SHAPE ANALYSIS VIA UNIFIED SEGMENTATION, SMOOTHING, AND REGISTRATION OF RIEMANNIAN STRUCTURES

By

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To Maggie,

for more than I asked for,

for more than I deserved,

for more than I can repay
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Locality is an important but oft-ignored aspect of shape asymmetry quantification, and segmentation is one method by which to make the locality of an analysis explicit. We have thus formulated an approach to surface registration and shape comparison which features an integrated segmentation component for the purpose of simultaneously identifying and separating regions by their evolving deformation characteristics. In the first instantiation, we achieve this effect through an adaptation of the Chan-Vese approach for image segmentation to the problem of segmenting the Riemannian structures of the very surfaces comprising the domain of the segmentation. We have successfully used the method’s output on hippocampal pairs in an epilepsy classification problem, demonstrating improvement over global measures. Noting that a Chan-Vese-based approach to simultaneous segmentation and registration is inherently limited, we have also developed a unified approach to segmentation, smoothing, and nonrigid registration of images via extension of the Mumford-Shah functional, devised in such a way as to be applicable symmetrically and consistently to multiple (two or more) inputs. To conclude, we propose an extension of this unified framework (dubbed “USSR”) to the previously considered problem of 2D surface shape analysis (asymmetry quantification and localization), conferring the benefits of unbiasedness, consistency, multiple input processing, and nontrivial data field reconstruction.
1.1 Local Surface Shape Asymmetry Analysis

The problem of quantitative shape analysis of two-dimensional simple closed surfaces appears in many applications, and is as such a well-studied problem. However, we note that while such applications could oftentimes benefit from characterizing shape differences on the basis of their location in addition to their extent, this is not the general practice. Moreover, among those more sophisticated of recent proposals which do explicitly determine shape disparity functions rather than global measures, there still lacks a framework for ready interpretation of this data. The provision of such a framework could advance the state of applications reliant on shape analysis, particularly those seeking to use the data to perform classification. We have successfully proposed and implemented one such framework, and applied it to a set of point clouds representing the hippocampal surfaces of patients belonging to either of two epileptic classes or a control group. Driven by the clinical observation that asymmetry between the two halves of a given patient’s hippocampus appears to differ between epileptics and controls, and given that this variation is generally described in terms of relative local enlargement and shrinkage, we devised a scheme based on simultaneous registration and segmentation of the Riemannian surface structures themselves. The registration component ensures that the surfaces are placed into sensible correspondence, while the Chan-Vese (CV)-style segmentation component identifies surface subregions which are particularly disparate under this correspondence. Using information from both the match disparity and the segmentation thereof, we were able to successfully differentiate patient class members via a fully automated system.

1.2 USSR: Unified Segmentation, Smoothing, and Registration

The issue of simultaneous segmentation and registration itself separately bears further consideration. As before, the segmentation scheme utilized in the first iteration
of the epilepsy classifier was based on the CV framework. In essence, this means that its resultant segmentation corresponds to the optimal piecewise-constant reconstruction of the field being segmented (the Riemannian surface structure in the above case, image intensities traditionally). As such, CV can be viewed as a scheme which not only segments a field but simultaneously reconstructs it. When coupled with a registration evolution, a system is then produced which simultaneously performs segmentation, registration, and reconstruction of its input. However, the piecewise constancy inherent in Chan-Vese limits the segmentation partially, and the reconstruction severely. This relates directly to the fact that the framework is simply a special (limiting) case of the minimization of the general Mumford-Shah (MS) functional for image reconstruction. We have thus proposed a general framework for and implemented a specific instantiation of a scheme for simultaneous segmentation, smoothing, and (nonrigid) registration of two-dimensional image suites via extension of the MS functional. Our implementation surpasses any comparable techniques known to us by virtue of possessing all of the following properties at once: (1) meaningful mutual assistance between all three image processing suboperations with nontrivial results in all, (2) applicability to input sets of size greater than two, (3) symmetry/lack of bias with respect to input arguments (including sets of size greater than two), and (4) lack of requirement of prior knowledge of region statistics, count, or shape. Results on sample image suites demonstrate successful mutual assistance of the three image processing operations under the simultaneous framework, by addressing cases under which the operations fail in isolation. (We henceforth refer to this method as “USSR”, for “Unified Segmentation, Smoothing, and Registration”.)

1.3 Application of USSR Framework to Shape Analysis

In the final instantiation, we apply the USSR method (an extended MS framework) to the shape asymmetry analysis problem in a manner analogous to our adaptation of the CV scheme. Even a cursory glance will reveal that the implementation of such an extension is more involved than in the special case of CV, but it carries with it the
following benefits: (1) the fact that MS involves a degree of freedom more than CV allows for the possibility of optimizing the free parameter (establishing the balance between data fidelity and smoothness of the reconstruction) for the application at hand, (2) the added argument-wise symmetry of the USSR framework adds a necessary consistency, and the multiple input capability allows for the possibility of batch processing in the absence of an explicit reference shape (using the example of the hippocampus dataset, one might wish to examine deviations between the left hippocampi of many patients, rather than the left-to-right asymmetry of each patient: for a structure such as the hippocampus, it might be especially prudent to avoid explicitly defining a notion of normalcy of shape through an atlas), and (3) theoretically, this is an interesting development: a proposal for simultaneous segmentation, nonrigid registration, and piecewise-smooth reconstruction of data defined over 2-manifolds is to our knowledge unprecedented in the literature, and the fact that the data in question is in fact the Riemannian structure of the manifold itself makes it more so; even in its current instantiation, our algorithm has immediate applications beyond comparing hippocampi.
CHAPTER 2
LOCAL SURFACE SHAPE ASYMMETRY ANALYSIS (APPLIED TO HIPPOCAMPUS)

2.1 Background: Hippocampal Shape Analysis

The temporal lobe’s role in memory and learning makes it a natural point of focus in the study of neurodegenerative disease progression. In particular, the medical imaging literature abounds with attempts to identify and correlate abnormalities of the hippocampus with Alzheimer’s disease, schizophrenia, and epilepsy. While early studies typically characterized hippocampal shape in terms of such simple global measures as volume, length, and surface area, it was shown as early as [30] that analysis of regional asymmetries could improve disease classification capability. Several methods for fine-grained regional hippocampal shape analysis have since been suggested. Gerig et al. [12] included a medial shape representation with age and drug treatment data in an exploratory statistical analysis of the hippocampus’s link to schizophrenia. Shen et al. [21] conducted a statistical analysis based on the spherical harmonic (‘SPHARM’) representation method. Styner et al. [23] tested the power of a SPHARM-based medial representation to separate monozygotic from dizygotic twins through lateral ventricular structure, and schizophrenics from normals through hippocampal and hippocampus-amygdalan structures. Davies et al. [9] devised a minimum description length framework for statistical shape modeling and extracted modes of variation between normal and schizophrenic populations. Bouix et al. [1] employed medial surfaces in a local width analysis. Using a viscous fluid flow model, Csernansky et al. [8] computed diffeomorphic maps of patient hippocampi onto a reference, producing a dense inward/outward deformation field over each hippocampal surface. The surface itself was additionally manually segmented to allow for the regional comparison of the deformation fields as part of an attempt to separate healthy individuals from those exhibiting dementia of the Alzheimer’s type. All reported results have indicated the benefit of incorporating regional information into the analysis. This is unsurprising, as
simple scalar measures of entire structures necessarily discard a wealth of information concerning the full characterization of those structures. However, the obtained statistical results are preliminary, and shape analysis results can actually appear contradictory across different studies (for instance, Styner notes in [24] the contrast between the primary abnormality localization in the hippocampal tail found in that work and the localization in the head reported in [7]). Furthermore, while statistically significant differences of hippocampal shape have been identified between diseased and normal sample populations, reliable classification of a sizeable number of patients with respect to those categories has not occurred previously.

2.2 Background: Simultaneous Registration and Segmentation

The first scheme for simultaneous segmentation and registration known to us was given by Yezzi et al. [34], who introduced the idea of explicit interdependence of segmentation and registration: registration methods can make use of the feature detection inherent in segmentation, while segmentation (particularly in the case of noisy or incomplete/occluded data) can utilize the redundancy provided by correctly registered images of the same structures to segment said structures more robustly jointly than individually. Their proposal involved the unification of mutual information (MI)-based flat 2D rigid image registration with a level set implementation of a piecewise constant (Chan-Vese) segmentation scheme through a variational principle. Wyatt et al. [32] accomplished much the same thing, solving instead a maximum a posteriori (MAP) problem in a Markov random field (MRF) framework. Xiaohua et al. [33] extended this to the nonrigid registration case, and Unal et al. [26] addressed the same problem using coupled partial differential equations (PDEs). In [29], F. Wang et al. presented a simultaneous nonrigid registration and segmentation technique for multi-modal data sets. Richard and Cohen [19] proposed a variational framework for combining region segmentation and registration (matching) using free boundary conditions. Jin et al. [13] simultaneously evolved a surface segmentation and a radiance-discontinuity-detecting
segmentation of the evolving surface itself. Young et al. [35] combined joint segmentation and (scaled rigid) registration with morphing active contours for segmentation of groups of CT images. Pohl et al. [17] produced an expectation maximization (EM)-based algorithm for simultaneous affine registration and subcortical segmentation of MRIs using a labeled cortical atlas.

### 2.3 Application of Registration and Segmentation to Shape Analysis

The need to identify and possibly isolate subregions of interest on the hippocampal surface suggests segmentation (henceforth synonymous with ‘parcellation’), and the need to compare hippocampal surfaces between and within (i.e. detection of asymmetry between halves of the same structure) patients represents an inherent requirement for registration. A unified segmentation and registration scheme is thus a natural approach to the problem at hand. As is to be seen, the applicability comes through viewing the Riemannian surface structures of the input surfaces themselves as the data to be operated upon. Our scheme then rests on the fundamental assumptions that: (1) the deformity of homologous anatomical structures can be quantified as the deviation from isometry of the deformation map between their surfaces, and (2) the evolution of the global correspondence must allow for a partial disconnection between the normal and abnormal regions, as these are by definition not expected to exhibit the same deformation patterns. As such, surface segmentation rests at the heart of our approach, which can be viewed both as a registration tool and an asymmetry localizer. For the topologically spherical surfaces considered here, our intrinsic approach leads naturally to an elegant 2D parametric representation of both the segmentation and registration. Our use of Riemannian surface structure in the matching criterion is similar in spirit to work done by Y. Wang et al. (e.g. [31]), in which the authors presented a technique to nonrigidly register surfaces using a 2D (parametric) diffeomorphic map constructed from Riemannian surface structure information. The integrated manifold segmentation is enough to distinguish our approach from this work, but additionally, our definition of
system energy in terms of piecewise isometry (i.e. using first fundamental forms in the matching criterion) inherently balances the consideration that the map should accurately match Riemannian surface characteristics with the idea that the map should incur as little deformation as possible in doing so. As such we eliminate the need for additional regularization of the map gradient. In fact, the real novelty of the algorithm is more fundamental: this is the first proposal for simultaneous nonrigid surface registration and segmentation wherein the segmentation is driven by the evolving registration map itself. (Though, we do note that there exist works of similar spirit. For example, see the “deformotion” concept of Soatto and Yezzi [22] for separating physical motion from surface deformation in registration of correspondent regions.)

2.4 Derivation of Framework for Closed 2D Surfaces

To begin the explanation of our framework, we first turn attention to the already-studied problem of 2D image pair segmentation and nonrigid registration. A solution to this problem (regardless as to how one obtains it) clearly takes the form of (1) a differentiable one-to-one and onto mapping function which shares its domain with one of the images and its range with the other, and (2) a segmenting curve which lies within one image domain and is carried into the other through the mapping function. With images, there is a natural and obvious basis for computing this solution: the image intensity values. Intensity agreement represents the matching criterion by which algorithms such as mutual information maximization estimate pairwise image registration. Statistics derived from intensity values form the driving force of region-based segmentation methods (in Chan-Vese [3], for instance, the statistic is the mean of each curve-defined region). Further, differential qualities (e.g. gradient, Laplacian) of the intensity profiles can prove useful in identifying edges, corners, and other useful features. When conducting pairwise segmentation and registration of shapes rather than images, however, the appropriate measures are perhaps less obvious. It remains to us to define them.
Figure 2-1. Illustration of shape analysis framework: The input to the algorithm is a pair of homologous shapes $S_1$ and $S_2$. Use of appropriate boundary conditions allows us to associate the shapes with flat rectangular patches $P_1$ and $P_2$ respectively. These patches, now considered as parametric domains, store all Riemannian metric information $g_1$ and $g_2$ of their respective surfaces in terms of the first fundamental form matrices $G_1$ and $G_2$. The metric information is then matched between $P_1$ and $P_2$ through a variational principle which drives both a homeomorphic map $f$ of $P_1$ onto $P_2$ (diffeomorphic except possibly on the curves $\gamma$) and a segmenting curve $\gamma_1$ in $P_1$ which is carried by $f$ into $P_2$ as $\gamma_2$. This process registers and segments the surfaces $S_1$ and $S_2$ via the parametric correspondence. Note that the map $f$ between the parametric domains can be visualized as the deformation of a regular grid representing the left parametric domain $P_1$, and that the segmentation interiors (red) and exteriors (blue) can be represented on $S_1$ through shading.

Before defining a matching criterion, we first show how it is possible to cast the shape registration and segmentation problem into a computational framework similar to the image framework described above (save for the matching criterion itself) through appropriate parametrization. To accomplish this, one need note only the following simple topological facts: (1) the surfaces of hippocampi can be thought of as 2D Riemannian manifolds embedded in $\mathbb{R}^3$, (2) these manifolds are topologically equivalent to the sphere (closed and genus zero), (3) the exclusion of a pair of poles from the sphere produces
a cylindrical topology, and (4) any surface topologically equivalent to the cylinder can be parametrized by a class of functions whose common domain is a single rectangular patch with periodic boundary conditions at a pair of opposite sides. Note that this last fact directly implies that any function defined on the surface in question can be fully represented as a function over a closed rectangular domain with the stated boundary conditions. In particular, a smooth map between two hippocampi can be represented as a smooth map (deformation field) between two parametric rectangular domains satisfying the boundary conditions. Through this simple and natural representation of the surfaces, our solution format becomes quite similar to that of the 2D image segmentation and registration problem in that all of the computations are performed on and between rectangular domains. In contrast, when computation takes place in the embedding (3D) space before being restricted to an approximation to the surface within that space, one cannot draw nearly so direct a parallel (though one may take this approach if one wishes: see for example [4]). The approach can be visualized in the manner depicted in Fig. 2-1.

Clearly, the matching criterion which we define in place of image intensity must communicate shape information about the surfaces being registered and segmented. We now take the opportunity to define the first and second fundamental forms of a parametrized surface, and subsequently motivate their consideration:

\[
FFF = \begin{pmatrix}
X_u \cdot X_u & X_u \cdot X_v \\
X_u \cdot X_v & X_v \cdot X_v
\end{pmatrix}
\]  
(2–1)

\[
SFF = \begin{pmatrix}
N \cdot X_{uu} & N \cdot X_{uv} \\
N \cdot X_{uv} & N \cdot X_{vv}
\end{pmatrix}
\]  
(2–2)

where \(X\) is the 3D surface map, \(u\) and \(v\) are the map parameters (equivalently, the coordinate system of the parametric domain), \(N\) is the surface normal (defined as \(\frac{X_u \times X_v}{|X_u \times X_v|}\)), and subscripts denote differentiation.

But a well-known theorem due to Bonnet (as stated in [10]) reads as follows:
Theorem 1. Let \( E, F, G, e, f, g \) be differentiable functions, defined in an open set \( V \subset \mathbb{R}^2 \), with \( E > 0 \) and \( G > 0 \). Assume that the given functions satisfy formally the Gauss and Mainardi-Codazzi equations and that \( EG - F^2 > 0 \). Then, for every \( q \in V \) there exists a neighborhood \( U \in V \) of \( q \) and a diffeomorphism \( x : U \to x(U) \subset \mathbb{R}^3 \) such that the regular surface \( x(U) \subset \mathbb{R}^3 \) has \( E, F, G \) and \( e, f, g \) as coefficients of the first and second fundamental forms, respectively. Furthermore, if \( U \) is connected and if

\[
\bar{x} : U \to \bar{x}(U) \subset \mathbb{R}^3
\]

is another diffeomorphism satisfying the same conditions, then there exists a translation \( T \) and a proper linear orthogonal transformation \( \rho \) in \( \mathbb{R}^3 \) such that \( \bar{x} = T \circ \rho \circ x \).

(NOTE: In this statement of the theorem, the form coefficients \( E, F, G, e, f, g \) correspond elementwise to the quantities in the definitions of Eqns. 2–1 and 2–2 as follows:)

\[
FFF = \begin{pmatrix} E & F \\ F & G \end{pmatrix}
\]

\[
SFF = \begin{pmatrix} e & f \\ f & g \end{pmatrix}
\]

Though Thm. 1 may appear arcane in its given form, a brief explanation should illuminate its application to the problem in question. The set \( V \) in the theorem represents a patch of a two-dimensional parametric domain (such as \( P_1 \) and \( P_2 \) in Fig. 2-1). \( E > 0 \) and \( G > 0 \) are necessary conditions for those quantities to be valid coefficients in the FFF of any surface parametrized on \( V \), which is obvious from the definition in Eqn. 2–1. \( EG - F^2 > 0 \), the requirement that the FFF matrix be positive definite, is necessary for map regularity, which is in turn a necessary condition for diffeomorphism, the only class of maps with which we are here concerned. (Since \( \sqrt{EG - F^2} \) can in fact be interpreted
as the size of an area element on the surface corresponding to a parametric patch\(^1\), for, the requirement of FFF positive definiteness everywhere dictates that any subset of the parametric domain with positive area must map to a subregion of the parametrized surface with nonzero area: this is one way of obtaining a rudimentary intuition for the notion of regularity. We will make use of this relationship in the implementation as well.) These stipulations all exist simply to ensure that \(E\), \(F\), and \(G\) are possible FFF coefficients of a surface parametrized by a diffeomorphism defined on \(V\). The theorem then goes on to mention the Gauss and Mainardi-Codazzi equations (known as the ‘compatibility equations’ of surface theory): without taking the clarification too far tangentially, we can say that satisfaction of these equations means that the combined information content of the FFF and SFF is self-consistent.\(^2\) This completes the stipulation portion of the theorem: essentially, \(E\), \(F\), and \(G\) must be suitable FFF coefficients, and \(e\), \(f\), and \(g\) must not contradict them as SFF coefficients.

What the theorem then tells us is that if these conditions are satisfied, then \(V\) is everywhere locally parametrizing some regular surface through a diffeomorphic map \(x\) with FFF and SFF specified by \(E\), \(F\), \(G\), \(e\), \(f\), and \(g\). In fact, not only does such a surface exist, it is determined up to rigid motion, and as such it is unique in the sense of shape. Therefore, if we consider only maps \(x\) which are diffeomorphic over the entirety of their (connected) domains, then for given topology we can state that the FFF and SFF contain all possible shape information. These tensor fields are thus a natural choice for the matching criterion sought.

---

\(^1\) Consult reference [11], Chs. 2-5 and 2-8, for a detailed discussion of why this is so.

\(^2\) (Note that this implicitly broaches the fact that the shape information of the two forms is not entirely independent. For instance, the Gaussian curvature, the product of the minimum and maximum curvatures, a.k.a. principal curvatures, can be expressed either in terms of both the FFF and SFF coefficients, or, remarkably, in terms of FFF coefficients alone. This is the content of Gauss’s Theorema Egregium, or Remarkable Theorem. Thus, there are SFF matrices which cannot coexist with given FFF matrices.)
There are different ways to conceptually distinguish the two quantities. For one, the geometric information contained in the FFF is of first order, while that of the SFF is second order in nature. Alternately, the FFF encodes the intrinsic geometry of the parameterized surface while the SFF encodes the extrinsic geometry, or, the FFF encodes local length information, the SFF surface curvature (the mean curvature, an extrinsic characteristic, is used as a matching criterion in [31]). Because of their lower differential order (and thus greater relative stability) and intrinsic nature, we choose to base our approach on comparison between the FFF tensors of the two surface parametric domains. Given topological equivalence, FFF equivalence everywhere through a bijective deformation map defines that map as a global isometry: this indicates that the extent of failure to match this characteristic serves well as a measure of shape dissimilarity.

2.4.1 Practical Differences Between FFF and SFF as Matching Criterion

We have established that consistent FFF and SFF fields are sufficient to specify the shape of a surface of given topology. Unsurprisingly, their information is non-redundant, i.e., one form does not fully specify the other. Yet we have chosen to use only the FFF tensor fields in the matching criterion, meaning that we are ignoring some aspect of shape in a shape analysis algorithm. This fact bears consideration: we will illuminate what our perspective retains and what it discards, and justify our decision.

The sort of information contained in the SFF (and thus discarded in omitting it from the matching criterion) can be visualized through pairs of surfaces with SFF fields which differ under parametrizations which match their FFF fields exactly. Fig. displays such a case. In this example, it is easy to see that there is an isometric mapping between the two surfaces (and thus zero energy in the sense of deviation from isometry), and what that isometric mapping is. However, under said mapping, the SFF fields differ over the portion of the domain parametrizing the two protrusions (inverted relative to one another): all curvatures are equal but opposite in sign. Viewed sitting in its embedding space ($\mathbb{R}^3$), this fact is obvious, but to the proverbial “bug on the surface”,
it is imperceptible: in the sense of measures, the surfaces are indistinguishable, which is precisely what defines isometry. All such distinction is represented in the SFF fields, which include the surface normals (a concept inherently tied to the embedding space).

![Figure 2-2](image)

Figure 2-2. A pair of cylindrical surfaces with identical “Gaussian bumps” of opposite orientation. The extrinsic orientation information is discarded in using the FFF as the matching criterion: in this framework, the two surfaces are considered identical.

However, the quality of the match criterion is application-dependent. For instance, when considering the case of hippocampus pairs, we seek to identify regions of localized growth or shrinkage (in line with the clinical motivation). The sort of shape difference represented in Fig. 2-2 does not represent such a case. Further, an example of the drawback of SFF (essentially, curvature) matching is illustrated in Fig. 2-3.

As regards this application and many others, any difference between the two surfaces is attributable to high-frequency noise in the point cloud data: that is, there is no meaningful difference between the surfaces. However, the trivial isometric map (identity) between the noise-free versions of the surfaces will by no means provide a low-magnitude difference in their second fundamental forms under that mapping. In fact, infinitesimal perturbations in surface data are theoretically capable of producing surface curvatures of arbitrarily large magnitudes at the perturbation locations. The problem with using the curvature structure of a surface as an identifying characteristic is the fact that it is inherently of high order: matching maps through their second derivatives cannot be
expected to be stable, and, worse still, in many cases cannot be expected to be physically meaningful even with numerical considerations cast aside. This fact is especially true for the inputs considered herein, which explains our omission of this aspect of the surface characterization.

2.5 System Energy Functional

We now formalize the problem and fully specify its solution as follows: Let $S_1$ and $S_2$ be two surfaces in $\mathbb{R}^3$. The Euclidean metric in $\mathbb{R}^3$ induces Riemannian metrics $g_1$ and $g_2$ on $S_1$ and $S_2$, respectively. The goal is to segment regions in $S_1$ and their corresponding regions in $S_2$ while mapping those correspondents so as to optimally match their metric structures. Specifically, the algorithm must compute a homeomorphism $f$ between $S_1$ and $S_2$ and a set of closed curves $\gamma_1$ on $S_1$ and $\gamma_2$ in $S_2$. The restriction of $f$ to the complement $S_1 \setminus \gamma_1$ is a diffeomorphism between $S_1 \setminus \gamma_1$ and $S_2 \setminus \gamma_2$. If $U_1, \cdots, U_n$ and $V_1, \cdots, V_n$ denote the collections of open components (topological discs) of $S_1 \setminus \gamma_1$ and $S_2 \setminus \gamma_2$, respectively, then $f$ maps $U_i$ diffeomorphically onto $V_i$ for each $i$ while matching the Riemannian structures of each $U_i$ and $V_i = f(U_i)$.

We solve the simultaneous segmentation and registration problem outlined above using a variational framework. The energy functional $E$ is defined as a functional of a pair of functions.
$(f, \gamma_1)$, where $\gamma_1$ is a set of closed curves on $S_1$ and $f : S_1 \rightarrow S_2$ is a homeomorphism which is $C^\infty$ on $S_1 \setminus \gamma_1$. Let $f^*g_2$ denote the pull-back metric on $S_1$. $E(f, \gamma_1)$ is then defined as

$$E(f, \gamma_1) = \int_{S_1 \setminus \gamma_1} e \, dA + \alpha \int |d\gamma_1(t)/dt|_{g_1} \, dt + \beta \left( \int_{S_{1in}} |e - \bar{e}_{in}|_{g_1}^2 \, dA + \int_{S_{1out}} |e - \bar{e}_{out}|_{g_1}^2 \, dA \right)$$

(2–3)

where

$$e = |f^*g_2 - g_1|_{g_1},$$

(2–4)

$$\bar{e}_{in} = \frac{\int_{S_{1in}} e \, dA}{\int_{S_{1in}} dA}, \quad \bar{e}_{out} = \frac{\int_{S_{1out}} e \, dA}{\int_{S_{1out}} dA},$$

(2–5)

$S_{1in}$ and $S_{1out}$ are the regions of $S_1$ inside and outside of $\gamma_1$ respectively, and $\alpha, \beta$ are positive weighting constants.

The quantity $e$ defined in Eqn. 2–4 provides, at each point on $S_1$, a measure of similarity between the Riemannian structures $g_1$ and $g_2$ as they correspond under the current estimate of $f$. As such it represents the local deviation from isometry, and its integral over the domain (we have adopted the convention of using $P_1$, the parametric domain of $S_1$) provides a global measure of how far the given deformation $f$ is from an isometry. This is the first term in the system energy represented in Eqn. 2–3. In local (parametric) coordinates, it is given by the distance between the two matrices $G_1$ and $J^tG_2J$:

$$|J^tG_2J - G_1|_{g_1}^2,$$

(2–6)

where $J$ is the Jacobian of $f$ when $f$ is expressed from the local coordinate system of $P_1$ onto that of $P_2$, and $G_1$ and $G_2$ are $2 \times 2$ positive definite matrices expressing the metrics $g_1$ and $g_2$ in the coordinate systems of $P_1$ and $P_2$ respectively, as in Eqn. 2–1. The second term in $E$ restricts the length of the segmenting curve, so as to achieve the
minimal necessary segmentation (shorter length and/or fewer open components) under the other energy constrains. Finally, the third and fourth terms place a homogeneity constraint on $e$ within each connected component defined by the segmenting curve. This corresponds to our notion that the curve should partition the surfaces in such a way as to isolate regions exhibiting one sort of deformation characteristic from those exhibiting another. Note that these terms are defined in terms of both $f$ and $\gamma_1$, and as such the minimization of their energy will drive both segmentation and registration. This coupling represents the fundamental difference between a simultaneous framework and one in which the segmentation follows the registration (in series).

2.6 Implementation Details

The minimization of the system energy defined by Eqn. 2–3 is achieved by alternating between the deformation map and segmentation estimation processes. When optimization steps are constrained to be sufficiently small, this alternation approximates minimization of $\mathcal{E}$ simultaneously with respect to both $f$ and $\gamma_1$. The Euler-Lagrange equation for Eqn. 2–3 can be derived through standard calculus of variations. The equation for $f$ contains a fourth-order expression involving $f$ and its derivatives. Because of the high differential order of this analytic expression, we instead choose to minimize the functional $\mathcal{E}$ with respect to $f$ directly through a constrained optimization process (more detail in Secs. 2.6.2 and 2.6.3). Minimization with respect to $\gamma_1$, on the other hand, is implemented via a level set segmentation as described by Chan and Vese in [3]. The only difference between this module of our method and that described therein is our generalization of the process to genus zero 2-manifolds, which entails respecting nonuniform surface length and area elements in accordance with the metric information.

2.6.1 Hippocampal Surface Data Acquisition and Formatting

To produce the surface data used in this study, MRI scans were first acquired with a 1.5 T MRI scanner (Siemens Magnetom 42SPA, Siemens Medical Systems, Iselin, NJ) using 3D magnetization-prepared rapid gradient echo (MPRAGE) sequences. The gradient
echo images were transferred for post-processing. The sequence parameters included the following: obtained in sagittal plane, 250 mm field of view, repetition/echo times of 10/4 ms, T1 weighted sequence of 300 ms, 10° flip angle, 130x256 matrix, and 180 mm slab with 128 partitions producing 1.25 mm gapless sections. The segmentation of the hippocampal surfaces from the MRI data was done manually by a trained neuroanatomist, and a smooth surface was fit to the marked points by using a deformable pedal surface as described by Vemuri and Guo in [28]. Each segmented hippocampal surface obtained from the application of this technique was represented by a 40 × 21 mesh of points on that surface, periodic in one direction. (This mesh is the instantiation of the surface parametric domain as discussed in Sec. 2.4.) Homologous hippocampi were brought into rigid alignment through application of the iterative closest points (ICP) algorithm following the segmentation. The rigid alignment step included extraction of volume data and normalization of the shapes with respect to this data.

2.6.2 Discrete Representation of the Deformation Map and the Segmentation Curve

An intrinsic map between two structures that are topologically equivalent to cylinders can be stored (in discretized form) as a 2-vector function over an $m \times n$ grid, where $m$ and $n$ are the grid point counts in the $u$ and $v$ parameter directions. The 2-vector at each grid point on the domain grid represents the displacement vector between the domain grid point and the corresponding target grid point. Since we are representing a cylinder, which is periodic in one direction, with a grid, we must enforce this periodicity through a modulo arithmetic implemented with respect to one of the grid coordinate directions (‘$u$’). The grid must then be bounded in the remaining (‘$v$’) parametric direction, with the two bounded ends corresponding to the “top” and “bottom” of the cylinder being parameterized. The $v$-coordinates at the poles (columns 1 and $n$ of the grid) are excluded from the map evolution process, and so we conduct energy minimization with respect to $mn$ displacement $u$-coordinates and $m(n - 2)$ displacement
Figure 2-4. Hessian sparsity structure.

$v$-coordinates, which can be stored as a single variable vector of size $2m(n - 1)$. Note that this optimization is constrained in that the $v$ component must everywhere respect the boundaries corresponding to the top and bottom of the cylinder. We choose also to restrict the modular family of solutions for the $u$ components (corresponding to 360° rotations around the cylindrical axis) to falling within a modulus above or below the initial conditions. We evolve the system under these constraints using the LargeScale version of Matlab’s ‘fmincon’ function, a subspace trust region method based on the interior-reflective Newton method described in [6]. We exploit the inherent sparsity structure of the Hessian (as described in Sec. 2.6.3). The Riemannian characteristics themselves, whose values of course drive the energy minimization process, are calculated through analytical differentiation of a bicubic surface fit to the given points.

The segmenting curve is simply stored and evolved according to well established level set curve evolution techniques (see [20] for extensive discussion). The one modification present in our case is that length and area computations respect the distances between grid points by using the appropriate Riemannian metric at each point (as opposed to the uniform Euclidean metric used for flat domains and targets).

2.6.3 Sparsity of the Hessian

The trust region method which conducts the energy minimization with respect to the map (for a fixed segmentation) can exploit the fact that local changes in the map can only effect local changes in the induced metric structure. To make this point clear, let us consider the fact that the FFF is defined as in Eqn. 2–1, where $X$ is the surface
spatial location 3-vector and $u$ and $v$ are the 2D parametric coordinates. The FFF at a point depends only on the derivatives of $X$ with respect to $u$ and $v$. Furthermore, for a fixed segmentation, the energy itself is a function only of the FFF structure on the target manifold imposed by the evolving map. But consider each component of the $2m(n - 1)$-vector which we are evolving: infinitesimal variation in one of these components can only perturb the metric structure on the target manifold locally, as the FFF definition shows that it depends only on $X_u$ and $X_v$ at the given $(u, v)$, and these partial derivatives themselves are inherently local. It should then be immediately clear that mixed second partials can only be nonzero when both variables being differentiated against lie within the same neighborhood (as defined by the differentiation scheme): this leads to the Hessian structure depicted in Fig. 2-4, wherein the solid diagonals represent the nonzero matrix elements (and the map variable vector structure has been explained in Sec. 2.6.2). Exploitation of this sparsity structure is necessary for a second-order optimization scheme to achieve computational feasibility.

2.7 Results

To illustrate the method’s performance, we first present a visual display of its output and follow by demonstrating the classification power of the extracted features. The input dataset consisted of 60 L/R hippocampus pairs, collected and formatted as described in Sec. 2.6.1. Clinicians provided a trinary classification of this dataset: LATL (left atrophied temporal lobe) epileptic, RATL (right atrophied temporal lobe) epileptic, and Control. In this step, 6 samples were discarded due to ambiguity of their clinical class membership, leaving the final set at 54 patient samples (15 LATL, 16 RATL, 23 Control).

2.7.1 Output Visualization

Our result visualization format can best be understood through reference to Fig. 2-1, in which we present a holistic schematic. For each sample, we illustrate (1) the deformation map $f$ as the warp necessary to carry points in $P_1$ to their correspondents in $P_2$, (2) the segmentation $\gamma_1$ and the deformation energy function $e$ as they exist in $P_1$,.
and (3) the segmentation $\gamma_1$ as it exists on $S_1$ through the parametrizing function from $P_1$ (there is a corresponding region on $S_2$ as well, not depicted here but fully specified for given $f$ and $\gamma_1$ through the parametrizing functions: $\gamma_2$ is specified by $f$ and $\gamma_1$). As noted in Fig. 2-1, the segmentation is depicted on $S_1$ through the shading of its interior and exterior with red and blue respectively: the curve(s) itself is then of course the boundary between the shaded regions.

We first consider the case of a pair of cylinders, where one member of the pair has had its surface distorted according to an outward normal vector field of magnitude dictated by a Gaussian function confined to a known support. Since the support is known, we can use its segmentation as one quantification of accuracy. Fig. 2-5 illustrates the obtained results. Here, $86\%$ of the distortion area is included in the segmentation, and $0\%$ of the undistorted area is included in the segmentation. This means that the segmenting curves lie slightly within the distorted regions, as can be seen in the figure. Even the $14\%$ “misclassification” cannot necessarily be seen completely as error in this context: since the protrusions are far less severe towards the boundaries of their support (the feet of the hills), they are accordingly far more similar in deformation level to the undistorted regions than to the heavily distorted regions towards the support centers. As such, the segmentation arrived at is in compliance with the two-mean framework (which produces a high/low segmentation). While we do later note that it is possible to refine this framework (see conclusion), it is important to understand that this manner of grouping is not inherently a limitation, particularly considering the intended application. We follow by demonstrating robustness of the segmentation in the face of reparametrization of the surface of comparison and random noise in the surface point cloud (see caption for parameter levels). When repeated under these conditions, the segmentation remains $100\%$ consistent at the pixel resolution with that obtained prior.

For the real cases described previously, we present the results obtained on one member of each class (LATL, RATL, and control), as Figs. 2-6, 2-7, and 2-8, respectively.
Figure 2-5. Synthetic results: case of cylinders with surface distortion.

(a) Segmentation of prominent distortions from cylindrical surface (left: init.; middle: 5 its.; right: 20 its.).

(b) Performance under surface noise ($\sigma = 0.01$ for unit cylinder) and reparametrization (left: warp field, $\sigma = 0.2$ in each component; middle: segmented energy field in parametric domain; right: converged result, at 20 its.).

Each figures depicts a mirrored left hippocampus on the left side of each panel and the corresponding right hippocampus to which it is compared on the right side. This aspect of the study (inherently) lacks ground truth, and a “correct” result is thus one in which the segmented regions indeed appear to be the portions of one hippocampus which are shaped least like the corresponding portions of the hippocampus to which they are being compared. This is readily evident in the presented figures. The provision of an input device to allow for expert manual segmentation (closed curves confined to the surface) for validation is itself a research problem. This does not mean, however, that we cannot evaluate the quality of the data extracted from the real set. Immediately following, we present a classification analysis as evidence.

2.7.2 Classification Analysis

As it is clear that volume does indeed contain information relevant in the shape comparison of homologues, it is unsurprising that it is a significant feature in distinguishing epileptics from controls. However, it is equally evident that the complex process of
Figure 2-6. Segmentation of distortion between hippocampi of LATL set member.

(a) Top left: naive initialization; Top right: 3 iterations; Middle left: 10 iterations; Middle right: 20 iterations; Bottom: final (50 iterations)

(b) Warp of parametric domain induced by evolution process.

(c) Final segmented distortion energy graph as function over parametric domain (arbitrary units, intercomparable between patients).
Figure 2-7. Segmentation of distortion between hippocampi of RATL set member.

(a) Top left: naive initialization; Top right: 3 iterations; Middle left: 10 iterations; Middle right: 20 iterations; Bottom: final (50 iterations)

(b) Warp of parametric domain induced by evolution process.

(c) Final segmented distortion energy $e$ graph as function over parametric domain (arbitrary units, intercomparable between patients).
Figure 2-8. Segmentation of distortion between hippocampi of control set member. (The smaller of the two segmented regions has been chosen for enlargement.)

(a) Top left: naive initialization; Top right: 3 iterations; Middle left: 10 iterations; Middle right: 20 iterations; Bottom: final (50 iterations)

(b) Warp of parametric domain induced by evolution process, in parametric pixels.

(c) Final segmented distortion energy $e$ graph as function over parametric domain (arbitrary units, intercomparable between patients).
neurodegeneration associated with epilepsy cannot be captured in full so simply. The utility of volume as a sole basis for classification is thus inherently limited. We have provided a fine-grained shape difference characterization with a mind to overcoming precisely this limitation. The test performance of a classifier should thus increase when our extracted features are included as input. Note that we have worked with volume-normalized data to deliberately orthogonalize the extracted features with respect to volume. Since volume itself is assumed to be a relevant feature, it is possible that the best test classification accuracy will be obtained by using volume and local shape difference information in conjunction.

Since we produce a segmented energy function over the entire surface (parametric) domain, we have options as to how to present features to the classifier algorithm. We can collect summary statistics that profit from both segmentation and deformation quantification, such as mean deformation energy ($e$ as defined in Eqn. 2–4) inside and outside of the segmenting curves. Alternately, once the shapes have been volume-normalized and mutually registered, we can compare the (quasi-)continuous $e$ functions across patients by forming feature vectors from the $e$ values at corresponding sample locations. There is also an array of supervised learning methods from which to select a preferred classification algorithm: we choose the ones observed to give best test performance on all feature sets examined, including volume alone (as further detailed below).

Fig. 2-10a demonstrates the success rates in classifying controls vs. epileptics (LATL and RATL groups combined into a single set) and Fig. 2-10b demonstrates analogous results for the problem of separating RATL and LATL members. The training and test statistics were collected through a standard leave-one-out cross validation procedure: test results should be regarded as the reliable performance indicators (and high discrepancy between training and test percentages as evidence of overfitting). The feature vectors being compared are (1) volume alone, (2) volume with a set of 6 summary statistics derived from the algorithm output (total area within the curve(s), total area outside of
Figure 2-9. Epilepsy classification results: optimal test accuracies over all classifiers shown in bold.

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<th>Feature Vector</th>
<th>SVM w/ PB</th>
<th>SVM w/ RB</th>
<th>KFD w/ PB</th>
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</thead>
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<tr>
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<td>79.59</td>
<td>79.63</td>
<td>77.81</td>
</tr>
<tr>
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<td>88.68</td>
<td>80.43</td>
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<td>94.51</td>
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<td>E+VOL+6SUM</td>
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<td>99.97</td>
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</table>

(a) Control vs. epilepsy

<table>
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<th>Feature Vector</th>
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<th>SVM w/ RB</th>
<th>KFD w/ PB</th>
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<tr>
<td>E</td>
<td>100.00</td>
<td>100.00</td>
<td>99.89</td>
</tr>
<tr>
<td>E+VOL+6SUM</td>
<td>100.00</td>
<td>100.00</td>
<td>93.23</td>
</tr>
</tbody>
</table>

(b) LATL vs. RATL

The curve(s), total energy $e$ within the curve(s), total energy $e$ outside of the curve(s), mean energy $e$ within the curve, mean energy $e$ outside of the curve), (3) the function $e$ at 600 sampled locations, and (4) a concatenation of feature vectors (1), (2), and (3). The chosen abbreviations for these feature vectors are ‘VOL’, ‘VOL+6SUM’, ‘E’, and ‘E+VOL+6SUM’, respectively. The classifiers used are (1) support vector machine with polynomial basis (SVM w/ PB), (2) support vector machine with radial basis (SVM w/ RB), and (3) kernel Fisher discriminant with polynomial basis (KFD w/ PB).

The results of both studies make the method’s success clear. In the case of distinguishing epileptics from controls, all experimental feature vectors demonstrated superior test classification accuracy, with a maximum performance of 88.89% on ‘E+VOL+6SUM’ as classified by SVM w/ PB, as compared to 79.56% for ‘VOL’ as classified by SVM w/ RB. Note in particular that feature vector ‘E’ outperforms ‘VOL’ as well, despite the fact that ‘E’ contains no volume information. This portion of the experiment confirms our notion that local distortion information is an important complement to volume data, and suggests that it can in some cases outweigh volume in independent relevance.
While we do not observe this latter phenomenon in the LATL vs. RATL case, we do still observe substantial outperformance of ‘VOL’ by ‘VOL+6SUM’ (90.32% vs. 80.65%), again demonstrating the algorithm’s extraction of clinically relevant shape information not represented by volume. The large difference between training and test accuracies for the other two feature vectors indicates overfitting, which is enabled by their higher dimensionalities. This indicates that an overly fine-grained approach can confound classification on this set, but that subregion identification and characterization remains nonetheless crucial. We have not come across clinical test classification accuracies of this level in any of the literature we have surveyed.
3.1 Motivation for Unification of Image Processing Operations

We have previously discussed the history of the simultaneous segmentation and registration subfield, and noted the nature of the interdependence which motivates unifying the two image processing operations (IPO) in the first place. But given that smoothing (a.k.a. reconstruction) is often performed as a preprocessing step for the benefit of other operations, we might wish to consider whether it too has a rightful place in a unified processing framework. In fact, the natural interdependence of segmentation and smoothing was elucidated in the seminal paper by Mumford and Shah[15], in which the optimal image reconstruction is defined in terms of a segmenting curve or curves specifying the boundaries of the piecewise smooth regions. In the curve evolution implementation of Mumford-Shah minimization by Tsai et al.[25], the mutual assistance of segmentation and smoothing is made explicit: a preliminary solution to each alternately serves as input to the other. As such, tentative smooth reconstructions enable the curve to find object boundaries, and the identification of object boundaries by the curve enables the smooth reconstruction to remain maximally faithful to the initial data. But as already pointed out, registration (indeed, any data-driven IPO) depends on smoothing results, as a smoothing process by definition reconstructs the data upon which it operates. Based on the observations made prior, we can observe the system relationship depicted in Fig. 3-1. All interdependences are direct with the exception of the dependence of smoothing on registration. However, considered in the context of simultaneous segmentation, this relationship clearly emerges.

Despite inherent connection between the three IPOs, and the evident option of unifying them under a single variational principle, such proposals are recent and limited in number (some references which could be considered to fall into this category are Vemuri et al.[27], Pohl et al.[18], and Unal et al.[26]). Moreover, each of the works just cited
contains at least one of the following restrictions, simplifications, or limitations: (1) reliance on explicit atlasing, (2) low-dimensional registration parametrization (i.e. rigid or affine), (3) lack of guaranteed symmetry/unbiasedness with respect to input arguments or consistency (through composition) of registration maps, (4) reliance on piecewise constant reconstruction model, and/or (5) restriction to image pairs (rather than larger sets). Through an extension to the Mumford-Shah functional and the formulation of the problem on a single canonical domain, we have devised a variational framework for simultaneous segmentation, smoothing, and (nonrigid) registration of image suites which eliminates all of the aforementioned limitations. The energy functional can accurately be summarized as a linear combination of an intensity-based matching penalty on the evolving registration maps with Mumford-Shah functionals of each image through the correspondences defined by those maps. An important aspect is that instead of working with $K$ different segmentation/smoothing problems, the registration allows us to solve a single joint smoothing and segmentation problem on a shared domain ($\mathcal{D}$), as dictated by the evolving correspondences.

![Figure 3-1. Nature of interdependence between segmentation, smoothing, and registration.](image)
3.2 Derivation of Framework for Flat Image Sets

In this section, we present our unified variational framework. Let \(\{I_1, \ldots, I_K\}\) denote the set of \(K\) input images, and \(\hat{I}_i\) a smooth reconstruction of the corresponding \(I_i\). The outputs of the USSR algorithm are

1. A collection \(\Phi\) of registration maps (diffeomorphisms) \(\Phi_{ij}\) between \(I_j\) and \(I_i\), \(1 \leq i, j < K\).
2. A collection of smoothed images \(\hat{I}_i\), \(1 \leq i \leq K\).
3. A collection of segmenting contours \(C_1, \ldots, C_K\) on \(I_1, \ldots, I_K\), respectively.

The outputs are subject to the following compatibility constraints:

1. \(C_i = \Phi_{ij}(C_j)\), for \(1 \leq i, j \leq K\).
2. \(\Phi_{ij} = \Phi_{ik}\Phi_{kj}\).
3. \(\Phi_{ii} = \text{id}\) is the identity map for all \(i\).

The compatibility constraints ensure the consistency among all \(K\) images of the results of all three operations given the registration maps \(\Phi_{ij}\) and the segmenting contour \(C_i\). A general form of the energy functional can be written down immediately as

\[
E(\Phi, C, \hat{I}) = \sum_{i=1}^{K} \int_{I_i} |I_i - \hat{I}_i|^2 dA + \sum_{i=1}^{K} \int_{C_i} |\nabla \hat{I}_i|^2 dA \\
+ \sum_{i} \text{Length}(C_i) + \text{MatchingCost}(\Phi, \hat{I}) \\
+ \text{Regularization}(\Phi).
\]

The first three (summation) terms represent the combination of Mumford-Shah functionals of each of the input images. As discussed previously, these terms represent an integrated approach to assessing the quality of smoothing and segmentation on each image. The last two terms measure the quality of registration and its smoothness. The interaction between the registration and segmentation is explicit in the compatibility constraint above, while the interaction between the registration and smoothing is given by \(\text{MatchingCost}(\Phi, \hat{I})\), which measures the quality of the registration using the smoothed versions of the input images. The interdependence of the three operations thus appears naturally.
A direct approach to solving this constrained variational problem is difficult. However, a slight adjustment in perspective allows for a much more efficient computational strategy (Figure 3-2). Instead of computing the $K^2$ registration maps $\Phi_{ij}$ and $K$ contours $C_i$, we will compute $K$ diffeomorphisms $\Phi_i$ between $I_i$ and a canonical domain $D_1$ and a segmenting contour $C$ on $D$. This simplifies the computation enormously while simultaneously satisfying all of the constraints mentioned above: the registration $\Phi_{ij}$ is defined as $\Phi_i \circ \Phi_j^{-1}$ and the segmenting contour on $I_i$ is defined as $C_i = \Phi_i(C)$.

Figure 3-2. Unbiasedness and compatibility can be guaranteed by performing all computations on a canonical domain $D$.

Mathematically, the canonical domain $D$ and the $K$ registration maps provide parameterizations for the $K$ input images with a common domain. Computations on each individual image can now be covariantly formulated on $D$ using the concept of pullbacks.

1 In practice, $D$ is taken to be a rectangular domain with the same size as the images.
For instance, the integral on the left
\[
\int_{I_i \setminus C_i} |\nabla \hat{I_i}|^2 dx dy = \int_{D \setminus C} |\nabla \hat{I_i}(\Phi_i)|^2_{g_i} J_{\Phi_i} dx dy
\] (3–1)
can be calculated on D as on the right hand side. There are two additional factors: the pull-back metric \(g_i^2\), and \(J_{\Phi_i}\) the determinant of the Jacobian of \(\Phi_i\).

Through the \(K\) registration maps \(\Phi_1, \cdots, \Phi_K\), we can “pull all of the integrals back” to the canonical domain, giving an energy function that can be evaluated on D:
\[
E(\Phi, C, \hat{I}) = \sum_{i=1}^{K} \int_{D} |I_i(\Phi(x)) - \hat{I}_i(\Phi(x))|^2 J_{\Phi_i} dA + \sum_{i=1}^{K} \int_{D \setminus C} |\nabla \hat{I}_i(\Phi(x))|^2 J_{\Phi_i} dx + \sum_{i=1}^{K} \text{Length}(\Phi_i(C)) + MC(\Phi, \hat{I}) + \text{Reg}(\Phi).
\]
where \(J_{\Phi_i}\) is the determinant of the Jacobian of \(\Phi_i\). We choose the thin-plate spline (TPS) to model the nonrigid deformation. The \(l\) control points \(\mu \equiv \{c_1, \cdots, c_l\}\) are fixed in \(D\), and each registration map \(\Phi_i\) is uniquely specified by the images of these control points in \(I_i\). One way to regularize the deformation is to use the bending energy
\[
\int \int_{D} \left[ (\frac{\partial^2 \Phi_i}{\partial x^2})^2 + 2(\frac{\partial^2 \Phi_i}{\partial x \partial y})^2 + (\frac{\partial^2 \Phi_i}{\partial y^2})^2 \right] dxdy. \] (3–2)

For the sake of computational efficiency, we use a sum of square intensity differences from the mean smoothed image for the matching cost term. (A preferable but far more computationally costly alternative would be to insert an information theoretic registration penalty based on the joint probability density function defined by the registration maps \(\Phi\) and the smoothed images \(\hat{I}\). However, devising a sufficiently accurate approximation

\[2\] We refer the reader to [16] for the definition of pull-back metric \(g_i\). Briefly, the metric \(g_i\) is the distorted version of the standard Euclidean metric on \(C_i\) under the nonlinear map \(\Phi_i\).
which scales well with \( K \) is itself an ongoing research problem. As such we for now confine ourselves to a single modality.) Finally, the length of the contour \( \Phi_i(C) \) can be computed directly on \( D \) as well. Let \( \gamma(t) \) denote a parametrization of the contour \( C \): the length of the contour \( C_i = \Phi_i(C) \) is then given by the integral

\[
\int_C ds_{\Phi} = \int \sqrt{< d\Phi_i(\dot{\gamma}), d\Phi_i(\dot{\gamma})>} dt,
\]  

(3-3)

where \( d\Phi_i \) is the Jacobian of \( \Phi_i \). The full energy function is thus given as

\[
\mathcal{E}(\Phi, C, \hat{I}) = \sum_{i=1}^{K} \int_D |I_i - \hat{I}_i|^2 J_{\Phi_i} dx \\
+ \beta_1 \sum_{i=1}^{K} \int_D |\nabla \hat{I}_i(\Phi_i(x))|^2 J_{\Phi_i} dx \\
+ \beta_2 \sum_{i=1}^{K} \int_D |\hat{I}_i - \frac{1}{K} \sum_{j=1}^{K} \hat{I}_j|^2 J_{\Phi_i} dx \\
+ \beta_3 \sum_{i=1}^{K} \int_C ds_{\Phi_i} + \beta_4 \text{Reg}(\Phi).
\]

### 3.3 Derivation of System Euler-Lagrange Equations

Let \( I_1, \cdots, I_K \) denote the \( K \) input images. Also we let \( D \) denote the common domain and \( \Phi \) the registration map. \( \Phi = (\Phi_1, \cdots, \Phi_K) \), where \( \Phi_i \) is the registration map \( \Phi_i : D \to I_i \). The notation here is slightly confusing: we are using \( I_i \) to denote both the image (which is a function) and the image domain (which is the function domain). However, the confusion should be easily resolved when the context is clear.

The energy functional is given by

\[
\mathcal{E}(\Phi, C, \hat{I}) = \sum_{i=1}^{K} \int_D |I_i - \hat{I}_i|^2 J_i dxdy + \beta_1 \sum_{i=1}^{K} \int_D |\nabla \hat{I}_i|^2 J_i dxdy + \beta_2 \sum_{i=1}^{K} \int_D |\hat{I}_i - \frac{1}{K} \sum_{j=1}^{K} \hat{I}_j|^2 J_i dxdy \\
+ \beta_3 \sum_{i=1}^{K} \int_C ds_{\Phi_i} + \beta_4 \text{Reg}(\Phi).
\]  

(3-4)

The last two terms do not explicitly depend on \( \hat{I}_i \) and therefore, they do not contribute to the Euler-Lagrange equations. In the equation above, we have abused the notation
slightly using $I_i$ (and likewise for $\hat{I}_i$) to denote $I_i(\Phi_i(x))$. $J_i$ denotes the determinant of the Jacobian of $\Phi_i$, and its appearance in the equation above is to ensure that each integral is equal to the corresponding (push-forward) integral on the images $I_i$. $g_i$ denotes the pull-back of the usual Euclidean metric under $\Phi_i$, and $|\nabla \hat{I}_i|_{g_i}^2$ is the squared norm of the gradient of $I_i$ using this pull-back metric.

The Euler-Lagrange equation will be a system of $K$ equations (treating each $\hat{I}_i$, $1 \leq i \leq K$ as a variable). We will use the formula that the EL equations for an energy functional of the form

$$\int \mathcal{F}(f^1, \ldots, f^K, f^1_x, \ldots, f^K_x f^1_y, \ldots f^K_y) dx dy$$

are the $K$ equations (for $1 \leq i \leq K$)

$$\frac{\partial \mathcal{F}}{\partial f^i} - \frac{\partial}{\partial x} \frac{\partial \mathcal{F}}{\partial f^i_x} - \frac{\partial}{\partial y} \frac{\partial \mathcal{F}}{\partial f^i_y} = 0$$

Back to Equation 1, the contributions from the first and third terms are easy to see since these terms do not have derivatives of $\hat{I}_i$. We compute the EL equation for $I_h$ with $1 \leq h \leq K$. The contribution from the first term

$$\int_D |I_h - \hat{I}_h|^2 J_h dxdy$$

is simply $-2(I_h - \hat{I}_h)J_h$. The contribution from the third term is (let $mI = \frac{\sum_{j=1}^K \hat{I}_j}{K}$)

$$2\beta_2 ( \sum_{i=1, i \neq h}^K - \frac{1}{K} (\hat{I}_i - mI)J_i + (1 - \frac{1}{K})(\hat{I}_h - mI)J_h).$$

Finally, we consider the contribution from the second term. We note that the volume form for the metric $g_i$ is $J_i dx dy$. The second term can be rewritten as

$$T(\hat{I}_h) = \beta_1 \int_D |\nabla \hat{I}_h|_{g_h}^2 J_h dxdy = \beta_1 \int_D |\nabla \hat{I}_h|_{g_h}^2 d\omega.$$
where $\Delta_{g_i}$ is the Laplace-Beltrami operator

$$\Delta_{g_i} \hat{I}_i = g^{ij} \left( \frac{\partial^2 \hat{I}_i}{\partial x_i \partial x_j} - \Gamma^k_{ij} \frac{\partial \hat{I}_i}{\partial x_k} \right),$$

and $\Gamma^k_{ij}$ are the Christoffel symbols. The equation $\Delta_{g_i} \hat{I}_i = 0$ is an elliptic partial differential equation. The first term contains second partial derivatives of $\hat{I}_i$, and the nonlinear second term contains the first partial derivatives of $\hat{I}_i$ and the metric tensor $g_i$.

Putting everything together, the Euler-Lagrange Equation for $\hat{I}_h$ is

$$\beta_1 \Delta_{g_i} \hat{I}_h + (I_h - \hat{I}_h) J_h + \beta_2 \left( \sum_{i=1}^{K} \frac{1}{K} (\hat{I}_i - mI) \frac{J_i}{J_h} - (1 - \frac{1}{K}) (\hat{I}_h - mI) \right) = 0.$$  

We note that for every $i$, $J_i > 0$ everywhere.

### 3.4 Computational Framework Details

Working in the canonical domain $D$ ensures the satisfaction of all compatibility constraints, and the absence of bias towards any image or images. To solve the variational problem, we follow the standard approach of using appropriate gradient flows. In the following, $g_i$ will denote the Riemannian metric on $D$ given as the pullback of the Euclidean metric on $I_i$ by the diffeomorphism $\Phi_i$. The gradient flow with respect to the contour is given by

$$\frac{\partial C}{\partial t} = \frac{1}{2} \sum_{i=1}^{K} \left( (I_i - \hat{I}_i)^{out} - (I_i - \hat{I}_i)^{in} \right)^2 J_{\Phi_i} N$$

$$+ \beta_1 \sum_{i=1}^{K} \left( \|\nabla_{g_i} \hat{I}_i^{out} \|_{g_i}^2 - \|\nabla_{g_i} \hat{I}_i^{in} \|_{g_i}^2 \right) J_{\Phi_i} N$$

$$+ \beta_2 \sum_{i=1}^{K} \left( \hat{I}_i^{out} - \frac{\sum_{j=1}^{K} \hat{I}_j^{out}}{K} \right)^2 \hat{I}_i^{in} - \frac{\sum_{j=1}^{K} \hat{I}_j^{in}}{K}, J_{\Phi_i} N$$

$$- \beta_3 \kappa_i \eta_i N,$$

where $N$ is the unit normal field of the contour, and $\hat{I}_i^{in}, \hat{I}_i^{out}$ are the smoothed images inside and outside of the contour $C$, respectively. In the above, $\kappa_i$ is the curvature of $C$ with respect to the Riemannian metric $g_i$, and $\eta_i$ is a function that captures (details
omitted) the difference between the metric $g_i$ and the standard Euclidean metric on $D$, with respect to which the contour will be evolving (i.e., the normals of the contour $C$ are defined by the Euclidean metric). Both quantities $\kappa_i, \eta_i$ can be easily incorporated into a level set framework. In particular, the curvature $\kappa_i$ with respect to the metric $g$ is given as

$$(\phi \text{ is the level-set function})$$

$$\nabla_g \cdot \frac{\nabla g \phi}{|\nabla g \phi|_g}$$

with the gradient and divergence for the Riemannian metric $g$ given by,

$$(\nabla_g \phi)^i = \sum_k g^{ik} \frac{\partial \phi}{\partial x^k}, \quad \nabla_g V^i = \frac{1}{\sqrt{g}} \sum_k (\sqrt{g} V^k \frac{\partial x^k}{\partial \phi})$$

where $g^{ij}$ are the components of the local metric tensor.

Next, we derive the gradient for the deformation parameter $\mu$ of the thin-plate spline basis function

$$\frac{\partial E}{\partial \mu} = \beta_2 \sum_{i=1}^K \int_D \frac{\partial |\hat{I}_i - \sum_{j=1}^K \hat{I}_j| J_\Phi}{\partial \mu} dx + \beta_3 \sum_{i=1}^K \int_C \frac{\partial \int_C ds_\Phi}{\partial \mu} + \beta_4 \frac{\partial \text{Reg}(\Phi)}{\partial \mu}. \quad (3–6)$$

The last summand can be computed in closed form using Equation 3–2, and the derivative inside of the first integral can be evaluated using the chain rule, resulting in a formula containing image gradients of $\hat{I}_i$. The derivative with respect to the curve length can be derived using Equation 3–3:

$$\frac{\partial \int_C ds_\Phi}{\partial \mu} = \int_t \frac{<\frac{\partial \Phi_i(\gamma)}{\partial \mu}, d\Phi_i(\gamma)>}{\sqrt{<d\Phi_i(\gamma), d\Phi_i(\gamma)>}} dt. \quad (3–7)$$

The latter integral can be evaluated on the contour $C$ for each $\Phi_i$. Finally, the Euler-Lagrange Equation for $\hat{I}_i, 1 \leq i \leq K$ is

$$\beta_1 \Delta_g \hat{I}_i + (I_i - \hat{I}_i) + \beta_2 \omega(J_\Phi, \hat{I}) = 0, \quad (3–7)$$

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where \( \omega(J_{\Phi}, \hat{I}) \) is a term involving the smoothed images \( \hat{I}_i \) and the determinants of the Jacobians \( J_{\Phi_i} \), and \( \Delta g_i \) is the Beltrami-Laplacian operator for the metric \( g_i \). The above equations give a system of nonlinear second-order elliptic equations, which can be solved using an iterative method, such as quasi-Newton [14]. The entire algorithm is summarized below.

**Algorithm Summary**

For an input set of \( K \) intensity images \( \{I_1, \cdots, I_K\} \), the output of the algorithm is a set of \( K \) piecewise-smooth images \( \{\hat{I}_1, \cdots, \hat{I}_K\} \), a set of \( K \) registration maps \( \{\Phi_1, \cdots, \Phi_K\} \) between the corresponding \( I_i/\hat{I}_i \) and the shared domain \( D \), and a closed contour \( C \) in \( D \).

Those output parameters are evolved as follows:

1. Optimize motion parameters \( \mu \) using gradient descent with gradient given by Equation 3–6. Update the deformation fields \( \Phi_i \).
2. Evolve the level set function \( \phi \) using Equation 3–5. The contour is updated as the zero level set of \( \phi \).
3. Solve the nonlinear elliptic PDEs in Equation 3–7. This updates the smoothed images \( \hat{I}_i \).
4. If the difference between consecutive iterates is below a pre-chosen tolerance, stop, else go to Step 1.

### 3.5 Results on Image Sets

We here present visual examples of simultaneous smoothing, segmentation, and registration, obtained through a level set contour evolution and thin-plate spline implementation. In Figures 3-3 and 3-4, the inputs are trios of hands making similar gestures, but in visibly different places and manners\(^3\). The segmentation, smoothing, and registration results are all contained directly in the displayed outputs: recall as well that the segmentation curves correspond across all images in a set through the registration maps. The examples shown here therefore demonstrate the success of the registration in matching even such fine features as single fingers, when those fingers are undergoing the

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\(^3\) Thanks to M.R.C. Tarrosa for acquiring the input images.
fairly complex motion of spreading apart from one another. The (correct) identification of
the fingers in the first column of Figure 3-4 is possible only because of the coupling of the
segmentation of that image to the segmentation of the other two (in which the gaps are
more readily distinguished).

Of course, dense correspondence is also implicitly being obtained between the hand
features (e.g. the crease of the palm) as part of this process. The third sample case
makes this correspondence explicit: in Fig. 3-5, we see the isolation of brain ventricles
in human MRI cross sections. Note that ventricle identification in the second (middle)
image is far more difficult than in the first or third, and yet an excellent result is achieved,
thanks to the process coupling. In Fig. 3-6, two alternate methods of visualizing the dense
registration alignment are provided. On the left, we have the effect of each $\Phi_i$ on a set
of concentric circles resident in shared domain $\mathbf{D}$. Since $\Phi_{ij} = \Phi_i \circ \Phi_j^{-1}$ for all $i$ and $j$,
this display provides a visual intuition of how structures in one image are carried onto
another. On the right, we have the initial and final average image as seen from domain
$\mathbf{D}$ (i.e. the average pullback). As $\mathbf{D}$ is the domain of correspondence, note that the
sharpness of the pullback is a visual method of assessing the success of the registration in
matching intensity values. As we are here segmenting the ventricles, it is the improvement
in clarity and definition of the ventricles with which we are concerned. This effect is
clearly evidenced. (Note that the unusual warping of the ventricles as seen from this
perspective is of no significance. The “implicit atlas” which resides in the domain is
merely a correspondent intermediary between input images, and need not represent a
realistic input, merely a smooth deformation of a realistic input.)
Figure 3-3. Processing of variations on a hand position. Each column represents the evolution for a single input image in the trio, with the original input on top, and the final smooth reconstruction at bottom. The evolving segmentation and registration are depicted through the red curve and blue grid, respectively. Note the success in corresponding the fingers between all images.
Figure 3-4. Processing of various openings of the hand, organized as in Figure 3-3. Note the identification of the fingers in the first column despite their near closure in the input image, under heavy noise. This owes to the “implicit atlasing” inherent in the method.
Figure 3-5. Identification of the ventricles of the brains of three different patients (in respective rows) in cross-sectional MRI scans. Note the successful identification in the case of the second input, despite the relative fineness of the structure. The final smoothed images are displayed in the last column.

Figure 3-6. Display and analysis of inter-image registrations for brain MRI dataset displayed in Fig. 3-5.

(a) Concentric circles as warped by the final $\Phi_i$ for each $i$ (row) in Fig. 3-5, $i$ increasing from left to right. This is one method of visualizing the consistent correspondences between all three pairs.

(b) An alternate visualization of registration success, as the sharpness of the average intensity pullback of the structure being segmented. On the left is the initial average (under identity mappings), on the right the converged result. The sharpening of the ventricle is dramatic.
4.1 Motivation for Extending USSR to Shape Analysis Problem

In Ch. 2, we have presented a novel approach to shape comparison which benefits from being conceptualized around local notions of shape characterization. As such we are able to provide a fineness of granularity beneficial in many applications, including the chosen example of hippocampal asymmetry measurement. However, one could note some aspects of the proposed framework which invite either extension or improvement:

- **Lack of theoretical symmetry/unbiasedness.** A glance at the variational principle used in the previously proposed method, Eqn. 2–3, will reveal the fact that of the two surfaces accepted as inputs to the algorithm, one is arbitrarily chosen as the domain of energy evaluation and integration. A measure of asymmetry between two surfaces should itself be theoretically symmetric with respect to permutation of those surfaces, yet the principle cannot guarantee this as stated.

- **Restriction to two-mean framework.** As an extension to the Chan-Vese image segmentation scheme, the proposed method can parcellate surfaces only into regions of either high or low asymmetry, modelling each only on the basis of its mean asymmetry level. While there have been proposals for the extension of the mean-value framework for image segmentation to a fixed number of regions greater than two (Chung and Vese, [5]) and a varying number of regions which is itself optimized over the course of the minimization (Brox and Weickert, [2]), and while it is true that such implementations could be applied directly to the scheme under discussion, they would still not eliminate the simplification of mean-based modelling.

- **Restriction to surface pairs.** There is no reason to presume that a user of a surface shape comparison method will be interested only in comparing pairs of objects. One might instead wish to identify regions of high shape variability across subpopulations. Yet the algorithm as described does not admit more than two inputs, and there is no trivial extension to the multiple input case.

Fortunately, the above list is a subset of the limitations of the prior art in unified smoothing, segmentation, and registration as enumerated in Sec. 3.1. That set of limitations is precisely what the USSR algorithm of Ch. 3 was designed to address, in the domain of image processing. Since our original proposal for local shape asymmetry analysis was based on moving standard image processing techniques to the manifold and having them operate on the metric structures of those manifolds, we might look to adapt
USSR similarly. In so doing, we would produce a technique similar in functionality to the proposal of Ch. 2, but with certain practical and theoretical superiorities.

4.2 Modification of Variational Principle

Since the proposals of Chs. 2 (local asymmetry analysis) and 3 (USSR) are both variational in nature, their synthesis will be as well. The bulk of the task of defining the extended shape analysis method is the specification of the variational principle to be minimized. The first shape analysis proposal was defined through Eqns. 2–3 and 2–4, which set the form of the functional in terms of the deformation energy, and defined that deformation energy, respectively. The modified proposal will be defined through two analogous equations, as follows.

4.2.1 Movement of Computation to Shared Canonical Domain

Glancing at the variational principle driving the original asymmetry analysis, Eqn. 2–3, one sees that of the two surfaces input to the method, $S_1$ and $S_2$, $S_1$ is arbitrarily chosen as a reference for computation and integration of the asymmetry function ($e$, Eqn. 2–4), and its parametric domain $P_1$ used as the domain of computation. Symmetry with respect to the order of the surfaces therefore cannot be guaranteed. One might note that there is at least one trivial solution to this issue, which is to change the functional to the following:

$$
\int_{S_1 \setminus \gamma_1} e \, dA + \alpha \int |d\gamma_1(t)/dt|_{g_1} \, dt + \beta \left( \int_{S_{1\text{in}}} |e - \bar{e}_{in}|_{g_1}^2 \, dA + \int_{S_{1\text{out}}} |e - \bar{e}_{out}|_{g_1}^2 \, dA \right)
$$

$$+
\int_{S_2 \setminus \gamma_2} e \, dA + \alpha \int |d\gamma_2(t)/dt|_{g_2} \, dt + \beta \left( \int_{S_{2\text{in}}} |e - \bar{e}_{in}|_{g_2}^2 \, dA + \int_{S_{2\text{out}}} |e - \bar{e}_{out}|_{g_2}^2 \, dA \right)
$$

However, repeating computation and integration of all quantities on the other domain is needlessly expensive and lacks theoretical elegance. Another price paid for such a heuristic solution would be the lack of a clear extension to the case of multiple (more than two) input surfaces. Something more general must be sought.

Facing the same issue regarding multiple image registration, USSR took the approach of moving all function computations and integration to a shared canonical domain. But
thanks to the deliberate casting of the shape analysis problem into a framework essentially equivalent to that of image segmentation and registration, as discussed in Sec. 2.4, USSR’s approach can be applied without much difficulty. Pictorially, the framework will change from the depiction in Fig. 2-1 to that of Fig. 4-1.

Figure 4-1. The reformulated version of the local asymmetry analysis proposal: All $K$ input surfaces are corresponded through the $K$ maps $f_i$ between their parametric domains $P_i$ and shared canonical domain $D$. Using the metric information of the $g_i$, we can define an energy function over $D$ which can be simultaneously segmented via $\gamma$, which in turn appears on each $S_i$ as $\gamma_i$ through $f_i$, segmenting the surfaces themselves based on group asymmetry structure.
At its most general level, the system energy functional will now take the form

\[ \mathcal{E}(f_1, \cdots, f_K, \gamma) = \int_{D \setminus \gamma} e \, dA + \alpha \int_D \text{SegmentationEnergy} \, dA + \beta \int_\gamma \text{CurveRegularizationEnergy} \, ds \]

which by design accepts input of an arbitrary number of surfaces \( K \). To complete the specification of the modified variational principle, we will need to redefine asymmetry measure \( e \) so as to be computable from domain \( D \) and unbiased with respect to the multiple surfaces parametrized by \( D \), and specify the scheme by which the \( e \) field will be segmented over \( D \). Secs. 4.2.2 and 4.2.3 discuss the manner in which this is done.

### 4.2.2 Functional Form

The form of the original shape analysis functional (omitting explicit domain metric notation, weighting constants, etc.) is

\[
\int_{\Omega \setminus C} e \, dA + \int_{\Omega_{in}} |e - \bar{e}_{in}|^2 dA + \int_{\Omega_{out}} |e - \bar{e}_{out}|^2 dA + \int_{C} ds
\]

for domain \( \Omega \), curve \( C \), and asymmetry function \( e \). The second and third term represent the data terms from Chan-Vese segmentation as applied to the surface asymmetry parcellation problem: as such they are the source of the two-mean restriction. In the USSR framework, this restriction was avoided by recognizing the commonly-used Chan-Vese (“cartoon”) model as a limiting case of the general Mumford-Shah functional, and implementing the minimization of the latter instead. Using the above terminology, the form of Mumford-Shah is

\[
\int_{\Omega \setminus C} |\nabla \hat{e}|^2 dA + \int_{\Omega} |\hat{e} - e|^2 dA + \int_{C} ds
\]

with the first term representing reconstruction smoothness, the second data fidelity, and the third curve length. The data terms in Chan-Vese correspond directly to the data
and smoothness terms in the above (there is no separate smoothness term in Chan-Vese because the limiting assumption of piecewise constancy has already been incorporated through the region means). Therefore the basic form of our revised functional will be produced by substituting the latter pair for the former:

\[
\int_{\Omega \setminus C} e \, dA + \int_{\Omega \setminus C} |\nabla \hat{e}|^2 dA + \int_{\Omega} |\hat{e} - e|^2 dA + \int_{C} ds
\]

### 4.2.3 Asymmetry Measure

In Sec. 4.2.2, we have specified the form of the extended shape analysis functional in terms of the function representing local deviation from isometry, \( e \). However, the definition of \( e \) given by Eqn. 2–4 is binary in the sense of input arguments, implicitly accepting exactly two locations, one from each parametric domain of the surfaces under comparison. This leads to the restriction of the entire algorithm to input pairs, as indicated in Sec. 4.1.

Since we seek to enable the processing of arbitrarily sized “populations” of surfaces, \( e \) must be redefined in terms of a variable number of arguments. But USSR has already addressed a similar issue, in defining a single registration data term for arbitrarily many images (a term in Eqn. 3–4). For (smoothed) images \( \hat{I}_1 \cdots \hat{I}_K \), this data penalty function was given as the sum of differences of individual images from the mean intensity image:

\[
\sum_{i=1}^{K} |\hat{I}_i - \frac{\sum_{j=1}^{K} \hat{I}_j}{K}|^2
\]

Using USSR’s registration data term as inspiration, we can replace Eqn. 2–4 with

\[
e = \sum_{i=1}^{K} |J^i G_i J - \sum_{j=1}^{K} \frac{J^j G_j J}{K}|^2
\]

(4–1)

Just as, in the case of images, Eqn. 2–4 is designed to assign high registration energy to pixel locations (as corresponded through the maps) with high variation in intensity across images, Eqn. 4–1 will, in the shape comparison context, tend to penalize regions of large FFF difference across the \( K \) (arbitrarily many) input shapes.
Related to Eqn. 4–1, there exists an important question regarding the weighting (area) of the pixel elements in the canonical domain. In the version of the algorithm in Ch. 2 (as specified by Eqn. 2–3) the issue is rendered trivial by the fact that integration is arbitrarily carried out on $S_1$, which, under a piecewise constant quadrature scheme, simply equates to summing the per-pixel deformation energies on parametric domain $P_1$ under weighting by the areas of the corresponding surface elements. Since those areas are simply given by the determinants of the FFFs stored at the pixels, in discrete form the integral $\int_{S_1} f dA$ becomes the summation term $\sum_{P_1} f|G|$, where the summation is of course over domain pixel elements, with implicit correspondence between the pixels and FFF matrices $G$. Under that scheme, the presence of a bias in the energy calculation/integration was allowed for. However, this is precisely one of the listed limitations to be eliminated in the modification we are proposing (and is partly the motivation of the use of the canonical domain in Sec. 4.2.1). The new $e$ function and the integration thereof must therefore remain unbiased with respect to the input surfaces while capturing in an intuitive manner the fact that canonical pixels represent comparisons between surface patches of variable size (which should be accounted for in their integration). Weighting the domain pixels by the mean element area is such a solution, and is the one we adopt. As such, the quadrature scheme will involve the factor $\sum_{j=1}^{K} |J^T G_j J|$.  

4.2.4 Final System Variational Principle

Using the developments of Sec. 4.2.1, 4.2.2, and 4.2.3, and in particular the definition of $e$ given by Eqn. 4–1, we can state the modified system variation principle
as follows:

\[ E(f_1, \cdots, f_K, \gamma) = \int_{D \setminus \gamma} e \sum_{j=1}^{K} \frac{J'G_j J}{K} |dA| + \beta_1 \int_{D \setminus \gamma} |\nabla \hat{e}|^2 \sum_{j=1}^{K} \frac{J'G_j J}{K} |dA|
\]
\[ + \beta_2 \int_{D} |\hat{e} - e|^2 \sum_{j=1}^{K} \frac{J'G_j J}{K} |dA|
\]
\[ + \beta_3 \int_{C} ds \]  \hspace{1cm} (4–2)

where \(dA\) is the trivial uniform metric on \(D\). To represent the areas of the surface patches under comparison in an unbiased manner, this metric is weighted by \(\sum_{j=1}^{K} |\frac{J'G_j J}{K}|\).

4.3 Extension of Mumford-Shah to Case of Weighted Integral

4.3.1 Necessity

In Sec. 4.2.2, we have noted the fact that a Mumford-Shah-based segmentation scheme forms a component of the modified asymmetry analysis proposal as codified in the variational principle of Eqn. 4–2. Theory and practice for the minimization of the classic Mumford-Shah functional are well-established (as in [15] and [25]), but the terms appearing in the variational principle proposed here carry an important alteration which demands a corresponding alteration in the theory. As discussed in Sec. 4.2.3, the domain of computation \(D\) cannot be justifiably considered “flat” in that each pixel corresponds to \(K\) (number of input surfaces) different surface patches, each in general of different area. As such, the integrals (approximated by summations in the discrete case) are weighted by a function defined over \(D\) chosen to reflect the sizes of the area elements under correspondence (as detailed, again, in Secs. 4.2.3 and 4.2.4). The price of being able to perform Mumford-Shah minimization over the familiar two-dimensional pixel grid is that the Euler-Lagrange equation used in the curve evolution of [25] must be rederived for the case of weighted integration. This is done below, in Sec. 4.3.2. An interesting consequence of the extension is revealed, in that an additional term (henceforth referred to
as the “metric correction operator”) arises in the differential equation. Additionally, it is
easily noted why this operator reduces to zero in the special case of unity weighting.

### 4.3.2 Derivation

We will here derive the Euler-Lagrange equation for the general case of a functional
consisting of the Mumford-Shah data and smoothness terms with weighted integration:

$$
E(\hat{I}) = \beta \int_{D} |I - \hat{I}|^2 g dA + \alpha \int_{D \setminus C} |\nabla \hat{I}|^2 g dA,
$$

where $g$ represents the arbitrary weighting function defined over the domain of integration
$D$, within which exists closed partitioning curve(s) $C$. The derivation is a simple matter of
using standard variational calculus in conjunction with a simple application of the formula
for integration by parts in $\mathbb{R}^2$:

$$
\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\partial \Omega} u \nabla v \cdot \nu d\sigma - \int_{\Omega} u \Delta v dx,
$$

where $u$ and $v$ are two functions defined on domain $\Omega$, and $\nu$ is the normal vector of the
boundary $\partial \Omega$ of $\Omega$. This equation is also referred to as the first Green’s identity.

Now, we seek a critical point of $E$ with respect to $\hat{I}$, in a functional sense. That is, for
scalar $t$ and any function $h$ defined on $D$,

$$
\lim_{t \to 0} \frac{E(\hat{I} + th) - E(\hat{I})}{t} = 0.
$$

We can easily write a first-order approximation to $E(\hat{I} + th)$ in $t$ around $t = 0$:

$$
E(\hat{I} + th) \approx E(\hat{I} + th)|_{t=0} + t \left[ \frac{\partial E(\hat{I} + th)}{\partial t} \right]_{t=0}
$$

$$
= E(\hat{I}) - \beta t \left[ 2 \int_{D} g(I - \hat{I} - th) h dA \right]_{t=0} + \alpha t \left[ 2 \int_{D \setminus C} g(\nabla(\hat{I} + th)) \nabla h dA \right]_{t=0}
$$

$$
= E(\hat{I}) - 2\beta t \int_{D} g(I - \hat{I}) \cdot h dA + 2\alpha t \int_{D \setminus C} g \nabla \hat{I} \cdot \nabla h dA
$$
Substituting into the critical point equation, we have

\[
\lim_{t \to 0} \frac{\mathcal{E}(\hat{I} + th) - \mathcal{E}(\hat{I})}{t} = \lim_{t \to 0} \frac{\mathcal{E}(\hat{I}) - 2\beta t \int_{D} g(I - \hat{I}) \cdot h dA + 2\alpha t \int_{D \setminus c} g \nabla \hat{I} \cdot \nabla h dA - \mathcal{E}(\hat{I})}{t} \\
= 2 \int_{D \setminus c} (\alpha g \nabla \hat{I} \cdot \nabla h - \beta g(I - \hat{I})h) dA \\
= 0
\]

Now, we make use of the first Green’s theorem, through appropriate identification of factors in the above expressions with those in the theorem’s general form as given:

\[
\nabla v = g \nabla \hat{I} \\
u = h
\]

Then Green’s theorem tells us that

\[
\int_{\partial D} hg \nabla \hat{I} \cdot v d\sigma = \int_{D} (h \nabla \cdot (g \nabla \hat{I}) + \nabla h \cdot g \nabla \hat{I}) dA
\]

But if we impose a Neumann boundary condition at the region boundaries, then by definition we have

\[
\nabla \hat{I} \cdot v d\sigma = 0
\]

and so

\[
\int_{D} (h \nabla \cdot (g \nabla \hat{I}) + \nabla h \cdot g \nabla \hat{I}) dA = 0 \\
\int_{D} h \nabla \cdot (g \nabla \hat{I}) dA = -\int_{D} \nabla h \cdot g \nabla \hat{I} dA
\]

This allows a substitution into the expression derived from the critical point equation:

\[
\int_{D} (\alpha g \nabla \hat{I} \cdot \nabla h - \beta g(I - \hat{I}) \cdot h) dA = \int_{D} (-\alpha h \nabla \cdot (g \nabla \hat{I}) - \beta g(I - \hat{I})h) dA = 0
\]

Using the general fact that for any functions \( u \) and \( v \),

\[
\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \Delta v
\]
(a two-dimensional version of the product rule), we can restate this as

$$\int_D (\beta gh \hat{I} - \beta gh I - \alpha h \nabla \hat{I} \cdot \nabla g - \alpha gh \Delta \hat{I}) dA = \int_D h(\beta g \hat{I} - \beta g I - \alpha \nabla \hat{I} \cdot \nabla g - \alpha g \Delta \hat{I}) dA = 0$$

Since this equation must hold for arbitrary $h$, the other term in the integrand must be zero identically:

$$\beta g \hat{I} - \beta g I - \alpha \nabla \hat{I} \cdot \nabla g - \alpha g \Delta \hat{I} = 0$$

This is the Euler-Lagrange equation for the general case of the Mumford-Shah terms under integration weighted by function $g$. For the case in which $g$ represents an area element size (positive definite), division by $g$ is well-defined, and we can write the expression in the form

$$\beta \hat{I} - \beta I - \alpha \frac{\nabla g}{g} \cdot \hat{I} - \alpha \Delta \hat{I} = 0$$

$$\hat{I} - \frac{\alpha}{\beta} \frac{\nabla g}{g} \cdot \nabla \hat{I} - \frac{\alpha}{\beta} \Delta \hat{I} = I$$

$$(1 - \frac{\alpha}{\beta} \frac{\nabla g}{g} \cdot \nabla - \frac{\alpha}{\beta} \Delta) \hat{I} = I \quad (4-3)$$

Note that in the special case in which $g$ is a constant function (e.g. unity), the above reduces to

$$(1 - \frac{\alpha}{\beta} \Delta) \hat{I} = I,$$

which is the Euler-Lagrange equation used in the descent in [25]. The operator $\frac{\alpha}{\beta} \frac{\nabla g}{g} \cdot \nabla$, which acts upon $\hat{I}$, can be thought of as the metric correction operator for the more general case.

4.4 Changes in Implementation

Though the proposal under discussion involves some important conceptual distinctions from the original shape analysis method of Ch. 2, the conversion of the implementation of the latter into that of the former can be structured so as to involve a minimal number of relatively confined changes. One obvious (and trivial, in the sense of implementation) point of alteration is that the definitions used in the computation of quantities $\mathcal{E}$ and $e,$
given by Eqns. 2–3 and 2–4, must be replaced by Eqns. 4–2 and 4–1 respectively. Beyond this, the only noteworthy structural changes to the program involve the allowance of a variable number $K$ of input surfaces (as point clouds, all preprocessed identically with resultant parameterizations as described in Sec. 2.6.1), the maintenance and calculation of the $K$ map functions from shared domain $D$ to each of these surfaces, the change in the optimizer’s Hessian sparsity structure to account for those changes in the structure of the map vector, and the replacement of the Chan-Vese-based curve evolution module with the Mumford-Shah-based approach appropriate for the new system energy.

4.4.1 Alteration of Map Representation and Hessian Structure

The essential nature of the map representation remains the same as was initially presented in Sec. 2.6.2: we are still working with maps between intrinsically two-dimensional surfaces by representing them as discrete vector fields over the parameterizing domains. In the framework of Ch. 2, there existed a single map from $S_1$ to $S_2$ represented as a 2-vector function over the parametric domain of $S_1$, $P_1$, representing the corresponding locations of each pixel on $P_2$. With a discrete sampling grid of size $m \times n$, two components in the function, and a zero-order boundary condition on the first and last columns, the map variables could be represented as a single $2m(n - 1)$-vector (the concatenation of the columnwise-vectorized row $(u)$ components followed by likewise vectorization of the variable column $(v)$ components). Of course, with known sampling size and map variable vector structure, it is trivial to convert back and forth between the vector (for input to the optimization method) and grid (for interpolation) representations as necessary.

To represent the map variables in the new version of the algorithm, little need change: instead of representing a single map from $P_1$ to $P_2$, we must now represent $K$ maps, each from canonical domain $D$ to one of the $K$ $P_i$. Each of these maps can be structured in a manner identical to the above, producing a total of $2Km(n - 1)$ components. For convenient reshaping between the vector and grid representations, we order these components via successive concatenations of columnwise-vectorized $u$-component grids.
through $K$, followed likewise by concatenations of the vectorized $v$-component grids, as in Fig. 4.2.

![Diagram](image)

Figure 4.2. The structure of the argument map vector, used as input to the optimization method and with respect to which the system Hessian is defined (shown transposed into a row vector). The $u$ coordinates of the $K$ maps ($K$ segments with $mn$ elements each) are concatenated first, followed by the variable $v$ coordinates likewise ($K$ segments of $m(n-2)$ elements), for a vector of total length $2Kmn(n-1)$.

However, alteration of the map vector structure also necessitates alteration of the Hessian sparsity structure input to the optimizer for efficiency purposes (as explained and diagrammed in Sec. 2.6.3). Following the earlier logic, however, the correction is not difficult. We can first partition the Hessian into $(2K)^2$ submatrices, where each submatrix is defined by having its rows correspond to one component (either $u$ or $v$) of a single map $f_i$, and likewise its columns to one component of $f_j$ ($j$ possibly equal to $i$). Of these submatrices, we can then identify four types: (1) the row coordinates of the entries correspond to $u$ coordinates of $f_i$ and the column coordinates of the entries correspond to $u$ coordinates of $f_j$, (2) the row coordinates of the entries correspond to $u$ coordinates of $f_i$ and the column coordinates of the entries correspond to $v$ coordinates of $f_j$, (3) the row coordinates of the entries correspond to $v$ coordinates of $f_i$ and the column coordinates of the entries correspond to $u$ coordinates of $f_j$, and (4) the row coordinates of the entries correspond to $v$ coordinates of $f_i$ and the column coordinates of the entries correspond to $v$ coordinates of $f_j$. Let us call each of these submatrices $H_{uu}, H_{uv}, H_{vu}, H_{vv}$, respectively: their structures are the top left, top right, bottom left, and bottom right submatrices of the matrix depicted in Fig. 2.4. The new system Hessian is obtained from the old through replacement of each submatrix with a $K \times K$ block of replications of that matrix.
4.4.2 Switch from Chan-Vese to Mumford-Shah Segmentation

The most involved implementation alteration necessitated by the modified proposal is the replacement of the Chan-Vese-based curve evolution method with one based on Mumford-Shah energy minimization. Tsai and Yezzi have demonstrated in [25] how a level set curve evolution implementation of Mumford-Shah minimization can be formulated for image functions over bounded rectangular domains (with naturally uniform associated metrics). In Sec. 4.3.2, we have demonstrated that the difference between the descent equation for the image reconstruction (for fixed curve position) in the general case of integral weighting by a general $C^1$ function and the special (and common) case of uniform unit weighting is the appearance of a single additional linear operator defined in terms of the weighting function and its first derivative. This means that we can easily base the implementation of the curve evolution (and now, simultaneous energy reconstruction) module of the modified asymmetry analysis proposal on the Tsai-Yezzi implementation through appropriate inclusion of the metric correction operator derived in the referenced section, and implementation of the $u$-periodicity demanded by the cylindrical topology of the parametric domain. Since [25] omits most of the implementation details of functional minimization, and since this method forms the core of the curve evolution and piecewise-smooth energy reconstruction module, we will first discuss our chosen implementation approach in Sec. 4.4.2.1, and the inclusion of the metric correction operator in 4.4.2.2.

4.4.2.1 Implementation Details for Mumford-Shah Minimization

For an image function $I$, we have already seen that for fixed curve $C$, the equation for the optimal piecewise smooth reconstruction $\hat{I}$ is

$$(1 - \frac{\alpha}{\beta} \Delta_{dis})\hat{I} = I,$$

where $\hat{I}$ and $I$ are in columnwise-vectorized form, $1$ is the appropriately sized identity matrix, and $\Delta_{dis}$ is a modified version of the usual discrete Laplacian operator. It is easy
to see that once that modified Laplacian has been specified, \( \hat{I} \) can be solved for directly (or, for the sake of computational efficiency, approximated by an iterative method to a prespecified error tolerance: the method of conjugate gradients serves this purpose well). The formation of \( \Delta_{\text{dis}} \), then, is the only nontrivial portion of optimal image reconstruction, for a fixed curve position.

\( \Delta_{\text{dis}} \) differs from the usual Laplacian operator \( \Delta \) in that it must respect the Neumann boundary condition across the curve \( C \). In a discrete implementation, over a regular grid, a sensible way of enforcing this condition is to nullify any finite difference derivatives which involve quantities from opposite sides of the segmenting curve. This discrete notion of the Neumann condition is what produces the difference between \( \Delta \) and \( \Delta_{\text{dis}} \). However, it is easy to see that the ordinary Laplacian can be formed through the linear combination of standard finite difference operators:

\[
\Delta = D_{FU} - D_{BU} + D_{FV} - D_{BV}
\]

where the four terms on the right side are the forward difference in \( u \), backward difference in \( u \), forward difference in \( v \), and backward difference in \( v \), respectively. But these four matrices can easily be modified to respect the discrete Neumann condition through left multiplication by a diagonal boolean matrix indicating pixel locations at which the difference operator crosses the curve boundary: in the level set representation, a curve crossing simply means a change of sign of the level set function at the two pixels used in the difference equation (the nonzero entries in the corresponding difference operator row). These “zeroing” matrices are of course a function of the current position of the curve, and must thus be recalculated at each step. For each curve position, there are four such matrices, one to act on each of the four derivative matrices used in forming the Laplacian, and we can then simply recalculate the disconnected Laplacian for the current curve position as

\[
\Delta_{\text{dis}} = Z_{FU} D_{FU} - Z_{BU} D_{BU} + Z_{FV} D_{FV} - Z_{BV} D_{BV}
\]
Solving for \( \hat{I} \) then yields the optimal piecewise smooth function reconstruction, in accordance with weighting constants \( \alpha \) and \( \beta \) as set in the variational principle.

For any curve position, then, we are able to calculate \( \hat{I} \), which allows evaluation of the system energy. This in turn enables us to implement the descent with respect to the curve position, which is

\[
\frac{\partial C}{\partial t} = \frac{\beta}{2} \left( (I - \hat{I}_{\text{out}})^2 - (I - \hat{I}_{\text{in}})^2 \right) N \\
+ \frac{\alpha}{2} \left( \| \nabla \hat{I}_{\text{out}} \|^2 - \| \nabla \hat{I}_{\text{in}} \| \right) N \\
- \gamma \kappa N
\]

in continuous form. The computation of curvature \( \kappa \) can be done in the level set framework as discussed in detail in [20]. The computation of the other terms is straightforward once a method for calculating \( \hat{I}_{\text{out}} \) and \( \hat{I}_{\text{in}} \) has been determined. These terms refer to the reconstructed function values at the curve boundary if they are considered to lie outside or inside of the curve, respectively. In the discrete implementation, of course, function values are only provided at pixel locations, and these will in general not lie “on” the curve: our concern is whether at any step, a pixel adjacent to the curve should be moved to the other side of it. A sufficiently accurate and fast approach to computing (defining, really) the terms \( \hat{I}_{\text{out}} \) and \( \hat{I}_{\text{in}} \), then, is to advance the curve outward everywhere by one pixel (bringing all border-adjacent outside pixels to the interior), recompute \( \hat{I} \) to serve as \( \hat{I}_{\text{in}} \), and to repeat the process using inward curve motion for computation of \( \hat{I}_{\text{out}} \). With these quantities, the energies of inward and outward motion can be compared for all bordering pixels, and the curve updated accordingly. Again, numerical methods for level set curve maintenance and updating are found in the provided reference.

4.4.2.2 Inclusion of Periodicity and Metric Correction Operator

Relative to the implementation described in Sec. 4.4.2.1, there is nothing to do to produce the curve evolution and piecewise smooth function reconstruction module of the modified proposal but to implement the \( u \) periodicity necessitated by the domain topology
and the metric correction operator derived in Sec. 4.4.2.2. The former is simply a matter of appropriately altering the finite difference operators throughout the program (including in the construction of $\nabla_{\text{dis}}$): call the modified difference operator matrices $D_{F_U\text{per}}$ and $D_{B_U\text{per}}$. For the latter task, the continuous operator $\frac{\nabla_{\text{dis}} g}{g} \cdot \nabla_{\text{dis}}$ is given in discrete form as

$$\frac{Z_{F_U\text{per}} D_{F_U\text{per}} + Z_{B_U\text{per}} D_{B_U\text{per}}}{2} g Z_{F_U\text{per}} D_{F_U\text{per}} + Z_{B_U\text{per}} D_{B_U\text{per}} 2 + \frac{Z_{F_V} D_{F_V} + Z_{B_V} D_{B_V}}{2} g Z_{F_V} D_{F_V} + Z_{B_V} D_{B_V} 2$$

We then need only include this term in the discrete implementation of solution of Eqn. 4–3, and otherwise proceed as directed in Sec. 4.4.2.1.
CHAPTER 5
CONCLUDING REMARKS

In this dissertation, we have presented a method for local shape analysis of homologous structures, a general image processing framework for groups of images in which suboperations provide mutual assistance, and a proposal for improving the former using the insights of the latter. Through visual observation and statistical validation, we were able to demonstrate the practical benefits of the novel algorithms. If there is an overarching theme in their formulation, it is generality: while the implementation details given herein (certainly in the case of Ch. 2) are specific to the preprocessing steps and formatting/representation of the data, the formulations themselves remain quite broadly applicable. Far from being specific to the hippocampi of epileptics, the shape analysis proposal is conceived around general genus zero surfaces for which symmetry is expected. This of course covers a wide variety of practical applications, in addition to the theoretical contribution. As we have noted in Sec. 2.4.1, the particular application and representative dataset chosen did influence formulation in that the higher-frequency SFF information was completely discarded, as both numerically problematic and somewhat conceptually irrelevant in the case of that application. However, if desired for either theoretical or practical reasons, one could reincorporate this aspect of the shape characterization into the formulation in a straightforward manner. (Again, the caveat is that to do so implicitly places substantial trust both in the accuracy of the curvature data represented by the input shape and in the importance of that data in characterizing the shape.) Likewise, the USSR proposal is arguably the most general image processing framework of its sort heretofore suggested. While certain aspects of the “theoretical implementation” we have used, such as the use of thin-plate spline regularization and sum-of-square-differences intensity data penalty on the registration component, do influence/limit the method’s performance and scope of applicability, they can easily be modified as seen fit. For instance, if one wishes to pay the higher computational cost of an information-theoretic
match measure in exchange for the multi-modality it would afford, the substitution is trivial. It is perhaps due to the general applicability of these methods in their respective (and distinct) domains that their synthesis can be formulated so naturally. But in emphasizing generality, we should not neglect the role of recognition of interdependence. As the field of computer vision has matured, its individual subareas have become increasingly (and at times exhaustingly) well studied and understood. But there remains a good deal of unexplored space between these topics, and with it a chance to yield a whole greater than the sum of its parts.
REFERENCES


BIOGRAPHICAL SKETCH

Nicholas Lord has been, above all things, poor at waking up to alarm clocks. He did not grow up with any particular academic direction, and to the best of his knowledge still does not possess one. He likes the fact that ‘Caucasian’ rhymes with ‘Jamaican’. He can do the Frug, Robocop, and Freddie, but cannot do the Smurf. He is seldom accused of excessive reverence. He is unsure whether there exists a finer food than fried red snapper with chili garlic sauce. He interprets praise of Richard Linklater movies as a deliberate attempt to get on his bad side, and takes David Cronenberg more seriously than most. He has performed tasks for the purpose of securing sustenance, and will no doubt do so again at some point in the future. Finally, he is a bit old-fashioned in that he draws a distinction between close friends and total strangers, and as such believes that anyone wishing to hear meaningful details of his life’s history can either offer him a job or buy him a drink.

Figure 5-1. Your author: a wascawwy wabbit.