FRACTURE TOUGHNESS OF CELLULAR MATERIALS USING FINITE ELEMENT BASED MICROMECHANICS

By

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To my parents, Shiming Wang and Yuezhen Jing, and my wife, Baoning Zhang
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FRACTURE TOUGHNESS OF CELLULAR MATERIALS USING FINITE ELEMENT BASED MICROMECHANICS

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A finite element method based micromechanical analysis is used to understand the fracture behavior of homogeneous and functionally graded foams. Both rectangular prism and tetrakaidecahedral unit cells are studied. Two approaches of predicting fracture toughness of foams and other cellular materials are used in this study. In one approach, the finite element analysis uses a micromechanical model in conjunction with a macro-mechanical model in order to relate the stress intensity factor to the stresses in the struts of the foam. The stress intensity factor at the crack tip of the macro-mechanical model can be evaluated using either the $J$-contour integral or the stresses in the singularity-dominated zone. The other approach is to directly apply displacements based on the $K$-field on the boundary of the micromechanical model.

Using the first approach, the mode I fracture toughness is evaluated for various crack positions and length. Both homogeneous foam and graded foam are studied to investigate the effect of stress gradients in the vicinity of the crack-tip on the fracture toughness. Various types of loading such as remotely applied displacements, remote traction are studied. Preliminary results of this study show that the stress gradient has slight effects on the fracture toughness. However, since the effects are relatively small, $K_{lc}$ can be defined as a material property. Then
the relationship between the fracture toughness of the graded foam and the local density at the crack tip is studied.

The second approach is easy to apply in predicting the fracture toughness of homogeneous foam. By using this approach, convergence study of a micromechanical model is conducted. Also, an analytical model for the mode I fracture toughness of foams with rectangular prism cells is introduced. The mode I and mode II fracture toughness of homogeneous foam consisting of tetrakaidecahedral unit cells are predicted. A parametric study is performed to understand the effect of the geometric parameters of the unit cell and tensile strength of the foam ligament and also dislocation imperfection in the foam.
CHAPTER 1
INTRODUCTION

Background

Cellular materials are made up of a network of beam or plate like structures. There are a number of cellular materials that occur in nature, such as honeycombs, wood, bone, and cork. Cellular materials can offer high thermal resistance, low density, and high energy-absorption. Foams are a class of cellular solids, generally made by dispersing gas into a liquid material and then cooling it to solidify. Foams are categorized as open-cell and closed-cell foams. According to the materials made into foams, foams are also categorized as polymeric, metallic, and ceramic foams, e.g., carbon foams. Due to rapid developments in material science and manufacturing techniques, a wide variety of foams have been developed and used in automobiles, aircrafts, and space vehicles. A special example is the thermal protection system (TPS) of space vehicles, e.g., Space Shuttle.

Traditional TPS cannot bear loads as they are designed for very low thermal conductivity, and are easy to damage, which increases the risk of flight. For instance, a disassembled tile of old TPS caused the tragedy of the Shuttle Columbia in 2003[1]. NASA has started the study of novel TPS concepts for the Crew Exploration Vehicle, which is essentially a replacement for the Space Shuttle. An Integral Thermal Protection System (ITPS) concept is a new idea in which the load-bearing function and insulation are combined into a single structure. This new concept can be achieved by using foams as core of the sandwich structures since foams can be tailored to obtain optimum performance. Under such conditions foams are subjected to various mechanical loads and extreme heat loads.

Thus there exists an urgent need for the study on fracture toughness and other material properties of foams.
Literature Review

Fracture Toughness

The most important parameter of a cellular material is the relative density $\rho^*/\rho_s$, where $\rho^*$ is the density of the cellular material or foam and $\rho_s$ the solid density, which is the density of the strut or ligament material. The relative density is a measure of solidity, and most of the material properties depend on the relative density. Analytical methods for determining the mechanical and thermal properties of cellular solids are well documented. However, research on fracture behavior of foams is still at its infancy. Maiti, Ashby, and Gibson [2] found that Mode I fracture toughness $K_{IC}$ is proportional to $(\rho^*/\rho_s)^3$ for open cell and to $(\rho^*/\rho_s)^2$ for closed cell foams. Huang and Gibson [3, 4] studied several open-cell foams with short crack and further confirmed the above conclusion. Brezny and Green [5, 6] experimentally verified the factors that determined the fracture toughness in the theoretical model. Gibson and Ashby [7] summarized the formulations for Mode I fracture toughness. Recently, Choi and Sankar [8, 9], and Lee [10] presented new results on fracture toughness of open-cell foams.

In a homogeneous continuum the near-tip stress and displacement fields uniquely depend on the stress intensity factor (SIF). It is important to obtain accurate SIF value, which could be calculated from crack-tip stresses. However, it is difficult to obtain accurate stress fields by using FEM because of the existence of the singularity. In order to improve the accuracy, more elements are needed near the crack tip, which causes more computational cost. Another way to calculate SIF is based on the relation between the SIF and the $J$-integral. For homogeneous materials, the $J$-integral is path independent, which allow us to get accurate $J$ along a path away from the crack tip.
Rice[11] introduced the path-independent \( J \) integral for elastic solids under isothermal conditions. A general form of the \( J \) integral, suitable for elastic or elastic-plastic thermal crack problems, is defined by Aoki et al.[12]. Jin [13] used this integral to solve thermal fracture problems of inhomogeneous materials. However, this form is not a standard \( J \)-integral. Shih et al. [14] provided a domain integral of \( J \), and it has been proved to be more efficient and more accurate than the direct calculation of the \( J \)-integral and is suitable for elastic, thermal elastic, and plastic materials. Gu et al. [15] applied this domain integral to evaluate the crack-tip field in inhomogeneous materials, such as functionally graded materials (FGM). The commercial software ABAQUS also uses this domain integral method to calculate the \( J \) integral.

Another approach to investigate the fracture toughness is applying displacement boundary conditions corresponding to a given SIF. Choi and Sankar[8, 9] first used this method to study the fracture toughness of some carbon foams. Most recently Fleck and Qiu[16] have used this method to study the damage tolerance of elastic-brittle, 2-D isotropic lattices.

**Functionally Graded Foam**

One should distinguish functionally graded foam (FGF) from functionally graded materials (FGM). FGMs are a combination of two materials, e.g., a mixture of metals or ceramics, to create a desired composite. However in our study, we assume the material properties of the solid material are isotropic and only the cell size or the strut thickness varies along one direction in the cellular medium. However, both FGF and FGM have thermal and mechanical inhomogeneities, and the computational methods used to analyze FGMs are suitable for FGFs also. Some of the results and conclusions on the behavior of FGMs also apply to FGFs.
There are a large amount of analytical studies available on FGMs. Erdogan and his co-workers[17-19] provided analytical solutions of some fracture problems for FGM. They found the square root singularity of crack-tip stress is the same as that in a homogeneous material. Jin and Noda[13] showed that temperature distribution, and elastic or plastic crack-tip singular fields of nonhomogeneous materials are the same as those of homogeneous materials. Gu and Asaro[20] analytically studied a semi-infinite crack of a FGM. They concluded that material gradients do not affect the order of the singularity and the angular function, but do affect stress intensity factors (SIF). The near-tip stresses have the same form as that for a homogeneous material and the propagation direction is the direction of maximum energy release rate. Sankar[21] derived an elasticity solution for functionally graded beams with the conclusion that the stress concentrations occur in short or thick beams. They are less than that in homogeneous beams, when the softer side of FG beam is loaded and the reverse is true when the stiffer side loaded.

**Tetrakaidecahedral Foam**

It has been accepted for a long time that tetrakaidecahedron, packed in the BCC structure, satisfies the minimum surface energy for mono-dispersed bubbles [22]. Only in 1994 a little better example with smaller surface energy was found by Wearire and Phelan [23]. The tetrakaidecahedral foams have held the interest of researchers for decades. Microcellular graphitic carbon foams was first developed at the US Air Force Research Laboratory in the 1990s [24]. The repeating unit cells of this foam can be approximated by a regular tetrakaidecahedron[25]. Micromechanical models have been used to predict mechanical properties such as Young’s modulus, bulk modulus, yield surface, etc. Warren and Kraynik [26] studied the linear elastic behavior of a low-density Kelvin foam. Zhu [27] provided an analytical solution of the elastic moduli. Li and Gao et al. [25, 28] developed some micromechanics models to analyze the homogeneous material properties and simulate the macroscopic mechanical
behavior under compressive loading. Laroussi et al.[29] studied the compressive response of foams with periodic tetrakaidecahedral cells. A failure surface is defined in macroscopic stress space by the onset of the first buckling-type instability encountered along proportional load paths. Ridha et al. [30] obtained a fracture model for rigid polyurethane foam based on the first tensile failure of any strut in the cell. However, fracture toughness prediction of tetrakaidecahedral foam is a new field, and there is no published work available in this topic.

**Objectives**

In this research, we plan to study open-cell foams with the unit cell shown as in Figure 1-1. Since this is one of the simplest unit cells, it is easy to model and expected to be helpful in understanding the fracture behavior of cellular solids. Our focus is the effect of stress gradients on the fracture toughness. Both homogeneous and graded foams are investigated. We calculate homogeneous material properties based on the cell geometry and its material properties. And then the fracture behaviors of an edged-crack specimen with the homogeneous material properties under different mechanical or thermal loadings are studied. A commercial FEM software – ABAQUS is used for FEM calculations and the input files of FEM are generated by MATLAB.

Since the unit-cells of many foams such as the carbon foam in Figure 1-2 could be well approximated by tetrakaidecahedrons (Figure 1-3), we shall do further study on the foam made of this unit cell.

**Scope**

This research reviews some background information on cellular materials/foams including fracture toughness determination, functionally graded foam analysis, and tetrakaidecahedral foam study. We discuss two approaches to determine fracture toughness of foams which are used in our study, and describe the finite element analysis of homogeneous and functionally graded
foams under different types of loading. We develop an analytical model for fracture toughness
and use it to compare the FEM results. We provide parametric study of fracture toughness of
tetrakaidecahedral foams, and analyze dislocation imperfection effects on material properties
such as elastic modulus and fracture toughness. We briefly discuss some plastic deformation in
the struts near crack tip in ductile foams. Concluding remarks and future work are included.
Figure 1-1. Microstructure of a cellular medium with rectangular unit cells: unit cell with cell lengths $c_1$, $c_2$ and $c_3$.

Figure 1-2. Micrograph of an AFRL carbon foam[24].

100 µm
Figure 1-3. Three tetrakaidecahedral cells with strut length \( l \) and thickness \( t \) in a BCC lattice.
CHAPTER 2
APPROACHES FOR PREDICTING FRACTURE TOUGHNESS

Our approach is a global-local approach wherein the microstructure is modeled in detail near the crack tip (inner region), and boundary conditions are applied at far away points (outer region) according to continuum fracture mechanics. The foam in the outer region is modeled as a homogeneous orthotropic material. We also use two crack propagation criteria, one at the micro-scale and one at the macro-scale. For brittle foams, once we know the stress intensity factor at macroscale and the corresponding maximum tensile stress (microscale) in the struts ahead of the crack, we can calculate the fracture toughness of the foam by the following equation:

\[
\frac{K_I}{K_{IC}} = \frac{\sigma_{tip}}{\sigma_u} \quad \text{or} \quad K_I = \frac{K_{IC}}{\sigma_{tip}} \sigma_u
\]

where \(K_I\) is Mode I stress intensity factor, \(K_{IC}\) is Mode I fracture toughness, \(\sigma_u\) the tensile strength of struts or the foam ligaments, and \(\sigma_{tip}\) the maximum tensile stress in the first unbroken strut ahead of the crack tip.

There are two approaches of predicting the fracture toughness of foams used in this study.

**Approach 1**

As an example, we study an edge-cracked plate and impose the displacements around the outer region surrounding the crack. The maximum tensile stress in the microstructure is obtained from a local model of the inner region. The stress intensity factor is obtained from the macro model of the edge-cracked plate.

The edge-cracked plate is shown in Figure 2-1. The plate is comprised of microstructure with the unit cell shown as Figure 1-1. Due to the symmetry of the geometry and loads, only one-half of the plate is analyzed (Figure 2-2). A multi-scale modeling approach consisting of three different length scales is used. Three models (Figure 2-2C) are used and they are: macro model,
macro sub-model, and micro model. The macro sub-model and micro model are attached to the macro model. The boundary condition (BC) of the macro sub-model is obtained from the macro model results and the BC of the micro model is obtained from the macro sub-model. In other words, the displacements of the nodes on the boundary of the macro sub-model and the micro model are the same as those values at the same position of the macro model and macro sub-model, respectively. The values are automatically obtained by ABAQUS.

In the macro model, namely a model in macro scale, different loads are applied to investigate the crack-tip field. The material properties of this model are calculated through homogenization (see equations in Chapter 3). Due to the stress singularity near the crack-tip, more elements are needed in this area to obtain accurate crack-tip fields. The macro sub-model plays such a role that allows us to increase the number of elements near the crack-tip.

The micro model is used to calculate the maximum tensile stresses in the unbroken strut ahead of the crack tip. Figure 2-3 shows the resultant force and bending moments in the strut of rectangular foam. The maximum tensile stress is given by

\[
\sigma_{\text{tip}} = \sigma_{\text{bend}} + \sigma_{\text{ten}} = \frac{M_{\text{tip}}}{I_{\text{tip}}} \frac{h_{\text{tip}}}{2} + \frac{F_{\text{tip}}}{A_{\text{tip}}} \frac{6M_{\text{tip}}}{h^3} + \frac{F_{\text{tip}}}{h^2}
\]  

(2.2)

Mode I stress intensity factor \((K_I)\) can be determined by:

\[
K_I = \lim_{r \to 0} \sigma_{22}(r,0) \sqrt{2\pi r}
\]  

(2.3)

where \(\sigma_{22}(r,\theta)\) is the stress in the \(y\)-direction near the crack tip, and is a function of \(r\) and \(\theta\) (see Figure 2-4).

The stress intensity factors can also be calculated from the \(J\)-integral or energy release rate. Sih and Liebowitz [31] presented such a relation for orthotropic materials.
\[ G = K_i \left( \frac{a_{11}a_{22}}{2} \right)^\frac{1}{2} \left[ \frac{a_{22}}{a_{11}} \right]^\frac{1}{2} \left( \frac{1}{2a_{12} + a_{66}} \right)^\frac{1}{2} \]  

(2.4)

where \( a_{11} = \frac{1}{E_i^*} \), \( a_{22} = \frac{1}{E_2^*} \), \( a_{33} = \frac{1}{E_3^*} \), and in present case \( a_{12} = a_{23} = a_{31} = 0 \), and

\[ a_{44} = \frac{1}{G_{23}^*} \quad a_{55} = \frac{1}{G_{33}^*} \quad a_{66} = \frac{1}{G_{12}^*} \]

\( E_i^* \), \( E_2^* \), \( E_3^* \) are Young’s moduli in \( x \), \( y \) and \( z \) directions respectively, and \( G_{12}^* \), \( G_{23}^* \), \( G_{32}^* \) are shear moduli in \( x-y \), \( y-z \), and \( z-x \) planes.

In this study we use domain integral in ABAQUS software to calculate the energy release rate. In the case of graded foams the \( J \)-integral is not path-independent. This is because the graded foam is an inhomogeneous material in macro-scale. Hence, we will use an extrapolation technique to calculate the energy release rate in graded foams. The stress near the crack tip is underestimated in the macro model and the macro submodel can capture the square root singularity of the crack-tip stress. SIF calculated by Eq (2.3) expected to agree well with that based on \( J \)-integral.

**Approach 2**

Sih et al. [31] determined the \( K \)-field in the vicinity of a crack tip in homogeneous orthotropic materials. We can directly apply displacements based on the \( K \)-field on the boundary of the microstructure.

The displacement fields near the crack tip for Mode I:

\[ u_i = K_i \sqrt{\frac{2r}{\pi}} \text{Re} \left\{ \frac{1}{s_i - s_2} \left[ s_i p_2 \left( \cos \theta + s_2 \sin \theta \right)^{1/2} - s_2 p_i \left( \cos \theta + s_i \sin \theta \right)^{1/2} \right] \right\} \]

\[ u_2 = K_i \sqrt{\frac{2r}{\pi}} \text{Re} \left\{ \frac{1}{s_i - s_2} \left[ s_i q_2 \left( \cos \theta + s_2 \sin \theta \right)^{1/2} - s_2 q_i \left( \cos \theta + s_i \sin \theta \right)^{1/2} \right] \right\} \]

(2.5)

The displacement fields near the crack tip for Mode II:
\[ u_i = K_H \frac{2r}{\pi} \text{Re} \left\{ \frac{1}{s_1 - s_2} \left[ \frac{p_2 (\cos \theta + s_2 \sin \theta)^{1/2}}{s_2} - p_1 \left( \frac{\cos \theta + s_1 \sin \theta}{s_1} \right)^{1/2} \right] \right\} \]

\[ u_2 = K_H \frac{2r}{\pi} \text{Re} \left\{ \frac{1}{s_1 - s_2} \left[ \frac{q_2 (\cos \theta + s_2 \sin \theta)^{1/2}}{s_2} - q_1 \left( \frac{\cos \theta + s_1 \sin \theta}{s_1} \right)^{1/2} \right] \right\} \]  

(2.6)

The parameters \( p, q \) and \( s \) are dependent on material elastic constants and they are given in Appendix A.

After we find the maximum tensile stress in the struts near the crack tip, we can use Eq (2.1) to obtain the fracture toughness of the foam.

**Comparison of the Two Approaches**

Approach 2 is easier to use since only a micromechanical model is involved. Hence, this approach is good for convergence tests. However, this approach is related to a stress intensity factor for homogeneous foams so that it cannot be used to predict the fracture toughness of functionally graded foams. This simple expression for stress intensity factor hinders the use of the approach in the stress gradient effects analysis.

On the other hand, Approach 1 requires a macro model except for the micromechanical model. The stress intensity factor is needed to be determined from the macro model. So there is much more effort involved in preparing the finite element models and calculations.

Since both approaches have advantages and disadvantages, the selection of the right approach depends on the needs of the research task.
Figure 2-1. An edge-cracked plate: \( H \)-height; \( a \)-crack length; \( W \)-width.

Figure 2-2. Finite element models: A) An edged-crack plate under remote prescribed displacement; B) Microstructure of the plate; C) Half model of the plate

Figure 2-3. Crack tip in micromechanical model: Left- crack tip in microstructure; right- actual foam with resultant force and bending moment.
Figure 2-4. Stress field near crack-tip
CHAPTER 3
HOMOGENEOUS AND FUNCTIONALLY GRADED FOAMS

Homogeneous Material Properties

At first some notation should be specified. Symbols with * denote properties belonging to macrostructure or foam; symbols with a subscript s are of the strut/ligament material.

The material of the foam is orthotropic and so nine independent parameters are required to be determined. These nine parameters are Young’s moduli in $x$, $y$ and $z$ directions ($E_1^*$, $E_2^*$, $E_3^*$), shear moduli in $x$-$y$, $y$-$z$, and $z$-$x$ planes ($G_{12}^*$, $G_{23}^*$, $G_{32}^*$), and Poisson’s ratios in $x$-$y$, $y$-$z$, $z$-$x$ planes ($\nu_{12}^*$, $\nu_{23}^*$, $\nu_{31}^*$). We choose the same carbon foam as Choi studied in [9]. The material properties of microstructure are listed in Table 3-1.

The Young’s modulus in $y$ direction could be derived as depicted Figure 3-1. When the foam is loaded in the $y$ direction, equilibrium requires the force in the unit area equal to that in the strut.

\[ \sigma_y h^2 = \sigma^* c_2 c_3 \Rightarrow \varepsilon_y E_2^* h^2 = \varepsilon^* E^* c_2 c_3 \]  

(3.1)

where $h$ is the strut thickness, $c_2$ and $c_3$ are the cell length in $y$ and $z$ directions.

Since the strains $\varepsilon_y^*$, $\varepsilon^*$ of micro and macro structure in $y$ direction are equal. We have

\[ E_2^* = E_s \frac{h^2}{c_2 c_3} \]  

(3.2)

Similarly we can obtain:

\[ E_1^* = E_s \frac{h^2}{c_2 c_3}, \ E_3^* = E_s \frac{h^2}{c_1 c_2} \]  

(3.3)

The derivation of shear modulus $G_{12}^*$ is illustrated in Figure 3-2. Because of symmetry, there is no curvature at the half-length of the strut. And thus we can use a half beam to solve for $\delta_2$. 

28
\[
\delta_2 = \frac{F \left( \frac{c_2}{2} \right)^3}{3E_s I}
\]  
(3.4)

Again equilibrium requires:

\[
F = \tau c_2 c_1
\]  
(3.5)

Substitute Eq.(3.5) into Eq.(3.4), we obtain \( \delta_2 \) as

\[
\delta_2 = \frac{\tau c_2 c_3 c_1^3}{24E_s I}
\]  
(3.6)

In the same manner, \( \delta_1 \) is

\[
\delta_1 = \frac{\tau c_2 c_1 c_3^3}{24E_s I}
\]  
(3.7)

The shear strain is given as

\[
\gamma_{12} = \frac{2\delta_1}{c_1} + \frac{2\delta_2}{c_2} = \frac{\tau c_2 c_3 (c_1 + c_2)}{12E_s I}
\]  
(3.8)

And the shear modulus \( G_{12}^* \) can be derived as

\[
G_{12}^* = \frac{\tau}{\gamma_{12}} = \frac{12E_s I}{c_1 c_2 c_3 (c_1 + c_2)}
\]  
(3.9)

Substitute the moment of inertia \( I = \frac{h^4}{12} \),

\[
G_{12}^* = \frac{h^4}{c_1 c_2 c_3 (c_1 + c_2)} E_s
\]  
(3.10)

The shear moduli in the other two planes can be obtained by cyclic permutation as

\[
G_{23}^* = \frac{h^4}{c_1 c_2 c_3 (c_2 + c_3)} E_s
\]  
(3.11)

\[
G_{31}^* = \frac{h^4}{c_1 c_2 c_3 (c_3 + c_1)} E_s
\]  
(3.12)
So far we have derived the shear and Young’s moduli of the foam, the three undefined parameters are Poisson’s ratios. Based on Figure 3-1, we can see that the strain in the x-direction is negligible and thus the Poisson’s ratio \( \nu_{12}^* \) is approximately zero. Finally we conclude that

\[
\nu_{12}^* = \nu_{23}^* = \nu_{31}^* = 0
\]  

(3.13)

The relative density \( \rho^*/\rho_s \) is an important parameter of foam, which is a measure of solidity. Based on the cell’s geometry, the relative density can be expressed as

\[
\frac{\rho^*}{\rho_s} = \frac{V_s}{V^*} = \frac{(c_1 + c_2 + c_3) h^2 - 2h^3}{c_1c_2c_3}
\]  

(3.14)

When cell length is much larger than strut thickness, the \( h^3 \) term can be neglected. Furthermore, when \( c_1=c_2=c_3=c \), the relative density is \( 3(h/c)^2 \).

**Material Models for Graded Foams**

Two types of functionally graded foams are studied independently, namely, foams with non-uniform strut thickness and with non-uniform cell length. They are defined respectively by

\[
h(x) = h_0 + \alpha x
\]  

(3.15)

\[
c_i^{i+1} = c_i^{i} + \beta
\]  

(3.16)

where \( \alpha \) and \( \beta \) are constants, and \( h_0 \) is the strut thickness at left edge of the foam. In the first kind of foam, strut thickness varies in the \( x \) direction and cell length is constant, and the reverse for the second kind of foam.

The orthotropic linear elastic material model is applied for the homogeneous foam. But more effort is needed for graded foams. The material properties of graded foams vary along \( x \)-direction since strut thickness or cell length varies in the direction. Instead of using graded elements as Santare[32], we divided the foam into small regions with constant material properties in each strip as Figure 3-3A shows. As long as the regions are small enough, the
gradient material properties of foam can be approximated by constant material properties; Figure 3-3B is an example for Young’s modulus.

**Loading Cases**

Our main objective is to investigate stress gradient effects on fracture behavior. Since different loads provide various stress gradient, we compare the results of the foams subjected to five types of loading (Figure 3-4) including: A. Prescribed remote displacement; B. Remote traction; C. Crack surface traction; D. Remote bending; and E. Thermal loads. In total, six cases of the five types of loading, listed in Table 3-2, are studied.

**An Analytical Model for Fracture Toughness**

Maiti, Ashby and Gibson [2] used a $K_\text{I}$ field to calculate the crack-tip stress (Eq.(2.2)) of homogeneous foam. The force and the bending moment in the strut were obtained by integration. They assumed the bending stress in Eq.(2.2) is dominant and they ignored the tensile stress part. However, we find that in some cases the tensile stress is greater than the bending stress. The ratio of bending stress over the tensile stress is a constant, 0.415 in the present case, and thus neither could be negligible. For the foam with a simple cubic cell ($c_1= c_2= c_3= c$, Figure 3-5), Choi and Sankar [9] introduced an effective length $l = \alpha c$, instead of using the actual cell length, as shown in Eq. (3.18) and Eq.(3.19).

\[
\sigma_{22} = \frac{K_\text{I}}{\sqrt{2\pi r}} \tag{3.17}
\]

\[
F = c \int_0^l \sigma_{22} \, dr = c \int_0^l \left( \frac{K_\text{I}}{\sqrt{2\pi r}} \right) \, dr \tag{3.18}
\]

\[
M = c \int_0^l \left( \frac{K_\text{I}}{\sqrt{2\pi r}} \right) r \, dr \tag{3.19}
\]
However, there is no reason to let the effective lengths in Eq.(3.18) and Eq.(3.19) to be equal. If the cell size is much smaller than the crack size, the homogeneous stress field represents the stress field of microstructure accurately. Then the homogeneous stress must be balanced by the tensile stress in the strut and thus we can get good results by setting $l = c$ in Eq.(3.18). More generally, in the case that the cell lengths in the three coordinate directions are not equal, this equation is rewritten as

$$F = c_3 \int_0^{c_3} \left( \frac{K_I}{\sqrt{2\pi r}} \right) dr = \sqrt{\frac{2}{\pi}} K_I \sqrt{c_1 c_3}$$  \hspace{1cm} (3.20)$$

Table 3-3 gives an example that Eq.(3.20) is a good approximation of the axial force in the first unbroken strut.

If $K_{lc}$ is a material property of the foam, $K_{lc}$ is a constant. And thus based on Eq.(2.1), the ratio $K_I / \sigma_{tp}$ must be a constant, which means $\sigma_{tp} = C K_I$ where $C$ is a constant. And therefore, the ratio $\sigma_{bend} / \sigma_{ten}$ is a constant as a result of Eq.(2.2) and Eq.(3.20). For convenience, denote the ratio as $\gamma$.

$$\gamma = \frac{\sigma_{bend}}{\sigma_{ten}} = \frac{(6M) / (Fh)}{(3.21)}$$

Substituting Eq.(2.2), Eq.(3.21), and Eq.(3.20) into Eq.(2.1), we obtain

$$K_{lc} = \frac{K_I \sigma_u}{\sigma_{bend} + \sigma_{ten}} = \frac{K_I \sigma_u}{\sigma_{ten} (1 + \gamma)} = \frac{K_I \sigma_u}{\sqrt{\frac{2}{\pi} K_I \sqrt{c_1 c_3}} (1 + \gamma)} = \frac{\sigma_u \sqrt{\frac{\pi}{2}} \frac{1}{1 + \gamma} \cdot \frac{h^2}{c_1 c_3}}{\frac{2}{1 + \gamma} \cdot \frac{h^2}{c_1 c_3}}$$  \hspace{1cm} (3.22)$$

The relative density $\rho^*/\rho_s$ can be related to the cell lengths and strut thickness with Eq.(3.14). And then we plot $K_{lc}$ versus relative density in Figure 3-6. It shows that the above equation agrees very well with Choi’s results (Choi 2005: Fig. 13. and Eq. 19) for homogeneous foam. We also can see that the relative density alone cannot determine $K_{lc}$, and $K_{lc}$ also strongly depends on cell size and shape. Figure 3-7 shows the comparison of current model with Choi’s
results in [8] and Gibson and Ashby[7]. Our current model is almost the same as Choi’s result and it give a little smaller fracture toughness. The relative error between our model and the experimental results is 3%.

**More Discussion on the Ratio of Bending Stress over the Tensile Stress.** The reason that the ratio is a constant lies in that the displacement fields in the vicinity of a crack tip in a homogeneous orthotropic material depend on the stress intensity factor as discussed in Eq. 15 of Choi 2005. The displacements of the boundary nodes in the micro model are equal to the displacements at the same place of the homogeneous material, if there are enough cells near the crack tip. Thus the ratio must be a constant. Figure 3-8 shows that the ratio converges to 0.409 as the number of beam elements increases.

Table 3-4 and Table 3-5 show that the ratio varies for different foams with different unit cells. These results are obtained from microstructures with more than 40000 beam elements.

The ratios vary a little. For the sake of simplification, a constant ratio $\gamma=0.409$ is used. The error between fracture toughness by using a fixed ratio and by using the ratio listed in Table 3-4 and Table 3-5 could be determined by following procedure. Using a Taylor series expansion, we can rewrite Eq.(3.22) in terms of $\gamma$ and $\tilde{\gamma}$ as

$$K_{IC} = \frac{A}{1+\gamma} = \frac{A}{1+\tilde{\gamma}+\gamma-\tilde{\gamma}} = \frac{A}{(1+\tilde{\gamma})\left(1+\frac{\gamma-\tilde{\gamma}}{1+\tilde{\gamma}}\right)} \approx \frac{A}{(1+\tilde{\gamma})}\left(1-\frac{\gamma-\tilde{\gamma}}{1+\tilde{\gamma}}\right)$$

(3.23)

$$\tilde{K}_{IC} = \frac{A}{1+\tilde{\gamma}}$$

(3.24)

The absolute value of relative error is
Corresponding to the largest ratio 0.427 and the smallest ratio 0.383 listed in the tables, the absolute value of relative errors are 1.28% and 1.85% respectively, which gives us confidence to use a fixed ratio in Eq.(3.22).

**Results and Discussion**

**Bending Loading Case**

We studied different loading cases; here we only show some detailed results of bending loading to illustrate some conclusions.

We investigate plates (Figure 2-1) with different aspect ratios: 1, 2 and 8 using ABAQUS. Figure 3-9a shows that the J-integral increases as crack size increases. Figure 3-9c shows the maximum tensile stress in the first unbroken strut ahead of the crack tip. Figure 3-9b gives the stress intensity factor calculated based on Eq.(2.4). Also, we compare the stress intensity with the analytical solution for \( H \to \infty \) by Eq.(3.26) [33]. The FEM results agree well with the analytical solution. Finally, the fracture toughness is calculated by Eq.(2.1) and listed in Table 3-6. The relative errors of fracture toughness are shown in Figure 3-9d, where the true value is evaluated by the mean value of fracture toughness of Case \( H/W = 8 \). The analytical solution by Eq.(3.22) is \( 4.55 \times 10^5 \text{ Pa} \cdot \text{m}^{0.5} \). The results in Table 3-6 show the aspect ratio has very little effect on the fracture toughness. In other words, the plate size does not change the fracture toughness of the foam.

\[
K_I = \sigma_\infty \sqrt{\pi a} \left( 1.12 - 1.39 \frac{a}{w} + 7.3 \frac{a^2}{w^2} - 13 \frac{a^3}{w^3} + 14 \frac{a^4}{w^4} \right)
\]  

(3.26)
Stress in Microstructure

As mentioned in section 4, the ratio $\gamma$, of maximum bending stress and tensile stress in the first unbroken strut ahead of the crack tip is a constant when the cell size is small. Table 3-7 and Table 3-8 show the variance of the ratio becomes less as the cell size decreases. Also by comparing the data in Table 3-9 with those in Table 3-7 we observe that the ratio varies for different crack sizes.

Figure 3-10 is an example of the total stress, bending stress and tensile stress in the struts ahead of the crack tip. The tensile stress is continuously distributed in the struts ahead of the crack tip. But the bending stress is discontinuous, especially for the first three struts. This indicates it is difficult to derive an analytical form for the bending stress in the first strut.

Stress Gradient Effects on Fracture Toughness of Homogeneous Foam

Figure 3-11 and Figure 3-12 show the fracture toughness calculated using Eq.(2.1) under different loads. Since the fracture toughness of remote displacement loading is almost constant, the fracture toughness is normalized with the mean value of the fracture toughness of Case 1-remote displacement loading. The fracture toughness of Case 2-Remote traction and Case 4-Bending are almost the same. Both cases correspond to remote traction. The results of Case 2-Surface traction and Case 5-Thermal 1 show similar trends as the crack size increases. The case of Thermal 1 is involved with a negative stress intensity factor. There is a contact pressure occurring in the crack surface. This is similar to a crack surface traction loading.

Comparing with Figure 3-11 and Figure 3-12, we can conclude that cell size does not change much of the distribution trends. But the relative difference of fracture toughness for foams with small cells is smaller than that of foams with large cells. Also we can see that the
stress ratio presented in previous section is not a constant. As a result of variable ratio $\gamma$, the fracture toughness varies a little.

**Fracture Toughness of Functionally Graded Foam with Non-uniform Strut Thickness**

We get the same conclusion as Lee[10] got for the remote displacement load case – the fracture toughness of graded foam is the same as that of homogeneous foam with the same cell shape and size at the crack tip. This can also be explained by the analytical Eq.(3.22). The fracture toughness depends on shape, size and material of the cell.

The conclusion is illustrated by the remote bending case. Figure 3-13 shows the fracture toughness of the foam under remote bending load with $h_0=10\,\mu m$ and $\alpha=2\,\mu m$. The ‘increasing $h$’ means that the crack propagates into higher density region, and vice versa. The fracture toughness is very close to the analytical result for homogeneous foam. Only the fracture toughness for ‘increasing $h$’ case is a slightly greater than the analytical one on the right side of the figure. This phenomenon is similar to the homogeneous problem discussed in a previous section: the fracture toughness is greater than the mean value for the small crack size.

The strut thickness $h$ is related to the relative density by Eq.(3.14). The fracture toughness is plotted with respect to the relative density in Figure 3-14, where fracture toughness is calculated using Eq.(3.22). It shows that the fracture toughness linearly depends on the relative density. As mentioned previously, when cell length is much greater than strut thickness and $c_1=c_2=c_3=c$, the relative density is $3(h/c)^2$. Then the fracture toughness can be written as shown in Eq.(3.27). This equation illustrates that $K_{IC}$ linearly depends on the relative density $(\rho^*/\rho_s)$.

\[
K_{IC} = \sigma_u \sqrt{\frac{\pi}{2}} \frac{1}{1+\gamma} \frac{h^2}{\sqrt{c_1c_3}} = \frac{1}{3} \sigma_u \sqrt{\frac{\pi c}{2}} \frac{1}{1+\gamma} \frac{\rho^*}{\rho_s} \quad (3.27)
\]
By substituting $\bar{\gamma} = 0.409$ into Eq.(3.27), the dimensionless fracture toughness takes the following simple form as shown in Eq.(3.28). However, Figure 3-15 shows that the relative error increases dramatically near zero relative density and approaches to 10% when the relative density is 0.05. And thus, this simple form does not work well.

$$\frac{K_{ic}}{\sigma_u \sqrt{c}} = \frac{1}{3} \sqrt{\frac{\pi}{2}} \frac{1}{1 + \bar{\gamma}} \frac{\rho^*}{\rho_s} = 0.2965 \frac{\rho^*}{\rho_s}$$

(3.28)

Based on Eq.(3.14), $h^2$ can be derived as

$$h^2 = \frac{\rho^*}{\rho_s} \frac{c^3}{3c - 2h} = \frac{\rho^*}{\rho_s} \frac{c^2}{3} \left(1 - \frac{2h}{3c}\right) \approx \frac{\rho^*}{\rho_s} \frac{c^2}{3} \left(1 + \frac{2h}{3c}\right) \approx \frac{\rho^*}{\rho_s} \left(1 + \frac{2}{3} \sqrt{\frac{\rho^*}{\rho_s}}\right)$$

(3.29)

And then the fracture toughness is

$$K_{ic} \approx \frac{1}{3} \sigma_u \sqrt{c} \frac{1}{1 + \gamma} \rho^* \left(1 + \frac{2}{3} \sqrt{\frac{\rho^*}{\rho_s}}\right) = 0.2965 \sigma_u \sqrt{c} \rho^* \left(1 + 0.3849 \sqrt{\frac{\rho^*}{\rho_s}}\right)$$

(3.30)

Comparing this form with the simple form Eq.(3.28), the relative error should equal to $0.3849 \rho^*/\rho_s$ and Figure 3-16 shows this conclusion.

The stress gradient effects on the fracture toughness are shown in Figure 3-17 and Figure 3-18. The fracture toughness is normalized with the analytical one (Eq.(3.22)). Similar to the homogeneous foam, the fracture toughness of Case 2-Remote traction and Case 4-Bending are almost the same. The results of Case 3-Surface traction and Case 5-Thermal 1 show similar trends as the crack size increases.

**Fracture Toughness of Functionally Graded Foam with Non-uniform Cell Length**

We obtain the same conclusion for the non-uniform cell length case: the fracture toughness of graded foam equals to the fracture toughness of homogeneous foams with the same cell as that
of the graded foam at the crack tip. However the fracture toughness does not linearly depend on the relative density for this case. $c_1$ can be derived from Eq.(3.14) as

$$c_1 = \frac{(c_2 + c_3)h^2 - 2h^3}{c_2c_3\left(\frac{\rho^*}{\rho_s}\right) - h^2}$$ (3.31)

and thus the fracture toughness is related to the relative density as follows:

$$K_{IC} = \frac{\sigma_u}{\sqrt{\pi}} \sqrt{\frac{1}{2 + \gamma}} \frac{h^2}{\sqrt{c_1c_3}} = \frac{\sigma_u}{\sqrt{\pi}} \sqrt{\frac{1}{2 + \gamma}} \sqrt{\left(\frac{c_2 \rho^* - h^2}{c_3 \rho_s} \right) \frac{h^2}{c_2 + c_3 - 2h}}$$ (3.32)

which shows the nonlinear relationship between the fracture toughness and the relative density.

The dimensionless fracture toughness can be derived as Eq.(3.33), which shows the dimensionless fracture toughness depends not only on the relative density but also on the geometry of the foam. However, the dimensionless fracture toughness linearly depends on the relative density of the foam since $c_2$, $c_3$ and $h$ are constant.

$$\frac{K_{IC}}{\sigma_u \sqrt{c_1}} = \sqrt{\frac{\pi}{2 + \gamma}} \frac{1}{c_2c_3} \frac{h^2}{\rho^* - h^2} = \sqrt{\frac{\pi}{2 + \gamma}} \frac{c_2c_3 \rho^*}{c_2 \rho_s} - h^2 \frac{h^2}{c_3 \rho_s} \frac{h^2}{c_2 + c_3 - 2h}$$ (3.33)

Figure 3-19 and Figure 3-20 show that the fracture toughness of the remote bending case depends on cell length or relative density.

The stress gradient effects on the fracture toughness are shown in Figure 3-21 and Figure 3-22. The fracture toughness is normalized with the analytical one (Eq.(3.22)). Similar to the homogeneous foam, the fracture toughness of Case 2-Remote traction and Case 4-Bending are almost the same. The results of Case 3-Surface traction and Case 5-Thermal 1 show similar trends as the crack size increases.
Conclusion

Through this study, we find that the fracture toughness of the foam could be predicted by the strength of the strut or ligament material and the shape and size of the cells that constitute the foam. The crack-tip singular fields of the graded foam, as a nonhomogeneous material, are the same as those of homogeneous foam. Different loading cases are studied by using a micro-macro combined method. The effect of stress gradients in the vicinity of the crack-tip on the fracture toughness is studied. Our results lead to the following conclusions:

- Except for remote displacement loading cases, the fracture toughness of the homogeneous foam decreases as the crack size increases.
- The aspect ratio of the plate does not have much effect on the fracture toughness.
- As the cell size become smaller, the fracture toughness of the homogeneous foam under different types of loads becomes uniform;
- Since the relative differences of the fracture toughness of the homogeneous foam under different loads are within ±5%, the fracture toughness can be treated as a material property;
- The analytical model matches well with numerical results for both homogeneous and graded foams; the fracture toughness of the analytical model agrees with that determined by the combined micro-macro-mechanics method;
- The fracture toughness of graded foam equals to the fracture toughness of homogeneous foam with the same cell as that of the graded foam at the crack tip;
- The fracture toughness does not simply depend on the relative density. It depends on both the material and the shape and size of the cell.
Table 3-1. Material properties of struts

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, $\rho_s$</td>
<td>1750 Kg/m$^3$</td>
</tr>
<tr>
<td>Elastic Modulus, $E_s$</td>
<td>207 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu_s$</td>
<td>0.17</td>
</tr>
<tr>
<td>Ultimate Tensile Strength, $\sigma_u$</td>
<td>3600 MPa</td>
</tr>
</tbody>
</table>

Table 3-2. List of load cases.

<table>
<thead>
<tr>
<th>Case #</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>A Remote displacement: $v_0=7\times10^{-5}$ m</td>
</tr>
<tr>
<td>2.</td>
<td>B Remote traction: $\sigma_0=5\times10^7$ Pa</td>
</tr>
<tr>
<td>3.</td>
<td>C Crack surface traction: $\sigma_0=5\times10^7$ Pa</td>
</tr>
<tr>
<td>4.</td>
<td>D Max remote bending stress: $\sigma_{\text{max}}=1\times10^6$ Pa</td>
</tr>
<tr>
<td>5.</td>
<td>E Temperature change: $\Delta T(x) = 75000x^2$ °C</td>
</tr>
<tr>
<td>6.</td>
<td>E Temperature change: $\Delta T(x) = -100x^2$ °C</td>
</tr>
</tbody>
</table>

Table 3-3. Axial forces for Unit cell: $c_1=c_2=c_3=200$ μm, $h=20$ μm; Load: remote traction

<table>
<thead>
<tr>
<th>$a/W$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Force (FEM) (N)</td>
<td>33.20</td>
<td>66.66</td>
<td>106.2</td>
<td>154.7</td>
<td>217.7</td>
<td>307.2</td>
<td>452.9</td>
</tr>
<tr>
<td>Analytical Force (N)</td>
<td>33.50</td>
<td>67.37</td>
<td>107.4</td>
<td>156.5</td>
<td>220.3</td>
<td>310.9</td>
<td>458.4</td>
</tr>
<tr>
<td>Relative error*</td>
<td>0.90</td>
<td>1.05</td>
<td>1.09</td>
<td>1.13</td>
<td>1.19</td>
<td>1.19</td>
<td>1.19</td>
</tr>
</tbody>
</table>

* Relative error = (Analytical – FEM)/Analytical × 100%

Table 3-4. The ratio $\gamma$ for cell size: $c_1=c_2=200$ μm, $h=20$ μm, and $c_1$ varies

<table>
<thead>
<tr>
<th>$c_1$ (μm)</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>0.383</td>
<td>0.399</td>
<td>0.409</td>
<td>0.415</td>
<td>0.420</td>
</tr>
</tbody>
</table>

Table 3-5. The ratio $\gamma$ for cell size: $c_1=c_2=200$ μm, $h$ varies

<table>
<thead>
<tr>
<th>$h$ (μm)</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>0.396</td>
<td>0.409</td>
<td>0.419</td>
<td>0.427</td>
</tr>
</tbody>
</table>

Table 3-6. Mode I fracture toughness($\times10^5$Pa·m$^{0.5}$)

<table>
<thead>
<tr>
<th>$a/W\times100%$</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H/W$</td>
<td>1.0</td>
<td>4.56</td>
<td>4.54</td>
<td>4.53</td>
<td>4.52</td>
<td>4.51</td>
<td>4.51</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>4.57</td>
<td>4.56</td>
<td>4.55</td>
<td>4.54</td>
<td>4.54</td>
<td>4.53</td>
</tr>
<tr>
<td></td>
<td>8.0</td>
<td>4.58</td>
<td>4.57</td>
<td>4.56</td>
<td>4.56</td>
<td>4.55</td>
<td>4.55</td>
</tr>
</tbody>
</table>
Table 3-7. Tip stress at first unbroken strut normalized with total tip stress for the case with cell: $c=200 \mu m$, $h=20 \mu m$ and $a/W=0.5$

<table>
<thead>
<tr>
<th>Load case #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalized bending stress</td>
<td>0.2797</td>
<td>0.2981</td>
<td>0.3000</td>
<td>0.2998</td>
<td>0.2936</td>
<td>0.3085</td>
</tr>
<tr>
<td>Normalized tensile stress</td>
<td>0.7203</td>
<td>0.7019</td>
<td>0.7000</td>
<td>0.7002</td>
<td>0.7064</td>
<td>0.6915</td>
</tr>
<tr>
<td>Ratio $\gamma$</td>
<td>0.388</td>
<td>0.425</td>
<td>0.429</td>
<td>0.429</td>
<td>0.416</td>
<td>0.446</td>
</tr>
</tbody>
</table>

Table 3-8. Stress in the first unbroken strut normalized by total crack tip stress for the case with cell: $c=50 \mu m$, $h=5 \mu m$ and $a/W=0.5$

<table>
<thead>
<tr>
<th>Load case #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalized bending stress</td>
<td>0.2875</td>
<td>0.2934</td>
<td>0.2944</td>
<td>0.2938</td>
<td>0.2919</td>
<td>0.2957</td>
</tr>
<tr>
<td>Normalized tensile stress</td>
<td>0.7125</td>
<td>0.7066</td>
<td>0.7056</td>
<td>0.7062</td>
<td>0.7081</td>
<td>0.7043</td>
</tr>
<tr>
<td>Ratio $\gamma$</td>
<td>0.404</td>
<td>0.415</td>
<td>0.417</td>
<td>0.416</td>
<td>0.412</td>
<td>0.420</td>
</tr>
</tbody>
</table>

Table 3-9. Stress in the first unbroken strut normalized by total crack tip stress for the case with cell: $c=200 \mu m$, $h=20 \mu m$ and $a/W=0.1$

<table>
<thead>
<tr>
<th>Load case #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normalized bending stress</td>
<td>0.2801</td>
<td>0.2900</td>
<td>0.3025</td>
<td>0.2908</td>
<td>0.2150</td>
<td>0.3009</td>
</tr>
<tr>
<td>Normalized tensile stress</td>
<td>0.7199</td>
<td>0.7100</td>
<td>0.6975</td>
<td>0.7092</td>
<td>0.7850</td>
<td>0.6991</td>
</tr>
<tr>
<td>Ratio $\gamma$</td>
<td>0.389</td>
<td>0.408</td>
<td>0.434</td>
<td>0.410</td>
<td>0.274</td>
<td>0.430</td>
</tr>
</tbody>
</table>
Figure 3-1. Micro- and Macro-stresses in an open-cell foam

Figure 3-2. Cell deformation by cell strut bending: A) the undeformed cell and deformed cell; B) the loads, moments in a strut; C) The loads and moment in a half strut
Figure 3-3. Example of graded foam with 50 strips and the discrete elastic modulus compared with the actual modulus

Figure 3-4. Five types of loading
Figure 3-5. Crack-tip forces and moments and corresponding crack tip stresses in the idealized homogeneous continuum. (Refer to (Choi 2005) Fig. 19)

Figure 3-6. Mode I fracture toughness as a function of relative density
Figure 3-7. Cubic foam with $c=1.8$mm, $\sigma_u=3.5805$ MPa

Figure 3-8. The bending-tensile ratio convergence test for cell length over strut thickness $c/h=10$
Figure 3-9. Results of homogeneous foams under the remote bending load: A) $J$-integral; B) Stress intensity factor; C) Tip stress; D) Relative difference in fracture toughness
Figure 3-10. Normalized stress in the struts ahead of the crack tip for remote prescribed displacement load case for the cell with $h=5 \, \mu m$, $c=50 \, \mu m$. 
Figure 3-11. Fracture toughness under different loads for the foam (\(c=200 \, \mu m, h=20 \, \mu m\))

Figure 3-12. Fracture toughness under different loads for the foam (\(c=50 \, \mu m, h=5 \, \mu m\))
Figure 3-13. Fracture toughness under remote bending load. (Plate size: $W=0.1$, $H/W=1$; Graded foam: $h_0=10$ μm; $\alpha=2$ μm; $c=c_1=c_2=c_3=200$μm)

Figure 3-14. Fracture toughness under remote bending load. (Plate size: $W=0.1$, $H/W=1$; Graded foam: $h_0=10$ μm; $\alpha=2$ μm; $c=c_1=c_2=c_3=200$μm)
Figure 3-15. Comparison of Eq(3.22) with Eq.(3.28)

Figure 3-16. Comparison of three forms: A) Eq(3.22): analytical, and Eq.(3.30): modified; B) Relative error is between Eq.(3.22) and Eq.(3.28); Relative error 1 is between Eq.(3.22) and Eq.(3.30).
Figure 3-17. $K_{lc}$ is normalized with the analytical value. (Plate size: $W=0.1\text{m}$, $H/W=1$; Graded foam: $h_0=30\ \mu\text{m}$; $\alpha=-2\times10^{-4}$; $c=c_1=c_2=c_3=200\ \mu\text{m}$)

Figure 3-18. Plate size: $W=0.1\text{m}$, $H/W=1$; Graded foam: $h_0=30\ \mu\text{m}$; $\alpha=-2\times10^{-4}$; $c=c_1=c_2=c_3=200\ \mu\text{m}$
Figure 3-19. Fracture toughness under remote bending load. (Plate size: $W=0.1\,\text{m}, \, H/W=1$; Graded foam: $c_0=200\,\mu\text{m}; \beta=-0.15023\,\mu\text{m}; \bar{h}=20\mu\text{m}$)

Figure 3-20. Fracture toughness under remote bending load. (Plate size: $W=0.1\,\text{m}, \, H/W=1$; Graded foam: $c_0=200\,\mu\text{m}; \beta=-0.15023\,\mu\text{m}; \bar{h}=20\mu\text{m}$)
Figure 3-21. $K_Ic$ is normalized with the analytical value; Plate size: $W=0.1m$, $H/W=1$; Graded foam: $c_0=200 \, \mu m$; $\beta=-0.15023 \, \mu m$; $h=20\mu m$

Figure 3-22. Plate size: $W=0.1m$, $H/W=1$; Graded foam: $c_0=200 \, \mu m$; $\beta=-0.15023 \, \mu m$; $h=20\mu m$
CHAPTER 4
FRACTURE TOUGHNESS PREDICTION OF A TETRAKAI DECAHEDRAL FOAM

FEM Model of a Unit Cell

The tetrakaidecahedral unit cell that we propose to study is a 14-sided polyhedron with six square and eight hexagonal faces. It is more precisely called truncated octahedron, since it is created by truncating the corners of an octahedron [34]. From a different viewpoint, it can be generated by truncating the corners of a cube [27]. All the edges of the cell are of equal length $L$ and cross sectional area $A$.

The tetrakaidecahedral foam has a BCC lattice. The axes of the BCC lattice are parallel to the axes of the cube. Due to the symmetry of the structure, the Young’s moduli of the foam in the lattice vector directions are equal:

$$E_{001}^* = E_{010}^* = E_{100}^*$$  \hfill (4.1)

Each strut of the cell is treated as a beam element. In our study, the cross section of the struts is assumed to be an equilateral triangle with side length $D$ (Figure 4-1). A reticulated vitreous carbon (RVC) foam will be studied and the material properties of the RVC are listed in Table 4-1.

**Elastic Moduli of Homogeneous Foam**

Zhu [27] obtained analytical expressions for the Young’s modulus and Poisson’s ratio based on the symmetry of the microstructure:

$$E_{100}^* = \frac{0.726E_s \rho^2}{1 + 1.09\rho}$$  \hfill (4.2)

$$\nu_{12} = 0.5 \left( \frac{1 - 1.514\rho}{1 + 1.514\rho} \right)$$  \hfill (4.3)
where $\bar{\rho}$ is the relative density, which is related to the side length $D$ and strut length $L$ as shown below:

$$\bar{\rho} = 0.4593(D/L)^2$$  \hspace{1cm} (4.4)

In our study, the cross section of struts is an equilateral triangle with side length $D$ (Figure 4-1). Using FEM, we verified the above equation for the Young’s modulus. By applying a compressive load (Figure 4-2), we calculated the nominal strain from the change in height of the structure and the original height, and the nominal stress is obtained by total resultant forces in $y$-direction per unit area. Figure 4-3 shows that the Young’s modulus converges to the analytical solution as the number of cells increases. In the following sections, the homogeneous material properties of the foam will be calculated with the above equations unless specified otherwise.

**Fracture Toughness**

We study the fracture toughness of plane strain problems for tetrakaidecahedral foams. Approach 2 is used to obtain fracture toughness, in other words, by imposing the displacements of $K_1$ field on the boundary to micromechanical model, we can obtain the maximum tensile stress near the crack tip from the ABAQUS results. Figure 4-4 gives an example of the deformation of the micromechanical model. However, in order to reduce the cost of computation and storage, we take advantage of the symmetry and model only one-quarter of the cellular medium.

Two convergence tests are conducted: Case 1 in which the cell number is increased gradually in both $x$ and $y$ directions; Case 2 in which the cell number in $x$ direction is increased and that in $y$ direction is kept constant. The results are listed in Table 4-2 and Table 4-3 and also shown in Figure 4-5.
Parametric Study

In the parametric study, the two parameters, $L$ and $D$ (see Figure 4-1), are varied to study their effects on the fracture toughness. At first, the detail results for Mode 1 fracture toughness will be presented. And the results of Mode 2 fracture toughness are also included.

Mode I fracture toughness

At first, we fixed the strut length at $L=1$ mm and varied the strut thickness. The effect of strut thickness on Mode I fracture toughness is shown in Table 4-4. The procedures were repeated for $L=2$ mm and the corresponding results are given in Table 4-5.

The results presented in Table 4-4 and Table 4-5 are also plotted in Figure 4-6. In general, we can conclude that the fracture toughness decreases as $L$ increases for a given strut thickness $D$. For the same $L$, the fracture toughness increases as $D$ increases.

The relationship between fracture toughness and relative density is shown in Figure 4-7 and Figure 4-8. We use power law for deriving an empirical relation as:

$$K_{IC} = c_1 \rho^{a_1} \quad (4.1)$$

$$\frac{K_{IC}}{\sigma_0 \sqrt{L}} = a_1 \rho^{a_2} \quad \text{(Non-dimensional form)} \quad (4.2)$$

The coefficients in the above relations are listed in the Table 4-6. Base on results presented in Table 4-6 and Figure 4-8, the relative differences of coefficients $a_1$ and $a_2$ for $L=1$ mm and $L=2$ mm are less than 1% and the two curves for two cases collapse into one curve.

Hence, we can conclude that the dimensionless fracture toughness of tetrakaidecahedral foam mainly depends on its relative density. The dimensionless fracture toughness increases as relative density increases.
Mode II fracture toughness

Mode II fracture toughness is obtained by Approach 2, that is, by imposing the displacements of $K_{II}$ field on the boundary to micromechanical model, we can obtain the maximum tensile stress near the crack tip from the ABAQUS results. At first, fixing the strut length at $L=1$ mm and varying the strut thickness, we obtain Mode II fracture toughness as shown in Table 4-7. And then we choose $L=2$ mm and follow the same procedure to obtain fracture toughness listed in Table 4-8. The results presented in Table 4-7 and Table 4-8 are plotted in Figure 4-9, and we can conclude that the fracture toughness decreases as $L$ increases for the foam with the same strut thickness $D$.

The relationship between fracture toughness and relative density is shown in Figure 4-10 and Figure 4-11. We use power law for deriving an empirical relation as:

$$K_{IIIc} = c_1 \rho^{\nu_2}$$  \hspace{1cm} (4.5)

$$\frac{K_{IIIc}}{\sigma_0 \sqrt{L}} = a_1 \rho^{\nu_2} \quad \text{(Non-dimensional form)}$$  \hspace{1cm} (4.6)

The coefficients in the above relations are listed in Table 4-9. Base on results presented in Table 4-9 and Figure 4-11, we can conclude that the normalized fracture toughness of tetrakaidecahedral foam mainly depends on its relative density.

Progressive Fracture and Crack Propagation

So far the fracture toughness we have presented is calculated based on the maximum stress in one strut near the crack tip. In this section we will study progressive fracture by continuously loading the plate and failing a series of struts.

In this study we assume that the crack is sufficiently long compared to the cell dimension $L$, and hence the crack propagation is considered under Mode I loading condition. After a strut fails, the failed strut is removed, and displacements corresponding to an arbitrary $K_I$ are applied
along the boundary of the model. In the present study we used $K_I=0.01$. The stresses in the struts in the vicinity of the crack tip are calculated. From the maximum stress, the stress intensity factor $K_{Ic}$ that will cause a strut to break is calculated using the relation. This procedure repeated until several struts fail in the vicinity of the crack tip. Figure 4-12 depicts the sequence in which the struts break in the FE model. It is interesting to see that the crack does not propagate in a self-similar manner (horizontally). Instead there are two kink cracks occurring in $45^\circ$ and $-45^\circ$ directions. Maximum stresses in the strut at each stage for $K_I=0.01$ (MPa·mm$^{0.5}$) and corresponding fracture toughness are listed in Table 4-10. One can note that the fracture toughness slightly increases as the kinked crack grows.

**Summary and Conclusion**

A finite element based method developed by Choi and Sankar has been used to study the fracture toughness of tetrakaidecahedral foam. We obtain the plain-strain fracture toughness of the foam by relating the fracture toughness to the tensile strength of the cell struts. Also, we have studied the effects of various geometric parameters that describe the cell. The fracture toughness decreases as strut length $L$ increases for the foam with the same strut thickness $D$. For the same strut length, as $D$ increases the fracture toughness increases. However, the dimensionless fracture toughness only depends on the relative density.
Table 4-1. Material properties of struts

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, $\rho_s$</td>
<td>1650 kg/m³</td>
</tr>
<tr>
<td>Elastic Modulus, $E_s$</td>
<td>23.42 GPa</td>
</tr>
<tr>
<td>Poisson’s ratio, $\nu_s$</td>
<td>0.33</td>
</tr>
<tr>
<td>Ultimate Tensile Strength, $\sigma_u$</td>
<td>689.5 MPa</td>
</tr>
</tbody>
</table>

Table 4-2. Convergence study of fracture toughness

<table>
<thead>
<tr>
<th>Number of Cell</th>
<th>$10\times5$</th>
<th>$16\times8$</th>
<th>$20\times10$</th>
<th>$25\times12$</th>
<th>$32\times16$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{Ic}$</td>
<td>0.405</td>
<td>0.399</td>
<td>0.397</td>
<td>0.395</td>
<td>0.393</td>
</tr>
</tbody>
</table>

Table 4-3. Convergence study of fracture toughness

<table>
<thead>
<tr>
<th>Number of Cell</th>
<th>$10\times10$</th>
<th>$20\times10$</th>
<th>$30\times10$</th>
<th>$40\times10$</th>
<th>$50\times10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{Ic}$</td>
<td>0.405</td>
<td>0.397</td>
<td>0.394</td>
<td>0.393</td>
<td>0.392</td>
</tr>
</tbody>
</table>

Table 4-4. Fracture toughness for strut length $L=1$ mm by using $40\times12$ cells

<table>
<thead>
<tr>
<th>$L$ (mm)</th>
<th>$D$ (mm)</th>
<th>Relative density</th>
<th>$K_{Ic}$ (MPa·mm$^{0.5}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0600</td>
<td>1.654×10$^{-3}$</td>
<td>9.09×10$^{-2}$</td>
</tr>
<tr>
<td>1</td>
<td>0.1000</td>
<td>4.593×10$^{-3}$</td>
<td>3.92×10$^{-1}$</td>
</tr>
<tr>
<td>1</td>
<td>0.1875</td>
<td>1.615×10$^{-2}$</td>
<td>2.23</td>
</tr>
<tr>
<td>1</td>
<td>0.2308</td>
<td>2.446×10$^{-2}$</td>
<td>3.87</td>
</tr>
<tr>
<td>1</td>
<td>0.2727</td>
<td>3.416×10$^{-2}$</td>
<td>5.98</td>
</tr>
<tr>
<td>1</td>
<td>0.3000</td>
<td>4.134×10$^{-2}$</td>
<td>7.63</td>
</tr>
</tbody>
</table>

Table 4-5. Fracture toughness for strut length $L=2$ mm by using $40\times12$ cells

<table>
<thead>
<tr>
<th>$L$ (mm)</th>
<th>$D$ (mm)</th>
<th>Relative density</th>
<th>$K_{Ic}$ (MPa·mm$^{0.5}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1200</td>
<td>1.654×10$^{-3}$</td>
<td>0.128</td>
</tr>
<tr>
<td>2</td>
<td>0.2000</td>
<td>4.593×10$^{-3}$</td>
<td>0.554</td>
</tr>
<tr>
<td>2</td>
<td>0.3750</td>
<td>1.615×10$^{-2}$</td>
<td>3.15</td>
</tr>
<tr>
<td>2</td>
<td>0.4615</td>
<td>2.446×10$^{-2}$</td>
<td>5.48</td>
</tr>
<tr>
<td>2</td>
<td>0.5455</td>
<td>3.416×10$^{-2}$</td>
<td>8.46</td>
</tr>
<tr>
<td>2</td>
<td>0.6000</td>
<td>4.134×10$^{-2}$</td>
<td>10.8</td>
</tr>
</tbody>
</table>

Table 4-6. Interpolation parameters for Mode I

<table>
<thead>
<tr>
<th>$L=1$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$a_1$ (MPa·mm$^{0.5}$)</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>494</td>
<td>1.31</td>
<td>7.17×10$^{-1}$</td>
<td>1.31</td>
</tr>
<tr>
<td>$L=2$</td>
<td>694</td>
<td>1.31</td>
<td>7.12×10$^{-1}$</td>
<td>1.31</td>
</tr>
</tbody>
</table>
### Table 4- 7. Mode II fracture toughness for strut length $L=1$ mm by using 30×17 cells

<table>
<thead>
<tr>
<th>$L$ (mm)</th>
<th>$D$ (mm)</th>
<th>Relative density</th>
<th>$K_{f2}$ (MPa·mm$^{0.5}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0600</td>
<td>1.654×10$^3$</td>
<td>3.37×10$^{-2}$</td>
</tr>
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<td>1.615×10$^{-2}$</td>
<td>1.10</td>
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<td>2.03</td>
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<td>3.416×10$^{-2}$</td>
<td>3.32</td>
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<tr>
<td>1</td>
<td>0.3000</td>
<td>4.134×10$^{-2}$</td>
<td>4.40</td>
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</table>

### Table 4- 8. Mode II fracture toughness for strut length $L=2$ mm by using 30×17 cells

<table>
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<tr>
<th>$L$ (mm)</th>
<th>$D$ (mm)</th>
<th>Relative density</th>
<th>$K_{f2}$ (MPa·mm$^{0.5}$)</th>
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<tr>
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<td>0.2000</td>
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<td>1.55</td>
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<tr>
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<td>4.70</td>
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<tr>
<td>2</td>
<td>0.6000</td>
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<td>6.22</td>
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### Table 4- 9. Interpolation parameters for Mode II

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<tr>
<th>$L$ (mm)</th>
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<th>$c_2$</th>
<th>$a_1$</th>
<th>$a_2$</th>
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<tbody>
<tr>
<td>$L=1$</td>
<td>486</td>
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<td>0.704</td>
<td>1.48</td>
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<td>1.48</td>
<td>0.704</td>
<td>1.48</td>
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Table 4-10. Maximum stress in the struts ahead of crack tip and Mode I fracture toughness for kinked cracks

<table>
<thead>
<tr>
<th>Sequence of analysis</th>
<th>Maximum stress (MPa) for $K_I=0.01$ (MPa·mm$^{0.5}$)</th>
<th>$K_{IC}$ (MPa·mm$^{0.5}$)</th>
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<tbody>
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<tr>
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<tr>
<td>8</td>
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<tr>
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<td>10</td>
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<td>0.361</td>
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<tr>
<td>11</td>
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</tr>
<tr>
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<td>15.1</td>
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<tr>
<td>15</td>
<td>17.6</td>
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</tr>
</tbody>
</table>

Mean Fracture Toughness (MPa·mm$^{0.5}$) 0.338

Standard Deviation (MPa·mm$^{0.5}$) 0.070 (21%)
Figure 4-1. A tetrakaidecahedral unit cell and the cross section of a strut

Figure 4-2. A structure with 27 (3×3×3) cells
Figure 4-3. Convergence study of Young’s modulus

Figure 4-4. Deformation of a micromechanical mode
Figure 4-5. Convergence study of Mode I fracture toughness

Figure 4-6. Mode I fracture toughness vs. strut thickness
Figure 4-7. Mode I fracture toughness vs. relative density

Figure 4-8. Normalized Mode I fracture toughness vs. relative density
Figure 4-9. Mode II fracture toughness vs. strut thickness

Figure 4-10. Mode II fracture toughness vs. relative density
Figure 4-11. Normalized Mode II fracture toughness vs. relative density

Figure 4-12. Crack development history: first broken strut is labeled 1 and the last broken strut is 15.
CHAPTER 5
IMPERFECTION EFFECTS

So far only idealized foams are studied. In reality there are always imperfections in foams:

- Dislocation of a vertex which connects several struts
- Non-uniform strut thickness or material properties
- Voids in the microstructure
- Inclusion in the microstructure

We shall study the first kind of imperfection effects on foams consisting of tetrakaidecahedral unit cells. Generally for the dislocation imperfection, a vertex is assumed to be somewhere within a sphere with radius $R$, the center of which corresponds to its perfect position. However, due to computer and software limitations, we will assume only in-plane dislocation, which means a vertex is within a circle of radius $R$ in the $x$-$y$ plane with the center of the circle at the perfect position. As Figure 5-1 shows, $O$ is the perfect position of a vertex, and $O'$ is the actual position. $R'$ and $\alpha$ are uniformly distributed in $[0, R]$ and $[0, 2\pi]$ respectively. We introduce a new parameter $R_a$.

\[ R_a = \frac{R}{L}, \quad R_a \in [0, 0.5] \]  \hspace{1cm} (5.1)

**Homogeneous Material Properties**

When we study imperfection problems, the microstructure is no longer symmetric. Hence, in this section we study the whole model instead of the half model. We conducted four simulations for each $R_a$ value, which means that there are four finite element models randomly generated and analyzed. The relative density of the foam is calculated and plotted in Figure 5-2. The results show that the relative density increases as $R_a$ increases. Since the tetrakaidecahedral unit cell has almost minimum surface area, as $R_a$ increase, the total length of struts increases and thus relative density becomes larger. In order to obtain the nodal displacements on the boundary
of the microstructure, we need to calculate the equivalent material properties \( E_1^* \), \( E_2^* \), \( \nu_{12}^* \), and \( G_{12}^* \) in Eq (A.1).

Symmetric displacement conditions are still applied in the nodes on the front and back surfaces. Detailed schemes to obtain these equivalent material properties are shown in Table 5-1. For a given \( R_a \), the inner nodes are randomly generated within a circle with radius \( R = R_aL \) and the center located at the perfect position, but the nodes on the boundary are located in the perfect position so that it is easy to apply BC. In order to observe the random parameter \( R_a \) effect at each value of \( R_a \), we conduct four FEM analyses, where four finite element models are randomly generated.

Figure 5-3 shows that the elastic modulus increases as \( R_a \) increases. This result confirms the general conclusion that higher the relative density larger its modulus. The Poisson’s ratio decreases as \( R_a \) increases (Figure 5-4). Generally, when \( R_a \) increases the deviation from the mean value of these material properties also increase.

**Fracture Toughness**

Once we obtain the homogeneous material properties using a finite element model, we use the same structure and break the elements at the crack (see Figure 5-5). The fracture toughness analysis is similar to that in Chapter 4. The only difference is that we use finite element model to calculate the equivalent material properties instead of the analytical solution. In previous section, we found that the deviation of material properties is small and negligible. However, the deviation of fracture toughness is large as shown in Figure 5-6. All these results are also listed in Table 5-2.

We also studied the foams with only one imperfection cell ahead of crack-tip as Figure 5-7 shows. The overall material properties were assumed not to be affected by the imperfection. We conducted three simulations for each \( R_a \) value. Figure 5-8 shows that the imperfection of this single cell has significant effect on the fracture toughness.
Summary and Conclusion

In this chapter, the dislocation imperfection effects are studied. Equivalent homogeneous material properties are obtained by finite element analysis. The fracture toughness of tetrakaidecahedral foams is analyzed with the same approach as in Chapter 4. We obtain the plane-strain fracture toughness of the foam by relating the fracture toughness to the tensile strength of the cell struts. We find that dislocation distance $R_a$ affects the elastic modulus slightly. But it has a huge effect on the fracture toughness. The deviation of the fracture toughness increases as the dislocation distance $R_a$ increases. The results of foams with one imperfect cell ahead of crack tip also confirm this conclusion.
Table 5-1. Equivalent material properties

<table>
<thead>
<tr>
<th>Material properties</th>
<th>Deformation and boundary condition</th>
<th>Equations</th>
</tr>
</thead>
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<tr>
<td>$E_1^*$</td>
<td><img src="image1.png" alt="Image" /></td>
<td>$\varepsilon_1 = \frac{\Delta L}{L}$, $\sigma_2 = \tau_{12} = 0$, $\sigma_1 \neq 0$ (calculated based on resultant force) $\Rightarrow E_1^* = \frac{\sigma_1}{\varepsilon_1}$</td>
</tr>
<tr>
<td>$E_2^*$</td>
<td><img src="image2.png" alt="Image" /></td>
<td>$\varepsilon_2 = \frac{\Delta L}{L}$, $\sigma_1 = \tau_{12} = 0$, $\sigma_2 \neq 0$ (calculated based on resultant force) $\Rightarrow E_2^* = \frac{\sigma_2}{\varepsilon_2}$</td>
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<td>$\nu_{12}$</td>
<td><img src="image3.png" alt="Image" /></td>
<td>$\varepsilon_2 = 0$, $\tau_{12} = 0$, $\sigma_1 \neq 0$ &amp; $\sigma_2 \neq 0$ (calculated based on resultant force) $\Rightarrow \nu_{12} = \frac{\sigma_2}{\sigma_1}$</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td><img src="image4.png" alt="Image" /></td>
<td>Applying Periodic BC: $u_{x_1} - u_{x_0} = 0$, $v_{y_1} - v_{y_0} = 0$, $u_{y_1} - u_{y_0} = \Delta L$, $v_{x_1} - v_{x_0} = 0$. Shear strain: $\gamma_{12} = \frac{\Delta L}{y_{1} - y_{0}}$ Strain energy density: $U = \frac{1}{2} \tau_{12} \gamma_{12} = \frac{1}{2} G_{12}^* (\gamma_{12})^2$ $\Rightarrow G_{12}^* = \frac{2U}{(\gamma_{12})^2}$</td>
</tr>
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Table 5-2. Numerical results with respect to $R_a$

<table>
<thead>
<tr>
<th>$R_a$</th>
<th>Relative density</th>
<th>$E_1^*$ (MPa)</th>
<th>$E_2^*$ (MPa)</th>
<th>$G_{12}^*$ (MPa)</th>
<th>$\nu_{12}^*$</th>
<th>Max tip stress (MPa)</th>
<th>$K_{IC}$ (MPa·mm$^{0.5}$)</th>
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<td>0.4664</td>
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</table>
Figure 5-1. Dislocation of a vertex

Figure 5-2. Relative density as a function of the dislocation distance $R_a$ of a vertex.
Figure 5-3 Effective moduli vs. $R_a$: A) Equivalent elastic modulus; B) Equivalent shear modulus

Figure 5-4 Equivalent Poisson’s ratio versus $R_a$
Figure 5-5. $R_a=0.50$: left – whole finite element model; right – scaled structure near the crack tip.

Figure 5-6. Fracture toughness versus $R_a$. 

Fracture toughness (MPa$\cdot$mm$^{0.5}$)
Figure 5-7. An example of the structure near the crack tip with only one imperfect cell ($R_a=0.5$) ahead of the crack tip.

Figure 5-8. Fracture toughness versus $R_a$ for foams with one imperfect cell ahead of the crack tip.
CHAPTER 6
PLASTIC DEFORMATION NEAR CRACK TIP

We have studied brittle foams in previous chapters. Now we will have a glimpse of fracture behavior of ductile materials, which means plastic deformation will occur when the principal stress is greater than the yield stress of the material. In order to use ABAQUS’s capability, pipe cross section (Figure 6-1) is chosen for the struts of cells in the rectangular prism foam. The outer radius is \( r \) and the thickness is \( t \). The cross section area is \( A \) and the moment of inertia is \( I \) (Eq.(6.1) and Eq.(6.2)).

\[
A = 2\pi (r - 0.5t)t
\]  
\[
I = \frac{\pi}{4} \left[r^4 - (r - t)^4\right]
\]

**Elastic Deformation Analysis**

Before studying plastic deformation, we first follow the same procedures in Chapter 3 to analyze fracture toughness of brittle material (Table 3-1). Here we use Approach 2 to obtain the fracture toughness.

Similar to derivation of Eq.(3.2), we obtained elastic homogeneous material properties:

\[
E_1^* = E_i \frac{A}{c_2c_3}, \quad E_2^* = E_i \frac{A}{c_1c_3}, \quad E_3^* = E_i \frac{A}{c_1c_2}
\]

The shear modulus \( G_{12}^* \) takes the same form as Eq.(3.9) and the Poisson’s ratio are given as Eq.(3.13).

The maximum tensile stress of the first unbroken strut ahead of the crack tip is calculated as the resultant force and bending moments on the strut are obtained.

\[
\sigma_{tip} = \sigma_{bend} + \sigma_{ten} = \frac{M_{tip}r}{I} + \frac{F_{tip}}{A}
\]
Case 1: input parameters are $K_I=500$ (MPa·mm$^{0.5}$), cell size $c_1=c_2=c_3=0.2$mm, cross section $r=c_1/5$ and $t=r/6$.

The elastic strain contour (Figure 6-2) shows the strains in the struts near crack tip are much larger than those in other struts as expected. We obtain $M_{tip}=0.122691$ N·mm and $F_{tip}=36.7165$ N. Hence,

$$\sigma_{tip} = \frac{M_{tip}}{I} + \frac{F_{tip}}{A} = 15,287 \text{ MPa}$$

And

$$K_{IC} = \frac{K_I \sigma_u}{\sigma_{tip}} = 61.23 \text{ MPa·mm}^{0.5}$$

The ratio of bending stress over the tensile stress $\gamma$ is

$$\gamma = \left( \frac{M_{tip}}{I} \right) \left( \frac{F_{tip}}{A} \right) = 0.2964$$

The analytical solution of fracture toughness is

$$K_{IC} = \sigma_u \sqrt{\frac{\pi}{2}} \frac{1}{1+\gamma} \frac{A}{\sqrt{c_1 c_3}} = 60 \text{ MPa·mm}^{0.5}$$

The numerical result of fracture toughness agrees well the analytical solution.

**Plastic Deformation Analysis**

To include elastic-plastic response, we need to specify the nodal forces $N$, $M_1$, $M_2$, and $T$ directly as functions of their conjugate plastic deformation variables. For elastic-perfectly plastic deformation, for each of the above nodal forces we need to provide the value at which plastic deformation sets in (denoted by $F_0$) and the force at which the section becomes fully plastic ($F_1$).
These are given in the form of a graph depicted Figure 6-3. In this figure $P_1$ is the plastic deformation per unit length corresponding to $F_1$ and $P_2$ is an arbitrarily big value.

The ultimate stress in Table 3-1 is taken as the yield stress for the elastic-plastic deformation. The other properties of the foam are the same as for Case 1 in Table 3-1. The forces $F_0$ and $F_1$ are calculated for each mode, extension, flexure and torsion, using mechanics of materials formulas, and are listed in Table 6-1.

ABAQUS assumes the displacement and rotation increments can be decomposed into elastic and plastic parts. Plastic strain will occur when the strain is larger than the yield strain given by $\frac{\sigma_Y}{E_s} = \frac{3600}{(207 \times 10^3)} = 0.01739$.

For this study we used the full micromechanical model. The displacements corresponding to a given $K_I$ are applied along the boundary of the model and they were increased incrementally starting from $K_I=0$. The strain in the crack tip strut is monitored for each increment. Figure 6-4 shows the elastic strain vs. $K_I$ plot. It shows the $K_I$ corresponding to the onset of plastic strain in the strut. This value will be approximately equal to the $K_{IC}$ obtained for brittle foams with the rupture strength equal to the yield stress. The elastic strain does not increase beyond the onset of yielding as we are using elastic-perfectly plastic model. However, the plastic strain increases as $K_I$ increases (Figure 6-5).

If maximum strain criterion is used, then the fracture toughness can be determined based on the curve in Figure 6-6 in which the $K_I$ is plotted against the total strain in the crack tip strut. For example, if strain to failure of the strut material is 0.2, then $K_{IC} = 207$ MPa·mm$^{0.5}$. 
Table 6-1. Forces and deformations for onset of yielding and fully plastic conditions

<table>
<thead>
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<th></th>
<th>$F_0$</th>
<th>$F_I$</th>
<th>$P_I$</th>
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<tbody>
<tr>
<td>*Plastic axial</td>
<td>5.5292 N</td>
<td>5.8057 N</td>
<td>0.04348</td>
</tr>
<tr>
<td>*Plastic $M_1$</td>
<td>0.09292 MPa</td>
<td>0.12544 MPa</td>
<td>0.5031 Rad</td>
</tr>
<tr>
<td>*Plastic $M_2$</td>
<td>0.09292 MPa</td>
<td>0.12544 MPa</td>
<td>0.5031 Rad</td>
</tr>
<tr>
<td>*Plastic Torque</td>
<td>0.01014 MPa</td>
<td>0.13685 MPa</td>
<td>0.6421 Rad</td>
</tr>
</tbody>
</table>

Figure 6-1. Pipe cross section
Figure 6-2. Contour of axial elastic strain in struts

Figure 6-3. Data points generated for the perfect plastic model
Figure 6-4. Stress intensity factor $K_I$ vs. elastic strain in the crack tip strut

Figure 6-5. Stress intensity factor $K_I$ vs. plastic strain in the crack tip strut
Figure 6-6. Stress intensity factor $K_I$ vs. total strain in the crack tip strut
CHAPTER 7
CONCLUDING REMARKS AND SUGGESTED FUTURE WORK

In this dissertation, we have studied two types of foams: foams with rectangular prism unit cells, including homogeneous foams and functionally graded foams, and tetrakaidecahedral foams. The geometry of first one is simple and easy to model which provide a means to understand fracture behavior of foams. The other one is close to reality as some carbon foams can be approximated to be tetrakaidecahedral foams. Our approach to study the fracture toughness of foams is a global-local approach wherein the microstructure was modeled in detail near the crack tip (inner region), and boundary conditions are applied at far away points (outer region) according to continuum fracture mechanics. Two crack propagation criteria, one at the micro-scale and one at the macro-scale, are used. The fracture toughness of brittle foam is calculated based on the stress intensity factor and the corresponding maximum tensile stress in the struts ahead of the crack.

We have studied stress gradient effects on the homogeneous and graded foams with rectangular prism unit cells. The fracture toughness of the foam could be predicted by the strength of the strut or ligament material and the shape and size of the cells that constitute the foam. An analytical model of fracture toughness was derived. Different loading cases were studied by using a micro-macro combined method. Fracture toughness of the homogeneous foam decreases as the crack size increases except for remote displacement loading cases. The aspect ratio of the plate does not have much effect on the fracture toughness. As the cell size becomes smaller, the fracture toughness of the homogeneous foams under different types of loads becomes uniform. Since the relative differences of the fracture toughness of the homogeneous foam under different loads are within ±5%, the fracture toughness can be treated as a material property. The fracture toughness of the analytical model agrees with that determined by the
combined micro-macro-mechanics method. It is found that the fracture toughness of graded foam equals to the fracture toughness of homogeneous foam with the same cell as that of the graded foam at the crack tip. The fracture toughness does not simply depend on the relative density. It also depends on both the material and the shape and size of the cell.

Approach 2 has been used to study the fracture toughness of tetrakaidecahedral foam. We obtain the plain-strain fracture toughness of the foam by relating the fracture toughness to the tensile strength of the cell struts. Also, we have studied the effects of various geometric parameters that describe the cell. The fracture toughness decreases as strut length $L$ increases for the foam with the same strut thickness $D$. For the same strut length, as $D$ increases the fracture toughness increases. However, the dimensionless fracture toughness only depends on the relative density. In the study of the dislocation imperfection effects, we find that dislocation distance $R_a$ has no significant effect on the elastic modulus. But it has a huge effect on fracture toughness. The deviation of the fracture toughness increase as the dislocation distance $R_a$ increases.

Finally, we have taken the first step to study plastic deformation near crack tip.

However, there are some supplement study and new areas needed to be studied. We only have one experimental result for homogeneous foam for comparing our results. Hence, experimental study could be an area of future study. Since foams other than brittle foams are widely used, large deformation of foams will be an interesting topic. There are also some research needs in open-cell foams used in cooling system wherein hot air/fluid flows through the foam. As energy-absorption function, foams are under compression and closed-cell foams are often used. Hence, fracture behavior under compression and research on closed-cell foams are good future topics.
APPENDIX A
CRACK TIP DISPLACEMENT FIELDS FOR ORTHOTROPIC MATERIALS

The stress-strain relation in the principal direction for plane stress problem is given as:

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} = [S]\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1/E_1^* \\
-\nu_{21}/E_2^* \\
-\nu_{12}/E_2^*
\end{bmatrix} - \begin{bmatrix}
0 \\
1/E_2^* \\
0
\end{bmatrix} = \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

(A.1)

The stress-strain relation can be transformed from the principal 1-2 coordinate system to the \(x-y\) coordinate system by using the transformation matrix \([T]\):

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} = [T]\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\]

(A.2)

where the transformation matrix is defined as:

\[
[T] = \begin{bmatrix}
\cos^2 \theta & \sin^2 \theta & 2\cos \theta \sin \theta \\
\sin^2 \theta & \cos^2 \theta & -2\cos \theta \sin \theta \\
-\cos \theta \sin \theta & \cos \theta \sin \theta & \cos^2 \theta - \sin^2 \theta
\end{bmatrix}
\]

(A.3)

The compliance matrix \([\bar{S}]\) in the \(x-y\) plane is

\[
[\bar{S}] = \begin{bmatrix}
\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\
\bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\
\bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66}
\end{bmatrix} = [T]^T [S] [T]
\]

(A.4)

The characteristic equation of the orthotropic material is given by Sih and Liebowitz(1968)

\[
\bar{S}_{11}\mu^4 - 2\bar{S}_{16}\mu^3 + \left(2\bar{S}_{12} + \bar{S}_{66}\right)\mu^2 - 2\bar{S}_{26}\mu + \bar{S}_{22} = 0
\]

(A.5)

There are four roots of the characteristic equation. We denote \(s_1\) and \(s_2\) as the two unequal roots with positive conjugate values:

\[
s_1 = \mu_1 = \alpha_1 + i\beta_1, \quad s_2 = \mu_2 = \alpha_2 + i\beta_2
\]

(A.6)

The constants \(p_j\) and \(q_j\) (j=1,2) are related \(s_1\) and \(s_2\) as bellows
The displacement field in the vicinity of crack tip is a function of the orthotropic material parameters \( p_1, p_2, q_1, q_2, s_1 \) and \( s_2 \) as shown in Eq(2.5).

For plane strain problem, the strain and stress relation is

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} = [Q_s] \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix},
\]

where

\[
C_{11} = \frac{1}{D} E_1 \left(1 - \frac{E_3}{E_2} v_{23}^2\right), \quad C_{12} = \frac{1}{D} E_2 \left(v_{12} + \frac{E_3}{E_2} v_{13} v_{23}\right),
\]

\[
C_{22} = \frac{1}{D} E_2 \left(1 - \frac{E_3}{E_1} v_{13}^2\right), \quad C_{21} = C_{12}, \quad C_{66} = G_{12},
\]

\[
D = \frac{E_1 E_2 E_3 - v_{23}^2 E_1 E_2^2 - v_{13}^2 E_2 E_3^2 - 2 v_{12} v_{13} v_{23} E_2 E_3^2 - v_{13}^2 E_2 E_3^2}{E_1 E_2 E_3}
\]

And then the compliance matrix \([S_s]\) in the 1-2 coordinates is the inverse matrix of \([Q_s]\):

\[
[S_s] = [Q_s]^{-1}
\]

In order to obtain the displacement field near the crack tip, simply replace \([S]\) in Eq.(A.4) with \([S_s]\) and then the solution takes the same form as plane stress problems.
Space frame elements are used in the study of tetrakaidecahedral foam. Forces and moments on a frame element in space are shown in Figure B-1. These forces and moments can be output at three nodes, that is, two end nodes and the middle node.

Since equilateral triangle is not a default cross section in ABAQUS, general cross section option is used in the frame element for tetrakaidecahedral foam. Area $A$, the moment of inertia $I_1$ and $I_2$, the polar moment of inertia $J$ are required for input data. Those values could be determined by the equations listed in Table B-1. Since those values are needed to calculate stress in the struts, Table B-1 also gives equations for other types of cross section used in this study.

If we ignore the shear stress, stress at a point in the section at the middle of a strut is given as

$$
\sigma_{\text{tan}} = \frac{N}{A} + \frac{M_1}{I_1} x_2 + \frac{M_2}{I_2} x_1
$$  \hspace{1cm} (B.1)

For rectangular prism foam, since $M_2=0$ this foam becomes

$$
\sigma_{\text{tan}} = \frac{N}{A} + \frac{M_1}{I_1} \frac{h}{2} = \frac{N}{h^2} + \frac{6M_1}{h^3}
$$  \hspace{1cm} (B.2)

In order to include elastic-plastic response, we need specify $N, M_1, M_2,$ and $T$ directly as functions of their conjugate plastic deformation variables. The plasticity is lumped at the element ends. There are no plastic strains as output in the frame element. Plastic displacements and rotations in the element coordinate system are output for plastic deformation. ABAQUS assumes the displacement and rotation increments can be decomposed into elastic and plastic parts. We can obtain plastic deformation in axial direction by adding the plastic displacements on element ends. Then the plastic strain is assumed to be the plastic deformation divided by the element
length. However, this simple method is an approximate method to evaluate the plastic deformation near the crack tip.

Table B-1. Cross section of frame element

<table>
<thead>
<tr>
<th>Cross section</th>
<th>Equations</th>
</tr>
</thead>
</table>
| Equilateral triangle | $A = \frac{\sqrt{3}}{4} D^2$  
$I_1 = \frac{\sqrt{3}}{18} A^2$  
$I_1 = I_2$  
$J = \frac{A^2}{5\sqrt{3}}$ |
| Pipe | $A = 2\pi (r - 0.5t)t$  
$I_1 = \frac{\pi}{4} (r^4 - (r - t)^4)$  
$I_1 = I_2$  
$J = \frac{\pi}{2} (r^4 - (r - t)^4)$ |
| Square | $A = h^2$  
$I_1 = \frac{h^4}{12}$  
$I_1 = I_2$  
$J = \frac{h^4}{6}$ |
Figure B-1. Forces and moments on a frame element in space.
LIST OF REFERENCES


BIOGRAPHICAL SKETCH

Junqiang Wang was born in China in 1973. He received his Bachelor of Engineering in mechanical engineering from University of Science and Technology Beijing in 1996. He worked for 2 years for Qinhuangdao Branch of Baotou Engineering and Research Corp. of Iron and Steel Industry, China. He received his master’s degree in the speciality of materials processing engineering in Tsinghua University, China. He also got a Master of Science in mechanical engineering at University of Florida. He is pursuing his doctoral degree at the Center for Advanced Composites in the Department of Mechanical and Aerospace Engineering, University of Florida.