and extracts maximum information from the data.

The major difference between EnBiD and FiEstAS is that, because it employs a minimum entropy criterion, EnBiD is more accurate at measuring the phase-space density when it is low. Since we are mainly interested in the evolution of the overall density profile and, in particular, the robustness of the regions with the highest phase-space density, we do not expect our results to be significantly altered by the choice of code. Furthermore, we compared the outputs of the Sheshdel code and the FiEstAS codes on some of our simulations and found the results to be qualitatively independent of code employed.

5.2 Previous Work on the Evolution of Phase-Space Density

As mentioned previously, the simplest method for estimating phase-space density, represented by \( Q(r) = \frac{\rho(r)}{\sigma(r)^3} \), was formulated by Taylor & Navarro (2001). They found that \( Q(r) \) is well approximated by a single power law, \( Q \propto r^{-1.87} \), over more than 2.5 orders of magnitude in radius.

Since this power-law profile was discovered, several studies have focused on understanding its origin and, in particular, whether the profile is: (a) a result of the hierarchical growth during structure formation, (b) the result of the relaxation processes such as violent relaxation, (c) the result of initial conditions, or some combination of the three.

Austin et al. (2005) studied the issue of the origin of the power-law profiles in \( Q \) by using semi-analytical extended secondary infall models. These models follow the evolution of collisionless spherical shells of matter that are initially set to be out of dynamical
equilibrium and are allowed to move only radially. They showed that the power-law behavior of the final phase-space density profile is a robust feature of virialized halos, which have reached equilibrium via violent relaxation. They arrived at this conclusion based on the fact that their semi-analytic simulations did include violent relaxation but did not include hierarchical mergers or non-radial mixing. The chief drawback of their simulations was that they were restricted to the study of the motion of spherically infalling shells and did not allow for non-radial oscillations or exchange of energy and were therefore not expected to be very realistic.

Barnes et al. (2006) demonstrated that the dynamical equilibrium of halos with velocity anisotropy is not enough to produce the power law for $\rho/\sigma^3$, but Navarro et al. (2004) and Sérsic density distributions in dynamical equilibrium do produce nearly scale-free $Q$ profiles. Barnes et al. (2007) extended their previous work and included constant anisotropies and anisotropy distributions, and they found that there is a distinct similarity between the shapes of radial density and anisotropy profiles.

MacMillan et al. (2006) argued that the radial orbit instability is the missing link that produces the power-law structure, and that it might also drive isolated systems to a common final state independent of initial perturbations.

Peirani et al. (2006) studied the evolution of $Q$ in the core of DM halos and showed that the $Q - \sigma$ relation depends on the mechanism by which halos acquire mass, and that halos that have undergone major merger events have lower phase-space densities than those that have accreted mass quietly. Their results were independent of the choice of the specific DM model chosen: cold dark matter models resulted in the same profiles as warm dark matter scenarios.

Peirani & de Freitas Pacheco (2007) defined a global value for the phase-space density, $Q$, which was found to decrease rapidly with time. This is considered to be a consequence of the randomization process of initial bulk motions after redistribution of energy following each successive merger. In other words, as halos merge, their orbital kinetic energy is
transferred to the internal kinetic energy of the particles, and this results in an increase in the phase volume occupied by the same number of particles and, consequently, a decrease in overall phase-space density.

One of the difficulties of using $Q$ as a measure of phase-space density is that it is a spherically averaged function and DM halos that result from cosmological simulations are rarely spherical.

In the sections that follow, we describe the results of our analysis of the evolution of phase-space density in the binary mergers of spherical potentials. The spherical potentials were chosen to have central density cusps with varying cusp slopes: cusps shallower than NFW ($\gamma = 0.2$), NFW ($\gamma = 1$), or steeper than NFW ($\gamma = 1.7$). The merger simulations were similar to those described in detail in Chapter 3. In addition to the simulations described in Chapter 3, we also briefly describe the results of four other simulations in which we study the effects of varying orbital parameters and the slope of the central cusp.

5.3 Numerical Methods

As described in § 5.1, the CBE determines the evolution of fine-grained phase-space density with time, and this equation implies that $f(t)$ is constant. The coarse-grained phase-space density is the only quantity that is actually measurable via realistic experiments, and this can be computed in a number of different ways. One way to compute the coarse-grained phase-space density profile is to average spherically, as in the case of the quantity $Q$, used by several other authors.

For the results described in this chapter and in Chapter 6, we use the FiEstAS algorithm which was kindly provided by Yago Ascasibar, which is described in greater detail in (Ascasibar & Binney 2005). Since we can only estimate the coarse-grained phase-space density $\bar{f}$ in a real dynamical system or in $N$-body simulations, and we can never measure the fine-grained phase-space density $f$, we simplify our notation and use $f$ to represent the coarse-grained phase-space density obtained from FiEstAS. The FiEstAS algorithm takes as input the phase-space coordinates $(x, v)$ of each particle in the $N$-body
Figure 5-1. Histograms of the change in phase-space density ($\Delta f = (f_{\text{final}} - f_{\text{initial}})/f_{\text{initial}}$) for all particles in shells 1 (top), 4 (middle), 7 (bottom). Particles from one of the two NFW halos (in run Bp1) are shown as a solid line, while particles in the same shells from the isolated NFW halo are shown in a dashed line.

The dashed curves are for particles in an isolated spherical NFW halo which is evolved in isolation for 15 Gyr, while the solid curves are for particles in the parabolic merger of two NFW halos. The simulation and gives as output the configuration-space density, $\rho(x)$, and phase-space density, $f(x, v)$, at the location of each particle. In order to understand how to interpret the information provided by $f(x, v)$ as it evolves with time, we compare histograms of the change in $f(x, v)$ from the beginning to the end of a simulation.

Figure 5-1 shows the distributions of the changes in $f$ ($\Delta f = (f_{\text{final}} - f_{\text{initial}})/f_{\text{initial}}$). The dashed curves are for particles in an isolated spherical NFW halo which is evolved in isolation for 15 Gyr, while the solid curves are for particles in the parabolic merger of two NFW halos.
spherical NFW halos (the Bp1 merger). In this figure, we use the same method to divide
the spherical NFW halo into 11 concentric spherical shells as discussed in Chapter 3 and
show all particles in radial shells 1, 4, and 7. The y-axis gives the fraction of the total
number of particles in a given shell.

The solid curves clearly show that, in the case of the merger of two halos, the peak
of the distribution is below zero, indicating that the median phase-space density has
decreased as expected during the merger.

It is important to note that, in the case of the isolated halo in equilibrium, \( \Delta f/f \)
phase mixing is expected to result in a slight overall decrease in coarse-grained phase-space
density when averaged over the entire system. When we follow the change in phase-space
density around a specific particle (as we do with FiEstAS) we see that the distribution of
\( \Delta f/f \) is almost Gaussian, since each particle in the equilibrium halo can travel to either
a higher or lower region of phase-space. We note, therefore, that it is necessary to obtain
either a mean or a median distribution of the FiEstAS output \( f \) to properly interpret
the changes we observe in the context of our standard expectation that coarse-grained
phase-space density decreases with evolution due to mixing.

In order to compare with results from previous work, we also compute the spherically
averaged phase-space density \( Q(r) \), employed by other authors. Throughout this study,
\( Q = \rho/\sigma^3 \) was estimated as follows: all particles in a given halo were sorted in their
separation \( r \) from the most bound particle (MBP) of that halo and then were binned in
100 radial shells of equal width. The mass density \( \rho \) was evaluated for each radial bin, and
the total velocity dispersion \( \sigma \) was calculated as \( \sigma = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2} \).

Figure 5-2 shows a comparison of \( Q(r) \) for the isolated NFW halo as well as the
median (dotted line) and the mean value (thin solid line) of \( f \) for this halo produced by
FiEstAS. The median and mean curves were computed for all the particles in each of the
100 spherical shells mentioned above. This figure illustrates that, in the case of NFW
halos, \( Q(r) \) is a power law over several orders of magnitude (up to the virial radius). It
Figure 5-2. Two measures of the phase-space density as a function of $r/r_{\text{vir}}$: $Q = \rho/\sigma^3$ (dot-dashed line) and $f$ (dotted line corresponds to the median values of $f$, while the thin solid line represents the mean - see text for explanations), for the control simulation of an isolated NFW halo at $t=0$ Gyr (top panel). A power-law fit $Q \propto r^{-1.87}$ (red line) is overplotted. Residuals for the fit in $Q$ are in the bottom panel.

deviates from power law at large radii with a bump in the distribution. This deviation from power law arises because of the necessity of setting up an initial distribution function that is in equilibrium and could also be because the NFW halos used here were designed to have isotropic velocity dispersion profiles at all radii, while cosmological NFW halos have large radial velocity anisotropy especially at large radii. The density profile of the isolated NFW profile falls off more rapidly than in the case of a simple extrapolation.
of the NFW profile. The median and mean values of $f(r)$ differ very little from each other, but neither is a simple power law. For reasons we describe later in this section, we prefer to use the median value of $f(r)$, since it is less sensitive to the extremes in the distribution. The $f(r)$ profiles have a continuously varying slope and show a knee-like feature at around $0.08 \, r_{\text{vir}}$, in addition to a small bump at large radii. On average, $Q(r)$ is a steeper function of $r$ than $f(r)$. A power-law fit to the profile for $Q$ ($Q \propto r^{-1.87\pm0.014}$) is shown as a solid red line. (A bootstrapping technique was used to determine the errors on the fitting parameters in the fits throughout this study.)

The time evolution of $Q$ and $f$ (not shown here) for the isolated case exhibits larger fluctuations with time for $Q$ than for $f$. For a system that is in dynamical equilibrium, both functions should remain the same with respect to time. This suggests that $f(r)$ is a more stable quantity numerically than $Q$, and it is better to use $f$ to monitor changes in phase-space density during a merger.

Figure 5-3 shows $Q$ (dashed line) and the median of $f$ (dotted line) for different epochs during the merger of two spherical NFW halos on a parabolic orbit (Bp1 merger), as a function of radius (in units of the virial radius at $t=0$ Gyr).

Both $Q$ and $f$ spread or extend to larger radii as the merger progresses. This is because (as we discussed in Chapter 5) material is ejected to large radii during the merger. For $t$ between two and three Gyr (three Gyr is not shown in the figure), we see a temporary shrinkage with radius and a large associated fluctuation in both $Q$ and $f$. These times are significant, as they correspond to the first pericenter passage and the first apocenter passage, respectively.

For $r < 0.9 \, r_{\text{vir}}$, $Q$ remains roughly power law throughout the evolution. Particles beyond that radius have higher values of $Q$ than would be expected from a simple extrapolation of the inner power law. This is significant because, as we shall note in Chapter 6, this deviation above the power-law distribution is not seen in cosmological profiles (see Figure 6-2).
Figure 5-3. $Q = \rho/\sigma^3$ (dot-dashed line) and $f$ (dotted line) as a function of $r/r_{\text{vir}}$ for different epochs for run Bp1.
The fact that $Q$ has a roughly universal power-law profile for cosmological halos, while our halos show deviations from power law after a binary merger, could be because the NFW halos used were designed to have isotropic velocity dispersion profiles at all radii (while we know that cosmological DM halos have radially anisotropic velocity dispersion profiles) and have an initial bump in $Q$ at large radii. The deviation is also likely to be a signature of binary major mergers, which are rare after $z = 2 - 3$ in the formation of galaxy-sized cosmological halos.

The power-law fit to the profile for $Q$ ($Q \propto r^{-1.87\pm0.014}$) for the isolated case (Figure 5-2) is also an excellent fit at $t = 0$ Gyr in this case and it continues to provide a good fit to $Q$ up to four Gyr.

Figure 5-4 shows $Q$ (blue dot-dashed line) and $f$ (blue dotted line) as a function of radius for the initial epoch (including particles from only one halo) and for the final one (black lines, including particles from both halos) for run Bp1 (top panel), for the Bp2 run (middle panel), and for the hBp2 run (bottom panel). The orange lines in the bottom panel represent the particles that, at $t=0$ Gyr, form the halo with a shallower cusp ($\gamma=0.2$), and the blue ones are for the halo with a steeper cusp ($\gamma=1.7$) in the hBp2 run. It is remarkable that both $Q$ and $f$ profiles are extremely similar for the initial configuration and for the remnant for radii less than 0.8 $r_{\text{vir}}$ in the Bp1 (‘NFW-NFW’) and Bp2 (‘shallow-shallow’) mergers. For the hBp2 case (‘shallow-steep’), the remnant profiles are very much identical with those of the steeper case, which is in agreement with results of Boylan-Kolchin & Ma (2004) and Kazantzidis et al. (2006).

In addition to knowing the mean or the median value of $f$ at a given radius, it is useful to know how the values of $f$ are distributed at each radius. In Figure 5-5, we plot isodensity contours to represent the number density of particles in the $(f, r)$ plane. In addition, the median value of $f$ at each radius is overplotted as a thick white line. The orange regions of the plot correspond to the highest number density of particles, and the black regions correspond to the lowest number density of particles. Contours are spaced
Figure 5-4. $Q = \rho/\sigma^3$ (blue dot-dashed line) and $f$ (blue dotted line) as a function of $r/r_{\text{vir}}$ for the initial epoch (including particles from only one halo) and for the final one (black lines, including particles from both halos) for run Bp1 (top panel), for the Bp2 run (middle panel), and for the hBp2 run (bottom panel). The blue lines in the bottom panel represent the particles that, at $t=0$ Gyr, form the halo with a shallower cusp ($\gamma=0.2$), and the orange ones are for the halo with a steeper cusp ($\gamma=1.7$) in the hBp2 run.
Figure 5-5. Isodensity contours for the phase-space density \( f \) as a function of radius for the Bp1 run, plotted for different times in its evolution. The orange contours correspond to the highest density regions and the black to the lowest density regions. Contours are spaced at logarithmic density intervals relative to the maximum density contour. The median value of \( f \) (white line) is evaluated in equally spaced logarithmic bins in radius. The presence of islands of higher particle density at different radii is clearly noticeable, suggesting that the system is not yet completely mixed.
at logarithmic density intervals relative to the maximum density contour. The presence of islands of higher phase-space density at different radii is clearly noticeable in the plots at eight and 10 Gyr, suggesting that the system is not yet completely mixed. The wiggles in the outer contours at large radii illustrate the formation of phase-space shells due to ejection of particles (Kazantzidis et al. 2006). As can be seen, some of these ejected shells have high phase-space density (orange regions). Similar wiggles were seen by Arad et al. (2004) (their Figure 7) and were attributed to sub-halos, but our simulations do not have subhalos, and hence, we conclude that the features they observed could also be shells.

As discussed earlier in this section, the collisionless Boltzmann equation states that the fine-grained phase-space density $f$ around the phase-space point of a given particle always remains constant. So, $V(f)df$, the volume of phase-space occupied by phase-space elements whose density lies between $(f, f + df)$, is also conserved. However, it is important to note that the coarse-grained phase-space density $\bar{f}$ and its associated phase-space volume $\bar{V}(\bar{f})$ are not conserved. On the other hand, there are simple relationships that exist between the initial $V(f)$ and $\bar{V}(\bar{f})$ at a later time that arise from the Mixing Theorem (Tremaine et al. 1986; Mathur 1988). The Mixing Theorem states that mixing reduces $\bar{f}$ such that, at any time, $\bar{f} < \bar{f}_{\text{max}}$, the maximum phase-space density at the start. In addition, Mathur (1988) showed that $\bar{V}(\bar{f})$ is always greater than $V(f)$. $V(f)$ is easier to visualize than $f$, both analytically and numerically. $V(f)$ is formally defined as

$$V(f = f_0) \equiv \int \delta[f(x, v, t) - f_0]dxdv ,$$

the volume of phase space occupied by phase-space elements whose density lies in the range $(f, f + df)$. As shown by Arad et al. (2004) for most spherical density profiles which are self-similar, if $f = f(E)$, i.e. if the distribution function can be described as a function of energy alone, then the volume distribution function $V(f)$ follows a power law ($V(f) \propto f^{-\beta}$). They also provided simple equations to relate $\beta$ to $\alpha$, the exponent of the density profile. Furthermore, they found that $V(f) \propto f^{-2.5}$ over more than four
decades in $f$ for the relaxed halos resulting from cosmological $N$-body simulations. We now investigate the evolution of the coarse-grained volume element of phase-space $V(f)$.

Numerically, $V(f)$ is evaluated by binning the particles in logarithmically-spaced bins of $f$. If $n_{\text{bin}}$ is the total number of particles in the $i^{\text{th}}$ bin (the model assumes equal mass particles), the density of the bin is (following Sharma & Steinmetz (2006)):

$$V(f_{\text{bin}}) = (n_{\text{bin}}/f_{\text{bin}})/(f_{i+1} - f_i),$$

where $f_{\text{bin}} = (f_{i+1} + f_i)/2$ \hspace{1cm} (5-4)

The plot in Figure 5-6 shows $V(f)$ profiles for the Bp1 run at each time in the evolution of the merger. In addition, a best-fit power law ($V(f) \approx [f]^{-2.45^{\pm0.012}}$), over the range $10^{-10} < [f] < 10^{-7.5}$ (where $[f]$ is $f_{\text{bin}}$ defined in Equation 5-5) is plotted at all epochs. We note that, since there is insignificant mixing in the isolated halo, $V(f)$ for the isolated halo (not plotted) does not change with time.

The volume of phase-space density $V(f)$ associated with low phase-space density $[f]$ increases after the merger is complete (six Gyr). However, we note that the volume of phase-space density $V(f)$ associated with higher phase-space density values exhibits very little change during the evolution. The increase in volume of phase-space at low phase-space densities is predicted to be a consequence of the Mixing Theorem. However, the preservation of the power-law distribution of $V(f)$ over 2.5 orders of magnitude in $[f]$ during the evolution is surprising. Is the conservation of the $V(f)$ profile shape the result of conservation of the density profile (which remains a NFW profile) at various radii, or is there something more fundamental that results in this evolution?

We investigate the robustness of the power-law profile in $V(f)$ by over-plotting the $V(f)$ profiles separately for particles in four different radial bins (Figure 5-7). $V(f)$ of particles in the first bin (inner 10% by radius) is plotted with a solid line. The particles in the next three radial bins are combined and represented by the dotted line. The particles in bins five, six, and seven are represented by a dashed-dotted line, and $V(f)$ for all
Figure 5-6. The volume distribution function $V(f)$ (see Equation 5-3), for different epochs in the evolution of the Bp1 run. A power-law fit $V(f) \propto f^{-2.45}$ is shown on top of each curve for the values of $[f]$ in the range of $[10^{-10}, 10^{-7.5}]$. 
Figure 5-7. The volume distribution function $V(f)$, for different epochs and for different bins in the evolution of the Bp1 run. The solid line represents the particles that, at $t=0$ Gyr, were located within 0.1 $r_{\text{vir}}$; the dotted line is for [0.1-0.4] $r_{\text{vir}}$; dashed-dot for [0.4-0.7] $r_{\text{vir}}$; long dashes for all particles beyond 0.7 $r_{\text{vir}}$. 
particles outside the seventh bin is plotted by a long-dashed line. From top to bottom, the curves in each of the radial bins as defined above are plotted as a function of time (indicated by the time labels).

At early times, for individual bins, \( V(f) \) is not power-law, and \([f]\) in a given bin occupies a very small range of values, except in the first bin, which has the highest range of \([f]\) values. The spread of \([f]\) in the innermost bin is nearly four orders of magnitude which is in agreement with Figure 4-1, top panel, which shows where particles that were originally in the cusp are redistributed at the end of the merger. About 57% remain in the innermost shell, and the remaining 43% spread to all the other shells. It is this 43% which leave the cusp during the merger that cause the spread over four orders of magnitude in \([f]\). It is clear that the large low phase-space density regions observed in Figure 5-6 originate from the particles situated outside the seventh bin (long-dashed line).

Figure 5-8 plots \( f \) as a function of the total energy \((E)\) and total angular momentum \((J)\) of the particles at \( t = 0 \text{ Gyr} \) for the isolated run. It shows contours of equal \( f \) for particles as a function of their \((E, J)\). We see that the highest phase-space-density particles have the most negative energies. At each energy there is a spread of angular momenta. This initial distribution arises because the original NFW halos were set up to have distribution functions dependent solely upon the energy, \( f = f(E) \). So, at each energy, particles were distributed uniformly with angular momentum. Thus, the lower the phase-space density of the particles is, the larger their energy \( E \) and the greater their spread in \( J \).

In Figure 5-9 we plot \( f \) as a function of the total energy \((E)\) and total angular momentum \((J)\) of all the particles at different epochs for the Bp1 run. Figure 5-9 shows contours of equal \( f \) for particles as a function of their \((E, J)\). At the initial epoch we see that the highest phase-space-density particles have the most negative energies. At each energy there is a spread of angular momenta. This initial distribution arises because the original NFW halos were set up to have distribution functions dependent solely upon the
Figure 5-8. Isodensity contours for the phase-space density ($f$) as a function of the total energy $E$ and total angular momentum $J$ for the isolated run, plotted at $t = 0$ Gyr. The purple contours correspond to the highest density regions and the black to the lowest density regions. Contours are spaced at logarithmic density intervals relative to the maximum density contour.

energy, $f = f(E)$. So, at each energy, particles were distributed uniformly with angular momentum. Thus, the lower the phase-space density of the particles is, the larger their energy $E$ and the greater their spread in $J$. As the merger progresses, there is significant spreading and mixing in energy and angular momentum as well as an increase in the volume of phase-space occupied by particles (as represented by the green to black colors in the contour levels, indicating low phase-space density). However, in the final epoch, the particle distribution regains its triangular distribution with the phase-space density of
Figure 5-9. Isodensity contours for the phase-space density ($f$) as a function of the total energy $E$ and total angular momentum $J$ for the Bp1 run, plotted for different times in its evolution. The purple contours correspond to the highest density regions and the black to the lowest density regions. Contours are spaced at logarithmic density intervals relative to the maximum density contour.
particles correlating strongly with their energy. The final distribution, while not strictly $f(E)$, is very similar to the $f(E)$ distribution at the initial epoch for the isolated halo.

Figure 5-10 shows the separation between the most bound particle (MBP) of each halo in the mergers, for different orbital parameters. The top panel shows the case of the radial (Br1) case, the middle panel is for the parabolic (Bp1) case, and the bottom panel is for the circular (Bc1) run. We note that the pericenter passages in the radial case are not as frequent as in the parabolic merger, and for the circular situation, they are absent.

Figure 5-11 shows $f(E, J)$ at the final epoch for mergers with different orbital parameters - i.e. we monitor the effect of varying the orbit from purely radial to purely circular on the final $f(E, J)$ distribution. The final epoch is defined, in each case, to be the time when each system is globally relaxed, which is a different time in each simulation, as it can be seen from Figure 5-10. The times for each case are $t \approx 6, 7, \text{ and } 25 \text{ Gyr}$, respectively. The parabolic case has the highest incidence of low phase-space regions at high energies, which are indicative of unbound particles. The largest spread in angular momentum at high energies occurs for the circular case.

5.4 Summary

The main results can be summarized as follows:

- The FiEstAS code allowed us to monitor the change in coarse-grained phase-space density along the trajectory of a given particle. Figure 5-1 shows that in an equilibrium halo particles can move from regions of low phase-space density to high phase-space density with a probability distribution that is a Gaussian centered on zero - indicating that the coarse-grained phase-space density around the median (or mean) particle does not change. During a merger the coarse-grained phase-space-density around the median particle decreases.

- In Figure 5-2 we showed that $Q(r)$ for spherical isotropic NFW halos is a power-law with a slope of $-1.87$ over 2.5 orders of magnitude in radius. $f(r)$ is not a power-law but has a slope that is slowly changing with radius on a logarithmic scale with a “knee”-like feature at $0.8 \, r_{\text{vir}}$. Both $Q(r)$ and $f(r)$ have a bump at $r \sim r_{\text{vir}}$ which is not seen in cosmological NFW halos. We attribute this feature to the fact that our NFW halos are designed to have isotropic velocity dispersion profiles with and exponential cut off in density and are designed to be stable in isolation while NFW
Figure 5-10. The separation between the most bound particles of the two merging NFW halos for the Br1 run (top panel), the Bp1 run (middle panel), and for the Bc1 run (bottom panel).
Figure 5-11. Isodensity contours for the phase-space density ($f$) as a function of the total energy $E$ and total angular momentum $J$ for the Br1 run (top row), for the Bp1 run (middle row), and for the Bc1 run (bottom row).
halos from cosmological simulations have radially anisotropic velocity dispersion profiles and are not in isolation and have positive densities to much greater radii.

- In Figures 5-3 and 5-5 we showed that both $Q(r)$ and $f(r)$ are remarkably robust and are preserved during the merger of two NFW halos.

- The volume distribution function $V(f)$ is found to increase at low phase-space densities due to the ejection of matter during the merger. However, over two orders of magnitude, this function is found to be surprisingly stable during the merger of two NFW halos. The robustness of $V(f)$ is not a consequence of similar levels of mixing at different radii - in fact, $V(f)$ in the innermost shell is found to be essentially constant, demonstrating, once again, the robustness of the cusp. Much of the increase in $V(f)$ at low values of $f$ is the result of matter ejected from the outer 30% of the halos during the merger.

- By design, the initial distribution functions used in our simulations have a form that is approximately $f(E)$ (Figure 5-8). During the major merger of two NFW halos on a parabolic or a radial orbit the final distribution is found to have a distribution that is remarkably similar to the initial distribution function - at a given energy $E$ particles are almost uniformly distributed in angular momentum $J$ except for the outer most regions of the halo (lowest phase-space density regions).
CHAPTER 6
PHASE-SPACE DENSITY EVOLUTION IN COSMOLOGICAL MERGERS

The ΛCDM (Lambda-Cold Dark Matter) model is frequently referred to as the concordance model of Big Bang cosmology, since it offers an explanation for cosmic microwave background observations, as well as large-scale structure and supernova observations of the accelerating expansion of the universe. It is the simplest known model that is in general agreement with observed phenomena.

Λ (Lambda) stands for the cosmological constant, a term that allows for the current accelerating expansion of the Universe. The cosmological constant is often described in terms of ΩΛ, the fraction of the energy density of a flat universe in the form of the cosmological constant. Currently, ΩΛ ≃ 0.74, implying 74% of the energy density of the present universe is in this form.

Cold dark matter (CDM) is the model where the dark matter (DM) is explained as being cold (its velocity is non-relativistic (v ≪ c) at the epoch of radiation-matter equality), non-baryonic, dissipationless (can not cool by radiating photons) and collisionless. This component makes up 23% of the energy density of the present Universe. The remaining 3% is all of the matter (and energy) that makes up the atoms (and photons) that are the building blocks of planets, stars, and gas clouds in the Universe.

The model has several basic and some derived parameters. The Hubble constant determines the rate of expansion of the universe, as well as the critical density for closure of the universe, ρc. Densities for baryons, DM and dark energy are expressed as Ωs, which are the ratio of the true density to the critical density, e.g. Ωb = ρb/ρc (baryon density). Since the ΛCDM model assumes a flat universe, these densities sum to one, and the density of dark energy is not a free parameter. Table 6-1 includes a list of some of the ΛCDM model parameters. The errors quoted are 1σ, and they have been derived using a Markov chain Monte Carlo analysis by the WMAP collaboration (Spergel et al. 2007).
Table 6-1. Some WMAP parameters of the $\Lambda$CDM model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>Present expansion rate</td>
<td>$0.732^{+0.031}_{-0.032}$</td>
</tr>
<tr>
<td>$\Omega_b$</td>
<td>Baryon density</td>
<td>$0.044 \pm 0.001$</td>
</tr>
<tr>
<td>$\Omega_c$</td>
<td>Cold dark matter density</td>
<td>$0.196 \pm 0.014$</td>
</tr>
<tr>
<td>$\Omega_\Lambda$</td>
<td>Dark energy density</td>
<td>$0.759 \pm 0.034$</td>
</tr>
<tr>
<td>$\Omega_m$</td>
<td>Matter density</td>
<td>$0.241 \pm 0.034$</td>
</tr>
<tr>
<td>$\sigma_8$</td>
<td>Matter fluctuation amplitude</td>
<td>$0.761^{+0.049}_{-0.048}$</td>
</tr>
</tbody>
</table>

Cosmological $N$-body simulations are important tools that allow us to reconstruct how gravitationally bound objects like galaxies and clusters form and evolve in the Universe.

As was discussed in Chapter 5, one of the interesting findings of structure formation simulations ($\Lambda$CDM, standard CDM), as well as more simple collapse models such as the spherical infall models (Bertschinger 1985), has been the discovery that the coarse-grained phase-space density $Q(r)$ (see Equation 5–2) follows a universal power-law profile over 2.5 orders of magnitude in radius (Taylor & Navarro 2001) and the volume density of phase-space $V(f)$ also has a power-law profile over nearly four orders of magnitude in $f$ (Arad et al. 2004). As we saw in Figures 5-3 and 5-6 in Chapter 5, major mergers (those which are the most violent and therefore likely to result in the greatest amount of mixing in phase-space) preserve both profiles over a significant range in both $r$ and $f$ but deviate from the cosmological profiles at low phase-space densities $f$ (i.e., large $r$). While major mergers such as those studied in Chapter 5 are instructive since they represent the most extreme form of mixing, they are not the most common mode of mass accretion in the Universe. In fact, on the scales of galaxy-sized DM halos, most halos are expected to have had their last major merger before $z \approx 2$. Thus, most of the mass currently in galaxy-sized DM halos was not acquired in a major merger. If we are to understand better the origin of the universal profiles in $Q(r)$ and $V(f)$, it is vital to follow the evolution of these quantities in cosmological structure formation simulations.
Hoffman et al. (2007) studied the evolution of phase-space density profiles in DM halos through constrained simulations designed to control the merging history of a given halo. They showed that halos evolve through a series of quiescent phases of slow mass accretion interrupted by occasional major mergers. During the quiescent phases of slow accretion, the density profile closely follows that of a NFW halo and the phase-space density $Q(r)$ is well represented by a power-law $r^{-\alpha}$ with $\alpha = 1.9 \pm 0.1$ over a Hubble time. They showed that the profiles deviate from power-law most strongly during major mergers but recover their power-law profile thereafter. They conclude that, while the physical origin of the NFW density profile and the power-law phase-space density profile is still unknown, virial equilibrium in the quiescent phases implies that a DM halo evolves along a sequence of NFW profiles with constant energy per unit volume within the scale radius $r_s$. As discussed in Chapter 5, $Q(r)$ is really not a true measure of phase-space density despite the fact that it has the dimensions of phase-space density. It is therefore worthwhile to investigate the evolution of the true phase-space density $f(x, v, t)$, the quantity that appears in the collisionless Boltzmann equation (CBE), as well as projections of this quantity, such as $f(r)$ and $f(E, J)$, as a function of time.

A second motivation for studying the evolution of phase-space density in cosmological $N$-body simulations is to test some of the assumptions that have been used to put constraints on the properties of DM particles. Tremaine & Gunn (1979) were the first to use phase-space density arguments based on the observed central mass densities and central velocity dispersions of kinematical tracers in objects on various scales: massive galactic halos, binary galaxy pairs such as the Milky Way and Andromeda, groups of galaxies, and clusters. They used kinematical estimates of maximum central phase-space density in these systems to put constraints on the phase-space density of the DM particle. They argued that if structure formation is hierarchical, and if the DM that dominates the formation of structure is collisionless, then the CBE requires that the maximum fine-grained phase-space density is conserved. Since the maximum coarse-grained
phase-space density must decrease with time, higher mass systems should have higher central velocity dispersion and smaller cores than less massive structures. They showed that the phase-space density of DM in a binary galaxy and in groups of galaxies today is too large to permit neutrinos (neutral leptons in general) to be the DM particle. This argument was revised to put constraints on possible properties of self-interacting DM particles, based on recent models for warm dark matter (WDM) particles (Hogan & Dalcanton 2000) as well as on updated observational constraints on the distribution of DM in dwarf galaxies, low-surface brightness galaxies, and on more massive objects (Dalcanton & Hogan 2001).

The central premise of these arguments is the assumption that, while collisionless particles decrease their coarse-grained phase-space density during the process of hierarchical assembly of structure, the central cusps that form in CDM simulations are the result of low-entropy material that sinks to the center of the cluster halo and preserves the primordial entropy as well as primordial phase-space density of DM. This assumption has not yet been tested via detailed cosmological simulations.

In this chapter, our main goal is to investigate the evolution of phase-space density in the formation and evolution of four Milky-Way-sized halos in a ΛCDM cosmology \(((\Omega_m, \Omega_\Lambda, h, \sigma_8) = (0.3, 0.7, 0.7, 0.9))\), and in particular, we focus on the distribution of the matter with the highest phase-space density and its relationship to its primordial values. This is a work in progress and we present preliminary results.

The simulations analyzed below were described in greater detail in the works of Kravtsov et al. (2004); Gnedin & Kravtsov (2006). The simulations were carried out using the Adaptive Refinement Tree N-body code (ART, Kravtsov et al. (1997)). The simulation starts with a uniform 256³ grid covering the entire computational box. This grid defines the lowest (zeroth) level of resolution. Higher force resolution is achieved in the regions corresponding to collapsing structures by recursive refining of all such regions by using an adaptive refinement algorithm. Each cell can be refined or de-refined individually. The
cells are refined if the particle mass contained within them exceeds a certain specified value. The grid is thus refined to follow the collapsing objects in a quasi-Lagrangian fashion.

Three of the galactic halos were simulated in a comoving box of 25 $h^{-1}$ Mpc (hereafter L25); they were selected to reside in a well-defined filament at $z = 0$. Two halos are neighbors, located 425 $h^{-1}$ kpc from each other. The third halo is isolated and is located 2 Mpc away from the pair. Hereafter, we refer to the isolated halo as G1 and the halos in the pair as G2 and G3. The virial masses and virial radii for the halos studied are given in Table 6-2. The virial radius (and the corresponding virial mass) was chosen as the radius encompassing the density of 180 times the mean density of the Universe. The masses of the DM halos are well within the range of possibilities allowed by models for the halo of the Milky Way galaxy (Klypin et al. 2002).

The fourth galactic halo was simulated in a comoving box of 20 $h^{-1}$ Mpc box (hereafter L20) and it was used to follow the evolution of the Milky Way-sized halo with high resolution.

### 6.1 Evolution of $f$ and $Q$ with redshift

The halos were identified at $z = 0$, and all the particles that lie inside twice the virial radius at $z = 0$ were then tracked back up to $z = 9$. In total there were $1.6 \times 10^6$ particles in the G1 halo, which is the halo that is isolated at $z = 0$. The other two halos, G2 and G3 from the L25 simulation, were closer to each other at $z = 0$ (1.5 virial radii apart), and thus, we use just the virial radius to define the outer radius of these two halos. The particles in the L20 simulation were selected and tracked in the same way as for the

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Table 6-2. Main Properties of the Halos

<table>
<thead>
<tr>
<th>Halo</th>
<th>$M_{\text{vir}}(h^{-1}M_\odot)$</th>
<th>$r_{\text{vir}}(h^{-1}\text{kpc})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>G1</td>
<td>$1.66 \times 10^{12}$</td>
<td>298</td>
</tr>
<tr>
<td>G2</td>
<td>$1.24 \times 10^{12}$</td>
<td>278</td>
</tr>
<tr>
<td>G3</td>
<td>$1.19 \times 10^{12}$</td>
<td>281</td>
</tr>
<tr>
<td>L20</td>
<td>$1.4 \times 10^{12}$</td>
<td>231</td>
</tr>
</tbody>
</table>
G1 halo, and at $z = 0$ there were $3 \times 10^6$ inside twice the virial radius. The position and velocity data for all the particles identified as belonging to a given halo at $z = 0$ were analysed in physical coordinates (distances are always in kpc, velocities in km/s). In this section, we will only present the results obtained for the G1 halo in the L25 simulation, since the results for the other two halos were very similar.

Figure 6-1 shows the spherically averaged $Q$ (dashed-dotted line) and both the median and mean values of $f$ (thick and thin solid lines, respectively) in 100 concentric radial bins at $z = 0$. The computational methods used throughout this chapter were the same as in Chapter 5. The large fluctuations in the mean value of $f$ (thin line) are the result of the presence of substructure, which (as we shall see later in this section) can have extremely high values of $f$, which dominate the mean distribution of $f$ at radii beyond $\sim 0.1 \ r_{\text{vir}}$. The median value of $f$ in concentric radial bins is much smoother. In what follows, we will use the median value of $f$ in concentric radial bins to follow the evolution of $f$ in the main halo, since it is less sensitive to substructure. We note that, as for the case of the spherical, isotropic NFW halo models we analyzed in the previous chapter, $Q(r)$ resulting from the cosmological simulation is also a power-law over $\sim 2.5$ orders of magnitude in $r$. A power-law fit is also plotted for $f \propto r^{-1.46 \pm 0.01}$ for a range of $[0.004–0.562]$ in $r_{\text{vir}}$.

Figure 6-2 shows the spherically averaged $Q(r)$ (dashed-dotted line) and median of $f$ as a function of redshift. The power-law fit $Q \propto r^{-1.84 \pm 0.01}$ at $z = 0$ provides a good fit only up to $z = 2$. For higher values of $z$, $Q(r)$ deviates significantly from a power-law. This behavior should be contrasted with the findings of Hoffman et al. (2007), who found that $Q$ was fitted by a power-law as far back in time as $z = 5.3$, although we note that their plots show significant deviations from the power-law fit beyond one scale radius (approximately $0.1 \ r_{\text{vir}}$) at this redshift.

As we showed in Figure 6-1, $f(r)$ deviates from power-law at $z = 0$. This deviation is even stronger at other redshifts. In particular, the importance of high phase-space density
Figure 6-1. $Q = \rho / \sigma^3$ (dot-dashed line) and $f$ (dotted line corresponds to the median values of $f$, while the thin solid line represents the mean - see text for details), for the L25 halo at $z=0$ (top panel). A power-law fit $Q \propto r^{-1.84}$ (red line) and $f \propto r^{-1.46}$ (green line) are overplotted. Residuals for the fit in $Q$ are shown in the middle panel, and residuals for the fit in $f$ are shown in the bottom panel.
Figure 6-2. $Q = \rho/\sigma^3$ (dot-dashed line) and median $f$ (dotted line) as a function of $r/r_{\text{vir}}(z=0)$ for different redshifts for the L25 run. The dashed vertical line is situated at the $r_{\text{vir}}$ of the main progenitor at each redshift.
substructure is seen to increase with increasing \( z \), and this results in both the \( f \) and \( Q \) profiles in Figure 6-2 becoming flatter with radius.

At \( z = 9 \) (the earliest redshift at which we are able to reliably track the particles back in time), \( Q \) and \( f \) are essentially constant at all radii beyond the inner 0.01 \( r_{\text{vir}} \) and have mean values as high as or higher than their maximum values at \( z = 0 \). Smooth profiles develop only for \( z \leq 2 \).

Figure 6-3 shows contours of constant particle number per unit area on the plane \((\log(f), \log(r))\), while the solid white curve shows the median value of \( f \) in logarithmically spaced bins in \( r \). As expected, the median value of \( f \) traces the regions of largest particle numbers. At radii \( r > 0.05 r_{\text{vir}} \), the high \( f \) values appear as spikes that are well concentrated in radius. At \( z \approx 3 \) and towards \( z=0 \), it is clear that these spikes in \( f \) (which have orange central cores) also have large particle numbers and consequently correspond to DM sub-halos. The sub-halos merge continuously and are heated and mix. As the infall of subhalos proceeds, there is an overall lowering of the median \( f \) curve as well as a decline in the lower envelope of the distribution to smaller \( f \) values. This is a demonstration of the decrease of coarse-grained phase-space density expected to result from the mergers. The most important feature of the distributions in Figure 6-3 is that the peaks in \( f \) seen in the subhalos at all redshifts remain high at values of \( f \approx 10^{-5} \) from \( z = 9 \) all the way to \( z = 0 \). However, during that time the median value of \( f \) at the smallest radius drops from \( 10^{-7} \leq f \leq 10^{-6} \) at \( z = 9 \) down to \( 10^{-10} \leq f \leq 10^{-7} \) at \( z = 0 \). Furthermore, the highest values of \( f \) at the center of the halo also drop by almost an order of magnitude. The central value of \( f \) in a Milky-Way-sized DM halo is therefore less representative of the primordial phase-space density than the central value of \( f \) in the high phase-space density subhalos.

Another striking feature of the distribution of \( f \) values at larger radii is that, at several redshifts (e.g. at \( z = 3 \)), there are particles situated in the same radius which
Figure 6-3. Isodensity contours for the particle density in the plane ($\log(f), \log(r)$) for the L25 run, plotted for different redshifts in its evolution. The orange contours correspond to the regions with the highest particle densities and the black to the lowest density regions. Contours are spaced at logarithmic density intervals relative to the maximum density contour. The median value of $f$ (white line) is evaluated in equally spaced logarithmic bins in radius.
Figure 6-4. Tracks those particles that, at $z = 2$, are situated in the same region in configuration space, but whose phase-space values differ by three orders of magnitude. The particles with high values of $f$ are shown in orange, while those with low values of $f$ are in blue.

produce both a high $f$ spike (orange arrow) corresponding to a subhalo and a low $f$ spike (blue arrow). The phase-space values of these spikes differ by 5-7 orders of magnitude.

In Figure 6-4, we identify the particles associated with these extremely low values of $f$ and determine their relationship to the particles with the highest values of $f$. In particular, we track all particles with radius $0.4 \leq r/r_{\text{vir}} < 0.5$ at $z = 3$. The particles with high phase-space densities ($f \geq 10^{-6}$) are shown in orange, and particles with low phase-space densities ($f \leq 10^{-10}$) are shown in blue. The particles are identified at $z = 3$ (middle left-hand panel) and then tracked starting at an earlier time $z = 3.3$ (top row) through to a later time $z = 2.6$ (bottom row). The middle column shows the evolution of the particles in two spatial coordinates ($x$ and $y$), and the rightmost column shows
the evolution in two velocity components ($v_x$ and $v_y$). The middle column clearly shows that the feature identified at $z = 3$ in fact consists of three different subhalos which are interacting with each other. The most extended object (seen at $x = 0, y = -180$ in the center) is moving the slowest, and the two less extended objects fall toward it from the right in the top middle column and are moving out to the left by $z = 2.6$. The low phase-space density blue particles in the center plot form a halo of particles around the three interacting subhalos at $z = 3$. This is also clearly seen as three narrow orange stripes in the left column, both at $z = 3.3$ and $z = 2.6$. The right column shows that the blue particles have a much greater velocity dispersion than the orange ones, and it is this that contributes to their very low values of $f$. The left column shows that the blue particles are only coincidentally overlapping spatially with the orange ones at $z = 3$, and the fact that they temporarily found themselves in the deeper potential well of the three interacting subhalos lead to an increase in their random velocities. The mean value of $f$ of the blue particles is seen to increase back to the median value at that radius following the encounter, and this increase is accompanied by a decrease in the $y$ component of their velocity. There is also an indication in the top left plot that some fraction of the blue particles belonged to a loosely bound subhalo (seen as a faint blue stripe at $r \sim 0.6$ at $z = 3.3$) and that the interaction with the three tightly bound subhalos at $z = 3$ caused the complete disruption and heating of this structure. Figure 6-4 gives a graphical illustration of how the subhalos interact with each other as well as DM particles in the general halo and cause mixing in phase-space and lead to a smooth distribution in $f$. It is still unclear what gives rise to the observed final distribution of $f$, but the microscopic process of heating via mergers of small subhalos is apparently not very different from the types of heating and mixing we observed in major mergers in Chapter 5.

We saw in Figure 6-3 that the highest phase-space density regions, both at the center of the main halo and within subhalos, have values of $f \geq 10^{-7}$. In Figure 6-5 we determine in a more quantitative manner how representative of the primordial value of
Figure 6-5. Histograms of the phase-space density as a function of redshift. The thick red dashed line follows those particles which, at $z = 0$, are deviated from the Gaussian that approximates the distribution till $z = 1$. The thin red dashed line follows the same particles from $z = 0$, but the corresponding $y$-axis values are ten times higher for clarity.
The figure shows a distribution function of \( f \) on a logarithmic scale as a function of redshift (solid line). Initial distribution of \( \log(f) \) at \( z = 9 \) is close to Gaussian (indicating that \( f \) itself has a log-normal distribution). The peak of the Gaussian moves to lower values of \( \log(f) \) as the halo evolves from \( z = 9 \) to \( z = 0 \). This is a result of mixing due to mergers and phase mixing and results in a decrease in the overall coarse-grained phase-space density, as anticipated. It is important to note that, by approximately \( z = 1 \), the skewness of the distribution is significant due to the tail of particles with high values of \( f \) whose high phase-space densities are preserved despite the mixing. After \( z = 1 \), the degree of skewness increases by almost 50%, and the distribution deviates quite significantly from its Gaussian behavior at earlier redshifts. Overplotted with a red dashed line is the distribution of those particles that have the values of \( f \geq 10^{-7} \) (situated in the right tail of the histogram) at \( z = 0 \). As we saw in Figure 6-3, most of these high-\( f \) tail particles are in subhalos at \( z = 0 \), while some are also in the highest phase-space density particles in the central cusp of the halo at \( z = 0 \).

In each of the plots, the distribution of this population of particles is plotted as a dashed curve. We see that this sub-population has a Gaussian distribution at \( z = 9 \) with a mean \( \log(f) = -6.40 \) compared with the mean \( \log(f) = -6.53 \) for all the particles (in the solid curve) at \( z = 9 \). This implies that the material in the centers of DM subhalos at \( z = 0 \) has phase-space densities that are representative of the mean phase-space density of material even at \( z = 9 \). Our simulations do proceed back in time to higher redshifts, however we do not analyze them here because the mass resolution does not allow us to resolve the evolution of most of the objects beyond this epoch, but we anticipate that, at higher \( z \), the distributions become much more sharply peaked and probably move to even higher values of \( f \).

As we discussed in Chapter 5, while the fine-grained version of the volume \( V(f) \) of phase-space associated with material of phase-space density \( f \) is conserved by the CBE, the coarse-grained version of this quantity always increases due to mixing by the Mixing
Figure 6-6. The volume distribution function $V(f)$, for different redshifts in the evolution of the L25 run. A power-law fit $V(f) \propto f^{-2.51}$ is shown at $z = 0$ (red solid line).
Theorem (Mathur 1988). We now consider the evolution of $V(f)$, the coarse-grained volume density of phase-space. This quantity was defined in Chapter 5, Equation 5–3, to be the volume of phase-space occupied by phase-space elements whose density lies between $(f, f + df)$. Figure 6-6 shows the time-evolution of $V(f)$. At $z = 0$, Arad et al. (2004) showed that $V(f)$ exhibits a power-law profile over four orders of magnitude in $\log(f)$. A thin solid line with profile $y \propto f^{-2.51 \pm 0.07}$ is overplotted on the $V(f)$ curve at $z = 0$. As for the case of the major mergers we saw in Chapter 5 (e.g. the Bp1 merger in Fig. 5-6), there is a noticeable deviation from the power-law profile at low values of $f$. However, we note that, whereas in Figure 5-6 the volume of phase-space $V(f)$ associated with low values for $f < 10^{-10}$ continued to increase with decreasing $f$ at the end of the major merger, the opposite behavior is seen in the cosmological simulation at $z = 0$. In the bottom right panel of Figure 6-6, we find that, while upturns in $V(f)$ at low values of $f$ are seen from $z = 2$ till $z \approx 0.5$, the volume of phase-space associated with this low phase-space density material decreases thereafter and apparently “disappears” entirely by $z = 0$. This is a result of the fact that the halo is continuously accreting mass and growing in the simulation L25. The upturns in $V(f)$ in the simulation Bp1 as well as in the cosmological simulation arise following a significant accretion of matter (in the case of the L25 simulation, this was probably due to the simultaneous accretion of several subhalos and the ejection of material seen in the low phase-space density spikes in Figure 6-3 rather than a major merger). From $z = 0.7$ to $z = 0$, this low phase-space density matter is pulled back into the halo following an increase in the depth of its potential well by the continued accretion of matter.

6.2 Summary

In this chapter we used one more time the FiEstAS code in order to estimate the averaged phase-space density in cosmological simulations of DM particles. Successive minor mergers contribute at the formation of the profiles. The evolution consists both of infall of subhalos and processing of material of existing subhalos.
(stripping). There is also a noticeable amount of mixing when very massive objects are accreted and therefore the potential fluctuates significantly.

We summarize the main results as follows:

- The median of $f$ is less sensitive to substructure than mean of $f$ at all $z$. $f$ and $Q$ at $z = 0$ are both represented by power-laws over 2.5 orders of magnitude in $r$, $Q \propto r^{-1.84\pm0.01}$, $f \propto r^{-1.46\pm0.01}$ (Figure 6-1).

- From $z = 5$ onward, material within $r_{\text{vir}}(z)$ at that redshift follows a power-law in $Q$ and an approximate power-law in $f$. Beyond this radius, $Q$ decreases but $f$ is almost constant with radius. This is because $f$ is a measure of true phase-space density, and its constancy shows that material outside the virial radius still has not mixed. $Q$ decreases because the volume, $r^3$, is increasing rapidly with radius, while local density of material is approximately constant (Figure 6-2).

- The highest values of $f$ at the center of the halo is less representative of the initial phase-space density than the central value of $f$ in the high phase-space density subhalos. Regions of highest $f$ ($f \geq 10^{-7}$) are in the subhalos at all $z \geq 5$. Subhalos contribute minimally to the median $f$ at large radii (Fig6.3). In the center, the median of $f$ drops by almost one order of magnitude, from $f \approx 10^{-6}$ to $f \approx 10^{-7}$. At $r_{\text{vir}}$, the median of $f$ drops from $f \approx 10^{-6}$ to $f \approx 10^{-10}$, nearly four orders of magnitude (Figure 6-3).

- At all redshifts, we see regions of very low $f$ co-spatial with subhalos at large radii. These are found to be regions where multiple subhalos are undergoing gravitational interactions. Unbound particles are transiently heated as they are gravitationally accelerated by the interacting subhalos. This produces transient low $f$ spikes (Figure 6-4).

- Histograms of log($f$) are approximately Gaussian at all redshifts. At $z = 9$ the mean and median values of $f$ are approximately $10^{-6.46}$. The distribution is slightly skewed towards lower values of $f$. As time progresses, the mean and median of log($f$) shift to progressively lower values. At $z = 1$ the distribution develops a distinctive skewness due to the formation of a high $f$ tail. The high phase-space density particles at $z = 0$ have phase-space densities that are representative of the mean phase-space density of material at $z \approx 9$, and hence we learn about the nature of DM, since measuring $f$ now is the same as measuring it at $z \approx 9$ (Figure 6-5). The matter with the highest $f$ at $z = 0$ ($f \geq 10^{-7}$), when traced back to $z = 9$, is found to have the same mean as the overall distribution.

- While we find that the volume distribution function $V(f)$ is following the power-law profile that was first observed by Arad et al. (2004), it was also found that the its values associated with low phase-space density regions increase at earlier redshifts,
and then decrease by $z = 0$, unlike in the case of the binary merger. The difference is that some particles with low values of $f$ in the shoulder of $V(f)$ (Figure 6-6) at earlier times continue to collapse onto the main progenitor, and so they end up in regions of higher density and their $f$ increases. The high $V(f)$ regions were seen in Chapter 5 to be the result of ejection of material due to major mergers. This material is pulled back into the main halo by $z = 0$, due to the continued growth of the halo and due to the cosmic expansion.
CHAPTER 7
CONCLUSIONS

7.1 Summary

This study presents a multi-faceted approach to the analysis of mixing in the formation and evolution of structure in the Universe.

We started by trying to answer questions about why violent relaxation is as efficient as it is. What sort of scenaria might be expected to lead to incomplete violent relaxation? Understanding the role of chaos in violent relaxation is important for interpreting the numerical simulations of galaxy-galaxy encounters and for making predictions as to the structure and evolution of galaxy and cluster halos.

Chapter 2 discussed some ‘natural’ physical expectation regarding the possibility of transient chaos induced by time-dependent perturbations. First we considered what can happen when an integrable system is subjected to an undamped oscillatory perturbation with a single frequency, and then we extrapolated from this example to consider more realistic computations that deal with two important aspect associated with violent relaxation, i.e. allowing for damped oscillations and allowing for variable pulsation frequency.

An extension of the work presented in Chapter 2 involved searching for evidence of transient chaos in the context of fully self-consistent numerical simulations for self-gravitating systems. The obvious issue was whether the degree of chaotic mixing observed in a simulation of violent relaxation in a self-gravitating system correlates with the degree to which the bulk potential admits a significant time-dependent oscillatory component.

The models used in Chapter 2 (Plummer spheres) do not constitute especially realistic models of early-type galaxies; aside from the idealization of spherical symmetry, real galaxies often have a central cusp, and their densities fall off much more slowly at large radii. Also, Plummer spheres have constant-density cores, where all the orbits have
roughly the same oscillation frequency, and therefore, it is easier to induce resonance in a potential similar to the Plummer sphere than a realistic one like a NFW halo, and for this reason, the short-lived but violent and time-dependent oscillations following the mergers of two halos are investigated.

While Chapters 3 and 4 focus on simulations of the merger of two DM halos with NFW potentials, most of the results are generic to mergers of collisionless gravitating systems and are therefore applicable to dissipationless (“dry”) mergers of elliptical galaxies. In particular, the absence of large amounts of mixing in radius over and above that expected from phase mixing in an isolated halo supports previous work that indicates that radial gradients in stellar properties (such as metallicity gradients) are likely to survive intact even in major mergers (Boylan-Kolchin & Ma 2004). These results are also expected to be generically applicable to cosmological mergers. The work of Faltenbacher et al. (2006) - a study of the relaxation processes that operate in hierarchical mergers of DM halos in cosmological N-body simulations - independently arrives at the same conclusions, namely that the primary driver for mixing and relaxation in cosmological mergers is tidal shocking.

We find strong evidence that the processes that drive the mixing in phase-space and evolution to a dynamical equilibrium do not occur continuously due to the time-dependent potential of the system. In fact, strong mixing occurs primarily following episodic injection of energy and angular momentum into the internal energies of particles during pericenter passages. The injection of energy and angular momentum causes nearest particles in phase-space to separate in $E, J$.

Previous studies of orbital evolution in time-dependent and time-independent potentials have argued that the presence of large fractions of chaotic orbits could be the principal driver of the mixing and evolution to an equilibrium distribution. While chaotic mixing could well be occurring, we do not find evidence that such chaotic mixing is driving the relaxation. This is probably a consequence of the fact that the timescales
required for chaotic mixing to be relevant are much longer than the duration of the merger. It is also likely to be a consequence of the fact that the parametric resonance described in Chapter 2 occurs when the driving frequency of the oscillation is close to the orbital frequency for a large fraction of orbits. In NFW halos, the potential is steeply cusped, and consequently, there is a much wider range of orbital frequencies.

The results presented in Chapter 4 further highlight the fact that mass, in its standard virial definition, is not additive in mergers. Models in which the mergers double the mass within the virial radius of the remnant greatly overestimate both the mass and density of the merger product. One of the main findings of our analysis in Chapter 4 is that particles in the central density cusps of steeply cusped dark matter (DM) profiles (like the NFW profile) retain a strong memory of their initial conditions and that mixing occurs primarily due to redistribution of particles at larger radii.

As mentioned in the Introduction, one of the reasons why both particle physicists and cosmologists are interested in knowing about the phase-space density of DM particles in the current Universe is that there are currently several particle physics experiments searching for candidate DM particles.

In Chapter 5, we focused on the evolution of the phase-space density of collisionless (DM) particles following mergers of spherical potentials with different density profiles. We showed that both $Q(r)$, the “poor-man’s” phase-space density’s proxy, and $f(r)$, the distribution function provided by the field estimator developed by Ascasibar & Binney (2005), are surprisingly stable and are preserved during the merger of two NFW halos. The volume of phase-space occupied by phase-space elements whose density lies in the range $(f, f + df)$, $V(f)$ (the volume distribution function) is power-law over several orders of magnitude, confirming the results of Arad et al. (2004); Ascasibar & Binney (2005); Sharma & Steinmetz (2006). The departures from power-law are the result of matter ejected from the outer 30% of the halos during the merger.
Perhaps the most important finding from the cosmological simulations examined in Chapter 6 illustrated that, at present time, the high phase-space density particles have phase-space densities that are representative of the mean phase-space density of material at \( z \approx 9 \); this provides some insight into the nature of DM, as measuring \( f \) now is the same as measuring it at much earlier redshifts.

### 7.2 Future Work

Following up on the work in Chapter 3, where our analysis of the mergers stopped around 10 - 15 Gyr, slow mixing due to chaotic processes are likely to become important at this phase. This can be studied with the traditional tools of non-linear dynamics since the potential is only slowly changing after this time.

The major mergers studied in Chapter 3 are not the most typical mode of mass accretion in the Universe. It is therefore necessary to study multiple minor mergers in the growth of halos to understand the role of dynamical friction and tidal compression in hierarchical mergers. The tools developed can still be applied to this future study. New simulations of multiple minor mergers will need to be run and analysed. This will give us better insights into such questions as whether mixing is as effective when the merger is less violent, and on how dynamical friction and tidal compression and tidal stripping (which act in complementary ways) work in structure formation.

Chapter 4 showed that 40\% of the mass of the final remnant lies outside the formal virial radius in the case of a major merger. It is important to know if this is also true for minor mergers and successive mergers, or for mergers considering different orbital parameters or different inner density profiles, which were partially studied in Chapter 5.

The completion of Chapter 6 requires a necessary analysis of the distribution function’s dependence on the total energy and total angular momentum, \( f(E,J) \), similar to the one described at the end of Chapter 5, in order to monitor its changes as a function of halo mass on both small and large scales.
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BIOGRAPHICAL SKETCH

I was born in Bucharest, Romania, where I went to various schools of various educational levels.

I clearly remember my first lesson in astronomy, when I was three years old. The moon had just risen from behind the apartment building in front of ours, and it was red, and full, and incredibly immense. I asked my father, how was it possible for the moon to change colors. He explained to me that it wasn’t actually red, but that was the way we were seeing it due to the Earth’s atmosphere. If I waited long enough, I would see it change to yellow, and then when it got nearer to the top of the sky it would turn almost white. Just then somebody called and asked to talk to my father, and he said that he was doing astronomy with his daughter. The caller asked how old I was, and my father replied, “She’s three.” That is how I remember my age back then. I stayed awake to see the moon changing from red to yellow, just as my father had said it would. Later that night, I got up from my bed and asked my mother to take me to the window, and I saw the moon turning white when it reached the top of the sky.

Since then, life for me was quite full of experiences that produced equally wondrous, strong, ever-lasting impressions as the one I described. I did graduate research in Romania, Spain, and in the US. In the US, I did graduate research at the University of Florida, Northern Illinois University, and University of Chicago. I would like to think that, wherever the path of life will take me from now on, the moon will still change colors just for me.