MANIFESTATIONS OF ONE-DIMENSIONAL ELECTRONIC CORRELATIONS IN HIGHER-DIMENSIONAL SYSTEMS

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To my family
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In this work we have studied the fundamental aspects of transport and
thermodynamic properties of a one-dimensional (1D) electron system, and
have shown that these 1D correlations play an important role in understanding
the physics of higher-dimensional systems. The first system we studied is a
three-dimensional (3D) metal subjected to a strong magnetic field that confines the
electrons to the lowest Landau level. We investigated the effect of dilute impurities
in the transport properties of this system. We showed that the nature of electron
transport is one dimensional due to the reduced effective dimensionality induced
by the magnetic field. The localization behavior in this system was shown to be
intermediate, between a 1D and a 3D system. The interaction corrections to the
conductivity exhibit power law scaling, $\sigma \propto T^\alpha$ with a field dependent exponent.

Next we studied the thermodynamic properties of a one-dimensional
interacting system, where we showed that the next-to-leading terms in the specific
heat and spin susceptibility are nonanalytic, in the same way as they are for
higher-dimensional ($D = 2, 3$) systems. We obtained the nonanalytic, $T \ln T$ term
in the specific heat in 1D and showed that although the nonanalytic corrections
in all dimensions arise from the same source, there are subtle differences in the magnitude of the effect in different dimensions.

In the final part of this work we analyzed the nonanalytic corrections to the spin susceptibility ($\chi_s(H)$) in higher dimensional systems. We showed that, although there were contributions from non-1D scattering in these nonanalytic terms, the dominant contribution came from 1D scattering. We also showed that the second order ferromagnetic quantum phase transition is unstable both in 2D and 3D, with a tendency towards a first order transition.
CHAPTER 1
INTRODUCTION

One-dimensional interacting systems (Luttinger-liquids) exhibit many features which appear distinct from their higher-dimensional counterparts (Fermi-liquids). Our goal in this thesis is to highlight the similarities between higher-D and 1D systems. The progress in understanding of 1D systems has been greatly facilitated by the availability of exact or asymptotically exact methods (Bethe Ansatz, bosonization, conformal field theory), which typically do not work very well above 1D. The downside of this progress is that 1D effects, being studied by specifically 1D methods, look somewhat special and not really related to higher dimensions. We are going to argue that this is not true. Many effects which are viewed as the hallmarks of 1D physics, e.g., the suppression of tunneling conductance by the electron-electron interaction, do have higher dimensional counterparts and stem from essentially the same physics. In particular, scattering at Friedel oscillations caused by tunneling barriers and impurities is responsible for zero-bias tunneling anomalies in all dimensions. The difference lies in the magnitude of the effect, not in its qualitative nature. We illustrate this similarity by showing that 1D correlations play an important role in understanding the physics of higher dimensional systems. We studied three seemingly different problems, but as we will show, all three of them are connected by the common feature of 1D correlations. Our goal in the introduction is to provide a background for the physics discussed in the three chapters of this dissertation. We have set $\hbar$ and $k_B$ equal to unity everywhere.
1.1 Transport in Ultra Strong Magnetic Fields

The behavior of an isotropic three-dimensional (3D) metal in a high magnetic field has been a focus of attention of the condensed matter community for many decades. Due to Landau quantization of orbits, the energy of an electron in this system depends only on the momentum along the magnetic field,

\[ E = \frac{p_z^2}{2m} + \omega_c \left( n + \frac{1}{2} \right), \]

(1–1)

where \( \omega_c = eH/mc \) is the cyclotron frequency and \( n \) is the Landau level. Thus the system exhibits effects characteristic of one-dimensional (1D) metals, while being intrinsically a 3D system. This reduction of effective dimensionality of charge carriers from 3D to 1D is most pronounced in the ultra-quantum limit \( (n = 0, \text{ when only the lowest Landau level remains populated}) \) and is expected to result in a number of unusual phases. It is well known that the ground state of repulsively interacting electrons in the UQL is unstable with respect to the formation of a charge density wave [1–3], which has been observed experimentally in magneto-resistance measurements on graphite in high magnetic fields [4].

The most complete analysis of the CDW instability for the case of short range interactions was performed in Ref. [3], by solving the renormalization-group (RG) equations for the interaction vertex. On the other hand, it has recently been shown that for the case of long-range (Coulomb) interactions between electrons, a 3D metal in UQL exhibits Luttinger-liquid like (1D) behavior at energies higher than the CDW gap [5, 6]. Biagini et al. [5] and Tsai et al. [6] showed that in the UQL, the tunneling conductance has a power law anomaly (nonlinearities in I-V characteristics at small biases), which is typical for a one dimensional interacting system (Luttinger liquid). The magnetic-field-induced Luttinger liquid phase can be anticipated from the following simplified picture. In a strong magnetic field, electron trajectories are helices spiraling around the field lines. A bundle of
such trajectories with a common center of orbit can be viewed as a 1D conductor ("wire"). In the presence of electron-electron interactions, each "wire," considered separately, is in the LL state. Interactions with small momentum transfers among electrons on different "wires" do not change the LL nature of a single wire [7]. In chapter 2 of this dissertation we study the transport properties of a disordered 3D metal in the UQL, both with and without electron-electron interactions. Both the localization and interaction corrections to the conductivity show signatures typical for one-dimensional systems. Before we get into the details of our study, we will briefly review the physics of the interplay between the interaction effects and disorder induced localization in diffusive systems of low dimensionality.

At low temperatures, the conductivity of disordered conductors (normal metals and semiconductors) is determined by scattering of electrons off quenched disorder (e.g., impurities and defects). The residual conductivity is given by the Drude formula,

\[ \sigma = \frac{n e^2 \tau}{m}, \]

(1–2)

where \( n \) is the electron concentration, \( e \) is the electron charge, \( \tau \) is the transport mean free time, and \( m \) is the effective mass. The Drude formula neglects interference between electron waves scattered by different impurities, which occur as corrections to Eq.1–2, in the parameter \( (k_F \ell)^{-1} \ll 1 \) (where \( k_F \) is the Fermi momentum and \( \ell \) is mean free path). In low dimensions \( d \leq 2 \), these (interference) quantum corrections to the conductivity (QCC) diverge when the temperature \( T \) decreases and eventually, drive the system to the insulating regime. The quantum corrections to the conductivity are of substantial importance even for conductors that are far from the strong localization regime: in a wide range of parameters QCC, though smaller than the conductivity, determine all the temperature and field dependence of the conductivity. The systematic study of
QCC started almost three decades ago. A comprehensive review of the status of the problem from both theoretical and experimental viewpoints can be found in several papers [8–11].

According to their physical origin, QCC can be divided into two distinct groups. The correction of the first type, known as the weak localization (WL) correction, is caused by the quantum interference effect on the diffusive motion of a single electron. For low-dimensional \((d = 1, 2)\) infinite systems the WL QCC diverge at \(T \to 0\); this divergence is regularized either by a magnetic field or by some other dephasing (inelastic scattering) mechanism. We will elaborate on this type of QCC in section 1.1.1 below and also see how it changes for a 3D metal in UQL in chapter 2.

The second type of QCC, usually referred to as the interaction effects, is absent in the one-particle approximation; they are entirely due to interaction between electrons. These corrections can be interpreted as the elastic scattering of an electron off the inhomogeneous distribution of the density of the rest of the electrons. One can attribute this inhomogeneous distribution to the Friedel oscillations produced by each impurity. The role of the electron-electron interactions in this type of QCC is to produce a static self-consistent (and temperature dependent) potential which renormalizes the single particle density of states and the conductivity. Such a potential does not lead to any real transitions between single-electron quantum states (those require real inelastic scattering). Therefore, it does not break the time reversal invariance of the system and neither does it affect nor regularize the WL corrections. We will elaborate on this type of QCC in section 1.1.2, and also study it for our case of 3D metal in UQL in chapter 2.

However the interaction between electrons is by no means irrelevant to the WL QCC. Indeed, these interactions cause phase relaxation of the single electron
states, and thus result in the cut-off of the divergences in the WL corrections. This dephasing (described by the phase breaking time $\tau_{\phi}(T)$) requires real inelastic collisions between the electrons and can be obtained experimentally from the temperature dependence of magneto-resistance measurements. We will discuss the phase breaking time due to electron-electron interactions in section 1.1.3, of this introduction. Therefore there are two classes of interaction contribution to the conductivity: the genuine interaction corrections (elastic scattering of Friedel oscillation: Altshuler-Aronov corrections)-Class I, and corrections to WL QCC due to interactions (inelastic scattering-dephasing)-Class II.

1.1.1 Weak Localization QCC

These corrections are caused by the quantum interference effect in the diffusive motion of a single electron. In going from point A to point B a particle can travel along different trajectories (Fig.1–1). The total probability $W$ for a transfer from point A to point B is

$$W = |\sum_i A_i|^2 = \sum_i |A_i|^2 + \sum_{i \neq j} A_i A_j^*.$$  \hfill (1–3)

The first term in Eq.1–3 describes the sum of the probabilities for each path and the second term corresponds to interference of various amplitudes. The interference

![Figure 1–1. Weak localization corrections.](image-url)
term drops out when averaging over many paths because of its oscillatory nature. However, there exists special type of trajectories, i.e., the self-intersecting ones, for which interference cannot be neglected (see Fig.1–1). If $A_1$ is the amplitude for the clockwise motion around the loop and $A_2$ is the amplitude for the anticlockwise motion, then the probability to reach point $O$ is

$$W = |A_1|^2 + |A_2|^2 + 2ReA_1^*A_2 = 4|A_1|^2.$$ (1–4)

i.e., twice the value we would have obtained by neglecting interference. Enhanced probability to find the particle at a point of origin means reduced probability to find it at final point (B). Therefore this effect leads to a decrease in the conductivity (increase in resistivity) induced by interference.

The relative magnitude of weak localization QCC, $\delta\sigma/\sigma$, is proportional to the probability to form a loop trajectory

$$\frac{\delta\sigma}{\sigma} \sim -\int dP \sim -\int_\tau^{\tau_\phi} dt \frac{v\lambda^2}{(D_\tau^2)^{d/2}},$$ (1–5)

leading to $\delta\sigma \sim -\tau_\phi^{(2-d)/2} (\ln \tau_\phi$ for $d=2$), which diverges as $T$ is lowered for $d \leq 2$, leading to strong Anderson localization. Here $v$ is the electron velocity, $D$ is the diffusion coefficient, $\lambda$ is the electron wavelength and $\tau_\phi(T)$ is the phase breaking time. Phase coherence is destroyed by inelastic scattering (electron-electron, electron-phonon) or by magnetic and a.c electric fields. The temperature dependence of the WL correction is determined by $\tau_\phi(T)$. Typically, $\tau_\phi \sim T^{-p}$, where the exponent $p$ depends on the inelastic scattering mechanism (electron-electron, electron-phonon) and dimensionality. Interference effects occur for $\tau \ll \tau_\phi(T)$ i.e., at low temperatures.

In the language of Feynman diagrams, the WL QCC [12] is obtained by including the maximally crossed ladder diagram, the Cooperon (see Fig.1–2), in the conductivity diagram. The other type of ladder (vertex ) diagram, the Diffuson
(Fig. 1–2), when included in the conductivity diagram changes the elastic scattering
time to the transport time. In the field-theoretic language, the Weak Localization

\[
\begin{align*}
M(q;w) &= \frac{1}{2\pi} \int \frac{dQ}{Q} C(Q;w)
\end{align*}
\]

correction, which arises due to interference of time reversed paths is determined by
the “Cooperon” mode \( C(Q;\omega) \), i.e., the particle-particle diffusion propagator,

\[
C(Q;\omega) = \frac{1}{2\pi \nu \tau} \left( \frac{1}{-i\omega \tau + DQ^2 \tau} \right),
\]

to first order in \((k_f \ell)^{-1}\). Calculation of the singular contributions to conductivity
(interference effect) at small \( \omega, Q \) should include diagrams containing as an internal
block the graphs which yield after summation \( C(Q;\omega) \) (1–3). The WL QCC is

\[
\begin{align*}
\delta \sigma_{WL} &= -\frac{Dc^2 \tau}{2\pi} \int (dQ) C(Q;\omega),
\end{align*}
\]
which gives

\[ \frac{\delta \sigma_{WL}}{\sigma} \sim -\sqrt{\frac{\tau}{\tau}} \rightarrow -\frac{1}{\sqrt{\omega \tau}} (1D), \]  

(1-8)

\[ \sim -\frac{1}{k_F\ell} \ln\left(\frac{\tau}{\tau}\right) \rightarrow -\frac{1}{k_F\ell} \ln\left(\frac{1}{\omega \tau}\right) (2D), \]  

(1-9)

\[ \sim + \left(\frac{1}{k_F\ell}\right)^2 \sqrt{\frac{\tau}{\tau}} \rightarrow + \left(\frac{1}{k_F\ell}\right)^2 \sqrt{\omega \tau} (3D), \]  

(1-10)

in different dimensions. Perturbation theory breaks down in one- and two-dimensions

(for \( \omega \ll \frac{1}{\tau} \), in the diffusive limit or at low temperatures, \( \tau_\phi \sim T^{-p} \)) which suggests

strong localization in reduced dimensions. Anderson [13] had first shown that

at sufficiently high impurity concentration, electronic states become localized

and the system becomes an insulator. Mott and Twose [14] had predicted that

the conductivity for a one dimensional system should vanish in the limit of low

frequencies (Mott’s law) which was later rigourously proved by Berezinskii [15],

who showed that electron states in 1D are strongly localized and there is no

diffusive regime in 1D. The localization length in 1D is of the order of the mean

free path (\( \ell \)), therefore in 1D for length scales shorter than \( \ell \), the electron motion is

ballistic and for lengths longer than \( \ell \) then electron motion is localized. 2D systems

are also strongly localized but the localization length is very large (\( L_{loc} \sim \ell e^{k_F \ell} \)) as

compared to 1D (\( L_{loc} \sim \ell \)). Thus in 2D, the ballistic regime (\( L \ll \ell \)) crosses over to

the diffusive regime ( for \( \ell < L < L_{loc} \)) and then to the localized regime (\( L_{loc} < L \)).

Scaling theory of localization proposed by Abrahams et al. [16] describes

Localization in higher dimensions. This theory is based on the assumption that the

only parameter that determines the behavior of the system under renormalization

is the dimensionless conductance, \( g \) (in units of \( e^2/\hbar \)). The variation of \( g \) with the

system size obeys the Gell-Mann Low equation

\[ \frac{d \ln g}{d \ln L} = \beta(g). \]
In the metallic regime $g \gg 1$, the conductance shows ohmic behavior for which $\beta(g) = d - 2$. Corrections to $\beta(g)$ in the metallic regime are obtained by perturbation theory in $\frac{1}{g}$. For $g \ll 1$ (insulating regime), $\beta(g)$ is obtained from the simple argument that $g$ must decrease exponentially with the system size in this regime. In 1D and 2D, $g$ decreases with increasing system size, which means that the electron states are always localized. In 3D there is a continuous phase transition between metallic and insulating phases. This transition happens when $k_f \ell \sim 1$ (Anderson transition).

**Effect of magnetic field on WL QCC.** If the system is placed in a magnetic field $H$, the amplitude for a particle to pass the loop clockwise and anticlockwise (Fig. 1–1) acquire additional phase factors,

$$A_1 \rightarrow A_1 \exp\left(\frac{ie}{\hbar c} \int dlA\right) = A_1 e^{\frac{\sqrt{H}S}{\phi_0}},$$

$$A_2 \rightarrow A_2 \exp\left(-\frac{ie}{\hbar c} \int dlA\right) = A_2 e^{-\frac{\sqrt{H}S}{\phi_0}},$$

where $\phi_0 = \frac{hc}{2e}$ is the flux quantum. The phase difference between waves passing the loop clockwise and anticlockwise is $\delta \phi = 2\pi \phi / \phi_0$, $\phi = HS$ is the flux through the loop with cross-section $S$. Thus the magnetic field destroys interference, reducing the probability for a particle to return to a given point, and hence reduces the resistivity. This mechanism is responsible for negative magneto-resistance [17].

The characteristic time scale for phase breaking is $t_H \sim \frac{l_H^2}{D}$ where $l_H = \sqrt{c/eH}$ is the magnetic length. The typical magnetic field involved is $H \sim \frac{c}{eD\tau_\phi}$. At this field, the product $\omega_c \tau$ satisfies $\omega_c \tau \sim (E_F \tau_\phi)^{-1} \ll 1$ where $E_F$ is the Fermi energy and $\omega_c$ is the cyclotron frequency. Thus at the phase breaking field the classical magneto-resistance, determined by the value of $\omega_c \tau$ is still small.

A weak magnetic field destroys phase coherence and increases the conductivity. If the field is increased further, we reach the classical magneto-resistance regime,
where the conductivity decreases with the field. What happens at even higher fields when Landau quantization becomes important? We address this issue in chapter 2 of this dissertation. We show that a three-dimensional disordered conductor in the Ultra quantum limit, where only the lowest landau level is populated, exhibits a new phenomenon: *intermediate localization*. The quantum interference correction $\delta\sigma$ is of the order of the Drude conductivity $\sigma_D$ (as in 1D) which indicates a breakdown of perturbation theory. However, the conductivity remains finite at $T \to 0$ (as in 3D). It is demonstrated that the particle-particle correlator (Cooperon) is massive. Its mass (in units of the scattering rate) is of the order of the impurity scattering rate.

1.1.2 Interaction Correction to the Conductivity-Altshuler Aronov Corrections (class I)

The effect of electron-electron interaction in disordered systems makes it drastically different from that of pure metals, where the interaction at low temperatures manifests itself only in the renormalization of the electron spectral parameters [18] (the wave function renormalization $Z$, effective mass $m^*$, etc.). First we note that within the transport equation, electron-electron collisions can in no way affect the conductivity in the case of a simple band structure and in the absence of Umklapp processes, since electron-electron collisions conserve the total momentum of the electron system. Inclusion of the Fermi liquid corrections renormalizes the residual resistivity while not resulting in any dependence of the conductivity on the temperature and frequency. However one frequently encounters the situation that the resistivity scales as $T^2$. This dependence is often interpreted as the “Fermi-liquid” effect, arising from electron-electron scattering with characteristic time $\tau_{ee} \propto T^{-2}$. In fact, the resistivity is due to Umklapp scattering. In good metals, normal processes (which conserve the total electron momenta) and Umklapp processes (which conserve the momenta up to a reciprocal
lattice vector) are equally probable and the Umklapp scattering rate entering the resistivity also scales as $T^2$. Note that at low temperatures this resistivity due to electron-electron scattering (Umklapp) gives the dominant contribution because the electron-phonon contribution to the resistivity scales as $T^5$ (Bloch’s law, $\tau_{e-ph} \sim 1/T^5$).

As was mentioned previously, taking into account the interference of elastic scattering by impurities with the electron-electron interaction produces non trivial temperature and frequency dependences of the conductivity. This correction arises from coherent scattering of an electron from an impurity and the Friedel oscillation it creates [19]. We will first study this correction to the conductivity in the ballistic limit, ($T\tau \gg 1$, where $\tau$ is the elastic scattering lifetime) and then in the diffusive regime ($T\tau \ll 1$). In the diffusive limit an electron undergoes multiple collisions with impurities before it scatters from another electron, whereas in the ballistic limit the electron-electron collision rate is faster than electron-impurity rate, thus single impurity effects are important in the ballistic limit. In chapter 2 of this dissertation we will evaluate this interaction QCC in the ballistic limit in a 3D metal in the UQL. There has been a recent renewal of interest in the interaction QCC, (class I) due to the metal to insulator transition observed in two-dimensional (high mobility) Si-MOSFET samples [20]. The qualitative features of this transition was understood by Zala, Narozhny and Aleiner [19] who showed that the insulating (logarithmic upturn in the resistivity) behavior in the diffusive regime and metallic (linear rise in temperature) behavior of the resistivity in the ballistic limit (2D), are due to coherent scattering at Friedel oscillations. Below, we first outline their simple quantum mechanical scattering theory approach to show how temperature dependent corrections to conductivity arise for scattering at Friedel oscillations, and then extend their analysis to obtain the interaction QCC in
3D ballistic limit. In chapter 2 we evaluate this correction for a 3D system in the UQL.

![Diagram](image)

Figure 1–4. Scattering by Friedel oscillations.

**Scattering at Friedel Oscillations** Friedel oscillations in the electron density are created due to standing waves formed as a result of interference between incoming and backscattered electron waves (Fig.1–4). Consider an impurity at the origin; its potential $U_{\text{imp}}(\vec{r})$ induces a modulation of electron density around the impurity. In the Born approximation one can find the oscillating correction, $\delta n(r) = n(r) - n_0$ to the electron density $n(\vec{r}) = \sum_k |\psi_k(\vec{r})|^2$:

$$\delta n(r) \sim -g\nu\frac{\sin(2k_F r)}{r^d}. \quad (1–11)$$

Here $r$ is the distance from the impurity, $k_F$ is the Fermi momentum, $g = \int U_{\text{imp}}(\vec{r}) d\vec{r}$ is the matrix element for impurity scattering and $n_0$ is the electron density in the absence of impurities and $d$ is dimensionality. Taking into account electron-electron interactions $V_0(\vec{r}_1 - \vec{r}_2)$ one finds additional scattering potential due to the Friedel oscillations Eq.1–11. This potential can be presented as a sum of the direct (Hartree) and exchange (Fock) terms [21]

$$\delta V(\vec{r}_1, \vec{r}_2) = V_H(\vec{r}_1)\delta(\vec{r}_1 - \vec{r}_2) - V_F(\vec{r}_1, \vec{r}_2), \quad (1–12)$$
\[ V_H(\vec{r}_1) = \int d\vec{r}_3 V_0(\vec{r}_1 - \vec{r}_3) \delta \rho(\vec{r}_3), \quad (1-13) \]
\[ V_F(\vec{r}_1, \vec{r}_2) = \frac{1}{2} V_0(\vec{r}_1 - \vec{r}_2) \delta n(\vec{r}_1, \vec{r}_2), \quad (1-14) \]

where by \( \rho(\vec{r}) \) we denote diagonal elements of the one electron density matrix,

\[ n(\vec{r}_1, \vec{r}_2) = \sum_k \Psi^*_{k}(\vec{r}_1) \Psi_{k}(\vec{r}_2). \quad (1-15) \]

As a function of the distance from the impurity, the Hartree-Fock energy \( \delta V \) oscillates similarly to Eq.1–11. The leading correction to conductivity is a result of interference between two semi-classical paths shown in fig. 4. If an electron follows path “A,” it scatters off the Friedel oscillation created by the impurity and path “B” corresponds to scattering by the impurity itself. Interference is most important for scattering angles close to \( \pi \) (or for backscattering), since the extra phase factor accumulated by the electron on path “A” \( e^{i 2kR} \) relative to path “B” is canceled by the phase of the Friedel oscillation \( e^{-i 2k_F R} \), so that the amplitude corresponding to the two paths are coherent. As a result, the probability of backscattering is greater than the classical expectation (taken into account the Drude conductivity). Therefore, accounting for interference effects lead to a correction to the conductivity. We note that the interference persists to large distances, limited by temperature \( R \sim |k - k_F|^{-1} \leq v_F/T \). Thus there is a possibility for the correction to have nontrivial temperature dependence. The sign of the correction depends on the sign of the effective coupling constant that describes electron-electron interaction. First, we will study the contribution arising from the Hartree potential. Consider a scattering problem in the potential given in Eq.1–13. The particle’s wave function is a sum of the incoming plane wave and the outgoing spherical wave (3D),

\[ \Psi = e^{i \vec{k} \cdot \vec{r}} + f(\theta) e^{i k r} \frac{e^{i k r}}{r}, \]
where \( f(\theta) \) is the scattering amplitude, which we will determine in the Born approximation. For the impurity potential itself the amplitude \( f(\theta) \) weakly depends on the angle. At zero temperature it determines the Drude conductivity \( \sigma_D \), while the leading temperature correction is \( T^2 \) (when the scattering time energy dependent), as is usual for Fermi systems. We now show that this is not the case for the potential in Eq.1–12. In fact, taking into account Eq.1–12 leads to enhanced backscattering and thus to the conductivity correction which depends on temperature as \( \delta \sigma \propto T^2 \ln T \) (in 3D), \( \delta \sigma \sim T \) (in 2D) and, as we will see later \( \delta \sigma \sim T^{2\alpha} \) (in 3D UQL, \( \alpha \) is the interaction parameter) all for the ballistic limit.

Far from the scatterer the wave function of a particle can be found in the first order of perturbation theory as \( \Psi = e^{i\vec{k}.\vec{r}} + \delta \Psi(\vec{r}) \), where the correction is given by [22]

\[
\delta \Psi(\vec{r}) = -\frac{m}{2\pi} \int d\vec{r}_1 V_H(r_1) e^{i\vec{k}.\vec{r}_1} e^{i\vec{q}|\vec{r} - \vec{r}_1|}. 
\] (1–16)

Substituting the form of the Hartree potential from Eq:1–13, and introducing the Fourier transform of the electron-electron interaction \( V_0(q) \), we obtain for the scattering amplitude (at large distances from the impurity)

\[
f(\theta) = -\frac{m V_0(q)}{2\pi} \int d\vec{r} \delta n(r) e^{i\vec{q}.\vec{r}}, 
\] (1–17)

where \( \vec{q} = \vec{k} - k\vec{r}/r \) and \( |\vec{q}| = 2k \sin(\theta/2) \). We see that the scattering amplitude depends on the scattering angle (\( \theta \)), as well as the electron’s energy (\( \epsilon = k^2/2m \)). The density oscillation in 3D, with hard wall boundary condition at the origin (impenetrable impurity), is

\[
\delta n(r) = \int d\vec{k} f(k)[|\Psi_k(\vec{r})|^2 - |\Psi^0_k(\vec{r})|^2], 
\]

\[
= -\frac{2k_F}{r^2} \left( \frac{\sin(2k_F(r-a))}{2k_F(r-a)} - \frac{\sin(2k_F r)}{2k_F r} \right). 
\]
where $a$ is the size of the impurity and $f(k)$ is the Fermi distribution function. We make the $s$ wave scattering approximation (slow particles, $k_Fa \ll 1$) to obtain

$$\delta n(r) = \frac{(2k_F)^2a}{r^2} \left( \frac{\cos(2k_Fr)}{2k_Fr} - \frac{\sin(2k_Fr)}{(2k_Fr)^2} \right). \quad (1-18)$$

Substituting the density from Eq.1–18 in Eq.1–17, we obtain for the scattering amplitude

$$f(\theta) = \frac{-2mV_0(2k_F \sin(\frac{\theta}{2}))2k_Fa}{\sin(\frac{\theta}{2})} \left[ -\frac{1}{2} \sin(\frac{\theta}{2}) + \frac{1}{4} \ln \left| \frac{1 - \sin(\frac{\theta}{2})}{1 + \sin(\frac{\theta}{2})} \right| \cos^2(\frac{\theta}{2}) \right]. \quad (1-19)$$

In the limit $\theta \approx \pi + x$ where $x \ll 1$, the scattering amplitude behaves as $f(x) \sim V_0(2k_F)[x \ln x]$. The transport scattering cross section is now

$$A_{tr} = \int_0^{\pi} d\theta \sin(\theta) \int_0^{2\pi} d\phi (1 - \cos(\theta)) |f_0 + f(\theta)|^2, \quad (1-20)$$

where $f_0$ is the amplitude for scattering at the impurity itself (which does not depend on $\theta$ in the Born limit and gives a constant ($T$ independent) value for the Drude conductivity). The leading energy dependence comes from the interference (cross term), which is proportional to $f(\theta)$. The main contribution to the integral comes from $\theta \sim \pi$ (backscattering). Expanding near $\pi$, i.e., $\theta = \pi + \theta_1$ where $\theta_1$ is small [19], $\theta_1 \sim \sqrt{k - k_F/k_F} \sim \sqrt{\epsilon/E_F}$, we obtain for the scattering cross section and transport rate $((\tau_{tr})^{-1} \propto n_i v_F A_{tr} \sim \delta \rho)$

$$\frac{1}{\tau_{tr}(\epsilon)} \propto cV_0(2k_F)(\epsilon)^2 \ln(\epsilon). \quad (1-21)$$

Then one obtains the interaction QCC from the Hartree channel [23] in 3D (using $\delta \sigma/\sigma_D = -\delta \rho/\rho_D$)

$$\frac{\delta \sigma}{\sigma_D} \sim -\nu V_0(2k_F) \left( \frac{T}{E_F} \right)^2 \ln(\frac{E_F}{T}). \quad (1-22)$$

One obtains a similar contribution from the exchange (Fock) potential, except now the coupling constant in front of the $T^2 \ln T$ term is $V_0(0)$. The Hartree and exchange contribution come with opposite signs. In 2D the interaction QCC is
linear in temperature \([19]\)

\[
\frac{\delta \sigma}{\sigma_D} \sim \nu [2V_0(2k_F) - V_0(0)] \frac{T}{E_F}.
\]

(1–23)

In 1D Yue, Glazman and Matveev \([21]\) used the same approach and calculated the correction to the transmission coefficient due to scattering at the Friedel oscillation and obtained a logarithmic temperature correction at the lowest order

\[
\delta t = -t_0 \alpha \ln \left| \frac{E_F}{T} \right|,
\]

(1–24)

where \(\alpha \sim [V_0 - 2V_{2k_F}]/v_F\). Using a poor man renormalization group procedure, they showed that the first order logarithmic correction is in fact a weak coupling expansion of the more general power law scaling form of the transmission coefficient,

\[
t = t_0 \left( \frac{T}{W} \right)^\alpha.
\]

where \(W\) is the band width. The transmission coefficient is related to the conductance using the Landauer formula \(G \sim |t|^2\), which gives in 1D

\[
G = G_0 \left( \frac{T}{W} \right)^{2\alpha}.
\]

(1–25)

This result was also obtained independently (via bosnization) by Kane and Fisher \([24]\). Eq. 1–22, 1–23, and Eq. 1–25 give the interaction QCC in the ballistic limit in 3D, 2D and 1D systems respectively. In chapter 2 of this dissertation we show that in 3D UQL, this interaction correction to the conductivity behaves similar to that of a true 1D system.

The interaction correction to QCC in the diffusive limit also arises from the same physics (namely scattering at friedel oscillations) but now one has average over many impurities (diffusive motion). This correction to the conductivity was evaluated by Altshuler and Aronov in 3D \([8]\) and by Altshuler, Aronov and Lee in
\[ \delta \sigma \sim (2 - 2F) \ln(T\tau), (2D) \]  

where \( F \) is the depends on the strength of the interaction, and

\[ \delta \sigma \sim \left( \frac{4}{3} - \frac{3F}{2} \right) \sqrt{\frac{T}{D}}, (3D). \]  

Scattering at the Friedel oscillations also results in a singular energy (temperature) dependence of the local density of states which can be observed as a zero bias anomaly in tunneling. The local DOS can be obtained from the electrons Green’s function using \( \delta \nu(\epsilon) = -\text{Im} \int d\vec{p} \delta G^R(\vec{p}, \epsilon) \). The correction to the Green’s function can be evaluated the same way as we evaluated the correction to the wave function due to Friedel oscillation or it can also be evaluated diagrammatically by calculating the electron’s self energy in the presence of disorder and interaction \([8]\).

1.1.3 Corrections to WL QCC due to Electron-Electron Interactions: Dephasing (class II)

The basic feature underlying the quasi-particle description of electrons in metals and semiconductors is the small width of the one electron states. The minimum width of a wave packet and, hence, the minimum decay of a state are determined by the wave function phase relaxation time \( \tau_\phi \). For strongly inelastic processes this time coincides with the out-relaxation time. In degenerate Fermi systems where the energy transferred in each collision is of the order \( \epsilon \), i.e., of the order of the excitation energy measured from the Fermi level, the inverse excitation decay time is of the order of \( \epsilon^2/E_F \) and thus, is smaller than the excitation energy, which is \( \epsilon \). These considerations does not depend on the specific details of the electron interaction and originate from the fact that scattering of quasiparticles by one another is governed by large momentum transfers. Therefore the decay is determined only by the phase volume of final states. It was believed by analogy with the Fermi liquid, that in the case of weak disorder, \( k_F \ell \gg 1 \), the
excitation decay should likewise be proportional to $\epsilon^2$. It turns out, however, that
excitation in disordered systems decays faster, which raises the question of validity
of quasiparticle description of disordered conductors in low dimensional systems.
Apart from being important in the development of the theory, the decay time
for one electron excitations (the phase relaxation time), governs the temperature
dependence of the WL QCC.

It was shown by Altshuler and Aronov that for 3D disordered systems, the
phase relaxation time $\tau_\phi$ is governed by large energy transfer processes and in
this regime $\tau_\phi \sim \tau_{ee}$ (where $\tau_{ee}$ is the out relaxation time). The out relaxation
time can be calculated from the Bolzmann equation (with diffusive dynamics for
the electrons). This gives $\frac{1}{\tau_\phi} \sim (\frac{\epsilon}{D})^{3/2}$ in 3D, [8]. However in lower dimensions
($d = 1, 2$) electron-electron collisions with small energy transfers is the dominant
mechanism for dephasing. The Bolzmann approach (which is good for large energy
transfers) fails in 2D and 1D case. Technically, there would be divergences for
small energy transfers [8] both in 2D (logarithmic) and quasi-1D (power law)
in the Bolzmann-equation result for the out relaxation rate. These divergences
must be regularized in a self-consistent manner. The phase breaking time in
lower dimensions can also be obtained by solving the equation of motion for
the particle-particle (Cooperon) propagator in the presence of space and time
dependent fluctuating electromagnetic fields which model the small energy transfer
processes [26]. This gives $(\tau_\phi)^{-1} \sim T$ (in 2D) and $(\tau_\phi)^{-1} \sim T^{2/3}$ (in quasi-1D).

In true one-dimensional systems, this subject is controversial as true 1D
systems do not have a diffusive regime (the ballistic limit crosses over to the
localized regime) and the quasiparticle description breaks down for an interacting
1D system which is in the Luttinger liquid state. As a result one cannot define $\tau_{ee}$.
In a recent work on this subject [27], it was shown that even for a 1D disordered
Luttinger liquid, there exists a weak localization correction to the conductivity
whose temperature dependence is governed by the phase relaxation rate, $(\tau_0)^{-1} \propto \sqrt{T}$ (for spinless electrons in 1D) and $(\tau_0)^{-1} \propto T$ (for electrons with spin) and the WL QCC behaves as,

$$\sigma_{WL} \sim \sigma^D \left( \frac{\tau_\phi}{\tau} \right)^2 \ln \left( \frac{\tau}{\tau_\phi} \right),$$

(1–28)

where $\sigma^D = e^2 \nu v_F^2 \tau$ is the Drude conductivity in 1D, which depends on $T$ through a renormalization of static disorder, $\tau_0/\tau = (E_F/T)^{2\alpha}$. Here $\tau_0$ is non-interacting scattering time and $\tau$ is the renormalized (by Friedel oscillation) scattering time and $\alpha$ characterizes the interaction.

At present there are no theoretical predictions for $\tau_\phi$ in 3D UQL. The Fermi liquid approaches for calculating the phase breaking time are not expected to work here because the Cooperon is not a singular diagram (it acquires a mass in 3D UQL as shown in chapter 2) and, once again, there are no single particle like excitations as the ground state is a charge-density-wave and excitations above the ground state are Luttinger liquid like. However in chapter 2 we will show that some recent magneto-resistance measurements on graphite in UQL qualitatively agree with predictions of $\tau_\phi$ due to electron-phonon interactions in 1D.

1.2 Non-Fermi Liquid Features of Fermi Liquids: 1D Physics in Higher Dimensions

The universal features of Fermi liquids and their physical consequences continue to attract the attention of the condensed-matter community for almost 50 years after the Fermi-liquid theory was developed by Landau [28]. A search for stability conditions of a Fermi liquid and deviations from a Fermi liquid behavior, [29–32] particularly near quantum critical points, intensified in recent years mostly due to the non-Fermi-liquid features of the normal state of high $T_c$ superconductors[33] and heavy fermion materials[34].

The similarity between the Fermi-liquid and a Fermi gas holds only for the leading terms in the expansion of the thermodynamic quantities (specific heat
$C(T)$, spin susceptibility $\chi_s$ in the energy (temperature) or spatial (momentum) scales. Next-to-leading terms are singular (nonanalytic) and, upon a deeper look, reveal a rich physics of essentially 1D scattering processes, embedded into higher dimensional phase space.

In this introduction, we will discuss the difference between the “regular” processes which lead to the leading Fermi-liquid forms of thermodynamic quantities and “rare” 1D processes which are responsible for the nonanalytic (non-Fermi liquid) behavior. We will see that the role of these rare processes increases as the dimensionality is reduced and, eventually, the rare processes become normal in 1D, where the Fermi-liquid description breaks down.

In a Fermi gas, thermodynamic quantities form regular, analytic series as a function of either temperature $T$, or the inverse spatial scale $q$ of an inhomogeneous magnetic field. For $T \ll E_F$ and $q \ll k_F$,

$$C(T)/T = \gamma + aT^2 + bT^4 + \ldots,$$

$$\chi_s(T, q = 0) = \chi^0_s(0) + cT^2 + dT^4 + \ldots,$$

$$\chi_s(T = 0, q) = \chi^0_s(0) + eq^2 + fq^4 + \ldots,$$

where $\gamma = \pi^2\nu_F^2/3$, $\chi^0_s = g\mu_B^2\nu_F$ and $\nu_F \sim mk_F^{D-2}$ is the density of states (DOS) on the Fermi surface, $g$ is the Lande factor and $\mu_B$ is the Bohr magneton and $a \ldots f$ are some constants. Even powers of $T$ occur because of the approximate particle-hole symmetry of the Fermi function around the Fermi energy. The above expressions are valid in all dimensions, except $D = 2$. This is because the DOS is constant in $2D$, the leading correction to the $\gamma T$ term in $C(T)$ is exponential in $E_F/T$ and $\chi_s$ does not depend on $q$ for $q \leq 2k_F$. However this anomaly is removed as soon as we take into account a finite bandwidth of the electron spectrum, upon which the universal ($T^{2n}$ and $q^{2n}$) behavior is restored.
An interacting Fermi system is described by Landau’s Fermi-liquid theory, according to which the leading terms in $C(T)$ and $\chi_s$ are same as that of the Fermi gas with renormalized parameters (replace bare mass by effective mass $m^*$, bare $g$ factor by effective $g$-factor $g^*$ in the above Fermi gas results),

$$C(T)/T = \gamma^* = \gamma_0(1 + \langle \cos \theta F_c \rangle),$$

$$\chi_s(T, q) = \chi_s^*(0) = \chi_s^0 \left( \frac{1 + \langle \cos \theta F_c \rangle}{1 + \langle F_s \rangle} \right),$$

where $F_c, F_s$ are charge and spin harmonics of the Landau interaction function:

$$\hat{F}(\vec{p}, \vec{p}') = F_c(\theta) \hat{I} + F_s(\theta) \vec{\sigma}.\vec{\sigma'},$$

where $\vec{\sigma}$ are the Pauli matrices. The Fermi-liquid theory is an asymptotically low-energy theory by construction, and it is really suitable only for extracting the leading terms, corresponding to the first terms in the Fermi gas expression. Indeed, the free energy of the Fermi-liquid of an ensemble of quasiparticles interacting in a pairwise manner can be written as [35]

$$F - F_0 = \sum_k (\epsilon_k - \mu) \delta n_k + \frac{1}{2} \sum_{k,k'} f_{k,k'} \delta n_k \delta n_{k'} + O(\delta n^3_k),$$

where $F_0$ is the ground state energy, $\delta n_k$ is the deviation of the fermion occupation number from its ground state value, and $f_{k,k'}$ is the Landau interaction function. As $\delta n_k$ is of the order of $T/E_F$, the free-energy is at most quadratic in $T$, and so the corresponding $C(T)$ is at most linear in $T$. Consequently the Fermi-liquid (FL) theory (within the conventional formulation) does not say anything about the higher order terms.

Strictly speaking, a nonanalytic dependence of $f_{k,k'}$ on the deviations from the Fermi surface $k - k_F$, accounts for the non-analytic $T$ dependence of $C(T)$ [36]. Higher order terms in $T$ or $q$ can be obtained within microscopic models which specify particular interaction and employ perturbation theory. Such an approach is complimentary to the FL: the former works for weak interactions but at arbitrary temperatures whereas FL works both for weak and strong interactions,
but only in the limit of lowest temperatures. Microscopic models (Fermi gas with weak repulsion, electron-phonon interaction, paramagnon model, etc.) show that the higher order terms in the specific heat and spin susceptibility are nonanalytic functions of $T$ and $q$ [37–48]. For example,

\[ \frac{C(T)}{T} = \gamma_3 - \alpha_3 T^2 \ln(k_F / T)(3D), \]  
\[ \frac{C(T)}{T} = \gamma_2 - \alpha_2 T(2D), \]  
\[ \chi_s(q) = \chi_s(0) + \beta_3 q^2 \ln(k_F / |q|)(3D), \]  
\[ \chi_s(q) = \chi_s(0) + \beta_2 |q|(2D), \]

where all coefficients are positive for the case of electron-electron interaction.

As seen from the above equations the nonanalyticity becomes stronger as the dimensionality is reduced. The strongest nonanalyticity occurs is 1D, where-at least as long as single particle properties are concerned-the FL breaks down [49, 50]:

\[ \frac{C(T)}{T} = \gamma_1 - \alpha_1 \ln(k_F / T)(1D), \]  
\[ \chi_s(q) = \chi_s(0) - \beta_1 \ln(k_F / |q|)(1D). \]

These nonanalytic corrections to the specific heat and spin susceptibility in 1D are obtained in chapter 3. It turns out that the evolution of the non-analytic behavior with the dimensionality reflects an increasing role of special, almost 1D scattering processes in higher dimensions. Thus non-analyticities in higher dimensions can be viewed as precursors of 1D physics for $D > 1$.

We will first study the necessary condition to obtain a FL description and then see how relaxing these conditions lead to the nonanalytic form for the self-energy and thermodynamic properties. Within the Fermi liquid

\[ \text{Re}\Sigma^R(\varepsilon, k) = -A\varepsilon + B\xi_k + \ldots \]  
\[ \text{Im}\Sigma^R(\varepsilon, k) = C(\varepsilon^2 + \pi^2 T^2) + \ldots \]
Landau's argument for the $\varepsilon^2$ (or $T^2$) behavior of $\text{Im} \Sigma^R$ requires two conditions: (1) quasiparticles must obey Fermi statistics, i.e., the temperature is smaller than the degeneracy temperature $T_F = k_F v_F^*$, where $v_F^*$ is the renormalized Fermi velocity, (2) inter-particle scattering is dominated by processes with large (generally, of order $k_F$) momentum transfers. Once these two conditions were satisfied, the self-energy assumes a universal form, Eq.1–40 and Eq.1–41, regardless of a specific type of interaction (electron-electron, electron-phonon) and dimensionality. Consider the self-energy of an electron (1$^{st}$ order) as it interacts with some boson (see Fig. 1–5). The wavy line can be, e.g., a dynamic Coulomb interaction, phonon propagator, etc. On the mass shell ($\varepsilon = \xi_k$; where $\xi_k = k^2/2m - k_F^2/2m$) at $T = 0$ and for $\varepsilon > 0$

\[ a) \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1-5.png}
\caption{Self-energy at first order in interaction with a bosonic field}
\end{figure}

\[
\text{Im} \Sigma^R(\varepsilon) \sim \int_0^\varepsilon d\omega \int d^D q \text{Im} G^R(\varepsilon - \omega, \vec{k} - \vec{q}) \text{Im} V^R(\omega, q)
\]  
(1–42)

The constraint on energy transfers ($0 < \omega < \varepsilon$) is a direct manifestation of the Pauli principle. The potential term $V(r, t)$ is a propagator of some field which has a classical limit, so $V(r, t)$ is real, thus $\text{Im} V(q, \omega)$ is an odd function of $\omega$ and we write it explicitly as

\[
\text{Im} V^R(\omega, q) = \omega F(|\omega|, q).
\]  
(1–43)
As a function of $q$, $F$ has at least two characteristic scales. One is provided by the internal structure of the interaction (screening wave vector for the Coulomb potential) or by $k_F$ whichever is smaller. This scale, $Q$, does not depend on $\omega$ and provides the ultra-violet cutoff in the theory. In addition there is a second scale $|\omega|/v_F$, and, since $\omega$ is bounded from above by $\varepsilon$ and for low energies ($\varepsilon \to 0$), one can assume $Q \gg |\omega|/v_F$. Thus in a dimensionless form

$$\text{Im} V^R(\omega, q) = \frac{\omega}{Q} U \left( \frac{q}{Q}, \frac{|\omega|}{v_F Q} \right). \tag{1-44}$$

The angular integration over $\text{Im} G^R$ yields on the mass shell

$$\int d\theta \text{Im} G^R = -\pi \int d\theta \delta(\omega - v_F \cdot \vec{q} + q^2/2m) = \frac{1}{v_F q} A_D \left( \frac{\omega + q^2/2m}{v_F q} \right), \tag{1-45}$$

where the subscript $D$ stands for the dimensionality, and

$$A_3(x) = \theta(1 - |x|),$$
$$A_2(x) = \frac{\theta(1 - |x|)}{\sqrt{1 - x^2}}.$$ 

The function $A_D$ primarily serves to impose a lower cutoff $q \geq |\omega|/v_F$ and we can ignore the specific functional form. Using Eq.1-45 and Eq.1-44 into Eq.1-42, one obtains

$$\text{Im} \Sigma^R(\varepsilon) \sim \int_0^\varepsilon d\omega \omega \int_{q \geq |\omega|/v_F} dq q^{D-2} U \left( \frac{q}{Q}, \frac{|\omega|}{v_F Q} \right). \tag{1-46}$$

Now if the momentum integral is dominated by large momenta of the order of $Q$, then the function $U$ to leading order can be considered to be independent of frequency (since $Q \gg |\omega|/v_F$), and one can set $\omega = 0$ in $U$, and also replace the lower limit of the $q$ integral by zero. The momentum and frequency integrals then decouple, (the momentum integral gives a pre-factor and the frequency integral gives $\varepsilon^2$), and one obtains an analytic $\varepsilon^2$ dependence for $\text{Im} \Sigma$. Then the linear in $\varepsilon$ term in $\text{Re} \Sigma$ can be obtained by using the Kramers-Kronig relation. Thus
we see that large momentum (and energy independent) transfers and decoupling of the momentum and frequency integral are essential to obtain a FL behavior. The $\varepsilon^2$ result seems to be quite general under the assumptions made. When and why are these assumptions violated? Long-range interaction, associated with

\[
\text{Im} \Sigma \propto \omega^2: \quad Q \sim \Lambda \\
\Omega \sim \omega
\]

**Non-analytic part of Im $\Sigma$**

\[
Q \sim |\omega| / v_F \quad |Q - 2k_F| \sim |\omega| / v_F
\]

\[
\Omega \sim \omega
\]

Figure 1–6. Kinematics of scattering. (a) “Any-angle” scattering leading to regular FL terms in self-energy; (b) Dynamical forward scattering; (c) Dynamical backscattering. Processes (b) and (c) are responsible for nonanalytic terms in the self-energy.

small-angle scattering, is known to destroy the FL. For example, transverse long range (current-current [51] or gauge [52]) interactions which, unlike the Coulomb interaction are not screened, lead to the breakdown of the Fermi-liquid. But these interactions occur under special circumstances (e.g., near half-filling for gauge interactions). For a more generic case, it turns out that even if the bare interaction is of the most benign form, e.g., a delta-function in real space, there are deviations from a FL behavior. These deviations get amplified as the dimensionality is reduced, and, eventually, lead to a complete breakdown of the FL in 1D. Already for the simplest case of a point-like interaction, the second order self-energy shows
Figure 1–7. Non trivial second order diagrams for the self-energy

a nontrivial frequency dependence. For a contact interaction the two self-energy diagrams of Fig. can be lumped together (the second diagram is $-1/2$ the first one). Two given fermions interact via polarizing the medium consisting of other fermions. Hence the effective interaction at the second order is proportional to the polarization bubble, which just shows how polarizable the medium is,

$$\text{Im} V^R(\omega, q) = -U^2 \text{Im} \Pi^R(\omega, q).$$

For small angle scattering $q \ll 2k_F, \omega \ll E_F$, the particle-hole polarization bubble has the same scaling form in all three dimensions [53],

$$\text{Im} \Pi^R(q; \omega) \sim \nu_D \frac{\omega}{v_F q} B_D \left( \frac{\omega}{v_F q} \right),$$

where $\nu_D = a_D mk_F^{D-2}$ is the density of states in D dimensions ($a_3 = \pi^{-2}, a_2 = \pi^{-1}, a_1 = 2/\pi$) and $B_D$ is a dimensionless function whose main role is to impose a constraint $\omega \leq v_F q$ in 2D and 3D, and $\omega = v_F q$ in 1D. The above form of the polarization operator indicates Landau damping: Collective excitations (spin and charge density waves) decay into particle-hole pairs, this decay occurs only within the particle-hole continuum whose boundary for $D > 1$ is at $\omega = v_F q$ for small $\omega, q$, therefore, decay occurs for $\omega < v_F q$. Using the polarization operator in Eq.1–42 one
gets in 3D,
\[
\text{Im} \Sigma_R^{\varepsilon} \sim U^2 \int_0^\varepsilon d\omega \int_{|\omega|/v_F}^{Q-k_F} dq q^2 \frac{\omega}{v_F q v_F} \sim U^2 \int_0^\varepsilon d\omega \omega \left[ k_F - \frac{\omega}{v_F} \right],
\]
\[
\sim a\varepsilon^2 - b|\varepsilon|^3,
\]
where the first term originates from the large momentum transfer regime and is the Fermi-liquid result whereas the sub-leading second term originates from the small-momentum-transfer regime and is nonanalytic. The fraction of phase space for small angle scattering is small: most of the self-energy comes from large-angle scattering events \((q \sim Q)\), but we already start to see the importance for small angle processes. Applying Kramers-Kronig transformation to the non-analytic part \(|\varepsilon|^3\) in \(\text{Im} \Sigma_R^{\varepsilon}\), we get a corresponding non-analytic contribution to the real part as \(\text{Re} \Sigma_R^{\varepsilon, \text{non-an}} \propto \varepsilon^3 \ln|\varepsilon|\) and, finally, using the specific heat formula (see Eq. 3–14 in chapter 3) we get a nonanalytic \(T^3 \ln T\) contribution which has been observed experimentally both in metals \([54]\) (mostly heavy fermion materials) and \(\text{He}^3\) \([55]\).

Similarly in 2D
\[
\text{Im} \Sigma_R^{\varepsilon} \sim U^2 \int_0^\varepsilon d\omega \int_{|\omega|/v_F}^{k_F} dq q \frac{\omega}{v_F q v_F} \sim U^2 \varepsilon^2 \ln \frac{E_F}{|\varepsilon|},
\]
and \(\text{Re} \Sigma_R^{\varepsilon} \propto \varepsilon|\varepsilon|\) and this results in the \(T^2\) non-analyticity for the specific heat which has been observed in recent experiments on monolayers of \(\text{He}^3\) adsorbed on solid substrate \([56]\).

In 1D, as we show in chapter 3, the situation is slightly different. Even though the same power counting arguments lead to \(\text{Im} \Sigma_R \propto |\varepsilon|\) and \(\text{Re} \Sigma_R \propto \varepsilon \ln|\varepsilon|\) for the second order self-energy, \(C(T)\) is linear (analytic) in \(T\) at second order and the nonanalytic \(T \ln T\) shows up only at third order in interaction and only for fermions with spin. This difference is due to the fact that in 1D, small momentum transfers (here particle-hole continuum shrinks to a single line \(\omega = v_F q\), so decay of
collective excitations is possible only on this line) do not lead to the specific heat nonanalyticity which occurs solely from the nonanalyticity of the backscattering (at \( q \sim 2k_F \)) particle-hole bubble or the Kohn anomaly. Thus, we have the same singular behavior of the bubble (response functions) and the results for the self-energy differs because the phase volume \( q^D \) is more effective in suppressing the singularity in higher dimensions than in lower ones.

In addition to the forward scattering nonanalyticity, there is also a nonanalytic contribution to the self-energy and thermodynamics arising from \( q \approx 2k_F \), part of the response function, i.e., the Kohn anomaly. Usually, the Kohn anomaly is associated with the \( 2k_F \) nonanalyticity of the static particle-hole bubble and its most familiar manifestation is the Friedel oscillation in electron density produced by a static impurity (see section 1.1.2, of this thesis). Here the static Kohn anomaly is of no interest as we are dealing with dynamical processes. However, the dynamical bubble is also singular near \( 2k_F \), e.g., in 2D

\[
\text{Im}\Pi^R(q \sim 2k_F, \omega) \propto \frac{\omega}{\sqrt{k_F(2k_F - q)}} \theta(2k_F - q).
\] (1–51)

Due to the one-sided singularity in \( \text{Im}\Pi^R \) as a function of \( q \), the \( 2k_F \) effective interaction oscillates and falls off as a power law in real space. By power counting, since the static Friedel oscillation falls off as \( \frac{\sin(2k_F r)}{r^D} \), then the dynamical one behaves as:

\[
\tilde{U} \propto \frac{\omega \sin(2k_F r)}{r^{D-1}}
\] (1–52)

Dynamical Kohn anomaly results in the same kind of non-analyticity in the self-energy (and thermodynamics) as the forward scattering. The singularity now comes from \( |q - 2k_F| \sim \omega/v_F \), i.e., dynamic backscattering. Therefore the nonanalytic term in the self-energy is sensitive only to strictly forward or
backscattering events, whereas processes with intermediate momentum transfers
contribute to the analytic part of the self-energy.

\[
\begin{align*}
\text{(a) } & \quad \text{(b) } & \quad \text{(c)} \\
\begin{array}{c}
k_1 & \alpha & k_1 \\
\hline
k_2 & \beta & U(0) & k_2
\end{array} & \begin{array}{c}
k_1 & \alpha & k_1 \\
\hline
k_2 & \beta & U(2k_F) & k_2
\end{array} & \begin{array}{c}
k_1 & \alpha & k_1 \\
\hline
k_2 & \beta & U(2k_F) & k_2
\end{array}
\end{align*}
\]

Figure 1-8. Scattering processes responsible for divergent and/or nonanalytic
corrections to the self-energy in 2D. (a) “Forward scattering”—an analog
of the \( g_4 \) process in 1D (b) “Forward scattering” with anti-parallel
momenta—an analog of the \( g_2 \) process in 1D (c) “backscattering” with
antiparallel momenta— an analog of the \( g_1 \) process in 1D

We will now perform a kinematic analysis and show that the nonanalytic terms
in the self-energy and specific heat in 2D comes from only 1D scattering processes.
Consider the self-energy diagram of Fig.1-7.(a). The nonanalytic \( \varepsilon^2 \ln \varepsilon \) term in the
self-energy came from two \( q^{-1} \) singularities: one from the angular average of \( \text{Im} G_R \)
and the other one from the dynamic, \( \omega/v_F q \) part of the particle-hole bubble. This
form of the bubble arises only in the limit \( \omega \ll v_F q \),

\[
\text{Im}\Pi_R^R(\omega, q) \sim \text{Im} \int d^D \vec{p} d\varepsilon G(\varepsilon - \omega, \vec{p} - q) G(\varepsilon, \vec{p}) \sim \frac{\omega}{v_F q} \int d\theta \delta(\cos \theta - \frac{\omega}{v_F q}),
\]

\[
\sim \frac{\omega}{v_F q \sqrt{1 - \frac{\omega^2}{v_F^2 q^2}}} \rightarrow \frac{\omega}{v_F q} (\text{for } \omega \ll v_F q). \quad (1-53)
\]

From the delta function, \( \cos \theta = \omega/v_F q \ll 1 \), which means that the angle between \( \vec{p} \)
and \( q \) is \( \theta \sim \pi/2 \) or \( \vec{p} \) and \( q \) are perpendicular to each other. Similarly the angular
averaging of \( \text{Im} G_R^R(\vec{k} - \vec{q}, \varepsilon, \omega) \) also pins the angle between \( \vec{k} \) and \( q \) to 90 degrees.

\[
\langle \text{Im} G_R^R(\vec{k} - \vec{q}, \varepsilon, \omega) \rangle_{\theta_1} \sim \int d\theta_1 \delta(\varepsilon - \omega - qv_F \cos \theta_1)
\]

\[
\Rightarrow \cos \theta_1 \sim \frac{\varepsilon - \omega}{v_F q} \sim \frac{\omega}{v_F q} \ll 1 \Rightarrow \theta_1 \sim \pi/2
\]

Thus \( \vec{p} \) and \( \vec{k} \) (the two incoming momenta of the fermions) are almost perpendicular
to the same vector \( \vec{q} \). In 2D, this means that they are either almost parallel to each
other or anti-parallel to each other, and since the momentum transfer is either
small, $\vec{q} \sim 0$ or near $2k_F$, i.e., $|q - 2k_F| \sim 0$, we essentially have three 1D scattering processes (see Fig.1–8) which are responsible for the nonanalytic corrections to the self-energy. These three processes are (a) two fermions with almost parallel momenta ($\vec{k}_1 \sim \vec{k}_2$) collide and transfer a small momentum ($\vec{q} \sim 0$) and leave with outgoing momentum which are almost parallel to each other ($\vec{k}_1' \sim \vec{k}_2'$) and parallel to their incoming momenta ($\vec{k}_1' \sim \vec{k}_1$, $\vec{k}_2' \sim \vec{k}_2$): analogous to the “$g_4$” scattering mechanism in 1D (see Fig.1–8(a) and chapter 3) (b) two fermions with almost anti-parallel momenta collide ($\vec{k}_1 \sim -\vec{k}_2$) and transfer a small momentum ($\vec{q} \sim 0$) and leave with outgoing momentum which are almost anti-parallel to each other ($\vec{k}_1' \sim -\vec{k}_2'$) but parallel to their incoming momenta ($\vec{k}_1' \sim \vec{k}_1$, $\vec{k}_2' \sim \vec{k}_2$): analogous to the “$g_2$” scattering mechanism in 1D (see Fig.1–8(b) and chapter 3), (c) two fermions with anti-almost parallel momenta collide ($\vec{k}_1 \sim -\vec{k}_2$) and transfer a large momentum $\vec{q} \sim 2k_F$ and leave with outgoing momentum which are almost anti-parallel to each other ($\vec{k}_1' \sim -\vec{k}_2'$) and also anti-parallel to their incoming momenta ($\vec{k}_1' \sim -\vec{k}_1$, $\vec{k}_2' \sim -\vec{k}_2$): analogous to the “$g_1$” scattering mechanism in 1D (see Fig.1–8(c) and chapter 3). Therefore the nonanalytic $\varepsilon^2 \ln \varepsilon$ term in the self-energy in 2D comes from 1D scattering events, the only difference is that 2D trajectories do have some angular spread, which is of the order of $|\omega|/E_F$. It turns out (Ref. [44]), that out of the three 1D processes, the $g_2$ process and $g_1$ process are directly responsible for nonanalytic corrections (NAC) to $C(T)$ in 2D and only the $g_1$ process leads to NAC to $C(T)$ in 1D. The $g_4$ process although leads to a mass-shell singularity in the self-energy in both 2D and 1D, but does not give any NAC to thermodynamics.

In 3D the situation is slightly different, $\vec{p} \perp \vec{q}$ and $\vec{k} \perp \vec{q}$ mean that both $\vec{p}$ and $\vec{k}$ lie in the same plane. However, it is still possible to show that for the thermodynamic potential, $\vec{p}$ and $\vec{k}$ are either parallel or anti-parallel to each other. Hence, the nonanalytic term in $C(T)$ also comes from the 1D processes. In
addition, the dynamic forward scattering events (marked with a star in Fig.1-9.) which, although not being 1D in nature, does lead to a nonanalyticity in 3D. Thus the $T^3 \ln T$ anomaly in $C(T)$ comes from both 1D and non-1D processes [47]. The difference is that the former start already at the second order in interaction whereas the latter occur only at third order. In 2D, the entire $T^2$ nonanalyticity in $C(T)$ comes from 1D processes. The nonanalytic correction to the spin susceptibility will be the subject of discussion in chapter 4 of this thesis, where we will show that the nonanalyticity in $\chi_s$, both in 2D and 3D comes from both 1D and non 1D scattering processes.

Figure 1–9. Typical trajectories of two interacting fermions

Our kinematic arguments can be summarized in the following pictorial way. Suppose we follow the trajectories of two fermions, as shown in Fig.1–9. There
are several types of scattering processes. First, there is a “any-angle” scattering which, in our particular example, occurs at a third fermion whose trajectory is not shown. This scattering contributes a analytic, FL terms both to the self-energy and thermodynamics. Second, there are dynamic forward scattering events, when $q \sim |\omega|/v_F$. These are non-1D processes, as the fermions enter the interaction region at an arbitrary angle to each other. In 3D, a third order in such a process leads to a $T^3 \ln T$ term in $C(T)$. In 2D dynamic forward scattering does not lead to a nonanalyticity. Finally there are 1D scattering processes marked with a Sirius star where fermions conspire to align their momenta either parallel or anti-parallel to each other. These processes determine the nonanalytic part of $\Sigma$ and $C(T)$ in 2D and 1D.

Therefore the nonanalytic terms in the two-dimensional self-energy and thermodynamics are completely determined by 1D processes, 2D scattering does not play any role in the nonanalytic terms. As a result, if the bare interaction has some $q$ dependence, only two Fourier components matter: $U(0)$ and $U(2k_F)$ e.g., in 2D

\begin{align}
\text{Im}\Sigma^R(\varepsilon) &\propto [U^2(0) + U^2(2k_F) - U(0)U(2k_F)]\varepsilon^2 \ln |\varepsilon|, \quad (1-54) \\
\text{Re}\Sigma^R(\varepsilon) &\propto [U^2(0) + U^2(2k_F) - U(0)U(2k_F)]\varepsilon|\varepsilon|, \quad (1-55) \\
C(T)/T &\propto \gamma^* - a[U^2(0) + U^2(2k_F) - U(0)U(2k_F)]T, \quad (1-56) \\
\chi_s(Q,T) &\propto \chi_s(0) + bU^2(2k_F)\max[v_FQ, T, H], \quad (1-57)
\end{align}

where $a$ and $b$ are some coefficients. These perturbative results can be generalized for the Fermi-liquid case, when interaction is not weak. Then the vertices $U(0)$ and $U(2k_F)$, occurring in the perturbative expressions are replaced by scattering amplitude ($\Gamma$) at angle $\theta = \pi$, 

\begin{align}
\hat{\Gamma}(\vec{p},\vec{p}') &\propto \Gamma_c(\theta)\vec{l} + \Gamma_s(\theta)\vec{\sigma}\vec{\sigma}', \quad (1-58)
\end{align}
where $c$ and $s$ refer to the charge and spin sectors respectively. Thus in 2D \[45\],

$$C(T)/T = \frac{\gamma^* - a\left[\Gamma_c^2(\pi) + 3\Gamma_s^2(\pi)\right]}{(\ln T)^2} T \quad (1-59)$$

The additional $(\ln T)^2$ factor in the denominator comes from the Cooper channel renormalization of the backscattering amplitude \[47, 48\]. In 3D, the $T^3 \ln T$ nonanalyticity in the specific heat arises from both 1D (excitation of a single particle-hole pair) and non-1D (excitation of three particle-hole pairs) scattering processes \[47\].

\[
C(T) \sim \frac{T^3 \ln T}{(1 + g \ln T)^2} \left[ \Gamma_a^2(\pi)_{\text{1D, one p-h pair}} + \Gamma_{a,1}\Gamma_{a,0} + \Gamma_{a,1}^3 + \ldots \right] \quad (1-60)
\]

where subscript $a = c, s$ and 0, 1, 2... indicate the harmonics of the expansion.

Again, the additional $(1 + g \ln T)^2$ factor in the denominator comes from the Cooper channel renormalization of the backscattering amplitude \[47, 48\].

We saw that the nonanalytic corrections to the specific heat in $D = 2, 3$, arise from one dimensional scattering processes, (and they show up at second order in perturbation theory), and the degree of nonanalyticity increases with decrease in dimensionality. This predicts that the strongest nonanalyticity in the specific heat should occur in 1D. However, it was shown in Ref.\[57\], that the specific heat in 1D is linear in $T$, at least in second order in perturbation theory. In addition, the bosonization solution of a one-dimensional interacting system, predicts that the $C(T)$ is linear in $T$. We resolve this paradox by showing (in chapter 3) that the general argument for nonanalyticity in $D > 1$ at the second order in interaction breaks down in 1D, due to a subtle cancelation and the nonanalytic $T \ln T$ term in the specific heat occurs at third order and only for electrons with spin. This is verified by considering the RG flow of the marginally irrelevant operator in the Sine-Gordon theory (which appears in the bosonization scheme for fermions with spin). For spinless electrons we show that the nonanalyticities in particle-particle
and particle-hole channels completely cancel out and the resulting specific heat is linear in $T$ (the bosonized theory is gaussian). The singularity in the particle-hole channel results in a nonanalytic behavior for the spin-susceptibility $\chi_s \propto \ln \max[|Q|, |H|, T]$, present at the second order.

The spin susceptibility both in 2D and 3D gets nonanalytic contributions from both 1D and non-1D processes. These corrections will be described in detail in Chapter 4 of this thesis where we also study the nonanalytic corrections near a ferromagnetic quantum critical point.

1.3 Spin Susceptibility near a Ferromagnetic Quantum Critical Point in Itinerant Two and Three Dimensional Systems.

The physics of quantum phase transitions has been a subject of great interest lately. In contrast to the usual classical (thermal) phase transitions, quantum phase transitions occur at zero temperature as a function of some non-thermal control parameter (e.g., pressure or doping), and the fluctuations that drive the transition are quantum rather than thermal. Among the transitions that have been investigated are various metal-insulator transitions, the superconductor-insulator transition in thin metal films, and (the first one to be studied in detail and the subject of this thesis), the ferromagnetic transition of itinerant electrons that occurs as a function of the exchange coupling between the electron spins. In a pioneering paper, Hertz [58] derived a Landau-Ginzburg-Wilson (LGW) functional for this transition by considering a simple model of itinerant electrons that interact only via the exchange interaction. Hertz analyzed this LGW functional by means of the renormalization group (RG) methods that generalize the Wilson’s treatment of classical phase transitions. He concluded that the ferromagnetic order in an itinerant system sets in via a continuous (or 2$^{nd}$ order) quantum phase transition and the resulting state is spatially uniform. Furthermore, he showed that the critical behavior in the physical dimensions $d = 3$ and $d = 2$ is mean-field-like,
since the dynamical critical exponent $z = 3$, (which arises due to the coupling between statics and dynamics in a quantum problem), decreases the upper critical dimension from $d^+_{uc} = 4$ for the classical case to $d^+_{uc} = 1$ in the quantum case. Hertz’s theory which was later extended by Millis [59] and Moriya [60], (it is commonly referred as the Hertz Millis Moriya (HMM) theory), is believed to explain the quantum critical behavior in a number of materials [61]; however, there are other systems which do not agree with the HMM predictions and show a first order transition, (e.g., UGe$_2$), to the ordered state. This contradiction motivated the theorists to re-examine the assumptions made in the HMM theory.

The HMM scenario of a ferromagnetic quantum phase transition is based on the assumption that fermions can be integrated out so that the effective action involves only fluctuations of the order parameter. This assumption has recently been questioned, as microscopic calculations reveal non-analytic dependences of the spin susceptibility on the momentum ($q$), magnetic field ($H$), and for $D \neq 3$, temperature ($T$) [42, 44] both away and near the quantum critical point (see the discussion in section 1.2). For example, in 2D

$$\chi_s(H, Q, T) = \text{const.} + \max(|H|, |Q|, T), \quad (1-61)$$

and in 3D

$$\chi_s(H, Q) = \text{const.} + (q^2, H^2) \ln[\max(|H|, |Q|)], \quad (1-62)$$

where $H$, $q$, and $T$ are measured in appropriate units. The dependence on $T$ is nonanalytic in the sense that the Sommerfeld expansion for the Fermi gas can only generate even powers of $T$. Of particular importance is the sign of the nonanalytic dependence: $\chi_s(H, Q)$ increases both as a function of $H$ and $q$ (at 2$nd$ order in perturbation theory) for small $H,q$. As $\chi_s(H, Q)$ must definitely decrease for $H$ and $q$ exceeding the atomic scale, the natural conclusion is then it has a maximum
at finite $H$ and $q$. This means that the system shows a tendency either to a first order transition to a uniform ferromagnetic state (the metamagnetic transition as a function of the field), or ordering at finite $q$, (to a spiral state). The choice of the particular scenario is determined by an interplay of the microscopic parameters. In Chapter 4 of this thesis, we will obtain the nonanalytic corrections to $\chi_s(H)$ in second and third order in perturbation theory and show that these corrections oscillate between positive at 2nd order, (which points towards a metamagnetic transition), and negative at 3rd order (which points towards a continuous second order phase transition) values. Thus it is impossible to predict the nature of the phase transition by investigating the nonanalytic terms at the lowest order in perturbation theory. Furthermore, in real systems interactions are not weak and one cannot terminate the perturbation theory to a few low orders. To circumvent this inherent problem with perturbative calculations and to make predictions for realistic systems (e.g., He$^3$), we obtain the nonanalytic field dependence for a generic Fermi liquid by expressing our result in terms of the lowest harmonics of the Landau interaction parameters. We also describe the nonanalytic field dependence near the quantum critical point using the self-consistent spin-fermion model, and show that the sign of the corrections is metamagnetic. Here, in the introduction, we briefly review Hertz’s theory of the second order phase transition.

1.3.1 Hertz’s LGW Functional

Hertz considered the Hubbard model with the lagrangian $L$ given by

$$ L = \sum_{i,\sigma} C_{i,\sigma}^\dagger (\partial_\tau - \mu_0) C_{i,\sigma} - \sum_{l,l',\sigma} t_{l-l'} C_{l,\sigma}^\dagger C_{l',\sigma} + U 4 \sum_l (n_{l\uparrow} + n_{l\downarrow})^2 - U 4 \sum_l (n_{l\uparrow} - n_{l\downarrow})^2. $$

The partition function is obtained by performing a Hubbard-Stratonovitch transformation to decouple the four-fermion interaction in the charge and spin channel. The charge channel is assumed to be non-critical and is thus discarded,
whereas the partition function for the spin channel takes the following form;

\[ Z = \int D\phi DC^\dagger DC e^{-\int_0^\beta d\tau L(\phi, C^\dagger, C)}, \]  

(1–64)

where

\[ L(\phi, C^\dagger, C) = \sum_{i,\sigma} C^\dagger_{i,\sigma}(\partial_\tau - \mu_0)C_{i,\sigma} - \sum_{l,\nu,\sigma} t_{l-\nu}C^\dagger_{l,\sigma}C_{l,\nu} \]

\[ + \frac{U}{4} \sum_l \phi_l^2 l + \frac{U}{2} \sum_l \phi_l(n_{l\uparrow} - n_{l\downarrow}). \]  

(1–65)

The field \(\phi_l\) is the conjugate field to the \(n_{l\uparrow} - n_{l\downarrow}\), which can be considered as the magnetic field acting on the fermions. Performing the functional integration over the fermion operators \((C, C^\dagger)\) he arrived at the partition function

\[ Z = \int D\phi e^{-S_{eff}(\phi)}, \]

with the effective action;

\[ S_{eff}(\phi) = \frac{U}{4} \int_0^\beta d\tau \phi^2_l(\tau) - Tr \ln[(\partial_\tau - \mu - t_{l-\nu} + \sigma \frac{U}{2} \phi_l)]. \]  

(1–66)

The Mean-field-theory would correspond to the saddle point approximation to the functional integration with respect to \(\phi\). To deduce an effective (LGW) functional, one expands the interaction term \((Tr \ln\) term) in \(\phi_l\). The matrix \((M)\) in the \(Tr \ln\) term in Eq.1–66 in the Fourier space becomes

\[ (M)(k, i\omega_n, \sigma; k', i\omega_m, \sigma') = \delta_{\sigma\sigma'}[(-i\omega_n + \xi_k)\delta_{\omega_n, \omega_m}\delta_{k,\bar{k'}}] \]

\[ + \frac{\sigma U}{2\sqrt{\beta V}} \phi(\bar{k} - k', i\omega_n - i\omega_m)]. \]  

(1–67)
The first term on the right hand side of the above equation is the inverse Green’s function for free fermions \((-G^{-1}_0)\), the second term is the “interaction” \((V)\). Then

\[
Tr \ln[M] = Tr \ln[-G^{-1}_0(1 - G_0 V)] = Tr \ln[-G^{-1}_0] + Tr \ln[1 - G_0 V]
\]

\[
= Tr \ln[-G^{-1}_0] - \sum_{n=1}^{\infty} \frac{1}{n} Tr(G_0 V)^n.
\]

Expanding up to fourth order in \(V\) the effective action is

\[
S_{\text{eff}}(\phi) = \frac{1}{2} \sum_{\vec{q}, i\omega} v_2(\vec{q}, i\omega) \phi(\vec{q}, i\omega) \phi(-\vec{q}, -i\omega)
\]

\[
+ \frac{1}{4\beta V} \sum_{\vec{q}, i\omega} v_4(\vec{q}, i\omega) \phi(\vec{q}_1, i\omega_1) \phi(\vec{q}_2, i\omega_2) \phi(\vec{q}_3, i\omega_3)
\]

\[
\times \phi(\vec{q}_4, i\omega_4) \delta(\sum_{i=1}^{4} \vec{q}_i) \delta(\sum_{i=1}^{4} \omega_i).
\]

The coefficients \(v_m\) in Eq. 1–68 are the irreducible bare \(m\)-point vertices in the diagrammatic perturbation theory language. The quadratic coefficient is

\[
v_2(\vec{q}, i\omega) = 1 - U \chi_0(\vec{q}, i\omega),
\]

where \(\chi_0(\vec{q}, i\omega)\) is the free electron susceptibility given by the Lindhard function (Polarization bubble), which at small \(q\) and small \(\omega/qv_F\) behaves as

\[
\chi_0(\vec{q}, i\omega_m) = -\frac{1}{\beta V} \sum_{\vec{k}, i\varepsilon_n} G_0(\vec{k}, i\varepsilon_n) G_0(\vec{k} + \vec{q}, i\varepsilon + i\omega_m),
\]

\[
= \nu_F \left[ 1 - \frac{1}{3} \left( \frac{q}{2k_F} \right)^2 - \frac{\pi}{2} \left( \frac{q\omega_m}{qv_F} \right) + \ldots \right].
\]

Hertz assumed the all the higher order coefficients \(v_m\) starting with \(v_4\) can be approximated as constants as they vary on the scale of \(q \sim 2k_F\) and \(\omega \sim E_F\). In appropriate units Hertz’s form of the effective LGW functional is

\[
S_{\text{eff}}(\phi) = \frac{1}{2} \sum_{\vec{q}, i\omega_m} (r_0 + q^2 + \frac{|\omega_m|}{q}) |\phi(\vec{q}, i\omega_m)|^2 + \frac{u_0}{4\beta V} \sum_{\vec{q}, i\omega} \phi(\vec{q}_1, i\omega_1) \phi(\vec{q}_2, i\omega_2)
\]

\[
\times \phi(\vec{q}_3, i\omega_3) \phi(-\vec{q}_1 - \vec{q}_2 - \vec{q}_3, -i\omega_1 - i\omega_2 - i\omega_3)
\]

(1–70)
where \( r_0 = 1 - U\nu_F \equiv \xi^{-2} \) (\( \xi \) is the correlation length which diverges at the phase transition), is the distance from the critical point and \( u_0 = U^4\nu''_F/12 \) is a constant. Thus, Hertz’s effective action is almost of the same form as the classical LGW functional (for the \( \phi^4 \) theory), except for the presence of the frequency dependent term in \( v_2 \) which contains the essential information about the dynamics. The action therefore describes a set of interacting, weakly Landau-damped (due to the \( |\omega_m|/qv_F \) term) excitations: paramagnons.

Hertz then applied Wilson’s momentum shell renormalization group transformation to the above quantum functional. Here, \( q \) and \( \omega \) have to be re-scaled differently. This is due to the fact that in the paramagnon propagator \( (v_2^{-1}) \), \( q \) and \( |\omega_m| \) appear in a non-symmetric way. Therefore, the system is anisotropic in the time and space directions. As a result it becomes necessary to introduce a new parameter, the dynamical critical exponent \( z \) for scaling

\[
\omega \sim q^z.
\]  

(1–71)

For the quantum ferromagnetic transition which we study here, \( z = 3 \). In the RG procedure consists of the following steps (a) high energy states (with \( q \) and \( \omega \)) in the ”outer shell” (\( \Lambda > q > \Lambda/b; \Lambda > \omega > \Lambda/b; b > 1, \Lambda \) is a cut-off) are integrated out; (b) variables \( q \) and \( \omega \), are re-scaled as \( q' = qe^l \) and \( \omega' = \omega e^l \), with \( l \) being infinitesimal. (c) fields \( \phi \) are also re-scaled so that in terms of the new fields and re-scaled \( q \) and \( \omega \), the \( q^2 \) and \( |\omega|/q \) terms in the quadratic part of the action looks like those in the original functional. Performing all these steps, Hertz obtained the following RG equations

\[
\frac{dr}{dl} = 2r + 12uf^2, \quad \quad (1–72)
\]

\[
\frac{du}{dl} = \epsilon u - 18u^2f^4, \quad \quad (1–73)
\]
where $\epsilon = 4 - (d + z)$ and the expressions for $f^2$ and $f^4$ can be found in Ref. [58]. The second RG equation shows that the Gaussian fixed point, with $u = 0$, is stable if $\epsilon$ is negative, that is, if $d > 4 - z$. For $z = 3$, we should therefore expect a stable Gaussian fixed point and Landau exponents in $d = 2, 3$.

The two main assumptions that Hertz made in arriving at his LGW functional (Eq. 1–68 and 1–70) were: (1) the coefficients $v_{m,m\geq4}$ are nonsingular and can be approximated by constants and (2) the static spin susceptibility has regular $q^2$ momentum dependence. For the 2D ferromagnetic transition, nonanalytic terms in $v_m$ were found by Chubukov et al., [62], however, the authors claimed that these nonanalyticities do not give rise to an anomalous exponent in the spin susceptibility and therefore were not dangerous. In chapter 4 of this dissertation we examine the second assumption (2) more carefully. The reasoning behind Hertz’s second assumption was the belief that in itinerant ferromagnets the $q$ dependence of the $\phi^2$ term comes solely from fermions with high energies, of the order of $E_F$ or bandwidth, in which case the expansion in powers of $(q/p_F)^2$ should generally hold for $q \ll p_F$. This reasoning was disputed in Refs. [42, 44]. These authors considered a static spin susceptibility $\chi_s(q)$ in a weakly interacting Fermi liquid, i.e., far away from a quantum ferromagnetic transition, and argued that for $D \leq 3$ and arbitrary small interaction, the small $q$ expansion of $\chi_s(q)$ begins with a nonanalytic $|q|^{d-1}$ term, with an extra logarithm in $D = 3$. This nonanalyticity originates from the $2p_F$ singularity in the particle-hole polarization bubble [42–44] and comes from low energy fermions (in the vicinity of the Fermi surface), with energies of the order of $v_Fq \ll E_F$. These nonanalytic terms arise when one considers the reference action $S_0$ as the one which contains the particle-hole spin singlet channel interaction (charge channel) and the Cooper channel interaction, which were neglected in the Hertz model (Hertz’s reference action was just the noninteracting one). Furthermore, the pre-factor of this
term turns out to be negative, which indicates the breakdown of the continuous transition to ferromagnetism. Thus according to Ref. [42, 63] the modified effective action near the critical point should be

\[ S_{\text{eff}}(\phi) = \frac{1}{2} \sum_{\vec{q},i\omega_n} \left( r_0 - |q|^{D-1} + q^2 + \frac{\omega_m}{q} \right) |\phi(\vec{q}, i\omega_n)|^2 + b_4 \phi^4 + \ldots \]  

(1–74)

with an extra logarithm in \( D = 3 \). The weak point of this argument is that within the RPA, one assumes that fermionic excitations remain coherent at the quantum-critical point (QCP). Meanwhile, it is known [64] that upon approaching the QCP, the fermionic effective mass \( m^* \) diverges as \( \ln \xi \) in \( D = 3 \) and \( \xi^{3-D} \) in smaller dimensions. It can be shown that \( m/m^* \) appears as a prefactor of the \( |q|^{D-1} \) term; which would mean that the nonanalytic term vanishes at the QCP. This still does not imply that Eq.1–70 is valid at the transition because, as we show in chapter 4, the divergence in \( m^* \) does not completely eliminate the nonanalytic term, it just makes it weaker than away from the QCP.

Our approach will be to use the low-energy effective spin-Fermion hamiltonian, which is obtained by integrating the fermions with energies between the fermionic bandwidth \( W \) and a lower cut-off \( \Lambda \) (with \( \Lambda \ll W \)), out of the partition function [64, 65]:

\[ H = \sum_{p,\alpha} v_F (p - p_F) c_{p,\alpha}^\dagger c_{p,\alpha} + \sum_q \chi_{s,0}^{-1}(q) S_q S_{-q} + g \sum_{p,q} c_{p+q,\alpha}^\dagger \sigma_{\alpha,\beta} c_{p,\beta} S_q. \]  

(1–75)

Here \( S_q \) describe the collective bosonic degrees of freedom in the spin channel, and \( g \) is residual spin-fermion coupling. In Hertz’s approach, all fermions were integrated out, whereas in the Spin-Fermion model only the high-energy fermions are integrated out while keeping the low-energy ones. This will turn out to be important because the spin fluctuation propagator is renormalized by the fermions, and the fermion self energy is renormalized by interaction with bosons. This model has to be solved self-consistently as it takes into account the low-energy (mass)
renormalization of the spin fluctuation propagator. In chapter 4 of this dissertation we use this model to obtain the magnetic field dependence of the spin susceptibility near the quantum critical point, and analyze the stability of the second order quantum phase transition.
CHAPTER 2
CORRELATED ELECTRONS IN ULTRA-HIGH MAGNETIC FIELD: TRANSPORT PROPERTIES

One-dimensional systems exhibit unique physical properties which reflect the influence of strong correlations. The effective dimensionality of charge carriers in a bulk metal may be reduced from 3D to 1D by applying a strong magnetic field. It has recently been shown that this reduction leads to formation of a strongly correlated state, which belongs to the universality class of a Luttinger liquid [5]. The tunneling density of states exhibits a characteristic scaling behavior for the case of long-range repulsive interaction [5, 6]. This effect is most pronounced in the ultra-quantum limit (UQL), when only the lowest Landau level remains occupied. Here, in this chapter we investigate the effect of dilute impurities on the transport properties of the system. For good metals, the quantizing field is too high (of the order of $10^4$ Tesla), but semi-metals and doped semiconductors have a low carrier density and quantizing fields of the order of $1 - 10$ Tesla and allow for an experimental test of the theoretical predictions made here.

In section 2.1 we discuss localization effects for non-interacting electrons in the UQL. We find that the localization behavior is intermediate between 1D ($\delta\sigma \sim \sigma_D$: strong localization) and 3D ($\delta\sigma \ll \sigma_D$: weak localization). We show that the particle-particle correlator (Cooperon) is massive in the strong magnetic field limit. It’s “mass” (in units of the scattering rate) is of the order of the impurity scattering rate. Therefore, localization in the strong-field limit proceeds as if a strong phase-breaking process is operating as frequently as impurity scattering. Even at $T= 0$, this phase-breaking exists as it is provided by the magnetic field and as a result complete localization never occurs in 3D UQL. On the other
hand, the particle-hole correlator (the diffuson) remains massless, which means that normal quasi-classical diffusion takes place. Our findings are in agreement with previous work on this subject [66, 67], where the localization problem was analyzed for the case of long ranged disorder, whereas in our study we have analyzed the case of short ranged disorder. Our result for conductivity in the UQL is $\delta \sigma_{\text{coop}} + \sigma_{\text{Drude}} = \sigma_{\text{Drude}}/2$.

In section 2.2 we calculate the corrections to the conductivity due to electron-electron interactions using finite-temperature diagrammatic technique where disorder is treated in the ballistic limit. Due to this reduced effective dimensionality, to first order in interaction, the leading corrections are logarithmic in temperature. Another way of obtaining the conductivity is to calculate the interaction correction to the scattering cross-section through an impurity (in a Hartree-Fock approximation) and use a Drude relation between the cross-section and the conductivity. We show in section 2.3 that, to first order in the interaction, the two approaches are equivalent. This is important since, while a higher order calculation using the diagrammatic technique would be extremely lengthy, the interaction correction to the cross-section is obtained to all orders via an exact mapping on to a 1D problem of tunneling conductance of interacting electrons through a barrier [21]. We find that the Drude conductivities parallel ($\eta = +1$) and perpendicular ($\eta = -1$) to the magnetic field exhibit the scaling laws $\sigma_\eta \propto T^{\eta^{2\alpha}}$, where $\alpha$ is a function of the magnetic field. The physical reason for such a behavior of the conductivity is a nearly 1D form of the Friedel oscillation around an impurity in the strong magnetic field.

The ground state of repulsively interacting electrons in the UQL is known to be unstable to the formation of a charge-density wave (CDW) [1–3]. This has been confirmed, for example, by experiments on graphite in high magnetic fields [4]. Both the Hartree-Fock and the diagrammatic calculations presented here are done
without taking into account renormalization corrections for the interaction vertices themselves. This is justified at energies much larger than the CDW gap but breaks down at low enough energies. In order for our results to hold, there should exist an intermediate energy interval in which the renormalization of the interaction vertices due to CDW fluctuations is not yet important but the power-law renormalization of the scattering cross-section is already significant. That such an interval exists for the case of long-range electron-electron interaction was shown by solving the full RG equations for the vertices and for the cross-section. We have not included this discussion here for brevity. We discuss possible experimental verification of our results in section 2.4 and conclude in Section 2.5.

2.1 Localization in the Ultra Quantum Limit

In this section we analyze the localization effects for electrons in the UQL. As an external magnetic field is applied to the system, a question is whether the reduction of the effective dimensionality leads to re-entrance of interference effects, which were initially suppressed by a weak magnetic field. It seems plausible (and was indeed suggested by some authors in the past) that the application of a strong magnetic field may result in a strong localization of carriers, similarly to what happens in a truly 1D system. A physical argument rules out this possibility \cite{66,68}, at least for short-range impurities. In this case, while scattering at an impurity, an electron moves transverse to the field by a distance of the order of the magnetic length \( l_H = 1/\sqrt{eH} \). The flux captured by the electron’s trajectory is then of the order of the flux quantum, and thus interference is destroyed. However, an application of the standard cross-technique to the calculation of the conductivity in the UQL fails for the same reason that it does in 1D: all diagrams which go beyond the Boltzmann equation level give contributions of the order of the Drude conductivity \( \sigma_D \) itself. Therefore, perturbation theory breaks down. In 1D, a similar breakdown is a signal but not the proof of strong localization. In the
following subsection we illustrate the different scenario which arises for 3D electrons in the UQL.

2.1.1 Diagrammatic Calculation for the Conductivity

In this section we will use the zero temperature diagrammatic formalism to evaluate first the conductivity and then the WL type quantum corrections. We choose to work in the symmetric gauge, in which the eigenfunction of a three dimensional electron gas in UQL is \[ \psi_{p_z,m}(\rho, \phi, z) = e^{ip_z z} e^{-im\phi} \frac{R_m(\rho)}{\sqrt{2\pi}} \], where \( m(p_z) \) is the angular (linear) momentum quantum number in the direction of the field. In the UQL the allowed values of \( m \) are \( m = 0, 1, 2, 3 \ldots \) and

\[ R_m(\rho) = \frac{1}{l_H^{1+m}2^m} \rho^m \exp\left(-\frac{\rho^2}{4l_H^2}\right). \]

The single particle Green’s function, in the Matsubara representation is

\[ G(\mathbf{r}, \mathbf{r}'; i\varepsilon) = \sum_\lambda \frac{\psi_\lambda^*(\mathbf{r}') \psi_\lambda(\mathbf{r})}{i\varepsilon - \xi_p}, \]

where \( \xi_p = (p_z^2 - p_F^2)/2m + \omega_c/2, (\omega_c \) is the cyclotron frequency). The Green’s function in the mixed representation (momentum and position representation because the magnetic field breaks translational invariance) separates into a 1D and a transverse part

\[ G(p_z, p\phi, p'\phi'; i\varepsilon) = \frac{1}{i\varepsilon - \xi_p} \sum_{m=0}^{\infty} \left[ \frac{e^{-im(\phi - \phi')}}{2\pi} R_m(\rho) R_m^*(\rho') \right], \]

(2–1)

\[ \equiv G_{1d}(p_z; i\varepsilon) G_{\perp}(\mathbf{r}_\perp, \mathbf{r}_\perp'), \]

(2–2)

where \( (\mathbf{r}_\perp = \rho, \phi) \). The reason for this separability is the degeneracy of the Landau level; the energy does not depend on the transverse quantum numbers. The 1D part, \( G_{1d} \), is in the momentum space and the transverse part, \( G_{\perp} \), has been kept in real space. The disorder averaged Green’s function is obtained by doing
perturbation theory in the impurity potential $U$ (for weak disorder $k_F \ell \gg 1$, so that the small parameter is $1/k_F \ell$) and employing standard cross technique \cite{18} for disorder averaging. The perturbative (in $U$) solution of the Schrodinger equation for the Green’s function is

$$G(r, r'; i\varepsilon) = G_0(r, r'; i\varepsilon) + \int d^3r_1 G_0(r, r_1; i\varepsilon) U(r_1) G_0(r_1, r'; i\varepsilon)$$

$$+ \int \int d^3r_1 d^3r_2 G_0(r, r_1; i\varepsilon) U(r_1) G_0(r_1, r_2; i\varepsilon) U(r_2) G_0(r_2, r'; i\varepsilon) + \ldots$$

where $G(G_0)$ is the full (bare) Green’s function, $U(r)$ is the random impurity potential which is chosen to have a $\delta$ correlated gaussian distribution with

$$\langle U(r) \rangle = 0 \quad \text{and} \quad \langle U(r) U(r') \rangle = n_i u_0^2 \delta^3 (r - r').$$

The number density of impurities is $n_i$, and $u_0$ is the impurity strength (for short ranged point like impurities).

The leading contribution to the self-energy comes from the second order diagram. The first and third order corrections are zero as $\langle U(r) \rangle = 0$. We work in the Born limit, neglecting processes where an electron scatters from more than two (same) impurities. The second order correction is

$$\langle G_2(r, r'; i\varepsilon) \rangle = \int d^3r_1 d^3r_2 G_0(r, r_1; i\varepsilon) G_0(r_1, r_2; i\varepsilon) G_0(r_2, r'; i\varepsilon) \langle U(r_1) U(r_2) \rangle,$$

which gives

$$\langle G_2(p_z, p\phi, p'\phi'; i\varepsilon) \rangle = \left[ -n_i (u_0)^2 \nu H \pi \text{sgn}(\varepsilon) \right] \left[ \frac{1}{(i\varepsilon - \xi_p)^2} \right] \times$$

$$\left[ \sum_{m=0}^{\infty} \frac{e^{-im(\phi - \phi')}}{2\pi} R_m(\rho) R_m^*(\rho') \right],$$

and the fourth order correction gives

$$\langle G_4(p_z, p\phi, p'\phi'; i\varepsilon) \rangle = \left[ -n_i (u_0)^2 \nu H \pi \text{sgn}(\varepsilon) \right]^2 \left[ \frac{1}{(i\varepsilon - \xi_p)^3} \right] \times$$

$$\left[ \sum_{m=0}^{\infty} \frac{e^{-im(\phi - \phi')}}{2\pi} R_m(\rho) R_m^*(\rho') \right].$$
where the transverse part in each of these expressions is simply $G_\perp(r_\perp, r'_\perp)$. At fourth order, there are three diagrams, the rainbow diagram (Fig. 2-1, (b)) and the intersecting diagram (Fig. 2-1, (c)) are small by a factor $1/k_F\ell$ compared to the leading one ($\langle G_4 \rangle$) (Fig. 2-1,(a)) for short-range weak disorder. In the Born approximation, the scattering rate in a magnetic field, is $1/\tau = 2\pi n_i u_0^2 \nu_H$, where $\nu_H = 1/(2\pi^2 v_F l_H^2)$ is the 3D density of states in the presence of a magnetic field, and the self-energy is $\Sigma = -i\text{sgn}(\varepsilon)/2\tau$. The full Dyson’s series (Fig. 2–2) can be summed to give:

$$G(p_z, r_\perp, r'_\perp; i\varepsilon) = \left(\frac{1}{i\varepsilon - \xi_p}\right) G_\perp(r_\perp, r'_\perp) \left[1 + \frac{\Sigma}{i\varepsilon - \xi_p} + \left(\frac{\Sigma}{i\varepsilon - \xi_p}\right)^2 \ldots\right] = \left(\frac{1}{i\varepsilon - \xi_p + i\text{sgn}(\varepsilon)/2\tau}\right) G_\perp(r_\perp, r'_\perp)$$

(2-3)

Therefore, the effect of impurity scattering enters only in the 1D part of the Green’s function. Using the above form of the Green’s function and the Kubo formula we now evaluate the Drude conductivity. The Kubo formula for the longitudinal d.c. conductivity ($E \parallel H \parallel z$) in the kinetic equation approximation is

$$\sigma_{zz} = -\lim_{\omega \to 0} \frac{1}{\varepsilon m^2} \Im \left[ \int \frac{d\epsilon}{2\pi} \int d r'_\perp \int \frac{d p_z}{2\pi} \langle G(p_z, r_\perp, r'_\perp; i\varepsilon_+) \rangle \langle G(p_z, r'_\perp, r_\perp; i\varepsilon_-) \rangle \right]_{\omega \to \omega + i\delta}.$$ 

The diagram for the Drude conductivity is shown in (Fig. 2–3). Due to the
factorizability of the Green’s function, the conductivity is also separable into a 1D and a transverse part: \( \sigma_{zz} = \sigma_{1-d} \times \sigma_\perp \). The 1D part is in the standard form and gives the famous Einstein’s relation for the d.c conductivity \( \sigma_{1d} = e^2 \nu_1 D = e^2 \ell / \pi \), where \( D = v_F^2 \tau \) is the diffusion coefficient and \( \nu_1 = 1 / (\pi v_F) \) is the density of states in 1D. Using the orthonormality: \( \int d\rho \rho [R_m(\rho)]^2 = 1 \) and completeness: \( \sum_n [R_n(\rho)]^2 = 1 / (l_H^2) \) properties of the wave function, the transverse part can be shown to be equal to 1 with the degeneracy factor of the lowest landau level.

\[
\sigma_\perp = \sum_{m,n=0}^{\infty} \int d\rho' \int d\phi' \left[ \frac{e^{-im(\phi' - \phi)}}{2\pi} R_m(\rho) R_m^*(\rho') \right] \left[ \frac{e^{-in(\phi' - \phi)}}{2\pi} R_n(\rho) R_n^*(\rho') \right] \]
\[
= \frac{1}{2\pi l_H^2} \times 1.
\]

Thus \( \sigma_{zz} = e^2 D \nu_1 / (2\pi l_H^2) = e^2 D \nu_H \) is the Drude result in UQL. The Diffusion (or particle-hole correlator) corrections to the conductivity can be shown to be zero for a delta function impurity potential. In the next sub-section we show that the particle-particle correlator, or the Cooperon acquires a mass in a strong

\[
G = G_0 + dG
\]
\[
dG = \left< G_z \right> + \left< G_r \right> + \ldots
\]
magnetic field and evaluate the Cooperon correction to the conductivity. The
Diffuson remains massless which means normal quasiclassical diffusion occurs in the
particle-hole channel. Implications of these results on electron localization will be
discussed.

2.1.2 Quantum Interference Correction to the Conductivity

Every diagram for the conductivity, even higher order ones, can be split
into a 1D and a transverse part. The transverse part is always a number, \( c_n \), multiplied by the Landau level degeneracy factor \((1/2\pi l_H^2)^{n+1}\), where \( n \) denotes the order of the diagram (the number of dashed lines in the vertex diagrams), so that
\[
\delta \sigma_{zz} = \delta \sigma_{1D} \times \sigma_{\perp} \quad \text{and} \quad \sigma_{\perp} = c_n \times (1/2\pi l_H^2)^{n+1}
\]
We show that for the cooperon type diagrams, \( c_n = 1/(2n-1) \) and for the diffuson type \( c_n = 1 \). In 1D all quantum corrections to conductivity (QCC) are of the same order as the Drude value,
\[|\delta \sigma_{1D}| \sim \sigma_D.\] This indicates the breakdown of perturbation theory in 1D. Similarly, for 3D electrons in the UQL, \[|\delta \sigma_{zz}| \sim \sigma_H^2,\] because the transverse numbers \( c_n \) are of order one. Therefore perturbation theory also fails in the UQL. However, we will see below that these transverse numbers \( c_n \) make strong localization impossible in a strong field. In technical terms, they are responsible for generating a finite mass for the Cooperon.

![Figure 2–4. Third and second order fan diagram.](image)

We calculate these coefficients for the lowest order Cooperon diagrams (2nd and 3rd order fan diagram shown in Fig. 2–4) explicitly and then state the general argument by which these numbers can be obtained for all higher order diagrams.
For the second order fan diagram,

\[
\delta \sigma_{zz} = -\lim_{\omega \to 0} \frac{1}{\omega} \left( n_i u_0^2 \right)^2 \Im \left\{ \frac{e^2}{m^2} \left[ \int \frac{d \varepsilon}{2\pi} \int dr' \int dr_1 \int dp_2 \int dq_2 \int \frac{dp_1}{2\pi} \right] p_z (q_z - p_z) G(p_z, r_1, r_2; i\varepsilon_+) G(p_1, r_1, r_2; i\varepsilon_+) G(q_z - p_z, r_2, r'_1; i\varepsilon_+) \times G(q_z - p_z, r'_1, r_1; i\varepsilon_-) G(q_z - p_1, r_1, r_2; i\varepsilon_-) \times G(p_z, r_2, r_1; i\varepsilon_-) \right\},
\]

(2-4)

The one-dimensional part of Eq. (2-4) is given by

\[
\delta \sigma_{1D} = -\lim_{\omega \to 0} \frac{1}{\omega} \Im \left\{ \frac{e^2}{m^2} \left[ \int \frac{d \varepsilon}{2\pi} R(q) X(q, \omega) \right] \right\},
\]

(2-5)

where \( R(q) \) is the one-dimensional rung in the particle-particle channel and \( X(q, \omega) \) is the part containing the vertices:

\[
R(q) = \left( n_i u_0^2 \right)^2 \int \frac{dp_1}{2\pi} G_{1D}(p_1; i\varepsilon_+) G_{1D}(q - p_1; i\varepsilon_-),
\]

(2-6)

\[
X(q, \omega) = \int \frac{d \varepsilon}{2\pi} \int \frac{dp}{2\pi} p(q - p) G_{1D}(p; i\varepsilon_+) G_{1D}(q - p; i\varepsilon_-) \times G_{1D}(q - p; i\varepsilon_-) G_{1D}(p; i\varepsilon_-).
\]

(2-7)

We use the linear spectrum approximation, \( \int \frac{dp}{2\pi} (\ldots) = \nu' \int d\xi (\ldots) \) where \( \nu' = 1/2\pi v_F = \nu_1/2 \) and \( \nu_1 \) is the 1D density of states, and assume small (total) momentum, \( q \ll p_F \) to obtain for the rung (\( \omega > 0 \)):

\[
R(q) = \left( n_i u_0^2 \right)^2 2\pi \nu_1 \tau \left[ \frac{1 + \omega \tau}{(1 + \omega \tau)^2 + q^2 \ell^2} \right].
\]

(2-8)

For the vertex, linearizing and using \( q \ll p_F \), (we cannot set \( q = 0 \) in the vertex a priori because our cooperon will acquire a mass) followed by the pole integration in \( \xi \), and the \( \varepsilon \)-integration, we obtain for \( \omega > 0 \),

\[
X(q, \omega) = \frac{-2\nu_1 \tau^3 p_F^2 \omega}{(1 + \omega \tau) \left( (1 + \omega \tau)^2 + q^2 \ell^2 \right)}. \]

(2-9)
and finally for the 1D part of the conductivity

$$\delta \sigma_{1D} = \frac{-e^2 \ell}{4 \pi (2\pi l_H^2)^2}. \tag{2–10}$$

Note that $\delta \sigma_{1d} \sim \sigma_D$, so perturbation theory breaks down in the UQL. The transverse part of the conductivity is,

$$\sigma_\perp = \int dr'_\perp \int dr_{1\perp} \int dr_{2\perp} G_\perp (r'_\perp, r_{1\perp}) G_\perp (r_{1\perp}, r_{2\perp}) G_\perp (r_{2\perp}, r'_\perp) \times G_\perp (r'_\perp, r_{1\perp}) G_\perp (r_{1\perp}, r_{2\perp}) G_\perp (r_{2\perp}, r_\perp), \tag{2–11}$$

where $r_\perp = (\rho, \phi)$ and $G_\perp$ is defined in Eq. (2–2). After performing the azimuthal integrations, we obtain

$$\sigma_\perp = \sum_{l,m=0}^{\infty} \sum_{n=0}^{l+m} \frac{R_m^2(\rho)}{(2\pi)^3} \left[ \int d\rho' R_l^2(\rho') \right] \left[ \int d\rho_1 R_m(\rho_1) R_n(\rho_1) R_l(\rho_1) R_{n'}(\rho_1) \right]^2 \equiv \sum_{l,m=0}^{\infty} \sum_{n=0}^{l+m} \frac{R_m^2(\rho)}{(2\pi)^3} [A_{lmnn'}]^2. \tag{2–12}$$

where $n' = l + m - n$. Notice that the second order fan diagram has two radial integrations, likewise third order fan diagrams will have three radial integrations, and so on. Using the integral representation of the Gamma function, $A_{lmnn'}$ is

$$A_{lmnn'} = \frac{1}{2 l_H^2} \left( \frac{1}{2^{m+l}} \right) \left[ \frac{1}{m! n! l! n'}! \right]^{1/2} (m + l)!. \tag{2–13}$$

and the transverse part of the conductivity becomes:

$$\sigma_\perp = \frac{1}{4 (2\pi l_H^2)^3} \sum_{l,m=0}^{\infty} \sum_{n=0}^{l+m} \left( \frac{\rho}{l_H} \right)^{2m} \frac{e^{-\rho^2}}{2^{2l+2m+1} m!} \left[ \frac{1}{m! n! (l + m - n)!} \right] [(m + l)!]^2. \tag{2–14}$$
The sum over \( n \) is done using the binomial property and the sum over \( l \) is a tabulated sum \([69]\),

\[
\sigma_\perp = \frac{1}{4 (2\pi l_H^2)^3} \sum_{m=0}^{\infty} \left( \frac{\rho^2}{2l_H^2} \right)^m \frac{e^{-\rho^2/2l_H^2}}{2m!m!2^{(m+1)}},
\]

(2–15)

\[
= \frac{1}{2 (2\pi l_H^2)^3}.
\]

(2–16)

The coefficient of the transverse part of the second order fan diagram for the conductivity is \( c_2 = 1/2 \). Combining Eq. (2–10) and (2–16), the QCC from the fan diagram at second order is \( \delta\sigma_{zz} = -e^2 D\nu_H/8 \).

Similarly, the higher order diagrams can be evaluated. The third order fan diagram has the same vertex as the second order one but has one extra factor of the rung \( R(q) \). This gives for the one dimensional part: \( \delta\sigma_{1D} = -(3e^2\ell/16\pi)(2\pi l_H^2)^3 \). The transverse part now has three radial integrations (3 factors of \( A \)'s) is given as:

\[
\sigma_\perp = \sum_{q,m=0}^{\infty} \sum_{n,n'=0}^{m+q} \frac{R_m^2(\rho)}{(2\pi)^4} [A_{mnqs}] [A_{ns's'}] [A_{sqmn'}].
\]

(2–17)

where \( s' = m + q - n \) and \( s = m + q - n' \). The radial integrations can be performed as before to obtain the \( A \)'s. Performing the sums, we find

\[
\sigma_\perp = \frac{1}{(2\pi)^4 (2\pi l_H^2)^3} \sum_{m,q=0}^{\infty} \frac{R_m^2(\rho) (m+q)!}{m!q!2^{3(m+q)}} \sum_{n=0}^{m+q} \frac{(m+q)!}{n!(m+q-n)!} \sum_{n'=0}^{m+q} \frac{(m+q)!}{n'!(m+q-n')!}
\]

(2–18)

Thus for the third order fan diagram, \( c_3 = 1/4 \), and the QCC is \( \delta\sigma_{zz} = -3e^2 D\nu_H/64 \). The \( n \)th-order fan diagram has \( n \) radial integrations, each of which gives a factor of 1/2 so that one has a coefficient \((1/2)^n\). The summation over angular momentum indices gives a factor 2 regardless of the diagram’s order. So, the overall coefficient in the \( n \)th-order fan diagram due to the transverse integration is \( c_n = 1/2^{n-1} \).
Figure 2–5. Cooperon sequence for 3D electrons in the UQL. Unlike in 1D, each term in the series comes with a different coefficient $c_n$.

We construct the Cooperon sequence in UQL as shown in Fig. (2–5), with the prefactors indicating $c_n$, the numbers obtained after transverse integration at each order. These numbers are responsible for the mass of the Cooperon. The DOS factor at $n^{th}$ order is $(1/2\pi l_H^2)^{n+1}$. The dashed line in the figure denotes $g(g_1)$, the correlator in 3D (1D), where $g = 1/2\pi \nu_H \tau = n_i u_0^2 = 2\pi l_H^2/(2\pi \nu_1 \tau) = g_1 2\pi l_H^2$. $R$ is the one dimensional rung in the particle-particle channel (small total momentum) evaluated in the diffusive limit.

$$R = \int \frac{dp}{2\pi} G_{1D}(p; i\varepsilon_+) G_{1D}(q-p; i\varepsilon_-) = g_1^{-1} \left( 1 - D\tau q^2 - |\omega| \tau + ... \right), \quad (2–19)$$

For 3D electrons in the UQL, the cooperon sequence gives:

$$C(q; i\omega) = \frac{g}{(2\pi l_H^2)^2} + \frac{g^2}{(2\pi l_H^2)^3} \times \frac{R}{2} + \frac{g^3}{(2\pi l_H^2)^4} \times \frac{R^2}{4} + ...$$

$$= \frac{g_1}{2\pi l_H^2} \left( 1 + \frac{g_1 R}{2} + \frac{g_1^2 R^2}{4} + ... + \frac{(g_1 R)^{n-1}}{2^{n-1}} + ... \right), \quad (2–20)$$

and using Eq. (2–19), this becomes,

$$C(q; i\omega) = \frac{g_1 / (2\pi l_H^2)}{1/2 + D\|\tau q^2/2 - |\omega| \tau/2}. \quad (2–21)$$

In the limit $q, \omega \to 0$, $C$ becomes a constant. There are no infrared divergence, because we have a massive Cooperon. The mass in units of the scattering rate is a pure number $(1/2)$. It can be interpreted as $\tau/\tau_H^\phi$, so that $\tau_H^\phi$ is of the order of the impurity scattering time $\tau$. This indicates that localization in a strong field proceeds as if a strong phase-breaking process is operating simultaneously with impurity scattering. This is dephasing by the field and it persists even at $T \to 0$. 

\[= \]
We now contrast this situation which arises in the UQL with that of any other dimensions (1D,2D,3D) without the magnetic field. In the absence of the field the \( c_n \)'s are all one (for all dimensions) and the cooperon sequence is singular (there is a diffusion like pole for real frequencies, for \( q, \omega \sim 0 \)).

\[
C^{H=0}(q,i\omega) = g \left( 1 + gR + g^2 R^2 + \ldots \right) = \frac{g}{1 - gR} = \frac{g}{D\tau q^2 + |\omega|\tau}. \tag{2–22}
\]

This gives the weak-localization correction to the conductivity (WL QCC discussed in chapter 1) in 2D and 3D \([12]\). In 1D although the cooperon diagram has a pole, all non-cooperon diagrams are also of the same order, and one needs to sum over all the diagrams to get strong localization \([15]\).

In the ultra quantum limit the transverse numbers for the particle-hole diffusion propagator (the diffuson) are all equal to unity (\( c_n = 1 \)). Therefore diffuson remains massless in a strong field and normal quasi-classical diffusion occurs in the particle-hole channel. We will now evaluate the transverse part of the first and second order diffuson correction of Fig. (2–6), assuming a long ranged impurity potential such that \( \sigma_{1d} \neq 0 \). We do not attempt to calculate the longitudinal part of the conductivity, \( \sigma_{1d} \) as this will be more complicated due to the long range disorder potential. We just assume that the longitudinal part is finite. In the short ranged impurity case the diffuson correction to conductivity is zero (because the \( \sigma_{1d} = 0 \)). The transverse part for the first order Diffuson
correction is,
\[
\sigma_\perp = \int dr_1 \int dr'_1 G_\perp (r_\perp, r'_\perp) G_\perp (r'_\perp, r_\perp) G_\perp (r_\perp, r'_\perp) G_\perp (r'_\perp, r_\perp), \quad (2-23)
\]
where \(G_\perp\) is defined in Eq. (2-2). Performing the azimuthal integrations,
\[
\sigma_\perp = \frac{1}{(2\pi)^2} \sum_{m,n=0}^{\infty} [R_m (\rho)]^2 \int d\rho_1 \rho_1 [R_m (\rho_1)]^2 [R_n (\rho_1)]^2 \int d\rho' \rho' [R_n (\rho')]^2. \quad (2-24)
\]
Using orthonormality and completeness, we get \(\sigma_\perp = 1/ (2\pi l_H^2)^2 \times 1\), so that \(c_1 = 1\). For the second order diffusion we perform the azimuthal integrations and obtain,
\[
\sigma_\perp = \frac{1}{(2\pi)^3} \sum_{l,k,n=0}^{\infty} [R_l (\rho)]^2 \int d\rho_1 \rho_1 [R_k (\rho_1)]^2 [R_l (\rho_1)]^2 \times \\
\int d\rho_2 \rho_2 [R_k (\rho_2)]^2 [R_n (\rho_2)]^2 \int d\rho' \rho' [R_n (\rho')]^2, \quad (2-25)
\]
and using orthonormality and completeness, we obtain \(\sigma_\perp = 1/ (2\pi l_H^2)^3 \times 1\), and \(c_2 = 1\). Any \(n^\text{th}\)-order diagram can be calculated in the same way, giving \(c_n = 1\).

Therefore the longitudinal diffusion is free and the diffusion remains massless.

\[\text{Figure 2-7. Interference correction to conductivity}\]

Next we calculate the quantum interference correction to the conductivity in the ultra quantum limit (see Fig. (2-7)):
\[
\delta \sigma_{\text{coop}} = -\lim_{\omega \rightarrow 0} \frac{1}{\omega} \Im \left( \frac{e^2}{m^2} \left( \int \frac{dq}{2\pi} C(q; \omega) X(q; \omega) \right)_{\omega \rightarrow \omega + i\delta} \right), \quad (2-26)
\]
where the transverse integrations have been performed. The vertex part of this diagram has already been evaluated in Eq. (2–9). Using Eq. (2–21),

\[
\int \frac{dq}{2\pi} C(q; \omega) X(q; \omega) = \int \frac{dq}{2\pi} \left( \frac{g_1}{2\pi l_H^2} \right) \frac{g_1}{(1/2 + \omega/2 + Dq^2/2)} \times \left[ -2\nu_1\tau^3 p_F^2 \omega \frac{(1 + \omega)^2 + q^2\ell^2}{(1 + \omega/2)} \right],
\]

\[
= \frac{-2\nu_1\tau^3 p_F^2 \omega 2g_1}{(2\pi l_H^2)(1 + \omega/2)2\pi (D\ell^2)} \times \frac{\pi \ell^3 (\sqrt{1 + \omega/2} - 1)}{(1 + \omega/2) \omega/2}.
\]

The localization correction (in the diffusive limit \(\omega/2 \ll 1, q\ell \ll 1\)) is

\[
\delta \sigma_{coop} = -\lim_{\omega \to 0} \frac{1}{\omega} \text{Im} \left\{ \frac{e^2}{m^2} \left[ -\frac{(p_F\tau)^2}{2\pi \ell (2\pi l_H^2)} \frac{\omega (1 - \omega/2)}{(1 + \omega/2)^3} \right]_{\omega \to 0 + i\delta} \right\},
\]

\[
= -\frac{e^2\ell}{2\pi} \left( \frac{1}{2\pi l_H^2} \right) = -\frac{\sigma_D^H}{2}. \quad (2–27)
\]

The above result indicates that perturbation theory fails in the UQL \(\delta \sigma_{coop}/\sigma_D^H = -1/2\) in the same manner as it does in a one dimensional system. However, contrary to what happens in 1D, there is no strong localization in the UQL. The crossed diffuson diagram (next order in \(1/k_F\ell\), see Fig. 2–8), is also non singular and massive. The transverse coefficient for the lowest order crossed diffuson diagram is \(c = 1/3\sqrt{2}\). It is different from the transverse coefficients of the 2nd and 3rd order fan and diffuson diagrams. Therefore one cannot sum the perturbation series in the crossed diffuson diagram and obtain a simple geometric series. We propose that intermediate localization (as opposed to weak or strong) occurs in a 3D metal in the UQL, by which we mean that although all interference corrections are of the order of the Drude conductivity itself, the zero temperature value remains finite (finite suppression of the Drude conductivity):

\[
\sigma_{T=0} = \alpha \sigma_D^H \to \frac{1}{2} \leq \alpha < 1. \quad (2–28)
\]
The lower bound for $\alpha$ is based on the fact that $\delta \sigma_{\text{coop}} + \sigma^H_D = \frac{1}{2} \times \sigma^H_D$. The non-cooperon type diagrams, at least in the lowest order, have the opposite sign as that of the cooperon. They are also of the same order as the cooperon, so it is not clear what they will add up to. It may happen that all the non-Cooperon diagrams modify our prediction for $\alpha$ and may make $\alpha$ anywhere from $0 \rightarrow 1$. To obtain a better estimate for $\alpha$ one needs to generalize Berezinskii’s [15] diagram technique (developed for the 1D localization problem) to 3D UQL. Our results also agree with those obtained by the authors of Ref. [66, 67]. These authors considered long range disorder, $\ell_\perp \ll l_H$, and obtained $\sigma_\parallel \sim (\frac{\ell_\perp}{l_H})^2 \sigma_D$. If one uses the author’s formula for the short range impurity case, where $\ell_\perp \sim l_H$ then one obtains $\sigma_\parallel \sim \sigma_D$, which means that the number $\alpha \sim 1$.

In the next section we use the finite temperature diagrammatic technique to calculate the corrections to the conductivity due to electron-electron interactions (interaction QCC). We will show that these corrections are logarithmic in temperature and thus they confirm that the system behavior is one-dimensional.

![Crossed diffuson diagrams](image)

Figure 2-8. Crossed diffuson diagrams. Left, a double-diffuson diagram, which also acquires a mass. Right, a third-order non-cooperon diagram which, up to a number, gives the same contribution as the third order fan diagram.

### 2.2 Conductivity of Interacting Electrons in the Ultra-Quantum Limit: Diagrammatic Approach

In this section we study the electron interaction corrections (Altshuler-Aronov corrections) to the conductivity in UQL in the ballistic limit. We work in the Landau gauge and use the basis,

$$\psi_{p_x, p_z, n}(x, y, z) = \frac{e^{i(p_x x + p_z z)}}{\sqrt{L_z L_x}} \phi_n(y + p_x l_H^2)$$  \hspace{1cm} (2–29)
for the single-electron wave function, where
\[ \phi_n(u) = \frac{1}{(2^n n! \sqrt{\pi l_H})^{1/2}} e^{-u^2 / 2 l_H^2} H_n(u / l_H). \] (2–30)

Here \( l_H = 1 / \sqrt{eH} \) is the magnetic length and \( H_n \) is a Hermite polynomial. The Green’s function in Matsubara representation can be written as
\[ G(\varepsilon, p_z, p_x, y, y') = \sum_n \frac{\phi_n(y + p_x l_H^2)\phi_n(y' + p_x l_H^2)}{i \varepsilon - \xi_n(p_z)} \] (2–31)

where the sum is over all Landau levels, \( \xi_n(p_z) = (p_z^2 - p_F^2) / 2m + n\omega_c \), and \( \omega_c = eH/m \) is the electron cyclotron frequency. We will need only excitations near the Fermi level for our calculation, so in the UQL (\( E_F < \omega_c \)) contributions to the Green’s function coming from \( n > 0 \) terms in the sum in Eq. (2–31) are negligible due to the large mass term (of order \( \omega_c \)) in the denominator. Neglecting these terms, the total Green’s function is written as the product of a longitudinal and a perpendicular part
\[ G(\varepsilon, p_z, p_x, y, y') = G_{1D}(\varepsilon, p_z)G_{\perp}(p_x, y y'). \] (2–32)

with \( G_{\perp}(p_x, y y') = \phi_0(y + p_x l_H^2)\phi_0(y' + p_x l_H^2) \). As shown in the previous section the disorder-averaged longitudinal Green’s function corresponds to \( G_{1D}(\varepsilon, p_z) = 1/(i \varepsilon - \xi_{p_z} + i \text{sgn}(\varepsilon)/2\tau) \) where \( 1/\tau = -2 \text{Im} \Sigma \). Calculating the conductivity using this Green’s function gives the Drude formula, with density of states \( \nu_H = \nu_{1D} / 2\pi l_H^2 \).

The (dynamically) screened Coulomb potential in the ultra-quantum limit is given by [70]
\[ V^R(\omega, q) = \frac{4 \pi e^2}{q_z^2 + q_\perp^2 - \kappa^2 e^{-q_\perp l_H^2} \Pi^R(\omega, q_z) / \nu_{1D}}, \] (2–33)

where the screening wavevector is related to the density of states via the usual relation \( \kappa^2 = 4 \pi e^2 \nu_H \) and \( \Pi^R(\omega, q_z) \) is the polarization bubble of 1D electrons.

In what follows, we will need only some limiting forms of the potential. For
\[1/\tau \ll \omega \ll E_F \text{ and } 1/\ell \ll q \ll k_F,\]

\[\Pi^R(\omega, q_z) = -\nu_1D \frac{q_z^2}{q_z^2 - (\omega + i0)^2},\] (2–34)

independent of the temperature (up to \((T/E_F)^2\)-terms). In the static limit, when the transverse momenta are small \(q_\perp l_H^2 \ll 1\), the potential reduces to an isotropic form

\[V^R(0, q) = \frac{4\pi e^2}{q_z^2 + q_\perp^2 + \kappa^2},\] (2–35)

which differs from a corresponding quantity in the zero magnetic field only in that \(\kappa\) scales with \(H\) as \(\kappa \sim \sqrt{\nu_H} \sim H\). As it will be shown below, the leading correction to the conductivity (as well as to the tunneling density of states \([5, 6]\)) comes with processes with \(q_\perp \sim \kappa \ll l_H^{-1}\). Therefore, the Gaussian factor in the denominator of Eq. (2–33) can be replaced by unity for all cases of interest.

The polarization bubble exhibits a 1D Kohn anomaly for \(q_z\) near \(2k_F\). Such large momentum transfers are important only in Hartree diagrams, where the potential is to be taken at \(\omega = 0\) in the ballistic limit. Near the Kohn anomaly, the static polarization bubble can be written as

\[\Pi_{2k_F}(0, q_z) = -\frac{1}{2}\nu_1D \ln \frac{E_F}{\max\{|q_z - 2p_F|, T/v_F\}},\] (2–36)

to logarithmic accuracy.

Finally, the pole of the potential in Eq. (2–33) corresponds to a collective mode – magnetoplasmon. For \(\omega, qv_F \ll E_F\) and \(q_\perp l_H \ll 1\), the dispersion relation of the magnetoplasmon mode is given by

\[\omega_p^2 = \omega_{p0}^2 \frac{q_z^2}{q^2} + v_F^2 q_z^2 \equiv s^2 q_z^2,\] (2–37)

where \(\omega_{p0} = \sqrt{4\pi ne^2/m}\) is the plasmon frequency of a 3D metal in zero magnetic field and \(s \equiv v_F\sqrt{1 + \kappa^2/q^2}\) is the magnetoplasmon velocity. In all situations of interest for this problem, typical longitudinal momenta is much smaller than the
transverse ones, $|q_z| \ll q_\perp$, so that one can write

$$s = v_F \sqrt{1 + \frac{\kappa^2}{q_\perp}}. \quad (2.38)$$

Figure 2–9. First order interaction corrections to the conductivity where effects of impurities appear only in the disorder-averaged Green’s functions.

We now proceed to compute the first order interaction correction to the conductivity in the ballistic limit ($T\tau \gg 1$). This includes contributions from diagrams shown in Fig. 2–9, where effects of impurities appear only in the disorder-averaged Green’s functions. It also includes diagrams with one interaction line and one extra impurity line. These can be separated further into exchange (Fig. 2–10) and Hartree (Fig. 2–11) diagrams. In this section we show that diagrams 2–10(a), 2–10(b), 2–11(a) and 2–11(b) give a leading $-|\ln(T/E_F)|$ correction to the conductivity, whereas all other diagrams give sub-leading contributions.

### 2.2.1 Self-Energy Diagrams

Diagrams Fig. 2–9(a), 2–10(a), 2–10(c), and 2–11(a) involve corrections to the self-energy due to electron-electron interaction. Diagram 2–9(a) describes inelastic scattering of an electron on a collective mode (plasmon), which would have existed even for a system without disorder. As the electron-electron interaction cannot lead to a finite conductivity in the translationally invariant case, this diagram is canceled by the counter-correction of the vertex type [Fig. 2–9(b)]. Diagrams Fig. 2–10(a), 2–10(c), and 2–11(a) describe correction to the self-energy due to
Figure 2–10. Exchange diagrams that are first order in the interaction and with a single extra impurity line. The Green’s functions are disorder-averaged. Diagrams (a) and (b) give \( \ln T \) correction to the conductivity and exchange diagrams (c), (d) and (e) give sub-leading corrections to the conductivity.

interference between electron-electron and electron-impurity scattering. A general form of the correction to the conductivity for all diagrams of the self-energy type can be written as

\[
\delta \sigma = \frac{2e^2v_F^2}{2\pi l_H^2} \lim_{\Omega \to 0} \frac{1}{\Omega_m} \left[ \int \frac{dp_z}{2\pi} T \sum_{\varepsilon_n} G_{1D}^2 (\varepsilon_n, p_z) G_{1D} (\varepsilon_n - \Omega_m, p_z) \delta \Sigma_{1D} (\varepsilon_n, p_z) \right]_{\Omega_m \to \Omega + i0},
\]

where \( \delta \Sigma_{1D} (\varepsilon_n, p_z) \) is the correction to the (Matsubara) self-energy of the effective 1D problem, to which the original problem is reduced upon integrating out transverse coordinates. This is possible due to the fact that the Green’s functions are factorized into a 1D and a transverse part, as shown in Eq. (2–32), and the integrations over transverse variables can be carried out and simply give the
Figure 2–11. Hartree diagrams that are first order in the interaction and with a single extra impurity line. The Green’s functions are disorder-averaged. Both diagrams give $\ln T$ correction to the conductivity.

degeneracy factor $1/2\pi l_H^2$. In this effective 1D problem, electrons interact via an effective potential

$$\rho_{1D} (\omega, q) = \int \frac{d^2 q}{(2\pi)^2} V (\omega, q) e^{-q^2 l_H^2}, \quad (2–40)$$

whereas each impurity line carries a factor $n_i u_0^2 / 2\pi l_H^2 = v_F / 2\tau$, where $n_i$ is the concentration of impurities and $u_0$ is the impurity potential. The overall factor of 2 in Eq. (2–39) is the combinatorial coefficient associated with each diagram of the self-energy type.

Substituting (2–33) into (2–40) and using the condition $\kappa l_H \ll 1$, we obtain

$$\rho_{1D} (\omega, q) = e^2 \ln \frac{1}{l_H^2 [q^2 / 2\pi] \kappa^2 (\omega, q)} \rho_{1D}.$$

Performing the analytic continuation in Eq. (2–39), we obtain

$$\delta \sigma_{\Sigma} = \frac{e^2 v_F^2}{2\pi l_H^2} \int \frac{d\xi}{2\pi} \left( -\frac{\partial n_0}{\partial \xi} \right) \int \frac{dp}{2\pi} \left[ G_{1D}^R \right]^2 \left[ -\text{Im} G_{1D}^R \text{Im} \delta \Sigma_{1D}^R + \text{Re} G_{1D}^R \text{Re} \delta \Sigma_{1D}^R \right]. \quad (2–42)$$

where $G_{1D}^R = 1/(\xi - \xi + i/2\tau)$ and $\delta \Sigma_{1D}^R$ is the interaction correction to the retarded self-energy of the non-interacting electrons which is $\Sigma_0 = -i/2\tau$. (For brevity, we suppressed the arguments of $G_{1D}^R$ and $\delta \Sigma_{1D}^R$, which are $\xi, p$).
2.2.1.1 Diagram Fig. 2–10(a)

We start with the exchange diagram Fig. 2–10(a) which corresponds to a correction in the self-energy as shown in Figure 2–12. As \( \omega \) and \( q_z \) are expected to be large compared to \( 1/\tau \) and \( 1/\ell \), respectively, it suffices to replace the Green’s functions in the self-energy by those in the absence of disorder. In the rest of the diagram for the conductivity, the Green’s functions are taken in the presence of disorder. In 1D, it is convenient to separate the electrons into left- and right-movers described by the Green’s functions 

\[
G_{\pm}(\varepsilon_n, p) = \frac{1}{i\varepsilon_n \mp v_F p_{\pm} + \text{sgn}\varepsilon_n / 2\tau},
\]

where \( p_{\pm} = p_z \mp p_F \). Accordingly, there are also two self-energies \( \Sigma_{\pm} \), for left- and right- moving electrons. The contribution for \( \Sigma_+ \) is shown in Fig. 2–12. The Green’s functions of right/left electrons are labeled by \( \pm \) in the diagram. Processes in which an electron is forward-scattered twice at the same impurity do not contribute to the conductivity and are therefore not considered in this calculation. The diagram with backscattering both at an impurity and other electrons involves states far away from the Fermi surface and is thus neglected. The only important diagram is the one shown in Fig. 2–12 where the electron is backscattered by an impurity and forward scattered by other electrons.

Figure 2–12. The self-energy correction contained in diagram 2–10(a), denoted in the text as \( \Sigma_{+}^{(2–12)} \).

At first, we neglect the frequency dependence of the potential. The momentum carried by the interaction line is small, \( q_z \simeq \varepsilon/v_F \simeq T/v_F \), and at low temperatures, such that \( T/v_F \ll \kappa \), one can neglect \( q_z \) compared to \( \kappa \) in \( V_{1D} \) and replace \( V_{1D} \) by a
constant, $V_{1D} \rightarrow 2g_0v_F$, where

$$g_0 = \left( \frac{e^2}{v_F} \right) \ln \frac{1}{\kappa l_H}$$

is a dimensionless coupling constant. The perturbation theory is valid for $g_0 \ll 1$.

Having in mind that the retarded self-energy is obtained from the Matsubara one by the analytic continuation $i\varepsilon_n \rightarrow \varepsilon + i0$ for $\varepsilon_n > 0$, we choose $\varepsilon_n$ to be positive. Performing an elementary integration over $p'$, we arrive at

$$\Sigma_{+}^{(2-\text{12})}(\varepsilon_n, p) = \frac{2ig_0v_FT}{\tau} \sum_{\omega_m > \varepsilon_n} \int \frac{dq_z}{2\pi} \frac{1}{[i\omega + v_F q_z][i(\varepsilon_n - \omega_m) - v_F(p - q_z)]},$$

where $1/\tau = n_i u_0^2 \nu_{1D}/l_H^2$. Although the integral over $q_z$ is convergent at the upper limit, it is instructive to calculate it with an ultraviolet cut-off $q_{\text{max}} \sim p_F$. Doing so, we obtain

$$\Sigma_{+}^{(2-\text{12})}(\varepsilon_n, p) = 2g_0 \frac{T}{\tau} \sum_{\omega_m > \varepsilon_n} \frac{1}{i(2\omega_m - \varepsilon_n) + v_F p} \times$$

$$\left[ 1 - \frac{1}{\pi} \left\{ \tan^{-1} \frac{\omega_m}{v_F q_{\text{max}}} + \tan^{-1} \frac{\omega_m - \varepsilon_n}{v_F q_{\text{max}}} \right\} \right]. \quad (2-44)$$

Now we see that to logarithmic accuracy it is safe to cut the sum at $\omega_M \sim v_F q_{\text{max}} \sim E_F$ and omit the factor in the curly brackets in Eq. (2-44):

$$\Sigma_{+}^{(2-\text{12})}(\varepsilon_n, p) = 2g_0 \frac{T}{\tau} \sum_{\omega_m > \varepsilon_n} \frac{1}{i(2\omega_m - \varepsilon_n) + v_F p} \left[ \ln \frac{E_F}{2\pi T} - \Psi \left( \frac{1}{2} + \frac{\varepsilon_n - iv_F p}{4\pi T} \right) \right]. \quad (2-45)$$

Performing analytic continuation and separating real and imaginary parts, we obtain

$$\text{Re}\Sigma_{+}^{R(2-\text{12})}(\varepsilon, p) = \frac{g_0}{4\tau} \tanh \frac{\varepsilon + v_F p}{4\pi T}; \quad (2-46)$$

$$\text{Im}\Sigma_{+}^{R(2-\text{12})}(\varepsilon, p) = -\frac{g_0}{2\pi T} \left[ \ln \frac{E_F}{2\pi T} + \text{Re}\Psi \left( \frac{1}{2} - i\frac{\varepsilon + v_F p}{4\pi T} \right) \right]$$

$$= -\frac{g_0}{2\pi T} \ln \frac{E_F}{\max \{|\varepsilon + v_F p|, T\}}. \quad (2-47)$$
To obtain the real part in a form given in Eq. (2–46) we used an identity $|\Gamma(\frac{1}{2} - ix)|^2 = \pi / \cosh \pi x$, whereas the last line in Eq. (2–47) is valid to logarithmic accuracy. The self-energy of left-moving electrons is obtained from Eqs. (2–46), (2–47) by making a replacement $\varepsilon + v_F p \rightarrow \varepsilon - v_F p$.

We pause here to discuss the physical meaning of the results contained in Eqs. (2–46) and (2–47). Eq. (2–46) shows that the correction to the effective mass is $T$-dependent: for $|\varepsilon + v_F p| \ll T$, $\delta m \propto T^{-1}$. In principle, such a correction might result in an additional $T$-dependence of the conductivity. However, this $T$-dependence occurs only in the next-to-leading order in the parameter $(T\tau)^{-1} \ll 1$ of the ballistic approximation. The leading correction to the conductivity comes from the imaginary part of the self-energy, Eq. (2–47). This correction exhibits a characteristic 1D logarithmic singularity, which signals the breakdown of the Fermi liquid (to the lowest order in the interaction).

The main contribution to the conductivity comes from the correction to the imaginary part of the self-energy [Eq. (2–47)]. Substituting Eq. (2–47) into Eq. (2–42) and adding a similar contribution from the left-moving electrons, we obtain

$$\frac{\delta \sigma^{(2-10a)}}{\sigma} = -\frac{g_0}{\pi} \ln \frac{E_F}{T} = -\frac{e^2}{\pi v_F} \ln \frac{1}{\kappa l_H} \ln \frac{E_F}{T}$$ (2–48)

We note that the above result was obtained using the static form of the interaction potential. We now return to the full dynamic potential and show that the frequency dependence of the potential does not change the results given by Eqs. (2–46) and (2–47), to logarithmic accuracy. For a dynamic potential it is more convenient to perform the integration over $q_\perp$ at the very end so that the potential entering the calculation is of the 3D form

$$V(\omega_m, \mathbf{q}) = \frac{4\pi e^2}{q_z^2 + q_\perp^2 + \kappa^2 \frac{q_z^2}{\omega_m^2 + q_z^2}}$$ (2–49)

$$\approx \frac{4\pi e^2}{q_\perp^2 + \kappa^2 v_F^2 q_z^2 + \alpha^2 \omega_m^2},$$ (2–50)
where we used that $q_z \ll q_{\perp}$ and introduced $\alpha^2(q_{\perp}) = q_{\perp}^2/(q_{\perp}^2 + \kappa^2)$. The integral over $p'$ gives the same result as for the static potential. Integrating over $q_z$, we obtain for the effective 1D self-energy instead of (2–45),

$$\tilde{\Sigma}_+^{(2−12)}(\varepsilon_n, p) = \frac{e^2}{\tau} \int d^2 q_{\perp} \frac{e^{-q_{\perp}^2 l_H^2}}{(2\pi)^2 q_{\perp}^2 + \kappa^2} \frac{1}{\alpha(q_{\perp})} \left[ \ln \frac{E_F}{2\pi T} - \Psi \left( \frac{1}{2} + \frac{\varepsilon_n - iv_F p}{2\pi T [1 + \alpha(q_{\perp})]} \right) \right].$$

To log-accuracy, the integral over $q_{\perp}$ is solved by taking the limits $q_{\perp}/\kappa \to \infty$ and $q_{\perp} l_H \to 0$ in the integrand and cutting the integral at $q_{\perp} = \kappa$ and $q_{\perp} = l_H$ as the lower and upper limits, respectively. In this approximation, $\alpha(q_{\perp})$ is replaced by 1, and the result for $\tilde{\Sigma}_+^{(2−12b)}$ coincides with that obtained for the static potential, Eqs. (2–46) and (2–47). Coming back to Eqs. (2–49) and (2–50), we can interpret this result in the following way. The difference between the dynamic potential and the static one is in the presence of the dynamic polarization bubble multiplying $\kappa^2$ in the denominator of Eq. (2–49). If the potential is taken in the static form, typical frequencies are related to typical momenta as $\omega \sim v_F q_z$, which means that this factor is of order of unity and $\kappa$ must be replaced by $\kappa^* \sim \kappa$. But because the final result for $\Sigma$ depends on $\kappa$ only via a (large) logarithmic term, $\log(|\ln \kappa l_B|)$, such a renormalization of $\kappa$ is beyond the logarithmic accuracy of the calculation.

2.2.1.2 Diagram Fig. 2–11(a).

Figure 2–13. The self-energy correction contained in diagram 2–11(a), denoted in the text as $\Sigma_+^{(2−13)}$. 
Diagram Fig. 2–11(a) is a Hartree counter-part of the exchange diagram of Fig. 2–10(a). Separating the contributions of left- and right movers, the diagram corresponding to backscattering at the static impurity potential is shown in Fig. 2–13. Again, diagrams corresponding to forward scattering at the impurity potential do not contribute to the conductivity and do not need to be considered here. The diagram of Fig. 2–13 also includes backscattering at a Friedel oscillation. Although this diagram contains a particle-hole bubble, it is more convenient to label the momenta as shown in Fig. 2–13, integrate over \( p' \) first, and then over \( k \).

For backscattering, the 1D potential of Eq. (2–41) becomes

\[
V_{1D}^{2k_F}(\omega_m, q') = V_{1D}(\omega_m, q' + 2k_F)
\]

\[
= \frac{e^2}{\hbar} \ln \left\{ l_H^2 \left[ (2k_F)^2 + \kappa^2 \ln \frac{2k_F}{\max \{|q'|, |\omega_m|/v_F, T/v_F\}} \right] \right\}
\]

where the last line is valid to logarithmic accuracy. As a first approximation, we neglect the \( q' \)-dependence of the interaction potential, replacing \( V_{1D}^{2k_F} \) in Eq. (2–51) by a constant \( V_{1D}^{2k_F} \rightarrow 2g_{2k_F}v_F \). This results in

\[
\Sigma_+^{R(2−13)}(\varepsilon_n, p) = -\frac{2g_{2k_F}T}{\tau} \sum_{\omega_m > \varepsilon_n} \frac{1}{i(2\omega_m - \varepsilon_n) + v_Fp}, \quad (2–52)
\]

\[
= -\frac{g_{2k_F}}{2\pi i\tau} \left[ \ln \frac{E_F}{2\pi T} - \Psi \left( \frac{1}{2} + \frac{\varepsilon_n - iv_Fp}{4\pi T} \right) \right] \quad (2–53)
\]

which, up to a sign and overall factor of the coupling constant, is the same as the exchange contribution \( \Sigma_+^{R(2−13)} \) in Eq. (2–45).

When the dependence of \( V_{1D}^{2k_F} \) on \( q' \) is restored, the result in Eq. (2–53) changes only in that the coupling constant acquires a weak \( T \)-dependence

\[
g_{2k_F} \rightarrow g_{2k_F}(T) = \left( \frac{e^2}{2v_F} \right) \ln l_H^2 \left[ (2k_F)^2 + \kappa^2 \ln E_F/T \right]. \quad (2–54)
\]
Calculating the contribution of Eq. (2–53) with Eq. (2–54) to the conductivity, we find the correction to the conductivity from diagram Fig. 2–11(a) to be:

$$\frac{\delta\sigma^{(2--11a)}}{\sigma} = -\frac{e^2}{2\pi v_F} \ln \left[ \frac{4k_F^2 + \kappa^2 \ln E_F/T}{\kappa^2} \right] \ln \frac{E_F}{T}. \quad (2-55)$$

Notice that in the limit of very low $T$ and/or very strong fields, the screening wavevector drops out of the result and the net $T$-dependence of the conductivity becomes $\ln x \ln (\ln x)$, where $x \equiv E_F/T$.

### 2.2.2 Vertex Corrections

Two other important diagrams are the ones in Fig. 2–10(b) and Fig. 2–11(b). These are the vertex corrections counterparts of the self-energy diagrams in Fig. 2–10(a) and Fig. 2–11(b), correspondingly.

#### 2.2.2.1 Diagram 2–10(b)

Diagram 2–10(b) can be shown to give the same contribution as 2–10(a). In this Section we show this by reducing diagram 2–10(b) to 2–10(a) without doing explicit integrations over $q_z$ and Matsubara summations.

Decomposing diagram 2–10(b) into contributions from left and right fermions, we obtain

$$\delta\sigma^{(2--10b)} = -\frac{2v_F}{\tau} \frac{1}{2\pi T_F^2} \lim_{\Omega \to 0} \left( \frac{e^2 v_F^2}{\Omega_m} \right) T^2 \sum_{\omega_l} \sum_{\varepsilon_n} \int \frac{dq_z}{2\pi} M_+ M_- V_{1D}(\omega_l, q_z) |_{i\Omega_m - i\Omega + i\varepsilon_n}, \quad (2-56)$$

where $M_\pm$ are the vector vertices

$$M_\pm = \pm \int \frac{dp}{2\pi} G_\pm(\varepsilon_n, p) G_\pm(\varepsilon_n - \Omega_m, p) G_\pm(\varepsilon_n - \omega_l, p - q_z).$$

It is obvious that $M_- = -M_+^*$. When evaluating $M_\pm$, one needs to consider all choices of signs for Matsubara frequencies. For the cases

$$\varepsilon_n > 0, \varepsilon_n - \Omega_m < 0, \text{ and } \varepsilon_n - \omega_l < 0; \quad (2-57)$$

$$\varepsilon_n > 0, \varepsilon_n - \Omega_m < 0, \text{ and } \varepsilon_n - \omega_l > 0, \quad (2-58)$$
we have

\[ M_+ = -\frac{1}{v_F(\Omega_m + 1/\tau)[i\omega_l \mp v_F q_z + i/\tau]}; \]  

\[ M_- = \frac{1}{v_F(\Omega_m + 1/\tau)[i(\Omega_m - \omega_l) \mp v_F q_z + i/\tau]}, \]

respectively. For all other cases, the results can be shown either to vanish because of the locations of the poles or to cancel each other. In the ballistic limit, the product \( M_+ M_- \) can be simplified in both cases to

\[ M_+ M_- = -\frac{1}{(\Omega_m + 1/\tau)^2[\omega_l^2 + (v_F q_z)^2]}. \]

The subsequent integration of this expression gives a \(|\omega_l|^{-1}\)-singularity and it is this singularity which gives the \( \ln T \)-dependence of the correction to the conductivity.

Now we go back to diagram 2–10(a). In Sec. 2.2.1.1, we found the contribution of this diagram by evaluating the self-energy first and then substituting the result into the Kubo formula. To prove the equivalence of diagrams 2–10(a) and (b) it is convenient to consider the full diagram 2–10(a) without singling out the self-energy part. Summing up the contribution of left and right fermions, we obtain

\[ \delta \sigma^{(2-10a)} = -\frac{2v_F}{\tau} \frac{1}{2\pi l_H^2} \lim_{\Omega \to 0} \left( \frac{e^2 v_F^2}{\Omega_m} \right) T^2 \sum_{\omega_l} \sum_{\varepsilon_n} \sum_{\alpha = \pm} \int \frac{dq_z}{2\pi} P_\pm V_{1D}(\omega_l, q_z)|_{\Omega_m \to \Omega + i0^+}, \]

where

\[ P_\pm = \int \frac{dp}{2\pi} G_\pm^2 (\varepsilon_n, p) G_\pm (\varepsilon_n - \Omega_m, p) G_\pm (\varepsilon_n - \omega_l, p - q_z) \times \]

\[ \int \frac{dp'}{2\pi} G_\pm (\varepsilon_n, p') G_\pm (\varepsilon_n - \omega_l, p' - q_z). \]

A non-zero result for Eq. (2–62) is obtained only for the case given in Eq. (2–57), when
\[ P_+ + P_- = -\frac{2}{v_F^2 (\Omega_m + 1/\tau)^2 [i\omega_l + v_F q_z + i/\tau][i\omega_l - v_F q_z + i/\tau]} + Q, \quad (2\text{-}63) \]

with
\[ Q = \frac{i}{v_F^2} \left[ \frac{1}{(i\omega_l - v_F q_z + i/\tau)^2} \frac{1}{(i(\omega_l - \Omega_m) - v_F q_z)(i\omega + v_F q_z + i/\tau)} - \left( q_z \to -q_z \right) \right]. \]

Neglecting the \( q_z \)-dependence of \( V_{1D} \), we see that \( \int dq_z Q = 0 \). An expansion in \( q_z^2 \) results in non-divergent terms which do not bring any non-trivial \( T \)-dependence.

Making a ballistic approximation in the rest of Eq. (2\text{-}63), we see that it coincides with Eq. (2\text{-}61). The Matsubara summation goes over a twice smaller interval of frequencies compared to that in Eq. (2\text{-}56). We see that Eqs. (2\text{-}62) and (2\text{-}56) give the same result and thus
\[ \delta\sigma^{(2\text{-}10b)} = \delta\sigma^{(2\text{-}10a)}. \quad (2\text{-}64) \]

**2.2.2.2 Diagram Fig. 2–11(b)**

The diagram in Fig. 2–11(b) is a vertex correction counterpart of the Hartree self-energy diagram Fig. 2–11(a), and it gives the same contribution as Fig. 2–11(a). To see this, we compare the diagrams in Figs. 2–10 (b) and 2–11(b).
labeling them as shown in Fig. 2–14. For a \( q \)- and \( \omega \)-independent interaction, diagram Fig. 2–14(b) is of the same magnitude but opposite sign as diagram Fig. 2–14(a). For a \( q \)-dependent interaction, the \( T \)-dependent parts of these diagrams differ also in the overall factor of the coupling constant: diagram 2–14(a) contains \( g_0 \) whereas diagram 2–14(b) contains \( g_{2k_F} \). Electron-electron backscattering in diagram (a) and electron-electron forward scattering in diagram (b) give either sub-leading or \( T \)-independent contributions. Thus

\[
\delta \sigma(2-11b) = -\frac{g_{2k_F}}{g_0} \delta \sigma(2-10b) = -\frac{g_{2k_F}}{g_0} \delta \sigma(2-10a) = \delta \sigma(2-11a). \tag{2–65}
\]

### 2.2.3 Sub-Leading Diagrams

All other diagrams give sub-leading contributions. Diagram Fig. 2–10(c) gives a self-energy type contribution to the conductivity so we use Eq. (2–42). If the interaction potential is taken to be static, the contribution from this diagram is zero. Using the dynamical potential, the leading contribution from this diagram is a \( \ln T \)-correction to the conductivity

\[
\left| \frac{\delta \sigma(2-10c)}{\sigma} \right| = \frac{1}{2\pi \nu_F} e^2 \ln \frac{E_F}{T}. \tag{2–66}
\]

This contribution is smaller than that from diagrams Fig. 2–10(a) [Eq.(2–48)] and Fig. 2–11(a) [Eq.(2–55)] (and diagrams Fig. 2–10(b) and 2–11(b)) by a \( T \)-independent log-factor.

Diagrams 2–10(d) and 2–10(e) give mutually canceling contributions of the form:

\[
\frac{\delta \sigma(2-10d)}{\sigma} = \frac{e^2}{24\nu_F g_0} \frac{1}{T \tau}; \tag{2–67}
\]

\[
\frac{\delta \sigma(2-10e)}{\sigma} = -\frac{e^2}{24\nu_F g_0} \frac{1}{T \tau}. \tag{2–68}
\]

Each of these contributions is small since we are in the ballistic limit \( (T \tau \gg 1) \).
All the calculations shown here are done considering the dynamic interaction potential at small frequencies. At high frequencies, \( i.e. \), at frequencies close to the magnetoplasmon frequency, the contributions from all the conductivity diagrams cancel out. That this has to be the case was pointed out recently in Ref. \([19]\). This is a very useful result because each individual diagram, taken separately, may give singular corrections. In our case we have also explicitly checked that this cancellation indeed occurs. Contributions from diagrams (2–9a), (2–10a) and (2–10c) cancel each other. Contribution from (2–9b) cancels that of (2–10b), and finally (2–10d) and (2–10e) cancel each other.

### 2.2.4 Correction to the Conductivity

Adding up the results from Eqs.\((2–48)\), \((2–55)\), \((2–64)\), and \((2–65)\), we find the leading correction to the conductivity

\[
\frac{\delta \sigma}{\sigma} = \frac{\delta \sigma^{(2–10a)}}{\sigma} + \frac{\delta \sigma^{(2–11a)}}{\sigma} + \frac{\delta \sigma^{(2–10b)}}{\sigma} + \frac{\delta \sigma^{(2–11b)}}{\sigma}
\]

\[
= -\frac{e^2}{\pi v_F} \ln \left( \frac{4p_F^2 + \kappa^2 \ln E_F/T}{\kappa^2} \right) \ln \frac{E_F}{T}. \tag{2–69}
\]

Eq. \((2–69)\) is the main result of this Section.

### 2.2.5 Effective Impurity Potential

The fact that only four diagrams, Figs. 2–10(a,b) and Figs. 2–11(a,b), determine the leading correction to the conductivity suggests that there must be some simple reason for these diagrams to be the dominant ones. Indeed, only these diagrams arise if one considers scattering of electrons by “effective” impurities that consist of a combination of bare impurities and the Coulomb fields of electrons surrounding the bare impurities. For weak delta-function bare impurities, the effective impurity potential corresponds to “dressing” the impurity with the mean field of Hartree and exchange interactions (see Fig. 3–1).

\[
\tilde{V}_0(\varepsilon, p, p') = V_0 + V_H(p - p') + V_x(\varepsilon, p, p').
\]
The first term in this equation is the strength of a bare impurity, the second one is the Coulomb potential of electrons whose density is modulated due to the presence of the bare impurity, and the third one is an exchange potential for electrons interacting and scattering through a weak impurity.

\[
C = n_i |\tilde{V}_0(\varepsilon, p, p')|^2 = n_i V_0^2 + 2n_i V_0 [V_H(p - p') + V_x(\varepsilon, p, p')] + \mathcal{O}(g^2),
\]

(2–70)

where \( g = e^2/\nu_F \) is the interaction strength. Diagrammatically, \( C \) corresponds to a dashed line of the cross-technique [18]. The first term (bare impurities) is taken into account in the leading order in \( 1/E_F \tau \ll 1 \) by summing infinite series for the single-particle Green’s function and then using the Kubo formula for the conductivity. Because the bare impurities are short-range, there is only one diagram for the conductivity—the usual “handle” diagram; the vertex correction to this diagram vanishes. Corrections to the conductivity result from the Hartree and exchange terms in Eq. (2–70). To first order there are two diagrams, shown in Fig. 2–16. Although the bare impurity is point-like, the Hartree and exchange potentials it generates have slowly decaying tails and also oscillate in space. Thus the vertex correction, Fig. 2–16(b), is not zero. The self-energy diagram, Fig.

\[
\text{Figure 2–15. Effective impurity potential}
\]
2–16(a), corresponds to two diagrams: Fig. 2–10(a) and Fig. 2–11(a). Diagram Fig. 2–16(b) corresponds to the diagrams in Fig. 2–10(b) and Fig. 2–11(b). For an arbitrary impurity potential, it can be shown that contributions of 2–16(a) and 2–16(b) coming from forward scattering cancels each other. For backscattering, the contribution from 2–16(a) and 2–16(b) are the same.

2.3 Impurity Scattering Cross-Section for Interacting Electrons

In this Section we apply a different approach to the conductivity of interacting electrons in the UQL. In Section 2.2.5 we demonstrated that, to first order in the interaction, the only diagrams which are important for $\delta \sigma$ correspond to scattering at an effective impurity potential. This suggests that the result for $\delta \sigma$ can be obtained by calculating the interaction correction to the impurity scattering cross-section and then substituting the corrected cross-section into the Drude formula. In this section we show that to first order this procedure gives a result identical to that of the diagrammatic approach of Sec. 2.2. Unlike the diagrammatic series in the interaction for the conductivity, the perturbation theory for the scattering cross-section can be summed up to all orders via a renormalization group procedure. This will lead to a Luttinger-liquid-like power-law scaling of the conductivity, discussed at the end of this section.
2.3.1 Non-Interacting Case

For electron scattering off an impurity potential $V_{\text{imp}}(r)$ that is axially symmetric about the direction of the magnetic field, the component of the electron’s angular momentum is conserved. In particular, any spherically symmetric impurity satisfies this condition. For this reason, it is convenient to work in the symmetric gauge, where the basis of single-electron states is labeled by $p_z$, the momentum in the direction of the magnetic field, and $m_z$, the projection of the angular momentum in the magnetic field direction:

$$\Psi_{p_z,m_z}(r) = e^{ip_z z} \frac{1}{\sqrt{L_z}} \chi_{m_z}(\zeta),$$

where $\zeta = (x + iy)/l_H$ and

$$\chi_{m_z}(\zeta) = \frac{1}{l_H \sqrt{2^{m_z+1} \pi m_z}} \zeta^{m_z} \exp(-|\zeta|^2/4), \quad (2-71)$$

with $m_z = 0, 1, 2, \ldots$.

Electrons are restricted to the lowest Landau band and therefore there are only two types of scattering events: forward and backward. Only backscattering events contribute to the scattering cross-section, which can be written as $A \equiv \dot{N}$, where $\dot{N}$ is the number of electrons backscattered per unit time and $J$ is the total flux of incoming electrons. Using a Landauer-type scheme, the scattering cross-section in each channel of conserved $m_z$ can be related to a reflection coefficient in this channel via $A_{m_z} = 2\pi l_H^2 |r_{m_z}|^2$. The total cross-section is obtained from the sum of the cross-sections in each channel [71]:

$$A = 2\pi l_H^2 \sum_{m_z=0}^{\infty} |r_{m_z}|^2, \quad (2-72)$$
The coefficients \( r_{m_z} \) are the reflection amplitudes of 1D scattering problems, given by a set of 1D Schrödinger equations

\[
\left[-\frac{1}{2m} \frac{\partial^2}{\partial z^2} + V_{m_z}(z)\right] \phi(z) = (\varepsilon - \omega_c/2) \phi(z)
\]

with effective 1D impurity potentials \( V_{m_z}(z) = \langle m_z|V_{\text{imp}}(r)|m_z \rangle \) obtained by projecting the impurity potential on the angular momentum channel \( m_z \). The kinetic energy of the electron is denoted by \( \varepsilon = p_z^2/2m \). The cross-section \( A \) is related to the backscattering time via the usual relation, \( 1/\tau_H = n_i v_F A \), where \( n_i \) is the density of impurity scattering centers. When the electric field is along the magnetic field and for \( T = 0 \), the corresponding component of the conductivity is related to \( \tau_H \) via

\[
\sigma_{zz}^0 = e^2 v_1 D v_F^2 \tau_H / 2. \tag{2–73}
\]

An impurity of radius \( a \ll l_H \) can be modeled by a delta-function: \( V_{\text{imp}}(r) = V_0 \delta(r) \). For a delta-function potential, only the \( m_z = 0 \) component of \( V_{m_z}(z) \) is non-zero, \( V_{m_z}(z) = (V_0/2\pi l_H^2) \delta_{m_z,0} \delta(z) \). In this case, the scattering cross-section for non-interacting electrons is simply

\[
A = A_{m_z=0} = 2\pi l_H^2 |r_0|^2 = 2\pi l_H^2 \frac{(V_0/2\pi l_H^2)^2}{(V_0/2\pi l_H^2)^2 + v_z^2}, \tag{2–74}
\]

where \( v_z = p_z/m \). Consequently, at \( T = 0 \) the conductivity is given by

\[
\sigma_{zz}^0 = \frac{n}{n_i} \frac{e^2}{m v_F^2} \frac{1}{2\pi l_H^2} \left[ 1 + \left( \frac{2\pi l_H^2 v_F}{V_0} \right)^2 \right]. \tag{2–75}
\]

In the Born limit (when \( V_0 \ll 2\pi l_H^2 v_F \)) we recover the result for the conductivity as found by using the Kubo formula for weak, delta-correlated disorder \([\text{Eq. (2–73)}]\). In the opposite (unitary) limit \( A = 2\pi l_H^2 \) and

\[
\sigma_{zz,u}^0 = e^2 / 4\pi^2 n_i l_H^2. \tag{2–76}
\]
2.3.2 Interacting Case

Now we turn to the calculation of interaction correction to the scattering cross-section \(A\). For an effective delta-function impurity potential \(V_{\text{imp}}(r) = V_0 \delta(r)\), only the \(m_z = 0\) component is scattered. The free-electron wave function in this channel is given by:

\[
\Psi_{p_z, m_z}(r) = \frac{\psi_{p_z}(z)}{\sqrt{L_z}} \chi_{m_z=0}(\zeta)
\]

where far away from the impurity site, the asymptotic form of the \(z\)-component of the unperturbed wave-function is:

\[
\psi^0_{p_z}(z) = \begin{cases} 
  t_0 e^{ip_z z} & , \ z > 0 \\
  e^{ip_z z} + r_0 e^{-ip_z z} & , \ z < 0
\end{cases}
\]

\[
\psi^0_{-p_z}(z) = \begin{cases} 
  e^{-ip_z z} + r_0 e^{ip_z z} & , \ z > 0 \\
  t_0 e^{-ip_z z} & , \ z < 0
\end{cases}
\]

By calculating the electron-electron interaction correction to the wave function, one obtains the correction to amplitudes \(t_0\) and \(r_0\), and therefore to the scattering cross-section via Eq. (2–72). Since now the problem has been mapped onto a 1D scattering problem [21, 24], one can anticipate that this interaction correction has an infrared logarithmic singularity, as it does in the pure 1D case.

The 1D nature of the system in the UQL is also clearly manifested by the behavior of the Friedel oscillations around the impurity. The profile of the electron density around the impurity site is given by \(\delta n(r) = \int d\mathbf{r}' \Pi(r, \mathbf{r}') V_{\text{imp}}(\mathbf{r}')\), where \(\Pi(r, \mathbf{r}')\) is the polarization operator. For a weak delta impurity potential, we obtain

\[
n(r) = n_0 - \left( \frac{1}{2\pi l_H^2} \right) \frac{\sin(2pFz)}{z} \exp(-r_{\perp}^2/2l_H^2),
\]

which shows only a slow, \(1/z\) decay (see Fig. 2–17), characteristic of one-dimensional systems (in contrast to the \(1/r^3\) decay in 3D systems). Correspondingly, the Hartree \(V_H(r)\) and exchange \(V_{ex}(r, \mathbf{r}')\) potentials, that an incoming electrons feels
when being scattered from an impurity, also exhibit $2p_F$-oscillations and decay as $1/z$ away from the impurity and along the magnetic field direction. In the transverse plane, the density, and thus the potentials, have Gaussian envelopes which fall off on the scale of the magnetic length (see Fig. 2–17).

Figure 2–17. Profile of the Friedel oscillations around a point impurity in a 3D metal in the UQL. The oscillations decay as $1/z$ along the magnetic field direction and have a Gaussian envelope in the transverse direction.

The interaction correction to the wave function due to the Hartree and exchange potentials is

$$
\delta \Psi_{p_z, m_z=0}(r) = \int \! dr' G(r, r'; E) \int \! dr'' [V_H(r'')\delta(r' - r'') + V_{ex}(r', r'')] \Psi_{p_z, m_z=0}^{0}(r'').
$$

(2–79)

As discussed in the previous section, for the UQL the Green’s function is the product of a longitudinal (1D) and a transverse part, $G(r, r'; E) = G_{1D}(z, z'; p_z)G_{\perp}(r_\perp, r'_{\perp})$, where the asymptotic form of the longitudinal part as $z \to \infty$ is

$$
G_{1D}(z, z') = \frac{m}{ip_z} \begin{cases} 
  t_0 e^{ip_z(z-z')} & , \quad z' < 0 \\
  e^{ip_z(z-z')} + r_0 e^{ip_z(z+z')} & , \quad z' > 0
\end{cases}
$$

(2–80)
and, in the symmetric gauge, the transverse part is
\[
G_{\perp}(r_{\perp}, r'_{\perp}) = \sum_{m_z} \chi^*_{m_z}(r_{\perp}) \chi_{m_z}(r'_{\perp}) = \frac{1}{2\pi l_H^2} \exp \left\{ -\frac{(|\zeta|^2 + |\zeta'|^2 - 2\zeta^* \zeta')}{4} \right\} .
\] (2–81)

For \( z > 0 \), Eq. (2–79) directly gives the correction for the transmission amplitude \( t \). We first consider the exchange potential,
\[
V_{ex}(r, r') = -V(r - r') \sum_{m_z} \int \frac{dp_z}{2\pi} \left[ \psi^0_{p_z, m_z}(r')^* \psi^0_{-p_z, m_z}(r) \right] .
\] (2–82)

which can be factored as
\[
V_{ex}(r, r') = -V(r - r') f(z, z') G_{\perp}(r_{\perp}, r'_{\perp})
\] (2–83)

where
\[
f(z, z') = \int \frac{dp_z}{2\pi} \left\{ \left[ \psi^0_{p_z}(z) \right]^* \psi^0_{p_z}(z') + \left[ \psi^0_{-p_z}(z) \right]^* \psi^0_{-p_z}(z') \right\} .
\] (2–84)

From the form of \( f(z, z') \) one can see that the exchange potential also has terms with \( 1/(z - z') \) and \( 1/(z + z') \) decay. For example, for \( z, z' > 0 \),
\[
f(z, z') = \frac{\sin p_F(z - z')}{\pi(z - z')} + r_0 \frac{(e^{ip_F(z-z')} - 1)}{2\pi i(z + z')} - r_0^* \frac{(e^{-ip_F(z-z')} - 1)}{2\pi i(z + z')}
\] (2–85)

The \( 1/(z + z') \) decay leads to a log-divergent correction to \(|t|\). Decomposing the screened Coulomb potential \( V(r - r') \) into Fourier components, all the dependence of Eq. (2–79) on the transverse coordinates \( r_{\perp} \) can be collected into the factor
\[
T_{m_z=0}(r_{\perp}) = \int \frac{d\mathbf{r}'}{2\pi} G_{\perp}(r_{\perp}, r'_{\perp}) G_{\perp}(r'_{\perp}, r''_{\perp}) e^{-i\mathbf{q}_{\perp} \cdot (r'_{\perp} - r''_{\perp})} \chi_{m_z=0}(r'_{\perp}) .
\] (2–86)

Performing the integrals which appear in Eq. (2–86) for the exchange contribution, we find that the part containing perpendicular coordinates simply enters the interaction correction as a form factor:
\[
T_{m_z=0}(r_{\perp}) = \chi_{m_z=0}(r_{\perp}) \exp \left( -\frac{q_{\perp}^2 l_H^2}{2} \right) .
\] (2–87)
Therefore, the transverse part of the free wave function \( \chi_{m_z=0}(\mathbf{r}_\perp) \) simply remains unchanged in the rhs of Eq. (2–79). The remaining exponential term appears in the definition of the effective 1D potential, as in Eq. (2–40)

\[
V_{1D}(q_z) \equiv \int \frac{d^2 q_\perp}{(2\pi)^2} V(q_z, \mathbf{q}_\perp) \exp\left(-\frac{q_\perp^2 l_H^2}{2}\right).
\] (2–88)

The same result is obtained for the transverse part of the Hartree contribution in Eq. (2–79). Once the transverse part is solved and the effective 1D potential is defined, the rest of the calculation is exactly equivalent to the calculation of Yue et al. [21] for tunneling of weakly-interacting 1D electrons through a single barrier.

The interaction correction to the transmission amplitude \( t \) is directly obtained from the correction to the wave function, Eq. (2–79). Just as in 1D, a logarithmically divergent correction for \( t \) is obtained from the longitudinal part of this equation, after integrating over \( z \) and \( z' \).

It is straightforward to see why there is a log-divergent term. The Hartree term of Eq. (2–79), after integration of the transverse coordinates, is

\[
\delta \psi_{p_z}(z) = \int_0^\infty dz' G_{1D}(z, z') V_H(z') \psi_{p_z}(z')
\] (2–89)

where

\[
V_H(z) = \int_0^\infty dz' \int_{-\infty}^\infty \frac{dq_z}{2\pi} V_{1D}(q_z) e^{-iq_z(z-z')} \delta n(z')
\] (2–90)

Let’s consider for simplicity \(|r_0| \approx 1\) and \(|t_0| \ll 1\), in which case Eq. (2–80) gives \( G_{1D}(z, z') = (2m/p_z) \exp(ip_z z) \sin p_z z' \). The Hartree potential behaves as \( V_H(z) \approx V_{1D}(2p_F) \sin(2p_F z)/z \) so that Eq. (2–89) gives

\[
\frac{t - t_0}{t_0} \approx \int_0^\infty \sin(p_z z') V_{1D}(2p_F) \frac{\sin(2p_F z')}{z'} e^{ip_z z'}
\] (2–91)

The \( 1/z \) term gives a logarithmic singularity only in the limit \( p_z \to p_F \), so that \( \delta t/t_0 \propto V_{1D}(2p_F) \ln[1/(p_z - p_F)] \). The Hartree contribution corresponds
to enhancement of $t_0$. The exchange contribution has opposite sign and is proportional to $V_{1D}(0)$. The general answer, can be written as [72]

$$\frac{\delta t}{t_0} = -\alpha |r_0|^2 \ln \left( \frac{1}{t_H |p_z - p_F|} \right)$$

(2–92)

where $\alpha = (g_0 - g_{2k_F})/2\pi$, and $g_0$ and $g_{2k_F}$ are defined in Eqs. (2–43) and (2–54), respectively.

Figure 2–18. Renormalized conductivities parallel ($\sigma_{zz}$) and perpendicular ($\sigma_{xx}$) to the direction of the applied magnetic field. Power-law behavior is expected in the temperature region $1/\tau \ll T \ll W$.

The second-order contribution to the transmission amplitude was calculated explicitly in Ref. [21]. The higher-order contributions can be summed up by using a renormalization group (RG) procedure. Without repeating all the steps of Ref. [21], we simply state here that in our case the transmission amplitude satisfies the same RG equation, as in the purely 1D case. i.e.,

$$\frac{dt}{d\xi} = -\alpha t \left( 1 - |t|^2 \right)$$

(2–93)
where \( \xi \equiv \ln (1/|p - p_F|l_H) \) and \( t(\xi = 0) = t_0 \). The solution of Eq. (2–93) is

\[
t_0 = \frac{t_0^{(0)}(|p_z - p_F|l_H)^{\alpha}}{\sqrt{|r_0^{(0)}|^2 + |t_0^{(0)}|^2(|p_z - p_F|l_H)^{2\alpha}}}
\]

The renormalized cross-section is given by Eq. (2–72), but now written in terms of the renormalized reflection coefficient \(|r|^2 = 1 - |t|^2\). The final result for the conductivity can be cast in a convenient form by expressing the bare reflection and transmission coefficients via bare conductivities in the Born and unitary limits, \( \sigma_{zz}^0 \) and \( \sigma_{zz,U}^0 \), given by Eqs. (2–75) and (2–76), respectively:

\[
\sigma_{zz} = \sigma_{zz,U}^0 + \left( \sigma_{zz}^0 - \sigma_{zz,U}^0 \right) \left[ \frac{T}{W} \right]^{2\alpha}, \tag{2–94}
\]

where \( W \) is an ultraviolet cut-off of the problem and \( \alpha = (g_0 - g_{2k_F})/2\pi \), and \( g_0 \) and \( g_{2k_F} \) are defined in Eqs. (2–43) and (2–54), respectively which shows that \( \alpha \) scales with the magnetic field as \( \alpha \sim H \ln H \). We are interested in temperature dependence of the conductivities due to electron-electron interaction corrections and we assume here that the bare conductivities \( \sigma_{zz}^0 \) and \( \sigma_{zz,U}^0 \) have only weak \( T \)-dependence which can be neglected.

Eq. (2–94) is the main result of this Section and is shown in Fig. 2–18. It has a simple physical meaning: At \( T = W \), the conductivity is equal to its value for non-interacting electrons. At temperatures \( T \ll W \), the conductivity approaches its unitary-value limit, which means any weak impurity is eventually renormalized by the interaction to the strong-coupling regime. However, if the impurity is already at the unitary limit at \( T = W \), it is not renormalized further by the interactions. We emphasize that Eq. (2–94) is applicable only for high-enough temperatures, i.e., \( T \gg \max[1/\tau, \Delta] \). The first conditions is necessary to remain in the ballistic (single-impurity) regime, the second one allows one to consider only the renormalization of the impurity’s scattering cross-sections by the interaction without renormalizing the interaction vertex. The latter process leads
eventually for a charge-density-wave instability at a temperature $T \sim \Delta$, where $\Delta$ is the charge-density-wave gap [1–3]. For the power-law behavior [Eq. (2–94)] to have a region of validity, there should be an interval of intermediate energies in which the renormalization of the interaction coupling constants due to CDW fluctuations is not yet important but the corrections to the cross-section leading to the formation of power-law is already significant. Such an interval exists for a long-range Coulomb interaction ($|\kappa l_H| \ll 1$) both for the conductivity and the renormalization of the tunneling density of states [6].

The dissipative conductivity in a geometry when the current is parallel to the electric field but both are perpendicular to the magnetic field, $\sigma_{xx}$, occurs via jumps between adjacent cyclotron trajectories. In the absence of impurities, electrons are localized by the magnetic field and $\sigma_{xx} = 0$. In the presence of impurities, $\sigma_{xx}$ is directly, rather than inversely, proportional to the scattering rate. In particular, for short-range impurities, $\sigma_{xx} \propto 1/\tau \propto \sigma_{zz}^{-1}$ and the temperature dependence of $\sigma_{xx}$ is opposite to that of $\sigma_{zz}$. In the scaling regime, $\sigma_{zz} \propto T^{2\alpha}$ and $\sigma_{xx} \propto T^{-2\alpha}$. This situation is illustrated in Fig. 2–18, where $\alpha = (g_0 - g_{2k_F})/2\pi$, and $g_0$ and $g_{2k_F}$ are defined in Eqs. (2–43) and (2–54), respectively which shows that $\alpha(H) \sim H \ln H$. In the next section we discuss possible experimental studies for observing the localization and correlation effects mentioned in the first three sections.

2.4 Experiments

For experimental observation of the effects described here, the right choice of material is crucial. Firstly, a low-density material is needed so that the UQL may be achieved at feasible magnetic fields. For a good metal, the quantizing field is too high (of the order of $10^4$ Tesla). Semi-metals, such as bismuth and graphite, and doped semiconductors have low carrier density and quantizing fields of the order of $1 - 10$ Tesla. Another important condition for observing the Luttinger-liquid
like behavior is that the systems be relatively clean, so that there is a sizable range of temperatures in which the system is in the ballistic regime \((1/\tau \ll T \ll E_F)\). This rules out doped semiconductors \([73-75]\) since the charge carriers come from dopants which act as impurity centers in the system. An additional condition for occurrence of the power-law scaling behavior and formation of charge-density-wave or Wigner crystal, is that the electron-electron interaction is strong enough.

Bismuth crystals can be made extremely pure; however, the charge carriers in bismuth are extremely weakly interacting due to a large dielectric constant \((\sim 100)\) of the ionic background. Therefore, the log-corrections calculated here can be estimated to be very small and would be difficult to be observed experimentally.

Charge-density wave instability have been observed in graphite \([4]\) suggesting that interaction of charge carriers in this system is important in strong magnetic fields and at very low temperatures. Thus graphite would be an ideal material to observe the correlation and localization effects mentioned here. Below we present some recent experimental results of transport measurements in graphite first in weak magnetic fields \([76]\) and then in ultra quantum regime and try to interpret them in view of our findings.

Graphite has a low carrier density, high purity, relatively low Fermi-energy \((\sim 220K)\), small effective mass (along the c-axis) and an equal number of electrons and holes (compensated semi-metal). The metallic \(T\) dependence of the in-plane resistivity in zero field turns into an insulating like one when a magnetic field of the order of 10 mT is applied normal to the basal \((ab)\) plane. Using magnetotransport and Hall measurements, the details of this unusual behavior were shown \([76]\), to be captured within a conventional multiband model. The unusual temperature dependence displayed in (Fig. 2-19) can be understood for a simple two-band case
Figure 2–19. Temperature dependence of the ab-plane resistivity $\rho_{xx}$ for a graphite crystal at the c-axis magnetic fields indicated in the legend where $\rho_{xx}$ is given by \[ \rho_{xx} = \frac{\rho_1 \rho_2 (\rho_1 + \rho_2) + (\rho_1 R_2^2 + \rho_2 R_1^2) H^2}{(\rho_1 + \rho_2)^2 + (R_1 + R_2)^2 H^2}, \] (2–95) with $\rho_i$ and $R_i = 1/q_i n_i$ ($i = 1, 2$) being the resistivity and Hall coefficient of the two majority electron and hole bands, respectively. At not too low temperatures (where the measurements were performed) electron-phonon scattering is the main mechanism for the resistivity in the band. Assuming that $\rho_{1,2} \propto T^a$ with $a > 0$, we find that for perfect compensation, ($R_1 = -R_2 = |R|$), Eq. 2–95 can be decomposed into two contributions: a field-independent term $\propto T^a$ and a field-dependent term $\propto R^2(T) H^2 / T^a$. At high $T$, the first term dominates and metallic behavior ensues. At low $T$, $R(T) \propto 1/n(T)$ saturates and the second term dominates, giving insulating behavior $\propto T^{-a}$. Although this interpretation explains the qualitative features of the field induced metal-insulator behavior shown in Fig.2–19, the actual situation is somewhat more complicated due to the presence
of a third (minority) band, $T$ dependence of the carrier concentration and imperfect compensation between the majority bands. For more details see Ref. [76].

Let us now direct our attention on transport measurements in the ultra quantum regime in which we expect to see the power law conductivity behavior similar to what is shown in Fig. 2–18. Below we present some recent data on the same graphite samples on which the weak-field measurements were performed.

![Figure 2–20. Temperature dependence of the c-axis conductivity $\sigma_{zz}$ for a graphite crystal in a magnetic field parallel to the $c$ axis. The magnetic field values are indicated on the plot, with the field increasing downwards, the lowest plot corresponds to the highest field.](image)

Within the experimentally studied temperature range ($5K < T < 10K$), the c-axis conductivity exhibits an *insulating* linear temperature dependence, $\sigma_{zz} \propto T$ as shown in Fig. 2–20, whereas the transverse resistivity exhibits a metallic power law temperature dependence, $\rho_{xx} \propto T^{1/3}$, as shown in Fig. 2–21 and Fig. 2–22. Although the insulating sign of the temperature dependence (for $\sigma_{zz}$, see Fig.2–18) is consistent with the model of a field-induced Luttinger liquid, the independence of the exponent of the field is not. The slope (exponent) of both $\sigma_{zz}$ and $\rho_{xx}$ (for various magnetic fields), are independent of the magnetic field whereas in the
field-induces Luttinger liquid model the exponent of the power law should depend on the field (see Eq. 2.94, $\alpha \sim H \ln H$). Also, the exponents of the $T$ scaling in $\sigma_{zz}$ (which is 1), and $\rho_{xx}$ (which is $1/3$) are different (as seen in experiments) whereas they were predicted to be the same in the Luttinger liquid model.

We are going to argue that the unusual temperature behavior of $\sigma_{zz}$ and $\rho_{xx}$, can be understood within a model which includes phonon-induced dephasing of one-dimensional electrons (in the UQL) and the correlated motion in the transverse direction due to the memory effect of scattering at long ranged disorder. Before we get into the details of the model, let us keep in mind a few numbers for the system we are about to describe. For our graphite samples the Fermi energy is $E_F = 220K$, the Bloch-Gruniesen temperature (which separates the region of $T$ and $T^5$ contribution to the resistivity) is $\omega_0 = 2k_F s \sim 10K$ and the Dingle temperature (which gives the impurity scattering rate) is $3K$. Also the transport relaxation time is much longer (by a factor of 30), than the total scattering time (or life time) $\tau_{tr} \gg \tau$, indicating the long range nature of the impurities.

Figure 2-21. Temperature dependence (log-log scale) of the ab-plane resistivity scaled with the field $\rho_{xx}/B^2$ for a graphite crystal at the c-axis magnetic fields indicated in the legend.
Figure 2-22. Temperature dependence (on a log-log scale) of the ab-plane resistivity $\rho_{xx}/B^2$ at the highest attained c-axis magnetic field of 17.5T for the same graphite crystal.

We first outline an argument by Murzin [73] which shows that the transverse motion of the electron is correlated due to drift motion in a crossed magnetic and electric field. The disorder model is assumed to be ionized impurity type and is therefore long ranged. The transverse displacement (after a single scattering act) is assumed to satisfy $r_\perp \ll l_H \ll r_D$ ($r_D$ being the screening radius). Electrons are assumed to diffuse in the $z$ direction. An electron re-enters the region where the impurity’s electric field is strong, ($r_\perp < r_D$), many times as it moves in the transverse direction. Thus electron’s motion in the transverse direction is correlated. Only after an electron has traveled a distance greater than $r_D$ in the transverse direction, its motion becomes diffusive with the diffusion coefficient $D_{xx} \sim r_D^2/\tau_D$. We estimate $\tau_D$ (the time in which electron has moved a distance $r_D \perp \text{ to } H$) by finding the transverse displacement $\Delta X(t)$ for $\Delta X(t) < r_D$, and then obtain $\tau_D$ by setting $\Delta X(\tau_D) \sim r_D$. The probability to find an electron again
in the field of a given impurity is

\[ W_D \sim \frac{r_D}{\sqrt{D_{zz} t}} \]

The transverse drift velocity of the electron (in crossed \( E \) and \( H \) field) in time \( t \) is

\[ v_d = \frac{cE}{H} \times W_D = \frac{c e r_D}{\kappa r_D^2 H \sqrt{D_{zz} t}} \]

The electric field of the impurity is \( E \sim e/\kappa r_D^2 \). (here \( \kappa \) is the dielectric constant).

The transverse displacement is \( \delta x(t) \sim v_d t \). The number of impurities with which the electron interacts as it diffuses in the \( z \) direction is

\[ P(t) \sim N r_D^2 \sqrt{D_{zz} t}, \]

where \( N \) is the number density of randomly distributed impurities. The average square of the total transverse displacement is

\[ \langle \Delta X^2(t) \rangle \sim \delta x(t)^2 P(t) \sim \left( \frac{c e}{\kappa H} \right)^2 N D_{zz}^{-1/2} t^{3/2}. \]

We then set \( \langle \Delta X^2(\tau_D) \rangle \sim r_D^2 \) to get \( \tau_D \) and use the definition of the diffusion coefficient, \( D_{xx} \sim r_D^2 / \tau_D \) and the conductivity \( \sigma_{xx} \sim e^2 \nu_F D_{xx} \) to obtain

\[ D_{xx} \sim \left( \frac{c e}{\kappa H} \right)^{4/3} N^{2/3} r_D^{2/3} D_{zz}^{1/3}, \]

\[ \sigma_{xx} \sim \left( \frac{c e^3 \nu_F}{\kappa H} \right)^{4/3} (N r_D)^{2/3} \sigma_{zz}^{1/3}. \]

In an compensated semi-metal \( \nu_F \propto H \), for fields smaller than the metal-insulator transition field, so the pre-factor in \( \sigma_{xx} \) is field independent. The transverse
resistivity $\rho_{xx} \sim \frac{1}{\sigma_{xx}}$, is

$$\rho_{xx} \sim c\sigma_{zz}^{1/3}, \quad (2–96)$$

where $c = \frac{\kappa^{4/3}}{(ce^3)^{4/3}(N\tau_D)^{2/3}}$ is a material dependent constant which does not have any explicit field dependence.

Now we will show that the c-axis transport behavior (only the temperature dependence) can be explained within the context of a power-law hopping mechanism, in which phonons cause localized electrons to hop over distances of the order of the localization length with a frequency of $\frac{1}{\tau_0 - \tau_{ph}} \propto T$. This leads to the linear $T$ dependence of $\sigma_{zz}$ and then, using Eq. 2–96, one obtains the $T^{1/3}$ power law temperature dependence for $\rho_{xx}$. In a strongly localized system (happens only in 1D) one should expect the absence of static conductivity. When inelastic scattering is allowed the situation changes considerably as jumps between individual localized states become possible and are accompanied by phonon (or electron-hole pair) emission or absorption. Violation of localization leads to a finite conductivity of a diffusion type which is the called the super diffusion regime or Power law hopping regime [78]. In this regime electron’s move by diffusion but are trapped inside a localization length $\ell$. Inelastic scattering allows the electron to jump/hop over distances of the order of the localization length with a frequency of $(\tau_0(T))^{-1}$, the phase breaking rate. The diffusion constant is then $D = \ell^2/(\tau_0(T))$ and the conductivity $\sigma(T) = e^2\nu D$. This occurs for $\tau_i \ll \tau_0(T)$, where $\tau_i$ is the elastic scattering time. As the temperature is further lowered super diffusion gives way to Mott’s variable range hopping transport regime where $\sigma(T) \propto \sigma_D e^{-\sqrt{\frac{T}{\pi\tau_i}}}$ in 1D, for $T\tau_i \ll 1$. The conductivity behavior changes from activated VRH type to super diffusion type for $T \sim \frac{1}{\tau_i}$. The Dingle temperature in the graphite samples used was $3K$, so $\pi T_D \sim \frac{1}{\tau_i} \sim 3K$, and therefore the temperature range where
\(\sigma_{zz} \propto T, (5 - 15K)\) satisfies \(\frac{1}{\tau_i} < T\), which justifies the use of the super diffusion formula.

Figure 2–23. Phase breaking rate vs T due to electron-phonon scattering

The behavior of the phase breaking time due to electron-phonon scattering is illustrated schematically in Fig.2–23. In the low temperature regime \(T \ll 2k_Fs \sim 10K\), electron-phonon scattering is inelastic and the phase breaking time is of the order of the electron-phonon scattering time \(1/\tau_\phi \sim 1/\tau \propto T^3\). But the experimental plots were not in this \(T\) range. In the high temperature regime, \(T > 2k_Fs\), the electron-phonon scattering time is \(\frac{1}{\tau_{e-ph}} \sim (k_Fa_0\frac{m^*}{m})T \sim AT\), with a small numerical value for \(A = 0.065\) in graphite [76], due to its low density and small effective mass. The high \(T\) regime can be further subdivided into the ballistic regime \((\tau_{e-ph} \gg \Delta t \sim \frac{1}{2k_Fs} \Rightarrow 2k_Fs \ll T \ll \frac{2k_Fs}{A})\) and the diffusive regime \((\tau_{e-ph} \ll \Delta t \sim \frac{1}{2k_Fs} \Rightarrow \frac{2k_Fs}{A} \ll T \ll E_F\), where \(\Delta t\) is the duration of a single collision act, whereas \(\tau\) is the relaxation time. Notice that the ballistic regime exists only for (semi-metals) materials with \(A \ll 1\). For metals \(A \sim 1\) and the ballistic regime does not exist. In the high temperature diffusive limit the phase
breaking rate is given by \[26\]
\[
\frac{1}{\tau_\phi} \sim \frac{(2k_F s)^{2/3}}{(\tau_{e-ph})^{1/3}}
\] (2–97)
which corresponds to the $T^{1/3}$ temperature dependence in Fig. 2–23 and occurs at very high temperatures ($T > 200K$) in semi-metals due to the small value of $A$. In the ballistic regime $\tau_{e-ph} \gg \Delta t \gg (\omega_D)^{-1}$, where $\omega_D$ is the Debye frequency. This means that ions perform many oscillations during a single act of electron-phonon scattering. Thus scattering at moving ions works as dephasing due to dynamic potential \[79\] The phase-breaking time is then of the order of the relaxation time, i.e., $\tau_{e-ph} \sim \tau_\phi$. Therefore, the graphite samples in the ballistic regime ($5K < T < 15K$) should have
\[
\frac{1}{\tau_\phi} \sim AT,
\] (2–98)
and using the super diffusion formula one obtains the linear (insulating like) temperature dependence for the $c$-axis conductivity, $\sigma_{zz} \sim e^2 \nu l^2 / \tau_\phi \sim T$ which is seen in Fig. 2–20. After that using Eq. 2–96 one can obtain the $T^{1/3}$ dependence of $\rho_{xx}$ seen in Fig. 2–22. This explains the temperature dependence observed in the experiments. However, the field dependence of the resistivities are not well understood, e.g., Fig. 2–20, shows a positive magneto-resistance for $\rho_{zz}$ which is counter-intuitive because there cannot be a Lorentz force along the magnetic field ($z$ direction).

\section{2.5 Conclusions}

We presented here a study of transport properties of a 3D metal subjected to a strong magnetic field that confines the electrons to the lowest Landau level (UQL). We showed that the nature of electron transport is one dimensional due to the reduced effective dimensionality induced by the field. The first sign of this is that the localization corrections to the conductivity was of the order of
the bare Drude conductivity itself. Therefore perturbation theory breaks down just as it does in 1D. However, unlike in the 1D case, the conductivity remains finite at zero temperature. Therefore, we call this regime *intermediate localization*. The second important manifestation of electronic correlations and lower dimensionality is that the first order interaction correction to the conductivity is logarithmically divergent in temperature, just as for 1D systems. We then calculate the interaction correction to the cross-section of electrons scattering off a single impurity. Using a Drude relation to obtain the conductivity from the cross-section, we find that this result is equal to the result obtained from the full diagrammatic calculation. This suggests that the dominant diagrams for the conductivity can be described in terms of scattering off an effective “dressed” impurity potential. A renormalization-group calculation for the cross-section allows for the summation of a series of most divergent log-corrections at all orders in the interaction. Just as in 1D this summation in our case leads to power-law scaling. However in the system of 3D electron in the UQL this is a high-energy behavior which exists provided that the electron-electron interaction is sufficiently long-ranged. Some recent transport measurements in graphite were compared with the above theoretical findings and shown to disagree. To resolve the disagreement, we invoked a model with long ranged-disorder and phonon induced dephasing to explain the experimental observations.
CHAPTER 3
SINGULAR CORRECTIONS TO THERMODYNAMICS FOR A ONE DIMENSIONAL INTERACTING SYSTEM: EVOLUTION OF THE NONANALYTIC CORRECTIONS TO THE FERMI LIQUID BEHAVIOR

There has been substantial recent interest in the thermodynamics of a Fermi liquid [41–47, 80]. The revival of interest in the problem is twofold. On the experimental side, technical advances now allow one to measure the temperature dependence of the thermodynamic parameters such as specific heat and spin susceptibility of a two-dimensional (2D) Fermi liquid with short range interaction, such as monolayers of He3 [56], as well as, two-dimensional semiconductor structures with long range interaction and relatively low Fermi temperatures (∼1K). On the theory side, it turns out that the leading interaction corrections are nonanalytic functions of temperature or magnetic field, making the subject particularly interesting. The fate of these nonanalytic corrections in the spin susceptibility, near a quantum critical point is important for our understanding of the nature of paramagnetic to ferromagnetic quantum phase transition and we discuss this in detail in chapter 4.

As it has been mentioned in the introduction, naive power counting arguments suggest that the temperature dependence of any thermodynamic quantity, including the spin susceptibility and the specific heat coefficient \( C(T)/T = \gamma \), should start with terms quadratic in temperature. This conjecture is based on the observation that a thermodynamic quantity at finite temperature can be written as \( \int a(\epsilon)n_F(\epsilon)d\epsilon \), where \( n_F(\epsilon) \) is the Fermi distribution function and \( a(\epsilon) \) is some function. If the latter is smooth, the temperature dependence starts with a term of order \( T^2 \) [28]. Such a correction is called ”analytic”. This is also consistent
with the intuitive expectation of the one-to-one correspondence between the
noninteracting Fermi gas and the interacting Fermi liquid since in the Fermi gas,
the Sommerfeld expansion leads to a simple quadratic temperature corrections.

However, we also saw that the assumption about the analyticity of the
functions involved in the calculation of various thermodynamic properties of
the Fermi liquid is quite generally not justified, because in any Fermi liquid,
the dynamic interaction between particles gives rise to a nonanalytic energy
dependence of \( a(\epsilon) \). This leads to temperature corrections that do not scale as \( T^2 \)
and are therefore called "nonanalytic". Collecting these nonanalytic corrections is a
subtle theoretical problem and in this chapter we will evaluate these corrections for
a one-dimensional interacting system.

In a generic Fermi liquid, the fermionic self-energy behaves as
\[
\text{Re}\Sigma^R(\epsilon, k) = -A\epsilon + B\xi_k + \ldots
\]
and
\[
\text{Im}\Sigma^R(\epsilon, k) = C(\epsilon^2 + \pi^2 T^2) + \ldots
\].
This form of the self-energy implies that quasiparticles are well defined, and the dominant effect
of the interactions at low energies is the renormalization of the quasi-particle
mass and the residue of the quasiparticle Green’s function. This behavior has a
profound effect on observable quantities such as the specific heat and static spin
susceptibility, which have the same functional dependence as for free fermions, e.g.,
the specific heat \( C(T) \) is linear in \( T \), while the susceptibility \( \chi_s(Q, T) \) approach a
constant value at \( Q = 0 \) and \( T = 0 \).

It has been known for some time that the sub-leading terms in the \( \epsilon \) and \( T \)
expansions of the fermion self-energy do not form regular, analytic series in \( \epsilon^2 \)
or \( T^2 \) [81]. In particular, power counting (dimensional analysis) shows that the
first subleading term in the retarded on-shell (\( \epsilon = \xi_k \)) self-energy at \( T = 0 \), is
\[
\delta\Sigma^R(\epsilon) \propto \epsilon^3 \ln(-i\epsilon) \text{ in } 3D \text{ and } \delta\Sigma^R(\epsilon) \propto \epsilon^2 \ln(-i\epsilon) \text{ in } 2D.
\]
These singularities in the self-energy give corrections to the Fermi-liquid forms of the specific heat and
static spin susceptibility (\( C^{FL} \propto T, \chi^{FL}_s = \text{const.} \)), which are nonanalytic in \( D \leq 3 \).
and scale as $\delta C(T) \propto T^D$ and $\chi_s(Q) \propto Q^{D-1}$, with extra logarithms in $D = 3$ and 1. It was shown recently [44] that the non-analytic corrections to the specific heat and spin susceptibility in 2D originate from essentially one-dimensional scattering processes, embedded in a higher dimensional phase space. In particular, these 1D scattering events (strictly forward or backscattering) give non-analytic sub-leading terms in the electron self-energy, with the degree of non-analyticity increasing as the dimensionality is lowered, simply because the phase volume $(q^D)$ is more effective in suppressing the singularity in higher dimensions than in lower ones. Thus non-analyticities in higher dimensions can be viewed as precursors of 1D physics for $D > 1$ and the strongest nonanalyticity should occur in 1D.

The purpose of this work is to obtain the nonanalytic, $T \ln T$, correction to the specific heat in 1D, and to elucidate the similarities and differences between higher dimensional and one dimensional non-analyticities. The nonanalytic term in the specific heat in 1D, was missed in an earlier work [57], as the authors analyzed the self-energy up to second order in perturbation theory. We show that the specific heat and spin susceptibility in 1D acquire nonanalytic corrections from the singularities in the one dimensional bosonic response functions, just as they did for higher dimensions. The major difference between the non-analyticities in $C(T)$ in $D > 1$ and $D = 1$ is that the former occurs at the second order in interaction, whereas the latter starts only at third order (contrary to the expectation that the degree of nonanalyticity increases with reduction of dimensionality), and the nonanalyticity in 1D occurs only for fermions with spin. In higher dimensions the specific heat is nonanalytic even for spin-less (i.e., fully spin polarized) fermions. Naive power counting breaks down in 1D because the coefficient in front of the $T \ln T$ term in $C(T)$ vanishes in second order, and one has to go to third order. Although bosonization predicts that $C(T) \propto T$ in 1D, this is true only for spin-less fermions, in which case the third order diagrams for the non-analytic
temperature dependence exactly cancel out. For fermions with spin, the bosonized theory is of the sine-Gordon type with the non-gaussian \((\cos \phi)\) term coming from backscattering of fermions of opposite spins. Even though this term is marginally irrelevant and flows down to zero at the lowest energies, at intermediate energies it leads to a multiplicative \(\ln T\) factor in \(C(T)\) and a \(\ln \max [Q, T, H]\) correction to \(\chi_s\). The spin susceptibility is nonanalytic at second order in interaction as in higher dimensions. The advantage of using the fermionic diagrammatic approach in 1D, is that in addition to correctly predicting the non-analyticities it also elucidates the underlying physics: the non-analyticities arise from universal singularities in the bosonic response functions, thus establishing the connection with higher dimensions.

This chapter is organized as follows. In section 3.1, we discuss the main physics of a one dimensional interacting system and state the 1D model we used in our calculations which is the Tomonaga-Luttinger model with backscattering. In section 3.2, we evaluate the specific heat for a one-dimensional interacting system. In subsection 3.2.1, we explicitly obtain the nonanalytic forms of the 2nd order self-energy in 1D, but we show that these non-analyticities of the fermion self-energy does not lead to a nonanalytic correction (henceforth referred to as NAC) to the specific heat at second order in interaction. This is an important difference between higher dimensional and one dimensional NAC to the specific heat, and we show this in two different ways, first in subsection 3.2.1, where we calculate the specific heat from the fermion self-energy and then again from the thermodynamic potential in subsection 3.2.2, to firmly establish this difference. In subsection 3.2.3, we obtain the nonanalytic \(T \ln T\) term in the specific heat using the third order (nonanalytic) self-energy and show that this term is present only for fermions with spin. The absence of the nonanalytic \(T \ln T\) term at 2nd order and its presence at the 3rd order (for spinful fermions) is consistent with the
renormalization group treatment of the Sine-Gordon model, and this is shown in subsection 3.2.4. In section 3.3, we show that the nonanalytic \( \ln \max[Q, H, T] \) term in the static spin susceptibility is present at second order in perturbation theory similar to higher dimensions. We discuss possible experimental observations of the quantities studied in this chapter in section 3.4, and give our conclusions in section 3.5.

### 3.1 One-Dimensional Model

Fermi liquid theory is based on a picture of quasi-particles evolving out of the particles (and holes) of a Fermi gas upon adiabatically switching on the interactions\[^{28}\]. They are in one-to-one correspondence with the bare particles and, specifically, carry the same quantum numbers and obey Fermi-Dirac statistics. The free Fermi gas, thus is the solvable model on which Fermi liquid theory is built. The electron-electron interaction has three main affects: (1) it renormalizes the kinematic parameters of the quasi-particles such as the effective mass, and the thermodynamic properties (specific heat, spin susceptibility), described by the Landau parameters \( F^{a,s} \); (2) it gives quasiparticles a finite lifetime which diverges, however, as \( \tau \sim (E - E_F)^{-2} \) as the Fermi surface is approached, so that the quasi-particles are robust against small displacements away from \( E_F \); (3) it introduces new collective modes. The existence of quasi-particles formally shows up through a finite jump \( z_{K_F} \), of the momentum distribution function \( n(k) \) at the Fermi surface, corresponding to a finite residue of the quasi-particle pole in the electron’s Greens function.

In contrast, the properties of the one-dimensional interacting system, the \textit{Luttinger liquid}, are fundamentally different from two or three-dimensional Fermi liquids. In particular, the elementary excitations are not quasi-particles but rather bosonic collective charge and spin density fluctuations dispersing at different velocities. An incoming electron decays into charge and spin excitations which
then spatially separate with time (\textit{spin-charge separation}) \cite{82}. The correlations between these excitations are anomalous and show up as interaction-dependent \textit{non-universal power laws} in many physical quantities where as, those of ordinary metals (Fermi Liquids) are characterized by universal (interaction independent) powers.

To be more specific, a list of salient properties of such 1D interacting systems include: (1) a continuous momentum distribution function $n(k)$ varying as $|k - k_F|^{\alpha}$ with an interaction- dependent exponent $\alpha$, and a pseudogap in the single-particle density of states $\propto |\omega|^{\alpha}$, both of which are the consequences of the absence of fermionic quasi-particles; (2) similar power law behavior in all correlation functions, specifically in those for superconducting and spin or charge density wave fluctuations; (3) finite spin and charge response at small wave vectors, and a finite Drude weight in the conductivity; (4) spin-charge separation. All these properties can be described in terms of only two effective parameters ($K$, $u$ in Eq. 3–2) per degree of freedom, (spin and charge) which play the role of Landau parameters in 1D.

The reason for these peculiar properties, is the very special Fermi surface topology of 1D fermions, producing both singular particle-hole response functions and severe conservation laws. In a 1D system, there are two Fermi ”points” $\pm k_F$, and one has perfect nesting, namely one complete Fermi point can be translated into the other by a single wave vector $\pm 2k_F$. This produces a singular particle-hole response at $2k_F$. This type of response is assumed finite in Fermi liquid theory but, in 1D, is divergent for repulsive forward scattering, leading to a breakdown of the Fermi liquid description. In addition we have, as in 3D, the Bardeen-Cooper-Schrieffer (BCS) singularity for attractive interactions. In fact, none of these instabilities occur, the competition between the particle-particle (BCS) and particle-hole (at $2k_F$) channel leads to a critical like (power law)
behavior of the 1D correlation functions at zero temperatures. The one-dimensional electron gas is thus always at the verge of an instability without being able to order. One the other hand, the disjoint Fermi surface gives a well defined dispersion, i.e., particle-like character to the low energy particle-hole excitations in a free system. These particle-hole excitations are well defined, meaning they have well defined momenta and energy. They now can be taken as the building blocks upon which one can construct a description of the 1D low-energy physics. The density operator which is a superposition of the particle-hole excitations \( \rho(q) = \sum_k c_{k+q}^\dagger c_k \sim b_q \), is used as a bosonic basis in which the original four fermion interacting hamiltonian (Eq. 3–1) becomes quadratic and therefore exactly solvable. This is the essence of the bosonization theory which was used by Mattis and Lieb to solve the 1D Tomonaga-Luttinger model \[83\]. The notion of a “Luttinger liquid” was coined by Haldane to describe these universal low energy properties of gapless 1D quantum systems and to emphasize that an asymptotic \((\omega \to 0, q \to 0)\) description can be based on the Luttinger model in much the same way as the Fermi liquid theory in 3D is based on the free Fermi gas.

\[ \text{Figure 3–1. Interaction vertices} \]

Our model hamiltonian for calculating the nonanalytic corrections to the specific heat and spin susceptibility will be the standard Tomonaga-Luttinger model, extended to include backscattering vertices \[82\],

\[
\hat{H} = \sum_{k,r=+,-} v_F (r k - k_F) c_{r,k}^\dagger c_{r,k} + \frac{1}{2\Omega} \sum_{r,k,k',q} V(q) c_{r,k+q}^\dagger c_{r,k'}^\dagger c_{r,k'} c_{r,k} \quad (3-1)
\]
where $c_k^\dagger (c_k)$ is the electron creation (destruction) operator and $V(q)$ is the interaction potential. The linearization of the spectrum (which is essential for making the particle-hole excitations well defined) forces one to introduce two species of fermions: right movers ($r = +1$) and left movers ($r = -1$). One has to keep in mind that the most efficient processes in the interaction are the ones which acts close to the Fermi surface. The fact that in one-dimension the Fermi surface is reduced to two points ($+p_F$ and $-p_F$) allows one to decompose the important low energy processes of the interaction into three different sectors. These three interaction processes are shown in Fig. 3–1, where solid lines represent right movers and dashed lines denote left movers. The first process $g_4$ couples fermions on the same side of the Fermi surface. The second one $g_2$ couples fermions from different branches. However, each species stays on the same side of the Fermi surface after the interaction (both forward scattering). Finally, the last process $g_1$ corresponds to $\sim 2k_F$ scattering (backscattering) where the fermions exchange sides. We assume that the interaction potential is finite ranged ($g_1 \neq g_2$), and for generality we allow for different interactions between left and right moving fermions ($g_2 \neq g_4$) but we neglect the momentum dependence of the interaction coefficients, treating them as constants. The interaction part of the Hamiltonian is written in terms of operators $c_{+,\alpha} (c_{-,\alpha})$ denoting right (left) moving fermions as [84],

$$
\hat{H}_{int} = \frac{1}{L} \sum_{k_{1,2,p}} \sum_{\alpha,\beta} \left( g_{1\parallel} \delta_{\alpha\beta} + g_{1\perp} \delta_{\alpha,-\beta} \right) c_{+,k_{1+}k_{2-}p}^\dagger c_{-,k_{2+}k_{1+}2k_{F}+p\beta}^\dagger c_{+,k_{1+}k_{2+}+2k_{F}+p\beta} c_{-,k_{1+}k_{2+}+2k_{F}+p\beta} - k_{1+}2k_{F} - p\alpha
$$
$$
+ \frac{1}{L} \sum_{k_{1,2,p}} \sum_{\alpha,\beta} \left( g_{2\parallel} \delta_{\alpha\beta} + g_{2\perp} \delta_{\alpha,-\beta} \right) c_{+,k_{1+}k_{2-}p}^\dagger c_{-,k_{2+}k_{1+}2k_{F}+p\beta}^\dagger c_{+,k_{1+}k_{2+}+2k_{F}+p\beta} c_{-,k_{1+}k_{2+}+2k_{F}+p\beta} - k_{1+}2k_{F} - p\alpha
$$
$$
+ \frac{1}{2L} \sum_{k_{1,2,p}} \sum_{\alpha,\beta} \left( g_{4\parallel} \delta_{\alpha\beta} + g_{4\perp} \delta_{\alpha,-\beta} \right) c_{+,k_{1+}k_{2-}p}^\dagger c_{+,k_{2+}k_{1+}2k_{F}+p\beta} c_{+,k_{1+}k_{2+}+2k_{F}+p\beta} c_{-,k_{1+}k_{2+}+2k_{F}+p\beta} - k_{1+}2k_{F} - p\alpha
$$
$$
+ c_{-,k_{1+}k_{2-}p}^\dagger c_{-,k_{2+}k_{1+}2k_{F}+p\beta}^\dagger c_{+,k_{1+}k_{2+}+2k_{F}+p\beta} c_{-,k_{1+}k_{2+}+2k_{F}+p\beta} - k_{1+}2k_{F} - p\alpha \right).
$$
For spinless fermions with only forward scattering ($g_2$ and $g_4$ vertices, $g_1 = 0$), the Hamiltonian can be bosonized and transformed to a quadratic form [82]:

$$H = \frac{1}{2\pi} \int dx \left[ uK(\nabla \theta(x))^2 + \frac{u}{K} (\nabla \phi(x))^2 \right],$$

(3–2)

where $\theta$ and $\phi$ are the bosonic fields which satisfy the canonical commutation relations

$$[\phi(x_1), \nabla \theta(x_2)] = i\pi \delta(x_2 - x_1),$$

(3–3)

and $u$ and $K$ are the parameters renormalized by the interaction,

$$u = v_F \left[ (1 + y_4/2)^2 - (y_2/2)^2 \right]^{1/2},$$

(3–4)

$$K = \left( \frac{1 + y_4/2 - y_2/2}{1 + y_4/2 + y_2/2} \right)^{1/2},$$

(3–5)

with $y = g/(\pi v_F)$ being a dimensionless coupling constant. Thus the physics of the one-dimensional interacting spin-less fermionic system is completely described by free bosons. The energy spectrum is changed from $\epsilon(p) = v_F|p|$ (for a free fermionic system) to $\epsilon(p) = u|p|$ for the interacting system. The specific heat is

$$C = \frac{dE}{dT} = \frac{d}{dT} \sum_p \epsilon(p) f_B(\epsilon(p)) = \frac{u^2}{4T^2} \sum_p \frac{p^2}{\sinh^2(\beta up/2)} = \frac{T}{u} (L\pi/3).$$

(3–6)

The specific heat is linear in temperature even for an interacting system (for free fermions $C(T) = T (L\pi/3v_F)$). For fermions without spin, including backscattering amounts to replacing $g_2$ with $g_2 - g_1$. All the previous results still hold after making the change $g_2 \rightarrow g_2 - g_1$ in $u$ and $K$. The specific heat will remain linear in temperature with a new coefficient. For fermions with spin, including backscattering will lead to a sine-Gordon term in addition to the quadratic term in the spin part of the hamiltonian. The charge part retains its quadratic form but with new coefficients $u_c, K_c$. The specific heat for this model is analyzed in detail in section 3.2.4.
There are three distinct non-analyticities in the bosonic response functions, in one dimensions at $T = 0$ (as in any other dimensions \[53\]). These are the singularities in (1) the particle-hole polarization bubble for small momentum and frequency transfers, in (2) the particle-hole polarization bubble for momentum transfer near $2k_F$ and in (3) the particle-particle polarization operator for small total momentum. The free Green’s function for left ($-$) and right ($+$) movers are

$$G_\mp(k; i\varepsilon) = \frac{1}{i\varepsilon \pm kv_F}. \quad (3-7)$$

Here $k = p \pm p_F$ is the momentum counted from the corresponding Fermi point.

The particle-hole polarization bubble for left movers and small $q, \omega$ is

$$\Pi_{-\pm} (q; i\omega) = \int \frac{dk}{2\pi} \int \frac{d\varepsilon}{2\pi} G_{-\pm} (k; i\varepsilon) G_{-\pm} (k + q; i(\varepsilon + \omega)) = \frac{-q}{2\pi (i\omega + q)}, \quad (3-8)$$

and similarly for right movers

$$\Pi_{++} (q; i\omega) = \int \frac{dk}{2\pi} \int \frac{d\varepsilon}{2\pi} G_{++} (k; i\varepsilon) G_{++} (k + q; i(\varepsilon + \omega)) = \frac{q}{2\pi (i\omega - q)}, \quad (3-9)$$

where $v_F = 1$. Both the polarization operators have the same singularity in $q$:

$$\Pi (q; \omega) \sim \frac{\omega}{q}, \text{ for } q \ll k_F \text{ and } q \gg \omega. \text{ The above form of the polarization operator indicates Landau damping: } \text{Collective excitations (spin and charge density waves) decay into particle-hole pairs, this decay occurs only within the particle-hole continuum, which in 1D, shrinks to a single line } \omega = v_F q. \text{ As it was shown in the introduction, this singularity in the particle-hole polarization bubble resulted in non-analytic, sub-leading terms in the self energy and thermodynamics. We will show below (in subsection 3.2.1), that the forward scattering response functions in 1D gives nonanalytic terms in the self-energy, but these do not lead to NAC to } C(T) \text{ or } \chi_s.$$

The dynamical Kohn anomaly, which is the singularity in the (2) particle-hole response function (for $q \approx 2k_F$), also gives non-analytic sub-leading terms in the
self-energy and thermodynamics [53]. In 1D,

$$\Pi_{kF}^{2kF}(q; i\omega) = \int \frac{dk}{2\pi} \int \frac{d\varepsilon}{2\pi} G_-(k; i\varepsilon) G_+(k + q; i(\varepsilon + \omega))$$

$$= +\frac{1}{4\pi} \ln \left| \frac{(q + i\omega)(q - i\omega)}{(2\lambda)^2} \right|, \quad (3-10)$$

where $\lambda$ is the ultraviolet cut-off. Finally, the (3) particle-particle or Cooper bubble (for small total momentum) has the same non-analyticity but with an opposite sign as the $2k_F$ particle-hole channel in 1D

$$\Pi_{p-p}^{p-p}(q; i\omega) = \int \frac{dk}{2\pi} \int \frac{d\varepsilon}{2\pi} G_+(k + q; i(\varepsilon + \omega)) G_-(-k; -i\varepsilon)$$

$$= -\frac{1}{4\pi} \ln \left| \frac{(q + i\omega)(q - i\omega)}{(2\lambda)^2} \right|. \quad (3-11)$$

We will show that the above singularities give rise to nonanalytic sub-leading terms in the one-dimensional self-energy, for e.g. $\text{Im}\Sigma_+^R(\varepsilon; k = 0) \propto |\varepsilon|$ at second order and $\text{Im}\Sigma_+^R(\varepsilon; k = 0) \propto |\varepsilon| \ln \frac{|\varepsilon|}{\lambda}$ at third order in interaction. Unlike higher dimensions, the second order self energy, although non-analytic, gives a linear in $T$ contribution to the specific heat. We obtain a non-analytic $T \ln T$ term for the specific heat at third order, and show that this term survives only for fermions with spin. For spinless fermions all those diagrams which give a $T \ln T$ term in the specific heat exactly cancel out.

### 3.2 Specific Heat

In principle, the electron Green’s function defines all the thermodynamic properties of the system. The total number of particles in the system as a function of $\mu$ and $T$ is:

$$N(\mu, T) = \int G_{\alpha\alpha}(\vec{r}, \tau; \vec{r}, \tau + 0) d\vec{r} = 2VT \sum_\varepsilon \int_{\tau \to +0} \frac{d\vec{p}}{(2\pi)^3} G(\vec{p}, \varepsilon) e^{i\varepsilon \tau}. \quad (3-12)$$

From this function, one can determine the entropy by using the thermodynamic relation $\left( \frac{\partial S}{\partial \mu} \right)_T = \left( \frac{\partial N}{\partial T} \right)_\mu$. Following the steps of Ref. [18] we isolate the
temperature dependence of the number density and obtain for the entropy,

\[ \frac{S}{V} = -2 \frac{\partial}{\partial T} \left[ T \sum_\varepsilon \int \frac{d^3 \vec{p}}{(2\pi)^3} \ln G(T = 0; p, \varepsilon) \right] \]

Performing the Matsubara sum using a contour integration, the entropy becomes

\[ \frac{S}{V} = 2 \int dp \frac{1}{2\pi} \int_{-\infty}^{\infty} d\varepsilon \varepsilon \left( -\frac{\partial n_F}{\partial \varepsilon} \right) \ln \left( \frac{G_R}{G_A} \right), \quad (3\text{–}13) \]

where \( n_F \) is the Fermi distribution function. \( G_R(G_A) \) is the retarded (advanced) Green’s function at zero temperature. Using Dyson’s equation \( (G = \frac{1}{G_0 + i\Sigma}) \) and the thermodynamic relation \( C(T) = T \frac{\partial}{\partial T} \left( \frac{S}{V} \right) = C_{FG} + \delta C(T) \) the interaction correction to the specific heat (to lowest order in \( \Sigma \)) is \([18]\):

\[ \delta C(T) = -\frac{2T}{\pi} \frac{\partial}{\partial T} \left[ \frac{1}{T} \int \frac{dp}{2\pi} \int d\varepsilon \varepsilon \frac{\partial n_F}{\partial \varepsilon} \left( \text{Im} G^0_R \text{Re} \Sigma_R + \text{Re} G^0_R \text{Im} \Sigma_R \right) \right] \quad (3\text{–}14) \]

where \( \Sigma_R (\Sigma_A) \) is the retarded (advanced) self energy evaluated at zero temperature, and \( G^0_R (G^0_A) \) is the free retarded (advanced) Green’s function. Strictly speaking the above formula for the specific heat is valid only for the leading temperature dependence (see a discussion about this in Ref.\([45]\)). However, we are justified in using the zero temperature formalism in 1D, since the non-analytic \( T \ln T \) term in the specific heat grows faster than the analytic \( T \) term for low enough temperatures. In sub-section 3.2.1, we evaluate the specific heat from the second order self-energy (at \( T = 0 \)) and find only a regular, linear in \( T \) contribution.

In sub-section 3.2.2, we again evaluate the specific heat, only this time using the thermodynamic potential (for \( T \neq 0 \)) also at second order, to verify the absence of the nonanalytic temperature dependence at second order in perturbation theory. This is one major difference from the higher order nonanalyticities. In subsection 3.2.3, we show that the non-analytic \( T \ln T \) term arises only at third order in interaction, and only for fermions with spin. In this section the nonanalytic correction to the specific heat is obtained from the third order self energy
evaluated at zero temperature. We verified our 3rd order result by performing a renormalization group analysis of the sine-Gordon theory in subsection 3.2.4.

3.2.1 Specific Heat from the Second Order Self Energy

The only non-trivial second order self-energy diagrams are shown in Fig. 3–2. The dashed (solid) lines represent the Green’s function, $G_-$($G_+$) for left (right) moving fermions and the wiggly lines denote the interaction vertices. The rest of the second order and first order self energy diagrams [18] are constant and lead to a trivial shift of the chemical potential and thus result in a linear $T$ dependence for $C(T)$.

![Figure 3–2. Non-trivial second order self energy diagrams for right moving fermions](image)

The singularities in the $2k_F$ particle-hole polarization bubble and the particle-particle channel do not affect the $2^{nd}$ order self-energy (and this is the reason why we get an analytic contribution for the specific heat in 1D at $2^{nd}$ order), which can be solely written in terms of the forward scattering polarization bubbles $\Pi_{--}$ and $\Pi_{++}$. Thus there are only two distinct contributions from all of the above diagrams, the one in Fig. 3–2(a) and 3–2(c). The diagrams in Fig. 3–2(b),(d) and (e) which clearly have a backscattering particle-hole bubble can be shown equal up to a constant pre-factor to the diagram in Fig. 3–2(a). The diagram in Fig. 3–2(f)
is same as that of Fig. 3–2(c). For the self-energy diagram in Fig. 3–2 (a) we have,

$$\Sigma^{3-2} (i\varepsilon; \mathbf{k}) = -g_2^2 T \sum_{\omega} \int \frac{dq}{2\pi} G_+ (i (\varepsilon - \omega); k - q) \Pi_- (i\omega; q).$$

To calculate the specific heat correction using Eq. 3–14 one needs the imaginary and real part of the retarded self energy. Applying the spectral representation for the Green’s function and the polarization operator [85], followed by a simple pole integration (for \(T = 0\)) and analytic continuation to real frequencies, one gets

$$\text{Im}\Sigma^{3-2} (\varepsilon, \mathbf{k}) = -\frac{\pi}{4} \frac{g_2^2}{(2\pi)^2} (\varepsilon - k) \ln \left| \frac{k^2 - \varepsilon^2}{\lambda^2} \right| . \quad (3–17)$$

The real part of the self-energy is then obtained from the Kramers-Kronig relation

$$\text{Re}\Sigma^{3-2} (\varepsilon, \mathbf{k}) = \frac{1}{\pi} \mathcal{P} \int d\omega \frac{\text{Im}\Sigma^{R} (\omega, \mathbf{k})}{\omega - \varepsilon} = \frac{g_2^2}{(4\pi)^2} (\varepsilon - k) \ln \left| \frac{k^2 - \varepsilon^2}{\lambda^2} \right| , \quad (3–18)$$

where \(\lambda\) is a cut-off. Notice that the self energy (real part) is zero on the mass-shell \((\varepsilon = k)\) contrary to higher dimensions. In 1D, the entire self-energy comes from the processes \(\omega = -q\) (because the particle-hole continuum has shrunk to a single
line, \( \omega = \pm q \). This is the ultimate case of forward scattering, whose precursors in higher dimensions lead to non-analyticities in the specific heat [44]. However in 1D, these non-analyticities in the second order self energy will lead to a linear temperature dependence for the specific heat. Also note that \( \text{Im} \Sigma^R (\varepsilon, k = 0) \propto |\varepsilon| \), which is indicative of the poorly defined quasi-particles in 1D (\( \text{Im} \Sigma^R \) scales with \( \varepsilon \) in the same way as the energy of a free excitation above the Fermi level). In a conventional Fermi liquid, the condition for well defined quasi-particle excitations is, \( \text{Im} \Sigma^R (\varepsilon) \sim \varepsilon^2 \ll \text{Re} \Sigma^R \sim \varepsilon \). In 1D, \( \text{Re} \Sigma^R (\varepsilon, k = 0) \sim \varepsilon \ln |\varepsilon| \), which means that effective mass diverges as: \( m^* \sim \ln |\varepsilon| \) and, to this order the behavior is reminiscent of a marginal Fermi liquid [86].

The self-energy diagram of Fig. 3–2(c) is,

\[
\text{Im} \Sigma^R_{++} (\varepsilon; k) = -\frac{2g^4}{(2\pi)^2} \int dq \int_0^\varepsilon d\omega \text{Im} \left( G^R (\varepsilon - \omega; k - q) \right) \text{Im} \left( \Pi^R_{++} (\omega; q) \right),
\]

(3–19)

Substitute \( \text{Im} \Pi^R_{++} (\omega, q) = -\frac{1}{2} q \delta (\omega - q) \), from Eq. 3–9 and the Green’s function and performing the integrations we obtain

\[
\text{Im} \Sigma^R_{++} (\varepsilon; k) = -\frac{g^4}{8\pi} \varepsilon^2 \delta (\varepsilon - k).
\]

(3–20)

We see that the self-energy diverges on the mass-shell. This is the infra-red catastrophe [87] in 1D. The one dimensional electrons can emit infinite number of soft bosons: quanta of density excitations. The real part of the self-energy is found again from the Kramers-Kronig relation

\[
\text{Re} \Sigma^R_{++} (\varepsilon; k) = -A \int d\omega \omega^2 \delta (\omega - k) = A \frac{k^2}{\varepsilon - k},
\]

(3–21)

where \( A = \frac{g^4}{8\pi^2} \), and the total self-energy is

\[
\Sigma^{3--2c} = A \frac{\varepsilon^2}{\varepsilon - k + i\delta}.
\]

(3–22)
When this form of the self-energy is substituted in the Green’s function, there are two poles, which corresponds to dispersion of the the spin and charge mode. This, essentially non-perturbative and 1D effect is the spin-charge separation, which is confirmed by an exact solution (Dzyaloshinskii-Larkin [88] solution of Tomonaga-Luttinger model). The rest of the self-energy diagrams in Fig. 3–2 can be reduced to either of the two self energies evaluated above (up to a pre-factor) by relabeling the dummy variables, e.g., the self energy in Fig. 3–2(b) is

\[
\Sigma^{3-2(b)} = -g_1^2 \int q G_-(k - q) \Pi_{2kF} (q) = -g_1^2 \int q G_-(k - q) \int_{k'} G_+(k') G_-(k' - q)
\]

\[
= -g_1^2 \int_{k'} G_+(k') \int q G_-(k - q) G_-(k' - q) = -g_1^2 \int_{k'} G_+(k') \Pi_- (k - k')
\]

\[
= -g_1^2 \int q' G_+(k - q') \Pi_- (q') = \frac{g_1^2}{g_2^2} \Sigma^{3-2(a)}
\]

where \( \int q \equiv \int \frac{dq}{2\pi} \sum \omega \) and \( \int_{k'} \equiv \int \frac{dk'}{2\pi} \sum \epsilon' \). Similarly it can be shown that \( \Sigma^{3-2(f)} = -\frac{1}{2} \Sigma^{3-2(c)} \), the negative sign is due to one extra closed loop and the factor of two is due to spin sum in closed loop of Fig. 3–2(c). Also \( \Sigma^{3-2(d),(e)} = \frac{-g_1 g_2}{g_2^2} \Sigma^{3-2(a)} \). So the net self-energy at second order in interaction is:

\[
\Sigma^+ (\varepsilon; k) \equiv \left( -g_2^2 - g_1^2 + g_1 g_2 \right) \Sigma_1 + \left( -g_4^2 \right) \Sigma_2,
\]

\[
\text{Im}\Sigma_1 = \text{sgn}(\varepsilon)(\varepsilon - k)\Theta(\varepsilon - |k|),
\]

\[
\text{Re}\Sigma_1 = - (\varepsilon - k) \ln \frac{k^2 - \varepsilon^2}{\lambda^2},
\]

\[
\text{Im}\Sigma_2 = \varepsilon^2 \delta(\varepsilon - k),
\]

\[
\text{Re}\Sigma_2 = \frac{-k^2}{\varepsilon - k}
\]

where we have dropped all the numerical pre-factors. Now we can calculate the correction to the specific heat at second order in interaction using Eq. 3–14. Although the log singularity in \( \text{Re}\Sigma_1 \), suggests a nonanalytic term in \( \delta C(T) \); however, a careful analysis shows that this is not the case. Note that in Eq. 3–14, \( \text{Re}\Sigma \) is multiplied by \( \text{Im}G^R_+ (\varepsilon; k) \) which is \( \delta (\varepsilon - k) \), so doing the \( k \) integration
projects the $\text{Re} \Sigma_1 (\varepsilon; k)$ on the mass shell where, it is zero ($\text{Re} \Sigma_1 (\varepsilon, k = \varepsilon) = 0$).

Consider now the contribution from to $\text{Im} \Sigma_1$. The momentum integration gives,

$$P \int dk \frac{1}{\varepsilon - k} (\text{sgn}\varepsilon) (\varepsilon - k) \theta (|\varepsilon| - |k|) = \text{sgn}\varepsilon \int_{-|\varepsilon|}^{|\varepsilon|} dk = 2\varepsilon$$

which leads to a (analytic) linear in $T$ correction to $C(T)$. The net contribution from $\Sigma_1$ to $\delta C$ is

$$\delta C^{\Sigma_1} (T) \propto \left( +g_2^2 + g_1^2 - g_1 g_2 \right) T \frac{\partial}{\partial T} \frac{1}{T} \int d\varepsilon \varepsilon^2 \frac{\partial n}{\partial \varepsilon} \simeq T \left( -g_2^2 - g_1^2 + g_1 g_2 \right).$$

The linear temperature dependence in Eq. 3–23, can be seen by re-scaling $\varepsilon$ by $T$ and bringing the integral in a dimensionless form which gives a number of order one. Both the real and imaginary part of $\Sigma_2$ will contribute equally to the specific heat. Consider the momentum integral for $\Sigma_2$'s contribution to $C(T)$,

$$\int dk \left[ \text{Im} G^R \text{Re} \Sigma^R_2 + \text{Re} G^R \text{Im} \Sigma^R_2 \right] = \int dk \left[ -\pi \delta (\varepsilon - k) \frac{Ak^2}{\varepsilon - k} + \left( -\frac{A\pi}{2} \right) k^2 \delta (\varepsilon - k) \right] = 4A\pi \varepsilon.$$

Once again power-counting (dimensional analysis) gives $\delta C^{\Sigma_2} (T) \propto g_4^2 T$.

The specific heat remains linear in temperature in 1D at least in second order perturbation theory (unlike in $D > 1$, see Ref. [44]). The linear temperature behavior does not change even if we use the finite temperature formula for the self-energy. The result for the diagram in Fig.3–2(a) at finite temperature from Eq. 3–16 is,

$$\text{Im} \Sigma_3^{3-2(a)} \propto (\varepsilon - k) \left[ \coth \frac{\varepsilon - k}{4T} + \tanh \frac{\varepsilon + k}{4T} \right] = \Sigma_1 (\varepsilon; k; T)$$

The temperature dependence comes solely from the Matsubara sum, the polarization operator is still evaluated at zero temperature. The momentum integration in the specific heat formula Eq.3–14 gives a linear $\varepsilon$ dependence (the same result both for
zero and finite temperature).

\[ P \int \frac{dk}{k} \frac{\varepsilon - k}{\varepsilon - k} \left[ \coth \frac{\varepsilon - k}{4T} + \tanh \frac{\varepsilon + k}{4T} \right] = 4\varepsilon \]

Therefore, the specific heat contribution is linear in temperature. \( \text{Re}\Sigma_1 (\varepsilon; k = \varepsilon; T) = 0 \) on the mass-shell. The finite temperature result for the self-energy diagram of Fig. 3–2(c) is

\[ \text{Im}\Sigma_R^{3-2(c)}(\varepsilon; k) \propto (\varepsilon^2 + \pi^2T^2) \delta (\varepsilon - k) = \Sigma_2 (\varepsilon; k; T) . \quad (3–25) \]

Using the above form of the finite temperature self-energy the momentum integration in Eq. 3–14 gives a term proportional to \( \varepsilon \), which again leads to \( C(T) \propto T \). Thus, 1D is different from higher dimensions because the specific heat is analytic at second order in interaction. Before considering the third order self-energy contribution to the specific heat, we will calculate the specific heat from the thermodynamic potential at second order (see Fig. 3–3) and show that an apparent \( T \ln T \) contribution to the specific heat gets canceled, when we consider the temperature dependence of the polarization bubble.

3.2.2 Specific Heat from the Thermodynamic Potential at Second Order

![Second order diagrams for the thermodynamic potential with maximum number of explicit particle-hole bubbles](image)

Figure 3–3. Second order diagrams for the thermodynamic potential with maximum number of explicit particle-hole bubbles
Another way to obtain $C(T)$ beyond the leading term in $T$ is to find the thermodynamic potential $\Omega(T)$ within the Luttinger-Ward approach [89], and then use the thermodynamic relation $C(T) = -\frac{\partial^2 \Omega}{\partial T^2}$. The interaction correction to the thermodynamic potential can also be evaluated using the linked cluster expansion [85] (it is an approximate method, since only a certain subset of the diagrams are summed). Here in this section we will evaluate the thermodynamic potential diagrams directly to second order in perturbation theory, and verify the linear specific heat obtained via the self energy calculation of the previous sub-section. Fig. 3–3 shows the second order diagrams, which have maximum number of explicit particle-hole bubbles for the thermodynamic potential. Once again dashed (solid) lines denote left (right) movers. We will show that the forward scattering diagrams [(Fig. 3–3(a) and (c))] give a linear specific heat. Using the zero temperature form for the $2k_F$ particle-hole polarization bubble in the diagram in Fig. 3–3(b), one would get a $T \ln T$ term in the specific heat, however, such a term goes away when we use the full finite temperature result for the bubble. The first order and the other second order diagrams (non RPA type) for the thermodynamic potential give a linear temperature behavior for the specific heat. For the diagram in Fig. 3–3(c),

$$\Omega = -L \frac{g^2}{2} T \sum_{i,\omega} \int \frac{dq}{2\pi} \left( \Pi_{-+}(q; i\omega) \Pi_{++}(q; i\omega) \right)$$

We omit the constant factor of $-L/2$ and sum over the bosonic Matsubara frequencies using a contour integration [85] and get

$$\Omega^{3-3(c)} = g^2 \int \frac{dq}{2\pi} \int \frac{d\omega}{2\pi} \coth \left( \frac{\omega}{2T} \right) \text{Im} \left[ \Pi_{++}^R(q, \omega) \Pi_{--}^R(q, \omega) \right], \quad (3-26)$$

where $\text{Im} \left[ \Pi_{++}^R \Pi_{--}^R \right]$ should now be evaluated at a finite temperature. For forward scattering processes, both the finite temperature and zero temperature forms of the polarization operator $(\Pi_{++}, \Pi_{--})$ are the same. At finite temperature
from Eq. 3–8,

\[ \Pi_{--}(q; i\omega) = \frac{1}{i\omega + q} \int \frac{dk}{2\pi} (n_-(k) - n_-(k + q)) = -\frac{q}{2\pi (i\omega + q)}. \]

The result of the momentum \((k)\) integration is the same both at zero and at finite temperature. This can be seen using the form of the momentum distribution function for left and right movers, \(n_{\pm}(k) = (e^{(\pm k/T)} + 1)^{-1}\). At \(T = 0\), \(n_-(k) = \Theta(k)\) and the \(k\) integration gives \(-q\). Even for \(T \neq 0\), one can integrate by parts to show that the result for the polarization bubble does not change. Thus we can use the zero temperature forms, Eq. 3–8 and Eq. 3–9 to obtain, \(\text{Im} [\Pi_{--}^R \Pi_{++}^-] = \frac{q\pi}{8\pi} (\delta(\omega - q) - \delta(\omega + q))\). Then

\[ \Omega^{3--(c)}_3 = \frac{g_2^2}{4\pi (2\pi)^2} \mathcal{P} \int d\omega \coth \left( \frac{\omega}{2T} \right) \omega. \]

The entropy \((S = -\frac{\partial \Omega}{\partial T})\) and the specific heat then depend linearly on temperature

\[ S^{3--(c)} = + \frac{g_2^2 2T}{2\pi (2\pi)^2} \int_0^\infty dx \frac{x^2}{\sinh^2 x} = + \frac{g_2^2 T}{24\pi} \Rightarrow C^{3--(c)}_3(T) \propto T. \]  

(3–27)

Now for the diagram in Fig. 3–3(a),

\[ \Omega^{3--(a)}_3 = g_4^2 \int \frac{dq}{2\pi} \int \frac{d\omega}{2\pi} \coth \left( \frac{\omega}{2T} \right) \text{Im} [\Pi_{++}^R \Pi_{++}^-] \]

(3–28)

Using Eq. 3–9, one gets

\[ \text{Im} \left( (\Pi_{++}^2)^R \right) = -\frac{q^2 \delta(\omega - q)}{2\pi (\omega - q)}, \]

(3–29)

even for finite \(T\). Thus the thermodynamic potential becomes,

\[ \Omega^{3--(a)}_3 = + \frac{4g_4^2}{(2\pi)^3} \int_0^\infty d\omega \coth \left( \frac{\omega}{2T} \right) \omega, \]

(3–29)
and once again the entropy and the specific heat are linear in temperature.

\[ S^{3--3(a)} = + \frac{g_4^2 T}{(\pi)^3} \int_0^\infty dx \frac{x^2}{\sinh^2 x} = + \frac{g_4^2 T}{6\pi} \implies C^{3--3(a)}(T) \propto T. \quad (3-30) \]

The backscattering diagram of Fig.3–3(b) is,

\[ \Omega^{3--3(b)} = g_1^2 \int dq \int d\omega \coth \left( \frac{\omega}{2T} \right) \text{Im} \left( \frac{(\Pi_{2k_F})^R}{2\pi} \right) \quad (3-31) \]

We now show that using the zero temperature form of the 2\(k_F\) bubble from Eq. 3–10, we get a \(T \ln T\) term in the specific heat; however this nonanalytic term drops out when we use the finite temperature form of the bubble. At \(T = 0\),

\[ \Omega^{3--3(b)} = - \frac{g_1^2}{8\pi} \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} d\omega \coth \left( \frac{\omega}{2T} \right) \ln \left| \frac{\omega^2 - q^2}{(2\lambda)^2} \right| \text{sgn}(\omega) \Theta(\omega^2 - q^2) \]

\[ = - \frac{2g_1^2}{8\pi^3} \int_0^\infty d\omega \coth \left( \frac{\omega}{2T} \right) \left( \omega \ln \left| \frac{\omega}{\lambda} \right| - \omega \right) \]

The specific heat acquires a nonanalytic \(T \ln T\) term in addition to the linear in \(T\) term.

\[ S^{3--3(b)} = - \frac{T g_1^2}{2\pi^3} \left( -0.56 + \frac{\pi^2}{6} \ln \left| \frac{2T}{\lambda} \right| - \frac{\pi^2}{6} \right) \implies C^{3--3(b)}(T) \propto T \ln T \quad (3-32) \]

This non-analyticity is artificial and is removed (exactly canceled) when we substitute the finite temperature \(\text{Im}(\Pi_{2k_F})\) in the thermodynamic potential. That such a cancelation must occur can be seen easily: the diagram in Fig. 3–3(b) can be shown equivalent to the diagram in Fig. 3–3(c), (up to an overall multiplicative constant) by pairing different Green’s function to form the bubble, interchanging the order of integration of the dummy variables and relabelling. Since the diagram of Fig. 3–3(c) gives a linear specific heat (see Eq. 3–27), the double backscattering diagram of Fig. 3–3(b) must give a linear (regular) temperature correction for the specific heat as well. To resolve the apparent contradiction, we calculate explicitly
the finite-$T$ form of the backscattering bubble

$$\text{Im} [\Pi_{2k_F}^2 (q, \omega)]^R = \frac{1}{4} \left[ n_- \left( \frac{\omega - q}{2} \right) - n_+ \left( \frac{\omega + q}{2} \right) \right] \times \mathcal{P} \frac{dk}{2\pi} \left[ n_-(k) - n_+(k + q) \right] \frac{1}{k - (\omega - q)/2}.$$  (3–33)

The thermodynamic potential now becomes

$$\Omega = A \mathcal{P} \int d\omega \coth \left( \frac{\omega}{2T} \right) \int dq \left[ n_+ \left( \frac{\omega + q}{2} \right) - n_- \left( \frac{\omega - q}{2} \right) \right] \left[ \ln \left( \frac{\omega^2 - q^2}{(2\lambda)^2} \right) - \frac{2\pi^2 T^2}{3} \left( \frac{1}{(\omega - q)^2} + \frac{1}{(\omega + q)^2} \right) \right],$$  (3–34)

where $A$ is a numerical constant. Evaluating the specific heat, we find that the nonanalytic $T \ln T$ term drops out and the specific heat remains linear in temperature. This calculation is tedious, so we do not present it in the thesis.

This section verifies the result obtained in previous section, that the specific heat in 1D is analytic at 2$^{nd}$ order in perturbation theory unlike in higher dimensions ($D = 2, 3$) [44, 47]. In the next section we evaluate the specific heat from the third order self-energy and obtain a genuine nonanalytic $T \ln T$ contribution for spin-full fermions in 1D.

### 3.2.3 Specific Heat from Third Order Self Energy

We will consider all the self-energy diagrams at third order (for right movers), except those which reduce to self-energy insertions made in to second order diagrams. We will show that a nonanalytic $T \ln T$ term in the specific heat can arise only from those third order diagrams which have two $2k_F$ particle-hole bubbles or two Cooper (particle-particle) bubbles in them. The diagrams with self-energy insertions are omitted because they cannot have two $2k_F$ particle-hole bubbles or two particle-particle bubbles. The first set of 3$^{rd}$ order self-energy diagram, we evaluate, are shown in Fig.3–4. These diagrams explicitly contain two particle-hole polarization bubbles in them. There are four distinct possibilities
which can occur in diagrams with two polarization bubbles: (1) both polarization bubbles can be backscattering ones ($\Pi_{2k_F^2}$) as in Fig.3–4(a); (2) both polarization bubbles are forward scattering, with one of them being $\Pi_{++}$ and the other $\Pi_{--}$ as in Fig.3–4(b); (3) both polarization bubbles are forward scattering and $\Pi_{--}$ as in Fig.3–4(c); and, finally, (4) both forward scattering polarization bubbles are $\Pi_{++}$, as in Fig.3–4(d). All third order self-energy diagrams, which have two particle-hole polarization bubbles can be classified into the above four categories. We will explicitly evaluate all four of these diagrams and show that only when both the particle-hole polarization bubbles are of the backscattering type ($\Pi_{2k_F^2}$), one gets a nonanalytic $T \ln T$ dependence for the specific heat. The particle-particle channel has the same nonanalytic momenta and frequency dependence as the particle-hole backscattering bubble, so we expect a nonanalytic $T \ln T$ term in the specific heat from diagrams which have two Cooper bubbles as well. All the 3\textsuperscript{rd} order diagrams which give a non-analytic specific heat are shown in Fig.3–9. In this section we will be using the zero temperature forms of the bosonic response functions in evaluating the self energy and the specific heat. We remind ourselves that the $g_4$ couples fermions on the same side of the Fermi surface wheras $g_2$ couples fermions from different branches. However, each species stays on the same side of the Fermi surface after the interaction (both forward scattering). Finally, the $g_1$ process corresponds to $\sim 2k_F$ scattering (backscattering) where the fermions exchange sides. Once again, solid lines represent right movers and dashed lines denote left movers.

![Figure 3–4. The different choices for the 3rd order diagram.](image)
(1) Two backscattering bubbles:

\[ \Sigma_{3-4}^{3-4(a)} (k; i\varepsilon) = -g_1^3 \int \frac{dq}{2\pi} \int \frac{d\omega}{2\pi} G_+ (k - q; i(\varepsilon - \omega)) \left[ \Pi_{2k_F}^2 (q; i\omega) \right]. \]  

(3–35)

Using the spectral representation for the Green’s function and the polarization operator,

\[ \text{Im} \Sigma_{3-4}^{3-4(a)} = -\frac{2g_1^3}{(2\pi)^2} \int_{-\infty}^{\infty} dq \int_{0}^{\varepsilon} d\omega \text{Im} \left( G^R_- (k - q; \varepsilon - \omega) \right) \text{Im} \left[ \Pi_{2k_F}^2 (q; \omega) \right]. \]  

(3–36)

From Eq.3–10, we get

\[ \text{Im} \left[ \Pi_{2k_F}^2 (q; \omega) \right] = -\frac{1}{8\pi} \ln \left| \frac{q^2}{(2\lambda)^2} \right| |\text{sgn} (\omega)| \Theta \left( |\omega| - |q^2| \right). \]  

(3–37)

The real part of the self-energy is obtained from Kramers-Kronig relation and on the mass shell it is

\[ \text{Re} \Sigma_{3-4}^{3-4(a)} = \mathcal{P} \int d\omega \frac{(\omega - \varepsilon)}{\omega - \varepsilon} \times |\text{sgn} (\omega)| \Theta (|\omega| - |\varepsilon|) \left[ \ln \left| \frac{q^2}{(2\lambda)^2} \right| - 1 \right] = 0 \]  

(3–38)

because the integral is an odd function of \( \omega \). Thus the real part of the self-energy does not contribute to the specific heat. The contribution to the specific heat from \( \text{Im} \Sigma_R \) is non-analytic,

\[ \delta C (T)^{3-4(a)} = b \frac{-2T}{\pi} \frac{\partial}{\partial T} \left( \frac{1}{T} \int d\varepsilon \frac{\partial n_F}{\partial \varepsilon} |\text{sgn} (\varepsilon)| \times \int_{-\infty}^{\infty} \frac{dk}{2\pi} \Theta (|\varepsilon| - |k|) \left[ \ln \left| \frac{q^2}{(2\lambda)^2} \right| - 1 \right] \right), \]

where \( b = -(g_1^3)/(8 (2\pi)^2) \) is the constant coefficient. We re-scale \( \varepsilon \) by \( T \) to get a \( T \ln T \) factor multiplying a dimensionless integral which gives a number of order
\[ \delta C(T) \sim \propto g_1^3 T \ln(T/\lambda). \tag{3–39} \]

(2) Two forward scattering bubbles, with one bubble being \( \Pi_{--} \) and the other \( \Pi_{++} \):

\[ \text{Im} \Sigma_R^{3-4(b)}(k; \varepsilon) = -\frac{2g_2^2 g_4}{(2\pi)^2} \int_0^\varepsilon d\omega \int_{-\infty}^{\infty} dq \text{Im} G_+^R(k - q; \varepsilon - \omega) \times \text{Im} \left[ \Pi_{--}^R(q; \omega) \Pi_{++}^R(q; \omega) \right]. \tag{3–40} \]

Using the zero temperature forms for the above response functions one obtains,

\[ \text{Im} \Sigma_R^{3-4(b)}(k; \varepsilon) = \frac{+g_2^2 g_4}{4(2\pi)^2} \left( \varepsilon^2 \frac{\delta(\varepsilon - k)}{2} + \text{sgn}(\varepsilon) \left( \frac{\varepsilon - k}{4} \right) \Theta(|\varepsilon| - |k|) \right). \tag{3–41} \]

The above form for the self-energy consists of two parts each of which were earlier obtained for the 2nd order self energy diagrams in Eq. 3–17 and Eq.3–20. This is not unexpected because this 3rd order diagram is made up of second order pieces (see Fig.3–4(b)). Then from the second order specific heat analysis we can surely say that the above form of the self-energy gives a linear specific heat.

(3) Two polarization bubbles, both being \( \Pi_{--} \),

\[ \text{Im} \Sigma_R^{3-4(c)} = \frac{-2g_2^2 g_4}{(2\pi)^2} \int_0^\varepsilon d\omega \int_{-\infty}^{\infty} dq \text{Im} G_+^R(k - q; \varepsilon - \omega) \text{Im} \left[ \Pi_{--}^R(q; \omega) \right]^2. \tag{3–42} \]

Once again the self energy is the same as that for the second order diagram,(see Eq.3–17),

\[ \text{Im} \Sigma_R^{3-4(c)}(k; \varepsilon) = \frac{g_2^2 g_4}{2(2\pi)^2} \text{sgn}(\varepsilon) (\varepsilon - k) \Theta(|\varepsilon| - |k|), \tag{3–43} \]

From the 2nd order analysis we know that this form of the self-energy does not give a nonanalytic contribution to the specific heat.
(4) Two polarization bubbles, both are $\Pi_{++}$;

$$\text{Im}\Sigma_{R}^{3-4(d)} = \frac{-2g_{4}^{3}}{(2\pi)^{2}} \int_{0}^{\varepsilon} d\omega \int_{-\infty}^{\infty} dq \text{Im}G_{+}^{R}(k-q;\varepsilon-\omega) \text{Im} \left[ \Pi_{++}^{2}(q;\omega) \right],$$

$$= \frac{g_{4}^{2}}{(2\pi)^{2}} \varepsilon^{2} \delta(\varepsilon-k). \quad (3-44)$$

The above form of the self-energy is the same as that obtained for the second order diagram with a mass shell singularity (see Eq. 3-20). This also results in a linear in $T$ contribution to the specific heat. Therefore we have shown that the three forward scattering diagrams of Fig. 3-4(b), (c) and (d) all give only a linear in $T$ contribution to specific heat. The only diagram which gives a nonanalytic $T \ln T$ correction to the specific heat is the one with two backscattering bubbles, Fig. 3-4(a). From here on we will focus only on those diagrams which give a nonanalytic contribution to the specific heat.

The self energy diagrams in Fig. 3-4 contains two explicit particle-hole bubbles. There are several other (seven) self energy diagrams (see Fig. 3-5) which do not contain explicit particle-hole bubbles, but they can be shown equivalent to the ones shown in Fig. 3-4, by trivially re-labeling the dummy variables. These are the self energy diagrams which implicitly contain two particle-hole bubbles in them, Fig. 3-5(b)-(h). Thus all third order self energy diagrams which have two particle hole bubbles (explicit or implicit) fall into the four categories studied above. We show this next, only for the case of two backscattering bubbles because the nonanalytic $T \ln T$ term arises only from two $\Pi_{2k_{F}}$, bubbles. Consider the diagram in Fig. 3-5(b), which we will show is equal (up to a numerical pre-factor) to the diagram
in Fig. 3–4(a)

\[
\Sigma^{3\rightarrow-5(b)} (k; i \varepsilon) = +g_1^2 g_2 \int dq \int d\omega \int dk_1 \int d\varepsilon_1 \int dq_1 \int d\omega_1 \\
G_-(k - q; i (\varepsilon - \omega)) G_- (k_1; i \varepsilon_1) G_+ (k_1 + q; i (\varepsilon_1 + \omega)) \\
\times G_+ (k_1 + q + q_1; i (\varepsilon_1 + \omega + \omega_1)) G_- (k_1 + q_1; i (\varepsilon_1 + \omega_1)) \\
= +g_1^2 g_2 \int dq \int d\omega G_- (k - q; i (\varepsilon - \omega)) \\
\times \int dk_1 \int d\varepsilon_1 G_- (k_1; i \varepsilon_1) G_+ (k_1 + q; i (\varepsilon_1 + \omega)) \\
\times \int dq_1 \int d\omega_1 G_+ (k_1 + q + q_1; i (\varepsilon_1 + \omega + \omega_1)) G_- (k_1 + q_1; i (\varepsilon_1 + \omega_1)) \\
= +g_1^2 g_2 \int dq \int d\omega G_- (k - q; i (\varepsilon - \omega)) [\Pi_{2k_F} (q; i \omega)]^2.
\]

We see that, \( \Sigma^{3\rightarrow-5(b)} = -\frac{g_2}{g_1} \Sigma^{3\rightarrow-4(a)} \), and the specific heat is \( C (T)^{3\rightarrow-5(b)} \propto -g_1^2 g_2 T \ln T \). Similarly it can be shown that the all the self-energy diagram of Fig. 3–5 with two \( \Pi_{2k_F} \) bubbles give a \( T \ln T \) contribution to the specific heat. The diagrams of Fig. 3–5(b)-(h), can also be drawn with forward scattering bubbles (analogous to the diagrams in Fig. 3–4(b)-(d)). However, these diagrams give a linear \( T \) correction to the specific heat and we have omitted them in this chapter for lack of space. The nonanalytic contributions from all those self energy diagrams with two particle-hole (backscattering) bubbles are:

\[
\Sigma^{3\rightarrow-5(a)} \rightarrow C (T) \propto g_1^3 T \ln T \\
\Sigma^{3\rightarrow-5(b)+3\rightarrow-5(c)+3\rightarrow-5(d)} \rightarrow C (T) \propto -3g_1^2 g_2 T \ln T \\
\Sigma^{3\rightarrow-5(e)} \rightarrow C (T) \propto -g_2^3 T \ln T \\
\Sigma^{3\rightarrow-5(f)+3\rightarrow-5(g)+3\rightarrow-5(h)} \rightarrow C (T) \propto +3g_2^2 g_1 T \ln T
\]

The net nonanalytic contribution is \( C (T) \propto (g_1 - g_2)^3 T \ln T \). The non-analyticities
Figure 3–5. All 3rd order se diagrams for right movers which have two $\Pi_{2k_F}$.

seem to arise from $g_1$ (exact backscattering) and $g_2$ (exact forward scattering) interaction vertices, similar to $2D$, as shown in Ref. [44]. This result suggests that even if $g_1 = 0$ (long range interaction potential), the specific heat remains nonanalytic in 1D. This contradicts the bosonization result which states that for a Gaussian theory (with $g_4$ and $g_2$ interaction, see Eq. 3–2) the specific heat remains linear in temperature. Therefore our result cannot be correct and we must have overlooked some diagrams which must cancel (at least) the $g_2$ dependence of the NAC to $C(T)$. These are the particle-particle or Cooper diagrams, which have the same nonanalytic behavior as the $2k_F$ particle-hole bubble (compare Eq. 3–11 and Eq. 3–10). Therefore all the third order self energy diagrams with two Cooper bubbles in them (explicit or implicit), also give a $T \ln T$ term in the specific heat and may cancel (some or all) the nonanalytic contribution arising from the backscattering particle-hole bubbles. These diagrams are shown in Fig. 3–6. One can reduce these Cooper self energy diagrams to a $2k_F$ self-energy diagram to obtain a $T \ln T$ contribution for the specific heat.

The self-energy diagram in Fig. 3–6(e) has two particle-particle bubbles and it will be shown to be equal to the self-energy diagram with two backscattering
Figure 3–6. All third order self energy diagrams containing two Cooper bubbles and thus will give rise to a $T \ln T$ contribution to the specific heat.

\[
\Sigma^{3-6(e)} (k; i \varepsilon) = g_2^3 \int dq \int d\omega G_- (q - k; i (\omega - \varepsilon))
\times \int dk_1 \int d\varepsilon_1 G_+ (k_1 + q; i (\varepsilon_1 + \omega)) G_- (-k_1; -i \varepsilon_1)
\times \int dk_2 \int d\varepsilon_2 G_+ (k_2 + q; i (\varepsilon_2 + \omega)) G_- (-k_2; -i \varepsilon_2),
\]
\[
= g_2^3 \int dq \int d\omega G_- (q - k; i (\omega - \varepsilon)) \left[ \Pi^{p-p} (q; i \omega) \right]^2,
\]
\[
= -g_2^3 \int dq \int d\omega G_- (k - q; i (\varepsilon - \omega)) \left[ \Pi_{2k_F} (q; i \omega) \right]^2 = -\Sigma^{3-5(e)}
\]
\[
\implies C (T) \propto g_2^3 T \ln T.
\]

where we have used $\Pi^{p-p} (q; i \omega) = -\Pi_{2k_F} (q; i \omega)$, (see Eq. 3–10 and Eq. 3–11).

Since the self energies are equal and opposite, the $T \ln T$ corresponding terms in the specific heat are exactly canceled. This confirms our previous expectation that the NAC to $C(T)$ from the Cooper type self energy diagrams will cancel some (or all) of the nonanalytic contribution from the backscattering particle-hole self energy diagrams.

In order to systematically list all the third order self energy diagrams, one must start from the second order self energy diagrams and replace one of the interaction lines with a vertex (see Fig.3–7). All second order vertices with $g_2$
Figure 3–7. Effective third order self-energy diagrams (the double line is a vertex).

and $g_1$ interaction lines are shown in Fig. 3–8. We have left out the $g_4$ vertices as they do not result in a non-analytic self-energy. This is because a $g_4$ interaction line can only be paired with another $g_4$ line to form a $g_4^2$ vertex. There can not be a $g_4g_2$ or $g_4g_1$ vertex either. Therefore, although each of the vertices shown in Fig. 3–8, can be drawn with $g_4$ lines, they can be only be combined with another $g_4$ line in the self-energy making the overall coefficient in front of the diagram $g_4^3$. These diagrams can only have two forward scattering particle-hole bubbles in them and thus cannot lead to a non analyticity. Also notice that the vertex in Fig. 3–8(f) could have been drawn with two $g_2$ processes; however, such a vertex, when included in a self energy diagram comes with two forward scattering bubbles one being $\Pi_{--}$ and the other $\Pi_{++}$ and we have seen that this combination gives a linear temperature dependence of the specific heat. In Fig. 3–9 we show all the third order self-energies which either have two p-h ($2k_F$) bubbles or two p-p channels, and all of them give $T\ln T$ nonanalyticity to the specific heat. However, for fermions without spin there is an exact cancelation among the diagrams, making the specific heat linear in temperature. The nonanalytic $T\ln T$ term survives only for fermions with spin.

The first eight diagrams of Fig. 3–9((a)..(h)) arise when we replace the vertex (double line) in Fig. 3–7(c) by each of the eight vertices listed in Fig. 3–8. The next eight diagrams of Fig. 3–9((i)..(p)) arise when we replace the (double) interaction line in the second order self energy diagram of Fig. 3–7(a) by each of the eight vertices in Fig. 3–8. Now let us write the total nonanalytic specific heat
Thus for spinless fermions all the non-analytic contributions cancel out exactly. In particular, the diagrams in Fig. 3–9(a)+(d)+(e) and those in Fig.3–9(k)+(l)+(m) with two Cooper bubbles cancel with the diagrams in Fig.3–9(b)+(g)+(h) and Fig. 3–9(f)+(p)+(o); correspondingly. Similarly the diagram in Fig. 3–9(i) cancels with the one in Fig. 3–9(j) and the diagram in Fig. 3–9(c) cancels with the one in Fig. 3–9(n). Thus the specific heat is perfectly
Figure 3–9. All third order self-energy diagrams with two Cooper bubbles or two $\Pi_{2k_F}$ bubbles.

regular with a linear temperature dependence even at the 3rd order in interaction. This is consistent with the bosonization treatment since for fermions without spin (even with backscattering vertex $g_1$) one has a quadratic hamiltonian (see Eq. 3–2) where in $K, u$ one has to replace $g_2 \rightarrow g_2 - g_1$.

For fermions with spin the cancelation between Fig.3–9(c) and Fig. 3–9(n) is incomplete because of extra spin sum in the polarization bubble. The particle-particle diagram of Fig. 3–9(c) can only have $g_1||$ interaction vertex so $C(T)^{3-9(c)} \propto -g_1||^3 T \ln T$, but the double backscattering particle-hole diagram of Fig. 3–9(n) can have two choices. It can (1) have all three $g_1||$ interaction lines, for which $C(T)^{3-9(n)} \propto g_1||^3 T \ln T$ which will cancel the previous contribution, but it can also have (2) two $g_1\perp$ lines and one $g_1||$ line, and so $C(T)^{3-9(n)} \propto g_1||^2 g_1\perp^2 T \ln T$, a nonanalytic contribution which survives. Furthermore, if we assume, $g_2|| \neq g_2\perp$, then the three diagrams in Fig. 3–9(f),(o),(p) and the corresponding ones in Fig. 3–9(k),(l),(m) do not cancel. In the first set Fig. 3–9(f),(o),(p), the coefficient
in front of the $T \ln T$ term can be either $-(g_1^\parallel)^2g_2^\parallel$ or $-(g_1^\perp)^2g_2^\parallel$ whereas for the second set Fig. 3–9(k),(l),(m), the coefficients can be $(g_1^\parallel)^2g_2^\parallel$ (which cancel out) or $(g_1^\perp)^2g_2^\perp$, which do not cancel. All the rest of the diagrams cancel out completely even with spin. Thus the main result of this section is that the specific heat at third order in interaction is:

$$C(T) \propto T \leftarrow \text{spin-less fermions} \quad (3\text{–}45)$$
$$C(T) \propto g_1^\perp(g_1^\parallel - g_2^\parallel + g_2^\perp)T \ln T \leftarrow \text{spin-full fermions} \quad (3\text{–}46)$$

In the next section we show that this result is consistent with a renormalization group analysis of the sine-Gordon model which arises when one uses the boson representation to treat fermions with spin and with backscattering interaction vertex $g_1$. In the bosonization description, the $g_1^\perp$ term (backscattering with antiparallel spins) in the hamiltonian gives rise to a $\cos \phi$ term in spin part of the action. We will show that this sine-Gordon term leads to a nonanalytic ($T \ln T$), temperature dependence in the specific heat, with the same coefficient $(g_1^\perp g_1^\parallel)$ we predicted using the diagrammatic analysis.

### 3.2.4 Specific Heat from the Sine-Gordon Model

When each component of the fermionic operator (now fermions have spin) is bosonized, the hamiltonian separates into charge part and spin parts [82],

$$H = H_\rho + H_\sigma, \quad (3\text{–}47)$$

where

$$H_\rho = \frac{1}{2\pi} \int dx \left[ \frac{u_\rho}{K_\rho} (\nabla \phi_\rho)^2 + u_\rho K_\rho (\nabla \theta_\rho)^2 \right], \quad (3\text{–}48)$$
$$H_\sigma = \frac{1}{2\pi} \int dx \left[ \frac{u_\sigma}{K_\sigma} (\nabla \phi_\sigma)^2 + u_\sigma K_\sigma (\nabla \theta_\sigma)^2 \right] + \frac{2g_1^\perp}{(2\pi \alpha)^2} \int dx \cos \left( \sqrt{8\pi} \phi_\sigma \right), \quad (3\text{–}49)$$
where $\phi_\rho = (\phi_\uparrow + \phi_\downarrow) / \sqrt{2}$ and $\phi_\sigma = (\phi_\uparrow - \phi_\downarrow) / \sqrt{2}$ and a similar relation holds for the $\theta$ fields. We assume that $g_{4\parallel} = g_{4\perp}$, but $g_{2\parallel} \neq g_{2\perp}$, then $u_\sigma = v_F \sqrt{1 - (g_{1\parallel} - g_{2\parallel} + g_{2\perp})^2}$ and $K_\sigma = \sqrt{(1 + g_{1\parallel} - g_{2\parallel} + g_{2\perp}) / (1 - (g_{1\parallel} - g_{2\parallel} + g_{2\perp}))}$, where the interactions coefficients are made dimensionless $g \rightarrow \frac{g}{2\pi v_F}$. The spin part of the bosonic action is a sum of a Gaussian part ($S_0$) and a sine-Gordon part ($S_1$):

$$S = S_0 + S_1,$$

where

$$S_0 = \frac{1}{2K_\sigma} \int dx \int_0^\beta d\tau \left[ (\partial_\tau \phi_\sigma)^2 + (\partial_x \phi_\sigma)^2 \right], \quad (3\text{-}50)$$

$$S_1 = \frac{2g_{1\perp}}{(2\pi \alpha)^2} \int dx \int_0^\beta d\tau \cos \left( \sqrt{8\pi \phi_\sigma} \right). \quad (3\text{-}51)$$

We have set $u_\sigma = 1$, because it is 2nd order in the coupling constants, whereas $K_\sigma$, which is linear in the coupling constants is kept finite. Below, we drop the suffix $\sigma$ for the fields, as we will be solely considering the spin fields. Treating the sine-Gordon term perturbatively ($g_{1\perp} \ll 1$), one can evaluate corrections to the specific heat. Consider the Free-energy,

$$F = -T \ln Z = -T \ln \int D\phi \exp \left( -[S_0 + S_1] \right)$$

$$= -T \ln \int D\phi \exp \left( -S_0 \right) - T \ln \left( 1 + \frac{\int D\phi \exp \left( -S_0 \right) S_1^2}{2 \int D\phi \exp \left( -S_0 \right)} \right),$$

$$= F^0 + \delta F,$$

where we have expanded in $g_{1\perp}$ and kept terms up to second order, the first order term is zero (as $\langle S_1 \rangle = 0$). We Taylor expand the logarithm and obtain the correction to the free energy as

$$\delta F = -T \left[ \frac{1}{2} \frac{\int D\phi \exp \left( -S_0 \right) S_1^2}{\int D\phi \exp \left( -S_0 \right)} \right] = -T \langle S_1^2 \rangle$$

$$= -T \int dx \int dx' \int_0^\beta d\tau \int_0^\beta d\tau' A \left( |x - x'|, |\tau - \tau'| \right),$$
where,
\[
A (|x|, |\tau|) = \left( \frac{2g_{1\perp}}{2\pi\alpha} \right)^2 \langle \cos \sqrt{8\pi} \phi (x, \tau) \cos \sqrt{8\pi} \phi (0, 0) \rangle.
\]

To compute the above average, we will write the cosines as a sum of two exponentials and use the formula for the average of the product of the exponentials of Gaussian fields \[82],
\[
\langle e^{i\sum_j (A_j \phi(r_j) + B_j \theta(r_j))} \rangle = e^{-\frac{1}{2} \left( \sum_j (A_j \phi(r_j) + B_j \theta(r_j)) \right)^2}.
\]

We then get,
\[
A (|x|, |\tau|) = c g_{1\perp}^2 \left( \frac{1}{(1/\pi\alpha T)^2 \left[ \sinh^2 (\pi x T) + \sin^2 (\pi T \tau) \right]} \right)^{2K_{\sigma}} \tag{3–53}
\]

by using the finite temperature form of the \( \phi - \phi \) correlation function \[82],
\[
\langle [\phi (r) - \phi (0)]^2 \rangle = \frac{1}{4\pi} \ln \left( \frac{1}{\pi \alpha T} \right)^2 \left\{ \sinh^2 (\pi x T) + \sin^2 (\pi T \tau) \right\}. \tag{3–54}
\]

To evaluate the temperature dependence of the correction to the Free energy we will go to the relative and center of mass coordinate system for both \( \tau \) and \( x \), and scale out the temperature dependence by bringing the integral to a dimensionless form,
\[
\frac{\delta F}{L} = -2c \frac{g_{1\perp}^2}{\alpha^4} T \int d\bar{x} \int d\bar{\tau} \left( \frac{1}{(\beta/\pi\alpha)^2 \left[ \sinh^2 (\pi \bar{x}/\beta) + \sin^2 (\pi \bar{\tau}/\beta) \right]} \right)^{2K_{\sigma}} \tag{3–55}
\]

\[
\frac{\delta F}{L} \propto g_{1\perp}^2 T^2 T^{4(K_{\sigma} - 1)} = g_{1\perp}^2 T^2 \left( 1 + 4(g_{1\parallel} - g_{2\parallel} + g_{2\perp}) \ln T \right). \tag{3–56}
\]

where we have set \( K_{\sigma} - 1 = g_{1\parallel} - g_{2\parallel} + g_{2\perp} \), for \( g \ll 1 \). Now \( \delta C (T) = -T^{\frac{\partial^2 \delta F}{\partial T^2}} \).

Therefore there is a nonanalytic \( T \ln T \) term in the specific heat, with the same
coefficient \( (g_{1\perp}^2(g_{1\parallel} - g_{2\parallel} + g_{2\perp})) \) as obtained using the diagrammatic approach.

\[
C(T) \propto g_{1\perp}^2(g_{1\parallel} - g_{2\parallel} + g_{2\perp})T \ln T \tag{3–57}
\]

We see that the specific heat in 1D acquires nonanalytic temperature dependence starting at third order in interaction (unlike \( d = 2, 3 \) where they occur even at 2\textsuperscript{nd} order). This nonanalytic piece exists only for fermions with spin, because then the 1D hamiltonian with spin and backscattering interaction has a sine-Gordon term in addition to the gaussian term. This result was earlier obtained by Japaridze and Nersesyan \[49\] by an exact solution of the SU(2) Thirring model. In a recent work on this subject, Aleiner and Efetov \[48\] used a supersymmetric approach and obtained the nonanalytic corrections to the specific heat in a repulsive Fermi gas in all dimensions. However their work seem to suggest that the one-dimensional NAC to \( C(T) \), start at fourth order in perturbation theory whereas we just showed (using two different methods: diagrammatically as well as using bosonization) that the \( T \ln T \) nonanalyticity in 1D should occur at third order in interaction.

Next, we show that the singularity in the backscattering particle-hole channel causes a non-analyticity in the spin susceptibility already present at the second order in interaction just as in higher dimensions.

### 3.3 Spin Susceptibility

In the bosonized description of the one-dimensional interacting system, the uniform magnetic susceptibility is the response to:

\[
\hat{H} = -\hbar \int dx \frac{1}{2} [\rho_\uparrow(x) - \rho_\downarrow(x)] = \frac{\hbar}{\pi \sqrt{2}} \int dx \nabla \phi_\sigma(x),
\]

where \( h = g\mu_B H \), with \( H \) the magnetic field, \( \mu_B \) the Bohr magneton and \( g \) is the Lande factor. If we assume that \( g_{1\perp} = 0 \), then the spin Hamiltonian is quadratic (see Eq. 3–49 with \( g_{1\perp} = 0 \)) and the above field dependent term can be absorbed into the quadratic part by shifting the \( \phi_\sigma \) field by, \( \tilde{\phi}_\sigma = \phi_\sigma + hK_\sigma x/\sqrt{2}u_\sigma \). Then
the spin susceptibility is \(\chi_\sigma \sim \partial \langle (\rho_\uparrow - \rho_\downarrow) \rangle / \partial h\) is

\[
\chi_\sigma = \frac{K_\sigma}{2\pi u_\sigma}.
\]

(3–59)

A constant spin susceptibility with renormalized (by interactions) coefficients is "Fermi liquid" like. As we saw in the case of the specific heat, making \(g_{\perp}\) finite, led to a nonanalytic correction to the Fermi liquid form \((C_{FL}(T) \sim T)\) for the specific heat, in the same way we expect the spin susceptibility to acquire nonanalytic corrections which are proportional to \(g_{\perp}\). Indeed, we find that the singularity in the backscattering particle-hole channel (the dynamical-Kohn anomaly) is responsible for a non analyticity in the spin susceptibility, \(\delta \chi_\sigma \propto \ln \max (|Q|, |H|, |T|)\) at second order in interaction, and verify an earlier result of Dzyaloshinskii and Larkin [90]. The charge sector is still gaussian, hence charge susceptibility remains analytic.

We will evaluate the thermodynamic potential in a finite magnetic field, (at second order perturbation theory) and then obtain the spin susceptibility using the thermodynamic relation, \((\chi_\alpha(H) = -\partial^2 \Omega / \partial H^2)\). The free Green’s function in 1D (now in the presence of the magnetic field) is

\[
G_{0 \uparrow \downarrow}^\pm (p, i\varepsilon) = \frac{1}{i\varepsilon - \xi_{p\uparrow\downarrow}^\pm} = \frac{1}{i\varepsilon - (v_F (\pm p - p_F) \pm 2\mu_B H)}.
\]

(3–60)

The diagrams for the second order thermodynamic potential were considered before in Fig.3–3. Here fermions have spin therefore each of the diagrams of Fig. 3–3, will come in three varieties depending on whether the bubbles have parallel spins (either both bubbles have Green’s function with up spins, or both bubbles have Green’s function with down spins) or if the bubbles have antiparallel spins (one bubble has both spins up in its Green’s function and the other bubble has both spins down in its Green’s function). All these diagrams are shown in Fig.3–10. We will show that the forward scattering particle-hole polarization operators do not
have an explicit field dependence (they can only have an analytic magnetic field
dependence through the density of states). Therefore, all those thermodynamic
potential diagrams which have forward scattering polarization bubbles in them
for e.g., diagrams in Fig.3–10(a),(b),(c) and Fig.3–10(g),(h),(i) cannot give a
non-analytic contribution to the spin susceptibility.

Figure 3–10. Second order diagrams for the thermodynamic potential.
\[ \Pi_{++}^{\uparrow \uparrow} (q; i\omega) = \int \frac{dk}{2\pi} \int \frac{d\varepsilon}{2\pi} G_{\uparrow}^+ (k; i\varepsilon) G_{\uparrow}^+ (k + q; i(\varepsilon + \omega)), \]
\[ = \int \frac{dk}{2\pi} \int \frac{d\varepsilon}{2\pi} \left( \frac{1}{i\varepsilon - k + h/2} - \frac{1}{i(\varepsilon + \omega) - (k + q) + h/2} \right) \times \frac{1}{i\omega - q}, \]
\[ = \frac{1}{i\omega - q} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left( \Theta (h/2 - k) - \Theta (h/2 - k - q) \right) = \frac{q}{2\pi (i\omega - q)}, \]

where we have relabeled $4\mu_B H = h$ and set $v_F = 1$. $\Pi_{++}^{\downarrow \downarrow} (q; i\omega)$ can be obtained from the above result by letting $h \to -h$. Thus, $\Pi_{++}^{\downarrow \uparrow} (q; i\omega) = \Pi_{++}^{\uparrow \downarrow} (q; i\omega) = \Pi_{++} (q; i\omega)$. Similarly it can be shown that $\Pi_{--}^{\downarrow \uparrow} (q; i\omega) = \Pi_{--}^{\uparrow \downarrow} (q; i\omega) = \Pi_{--} (q; i\omega)$. Then the thermodynamic potential with forward scattering bubbles,

\[ \Omega/L = -\left( g_2^2 \sigma g_4^2 \right) \int \frac{dq}{2\pi} \int \frac{d\omega}{2\pi} \coth \left( \frac{\omega}{2T} \right) \text{Im} \left( \Pi_{++}^{\uparrow \downarrow \uparrow \downarrow} (q, \omega) \Pi_{--}^{\downarrow \uparrow \downarrow \uparrow} (q, \omega) \right) \]
\[ \implies \text{only analytic magnetic field dependence, through density of states} \]

Therefore there is no nonanalytic contribution to the spin susceptibility from diagrams in 3–10(a),(b),(c),(g),(h),(i). The backscattering polarization operator has an explicit non analytic field dependence.

\[ \Pi_{2kF}^{\downarrow \downarrow} (q, i\omega) = \int \frac{dk}{2\pi} \int \frac{d\varepsilon}{2\pi} G_{\downarrow}^\dagger (k, i\varepsilon) G_{\downarrow}^\dagger (k + q; i(\varepsilon + \omega)), \]
\[ = \frac{1}{4\pi} \ln \left| \frac{(q + h - i\omega)(q + h + i\omega)}{(2\lambda)^2} \right|. \quad (3–61) \]

Hence we expect all three backscattering thermodynamic potential diagrams (Fig.3–10(d),(e),(f)) to give a nonanalytic contribution to the spin susceptibility. However, it turns out, that only the thermodynamic potential diagram with antiparallel spins (Fig.3–10(f)) in the two backscattering particle-hole bubble will give a nonanalytic field dependence and the parallel spin diagrams (Fig.3–10(d),(e)) can again give an analytic contribution to the spin susceptibility.
The thermodynamic potential with parallel spins in the two backscattering bubbles is,

$$\Omega^{3-10(c)/L} = -g_1^2 \int dq \int d\omega \frac{d\omega}{2\pi} \coth(\omega/2T) \Im \left( \Pi_{2kF\downarrow\downarrow}^R (q, \omega) \right)^2$$

$$= \frac{g_1^2}{(4\pi)^2} \int_0^\infty dq \int_{-\infty}^{\infty} d\omega \coth(\omega/2T) \sgn(\omega)$$

$$\times \ln \left| \frac{\omega^2 - (q + h)^2}{(2\pi)^2} \right| \Theta \left( \omega^2 - (q + h)^2 \right),$$

\(3-62\)

Notice that in the momentum integration one can relabel the dummy variable, \(q + h = q'\), and then the field dependence drops out and therefore this diagram can at the most, make an analytic contribution. The same argument applies to the diagram where both backscattering bubbles have spin-up, which also cannot give a nonanalytic term in the spin susceptibility. Therefore the non-analyticity in the spin susceptibility, can only arises from a single diagram at second order; the one which has antiparallel spins in the two backscattering particle-hole bubbles, Fig. 3–10(f). This argument also explains why the remaining second order diagrams for the thermodynamic potential (single loop with two interaction lines), and the first order diagram (single loop with one interaction line), do not give a non analytic correction to the spin susceptibility; they cannot have antiparallel spins as the interaction line cannot flip the spin in the loop. The only nonanalytic contribution to the spin susceptibility at 2\(^{nd}\) order is from,

$$\Omega^{3-10(f)/L} = -\int dq \int d\omega \frac{d\omega}{2\pi} \coth(\omega/2T) \Im \left[ \Pi_{2kF\uparrow\downarrow} (q, \omega) \Pi_{2kF\uparrow\uparrow} (q, \omega) \right],$$

$$= \frac{(g_{1\downarrow})^2}{(4\pi)^2} \int dq \int_{-\infty}^{\infty} d\omega \coth(\omega/2T) \sgn(\omega)$$

$$\times \left[ \Theta(\omega^2 - (q + h)^2) \ln \left| \frac{\omega^2 - (q - h)^2}{(2\lambda)^2} \right| \right.$$

$$+ \Theta(\omega^2 - (q - h)^2) \ln \left| \frac{\omega^2 - (q + h)^2}{(2\lambda)^2} \right| \right).$$

\(3-63\)
Here relabeling the dummy momentum does not get rid of the field dependence, so there will be a finite nonanalytic contribution to \( \chi \). Performing the momentum integration we get,

\[
\Omega^{3-10(f)} = 4c \int_{0}^{\infty} d\omega \coth (\omega / 2T) \left( -4\omega + 2 \omega \ln \left| \frac{\omega^2 - h^2}{\Lambda} \right| + 2h \ln \left| \frac{\omega + h}{\omega - h} \right| \right) \tag{3-64}
\]

where \( c = \frac{(g_{1\perp})^2}{(4\pi)^2} \). The above form of the thermodynamic potential is only valid for \( h \gg T \). This is because there is a Kohn anomaly at \( q = 2k_F \), which can be regularized either by finite \( h \) (as done above) or by finite \( T \). In the limit \( T \gg h \), the bubble must be taken at finite \( T \). For the field dependence,

\[
\delta \Omega(h) = \Omega(h)^{3-10(f)} - \Omega(0) = 4c \int_{0}^{\infty} d\omega \coth (\omega / 2T) \left( 2 \omega \ln \left| \frac{\omega^2 - h^2}{\omega^2} \right| + 2h \ln \left| \frac{\omega + h}{\omega - h} \right| \right) \tag{3-65}
\]

Notice that the integral diverges logarithmically at the upper limit. We can cut it at \( \omega = E_F \) and do the rest of the calculation to log accuracy:

\[
\delta \Omega(h) = 8c h^2 \int_{0}^{E_F / h} dx \coth \left( \frac{xh}{2T} \right) \left( x \ln \left| \frac{x^2 - 1}{x^2} \right| + \ln \left| \frac{x + 1}{x - 1} \right| \right) \tag{3-66}
\]

Analyzing the above integral in the two limits a) \( h \gg T \) and b) \( h \ll T \), we find the leading contribution to the thermodynamic potential to be

\[
\delta \Omega(h) \sim c h^2 \ln \frac{E_F}{\max[h, T]} \tag{3-67}
\]

and

\[
\delta \chi \sim - (g_{1\perp})^2 \ln \frac{E_F}{\max[h, T]} \tag{3-68}
\]

This contribution arises solely from a single diagram the one in Fig. 3–10(f). The issue of a precise form of the function interpolating in between \( h \) and \( T \) under the log is outside the log accuracy. Thus we see that the spin susceptibility is a non-analytic function of \( h \) and \( T \) already at second order in interaction. The
nonanalytic momentum dependence is not manifest in the above method, (as one integrates over the momentum to get the thermodynamic potential). One can show that the spin susceptibility is a nonanalytic function of the bosonic momentum, magnetic field and temperature in 1D \([44, 90]\).

\[ \delta \chi_s \propto -g_1 \perp \ln \frac{E_F}{\max[v_FQ, h, T]} \cdot \]

Therefore the spin susceptibility increases as you increase the temperature or magnetic field in 1D.

### 3.4 Experiments

The specific heat nonanalyticity in higher dimensions has been observed experimentally. In 3D, the \(T^3 \ln T\) term in the specific heat shows up in various heavy fermion materials, e.g. UPt\(_3\) \([54]\) as well as bulk He\(^3\) \([55]\). A nonanalytic \(T^2\), term in the specific heat in 2D has been observed recently on monolayers of He\(^3\) adsorbed on solid substrate \([56]\). However, to the best of my knowledge, the \(T \ln T\) nonanalyticity in the one-dimensional specific heat has not been observed in experiments. The main reason for this could be the inherent difficulty associated with making specific heat measurements on real 1D systems for e.g., quantum wires and carbon nanotubes which are extremely small (mesoscopic) structures. One way to avoid this difficulty might be to measure the thermal expansion coefficient of a carbon nanotube. The Gruneisen law states that the ratio of the thermal expansion coefficient \(\alpha\) to the specific heat stays constant in the limit \(T \to 0\). At low temperatures, the specific heat is determined, mostly, by electrons, therefore \(\alpha/C_{el} = \text{const.}\). Measuring the thermal expansion coefficient of a carbon nanotube, one can try to detect the \(T \ln T\) behavior. A definitive test of our theory would be to see the polarization dependence of the \(T \ln T\) term (which should vanish for complete spin polarization). Graphite in the ultra-quantum limit (which clearly
shows signatures of 1D localization, chapter 2) might be a possible candidate for observing the $T \ln T$ term in $C(T)$.

\section{Conclusion}

We have shown that the next-to-leading terms in the specific heat and spin susceptibility in 1D are nonanalytic, in the same way as they are in higher dimensions ($D = 2, 3$). Thus, even though the low energy theory which describes a one-dimensional interacting system (Luttinger liquid theory) is different from the higher dimensional low energy theory (Fermi liquid), the sub-leading terms in the thermodynamic properties get nonanalytic corrections which arises from the same sources in all dimensions (singular response functions). 1D is similar to higher dimensions, at least in the context of nonanalytic corrections to thermodynamics. The only difference is that the non-analytic correction to the specific heat in 1D is present only for fermions with spin and occurs at 3\textsuperscript{rd} order in interaction: $C(T) \propto g_1^2 (g_1 - g_2 + g_2) T \ln T$ whereas in higher dimensions they occur even for spin-less fermions and start at 2\textsuperscript{nd} order. We also showed that non-analyticity in third order is natural in 1D, since in the sine-Gordon model-the correlation function of the cos $\phi$ term comes with two factors of the interaction, the third factor occurs by expanding the exponent to leading (first) order. The non-analyticity in the spin susceptibility occurs due to the dynamic kohn anomaly (in all dimensions), and can be seen at second order in interaction: $\delta \chi_s \propto -g_1^2 \ln (E_F / \max [v_F Q, h, T])$. 


CHAPTER 4
SPIN SUSCEPTIBILITY NEAR A FERROMAGNETIC QUANTUM CRITICAL POINT IN ITINERANT TWO AND THREE DIMENSIONAL SYSTEMS

The Landau Fermi liquid (FL) theory states that the low-energy properties of an interacting fermionic system are determined by states in the vicinity of the Fermi surface, and are similar to that of an ideal Fermi gas. At the lowest temperatures, when the decay of quasiparticles can be neglected, the specific heat $C(T)$, scales linearly with $T$ and spin susceptibility $\chi_s(T)$, approaches a constant value, as they do in a Fermi gas, the only difference being the renormalizations of the effective mass and $g$ factor \[18\]. However, this low temperature limit of the FL theory, considered by Landau, cannot tell whether the sub-leading terms in $T$ are analytic or not, and whether they come only from low-energy states (and are therefore described by the FL theory) or from the states far away from the Fermi surface.

For noninteracting fermions, the sub-leading terms in $C(T)/T$ and $\chi_s(T)$ scale as $T^2$ (come from Sommerfeld expansion) and come from high-energy states. However, it was found back in the 1960s that in 3D systems, the leading correction to $C(T)/T$ due to interaction with either phonons \[37\] or paramagnons \[38\] is nonanalytic in $T$ and comes from the states in the vicinity of the Fermi surface.

The same result was later shown to hold for the electron-electron interactions \[36, 40, 47, 48\]. More recently, it was shown by various groups \[41–46, 80\] that the temperature dependence of $C(T)/T$ is also nonanalytic in 2D and starts with a linear-in-$T$ term. Furthermore, it was shown in Ref. \[44, 45\] that the nonanalytic terms in the specific heat in 2D occurs exclusively from one dimensional scattering processes (where the incoming fermion momenta are anti-parallel, and momentum
transfers are either 0 or \(2k_F\) and, for a generic Fermi-liquid, the coefficient in front of the nonanalytic \((T)\) correction, is expressed in terms of the spin and charge components of the scattering amplitude at the scattering angle \(\theta = \pi\) (backscattering amplitude). However in 3\(D\), both 1\(D\) and non-1\(D\) scattering processes contribute to the \(T^2 \ln T\), correction to \(C(T)/T\) for a generic FL\[47\], and the coefficient in front cannot be expressed solely in terms of the backscattering amplitude. There are contributions from the angular averages (and not just \(\theta = \pi\)) of the scattering amplitude (or Landau function). The nonanalytic corrections to the specific heat have been observed experimentally both in 3\(D\) (heavy fermion materials like UPt\(_3\)) as well as in 2\(D\) (monolayers of He\(^3\) adsorbed on a solid substrate).

In this chapter we will be studying the nonanalytic corrections to the spin susceptibility in both 2\(D\) and 3\(D\). Until recently, the prevailing opinion had been that the nonanalytic, \(T^3 \ln T\) term in the specific heat is not paralleled by a similar nonanalyticity in \(\chi_s\) in 3\(D\). Crucial evidence for this view was provided by the results of Carneiro and Pethick \[36\] and Beal-Monod et al., \[91\] who found that the leading term in the spin susceptibility scales as \(T^2\) in 3\(D\). However, in an important paper Belitz et al. \[42\] demonstrated that the apparent analytic temperature dependence of \(\chi_s\) may be misleading. They performed a perturbative calculation of the momentum dependent spin susceptibility \(\chi_s(Q, T = 0)\) at small \(Q\) and found a nonanalytic \(Q^2 \ln Q\) behavior. Later, it was found \[92\] that the magnetic-field dependence of a nonlinear spin susceptibility parallels the \(Q\) dependence, i.e., \(\chi_s(Q = 0, T = 0, H) \propto H^2 \ln |H|\) which negated an earlier result of Beal-Monod \[93\], who found only an analytic magnetic field dependence \(\chi_s \propto H^2\) in 3\(D\).

Nonanalyticity of the spin susceptibility was also found for 2\(D\) systems by Millis and Chitov \[43\] and, later, by Chubukov and Maslov, \[44\], Galitski
et al. [46] and Betouras et al. [92]. These authors showed, (using second order perturbation theory), that $\chi_s(T, Q, H)$ scales linearly with the largest out of the three parameters (in proper units). Furthermore Chubukov and Maslov [44, 45] and Galitski et al. [46] have shown that in 2D, the nonanalytic term in $\chi_s$ can be solely expressed in terms of the backscattering amplitude. We are going show that this result is valid only if the backscattering amplitude is small. In general the nonanalytic correction to the spin susceptibility in 2D acquire contributions from the angular averages of the scattering amplitude (and not just $\theta = \pi$, as would have been the case for backscattering amplitude) (section 4.3).

Therefore, there is an important difference between the nonanalytic corrections to the specific heat and the spin susceptibility in 2D. The former solely comes from 1D scattering, expressed in terms of backscattering amplitude whereas the latter gets contributions from both 1D as well as non-1D scattering.

As it was mentioned in the Introduction, the sign of the nonanalytic dependence of $\chi_s(H, Q)$ is important in understanding the nature of the phase transition to the ferromagnetic state. All the known results for the nonanalytic dependence (both in $D = 2, 3$) gives an increase of $\chi_s(Q, H)$ with $Q, H$ which points towards a metamagnetic (first order) transition or ordering at finite $Q$. These results were for the second order in interaction. We show (in section 4.1 and 4.2) that the third order in interaction gives a decrease of $\chi_s(H)$ with $H$ (opposite to 2nd order) which favors, Hertz’s second order phase transition picture. However these signs oscillate at every order in perturbation and in general, it is impossible to determine the order (first or second) of the phase transition from the sign of a few low order in perturbation theory. To resolve this issue, we calculate $\chi_s(H)$ at the critical point using the spin-Fermion model and show that it is of the metamagnetic sign. Even experimentally, the situation is not clear, The ferromagnetic metallic alloys with low Curie Temperatures (“weak ferromagnets”)
do seem to show a variety of behaviors: in some of them, the QCP is of the first order, e.g., MnSi [94] and UGe$_2$, whereas other, e.g., Ni$_x$Pd$_{1-x}$ [95] show a second order transition to the lowest temperature measured.

In order to keep this discussion focused we will only talk about the nonanalytic magnetic-field dependence $\chi_s(H)$ in $D = 2, 3$. This chapter is organized as follows. We will obtain the nonanalytic corrections to $\chi_s(H)$, in 2D in section 4.1, at both second and third order in perturbation theory. Section 4.2 is devoted to the nonanalytic corrections in $\chi_s(H)$, in 3D. In section 4.3, we obtain $\chi_s(H)$ for a generic Fermi liquid in 2D, and show that it cannot be expressed only in terms of the backscattering amplitude. The same argument can be extended to show the importance of any- angle scattering in 3D (and not just $\theta = \pi$); however, due to lack of space we do not present the 3D calculation here. In section 4.4, we analyze the behavior of the nonanalytic terms in the spin susceptibility near the quantum critical point using the low energy effective spin-fermion model. We conclude in section 4.5.

### 4.1 Spin Susceptibility $\chi_s(H)$, in 2D

We consider the thermodynamic potential in a weak magnetic field in this section, perturbatively to second and third order in interaction, which we take to be momentum-independent. We obtain the spin susceptibility using the thermodynamic relation

$$\chi_s(H) = -\frac{\partial^2 \Xi(H)}{\partial H^2}. \quad (4-1)$$

We will be only interested in the spin effect of the magnetic field, but not the orbital effect. A magnetic field splits the Fermi surfaces for fermions with spins parallel and anti-parallel to the field. We will see that this splitting does not affect the $\xi \overline{q}$ nonanalyticity (see Introduction, section 1.2) of the polarization bubble at small $q$, if a particle and hole have the same spins (in this case a magnetic field just
shifts the chemical potential which can at most give an analytic $H^2$ contribution.
However, it has a nontrivial effect on a bubble composed of a particle and hole of opposite spins, $\Pi_{\uparrow\downarrow}(q, i\omega, H)$. At small momentum transfer ($q \sim 0$), part of the particle-hole polarization operator has an explicit magnetic field dependence only if the spins are antiparallel. To see this, first consider the case of parallel spins

$$
\Pi_{\uparrow\uparrow}(q, i\omega) = \int \frac{d^2 k}{(2\pi)^2} \int d\varepsilon G_{\uparrow}(\vec{k} + \vec{q}; i(\varepsilon + \omega))G_{\uparrow}(\vec{k}; i\varepsilon),
$$

$$
= \nu_F \int d\xi_k \int \frac{d\theta}{2\pi} \frac{1}{i\omega - \xi_{\uparrow+k+q} + \xi_{\uparrow+k}} \int \frac{d\varepsilon}{2\pi} \left[ \frac{1}{i\varepsilon - \xi_{\uparrow+k} + h/2} - \frac{1}{i(\varepsilon + \omega) - \xi_{\uparrow+k+q} + h/2} \right],
$$

$$
= \nu_F \int d\xi_k \int \frac{d\theta}{2\pi} \frac{1}{i\omega - \xi_{\uparrow+k+q} + \xi_{\uparrow+k}} (n_F(\xi_{\uparrow+k} - h/2) - n_F(\xi_{\uparrow+k+q} - h/2)),
$$

where we have labeled $2\mu_B H = h/2$, where $\mu_B$ is the Bohr magneton, and

$\nu_F = m/2\pi$ is the DOS in 2D. Expanding the fermionic dispersion near $q = 0$ as

$\xi_{\uparrow+k+q} = \xi_{\uparrow+k} + v_F q \cos \theta$, where $\theta$ is the angle between $k$ and $q$, Taylor expanding the Fermi functions, and using the zero temperature property that $(-\partial n_F / \partial \xi_k) = \delta(\xi_k)$ we get,

$$
\Pi_{\uparrow\uparrow}(q, i\omega) = \nu_F \int d\xi_k \int \frac{d\theta}{2\pi} \frac{\delta(\xi_k)}{i\omega - v_F q \cos \theta} \frac{v_F q \cos \theta + h/2 - h/2}{i\omega - v_F q \cos \theta},
$$

$$
= \nu_F \left[ -1 + \frac{\omega}{\sqrt{\omega^2 + (v_F q)^2}} \right].
$$

(4-2)

To arrive at the last expression we perform a standard contour integration. We see that the field dependence drops out. Similar analysis shows that $\Pi_{\downarrow\downarrow}(q, i\omega)$ is also given by Eq.4-2. Therefore all those thermodynamic potential diagrams which contains the $q \sim 0$ particle-hole bubbles with parallel spins do not give any nonanalytic field dependence in the spin susceptibility. Let us now consider these bubbles for anti-parallel spins .

$$
\Pi_{\uparrow\downarrow}(q, i\omega) = \int \frac{d^2 k}{(2\pi)^2} \int d\varepsilon G_{\uparrow}(\vec{k} + \vec{q}; i(\varepsilon + \omega))G_{\downarrow}(\vec{k}; i\varepsilon),
$$

$$
= \nu_F \int d\xi_k \int \frac{d\theta}{2\pi} \frac{n_F(\xi_{\downarrow+k} + h/2) - n_F(\xi_{\uparrow+k+q} - h/2)}{i\omega - \xi_{\uparrow+k+q} + \xi_{\downarrow+k} + h}.
$$
We now get an explicit field dependence. Following the same steps used to obtain Eq. 4–2, we get (at $T = 0$)

$$\Pi_{\uparrow \downarrow}(q, i\omega) = \nu_F \int_0^{2\pi} d\theta \frac{(-h + v_F q \cos \theta)}{2\pi i\omega - v_F q \cos \theta + h},$$

$$= \nu_F \left[ -1 + \frac{\omega}{\sqrt{(\omega - ih)^2 + (v_F q)^2}} \right] \quad (4–3)$$

We will use this response function to evaluate the thermodynamic potential at second and third order in interaction.

Consider the second order diagram for the thermodynamic potential shown in Fig.4–1. In this diagram one pairs a Green’s function from the top bubble with a Green’s function in the lower bubble to form $\Pi_{\uparrow \downarrow}(q, i\omega)$.

$$\Xi(H) = -\frac{g^2}{2} \int \frac{d^2 \mathbf{q}}{(2\pi)^2} T \sum_{\omega} [\Pi_{\uparrow \downarrow}(q, i\omega)]^2$$

Since we are only interested in the nonanalytic field dependence, we evaluate this diagram at zero temperature and in Matsubara frequencies. The nonanalytic part of the thermodynamic potential is

$$\Xi(H) = -\frac{g^2}{2(2\pi)^2} \int_{-E_F}^{E_F} d\omega \int_0^{k_F} dq q \frac{\omega^2}{(qv_F)^2 + (\omega - ih)^2},$$

$$= +\frac{(gv_F^2)^2}{2(2\pi)^2 v_F^2} \int_0^{E_F} d\omega \omega^2 \ln[\frac{(\omega^2 + h^2)/(E_F)^2}{\omega^2/(E_F)^2}],$$
where we have introduced ultraviolet cut-offs to regularize the formally divergent integrals. The nonanalytic terms arise from the lower limits of the integrals and are cut-off independent.

\[ \Xi(H) = B|H|^3, \quad (4-4) \]

where

\[ B = \frac{-16(g\nu_F)^2\mu_B^3}{3(2\pi)\nu_F^2} \]

is a cut-off independent pre-factor. The nonanalytic correction to the spin susceptibility in 2D at second order

\[ \chi_s(H) = |H|(\frac{+32(g\nu_F)^2\mu_B^3}{(2\pi)\nu_F^2}) . \quad (4-5) \]

We see that \( \chi_s \) increases as a function of the field. Now consider the third order correction shown in Fig. 4–2.

Figure 4–2. Particle-hole type third order diagram for the thermodynamic potential.

\[ \Xi(H) = \frac{g^3}{3} \int \frac{d^2\vec{q}}{(2\pi)^2} T \sum_{i\omega} [\Pi_{11}(q, i\omega)]^3 \]
Keeping the nonanalytic part of the response function (Eq. 4-3) and restricting our calculations to zero temperature, we obtain

\[
\Xi(H) = \frac{(g\nu_F)^3}{3(2\pi)^2v_F^2}\int_0^{E_F} d\omega \omega^3 \int_0^{E_F} dx \frac{1}{(x^2 + (\omega - i\hbar)^2)^{3/2}}
\]

\[
= \frac{(g\nu_F)^3}{3(2\pi)^2v_F^2} \int_{-E_F}^{E_F} d\omega \frac{\omega^3}{\omega - i\hbar} = + \frac{(g\nu_F)^3}{6(2\pi)v_F^2} |h|^3 + \ldots
\]  \hspace{1cm} (4-6)

where \ldots stand for analytic terms in \( h \). The nonanalytic correction to the spin susceptibility (at 3rd order) is,

\[
\chi_s(H) = -|H|(\frac{64(g\nu_F)^3\mu_B^3}{(2\pi)v_F^2}).
\]  \hspace{1cm} (4-7)

Here the spin susceptibility decreases, as the field is increased opposite to the behavior at second order. However at every higher order in interaction one gets an independent nonanalytic \( |H|^3 \) terms in the thermodynamic potential. Therefore one cannot predict the nature of the phase transition by looking at first few orders in perturbation theory. Since the nonanalytic contribution occurs for \( qv_F \sim \omega \), (see momentum integration at both second at third order), they come from any angle scattering events, (in order to have 1D scattering acts, the necessary condition was \( qv_F \gg \omega \), i.e., deep inside the particle-hole continuum).

### 4.2 Spin Susceptibility \( \chi_s(H) \), in 3D

As in 2D, nonanalytic correction to \( \chi_s(H) \) is obtained from the small momentum transfer \( (q \sim 0) \) bubbles in the thermodynamic potential, only if the bubbles have antiparallel spins.

\[
\Pi_{\uparrow\downarrow}(q,i\omega) = \int \frac{d^3k}{(2\pi)^3} \int d\varepsilon G_\uparrow(\vec{k} + \vec{q}; i(\varepsilon + \omega))G_\downarrow(\vec{k}; i\varepsilon),
\]

\[
= \nu_F \int d\xi_k \int_{-1}^1 d(\cos \theta) \frac{n_F(\xi_{\vec{k}} + h/2) - n_F(\xi_{\vec{k} + \vec{q}} - h/2)}{2i\omega - \xi_{\vec{k} + \vec{q}} + \xi_{\vec{k}} + h}.
\]  \hspace{1cm} (4-8)
Performing the same manipulations as in 2D case, we get for the particle-hole bubble in 3D at $T = 0$,

$$
\Pi_{11}(q, i\omega) = \nu_F \int_{-1}^{1} \frac{d(cos \theta)}{2} \frac{(-h + v_F q \cos \theta)}{i\omega - v_F q \cos \theta + h},
$$

$$
= \nu_F \left[ -1 + \frac{i\omega}{2v_F q} \ln \left( \frac{i\omega + v_F q + h}{i\omega - v_F q + h} \right) \right].
$$

(4–10)

Using this form of the bubble we get a nonanalytic contribution for the thermodynamic potential at every order in interaction, starting at second order (see Fig. 4–1).

$$
\Xi(H) = -\frac{g^2}{2} \int \frac{d^3q}{(2\pi)^3} T \sum_{i\omega} \left[ \Pi_{11}(q, i\omega) \right]^2
$$

$$
\Xi(H) = -\frac{(g\nu_F)^2}{(2\pi v_F)^3} \int_{-E_F}^{E_F} d\omega \int_{0}^{E_F} dq q \int_{-1}^{1} dy \left[ -1 + \frac{i\omega}{2q} \ln \left( \frac{i\omega + q + h}{i\omega - q + h} \right) \right]^2,
$$

where we have already rescaled $qv_F \to q$, therefore the $q$ integral has dimensions of energy. Let’s re-scale the dummy variables ($\frac{\omega}{q} = y, \frac{h}{q} = \tilde{H}$) and write the integrand in a dimensionless form

$$
\Xi(H) = -\frac{(g\nu_F)^2}{(2\pi v_F)^3} \int_{0}^{E_F} dq q \int_{-1}^{1} dy \left[ -1 + \frac{iy}{2} \ln \left( \frac{iy + \tilde{H} + 1}{iy + \tilde{H} - 1} \right) \right]^2,
$$

Since we are looking for a $H^4 \ln H$ term (resulting in a $H^2 \ln H$ term in $\chi_s(H)$) we can Taylor expand the integrand, for small $\tilde{H}$ ($\tilde{H} \ll y \sim 1$) up to fourth order in $\tilde{H}$.

$$
\Xi(H) = A \int_{0}^{E_F} dq q^3 \tilde{H}^4 \int_{-\infty}^{\infty} dy (6y^5 \tan^{-1}(1/y) + 3y^4 - 6y^3 \tan^{-1}(1/y) + 4y^2),
$$

(4–11)

The $y$ integral converges, so that we can extend the integration limits to infinity, upon which it gives a number. The remaining $q$ integral is log-divergent. As we had performed an expansion in $\tilde{H} = h/q \ll 1$, it is legitimate to cut the divergent integral at $q = h$. To logarithmic accuracy the arbitrariness in choosing the
the numerical prefactor in the cut-off does not affect the result.

\[
\Xi(H) = \frac{2\pi A}{12} \int_{-\hbar}^{\hbar} dq^2 \frac{h^4}{q^4} \ln \frac{E_F}{|h|},
\]  

(4–12)

where \( A = -(g\nu_F)^2/(2\pi v_F)^3 \). The non-analytic correction to the spin susceptibility is,

\[
\chi_s(H) = \frac{(g\nu_F)^2(4\mu_B)^4}{(2\pi^2)^3} H^2 \ln \frac{4\mu_B |H|}{E_F}.
\]  

(4–13)

Following the above procedure one can evaluate the third order thermodynamic potential (see Fig. 4–2),

\[
\Xi(H) = + \frac{2(g\nu_F)^3}{3(2\pi v_F)^3} \int_{-E_F}^{E_F} d\omega \int_{0}^{E_F} dq^2 \left[ -1 + \frac{i\omega}{2q} \ln \left( \frac{i\omega + q + h}{i\omega - q + h} \right) \right]^3,
\]

Performing the same steps as for the second order we find the non-analytic correction to the spin susceptibility at third order is

\[
\chi_s(H) = \frac{(2\pi)(\pi^2 - 8)(4\mu_B)^4 (g\nu_F)^3}{4(2\pi v_F)^3} H^2 \ln \frac{4\mu_B |H|}{E_F}.
\]  

(4–14)

Note that the third order sign is opposite to the second order sign. However, at every higher order 4th, 5th, there will be an independent nonanalytic contribution whose sign will oscillate and thus one cannot definitely predict the sign of the nonanalytic term by looking at few low orders in perturbation theory.

Furthermore in real systems interactions are not weak and one cannot terminate the perturbation series expansion to the lowest orders. To circumvent this inherent problem with perturbative calculations and to make predictions for realistic systems (e.g., He\(^3\)), we obtain the nonanalytic field dependence of the spin susceptibility for a generic Fermi liquid in 2D.

### 4.3 Spin Susceptibility for a Fermi Liquid in 2D

The Luttinger-Ward expression for \( \Xi \) reads [89]

\[
\Xi = -Tr \ln[G_0^{-1} - \Sigma] + \Sigma G + \Xi_{\text{skel}}
\]

(4–15)
with $\Sigma$ being the exact self-energy, and $\Xi_{\text{skeel}}$ is the set of skeleton diagrams for all interaction corrections to $\Xi$ that are not accounted for by the first two terms. The trace is taken over space, time, and spin variables. The skeleton diagram for the thermodynamic potential is shown in Fig. 4–3, where the bare interaction lines $U(q)$, are replaced with fully dressed vertices, $\Gamma(p - k)$. The vertex $\Gamma(p - k)$, has an expansion in terms of particle-hole bubbles [18], which to lowest order gives a diagram which is the paramagnon diagram (at 2nd order, see Fig. 4–1), except with momentum dependent interaction lines, the analytic expression for which is

$$
\Xi(H) = T \sum_{i\omega} \int \frac{d^2q}{(2\pi)^2} T \sum_{i\Omega} \int \frac{d^2k}{(2\pi)^2} T \sum_{i\varepsilon} \int \frac{d^2p}{(2\pi)^2} G_\uparrow(\vec{p} + \vec{q}, i(\varepsilon + \omega)) \nonumber $$

$$
G_\downarrow(\vec{k}, i\Omega) G_\uparrow(\vec{p}, i\varepsilon) \Gamma(\vec{p} - \vec{k})
$$

(4–16)

Figure 4–3. The skeleton diagram for the thermodynamic potential.

A next-to-leading order expansion of the vertex, gives a diagram which is the third order paramagnon diagram, again with momentum dependent interaction lines.

$$
\Xi(H) = T \sum_{i\omega} \int \frac{d^2q}{(2\pi)^2} T \sum_{i\Omega} \int \frac{d^2k}{(2\pi)^2} T \sum_{i\varepsilon} \int \frac{d^2p}{(2\pi)^2} T \sum_{i_1\varepsilon_1} \int \frac{d^2k_1}{(2\pi)^2} \nonumber $$

$$
G_\uparrow(\vec{p} + \vec{q}, i(\varepsilon + \omega)) G_\uparrow(\vec{k} + \vec{q}, i(\Omega + \omega)) G_\downarrow(\vec{k}_1 + \vec{q}, i(\varepsilon_1 + \omega)) \nonumber $$

$$
G_\uparrow(\vec{k}_1, i\varepsilon_1) G_\uparrow(\vec{k}, i\Omega) G_\uparrow(\vec{p}, i\varepsilon) \Gamma(\vec{p} - \vec{k}) \Gamma(\vec{k}_1 - \vec{p}) \Gamma(\vec{k} - \vec{k}_1)
$$

(4–17)
Below we evaluate the spin susceptibility arising from these lowest order expansion of the skeleton diagrams. For the second order diagram given by Eq. 4–16, we get

\[ \Xi(H) = (\nu_F)^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{dq}{2\pi} \int_{0}^{2\pi} \frac{d\theta_p}{2\pi} \left[ 1 + \frac{i\omega}{v_Fq} \right] \]

\[ \times \int_{0}^{2\pi} \frac{d\theta_k}{2\pi} \left[ 1 + \frac{i\omega}{v_Fq} \right] \Gamma^2(p_F(\theta_k - \theta_p)) \]

Now we will expand the angle dependent vertex in a Fourier series,

\[ \Gamma(\theta_k - \theta_p) = \sum_{n=-\infty}^{\infty} \Gamma_n e^{i n(\theta_k - \theta_p)}. \]

We will focus on the nonanalytic part of the thermodynamic potential, so we drop the constant term in the brackets, (the factors of 1) which do not lead to the nonanalyticity. Also, for convenience we will label \( i\omega/v_Fq = a \) and \( \omega + ih/v_Fq = b \).

Then the nonanalytic part of the thermodynamic potential becomes,

\[ \Xi(H) = (\nu_F)^2 \int_{0}^{\infty} \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{dq}{2\pi} (a)^2 \sum_{n,m=-\infty}^{\infty} \Gamma_n \Gamma_m A_{nm} \]

\[ + (\nu_F)^2 \int_{-\infty}^{0} \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{dq}{2\pi} (a)^2 \sum_{n,m=-\infty}^{\infty} \Gamma_n \Gamma_m B_{nm}, \]

where

\[ A_{nm} = \int_{0}^{2\pi} \frac{d\theta_p}{2\pi} \frac{e^{-i\theta_p(n+m)}}{\cos\theta_p - ib} \int_{0}^{2\pi} \frac{d\theta_k}{2\pi} \frac{e^{+i\theta_k(n+m)}}{\cos\theta_k - ib}, \text{ (for } \omega > 0) \]  \hspace{1cm} (4–18)

\[ B_{nm} = \int_{0}^{2\pi} \frac{d\theta_p}{2\pi} \frac{e^{-i\theta_p(n+m)}}{\cos\theta_p - ib} \int_{0}^{2\pi} \frac{d\theta_k}{2\pi} \frac{e^{+i\theta_k(n+m)}}{\cos\theta_k - ib}, \text{ (for } \omega < 0) \]  \hspace{1cm} (4–19)

Using the relation

\[ (\cos\theta_p - ib)^{-1} = \text{sgn}(b)i \int_{0}^{\infty} d\lambda e^{-i\lambda\text{sgn}(b)(\cos\theta_p - ib)} \]
Using the Bessel function property,
\[ \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-i \lambda \cos \theta + i (n+m) \theta} = J_{-n-m}(\lambda) e^{-i(n+m)3\pi/2}, \]
and \( J_{-n-m}(\lambda) = (-1)^{n+m} J_{n+m}(\lambda) \), we get
\[ A_{nm} = - \left[ \int_0^\infty d\lambda e^{-\lambda b} J_{n+m}(\lambda) \right]^2 (-1)^{n+m}, \]
\[ B_{nm} = - \left[ \int_0^\infty d\lambda e^{+\lambda b} J_{n+m}(-\lambda) \right]^2 (-1)^{n+m}, \]

Therefore the nonanalytic part of the Thermodynamic potential becomes
\[
\Xi(H) = -\left( \nu_F \right)^2 \int_0^\infty d\omega \int_0^\infty dq \left( \frac{v_F}{q} \right)^2 \sum_{n,m=\infty}^{\infty} \Gamma_n \Gamma_m \left[ \int_0^\infty d\lambda e^{-\lambda b} J_{n+m}(\lambda) \right]^2 (-1)^{n+m} \]
\[
= \Xi_1(H) + \Xi_2(H) \]

Let us first evaluate \( \Xi_1(H) \). It is convenient to split the sum over the integers \( n, m \) into two sums, one with \( n + m > 0 \) and the other with \( n + m < 0 \)
\[
\Xi_1(H) = \left( \frac{\nu_F}{2\pi} \right)^2 \int_0^\infty d\omega \int_0^\infty dq \left( \frac{v_F}{q} \right)^2 \left( \sum_{n,m=-\infty}^{\infty} C_{nm} \right)^2 + \sum_{n,m=-\infty}^{\infty} (C_{nm})^2 \Gamma_n \Gamma_m (-1)^{(n+m)}, \]
where \( C_{nm} \) are the integrations over the Bessel functions, which can yield two results depending on whether their arguments are positive or negative \([69]\),
\[
C_{nm} = \int_0^\infty d\lambda e^{-\lambda b} J_{m+n}(\lambda) = \left[ \Theta(m+n) \frac{(\sqrt{1+b^2} - b)^{m+n}}{\sqrt{1+b^2}} \right] + \left[ \Theta(-n-m)(-1)^{-n-m} \frac{(\sqrt{1+b^2} - b)^{-n-m}}{\sqrt{1+b^2}} \right] \]
We pull out the common denominator from both the sums which yields

\[ \Xi_1(H) = \frac{(\nu_F)^2}{2\pi} \int_0^\infty d\omega \omega^2 \int_0^\infty dq q \left( \frac{(\omega + ih)^2 + 1 - \left( \frac{\omega + ih}{v_F q} \right)^2}{v_F q} \right)^{2n+2m} \times \left[ \sum_{\substack{n,m = -\infty \to -n-m > 0}}^{\infty} (\omega + ih)^{2n+2m} \right] \Gamma_n \Gamma_m (-1)^{(n+m)} \]

Note that the integral is log divergent, therefore to log accuracy one can use a lower cut-off, \((\omega + ih)/v_F\) in the integral and everywhere inside the square brackets one can set, \(q \gg (\omega + ih)/v_F\). Then the terms inside the round brackets becomes equal to unity, and we get

\[ \Xi_1(H) = -\left( \frac{\nu_F}{2\pi v_F} \right)^2 \int_0^\infty d\omega \omega^2 \ln\left( \frac{\omega + ih}{E_F} \right) \Gamma^2_{BS} \] (4.21)

where the backscattering amplitude (scattering amplitude at the scattering angle \(\theta = \pi\)) is defined as, \(\Gamma_{BS} = \sum_n (-1)^n \Gamma_n\). Similarly one can show that,

\[ \Xi_2(H) = -\left( \frac{\nu_F}{2\pi v_F} \right)^2 \int_0^\infty d\omega \omega^2 \ln\left( \frac{\omega - ih}{E_F} \right) \Gamma^2_{BS} \]

The nonanalytic part of the net thermodynamic potential at second order is,

\[ \Xi(H) = -\left( \frac{\nu_F}{2\pi v_F} \right)^2 \Gamma^2_{BS} \int_0^\infty d\omega \omega^2 \ln\left( \frac{\omega^2 + h^2}{(E_F)^2} \right) = -\left( \frac{\nu_F}{2\pi v_F} \right)^2 \Gamma^2_{BS} \left( \frac{\pi}{6} \right) |4\mu_B H| \] (4.22)

where we have multiplied with an overall factor of \(-1/2\), which is the second order combinatorial coefficient. Therefore, the second order skeleton diagram for the spin susceptibility can be expressed solely in terms of the backscattering amplitude.
This result is similar to that for the specific heat \[45, 47\],

\[
\chi_s(H) = c\Gamma^2_{BS} \times |H| \quad (4-23)
\]

where \(c\) is a positive number. However, we show that the next-to-lowest order expansion of the vertex, in the skeleton diagram, acquires contributions which cannot be expressed in terms of the backscattering amplitude. This suggests the importance of non-one-dimensional scattering processes. The third order skeleton diagram reads

\[
\Xi(H) = T \sum_{\omega} \int \frac{d^2 \vec{q}}{(2\pi)^2} T \sum_{i \Omega} \int \frac{d^2 \vec{k}}{(2\pi)^2} T \sum_{i \varepsilon} \int \frac{d^2 \vec{p}}{(2\pi)^2} T \sum_{i \varepsilon_1} \int \frac{d^2 \vec{k}_1}{(2\pi)^2}
\]

\[
G_1(\vec{p} + \vec{q}, i(\varepsilon + \omega))G_1(\vec{k} + \vec{q}, i(\Omega + \omega))G_1(\vec{k}_1 + \vec{q}, i(\varepsilon_1 + \omega)) \times
\]

\[
G_1(\vec{k}_1, i\varepsilon_1)G_1(\vec{k}, i\Omega)G_1(\vec{p}, i\varepsilon)\Gamma(\vec{p} - \vec{k})\Gamma(\vec{k}_1 - \vec{p})\Gamma(\vec{k} - \vec{k}_1)
\]

The nonanalytic part at third order comes from

\[
\Xi(H) = -(\nu_F)^3 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{dq}{2\pi} \int_{0}^{2\pi} \frac{d\theta_p}{2\pi} \left( \frac{i\omega}{v_F q} \right) \int_{0}^{2\pi} \frac{d\theta_k}{2\pi} \left( \frac{i\omega}{v_F k} \right) \int_{0}^{2\pi} \frac{d\theta_{k_1}}{2\pi} \left( \frac{i\omega}{v_F k_1} \right)
\]

\[
\sum_{n,m,l=-\infty}^{\infty} \Gamma_n \Gamma_m \Gamma_l e^{in(\theta_p - \theta_k) + im(\theta_{k_1} - \theta_p) + il(\theta_k - \theta_{k_1})}
\]

The same manipulations which were performed at the second order leads

\[
\Xi(H) = \Xi_1(H) + \Xi_2(H), \quad (4-24)
\]

where,

\[
\Xi_1(H) = \frac{(\nu_F)^3}{(2\pi)^2} \int_{0}^{\infty} \frac{d\omega}{2\pi} \int_{0}^{\infty} \frac{dq}{2\pi} \int_{0}^{\infty} \frac{d\omega}{(v_F q)^2 + (\omega + ih)^2} \sum_{n,m,l=-\infty}^{\infty} \Gamma_n \Gamma_m \Gamma_l A_{nml}, \quad (4-25)
\]
where,
\[
A_{nm\ell} = \left[ \Theta(m - n)(\sqrt{1 + b^2} - b)^{m-n} + \Theta(n - m)(-1)^{n-m}(\sqrt{1 + b^2} - b)^{n-m} \right] \\
\times \left[ \Theta(n - \ell)(\sqrt{1 + b^2} - b)^{n-\ell} + \Theta(\ell - n)(-1)^{\ell-n}(\sqrt{1 + b^2} - b)^{\ell-n} \right] \\
\times \left[ \Theta(\ell - m)(\sqrt{1 + b^2} - b)^{\ell-m} + \Theta(m - \ell)(-1)^{m-\ell}(\sqrt{1 + b^2} - b)^{m-\ell} \right],
\]
with \( b = \omega + ih/v_Fq \), and
\[
\Xi_2(H) = \frac{(\nu_F)^3}{(2\pi)^2} \int_0^\infty d\omega \omega^3 \int_0^\infty dq q \int_0^\infty dq q \sum_{n,m,\ell = -\infty}^\infty \Gamma_n \Gamma_m \Gamma_\ell B_{nm\ell}, \quad (4-26)
\]
where,
\[
B_{nm\ell} = \left[ \Theta(m - n)(-1)^{n-m}(\sqrt{1 + (b_1)^2} - b_1)^{m-n} + \Theta(n - m)(\sqrt{1 + (b_1)^2} - b_1)^{n-m} \right] \\
\times \left[ \Theta(n - \ell)(-1)^{\ell-n}(\sqrt{1 + (b_1)^2} - b_1)^{n-\ell} + \Theta(\ell - n)(\sqrt{1 + (b_1)^2} - b_1)^{\ell-n} \right] \\
\times \left[ \Theta(\ell - m)(-1)^{m-\ell}(\sqrt{1 + (b_1)^2} - b_1)^{m-\ell} + \Theta(m - \ell)(\sqrt{1 + (b_1)^2} - b_1)^{m-\ell} \right],
\]
with \( b_1 = \omega - ih/v_Fq \). Notice that the momentum \( (q) \) integral in Eq.4-25 and Eq.4-26, has a power law singularity, so one cannot cut it at \( qv_F > \omega + ih \) and set \( qv_F \gg \omega + ih \) in \( A_{nm\ell} \) and \( B_{nm\ell} \), as we did in second order case where there was a log singularity. This is the essential difference between the second order and third order calculations which makes it extremely difficult to arrive at a closed form (for arbitrary \( n, m, l \)) at third order. We will obtain the coefficient in front of the \( |H| \) term in the spin susceptibility, for the few lowest harmonics \( 0 \leq n, m, l \leq 1 \) and show that it is not possible to write the third order result solely in terms of the backscattering amplitude. Several cases need to be considered separately. Case(a):
\[m = n = l = 0, \text{ which means } A_{nm} = 1 = B_{nm}\]

\[
\Xi_1(H) = \frac{(\nu_F \Gamma_0)^3}{(2\pi v_F)^2} \int_0^\infty d\omega \omega^3 \int_0^\infty dq \frac{dq}{\sqrt{(q^2 + (\omega + ih)^2)^{3/2}}}
\]

\[
= \frac{(\nu_F \Gamma_0)^3}{(2\pi v_F)^2} \int_0^\infty d\omega \frac{\omega^3}{\omega + ih}
\]

The result for \(\Xi_2(H)\) can be obtained from the above by letting \(h \rightarrow -h\). Adding the two contributions we get,

\[
\Xi^{000}(H) = \frac{(\nu_F \Gamma_0)^3}{(2\pi v_F)^2} \int_0^\infty d\omega \frac{2\omega^4}{\omega^2 + h^2} = \frac{(\nu_F \Gamma_0)^3}{(2\pi v_F)^2} \times \pi |4\mu_B H|^3, \quad (4-27)
\]

where the superscript denotes the harmonics.

Case(b): \(m = 1, n = 0, l = 0\). Then \(A_{100} = (-1)(\sqrt{1 + b^2} - b)^2\) and \(B_{100} = (-1)(\sqrt{1 + (b_1)^2} - b_1)^2\).

\[
\Xi_1(H) = -\frac{(\nu_F)^3 \Gamma_1 \Gamma_2^0}{(2\pi v_F)^2} \int_0^\infty d\omega \omega^3 \int_0^\infty dq \frac{dq}{q^2 [q^2 + (\omega + ih)^2]^{3/2}} \left(\sqrt{q^2 + (\omega + ih)^2} - (\omega + ih)^2\right)
\]

\[
= -\frac{(\nu_F)^3 \Gamma_1 \Gamma_2^0}{(2\pi v_F)^2} (2\ln(2) - 1) \int_0^\infty d\omega \frac{\omega^3}{\omega + ih}
\]

Similarly adding the contribution from \(\Xi_2(H)\), we get

\[
\Xi^{100}(H) = -\frac{(\nu_F)^3 \Gamma_1 \Gamma_2^0}{(2\pi v_F)^2} (2\ln(2) - 1) \int_0^\infty d\omega \frac{2\omega^4}{\omega^2 + h^2}
\]

\[
= -\frac{(\nu_F)^3 \Gamma_1 \Gamma_2^0}{(2\pi v_F)^2} (2\ln(2) - 1) \times \pi |4\mu_B H|^3
\]

Similarly the results for the case(c): \(m = 0, n = 1, l = 0\) and case(d): \(m = 0, n = 0, l = 1\) can be shown equal to the above form, so the net result for this harmonic needs to be multiplied by a factor of 3.

\[
\Xi^{100}(H) = -3\frac{(\nu_F)^3 \Gamma_1 \Gamma_2^0}{(2\pi v_F)^2} (2\ln(2) - 1) \times \pi |4\mu_B H|^3 \quad (4-28)
\]

Case(e): \(m = 1, n = 1, l = 0\). Then \(A_{110} = (-1)(\sqrt{1 + b^2} - b)^2\) and \(B_{110} = (-1)(\sqrt{1 + (b_1)^2} - b_1)^2\), which makes it identical to case(b), except now the
pre-factor is $\Gamma^2 \Gamma_0$. The extra factor of 3 arises due to the case(f): $m = 1, n = 0, l = 1$ and the case(g): $m = 0, n = 1, l = 1$ which gives the same contribution.

$$\Xi^{110}(H) = -\frac{3(\nu_F)^3 \Gamma_0 \Gamma^2_1}{(2\pi v_F)^2} (2 \ln(2) - 1) \times \pi |4 \mu_B H|^3.$$ \hspace{1cm} (4–29)

Case(h): $m = 1, n = 1, l = 1$, here $A_{111} = B_{111} = 1$. The integrals are identical to case(a) and

$$\Xi^{111}(H) = +\frac{(\nu_F \Gamma_1)^3}{(2\pi v_F)^2} \times \pi |4 \mu_B H|^3 \hspace{1cm} (4–30)$$

Adding the result for the first few harmonics we get for the spin susceptibility,

$$\chi_s(H) = - (\Gamma^3_0 - 3(2 \ln(2) - 1)\Gamma_1 \Gamma^2_0 - 3(2 \ln(2) - 1)\Gamma_0 \Gamma^2_1 + \Gamma^3_1) \times \frac{(4 \mu_B)^3 \pi (\nu_F)^3}{(2\pi v_F)^2} |H| \hspace{1cm} (4–31)$$

Note that the numerical factor of $2 \ln(2)$ (and also the signs) makes it impossible to represent the 3rd order result in terms of backscattering amplitude. If the third order result had an expansion in terms of $\Gamma^3_{BS}$, then our lowest harmonics results would have been $(\sum_n (-1)^n \Gamma_n)^3 = \Gamma^3_0 - 3\Gamma_0 \Gamma_1 + 3\Gamma_1 \Gamma_0 - \Gamma^3_1$ which is different from what we obtained.

Parameters $\Gamma_0$, $\Gamma_1$, can be estimated using the relation between the harmonics of the amplitudes and the Landau functions of a 2D Fermi liquid [47]

$$\Gamma_n = \frac{F_n}{1 + \frac{F_n}{2 - \delta_{n,0}}} \hspace{1cm} (4–32)$$

The Landau parameters can be measured from the renormalizations of the leading (analytic) terms in thermodynamic quantities. In bulk He$^3$,

$$F_0 \approx -0.7 \hspace{1cm} (4–33)$$

$$F_1 \approx -0.55 \hspace{1cm} (4–34)$$
in a wide interval of parameters [55]. Assuming that the Fermi-liquid parameters are the same in bulk He\(^3\) and in a 2D layer, we can estimate \(\Gamma_0\) and \(\Gamma_1\) to be

\[
\begin{align*}
\Gamma_0 &= -2.3 \quad (4-35) \\
\Gamma_1 &= -0.76 \quad (4-36) \\
\end{align*}
\]

In this approximation, the square of the backscattering amplitude

\[
\Gamma^2_{BS} = (\sum_n (-)^n \Gamma_n)^2 \approx 2.5 \quad (4-38)
\]

whereas the coefficient of the third order which we found is \(\Gamma^3 = 0.05\). Therefore, in this approximation the backscattering amplitude still dominates the result.

### 4.4 Spin Susceptibility near the Quantum Critical Point

Within the Spin-Fermion model, the thermodynamic potential is given by the Eliashberg formula [62],

\[
\Xi(H) = \Xi_{FG} + \frac{1}{2} \int \frac{d^Dq}{(2\pi)^D} \int \frac{d\Omega}{2\pi} \ln((\chi_{s,0})^{-1} + g^2 \Pi_{\pm}), \quad (4-39)
\]

where \(\Xi_{FG}\) is the thermodynamic potential for the Fermi gas, \(g\) is the spin-Fermion coupling constant, and \(\chi_{s,0}\) is the bare spin susceptibility, which is analytic at \(q \ll p_F\) and is given by the Ornstein-Zernicke form

\[
\chi_{s,0} = \frac{A}{\xi^{-2} + q^2}. \quad (4-40)
\]

Notice that \(A\xi^2\) is the \(q = 0\) susceptibility, which diverges at the second-order QCP. \(\Pi_{\pm}\) is the particle hole polarization operator, in a finite magnetic field or in a system with finite magnetization

\[
\Pi_{\pm}(q, i\Omega) = 2 \int \frac{d^Dk}{(2\pi)^D} \int \frac{d\omega}{2\pi} G_+(\vec{k} + \vec{q}, i(\omega + \Omega))G_-(-\vec{k}, i\omega), \quad (4-41)
\]
where \((G_\pm)\), are the dressed Green’s function containing the fermion self-energies. \(\chi_{s,0}\) is a phenomenological input of the theory. It is assumed that the high energy states (away from the Fermi surface) are already integrated out, and their effect is to drive the system to the vicinity of the QCP. The Spin-Fermion model provides an accurate description of the feedback from the low-energy states on the properties of the QCP.

One can perform the calculations either in a finite magnetic field, in which case \(G_\pm \to G_{\uparrow\downarrow}\), or one can work with a system with finite polarization in a zero magnetic field assuming spin up and spin-down electrons have different densities \(n_\pm\) and different chemical potentials \(\mu_\pm\)

\[
G_\pm(k, i\omega) = \frac{1}{i\omega - \xi_k - \Sigma(\omega) + \mu_\pm}. \tag{4–42}
\]

The fermion self-energies (see Fig.4–4) are calculated within the Eliashberg approximation, where one assumes a momentum independent self-energy and neglects the vertex corrections. The double line indicates the dressed Green’s function and the double wavy line indicates the spin fluctuation propagator, renormalized by the bosonic self-energy. It can be shown that the Eliashberg approximation works as long as \(\lambda \gg 1\) [62], where \(\lambda\) is the defines as follows,

\[
\Sigma(\omega) = -i\tilde{\Sigma}(\omega), \text{with}
\]

\[
\tilde{\Sigma}(\omega) = \lambda \omega, \tag{4–43}
\]

where \(\lambda\) (the mass renormalization factor, \(m^* \sim 1 - \lambda\)) takes different functional forms in the Fermi-liquid (far from the QCP) and the non-Fermi liquid (near the QCP) regimes, and also depending on whether one is in 2D or 3D. In 2D [62],

\[
\lambda = \alpha(\xi_{p_F}) \leftarrow \text{In the FL regime} \tag{4–44}
\]

\[
\lambda = \omega_0^{1/3}|\omega|^{-1/3} \leftarrow \text{In the non-FL regime} \tag{4–45}
\]
where \( \alpha \sim g^2 A/E_F \) is a small parameter, \( \xi \) is the correlation length which is assumed to be much larger than the interatomic distance and \( \omega_0 = \alpha^2 E_F \) is a constant. In 3D,

\[
\lambda = c \ln(p_F \xi) \leftarrow \text{In the FL regime,} \tag{4-46}
\]
\[
\lambda = c \ln \left( \frac{\epsilon_0}{|\omega|} \right) \leftarrow \text{In the non-FL regime,} \tag{4-47}
\]

where \( c \) is a small parameter, and \( \epsilon_0 \sim E_F \) is again a constant.

\[
\Pi_{\pm}(q, i\Omega) = \frac{2\nu_F}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega \int_{0}^{2\pi} d\theta \int_{-\infty}^{\infty} d\xi_k \left( \frac{1}{i(\omega + \Sigma(\omega)) - \xi_k + \mu_-} \right) \times \left( \frac{1}{i(\omega + \Omega + \Sigma(\omega + \Omega)) - \xi_k - v_F q \cos \theta + \mu_+} \right)
\]
In the non-perturbative regime ($\lambda \gg 1$) we have $\tilde{\Sigma}(\omega) \gg \omega$, so the Matsubara frequencies $i\omega$ and $i\omega + i\Omega$ in the Green’s function can be neglected compared to $\tilde{\Sigma}(\omega)$ and $\tilde{\Sigma}(\omega + \Omega)$. And performing a simple pole integration in $\xi_k$, we arrive at (for $\Omega > 0$)

$$\Pi_{\pm}(q, i\Omega) = 2\nu_F \int_{-\infty}^{\infty} d\omega \Theta(\tilde{\Sigma}(\omega + \Omega)) \Theta(-\tilde{\Sigma}(\omega)) \int_0^{2\pi} \frac{d\theta}{\cos \theta - \left(\frac{m + \Delta \Sigma}{\nu_F q}\right)},$$

where we have defined $m = \mu_+ - \mu_-$. The theta functions arise when we do the $\xi_k$-integration because the poles have to lie on opposite sides of the real axis. The self-energies have the property that $\tilde{\Sigma}(\omega) \sim \text{sgn}(\omega)$, so the theta functions cut-out a finite interval in the $\omega$ integration. In the above equation we have labeled $\Delta \Sigma = \tilde{\Sigma}(\omega + \Omega) - \tilde{\Sigma}(\omega)$. Performing the angular integration using a standard contour (and considering the case $\Omega < 0$), we get

$$\Pi_{\pm}(q, i\Omega) = 2\nu_F \int_{\text{max}[0,-\Omega]}^{\text{min}[-\Omega,0]} d\omega \frac{d\omega}{\sqrt{(\nu_F q)^2 + (\Delta \Sigma - im)^2}}.$$ (4–48)

Changing the sign of $\Omega$, is the same as $m \rightarrow -m$, which is equivalent to complex conjugation. Then the thermodynamic potential (Eq.4–39, non Fermi-gas part), becomes

$$\Xi(m) = 2B Re \int_0^{E_F} dq \int_0^{E_F} d\Omega \ln \left[\frac{q^2}{A} + 2g^2\nu_F \int_{-\Omega}^{0} d\omega \sqrt{(q)^2 + (\Delta \Sigma - im)^2}\right]$$ (4–49)

where $B = 1/2(2\pi v_F)^2$ is constant pre-factor and where we have scaled out the $v_F$ dependence by relabeling $v_F q \rightarrow q$. The first term inside the log comes from the Ornstein-Zernicke form of the bare susceptibility and we have set $\xi \rightarrow \infty$.

First, we analyze the nonanalytic part of the thermodynamic potential in the Fermi-liquid regime, where $\tilde{\Sigma}(\omega) = \lambda \omega$ with $\lambda = \alpha(\xi p_F)$. Then, $\Delta \Sigma = \lambda \Omega$, and the $\omega$ integral is trivial. Keeping just the dynamic part of the bubble we get

$$\Xi^{FL}(m) = 2B Re \int_0^{E_F} dq \int_0^{E_F} d\Omega \ln \left|\frac{g_1 \Omega}{\sqrt{q^2 + (\lambda \Omega - im)^2}}\right|,$$ (4–50)
where \( g_1 = 2g^2 \nu_F \). Re-scaling, \( \lambda \Omega = \Omega_1 \) we get

\[
\Xi^{FL}(m) = \frac{2B}{\lambda} \text{Re} \int_0^{E_F} d\Omega_1 \int_0^{E_F} dqq \ln \left[ \frac{\Omega_1}{\sqrt{q^2 + (\Omega_1 - im)^2}} \right]
\]

\[
= \frac{B}{\lambda} \text{Re} \int_0^{E_F} d\Omega_1 (\Omega_1 - im)^2 \ln \left| \frac{\Omega_1 - im}{\Omega_1} \right| = -\frac{\pi B}{6\lambda} |m|^3,
\]

where we have dropped the analytic in \( m \) terms. We see that the nonanalytic term in the thermodynamic potential comes with a negative pre-factor which indicates the possibility of a first order transition. The Free energy is given by,

\[
F(m) = \Xi(m) + \sum_{\sigma=\pm} n_{\sigma} \mu_\sigma.
\]

We will now perform a thermodynamic analysis and find the dependence on the magnetic field. The Gibbs free energy is

\[
G(m, H) = F(m) - mH,
\]

and

\[
F(m) = \frac{m^2}{2\xi^2} - \frac{c|m|^3}{3\alpha \xi} + \frac{bm^4}{4},
\]

where \( b \) and \( c \) are constants and where we have kept the next analytic \( (m^4) \) terms in \( F \), which stabilizes the phase. The units are chosen in such a way that \( F \) and \( m \) and dimensionless. The minimum of \( G(m, H) \) with respect to the magnetization gives the equation of state,

\[
\frac{\partial F}{\partial m} = \frac{m}{\xi^2} - \frac{cm|m|}{\alpha \xi} + bm^3 = H
\]

Solving this equation

\[
m = H\xi^2 + \frac{c}{\alpha} \xi^5 H|H| - b\xi^3 H^3 + \ldots
\]
Therefore the spin susceptibility is given by
\[ \chi_s(H) = \frac{\partial m}{\partial H} = \xi^2 + \frac{c}{\alpha} \xi^5 |H| - 3b \xi^8 H^2 + \ldots \]  
which diverges at the QCP as \( \xi \to \infty \).

Next we analyze the stability of the QCP, where the field dependence (of \( \chi_s \)), changes to \( |H|^{3/2} \). To analyze the behavior near the QCP we start from Eq. 4-49. Notice that if we use the non-FL form of the self-energy in 2D (\( \tilde{\Sigma}(\omega + \Omega) \sim \text{sgn}(\omega + \Omega)|\omega + \Omega|^{2/3} \)), in \( \Delta \Sigma \), we get the following scaling behavior
\[ \Omega^{2/3} \sim m, \]  
(4–58)

near the QCP. Also, since in the quantum critical regime \( \lambda, (\Delta \Sigma) \) is a function of \( \omega \), one can not do the \( \omega \) integration trivially. In Eq. 4–49, the nonanalytic field dependence arises from the dynamic bubble so that one can scale out the static part,
\[ \Xi(m) = 2 \operatorname{Re} \int_0^{E_F} dq \int_0^{E_F} d\Omega \ln \left[ 1 + \frac{2g^2 \nu_F A}{q^3} \int_{-\Omega}^0 \frac{d\omega}{\sqrt{1 + \left( \frac{(\Delta \Sigma - im)}{q^2} \right)^2}} \right] \]

We keep only the dynamic part, and expand the square root function inside the integral \((\Delta \Sigma - im)/q \ll 1\), we get
\[ \Xi(m) = 2 \operatorname{Re} \int_0^{E_F} dq \int_0^{E_F} d\Omega \ln \left[ \frac{\tilde{g}}{q^3} \int_{-\Omega}^0 d\omega (1 - \frac{1}{2} \left( \frac{\Delta \Sigma - im}{q} \right)^2) \right] \]  
(4–59)

where \( \tilde{g} = 2g^2 \nu_F A \),
\[ \Xi(m) = 2 \operatorname{Re} \int_0^{E_F} dq \int_0^{E_F} d\Omega \ln \left[ \frac{\tilde{g} \Omega}{q^3} - \frac{\tilde{g}}{2q^5} \int_{-\Omega}^0 d\omega (\Delta \Sigma - im)^2 \right] \]  
(4–60)

Once again we scale out the first term which does not depend on the magnetic field. Defining an average self-energy square, \( (\Sigma(\Omega, m))^2 = -1 \int_{-\Omega}^0 d\omega (\Delta \Sigma - \ldots \)
\( \frac{im^2}{2\Omega} \) we arrive at,

\[
\Xi(m) = 2BRe \int_0^{E_F} d\Omega \int_\delta^{E_F} dq q \ln[1 + \left(\frac{\Sigma}{q^2}\right)^2],
\]

\[
= -BRe \int_0^{E_F} d\Omega (\Sigma(\Omega, m))^2 \ln \left(\frac{\delta^2 + (\Sigma)^2}{(E_F)^2}\right),
\]  

(4–61)

where we will take the limit \( \delta \to 0 \) in the end. The average self-energy square is,

\[
2(\Sigma(\Omega, m))^2 = -\frac{1}{\Omega} \int_{-\Omega}^{\Omega} d\omega [(\omega_0)^{1/3} \text{sgn}(\omega + \Omega)(\omega + \Omega)^{2/3} - (\omega_0)^{1/3} \text{sgn}(\omega)|\omega|^{2/3} - im]^2
\]

\[
= -\Omega^{4/3}(\omega_0)^{2/3} \int_0^1 dx [(1 - x)^{2/3} + x^{2/3} - ia]^2
\]  

(4–62)

where \( a = m/(\Omega^{2/3}(\omega_0)^{1/3}) \). Substituting Eq.4–62, in Eq. 4–61, we perform a scaling analysis to get

\[
\Xi(m) \sim -|m|^{7/2} \implies \chi_s(H) \sim -|H|^{3/2},
\]  

(4–63)

We see that the Free-energy (thermodynamic potential) become negative, which signals an instability of the second order quantum critical point.

### 4.4.2 3D

In 3D we will show that the nonanalytic, \( H^4 \ln |H| \), term in the thermodynamic potential (see perturbative calculation) becomes \( \Xi(H) \sim \xi^8 H^4 \ln |\xi^2 H|/(\ln \xi) \) as you approach the QCP from the Fermi-liquid regime, but at the QCP the nonanalyticity becomes considerably weaker, \( \Xi(H) \sim |H|^4 \ln(\ln |H|) \). The polarization operator in 3D, (from Eq 4–41),

\[
\Pi_{\pm} = \frac{\nu}{(2\pi)} \int_{-\infty}^{\infty} d\omega \int_{-1}^{1} d\cos \theta \int_{-\infty}^{\infty} d\xi_k \left( \frac{1}{\xi_k - \mu_+ - i\Sigma(\omega)} \times \frac{1}{\xi_k + \nu F q \cos \theta - \mu_+ - i\Sigma(\omega + \Omega)} \right)
\]

\[
= \frac{\nu F i}{-\nu F q} \int_{\min[-\Omega,0]}^{\max[0,-\Omega]} d\omega \int_{-1}^{1} d(\cos \theta) \left( \frac{m+i\Delta \Sigma}{\nu F q} \right),
\]
where we have performed the same manipulations as in 2D. Performing the angular integration which gives a log, and integrating by parts we obtain for the leading term in the polarization operator, \((\Omega > 0)\),

\[
\Pi_{\pm}(q, i\Omega) = g^2\nu_F \frac{i\Omega}{v_F q} \ln \left[ \frac{v_F q + m - i\tilde{\Sigma}(-\Omega)}{-v_F q + m - i\tilde{\Sigma}(-\Omega)} \right]
\] (4–64)

Using the above form of the polarization operator (and also considering the case \(\Omega < 0\)), the formula for the thermodynamic potential, (non Fermi gas part), (Eq.4–39).

\[
\Xi(m) = \frac{1}{(2\pi v_F)^3} 2\text{Re} \int_{0}^{E_F} dq q^2 \int_{0}^{E_F} d\Omega \ln \left[ g^2
\nu_F \left( \frac{i\Omega}{q} \right) \ln \left( \frac{q + m - i\tilde{\Sigma}(-\Omega)}{-q + m - i\tilde{\Sigma}(-\Omega)} \right) \right],
\] (4–65)

where we have kept just the dynamic part and have dropped the static part (coming from the Ornstein-Zernicke form of the bare susceptibility). In the FL regime, \(\tilde{\Sigma}(-\Omega) = -\lambda\Omega\), where \(\lambda = c \ln(\xi k_F)\), is a constant. Therefore,

\[
\Xi_{\text{FL}}(m) = \frac{1}{\lambda (2\pi v_F)^3} 2\text{Re} \int_{0}^{E_F} dq q^2 \int_{0}^{E_F} d\Omega \ln \left[ g^2
\nu_F \left( \frac{i\Omega}{q} \right) \ln \left( \frac{q + m + i\lambda\Omega}{-q + m + i\lambda\Omega} \right) \right]
\]

Now one can re-scale \(\lambda\Omega = \Omega_1\) to get,

\[
\Xi_{\text{FL}}(m) = \frac{1}{\lambda (2\pi v_F)^3} 2\text{Re} \int_{0}^{E_F} dq q^2 \int_{0}^{E_F} d\Omega_1 \ln \left[ \left( \frac{i\Omega_1}{q} \right) \ln \left( \frac{q + m + i\Omega_1}{-q + m + i\Omega_1} \right) \right]
\] (4–66)

Writing the integrand in terms of dimension-less variables, \(\frac{\Omega_1}{q} = y\) and \(\frac{m}{q} = d\) and Taylor expanding the integrand for \(d \ll 1\), and extracting the coefficient of the \(d^4\) term (similar to what we did in perturbation theory) we get,

\[
\Xi_{\text{FL}}(m) = \frac{1}{\lambda (2\pi v_F)^3} 2\text{Re} \int_{0}^{E_F} dq q^3 \int_{0}^{E_F} dy \ln \left[ \left( \frac{1 + d + iy}{-1 + d + iy} \right) \ln \left( \frac{4 - \pi^2}{3\pi^3} \right) \right] - c_1 \frac{m^4}{\lambda} \ln \left( \frac{E_F}{m} \right)
\] (4–67)

\[
\to \frac{1}{\lambda (2\pi v_F)^3} m^4 \int_{m}^{E_F} dq \frac{q}{m} \times \left( \frac{4 - \pi^2}{3\pi^3} \right) \to -c_1 \frac{m^4}{\lambda} \ln \left( \frac{E_F}{m} \right)
\] (4–68)
where $c_1$ is a positive constant. Therefore performing the same thermodynamic analysis as was done for 2D, one gets

$$\chi_s^{FL}(H) \propto \frac{\xi^8 H^2 \ln |\xi^2 H|}{\ln \xi}, \quad (4-69)$$

where we have substituted, $\lambda \sim (\ln \xi)$. As one approaches the QCP, $\lambda$ becomes frequency dependent and that changes the field dependence at the critical point to, $\chi_s^{QCP}(H) \propto H^2 \ln(|H|))$. To analyze the field dependence in the non-FL regime, we start from Eq.4–65 and use $\Sigma(-\Omega) = -c \Omega \ln(\frac{\epsilon_0}{|\Omega|})$.

$$\Xi_{nFL}(m) = 2BRe \int_0^{E_F} dq q^2 \int_0^{E_F} d\Omega \ln\left[\frac{i\Omega}{q} \ln\left(\frac{q + m + i\Omega \ln(\frac{\epsilon_0}{|\Omega|})}{-q + m + i\Omega \ln(\frac{\epsilon_0}{|\Omega|})}\right)\right]$$

$$= 2BRe \int_0^{E_F} dq q^3 \int_0^{E_F} dy \ln[iy\ln\left(\frac{1 + d + iy \ln(a) - iy \ln(y)}{-1 + d + iy \ln(a) - iy \ln(y)}\right)\right]$$

where once again we have re-scaled the dummy variables as, $y = \Omega/q$, $d = m/q$ and $a = \epsilon_0/q$ and $B^{-1} = (2\pi v_F)^3$ is a constant pre-factor. To obtain a scaling form for the thermodynamic potential we neglect the $iy \ln(y)$ terms inside the logarithm.

$$\Xi_{nFL}(m) \sim 2BRe \int_0^{E_F} dq q^3 \int_0^{E_F} dy \ln[iy\ln\left(\frac{1 + d + iy \ln(a)}{-1 + d + iy \ln(a)}\right)\right]$$

$$\sim 2BRe \int_0^{E_F} dq q^3 \int_0^{E_F} dy dy_1 \ln[iy_1\ln\left(\frac{1 + d + iy_1}{-1 + d + iy_1}\right)\right]$$

which is of the same form as Eq.4–67. Once again Taylor expand for $d \ll 1$ and integrate (over $y_1$) the coefficient of the $m^4$ term.

$$\Xi_{nFL}(m) \sim Bm^4 \int_m^{E_F} dq \frac{q}{q \ln(\epsilon_0/q)} \times \left(\frac{4 - \pi^2}{3\pi^3}\right)$$

$$\sim -bm^4 \ln(\ln(\epsilon_0/m)) \quad (4–70)$$

where $b$ is a positive number. The second order QCP is unstable even in 3D, with a tendency towards first order transition. The spin susceptibility at the QCP is $\chi_s^{nFL}(H) \propto H^2 \ln(|H|))$. 


4.5 Conclusions

We analyzed the nonanalytic magnetic field dependence of the spin susceptibility in higher dimensional systems, and showed that they arise from both 1D, as well as non-1D processes. We obtained \( \chi_s(H) \) for a generic Fermi-liquid in 2D. Our result can be compared with the experimental studies on a two-dimensional He\(^3\) layer. We also studied the spin susceptibility in the vicinity of the ferromagnetic quantum critical point using the spin-Fermion model and showed that the second order quantum critical point is unstable in both 2D and 3D, with a tendency towards first order transition.
CHAPTER 5
CONCLUSIONS

In this work we have investigated the role of one-dimensional electronic correlations in the transport and thermodynamic properties of higher dimensional ($D = 2, 3$) systems.

In the first part of this work we presented a study of transport properties of a three-dimensional metal subjected to a strong magnetic field that confines the electrons to the lowest Landau level (UQL). We showed that the nature of electron transport is one dimensional due to the reduced effective dimensionality induced by the field. The first sign of this is that the localization corrections to the conductivity was of the order of the bare Drude conductivity itself. Therefore perturbation theory breaks down just as it does in 1D. However, unlike in the 1D case, we showed that the conductivity remains finite at zero temperature. Therefore, we call this regime intermediate localization. The second important manifestation of electronic correlations and lower dimensionality is that the first order interaction correction to the conductivity is log. divergent in temperature, just as for 1D systems. The physical reason for such a behavior of the conductivity is a nearly 1D form of the Friedel oscillation around an impurity in the strong magnetic field. A renormalization group calculation of the transmission amplitude through a single barrier allowed for a summation of a series of the most divergent log-corrections at all orders in the interaction. Just as in 1D this summation in our case led to power law (temperature) scaling behavior for the conductivity. Some recent transport measurements in graphite were compared with the above theoretical findings and shown to disagree. To resolve the disagreement, we
invoked a model with long ranged-disorder and phonon induced dephasing (1D phenomenon) to explain the experimental observations.

Previous work on the thermodynamic properties of higher dimensional systems had indicated the special role played by one-dimensional scattering events, in the nonanalytic corrections to the specific heat and spin susceptibility. In the second part of this work, we have shown that the next-to-leading terms in the specific heat and spin susceptibility in 1D are nonanalytic, in the same way as they are in higher dimensions ($D = 2, 3$). Thus, even though the low energy theory which describes a one-dimensional interacting system (Luttinger liquid theory) is different from the higher dimensional low energy theory (Fermi liquid), the sub-leading terms in the thermodynamic properties get nonanalytic corrections which arises from the same sources in all dimensions. The only difference is that the non-analytic correction to the specific heat in 1D is present only for fermions with spin, and it occurs at 3$^{rd}$ order in interaction; $C(T) \propto g_{1\perp}^2(g_{1\parallel} - g_{2\parallel} + g_{2\perp}) T \ln T$, whereas in higher dimensions they occur even for spin-less fermions and start at 2$^{nd}$ order. The non-analytic corrections to the spin susceptibility occur at second order in interaction in all dimensions $D = 1, 2, 3$. Thus, we have shown that 1D systems are similar to higher dimensional systems, at least in the context of nonanalytic corrections to thermodynamics.

In the third part of this work, we performed a detailed investigation of the nonanalytic magnetic field dependence of the spin susceptibility, in higher dimensional systems, and showed that they arise from both 1D, as well as non-1D processes. We obtained $\chi_s(H)$ for a generic Fermi-liquid in 2D. Our result can be compared with the experimental results on a two-dimensional He$^3$ layer. We also studied the spin susceptibility in the vicinity of the ferromagnetic quantum critical point using the spin-Fermion model and showed that the second order critical point is unstable both in 2D and 3D with a tendency towards first order transition.
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BIOGRAPHICAL SKETCH

Ronojoy Saha was born on July 17, 1978, in Calcutta, India. He spent the early years of his childhood in Calcutta, and a few years in north Bengal, before moving to New Delhi for his high school. After completing high school in 1995, he came back to Calcutta and joined the Physics Department of Presidency College for his undergraduate studies. During his undergraduate days in Presidency College, he met his future wife Sreya. He received his Bachelor of Science degree in 1998 and moved to New Delhi to pursue his master’s in physics at the Jawaharlal Nehru University (JNU), which he completed in summer of 2000. During his master’s at JNU he developed a keen interest in theoretical physics.

In Fall 2000, he joined the graduate program in physics at the University of Florida. Since Fall 2001, he has worked with Professor Dmitrii Maslov on various problems in strongly correlated electron systems. He received his Ph.D. in August 2006.