PROJECTION OPERATOR FORMALISM FOR QUANTUM CONSTRAINTS

By

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To my wonderful wife and soul mate, Megan.

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Motivated by several theoretical issues surrounding quantum gravity, a course of study has been implemented to gain insight into the quantization of constrained systems utilizing the Projection Operator Formalism. Throughout this dissertation we will address several models and techniques used in an attempt to illuminate the subject. We also attempt to illustrate the utility of the Projection Operator Formalism in dealing with any type of quantum constraint.

Quantum gravity is made more difficult in part by its constraint structure. The constraints are classically first-class; however, upon quantization they become partially second-class. To study such behavior, we will focus on a simple problem with finitely many degrees of freedom and will demonstrate how the Projection Operator Formalism is well suited to deal with this type of constraint.

Typically, when one discusses constraints, one imposes regularity conditions on these constraints. We introduce the "new" classification of constraints called "*highly irregular*" constraints, due to the fact these constraints contain both regular and irregular solutions. Quantization of irregular constraints is normally not considered; however, using the Projection Operator Formalism we provide a satisfactory quantization. It is noteworthy that irregular constraints change the observable aspects of a theory as compared to strictly regular constraints. More specifically, we will attempt to use the tools of the Projection Operator Formalism to study another gravitationally inspired model, namely

the Ashtekar-Horowitz-Boulware model. We will also offer a comparison of the results obtained from the Projection Operator Formalism with that of the Refined Algebraic Quantization scheme.

Finally, we will use the Projection Operator Method to discuss time-dependent quantum constraints. In doing so, we will develop the formalism and study a few key time-dependent models to help us obtain a larger picture on how to deal with reparameterization invariant theories such as General Relativity.

CHAPTER 1
INTRODUCTION

"It is very important that we do not all follow the same fashion... Its necessary to increase the amount of variety .... the only way to do it is to implore you few guys to take a risk ..." -Richard Feynman

The Standard Model is the archetype of the kind of success physics has had in describing the physical universe. The theory provides an explanation of the interactions of matter with the electroweak and strong forces on a fundamental level. The way these forces enter into the theory is based on Yang-Mills theory, a generalization of Maxwell's theory of electromagnetism. As is the case with electromagnetism, the equations of motion for a Yang-Mills field contain constraints that reduce the number of degrees of freedom [1]. This is a key characteristic of constrained systems. The process of converting a classical theory to a quantum theory is made more difficult by the presence of these constraints. Commonly used techniques to deal with these systems have been inadequate in providing a description of the low momentum behavior of the strong force, which is associated with the mass-gap conjecture [2].

General relativity, like the Standard Model, is another example of a constrained system. The quantization of gravity has presented theoretical physics a cornucopia of problems to solve for the past 50 years. To answer these deep theoretical questions, physicists have employed several and seemingly conflicting viewpoints. These perspectives, range from Superstrings [3], the main goal is the unification of all forces in one quantum mechanical description, Loop Quantum Gravity [4], in which the main objective of this is a consistent background independent description of quantum gravity, to Causal Sets [5] in which the approach preassumes that space-time is discretized, and the Affine Quantum Gravity Program [6], in which the aim of this approach is to solve quantum constraint problems with the Projection operator formalism. There are several deep underlying theoretical issues surrounding the quantization of gravity, one of which is that gravity

is non-renormalizable. Therefore traditional quantum field theory techniques [1], appear to be useless when approaching this subject. Canonical quantization schemes of gravity are also made difficult by the theory's constraint classification [7]. Classically, gravity's constraints are one algebraic class, but upon quantization the constraints morph into another type[1] . Conventional techniques are unsuited for this type of quantum system. The construction of these techniques does not assume a change from one type of constraint to another when the system is quantized. The projection operator method is well suited to handle this situation since all constraints are treated in the same theoretical framework. Both the Yang-Mills and quantum gravity serve as the primary motivation for this dissertation. It is hoped that studying simpler models will eventually aid us in studying more realistic quantum theories.

1.1 Philosophy

When faced with a particular theoretical problem, it has been our approach to follow the preceding philosophy to obtain an appropriate physical answer. In our analysis we have followed the time-honored principles that: (1.) Mathematics will give all possible solutions with no regard to the physics; (2.) When the mathematics leads to a choice, physics should be the guide in choosing the next step. We will not deviate from this long-standing point of view in this dissertation.

We also approach problems with the point of view that a "complete" description of the universe must be a quantum mechanical one. Therefore, a quantum mechanical description of a particular model will always supercede the classical description. This is the primary reason that we cite the mantra, quantize first, reduce second over and over in this dissertation.

[1] This will be discussed further in Chapters 2 and 4

1.2 Outline of the Remaining Chapters

Chapters 2, 3 and 4 serve as the background for the dissertation. The main topic of discussion in Chapter 2 is the introduction of constraint dynamics as well as the description of the Dirac procedure to deal with quantum constraints. The primary goal of Chapter 3 is to introduce the reader to three other alternative programs to deal with the problem of quantum constraints. These methods are the Fadeev-Popov procedure, the Refined Algebraic Quantization Program, and the Master Constraint Program, each of which has its own distinct strengths and weaknesses. The goal of Chapter 4 is to examine the projection operator formalism. In this chapter we will also exam three distinct constraint examples in this formalism. One of the constraint models is a system where the constraints are classically first class; however, upon quantization they become partially second class, similar to the constraints of gravity. This particular model served as the basis of [8]. Whenever encountered in the dissertation, repeated indices are to be summed.

The primary goal of Chapter 5 is to introduce a "new" classification of constraints called *highly irregular constraints* and also illustrate techniques used to deal with quantum versions of these constraints. The basis of this chapter comes from [9] and [10]. Using the techniques gained from Chapter 5, in Chapter 6 we offer a complete discussion of the quantization of the Ashtekar-Horowitz-Boulware Model [11]. The Ashtekar-Horowitz-Boulware model is a mathematical model also inspired by the constraints of gravity. This chapter is based on the results obtained in [9]. We also compare the results obtained by the Refined Algebraic Quantization program with the Projection Operator formalism. This comparison leads to the conclusion that the two methods are not compatible dealing with all constraints.

The remaining chapters are devoted to the topic of time-dependent quantum constraints. Until now, the methods used to delve into this topic [12] have been unsatisfactory due to the fact that these methods avoid solving for the quantum constraints. The aim

of this chapter is to use the Projection Operator Formalism to give a more complete description of the topics by solving for the constraints. Chapter 8 is devoted to the development of the time-dependent formalism and the comparison of the approach found in [12] with the Projection Operator. In Chapter 9, we will examine two time-dependent constraints, one first-class and one second-class. We will conclude with a brief summary and a possible look forward to future research problems.

CHAPTER 2
CONSTRAINTS AND THE DIRAC PROCEDURE

The primary goal of this chapter is to introduce the reader to the concept of constraints in classical physics. We will discuss the quantization of these classical systems in the framework proposed by Dirac, [7], as well as, discuss deficiencies in the method, which will help motivate the development of the projection operator formalism in Chapter 4.

2.1 Classical Picture

A natural starting point for the discussion of constraints is from a classical perspective. We will begin in the Lagrangian formulation of classical mechanics. In this formulation, we begin with the action functional

$$I = \int_{t_1}^{t_2} dt L(q_a, \dot{q}_a, t) \tag{2–1}$$

where L is the Lagrangian, t is a continuous parameter (often associated with time), $q_a \in \mathcal{Q}$, where $\mathcal{Q}$ is some configuration space, $a \in \{1, 2, ...N\}$, $\dot{}$ denotes the derivative with respect to t, and $\dot{q}_a$ is an element of a fiber of the tangent bundle $\mathcal{Q}$, $\mathbb{T}\mathcal{Q}$. Formally, we can write the Lagrangian functional as

$$L : \mathbb{T}\mathcal{Q} \to \mathbb{R}. \tag{2–2}$$

Later in the discussion we will define the configuration space, but for now it is just some C^∞-manifold. The goal[1] of classical mechanics is to determine the equations of motion. The equations of motion are determined by varying the action (2–1) and determining its

[1] This of course is from our point of view.

stationary points. The result, which is well-known, is the Euler-Lagrange equations [2]

$$\delta S = 0 \tag{2–3}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = 0 \tag{2–4}$$

While this formulation is useful in determining a wide variety of physical quantities [15], it is not as sensitive to particular features of a given classical theory as we need.[3] In order to illuminate these features we must first pass to an equivalent formulation of classical mechanics, namely the Hamiltonian formulation. In making the transition from Lagrangian to the Hamiltonian, we must first identify the conjugate momentum,

$$p_a \equiv \frac{\partial L}{\partial \dot{q}_a}. \tag{2–5}$$

This can be recognized as the fiber derivative from the tangent bundle of $\mathcal{Q}$ to the cotangent bundle of $\mathcal{Q}$ [15] (otherwise known as the tangent bundle's dual)

$$p_a : \mathbb{T}\mathcal{Q} \to \mathbb{T}^*\mathcal{Q}. \tag{2–6}$$

The next step in the procedure is to identify the Hamiltonian, which follows from a Legendre transformation of the Lagrangian

$$H = p_a q^a - L\big|_{\dot{q}=\dot{q}(p,q)}, \tag{2–7}$$

[2] This result can be generalized if L is a functional of $(q_a, \dot{q}_a, \dddot{q}_a, \dots q_a^k)$ where k is a finite number. The variation, which is determined by the functional derivative, of this equation is given by $\frac{\delta L}{\delta q_a} = \delta(t-t')\frac{\partial L}{\partial q_a} + \dot{\delta}(t-t')\frac{\partial L}{\partial \dot{q}_a} + \cdots + \delta^{(k)}(t-t')\frac{\partial L}{\partial q_a^k}$ [14]. Integrating this equation with respect to t' and setting the result to zero will yield the stationary points of the corresponding action.

[3] Obviously, if we express 2–4 in terms of a second order differential equation, the subtle point we are about to make becomes clearer. [16]

which can be accomplished as long as the conjugate momentum (2–5) is invertible in terms of $\dot{q}_a$. This condition is satisfied by the Hessian condition

$$\det \frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b} \neq 0. \tag{2–8}$$

We will return to the case when (2–8) fails shortly. Until then we will discuss the Hamiltonian formalism in more detail. For a more complete account see [17] and [15]. At this point we will no longer mention the cotangent bundle, but rather we will note that this space is symplectomorphic to the more familiar space, phase space $\mathcal{M}$, i.e.,

$$\mathbb{T}^*\mathcal{Q} \simeq \mathcal{M} \tag{2–9}$$

The geometric framework of the Hamiltonian framework is a rich and beautiful subject. However, for the sake of brevity we will only recount the most crucial elements to the development of the constraint picture. For a more complete description, we point the readers to [17], [15], and [18].

2.1.1 Geometric Playground

The natural geometric framework for Hamiltonian dynamics is a 2 n dimensional, real, symplectic manifold called phase space, where n is the number of degrees of freedom for a system.[4] The coordinates on the manifold are determined by the equations of motion. As is well known, the equations of motion for an unconstrained Hamiltonian (2–7) are given by

$$\dot{q}^i = \{q^i, H\}, \tag{2–10}$$

$$\dot{p}_j = \{p_j, H\}, \tag{2–11}$$

[4] This definition of the phase-space manifold is true when the system has a finite number of degrees of freedom. Extra care must be taken in the definition when dealing with a case with an infinite number of degrees of freedom.

where H is the Hamiltonian and $\{\cdot,\cdot\}$ are the classical Poisson brackets. The Poisson brackets are defined by the following:

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \tag{2–12}$$

where $f, g \in C^2(\mathcal{M})$. The Poisson brackets have the following properties for any f, g, and $h \in C^2(\mathcal{M})$:

$$\{f,g\} = -\{g,f\}, \tag{2–13}$$

$$\{f,gh\} = \{f,g\}h + \{f,h\}g, \tag{2–14}$$

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0. \tag{2–15}$$

Equation (2–13) illustrates that the Poisson brackets are antisymmetric in respect to its arguments. Equation (2–14) serves as the connection of the Poisson bracket with point-wise multiplication of the functions over the phase space. Finally, (2–15) demonstrates that the Poisson bracket obeys the famous Jacobi identity. With these three properties it is possible to show that the classical functions over phase space form a Lie algebra with respect to the Poisson bracket.[14][5]

The closed, non-degenerate, symplectic two-form is defined by the Poisson brackets of the dynamical variables,

$$\omega \equiv \{q^i, p_j\} = \delta^i_j. \tag{2–16}$$

The symplectic form is a crucial element when we move from a classical discussion of a system to its quantum analogue.

[5] Technically, a Lie algebra would only require the first and the third properties. One must also show that the functions in C^∞ form a vector space, in order to be classified an algebra.

2.1.2 Constraints Appear

Now let us move to the case in which (2–8) fails[6] that is

$$\det \frac{\partial^2 L}{\partial \dot{q}^a \dot{q}^b} = 0. \tag{2–17}$$

If (2–17) occurs, then it arises because the conjugate momenta are not all independent [14], since there exist redundant variables in the dynamical variables. In other words, there exist relations that are associated with the definition of the conjugate momemtum (2–7)

$$\phi_a(p, q) = 0 \tag{2–18}$$

where $a \in \{1, \ldots A\}$. These relations (2–18) are known as primary constraints [14]. It is an important to note that the primary constraints are not determined by the equations of motion. The set of equations (2–18) define a subspace of the phase space called a primary constraint sub-manifold, whose dimension are $2N - A$. Technically, we are assuming that the constraints obey a regularity condition [14]. We will examine this regularity condition, and instances when it fails in Chapter 5. Until then, we will assume and only consider examples in which these conditions are satisfied.

We can also relate the presence of constraints by considering the Noether theorem. When a global transformation exists that leaves the action invariant, the result is a conserved quantity. However, when this is a local transformation, the result is a constraint. See [19] for details.

It is clear that Hamilton's equations (2–10) and (2–11) are no longer valid if primary constraints are present. All the dynamics should take place on the primary constraint surface. We can achieve this by making the following modification to the Hamiltonian,

[6] While Dirac may have not been the first to consider this case, his seminal work on the topic [7] serves as the modern inspiration of the topic

which is defined on all of $\mathcal{M}$

$$H_E(p,q) = H(p,q) + \lambda^a \phi_m \tag{2–19}$$

where λ^a are Lagrange multipliers, that enforces the dynamics of the system to occur only on the sub-manifold. Therefore the equations of motion are given by the following

$$\dot{q}^i = \{q^i, H\} + \lambda^a \{q^i, \phi_a\}, \tag{2–20}$$

$$\dot{p}_j = \{p_j, H\} + \lambda^a \{p_j, \phi_a\}, \tag{2–21}$$

$$\phi_a = 0. \tag{2–22}$$

Since the set of primary constraints must be satisfied for all t, it follows that

$$\dot{\phi}_a = \{\phi_a, H\} + \{\phi_a, \phi_b\}\lambda^b \approx 0. \tag{2–23}$$

where $\approx$ is defined as weakly equal to, or equal to on the constraint sub-manifold. A direct consequence of (2.1.2) is that the solutions to (2.1.2) may not be independent of the set of primary constraints (2–18). If this is the case, we define a set of new constraints (χ_b), $b \in \{1, \ldots B\}$ which also satisfies (2.1.2). This set of constraints is called secondary constraints. We repeat the process of solving the consistency equation (2.1.2) to uncover all the constraints. With that being said we will always assume that all constraints have been uncovered. This statement is often referred to as the set of constraints is complete [14].

Now assuming that the set of constraints is complete, equation (2.1.2) also serves as the starting point of the discussion of the classification.[7] One possibility for (2–22) to be valid on the constraint surface is to allow each Poisson bracket to be separately zero by being proportional to a constraint. This hypothesis leads to our first classification: when

$$\{\phi_a, H\} = h^b_a \phi_b, \tag{2–24}$$

$$\{\phi_a, \phi_b\} = c^c_{ab} \phi_c, \tag{2–25}$$

hold true, we categorize this type of constraint as first class. We can make a further refinement of this class by considering the nature of the structure coefficients, h^b_a and c^c_{ab}. If the coefficients are constants, then the constraints are closed first class. If they are functions over phase space, then the constraints are open first class. As one can deduce from (), the Lagrange multipliers are undetermined by the equations of motion and thus can be arbitrarily chosen, a phenomenon called "choosing a gauge". Therefore, first-class systems are said to be gauge systems. Based upon this definition of a first-class system, we can assert that once the dynamics are restricted to the constraint surface, initially, they will always remain on the surface. Well known examples of first-class systems include Yang-Mills theories and General Relativity, with the former being closed and the latter being open.

If $\det\{\phi_a, \phi_b\} \neq 0$, the constraints are classified as second class [16]. No longer having the availability of the preceding criteria of (2–24) and (2–25), it follows that the Lagrange multipliers are determined by the equations of motion so that (2–22) is satisfied. The Lagrange multipliers force the dynamics to remain on the constraint surface for a second-class system. Namely the Lagrange multipliers can be determined by the following equations

$$\lambda^b = -[\{\phi_a(p,q), \phi_b(p,q)\}]^{-1}\{\phi_a(p,q), H(p,q)\}. \tag{2–26}$$

2.1.3 Another Geometric Interlude

When the dynamics are restricted to a sub-manifold in the phase space, some of the mathematical structures present in the entire phase space are no longer present. Most notably is the symplectic 2-form ω. One could imagine looking at a particular coordinate patch of the sub-manifold and determining the symplectic form for that particular patch. However, if we attempt to repeat this process for the entire sub-manifold, we would find that there exist some regions in which the 2-form is degenerate. Often this degenerate

2-form is referred to as a presymplectic form. Consequently, it is impossible[7] to define the Poisson brackets for the constraint sub-manifold. This particular issue is one of the reasons why the quantization procedure becomes extremely hazy when working with these systems. We refer the reader to [14] for a more detailed account of the pre-symplectic form. A more in depth discussion of the geometry of constraint surface will occur in the following chapter.

2.1.4 Observables

Before proceeding to the quantization of these classical systems, we will offer a brief discussion of observables, a topic which will be revisited for a more complete discusion in subsequent chapters. A Dirac observable is defined as a function over phase space that has a weakly vanishing Poisson bracket with every constraint

$$\{o, \phi_a\} \approx 0 \qquad \text{for all} \quad a \tag{2–27}$$

where $o \in C^\infty(\mathcal{M})$. In the context of first-class constraints (2–27) is a sufficient condition that guarantees a gauge invariant function [14]. If o is an observable, it is clear by the definition of an observable function (2–27) that $o + \lambda^a \phi_a$ is also an observable. Using this observation it is possible to partition the set of functions $C^\infty(\mathcal{M})$ by virtue of this equivalence relation into observable functions and non-observable functions. In taking the discussion further, if we were to consider the vector space of $C^\infty(\mathcal{M})$ equipped with the Poisson brackets, which defines a Lie algebra, along with pointwise multiplication, we can then identify the functions that vanish on the constraint sub-manifold (i.e. the constraints) as the ideal $\mathcal{N}$ in $C^\infty(\mathcal{M})$. We can classify algebra of observable fuctions as the quotient algebra $C^\infty(\mathcal{M})/\mathcal{N}$. [14] This identification of the algebraic structure of the observables

[7] It is possible to write a symplectic form if the constraints are all classical second-class. [14]

will not guarantee the Poisson bracket structure over the constraint sub-manifold for the reasons stated in the preceding section.

2.2 Quantization

2.2.1 Canonical Quantization Program

We proceed now with the standard canonical quantization procedure, described by Dirac [20]. The goal of the canonical quantization procedure is to find a rule that associates phase space functions with self-adjoint operators. Knowing the goal of the program, let's begin the implementation. First, with we must insist that the coordinates of phase space be flat coordinates, which implies there must be a global Cartesian coordinate patch for the entire phase space manifold.[8] The key ingredient to the quantization scheme is the so called quantization map. This map takes real, phase space functions and maps them to self-adjoint operators acting on an abstract, separable complex Hilbert space. More precisely, we require a rule that intertwines the Poisson algebra of the observables[9] with the algebra defined by the commutator bracket and the self-adjoint operators [22] and [23].

$$Q : C^\infty(\mathcal{M}) \rightarrow SA(\mathcal{H}) \tag{2–28}$$

$$Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)] \tag{2–29}$$

$$\tag{2–30}$$

where $SA(\mathcal{H})$ is the set of self-adjoint operators acting on the Hilbert space $\mathcal{H}$ and $f, g \in C^\infty(\mathcal{M})$. The self-adjoint operator $Q(f)$ should be recognized as the quantum

[8] Even though classical mechanics requires no such structure on phase space, quantum mechanics requires it. This was first pointed to by Dirac [20], and more recently by Klauder [21]. According to Klauder [21], this metric structure comes in the form of a "shadow" metric which is proportional to $\hbar$.

[9] Without the presence of constraints observables refer to any differentiable function over phase space.

observable corresponding to the classical observable f. An immediate consequence of (2–29), is the following

$$\begin{aligned} Q(\{q^i, p_j\}) &= \frac{1}{i\hbar}[Q(q^i), Q(p_j)] \\ \Rightarrow [Q(q^j), Q(p_j)] &= i\hbar\delta^i_j \mathbb{1} \end{aligned} \tag{2–31}$$

where (q^j, p_j) are the phase space canonical coordinates and $\mathbb{1}$ is the identity operator on the Hilbert space $\mathcal{H}$. We should note that there is not a definitive method by which to pick the quantization map since the quantization map is not a homomorphism between the two algebra. Ambiguity exists in the process because that there are some phase space functions (e.g. p^4q) that would correspond to multiple self-adjoint operators (e.g. P^2QP^2 or $(P^4Q + QP^4)/2$). It has also shown by [50] there does not exist a quantization map that can be defined for all elements from the full algebra of the classical observables. Despite these difficulties we will proceed, noting potential problems due to these ambiguities in the procedure as necessary.

Thus, for the remaining sections in this work we will assume that we have a quantization map and are free to use it. The notation that we will use is as follows: (p_j, q^i) represent the real c-number of phase space coordinates, while (P_j, Q^i) represent the set of irreducible, self-adjoint operators in which the canonical coordinates are mapped. The commutator of the (P_j, Q^j) follows directly (2–31)

$$[Q^i, P_j] = i\hbar\delta^i_j \mathbb{1}. \tag{2–32}$$

Secondly, we promote quantizable phase space observables (f) to self-adjoint operators F:

$$f(p_j, q^i) \mapsto F(P_j, Q^i). \tag{2–33}$$

If there exists factor order ambiguity, we will appeal to experiment to select the proper definition of F. However, as mathematically precise as the quantization procedure is for

unconstrained systems, we must note that, if our system contains constraints, we cannot apply the standard quantization techniques. [12].

2.2.2 What About Constraints? The Dirac Method

An important point to make is that the procedure described in the preceding subsection is done so without the presence of constraints. If we have constraints in a particular classical system that we are now attempting to quantize, we may not have all of the mathematical structures required to give us a quantization rule. To address this important issue Dirac proposed the following procedure. Quantize the entire classical theory first, then reduce the Hilbert space to the physically relevant subspace[10] . The pertinent question in this method is, "What is the quantum analog to the constraint equation (2–18)?" In response to this query, we will use Dirac's procedure [7]. To initiate this procedure, we begin by promoting the constraints to self-adjoint operators,

$$\phi_a(p,q) \mapsto \Phi_a(P,Q). \tag{2–34}$$

The next step is to determine the kernel of Φ_a, known as the physical Hilbert space [25]

$$\mathcal{H}_P = \{|\psi\rangle_P |\, \Phi_a|\psi\rangle_P = 0, \forall \text{ a}\}. \tag{2–35}$$

If the constraint possesses a zero in continuum of the spectrum, (i.e. suppose that the constraint $\Phi = P_1$), then we immediately encounter a potential difficulty in implementing this procedure. Based solely on the construction of the physical Hilbert space we cannot guarantee that $|{}_P\langle\psi|\psi\rangle_P| < \infty$.

[10] The philosophy of quantize first reduce second serves as a major motivation to the rest of the dissertation.

One should also check the quantum equivalent to the consistency equation (2.1.2). In essence this means we must consider

$$[\Phi_a(P,Q), H(P,Q)]\,|\psi\rangle_P = 0, \tag{2–36}$$

$$[\Phi_a(P,Q), \Phi_b(P,Q)]\,|\psi\rangle_P = 0, \tag{2–37}$$

where $H(P,Q)$ is the unconstrained, self-adjoint Hamiltonian operator. Once again, we are faced with a possible deficiency of the Dirac procedure. In general, we cannot attest to the validity of these equations, but if we restrict our arguments to considering only closed, first-class systems (2–36) and (2–37) will hold true. In the case of closed, first-class systems, the Poisson brackets transforms into the commutator brackets, which are expressed in the following form:

$$[\Phi_a(P,Q), H(P,Q)] = i\hbar h_a^b \Phi_b(P,Q), \tag{2–38}$$

$$[\Phi_a(P,Q), \Phi_b(P,Q)] = i\hbar c_{ab}^c \Phi_c(P,Q). \tag{2–39}$$

If Equation (2–36) or (2–37) fails and the classical system is classified as first class, the quantum system is said to have an anomaly. We will examine such a system in Chapter 4. Furthermore, we find that our definition for the physical Hilbert space may be vacuous when considering classically open, first-class or second-class systems since there may not be a zero in the spectrum [13]. Dirac attempts to remedy the problem of second-class constraints by redefining the Poisson bracket [7]. Therefore, the standard approach in the Dirac procedure prefers closed, first-class systems. We will return to a discussion of the Dirac bracket in Chapter 8.

Another deficiency to note is the fact the Dirac procedure does not offer a definition of the inner product of the physical Hilbert space. This, along with some of the other deficiencies that are illustrated in this chapter, will serve as the primary motivation for the discussion of the more modern methods to quantize constraint systems discussed in

Chapter 3, as well as, the motivation behind the developement of the projection operator formalism [13] in Chapter 4.

CHAPTER 3
OTHER METHODS

The primary objective of this chapter is to examine three distinct methods to deal with quantum constraints. These methods are the Faddeev-Popov procedure, the Refined Algebraic Quantization Program, and the Master Constraint Program, each of which has its own distinct strengths and weaknesses. During this chapter we will use the notation that is standard in literature, while also noting deficiencies of the methods in order to offer more motivation for the study of the projection operator method, which is the topic of Chapter 4.

3.1 Faddeev-Popov Method

3.1.1 Yet Another Geometric Interlude from the Constraint Sub-Manifold

Before discussing the Faddeev-Popov method, it is important to discuss some properties of the constraint sub-manifold that were neglected in the previous chapter. Let us begin with the following classical action,

$$I = \int_{t_1}^{t_2} dt(p_j \dot{q}^j - H(p,q) - \lambda^a \phi_a), \tag{3–1}$$

where $j \in \{1, \ldots, N\}$, $a \in \{1, \ldots, A\}$, and λ^a are Lagrange mulipliers which enforce the constraints ϕ_a[1] . We will consider the situation when the constraints that are present in a classical system are closed, first-class. The constraint submanifold can also be identified as the space of gauge orbits [14]. A gauge orbit is defined by the following: consider that F defines a particular physical configuration,[2] and the gauge orbit of F consists of all gauge-equivalent configurations to F. [12] A gauge transformation is defined as

$$\delta_\nu F = \nu^a \{F, \phi_a\}, \tag{3–2}$$

[1] We are still under the assumption from Chapter 2 that the constraints are regular (which will be defined in Chapter 5) and complete.

[2] One could consider F to be a dynamical variable.

where ν^a represents an arbitrary function. It is clear that these transformations, actually define an equivalence relation which implies that the set of gauge orbits can be identified as a quotient space [14]. If one defines a set of surface forming vectors, the gauge orbits would correspond with the null vectors.[14]

To avoid this rather complicated situation of the quotient, it is often suggested that one must impose a gauge choice to eliminate the redundancy. A gauge choice (χ_a) has the following property

$$\chi_a(p,q) \approx 0, \tag{3–3}$$

where $a \in \{1, \ldots, A\}$ [14]. We must also choose such a function that intersects the gauge orbits once and only once. A word of warning–One can guarantee this is the case locally; however, it may not be guaranteed globally, (i.e. for the entire constraint surface). This is known as the Gribov problem [16]. However, we are considering the ideal case for this discussion.

With this mathematical description established, we can now properly discuss the Fadeev-Popov procedure. [27]

3.1.2 Basic Description

This method requires us to depart from the canonical quantization scheme as described in Chapter 2. The philosophy of this method is to reduce the classical theory first, and quantize second, which is yet another departure from the Dirac procedure from Chapter 2. For a first-class system, the formal path integral is given by

$$\int e^{(i/\hbar)\int[p_a\dot{q}^a - H(p,q) - \lambda^b\phi_b]dt}\mathcal{D}p\mathcal{D}q\mathcal{D}\lambda. \tag{3–4}$$

To solve the constraint problem in this framework, we assert the constraints are satisfied classically within the functional integral by imposing a δ-functional of the constraints. Since the resulting integral may be divergent, we suppress this possibility, by requiring a choice of an auxiliary condition called a gauge fixing term of the form $\chi^a(p,q) = 0, a \in \{1, .., J\}$. With this choice we have lost canonical covariance, which can be restored

formally with the aid of the Faddeev-Popov determinant, $\det(\{\chi^b, \phi_c\})$. By determining a particular gauge fixing term, the hope is to integrate overall gauge orbits. The ensuing path integral becomes

$$\int e^{(i/\hbar)\int(p_a\dot{q}^a - H(p,q))dt}\Pi_b\Pi_c\delta\{\chi^b\}\delta\{\phi_b\}\det(\{\chi^b, \phi_c\})\mathcal{D}p\mathcal{D}q. \tag{3–5}$$

Expression (3–4) serves as a motivation to the introduction of (3–5), but they are not to be viewed as equivalent statements. The result of (3–5) could then be expressed as a path integral over the reduced phase space,

$$\int \exp\{\frac{i}{\hbar}\int_0^T [p^*_j q^{*j} - H^*(p^*, q^*)]dt\}\mathcal{D}p^*\mathcal{D}q^* \tag{3–6}$$

where p^* and q^* are reduced phase coordinates and $H^*(p^*, q^*)$ is the Hamiltonian of the reduced phase space. Since we have satisfied the constraints classically, we are no longer confident that our formal path integral is defined over Euclidean space. This presents a dilemma since the formal path integral is ill-defined over non-Euclidean spaces [13]. As with the Dirac Procedure, the Faddeev method can be modified to accommodate second-class constraints [28].

3.1.3 Comments and Criticisms

While the Faddeev-Popov procedure has yielded some of the most fruitful results in physics [1], it is not without its flaws. One of the most glaring flaws is the fact that one must first reduce the classical theory and then quantize it. The universe[3] is quantum mechanical; therefore, there may be some quantum mechanical correction to the classical theory. Let us consider the following simple model to illustrate this fact. Consider the

[3] At least up to the GUT energy scales [3]. We are not so bold to say that quantum mechanics may be superceded by a more complete description of nature. Of course, we assume however quantum mechanics is the proper route to look at nature until more evidence is discovered.

following classical action

$$I = \int dt(p^j \dot{q}_j - \lambda^a \bar{\phi}_a), \tag{3–7}$$

where $j \in \{1, \ldots, N\}$ and $a \in \{1, \ldots, N\}$. This system is purely constraint[4] . The definition of the constraints are

$$\bar{\phi}_a \equiv f(p,q)\phi_a(p,q), \tag{3–8}$$

where f is a non-vanishing function over the phase space and $\{\phi_a\}_{a=1}^{A}$ defines a closed-first class constraint, i.e.

$$\{\phi_a, \phi_b\} = c^c_{ab}\phi_c \tag{3–9}$$

where c^c_{ab} is a constant. It is clear while (3–8) defines the same constraint sub-manifold as the case in which ϕ_a are constraint, the constraints $\bar{\phi}_a$ are an open, first-class system[5] . We realize that the arbitrary function can be classically absorbed into the definition of the Lagrange multipliers; however, we are ignoring this to emphasize the quantum mechanical behavior. The Fadeev-Popov method for this model begins with the following expression:

$$\int e^{(i/\hbar)\int [p_j \dot{q}^j - \lambda^a \bar{\phi}_a] dt} \mathcal{D}p\mathcal{D}q\mathcal{D}\lambda, \tag{3–10}$$

is replaced with the gauge-fixed expression

$$\int e^{(i/\hbar)\int p_j \dot{q}^j dt} \Pi_b \Pi_c \delta\{\chi^b\}\delta\{f\phi_b\} \det\{\chi^b, f\phi_c\}\mathcal{D}p\mathcal{D}q, \tag{3–11}$$

where $\chi^b(p,q)$ is some appropriate gauge choice. A simple identity leads to

$$\int e^{(i/\hbar)\int p_a \dot{q}^a dt} \Pi_b \Pi_c \delta\{\chi^b\} \frac{\delta\{\phi_b\}}{\Pi_t f^n} \det(\{\chi^b, f\}\phi_c + \{\chi^b, \phi_c\}f)\mathcal{D}p\mathcal{D}q. \tag{3–12}$$

[4] Gravity is such a system [8]

[5] This system is similar to the one discussed in [8], which we will return to in the next chapter.

The first term in the determinant is zero by the δ functional of the ϕ's. The second term is an $N \times N$ matrix multiplied by a scalar f, and therefore becomes.

$$\int e^{(i/\hbar)\int p_a \dot{q}^a dt} \Pi_b \Pi_c \delta\{\chi^b\} \frac{\delta\{\phi_b\}}{\Pi_t f^n} \Pi_t f^n \det\{\chi^b, \phi_c\} \mathcal{D}p\mathcal{D}q. \tag{3–13}$$

We observe that all the factors of f completely cancel. As one can see, the Faddeev method is insensitive to the definition of f, as long as it be non-zero. Hence, this method considers the ϕ_a's and $\bar{\phi}_a$'s as identical constraints. We will examine a similar model in Chapter 4 [8], which demonstrates that in order to understand the entire theory, one must also consider the quantum mechanical corrections.

Another difficulty in this method derives from the selection of gauge choice χ_a. As we noted in the previous subsection, the choice of gauge is only guaranteed locally. In more complicated gauge theories, such as Yang-Mills it is well known [29] that there does not exist a gauge choice that slices the gauge orbits once and only once, a fact which limits the effectiveness to probe the non-perturbative regime of these gauge theories. [26]

3.2 Refined Algebraic Quantization

The Refined Algebraic Quantization Program (RAQ) is in stark constrast to the Fadeev-Popov method mentioned in the previous section. RAQ attempts to quantize the entire classical theory first including the constraints, then attempts to impose the quantum constraints in order to determine a Physical Hilbert space. In this respect, the RAQ attempts to extend and resolve some of the ambiguities of the Dirac Procedure namely, "How is the the inner product imposed on the physical Hilbert space?" and "Which linear space do the linear constraints act on?". [30] Refined Algebraic Quantization comes in two main varieties, Group Averaging and a more rigorous version that is based on the theory of rigged Hilbert spaces. In this chapter we will focus on the former rather than the latter because most experts will agree that there does not exist a group averaging technique for all constraints in this formalism. See [30] and [31] for more complete discussions on the failures of group averaging.

3.2.1 Basic Outline of Procedure

The prescription that RAQ follows begins with the basic treatment of canonical quantization that was described in Chapter 2, although, in general, this prescription generally relaxes the Cartesian coordinate requirement. First, one must represent the constraints C_i as self-adjoint operators (or their exponential action, as Unitary operators) that act on an auxiliary Hilbert space,$\mathcal{H}_{aux}$, which in turn is the prerequisite linear space. Since, in general, the constraints, C_i, have continous spectrum, it follows that the solutions of the constraints could be generalized vectors. Thus, we will consider a dense, subspace of $\mathcal{H}_{aux}$ ($\Phi \subset \mathcal{H}_{aux}$) which can be equipped with a topology finer than that of the regular auxiliary Hilbert space. The distributional solutions of the constraints are contained in the algebraic dual Φ^*, (space of all linear maps $\Phi \to \mathbb{C}$). The topology of Φ^* is that of point wise convergence, which is to say a sequence $f_n \in \Phi^*$ converges to $f \in \Phi^*$ if and only if $f_n(\phi) \to f(\phi)$ for all $\phi \in \Phi$. This concept, as we mentioned in the introduction of the section, is based on the theory of Rigged Hilbert spaces. The subspace Φ is chosen based upon the condition that it is left invariant by the constraints C_i or the exponentiated action, or that it can be determined based on physical choices. [31] Another technical requirement is that for every $A \in \mathcal{A}_{obs}$,(i.e. the algebra[6] of observables), which commutes with C_i, A as well as its adjoint $A^\dagger$, are defined on Φ and map Φ to itself. We will attempt to describe this particular requirement briefly.

The final stage of the RAQ procedure entails constructing an anti-linear map called a rigging map,

$$\eta : \Phi \to \Phi^*, \tag{3–14}$$

[6] This is a *-algebra on the $\mathcal{H}_{aux}$.

that satisifies the following condition: For every $\phi_1, \phi_2 \in \Phi$ then $\eta(\phi_1)$ is a solution of the constraint equation,

$$(C_i(\eta\phi_1))[\phi_2] = (\eta\phi_1)[C_i\phi_2] = 0. \tag{3–15}$$

In addition to (3–15) , the rigging map (3–14) must also satisfy the following two conditions, which are true for every $\phi_1, \phi_2 \in \Phi$:

1. The rigging map η is real and positive semi-definite

   $(\eta\phi_1)[\phi_2] = (\eta\phi_2)[\phi_1],^*$

   $(\eta\phi_1)[\phi_1] \geq 0.$

2. The rigging map inteterwines with the representations of the obervable algebra

   $\mathcal{O}(\eta\phi_1) = \eta(\mathcal{O}\phi_1),$

where $\mathcal{O} \in \mathcal{A}_{obs}$.

Once the rigging map has been determined, the vectors $\eta\phi$ that span the solution space are Cauchy completed with respect to the following inner-product

$$\langle \eta\phi_1 | \eta\phi_2 \rangle \equiv (\eta\phi_1)[\phi_2], \tag{3–16}$$

for every $\phi_1, \phi_2 \in \Phi$ and $\langle \cdot | \cdot \rangle$ is the inner product of the auxiliary Hilbert space. Thus, we define the physical Hilbert space derived by the techniques of the RAQ. We will revisit the Refined Algebraic Quantization program in Chapter 6 in the context of the Ashtekar-Horowitz-Boulware model [32].

3.2.2 Comments and Criticisms

While the RAQ does resolve some of the ambiguities of the Dirac procedure, the resolution is not without cost. One of the prices that we must pay is that we must also have an additional mathematical structure on particular subspaces on the Hilbert space. Namely, we require that the invariant subspace must also be equipped with a topology that is finer than the one inherited by the auxiliary Hilbert space. As is well

known, this choice of the invariant subspace can lead to non-physical results, such as, super-selection sectors[7] in a variety of constraint models where these structures are not motivated.[32] For this primary reason the RAQ procedure has difficulty[8] when dealing with constraints that have zeroes in the continuum. The RAQ method also has difficulty when the constraint algebra produces a quantum anomaly, as well as cases in which an infinite number of constraints are present. This leads us to the third and final method that we will discuss in this chapter the Master Constraint Program.

3.3 Master Constraint Program

The third and final method that is discussed in this chapter is known as the Master Constraint Program (MCP). Like the RAQ, MCP follows the mantra of the Dirac procedure in that one must quantize first and reduce second. We notice the same basic philosophy in the next chapter when we discuss the Projection operator formalism. The Master Constraint Program was developed by Thiemann, et al. [33], in an attempt to overcome situations in which the RAQ procedure fails. These "failures" include, but are not limited to, cases in which an infinite number of constraints are present, as well as when the structure functions are not constants, but rather are functions over the phase space. This program also attempts to eliminate other ambiguity from the RAQ procedure, namely the requirement of additional input into the physical theory. As mentioned before, this additional input is a dense and invariant subspace which is equipped with a finer topology than that of the Hilbert space in which it is embedded. [34] During this section, since we only intend to give a heuristic acount of the Master Constraint Program, we will

[7] See chapter 6 for further details.

[8] When we say difficulty, we mean conventional approaches such as group averaging techniques fail. Extra mathematical constructs must be implemented. [31]

eliminate the many technical details surrounding the mathematical machinery used[9] . We will motivate this quantum constraint program in much the same manner the original authors did, [33]. Particularly, we will describe the classical analog and then discuss the quantization of the classical theory.

3.3.1 Classical Description

Given a phase space $\mathcal{M}$ and a set of constraints functions $C_j(p,q)_{j\in I}$, where I is some countable index set and $(p,q) \in \mathbb{R}^{2N}$ and $2N$ is the dimensionality of the phase space, the master constraint replaces this set of constraints with a single expression, which is sum of the square of the constraint operators in a strictly positive semi-definite form, as shown in the following expressions:

$$M = \frac{1}{2}\sum_{j,k\in I} C_j(p,q) g^{jk} C_k(p,q) \tag{3–17}$$

where g^{ij} is chosen to be positive definite.[10] We will attempt to justify this act of adding the squares of constraint functions in the next chapter, as well as discuss some of the potential pitfalls of this procedure. The set of constraint equations $C_j = 0$ for all $j \in I$ has now been reduced to a single equation, $M = 0$. Despite being a great simplification another difficulty immediately presents itself. Namely, how can we recognize observables in the theory? As we noted in Chapter 2, observables are functions over the phase space that commute weakly with all the constraint functions

$$\{O, C_i(p,q)\} \approx 0 \tag{3–18}$$

[9] For a technical account of this program we refer the reader to the seminal works on this program namely [33] and [34]

[10] We should also note we can make a further modification on (3–17) if the constraints are actually fields. If this is the case, we must smear them over some set of test functions. For more details on this procedure see [33].

where $O \in C^{\infty}(\mathcal{M})$. However, after eliminating the need to deal with each constraint separately, it is immediately apparent that

$$\{f, M\} \approx 0 \tag{3–19}$$

is not valid for just observable functions but any general function, f, over the phase space. Thiemann amended this deficit in [33] by offering the following modification of the identification of an observable to the previous known scheme,

$$\{O, \{O, M\}\}|_{M=0} \approx 0 \tag{3–20}$$

where O is a twice differentiable function. In fact with the scheme, [34], all observables in a given theory can aslo be identified in the following manner. Suppose, using Thiemann's notation, we let α_t^M denote the one-parameter group of automorphisms over the phase space $\mathcal{M}$, which is defined as the time evolution of the master constraint, it follows that we can define the *ergodic mean* [33] of any $O \in C^{\infty}(\mathcal{M})$,

$$\begin{aligned} \bar{O} &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, \alpha_t^M(O). && (3–21) \\ &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, e^{it\{M,\cdot\}} O(p,q) \\ &\equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Sigma_{m=0}^{\infty} dt \frac{(t)^m}{m!} \{O, \{O, \cdots \{O, M\} \cdots\}\} && (3–22) \end{aligned}$$

If we assume that we can commute the integral with the Poisson brackets, then it is easy to see that(3–21) will satisfy (3–20). We will end the discussion with the classical considerations of the master constraint program on that particular note and address the issue of quantum observables later in Chapter 5 in the context of the Projection Operator Formalism.

3.3.2 Quantization

The *modus operandi* of the Master Constraint program is to use well- known and well-established theorems of self-adjoint operators in Operator theory to construct the

physical Hilbert space on a wide variety of constraint systems. This program has been "tested" in systems that included, but are not limited to, simple quantum mechanical constraints that form a non-compact algebra (like $sl(2,\mathbb{R})$) [36] to a fully interacting quantum field theory [37]. Therefore, in order to proceed with the quantization of the MCP, one must first promote the master constraint (3–17) to a self-adjoint operator that acts on an auxiliary Hilbert space

$$M \mapsto \hat{M}. \tag{3–23}$$

The main difference, at this point in the discussion, between the auxiliary Hilbert space of MCP and that of RAQ, is that MCP requires the Hilbert space to be separable.

At first glance it may appear that one has eliminated the possible quantum anomaly because a commutator of any operator with the same operator is zero, $[\hat{M}, \hat{M}] = 0$. While this is a true statement, the quantum anomaly has only been reformulated in another manner, *videlicet* the spectrum of $\hat{M}$ may not contain zero. An example that illustrates this point more clearly is as follows: Consider a classical system with a classical phases space, $\mathbb{R}$, with two constraints,

$$\begin{aligned} C_1 &= p_1 \\ C_2 &= q_1. \end{aligned}$$

Using the classification system we described in Chapter 2, we can identify this system as a second-class system. We should note that this system will not have a quantum anomaly as defined by Chapter 2 and [8]. However, it will provide a distinct property that we are attempting to illustrate, which is the quantum master constraint need not posses a 0 in the spectrum. The corresponding master constraint of this system can be written as the following,

$$M = p_1^2 + q_1^2. \tag{3–24}$$

We can quickly quantize this system by promoting the p's and q's as irreducible self adjoint operators. As noted before $[\hat{M}, \hat{M}] = 0$. However, as metioned before, 0 is not in the spectrum of $\hat{M}$ since the least eigenvalue of $\hat{M}$ is $\frac{\hbar}{2}$. Thiemann [33] offered a means to rectify this by suggesting a modification of the quantum master constraint by the following:

$$\hat{\tilde{M}} = \hat{M} - \lambda \mathbb{I} \tag{3–25}$$

where $\lambda = \inf\{spectrum(\hat{M})$ and $\mathbb{I}$ is the identity operator on the auxiliary or a kinematical Hilbert space. According to Thiemann, [[33]], (3–25) will still have the same classical limit of the master constraint because $\lambda \propto \hbar$. In general, if a system contains a constraint that is classically an open, first-class constraint, like gravity or the system that we will disscuss in Chapter 4, where λ would be proportional to $\hbar^2$.

Assuming the operator $\hat{\tilde{M}}$ is a densely defined self-adjoint operator, we can now proceed with the quantization by first addressing the auxiliary Hilbert space. Using the fact that $\hat{\tilde{M}}$ is a self-adjoint operator with a positive semi-definite spectrum, the auxiliary Hilbert space can be written as the following direct integral [34];

$$\mathcal{H}_{aux} \equiv \int_{\mathbb{R}^+}^{\oplus} d\mu(x) \mathcal{H}_{aux}^{\oplus}(x) \tag{3–26}$$

where $d\mu(x)$ is the spectral measure [33] of the master constraint operator (3–25). Each addend contribution to the sum, $\mathcal{H}_{aux}^{\oplus}(x)$, in (3–26) is a separable Hilbert space with the inner product induced by the auxiliary Hilbert space, $\mathcal{H}_{aux}$.

Using this particular construction we are now able to address the task of solving the quantum master constraint equation $\hat{\tilde{M}} = 0$. By the mathematical description of the auxiliary Hilbert space (3–26), it follows that the action of $\hat{\tilde{M}}$ on $\mathcal{H}_{aux}^{\oplus}(x)$ is simply multiplication of x. We can solve the quantum master constraint equation by identifying the physical Hilbert space by the following

$$\mathcal{H}_{phys} = \mathcal{H}_{aux}^{\oplus}(0). \tag{3–27}$$

Notice the inner product of the physical Hilbert space is inherited based upon the properties of the auxiliary Hilbert space. This will conclude our basic description of the Master Constraint Program. We now attempt to implement a simple quantum mechanical constraint, namely, we will consider a system whose constraints form the Lie algebra of $so(3)$, which will allow us to properly compare the results we obtain with the MCP and the projection operator formalism in Chapter 4.

3.3.3 MCP Constraint Example

Consider a classical system[11] where the phase space is $\mathbb{R}^6$ and is subject to the following three constraints:

$$J_i = \epsilon_{ijk} q_i p_k \tag{3–28}$$

where $i \in \{1, 2, 3\}$ q_i and p_i are the canonical position and canonical momentum respectively[12] . We can clearly recognize that the Poisson brackets of the constraints defined in (3–28) form a closed Lie Algebra that we can identify with the algebra of $so(3)$:

$$\{J_i, J_j\} = \epsilon_{ijk} J_k. \tag{3–29}$$

The classical master constraint (3–17), corresponds to the Casmir operator of the group

$$M = J_i J_i. \tag{3–30}$$

The quantization of this model is straightforward. The auxiliary Hilbert space is the set of all square integrable functions over $\mathbb{R}^3$, also known as $L^2(\mathbb{R}^3)$. The canonical position q_i is promoted to a self-adjoint operator Q_i in which the action on the auxiliary Hilbert space is multiplication by q_i. The conjugate momentum p_i is promoted to a self-adjoint operator P_i in which the action on $L^2(\mathbb{R}^3)$ is differentiation, $P_i = -i\hbar \frac{\partial}{\partial q_i}$. The

[11] We should note that the example covered in this subsection was first considered in [35].

[12] Einstein summation convention is used, as always.

Poisson brackets are replaced with the familar commutator brackets defined in Chapter 2. The classical master constraint (3–30) is promoted to a corresponding self-adjoint operator

$$\hat{M} = \hat{J}_i^\dagger \hat{J}_i. \tag{3–31}$$

Since there is no ordering ambiguity in (3–31), one does not need to subtract λ to obtain a zero in the spectrum of (3–31). Using the techniques that are well known in quantum mechanics [38], we will use spherical coordinates to determine the eigenvalues of the quantum master constraint (3–31)[13]

$$\hat{M} \equiv -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})\right), \tag{3–32}$$

where $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. The eigenvalues and eigenfunctions of (3–32) are well known [38]. The eigenvalues of (3–32) are $\hbar^2 l(l+1)$ where $l \in \mathbb{N}$ and the eigenfunctions are the sphercial harmonic functions $Y_{lm}(\theta, \phi)$, where $-l \leq m \leq l$. [38] A generalized eigenfunction could be written as the the product of a general element in $L^2(\mathbb{R}^+, r^2 dr)$, which we will denote by $R(r)$ with the spherical harmonic functions $Y_{lm}(\theta, \phi)$. Using this fact we can proceed to the next step of the MCP, which is to rewrite the auxiliary Hilbert space in terms of a direct integral decomposition

$$\mathcal{H}_{aux} = \sum_{l=0}^{\infty} cl\{span\{R(r)Y_{lm}(\theta, \phi)| - l \leq m \leq l\} \tag{3–33}$$

where *cl* donotes the closure of the set, and *span* is the linear span of the set of vectors defined in the brackets. The physical Hilbert space and the induced inner product come directly from selecting the subspace that corresponds with the $l = 0$ eigenvalue.

[13] We should note this conversion to spherical coordinates is done after quantization not before, as we noted the previous chapter of the potential difficulties that arises from the reverse [38].

$$\mathcal{H}_{phys} = cl\{span\{R(r)Y_{00}(\theta,\phi)\}\}. \tag{3–34}$$

The inner product of this physical Hilbert space is inherited from the $L^2(\mathbb{R}^+, r^2dr)$. Thus completes the quantization of this simple constraint model in the Master Constraint Program [14] . We will return to this model in the next chapter, when we will discuss it in the context of the projection operator formalism.

3.3.4 Comments and Criticisms

Despite the many successes [35], [36], and [37] that the program has had in resolving several of the ambiguities associated with the Refined Algebraic Quantization, it still may not be the perfect choice to use for all constraints. If the constraint's spectrum contains a zero in the continuum, then particular care, in the form of rather cumbersome mathematical machinery, must be used. Not that this yields an incorrect result, however it almost appears to be extraneous to the material. This is somewhat of a biased opinion because as we will see in the next chapter the projection operator formalism's answer to this seems more satisfactory. Again though, we emphasize the fact that the results have been shown to be equivalent to the results found in [13]. Another possible criticism of this program is not a criticism of the program, but instead, a criticism of its implemetation. That is to say that authors tend to display a heavy reliance on the classical analysis of groups to solve constraints [35]. The main critique of this point comes from the fact that most of the work done with groups such as $sl(2,\mathbb{C})$ neglect the zero representation [41], which should be the representation corresponding to physical Hilbert space. However, just as the authors pointed out in [35], this particular constraint is not physically realizable, and therefore not subject to experiment.

[14] Actually, it is not the very end of this discussion. We must also include a discussion of the quantum observables in the theory. We will simply point the reader to [35] for a discourse on that topic.

3.4 Conclusions

In this chapter, we have examined three distinct constraint quantization programs. All of these programs have their distinct advantages and disadvantages depending on the particular constraint under consideration. In the following chapter we will examine the projection operator method and examine the tools of that formalism and how it attempts to overcome the difficulties of the preceding methods.

CHAPTER 4
PROJECTION OPERATOR FORMALISM

The primary goal of this chapter is to introduce and motivate the Projection Operator Formalism (POF). The projection operator method is a relatively new procedure for dealing with quantum constraints [13] [26].The philosophy of this formalism is to first quantize the entire theory, and then reduce the quantum theory by using the constraints. We will attempt to illustrate how the POF attempts to remedy some of the deficiencies of the methods discussed in Chapter 3. In the final section of the chapter we will examine three constraint models. The first is a constraint that has a zero in the continuum, whereas the second and the third are models that were examined in [8]. They help illustrate the power of the projection operator formalism in dealing with all classifications of constraints. In this chapter it is understood that $\hbar = 1$ unless stated elsewise.

4.1 Method and Motivation

Following the Dirac procedure's initial footsteps, we canonically quantize the unconstrained classical theory as described in the preceding section. We then deviate from the Dirac method by introducing a projection operator, $\mathbb{E}$, which takes vectors from the unconstrained Hilbert space to the constraint subspace (i.e. the physical Hilbert space or even better the regularized[1] physical Hilbert space) [13]

$$\mathcal{H}_P \equiv \mathbb{E}\mathcal{H}. \tag{4–1}$$

We require $\mathbb{E}$ to be Hermitian which satisfies the relation $\mathbb{E}^2 = \mathbb{E}$ (idempotent), these are basic properties of a projection operator. More precisely, suppose that B_1 and B_2 denote measurable[2] sets on the Hilbert space. The product of two operators yields a projection

[1] We will explain this more clearly a little later in the chapter.

[2] Borel measurable [33]

operator that projects onto the intersection of the two sets,

$$\lim_{n\to\infty}(\mathbb{E}(B_1)\mathbb{E}(B_2))^n = \mathbb{E}(B_1 \cap B_2). \tag{4–2}$$

If $B_1 \cap B_2 = \emptyset$, then $\mathbb{E}(\emptyset) = 0$. We will use this property in Chapters 5, 6 and 8.

Reverting to the Dirac prescription of the physical Hilbert space it is defined as

$$\mathcal{H}_P = \{|\psi\rangle_P |\, \Phi_a|\psi\rangle_P = 0,\, \forall a\} = \bigcap_a^A \{\ker \Phi_a\} \tag{4–3}$$

where Φ_a is the quantum analogue to the classical constraint ϕ_a, and $a \in \{1, \ldots, A\}$. In an ideal situation[3] , (4–3) is equivalent to the following

$$\mathcal{H}_P \equiv \bigcap_a^A \{\ker \Phi_a\} = \{\ker \Phi_a \Phi_a\}. \tag{4–4}$$

The fact that (4–4) will not always lead to a non-trival result, is a clue on how to arrive at the true answer. Assuming that $\Phi_a\Phi_a$ is self-adjoint acting on a Hilbert space, we can use the following result from spectral theory to obtain our desired projection operator, $\mathbb{E}$. Namely, the operator $\Phi_a\Phi_a$ can be written in the following representation [39]

$$\Phi_a\Phi_a = \int_0^\infty \lambda d\mathbb{E} \tag{4–5}$$

where $d\mathbb{E}$ is the so-called projection valued measure [33] on the spectrum of $\Phi_a\Phi_a$, which was denoted by λ which contained a spectral range of 0 to ∞[4] The projection operator that was used in (4–1) can be introduced based on the result of (4–5)

$$\mathbb{E}(\Phi_a\Phi_a \leq \delta(\hbar)^2) \equiv \int_0^{\delta(\hbar)^2} \lambda d\mathbb{E} \tag{4–6}$$

[3] Which is generally not the case for reasons mentioned in Chapter 3 and later in this chapter.

[4] This range may or may have not included 0.

where $\delta(\hbar)^2$ is a regularization[5] parameter. As it is often emphasized by Klauder, $\delta(\hbar)^2$ is only a small parameter, not a Dirac δ-functional [40]. Equation (4–6) projects onto a subspace of the Hilbert space with a spectral measure of $\Phi_a\Phi_a$ from 0 to $\delta(\hbar)^2$. The true physical Hilbert space (4–1) is determined when the limit as $\delta(\hbar)^2 \to 0$ will be taken[6] in an appropriate[7] manner. In the following subsection we will offer functional form of 4–6. We will now turn our attention to further motivating the process of squaring the constraint.

4.1.1 Squaring the Constraints

Much like the Master Constraint Program (MCP), the Projection Operator Formalism (POF) also relies on the summing of the square of constraints to replace a set of constraints $\{\phi_a\}_{a=1}^{A}$, with a single term. Unlike the MCP, the POF offers further justification for only dealing with the sum of the squares instead of appealing to simplicity arguments. By simplicity arguments we mean, why stop at a second-order polynomial expression of the constraints, why not consider fourth-order or higher? The authors of [34] only mention that second order was chosen because it is the simplest expression. Instead, we attempt to offer some mathematical arguments that indicate that the sum of the squares of the quantum mechanical constraints is sufficient.

4.1.2 Classical Consideration

Before moving to the quantum mechanical description of the constraint story, it is important for us to be able to motivate the tale classically first. Let us consider a set of A

[5] Hence a regularized physical Hilbert space!!

[6] Or taken depending on the type of constraint. [13]

[7] More on this later.

constraints which satisfy the following A equations:

$$\begin{aligned}
\phi_1 &= 0 \\
\phi_2 &= 0 \\
&. \\
&. \\
&. \\
\phi_A &= 0.
\end{aligned} \tag{4–7}$$

In order to determine the constraint subspace in the phase space, all A equations must be satisfied simultaneously. This set of equations would be at least classically equivalent to the following set:

$$\begin{aligned}
\phi_1^2 &= 0 \\
\phi_2^2 &= 0 \\
&. \\
&. \\
&. \\
\phi_A^2 &= 0.
\end{aligned} \tag{4–8}$$

Finally, if we add all of the preceding equations together, we arrive at the conclusion that

$$\Sigma_a^A \phi_a^2 = 0 \tag{4–9}$$

is equivalent to the set of A equations (4–7). As we stated before, this is classically equivalent, but are we certain that this will be justified quantum mechanically?

4.1.2.1 Quantum Consideration

When moving to operators, the next point of concern is whether or not the procedure of summing the squares of operators is well defined. Since we are assuming Φ_a is

self-adjoint, it follows that $\Sigma\,\Phi_a^2$ is a symmetric operator with a lower bound, and, therefore, it has a well defined self-adjoint extension. This statement assumes that we only have a finite number of degrees of freedom.[8]

4.1.2.2 Projection Operator Justification

All of the preceding arguments could have been the same as those employed by practitioners of the MCP for squaring the constraints. We will now depart from this line of thinking to employ an arguement used by Klauder in [40].

The formal path integral form of the projection operator is the following expression.

$$\mathbb{E} = \int T e^{-i\int_{(n-1)\epsilon}^{n\epsilon} \lambda^a \Phi_a dt} DR(\lambda), \tag{4–10}$$

where T is the time-ordered product and $R(\lambda)$ is the formal measure over the c-number Lagrange multipliers $\{\lambda(t)\}$. As shown in [40] the projection operator (4–10), is constructed in two main steps. The time interval is defined as a positive real value equal to $t_2 - t_1$. The first step is to construct a Gaussian measure that would cause the odd-moments of the Lagrange multipliers to vanish (i.e. $\int \lambda^a(t) D\lambda(t) = 0$), while keeping the even moments (i.e $\int \lambda^a(t)\lambda^b(t')D\lambda = \frac{2\gamma}{\epsilon'} M^{ab}$, where (ϵ') is a small parameter corresponding to a time step, M^{ab} is a positive matrix, and γ is a real, positive integration parameter.)

$$\mathcal{N} \int T e^{-i\int \lambda^a(t)\Phi_a dt - \frac{i}{4\gamma}\int (\lambda^a(t)^2 dt)} \prod_a D\lambda^a = e^{i\gamma \Phi_a M^{ab} \Phi_b (t_2 - t_1)} \tag{4–11}$$

where $\mathcal{N}$ is the formal normalization of the path integral (4–11). The final step is to integrate (4–11) over γ. To accomplish this feat we will introduce a conditionally

[8] If we were to move to a problem with an infinite number of degrees of freedom, *per se* a quantum field theory, extra caution must be employed to establish a proper definition [33].

convergent integral namely,

$$\mathbb{E}(\Phi_a M^{ab}\Phi_b \leq \delta(\hbar)^2) = \lim_{\zeta\to 0^+}\int_{-\infty}^{\infty} d\gamma\, \frac{\sin[(\delta(\hbar)^2+\zeta)\gamma(t_2-t_1)]}{\pi\gamma} e^{(i\gamma(t_2-t_1)\Phi_a M^{ab}\Phi_b)}, \tag{4–12}$$

where the conditionally convergent integral is defined by the following [40];

$$\lim_{\zeta\to 0^+}\int e^{-ix\gamma}\frac{\sin[(\delta(\hbar)^2+\zeta)\gamma]}{\pi\gamma}d\gamma = \begin{cases} 1 & \text{if } |x|\leq\delta, \\ 0 & \text{if } |x|>\delta. \end{cases}$$

Equation (4–12) is true assuming that the constraints are not explicitly dependent on time. We examine that case in Chapters 7 and 8. The matrix M^{ab} is the most general case, but for our current purposes, we are free to select $M^{ab} = \delta^{ab}$, which would yield the desired form of squaring the constraints. Having illustrated the motivation behind considering squaring the constraints, we will divert our attentions to some of the mathematical tools that are required in implementing the POF.

4.2 Tools of the Projection Operator Formalism

4.2.1 Coherent States

As one may recall from Chapter 2, one of the limitations of the Dirac procedure is the lack of assurance of a normalizable state. To address this concern using the projection operator method, let us consider the coherent state as a suitable Hilbert space representation. Let P^i and Q_j denote the standard Heisenberg self-adjoint operators that obey the commutation relation

$$[Q_i, P^j] = i\delta_i^j \mathbb{1}_{\mathcal{H}}. \tag{4–13}$$

The Weyl (canonical) coherent state may be defined as

$$|p,q\rangle = e^{-(i/\hbar)q_i P^i}\, e^{(i/\hbar)p^j Q_j}|0\rangle \tag{4–14}$$

for a finite number of degrees of freedom, $(p,q) \in \mathbb{R}^{2N}$, and the states are strongly continuous in the labels (p,q), $|0\rangle$ is some fiducial vector often taken to be the ground

state of a harmonic oscillator[9] . The additional requirement that (4–14) truly are coherent states is that they possess a resolution of unity [41]:

$$\mathbb{1} = \int\int \frac{dq_i dp^i}{(2\pi)^N} |p,q\rangle\langle p,q| \tag{4–15}$$

We will offer (4–15) as an accepted truth without proof [41].

These coherent states also offer a connection to the classical limit of quantum operators. This property is known as the "weak correspondence principle". [41] We exploit, and also state more carefully, this property of coherent states in a subsequent chapter.

The coherent states are convenient because they form an overcomplete basis of the Hilbert space. Using this representation, we can express a dense set of vectors in the functional Hilbert space in terms of the coherent state overlap,

$$\psi(p,q) = \langle p,q|\psi\rangle = \sum_{n=1}^{N} \alpha_n \langle p,q|p_n,q_n\rangle, \qquad N < \infty, \tag{4–16}$$

where $\alpha_n \in \mathbb{C}$. The inner product of such vectors can be expressed as the following,

$$(\psi,\eta) = \sum_{n,m=1}^{N,M} \alpha_n^* \beta_m \langle p_n,q_n|p_m\prime,q_m\prime\rangle, \tag{4–17}$$

where η is an element of the dense set. The completion of such a set of vectors leads to the unconstrained Hilbert space, which leads us to the topic of the Reproducing Kernel Hilbert space.

4.2.2 Reproducing Kernel Hilbert Spaces

Reproducing Kernel Hilbert space is well established, yet it is an under utilized mathematical technique to describe functional Hilbert spaces. If a reproducing kernel can be defined, it will completely define the space. One such example of reproducing kernel

[9] However, one could select another fiducial vector depending on the situation.

is that which is defined by the coherent states overlap $\langle p', q'|p, q\rangle$, which also defines the inner product of the Hilbert space it defines. Using the fact that $\langle p', q'|\mathbb{E}|p, q\rangle$ is a function of positive type[10] , so it can be used as a reproducing kernel,[11]

$$\mathcal{K}_\Phi(p', q'; p, q) \equiv \langle p', q'|\mathbb{E}|p, q\rangle. \tag{4–18}$$

As in the case of the unconstrained Hilbert space, we can express a dense set of vectors in the functional constraint subspace as

$$\psi_P(p, q) = \sum_{n=1}^{N} \alpha_n \mathcal{K}_\Phi(p, q; p_n, q_n) \quad N < \infty. \tag{4–19}$$

The inner product for these vectors is given by

$$(\psi, \eta)_P = \sum_{m=1}^{N} \sum_{n=1}^{N} \alpha_n^* \beta_n \mathcal{K}_\Phi(p_m, q_m; p_n, q_n), \tag{4–20}$$

where η is also an element of the dense set of vectors. Using basic properties of the reproducing kernel and coherent states, we know that the norm defined by the inner product of these vectors will be finite [13]. This guarantees that the norm of vectors in the completion will also be finite. If we multiply a reproducing kernel $\mathcal{K}$ by a constant, the reproducing kernel $\mathcal{K}$ still corresponds to the same functional space. This is a key point and one that we exploit in the next section when we deal with a constraint that possesses a zero in the continuum.

[10] That is that $\Sigma_{j,k=1}^{N} \alpha_j^* \alpha_k \langle p_j, q_j|\mathbb{E}|p_k, q_k\rangle \geq 0$, for all $N < \infty$ and arbitrary complex numbers $\{\alpha_j\}$ and label sets $\{p_j, q_j\}$.

[11] If the projection operator is equal to unity, then we are left with the unconstrained Hilbert space.

4.3 Constraint Examples

4.3.1 Constraint with a Zero in the Continuous Spectrum

Consider a quantum system whose unreduced space correpsonds to $N = 1$ degrees of freedom, such as the quantum constraint of the system of the following form

$$\Phi_1 = P - 2, \tag{4–21}$$

where Φ_1 is the only constraint present. It is clear that the spectrum of Φ_1 is the real line and that it possesses a zero in that continuum. This implies that $\lim_{\delta\to 0}\mathbb{E} \equiv 0$, which is an unacceptable result. To resolve this quandary one should look at the projection operator overlap with a set of coherent states of the form of (4–16), which follows

$$\langle p'', q''|\mathbb{E}(-\delta \leq P - 2 \leq \delta)|p', q'\rangle \tag{4–22}$$

Using the Taylor series by expanding (4–22) as a function of δ, utilizing the resolution of unity and functional coherent states overlap we find that,

$$\langle p'', q''|\mathbb{E}(-\delta \leq P - 2 \leq \delta)|p\prime, q\prime\rangle \propto 2\delta. \tag{4–23}$$

As suggested in [13], we will multiply (4–22) by $\frac{1}{2\delta}$ to extract the *germ* of the reproducing kernel. We must emphasize that this is still the same functional space described by (4–22). The functional form of the reproducing kernel is expressed by the following:

$$\frac{1}{2\delta}\langle p'', q''|\mathbb{E}(-\delta \leq P - 2 \leq \delta)|p', q'\rangle = \int_{-\delta+2}^{\delta+2} dk\, e^{-(k-p'')^2/2+ik(q''-q')-(k-p')^2/2} \tag{4–24}$$

At this juncture we will not discuss how to evaluate this integral, since the topic is discussed in depth in subsequent chapters. However, we will state the result:

$$\lim_{\delta\to 0}\mathcal{K} = e^{-\frac{1}{2}((p''-2)^2+(p'-2)^2)+2i(q''-q')} \tag{4–25}$$

A characteristic of this reduced reproducing kernel (4–25) is that it does not define the same functional space as the unreduced reproducing kernel (4–22). This reproducing

kernel defines a one dimensional Hilbert space. We should note that (4–25) is gaussian peaked at the classical solution $p' = p'' = p = 2$. As stated we will return to several more examples of constraints with zeros in the continuum when we delve into this topic during Chapters 5 and 6.

4.3.2 Closed, First-Class Constrant

The next constraint system under consideration is a set of constraints that force the angular momentum j_i, $i \in \{1, 2, 3\}$ to vanish. With the angular momentum $j_i \equiv \epsilon_{ijk} q^j p_k$, the action integral we choose is

$$I_1 = \int \left(p_a \dot{q}^a - \lambda^b j_b \right) dt, \tag{4–26}$$

where λ^b denotes the Lagrange multipliers to enforce the constraints. Note that the Hamiltonian has been chosen to be zero for simplicity, so we can focus directly on the issues surrounding the constraints.

From the definition of the j_i's, one can immediately determine the Poisson algebra, given as usual by

$$\{j_i, j_j\} = \epsilon_{ijk} j_k \ . \tag{4–27}$$

Since this bracket yields a Lie algebra, our system is clearly a closed first-class constraint system [14].

The quantization of this model is straight forward and we promote the dynamical variables (p_j, q^i) to the set of irreducible self adjoint operators (P_j, Q^i), which obey the standard Heisenberg relation. The constraint j_i are promoted to self-adjoint operators J_i

$$j_i \mapsto J_i = \epsilon_{ijk} Q_j P_k. \tag{4–28}$$

The projection operator of these constraints (4–28) takes the form of

$$\mathbb{E}(J_i J_i \leq \delta(\hbar)^2). \tag{4–29}$$

Unlike the previous case considered in the subsection, δ can be taken to zero to yield a non-trivial result. We will still utilize the reproducing kernel to discover the functional nature of the physical Hilbert space. Before proceeding it will be convenient to introduce the equivalent form of the canonical coherent states;

$$|\vec{z}\rangle = e^{\frac{1}{2}\vec{z}\cdot(\vec{Q}-i\vec{P})-\frac{1}{2}\vec{z}^*\cdot(\vec{Q}+i\vec{P})}|0\rangle \tag{4–30}$$

where $\vec{z} \equiv (\vec{q}+i\vec{p})/\sqrt{2\hbar}$ and $\cdot$ denotes the standard Euclidean dot product [41].

A coherent state path integral can also be used to calculate the matrix elements of the projector as shown in previous works [13]. Let us begin with a preliminary equation, namely,

$$\begin{aligned}
&\langle \vec{z}''|Te^{-(i/\hbar)\int \lambda^a J_a dt}|\vec{z}'\rangle \\
&\qquad = \mathcal{M}\int \exp\{(i/\hbar)\int((p_a\dot{q}^a - q^a\dot{p}_a)/2 - \lambda^a j_a)dt\}\mathcal{D}p\mathcal{D}q \\
&\qquad = N''N'\exp\{\vec{z}''^* \cdot e^{-(i/\hbar)\vec{\theta}\cdot\vec{j}}\vec{z}',\rangle
\end{aligned}$$

where $\mathcal{M}$, N'', and N' are normalization factors, $\vec{z} \equiv (\vec{q}+i\vec{p})/\sqrt{2\hbar}$, $\vec{j}$ is a 3×3 matrix representation of the rotation algebra, T denotes time ordering, and $\vec{\theta}$ is a suitable functional of $\{\lambda^a(\cdot)\}$.

Following [13], we could integrate over $\vec{\lambda}$ with respect to a suitable measure $R(\vec{\lambda})$ to create the desired projection operator. However, it is equivalent and simpler to proceed as follows,

$$\begin{aligned}
\mathcal{K}_J(\vec{z}'';\vec{z}') &\equiv \langle \vec{z}''|\mathbb{E}(J^2 \le \hbar^2)|\vec{z}'\rangle \\
&= \int \langle \vec{z}''|e^{-(i/\hbar)\vec{\theta}\cdot\vec{J}}|\vec{z}'\}d\mu(\theta),
\end{aligned}$$

where $d\mu(\theta)$ is the normalized Haar measure of $SO(3)$. Consequently,

$$
\begin{aligned}
\mathcal{K}_J(\vec{z}'';\vec{z}') &= (N''N'/2)\int \exp\{\sqrt{\vec{z}''^{*2}}\sqrt{\vec{z}'^2}\cos\theta\}d\cos\theta \\
&= N''N''\,\frac{\sinh\sqrt{\vec{z}''^{*2}\vec{z}'^2}}{\sqrt{\vec{z}''^{*2}\vec{z}'^2}} \qquad (4\text{–}31)\\
&= N''N'[1+\frac{\vec{z}''^{*2}\vec{z}'^2}{3!}+\frac{(\vec{z''}^{*2}\vec{z}'^2)^2}{5!}+\ ...] \qquad (4\text{–}32)\\
& \qquad (4\text{–}33)
\end{aligned}
$$

From (4–32) we can deduce that the physical Hilbert space for every even particle sector is one-dimensional. The Hilbert space found using this method is unitarily equivalent to the one determined by the Master Constraint Program.

4.3.3 Open, First Class constraint

The next constraint model's inspiration is that of gravity (General Relativity). It is well known that the constraint algebra of gravity,

$$
\begin{aligned}
\{H_a(x),H_b(y)\} &= \delta_{,a}(x,y)H_b(x)-\delta_{,b}(x,y)H_a(x), \qquad (4\text{–}34)\\
\{H_a(x),H(y)\} &= \delta_{,a}(x,y)H(x), \qquad (4\text{–}35)\\
\{H(x),H(y)\} &= \delta_{,a}(x,y)g^{ab}(x)H_b(x), \qquad (4\text{–}36)
\end{aligned}
$$

is classically first-class; however, upon quantization the constraints transmute to partially second class. [8] The analysis of the model we are about to examine served as the primary motivation behind [8]. This model is also a type of constraint that that we considered in Chapter 3, in terms of the Fadeev-Popov procedure. The action for our choice of the modified system is very similar in form, i.e.,

$$
I_2=\int\left(p_a\dot{q}^a-\lambda^b l_b\right)dt\ , \qquad (4\text{–}37)
$$

where the essential change resides in the definition of the variables l_i. For some smooth, non-vanishing function, f, we define (note: $q_1 \equiv q^1$, etc.)

$$l_i \equiv f(p_1, p_2, q_1, q_2)\, j_i, \tag{4–38}$$

for all i, and choose for further study the particular example for which

$$f(p_1, p_2, q_1, q_2) \equiv \alpha + (\beta/\tilde{\hbar})(p_1^2 + q_1^2) \ + (\gamma/\tilde{\hbar})(p_2^2 + q_2^2). \tag{4–39}$$

The symbol $\tilde{\hbar}$ is a fixed constant equal in value to the physical value of Planck's constant $\hbar$, namely 1.06×10^{-27} erg-sec. When the classical limit is called for, and thus Planck's constant $\hbar \to 0$, we emphasize that $\tilde{\hbar}$ retains its original numerical value. The reason for such a small divisor is to emphasize the quantum corrections; different divisors can be considered by rescaling β and γ. We recognize, in this simple case, that we could absorb the factor f by a redefinition of the Lagrange multipliers in (4–37). In more complicated systems (e.g., gravity) this simplification is either extremely difficult or perhaps even impossible. Therefore, as a further analog, we retain f as a part of l_a. A straight forward analysis leads to

$$\begin{aligned}
\{l_i, l_j\} &= \{f j_i, f j_j\} \\
&= f^2\{j_i, j_j\} + f\{j_i, f\}j_j + f\{f, j_j\}j_i + \{f, f\}j_i j_j \\
&= f\epsilon_{ijk} l_k + \{j_i, f\}l_j + \{f, j_j\}l_i \\
&= f\epsilon_{ijk} l_k + \epsilon_{iab}[-q^a \partial f/\partial q^b + p_b \partial f/\partial p^a]l_j \\
&\quad - \quad \epsilon_{jab}[-q^a \partial f/\partial q^b + p_b \partial f/\partial p^a]l_i \ .
\end{aligned}$$

Since $f > 0$, our modified set of constraints is classified as open, first-class.

The quantization of this model proceeds much like that of the case with J_i, namely we promote l_i to a suitable self-adjoint quantum operator L_i

$$\begin{aligned} l_i \quad \mapsto \quad & L_i = \alpha J_i + (\beta/2\hbar)\left[(P_1^2 + Q_1^2)\, J_i + J_i\,(P_1^2 + Q_1^2)\right] \\ & + \quad (\gamma/2\hbar)\left[(P_2^2 + Q_2^2)\, J_i + J_i\,(P_2^2 + Q_2^2)\right], \end{aligned} \tag{4–40}$$

where $\alpha + \beta + \gamma = 1$. In dealing with the quantum theory, we drop the distinction between $\tilde{\hbar}$ and $\hbar$. A quick calculation shows that any other factor-ordering of the definition of (4–40) will yield an equivalent result. The commutation of the L_i yields a surprising result, namely

$$\begin{aligned} [L_i, L_k] \quad &= \quad \epsilon_{ijk}(i\hbar F L_k - \hbar^2(\beta\epsilon_{k1l}(a_k^\dagger a_1 + a_1^\dagger a_k))) \\ &- \quad 2i\epsilon_{ia1} J_i((Q^a Q^1 + P^a P^1)F + \ldots \end{aligned} \tag{4–41}$$

where $F = \alpha + (\beta/\hbar)(P_1^2 + Q_1^2) \ + (\gamma/\hbar)(P_2^2 + Q_2^2)$. The second, third, fourth, etc. terms in (4–41) represent the anomaly in the quantum theory. This anomaly corresponds to a transmutation to a partially second class system. With this being noted we will continue the quantum analysis of the system.

Let us introduce conventional annihilation and creation operators represented by

$$a_j = (Q_j + iP_j)/\sqrt{2\hbar}\,, \tag{4–42}$$

$$a_j^\dagger = (Q_j - iP_j)/\sqrt{2\hbar}\,. \tag{4–43}$$

If we define

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \tag{4–44}$$

as the total number operator, it is evident that

$$[J_j, N] = 0\,, \qquad [L_j, N] = 0\,, \tag{4–45}$$

for all j, and thus both sets $\{J_i\}$ and $\{L_i\}$ are *number conserving*. We will revisit this particular technique in Chapter 9. This conservation implies that we can study the fulfillment of both sets of constraints in each of the number-operator subspaces independently of one another. We observe that the subspace for which $N = 0$ consists of just a single state, and this state is an eigenvector of each J_i as well as each L_i, $i \in \{1, 2, 3\}$, all with eigenvalue zero. In the interest of simplicity in this paper, we restrict our attention to the lowest nontrivial subspace in which the constraints $J_i = 0$ are satisfied on a non-vanishing subspace. In particular, we confine our attention to a subspace of the entire Hilbert space corresponding to an eigenvalue of the total number operator of two. Note that the subspace of interest is six-dimensional and that it is spanned by the six vectors given by the two representatives

$$|1,1,0\rangle = a_1^\dagger\, a_2^\dagger\, |0\rangle\ , \quad \text{etc.}, \tag{4–46}$$

$$|2,0,0\rangle = (1/\sqrt{2})\,(a_1^\dagger)^2\, |0\rangle\ , \quad \text{etc.}\ , \tag{4–47}$$

where as usual $|0\rangle\ (= |0,0,0\rangle)$ denotes the no particle state for which $a_j|0\rangle = 0$ for all j.

The first non-empty subspace that produces a non-trival result is the 2-particle subspace. With the additional simplification that $\gamma \equiv \beta/2$, we can express the eigenevector that corresponds to least eigenvalue in this 6-dimensional subspace as,

$$|O_L\rangle = \frac{1}{\sqrt{1 + d^2 + d'^2_1}}(\, d|2,0,0\rangle + d'|0,2,0\rangle + |0,0,2\rangle, \tag{4–48}$$

where $d = 1 - 2\beta + O(\beta^2)$ and $d' = 1 - \beta + O(\beta^2)$. The projection operator of this subspace is constructed as the following;

$$\mathbb{E}_2 = |O_L\rangle\langle O_L| \tag{4–49}$$

We can also construct the fundamental kernel for the modified case using these results. Specifically,

$$\mathcal{K}_L(\vec{z}\prime\prime;\vec{z}\prime) = \langle\vec{z}\prime\prime|0\rangle\langle 0|\vec{z}\prime\rangle + \langle\vec{z}\prime\prime|O_L\rangle\langle O_L|\vec{z}\prime\rangle + ... \tag{4–50}$$

$$= N''N'[1 + \frac{(dz_1\prime\prime^{*2} + d'z_2\prime\prime^{*2} + z_3\prime\prime^{*2})(dz_1\prime^2 + d'z_2\prime^2 + z_3\prime^2)}{2!(d^2 + d'^2 + 1)} + ...]. \tag{4–51}$$

One final note regarding this particular model, as with the case for other partially or fully quantum mechanical second class constraint systems, the limit as $\delta \to 0$ is not taken. The Hilbert space is determined by the space corresponding to the least eigenvalue.

4.4 Conclusions

In this chapter, we have discussed the Projection operator method to deal with quantum constraints. We have also discussed the quantization models of 3 distinctly different constraint models. For the remaining chapters of this dissertation, we will be using the Projection Operator to analyze various quantum constraint situations.

CHAPTER 5
HIGHLY IRREGULAR CONSTRAINTS

The primary goal of this chapter is to introduce regularity conditions on constraints, as well as present a "new" classification of constraints called *highly irregular constraints* and also illustrate techniques used to deal with quantum versions of these constraints. The basis of this chapter comes from [9] and [10].

5.1 Classification

In constrained dynamics one typically places regularity conditions on the constraint to insure linear independence. If we consider A classical constraints, ϕ_a, $a \in \{1, .., A\}$ the regularity condition can be stated in terms of the rank of the Jacobian matrix of the constraints [16]

$$\text{Rank}\frac{\partial \phi_a}{\partial(p^n, q_n)}\Big|_\Gamma = A, \tag{5–1}$$

where $n \in \{1, ..., M\}$, $2M$ is the dimensionality of phase space, and Γ is the constraint hypersurface ($\phi_a = 0$). If this condition fails, then the constraint (or set of constraints) is called irregular [16]. Irregular constraints can appear in following form

$$\phi_a^r, \tag{5–2}$$

where ϕ_a is a regular constraint and r is an exponent $r > 1$. In the literature [16] the measure of irregularity is based on the order of the zero on the constraint surface. For example, (5–2) is an rth order irregular constraint. We should note that while the constraints ϕ_a and ϕ_a^r are equivalent (i.e. the constraints generate the same constraint hypersurface), the dynamics and set of observables associated with each given system are *not* necessarily equivalent.

The term *highly irregular constraint* refers to a constraint function that involves both regular and irregular constraints or two or more constraints of varying order [9]. For

example, let us consider the following two constraints:

$$\phi_1 = q(1-q)^2, \tag{5–3}$$

$$\phi_2 = (q-3)^2(q-4)^3. \tag{5–4}$$

The first constraint is regular at $q = 0$ and irregular at $q = 1$ of order 2. The second coinstraint is irregular at both $q = 3$, of order 2, and at $q = 4$, of order 3. Both of these constraints are representative of the class of highly irregular constraints. Since the dynamics as well as observability [9] of a given system are potentially not the same for regular and irregular constraints, careful consideration must be observed when quantizing such systems. The projection operator formalism [13] seems to provide an appropriate framework to deal with systems with irregular constraints [9].

The usual form of the projection operator is given by

$$\mathbb{E}(\Sigma_a \Phi_a^2 \leq \delta^2(\hbar)), \tag{5–5}$$

where $\Sigma_a \Phi_a^2$ is the sum of the squares of the constraint operators and $\delta(\hbar)$ is a small regularization factor. The projection operator is then used to extract a subspace of the unconstrained Hilbert space, $\mathcal{H}$. If $\Sigma\Phi_a^2$ has a discrete isolated 0 then δ can be chosen to be an extremely small number. However, if $\Sigma\Phi_a^2$ has a 0 in the continuum, we can not choose an appropriate δ to select the proper subspace. We will discuss this distinct possibility shortly. In the limit as $\delta \to 0$ if appropriate, this subspace becomes the Physical Hilbert space,

$$\lim_{\delta\to 0} \mathbb{E}|\psi\rangle \equiv |\psi\rangle_{Phys}, \tag{5–6}$$

$$\lim_{\delta\to 0} \mathbb{E}\mathcal{H} \equiv \mathcal{H}_{Phys}. \tag{5–7}$$

However, if the constraint's spectrum contains a zero in the continuum then the projection operator vanishes as $\delta \to 0$ [13], which is unacceptable. To overcome this obstacle, this limit must be evaluated as a rescaled form limit. To accomplish this, we will need to introduce suitable bras and kets in the unconstrained Hilbert space. For this discussion it will be convenient to choose canonical coherent states ($|p, q\rangle$) to fulfill this choice. We regard the following expression as the rescaled form

$$S(\delta)\langle p', q'|\mathbb{E}|p, q\rangle, \tag{5–8}$$

where $S(\delta)$ is the appropriate coefficient needed to extract the leading contribution of $\langle p', q'|\mathbb{E}|p, q\rangle$, for $0 < \delta \ll 1$. For example, if $\langle p', q'|\mathbb{E}|p, q\rangle \propto \delta$ to leading order, then $S(\delta) \propto \delta^{-1}$, for small δ. The limit $\delta \to 0$ can now be taken in a suitable[1] fashion. The expression (5–8) is a function of positive semi-definite type and this means that it meets the following criteria

$$\lim_{\delta \to 0} \Sigma_{j,l=1}^{N} \alpha_j^* \alpha_l S(\delta)\langle p_j, q_j|\mathbb{E}|p_l, q_l\rangle \geq 0, \tag{5–9}$$

for all finite N, arbitrary complex numbers $\{\alpha_l\}$ and coherent state labels $\{p_l, q_l\}$. A consequence of the previous statement is that (5–8) can lead to a reduced reproducing kernel for the physical Hilbert space

$$\mathcal{K}(p', q'; p, q) \equiv \lim_{\delta \to 0} S(\delta)\langle p', q'|\mathbb{E}|p, q\rangle. \tag{5–10}$$

The reproducing kernel completely defines the physical Hilbert space [13]. The reproducing kernel makes it possible to express a dense set of vectors in the functional

[1] non-trivial

constraint subspace as

$$\psi_P(p,q) = \Sigma_{n=1}^{N} \alpha_n \mathcal{K}(p,q;p_n,q_n), \qquad N < \infty. \tag{5–11}$$

The inner product for these vectors is given by

$$(\psi,\eta)_P = \sum_{n,m=1}^{N,M} \alpha_n^* \beta_m \mathcal{K}(p_n,q_n;p_m,q_m), \tag{5–12}$$

where η is also an element of the dense set of vectors. The completion of these vectors in the sense of Cauchy sequences with the relevant inner product will yield the physical Hilbert space.

Without explicitly calculating the reproducing kernel, we will consider the following highly irregular quantum constraint

$$\Phi = Q^2(1-Q), \tag{5–13}$$

where Q acts as a multiplication operator. Clearly this constraint vanishes when $Q = 0$ and $Q = 1$. Assuming, $0 < \delta \ll 1$, the projection operator for this constraint can be written in the following form

$$\mathbb{E}(-\delta < \Phi < \delta) = \mathbb{E}(-\delta < Q^2 < \delta) + \mathbb{E}(-\delta < (1-Q) < \delta). \tag{5–14}$$

Since the zeros of this operator fall in the continuum, it is clear from the previous discussion we cannot take the limit $\delta \to 0$ in its present naked form. The reproducing kernel can be expressed as the following

$$\mathcal{K}_\diamond = \langle p',q'|\mathbb{E}(-\delta < Q^2 < \delta)|p,q\rangle + \langle p',q'|\mathbb{E}(-\delta < Q-1 < \delta)|p,q\rangle. \tag{5–15}$$

By construction these projection operators $\mathbb{E}(-\delta < Q^2 < \delta)$ and $\mathbb{E}(-\delta < Q-1 < \delta)$ project onto orthogonal spaces. To leading order in $\delta(\hbar)$ the reproducing kernel can be

approximated by

$$\mathcal{K}_{\diamond} \simeq \delta^{1/2}\mathcal{K}_{Q=0} + \delta\mathcal{K}_{Q=1}, \tag{5–16}$$

where $\mathcal{K}_{Q=0}$ and $\mathcal{K}_{Q=1}$ are leading order contributions to the reproducing kernels around the two solutions to the constraint equation. See [9] for further details. Unlike expression (5–8) there does not exist a single $S(\delta)$ to extract the leading order dependency for the entire Hilbert space. To address this difficulty we will consider the following argument [9].

Our previous example had constraint solutions around $Q = 0$ and $Q = 1$, we will now address this in a more general setting. We begin by determining the reproducing kernel for each solution in the constraint equation. Recall that the sum of reproducing kernels will produce a direct sum of the corresponding reproducing kernel Hilbert spaces if the spaces are mutually orthogonal. This will be the case for highly irregular constraints. So let $\mathcal{K}$ represent the ($\delta > 0$) reproducing kernel for the reproducing kernel Hilbert space $\mathcal{H}$

$$\mathcal{K} = \Sigma_{n=1}^{N}\mathcal{K}_n, \tag{5–17}$$

where $\mathcal{K}_n$ is the determined reproducing kernel for each unique solution of the constraint. The Hilbert space generated has the following form,

$$\mathcal{H} = \bigoplus_{n=1}^{N}\mathcal{H}_n, \tag{5–18}$$

where $\mathcal{H}_n$ corresponds to the $\mathcal{K}_n$ for each n. However, we have not taken the limit as $\delta \to 0$, and since the leading order δ dependency is potentially different for each reproducing kernel $\mathcal{K}_n$, there does not exist a single $S(\delta)$ that can be used to extract the leading order δ contribution of each reproducing kernel. To accomplish this task we define a (similarity) transformation S,

$$\begin{aligned} S : \mathcal{K}_n &\mapsto \hat{\mathcal{K}}_n, \\ \hat{\mathcal{K}}_n &= S_n(\delta)\mathcal{K}_n, \end{aligned}$$

where $S_n(\delta) > 0$ for all n which leads to

$$\hat{\mathcal{K}} = \Sigma_{n=1}^{N} S_n(\delta)\mathcal{K}_n. \tag{5–19}$$

The rescaled $\hat{\mathcal{K}}$ serves as the reproducing kernel for the Hilbert space $\hat{\mathcal{H}}$. Although the inner product of $\mathcal{H}$ and $\hat{\mathcal{H}}$ are different the set of functions are identical. The goal of this little exercise is of course to take a suitable limit $\delta \to 0$ to yield a function that can serve as a reproducing kernel for the physical Hilbert space. At this point, we can take such a limit.

$$\tilde{\mathcal{K}} \equiv \lim_{\delta \to 0} \hat{\mathcal{K}}, \tag{5–20}$$

$$\mathcal{H}_{phys} \equiv \tilde{\mathcal{H}} = \bigoplus_{n=1}^{N} \tilde{\mathcal{H}}_n, \tag{5–21}$$

where $\tilde{\mathcal{K}}$ is the reduced reproducing kernel for the physical Hilbert space $\mathcal{H}_{phys}$. Having discussed the basic theory behind this classification of constraint, in the next section we will consider a simple but robust toy model that demonstrates the strength of the Projecion Operator Formalism to deal with these kinds of constraints.

5.2 Toy Model

The model we choose to study involves just one configuration variable q, $-\infty < q < \infty$, and its conjugate variable p. The classical action is taken to be

$$I = \int [p\dot{q} - \lambda R(q)]dt, \tag{5–22}$$

where λ is a Lagrange multiplier designed to enforce the single constraint

$$R(q) = 0 \ . \tag{5–23}$$

The classical equations of motion for our simple system are given by

$$\dot{q} = 0, \tag{5–24}$$

$$\dot{p} = -\lambda R'(q), \tag{5–25}$$

$$R(q) = 0\ , \tag{5–26}$$

with solutions

$$q(t) = q_l = q(0), \tag{5–27}$$

$$p(t) = -R'(q_l)\int_0^t \lambda(t')dt' + p(0), \tag{5–28}$$

where q_l is a root of $R(q) = 0$. If $R'(q_l) = 0$ then the solution becomes

$$q(t) = q_l = q(0), \quad p(t) = p(0). \tag{5–29}$$

The function $\lambda(t)$ is not fixed by the equations of motion, which is normal for first-class constrained systems. To explicitly exhibit a solution to the classical equations of motion it is generally necessary to specify the function $\lambda(t)$, and this constitutes a choice of gauge. Gauge dependent quantities are defined to be unobservable, while gauge independent quantities are declared to be observable. In the present example, if $R'(q_l) \neq 0$, then $p(t)$ is gauge dependent, while if $R'(q_l) = 0$, $p(t)$ is, in fact, gauge independent. This behavior suggests that the momentum p in the subset of the reduced classical phase space for which $\{q : R(q) = 0,\ R'(q) \neq 0\}$ is unobservable, while the momentum p in the subset of the reduced classical phase space for which $\{q : R(q) = 0,\ R'(q) = 0\}$ is observable. We discuss this point further below. The reduced classical phase space is given by $\mathbb{R} \times \mathcal{Z}$, where

$$\mathcal{Z} = \{q : R(q) = 0\}. \tag{5–30}$$

Clearly, for the classical theory to be well defined, it is sufficient for $R(q) \in C^1$, namely that $R(q)$ and $R'(q)$ are both continuous. (Strictly speaking this continuity is

required only in the neighborhood of the zero set $\{q : R(q) = 0\}$; however, with an eye toward the Ashtekar Horowitz Boulware model, discussed in Chapter 6, we choose $R(q) \in C^1$ for all q.)

Our discussion will cover a wide class of R functions, and for convenience of explanation we shall focus on one specific example; generalization to other examples is immediate. The example we have in mind is given by

$$R(q) = q(q-2)^{3/2}\theta(2-q) + (q-3)^3\theta(q-3), \tag{5–31}$$

where

$$\theta(x) \equiv \begin{cases} 1, & x > 0 \\ 0, & x < 0. \end{cases} \tag{5–32}$$

For this example, the zero set is given by

$$\mathcal{Z} \equiv \{q = 0, q = 2, q = 3, \text{and}\, 2 \leq q \leq 3\}; \tag{5–33}$$

only for $q = 0$ is $R'(q) \neq 0$. (Although physically motivated models would typically not include intervals in the zero set of R, we do so to illustrate the versatility of our approach.)

In summary, the phase space for the unconstrained classical system is parameterized by $(p, q) \in \mathbb{R} \times \mathbb{R}$, and the phase space for the constrained system is parameterized by the points $(p, q) \in \mathbb{R} \times \mathcal{Z}$. This latter space consists of several one-dimensional lines and a two-dimensional strip. From the standpoint of this elementary example all elements of $\mathbb{R} \times \mathcal{Z}$ are equally significant.

We now turn to the quantization of this elementary example following the precepts of the projection operator formalism [13]. In this approach one quantizes first and reduces second. The ultimate reduction leads to a physical Hilbert space appropriate to the constrained system.

Quantization first means that our original variables p and q become conventional self-adjoint operators P and Q subject to the condition that

$$[Q, P] = i\mathbb{1} \tag{5–34}$$

in units where $\hbar = 1$. (When we eventually examine the classical limit, we shall restore the parameter $\hbar$ to various expressions as needed.) The projection operator of interest is given by

$$\mathbb{E}(R(Q)^2 \leq \delta^2) = \mathbb{E}(-\delta \leq R(Q) \leq \delta) = \mathbb{E}(-\delta < R(Q) < \delta) \ , \tag{5–35}$$

where $\delta > 0$ is a temporary regularization parameter that will eventually be sent to zero in a suitable manner. Since the limit $\delta \to 0$ will ultimately be taken as a form limit, we need to introduce suitable bras and kets in this original, unconstrained Hilbert space. For that purpose we will again choose canonical coherent states defined, for the present discussion, by

$$|p, q\rangle \equiv e^{ipQ} e^{-iqP} |0\rangle. \tag{5–36}$$

As usual, we choose $|0\rangle$ to satisfy $(Q + iP)|0\rangle = 0$; namely, $|0\rangle$ is the normalized ground state of an harmonic oscillator with unit frequency. Thus we are led to consider the complex function

$$\langle p'', q''|\mathbb{E}(-\delta < R(Q) < \delta)|p', q'\rangle \tag{5–37}$$

which is continuous (actually C^∞) in the coherent state labels and uniformly bounded by unity since $\mathbb{E} = \mathbb{E}^\dagger = \mathbb{E}^2 \leq \mathbb{1}$.

It is important to remark that the function (5–37) is a function of positive type, a criterion that means

$$\Sigma_{j,k=1}^{N} \alpha_j^* \alpha_k \langle p_j, q_j|\mathbb{E}|p_k, q_k\rangle \geq 0 \tag{5–38}$$

for all $N < \infty$ and arbitrary complex numbers $\{\alpha_j\}$ and label sets $\{p_j, q_j\}$; this property holds because $\mathbb{E}$ is a projection operator. As a consequence of being a continuous function

of positive type the function

$$\langle p'', q''|\mathbb{E}|p', q'\rangle \tag{5–39}$$

serves as a reproducing kernel for a reproducing kernel Hilbert space, a functional representation by continuous functions on the original phase space ($\mathbb{R} \times \mathbb{R}$), of the regularized (by $\delta > 0$) physical Hilbert space. Our goal is to take a suitable limit $\delta \to 0$ so as to yield a function that can serve as a reproducing kernel for the true physical Hilbert space for the present problem.

Clearly the limit $\delta \to 0$ of the given expression vanishes and that is an unacceptable result. Suppose we assume $0 < \delta \ll 1$, e.g., $\delta = 10^{-1000}$. Then it is clear (even for a much larger δ as well!), for the example at hand, that

$$\begin{aligned}
\mathbb{E}(-\delta < R(Q) < \delta) &= \mathbb{E}(-\delta < 2Q < \delta) \\
&+ \mathbb{E}(-\delta < (8(2-Q)^{3/2} < \delta) + \mathbb{E}(2 \leq Q \leq 3) \\
&+ \mathbb{E}(-\delta < 8(Q-3)^3 < \delta) \\
&\equiv \mathbb{E}_1 + \mathbb{E}_2 + \mathbb{E}_3 + \mathbb{E}_4,
\end{aligned} \tag{5–40}$$

where $\mathbb{E}_n, 1 \leq n \leq 4$, corresponds to the terms in the line above in order. By construction, for very small δ, it follows that these projection operators obey

$$\mathbb{E}_n \mathbb{E}_m = \delta_{nm} \mathbb{E}_n, \tag{5–41}$$

i.e., they project onto mutually orthogonal subspaces. In like manner the reproducing kernel decomposes into

$$\begin{aligned}
\mathcal{K}(p'', q''; p', q') &\equiv \langle p'', q''|\mathbb{E}(-\delta < R(Q) < \delta)|p', q'\rangle \\
&= \Sigma_{n=1}^4 \mathcal{K}_n(p'', q''; p', q'),
\end{aligned} \tag{5–42}$$

where

$$\mathcal{K}_n(p'', q''; p', q') \equiv \langle p'', q''|\mathbb{E}_n|p', q'\rangle. \tag{5–43}$$

Each function $\mathcal{K}_n(p'', q''; p', q')$ serves as a reproducing kernel for a reproducing kernel Hilbert space $\mathcal{H}_n$, and the full reproducing kernel Hilbert space is given by

$$\mathcal{H} = \bigoplus_{n=1}^{4} \mathcal{H}_n. \tag{5–44}$$

Since $\mathbb{E}_n \mathbb{E}_m = \delta_{nm} \mathbb{E}_n$ it follows, from the completeness of the coherent states, that

$$\int \mathcal{K}_n(p'', q''; p, q) \mathcal{K}_m(p, q; p', q') dp dq/(2\pi) = \delta_{nm} \mathcal{K}_n(p'', q''; p', q'). \tag{5–45}$$

This equation implies that the $\mathcal{H}_n$, $1 \leq n \leq 4$, form 4 mutually disjoint (sub) Hilbert spaces within $L^2(\mathbb{R}^2)$. For the present example with $\delta > 0$, each $\mathcal{H}_n$ is infinite dimensional.

Let us first consider

$$\begin{aligned} \mathcal{K}_2(p'', q''; p', q') &\equiv \langle p'', q'' | \mathbb{E}(-\delta < 8(2-Q)^{3/2} < \delta) | p', q' \rangle \\ &= \frac{1}{\sqrt{\pi}} \int_{2-\delta_2}^{2+\delta_2} e^{-(x-q'')^2/2 - i(p''-p')x} e^{-(x-q')^2/2} dx \end{aligned}$$

where $\delta_2 \equiv \delta^{2/3}/2$. This expression manifestly leads to a function of positive type. For very small δ (e.g., $\delta = 10^{-2000}$), we can assert that to leading order

$$\mathcal{K}_2(p'', q''; p', q') = \frac{2\delta_2}{\sqrt{\pi}} e^{-(2-q'')^2/2 - i(p''-p') - (2-q')^2/2} \frac{\sin \delta_2(p''-p')}{\delta_2(p''-p')}. \tag{5–46}$$

This function is already of positive type and is correct to $\mathcal{O}(\delta^2)$ [i.e., to $\mathcal{O}(10^{-4000})$!].

As discussed frequently before [13], we can extract the "germ" from this reproducing kernel by first scaling it by a factor of $\mathsf{O}(\delta^{2/3})$, say by $\pi/(2\delta_2)$, prior to taking the limit $\delta \to 0$. Consequently, we first define a new reproducing kernel

$$\hat{\mathcal{K}}_2(p'', q''; p', q') = \frac{\sqrt{\pi}}{2\delta_2} \mathcal{K}_2(p'', q''; p', q'). \tag{5–47}$$

We remark that the space of functions that make up the reproducing kernel Hilbert space $\hat{\mathcal{H}}_2$ (generated by $\hat{\mathcal{K}}_2$) is *identical* to the space of functions that make up the reproducing kernel Hilbert space $\mathcal{H}_2$ (generated by $\mathcal{K}_2$). Next, we take the limit as $\delta \to 0$ of the

function $\hat{\mathcal{K}}_2$, which leads to

$$\begin{aligned}\tilde{\mathcal{K}}_2(p'',q'';p',q') &\equiv \lim_{\delta\to 0}\hat{\mathcal{K}}_2(p'',q'';p',q') \\ &= e^{-[(2-q'')^2+(2-q')^2]/2-i(p''-p)}. \end{aligned} \tag{5–48}$$

This procedure leads to a new function $\tilde{\mathcal{K}}_2$, which, provided it is still continuous – which it is – leads to a reduced reproducing kernel and thereby also to a new reproducing kernel Hilbert space $\tilde{\mathcal{H}}_2$. Generally, the dimensionality of the space as well as the definition of the inner product are different for the new reproducing kernel Hilbert space; however, one *always* has the standard inner product definition that is appropriate for any reproducing kernel Hilbert space [42]. In the present case, it follows that $\tilde{\mathcal{K}}_2$ defines a *one*-dimensional Hilbert space $\tilde{\mathcal{H}}_2$. Note that even though the coordinate value for the constrained coordinate Q is now set at $Q = 2$ – as is clear from the special dependence of $\tilde{\mathcal{K}}_2(p'',q'';p',q')$ on p'' and p' – the range of the values q'' and q' is still the whole real line. The only remnant that q'' and q' have of their physical significance is that $\tilde{\mathcal{K}}_2(p'',q'';p',q')$ peaks at $q'' = q' = 2$. It is noteworthy that an example of this type of irregular constraint was considered previously by [40].

A similar procedure is carried out for the remaining components in the original reproducing kernel. Let us next consider

$$\begin{aligned}\mathcal{K}_1(p'',q'';p',q') &= \langle p'',q''|\mathbb{E}(-\delta < 2Q < \delta)|p',q'\rangle \\ &= \frac{1}{\sqrt{\pi}}\int_{-\delta_1}^{\delta_1} e^{-(x-q'')^2/2-i(p''-p')x}e^{-(x-q')^2/2}dx\end{aligned}$$

where $\delta_1 \equiv \delta/2$. To leading order

$$\mathcal{K}_1(p'',q'';p',q') = \frac{2\delta_1}{\sqrt{\pi}}e^{-(q''^2+q'^2)/2}\frac{\sin\delta_1(p''-p')}{\delta_1(p''-p')}, \tag{5–49}$$

which is a function of positive type. It is noteworthy to note that this constraint is of regular type. [14]

We rescale this function differently so that

$$\hat{\mathcal{K}}_1(p'', q''; p', q') \equiv \frac{\sqrt{\pi}}{2\delta_1}\mathcal{K}_1(p'', q''; p', q') \tag{5–50}$$

and then take the limit $\delta \to 0$ leading to

$$\begin{aligned} \tilde{\mathcal{K}}_1(p'', q''; p', q') &\equiv \lim_{\delta\to 0}\hat{\mathcal{K}}_1(p'', q''; p', q') \\ &= e^{-[q''^2+q'^2]/2}, \end{aligned} \tag{5–51}$$

a continuous function of positive type that characterizes the one-dimensional Hilbert space $\tilde{\mathcal{H}}_1$.

Our procedure of scaling the separate parts of the original reproducing kernel by qualitatively different factors (i.e., δ_1 and δ_2) has not appeared previously in the projection operator formalism. This difference in scaling is motivated by the goal of having each and every element of the reduced classical phase space represented on an equal basis in the quantum theory. It is only by this procedure that we can hope that the classical limit of expressions associated with the physical Hilbert space can faithfully recover the physics in the classical constrained phase space. Scaling of $\tilde{\mathcal{K}}_1$ and $\tilde{\mathcal{K}}_2$ by finitely different factors has been addressed in [9].

Let us continue to examine the remaining $\mathcal{K}_n$, $3 \leq n \leq 4$. For $\mathcal{K}_4$ we have

$$\begin{aligned} \mathcal{K}_4(p'', q''; p', q') &= \langle p'', q''|\mathbb{E}(-\delta < 8(Q-3)^3 < \delta)|p', q'\rangle \\ &= \frac{1}{\sqrt{\pi}}\int_{3-\delta_3}^{3+\delta_3} e^{-(x-q'')^2/2-i(p''-p')x}e^{-(x-q')^2/2}dx, \end{aligned}$$

where $\delta_3 \equiv [\delta/8]^{1/3}$. The now familiar procedure leads to

$$\tilde{\mathcal{K}}_3(p'', q''; p', q') = e^{-[(3-q'')^2+(3-q')^2]/2-2i(p''-p')} \tag{5–52}$$

corresponding to a one-dimensional Hilbert space $\tilde{\mathcal{H}}_3$. For $\mathcal{K}_3$ we are led to

$$\begin{aligned}\mathcal{K}_3(p'',q'';p',q') &= \langle p'',q''|\mathbb{E}(2\leq Q\leq 3)|p',q'\rangle \\ &= \frac{1}{\sqrt{\pi}}\int_3^4 e^{-(x-q'')^2/2-i(p''-p')x}e^{-(x-q')^2/2}dx, \end{aligned} \tag{5–53}$$

In this case, no δ appears and no infinite rescaling is needed, so we may simply choose

$$\tilde{\mathcal{K}}_3(p'',q'';p',q') = \mathcal{K}_3(p'',q'';p',q'). \tag{5–54}$$

Although we do not have an explicit analytic expression for $\tilde{\mathcal{K}}_3$, we do have a well-defined integral representation in (5–53). Furthermore, it follows that $\tilde{\mathcal{H}}_3$ is infinite dimensional.

Finally, we define the reproducing kernel for the physical Hilbert space as

$$\tilde{\mathcal{K}}(p'',q'';p',q') \equiv \Sigma_{n=1}^4 \tilde{\mathcal{K}}_n(p'',q'';p',q')\ . \tag{5–55}$$

In turn, the physical Hilbert space $\mathcal{H}_P$ is defined as the reproducing kernel Hilbert space $\tilde{\mathcal{H}}$ uniquely determined by the reproducing kernel $\tilde{\mathcal{K}}(p'',q'';p',q')$.

Observe, by our procedure, all elements of the reduced classical phase space $(\mathbb{R}\times\mathcal{Z})$ are represented on an equivalent basis in $\tilde{\mathcal{K}}$. This feature has been designed so that the classical limit of the expressions within $\mathcal{H}_P$ correspond to all aspects of the reduced classical phase space. We will now turn to a discussion of observables of this model.

5.3 Observables

Let us restrict our discussions of observables to those that are self-adjoint operators $\mathcal{O}$ in the unconstrained Hilbert space. We also limit to constraints that are both classically and quantum mechanically first class[2] We first discuss the situation in the case of a regularized ($\delta > 0$) enforcement of the constraint [Chapter 4]. In this case, all physical

[2] These arguments can be extended to include both quantum mechanical second-class and first class with an anomaly. The exclusion is made for the sake of the model under consideration.

observables must obey the following:

$$[\mathbb{E}, \mathcal{O}] = 0. \tag{5–56}$$

We note that even though $\mathbb{E}$ is a function of the squares of the constraints we are not bound to the same concerns Thieman faced in the Master Constraint program. The starting points of the discussion of observables in the two frameworks are different. The Master Constraint Program suffers from the fact that multiplication of functions in a classical space is commutative, hence the additional requirement of an observable function. However, in the Projection Operator Formalism we begin in the quantum regime where the multiplication of self-adjoint operators may not be commutative therefore (5–56) is sufficient. We can take a general operator $\mathcal{G}(P, Q)$ in the unconstrainted Hilbert space and define

$$\mathcal{G}^E(P, Q) \equiv \mathbb{E}\mathcal{G}(P, Q\mathbb{E}) \tag{5–57}$$

as its observable component since clearly $[\mathbb{E}, \mathcal{G}^E(P, Q)] = 0$. In fact, every observable can be expressed in the preceding form (5–57). The equation (5–56) is valid for $\delta > 0$. Howerver as long as $\delta > 0$ we have yet to capture the true physical Hilbert space of a given theory. Therefore the limit $\delta \to 0$ must be taken in a suitable fashion to discuss observables. If quantum constraint[3] Φ admits a discrete spectrum that includes zero, which is *not* the case for our particular model, then we can make the following claims. An operator $\mathcal{O}$ is observable if the following is valid

$$\lim_{\delta \to 0} [\mathbb{E}, \mathcal{O}] = 0 \longleftrightarrow [\Phi, \mathcal{O}]|\psi\rangle_{Phys} = 0. \tag{5–58}$$

[3] suppressing the index

The first part of the preceding if and only if statement has no classical analog; however, the second statement is related to the following weak classical equation

$$\{\phi, o\} \approx 0, \tag{5–59}$$

where o is the classical analog of $\mathcal{O}$. We consider (5–59) to be a weak equation because it needs to vanish on the constraint hypersurface. It is obvious that if (5–58) is true then $\mathcal{O}$ is gauge independent in the physical Hilbert space. In the Heisenberg picture the evolution of the operator is given by

$$\dot{\mathcal{O}}|\psi\rangle_{Phys} = \frac{i}{\hbar}[H^E, \mathcal{O}]|\psi\rangle_{Phys} \tag{5–60}$$

where H^E is the observable part of the Hamiltonian in the form of (5–57). Therefore all observables will stay in the physical Hilbert space as they evolve with time. The same type statement can also be said in the classical world. However, in our particular model, the limit $\delta \to 0$ must be taken as a form limit because Φ has a zero in the continuous spectrum. Observables in these instances must be handled at the level of the reproducing kernel. Recall from the previous section that the physical Hilbert space is isomorphic to an infinite direct sum of complex numbers. In this realization the projection operator is the unit operator, and therefore the observables correspond to general symmetric matrices.

We will now direct our attention to a calcuation of the coherent state matrix element of the physical conjugate momentum at the level of the reproducing kernel. Specifically, we first note that

$$\begin{aligned}
\langle p'', q''|P^E|p', q'\rangle &\equiv \langle p'', q''|\mathbb{E}P\mathbb{E}|p', q\rangle \\
&= \int dx \int dx' \langle p'', q''|\mathbb{E}|x\rangle\langle x|P|x'\rangle\langle x'|\mathbb{E}|p', q'\rangle \\
&= -i\hbar \int dx \int dx' \langle p'', q''|\mathbb{E}|x\rangle\delta'(x - x')\langle x'|\mathbb{E}|p', q'\rangle \\
&= -i\hbar \int dx \langle p'', q''|\mathbb{E}|x\rangle \frac{d}{dx}\langle x|\mathbb{E}|p', q'\rangle.
\end{aligned}$$

We implement the constraints by integrating over the appropriate intervals $\{I_n\}$.

$$\langle p'', q''|P^E|p', q'\rangle = -i\hbar\Sigma_n \int_{I_n} dx \langle p'', q''|\mathbb{E}|x\rangle [ip'\hbar + (q' - x)/\hbar] \langle x|\mathbb{E}|p', q'\rangle. \quad (5\text{–}61)$$

Similarly, it follows that

$$\langle p', q'|P^E|p'', q''\rangle^* = i\hbar\Sigma_n \int_{I_n} dx \langle p', q'|\mathbb{E}|x\rangle [ip''\hbar + (q'' - x)/\hbar] \langle x|\mathbb{E}|p'', q''\rangle. \quad (5\text{–}62)$$

Using a similar technique of Araki [43], we now determine the desired matrix elements by adding (5–61) and (5–62), and dividing by two leads to,

$$\begin{aligned} \langle p'', q''|P^E|p', q'\rangle &= \frac{1}{2}((5 - -61) + (5 - -62)) && (5\text{–}63) \\ &= \frac{1}{2}\Sigma_n \int_{I_n} dx \langle p'', q''|\mathbb{E}||x\rangle [p'' + p' + i(q'' - q')] \langle x|\mathbb{E}|p', q'\rangle && (5\text{–}64) \\ &= \frac{[p'' + p' + i(q'' - q')]}{2} \tilde{\mathcal{K}}(p'', q''; p', q'). && (5\text{–}65) \end{aligned}$$

Finally, if we so choose, we allow only the gauge independent matrix elements by hand selecting the portions of the reproducing kernel that correspond to the irregular constraints,

$$\langle p'', q''|P^E|p', q'\rangle = \frac{[p'' + p' + i(q'' - q')]}{2} \tilde{\mathcal{K}}'(p'', q''; p', q') \quad (5\text{–}66)$$

where $\tilde{\mathcal{K}}'$ is the reduced reproducing kernel except for the component corresponding to $\tilde{\mathcal{K}}_1$. In chapter 6, we will further discuss the concept of observables and the classical limit of quantum mechanical observables in the Ashtekar-Horowitz-Boulware model.

5.4 Observation and Conclusions

In this chapter we have defined and examined the classification of *highly irregular constraints*. In the following chapter we will examine the Ashteker-Horowitz-Boulware model and apply the techniques developed here in this more complicated model.

CHAPTER 6
ASHTEKAR-HOROWITZ-BOULWARE MODEL

The primary motivation of this chapter is to analyze the Ashtekar-Horowitz-Boulware model utilizing the Projection Operator Formalism. We will also compare the result obtained by the POF approach with that obtained via methods of the Refined Algebraic Quantization program. The basis of this chapter can be found in [10].

6.1 Introduction

The Ashtekar-Horowitz model [11] was formulated to mimic a particular property of the Hamiltonian constraint of General Relativity. In this simple model the constraint of the Hamiltonian system was such that the classical constraint subspace did not project down to all of the configuration space. Using the methods described by Dirac [7], the constraint of this simple quantum mechanical system was imposed. It was argued that by requiring the additional condition of normalization of the constraint solutions, there is quantum mechanical tunneling into classically forbidden regions. This model was originally formulated with the configuration space of a sphere [11].

Later, Boulware modified the constraint problem by noting the curvature of the configuration space plays no role in the analysis and altered the configuration space to a torus - a compact yet globally flat configuration space [44]. In the quantization of the modified model, the additional requirement of the self-adjoint property was imposed on the canonical momentum. Using this additional criterion, it was shown that no tunneling would occur into the classically forbidden regions for the physical states.

Recently, Louko and Molgudo investigated this model using techniques of the refined algebraic quantization program (RAQ) to determine its physical Hilbert space structure [32]. The methods they employed led to the existence of super-selection sectors in the physical Hilbert space. The basic formalism of RAQ is unable to determine a rigging map for a constraint that has both regular and irregular solutions. Modifications were made in definition of the rigging map to accommodate for the variety of solutions, (i.e.r solutions of

the constraint equations that are stationary (critical) points of the arbitrary function $R(y)$ in the constraint). These modifications resulted in the advent of super-selection sectors in this model.

Using the projection operator formalism [13], we are able to ascertain the physical Hilbert space of the Ashtekar-Horowitz-Boulware (AHB) model with techniques which we feel are closer to the essence of the Dirac procedure [7] than those in [32]. The physical Hilbert space of this model is shown not to decompose into super-selection sectors. We are inclined to take the point of view that super-selection sectors are based on physical principles not pure mathematics. The approach in which we ultimately employ is a similarity transformation. Physics is invariant under similarity transformations. We should also note the two methods (Projection Operator vs. RAQ) are not equivalent. We were able to generalize to a class of functions (i.e. functions that have interval solutions to constraint equation) that the previous work [32] can not analyze without further modifications. The previous work [9] serves as a guide for this present endeavor.

This chapter is organized as follows: Section 2 provides a brief introduction to the classical AHB model. Section 3 presents the canonical quantization of the model. Section 4 deals with constructing the physical Hilbert space using the projection operator formalism. Section 5 deals with defining super-selection sectors and determining whether or not the Physical Hilbert space obtained in Section 5 contains super-selection sectors. Section 6 deals with the classical limit of the constrained quantum theory and establishes that the classical limit is the classical theory of the original model. Section 7 contains an account of the RAQ approach to this model.

6.2 Classical Theory

The classical system of the Ashtekar-Horowitz-Boulware model is given by the following action,

$$I = \int (p_x \dot{x} + p_y \dot{y} - \lambda C) dt, \tag{6–1}$$

where λ is a Lagrange multiplier corresponding to the constraint C. The configuration space of the AHB model is $\mathcal{C} = \mathbb{T}^2 \simeq S^1 \times S^1$. The constraint has the following form

$$C \equiv p_x^2 - R(y), \tag{6–2}$$

where the function $R(y) \in C^1(S^1)$ is assumed to be positive somewhere. When the constraint equation is satisfied the classical solutions are limited to the regions of the configuration space where $R(y) \geq 0$. The constraint region in the 4 dimensional phase space will involve a proper subset of configuration space. Note that the Hamiltonian equals zero in this model to emphasize the role of the constraint.

The dynamics of this system are given by the following 5 equations of motion.

$$\begin{aligned} \dot{x} &= -2\lambda p_x, & \dot{y} &= 0, \\ \dot{p}_x &= 0, & \dot{p}_y &= \lambda \frac{dR(y)}{dy}, \\ & p_x^2 - R(y) = 0. \end{aligned}$$

From these equations of motion, we can make some statements on some observability properties of this theory. The dynamical variable x is gauge dependent for all p_x except for $p_x = 0$. The conjugate momentum of y also appears to be gauge dependent if the constraints are regular around a given set of y that satisfies the constraint equation in the phase space.

If y_0 satisfies the constraint equation and $\frac{dR(y)}{dy}|_{y=y_0} = 0$, then the constraint is an example of irregular constraint about $y = y_0$, whereas if $\frac{dR(y)}{dy}|_{y=y_0} \neq 0$, then the constraint is regular about y_0. If there are multiple solutions to the constraint equation $p_x^2 - R(y) = 0$ then we may have a condition where combinations of regular constraints and irregular constraints, this is the characteristic of a "highly irregular" constraint. For the most general analysis, we can then assume that the constraint equation contains solutions that

are both regular and irregular. In this general setting we can classify the AHB constraint as a highly irregular constraint.

6.3 Quantum Dynamics

We now proceed to canonically quantize the system (6–1). We will assume our chosen canonical coordinates are Cartesian ones suitable for quantization [20]. We then promote the canonical dynamical variables (x, y, p_x, p_y) to a set of irreducible self-adjoint operators (X, Y, P_x, P_y). Conjugate pairs corresponding to compact, periodic spatial components will not obey the standard Heisenberg-Weyl relationship [45] because the eigenvalues of the conjugate momentum operators are not continuous but discrete. Continuing with the canonical quantization procedure, we promote the constraint to a suitable function of self-adjoint operators

$$C \mapsto \hat{C} = P_x^2 - R(Y). \tag{6–3}$$

Note, there is no ordering ambiguity for this operator. We assume the constraint operator is a self-adjoint operator in the unconstrained Hilbert space. We can now implement the quantum constraint using the projection operator method.

6.4 The Physical Hilbert Space via the Reproducing Kernel

The projection operator for the Ashtekar-Horowitz-Boulware model is chosen to be

$$\mathbb{E}(\hat{C}^2 \leq \delta^2) = \mathbb{E}(-\delta \leq \hat{C} \leq \delta) = \mathbb{E}(-\delta < \hat{C} < \delta). \tag{6–4}$$

Since the function $R(y)$ is a continuous function, we must introduce an appropriate set of bras and kets to deal with the subtleties described in section 2.

6.4.1 The Torus $\mathbb{T}^2$

Before constructing the model with the configuration space of a torus we must determine the correct coherent states to use. We wish to use the coherent states not only for computational ease, but also to determine the classical limit, which will be addressed later in this chapter. The torus is the Cartesian product of 2 circles. It follows that the

coherent states for the unconstrained Hilbert space can be written as the direct product of 2 coherent states on different circles.

Coherent states on a circle can be generated by coherent states of a line with the use of the Weil-Berezin-Zak (WBZ) transformation [45]. We shall use X and Y to denote the characteristic lengths of the x and y coordinates, respectively. The WBZ transform, T, is a unitary map from $L^2(\mathbb{R})$ to $L^2(S^1 \times S^{1*})$, where S^{1*} is the dual to S^1. The transformation is given by the following

$$(T\psi)(x,k) \equiv \Sigma_{n\in\mathbb{Z}} e^{in\mathsf{X}k}\psi(x - n\mathsf{X}) \tag{6–5}$$

where $\psi \in L^2(\mathbb{R})$, $x \in S^1$, and $k \in S^{1*}$ or stated otherwise $k \in [0, \frac{2\pi}{\mathsf{X}})$. We project a corresponding fiber of $L^2(S^1 \times S^{1*})$ onto $L^2(S^1)$ by fixing a value of k. Using the standard canonical coherent states in $L^2(\mathbb{R})$, it has been shown the coherent states on a circle have the following form ($\hbar = 1$)

$$\begin{aligned} \eta^{(k)}_{x,p}(x') &= \frac{1}{\pi^{1/4}} \exp(\frac{1}{2}p(x+ip)) \exp(-\frac{1}{2}(x+ip-x')^2)\Theta(i\frac{\mathsf{X}}{2}(x+ip-x'-ik);\rho_1), \\ &\equiv \langle x'|x,p,k\rangle \end{aligned} \tag{6–6}$$

where $\rho_1 = \exp(-\frac{\mathsf{X}^2}{2})$ and

$$\Theta(z) = \Sigma_{n\in\mathbb{Z}} \rho^{n^2} e^{2inz}, \tag{6–7}$$

$|\rho| < 1$, is the Jacobi theta function. These states are not normalized [45]. For each value of k these states satisfy the minimal axioms of generalized coherent states; i.e., a continuous labeling of the states where the label set has a topology isomorphic to $\mathbb{R}^2$ and a resolution of unity [40].

We can express the coherent states on $\mathbb{T}^2$ as the following,

$$|x,p_x,k_x;y,p_y,k_y\rangle = |x,p_x,k_x\rangle \otimes |y,p_y,k_y\rangle, \tag{6–8}$$

where x, $y \in S^1$ and $k_x \in S_x^{1*}$,and $k_y \in S_y^{1*}$. For simplification we will choose X, $\mathsf{Y} = 2\pi$. We will make a further simplification by choosing a value for k_x and k_y. We justify such a choice by noting that the spectrum of the momentum operator is shifted from the expected value by k [45], effectively, we can set the new ground state at k. Therefore, we can safely choose zero for both k_x and k_y. Thus we will make the following notational change

$$|x, p_x; y, p_y, \rangle_0 = |x, p_x, \rangle_0 \otimes |y, p_y\rangle_0, \tag{6–9}$$

The construction of the reproducing kernel is based on properties of the constraint operator as well as the coherent states (6–9). The constraint operator and the compactness of x restrict the spectrum of its conjugate momentum P_x and thereby of $R(Y)$. Allowed values of y are determined by the following equation

$$_0\langle n|R(Y)|n\rangle_0 =_0 \langle n|(n)^2|n\rangle_0 = n^2 \quad n \in \mathbb{Z}. \tag{6–10}$$

where $|n\rangle_0$ is the orthonormal basis for $L^2(S^1)$. We will proceed with the quantization of this model by implementing the method discussed in Section 2 for each n sector of the theory. Since we are not choosing a particular $R(y)$, we will only be discussing the physical Hilbert space in general. We consider the following two types of solutions to the constraint equation.

I.) (Point Solutions) The solution $y = y_m$ is a point value solution to the equation (6–10) for a given value of n. The index m corresponds to multiple values of the y that satisfies the equation for a given value of n.

II.) (Interval Solutions) The solutions $y = y_{m'}$ satisfy the equation (6–10) for all elements in an interval $I(m')$. This classification of solutions also includes a countable union of disjoint intervals. Although physically motivated models exclude such constraint solutions, we include them to illustrate the versatility of our approach

$$\{y_{m'}\} = \{y_{m'}|R(y_{m'}) = (n)^2 \quad \forall y_{m'} \in I_{m'}\}. \tag{6–11}$$

For simplification, we will only assume that R will only contain the first type of solution. We point the interested reader to the previous chapter or [9] to determine the physical Hilbert space contribution for type II solutions. The calculation of the reproducing kernel can be decomposed into portions corresponding to each value $n \in \mathbb{Z}$ in the following manner:

$$
\begin{aligned}
& {}_0\langle x', p'_x; y', p'_y | \mathbb{E}(-\delta \leq R(Y) - P_x^2 \leq \delta) | x, p_x; y, p_y \rangle_0 \\
= \;& \Sigma_{n=-\infty}^{\infty} {}_0\langle x', p'_x; y', p'_y | \mathbb{E}(P_x = n) \mathbb{E}(-\delta \leq R(Y) - P_x^2 \leq \delta) | x, p_x; y, p_y \rangle_0 \\
= \;& \Sigma_{n=-\infty}^{\infty} \langle x', p'_x; y', p'_y || \mathbb{E}(-\delta \leq R(y) - n^2 \leq \delta) | x, p_x; y, p_y \rangle_0 \\
= \;& \Sigma_{n=-\infty}^{\infty} {}_0\langle x', p'_x; y', p'_y | \mathbb{E}(-\delta \leq R(Y) - n^2 \leq \delta) | x, p_x; y, p_y \rangle_0 .
\end{aligned}
$$

To determine the point solution contribution, we fix a value for n and proceed as follows

$$
\begin{aligned}
& \mathcal{K}_m(x', p'_x, y', p'_y, ; x, p_x, ; y, p_y)|_{n=\text{constant}} \\
= \;& \int dy' \int dy_0 \langle x', p'_x; y', p'_y | y' \rangle \langle y' | \mathbb{E}(-\delta \leq R(Y) - n^2 \leq \delta) | y \rangle \langle y || x, p_x; y, p_y \rangle_0 . \\
= \;& \int dy' \int dy_0 \langle x', p'_x; y', p'_y | \mathbb{E}(-\delta \leq R(y) - n^2 \leq \delta) || x, p_x; y, p_y \rangle_0 . \delta(y' - y) \\
= \;& \int_{y_m - 1/S_m(\delta)}^{y_m + 1/S_m(\delta)} dy'' \langle y', p'_y | y'' \rangle \langle y'' | y, p_y \rangle \langle x', p'_x | n, k_x \rangle \langle n | x, p_x \rangle ,
\end{aligned}
$$

where $1/S_m(\delta)$ is the leading δ dependency as described in Section 2. For small $1/S_m(\delta)$ values, the integral can be approximated as follows

$$
\begin{aligned}
& \mathcal{K}_m(x', p'_x, y', p'_y, ; x, p_x, ; y, p_y,)|_{n=\text{constant}} \\
= \; & \frac{2\sin(1/S_m(\delta)(y-y'))}{2\pi(y-y')} e^{i(p_x x - p'_x x')/2 + p_x'^{2/2 + p_x^2/2}} \\
\times \; & \exp[-(y_m - y)^2/2 - i y_m(p'_y - p_y) - (y_m - y')^2/2] \\
\times \; & \Theta^*(i\pi(y' + i p'_y - y_n^m; \rho_1)\Theta(i\pi(y + i p_y - y_n^m; \rho_1) \\
\times \; & \exp[-n^2]\exp[in((x'-x) + i(-p'_x - p_x)], \qquad (6\text{–}12)
\end{aligned}
$$

where $\rho_1 = \exp[-2\pi^2]$. Following the prescription set forth in Section 2, we perform the required similarity transformation to extract the leading δ dependency of the reproducing kernel.

$$
\begin{aligned}
\hat{\mathcal{K}}_{m\,n=\text{constant}} = \; & S_m(\delta)\frac{2\sin(1/S_m(\delta)(y-y'))}{2\pi(y-y')} e^{i(p_x x - p'_x x')/2 + p_x'^2/2 + p_x^2/2} \\
\times \; & \exp[-(y_m - y)^2/2 - i y_m(p'_y - p_y) - (y_m - y')^2/2] \\
\times \; & \Theta^*(i\pi(y' + i p'_y - y_n^m; \rho_1)\Theta(i\pi(y + i p_y - y_n^m; \rho_1) \\
\times \; & \exp[-n^2]\exp[in((x'-x) + i(-p'_x - p_x)], \qquad (6\text{–}13)
\end{aligned}
$$

The limit $\delta \to 0$ can now be taken in a suitable manner to determine the reduced reproducing kernel for this portion of the physical Hilbert space [13] which then reads

$$
\begin{aligned}
\tilde{\mathcal{K}}_{m\,n=\text{constant}} = \; & \frac{1}{\pi} e^{i(p_x x - p'_x x')/2 + -(p'_x)^2/2 + (p_x)^2/2} \\
\times \; & \Sigma_m \exp[-(y_n^m - y)^2/2 - i y_n^m(p'_y - p_y) - (y_n^m - y')^2/2] \\
\times \; & \Theta^*(i\pi(y' + i p'_y - y_m; \rho_1)\Theta(i\pi(y + i p_y - y_m; \rho_1) \\
\times \; & \exp[-n^2]\exp[2in\pi((x'-x) + i(p'_x - p_x))] \qquad (6\text{–}14)
\end{aligned}
$$

for each value of m. Each of these reduced reproducing kernel Hilbert spaces is isomorphic to a one-dimensional Hilbert space (i.e. $\tilde{\mathcal{H}} \approx \mathbb{C}$). We continue the procedure for each whole number value of n until the maximum allowed value (of n) is reached.

Once this calculation is performed for all values of n, consistent with (6–10), then we can write the reproducing kernel for the physical Hilbert space in the following manner

$$\tilde{\mathcal{K}} = \Sigma_n^{n_{max}} \Sigma_m \tilde{\mathcal{K}}_{nm} \tag{6–15}$$

Similarly, the physical Hilbert space can be written as

$$\mathcal{H}_{phys} = \bigoplus_n^{n_{max}} \bigoplus_m \tilde{\mathcal{H}}_{nm}. \tag{6–16}$$

The support of the reproducing kernel is only in the classically allowed regions. This implies there is no tunneling into classically forbidden regions as reported by Boulware [44].

6.5 Super-selection Sectors?

Before determining whether or not the physical Hilbert space calculated in the preceding section contains super-selection sectors, let us first divert the question and discuss what is formally meant by super-selection sectors. Suppose a physical Hilbert space is given by the following,

$$\mathcal{H}_{phys} = \bigoplus_i \mathcal{H}_i, \tag{6–17}$$

which is to say that the physical Hilbert space is given by the direct sum of individual Hilbert spaces. The physical Hilbert space is said to decompose into super-selection sectors if for any two states $|\phi_1\rangle, |\phi_2\rangle$ that belong to two different sectors $\mathcal{H}_i$ and $\mathcal{H}_j$, respectively, and for any observable O in $\mathcal{A}_{obs}$, where $\mathcal{A}_{obs}$ the *-algebra of all observables, the following holds,

$${}_{phys}\langle\phi_i|O|\phi_j\rangle_{phys} = 0. \tag{6–18}$$

In (6–18), O denotes a generic self-adjoint operator in the unreduced Hilbert space. In previous works using the RAQ procedure [31], super-selection sectors arose because each sector had a different degree of divergence. Since O is a self-adjoint operator in the unreduced space, (6–18) is forced to vanish to avoid a contradiction from the varying degrees of divergency [46]. As we have shown in the preceeding section the physical

Hilbert space determined via the projection operator method can be written as the direct sum of n_{max} one-dimensional Hilbert spaces. It follows that the physical Hilbert space is isomorphic to the direct sum of n_{max} copies of the complex numbers. There the projection operator is merely the unit matrix on a finite dimensional space and thus observables correspond to general Hermitian matrices. Therefore, (6–18) will only hold if and only if the operator is proportional to the projection operator. In general (6–18), does not hold, therefore the phyical Hilbert space (6–17) does not decompose into super-selection sectors. We will now discuss the classical limit of the quantized AHB model.

6.6 Classical Limit

We must recall the general rule given by diagonal coherent state matrix elements

$$\langle p, q|\mathcal{O}|p, q\rangle = O(p, q; \hbar), \tag{6–19}$$

where $|p, q\rangle$ are canonical coherent states. This provides the connection between an operator $O(P, Q)$ and an associated function on the classical phase space manifold. In the limit, $\hbar \to 0$, we find this function reduces to the classical function that corresponds to the weak correspondence of quantum operator. This statement can easily be seen if $\mathcal{O}$ is a polynomial, however, this condition is not necessary. This result can be generalized to any number of phase space variables as will be demonstrated below.

Before evaluating the classical limit of the model, we must discuss the fundamental difference between quantum mechanics on a compact configuration space and that of an unbounded space. The conjugate momentum operator (P_x) has a discrete spectrum if the configuration space is compact. Therefore the standard canonical commutation relation

$$[X, P_x] = i\hbar, \tag{6–20}$$

is inappropriate. To alleviate this problem we consider the "angle" operator [45]

$$U_x = \exp(i\hbar x). \tag{6–21}$$

This unitary operator acts to translate the operator P_x in the following manner

$$U_x P_x U_x^\dagger = P_x - \hbar. \tag{6–22}$$

As observed in [9], the observable part of an operator can always be expressed as

$$\mathcal{O}^E = \mathbb{E}\mathcal{O}\mathbb{E}, \tag{6–23}$$

where $\mathcal{O}$ is a self-adjoint operator in the unconstrained Hilbert space.

The observable part of the Hermitian combination of U_x and $U_x^\dagger$ is

$$W_x = U_x^E U_x^{\dagger E} = \mathbb{E}U_x\mathbb{E}U_x^\dagger\mathbb{E}. \tag{6–24}$$

By observation, we note

$$W_x = \mathbb{E}(-\delta < P_x^2 - R(\mathbf{Y}) < \delta)\mathbb{E}(-\delta < (P_x - \hbar)^2 - R(\mathbf{Y}) < \delta). \tag{6–25}$$

These projection operators are acting on mutually orthogonal subspaces; therefore, the operator is identically zero. This result informs us that this is a gauge dependent question which is consistent with the classical picture. Recall from Section 3 the x dynamical variable is gauge independent only when $p_x = 0$. Quantum mechanically, we have posed the question to find a "physical" wave function that has support on both a gauge independent sector and gauge dependent sector. This is impossible.

If we were to examine the same query for the corresponding Hermitian combination of the "angle" operator for the Y coordinate, we would obtain the unit operator. The classical limit of this operator is again in complete agreement with the classical theory. As we have previously observed the classical dynamical variable y is always gauge independent.

Now we consider the following quotient to establish the classical limit of the Y "angle operator" U_y

$$\frac{\langle x, p_x; y, p_y, |\mathbb{E}U_y\mathbb{E}|x, p_x; y, p_y\rangle}{\langle x, p_x, ; y, p_y, |\mathbb{E}|x, p_x, ; y, p_y, \rangle}$$

$$= \exp[iy - \frac{\hbar}{2\pi}^{2}]\frac{\Theta(\frac{\pi}{\hbar}(p_y) - \frac{\pi}{2}); \exp(-\pi)^2/(\hbar))}{\Theta(\frac{\pi}{\hbar}(p_y - \hbar); \exp(-\pi^2/\hbar))}. \tag{6–26}$$

As $\hbar \to 0$ this expression becomes

$$\exp[i2y], \tag{6–27}$$

where y is subject to the condition $R(y) = p_x^2$. While this expression is imaginary, we can extract from it the classical reduced phase space coordinate y.

Now we direct our attention to the expectation value of the physical conjugate momentum, P_x

$$\frac{\langle x, p_x; y, p_y | \mathbb{E} P_x \mathbb{E} | x, p_x; y, p_y \rangle}{\langle x, p_x; y, p_y | \mathbb{E} | x, p_x; y, p_y \rangle}$$

$$= \frac{-i\hbar}{\tilde{\mathcal{K}}} \int dy' \int dx' \langle x, p_x; y, p_y | x', y' \rangle \frac{\partial}{\partial x'} \langle x', y' | x, p_x; y, p_y \rangle. \tag{6–28}$$

We implement the constraints by integrating over the appropriate intervals as described in Section 5. We can continue this calculation in a similar manner to that which is performed in [9].

$$\frac{\langle x, p_x; y, p_y | \mathbb{E} P_x \mathbb{E} | x, p_x; y, p_y \rangle}{\langle x, p_x; y, p_y | \mathbb{E} | x, p_x, ; y, p_y, \rangle} = p_x + \frac{\Theta'(\frac{\pi}{\hbar} p_x; \exp(-\pi^2/(\hbar)))}{2\Theta(\frac{\pi}{\hbar}(p_x \hbar); \exp(-\pi^2/(\hbar)))}, \tag{6–29}$$

where

$$\Theta'(z; \rho) = 2i \sum_{n=-\infty}^{\infty} n \rho^{n^2} e^{inz} e2inz. \tag{6–30}$$

As $\hbar$ approaches 0, the second term vanishes which can be seen in the definition of the Jacobi theta function (6–7) [45], thus recovering this aspect of the classical theory from its quantum analog. Using the same technique, we can also calculate the classical limit of the expectation value of the P_y operator. The projection operator formalism is well suited to not only properly impose quantum constraints, but also allow one to return to the proper classical theory in the limit $\hbar \to 0$.

6.7 Refined Algebraic Quantization Approach

Before proceeding with the Refined Algebraic Quantization (RAQ) of this model, we must first impose some additional technical issues on the constraint (6–2). First we must assume the constraint (6–2) contains a finite number of zeros and that all stationary points (i.e. $R'(y) = 0$, $R^n(y) = 0$, [i.e. the nth dervative of R with respect to y]) only to have a finite order that no zeros of (6–2) are to be stationary points. As with the analysis in the preceding sections we must also require that $R(y)$ be positive at least somewhere.

Following the program described in Chapter 3 we must first choose an auxiliary Hilbert space, $\mathcal{H}_{aux}$. The auxiliary Hilbert space of choice is the Hilbert space of square-integrable complex functions over the configuration space. The canonical inner-product is given by the following;

$$(\phi_1, \phi_2)_{aux} = \int\int dxdy\phi_1^*(x,y)\phi_2(x,y), \tag{6–31}$$

where $(\cdot)^*$ denotes complex conjugation. The classical constraint is promoted to an operator that acts on the auxiliary Hilbert space,

$$\hat{C} = -\frac{\partial^2}{\partial x^2} - R(Y), \tag{6–32}$$

where $R(Y)$ acts as a multiplication operator namely, $R(Y)\phi(x,y) = R(y)\phi(x,y)$ for all $\phi(x,y) \in \mathcal{H}_{aux}$. The operator, $\hat{C}$ is an essentially self-adjoint operator on $\mathcal{H}_{aux}$, therefore the operator will exponentiate to the one parameter unitary operator via Stone's theorem,

$$U(t) = e^{-it\hat{C}} \qquad\qquad t \in \mathbb{R}. \tag{6–33}$$

Keeping in line with the RAQ program, we must now choose a test space $\Phi \subset \mathcal{H}_{aux}$. In this model the convenient choice is the set of functions of the form;

$$f(x,y) = \Sigma_{m\in\mathbb{Z}} e^{imx} f_m(y), \tag{6–34}$$

where $f_m(y) : S^1 \to \mathbb{C}$, and only a finite number of f_m's are different from zero for all $f \in \Phi$. It is clear that this set is a dense set in the space of $\mathcal{H}_{aux}$. By the definition of $f \in \Phi$ the action under the unitary operator[1] (6–33) is as follows,

$$U(t)f(x,y) = \Sigma_{m\in\mathbb{Z}} e^{-it(m^2-R(y))} e^{imx} f_m(y) \in \Phi, \tag{6–35}$$

therefore by this calculation Φ is invariant under the action of $U(t)$. One further comment must be made before proceeding with the rest of the procedure, if $\mathcal{O} \in \mathcal{A}_{obs}$ then $\mathcal{O}$ commutes with $U(t)$ and is densely defined in Φ. The final phase of the RAQ procedure is to determine the anti-linear rigging map via the group average map,

$$\eta : \phi \mapsto \int_{-\infty}^{\infty} \phi^* U(t) dt \tag{6–36}$$

or equivalently we can discuss the map through the matrix elements [30]

$$\eta(\phi_1)[\phi_2] = \int_{-\infty}^{\infty} (\phi_1, U(t)\phi_2)_{aux} dt. \tag{6–37}$$

At this point a deviation from the standard RAQ approach is required. [46] Since (6–37) is not absolutely convergent, this is due to the fact the gauge group generated by $U(t)$ is a non-compact group. Formally, it was established [32] that the rigging map could be written as the following equation,

$$\eta(f)(x,y) = 2\pi \Sigma_{m\in\mathbb{Z}} e^{-imx} f_m^*(y) \delta(m^2 - R(y)) \tag{6–38}$$

or equivalently,

$$\eta(f)(x,y) = 2\pi \Sigma_{mj} \frac{e^{-imx} f_m^*(y)}{|R'(y_{mj}|} \delta(y, y_{mj}) \tag{6–39}$$

where y_{mj}s are solutions to

$$m^2 = R(y), \tag{6–40}$$

[1] The operator $\hat{C}$, as well as $U(t)$, is densely defined in Φ.

and the delta functions in (6–38) and (6–39) are the delta functions for $\mathbb{R}$ and S^1, respectively. Assuming that (6–40) has solutions then it was shown in [32] that (6–39) does satisfy the axioms of the rigging map. A key component of the verification of the aforementioned axioms is that η induces a representation of $\mathcal{A}_{obs}$ on the physical Hilbert space. This can be stated in terms of the matrix elements

$$\eta(A\phi_1)[\phi_2] = \eta(\phi_1)[A^\dagger\phi_2], \tag{6–41}$$

for all $\phi_1, \phi_2 \in \Phi$ and $A \in \mathcal{A}_{obs}$. It can be shown [32] that the representation of $\mathcal{A}_{obs}$ on $\mathcal{H}_{RAQ}$ is irreducible and is transitive.

In the preceding discussion the physical Hilbert space did not decompose into super-selection sectors. The advent of super-selection sectors appears to be a direct result of relaxing the condition to allow for solutions of (6–2) to include stationary points. With this relaxed condition, a further modification of the rigging map is required to avoid divergences in (6–39). This is accomplished by replacing the denominator with fractional powers of higher derivatives, which depends on the order of the stationary point. The replacement of the denominator can be thought of as a renormalization of the averaging procedure. Each of these renormalized rigging maps can be shown [32] to carry a transitive representation of $\mathcal{A}_{obs}$. The total Hilbert space $\mathcal{H}^{tot}_{RAQ}$ can be regarded as the direct sum of individual Hilbert spaces. The representation of $\mathcal{A}_{obs}$ also decomposes into the representation of the summands. Which in turn implies the presence of super-selection sectors in $\mathcal{H}^{tot}_{RAQ}$.

6.8 Commentary and Discussion

As a direct result of the gauge group being non-compact a deviation from the standard RAQ method was required to deal with this particular model. Instead of the standard group averaging approach, the author chose to pursue a formally equivalent rigging map. The rigging map (6–39) has definitive connection to the reduced phase space method for quantization. The authors of [32] also comment on the close connection to

the classical reduced phase space, by noting that the super-selection sectors are related to the classical singularities in the classical phase space. This dependence on the classical regime to determine quantum behavior is rather disturbing. This dependency should be reversed. The main difference in the RAQ procedure and the Projection Operator method is when $\delta(\hbar)$ is taken to zero. In the Projection Operator method the limit is taken after the evaluation of the matrix elements, while the RAQ method requires the limit to be taken before the evaluation. As we have demonstrated in this chapter, these two methods are related, however their implementation is different and the results obtained in this particular type of model are not the same.

CHAPTER 7
PROBLEM WITH TIME

"... beyond all day-to-day problems in physics, in the profound issues of principle that confront us today, no difficulties are more central than those associated with the concept of time ..." - John Archibald Wheeler

Time is a crucial element to any dynamical system; it is the evolution parameter of such a system. The nature of time is an extremely popular topic covered by many physicists, as well as, philosophers [22]. While the physical (or meta-physical) nature of time is outside the main focus of this dissertation, time-dependency in quantum mechanics offers us an interesting caveat to explore and study.

In the methodologies developed and discussed in the previous chapters, the primary goal was to solve quantum mechanical time-independent constraints. The exclusion of time was made primarily out of simplification. In most of the literature about constraints [12] the topic of time-dependent constraints is either briefly covered or it is not covered at all. However, it is clear that for a more complete discussion of constraint dynamics we must also include constraints that are explicitly dependent on time. Time dependence can enter a dynamical system through the Hamilitonian, constraints, or in the most general case or combination of the two. The inclusion of explicitly time-dependent constraints offers not only an interesting academic exercise but also gives physicists the tools required to examine more physical theories than those that previously could be discussed.

This is not the first occasion on which the projection operator has been used to deal with the case of time-dependent constraints. In [47], Klauder derived an expression for evolution operator of time-dependent constraint. The construction of this expression was based on modifying the expression for the time-independent case. Although this expression seemed to be correct the author chose not to pursue this subject matter further. Primarily, he made this choice because the formula did not reduce to a simpler operator expression. While the projection operator will be the primary mode of exploration

throughout this project, the expression derived in [47] will not be the starting point for our investigation.

We will, however, advocate the use of the reparameterization invariant description to discuss systems with time-dependent constraints. We should mention that this starting point is not a new approach to deal with time-dependent constraints. We will alter past efforts on this topic by exploring the "non-local" point of view. The phrase "non-local" point of view was coined by Gitman [48] when describing a physical system in which one assumes a reparameterization invariant form of a theory. However, it is well known, that if an action is a reparameterization invariant then the Hamiltonian vanishes on the constraint surface[1] . Physics described in a reparameterization invariant form is not dependent on the frame of reference [48]. We should mention that using the reparameterization invariant approach is not new, however, the implementation of this symmetry with the projection operator would seem to be new.

In the next two chapters we will discuss some of the facets of the problem with time-dependent constraints. In Chapter 8 we will motivate and develop the techniques in which one can study constraints with an explicit time-dependent feature. We will also give a brief introduction to an alternative to the projection operator, which is the approach used by Gitman, [12] and compare and contrast the two approaches. The primary goal of Chapter 9 is to implement the formalism developed in Chapter 8, in a few examples of time-dependent constraints.

[1] See Appendix for this result.

CHAPTER 8
TIME DEPENDENT CONSTRAINTS

8.1 Classical Consideration

8.1.1 Basic Model

Our discussion will begin considering a classical regular system with a single degree of freedom, whose canonical variables are named p and q. Such a system can generally be described by the action functional.

$$I = \int_{t_1}^{t_2} (p\dot{q} - H(p,q))dt \tag{8–1}$$

where $\dot{q} = dq/dt$ and $H(p,q)$ is the Hamiltonian of the system. The evolution of the system is obtained by varying the functional with respect to the dynamical variables, this reads as,

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{8–2}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{8–3}$$

subject to the suitable boundary conditions. As stated above, this system is purely dynamical, however, it is well known any action can be converted into an equivalent action that is a reparameterization invariant. [48] Let us begin this conversion by promoting the dependent parameter t to a dynamical variable. This is appealing from a relativist's perspective because the spatial and temporary coordinates are treated symmetrically. We also must introduce the formal momentum p_t conjugate to t. The integration variable in (8–1) is now replaced by a new independent parameter τ, which corresponds to proper time or a more general function of time [1] . We can express the reparameterization

[1] Proper time is the time seen by an observer in the rest frame of a system.[6] However, for our purposes we can consider the Lagrange multiplier are not strictly increasing see Appendix

invariant form of the action integral in the following manner.

$$I' = \int_{\tau_1}^{\tau_2} (p_t t^* + pq^* - \lambda(\tau)[p_t + H(p,q)])d\tau \tag{8–4}$$

where $(\cdot)^*$ denotes the derivative with respect to τ. The price paid in promoting t to a dynamical variable is that the Hamiltonian vanishes weakly in the extended phase space. We have identified the primary unexpressable velocity t^* as the Lagrange multiplier that enforces the (first-class) constraint $p_t + H(p,q) = 0$. Therefore, we have turned a theory that was a dynamical system into one that is purely gauge. Effectively, we have recast the original theory in such a manner that it can be related in any temporary reference frame[48]. The equations of motion of (8–4) are as follows:

$$\frac{dq}{d\tau} = \lambda \frac{\partial H}{\partial p}, \qquad \frac{dp}{d\tau} = -\lambda \frac{\partial H}{\partial q} \tag{8–5}$$

$$\frac{dt}{d\tau} = \lambda(\tau), \qquad \frac{dp_t}{d\tau} = 0 \tag{8–6}$$

$$p_t + H(p,q) = 0 \tag{8–7}$$

As the equation of motion appear above q, momentum p, and the physical time t, measured are gauge dependent quantities. However, p_t is gauge independent and therefore an observable quantity in this theory. By identifying the usual time as the gauge dependent quantity

$$dt = \lambda(\tau)d\tau \tag{8–8}$$

we can quickly reduce the preceding equations of motion (8–4) to the familiar parameterized form.

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{8–9}$$

$$\dot{p} = -\frac{\partial H}{\partial q}, \tag{8–10}$$

The dynamics of this system arises from imposing the constraints.

We can use the preceding discussion as motivation in the discussion of systems in which the Hamiltonian and the constraints are time dependent. Consider the classical action functional,

$$I = \int_{t_1}^{t_2} dt[p_j \dot{q}^j - H(p,q,t) - \lambda^a \phi_a(p,q,t)] \tag{8–11}$$

where $j \in \{1, ..., N\}$ and $a \in \{1, ..., A\}$. The dynamics this system is given by the familiar Hamiltonian's equations

$$\frac{dq^j}{dt} = \{q^j, H\} + \lambda^a \{q^j, \phi_a\} \tag{8–12}$$

$$\frac{dp_j}{dt} = \{p_j, H\} + \lambda^a \{p_j, \phi_a\} \tag{8–13}$$

$$\phi_a(p,q,t) = 0 \text{ for all } a \in \{1, .., A\} \tag{8–14}$$

At this point, the equations of motion are identical to those that appear in the time independent case. The distinction appears when we force the dynamics to lie on the constraint surface (i.e. subspace in the phase space defined by $\phi_a = 0$) for all time t.

$$\frac{d\phi_a}{dt} = \frac{\partial \phi_a}{\partial t} + \{\phi_a, H\} + \lambda^b \{\phi_a, \phi_b\} \approx 0 \tag{8–15}$$

where ≈ 0 implies (8–15) vanishes on the constraint surface. For simplicity, we will assume that the set of constraints are complete to all orders (e.g. secondary, tertiary, etc.) of constraints have been uncovered using the Dirac procedure [14]. The distinction between first and second class constraints is made based on the algebra of the Poisson brackets. However, we forgo this distinction for the moment for the sake of generality.

The cost of explicit time dependence in the constraints in (8–14) is the presence of the partial time derivative in equation (8–15). Despite the additional term in equation (8–15), we can maintain the usual structure of time-independent constraints, by following the procedure described in the preceeding subsection. As before we will promote t to a dynamical variable, and introduce its formal conjugate momentum p_t. By introducing additional dynamical variables and conjugate pairs to the phase-space, we also have

extended, in a natural manner, the symplectic two-form

$$\Omega = \omega + dt \wedge dp_t \tag{8–16}$$

where ω is the symplectic form of the original parameterized space ([17]), defined in Chapter 2. The Poisson bracket $\{\cdot,\cdot\}$ which is defined by the symplectic form, should be understood unless otherwise specified to be that of the extended space. The equivalent action can be written in the following manner:

$$I' = \int_{\tau_1}^{\tau_2} d\tau [p_j q^{j*} + p_t t^* - \lambda(p_t + H(p,q,t)) - \tilde{\lambda}^a \phi_a] \tag{8–17}$$

where $\tilde{\lambda}^a = \lambda(\tau)\lambda^a$ which is merely a redefinition of the Lagrange multiplier. Notice once again the canonical Hamiltonian vanishes. As in the previous section the dynamics of the system arises from implementing the constraints.

$$\frac{dq^j}{d\tau} = \lambda\{q^j, p_t + H\} + \tilde{\lambda}^a\{q^j, \phi_a\} \tag{8–18}$$

$$\frac{dp_j}{d\tau} = \lambda\{p_j, p_t + H\} + \tilde{\lambda}^a\{p_j, \phi_a\} \tag{8–19}$$

$$\phi_a(p,q,t) = 0 \text{ for all } \quad a \in \{1,..,A \tag{8–20}$$

$$\frac{d\phi_a}{d\tau} = \lambda\{\phi_a, p_t + H\} + \tilde{\lambda}^b\{\phi_a, \phi_b\} \approx 0 \tag{8–21}$$

$$\tag{8–22}$$

8.1.2 Commentary and Discussion

Inspired by reparameterization invariant theories [48] along with other models proposed by other authors [49], [50], we have arrived at a starting position to deal with time-dependent constraints. This was done by changing the dimension of the entire unconstrained phase-space from $\mathbb{R}^{2N}$ to $\mathbb{R}^{2N+2}$, which was accomplished by promoting t to a dynamical variable, and introducing its conjugate momentum p_t. The consequence of

this change of space[2] , is that the new Hamiltonian vanishes, and we have one additional constraint. However, constraints with explicit time dependence now pose the same mathematical structure of time-independent constraints in the Dirac procedure [12] with the aid of the extended symplectic form. We should also note that we have refrained from introducing a temporal gauge fixing term such as a chronological fixing gauge in our action. This is a point of divergence from the previous authors on the subject. As is well known, a gauge-fixing term has the potential to introduce topological obstructions that can cause difficulty in the analysis of the quantum system. This technique of introducing a gauge is used in quantization schemes such as Faddeev-Popov [27] which advocates reduction before quantization. Since one of the main philosophies of the projection operator formalism is to quantize the entire dynamical space and reduce second (i.e. eliminate the redundant variables), there is no need to introduce such a term in the action. Dirac observables are phase-space functions that commute weakly with all of the constraints. An extensive amount of literature has been devoted to the task of identifying observables in systems such as General Relativity and other generally covariant systems [51]. If o is a classical observable in a system with time-dependent constraint then the following must be true:

$$\frac{do}{d\tau} = \lambda\{o, p_t + H\} + \tilde{\lambda}^a\{o, \phi_a\} \approx 0 \tag{8–23}$$

where $\{\cdot, \cdot\}$ are understood to be the Poisson brackets for the extended space. Therefore, o is a constant of motion on the constraint surface in the extended phase-space, which implies that an observable is independent of a choice of reference frame or gauge. Since we will not make any further use of the concept of an observable in the discussion of systems with time-dependent constraints, we will defer this discussion to a future project. We

[2] An additional requirement of a global Cartesian coordinate system must be imposed when we proceed to the quantization of the described system ala Dirac

will now turn our attention to discussion of the quantum analysis of the afformentioned system.

8.2 Quantum Considerations

8.2.1 Gitman and Tyutin Prescription for Time-Dependent Second-Class Constraints

Before proceeding with the discussion of the projection operator formalism, we will briefly describe the method Gitman and Tyutin prescribed for dealing with time-dependent second-class constraints. [12] For simplicity, we will limit this discussion to include bosonic variables, however, one could extend any of the following arguments to include fermionic degrees of freedom as well. Also, for convenience we will use the notation used by the original authors, namely, $\eta = (q, p)$ which can explicitly depend on time, as well as, $\{\cdot,\cdot\}_{D(\phi)}$ represents the Dirac bracket with respect to a set of second-class constraints $\phi_a(\eta, t)$. The Dirac brackets are defined in the following manner

$$\{f, g\}_{D(\phi)} \equiv \{f, g\} - \{f, \phi_a\} C^{ab} \{\phi_b, g\} \tag{8–24}$$

where $\{\cdot,\cdot\}$ is the Poisson brackets, ϕ_a is a constraint, and C^{ab} is an invertible matrix [12]. Whenever encountered the Dirac bracket is taken assumed to be defined for the extended space, $(\eta; t, p_t)$, as described in the preceding section.

Consider a classical Hamiltonian system with a set of second-class constraints $\phi_a(\eta, t)$ and with a Hamiltonian $H(p, q, t)$. The Dirac brackets [7] are used to avoid having to solve the constraints. Therefore, the evolution of the canonical variables is given by

$$\frac{d\eta}{dt} = \dot{\eta} = \{\eta, H + p_t\}_{D(\phi)} \;\; \phi_a(\eta, t) = 0. \tag{8–25}$$

The quantization of the classical system, follows in the Schrödinger picture, in which the canonical variables η are assigned to operators η_S that satisfy the equi-time commutation relations;

$$[\eta_S, \eta'_S] = \frac{i}{\hbar}\{\eta, \eta'\}_{D(\phi)}, \qquad \Phi_a(\eta_S, t) = 0. \tag{8–26}$$

In the Schrödinger picture it is stated vectors evolve in time, where the time evolution is generated by a unitary operator. Operators in this picture are stationary, which implies the operators are time-independent. However, in this system the canonical operators η_S carry over an explicit time-dependence from their classical analogues, therefore these operators evolve in time. This is a departure from the traditional Schrödinger picture. In a later work, the authors recognize this distinction by calling this picture "rule" exist..

At this juncture we realize this current picture is unable to illustrate the full time evolution of the system. In order to fully obtain the time evolution we will move to a unitarily equivalent picture, the Heisenberg picture. In the Heisenberg picture of quantum mechanics, the state vectors remains fixed while the operators evolve in time. [38] In the Heisenberg picture the operators η_H are related to the operators η_S by $\eta_H = U^{-1}\eta_S U$, where U is the time evolution operator. The operator U is related to the Hamiltonian H_S by the differential equation,

$$\frac{\partial U}{\partial t} = \frac{-i}{\hbar} H_S U \tag{8–27}$$

We can evaluate the total time derivative of η_H by the following

$$\frac{d\eta_H}{dt} = \frac{d(U^{-1}\eta_S U)}{dt} \tag{8–28}$$

$$= U^{-1}(\frac{-i}{\hbar}[\eta_S, H_S] + \{\eta, p_t\}_{D(\Phi)})U \qquad \eta \mapsto \eta_S \tag{8–29}$$

Equation (8–29) establishes the connection between the quantum equations of motion and the classical equations of motion namely,

$$\frac{d\eta_H}{dt} = \{\eta, H + p_t\}_{D(\phi)} \qquad \eta \mapsto \eta_H \tag{8–30}$$

In the most general setting, the above described evolution is not considered "unitary", because in general no "Hamiltonian" exists whose commutator would result in the total derivative. The principal agent for this non-unitary character is the second term in the left-hand side of equation (8–29), which is time variation of η_S. Therefore, the dynamics are evolving, as well as the constraint surface.

8.2.2 Canonical Quantization

As mentioned in Chapter 2, the canonical quantization program requires the space to be a globally flat space. Therefore, we will state that the extended phase manifold not only has the topology of $\mathbb{R}^{2N+2}$, but is endowed with a globally flat Cartesian metric to ensure proper quantized results as described by Dirac [20]. Following the conventional program we "promote" the phase-space $(q_j, p^j; q_0 = t, p^0 = p_t)$ coordinates to irreducible, self-adjoint operators $(Q_j, P^j;\, Q_0 = T, P^0 = P_t)$. The non-vanishing commutation bracket follows the structure of the classical extended Poisson bracket i.e.

$$[Q_\mu, P^\nu] = \frac{\hbar}{i}\{q_\mu, p^\nu\} = \frac{\hbar}{i}\delta^\nu_\mu. \tag{8–31}$$

A possible objection that the reader may have is to question the self- adjoint nature of the T operator. If the spectrum of T is equal to the entire real line, as expected, this would imply that the spectrum of P_t would also be unbounded. However, as is well known, if we identify P_t with the energy E then P_t must be bounded from below, which would imply T is not a self-adjoint operator. [52]. This, however, assumes that P_t has identified or forced to become the negative Hamiltonian, which is a constraint. The contradiction is averted because we have not imposed the constraints, only quantized the entire classical system.

We follow the belief that abstract operator formulation of quantum mechanics is fundamental, as well as correct [53] . Therefore, preceeding with the canonical quantization of the classical theory, by promoting the classical constraints to self-adjoint functions of the irreducible operators.

$$\phi_a(p, q, t) \;\mapsto\; \Phi_a(P, Q, T) \tag{8–32}$$

$$\phi_0 = p_t + H(p, q, t) \;\mapsto\; P_t + \mathcal{H}(P, Q, T) = \Phi_0 \tag{8–33}$$

One possible objection at this juncture is there exist many ways of quantizing a given classical system. While this is certainly true, we will assert that we can appeal to

experiment to obtain the true result. Therefore we will not dwell on this ambiguity and assume that we have the correct realization of the quantum system.

8.2.3 Dirac

We will now follow the Dirac procedure as described in Chapter 2. The physical subspace of Hilbert space is selected to include only the elements of the original space that are annihilated by the constraints i.e.

$$\Phi_a|\psi\rangle_{phys} = 0 \tag{8–34}$$

$$\Phi_0|\psi\rangle_{phys} = 0 \tag{8–35}$$

for all $a + 1$ constraints. However, this procedure will only work for a select type of first-class constraints. In fact, if one adheres strictly to the Dirac procedure (8–35) will result in a trivial solution since the constraint Φ_0 is linear in P_t which implies that its spectrum will contain a zero in the continuum, thereby causing the physical Hilbert space to be comprised of only the 0 element, which is undesirable and unacceptable. We will therefore appeal to the use of the projection operator formalism [13] to circumvent these possible dilemmas.

8.2.4 Projection Operator Formalism

The Projection Operator Formalism [13] deviates from the Dirac method by introducing a projection operator $\mathbb{E}$, which takes vectors from the unconstrained Hilbert space ($\mathcal{H}$) to the constraint subspace (i.e. the physical Hilbert space or even better the regularized physical Hilbert space which will be described shortly.)

$$\mathcal{H}_{phys} \equiv \mathbb{E}\mathcal{H} \tag{8–36}$$

The general form of the projection operator is the following:

$$\mathbb{E} = \mathbb{E}(\Sigma_a \Phi_a^2 \leq \delta^2(\hbar)) \tag{8–37}$$

where $\delta(\hbar)$ is a regularization parameter. We require the projection operator to possess the properties of all projection operators namely Hermitian $\mathbb{E}^\dagger = \mathbb{E}$ and idempotent $\mathbb{E}^2 = \mathbb{E}$. The relation (8–37), implies that the operator projects onto the spectral interval $[0, \delta^2(\hbar)]$. The projection operator formalism allows us to deal with all constraints simultaneously and to place all types of constraints on equal footing.

8.2.5 Time-Dependent Quantum Constraints

The projection operator of the time-dependent quantum constraints, follows the same form suggested by Klauder in [13], namely,

$$\mathbb{E} = \lim_{L\to\infty} \lim_{\xi\to 0^+} \int_{-L}^{L} d\lambda e^{-i\lambda(\Phi_0^2+\Sigma_a\Phi_a^2)} \frac{\sin[\delta^2(\hbar)+\xi]\lambda}{\pi\lambda} \tag{8–38}$$

To obtain further insight it will be convenient to use the canonical coherent states of the unconstrained Hilbert space.

$$|\vec{p},\vec{q},p_t,t\rangle = \exp[i\alpha(p,q,p_t,t)]e^{-iq^jP_j}e^{ip_jQ^j}e^{-itP_t}e^{ip_tT}|\eta\rangle \tag{8–39}$$

where $|\eta\rangle$ is a normalized fiducial vector in $\mathcal{H}$. These coherent states admit a resolution of unity given as

$$\mathbb{1} = \int |\vec{p},\vec{q},p_t,t\rangle\langle\vec{p},\vec{q},p_t,t| \frac{dp_j dp^j}{2\pi}^N \frac{dt dp_t}{2\pi} \tag{8–40}$$

where the domain of integration is the entire extended phase space. The overlap of these vectors are given by the following:

$$\begin{aligned}
\langle\vec{p}',\vec{q}',p_t',t'|\vec{p},\vec{q},p_t,t\rangle &= \langle\vec{p}',\vec{q}'|\vec{p},\vec{q}\rangle\langle p_t',t'|p_t,t\rangle \\
&= \exp\{-\frac{1}{4\hbar}[|p'-p|^2-|q'-q|^2+\frac{i}{2\hbar}|p_j'q'^j-p_jq^j|\} \\
&\times \exp\{-\frac{1}{4\hbar}[|p_t'-p_t|^2-|t'-t|^2+\frac{i}{2\hbar}|p_t't'-p_tt|\}
\end{aligned} \tag{8–41}$$

Expression (8–41) defines a positive definite functional which can be chosen as the reproducing kernel and used to define a reproducing kernel Hilbert space $\mathcal{H}$.

Let us examine the extended-coherent state overlap of an equivalent form of the projection operator [13]

$$\mathbb{E} = T \int e^{-i\xi^0\Phi_0 - i\xi^a\Phi_a} f(\xi) d\xi \tag{8–42}$$

where $f(\xi)$ is some function chosen to insure (8–42) converges absolutely and T is the time ordered product.

$$\begin{aligned}\langle p'', q'', p''_t, t''|\mathbb{E}|p', q', p'_t, t'\rangle &= \langle p'', q''| \otimes \langle p''_t, t''|\mathbb{E}|p'_t, t'\rangle \otimes |p', q'\rangle \\ &= \lim \langle p'', q''| \otimes \int \prod_{k=0}^{N} \langle p_t^{k+1}, t_{k+1}|\mathbb{E}|p_t^k t_k\rangle \prod_{k=0}^{N-1} d\mu(p_t^k, t_k) \otimes |p', q'\rangle\end{aligned} \tag{8–43}$$

We can generalize the result from [54] namely

$$\begin{aligned}&\lim \delta \to 0 \langle p'', q'', p''_t, t''|\mathbb{E}(-\delta < \mathcal{H}(\mathcal{P}, \mathcal{Q} + P_t) < \delta)|p', q', p'_t, t'\rangle \\ &= \langle p'', q'', 0, 0|e^{-(p''_t + \mathcal{H}(P,Q))/2} e^{-i\mathcal{H}(t''-t')} e^{-(p'_t + \mathcal{H}(P,Q))/2}|p', q', 0, 0\rangle\end{aligned} \tag{8–44}$$

to include time dependent Hamiltonians.

$$\begin{aligned}&\langle p'', q'', p''_t, t''|\mathbb{E}(-\delta < \mathcal{H}(\mathcal{P}, \mathcal{Q}, \mathcal{T}) + P_t) < \delta)|p', q', p'_t, t'\rangle \\ &= \mathbb{T}\langle p'', q'', 0, 0|e^{-(p''_t + \mathcal{H}(P,Q),T)/2} e^{-i\int_{(n-1)\epsilon}^{n\epsilon} \mathcal{H}} dt e^{-(p'_t + \mathcal{H}(P,Q,T))/2}|p', q', 0, 0\rangle\end{aligned} \tag{8–45}$$

As in (8–44), we observe the variables p_t and t are not needed to span the reduced Hilbert space, therefore we can integrate p''_t and p'_t without altering the physics.

Therefore, the most general statement we can make about a system with time-dependent constraint is encapsulated in the following

$$\lim_{\epsilon \to o} \mathbb{T} e^{-i\int_{(N-1)\epsilon}^{N\epsilon} \mathcal{H}(t)dt} \mathbb{E}_{N-1} e^{-i\int_{(N-2)\epsilon}^{(N-1)\epsilon} \mathcal{H}(t)dt} \mathbb{E}_{N-2} \cdots \mathbb{E}_1 e^{-i\int_0^{\epsilon} \mathcal{H}(t)dt} \mathbb{E}_0 \tag{8–46}$$

where

$$\mathbb{E}_n = \int T e^{-i\int_{n\epsilon}^{(n+1)\epsilon} \lambda^a(t)\Phi_a(t)dt} DR(\lambda) \tag{8–47}$$

and $DR(\lambda)$ is the weak measure defined in Chapter 4. This result agrees with the result obtained in [47]. This of course is assuming that the constraints are continuous in t. This statement is also applicable if ϕ_a are second-class constraints.

8.2.6 Observations and Comparisons

Despite the fact that the Projection Operator Formalism and the approach used by Gitman [12] start on very similar grounds, the approaches end on very different grounds. The Gitman approach advocates the use of Dirac Brackets, which is a method used to avoid solving for second-class systems, while the Projection Operator Formalism treats all constraints on equal theoretical footing. In the proceeding chapter we will examine two different constraint models with the aid of the projection operator formalism.

CHAPTER 9
TIME-DEPENDENT MODELS

In the preceding chapter, we have developed an approach to contend with explicit time-dependency in constraints within the projection operator formalism. Despite this development, some looming questions persist. The primary purpose of this chapter is to elucidate these unresolved questions by considering some simple quantum mechanical models. One of the most pressing questions is whether or not the physical Hilbert space of a time-dependent constraint is trivial [1] As shown in the previous chapter, (8–46), the "evolution operator" for time-dependent constraints can be written as an infinite product of projection operators. However, as we will illustrate in our first model, even with the requirement of a stringent polarization of the states from the total Hilbert space, the physical Hilbert space is non-trivial. The second model is designed to demonstrate how a second-class system should be considered within this context.

9.1 First-Class Constraint

We deviate from the prescription described in the previous chapter by not pursuing the reparameterization invariant form of the model discussed briefly. The primary motivation of this model is stated above. Let us begin with the simple 3 degree-of-freedom classical extended Hamiltonian.

$$H_E = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2) + \lambda(t)(j_1 \sin(\frac{\pi}{2}t) + j_3 \cos(\frac{\pi}{2}t)) \tag{9–1}$$

where $j_1 = q_2 p_3 - q_3 p_2$ and $j_3 = p_2 q_1 - q_2 p_1$ and $\lambda(t)$ is the Lagrange multiplier that enforces the single first-class constraint

$$\phi(p, q) = j_1 \sin(\frac{\pi}{2}t) + j_3 \cos(\frac{\pi}{2}t). \tag{9–2}$$

[1] Which is to say that the physical Hilbert space contains only the zero vector.

The constraint will clearly commute with the Hamiltonian, $H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2)$ therefore this truly is a first-class system. For the case of this analysis we will restrict the allowable values of t to a compact subspace of $\mathbb{R}$ namely, $[0, 1]$. It is easy to observe that the constraint surface initially is defined by $j_3 = 0$ but evolves in a smooth fashion into the vanishing loci of j_1. Moving to to quantum analog of this system[2] the issue surrounding the alternating constraints is potentially very interesting since J_3 and J_1 are examples of incompatible observables, therefore it is impossible to diagonalize them simultaneously. Utilizing the technique in which we established the physical Hilbert space for the Casmir operator of $\mathfrak{su}(2)$ in Chapter 4, Section 3.2, we will use a similar technique to analyze this model. As before, let us introduce conventional annihilation and creation operators given by

$$a_j = (Q_j + iP_j)/\sqrt{2\hbar}, \tag{9–3}$$

$$a_j^\dagger = (Qj - iPj)/\sqrt{2\hbar}. \tag{9–4}$$

If we define the number operator

$$N = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3 \tag{9–5}$$

it is obvious that

$$[J_1, N] = 0 = [J_3, N]. \tag{9–6}$$

Based on this conservation, we can study the fulfillment of the time-dependent constraints in each of the number-operator subspaces independently of one another. Based on this information we can proceed with the following analysis:

[2] Since this model is similar to the example discussed in Chapter 4, we will forgo the formal arguments of the quantization scheme.

0-particle subspace[3]

$$\mathbb{E}^0(-\delta < J_1 \sin(\frac{\pi}{2}t) + j_3 \cos(\frac{pi}{2}t) < \delta) = |0,0,0\rangle\langle 0,0,0| \tag{9–7}$$

1-particle subspace

$$\mathbb{E}^1(-\delta < J_1 \sin(\frac{\pi}{2}t) + j_3 \cos(\frac{pi}{2}t) < \delta)$$

$$= \frac{1}{2}(\sin t|1,0,0\rangle + \cos t|0,0,1\rangle)(\langle 1,0,0|\sin t + \langle 0,0,1|\cos t) \tag{9–8}$$

2-particle subspace

$$\mathbb{E}^2(-\delta < J_1 \sin(\frac{\pi}{2}t) + j_3 \cos(\frac{\pi}{2}t) < \delta)$$

$$\begin{aligned} = & \ \frac{1}{3}(|2,0,0\rangle + |0,2,0\rangle + |0,0,2\rangle)(\langle 2,0,0| + \langle 0,2,0| + \langle 0,0,2|) \\ + & \ \frac{1}{3}((\sin(\frac{\pi}{2}t) + 1)|2,0,0\rangle + |0,2,0\rangle + 1 + \cos(\frac{\pi}{2}t)|0,0,2\rangle) \\ \times & \ (\sin(\frac{\pi}{2}t) + 1)\langle 2,0,0| + \langle 0,2,0| + (1 + \cos(\frac{\pi}{2}t))\langle 0,0,2|) \end{aligned} \tag{9–9}$$

The construction of the higher numbered projection operators continues in a similar fashion. The key observation at this juncture is that the projection operator decomposes into a time-dependent part and a time-independent portion. The time-independent part is asociated with the Casmir operator from the full $su(2)$ algebra, while the time-dependent portion is attributed to the remainder of the constraint modulo contribution from $\Sigma_i L_i^2 = 0$. Therfore we can write the full projection operator as the following:

[3] The superscript on the left-hand side of the equations designates the number subspace

$$\mathbb{E}(-\delta < J_1 \sin(\frac{\pi}{2}t) + J_3 \cos(\frac{\pi}{2}t) < \delta)$$

$$\begin{aligned} &= \mathbb{E}(\Sigma_i L_i^2 < \delta^2(\hbar)) \\ &+ \mathbb{E}_t(-\delta < J_1 \sin(\frac{\pi}{2}t) + J_3 \cos(\frac{pi}{2}t) < \delta \,/ \Sigma L^2 \delta^2) \end{aligned} \tag{9–10}$$

where $\mathbb{E}_t$ represents the explicit time-dependent nature of the second operator on the right-hand side of equation (9–10). Having discovered the full nature of the projection operator let us digress a bit to briefly discuss simplifications to equation (8–46), with the given description of the model. As we can easily observe, the Hamiltonian H commutes strongly with the constraint $\phi(p, q)$, it follows that;

$$[\mathbb{E}_{\mathrm{n}}, H] = 0. \tag{9–11}$$

This equation holds for all time-slices and therefore all n. Equation (8–46) then reduces in the following manner;

$$\begin{aligned} &\langle \cdot | e^{i\mathcal{H}/\hbar T} \mathbb{E}_{N-1} \cdots \mathbb{E}_{)} | \cdot \rangle \\ = \; &\langle \cdot | e^{i\mathcal{H}/\hbar T} \mathbb{E}_{N-1} \delta_{N-1,0} | \cdot \rangle \end{aligned} \tag{9–12}$$

where $\mathbb{E}_n$ is defined in equation (8–47). As we can determine from (9–12), the infinite product of projection operators will merge into one projection operator. In turn this will project onto the set which is the intersection of the initial projected space (i.e. $J_3 = 0$) and the final (i.e. $J_1 = 0$). This operator will of course project onto the sub-space that carries the trivial representation of this algebra, which is the time-independent portion of the projection operator. The conclusion that we can draw from (9–12) is that the physical Hilbert space for this model is not trivial. While this model may not be conclusive proof that the physical Hilbert space for a general time-dependent constraint is not trivial. It with the help of various generalizations of this model assist in answering the full query.

9.2 Second Class Constraint

The second and final model we will consider in this chapter is inspired by the work of J. Antonio Garcia, J. David Vergara and Luis F. Urrutia [49]. In this work, the authors extend the BRST-BFV method [14], to deal with non-stationary systems (i.e. time-dependent systems). For this dissertation, we chose not to discuss the BRST-BFV method, however for a description of the method see [14] and [49].

The model used to illustrate the author's technique was a two-dimensional rotor with a time-dependent radius. However, for this discussion we abate the model in [49], by reducing the number of degrees of freedom from 3 to 2, as well as, setting the Hamiltonian equal to zero to emphasize the constraints. Consider the following time-dependent classical constraints:

$$\phi_1 = q - ct, \tag{9–13}$$

$$\phi_2 = p - c, \tag{9–14}$$

$$\phi_3 = p_t, \tag{9–15}$$

where c is a positive constant and q, p, p_t, are the canonical position, its corresponding conjugate momentum and conjugate momentum corresponding to the temporal coordinate. Based on the the Poisson bracket of the constraints, this constraint system is a second-class system.

The canonical quantization of this model is straight forward. We simply follow the same procedure as stated in the preceding chapter, which implies that we promote all of the canonical coordinates $(p, q; p_t, t)$ to irreducible self-adjoint operators$(P, Q; P_t, T)$. We promote the constraints to self-adjoint operators as indicated by the following:

$$\phi_1 \mapsto \Phi_1 = Q - cT, \tag{9–16}$$

$$\phi_2 \mapsto \Phi_2 = P - c, \tag{9–17}$$

$$\phi_3 \mapsto \Phi_3 = P_t \tag{9–18}$$

The coherent states, that will be again useful in this analysis, satisfy $|p, q, p_t, t\rangle = |p, q\rangle \otimes |p_t, t\rangle$. In order to determine our projection operator $\mathbb{E}$ we need to determine a set of normalized vector states $|\psi\rangle$ to minimize the following relationship

$$\langle\psi|(P_t^2 + (Q - cT)^2 + (P - c)^2|\psi\rangle \propto \hbar \tag{9–19}$$

Using the logic employed by Klauder in [13], the state that minimized (9–19) would be the following state $|ct, c\rangle \otimes |0, t\rangle$ Using the theory of Weyl operators we can construct a representation of the desired projection $\mathbb{E}$, given by

$$|c, ct\rangle \otimes |0, t\rangle\langle 0, t| \otimes \langle c, ct| = \int e^{-i\lambda(P_t)-i\xi(Q-cT)-i\gamma(P-c)} e^{(-\lambda^2-\xi^2-\gamma^2)/4} \frac{d\lambda d\xi d\gamma}{(4\pi)^{3/2}}. \tag{9–20}$$

We should observe that in this case $\int e^{(-\lambda^2-\xi^2-\gamma^2)/4} \frac{d\lambda d\xi d\gamma}{4(\pi)^{3/2}} = 2$, rather than one as would be the case for normalized measure. The consequences of this projection operator can immediately be discerned following the procedure described in [13].

9.3 Conclusions

In this chapter, we successfully dealt with two systems that have explicit time dependence. It is hoped that techniques that we developed in the past two chapters will help us have a better understanding on how to deal with quantization of theories that are reparameterization invariant, such as General Relativity. The methods discussed and developed in the last two chapters can be extended to include field theories.

It is well known that a field theory that is not reparameterization invariant can be transmuted into one reparameterization invariant by a similar technique to that employed in Chapter 8 [55]. Namely this can be accomplished by changing the space-time coordinates (x^μ where $\mu \in \{1 \ldots, N\}$, and N is the number of space time dimensions) to functions over space-time,which in essence introduces N scalar fields to the field theory,

$$x^\mu \to y^\mu(x). \tag{9–21}$$

This step corresponds, in the preceding chapter, to promoting t to a dynamical variable. Using these techniques we will be able to discuss the quantization of gauge field theories with a non-stationary dynamical source, to name only one possibility.

CHAPTER 10
CONCLUSIONS AND OUTLOOK

"Mathematical study and research are very suggestive of mountaineering. Whymper made several efforts before he climbed the Matterhorn in the 1860's even then it cost the life of four of his party. Now, however, any tourist can be hauled up for a small cost, and perhaps does not appreciate the difficulty of the original ascent. So in mathematics, it may be found hard to realize the great initial difficulty of making a little step which now seems so natural and obvious, and it may not be surprising if such a step has been found and lost again." - Louis Joel Mordell (1888-1972; Three Lectures on Fermat's Last Theorem, p.4)

10.1 Summary

In Chapters 2, 3 and 4 we gave modest account of the background information needed for the remaining chapters of the dissertation. In chapter 4, we successfully analyzed a constraint that mimicked the aspect of the gravitational constraint, that it was classically a first-class system; however, upon quantization it became a partially second-class system. In the same chapter we also analyzed a closed, first-class quantum system, as well as a first-class system with a zero in the continuum.

Chapter 5 introduces the classification of constraints called "highly irregular" constraints. During this chapter, we described a general procedure to solve the quantum analog to the "highly irregular" constraints utilizing the Projection Operator Formalism. We also successfully analyzed a simple example of this type of constraint using only the Projection Operator Formalism.

In Chapter 6, we used the mathematical tools established in Chapter 5 to give a complete account of the quantization of the Ashtekar-Horowitz-Boulware model.[11] This model was inspired by the Hamilitonian constraint of General Relativity to answer whether or not there could be quantum mechanical tunneling into classically forbidden regions of phase space. During the course of this chapter we compare the results obtained by the Projection Operator Method with that of the Refined Algebraic Quantization

Scheme. This comparison has left us with the conclusion that these methods are different and incompatible with each other. To the author's knowledge, this is the first time the Projection Operator Formalism has been used when the configuration space has a non-trival topology.

The remaining chapters were devoted to the topic of time-dependent quantum constraints. We developed the formalism in which the topic can be approached in the context of the Projection Operator. This was accomplished by extending the classical phase space of the time-dependent system, thereby elevating the time parameter to a dynamical variable. In the same chapter, we compared the Projection Operator Formalism to the approach that was first discussed in[12]. While these methods start from the same point (i.e. an extended phase space) the conclusions reached are very different. In the following chapter we were sucessful in analyzing two examples of time-dependent constraints.

The story of classical and quantum constraints that we have presented within this dissertation is by no means a complete account. In fact it is impossible to give a complete account of any research. By researching we merely point the direction to new research, leading to new questions to ask and to attempt to answer. The topics in physics are always bigger than the individual physicist. However, this is the beauty of the subject, that things we leave unresolved can be picked up in the future generations.

There are several unresolved issues left from this dissertation that can be addressed by the author or future researchers. These include but are not limited to, "How do the methods of the Projection Operator generalize to a full quantum field theory?", "What lessons learned from the simple models that we analyzed in this dissertaion can be applied in more realistic theories such as Quantum Gravity?", "Can we use the formalism obtained in Chapter 8 to examine more realistic theories?"

10.2 Ending on a Personal Note

Becoming a physicist has been a dream of mine since I was fourteen years old. I would read about these famous men and women who tackled these deep theoretical problems and actually made progress and ultimately furthered our understanding on how the universe operated. I was completely determined to pursue this goal; however, no one from my high school had ever pursued a career in physics. When I was a junior in high school, there was a pull from external forces trying to pursuade me to pursue the "more traditional" route of becoming an engineer. I decided to end this dissertation by mentioning the poem by Robert Frost that aided me in cementing my decision of becoming a physicist, "Two Roads Diverged in a Yellow Wood." No matter how difficult this journey has been, I have enjoyed every step of it. This poem is also fitting because each new problem is like the fork in the road described by Frost, "should I pursue this problem with the same approach as everyone else or should I choose the problem nobody knows how to solve?" I cannot wait to see what new problems and new approaches will appear a little further down the road. I feel like I took the road less traveled, and it has made all the difference.

APPENDIX A
REPARAMETERIZATION INVARIANT THEORIES

What are the consequences if an action $I[q(t)]$ is invariant under an infinitesmal temporal transformation, which is the characteristic of a reparametrization invariant theory, i.e.

$$I[q(t)] = I[q(t+\epsilon(t))] \text{for } \epsilon \ll 1? \tag{A–1}$$

Given the infinitesimal transformation[1]

$$t \rightarrow t+\epsilon(t), \quad \epsilon(t_1) = \epsilon(t_2) = 0, \tag{A–2}$$

$$\Rightarrow \delta t = \epsilon, \tag{A–3}$$

$$\Rightarrow \delta q = \frac{dq}{dt}\epsilon = \dot{q}\epsilon \tag{A–4}$$

By our assumption that $I[q(t)] = I[q(t+\epsilon(t))]$ it follows that

$$\delta I = \int_{t_1}^{t_2} (\delta L) dt = 0 \tag{A–5}$$

$$= \int_{t_1}^{t_2} (\frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}) dt \tag{A–6}$$

$$= \int_{t_1}^{t_2} (\frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\dot{q}\epsilon + \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\dot{q}\epsilon) dt \tag{A–7}$$

$$= \int_{t_1}^{t_2} (\frac{dL}{dt}\epsilon + \frac{\partial L}{\partial \dot{q}}\dot{q}\frac{d}{dt}\epsilon) dt \tag{A–8}$$

Integrating by parts,

$$= -\int_{t_1}^{t_2} (L\frac{d\epsilon}{dt} - \frac{\partial L}{\partial \dot{q}}\dot{q}\frac{d\epsilon}{dt}) dt, \tag{A–9}$$

$$= \int_{t_1}^{t_2} ((\frac{\partial L}{\partial \dot{q}}\dot{q} - L)\frac{d\epsilon}{dt}) dt. \tag{A–10}$$

$$\tag{A–11}$$

[1] In general, we do not have to assume that $\frac{d\epsilon}{dt} > 0$, we point the reader to [6], for a discussion on this matter, however, we must insist that $\frac{d\epsilon}{dt} \neq 0$ almost everywhere.

We can conclude based on our assumption that $\frac{d\epsilon}{dt} \neq 0$, a.e., that the statement following must be true,

$$\frac{\partial L}{\partial \dot{q}}\dot{q} - L = 0 \qquad \text{a.e.} \tag{A–12}$$

$$\Rightarrow H = 0. \square \tag{A–13}$$

Therefore, in all reparameterization invariant theories the Hamiltonian vanishes.

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BIOGRAPHICAL SKETCH

Jeffrey Scott Little was born Janurary 10, 1980, in the small eastern Kentucky town of Pikeville. He is the eldest of three children and the only son of Jeff and Linda Little. His strong interest in science was evident at a very early age. When Scott was 14, he discovered his passion for quantum physics and read nearly everything he could find on the subject. Scott graduated from Shelby Valley High School in 1998, as class valedictorian. After high school, Scott set off to matriculate at Western Kentucky University in Bowling Green, Kentucky. While at Western he became extremely interested in the study of formal mathematics. During his senior year he entertained the idea of attending graduate school in mathematics; however, he realized that it would be possible to pursue both passions through a physics career. Scott graduated from Western Kentucky with a double bachelors degree in physics and mathematics in the Spring of 2002.

After finishing his undergraduate career he accepted an Alumni Fellowship from the University of Florida to continue his studies of physics. Though moving from a relatively small department at Western to the much larger Physics Department at Florida was initially daunting, Scott overcame his fears and succeeded in his course work. In the Spring of 2004, Scott became a student of John Klauder. Dr. Klauder allowed Scott to not only study physics but also allowed him to stay connected to the formal mathematics that he had grown fond of during his stay at Western. Under Dr. Klauder's tutelage, Scott was able to research and publish three papers on quantum constraints.

In June of 2006, at the age of 26, Scott married the love of his life Megan (Carty) Little. Scott obtained his Ph.D. in Physics in the Fall of 2007. Scott and Megan currently reside in Louisville, KY, where Scott is continuing to research a wide variety of theoretical problems and is an instructor at the University of Louisville.