# STOCHASTIC INVENTORY CONTROL IN DYNAMIC ENVIRONMENTS

By

JIE CAO

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2005

To my parents and my wife, for their love and support

## ACKNOWLEDGMENTS

I wish to express my sincere gratitude to the members of my supervisory committee, Dr. J.P. Geunes, Dr. A. Paul, Dr. H.E. Romeijn, Dr. Z.J. Shen and Dr. S. Urasev, for their assistance and guidance. I would also like to thank Dr. Aydin Alptekinoglu, for taking time to review my dissertation.

Especially, I am grateful to my chair, Dr. H. Edwin Romeijn, for his support, encouragement and patience throughout the study. As I will pursue my own career, nothing is more beneficial than having his fine example of a just person, an industrious and thorough scholar, and a responsible advisor, to look up to.

My parents and wife have given me tremendous support for my study abroad. Their trust and love are the invaluable wealth in my life, but no acknowledgement could possibly state all that I owe to them.

TABLE OF CONTENTS

LIST OF TABLES

Table | Page

LIST OF FIGURES

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

STOCHASTIC INVENTORY CONTROL IN DYNAMIC ENVIRONMENTS

By

Jie Cao

August 2005

Chair: H. Edwin Romeijn
Major Department: Industrial and Systems Engineering

This dissertation studies some issues in stochastic inventory control.

The first focus of the dissertation is on stochastic continuous-time inventory control problems for a single item in dynamic environments. The demand process is modeled as a semi-Markov chain modulated Poisson process. It is shown that a myopic policy is optimal if the products can be purchased or bought-back at a single price. Conditions on the semi-Markov chain under which products will never be returned is derived. An algorithm to dynamically compute the optimal policy for a special case of the model is also provided. This demand model is next extended to a semi-Markov modulated renewal process, and several results are generalized to this more realistic model.

The next focus of the dissertation is a class of Markov modulated Poisson demand processes in which the transitions between the different states of the world is unobservable. A basic model with two demand states is first studied, and the optimal inventory policy is derived. An algorithm to compute this policy is also provided. Next the basic model is extended to multiple states, and a recursive formula is given which can be used to compute the optimal policy.

The inventory models with simultaneous ordering and pricing decisions are studied next. The demand process is dependent on the price. The joint pricing and inventory model under a price-sensitive Poisson demand environment is studied, and an algorithm to compute the optimal solution is given. Next the study is extended to the semi-Markov modulated Poisson demand environment, and it is shown that with certain approximation, the model can be solved in the similar way as in a Poisson demand environment.

The other focus of the dissertation is on stochastic inventory models for multiple items with both equal and unequal replenishment intervals under limited warehouse capacity. The optimality condition for equal replenishment intervals case is given, three heuristics are implemented, and it is proved that these heuristics provide the optimal solutions in the case of equal replenishment intervals. Extensive numerical tests are conducted, and the heuristics yield high quality solutions in very limited time.

CHAPTER 1
INTRODUCTION

## 1.1 General Description

Stochastic inventory control has long been one of the central issues in supply chain management. This is in part because efficient inventory management can both maintain a high customer service level and reduce unnecessary over-stock expenses which may take up a significant part of an organization's total costs. Even after over 50 years' study and thousands of papers published in this area, inventory problems still continue to provide many new and challenging fields for researchers to explore. One modern direction is to study the optimal inventory behavior under more complicated but more realistic demand environments. In addition, more recent research starts studying the effects of including pricing decisions into the traditional inventory problems.

Distribution systems often contain a set of regional warehouses, each of which stores a variety of items supplied by multiple manufacturers in order to serve a regional population of customers. Effectively managing the inventory of multiple items under limited warehouse storage capacity is critical for ensuring good customer service without incurring excessive inventory holding costs. Each regional warehouse manager thus faces the challenge of coordinating the inventory levels and deliveries of multiple items in order to meet desired service levels while obeying warehouse capacity limits. Suppliers to such regional warehouses must efficiently manage the tradeoffs they face between inventory and transportation costs, which often leads different suppliers to prefer different warehouse replenishment frequencies. These different replenishment frequency preferences, combined with varying degrees of demand uncertainty, further compound the challenges the warehouse manager faces in effectively utilizing limited warehouse capacity.

## 1.2 Literature Review

Stochastic inventory control models can be roughly divided into two major classes based on the nature of the demand process. The first class deals with the stationary behavior of an inventory system and its corresponding control policies. In the literature, the control policies considered for such systems are often continuous time review policies, and the long-run average cost is often used as the performance measure. The demand process is either assumed to be stationary, or at least assumed to have a limiting distribution. These types of problems have been worked on extensively since the 1950s, and it has been well-known for decades that an $(s, S)$ policy, with reorder point $s$ and order-up-to level $S$, is optimal under mild conditions on the cost structure (Beckmann [6], Hadley and Whitin [29]). Under some special demand models, e.g., Poisson demand process, or in case of linear ordering costs, an $(s, S)$ policy can be simplified to a base stock policy $(s, Q)$ where $s$ is again the reorder point, and $Q$ is the order quantity. Earlier works in this area are well summarized by Lee and Nahmias [41]. Recent developments in this area have mainly been focused on the determination of the optimal parameters, or the design of good heuristics that result in near optimal solutions, e.g., Federgruen and Zheng [20] and Gallego [25].

The second class of problems deals with time-dependence and adaptive decision making in a dynamic demand environment. Most of the research work done in this area deals with discrete-time models, where the dynamic nature of the demand process is readily represented via a dynamic programming approach. The first mathematical formulation of problems of this type was introduced in Arrow et al. [3], and later enriched by Bellman et al. [7]. Karlin [36] extends their results by studying inventory models where demands are independent, but not necessarily identically distributed, over time. He showed that a state-dependent base stock policy is optimal. Moreover, in any set of consecutive periods for which the sequence of demand distributions decreases stochastically, the optimal base stock level also decreases. In these earlier works, the demands in different periods are assumed to be independent. Veinott [56] extends these results to an infinite horizon, discrete-time, multi-product dynamic nonstationary inventory problem. The demands in different periods are not necessarily independent. Under linear ordering costs and the assumption that disposal of excess inventory is allowed at the same price as replenishment of inventory, Veinott derives conditions under which a myopic base stock ordering policy is optimal. Lovejoy [44] considers a periodic review, dynamic, single-product inventory model with linear ordering costs. He considers both disposal and nondisposal models, and derives bounds on the relative loss compared to the optimal cost that is incurred by restricting consideration to the class of myopic inventory policies.

Recent studies model the dependence of demand in disjoint time intervals as resulting from the effect of some underlying events. These underlying events occur as time passes, and they may affect the properties of the current and future demand process. Song and Zipkin [52] and Zipkin [63] provide some examples of effects that may characterize the state of the world, such as weather, economy, technology, customer status, etc. They usually model the underlying events as a Markov process, either in continuous or discrete time. In particular, Song and Zipkin [52] assume a demand process that is governed by an underlying *core process*, called the *world*, which is a continuous-time Markov chain with discrete state space. The demand process is then a Poisson process whose rate depends on the current state of the world. They show that if the ordering costs are linear in the quantity ordered, then a state-dependent base stock policy is optimal. If a fixed ordering cost is incurred, then a state-dependent $(s, S)$ policy is optimal. They also show that if the demand process satisfies a certain monotonicity property, the optimal policy will inherit this monotonicity. They also construct an iterative algorithm to approximate the optimal policies, and an exact algorithm for the linear cost model. Later, Song and Zipkin [53] utilize the similar model to show how to manage inventory under a deteriorating demand environment.

The memoryless property inherent in the model of Song and Zipkin [52] means that ordering decisions are only made in response to an event, i.e., a change of the state of the world or the occurrence of a demand. This property allows the transformation of the continuous-time model into an equivalent discrete-time model, and they then employ a discrete-time dynamic programming approach. However, the implicit assumption in these models that the time between state transitions is memoryless is not always reasonable. For example, if the properties of the demand process are weather-related, e.g. in the case of seasonal demands, this assumption is clearly not satisfied. Knowledge of the amount of time that we have spent in a given season generally provides us with some information on how soon this season will end and what the next season will be.

The model we propose is closely related to the model of Song and Zipkin [52]. In particular, we extend their *world* model by relaxing the assumption that the time between state transitions is memoryless. This means that the changes in the demand process are described by a semi-Markov process instead of an ordinary Markov process. The main effect of this relaxation is that at every point in time in a given state, the process of future demands is different. Thus, an optimal policy may require making ordering decisions continuously in time. This means that the elapse of time itself provides us with information about the future. For example, the *absence* of demands for a certain amount of time may cause us to adjust the inventory position. Heyman and Sobel [31] also studied semi-Markov decision processes. But not like us, they restricted that the decisions can be made only at the epoch of state transitions and concluded that the optimization of infinite horizon discounted model is essentially the same as the optimization of discrete-time Markov decision processes. Obviously, allowing decision makings at any time, as we do here, makes the problem more complicated, and the behavior of the optimal policy also differs from that for MDPs, as we will show. To be able to deal with this additional complexity, we mainly focus on a model in which disposal of excess inventory is allowed at the same price as replenishment of inventory (analogous to the discrete-time model of Veinott [56]). For this case, we derive conditions under which a myopic base stock ordering policy is optimal.

Most real life inventory control problems face only partially observable demand. The true underlying distribution of the demand is not directly observed, and only demand occurrences are observed. Scarf [50, 51] studied an inventory problem in which the parameter of the demand is unknown, but a priori Bayesian distribution is chosen for the parameter. He used Bayesian methods to solve the inventory control problems and character the optimal ordering policy. Azoury [4] extended the result of Scarf [51] by studying dynamic inventory models under various families of demand distributions with unknown parameters. He derived the optimal Bayesian policy, and showed its computation is no more difficult than the corresponding computation when the demand distribution is known. Lovejoy [43] studied inventory models with uncertain demand distributions where estimates of the unknown parameter are updated in a statistical fashion as demand is observed through time. He showed that a simple inventory policy based upon a critical fractile can be optimal or near-optimal. Later, Lovejoy [44, 45] extended the study by showing the robustness of bounds on the value loss relative to optimal cost of myopic policies which may be stopped earlier.

Kurawarwala and Matsuo [38] gave a combined forecasting and inventory model according to the characteristics of short-life cycle products. They proposed a seasonal trend growth model and used optimal control theory to get the optimal inventory policy. Treharne and Sox [55] studied a partially observable Markov modulated demand model in which the probability distributions for the demand in each period is determined by the state of an underlying discrete-time Markov chain, and partially observed. They showed that that some suboptimal control policies, open-loop feedback control and limited look-ahead control, which account for more of the inherent uncertainty in the demand processes, almost always achieve much better performance than the typically used CEC (certainty equivalent control) policy.

Most traditional inventory problems concern the determination of optimal replenishment policies in different types of environments, where the demand process is often assumed to be given. In these problems, product selling prices are not a decision variable, but given as known parameters, although they may change from period to period. Therefore, the aim is to minimize the expected operating costs, because the expected revenues are not controllable.

More recent developments in industrial practice combining pricing and inventory management have shown great success, and have stimulated the need for research into combining pricing and inventory control policies. Whitin [61] proposed including the pricing into inventory planning decisions. He studied the single period newsvendor model with price dependent demand and considered the problem of simultaneous determination of a single price and ordering quantity. Thomas [54] considered a single item, periodic review, finite horizon model with a fixed ordering cost and price sensitive demand process. He conjectured that a $(s, S, p)$ policy was optimal: the inventory replenishment is governed by a dynamic $(s, S)$ policy, where the optimal price depends on the inventory level at the beginning of a review period. He also constructed a counterexample which demonstrates that if the available price choice is restricted to a discrete set, this policy may not be optimal.

Petruzzi and Dada [47] provided an excellent review on pricing decisions in the newsvendor problem, and in addition extended the single period model to a multi-period one. They concluded that in most papers on pricing the randomness in demand is assumed independent of the item price and can be modelled either in an additive or a multiplicative way. They pointed out that a difficulty in multi-period models results from the assumption that inventory leftovers cannot be disposed of. They show how revising this assumption and allowing for the possibility of salvaging leftovers is sufficient to yield a stationary myopic policy for the multiple period problem. By cutting off the links between periods, all results and managerial insight available for the single period model apply directly to the multiple period model.

Federgruen and Heching [19] analyzed a single item periodic review model where demands depend on the item's price, ordering costs are linear in the ordered amount, and all stockouts are backlogged. They studied both finite and infinite horizon models, using both expected discounted and time averaged profit criteria. They derived the structure of an optimal combined pricing and inventory strategy for all their models and developed an efficient value iteration method to compute the optimal strategies. They showed that a base-stock list price policy is optimal for their model: in each period the optimal policy is characterized by an order-up-to level and a price which depends on the starting inventory level before ordering at the beginning of each period. If the starting inventory level before ordering is below the order-up-to level, an order is placed to raise the inventory level to the level. Otherwise, no order is placed and a discount price is offered. The discount price is a non-increasing function of the starting inventory level.

Recently, Chen and Simchi-Levi [12, 13] generalized the above model by incorporating a fixed cost component. They show that the $(s, S, p)$ policy proposed by Thomas [54] is indeed optimal for additive demand functions when the planning horizon is finite; when the planning horizon is infinite, this policy is optimal for both additive and general demand processes under both discounted and average profit criteria. They also introduce the concept of symmetric $k$-convex functions and use this to provide a characterization of the optimal policy.

Though periodical review models have been studied quite extensively, continuous-review joint pricing and inventory control problems have received far less attention in the literature. Li [42] considered a continuous time integrated pricing and inventory planning strategies model where demand and production are both Poisson processes. The intensity of the demand process depends on the item's chosen price. He showed that if ordering and holding costs are both linear a barrier policy is optimal. He also gave an implicit characterization of the optimal pricing policy when dynamic pricing is allowed. Feng and Chen [23] studied a continuous review model that is related to ours where the demand is modelled as price-sensitive Poisson process. They restrict the available prices to a given finite set (specifically, only two candidate prices), and assume zero lead times. They show that a $(s, S, p)$ policy is optimal when fixed ordering costs are present.

In our problem, we model the demand as a Markov modulated Poisson process (see also Song and Zipkin [52]). In particular, the demand process is a Poisson process whose rate is governed by an underlying Markov chain that represents the state of the world. We introduce pricing flexibility into this model by allowing the rate of the Poisson process in each state to depend on the price of the product. Recently, Chen and Simchi-Levi [12, 13] generalized the above model by incorporating a fixed cost component. They show that the $(s, S, p)$ policy proposed by Thomas [54] is indeed optimal for additive demand functions when the planning horizon is finite; when the planning horizon is infinite, this policy is optimal for both additive and general demand processes under both discounted and average profit criteria. They also introduce the concept of symmetric $k$-convex functions and use this to provide a characterization of the optimal policy.

Distribution systems often contain a set of regional warehouses, each of which stores a variety of items supplied by multiple manufacturers. Effectively managing the inventory of multiple items under limited warehouse storage capacity is critical for ensuring good customer service without incurring excessive inventory holding costs. Suppliers to such regional warehouses must efficiently manage the tradeoffs they face between inventory and transportation costs, which often leads different suppliers to prefer different warehouse replenishment frequencies. For example, manufacturers who supply items with a high value-to-weight ratio typically find it more economical to send relatively frequent shipments in small quantities, while those who supply items with a low value-to-weight ratio often prefer to delivery large quantities less frequently (see Ballou [5]). These different replenishment frequency preferences, combined with varying degrees of demand uncertainty, further compound the challenges the warehouse manager faces in effectively utilizing limited warehouse capacity.

Stochastic inventory models involving (production) capacity constrained periodic-review policies have attracted the attention of many researchers. Evans [18] first considers this issue by modeling periodic-review production and inventory systems with multiple products, random demands and a finite planning horizon. He develops the form of the optimal policy for multi-product control for such a system. Since then, much of the literature has studied periodic-review, single-product systems with production capacity constraints. Florian and Klein [24] and De Kok et al. [37] characterize the structure of the optimal solution to a multi-period, single-item production model with a capacity constraint. Federgruen and Zipkin [21, 22] show that a modified base-stock policy is optimal under both discounted and average cost criteria and an infinite planning horizon. The modified base-stock policy requires that, when initial stock is below a certain critical number, we produce enough to bring total stock up to that number, or as close to it as possible, given the limited capacity; otherwise, we do not produce. They also characterize the optimal policy by deriving expressions for the expected costs of modified base-stock policies. Kapuscinski and Tayur [35] provide a simpler proof of optimality than Federgruen and Zipkin [21] for the infinite-horizon discounted cost case, based on results from Bertsekas [8]. Ciarallo et al. [15] and Wang and Gerchak [58] analyze a production model with variable capacity in a similar environment as Federgruen and Zipkin [21]. Wang and Gerchak [57] also incorporate variable capacity explicitly into continuous review models.

DeCroix and Arreola-Risa [16] study an infinite-horizon version of the capacitated multi-product case. They establish the optimal policy for the case of homogeneous products, and propose a heuristic policy for heterogeneous products by generalizing the optimal policy for the homogeneous product case. Products are called *homogeneous* if they have identical cost parameters and their demands are identically distributed. Glasserman [27] addresses a similar problem to DeCroix and Arreola-Risa [16] in a continuous-review system. He presents a procedure for choosing base-stock levels and capacity allocation that is asymptotically optimal, but assumes that a fixed proportion of total capacity is dedicated exclusively to each product. The use of asymptotic analysis is similar in spirit to Anantharam [1]. His static allocation problem contrasts with the dynamic scheduling problem addressed in Wein [60] and Zheng and Zipkin [62] and the priority scheme in Carr et al. [11]. Lau and Lau [39, 40] present formulations and solution procedures for handling a multi-product newsboy problem under multiple resource constraints. Nahmias and Schmidt [46] also investigate several heuristics for a single-period, multi-item inventory problem with a resource constraint.

Anily [2] and Gallego et al. [26] study a multi-item replenishment problem with deterministic demand. Anily [2] investigates the worst-case behavior of a heuristic for the multi-item replenishment and storage problem and derives a lower bound on the optimal average cost over all policies that follow stationary demand and cost parameters. Gallego et al. [26] consider two economic order quantity models where multiple items use a common resource: the tactical and strategic models. They derive a lower bound on the peak resource usage that is valid for any feasible policy, use this to derive lower bounds on the optimal average cost for both models, and show that simple heuristics for either model have bounded worst-case performance ratios. Additional literature, e.g., Rosenblatt and Rothblum [48], Goyal [28], Hartley and Thomas [30], Jones and Inman [34] and Dobson [17], deals with deterministic inventory models with warehousing constraints.

Although much of the literature is devoted to multi-item, periodic-review systems with a production capacity constraint, little has been done for stochastic inventory models with a warehouse-capacity constraint. Veinott [56] first considers a multi-product dynamic, nonstationary inventory problem with limited warehouse capacity. He provides conditions that ensure that the base stock ordering policy is optimal in a periodic-review inventory system with a finite horizon. Ignall and Veinott [33] show that, in the stationary demand case, a myopic ordering policy is optimal for a sequence of periods under all initial inventory levels. Recently, Beyer et al. [9, 10] use a dynamic programming approach to derive the optimal ordering policy for the average cost problem and show the convexity of the cost function, as we did in this study coincidentally. They also show the optimality of the modified base-stock policy in the discounted cost version of the problem. In this paper, we extend their results by explicitly characterizing the optimal myopic policy under relaxed assumptions on the demand distributions.

## 1.3 Outline of Dissertation

In Chapter 2, we propose an inventory model under a semi-Markov modulated Poisson demand environment. We give the description of the model, and some properties of the model. Then we give the optimal inventory policy. We show that if the demand process observes some monotone property, then the optimal inventory position will also show a similar pattern. For a special phase type lead time distribution, we give one algorithm to actually compute the optimal inventory positions. We also extend the model to a more general case where renewal process is in place of Poisson process for the demand process.

In Chapter 3, we restudy the inventory model in Chapter 2. But this time the state of the underlying world is no longer observable, and we can only observe the actual demand arrivals. We first study a two world states model, and give the form of the optimal inventory policy for this model, and propose one algorithm to solve it. We then extend the study to a multiple world states model and give a recursive formula to determine the probability of the underlying world in each state, and help determine the optimal inventory policy.

In Chapter 4, we include the pricing decisions at the same time that the inventory strategy is determined. The price will affect the demand process, and we are maximizing the total expected discounted profits in an infinite horizon. We first study the joint pricing and inventory model under a price-sensitive Poisson demand environment without Markov modulation. In the case where the price can only be set once at the beginning, we give some properties that can be used to determine the optimal solution, and derive an algorithm to compute the optimal solution. We then study the model where price can be continuously set. Next we extend the study to the semi-Markov modulated Poisson demand environment, and show that with certain approximation, the model can be solved in the similar way as in a Poisson demand environment.

In Chapter 5, we study discrete time stochastic inventory models for multiple items with both equal and unequal replenishment intervals under limited warehouse capacity. We propose three efficient and intuitively attractive heuristics. We show that these heuristics provide the optimal replenishment quantities in the case of equal replenishment intervals. For the general model, a numerical comparison of the heuristic solutions to the optimal solutions shows that the heuristics yield high quality solutions in very limited time.

CHAPTER 2
INVENTORY CONTROL IN A SEMI-MARKOV MODULATED DEMAND ENVIRONMENT

## 2.1 Introduction

This chapter is organized as follows. In Section 2.2, we formulate our model and study a special case of our demand process that reduces to a Markov modulated demand environment. Then, in Section 2.3, we show that for our general demand process a myopic policy is optimal, and characterize the optimal policy parameters. In Section 2.4, we derive sufficient conditions on the demands that imply that the disposal option will never be used and the myopic policy is thus optimal even in the case where disposal is not allowed. In Section 2.5, we propose an algorithm to compute the optimal inventory policy for a special case of our inventory model. In Section 2.6, we propose an extension of the model, where the demand process in a given state of the world is a general renewal process instead of a Poisson process. We end the chapter in Section 2.7 with some concluding remarks.

## 2.2 Model Formulation

### 2.2.1 The Demand Process

We consider inventory systems for managing the demand for a single product. Denote the stochastic process representing the cumulative demand for this product at each point in time by $\{D(t), t \geq 0\}$. We also assume that there is some underlying core stochastic process that models the state of the world, which is denoted by $\{A(t), t \geq 0\}$. Although we provide our basic analysis of the inventory model for general demand processes governed by such an underlying core process, the main results of this chapter assume that the core process is a continuous-time semi-Markov process. The embedded Markov chain's state space is denoted by $I \subseteq \{0, 1, 2, \ldots\}$, and the transition probability matrix is $P = (p_{ij})_{i,j \in I}$. Given a current state $i$ and a next state $j$, the distribution function of the transition time from $i$ to $j$ is denoted by $G_{ij}$. We assume the transition times are independent of each other. When the core process is in state $i$, the actual demand process follows a Poisson process with rate $\lambda_i$, where we assume that $\bar{\lambda} = \sup_{i \in I}\{\lambda_i\}$ is finite. We call this demand process a *semi-Markov modulated Poisson demand process.* This demand process is exogenous and is not affected by any ordering decisions. Clearly, if the distributions $G_{ij}$ are exponential distributions whose rate only depends on $i$, the demand process reduces to a Markov modulated Poisson demand process, as introduced by Song and Zipkin [52]. Before we describe and analyze our model, we first briefly discuss a case in for which the transition times are not exponentially distributed, but nevertheless results in a Markov modulated Poisson process.

### 2.2.2 Gamma Distributed State Transition Times with Observable Stage Transitions

A special case of our model is obtained if the time until a transition from state $i$ takes place has a Gamma distribution with parameters $(r_i, \nu_i)$ (where $r_i$ is a positive integer), independent of which state is visited next. That is, $G_{ij}$ is the Gamma distribution with parameters $(r_i, \nu_i)$ for all $j \in I$. Now recall that we can write a Gamma$(r_i, \nu_i)$ random variable as the sum of $r_i$ independent random variables which are exponentially distributed with parameter $\nu_i$. So, if we have the ability to observe not only the state transitions, but also the "stage" changes between the $r_i$ successive stages of the Gamma distributed transition times directly, then we can transform the underlying semi-Markov process to a continuous-time Markov process using the following method. We redefine the state space of of the underlying core process to be $I \times \{1, \ldots, r_i\}$. If the core process is in state $(i, k)$, we are currently in world state $i$, and have completed $k - 1$ stages of exponentially distributed duration with parameter $\nu_i$ in this state. The new embedded Markov chain has one-step transition probabilities given by

$$\begin{aligned} p_{(i,k),(i,k+1)} &= 1 \quad k = 1, \ldots, r_i - 1; i \in I \\ p_{(i,r_i),(j,1)} &= p_{ij} \quad i, j \in I. \end{aligned}$$

All other transition probabilities are zero. It is clear that we now have a Markov modulated Poisson demand process that falls within the framework of Song and Zipkin [52]. However, if we can not directly observe the stage changes of the core process in the above model, then their results cannot be applied.

### 2.2.3 Definitions and Notation

For convenience, we merge the two stochastic processes describing the demands and the state of the world into a single *demand history* process $\{H(t), t \geq 0\}$, where

$$H(t) = (A(t), D(t)).$$

The entire history *up to* time $t$, for all stochastic processes, is denoted by

$$\begin{aligned} \bar{A}(t) &= \{A(u) : 0 \leq u \leq t\} \\ &= \text{core state history up to and including time } t \\ \bar{D}(t) &= \{D(u) : 0 \leq u \leq t\} \\ &= \text{cumulative demand history up to and including time } t \\ \bar{H}(t) &= \{H(u) : 0 \leq u \leq t\} \\ &= (\bar{A}(t), \bar{D}(t)) \\ &= \text{history up to and including time } t. \end{aligned}$$

The observations of the history process are important because they provide information on the future of the demand process, and thus may affect the ordering decisions. Note that we can extract from the demand history process another stochastic process that represents, at each point in time $t$, the amount of time that has been spent in the current state $A(t)$ since the last state transition, say $S(t)$, and we let

$$\bar{S}(t) = \{S(u) : 0 \leq u \leq t\}.$$

Finally, we let $\mathcal{H}(t)$ denote the sample space of the demand history up to time $t$, $\bar{H}(t)$.

We assume that the inventory level is reviewed continuously, and we need to decide on how to adjust the inventory position at each point in time. We assume that both inventory ordering (i.e., and upwards adjustment of the inventory position) and inventory disposal (i.e., a downwards adjustment of the inventory position) are possible, both at the same price (see also Veinott [56]). This will be a reasonable assumption in the context of consignment sales. This is an increasingly popular business arrangement where for example the retailer does not pay its supplier until the items are sold. Ownership of the goods is therefore retained by the supplier. Such an arrangement is widely used in distribution channels for arts and crafts, as well as industrial and consumer goods such as electronics, furniture, food, books, journals and newspapers, etc. (see also Wang et al. [59]). It may also be applicable to settings where suppliers promise to buy back unsold goods as a service to their customers, in order to build a better relationship and improve the efficiency of the entire supply chain. Finally, this strategy may be attractive for producers of copyrighted products, i.e., books, software, music CDs, etc. The value of these products lies in their content or knowledge, while the costs of producing the media are relatively low. The risk of promising buyback is not high, while the chances of selling can be greatly increased by attracting more retailers to distribute the good. In the remainder of this paper, we will simply talk about the placement of orders, with negative values corresponding to disposal. Although we allow disposal, we assume that it is not possible to cancel or change an order that is already placed but which has not yet been delivered.

Inventories incur holding costs, whereas unsatisfied demand is backlogged and incurs a penalty cost. In our model, we assume that holding and penalty costs are linear with stationary rates $h$ and $p$, respectively. We can combine these two kinds of costs together, refer to them simply as inventory costs, and represent them by the following cost function:

$$\begin{aligned}\hat{C}(x) &= \text{the inventory cost rate when the inventory level is } x \\ &= \begin{cases} hx & \text{if } x \geq 0 \\ -px & \text{if } x < 0. \end{cases}\end{aligned}$$

Orders placed will arrive after a potentially stochastic lead time $L$ with distribution $F_L$. As in Song and Zipkin [52], we assume that the ordering (purchasing) costs, say $\bar{c}$ are paid when the order is received (i.e., after the lead time). All costs are discounted at a rate of $\alpha$. At the time when an ordering decision is made, the observed unit ordering cost is thus a discounted one, which we denote by $c = \bar{c}E[e^{-\alpha L}]$.

We will often be using the total demand occurring during the lead time. Note that this total demand depends on the observed demand history, but not the history of the inventory position, since we assumed that the demand process is exogenous. Also, as remarked in Song and Zipkin [52], if we require that the lead times do not cross in time, that is, orders that are placed earlier than other ones can never arrive later than these, then they are not independent. Following Song and Zipkin [52], we ignore the impact of lead time history in making ordering decisions because we lack the ability to collect and process such information, and treat them by standard approach proposed by Hadley and Whitin [29]. So, we define $D_L^{t,\bar{h}(t)}$ to be the random variable representing the total demand occurring during the time interval $(t, t+L]$, given demand history information $\bar{h}(t) \in \mathcal{H}(t)$. For fixed $\ell$, denote the distribution function of $D_\ell^{t,\bar{h}(t)}$ by $F_{D_\ell^{t,\bar{h}(t)}}(z)$, and define

$$\tilde{F}_{D_L^{t,\bar{h}(t)}}(z) = \int_0^\infty e^{-\alpha\ell} F_{D_\ell^{t,\bar{h}(t)}}(z) dF_L(\ell).$$

Now the conditional expected discounted holding and shortage cost rate, at the end of a lead time starting from the current time $t$, and viewed at time $t$, given that the demand history is $\bar{h}(t)$, and the current inventory position (after ordering decision) is $y$, can be written as

$$\begin{aligned} C(y,t,\bar{h}(t)) &= E\left[e^{-\alpha L}\hat{C}(y - D_L^{t,\bar{h}(t)})\right] \\ &= \int\int e^{-\alpha \ell}\hat{C}(y-z)dF_{D_\ell^{t,\bar{h}(t)}}(z)dF_L(\ell) \\ &= \int \hat{C}(y-z)d\tilde{F}_{D_L^{t,\bar{h}(t)}}(z). \end{aligned}$$

### 2.2.4 Problem Formulation

In our problem, we consider three types of costs: ordering, holding, and shortage costs, and our objective is to minimize the total expected discounted costs over the infinite horizon. With a slight abuse of notation, we define an *ordering policy* to be a family of functions that prescribes, for each time $t$ and each potential history observed up to that time, the desired inventory position at that time. Note that we have assumed that the demand process does not depend on any ordering decisions that we make. This, together with the fact that we may place negative and positive orders at the same price, implies that the ordering policy does not depend on the inventory position immediately preceding an ordering decision. More formally, we define an ordering policy $\mathbf{y}$ to be a family of functions, say $\{y(t,\cdot) : \mathcal{H}(t) \to \mathbb{R}; t \geq 0\}$, where $y(t,\cdot)$ prescribes that, if we have observed history $\bar{h}(t)$ up to time $t$, we place a (possibly negative) order that brings the inventory position to $y(t,\bar{h}(t))$, for every $t \geq 0$. It will be convenient to refer to the function $y(t,\cdot)$ as the *policy for time $t$*. We say that a policy $\mathbf{y}^*$ is optimal if this policy minimizes the expected total future discounted costs over all policies.

If the initial inventory position is $x$, the initial demand history $\bar{h}(0)$ is observed, and some ordering policy $\mathbf{y}$ is followed, the total costs may be expressed as

$$\begin{aligned}
& W(x, \bar{h}(0)|\mathbf{y}) \\
= \quad & E\left[c(y(0,\bar{h}(0)) - x) + \int_0^\infty e^{-\alpha t} cd(y(t,\bar{H}(t)) + D(t)) + \right. \\
& \left. \int_0^\infty e^{-\alpha t} C(y(t,\bar{H}(t)), t, \bar{H}(t)) dt\right] \qquad (2.1) \\
= \quad & -cx + E\left[cy(0,\bar{h}(0)) + \int_0^\infty e^{-\alpha t} cd(y(t,\bar{H}(t)))\right] + \\
& E\left[\int_0^\infty e^{-\alpha t} cd(D(t))\right] + E\left[\int_0^\infty e^{-\alpha t} C(y(t,\bar{H}(t)), t, \bar{H}(t)) dt\right]. \qquad (2.2)
\end{aligned}$$

In equation (2.1), the expectation is taken over the entire demand process from time 0 to the infinite horizon. Inside the expectation, the first term represents the ordering cost at time 0; the second term represents the ordering costs for replenishing inventory during the entire infinite horizon since the rate at which we order at time $t$ is $d(y(t,\bar{h}(t)) + D(t))$; and the third term represents the total inventory holding and shortage costs. We consider only those policies that make the total costs finite. We will show later that such policies do indeed exist.

In equation (2.2), note that the total expected replenishing costs of all the demands,

$$E\left[\int_0^\infty e^{-\alpha t} cd(D(t))\right]$$

as well as the value of the initial inventory position (i.e., the term $cx$) are not influenced by the choice of policy $\mathbf{y}$. Therefore, we will omit these terms in our analysis and redefine the cost function as

$$\begin{aligned}
& W(x, \bar{h}(0)|\mathbf{y}) = \\
& \quad E\left[cy(0,\bar{h}(0)) + \int_0^\infty e^{-\alpha t} cd(y(t,\bar{H}(t)))\right] + E\left[\int_0^\infty e^{-\alpha t} C(y(t,\bar{H}(t)), t, \bar{H}(t)) dt\right].
\end{aligned}$$

Now consider a realization of the expression inside the first expectation of the above cost function, i.e., we consider a fixed history path $\bar{h}(t)$ starting from $\bar{h}(0)$. This expression can then be simplified as follows:

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t} cd(y(t,\bar{h}(t))) + cy(0,\bar{h}(0)) \\
&= c\int_0^\infty \int_t^\infty \alpha e^{-\alpha\tau} d\tau d(y(t,\bar{h}(t))) + y(0,\bar{h}(0)) \\
&= c\int_0^\infty \alpha e^{-\alpha\tau} \int_0^\tau d(y(t,\bar{h}(t)))d\tau + y(0,\bar{h}(0)) \\
&= c\int_0^\infty \alpha e^{-\alpha\tau} (y(\tau,\bar{h}(\tau)) - y(0,\bar{h}(0))d\tau + y(0,\bar{h}(0)) \\
&= c\int_0^\infty \alpha e^{-\alpha\tau} y(\tau,\bar{h}(\tau))d\tau - \int_0^\infty \alpha e^{-\alpha\tau} y(0,\bar{h}(0))d\tau + y(0,\bar{h}(0)) \\
&= c\int_0^\infty \alpha e^{-\alpha\tau} y(\tau,\bar{h}(\tau))d\tau - y(0,\bar{h}(0))\int_0^\infty \alpha e^{-\alpha\tau} d\tau + y(0,\bar{h}(0)) \\
&= c\int_0^\infty \alpha e^{-\alpha\tau} y(\tau,\bar{h}(\tau))d\tau \\
&= \int_0^\infty \alpha c e^{-\alpha t} y(t,\bar{h}(t))dt. \qquad (2.3)
\end{aligned}$$

So we obtain that

$$E\left[cy(0,\bar{h}(0)) + \int_0^\infty e^{-\alpha t} cd(y(t,\bar{H}(t)))\right] = E\left[\int_0^\infty e^{-\alpha t} \alpha c y(t,\bar{H}(t))dt\right]$$

and the cost function reduces to

$$\begin{aligned}
&W(x,\bar{h}(0)|\mathbf{y}) \\
&= E\left[\int_0^\infty e^{-\alpha t}\alpha c y(t,\bar{H}(t))dt\right] + E\left[e^{-\alpha t} C(y(t,\bar{H}(t)),t,\bar{H}(t))dt\right] \\
&= \int_0^\infty e^{-\alpha t} E\left[C(y(t,\bar{H}(t)),t,\bar{H}(t)) + \alpha c y(t,\bar{H}(t))\right] dt. \qquad (2.4)
\end{aligned}$$

## 2.3 Model Analysis

### 2.3.1 Optimal Policy

In this section, we will show that under our model assumptions, the optimal ordering policy is a *myopic* one, in which, at each point in time, the optimal inventory position is found by solving a single one-dimensional optimization problem. Our first lemma constructs such a myopic policy for each time $t$.

**Lemma 2.3.1** *For every fixed $t$, and every possible history $\bar{h}(t) \in \mathcal{H}(t)$, let $y^*(t, \bar{h}(t))$ denote an optimal solution to the problem*

$$\min_y C(y, t, \bar{h}(t)) + \alpha c y$$

*if it exists. Then $y^*(t, \cdot)$ is a policy for time $t$ that minimizes*

$$E\left[C(y(t, \bar{H}(t)), t, \bar{H}(t)) + \alpha c y(t, \bar{H}(t))\right].$$

**Proof:** For any fixed $t$ and every possible $\bar{h}(t) \in \mathcal{H}(t)$, following any other policy for time $t$, say $y'(t, \cdot)$ will result in the inventory position $y'(t, \bar{h}(t))$, and

$$C(y^*(t, \bar{h}(t)), t, \bar{h}(t)) + \alpha c y^*(t, \bar{h}(t)) \le C(y'(t, \bar{h}(t)), t, \bar{h}(t)) + \alpha c y'(t, \bar{h}(t))$$

by the definition of $y^*(t, \cdot)$. Therefore, by denoting the distribution of the history up to time $t$ by $F_{\bar{H}(t)}$, we have

$$\begin{aligned}
& E\left[C(y^*(t, \bar{H}(t)), t, \bar{H}(t)) + \alpha c y^*(t, \bar{H}(t))\right] \\
&= \int \left[C(y^*(t, \bar{h}(t)), t, \bar{h}(t)) + \alpha c y^*(t, \bar{h}(t))\right] dF_{\bar{H}(t)}(\bar{h}(t)) \\
&\le \int \left[C(y'(t, \bar{h}(t)), t, \bar{h}(t)) + \alpha c y'(t, \bar{h}(t))\right] dF_{\bar{H}(t)}(\bar{h}(t)) \\
&= E\left[C(y'(t, \bar{H}(t)), t, \bar{H}(t)) + \alpha c y'(t, \bar{H}(t))\right]
\end{aligned}$$

which shows the desired result. $\square$

We call $y^*(t, \cdot)$ a myopic optimal policy for time $t$ in the sense that it only seeks to minimize the "current" discounted cost rate at time $t$ rather than the expected total future discounted costs. The following theorem shows that the optimal policy coincides with the myopic policy under our model assumptions.

**Theorem 2.3.2** *Let $y^*(t,\cdot)$ be defined as in Lemma 2.3.1. If it exists for all $t \geq 0$, then the policy $\mathbf{y}^* = \{y^*(t,\cdot) : t \geq 0\}$ is an optimal ordering policy that minimizes $W(x, \bar{h}(0)|y)$ among all policies $\mathbf{y}$.*

**Proof:** Since the decision variables, i.e., the inventory positions at any time $t$, are unrestricted and independent of each other by the assumption that negative orders are allowed, minimizing $W(x, \bar{h}(0)|\mathbf{y})$ over all policies can be decomposed into minimization problems

$$\min_{y(t,\cdot)} E\left[C(y(t, \bar{H}(t)), t, \bar{H}(t)) + \alpha c y(t, \bar{H}(t))\right]$$

for all fixed $t \geq 0$. Thus the conclusion follows from Lemma 2.3.1. □

Note that so far we have not used any specific details of the demand process. So our expression of the cost function in fact holds for all continuous-review inventory models with linear ordering cost in which the demand process is independent of ordering decisions, as long as negative orders are allowed. In the next section, we return to the semi-Markov modulated Poisson demand process as introduced in Section 2.2.1. We will show that in that case the optimal policy only depends on the current state of the core process and the amount of time that has been spent in this state since the last state transition.

### 2.3.2 Optimal Policy with Semi-Markov Modulated Poisson Demands

The properties of the semi-Markov modulated Poisson demand process immediately imply that the lead time demand $D_L^{t,\bar{h}(t)}$ depends on the history only through the state that the process is currently in and how long it has been in that state. To reflect this fact, we rewrite the lead time demand as $D_L^{i,s}$ when the core process has been in state $i$ for $s$ time units. For given $y$, $t$, and $\bar{h}(t)$, since the lead time demand can be simplified, we can (with a slight abuse of notation) also simplify the inventory cost rate function to

$$C(y, i, s) = E\left[e^{-\alpha L}\hat{C}(y - D_L^{i,s})\right]$$

which is equivalent to $C(y, t, \bar{h}(t))$ if the history $\bar{h}(t)$ says that $A(t) = i$ and $S(t) = s$. We can then replace $C(y(t, \bar{H}(t)), t, \bar{H}(t))$ in the total expected discounted cost formula (2.4) by $C(y(t, \bar{H}(t)), A(t), S(t))$. The total expected cost function for our semi-Markov modulated Poisson demand model thus reduces to

$$W(x, \bar{h}(0)|\mathbf{y}) = \int_0^\infty e^{-\alpha t} E\left[C(y(t, \bar{H}(t)), A(t), S(t)) + \alpha c y(t, \bar{H}(t))\right] dt.$$

Let us now define the function

$$f_{i,s}(y) = C(y, i, s) + \alpha c y$$

for every fixed $i$ and $s$, which can be viewed as the cost rate function if the inventory position is $y$ at the time when the core process has been in state $i$ for $s$ time units. With this definition, the objective function can be written as

$$W(x, \bar{h}(0)|\mathbf{y}) = \int_0^\infty e^{-\alpha t} f_{A(t),S(t)}(y(t, \bar{H}(t), t)) dt.$$

Due to the discrete nature of the demand process, the functions $C(\cdot, i, s)$ and $f_{i,s}(\cdot)$ will not be everywhere differentiable. Therefore, it will be convenient to define for every $i$ and $s$ the right derivatives of $C(\cdot, i, s)$ and $f_{i,s}(\cdot)$:

$$\begin{aligned} C'_+(y, i, s) &= \lim_{\varepsilon \downarrow 0} \frac{C(y + \varepsilon, i, s) - C(y, i, s)}{\varepsilon} \\ (f'_{i,s})_+(y) &= \lim_{\varepsilon \downarrow 0} \frac{f_{i,s}(y + \varepsilon) - f_{i,s}(y)}{\varepsilon} \\ &= C'_+(y, i, s) + \alpha c. \end{aligned}$$

Also, let

$$y_i^*(s) = \inf\{y : (f'_{i,s})_+(y) \geq 0\}.$$

Note that since the leadtime demand can only assume integral values, all points at which the functions $C(\cdot, i, s)$ and $f_{i,s}(\cdot)$ are nondifferentiable are integral. In addition, $y_i^*(s)$ is integral as well. We will now give some properties of the cost functions and optimal policy; these properties are similar to the ones obtained by Song and Zipkin [52] for the Markov modulated demand model.

**Lemma 2.3.3**

*(a) $C(y, i, s)$ and $f_{i,s}(y)$ are both convex in $y$ for all $i$ and $s$, so that $y_i^*(s)$ minimizes $f_{i,s}(y)$.*

*(b) If $\alpha\bar{c} < p$, then $y_i^*(s)$ is finite and nonnegative for all $i$ and $s$. In addition, $f_{i,s}(y)$ is nonnegative for all $i$ and $s$.*

*(c) If $\alpha\bar{c} \geq p$, then $y_i^*(s) = -\infty$ for all $i$ and $s$.*

**Proof:**

(a) $C(y, i, s)$ is a convex function in $y$ because $\hat{C}(x)$ is a convex function in $y$, and $C(y, i, s)$ is a positive weighted average of convex functions. The convexity of $f_{i,s}(y)$ and optimality of $y_i^*(s)$ then follow immediately.

(b) Note that

$$
\begin{aligned}
&C(y, i, s) \\
&\quad = E[e^{-\alpha L}\hat{C}(y - D_L^{i,s})] \\
&\quad = \int_0^\infty e^{-\alpha\ell} \int_0^\infty \hat{C}(y - z)\, dF_{D_\ell^{i,s}}(z) dF_L(\ell) \\
&\quad = \int_0^\infty e^{-\alpha\ell} \left\{ \int_0^y h(y - z) dF_{D_\ell^{i,s}}(z) + \int_y^\infty p(z - y) dF_{D_\ell^{i,s}}(z) \right\} dF_L(\ell)
\end{aligned}
$$

where $F_{D_\ell^{i,s}}(z)$ is the distribution function of the lead time demand when the demand process has been in state $i$ for $s$ time units, conditional on a lead time of $\ell$ time units. This implies that

$$\begin{aligned} C'_+(y,i,s) &= \int_0^\infty e^{-\alpha\ell}\{(h+p)F_{D_\ell^{i,s}}(y) - p\}dF_L(\ell) \\ &= (h+p)\int_0^\infty e^{-\alpha\ell}F_{D_\ell^{i,s}}(y)dF_L(\ell) - pE[e^{-\alpha L}]. \end{aligned} \tag{2.5}$$

So for $y < 0$,

$$C'_+(y,i,s) = -pE[e^{-\alpha L}]$$

and

$$\begin{aligned} (f'_{i,s})_+(y) &= C'_+(y,i,s) + \alpha c \\ &= (\alpha\bar{c} - p)E[e^{-\alpha L}]. \end{aligned}$$

If $\alpha\bar{c} - p < 0$ then $(f'_{i,s})_+(y) < 0$ for all $y < 0$ and all $i$ and $s$. So $y_i^*(s) \geq 0$ for all $i$ and $s$. Thus, for all $y$ we have

$$f_{i,s}(y) \geq f_{i,s}(y_i^*(s)) = C(y_i^*(s), i, s) + \alpha c y_i^*(s) \geq 0.$$

Furthermore, by equation (2.5) we have

$$\lim_{y\to+\infty} C'_+(y,i,s) = hE[e^{-\alpha L}] \geq 0$$

and

$$\lim_{y\to+\infty} (f'_{i,s})_+(y) = hE[e^{-\alpha L}] + \alpha c \geq 0.$$

So $y_i^*(s) < +\infty$ for all $s$ and $i$.

(c) If $\alpha\bar{c} - p \geq 0$, then

$$(f'_{i,s})_+(y) = (h+p)\int_0^\infty e^{-\alpha\ell}F_{D_\ell^{i,s}}(y)dF_L(\ell) + (\alpha\bar{c} - p)E[e^{-\alpha L}] \geq 0$$

for all $y$, $i$, and $s$, so $y_i^*(s) = -\infty$.

$\square$

By the result of Lemma 2.3.3, we conclude that we need to assume that

$$\alpha\bar{c} < p$$

to obtain a reasonable model. Intuitively, if this condition is not met, then we would always prefer to postpone ordering and pay the shortage penalty, and thus never place any orders. In the remainder of the paper, we will therefore assume that this condition is satisfied.

Before we continue, we derive a property of lead time demand.

**Lemma 2.3.4** *For any $i \in I$ and $s \geq 0$,*

$$D_L^{i,s} \leq_{st} D_L^{\bar{\lambda}}$$

*where $D_L^{\bar{\lambda}}$ represents the lead time demand when the demand process is a stationary Poisson process with rate $\bar{\lambda}$.*

**Proof:** For fixed lead time $\ell$, the conditional distribution of $D_\ell^{i,s}$, given a realization of the core process, is Poisson distributed (see, e.g., Ross [49]). Since the demand rates of all the states are bounded from above by $\bar{\lambda}$, the mean of this Poisson random variable is also bounded from above by $\bar{\lambda}\ell$, which is true for all the possible realizations of the core process. This implies that

$$D_\ell^{i,s} \leq_{st} D_\ell^{\bar{\lambda}} \qquad \text{for all } s \geq 0 \text{ and } i \in I. \tag{2.6}$$

Since inequality (2.6) holds for all fixed $\ell$, the desired result follows for the case of a stochastic lead time $L$. $\square$

We are now ready to derive the form of the optimal policy for our system, which is a continuous-time analog of Theorem 6.1 in Veinott (1965).

**Theorem 2.3.5** *Under the semi-Markov modulated Poisson demand model, the myopic policy* $\mathbf{y}^*$ *defined by*

$$y^*(t, \bar{H}(t)) = y^*_{A(t)}(S(t)) \quad \textit{for all } t \geq 0$$

*exists and its total policy costs are finite. Thus, the optimal inventory position at time* $t$ *only depends on the state at time* $t$ *and the amount of time that has elapsed since the core process last entered that state.*

**Proof:** First, note that for every fixed $i$ and $s$ we have

$$\begin{aligned} f_{i,s}(0) &= C(0, i, s) \\ &= E[e^{-\alpha L}\hat{C}(-D_L^{i,s})] \\ &= E[e^{-\alpha L} p D_L^{i,s}] \\ &\leq E[e^{-\alpha L} p D_L^{\bar{\lambda}}] && (2.7) \\ &\leq p\bar{\lambda}E[L] && (2.8) \end{aligned}$$

where inequality (2.7) follows from Lemma 2.3.4, and inequality (2.8) holds since $e^{-\alpha L} \leq 1$. Then, by the definition of $y_i^*(s)$ and Lemma 2.3.3, $f_{i,s}(y_i^*(s)) \leq f_{i,s}(0) < \infty$ and $y_i^*(s) < \infty$. Thus, for every fixed $t$ and every $\bar{h}(t) \in \mathcal{H}(t)$ for which $A(t) = i, S(t) = s$, the optimal policy for time $t$ stated in Lemma 2.3.1 does exist, and

$$y^*(t, \bar{h(t)}) = y_i^*(s).$$

Thus, by Theorem 2.3.2, the policy $\{y^*(t, \bar{H}(t)) = y^*_{A(t)}(S(t)) : t \geq 0\}$ is an optimal policy.

Furthermore, the optimal total expected cost satisfies

$$\begin{aligned}
W(x, \bar{h}(0)|\mathbf{y}^*) &= \int_0^\infty e^{-\alpha t} E\left[C(y^*_{A(t)}(S(t)), A(t), S(t)) + \alpha c y^*_{A(t)}(S(t))\right] dt \\
&= \int_0^\infty e^{-\alpha t} E\left[f_{A(t),S(t)}(y^*_{A(t)}(S(t)))\right] dt \\
&\leq \int_0^\infty e^{-\alpha t} p\bar{\lambda} E[L] dt \\
&= \frac{1}{\alpha} p\bar{\lambda} E[L]
\end{aligned}$$

and is thus finite. □

In the next section, we will give a more explicit characterization of the optimal inventory position, which can in principle be used to compute the optimal policy, as well as the cost of the optimal policy.

### 2.3.3 Determination of the Optimal Inventory Position

Using equation (2.5), we have

$$\begin{aligned}
y^*_i(s) &= \arg\min\left\{y : (f'_{i,s})_+(y) \geq 0\right\} \\
&= \arg\min\left\{y : C'_+(y, i, s) + \alpha c \geq 0\right\} \\
&= \arg\min\left\{y : \int_0^\infty e^{-\alpha \ell}(h+p) F_{D^{i,s}_\ell}(y) dF_L(\ell) - pE[e^{-\alpha L}] \geq -\alpha c\right\} \\
&= \arg\min\left\{y : \int_0^\infty e^{-\alpha \ell} F_{D^{i,s}_\ell}(y)\, dF_L(\ell) \geq \frac{pE[e^{-\alpha L}] - \alpha c}{h+p}\right\}.
\end{aligned}$$

For notational convenience, we may define

$$\tilde{F}_{D^{i,s}_L}(y) = \int_0^\infty e^{-\alpha \ell} F_{D^{i,s}_\ell}(y) dF_L(\ell)$$

so that

$$y^*_i(s) = \arg\min\left\{y : \tilde{F}_{D^{i,s}_L}(y) \geq \frac{pE[e^{-\alpha L}] - \alpha c}{h+p}\right\}. \tag{2.9}$$

This means that the optimal policy depends on the cost parameters only through the ratio

$$\frac{pE[e^{-\alpha L}] - \alpha c}{h+p} = \frac{(p - \alpha\bar{c})E[e^{-\alpha L}]}{h+p}.$$

It is easy to see that this ratio is always between 0 and 1. In case the lead time is deterministic, the expression for the optimal policy can be simplified to

$$\begin{aligned} y_i^*(s) &= \arg\min\left\{y : e^{-\alpha L} F_{D_L^{i,s}}(y) \geq \frac{pe^{-\alpha L} - \alpha c}{h+p}\right\} \\ &= \arg\min\left\{y : F_{D_L^{i,s}}(y) \geq \frac{p - \alpha \bar{c}}{h+p}\right\}. \end{aligned} \tag{2.10}$$

We next derive a more explicit expression of the myopic inventory cost rate function:

$$\begin{aligned} &C(y,i,s) \\ &= \int_0^\infty e^{-\alpha \ell}\left\{h\int_0^y (y-z)dF_{D_\ell^{i,s}}(z) + p\int_y^\infty (z-y)dF_{D_\ell^{i,s}}(z)\right\} dF_L(\ell) \\ &= \int_0^\infty e^{-\alpha \ell}\left\{hyF_{D_\ell^{i,s}}(y) - \int_0^y hz dF_{D_\ell^{i,s}}(z) + \int_y^\infty pz dF_{D_\ell^{i,s}}(z) - py\overline{F}_{D_\ell^{i,s}}(y)\right\} dF_L(\ell) \\ &= \int_0^\infty e^{-\alpha \ell}\left\{(h+p)yF_{D_\ell^{i,s}}(y) - py - hE[D_L^{i,s}] + (h+p)\int_y^\infty z dF_{D_\ell^{i,s}}(z)\right\} dF_L(\ell) \\ &= (h+p)y\int_0^\infty e^{-\alpha \ell} F_{D_\ell^{i,s}}(y) dF_L(\ell) - pyE[e^{-\alpha L}] - hE[D_L^{i,s}] + \\ &\quad (h+p)\int_0^\infty e^{-\alpha \ell}\int_y^\infty z dF_{D_\ell^{i,s}}(z) dF_L(\ell). \end{aligned}$$

This expression may be used to determine the optimal cost rate by substituting the optimal policy in this expression:

$$\begin{aligned} &f_{i,s}(y_i^*(s)) \\ &= C(y_i^*(s), i, s) + \alpha c y_i^*(s) \\ &= (h+p)\int_0^\infty e^{-\alpha \ell}\int_{y_i^*(s)}^\infty z dF_{D_\ell^{i,s}}(z) dF_L(\ell) - hE[D_L^{i,s}] + \\ &\quad y_i^*(s)\left\{(h+p)\int_0^\infty e^{-\alpha \ell} F_{D_\ell^{i,s}}(y_i^*(s)) dF_L(\ell) - (pE[e^{-\alpha L}] - \alpha c)\right\}. \end{aligned} \tag{2.11}$$

Note that if the inequality in equation (2.9) or (2.10) is in fact an equality, the last term in equation (2.11) reduces to zero, and the optimal cost rate reduces to

$$f_{i,s}(y_i^*(s)) = (h+p)\int_0^\infty e^{-\alpha \ell}\int_{y_i^*(s)}^\infty z dF_{D_\ell^{i,s}}(z) dF_L(\ell) - hE[D_L^{i,s}].$$

However, this generally can only happen if the lead time demand distribution is continuous, which is not the case in our model.

Finally, we would like to stress the similarity of the expressions in equations (2.9) and (2.11) with the optimal policy and cost in the standard newsvendor problem. It turns out that we can find the optimal policy for our model by solving one newsvendor problem for each $i$ and $s$.

### 2.3.4 Total Policy Costs

In the previous sections, we have obtained the form of the optimal policy. However, the corresponding optimal expected total cost is very hard to evaluate. In this section we will employ the underlying semi-Markov structure of our demand model to derive an easier way to determine the optimal costs.

We assume that an inventory policy characterized by $y_i(s)$ is adopted. Then define $V_i(x)$ to be the expected total costs of this policy from the time when the core process just enters state $i$ when the initial inventory position is $x$, and discounted to the time of transition. The total costs can be divided into two components: the total costs during our current stay in state $i$, and the total costs after transitioning away from state $i$. The first component can be determined by conditioning on the time until the next transition, whose distribution function is equal to

$$G_i(t) \equiv \sum_{j \in I} p_{ij} G_{ij}(t).$$

Note that this distribution, and therefore the first cost component, does not depend on the next state visited. However, for the second component we need to condition on both the time of the transition as well as the next state itself. We can then express $V_i(x)$ in terms of the other values of this function as follows:

$$\begin{aligned} V_i(x) \quad = \quad & c(y_i(0) - x) + \int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} C(y_i(s), i, s) ds + \int_0^\tau e^{-\alpha s} c dy_i(s) \right\} dG_i(\tau) + \\ & \sum_{j \in I} p_{ij} \int_0^\infty e^{-\alpha \tau} V_j(y_i(\tau)) dG_{ij}(\tau). \end{aligned}$$

Since the initial inventory position $x$ is not affected by the ordering decisions made for state $i$ period, let

$$V_i = V_i(x) + cx.$$

Using this definition for all states and all inventory positions, we obtain

$$\begin{aligned} V_i &= cy_i(0) + \int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} C(y_i(s), i, s) ds + \int_0^\tau e^{-\alpha s} c dy_i(s) \right\} dG_i(\tau) + \\ &\quad \sum_{j \in I} p_{ij} \int_0^\infty e^{-\alpha\tau} V_j dG_{ij}(\tau) - \sum_{j \in I} p_{ij} \int_0^\infty e^{-\alpha\tau} c y_i(\tau) dG_{ij}(\tau) \\ &= cy_i(0) + \int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} C(y_i(s), i, s) ds + \right. \\ &\qquad \left. \int_0^\tau e^{-\alpha s} c dy_i(s) - c e^{-\alpha\tau} y_i(\tau) \right\} dG_i(\tau) + \\ &\quad \sum_{j \in I} p_{ij} \int_0^\infty e^{-\alpha\tau} V_j dG_{ij}(\tau) \\ &= \int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} [C(y_i(s), i, s) + \alpha c y_i(s)] ds \right\} dG_i(\tau) + \sum_{j \in I} p_{ij} E[e^{-\alpha T_{ij}}] V_j \\ &= \int_0^\infty \bar{G}_i(s) e^{-\alpha s} [C(y_i(s), i, s) + \alpha c y_i(s)] ds + \sum_{j \in I} p_{ij} E[e^{-\alpha T_{ij}}] V_j \end{aligned}$$

where $T_{ij} \sim G_{ij}$ denotes the time spent in state $i$ when the next state is $j$, and we have also used a similar derivation as in equation (2.3) to simplify the expression for the costs while in state $i$. Now observe that we can in principle compute, for all $i$, the total costs from the time of transition to state $i$ by solving a system of linear equations if we can compute the total costs while in state $i$ for all $i$. Computing the total costs while in state $i$ is clearly still nontrivial, but much easier than directly trying to compute the infinite horizon costs.

## 2.4 Monotonicity Results

In this section we will show that, if the demand process possesses certain monotonicity properties, the optimal inventory positions over time inherit these properties.

### 2.4.1 Monotonicity of Optimal Inventory Positions within a Given State

In order to be able to analyze the behavior of the optimal inventory positions $y_i^*(s)$ while in a given state $i \in I$, we will first derive a general stochastic dominance result.

We say that a random variable $X$ has a *Conditional Poisson* distribution with random parameter $\Lambda$, where $\Lambda$ is a nonnegative random variable, if the conditional random variable $X|\Lambda = \lambda$ has a Poisson distribution with parameter $\lambda$. The following lemma will then prove useful later in this section.

**Lemma 2.4.1** *Let*

$$\begin{aligned} X_1 &\sim \textit{Conditional Poisson}(\Lambda_1) \\ X_2 &\sim \textit{Conditional Poisson}(\Lambda_2). \end{aligned}$$

*If* $\Lambda_1 \leq_{st} \Lambda_2$, *then* $X_1 \leq_{st} X_2$.

**Proof:** Fix some $x \geq 0$, and define

$$\phi_x(\lambda) = \Pr(X_1 \geq x|\Lambda_1 = \lambda) = \Pr(X_2 \geq x|\Lambda_2 = \lambda).$$

This function is increasing in $\lambda$ by the fact that a Poisson random variable is stochastically increasing in its mean (see Example 9.2(b) in Ross [49]). Therefore, the assumption in the theorem says that $E[\phi_x(\Lambda_1)] \leq E[\phi_x(\Lambda_2)]$. Now denote the distribution of $\Lambda_n$ by $H_n$ $(n = 1, 2)$. Then, for $n = 1, 2$,

$$\begin{aligned} \Pr(X_n \geq x) &= \int_0^\infty \Pr(X_n \geq x|\Lambda_n = \lambda)\, dH_n(\lambda) \\ &= E[\Pr(X_n \geq x|\Lambda_n] \\ &= E[\phi_x(\Lambda_n)]\,. \end{aligned}$$

This yields the desired result. $\square$

Returning to the focus of this section, denote the state of the core process after $t$ time units if the process has currently been in state $i$ for $s$ time units by $[A(t+s)|A(s)=i]$. We will show that the following condition implies that the function $y_i^*$ is increasing in $s$:

**Condition 2.4.2** *For all $\ell \geq 0$ and all $0 \leq s < s'$,*

$$\int_0^\ell \lambda_{[A(t+s)|A(s)=i]}dt \leq_{st} \int_0^\ell \lambda_{[A(t+s')|A(s')=i]}dt.$$

The following lemma shows that Condition 2.4.2 ensures that the lead time demands are stochastically increasing while in a given state.

**Lemma 2.4.3** *If the demand process satisfies Condition 2.4.2 for some $i \in I$, then the lead time demand $D_L^{i,s}$ is stochastically increasing in $s$, i.e.,*

$$D_L^{i,s} \leq_{st} D_L^{i,s'} \quad \textit{for all } 0 \leq s < s'.$$

**Proof:** Let $i \in I$ be such that Condition 2.4.2 is satisfied. Fix $0 \leq s < s'$ and consider a fixed lead time $\ell$. By the theory of nonhomogeneous Poisson processes (see, e.g., Ross (1996)), the lead time demand is a Poisson random variable when conditioned on the core process $A$. This means that

$$\begin{aligned} D_\ell^{i,s} &\sim \text{Conditional Poisson}\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]}dt\right) \\ D_\ell^{i,s'} &\sim \text{Conditional Poisson}\left(\int_0^\ell \lambda_{[A(t+s')|A(s')=i]}dt\right). \end{aligned}$$

Condition 2.4.2 now implies that

$$\int_0^\ell \lambda_{[A(t+s)|A(s)=i]}dt \leq_{st} \int_0^\ell \lambda_{[A(t+s')|A(s')=i]}dt.$$

By Lemma 2.4.1, we then have

$$D_\ell^{i,s} \leq_{st} D_\ell^{i,s'}. \tag{2.12}$$

Since this inequality holds for all fixed lead times $\ell$, the desired result follows for the stochastic lead time case as well. $\square$

We are now able to show that the optimal inventory positions in a given state are increasing over time if the demand process in that state is stochastically increasing over time in the sense of Condition 2.4.2.

**Theorem 2.4.4** *If the demand process satisfies Condition 2.4.2 for some $i \in I$, then the optimal inventory position $y_i^*(s)$ is increasing in $s$.*

**Proof:** Let $i \in I$ be such that Condition 2.4.2 is satisfied. Observe that inequality (2.12) in the proof of Lemma 2.4.3 says that, for fixed lead time $\ell$, for all $0 \leq s < s'$, and for all $y \geq 0$

$$F_{D_\ell^{i,s}}(y) \geq F_{D_\ell^{i,s'}}(y).$$

By integration, we then obtain that for all $0 \leq s < s'$ and for all $y \geq 0$,

$$\tilde{F}_{D_L^{i,s}}(y) \geq \tilde{F}_{D_L^{i,s'}}(y).$$

The result now follows immediately from the expression for the optimal inventory position $y_i^*(s)$ in equation (2.9). $\square$

We will next discuss two examples to which this result applies.

**Examples**

1. If the underlying core process is a continuous-time *Markov process*, the memoryless property implies that the random variables

$$\int_0^\ell \lambda_{[A(t+s)|A(s)=i]}dt \text{ and } \int_0^\ell \lambda_{[A(t+s')|A(s')=i]}dt$$

both have the same probability distribution as

$$\int_0^\ell \lambda_{[A(t)|A(0)=i]}dt.$$

Therefore, it immediately follows that Condition 2.4.2 is satisfied for all $i \in I$, and therefore the optimal inventory position $y_i^*(s)$ is increasing in $s$. In fact, we can use appropriate modifications of Lemma 2.4.3 and Theorem 2.4.4 to show that the inventory position $y_i^*(s)$ is constant in $s$, which corresponds with the result of Song and Zipkin [52].

2. Suppose that the interarrival distributions $G_{ij}$ depend on $i$ only and, moreover, are increasing failure rate (IFR). In addition, suppose that transitions can only be made to states with a higher demand rate, that is, $p_{ij} > 0$ implies that $\lambda_i < \lambda_j$. Then Condition 2.4.2 is satisfied for all $i \in I$, and we obtain monotonicity of the inventory positions.

   **Proof:** Choose some state $i \in I$, and let $Z_i^s$ be the amount of time remaining in state $i$ given that the core process has been in state $i$ for $s$ time units, and denote its distribution by $G_i^s$. To show that Condition 2.4.2 holds, we need to show that for all $\ell \geq 0$ and all $x$,

$$\Pr\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]} dt \geq x\right)$$

   is increasing in $s$. Now fix arbitrary values of $\ell \geq 0$ and $x$, and define

$$\psi_s(z) = \Pr\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]} dt \geq x | Z_i^s = z\right).$$

   Since

$$E[\psi_s(Z_i^s)] = \Pr\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]} dt \geq x\right)$$

   we in fact need to show that $E[\psi_s(Z_i^s)]$ is increasing in $s$. Since $G_i$ is IFR, we know that the random variables $Z_i^s$ are stochastically decreasing in $s$. In the remainder of the proof, we will show that the function $\psi_s(z)$ is decreasing in $z$ and independent of $s$, which then implies the desired result (see Ross, [49].

   For any $i \in I$, let $J_i$ be a random variable that represents the next state reached from state $i$ by the core process. If $0 \leq z < \ell$, we can then rewrite the function

$\psi_s$ as follows:

$$\begin{aligned} \psi_s(z) &= \Pr\left(\lambda_i z + \int_0^{\ell-z} \lambda_{[A(t)|A(0)=J_i]} dt \geq x | Z_i^s = z\right) \\ &= \Pr\left(\lambda_i z + \int_0^{\ell-z} \lambda_{[A(t)|A(0)=J_i]} dt \geq x\right) \qquad (2.13) \\ &= \Pr\left(\int_0^{\ell-z} \left(\lambda_{[A(t)|A(0)=J_i]} - \lambda_i\right) dt \geq x - \lambda_i \ell\right) \end{aligned}$$

where equality (2.13) follows from the fact that the distribution of the remaining transition time $Z_i^s$ is independent of the next state visited. If $z \geq \ell$, we have

$$\psi_s(z) = \Pr\left(\int_0^{\ell} \lambda_i dt \geq x | Z_i^s = z\right) = \Pr\left(\lambda_i \ell \geq x\right) = 1_{\{\lambda_i \ell \geq x\}}$$

where $1_{\{\}}$ denotes the indicator function. Summarizing, we have

$$\psi_s(z) = \begin{cases} \Pr\left(\int_0^{\ell-z} \left(\lambda_{[A(t)|A(0)=J_i]} - \lambda_i\right) dt \geq x - \lambda_i \ell\right) & \text{if } z < \ell \\ 1_{\{\lambda_i \ell \geq x\}} & \text{if } z \geq \ell \end{cases}$$

which is clearly independent of $s$. Moreover, if $x \leq \lambda_i \ell$, the function $\psi_s$ is identically equal to 1 and therefore decreasing. If $x > \lambda_i \ell$, the assumption in this example says that $\lambda_{[A(t)|A(0)=J_i]} - \lambda_i \geq 0$, so that in that case the function $\psi_s(z)$ is decreasing as well.

Since we chose the values of $\ell \geq 0$ and $x$ arbitrarily, we conclude that Condition 2.4.2 is satisfied for state $i$. □

The next theorem shows that, under an additional mild regularity condition, the optimal inventory position in a given state is a step function of time that increases only by one unit in each step.

**Theorem 2.4.5** *Suppose that the demand process satisfies Condition 2.4.2 for some $i \in I$ and, in addition, the transition time distributions $G_{ij}$ from that state have continuous densities. Then the optimal inventory position function $y_i^*$ is a step function that can only have step size 1 in the inventory position space.*

**Proof:** Let $i \in I$ be such that Condition 2.4.2 is satisfied, and suppose that we have been in this state for $s$ time units. Then recall that, for a fixed lead time $\ell$, we have

$$D_\ell^{i,s} \sim \text{Conditional Poisson}\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]} dt\right).$$

By the assumptions on the transition time distributions, we conclude that the random variables

$$\int_0^\ell \lambda_{[A(t+s)|A(s)=i]} dt$$

have densities that are continuous as a function of $s$. This implies that, for all $d = 0, 1, 2, \ldots$, the probability $\Pr(D_\ell^{i,s} = d)$ is continuous as a function of $s$. Thus, for a stochastic lead time $L$, $\tilde{F}_{D_L^{i,s}}(y)$ is continuous in $s$ for all $y \geq 0$ as well. In addition, $\tilde{F}_{D_L^{i,s}}(y)$ is a decreasing function of $s$ by the proof of Theorem 2.4.4.

Since the lead time demand is a discrete random variable that has strictly positive probability at every nonnegative integer $d$, we conclude that, for fixed $s$, $\tilde{F}_{D_L^{i,s}}(d)$ is a strictly increasing function of $d$ for $d = 0, 1, 2, \ldots$. This means that all functions in the family $\{\tilde{F}_{D_L^{i,s}}(y) : y \geq 0\}$, viewed as functions of $y$, are step functions that strictly increase at each integral value of $y$.

We conclude that $y_i^*(s)$ is a step function that, at each step, increases by exactly 1. □

As a final remark, note that, with probability 1, each order either replenishes a reduction in inventory position due to demand, or is due to an increase in the optimal inventory position. Therefore, under the conditions of Theorem 2.4.5, as long as the core process remains in a given state, each order is, with probability 1, for a single unit only.

### 2.4.2 Monotonicity of Optimal Inventory Positions between States

We will next analyze the relationship between the optimal inventory positions in different states. In particular, we will show that if the demand process satisfies the following condition for a pair of states $i, j \in I$, then the inventory position never decreases if a transition is made from state $i$ to state $j$.

**Condition 2.4.6** *For all $\ell \geq 0$ and all $s \geq 0$,*

$$\int_0^{\ell} \lambda_{[A(t+s)|A(s)=i]} dt \leq_{st} \int_0^{\ell} \lambda_{[A(t)|A(0)=j]} dt.$$

The following lemma shows that Condition 2.4.6 implies a monotonicity relationship between the lead time demands in different states.

**Lemma 2.4.7** *If the demand process satisfies Condition 2.4.6 for states $i, j \in I$, then*

$$D_L^{i,s} \leq_{st} D_L^{j,0} \quad \textit{for all } s \geq 0.$$

**Proof:** Fix $s \geq 0$, and consider a fixed lead time $\ell$. We then have

$$\begin{aligned} D_\ell^{i,s} &\sim \text{Conditional Poisson}\left(\int_0^{\ell} \lambda_{[A(t+s)|A(s)=i]} dt\right) \\ D_\ell^{j,0} &\sim \text{Conditional Poisson}\left(\int_0^{\ell} \lambda_{[A(t)|A(0)=j]} dt\right). \end{aligned}$$

Condition 2.4.6 and Lemma 2.4.1 then imply that

$$D_\ell^{i,s} \leq_{st} D_\ell^{j,0}. \tag{2.14}$$

Since this inequality holds for all fixed lead times $\ell$, the desired result follows for the stochastic lead time case as well. □

We are now able to show that the optimal inventory positions are increase when a state transition is made if the demand process satisfies Condition 2.4.6 for that transition.

**Theorem 2.4.8** *If the demand process satisfies Condition 2.4.6 for states $i, j \in I$ then*

$$y_i^*(s) \leq y_j^*(0) \quad \textit{for all } s \geq 0.$$

**Proof:** Observe that inequality (2.14) in the proof of Lemma 2.4.7 says that, for fixed lead time $\ell$, and for all $y \geq 0$

$$F_{D_\ell^{i,s}}(y) \geq F_{D_\ell^{j,0}}(y).$$

By integration, we then obtain that for all $y \geq 0$,

$$\tilde{F}_{D_L^{i,s}}(y) \geq \tilde{F}_{D_L^{j,0}}(y).$$

The result now follows immediately from the definition of the optimal inventory position $y_i^*(s)$ in equation (2.9). □

We will next discuss two examples to which this result applies.

**Examples**

1. If the underlying core process is a continuous-time *Markov process* and, in addition,

$$[A(t)|A(0) = i] \leq [A(t)|A(0) = j] \text{ w.p. 1, for all } t \geq 0$$

for all $i, j \in I$ such that $\lambda_i < \lambda_j$ it immediately follows that Condition 2.4.6 is satisfied for such pairs of states. Theorem 2.4.8 then corresponds to Theorem 8 in Song and Zipkin [52].

2. If the transition time distributions are arbitrary, but states are always visited in a predetermined sequence, i.e., $p_{i,i+1} = 1$ for all $i \in I$, and, in addition, $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, then Condition 2.4.6 is satisfied for all $i, j \in I$ such that $j > i$.

   **Proof:** Choose some state $i \in I$, and fix arbitrary values of $\ell \geq 0$ and $x$. We need to show that

$$\Pr\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]} dt \geq x\right) \leq \Pr\left(\int_0^\ell \lambda_{[A(t)|A(0)=i+1]} dt \geq x\right).$$

Now note that

$$\Pr\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]}dt \geq x|Z_i^s = 0\right) = \Pr\left(\int_0^\ell \lambda_{[A(t)|A(0)=i+1]}dt \geq x\right).$$

Using the notation of the proof of Example 2 in Section 2.4.1 we have

$$\begin{aligned}
\psi_s(0) &= \Pr\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]}dt \geq x|Z_i^s = 0\right) \\
E[\psi_s(Z_i^s)] &= \Pr\left(\int_0^\ell \lambda_{[A(t+s)|A(s)=i]}dt \geq x\right) \\
E[\psi_s(Z_i^s)] &\leq \psi_s(0)
\end{aligned}$$

where we have used the fact that $J_i \equiv i+1$, and the last inequality follows from the fact that the function $\psi_s$ is decreasing. Since we chose the values of $\ell \geq 0$ and $x$ arbitrarily, this implies that Condition 2.4.6 is satisfied for $i, i+1 \in I$. □

### 2.4.3 Implications of the Monotonicity Results

The results of the previous two sections can now be used to conclude our main monotonicity result:

**Theorem 2.4.9** *Assume that the demand process satisfies Condition 2.4.2 for all $i \in I$ and, in addition, Condition 2.4.6 is satisfied for all $i, j \in I$ such that $p_{ij} > 0$. Then the optimal policy results in a sequence of inventory positions that is nondecreasing.*

**Proof:** Theorem 2.4.4 says that the optimal inventory position never decreases as long as we are in a given state. Since Condition 2.4.6 is satisfied whenever it is possible to transition from state $i$ to state $j$, and Theorem 2.4.8 says that the optimal inventory position never decreases when we move to a new state. □

So far, we have dealt entirely with a situation where disposal of inventory is allowed at the purchase price. However, this may not always be a reasonable assumption. If disposal is not possible, then at each point in time the inventory position is bounded from below by the inventory position immediately preceding an ordering decision. In that case, the myopic policy may no longer be feasible, and therefore clearly not optimal. However, note that under the conditions of Theorem 2.4.9, the myopic inventory position will always increase. The following theorem now provides a sufficient condition under which disposal is never desirable, so that the myopic policy remains optimal even if disposal is not allowed.

**Corollary 2.4.10** *Consider the case where disposal of inventory is not allowed. Assume that the demand process satisfies Condition 2.4.2 for all $i \in I$ and, in addition, Condition 2.4.6 is satisfied for all $i, j \in I$ such that $p_{ij} > 0$. If, initially (at time 0), the core process has been in state $i \in I$ for $s$ time units, and the initial inventory $x$ is no larger than $y_i^*(s)$, then the myopic policy is the optimal policy.*

**Proof:** Since $x < y_i^*(s)$, the initial myopic inventory position $y_i^*(s)$ can be reached even if disposal is not allowed. At each subsequent point in time, an ordering decision should either replenish a demand or adjust the inventory position according to an optimal policy. Since Theorem 2.4.8 implies that the myopic policy will never prescribe a reduction in inventory position, it remains optimal even when disposal is not allowed. $\square$

Our next result derives the optimal policy if the condition in Corollary 2.4.10 on the initial inventory level is violated. This theorem is a continuous analog of Theorem 6.2 in Veinott [56].

**Corollary 2.4.11** *Consider the case where disposal is not allowed. Assume that the demand process satisfies Condition 2.4.2 for all $i \in I$ and, in addition, Condition 2.4.6 is satisfied for all $i, j \in I$ such that $p_{ij} > 0$. Then the policy, say $\tilde{\mathbf{y}}$, that does not order until the inventory position drops below the level that is prescribed by the myopic policy, and follows the myopic policy after that, is the optimal policy.*

**Proof:** Let $T$ denote the earliest time at which the inventory position drops to or below the level prescribed by the myopic policy when applying policy $\tilde{\mathbf{y}}$ (where we let $T = +\infty$ if this event never occurs). Up to time $T$, the inventory positions resulting from this policy will be the lowest among all feasible policies since $\tilde{\mathbf{y}}$ does not place any orders. Since the cost rate function at each point in time is convex, and the myopic policy is the smallest minimizer of the cost rate function, policy $\tilde{\mathbf{y}}$ will minimize the total cost over the interval $0 \le t \le T$ among all feasible policies. After time $T$, Corollary 2.4.10 implies hat the myopic policy will be optimal. So the policy described in this theorem is an optimal policy. □

## 2.5 An Algorithm to Compute the Optimal Inventory Policy

Recall that the expression for the optimal inventory position after spending $s$ time units in state $i$ is given by

$$y_i^*(s) = \arg\min\left\{y : \tilde{F}_{D_L^{i,s}}(y) \ge \frac{pE[e^{-\alpha L}] - \alpha c}{h+p}\right\}. \tag{2.9}$$

Note that the optimal inventory policy is thus a collection of functions, one for each state of the world. We therefore cannot expect to be able to compute in finite time (or represent using finite storage space) the entire optimal inventory policy for our model. In addition, $s$ is a continuous variable, which further complicates the a priori computation of the optimal inventory policy. Instead, we will in this section develop an algorithm that constructs parts of the optimal policy as needed for a special case of our model. In addition, we denote $\otimes$ and $\oplus$ to represent the Kronecker product and Kronecker sum, respectively.

### 2.5.1 Continuous Phase-Type Distriuted World Transition Time and Lead Time

From equation (2.9) we see that the key is the lead time demand distribution functions $\tilde{F}_{D_L^{i,s}}(y)$. However, in general it is very difficult to compute these lead time demand distributions directly (see Zipkin [63]). Song and Zipkin [52] designed an algorithm to compute the myopic policy for a special case of their Markov modulated Poisson demand model. Specifically, they devised a way to compute the cost rate function by assuming that the lead time has a continuous phase-type (CPH) distribution, which can be modeled by the time until absorption of a continuous-time Markov chain. Then they study the behavior of a joint process consisting of four Markov processes: the world process, demand process, lead time process, and the process used to represent continuous-time discounting. After some nontrivial transformations, they can compute the lead time demand distributions and cost rate function.

In this section, we apply the idea of this algorithm to our more complicated demand model when the lead time $L$ has a continuous phase type distribution. In addition, we assume that the transition time for leaving each world state is also continuous phase type distributed. Assume that we *cannot* observe the phase changes within this transition period. (Recall from Section 2.2.2 that, if the phase changes of the *Erlang* transition distributions, which are special CPH distributions, are observable, then we can transform the core process into a Markov process by using an extended state representation $(A(t), r(t))$ and a transformed transition probability matrix.) We first denote the probability mass function of $D_\ell^{i,s}$ for fixed $\ell$, given $i, s$, by

$$b_{i,s}(d|\ell) \quad = \quad \Pr(D_\ell^{i,s} = d)$$

then for random lead time $L$, we define

$$q_{i,s}(d) \quad = \quad E_L[e^{-\alpha L} b_{i,s}(d|L)] = \int_0^\infty e^{-\alpha\ell} b_{i,s}(d|\ell) dF_L(\ell).$$

With these two notations, we can express the lead time demand distribution function as

$$F_{D_\ell^{i,s}}(y) = \sum_{d=0}^{y} b_{i,s}(d|\ell)$$

for integer values of $y$. In addition, we can write

$$\tilde{F}_{D_L^{i,s}}(y) = \int_0^\infty e^{-\alpha\ell} F_{D_\ell^{i,s}}(y) dF_L(\ell) = \int_0^\infty e^{-\alpha\ell} \left( \sum_{d=0}^{y} b_{i,s}(d|\ell) \right) dF_L(\ell) = \sum_{d=0}^{y} q_{i,s}(d)$$

for integer values of $y$. It is easy to see that

$$\sum_{d=0}^{\infty} q_{i,s}(d) \quad = \quad E_L[e^{-\alpha L}].$$

Now we can see that the task of computing $\tilde{F}_{D_L^{i,s}}(y)$ can be accomplished by computing $q_{i,s}(d)$. In addition, we can express the discounted cost rate function in terms of $q_{i,s}(d)$ as

$$\begin{aligned}
C(y,i,s) &= E\left[e^{-\alpha L}\hat{C}(y - D_L^{i,s})\right] \\
&= E_L\left[e^{-\alpha L} h \sum_{d=0}^{y}(y-d)b_{i,s}(d|L)\right] + E_L\left[e^{-\alpha L} p \sum_{d=y}^{\infty}(d-y)b_{i,s}(d|L)\right] \\
&= h\sum_{d=0}^{y}(y-d)q_{i,s}(d) + p\sum_{d=y}^{\infty}(d-y)q_{i,s}(d).
\end{aligned}$$

We adopt the same assumptions as in Song and Zipkin [52] to develop our algorithm. Define $e$ to be a column vector whose elements are all 1, while $e_i$ is a unit column vector where the $i^{th}$ element is 1 and all other elements are 0. For completeness sake, we next briefly review some results from Song and Zipkin [52] that we need to further develop our algorithm. We assume that the leadtime $L$ has a continuous phase-type distribution $(\tau, M)$, where $\tau$ is row vector with $\rho$ nonnegative components whose sum is no longer than 1, and $M$ is an $\rho \times \rho$ nonsingular matrix whose off-diagonal entries are all nonnegative and whose diagonal entries as well as row sums are all nonpositive. Let $U$ be a continuous-time Markov chain with $\rho + 1$ states, where the last state is an absorbing state, initial probability distribution $[\tau, 1 - \tau e]$, and generator

$$\begin{bmatrix} M & -Me \\ 0 & 0 \end{bmatrix}.$$

Then the time until absorption of this chain is distributed as $L$. We assume $\tau e = 1$, so that $L$ has no mass at zero, i.e., $L \neq 0$.

In addition, we assume that the world transition time from state $i$ is a continuous phase-type distribution $(\varsigma_i, H_i)$, which can represented as the time until absorption of a continuous-time Markov chain $V$ with $r_i$ transient states and one absorbing states. We assume that for all $i$, $\varsigma_i e = 1$. If we can observe the phase changes of this Markov chain, we can translate the world process $A$ into a Markov chain with state $(i, r)$, where $r$ is the phase of the $V$. Let $Q$ denote the generator of the transformed world process of dimension $\sum_{i=1}^{m} r_i$. For example, for $i \neq j$, the rate $q_{(i,r');(j,r'')} = -(H_i e)^T e_{r'} p_{ij} (\varsigma_j)^T e_{r''}$, where $(H_i e)^T$ represents the transpose of matrix $H_i e$.

For a continuous phase type distribution $(\varsigma_i, H_i)$, the probability that it is in each state after $s$ time units, denoted by $\pi_{i,s}$, is the solution to the following differential equation(s):

$$\pi'_{i,s} = \pi_{i,s} H_i \tag{2.15}$$

with boundary condition $\pi_{i,0} = \varsigma_i$. It is easy to solve that $\pi_{i,s} = \varsigma_i e^{H_i s}$. Thus given that the last transition was into world state $i$, the conditional probability that the phase of the CPH distribution after spending $s$ time units in the current state, denoted by $r(s)$, is equal to $r$, denoted by $R_{i,s}(r)$, can be computed as

$$R_{i,s}(r) = \frac{\pi_{i,s} e_r}{\sum_{r=1}^{r_i} \pi_{i,s} e_r}.$$

It is easy to verify that $R_{i,0}(r) = \varsigma_i e_r$, the $r^{th}$ element of the initial probability distribution $\varsigma_i$ for state $i$.

Since a CPH distribution is interpreted in term of a Markov chain, given the current state of the Markov chain (i.e., the phase of the CPH), the time that has elapsed since the CPH distribution started, $s$, becomes irrelevant due to the memoryless property of a continuous-time Markov chain. So we can define

$$\begin{aligned}\Pr(D_\ell^{i,s} = d|r(s) = r) &= \Pr(D_\ell^{i,0} = d|r(0) = r) \\ &\equiv b_i(d|\ell, r) \qquad (2.16)\end{aligned}$$

as well as

$$\begin{aligned}\int_0^\infty e^{-\alpha\ell} \Pr(D_\ell^{i,s} = d|r(s) = r)dF_L(\ell) &= \int_0^\infty e^{-\alpha\ell} \Pr(D_\ell^{i,0} = d|r(0) = r)dF_L(\ell) \\ &\equiv q_i(d|r). \qquad (2.17)\end{aligned}$$

By conditioning on the phase of the CPH world transition time, we have

$$\begin{aligned}b_{i,s}(d|\ell) &= \sum_{r=1}^{r_i} R_{i,s}(r) \Pr(D_\ell^{i,s} = d|r(s) = r) \\ &= \sum_{r=1}^{r_i} R_{i,s}(r) b_i(d|\ell, r)\end{aligned}$$

where we have used equation (2.16) and

$$\begin{aligned}q_{i,s}(d) &= \int_0^\infty e^{-\alpha\ell} \sum_{r=1}^{r_i} R_{i,s}(r) \Pr(D_\ell^{i,s} = d|r(s) = r)dF_L(\ell) \\ &= \sum_{r=1}^{r_i} R_{i,s}(r) \int_0^\infty e^{-\alpha\ell} \Pr(D_\ell^{i,s} = d|r(s) = r)dF_L(\ell) \\ &= \sum_{r=1}^{r_i} R_{i,s}(r) q_i(d|r)\end{aligned}$$

where we have used equation (2.17). Now we can use Song and Zipkin's approach to compute the function $q_i(d|r)$ for every $i$ and $r$. The difference here is that we use a composite state $(i, r)$ to replace the world state $i$ in Song and Zipkin's model, and change our world process into a Markov process, as we did in Section 2.2.2.

We assume that the world process $A$ (note we have converted the state of this process into $(i, r)$) and the demand process $D$ stop changing and remain fixed when $U(t) = \rho + 1$. Thus for any realization of the process $A, U$ and $D$, the final value of the first of $D$ is precisely a realization of the lead time demand. To incorporate the discount factor, we construct an auxiliary continuous-time Markov chain $J$, *independent* of $A, U$ and $D$, with two states, an initial state 0 and a "killing" state 1. We state with $J(0) = 0$, and state 1 is absorbing. While $U(t) \leq \rho$, the transition rate from state 0 to 1 is the discount factor $\alpha$; when $U(t) = \rho + 1$, the process $J$ stops changing and remains fixed. Thus the probability that the process $J$ is not killed by the end of the leadtime is

$$\Pr(X > L) = \int_0^\infty \Pr(X > \ell | L = \ell) dF_L(\ell) = \int_0^\infty e^{-\alpha \ell} dF_L(\ell) = E[e^{-\alpha L}].$$

For fixed $i, s$, if the world is in the phase $r$ of its CPH distribution, then we consider the joint chain $\{D, A, U, J\}$. Using the generator of the joint process, we can write differential equations to represent the dynamics of the system, and solve them. Denote $I$ as the identity matrix of size $\sum_{i=1}^m r_i$, and

$$\begin{aligned}
\Lambda &= diag(\lambda_i) \\
K_\alpha &= -[Q \oplus M - \Lambda \otimes I - \alpha I \otimes I] \\
H_\alpha &= K_\alpha^{-1}[\Lambda \otimes I].
\end{aligned}$$

Then similarly as in Song and Zipkin [52], we can get

$$q_i(d|r) = \Pr(D(L) = d, J(L) = 0 | A(0) = i, r(0) = r).$$

After several steps of transformations, we can get

$$q_i(d|r) = (e_{ir} \otimes \tau) H_\alpha^d [I - H_\alpha - \alpha K_\alpha^{-1}][e \otimes e].$$

Now we can express the discounted cost rate function as

$$
\begin{aligned}
C(y,i,s) &= h\sum_{d=0}^{y}(y-d)q_{i,s}(d) + p\sum_{d=y}^{\infty}(d-y)q_{i,s}(d) \\
&= h\sum_{d=0}^{y}(y-d)\sum_{r=1}^{r_i}R_{i,s}(r)q_i(d|r) + p\sum_{d=y}^{\infty}(d-y)\sum_{r=1}^{r_i}R_{i,s}(r)q_i(d|r) \\
&= \sum_{r=1}^{r_i}R_{i,s}(r)\{h\sum_{d=0}^{y}(y-d)q_i(d|r) + p\sum_{d=y}^{\infty}(d-y)q_i(d|r)\} \\
&= \sum_{r=1}^{r_i}R_{i,s}(r)\Bigg\{p(e_{ir}\otimes\tau)(I-H_\alpha)^{-2}(I-H_\alpha-\alpha K_\alpha^{-1})(e\otimes e)- \\
&\qquad p(y-1)\tau(\alpha I-M)^{-1}Me(p+h)(e_{ir}\otimes\tau)\cdot \\
&\qquad \left[\sum_{d=0}^{y}(y-d)H_\alpha^d\right](I-H_\alpha-\alpha K_\alpha^{-1})(e\otimes e)\Bigg\} \\
&= \sum_{r=1}^{r_i}R_{i,s}(r)\tilde{C}(y,i,r) \qquad (2.18)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{C}(y,i,r) &= h\sum_{d=0}^{y}(y-d)q_i(d|r) + p\sum_{d=y}^{\infty}(d-y)q_i(d|r) \\
&= (h+p)\sum_{d=0}^{y}(y-d)q_i(d|r) + p\sum_{d=0}^{\infty}(d-y)q_i(d|r) \\
&= (h+p)\sum_{d=0}^{y}(y-d)q_i(d|r) + p\sum_{d=0}^{\infty}dq_i(d|r) - pyE_L[e^{-\alpha L}] \\
&= p(e_{ir}\otimes\tau)(I-H_\alpha)^{-2}(I-H_\alpha-\alpha K_\alpha^{-1})(e\otimes e) - \\
&\qquad p(y-1)\tau(\alpha I-M)^{-1}Me + (p+h)(e_{ir}\otimes\tau)\cdot \\
&\qquad \left[\sum_{d=0}^{y}(y-d)H_\alpha^d\right](I-H_\alpha-\alpha K_\alpha^{-1})(e\otimes e).
\end{aligned}
$$

We see that $C(y,i,s)$ is represented as a convex combination of the functions $\tilde{C}(y,i,r)$ for all $r=1,\ldots,r_i$, where the weights depend on the value of $s$, as its right and left derivatives. Thus, the left and right derivatives of $f_{i,s}(y) = C(y,i,s)+\alpha cy$ at integer point $y$ are simply $(C(y,i,s))'_+ + \alpha c$ and $(C(y,i,s))'_- + \alpha c$, respectively. As shown in Section 2.3, the optimal inventory position $y^*$ will minimize $f_{i,s}(y)$ if at $y^*$

the right derivative is greater than or equal to 0, while the left derivative is smaller than 0.

We need a result regarding the changes of optimal inventory position when each of the world transition time distribution is continuous. The proof is very similar to that of Theorem 2.4.5, thus omitted here.

**Theorem 2.5.1** *Suppose that the transition time distribution $G_{ij}$ from any state $i$ to state $j$, $i, j \in I$, have continuous density. Then if the world does not change state, every time that the optimal inventory position function $y_i^*$ changes, it will either increase by 1 or decrease by 1.*

If now the world has been in state $i$ for $s_0$ time units, and the current inventory position is optimal, then as time $s$ increases, the optimal inventory position may change. By Theorem 2.5.1, as long as the world state remains unchanged, it will change by one, either increasing or decreasing. This situation is illustrated in Figure 2–1 through Figure 2–3. So to determine after how long the optimal inventory position will change to a different value, we need to compute the left and right derivatives of $\tilde{C}(y, i, r)$ at $y^*$ for all $r$, and making use of equation (2.18). In other words, we need to solve each of the following two equations,

$$\sum_{r=1}^{r_i} R_{i,s}(r) \left(\tilde{C}(y^*, i, r)\right)'_+ + \alpha c = 0 \tag{2.19}$$

$$\sum_{r=1}^{r_i} R_{i,s}(r) \left(\tilde{C}(y^*, i, r)\right)'_- + \alpha c = 0. \tag{2.20}$$

Only the solutions for $s$ to the above two equations are candidate times at which the optimal inventory position will change. Let $s_1^+ \leq s_2^+ \leq \cdots$ and $s_1^- \leq s_2^- \leq \cdots$ be the solutions to equations (2.19) and (2.20) that are strictly greater than $s_0$, respectively. Only these solutions are candidate times at which the optimal inventory position will change. Note that it is possible that either of these equations does not have such a solution. If neither equation has such a solution, we know that the current optimal inventory position will continue to be optimal as long as the world state does not change. For equation (2.19), we check its candidate solutions as described above in increasing order, to find out the smallest one at which the right-hand-side of the equation has a negative derivative, and denote it by $s'$. If no such solution exist, we let $s' = \infty$. We follow a similar procedure for equation (2.20), except that we now choose the solution at which the derivative is greater than 0. Denote that solution by $s''$. If $s' < s''$, then after $s'$ time units the optimal inventory position will increase by one; if $s' > s''$, then after $s''$ time units the optimal inventory position will decrease by one. If both $s'$ and $s''$ are infinite, then the optimal inventory position will remain unchanged unless the world state changes. Note that it is not possible that $s' = s'' < \infty$, which would mean that at time $s' = s''$ the function $C(y,i,s)+\alpha cy$ has a positive left derivative and a negative right derivative, which contradicts the fact that it is convex.

Each time the world just enter a new state $i$, we then know probability that the world is in each state, which is derived directly from the initial distribution of the world transition time, i.e., $R_{i,0}(r) = \varsigma e_r$, and we compute the optimal inventory position for this time point. We can then repeat the procedures described above to compute when the optimal inventory positions will change.

Now turn to the calculation of the left and right derivative of $\tilde{C}(y,i,r)$ with respect to $y$ for some fixed $i$, $r$. The right derivative at integer value $y$ (recall that we have proved the optimal inventory positions can only be integers) is

$$\left(\tilde{C}(y,i,r)\right)'_+ = -pE[e^{-\alpha L}] + (p+h)\sum_{d=0}^{y} q_i(d|r)$$

and the left derivative is

$$\left(\tilde{C}(y,i,r)\right)'_- = -pE[e^{-\alpha L}] + (p+h)\sum_{d=0}^{y-1} q_i(d|r).$$

If $y < 0$, then the right derivative at $y$ is

$$-pE[e^{-\alpha L}] = p\tau(\alpha I - M)^{-1}Me;$$

if $y \geq 0$, the right derivative can be written as

$$-pE[e^{-\alpha L}] + (p+h)(e_{ir} \otimes \tau)[\sum_{d=0}^{y} H_\alpha^d](I - H_\alpha - \alpha K_\alpha^{-1})(e \otimes e).$$

To summarize, we can compute the optimal inventory levels and the time when the optimal inventory levels change by applying the following algorithm.

- Step 1. At the beginning time of a new world state $i$, i.e. $s = 0$. If the world has been in state $i$ before, retrieve the stored optimal inventory position curve for state $i$. Otherwise, solve the minimization of $C(y, i, 0) + \alpha cy = \sum_{r=1}^{r_i} \varsigma e_r \tilde{C}(y, i, r) + \alpha cy$. Use the method of Song and Zipkin [52] as described above to compute $\tilde{C}(y, i, r)$ and the optimal value, and denote it by $y_0^*$. Set $k = 0$, and denote $s_0^* = 0$.
- Step 2. If $s_k^* = \infty$, go to Step 4 directly. Otherwise, compute the left and right derivative of $\tilde{C}(y, i, r)$ at $y_k^*$ for all $r = 1$ to $r_i$ as

$$\left(\tilde{C}(y_k^*, i, r)\right)'_+ = -pE[e^{-\alpha L}] + (p+h) \sum_{d=0}^{y_k^*} q_i(d|r)$$

$$\left(\tilde{C}(y_k^*, i, r)\right)'_- = -pE[e^{-\alpha L}] + (p+h) \sum_{d=0}^{y_k^*-1} q_i(d|r).$$

Solve the following equations (2.21) and (2.22)

$$\sum_{r=1}^{r_i} R_{i,s}(r) \left(\tilde{C}(y_k^*, i, r)\right)'_+ + \alpha c = 0 \tag{2.21}$$

$$\sum_{r=1}^{r_i} R_{i,s}(r) \left(\tilde{C}(y_k^*, i, r)\right)'_- + \alpha c = 0. \tag{2.22}$$

Let $s_1^+ \leq s_2^+ \leq \ldots$ be the solutions to equation (2.21) that are strictly greater than $s_k^*$. Check them in increasing order, and let $s'$ be the smallest solution at

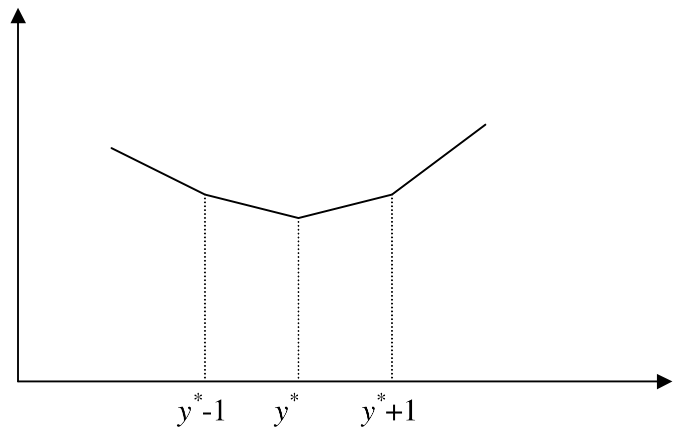


Figure 2–1: Optimal inventory position at $y^*$

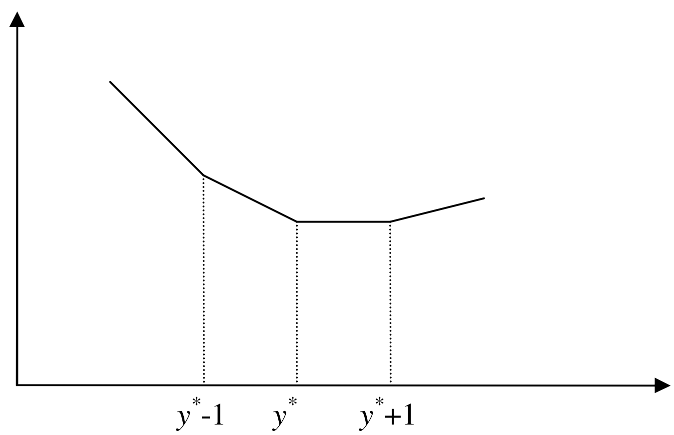


Figure 2–2: Optimal inventory position at $y^* + 1$

which the left-hand-side of equation (2.21) has a negative derivative, let $s' = \infty$ if no such solution exists.

Similarly, let $s_1^- \leq s_2^- \leq \ldots$ be the solutions to equation (2.22) that are strictly greater than $s_k^*$. Let $s''$ be the smallest solution at which the left-hand-side of equation (2.22) has a positive derivative, let $s'' = \infty$ if no such solution exists.

- Step 3. If $s' < s''$, let $s_{k+1}^* = s'$ and $y_{k+1}^* = y_k^* + 1$; if $s' > s''$, let $s_{k+1}^* = s''$ and $y_{k+1}^* = y_k^* - 1$. If both $s' = \infty$ and $s'' = \infty$, let $s_{k+1}^* = \infty$ and $y_{k+1}^* = y_k^*$. Store that $y_i^*(s) = y_k^*$ for $s_k^* \leq s < s_{k+1}^*$. Let $k = k + 1$.
- Step 4. If the world does not change to a different state, go back to Step 2; if a new state is encountered, go back to Step 1.

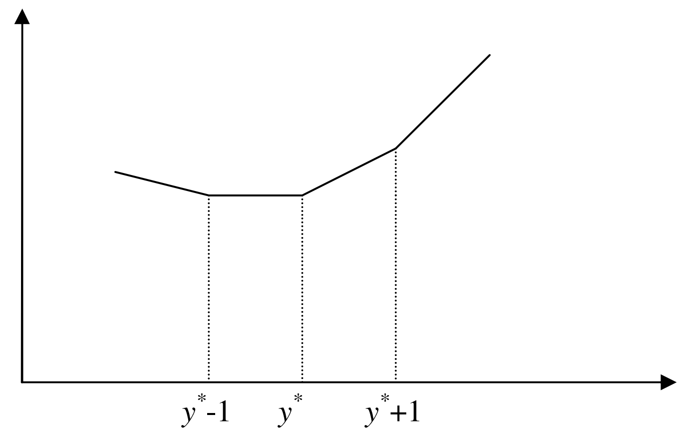


Figure 2–3: Optimal inventory position at $y^* - 1$

It is obvious that the computation involves the evaluation of the probability $R_{i,s}(r)$. And for general CPH world transition time, it is difficult to handle $R_{i,s}(r)$. In the next section, we will give an implementable algorithm by considering the world transition time as *Erlang* distributed.

### 2.5.2 *Erlang* Distributed World Transition Time and Continuous Phase-Type Distributed Lead Time

Suppose that the world transition for every state is a special case of CPH distribution, *Erlang* distribution. We first prove two useful lemmas regarding the property of the *Erlang* distribution.

**Lemma 2.5.2** *If $H_i$ is the generator of* $\mathrm{Erlang}(i, r_i, \nu_i)$ *distribution, then the $jr^{th}$ element of $H_i^l$, $1 \le j, r \le r_i$ and $l \ge 0$, is*

$$h_{jr}^l = \begin{cases} \nu_i^l (-1)^{l-r+j} \begin{pmatrix} l \\ r-j \end{pmatrix} & \textit{if } r \ge j \textit{ and } r-j \le l \\ 0 & o/w \end{cases} .$$

**Proof:** If a CPH distribution is an $Erlang(i, r_i, \nu_i)$ distribution, then the initial state distribution is $\varsigma_i = [1, 0, \ldots, 0]^T$, and the $jr^{th}$ element of $H_i$, $h_{jr}$, $1 \le j, r \le r_i$, has the following form

$$h_{jr} = \begin{cases} -\nu_i & \text{if } r = j \\ \nu_i & \text{if } r = j+1 \\ 0 & \text{o/w} \end{cases} .$$

We prove the lemma by induction. It can be verified easily that the lemma holds for $l = 1, 2$. Now suppose it is true for $l$, and we are considering the $jr^{th}$ element of $H_i^{l+1}$ for $1 \leq r, j \leq r_i$,

$$h_{jr}^{l+1} = \sum_{k=1}^{r_i} h_{jk}^{l} h_{kr}$$

If $r < j$ or $r > j+l+1$, it is easy to check that $h_{jr}^{l+1} = 0$; if $r = j$, it is straightforward that

$$h_{jr}^{l+1} = \nu_i^l(-1)^l 1(-\nu_i) = \nu_i^{l+1}(-1)^{l+1} \begin{pmatrix} l+1 \\ 0 \end{pmatrix}$$

if $r > j$ and $r \leq j+l+1$, then

$$\begin{aligned} h_{jr}^{l+1} &= h_{j,r-1}^{l} h_{r-1,r} + h_{jr}^{l} h_{rr} \\ &= \nu_i^l(-1)^{l-(r-1)+j} \begin{pmatrix} l \\ r-1-j \end{pmatrix} \nu_i + \nu_i^l(-1)^{l-r+j} \begin{pmatrix} l \\ r-j \end{pmatrix} (-\nu_i) \\ &= \nu_i^{l+1}(-1)^{l-r+j+1} \frac{l!(l-r+j+1) + l!(r-j)}{(l-r+j+1)!(r-j)!} \\ &= \nu_i^{l+1}(-1)^{l+1-r+j} \frac{(l+1)!}{(l+1-r+j)!(r-j)!} \\ &= \nu_i^{l+1}(-1)^{l+1-r+j} \begin{pmatrix} l+1 \\ r-j \end{pmatrix} . \end{aligned}$$

To summarize, we have

$$h_{jr}^{l+1} = \begin{cases} \nu_i^{l+1}(-1)^{l+1-r+j}\begin{pmatrix} l+1 \\ r-j \end{pmatrix} & \text{if } r \geq j \text{ and } r-j \leq l+1 \\ 0 & \text{o/w} \end{cases}.$$

So the lemma holds for $l+1$. By induction, the lemma holds for all $l$. □

**Lemma 2.5.3** *If a CPH distribution is an*$\mathrm{Erlang}(i, r_i, \nu_i)$ *distribution, then*

$$\pi_{i,s}e_r = \frac{(\nu_i s)^{r-1}e^{-\nu_i s}}{(r-1)!}.$$

**Proof:** Following the result of lemma 2.5.2, we have

$$\begin{aligned}
\pi_{i,s}e_r &= \varsigma_i e^{H_i s} e_r \\
&= \varsigma_i \sum_{l=0}^{\infty} \frac{(H_i s)^l}{l!} e_r \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} h_{1r}^l s^l \\
&= \sum_{l=r-1}^{\infty} \frac{1}{l!}(-1)^{l-r+1}\nu_i^l \begin{pmatrix} l \\ r-1 \end{pmatrix} s^l \\
&= \sum_{l=r-1}^{\infty} \frac{1}{l!}(-1)^{l-r+1}(\nu_i s)^{r-1}(\nu_i s)^{l-r+1}\frac{l!}{(r-1)!(l-r+1)!} \\
&= \frac{(\nu_i s)^{r-1}}{(r-1)!}\sum_{l=r-1}^{\infty}\frac{(-\nu_i s)^{l-r+1}}{(l-r+1)!} \\
&= \frac{(\nu_i s)^{r-1}}{(r-1)!}e^{-\nu_i s}.
\end{aligned}$$

□

From the previous two lemmas, for *Erlang*$(i, r_i, \nu_i)$ distribution, we can get that

$$\begin{aligned} R_{i,s}(r) &= \frac{\pi_{i,s} e_r}{\sum_{r=1}^{r_i} \pi_{i,s} e_r} \quad (2.23) \\ &= \frac{\frac{(\nu_i s)^{r-1} e^{-\nu_i s}}{(r-1)!}}{\sum_{k=1}^{r_i} \frac{(\nu_i s)^{k-1} e^{-\nu_i s}}{(k-1)!}} \\ &= \frac{\frac{(\nu_i s)^{r-1}}{(r-1)!}}{\sum_{k=1}^{r_i} \frac{(\nu_i s)^{k-1}}{(k-1)!}}. \quad (2.24) \end{aligned}$$

By replacing $R_{i,s}(r)$ by the values in equations (2.19) and (2.20)

$$\begin{aligned} 0 &= \sum_{r=1}^{r_i} \frac{(\nu_i s)^{r-1}}{(r-1)!} \left(\tilde{C}(y^*, i, r)\right)'_+ + \alpha c \sum_{k=1}^{r_i} \frac{(\nu_i s)^{k-1}}{(k-1)!} \\ &= \sum_{r=1}^{r_i} \frac{(\nu_i)^{r-1} \left[\left(\tilde{C}(y^*, i, r)\right)'_+ + \alpha c\right]}{(r-1)!} s^{r-1} \quad (2.25) \\ 0 &= \sum_{r=1}^{r_i} \frac{(\nu_i s)^{r-1}}{(r-1)!} \left(\tilde{C}(y^*, i, r)\right)'_- + \alpha c \sum_{k=1}^{r_i} \frac{(\nu_i s)^{k-1}}{(k-1)!} \\ &= \sum_{r=1}^{r_i} \frac{(\nu_i)^{r-1} \left[\left(\tilde{C}(y^*, i, r)\right)'_- + \alpha c\right]}{(r-1)!} s^{r-1}. \quad (2.26) \end{aligned}$$

At the time when the world just enter a new state $i$, i.e., $s = 0$, we know that the world must be in the first stage of the *Erlang* distribution, so $R_{i,0}(1) = 1$, and $R_{i,0}(r) = 0$ for all $r = 2, \ldots, r_i$. Thus, for $s = 0$, we have $C(y, i, 0) = \tilde{C}(y, i, 1)$. We can then repeat the procedures described above to compute when the optimal inventory positions will change.

We have the following algorithm by making the according changes.

- Step 1. At the beginning time of a new world state $i$, i.e. $s = 0$. If the world has been in state $i$ before, retrieve the stored optimal inventory position curve for state $i$. Otherwise, solve the minimization of $C(y, i, 0) + \alpha c y = \tilde{C}(y, i, 1) + \alpha c y$. Using the method of Song and Zipkin [52] as described above, compute the optimal value and denote it by $y_0^*$. Set $k = 0$, and denote $s_0^* = 0$.

- Step 2. If $s_k^* = \infty$, go to Step 4 directly. Otherwise, compute the left and right derivative of $\tilde{C}(y,i,r)$ at $y_k^*$ for all $r = 1$ to $r_i$ as

$$\left(\tilde{C}(y_k^*, i, r)\right)'_+ = -pE[e^{-\alpha L}] + (p+h)\sum_{d=0}^{y_k^*} q_i(d|r)$$

$$\left(\tilde{C}(y_k^*, i, r)\right)'_- = -pE[e^{-\alpha L}] + (p+h)\sum_{d=0}^{y_k^*-1} q_i(d|r).$$

Solve the following equations (2.27) and (2.28)

$$\sum_{r=1}^{r_i} \frac{(\nu_i)^{r-1}\left[\left(\tilde{C}(y_k^*, i, r)\right)'_+ + \alpha c\right]}{(r-1)!} s^{r-1} = 0 \tag{2.27}$$

$$\sum_{r=1}^{r_i} \frac{(\nu_i)^{r-1}\left[\left(\tilde{C}(y_k^*, i, r)\right)'_- + \alpha c\right]}{(r-1)!} s^{r-1} = 0. \tag{2.28}$$

Let $s_1^+ \leq s_2^+ \leq \ldots$ be the solutions to equation (2.27) that are strictly greater than $s_k^*$. Check them in increasing order, and let $s'$ be the smallest solution at which the left-hand-side of equation (2.27) has a negative derivative, i.e.,

$$\sum_{r=2}^{r_i} \frac{(\nu_i)^{r-1}\left[\left(\tilde{C}(y_k^*, i, r)\right)'_+ + \alpha c\right]}{r!} (s')^{r-2} < 0.$$

let $s' = \infty$ if no such solution exists.

Similarly, let $s_1^- \leq s_2^- \leq \ldots$ be the solutions to equation (2.28) that are strictly greater than $s_k^*$. Let $s''$ be the smallest solution at which the left-hand-side of equation (2.28) has a positive derivative, i.e.,

$$\sum_{r=2}^{r_i} \frac{(\nu_i)^{r-1}\left[\left(\tilde{C}(y_k^*, i, r)\right)'_- + \alpha c\right]}{r!} (s')^{r-2} > 0.$$

let $s'' = \infty$ if no such solution exists.

- Step 3. If $s' < s''$, let $s^*_{k+1} = s'$ and $y^*_{k+1} = y^*_k + 1$; if $s' > s''$, let $s^*_{k+1} = s''$ and $y^*_{k+1} = y^*_k - 1$. If both $s' = \infty$ and $s'' = \infty$, let $s^*_{k+1} = \infty$ and $y^*_{k+1} = y^*_k$. Store that $y^*_i(s) = y^*_k$ for $s^*_k \leq s < s^*_{k+1}$. Let $k = k + 1$.
- Step 4. If the world does not change to a different state, go back to Step 2; if a new state is encountered, go back to Step 1.

## 2.6 An Extension: Demand Arrives Following a General Renewal Process

### 2.6.1 Generalization of the Demand Process Model

In this section, we will discuss an important extension to the the models we studied so far. Instead of assuming that the demand arrivals in each world state follows a Poisson process, we now relax the demand process by allowing the interarrival time between demands within a world state to have a general world-dependent distribution. In other words, when the world is in state $i$, the actual demand process follows renewal process, with the distribution of the time between successive demands denoted by $K_i$. We call this demand process a *semi-Markov modulated renewal demand process*. Again the demand process is exogenous and is not affected by any ordering decisions.

By keeping all the other model assumptions as before, the results developed up to Section 2.3.1 continue to hold without any changes since the Poisson nature of the demand process is not used, i.e., the cost function and the general form of the optimal policy (as a function of the entire history) remain the same. However, now the history can no longer be summarized by the current world state $i$ and the time spent in this state $s$ only. The properties of the semi-Markov modulated renewal demand process imply that $D_L^{t,\bar{h}(t)}$ depends on the history through not only the state that the process is currently in ($i$) and how long it has been in that state ($s$), but also how long it has been since the last demand occurred, which we will denote by $B(t)$. To reflect this fact, we rewrite the lead time demand as $D_L^{i,s,b}$ when the core process has been in state $i$ for $s$ time units, and $b$ time units have elapsed since the last demand.

One important issue that we need to pay attention to is that before the first occurrence of a demand within a world state, the amount time since the last demand $b$ generally is not equal to the amount of time $s$ spent in the current state, since the last demand will likely have occurred while in the preceding state (or even earlier, if no demand occurred while in the preceding state). Thus, $b$ not only includes time spent in the current state, but also some amount of time spent in previous state(s). However, since the distribution of the time between demands changes between states of the world, neither $s$ nor $b$ seems to be an accurate measure of the time since the last demand for the current interarrival time distribution.

To handle this situation, we should in fact let the demand process in a given state be a *delayed* renewal process, where the distribution of the first interarrival time depends on the time since the last demand as well as the previous state visited. In particular, suppose we are currently transitioning from state $j$ into state $i$ with generic interarrival time $X_i \sim K_i$, and let the time since the last demand be $b$. We then let the first interarrival time be distributed as

$$X_i - \phi_{j,i}(b) | X_i > \phi_{j,i}(b)$$

where $\phi_{j,i}$ is some function that transforms the amount of time that has elapsed since the last demand. For convenience, we will in fact simply redefine $b$ to be equal to $\phi_{j,i}(b)$ at the moment of transition from state $j$ to state $i$. Note that the previous demand might actually have occurred in a state that was visited before state $j$. In that case, by recursively updating the time since the last demand using the appropriate transformation functions $\phi_{\cdot,\cdot}$ will appropriately define the first interarrival time distribution in each state.

Intuitively, it seems clear that we should choose the functions $\phi_{\cdot,\cdot}$ to be nondecreasing. Two interesting extreme cases are $\phi_{\cdot,\cdot}(b) = 0$, where we simply "forget" the time since the last demand at the moment of transition between world states, and $\phi_{\cdot,\cdot}(b) = b$, where we ignore the fact that the interarrival time between demands is different in different states. If we define the generalized inverse of distribution function $K$ by

$$K^{\leftarrow}(p) = \min\{y : K(y) \geq p\}$$

then a more reasonable choice of the conversion function would be

$$\phi_{i,j}(b) = K_j^{\leftarrow}\left(K_i(b)\right).$$

### 2.6.2 The Optimal Inventory Policy

Now we are ready to characterize the optimal inventory policy for the semi-Markov modulated renewal demand process described above. We first introduce some notation similar to Section 2.3.2 to accommodate the changes in the demand model. For given $y$, $t$, and $\bar{h}(t)$, we can simplify the inventory cost rate function to

$$C(y, i, s, b) = E\left[e^{-\alpha L}\hat{C}(y - D_L^{i,s,b})\right]$$

which is equivalent to $C(y, t, \bar{h}(t))$ if the history $\bar{h}(t)$ says that $A(t) = i$, $S(t) = s$ and $B(t) = b$. The total expected cost function for our semi-Markov modulated renewal demand model thus reduces to

$$W(x, \bar{h}(0)|\mathbf{y}) = \int_0^\infty e^{-\alpha t} E\left[C(y(t, \bar{H}(t)), A(t), S(t), B(t)) + \alpha c y(t, \bar{H}(t))\right] dt.$$

In addition, we define

$$f_{i,s,b}(y) = C(y, i, s, b) + \alpha c y$$

for every fixed $i$, $s$ and $b$, which can be viewed as the cost rate function if the inventory position is $y$ at the time when the core process has been in state $i$ for $s$ time units, and $b$ time units have been passed since the last demand.

Next, we generalize some of the key results obtained for the semi-Markov modulated Poisson demand process that continue to hold for our new demand model. Denote the right derivatives of $C(\cdot, i, s, b)$ and $f_{i,s,b}(\cdot)$ by

$$\begin{aligned} C'_+(y, i, s, b) &= \lim_{\varepsilon \downarrow 0} \frac{C(y+\varepsilon, i, s, b) - C(y, i, s, b)}{\varepsilon} \\ (f'_{i,s,b})_+(y) &= \lim_{\varepsilon \downarrow 0} \frac{f_{i,s,b}(y+\varepsilon) - f_{i,s,b}(y)}{\varepsilon} \\ &= C'_+(y, i, s, b) + \alpha c. \end{aligned}$$

Also, let

$$y_i^*(s, b) = \inf\{y : (f'_{i,s,b})_+(y) \geq 0\}.$$

Note that since the lead time demand can only assume integral values, all points at which the functions $C(\cdot, i, s, b)$ and $f_{i,s,b}(\cdot)$ are nondifferentiable are integral. In addition, $y_i^*(s, b)$ is integral as well.

**Lemma 2.6.1**

(a) *$C(y, i, s, b)$ and $f_{i,s,b}(y)$ are both convex in $y$ for all $i$, $s$ and $b$, so that $y_i^*(s, b)$ minimizes $f_{i,s,b}(y)$.*

(b) *If $\alpha\bar{c} < p$, then $y_i^*(s, b)$ is finite and nonnegative for all $i$, $s$ and $b$. In addition, $f_{i,s,b}(y)$ is nonnegative for all $i$, $s$ and $b$.*

(c) *If $\alpha\bar{c} \geq p$, then $y_i^*(s, b) = -\infty$ for all $i$, $s$ and $b$.*

We omit the proof because the arguments are very similar to the ones in Lemma 2.3.3.

**Theorem 2.6.2** *Under the semi-Markov modulated Poisson demand model, the myopic policy $\mathbf{y}^*$ defined by*

$$y^*(t, \bar{H}(t)) = y^*_{A(t)}(S(t), B(t)) \qquad \textit{for all } t \geq 0$$

*exists and its total policy costs are finite. Thus, the optimal inventory position at time $t$ only depends on the state at time $t$ and the amount of time that has elapsed since the core process last entered that state.*

We also omit the proof to this theorem because the arguments are very similar to the ones in Theorem 2.3.5.

Finally, we will give a more explicit characterization of the optimal inventory position, which can in principle be used to compute the optimal policy, as well as the cost of the optimal policy. Define

$$\tilde{F}_{D_L^{i,s,b}}(y) = \int_0^\infty e^{-\alpha\ell} F_{D_\ell^{i,s,b}}(y) dF_L(\ell).$$

Then

$$\begin{aligned} y_i^*(s,b) &= \arg\min \left\{ y : (f'_{i,s,b})_+(y) \geq 0 \right\} \\ &= \arg\min \left\{ y : \tilde{F}_{D_L^{i,s,b}}(y) \geq \frac{pE[e^{-\alpha L}] - \alpha c}{h+p} \right\}. \end{aligned}$$

In case the lead time is deterministic, the expression for the optimal policy can be simplified to

$$\begin{aligned} y_i^*(s,b) &= \arg\min \left\{ y : e^{-\alpha L} F_{D_L^{i,s,b}}(y) \geq \frac{pe^{-\alpha L} - \alpha c}{h+p} \right\} \\ &= \arg\min \left\{ y : F_{D_L^{i,s,b}}(y) \geq \frac{p - \alpha\bar{c}}{h+p} \right\}. \end{aligned} \tag{2.29}$$

## 2.7 Summary

In this chapter, we have studied an inventory control model in a semi-Markov modulated demand environment. Under linear ordering, holding, and shortage costs, and assuming that both positive and negative orders are allowed, we have derived the optimal inventory policy. In addition, we have formulated sufficient conditions on the demand process for the myopic inventory positions to be increasing over time. In that case, the myopic policy remains optimal even if negative orders are not allowed. For a special case where both the world transition time distributions and lead time distribution are continuous phase type distributed, we give an algorithm to compute the optimal inventory positions. Finally, we extend the model by relaxing the demand process from a semi-Markov modulated Poisson process to a general semi-Markov modulated renewal process, and see that this relaxation really does not affect the form of the optimal policy.

CHAPTER 3

# MODELS WITH PARTIALLY OBSERVABLE WORLD STATES

## 3.1 Introduction

In this chapter, we extend our models studied in Chapter 2 to a more complex case in which the demand process is a state-dependent Poisson process, but the underlying core process (world) is not directly observable. What we can observe is only the arrival of the customer demands, and we can use that information to obtain inference on the state of the world. This scenario is very common in real situations, and thus of significant practical interest.

The chapter is organized as follows. In Section 3.2 we study a model with only two world states. We describe the difference between this model and our previous models and show how the fact that the world state is unobservable affects the optimal policy. We also give the form of this optimal inventory policy and provide an algorithm to incrementally determine the optimal policy. Then we extend the basic two-state model to a multiple-state one in Section 3.3 and derive a recursive formula to help determine the optimal inventory policy. In Section 3.3.4 we generalize this result to a more general multiple-state model. Finally, we summarize the chapter in Section 3.4 and provide some future research directions.

## 3.2 A Simple Model with Two World States

We start with a simple case in which there are only two world states, state 1 and 2. A state transition can only happen from state 1 to state 2, and once in state 2 the world will stay in that state forever. The transition time from state 1 to state 2 is a continuous random variable with distribution $G$. While in world state 1 or 2, the demand process is a Poisson process with rate $\lambda_1$ or $\lambda_2$, respectively. We assume that we know the values of $\lambda_1$ and $\lambda_2$, but we do not observe the transition from state 1 to state 2. As before, we assume that the demand process is independent of the replenishment decisions. Implicitly, we treat the system as if we start our observations when the world just enters state 1. Put differently, even if the inventory system started at some point in the past before observations start, we assume that we know the distribution $G$ of the time that the system will remain in state 2. Note that if there is a positive probability that the transition to state 2 has already happened at the time observations start this can be incorporated by defining $G$ to have a positive probability mass at 0.

Recall that the cumulative demand by time $t$ is denoted by $D(t)$. Then, at time $t$ we will have observed

$$\bar{D}(t) = \{D(u) : 0 \leq u \leq t\}.$$

Note that in this case the history information $H(t)$ and $\bar{H}(t)$ contains only the demand information since the world state is now unobservable. As in the previous chapter, we still assume that negative orders are allowed. Under this assumption, the notation and model up to Section 2.3.1 can be used without changes, except for the content of the history $\bar{H}(t)$.

### 3.2.1 Effects of the Unobservable World

In this section we address the effects of the fact that the world process is unobservable. The history information by time $t$ contains the past cumulative demand at any time point before $t$ and can be fully characterized by how many demands have occurred, $N(t)$, and the interarrival times between consecutive demands, $X_1, X_2, \ldots, X_{N(t)}$. So another way to represent the history is

$$\begin{aligned} \bar{H}(t) &= \{N(t), X_1, X_2, \ldots, X_{N(t)}\} \\ &= \{N(t), S_1, S_2, \ldots, S_{N(t)}\} \end{aligned}$$

where $S_k = \sum_{i=1}^{k} X_i$ is the arrival time of the $k^{th}$ demand.

We denote the state of the world by the stochastic process $\{\Lambda(t),\, t \geq 0\}$. In particular, $\Lambda(t)$ is a random variable that is equal to $\lambda_i$ if the world is in state $i$ $(i = 1, 2)$ at time $t$. Then $(\Lambda(t), \bar{H}(t))$ is a joint mixed random variable with joint probability density function $f(\lambda_i, \bar{h}(t))$. Denote the conditional probability that the world is in state $i$ at time $t$ for every $t$ given that the history information up to time $t$ is $\bar{H}(t) = \bar{h}(t)$, by $p(i, t, \bar{h}(t)) = f_{\Lambda|\bar{H}}(\lambda_i|\bar{h}(t))$. By conditioning on the world changing state at time $S = s$, and denoting the conditional density function of history by $f_{\bar{H}|S}(\bar{h}(t)|s)$, we obtain

$$
\begin{aligned}
p(1, t, \bar{h}(t)) &\equiv f_{\Lambda|\bar{H}}(\lambda_1|\bar{h}(t)) \\
&= \frac{f(\lambda_1, \bar{h}(t))}{f_{\bar{H}}(\bar{h}(t))} \\
&= \frac{\int_0^\infty f(\lambda_1, \bar{h}(t)|s)dG(s)}{\int_0^\infty f_{\bar{H}|S}(\bar{h}(t)|s)dG(s)} \\
&= \frac{\int_t^\infty f_{\bar{H}|S}(\bar{h}(t)|s)dG(s)}{\int_0^\infty f_{\bar{H}|S}(\bar{h}(t)|s)dG(s)} \\
&= \frac{\int_t^\infty f_{\bar{H}|S}(\bar{h}(t)|s)dG(s)}{\int_0^t f_{\bar{H}|S}(\bar{h}(t)|s)dG(s) + \int_t^\infty f_{\bar{H}|S}(\bar{h}(t)|s)dG(s)}.
\end{aligned} \tag{3.1}
$$

The meaning of each density (conditional density) function $f$ should be clear within its context.

Now let us look at the conditional density function of history $\bar{H}(t)$ given $S = s$ more closely. For $\bar{h}(t) = \{N(t) = n, S_1 = s_1, \ldots, S_n = s_n\}$, let $f_{\bar{H}|S}(n, s_1, \ldots, s_n|s)$ denote the condition density of $N(t) = n, S_1 = s_1, \ldots, S_n = s_n$ given that $S = s$, and let $f_{\bar{H}|N,S}(s_1, \ldots, s_n|n, s)$ denote the conditional density of $S_1 = s_1, \ldots, S_n = s_n$ given that $N(t) = n$ and $S = s$.

If $s > t$, the world is still in state 1 by time $t$, and

$$
\begin{aligned}
f_{\bar{H}|S}(\bar{h}(t)|s) &= \Pr(N(t) = n|S = s) f_{\bar{H}|N,S}(s_1, \ldots, s_n|n, s) \\
&= \frac{e^{-\lambda_1 t}(\lambda_1 t)^n}{n!} \frac{n!}{t^n} \\
&= e^{-\lambda_1 t}\lambda_1^n.
\end{aligned}
$$

If $s < t$, then the state transition has already happened before $t$. We denote the number of demands that occur while in state 1 and 2 by random variables $N_1$ and $N_2$ respectively. For a given $s$, these two numbers are known by $t$, and we denote them by $n(s)$ and $n - n(s)$ respectively, and the occurrence of demands in states 1 and 2 are independent! Then

$$\begin{aligned} f_{\bar{H}|S}(\bar{h}(t)|s) &= f_{\bar{H}|S}(n(s), x_1, \ldots, x_{n(s)}; \\ &\quad n - n(s), \sum_{i=1}^{n(s)+1} x_i - s, x_{n(s)+2}, \ldots, x_n|s) \\ &= f_{\bar{H}|S}(n(s), x_1, \ldots, x_{n(s)}|s) \cdot \\ &\quad f_{\bar{H}|S}(n - n(s), \sum_{i=1}^{n(s)+1} x_i - s, x_{n(s)+2}, \ldots, x_n|s) \\ &= e^{-\lambda_1 s}\lambda_1^{n(s)} \cdot e^{-\lambda_2(t-s)}\lambda_2^{n-n(s)}. \end{aligned}$$

If $s_k < s < s_{k+1}$, for $k = 0, 1, \ldots, n$, then $n(s) = k$, and

$$f_{\bar{H}|S}(\bar{h}(t)|s) = e^{-\lambda_1 s}\lambda_1^k e^{-\lambda_2(t-s)}\lambda_2^{n-k}.$$

So

$$\int_0^t f(\bar{h}(t)|S = s)dG(s) = \sum_{k=0}^{n} \lambda_1^k \lambda_2^{n-k} \int_{s_k}^{s_{k+1}} e^{-\lambda_1 s} e^{-\lambda_2(t-s)} dG(s)$$

where we let $s_0 = 0$ and $s_{n+1} = t$.

Now we have

$$\begin{aligned} p(1, t, \bar{h}(t)) &= \frac{\int_t^\infty e^{-\lambda_1 t}\lambda_1^n dG(s)}{\sum_{k=0}^n \lambda_1^k \lambda_2^{n-k} \int_{s_k}^{s_{k+1}} e^{-\lambda_1 s} e^{-\lambda_2(t-s)} dG(s) + \int_t^\infty e^{-\lambda_1 t}\lambda_1^n dG(s)} \\ &= \frac{e^{-\lambda_1 t}\lambda_1^n \overline{G}(t)}{\sum_{k=0}^n \lambda_1^k \lambda_2^{n-k} \int_{s_k}^{s_{k+1}} e^{-\lambda_1 s} e^{-\lambda_2(t-s)} dG(s) + e^{-\lambda_1 t}\lambda_1^n \overline{G}(t)} \\ &= \frac{e^{-\lambda_1 t}\lambda_1^n \overline{G}(t)}{\sum_{k=0}^n \lambda_1^k \lambda_2^{n-k} e^{-\lambda_2 t} \int_{s_k}^{s_{k+1}} e^{-(\lambda_1-\lambda_2)s} dG(s) + e^{-\lambda_1 t}\lambda_1^n \overline{G}(t)} \\ &= \frac{1}{1 + \left(e^{(\lambda_1-\lambda_2)t}/\overline{G}(t)\right) \sum_{k=0}^n (\lambda_2/\lambda_1)^{n-k} \int_{s_k}^{s_{k+1}} e^{-(\lambda_1-\lambda_2)s} dG(s)}. \end{aligned} \quad (3.2)$$

It is easy to see that this is a continuous function of $t$ if the transition time distribution function $G$ is continuous.

Now we compute the lead time demand distribution given history $\bar{h}(t)$. Let $g(s)$ be the density function of the transition time distribution, and $g(s|\bar{h})$ be the conditional density function of the transition time distribution given history $\bar{h}(t)$. Conditioning only on which state the world is in now is not enough, since we also need to know how long the world has been in the current state to determine the the remaining life time distribution. So what we need is to condition on the time of the state transition. For fixed lead time $\ell$,

$$\begin{aligned}
F_{D_\ell^{t,\bar{h}(t)}}(z) &= \int_0^\infty \Pr(D_\ell^{t,\bar{h}(t)} \le z|S=s)g(s|\bar{h}(t))ds \\
&= \int_0^t \Pr(D_\ell^{t,\bar{h}(t)} \le z|S=s)g(s|\bar{h}(t))ds + \\
&\quad \int_t^\infty \Pr(D_\ell^{t,\bar{h}(t)} \le z|S=s)g(s|\bar{h}(t))ds \\
&= \int_0^t F_{D_\ell^{2,t-s}}(z)\frac{f_{\bar{H}|S}(\bar{h}(t)|s)g(s)}{f_{\bar{H}}(\bar{h}(t))}ds + \int_t^\infty F_{D_\ell^{1,t}}(z)\frac{f_{\bar{H}|S}(\bar{h}(t)|s)g(s)}{f_{\bar{H}}(\bar{h}(t))}ds \\
&= \int_0^t F_{D_\ell^{2,t-s}}(z)\frac{f_{\bar{H}|S}(\bar{h}(t)|s)}{f_{\bar{H}}(\bar{h}(t))}g(s)ds + F_{D_\ell^{1,t}}(z)\int_t^\infty \frac{f_{\bar{H}|S}(\bar{h}(t)|s)}{f_{\bar{H}}(\bar{h}(t))}g(s)ds.
\end{aligned}$$

Recall that in our current model there are only two world states, and once the world enters state 2, it will remain in state 2 forever. Therefore, $F_{D_\ell^{2,t-s}}$ is equivalent to $F_{D_\ell^{2,0}}$, and can move out of the integral. (This argument does not generalize to a model with more than two states, as we will see later). It follows from equation (3.1) that

$$\begin{aligned}
F_{D_\ell^{t,\bar{h}(t)}}(z) &= F_{D_\ell^{2,0}}(z)\frac{\int_0^t f_{\bar{H}|S}(\bar{h}(t)|s)dG(s)}{f_{\bar{H}}(\bar{h}(t))} + F_{D_\ell^{1,t}}(z)\frac{\int_t^\infty f_{\bar{H}|S}(\bar{h}(t)|s)dG(s)}{f_{\bar{H}}(\bar{h}(t))} \qquad (3.3)\\
&= F_{D_\ell^{2,0}}(z)\left(1-p(1,t,\bar{h}(t))\right) + F_{D_\ell^{1,t}}(z)p(1,t,\bar{h}(t)).
\end{aligned}$$

When the lead time is stochastic, we obtain

$$\begin{aligned}\tilde{F}_{D_L^{t,\bar{h}(t)}}(z) &= p(1,t,\bar{h}(t))\int_0^\infty e^{-\alpha\ell}F_{D_\ell^{1,t}}(z)dF_L(\ell)\\ &\quad + (1-p(1,t,\bar{h}(t)))\int_0^\infty e^{-\alpha\ell}F_{D_\ell^{2,0}}(z)dF_L(\ell). \qquad (3.4)\end{aligned}$$

Then we can write

$$\begin{aligned}C(y,t,\bar{h}(t)) &= E\left[e^{-\alpha L}\hat{C}(y-D_L^{t,\bar{h}(t)})\right]\\ &= \int\int e^{-\alpha\ell}\hat{C}(y-z)dF_{D_\ell^{t,\bar{h}(t)}}(z)dF_L(\ell)\\ &= p(1,t,\bar{h}(t))\int\int e^{-\alpha\ell}\hat{C}(y-z)dF_{D_\ell^{1,t}}(z)dF_L(\ell)+\\ &\quad \left(1-p(1,t,\bar{h}(t))\right)\int\int e^{-\alpha\ell}\hat{C}(y-z)dF_{D_\ell^{2,0}}(z)dF_L(\ell)\\ &= p(1,t,\bar{h}(t))C(y,1,t)+\left(1-p(1,t,\bar{h}(t))\right)C(y,2,0). \qquad (3.5)\end{aligned}$$

It is obvious that the conditional probability function $p(1,t,\bar{h}(t))$ as well as the cost rate functions $C(y,1,t)$ and $C(y,2,0)$ play key roles in the partially unobservable model. To make the model tractable, we consider the case where the state transition time distribution $G$ is *exponential* with rate $\mu$, i.e., the world process is a Markov process, in more detail in the remainder of this section.

### 3.2.2 Exponential Transition Time Distribution

If the state transition time distribution $G$ is *exponential* with rate $\mu$, i.e., the world is a Markov process, then by conditioning on which state the world is currently in, the distribution of the lead time demand can be expressed as

$$\begin{aligned}F_{D_\ell^{t,\bar{h}(t)}}(z) &= \Pr(D_\ell^{t,\bar{h}(t)}\le z)\\ &= \Pr(D_\ell^{t,\bar{h}(t)}\le z|\Lambda(t)=\lambda_1)p(1,t,\bar{h}(t))\\ &\quad +\Pr(D_\ell^{t,\bar{h}(t)}\le z|\Lambda(t)=\lambda_2)(1-p(1,t,\bar{h}(t)))\\ &= F_{D_\ell^1}(z)p(1,t,\bar{h}(t))+F_{D_\ell^2}(z)(1-p(1,t,\bar{h}(t))) \qquad (3.6)\end{aligned}$$

where we define $D_L^i$ to be the random variable representing the total demand during a lead time, started from now, given the world is currently in state $i$. The last equality in (3.6) follows from the fact that the world process follows a Markov process and demand is a Poisson process, and if we know which state the world is in now, the time that has elapsed in the current state becomes irrelevant. We see that we obtain the same result as by conditioning on the transition time. And accordingly,

$$C(y,t,\bar{h}(t)) \;=\; p(1,t,\bar{h}(t))C(y,1) + \left(1 - p(1,t,\bar{h}(t))\right) C(y,2) \tag{3.7}$$

where $C(y,i)$ is defined as the conditional expected discounted holding and shortage cost rate, at the end of a lead time and viewed from, given that the world is currently in state $i$, and the current inventory position (after ordering decision) is $y$. (See also Song and Zipkin [52].)

When comparing equations (3.5) and (3.7)we see that in case $G$ is the exponential distribution the instantaneous cost rate function simplifies considerably, and the dependence on time and history is then restricted to the conditional probability function $p(1,t,\bar{h}(t))$. In the remainder of this section we focus on the computation and analysis of this function. These results will then be used in the next section to compute the optimal inventory policy.

Using the fact that $G(t) = 1 - e^{-\mu t}$ and defining $\lambda = \lambda_1 - \lambda_2 + \mu$ we obtain

$$\begin{aligned}
p(1,t,\bar{h}(t)) &= \frac{1}{1 + \left(e^{(\lambda_1-\lambda_2)t}/\overline{G}(t)\right)\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}\int_{s_k}^{s_{k+1}} e^{-(\lambda_1-\lambda_2)s}dG(s)} \\
&= \frac{1}{1 + \mu e^{(\lambda_1-\lambda_2+\mu)t}\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}\int_{s_k}^{s_{k+1}} e^{-(\lambda_1-\lambda_2+\mu)s}ds} \\
&= \frac{1}{1 + \mu e^{\lambda t}\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}\int_{s_k}^{s_{k+1}} e^{-\lambda s}ds}.
\end{aligned}$$

We now distinguish between the cases $\lambda \neq 0$ and $\lambda = 0$.

Case 1: If $\lambda \neq 0$ then

$$
\begin{aligned}
& p(1,t,\bar{h}(t)) \\
&= \frac{1}{1+(\mu/\lambda)e^{\lambda t}\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}(e^{-\lambda s_k}-e^{-\lambda s_{k+1}})} \\
&= \frac{1}{1+(\mu/\lambda)e^{\lambda t}\left(\sum_{k=0}^{n-1}(\lambda_2/\lambda_1)^{n-k}(e^{-\lambda s_k}-e^{-\lambda s_{k+1}})+e^{-\lambda s_n}-e^{-\lambda t}\right)} \\
&= \frac{1}{1-\mu/\lambda+(\mu/\lambda)e^{\lambda t}\left(\sum_{k=0}^{n-1}(\lambda_2/\lambda_1)^{n-k}(e^{-\lambda s_k}-e^{-\lambda s_{k+1}})+e^{-\lambda s_n}\right)} \qquad (3.8) \\
&= \frac{1}{1-\mu/\lambda+(\mu/\lambda)e^{\lambda(t-s_n)}\left(1+e^{\lambda s_n}\sum_{k=0}^{n-1}(\lambda_2/\lambda_1)^{n-k}(e^{-\lambda s_k}-e^{-\lambda s_{k+1}})\right)}.
\end{aligned}
$$

It is now easy to show that

$$
\lim_{t\to\infty} p(1,t,\bar{h}(t)) = \begin{cases} 0 & \text{if } \lambda > 0 \\ \frac{1}{1-\mu/\lambda} & \text{if } \lambda < 0. \end{cases}
$$

Moreover, if $\lambda > 0$ the function $p(1,t,\bar{h}(t))$ decreases monotonely in $t$. If $\lambda < 0$, the function $p(1,t,\bar{h}(t))$ decreases monotonely in $t$ if the constant

$$
1+e^{\lambda s_n}\sum_{k=0}^{n-1}(\lambda_2/\lambda_1)^{n-k}(e^{-\lambda s_k}-e^{-\lambda s_{k+1}})
$$

is positive, and it increases monotonely in $t$ if it is negative. More intuitively, it follows that the function $p(1,t,\bar{h}(t))$ decreases monotonely in $t$ if the probability that we are in state 1 at the time of the $n^{\text{th}}$ demand exceeds the limiting probability $1/(1-\mu/\lambda)$ and increases monotonely in $t$ otherwise.

Case 2: If $\lambda = 0$ then

$$
\begin{aligned}
p(1,t,\bar{h}(t)) &= \frac{1}{1+\mu e^{\lambda t}\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}\int_{s_k}^{s_{k+1}}e^{-\lambda s}ds} \\
&= \frac{1}{1+\mu\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}\int_{s_k}^{s_{k+1}}ds} \\
&= \frac{1}{1+\mu\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}(s_{k+1}-s_k)} \\
&= \frac{1}{1+\mu\left(t+\sum_{k=0}^{n-1}(\lambda_2/\lambda_1)^{n-k}(s_{k+1}-s_k)-s_n\right)}. \qquad (3.9)
\end{aligned}
$$

It is easy to see that, in this case, the function $p(1, t, \bar{h}(t))$ decreases monotonely and

$$\lim_{t \to \infty} p(1, t, \bar{h}(t)) \;=\; 0.$$

Figures 3–1 through 3–4 illustrate the different behaviors of the probability function $p(1, t, \bar{h}(t))$ with different parameters.

We close this section by providing a summary regarding the probability function $p(1, t, \bar{h}(t))$. This function is always monotone, but the nature of monotonicity depends on the sign of $\lambda = \lambda_1 - \lambda_2 + \mu$ as well as the observed history. In particular, if $\lambda \geq 0$ the probability always decreases monotonely to 0, while if $\lambda < 0$ the probability will converge monotonely to a positive limit.

### 3.2.3 Computation of the Optimal Inventory Position

We next study the optimal inventory policy that is given in Theorem 2.3.2 for the partially unobservable demand model. First, the following theorem shows an important property of the optimal policy.

**Theorem 3.2.1** *The optimal inventory policy is a step function with step sizes 1 and* $-1$ *as long as no demands occur.*

**Proof:** From equation (3.2), we see that $p(1, t, \bar{h}(t))$ is a continuous function of $t$ since the transition time distribution is exponential and thus continuous. From equations (3.6) and (3.4), it is obvious that $\tilde{F}_{D_L^{t,\bar{h}(t)}}(z)$ is continuous in $t$ also. Finally, the lead time demand is a a discrete random variable having strictly positive probability mass at every nonnegative integer value due to the nature of the Poisson process. So from Lemma 2.3.1, each time the optimal inventory position changes, it will change to a neighboring integer, i.e., either increase by 1 or decrease by 1. $\square$

If the lead time distribution is continuous phase-type, we can now use the probability function we computed in the previous section as well as the analyses in Section 2.5 and in Song and Zipkin [52] to compute the optimal inventory policy for the

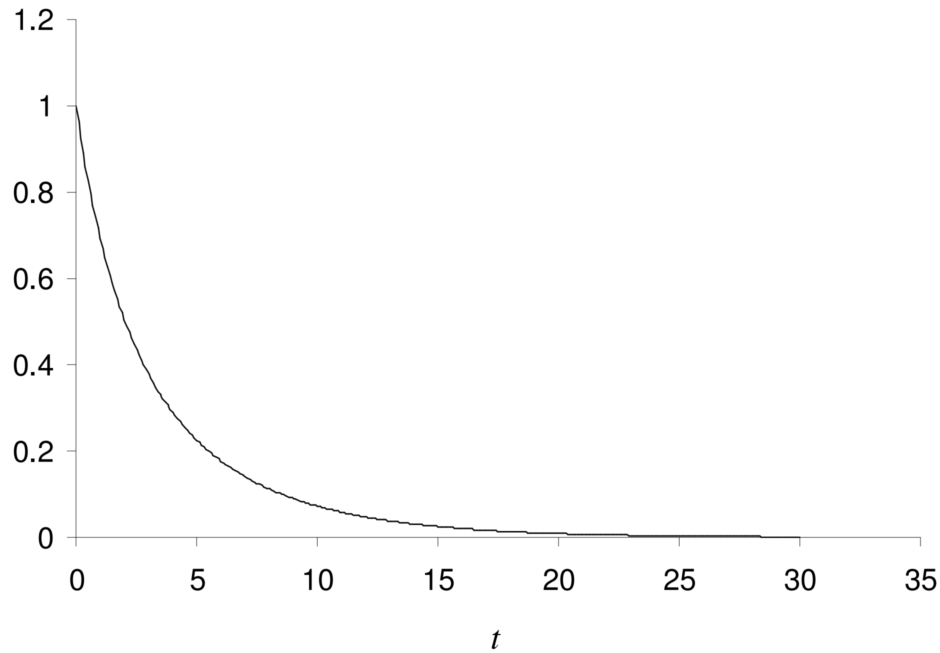


Figure 3–1: $\lambda_1 = 1$, $\lambda_2 = 1.2$,$\mu = 0.4$, and no demand before $t$

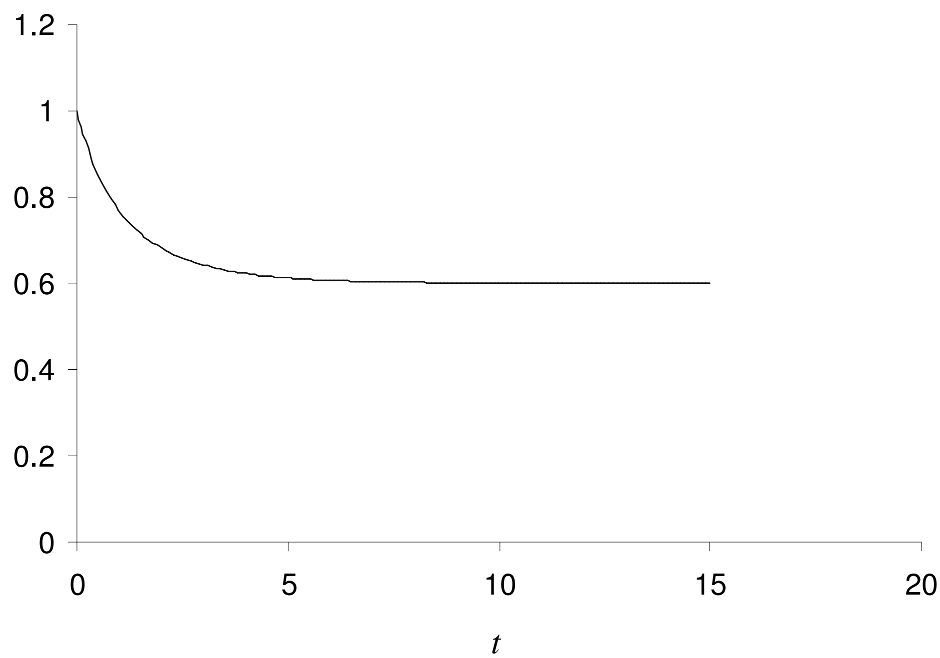


Figure 3–2: $\lambda_1 = 1$, $\lambda_2 = 2$,$\mu = 0.4$, and no demand before $t$

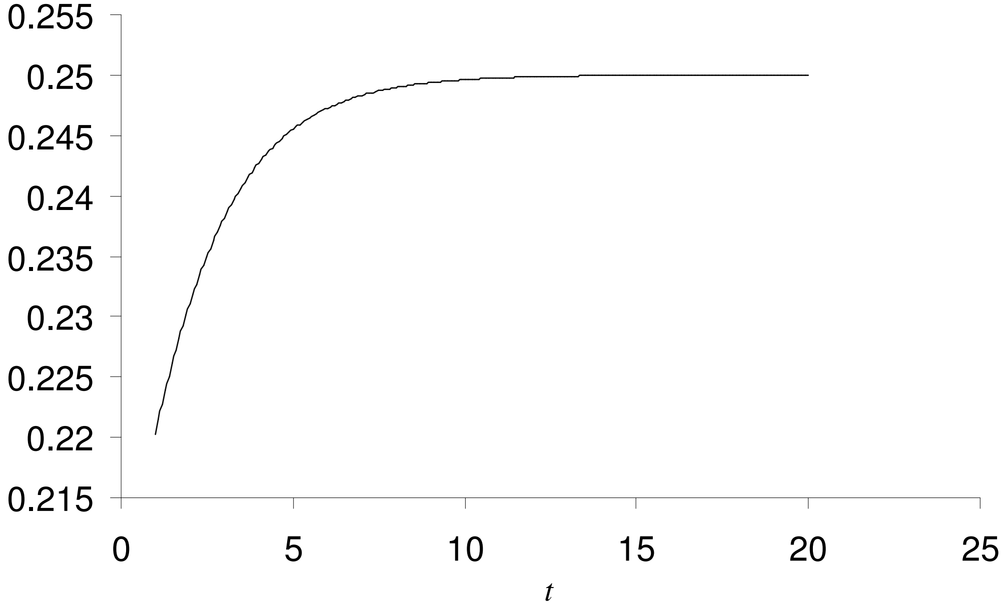


Figure 3–3: $\lambda_1 = 1$, $\lambda_2 = 3$,$\mu = 1.5$, and 1 demand before $t$, $s_1 = 1$

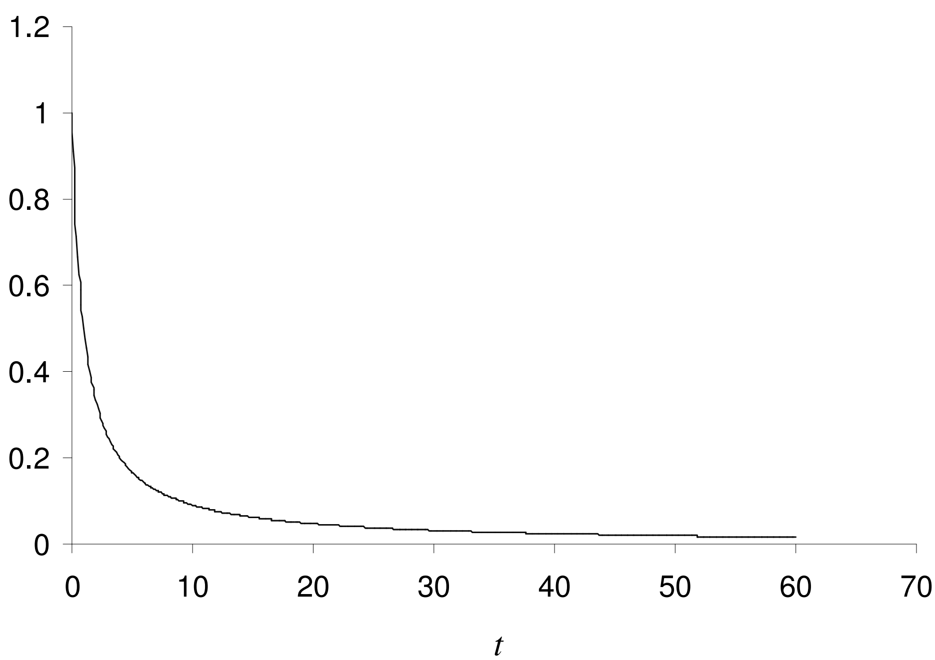


Figure 3–4: $\lambda_1 = 1$, $\lambda_2 = 2$,$\mu = 1$, and no demand before $t$

partially unobservable demand model. From Theorem 2.3.2, the optimal policy minimizes the function $C(y,t,\bar{h}(t)) + \alpha c y$ for every $t$. We want to find $y^*(t,\bar{h}(t))$, and we can use the method in Section 2.5 to compute the optimal inventory levels. Note that $y^*(t,\bar{h}(t))$ is just the point at which the right derivative of $C(y,t,\bar{h}(t)) + \alpha c y$ is no smaller than 0, while the left derivative is smaller than 0. Using equation (3.7), it is easy to see that

$$\begin{aligned}
\left(C(y,t,\bar{h}(t))\right)'_+ &= p(1,t,\bar{h}(t))\,(C(y,1))'_+ + (1-p(1,t,\bar{h}(t)))\,(C(y,2))'_+ \\
\left(C(y,t,\bar{h}(t))\right)'_- &= p(1,t,\bar{h}(t))\,(C(y,1))'_- + (1-p(1,t,\bar{h}(t)))\,(C(y,2))'_-
\end{aligned}$$

where the left and right derivatives of $C(y,1)$ and $C(y,2)$ can be determined by the method discussed in Song and Zipkin [52].

At $t=0$ or at a point in time where a demand occurs, say $t$, we can compute the probability $p(1,t,\bar{h}(t)$ as well as the optimal inventory position $y^*(t,\bar{h}(t)$ at that point in time (for convenience denoted simply by $y^*$ in the following if the arguments are clear from the context). As we will show below, we are able to determine the time at which the optimal inventory position will change if no new demand occurs by that time. Using an analogous approach as in Section 2.5, we need to solve the following two equations to determine the time $t$ at which the optimal inventory position will change if no new demand occurs up to time $t$:

$$\begin{aligned}
p(1,t,\bar{h}(t))\,(C(y^*,1))'_+ + (1-p(1,t,\bar{h}(t)))\,(C(y^*,2))'_+ + \alpha c &= 0 \\
p(1,t,\bar{h}(t))\,(C(y^*,1))'_- + (1-p(1,t,\bar{h}(t)))\,(C(y^*,2))'_- + \alpha c &= 0
\end{aligned}$$

or, equivalently,

$$p(1,t,\bar{h}(t)) = \frac{\alpha c + (C(y^*,2))'_+}{(C(y^*,2))'_+ - (C(y^*,1))'_+} \tag{3.10}$$

$$p(1,t,\bar{h}(t)) = \frac{\alpha c + (C(y^*,2))'_-}{(C(y^*,2))'_- - (C(y^*,1))'_-}. \tag{3.11}$$

Denote the solution to equation (3.10) by $t'$ and the solution to equation (3.11) by $t''$. If $\lambda \neq 0$ we can use equation (3.8) to find these solutions explicitly:

$$t' = s_n + \frac{1}{\lambda} \ln \left( \frac{\frac{(C(y^*,2))'_+ - (C(y^*,1))'_+}{\alpha c + (C(y^*,2))'_+} - 1 + \mu/\lambda}{(\mu/\lambda)\left[1 + e^{\lambda s_n} \sum_{k=0}^{n-1} (\lambda_2/\lambda_1)^{n-k} (e^{-\lambda s_k} - e^{-\lambda s_{k+1}})\right]} \right) \quad (3.12)$$

$$t'' = s_n + \frac{1}{\lambda} \ln \left( \frac{\frac{(C(y^*,2))'_- - (C(y^*,1))'_-}{\alpha c + (C(y^*,2))'_-} - 1 + \mu/\lambda}{(\mu/\lambda)\left[1 + e^{\lambda s_n} \sum_{k=0}^{n-1} (\lambda_2/\lambda_1)^{n-k} (e^{-\lambda s_k} - e^{-\lambda s_{k+1}})\right]} \right). \quad (3.13)$$

Similarly, if $\lambda = 0$ we can use equation (3.9) to obtain

$$t' = s_n + \frac{1}{\mu} \left( \frac{(C(y^*,2))'_+ - (C(y^*,1))'_+}{\alpha c + (C(y^*,2))'_+} - 1 \right) - \sum_{k=0}^{n-1} (\lambda_2/\lambda_1)^{n-k} (s_{k+1} - s_k) \quad (3.14)$$

$$t'' = s_n + \frac{1}{\mu} \left( \frac{(C(y^*,2))'_- - (C(y^*,1))'_-}{\alpha c + (C(y^*,2))'_-} - 1 \right) - \sum_{k=0}^{n-1} (\lambda_2/\lambda_1)^{n-k} (s_{k+1} - s_k). \quad (3.15)$$

Only solutions for $t'$ and $t''$ which are larger than $s_n$ will be considered. If either or both of these solutions are less than or equal to $s_n$, it simply means that the optimal inventory position will not increase (or decrease, or neither) from time $s_n$ onwards if no new demand occurs. It will prove to be convenient to replace a value of $t'$ or $t''$ that does not exceed $s_n$ by $\infty$. Moreover, after obtaining $t'$ and $t''$, we also need to check the derivative of the right derivative at $t'$ and the derivative of left derivative at $t''$. If the derivative of the right derivative of the cost rate function at $t'$ is nonnegative then the optimal inventory position will not change to $y^* + 1$ and we set $t' = \infty$. Similarly, If the derivative of the left derivative of the cost rate function at $t''$ is nonpositive then the optimal inventory position will not change to $y^* - 1$ and so we set $t'' = \infty$.

Now, if $t' < t''$ then we conclude that at time point $t'$ the optimal inventory position will increase by 1 unit; if $t' > t''$ then at time point $t''$ the optimal inventory position will decrease by 1 unit. If $t'$ and $t''$ are both infinity then the optimal inventory position will remain unchanged until a new demand occurs. Note that it is not possible that $t' = t'' < \infty$ since this would mean that right after time $t'$ the left derivative at $y^*$ is positive and the right derivative at $y^*$ is negative, which violates the convexity of the cost rate function.

To summarize, we have the following algorithm

- Step 1. At the beginning time of world state 1, we know that $p(1, 0, \bar{h}(0)) = 1$. By using the same method as in Song and Zipkin [52], compute the optimal value, and denote it by $y_0^*$. Set $n = 0$ and $s_n = 0$. Also, set $m = 0$ and $t_m = 0$. ($n$ records the number of demands that have occurred so far, while $m$ records the number of times that the optimal inventory position has changed so far.)
- Step 2. Compute $t'$ and $t''$ according to either equations (3.12) and (3.13) or equations (3.14) and (3.15) (with $y^*$ replaced by $y_m^*$). If $t' < t_m$ or

  $$p'(1, t', \bar{h}(t'))\,(C(y_m^*, 1))'_+ + (1 - p'(1, t', \bar{h}(t')))\,(C(y_m^*, 2))'_+ + \alpha c \geq 0$$

  set $t' = \infty$; if $t'' < t_m$ or

  $$p'(1, t'', \bar{h}(t''))\,(C(y_m^*, 1))'_- + (1 - p'(1, t'', \bar{h}(t'')))\,(C(y_m^*, 2))'_- + \alpha c \leq 0$$

  set $t'' = \infty$.
- Step 3. If no new demand occurs before $\min\{t', t''\}$, then the optimal inventory position changes to $y_{m+1}^* = y_m^* + 1$ at $t'$ if $t' < t''$ and to $y_{m+1}^* = y_m^* - 1$ at $t''$ if $t' > t''$. Set $t_{m+1} = \min\{t', t''\}$, $m = m + 1$, and return to Step 2.

  If a new demand occurs at time $s_{n+1} < \min\{t', t''\}$ set $t_{m+1} = s_{n+1}$ and compute the optimal inventory position $y_{m+1}^*$. Let $n = n + 1$, $m = m + 1$, and go to step 2.

### 3.2.4 An Extension

We next consider a minor extension to the model discussed above. Suppose that we start observing the model at some point in time that is past the actual start of the system, but that we know that the probability that the world is in state 1 at the starting time is $p_1$ (so that the probability that the world is in state 2 is $1-p_1$) and let $G$ denote the conditional distribution of the time remaining in state 1 given that we are currently in that state. Then by conditioning on whether we are in state 1 at time 0 we obtain that equation 3.2 becomes

$$p(1,t,\bar{h}(t)) \;=\; p_1 \frac{1}{1+\left(e^{(\lambda_1-\lambda_2)t}/\overline{G}(t)\right)\sum_{k=0}^{n}(\lambda_2/\lambda_1)^{n-k}\int_{s_k}^{s_{k+1}} e^{-(\lambda_1-\lambda_2)s}dG(s)}.$$

Note that, in case the time that will be spent in state 1 has an exponential distribution with parameter $\mu$, we have that $G$ is that same distribution and $p_1 = e^{-\mu s}$ where $s$ is the amount of time that has elapsed since the start of the system until the start of the observations.

## 3.3 Multiple World States Models

In the previous section, we studied models with only two world states. However, there are usually more than two world states to consider in real life. Therefore, in this section we consider models with multiple world states, i.e., $m$ world states.

### 3.3.1 Models with Multiple World States Which are Visited in a Fixed Sequence

We assume that we know all the world states, and the sequence they will appear. One example for this is the seasonal change. Another example for this type of world is the life cycle of products. We also assume that the demand in each world state follows a Poisson process, and the parameters are all known. As before, we cannot observe directly when the transition of world occurs, but only the demand.

Let us start with $m = 3$, that is, there are 3 world states, and they will be encountered in the order 1,2,3. Once the world enters state 3, it will stay there forever. The transition time in state $i$, $i = 1,2$, is *exponentially* distributed, with rate $\mu_i$. In state $i$, the demand process possesses rate $\lambda_i$.

As in the previous section, we use $\Lambda(t)$ equal to $\lambda_i$ to indicate whether the world is in state $i$ at time $t$, $i = 1,2,3$. Denote the conditional probability that the world is in state $i$ at time $t$ for every $t$ given that the history information up to time $t$ is $\bar{H}(t) = \bar{h}(t)$, by $p(i, t, \bar{h}(t)) = f_{\Lambda|\bar{H}}(\lambda_i|\bar{h}(t))$. Following from Bayes' rule, we obtain

$$p(i, t, \bar{h}(t)) = \frac{f(\lambda_i, \bar{h}(t))}{f_{\bar{H}}(\bar{h}(t))}. \tag{3.16}$$

To compute this conditional probability, we need to condition on both possible world state transition times in states 1 and 2, namely $T_1$ and $T_1 + T_2$, where $T_1 \sim G_1$ and $T_2 \sim G_2$. The scenarios need to be considered are $T_1 > t$, $T_1 + T_2 > t > T_1$, and $t > T_1 + T_2$. Let us consider computing the history density function $f_{\bar{H}}(\bar{h}(t))$ first. By conditioning on $T_1 = \tau_1$ (and $T_2 = \tau_2$), and denote the conditional density function of history information by $f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1)$ (and $f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2)$), we obtain

$$\begin{aligned}
f_{\bar{H}}(\bar{h}(t)) &= \int_0^\infty f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1) \\
&= \int_0^t f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1) + \int_t^\infty f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1) \\
&= \int_0^t \int_0^\infty f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2) dG_2(\tau_2) dG_1(\tau_1) + \\
&\quad \int_t^\infty f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1) \\
&= \int_0^t \left[ \int_0^{t-\tau_1} f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2) dG_2(\tau_2) + \right. \\
&\quad \left. \int_{t-\tau_1}^\infty f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2) dG_2(\tau_2) \right] dG_1(\tau_1) + \\
&\quad \int_t^\infty f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1) \\
&= \int_0^t \int_0^{t-\tau_1} f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2) dG_2(\tau_2) dG_1(\tau_1) + \\
&\quad \int_0^t \int_{t-\tau_1}^\infty f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2) dG_2(\tau_2) dG_1(\tau_1) + \\
&\quad \int_t^\infty f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1).
\end{aligned} \tag{3.17}$$

since the different world state transition time are independent.

It is obvious that the density function of the history $\bar{h}(t)$ can be decomposed as

$$f_{\bar{H}}(\bar{h}(t)) = f(\lambda_1, \bar{h}(t)) + f(\lambda_2, \bar{h}(t)) + f(\lambda_3, \bar{h}(t))$$

and

$$\begin{aligned}
f(\lambda_1, \bar{h}(t)) &= \int_t^\infty f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1) \\
f(\lambda_2, \bar{h}(t)) &= \int_0^t \int_{t-\tau_1}^\infty f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2) dG_2(\tau_2) dG_1(\tau_1) \\
f(\lambda_3, \bar{h}(t)) &= \int_0^t \int_0^{t-\tau_1} f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1, \tau_2) dG_2(\tau_2) dG_1(\tau_1)
\end{aligned}$$

The following computation and the notations used are similar as those in Section 3.2.1. For example, for history $\bar{h}(t) = \{N(t) = n, X_1 = x_1, \ldots, X_n = x_n\} = \{N(t) = n, S_1 = s_1, \ldots, S_n = s_n\}$ where $X_i$ is the interarrival time of the $i^{th}$ demand and $S_i$ is the arrival time of the $i^{th}$ demand, let $f_{\bar{H}|T_1}(n, s_1, \ldots, s_n|\tau_1)$ denote the conditional density of $N(t) = n, S_1 = s_1, \ldots, S_n = s_n$ given that $T_1 = \tau_1$, and let $f_{\bar{H}|N,T_1}(s_1, \ldots, s_n|n, \tau_1)$ denote the conditional density of $S_1 = s_1, \ldots, S_n = s_n$ given that $N(t) = n$ and $T_1 = \tau_1$. For additional conditions on $T_2$, $T_3$, the notations are straightforward.

If $\tau_1 > t$, then the world is in state 1 at time $t$, and

$$\begin{aligned}
f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) &= \Pr(N(t) = n|T_1 = \tau_1) f_{\bar{H}|N,T_1}(s_1, \ldots, s_n|n, \tau_1) \\
&= \frac{e^{-\lambda_1 t}(\lambda_1 t)^n}{n!} \frac{n!}{t^n} \\
&= e^{-\lambda_1 t} \lambda_1^n.
\end{aligned}$$

So

$$\begin{aligned}
f(\lambda_1, \bar{h}(t)) &= \int_t^\infty f_{\bar{H}|T_1}(\bar{h}(t)|\tau_1) dG_1(\tau_1) \\
&= e^{-\lambda_1 t} \lambda_1^n \bar{G}_1(t). \qquad (3.18)
\end{aligned}$$

If $\tau_1 < t < \tau_1 + \tau_2$, then the world is in state 2 at time $t$. As before, we use random variables $N_1$ and $N_2$ to represent the number of demands in state 1 and 2 respectively. At time $t$, for given $\tau_1$ and $\tau_2$, these numbers are known, denoted by $n_1(\tau_1)$ and $n - n_1(\tau_1)$ respectively, and the occurrences of demands in different world states are independent! Then we obtain

$$\begin{aligned} f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1,\tau_2) &= f_{\bar{H}|T_1,T_2}(n_1(\tau_1), x_1, \ldots, x_{n_1(\tau_1)}; \\ &\quad n - n_1(\tau_1), \sum_{i=1}^{n_1(\tau_1)+1} x_i - \tau_1, x_{n_1(\tau_1)+2}, \ldots, x_n|\tau_1,\tau_2) \\ &= f_{\bar{H}|T_1,T_2}(n_1(\tau_1), x_1, \ldots, x_{n_1(\tau_1)}|\tau_1,\tau_2) \cdot \\ &\quad f_{\bar{H}|T_1,T_2}(n - n_1(\tau), \sum_{i=1}^{n_1(\tau_1)+1} x_i - \tau_1, x_{n_1(\tau_1)+2}, \ldots, x_n|\tau_1,\tau_2) \\ &= e^{-\lambda_1\tau_1}\lambda_1^{n_1(\tau_1)} \cdot e^{-\lambda_2(t-\tau_1)}\lambda_2^{n-n_1(\tau_1)}. \end{aligned}$$

If $s_k < \tau_1 < s_{k+1}$, then $N_1(\tau_1) = k$ and $N_2(t-\tau_1) = n-k$, and we obtain

$$f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1,\tau_2) = e^{-\lambda_1\tau_1}\lambda_1^k e^{-\lambda_2(t-\tau_1)}\lambda_2^{n-k}.$$

So

$$\begin{aligned} f(\lambda_2, \bar{h}(t)) &= \int_0^t \int_{t-\tau_1}^{\infty} f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1,\tau_2)dG_2(\tau_2)dG_1(\tau_1) \\ &= \int_0^t \bar{G}_2(t-\tau_1) f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1,\tau_2)dG_1(\tau_1) \\ &= \sum_{k=0}^{n} \lambda_1^k\lambda_2^{n-k} \int_{s_k}^{s_{k+1}} \bar{G}_2(t-\tau_1)e^{-\lambda_1\tau_1}e^{-\lambda_2(t-\tau_1)}dG_1(\tau_1) \qquad (3.19) \end{aligned}$$

where we let $s_0 = 0$ and $s_{n+1} = t$.

Similarly, if $t > \tau_1 + \tau_2$, then the world is in state 3 at time $t$. We denote the numbers of demands occurred in each state by $N_1$, $N_2$ and $N_3$ respectively. For given $\tau_1$, $\tau_2$ and $\tau_3$, these numbers are known, denoted by $n_1(\tau_1)$, $n_2(\tau_2)$ and $n - n_1(\tau_1) - n_2(\tau_2)$ respectively. Then by following the similar procedures as above, we obtain

$$f_{\bar{H}|T_1,T_2}(\bar{h}(t)|\tau_1,\tau_2) = e^{-\lambda_1\tau_1}\lambda_1^{n_1(\tau_1)} \cdot e^{-\lambda_2\tau_2}\lambda_2^{n_2(\tau_2)} \cdot e^{-\lambda_3(t-\tau_1-\tau_2)}\lambda_3^{n-n_1(\tau_1)-n_2(\tau_2)}.$$

If $s_k < \tau_1 < s_{k+1}$, $s_{k+j} < \tau_1 + \tau_2 < s_{k+j+1}$, where $k, j \geq 0$ and $k + j \leq n$, then $n_1(\tau_1) = k$, $n_2(\tau_2) = j$, and $n_3(t - \tau_1 - \tau_2) = n - j - k$. Then

$$\begin{aligned}
f(\lambda_3, \bar{h}(t)) &= \int_0^t \int_0^{t-\tau_1} f(\bar{h}(t)|T_1 = \tau_1, T_2 = \tau_2) dG_2(\tau_2) dG_1(\tau_1) \\
&= \sum_{k=0}^{n} \int_{s_k}^{s_{k+1}} \int_0^{t-\tau_1} f(\bar{h}(t)|T_1 = \tau_1, T_2 = \tau_2) dG_2(\tau_2) dG_1(\tau_1) \\
&= \sum_{k=0}^{n} \int_{s_k}^{s_{k+1}} \Big[ \int_0^{s_{k+1}-\tau_1} e^{-\lambda_1 \tau_1} \lambda_1^k \cdot e^{-\lambda_2 \tau_2} \cdot e^{-\lambda_3 (t-\tau_1-\tau_2)} \lambda_3^{n-k} dG_2(\tau_2) + \\
&\qquad \sum_{j=1}^{n-k} \int_{s_{j+k}-\tau_1}^{s_{j+k+1}-\tau_1} e^{-\lambda_1 \tau_1} \lambda_1^k \cdot e^{-\lambda_2 \tau_2} \lambda_2^j \cdot e^{-\lambda_3 (t-\tau_1-\tau_2)} \cdot \\
&\qquad \lambda_3^{n-j-k} dG_2(\tau_2) \Big] dG_1(\tau_1) \\
&= \sum_{k=0}^{n} \int_{s_k}^{s_{k+1}} \Big[ \int_0^{s_{k+1}-\tau_1} e^{-\lambda_1 \tau_1} \lambda_1^k \cdot e^{-\lambda_2 \tau_2} \cdot e^{-\lambda_3 (t-\tau_1-\tau_2)} \lambda_3^{n-k} dG_2(\tau_2) + \\
&\qquad \sum_{j=1}^{n-k} \int_{s_{j+k}-\tau_1}^{s_{j+k+1}-\tau_1} e^{-\lambda_1 \tau_1} \lambda_1^k \cdot e^{-\lambda_2 \tau_2} \lambda_2^j \cdot e^{-\lambda_3 (t-\tau_1-\tau_2)} \cdot \\
&\qquad \lambda_3^{n-j-k} dG_2(\tau_2) \Big] dG_1(\tau_1). \qquad (3.20)
\end{aligned}$$

Now substitute $G_1(G_2)$ with the *exponential* distributions with rate $\mu_1(\mu_2)$ in equalities (3.18), (3.19) and (3.20), we can perform the following computations:

$$f(\lambda_1, \bar{h}(t)) = e^{-\lambda_1 t} \lambda_1^n e^{-\mu_1 t} \qquad (3.21)$$

$$\begin{aligned}
f(\lambda_2, \bar{h}(t)) &= \sum_{k=0}^{n} \lambda_1^k \lambda_2^{n-k} \int_{s_k}^{s_{k+1}} \bar{G}_2(t - \tau_1) e^{-\lambda_1 \tau_1} e^{-\lambda_2 (t-\tau_1)} dG_1(\tau_1) \\
&= \sum_{k=0}^{n} \lambda_1^k \lambda_2^{n-k} \mu_1 e^{-(\lambda_2+\mu_2)t} \frac{1}{(\lambda_1 + \mu_1) - (\lambda_2 + \mu_2)} \\
&\qquad \left[ e^{-((\lambda_1+\mu_1)-(\lambda_2+\mu_2))s_k} - e^{-((\lambda_1+\mu_1)-(\lambda_2+\mu_2))s_{k+1}} \right] \qquad (3.22)
\end{aligned}$$

and

$$
\begin{aligned}
& f(\lambda_3, \bar{h}(t)) \\
= & \sum_{k=0}^{n} \int_{s_k}^{s_{k+1}} \Bigg[ \int_0^{s_{k+1}-\tau_1} e^{-\lambda_1\tau_1}\lambda_1^k \cdot e^{-\lambda_2\tau_2} \cdot e^{-\lambda_3(t-\tau_1-\tau_2)}\lambda_3^{n-k} dG_2(\tau_2) + \\
& \qquad \sum_{j=1}^{n-k} \int_{s_{j+k}-\tau_1}^{s_{j+k+1}-\tau_1} e^{-\lambda_1\tau_1}\lambda_1^k \cdot e^{-\lambda_2\tau_2}\lambda_2^j \cdot e^{-\lambda_3(t-\tau_1-\tau_2)} \cdot \\
& \qquad\qquad \lambda_3^{n-j-k} dG_2(\tau_2) \Bigg] dG_1(\tau_1) \\
= & \sum_{k=0}^{n} \int_{s_k}^{s_{k+1}} \Bigg[ e^{-\lambda_1\tau_1}\lambda_1^k e^{-\lambda_3(t-\tau_1)}\lambda_3^{n-k} \frac{\mu_2}{\lambda_2+\mu_2-\lambda_3}[1-e^{-(\lambda_2+\mu_2-\lambda_3)(s_{k+1}-\tau_1)}] + \\
& \qquad \sum_{j=1}^{n-k} e^{-\lambda_1\tau_1}\lambda_1^k\lambda_2^j e^{-\lambda_3(t-\tau_1)}\lambda_3^{n-j-k} \frac{\mu_2}{\lambda_2+\mu_2-\lambda_3} \cdot \\
& \qquad\qquad [e^{-(\lambda_2+\mu_2-\lambda_3)(s_{j+k}-\tau_1)} - e^{-(\lambda_2+\mu_2-\lambda_3)(s_{j+k+1}-\tau_1)}] \Bigg] dG_1(\tau_1) \\
= & \sum_{k=0}^{n} \Bigg[ \lambda_1^k e^{-\lambda_3 t}\lambda_3^{n-k} \frac{\mu_2}{\lambda_2+\mu_2-\lambda_3} \Bigg[ \frac{\mu_1}{\lambda_1+\mu_1-\lambda_3} \cdot \\
& \qquad (e^{-(\lambda_1+\mu_1-\lambda_3)s_k} - e^{-(\lambda_1+\mu_1-\lambda_3)s_{k+1}}) - \\
& \qquad \frac{\mu_1}{\lambda_1+\mu_1-\lambda_2-\mu_2} \cdot \\
& \qquad (e^{-(\lambda_1+\mu_1-\lambda_2-\mu_2)s_k} - e^{-(\lambda_1+\mu_1-\lambda_2-\mu_2)s_{k+1}}) \Bigg] + \\
& \quad \sum_{j=1}^{n-k} \lambda_1^k\lambda_2^j e^{-\lambda_3 t}\lambda_3^{n-j-k} \frac{\mu_2}{\lambda_2+\mu_2-\lambda_3} \frac{\mu_1}{\lambda_1+\mu_1-\lambda_2-\mu_2} \cdot \\
& \qquad \Big[e^{-(\lambda_2+\mu_2-\lambda_3)s_{j+k}} - e^{-(\lambda_2+\mu_2-\lambda_3)s_{j+k+1}}\Big] \cdot \\
& \qquad \Big[e^{-(\lambda_1+\mu_1-\lambda_2-\mu_2)s_k} - e^{-(\lambda_1+\mu_1-\lambda_2-\mu_2)s_{k+1}}\Big] \Bigg]. \qquad (3.23)
\end{aligned}
$$

By time $t$ all the demand occurrence times $s_k, k = 0, n$ are known, and we can compute (3.21), (3.22) and (3.23). Plugging the results into equation (3.16), we can compute the conditional probability that the world is in state $i$ $(i = 1, 2, 3)$ at time $t$

given $\bar{h}(t)$,

$$\begin{aligned} p(i,t,\bar{h}(t)) &= \frac{f(\lambda_i,\bar{h}(t))}{f(\bar{h}(t))} \\ &= \frac{f(\lambda_i,\bar{h}(t))}{f(\lambda_1,\bar{h}(t))+f(\lambda_2,\bar{h}(t))+f(\lambda_3,\bar{h}(t))}. \end{aligned}$$

It is obvious that as the number of states increases, the computations in this section will be more complicated. Even the results will be too long to represent. We next seek a simple way to represent the (conditional) density functions of the history.

### 3.3.2 A Recursive Formula

To analyze the more general multiple-state model, i.e., $m$ states in total, the above way of multiple conditioning is not applicable. Another approach is to derive a recursive formula. For notational convenience in the recursion, we re-index the world states and number them from $m$ through 1 in the order of their occurrence as in the previous section. To use the history information into the recursive fashion, we let

$$\bar{H}(t) = \{D_m(u) : 0 \le u \le t\}$$

where the subscript $m$ represents the observation of the cumulative demand curve starts in state $m$.

Denote $f_k(\bar{h}(t)) = f_k(D_k(u) : 0 \le u \le t)$ to represent the density function that part of the history which starts at the starting time of world state $k$ to take the instance $\{D_k(u) : 0 \le u \le t\}$. We also denote $f_k(\bar{h}(t), j)$ to represent the (joint) density function of history information up to time $t$, $\bar{h}(t)$, and the world state at time $t$, $j$, given that the history observation starts at the starting time of state $k$.

Now by conditioning on the first state transition time $T_m = \tau_m$, we get two possibilities: if $t < \tau_m$, the world is still in state $m$ at time $t$; if $t > \tau_m$, we consider it a problem with $m-1$ states, starting at time $\tau_m$. Denote the conditional density function of history which starts in world state $m$ given $T_m = \tau_m$ by $f_m(\bar{h}(t)|\tau_m)$, we obtain

$$\begin{aligned} f_m(\bar{h}(t)) &= \int_0^\infty f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) \\ &= \int_0^t f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) + \int_t^\infty f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) \\ &= \int_0^t f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) + e^{-\lambda_m t} \lambda_m^n \bar{G}_m(t) \end{aligned}$$

where for $\tau_m > t$, $f_m(\bar{h}(t)|\tau_m) = e^{-\lambda_m t} \lambda_m^n \bar{G}_m(t)$, which is derived from the results of the previous section.

For the case $\tau_m < t$, the total past history information information by $t$ can be divided into two parts: the observations before $\tau_m$, i.e., $\bar{h}(\tau_m)$, which followed a stationary Poisson process with parameter $\lambda_m$ and $n_m(\tau_m)$ demands occurred in state $m$, and the observations after $\tau_m$. Since the world process is Markovian, these two parts of observations are independent. And for the history after $\tau_m$ and before $t$, it is exactly the same as the observations between time 0 and time $t - \tau_m$ for a $(m-1)$-state world model. By using the notations defined above, we get for $\tau_m < t$,

$$\begin{aligned} & f_m(\bar{h}(t)|\tau_m) \\ &= f_m(\bar{h}(\tau_m), m) \cdot f_{m-1}\left(D_m(\tau_m + u) - D_m(\tau_m) : 0 \le u \le t - \tau_m\right) \\ &= e^{-\lambda_m \tau_m} \lambda_m^{n_m(\tau_m)} \cdot f_{m-1}\left(\{D_{m-1}(u) : 0 \le u \le t - \tau_m\}\right) \end{aligned}$$

where we denote

$$\{D_{m-1}(u) : 0 \le u \le t - \tau_m\} = \{D_m(\tau_m + u) - D_m(\tau_m) : 0 \le u \le t - \tau_m\}$$

to be the observation of history from the entering time of state $m-1$ until $t - \tau_m$ time later, and $n_m(\tau_m)$ to be the number of demands occurred in $\tau_m$ time units in world state $m$.

So the general recursive formula can be written as

$$\begin{aligned}
& f_k(D_k(u) : 0 \leq u \leq t) \\
&= \int_0^t e^{-\lambda_k \tau_k} \lambda_k^{n_k(\tau_k)} \cdot f_{k-1}\left(D_{k-1}(u) : 0 \leq u \leq t - \tau_k\right) dG_k(\tau_k) + \\
&\quad f_k(\{D_k(u) : 0 \leq u \leq t\}, k).
\end{aligned}$$

We have

$$f_k(\{D_k(u) : 0 \leq u \leq t\}, k) = e^{-\lambda_k t} \lambda_k^n \bar{G}_k(t) \quad \text{for } 1 \leq k \leq m$$

and

$$\begin{aligned}
& f_k(\{D_k(u) : 0 \leq u \leq t\}, k-j) \\
&= \int_0^t e^{-\lambda_k \tau_k} \lambda_k^{n_k(\tau_k)} \cdot f_{k-1}\left(\{D_{k-1}(u) : 0 \leq u \leq t - \tau_k\}, k-j\right) dG_k(\tau_k) \\
&\qquad \text{for } 1 \leq k \leq m \text{ and } 1 \leq j < k
\end{aligned}$$

We apply this recursive formula to the case $m = 3$

$$\begin{aligned}
& f_3(\bar{h}(t)) \\
&= f_3(D_3(u) : 0 \leq u \leq t) \\
&= \int_0^t e^{-\lambda_3 \tau_3} \lambda_3^{n_3(\tau_3)} \cdot f_2\left(D_2(u) : 0 \leq u \leq t - \tau_3\right) dG_3(\tau_3) + \\
&\quad f_3(\{D_3(u) : 0 \leq u \leq t\}, \Lambda(t) = 3) \\
&= \int_0^t e^{-\lambda_3 \tau_3} \lambda_3^{n_3(\tau_3)} \cdot \left\{ \int_0^{t-\tau_3} e^{-\lambda_2 \tau_2} \lambda_2^{n_2(\tau_2)} \cdot f_1\left(D_1(u) : 0 \leq u \leq t - \tau_3 - \tau_2\right) dG_2(\tau_2) \right. \\
&\quad \left. + e^{-\lambda_2 (t-\tau_3)} \lambda_2^{n - n_3(\tau_3)} \bar{G}_2(t) \right\} dG_3(\tau_3) + e^{-\lambda_3 t} \lambda_3^n \bar{G}_3(t) \\
&= \int_0^t e^{-\lambda_3 \tau_3} \lambda_3^{n_3(\tau_3)} \int_0^{t-\tau_3} e^{-\lambda_2 \tau_2} \lambda_2^{n_2(\tau_2)} e^{-\lambda_1 (t - \tau_3 - \tau_2)} \lambda_1^{n - n_3(\tau_3) - n_2(\tau_2)} dG_2(\tau_2) dG_3(\tau_3) \\
&\quad + \int_0^t e^{-\lambda_3 \tau_3} \lambda_3^{n_3(\tau_3)} e^{-\lambda_2 (t-\tau_3)} \lambda_2^{n - n_3(\tau_3)} \bar{G}_2(t) dG_3(\tau_3) + e^{-\lambda_3 t} \lambda_3^n \bar{G}_3(t)
\end{aligned}$$

and the recursive formula gives the same results as those given by direct method in the previous section.

We need to point out that this recursive formula only eases the representation of the conditional probability functions given history $\bar{h}(t)$,

$$p(i,t,\bar{h}(t)) = \frac{f_m(\bar{h}(t),i)}{f_m(\bar{h}(t))},$$

but to compute the probability, we still need to express it extensively, as in Section 3.3.1, and the computation remains the same complicated.

### 3.3.3 Optimal Inventory Position

For fixed lead time $\ell$, by conditional on which state the world is currently in, the distribution of the lead time demand can be expressed as

$$\begin{aligned} F_{D_\ell^{t,\bar{h}(t)}}(z) &= \Pr(D_\ell^{t,\bar{h}(t)} \le z) \\ &= \sum_{i=1}^{m} \Pr(D_\ell^{t,\bar{h}(t)} \le z \mid \Lambda(t) = \lambda_i) p(i,t,\bar{h}(t)) \\ &= \sum_{i=1}^{m} F_{D_\ell^i}(z) p(i,t,\bar{h}(t)). \end{aligned} \quad (3.24)$$

The last equation follows from the fact that the world is a Markov process and demand process is a Poisson process. For stochastic lead time,

$$\tilde{F}_{D_L^{t,\bar{h}(t)}}(z) = \sum_{i=1}^{m} p(i,t,\bar{h}(t)) \int_0^\infty e^{-\alpha\ell} F_{D_\ell^i}(z) dF_L(\ell). \quad (3.25)$$

Then we can write

$$\begin{aligned} C(y,t,\bar{h}(t)) &= E\left[e^{-\alpha L}\hat{C}(y - D_L^{t,\bar{h}(t)})\right] \\ &= \int\int e^{-\alpha\ell}\hat{C}(y-z) dF_{D_\ell^{t,\bar{h}(t)}}(z) dF_L(\ell) \\ &= \sum_{i=1}^{m} p(k,t,\bar{h}(t)) \int\int e^{-\alpha\ell}\hat{C}(y-z) dF_{D_\ell^i}(z) dF_L(\ell) \\ &= \sum_{i=1}^{m} p(i,t,\bar{h}(t)) C(y,i) \end{aligned}$$

where $C(y,i)$ has the same definitions as in Song and Zipkin [52].

In principal we can find the optimal inventory positions by using the similarly algorithm as in Section 3.2.3, but now the form of the probability function $p(i,t,\bar{h}(t))$ becomes much more complicated, and to compute the $t$s at which the left or right derivatives of $C(y,t,\bar{h}(t))$ are 0 is not an easy task now.

To compute the optimal inventory position, we again need to assume that the lead time distribution is continuous phase-type. The optimal inventory level $y^*(t,\bar{h}(t))$ is just the point at which the right derivative of $C(y,t,\bar{h}(t)) + \alpha c y$ is no smaller than 0, while the left derivative is smaller than 0. It follows

$$\big(C(y,t,\bar{h}(t))\big)'_+ = \sum_{i=1}^{m} p(i,t,\bar{h}(t))\,(C(y,i))'_+$$

$$\big(C(y,t,\bar{h}(t))\big)'_- = \sum_{i=1}^{m} p(i,t,\bar{h}(t))\,(C(y,i))'_- \,.$$

At the starting time, we know the probability of $p(i,t,\bar{h}(t))$ for all $i = 1,\ldots,m$, and we can compute the optimal inventory position as $y^*$. As time goes on, if no demand occurs, we can determine the next time when the optimal inventory position will change given that no new demand occurs by that time by solving the following two equations separately,

$$\begin{aligned}
\sum_{i=1}^{m} p(i,t,\bar{h}(t))\,(C(y,i))'_+ + \alpha c &= 0 \\
\sum_{i=1}^{m} p(i,t,\bar{h}(t))\,(C(y,i))'_- + \alpha c &= 0.
\end{aligned}$$

Solve these two equations separately, and denote the solutions by $t'$ and $t''$ respectively. Then we can follow the procedure in Section 3.2.3, and the details are omitted here. The major difficulty here is to get the solution $t'$ and $t''$.

Every time when a new demand occurs, we should update all the probabilities $p(i,t,\bar{h}(t))$ at the time $t$, and get the new optimal inventory position at that time accordingly. Then we repeat the procedure to compute the next time point that the optimal inventory position changes if no demand occurs.

### 3.3.4 More General Multiple States World Models

The multiple states world models we considered in the previous section require that the world states are visited in a fixed order. We next consider a generalization of that model: When the world is in certain state, instead of only one possible state to visit next, the world process may go to any state that appears after the current state in the sequence. Once the world is in a state, it cannot go back to a previous state.

We study a multiple-state model with $m$ states. As in the previous section, we re-index the world states and number them from $m$ through 1. Also, we let $\{D_m(u) : 0 \leq u \leq t\}$ represent the observation of the cumulative demand curve between time 0 and $t$ where the state starts in state $m$. Denote $f_k(\bar{h}(t)) = f_k(D_k(u) : 0 \leq u \leq t)$ to represent the density that part of the history which starts at the starting time of world state $k$ to take the instance $\{D_k(u) : 0 \leq u \leq t\}$.

If the history starts in world state $m$, conditioning on the first state transition time $T_m = \tau_m$, and the next world state visited, $i$, we get two possibilities: if $t < \tau_m$, the world is still in state $m$ at time $t$; if $t > \tau_m$, the history information information by $t$ can be divided into two parts: the observations before $\tau_m$, i.e., $\bar{h}(\tau_m)$, which followed a stationary Poisson process with parameter $\lambda_m$ with $n_m(\tau_m)$ demand occurrences, and the observations after $\tau_m$. Since the world process is Markovian, these two parts of observations are independent. For the history after $\tau_m$ and before $t$ where the world process enters state $i$ at time $\tau_m$, it is exactly the same as the observations between time 0 and time $t-\tau_m$ for a $i$-state world model. Denoting that the conditional density function of history $\bar{h}(t)$ given the first state transition time $\tau_m$ by $f_m(\bar{h}(t)|\tau_m)$, while the conditional density function given the first transition time $\tau_m$ and the next state $i$ by $f_m(\bar{h}(t)|i, \tau_m)$, we obtain

$$
\begin{aligned}
& f_m(\bar{h}(t)) \\
& \quad = \int_0^\infty f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) \\
& \quad = \int_0^t f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) + \int_t^\infty f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) \\
& \quad = \int_0^t \sum_{i=1}^{m-1} p_{mi} f_m(\bar{h}(t)|i, \tau_m) dG_m(\tau_m) + \int_t^\infty f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) \\
& \quad = \sum_{i=1}^{m-1} p_{mi} \int_0^t f_m(\bar{h}(t)|i, \tau_m) dG_m(\tau) + \int_t^\infty f_m(\bar{h}(t)|\tau_m) dG_m(\tau_m) \\
& \quad = \sum_{i=1}^{m-1} p_{mi} \int_0^t f_m(\bar{h}(t)|i, \tau_m) dG_m(\tau) + e^{-\lambda_m t} \lambda_m^n \bar{G}_m(t). \qquad (3.26)
\end{aligned}
$$

Using the notation defined above, we obtain for $\tau_m < t$

$$
f_m(\bar{h}(t)|i, \tau_m) \;=\; e^{-\lambda_m \tau_m} \lambda_m^{n_m(\tau_m)} \cdot f_i\left(\{D_i(u) : 0 \le u \le t - \tau_m\}\right)
$$

where we denote

$$
\{D_i(u) : 0 \le u \le t - \tau_m\} = \{D_m(\tau_m + u) - D_m(\tau_m) : 0 \le u \le t - \tau_m | \Lambda(\tau_m +) = i\}
$$

as the observation of history from the entering time of state $i$ until $t - \tau_m$ time later.

So the general recursive formula can be written as

$$
\begin{aligned}
& f_k(D_k(u) : 0 \le u \le t) \\
& \quad = \sum_{i=1}^{k-1} p_{ki} \int_0^t e^{-\lambda_k \tau} \lambda_k^{n_k(\tau)} \cdot f_i\left(D_i(u) : 0 \le u \le t - \tau\right) dG_k(\tau) + \\
& \qquad f_k(\{D_k(u) : 0 \le u \le t\}, k)
\end{aligned}
$$

We have

$$
f_k(\{D_k(u) : 0 \le u \le t\}, k) \;=\; e^{-\lambda_k t} \lambda_k^n \bar{G}_k(t) \quad \text{for } 1 \le k \le m
$$

and

$$\begin{aligned}
&f_k(\{D_k(u) : 0 \le u \le t\}, j) \\
&= \int_0^t e^{-\lambda_k \tau} \lambda_k^{n_k(\tau)} \sum_{i=j}^{k-1} p_{ki} f_i\left(\{D_i(u) : 0 \le u \le t - \tau\}, j\right) dG_k(\tau) \\
&\qquad \text{for } 1 \le k \le m \text{ and } 1 \le j < k
\end{aligned}$$

This enables us to compute the probability functions

$$p(i, t, \bar{h}(t)) = \frac{f_m(\bar{h}(t), i)}{f_m(\bar{h}(t))}.$$

The algorithm to compute the optimal inventory position is very similar to that in previous section. The only difference lies in the probability function $p(i, t, \bar{h}(t))$.

## 3.4 Summary

In this chapter, we have studied inventory models with partially unobservable demand process. For a two-world-state model, we have shown some properties of the model, derived the optimal inventory policy and given an algorithm to compute the optimal policy. For a multiple state model, we give a recursive formula to help represent and compute the optimal inventory policy. We leave the computation of the optimal policy of a more general demand models for future research.

# CHAPTER 4
# JOINT PRICING AND INVENTORY CONTROL IN DYNAMIC ENVIRONMENT

## 4.1 Introduction

In the previous chapters, we assume that the demand process is exogenously determined and cannot be changed by human activities. Selling prices are not part of the decisions, and the goal is to minimize the expected costs, since the total expected revenue is not controllable and thus omitted. This chapter studies the joint pricing and inventory control problems under Markov and semi-Markov modulated demand environment, which introduces pricing flexibility by allowing the rate of the Poisson process in each state to depend on the price of the product. The decisions now need to make here are not only when and how much to place a replenishment order, but also the optimal prices to set. The objective is to maximize the total expected profits.

The chapter is organized as follows. In Section 4.2, we study the joint pricing and inventory model under a price-sensitive Poisson demand environment without Markov modulation. We first study the model where price can only be set once at the beginning, and give some properties that can be used to determine the optimal solution. We also give an algorithm to compute the optimal solution. Next we study the model where price can be continuously set. In Section 4.3, we extend the study to the semi-Markov modulated Poisson demand environment, and show that with certain approximation, the model can be solved in the similar way as in a Poisson demand environment. In all the models, we assume that the ordering cost is linear, and there is a positive order lead time.

## 4.2 Joint Pricing and Inventory Control in Price Sensitive Poisson Demand Environment

In this section, we assume that the demand process is a price sensitive Poisson process. The rate of the Poisson process depends on the price set. We first study the optimal policy if we can only set the price once at the start, then relax this condition and assume that we can continuously change the price.

The inventory level (position) is reviewed continuously, and an order can be placed any time. Orders placed will arrive after a fixed positive lead time $\ell$, so the discounted ordering cost becomes $c = \overline{c}E[e^{-\alpha\ell}]$. When a customer demand occurs, it is fulfilled immediately if the product is in stock, and put on back order if the product is currently stocked out. In both cases, we assume that the customer pays at the time when the demand occurs at the prevailing price.

The demand is sensitive to the prices: if the price is $\rho$, then the demand rate will be $\lambda(\rho)$. We will often be using the total demand occurring during the lead time. Note that this total demand depends on the pricing decisions during the lead time. If the price cannot be changed after the initial choice $\rho$, then this lead time demand is Poisson random variable with mean $\rho\ell$. If the price can be changed continuously, then the lead time demand depends on the entire price function during the lead time.

We consider the revenue from selling the product and three types of costs: ordering, holding, and shortage costs, and our objective is to maximize the total expected discounted profits over the infinite horizon.

### 4.2.1 The Model that Price Can Only Be Set Once

We first assume that the price can be set only once at the start, i.e., $\rho$, and then remain unchanged throughout the horizon. Thus the demand process will be a stationary Poisson process with rate $\lambda(\rho)$. So we can define $D^{\ell,\rho}$ to be the random variable representing the total demand occurring during the lead time, and let $F_{D^{\ell,\rho}}(x)$ be its cumulative distribution function.

Now the conditional expected discounted holding and shortage cost rate, at the end of a lead time, and viewed now, given that the current inventory position (after ordering decision) is $y$, can be written as

$$
\begin{aligned}
C(y) &= E\left[e^{-\alpha\ell}\hat{C}(y - D^{\ell,\rho})\right] \\
&= \int e^{-\alpha\ell}\hat{C}(y-z)dF_{D^{\ell,\rho}}(z).
\end{aligned}
$$

For a fixed price, the problem reduces to an inventory problem, and it is well known that an order-up-to policy is optimal. We denote $v^{\pi}(x|\rho)$ to be the expected total discounted profit over the infinite horizon under control policy $\pi$ when the initial inventory level is $x$, after serving the demand at that time (if any), given that the price is set at level $\rho$. Let $v^*(x|\rho)$ be the optimal profit associated with an optimal inventory policy $\pi^*$. If now an order is placed and the inventory position is raised up to $y$, then in a stationary Poisson environment, nothing changes until the next demand. We get the following recursion

$$
v^*(x|\rho) = \max_y \left\{ -c(y-x) - \frac{1}{\alpha+\lambda(\rho)}C(y) + \frac{\lambda(\rho)}{\alpha+\lambda(\rho)}(\rho + v^*(y-1|\rho)) \right\}
$$

Let $\Pi^*(x|\rho) = v^*(x|\rho) - cx$, and we get

$$
\begin{aligned}
\Pi^*(x|\rho) &= \max_y H(y|\rho) \\
H(y|\rho) &= -\frac{\alpha}{\alpha+\lambda(\rho)}cy - \frac{1}{\alpha+\lambda(\rho)}C(y) + \frac{\lambda(\rho)}{\alpha+\lambda(\rho)}(\rho - c) + \frac{\lambda(\rho)}{\alpha+\lambda(\rho)}\Pi^*(y-1).
\end{aligned}
$$

Since return of unwanted inventory is allowed, $\Pi^*(x|\rho)$ is independent of $x$, which can be simplified to $\Pi^*(\rho)$, and we can get

$$
\Pi^*(\rho) = \max_y \frac{1}{\alpha}\left\{\lambda(\rho)(\rho - c) - C(y) - \alpha c y\right\}. \tag{4.1}
$$

The optimal price should be chosen to maximize $\Pi^*(\rho)$, and the maximum profit over the infinite horizon $\Pi^*$ is

$$\Pi^* \;=\; \max_\rho \Pi^*(\rho) = \max_\rho \max_y \frac{1}{\alpha}\left\{\lambda(\rho)(\rho - c) - C(y) - \alpha c y\right\}. \tag{4.2}$$

For a fixed price $\rho$, it is easy to show (ref. Song and Zipkin [52]) that the myopic inventory control policy which minimize $\alpha c y + C(y)$ is optimal, i.e., the optimal inventory position can be determined as

$$y^*(\rho) = \arg\min_y \left\{F_{D_\ell^\rho}(y) \geq \frac{p - \alpha\bar{c}}{h+p}\right\}. \tag{4.3}$$

Before we proceed further, we make some common assumptions about the demand rate with regard to the price:

**Assumption 4.2.1** *The allowable price set $\Upsilon$ is a finite discrete set, or a compact subset of nonnegative real number, with a lower bound greater than the unit cost $c$.*

**Assumption 4.2.2** *The demand rate function $\lambda(\rho)$ is differentiable and decreasing.*

Obviously, for infinite horizon stationary problem, the price should not be set below cost $c$, otherwise we will not ever enter the market. The demand rate is also bounded above from Assumption 4.2.1, denoted by $\Lambda$.

For a fixed price $\rho$, the lead time demand is a Poisson distributed random variable with rate $\lambda(\rho)\ell$. Without loss of generality, we can scale the lead time to be 1, and do not consider the $\ell$ explicitly in the following. And from 4.2.2, we assume that $\lambda(\rho)$ is strictly decreasing, thus $\rho$ and $\lambda$ have one-to-one correspondence. So we will use $\lambda$ as the control variable instead of $\rho$ throughout the chapter. We use $D_\lambda$ instead of $D^{\ell,\rho}$ to represent the lead time demand, and $F_\lambda$ instead of $F_{D^{\ell,\rho}}$ to represent its cumulative distribution function (*cdf*). We also use $y^*(\lambda)$ instead of $y^*(\rho)$ to represent the optimal inventory position corresponding to price $\rho$ (or equivalently, demand rate $\lambda$). We make the following assumptions regarding $\lambda$.

**Assumption 4.2.3** *Function $\lambda(\rho(\lambda) - c)$ is concave in $\lambda$.*

It is easy to verify that $\lambda(\rho(\lambda) - c)$ is concave for two most frequently used cases of demand rate functions $\lambda(\rho)$: linear function, $\lambda(\rho) = \Lambda(1 - b\rho)$; and exponential function, $\lambda(\rho) = \Lambda e^{-b\rho}$.

With all these assumptions, the lead time demand is a $Poisson(\lambda)$ random variable, and the *cdf* is

$$\begin{aligned} F_\lambda(y) &= \sum_{x=0}^{y} \Pr(D_\lambda = x) \\ &= \sum_{x=0}^{y} \frac{e^{-\lambda}\lambda^x}{x!}. \end{aligned}$$

For fixed $y$, it is a continuous function of $\lambda$. And we can easily show it is decreasing in $\lambda$ for $\lambda > 0$ by taking the first order derivative

$$\begin{aligned} \frac{dF_\lambda(y)}{d\lambda} &= -e^{-\lambda} + \sum_{x=1}^{y} \frac{1}{x!}(-e^{-\lambda}\lambda^x + xe^{-\lambda}\lambda^{x-1}) \\ &= -e^{-\lambda} + \sum_{x=1}^{y} \frac{1}{x!} e^{-\lambda}\lambda^{x-1}(x - \lambda) \\ &= -e^{-\lambda}\frac{\lambda^y}{y!} \\ &= -\Pr(D^\lambda = y) \\ &< 0. \end{aligned}$$

And for fixed $y > 0$

$$\begin{aligned} \frac{d^2F_\lambda(y)}{d^2\lambda} &= \frac{d\left[-e^{-\lambda}\frac{\lambda^y}{y!}\right]}{d\lambda} \\ &= \frac{1}{y!}e^{-\lambda}\lambda^{y-1}(\lambda - y) \end{aligned}$$

So the *cdf* of lead time demand is convex for $\lambda > y$, and concave for $\lambda < y$. As a special case, for $y = 0$, $F_\lambda(y)$ is always convex in $\lambda$.

Now the following result is straightforward

**Proposition 4.2.4** *$y^*(\lambda)$ is a step function of $\lambda$ with step size one.*

**Proof:** It follows directly from equation (4.3), and the fact that $F_\lambda(y)$ is a continuous and decreasing function of $\lambda$. □

Note that the cost rate function can be expressed as

$$\begin{aligned} C(y,\lambda) &= e^{-\alpha}\left[\int_0^y h(y-z)dF_\lambda(z) + \int_y^\infty p(z-y)dF_\lambda(z)\right] \\ &= e^{-\alpha}\left[(h+p)yF_\lambda(y) - py - \int_0^y (h+p)z dF_\lambda(z) + p\lambda\right]. \end{aligned}$$

For integer $y$,

$$\begin{aligned} C(y,\lambda) &= e^{-\alpha}\left[(h+p)y\sum_{x=0}^{y}\Pr(D_\lambda = x) - (h+p)\sum_{x=0}^{y} x\Pr(D_\lambda = x) + p\lambda - py\right] \\ &= e^{-\alpha}\left[(h+p)\sum_{x=0}^{y-1}(y-x)\Pr(D_\lambda = x) + p\lambda - py\right] \\ &= e^{-\alpha}\left[(h+p)\sum_{x=0}^{y-1}\sum_{j=x}^{y-1}\Pr(D_\lambda = x) + p\lambda - py\right] \\ &= e^{-\alpha}\left[(h+p)\sum_{j=0}^{y-1}\sum_{x=0}^{j}\Pr(D_\lambda = x) + p\lambda - py\right] \\ &= e^{-\alpha}\left[(h+p)\sum_{j=0}^{y-1}F_\lambda(j) + p\lambda - py\right]. \end{aligned}$$

As a result of Proposition 4.2.4, the function $C(y^*(\lambda),\lambda)$ is discontinuous in $\lambda$. Denote the discontinuity points of $y^*(\lambda)$ by $\lambda^1 < \lambda^2 < \cdots$. For each such range of $\lambda$, $y^*(\lambda)$ is constant, and $C(y^*(\lambda^{m+1}),\lambda)$ is continuous in $\lambda$. For example, for $\lambda \in (\lambda^m, \lambda^{m+1}]$, $y^*(\lambda) = y^*(\lambda^{m+1})$, where $y^*(\lambda^{m+1})$ satisfies

$$F_{\lambda^{m+1}}(y^*(\lambda^{m+1})) = \frac{p-\alpha\bar{c}}{h+p}. \tag{4.4}$$

And for $\lambda \in (\lambda^m, \lambda^{m+1})$,

$$F_\lambda(y^*(\lambda^{m+1})) > \frac{p-\alpha\bar{c}}{h+p}.$$

The step function $y^*(\lambda)$ is illustrated in Figure 4–1.

Though $C(y^*(\lambda),\lambda)$ is discontinuous in $\lambda$, we can show that the following proposition holds.

**Proposition 4.2.5** *$C(y^*(\lambda),\lambda)+\alpha c y^*(\lambda)$ is continuous in $\lambda$ everywhere.*

**Proof:** We study the right continuity of $C(y^*(\lambda),\lambda)+\alpha c y^*(\lambda)$ at each discontinuous point of $y^*(\lambda)$, i.e., $\lambda^m$ for any $m>0$, which is the only possible discontinuity of $C(y^*(\lambda),\lambda)+\alpha c y^*(\lambda)$.

$$
\begin{aligned}
&\lim_{\lambda\downarrow\lambda^m}\{C(y^*(\lambda),\lambda)+\alpha c y^*(\lambda)\} - C(y^*(\lambda^m),\lambda^m)+\alpha c y^*(\lambda^m)\\
&= \lim_{\lambda\downarrow\lambda^m}\left\{C(y^*(\lambda^{m+1}),\lambda)+\alpha c y^*(\lambda^{m+1})\right\} - C(y^*(\lambda^m),\lambda^m)-\alpha c y^*(\lambda^m)\\
&= \lim_{\lambda\downarrow\lambda^m}\left\{e^{-\alpha}\left[(h+p)\sum_{j=0}^{y^*(\lambda^{m+1})-1} F_\lambda(j)+p\lambda-py^*(\lambda^{m+1})\right]\right\}+\alpha c\\
&\quad -e^{-\alpha}\left[(h+p)\sum_{j=0}^{y^*(\lambda^{m})-1} F_{\lambda^m}(j)+p\lambda^m-py^*(\lambda^{m})\right]\\
&= e^{-\alpha}[(h+p)F_{\lambda^m}(y^*(\lambda^m))-p]+\alpha c\\
&= 0
\end{aligned}
$$

where the second to last equation follows equation (4.4). So $C(y^*(\lambda),\lambda)+\alpha c y^*(\lambda)$ is indeed continuous everywhere in $\lambda$ for its allowable range. $\square$

In addition, for $\lambda\in(\lambda^m,\lambda^{m+1}]$,

$$
\begin{aligned}
\frac{d[C(y^*(\lambda^{m+1}),\lambda)+\alpha c y^*(\lambda^{m+1})]}{d\lambda} &= e^{-\alpha}\left[-(h+p)\sum_{j=0}^{y^*(\lambda^{m+1})-1} e^{-\lambda(\lambda)}\frac{\lambda^j}{j!}+p\right]\\
&= e^{-\alpha}\left[-(h+p)F_\lambda(y^*(\lambda^{m+1})-1)+p\right]\\
&> 0
\end{aligned}
$$

where the last inequality follows again the definition of $y_i^*(\lambda^{m+1})$ in equation (4.4). We summarize this result in the following proposition:

**Proposition 4.2.6** *For each $m=1,2,\ldots,$ and for $\lambda\in(\lambda^m,\lambda^{m+1}]$, $C(y^*(\lambda^{m+1}),\lambda)+\alpha c y^*(\lambda^{m+1})$ is an increasing function of $\lambda$.*

The second order derivative with respect to $\lambda$ for $\lambda \in (\lambda^m, \lambda^{m+1}]$ for $m = 1, 2, \ldots$ equals

$$\begin{aligned} & \frac{d^2[C(y^*(\lambda^{m+1}), \lambda) + \alpha c y^*(\lambda^{m+1})]}{d\lambda^2} \\ = \ & e^{-\alpha}(h+p)\Pr(D_\lambda = y^*(\lambda^{m+1}) - 1) > 0. \end{aligned}$$

This leads to the following proposition:

**Proposition 4.2.7** $C(y^*(\lambda^{m+1}), \lambda) + \alpha c y^*(\lambda^{m+1})$ *is piecewise convex in every range* $\lambda \in (\lambda^m, \lambda^{m+1}]$.

Now we study some properties of $\Pi^*(\lambda)$ (same as $\Pi^*(\rho)$ in equation (4.3)) to help determine the optimal demand rate $\lambda^*$, and thus the optimal price $\rho^*$. Recall that we assume that $\lambda(\rho)$ is strictly decreasing, the reverse function exists, and denoted by $\rho(\lambda)$. Let $R(\lambda) = \lambda(\rho(\lambda) - c)$. Now we want to maximize $MP(\lambda) = \alpha\Pi^*(\lambda) = R(\lambda) - C(y^*(\lambda), \lambda) - \alpha c y^*(\lambda)$ over all feasible $\lambda$. By assumption 4.2.3, $MP(\lambda)$ is a piecewise concave function for each range $(\lambda^m, \lambda^{m+1}]$. And for some $m$, and $\lambda \in (\lambda^m, \lambda^{m+1})$,

$$\begin{aligned} MP(\lambda) &= R(\lambda) - C(y^*(\lambda^{m+1}), \lambda) - \alpha c y^*(\lambda^{m+1}) \\ &= R(\lambda) - e^{-\alpha}\left[(h+p)\sum_{j=0}^{y^*(\lambda^{m+1})-1} F_\lambda(j) + p\lambda - p y^*(\lambda^{m+1})\right] - \alpha c y^*(\lambda^{m+1}). \end{aligned}$$

The first order derivative with regard to $\lambda$ within range $\lambda \in (\lambda^m, \lambda^{m+1})$ is

$$\begin{aligned} MP'(\lambda) &= R'(\lambda) - e^{-\alpha}\left[(h+p)\sum_{j=0}^{y^*(\lambda^{m+1})-1}(-\Pr(D_\lambda = j)) + p\right] \\ &= R'(\lambda) - e^{-\alpha}\left[-(h+p)F_\lambda(y^*(\lambda^{m+1}) - 1) + p\right]. \end{aligned}$$

At the two end points of the range, the derivative does not exist, but the left and right derivatives are well defined. The right derivative at the left end point $\lambda^m$, denoted by $MP'_+(\lambda^m)$, can be determined as

$$MP'_+(\lambda^m) = R'(\lambda^m) - \alpha c \tag{4.5}$$

while the left derivative at the right end point $\lambda^{m+1}$, denoted by $MP'_{-}(\lambda^{m+1})$, is

$$MP'_{-}(\lambda^{m+1}) \;=\; R'(\lambda^{m+1}) - \alpha c - e^{-\alpha}(h+p)\Pr(D_{\lambda^{m+1}} = y^*(\lambda^{m+1})) \qquad (4.6)$$

where we use the fact that

$$\begin{aligned} F_{\lambda^{m+1}}(y^*(\lambda^{m+1}) - 1) &= F_{\lambda^{m+1}}(y^*(\lambda^{m+1})) - \Pr(D_{\lambda^{m+1}} = y^*(\lambda^{m+1})) \\ &= \frac{p - \alpha\bar{c}}{h+p} - \Pr(D_{\lambda^{m+1}} = y^*(\lambda^{m+1})). \end{aligned}$$

Since $R(\lambda)$ is concave, $R'(\lambda^m) > R'(\lambda^{m+1})$. So $MP'_{+}(\lambda^m) > MP'_{+}(\lambda^{m+1})$, i.e., the right derivative of $MP$ at the left end point of each piece of $\lambda$ is decreasing.

We can obtain the following proposition

**Proposition 4.2.8** *The optimal price must lie in the intervals* $(\lambda^m, \lambda^{m+1}]$ *where the two end points satisfies*

$$\begin{aligned} MP'_{+}(\lambda^m) &> 0 \\ MP'_{-}(\lambda^{m+1}) &\leq 0. \end{aligned}$$

**Proof:** Note that the right derivative at $\lambda^{m+1}$ is

$$MP'_{+}(\lambda^{m+1}) = R'(\lambda^{m+1}) - \alpha c,$$

which is always greater than the left derivative at the same point, which is determined in equation (4.6). Also note that $MP'_{+}(\lambda^m)$ is decreasing in $m$, i.e., $MP'_{+}(\lambda^m) > MP'_{+}(\lambda^{m+1})$. Once $MP'_{+}(\lambda^m)$ drops to less than or equal to 0, the value of $MP(\lambda)$ will keep decreasing, thus cannot be optimal after. And if $MP'_{-}(\lambda^{m+1}) > 0$, the value of $MP(\lambda)$ will keep increasing within this range, thus cannot be optimal. So our search for the optimal price is restricted to the intervals of $(\lambda^m, \lambda^{m+1}]$ where

$$\begin{aligned} MP'_{+}(\lambda^m) &> 0 \\ MP'_{-}(\lambda^{m+1}) &\leq 0. \end{aligned}$$

□

From this theorem, once the right derivative at the left end point of a piece of $\lambda$ drops to below 0, we do not need to consider any intervals after this point, since the profit function value will keep decreasing onwards. However, we cannot exclude the intervals before an end point at which the right derivatives is positive, because the optimal solution may lie in those intervals. Figure 4–2 illustrates the possible shape of the profit function. We see that the optimization problem is a global optimization problem, and a search over all the intervals satisfying Proposition 4.2.8 may be necessary. We next introduce such an algorithm to find the optimal price $\lambda$.

### 4.2.2 Algorithm to Compute the Optimal Price and Inventory Position

Based on these analysis, we can compute the optimal order-up-to level and price. Assume that the allowable price set is $[\underline{\lambda}, \overline{\lambda}]$ where $\underline{\lambda} \geq c$. We can determine an upper bound for the optimal price, $\hat{\lambda}$. We first solve $R'(\tilde{\lambda}) - \alpha c = 0$ for $\tilde{\lambda}$. We claim that $\tilde{\lambda}$ resides in the last interval whose two end points satisfy

$$\begin{aligned} MP'_+(\lambda^m) &> 0 \\ MP'_-(\lambda^{m+1}) &\leq 0. \end{aligned}$$

This is true because $R'(\lambda)$ is decreasing in $\lambda$, and thus for the next interval,

$$MP'_+(\lambda^{m+1}) = R'(\lambda^{m+1}) - \alpha c < R'(\tilde{\lambda}) - \alpha c = 0.$$

And since

$$\begin{aligned} MP'(\tilde{\lambda}) &= R'(\tilde{\lambda}) - e^{-\alpha}\left[-(h+p)F_{\tilde{\lambda}}(y^*(\lambda^{m+1}) - 1) + p\right] \\ &\leq R'(\tilde{\lambda}) - e^{-\alpha}\left[-(h+p)F_{\lambda^m}(y^*(\lambda^{m+1}) - 1) + p\right] \\ &= R'(\tilde{\lambda}) - \alpha c \\ &= 0 \end{aligned}$$

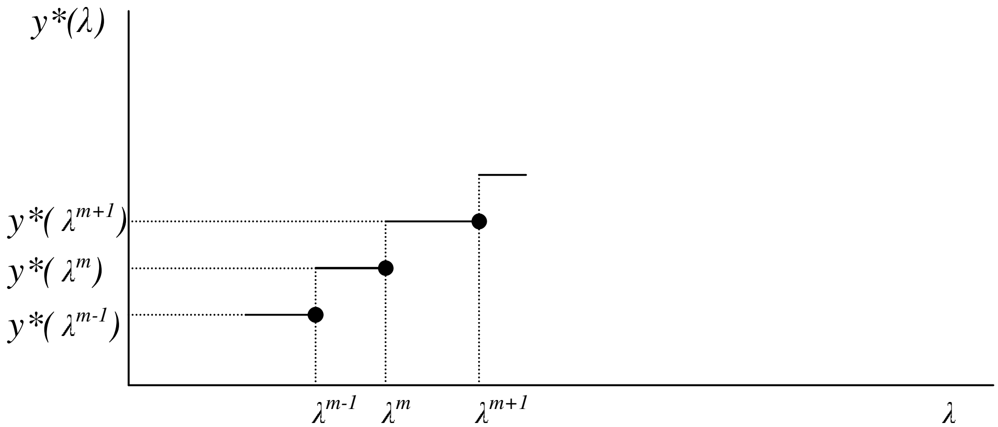


Figure 4–1: Optimal inventory position $y^*(\lambda)$

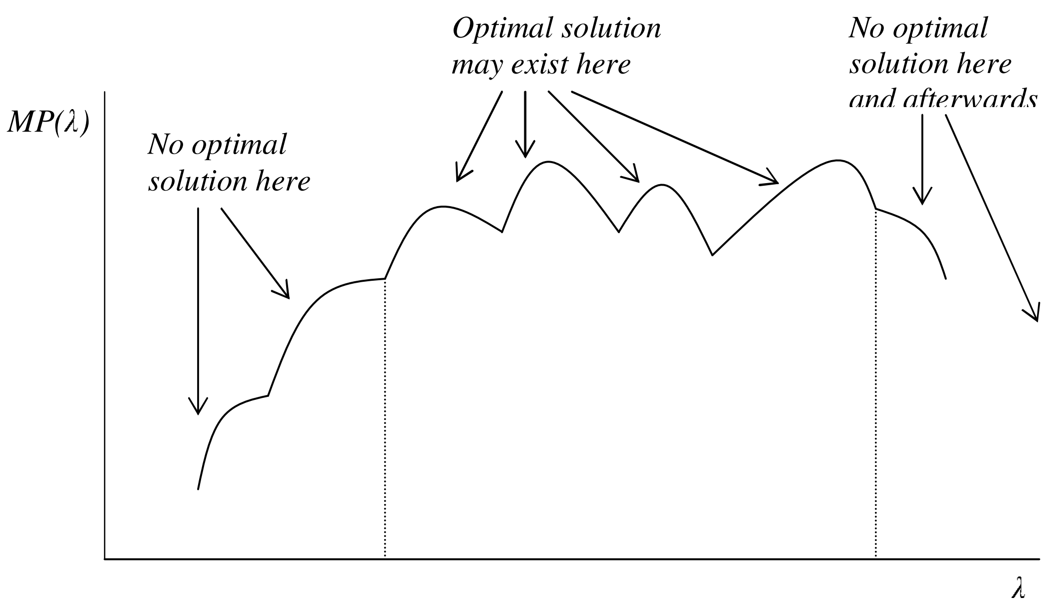


Figure 4–2: Profit function

we know that the stationary point in $(\lambda^m, \lambda^{m+1}]$ is no greater than $\tilde{\lambda}$. So we choose the smaller one of $\tilde{\lambda}$ and $\bar{\lambda}$ to be our new search upper bound for the optimal $\lambda$, denoted by $\hat{\lambda}$.

Next step is to determine all the discontinuous points of $y^*(\lambda)$ for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$, which satisfy

$$F_\lambda(y^*(\lambda)) = \frac{p - \alpha\bar{c}}{h + p}.$$

We start from the upper bound $\hat{\lambda}$ to find these discontinuous points. We first get $y^0 = \arg\min_y\{F_{\hat{\lambda}}(y) > \frac{p-\alpha\bar{c}}{h+p}\}$, then let $y^{-1} = y^0 - 1$. We then want to determine the value of $\lambda$ such that $F_\lambda(y^{-1}) = \frac{p-\alpha\bar{c}}{h+p}$, denoted by $\lambda^{-1}$. This $\lambda^{-1}$ is unique since $F_\lambda$ is a strictly decreasing function. To find out the $\lambda^{-1}$, we apply the Chernoff Bounds for Poisson random variables (ref. Ross [49]) to find a $\check{\lambda}$, for a fixed $t > 0$, such that

$$F_{\check{\lambda}}(y^m) \;\; \geq \;\; 1 - e^{\check{\lambda}(e^t-1)-ty^m} > \frac{p - \alpha\bar{c}}{h + p}.$$

Then we know that $\lambda^{-1}$ is between $(\check{\lambda}, \lambda^0)$, and we can use binary search method to locate the $\lambda^{-1}$ that satisfies $F_\lambda(y^{-1}) = \frac{p-\alpha\bar{c}}{h+p}$ within the interval $(\check{\lambda}, \lambda^0)$, since $F_\lambda$ is decreasing in $\lambda$. Next we let $y^{-2} = y^{-1} - 1$, and find $y^{-2}$ which satisfies $F_\lambda(y^{-2}) = \frac{p-\alpha\bar{c}}{h+p}$ in the same way. We continue this procedure until we reach the lower bound $\underline{\lambda}$. Thus we find out all the discontinuous points of $y^*(\lambda)$ between $\underline{\lambda}$ and $\hat{\lambda}$.

For interval $[\lambda^m, \lambda^{m+1}]$, we first check whether the both end points satisfy the optimality necessary condition

$$\begin{aligned} MP'_+(\lambda^m) &> 0 \\ MP'_-(\lambda^{m+1}) &\leq 0. \end{aligned}$$

It the condition is satisfied, we use binary search method to find out the unique root for $MP'(\lambda) = 0$ (stationary point) within $[\lambda^m, \lambda^{m+1}]$ since $MP'(\lambda)$ is strictly decreasing. If the condition is not satisfied, we discard this interval and continue to the next one. Evaluate the values of profits at all the stationary points, and pick up the best one. So the optimal price should be set to the value corresponds to this optimal demand rate, and solve a corresponding optimal order-up-to level. We summarize these procedures in the following algorithm.

Step 0. Solve $R'(\tilde{\lambda}) - \alpha c = 0$ for $\tilde{\lambda}$, and let $\hat{\lambda} = \min\{\tilde{\lambda}, \bar{\lambda}\}$.

Step 1. Let $k = 0$, $\lambda^k = \hat{\lambda}$, and set $\lambda^* = \hat{\lambda}$ and $MP^* = MP(\lambda^*)$. Solve $y^k = \arg\min_y\{F_{D_{\lambda^k}}(y) \geq \frac{p-\alpha\bar{c}}{h+p}\}$.

Step 2. If $\lambda^k < \underline{\lambda}$, set $\lambda^* = \underline{\lambda}$ and $MP^* = MP(\lambda^*)$. Go to Step 6. Let $y^{k-1} = y^k - 1$. For a fixed $t > 0$, solve for $\check{\lambda}$ that satisfies $1 - e^{\check{\lambda}(e^t-1)-ty^{k-1}} > \frac{p-\alpha\bar{c}}{h+p}$. Let $k = k-1$.

Step 3. Use binary search method to locate the $\lambda^k$ that satisfies $F_{D_{\lambda^k}}(y^k) = \frac{p-\alpha\bar{c}}{h+p}$ within the interval $(\check{\lambda}, \lambda^{k+1})$.

Step 4. Check whether the following two inequalities hold:

$$
\begin{aligned}
MP'_+(\lambda^k) &> 0 \\
MP'_-(\lambda^{k+1}) &\leq 0.
\end{aligned}
$$

It so, go to Step 5; if not, go back to Step 2.

Step 5. Use binary search method to determine the unique root for $MP'(\lambda) = 0$ (stationary point) within $[\lambda^k, \lambda^{k+1}]$, denote it by $\dot{\lambda}^k$. Let $MP^* = \max\{MP^*, MP(\dot{\lambda}^k)\}$, and let $\lambda^* = \arg\max\{MP^*\}$. Go back to Step 2.

Step 6. Output $\lambda^*$ as the optimal demand rate, and $\rho(\lambda^*)$ as the optimal price, and $y^*(\lambda^*)$ as the optimal order-up-to level. The optimal profit is $\frac{1}{\alpha}MP^*$.

### 4.2.3 The Model that Price Can Be Set Continuously

Now we suppose that the price can be continuously changed along the time. For a Poisson process, for the time point at which a demand occurs, and at any time point between this occurrence and next occurrence of demands, looking into the future, nothing has changed due to the memoryless property. So we can restrict our decision epoch to be the time point when a demand occurs.

One key difference between this model and the previous model where price can only be set once is that, during the lead time, there maybe multiple prices taking effect, and thus the demand rate will change. But to our knowledge, no one has tried this model formulation due to its complexity. Federgruen and Heching [19] discussed a heuristic treatment when a lead time is considered. They assumed that the price selected is maintained over the next order lead time. We adopt this assumption, and thus our model is really an approximation of the precise model.

Similarly as in Section 4.2.3, by discretizing time at the event epoch of demand occurrence, we get the following recursion

$$v^*(x) \quad = \quad \max_{\rho\in\Psi}\max_{y}\left\{-c(y-x)-\frac{1}{\alpha+\lambda(\rho)}C(y,\rho)+\frac{\lambda(\rho)}{\alpha+\lambda(\rho)}(\rho+v^*(y-1))\right\}.$$

Note that now the inventory holding and shortage cost rate function $C$ is dependent on the the inventory position and pricing decision now, which will be maintained over the lead time. Let $\Pi^*(x) = v^*(x) - cx$, and we can get

$$\begin{aligned}\Pi^*(x) &= \max_{\rho\in\Psi}\max_{y} H(y,\rho)\\ H(y,\rho) &= -\frac{\alpha}{\alpha+\lambda(\rho)}cy-\frac{1}{\alpha+\lambda(\rho)}C(y)+\frac{\lambda(\rho)}{\alpha+\lambda(\rho)}(\rho-c)+\frac{\lambda(\rho)}{\alpha+\lambda(\rho)}\Pi^*(y-1).\end{aligned}$$

Since return of unwanted inventory is allowed, $\Pi^*(x)$ is independent of the inventory position before ordering, $x$, we denote it by $\Pi^*$, and get

$$\Pi^* \quad = \quad \max_{\rho}\Pi^*(\rho) = \max_{\rho}\max_{y}\frac{1}{\alpha}\left\{\lambda(\rho)(\rho-c)-C(y)-\alpha cy\right\}$$

which is exactly the same profit function as the case where price can be set only once. So under our model assumption, this model can be solved in the exactly same way as the model where the price can only be set once is solved.

## 4.3 Semi-Markov Modulated Price-Sensitive Poisson Demand

In this section, we extend the demand in the previous section from *Poisson* demand environment to semi-Markov modulated Poisson demand process. We first introduce the demand model, and define some new notations. Then we analyze the model, and show under certain assumptions, the semi-Markov modulated Poisson demand model can be decomposed into different state-dependent *Poisson* demand models, and the similar results from the previous section follow.

### 4.3.1 The Model

We start by introducing the pricing factor into the demand model we have studied in Chapter 2, which is a *semi-Markov modulated price-sensitive Poisson demand process*. As before, the underlying core process which represents the state of the world is a continuous-time semi-Markov process with state space $I \subseteq \{0, 1, 2, \ldots\}$ and the transition probability matrix is $P = (p_{ij})_{i,j \in I}$ (where $p_{ii} = 0$ for each $i \in I$). We assume that the transition probabilities are not affected by price. When the core process is in state $i$ and the price for the product is $\rho$, the actual demand process follows a Poisson process with rate $\lambda_i(\rho)$. As a special case, the core process is a continuous-time Markov process, with rate $\mu_i$ for state $i$. Given current state $i$ and next state $j$, the transition time distribution of the semi-Markov process is $G_{ij}$, whose expected value is $1/\mu_{ij}$. Note that the time the world stays in state $i$ before leaving it has distribution $G_i$, where $G_i(x) = \sum_{j \in I} p_{ij} G_{ij}(x)$, and its expected value is $\sum_{j \in I} p_{ij}/\mu_{ij}$.

We want to determine a joint pricing and inventory control policy that maximizes the total expected discounted profits over an infinite horizon. The inventory level (position) is reviewed continuously, and an order can be placed any time. We keep all the other assumptions about the demand models unchanged as the Poisson demand model, except that now they are all state-dependent. We first study the model where price can be set only once at the beginning of each world state, and then briefly generalize the result to the case where price can be continuously set.

### 4.3.2 The Price Can Only Be Set Once for Each State

We first assume that the price can only be changed once at the beginning of each new world state, and remain constant throughout that state. If we set the price to be $\rho_i$ at the beginning of world state $i$, we can define a price vector $\bar{\rho} = \{\rho_i\}_{i \in I}$, with each element representing the price chosen for every state. The demand process within world state $i$ will then be a Poisson process with rate $\lambda_i(\rho_i)$.

For a fixed price vector $\hat{\rho}$, we define $D_\ell^{i,s,\hat{\rho}}$ to be the random variable representing the lead time demand given that the world has been in state $i$ for $s$ time units, and the price vector is $\hat{\rho}$. And the conditional expected discounted inventory and backlogging cost rate at the end of a lead time, as viewed from now, given that the current world state is $i$ and inventory position (after ordering) is $y$, and price is $\rho_i$, can be written as $C(y, i, s, \hat{\rho})$.

For a fixed price vector, the expected revenues will be fixed. The optimal inventory policy minimizes the total discounted expected costs. It depends on the current state of the world and how long it has been in the current state, and the entire price vector which determine the demand rate within each state, but nothing else since the world is a semi-Markov process. We can represent this inventory policy as $\mathbf{y}_i^*(s, \bar{\rho})$. We have solved for this policy before under the assumption if the return is allowed.

Define $\Pi_i^*(x, \bar{\rho})$ to be the optimal expected total profits by following the optimal inventory policy starting from the time when the core process just enters state $i$ when the initial inventory position is $x$, and discounted to the time of the occurrence of state transition, for a fixed price vector $\bar{\rho}$. The total profits can be divided into two components: the total profits during the world's current stay in state $i$, and the total profits after transitting into a different world state. The first component can be determined by conditioning on the time until the next transition, whose distribution function is equal to

$$G_i(t) \equiv \sum_{j \in I} p_{ij} G_{ij}(t).$$

Note that this distribution, and therefore the first profit component, does not depend on the next state visited. However, for the second profit component we need to condition on both the time of the transition as well as the next state itself.

We can compute the expected revenue earned in state $i$ discounted to the starting time of the state in this way: suppose there are $n$ demands occurred in state $i$, and they occurs at time $s_1, s_2, \ldots, s_n$ respectively, where the time is relative to the starting time of this state. Then the total revenue in state $i$ is

$$\rho_i(e^{-\alpha s_1} + e^{-\alpha s_2} + \ldots + e^{-\alpha s_n}).$$

Suppose the world stays in state $i$ for $\tau$ time, then

$$\begin{aligned}
& E[\text{state } i \text{ revenue in } \tau \text{ time} | n \text{ demands in state } i] \\
= \; & \rho_i n E[e^{-\alpha S}] \\
= \; & \rho_i n \frac{1}{\alpha\tau}[1 - e^{-\alpha\tau}]
\end{aligned}$$

where $S$ is uniformally distributed on $[0, \tau]$. It is true since given $n$ demands occurred, the distribution of arrival time of Poisson process is just like the order statistics of $n$ independent uniformally distributed random variables. And the order of demands becomes irrelavant when all of them are summed up.

Then the expected revenue in state $i$ can be computed as

$$\begin{aligned}
& \int_0^\infty E[\text{state } i \text{ revenue in } \tau \text{ time}] dG_i(\tau) \\
= \; & \int_0^\infty E\left[E[\text{state } i \text{ revenue in } \tau \text{ time} | N \text{ demands in state } i]\right] dG_i(\tau) \\
= \; & \int_0^\infty \rho_i E[N|\tau] \frac{1}{\alpha\tau}[1 - e^{-\alpha\tau}] dG_i(\tau) \\
= \; & \int_0^\infty \rho_i \lambda_i(\rho_i) \tau \frac{1}{\alpha\tau}[1 - e^{-\alpha\tau}] dG_i(\tau) \\
= \; & \rho_i \lambda_i(\rho_i) \frac{1}{\alpha}[1 - \tilde{G}_i(\alpha)]
\end{aligned}$$

where $\tilde{G}_i(\alpha)$ is the laplace transform of $G_i$,

$$\tilde{G}_i(\alpha) = \int_0^\infty e^{-\alpha\tau} dG_i(\tau).$$

Thus the optimal profits for fixed price vector $\bar{\rho}$ can be expressed in the following recursive way:

$$\begin{aligned}
\Pi_i^*(x,\bar{\rho}) \;&=\; (\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha}[1 - \tilde{G}_i(\alpha)] - c(y_i^*(0,\bar{\rho}) - x) - \\
&\int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} C(y_i^*(s,\bar{\rho}), i, s, \bar{\rho}) ds + \int_0^\tau e^{-\alpha s} c dy_i^*(s,\bar{\rho}) \right\} dG_i(\tau) \\
&+ \sum_{j\in I} p_{ij} \int_0^\infty e^{-\alpha\tau} \Pi_j^*(y_i^*(\tau,\bar{\rho})) dG_{ij}(\tau).
\end{aligned}$$

Let $\Pi_i^*(\bar{\rho}) = \Pi_i^*(x,\bar{\rho}) + cx$, and we can obtain

$$\begin{aligned}
\Pi_i^*(\bar{\rho}) \;&=\; (\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha}[1 - \tilde{G}_i(\alpha)] - c y_i^*(0,\bar{\rho}) - \\
&\int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} C(y_i^*(s,\bar{\rho}), i, s, \bar{\rho}) ds + \int_0^\tau e^{-\alpha s} c dy_i^*(s,\bar{\rho}) \right\} dG_i(\tau) + \\
&\sum_{j\in I} p_{ij} \int_0^\infty e^{-\alpha\tau} (\Pi_j^*(\bar{\rho}) - c y_i^*(\tau,\bar{\rho})) dG_{ij}(\tau) \\
&= (\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha}[1 - \tilde{G}_i(\alpha)] - \\
&\int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} [C(y_i^*(s,\bar{\rho}), i, s, \bar{\rho}) + \alpha c y_i^*(s,\bar{\rho})] ds \right\} dG_i(\tau) + \\
&\sum_{j\in I} p_{ij} \int_0^\infty e^{-\alpha\tau} \Pi_j^*(\bar{\rho}) dG_{ij}(\tau) \\
&= (\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha}[1 - \tilde{G}_i(\alpha)] - \\
&\int_0^\infty \left\{ \int_0^\tau e^{-\alpha s} [C(y_i^*(s,\bar{\rho}), i, s, \bar{\rho}) + \alpha c y_i^*(s,\bar{\rho})] ds \right\} dG_i(\tau) + \\
&\sum_{j\in I} p_{ij} E[e^{-\alpha T_{ij}}] \Pi_j^*(\bar{\rho})
\end{aligned}$$

where $T_{ij} \sim G_{ij}$ denotes the time spent in state $i$ when the next state is $j$, and we have also used a similar derivation as **before** to simplify the expression for the costs while in state $i$.

And the optimal profits over the infinite horizon can be determined as

$$\Pi_i^* = \max_{\bar{\rho}} \Pi_i^*(\bar{\rho}).$$

The inventory and backlogging cost rate function $C(y_i^*(s,\bar{\rho}), i, s, \bar{\rho})$ depends on the distribution of the lead time demand. The distribution is severely complicated by the fact that the underlying world may change to a new state during the lead time. Therefore, to achieve tractability of our model, we will approximate the actual expected cost rate by replacing the true lead time demand variable at any point in time by the lead time demand by assuming that no core process state changes take place during the lead time. Note that this approximation leads to a quite accurate estimation of total costs if the lead time is typically much smaller than the expected time spent in a state, i.e., if $l \ll E[T_{ij}]$ where $T_{ij} \sim G_{ij}$ for all $i, j \in I$ and $p_{ij} > 0$. In the following we will only study this approximate model.

### 4.3.3 Approximate Models for Semi-Markov Modulated Poisson Demand

With the approximation that the lead time never goes beyond the current world state, we can simplify the lead time demand to $D_L^{i,s,\rho_i}$, given that the world has been in state $i$ for $s$ time units, and the price is $\rho_i$. And correspondingly, the conditional expected discounted inventory and backlogging cost rate at the end of a lead time, as viewed from now, given that the current world state is $i$ and inventory position is $y$, and price is $\rho_i$, can be written as $C(y, i, s, \rho_i)$. With the approximation of the lead time, it is obvious that the optimal inventory policy for one world state for fixed price vector now depends only on the pricing decision within that state, i.e., $y_i^*(s,\bar{\rho})$ is reduced to $y_i^*(s,\rho_i)$.

The optimal expected total profits of the best joint pricing and inventory policy from the time when the core process just enters state $i$ when the initial inventory position is $x$, and discounted to the time of the occurrence of state transition, $\Pi_i^*(x)$, can be represented as the following recursive way:

$$\begin{aligned}\Pi_i^*(x) &= \max_{\rho_i}\Bigg\{(\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha}[1-\tilde{G}_i(\alpha)] - c(y_i^*(0,\rho_i)-x) - \\ &\qquad \int_0^\infty \left\{\int_0^\tau e^{-\alpha s}C(y_i^*(s,\rho_i),i,s,\rho_i)ds + \int_0^\tau e^{-\alpha s}cdy_i^*(s,\rho_i)\right\} dG_i(\tau) \\ &\qquad + \sum_{j\in I} p_{ij}\int_0^\infty e^{-\alpha\tau}\Pi_j^*(y_i^*(\tau,\rho_i))dG_{ij}(\tau)\Bigg\}.\end{aligned}$$

Let $\Pi_i^* = \Pi_i^*(x) + cx$, and we can obtain

$$\begin{aligned}\Pi_i^* &= \max_{\rho_i}\Bigg\{(\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha}[1-\tilde{G}_i(\alpha)] - \\ &\qquad \int_0^\infty \left\{\int_0^\tau e^{-\alpha s}[C(y_i^*(s,\rho_i),i,s,\rho_i) + \alpha c y_i^*(s,\rho_i)]ds\right\} dG_i(\tau) + \\ &\qquad \sum_{j\in I} p_{ij}E[e^{-\alpha T_{ij}}]\Pi_j^*\Bigg\} \\ &= \max_{\rho_i}\Bigg\{\int_0^\infty \Bigg[(\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha} - \int_0^\tau e^{-\alpha s}[C(y_i^*(s,\rho_i),i,s,\rho_i) + \\ &\qquad \alpha c y_i^*(s,\rho_i)]ds\Bigg] dG_i(\tau) + \sum_{j\in I} p_{ij}E[e^{-\alpha T_{ij}}]\Pi_j^*\Bigg\}.\end{aligned}$$

Since the total profits from the next world state onwards do not depend on $y_i, \rho_i$, $\Pi_i^*$ can be achieved by maximizing

$$(\rho_i - c)\lambda_i(\rho_i)\frac{1}{\alpha} - \int_0^\infty\int_0^\tau e^{-\alpha s}[C(y_i^*(s,\rho_i),i,s,\rho_i) + \alpha c y_i^*(s,\rho_i)]dsdG_i(\tau).$$

By the assumption that the leadtime is much smaller than every world state duration time, the leadtime demand $D_L^{i,s,\rho_i}$ is a Poisson random variable with mean $\lambda_i(\rho_i)$, independent of $s$. Thus $y_i^*(s,\rho_i)$ is also independent of $s$, and can be reduced to $y_i^*(\rho_i)$. Similarly, the inventory cost rate $C(y_i^*(\rho_i),i,s,\rho_i)$ does not depend on $s$ any more for our approximation model. So they can solved in the same way as the following Markov-modulated demand model.

For the case where the price is allowed to change continuously, if we make the approximation that the price selected at the beginning of a lead time is maintained over the lead time, then the optimal pricing is again to set the price only once at the beginning of each world state, the same as for the Poisson demand case, and the solution procedures remain unchanged.

### 4.3.4 Approximate Model that the World Process is Markovian and One Price for Each State

Now suppose we have a Markov modulated demand model with pricing decisions. The recursive relationship of the optimal profits can be simplified as:

$$\Pi_i^* = \max_{\rho_i} \frac{1}{\alpha + \mu_i} \left\{ (\rho_i - c)\lambda_i(\rho_i) - C(y_i^*(\rho_i), i, \rho_i) - \alpha c y_i^*(\rho_i) + \mu_i \sum_{j \in I} p_{ij} \Pi_j^* \right\}. \tag{4.7}$$

We see that in equation (4.7), the total profits from the next world state onwards do not depend on $y_i, \rho_i$. So determining the optimal price for state $i$ is equivalent to solve

$$\max_{\rho_i} (\rho_i - c)\lambda_i(\rho_i) - C(y_i^*(\rho_i), i, \rho_i) - \alpha c y_i^*(\rho_i). \tag{4.8}$$

This is exactly the same profit function we are seeking to maximize for the stationary Poisson demand model. Thus with our assumption of the approximate lead time, and the possibility to return, we can decompose the Markov modulated Poisson demand model into separate Stationary Poisson demand models, one for each world state.

## 4.4 Summary

In this chapter, we study the simultaneous pricing and inventory control decisions. We first study the joint pricing and inventory model under a price-sensitive Poisson demand environment without Markov modulation. In the case where the price can only be set once at the beginning, we give some properties that can be used to determine the optimal solution, and derive an algorithm to compute the optimal solution. We then study the model where price can be continuously set. Next we extend the study to the semi-Markov modulated Poisson demand environment, and show that with certain approximation, the model can be solved in the similar way as in a Poisson demand environment.

Future research will focus on the design of more effective algorithms for finding the optimal price in each state based on the deeper analysis of the profit function. We will also study the precise model for semi-Markov modulated Poisson demand process model instead of the approximate model we studied in this chapter. Finally, we will extend our models to the situations where returning excess inventory is not allowed.

# CHAPTER 5
# A STOCHASTIC MULTI-ITEM INVENTORY MODEL WITH UNEQUAL REPLENISHMENT INTERVALS AND LIMITED WAREHOUSE CAPACITY

## 5.1 Introduction

This chapter continues and extends a line of research that was started by Choi [14]. We first discuss a stochastic multi-item inventory model under both equal and unequal replenishment intervals with limited warehouse capacity in Section 5.2. In Section 5.3, we provide the optimal policy for the case of equal replenishment intervals under limited warehouse capacity, extending results by Choi [14] and Beyer [10] from continuous to general demand distributions. We then refine and generalize the three heuristics that were proposed by Choi [14] to approximate the optimal replenishment quantities. We also prove that in the equal replenishment intervals case, all the heuristics provide the optimal solution to the problem. Section 5.4 presents numerical results illustrating and comparing the performance of the heuristics. In Section 5.5, we discuss conclusions from our study and briefly summarize potential future research directions.

## 5.2 Model Assumptions and Formulation

Consider a single capacitated warehouse that serves multiple customers with multiple products. Customer demands for items are random variables, and back-logging is allowed at the warehouse at a per unit penalty cost. The probability distribution of demand for each product is assumed to be stationary. We assume that there essentially is zero replenishment lead time between manufacturers and the warehouse, which in a discrete time model implies that all replenishment orders in a period are received at the beginning of the following replenishment period. Items are replenished periodically by manufacturers, but individual items have different replenishment cycles. Since replenishment cycles are fixed and a delivery occurs in each replenishment cycle for every supplier, we assume any fixed delivery costs are constant and therefore outside the control of the warehouse decision maker. Our model does not therefore consider fixed ordering costs. In addition, since all demands will be satisfied we will also not consider any variable ordering costs. We consider a finite planning horizon. We first state our model assumptions and then formulate the expected cost function to determine an optimal replenishment policy. We use the following notation to describe this capacity constrained multi-item inventory model with unequal replenishment intervals:

**Parameters**

$V$ : fixed storage capacity of the warehouse

$\ell$ : number of items stocked at the warehouse

$T$ : length of the time horizon

$m$ : total number of *time instants* at which a replenishment occurs

$\tau_j$ : time instant at which the $j^{\text{th}}$ replenishment occurs

$R_j$ : set of items that are replenished at time $\tau_j$

$T_i$ : length of replenishment interval for item $i$

$h_i$ : nonnegative unit holding cost per replenishment interval of item $i$

$p_i$ : nonnegative unit backorder cost per replenishment interval of item $i$

**Random Variables**

$d_{i,t}$ : random demand of item $i$ during time $(t-1, t]$

$D_i$ : random variable for demand of item $i$ over the length of its replenishment interval, i.e., $D_i = \sum_{t=1}^{T_i} d_{i,t}$

**Decision variables**

$I_{i,j}$ : inventory level of item $i$ at time $\tau_j$ just before the replenishment

$Q_{i,j}$ : replenishment quantity of item $i$ at time $\tau_j$

Observe that each replenishment *time instant*, $\tau_j$, for $j = 1, \ldots, m$, corresponds to the beginning of a time period. We assume that the demands for a given item $d_{i,t}$ ($t = 1, \ldots, T$) are independent and identically distributed random variables, and that the demands of different items are independent (but not necessarily identically distributed). Furthermore, we denote the cumulative distribution function of the demand $D_i$ by $F_i$.

Define the vector of decision variables $Q_j = (Q_{i,j}; i \in R_j)$ consisting of the replenishment quantities of the items in $R_j$ that are replenished at time $\tau_j$. It is easy to see that the optimal replenishment quantity of an item depends not only on the item's own inventory level, but also on the current inventory level of all other items as well as the warehouse capacity. Figure 5–1 illustrates the tracking of inventory levels of all items, as well as the replenishment time instants.

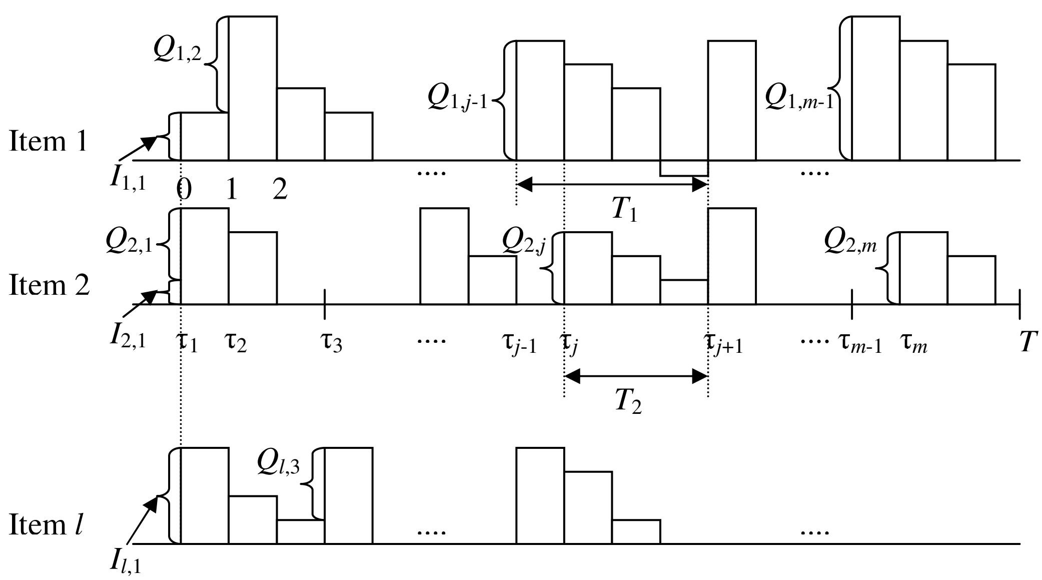


Figure 5–1: A multi-item inventory system.

In order to formulate our problem as a stochastic dynamic programming problem, we define $I_j = (I_{1,j}, \dots, I_{\ell,j})$ to be the state of the system in stage $j$ (i.e., at time $\tau_j$). For a given state $I_{j-1}$, the current inventory level of individual items in stage $j$ can be expressed recursively as follows:

$$I_{i,j} = \begin{cases} I_{i,j-1} + Q_{i,j-1} - \sum_{t=\tau_{j-1}+1}^{\tau_j} d_{i,t} & \text{if } i \in R_{j-1} \\ I_{i,j-1} - \sum_{t=\tau_{j-1}+1}^{\tau_j} d_{i,t} & \text{otherwise.} \end{cases}$$

We then define $g_j(I_j, Q_j)$ to be the expected holding and penalty costs for all items that are replenished in stage $j$ over their respective replenishment intervals, given initial inventories $I_j = (I_{1,j}, \dots, I_{\ell,j})$ and replenishment sizes $Q_j = (Q_{i,j};\, i \in R_j)$:

$$g_j(I_j, Q_j) = \sum_{i \in R_j} E\left[h_i(I_{i,j} + Q_{i,j} - D_i)^+ + p_i(D_i - I_{i,j} - Q_{i,j})^+\right].$$

Denoting the *minimum* expected holding and penalty costs for stages $j, \dots, m$ given starting inventory levels $I_j$ by $G_j(I_j)$ we obtain the following dynamic programming recursion, the solution of which provides an optimal policy for the problem:

$$G_j(I_j) = \min_{Q_j : \sum_{i \in R_j}(I_{i,j}+Q_{i,j})^+ + \sum_{i \notin R_j} I_{i,j}^+ \le V} \left\{ g_j(I_j, Q_j) + E\left[G_{j+1}(I_{j+1})\right] \right\}. \qquad (5.1)$$

We define the terminal costs $G_{m+1}(\cdot) = 0$ to be 0 for all possible final inventory levels remaining at the end of horizon. The dynamic programming recursion (5.1) includes the full costs of all replenishment cycles that have not been completed at the time horizon, $T$. If the finite horizon represents a truncation of an underlying infinite horizon problem, this mitigates end-of-study effects. However, in the case of a truly finite horizon problem, we may truncate all replenishment cycles at the horizon by defining a terminal replenishment instant $\tau_{m+1} = T + 1$ and corresponding $R_{m+1} = \{1, \dots, \ell\}$. Note that this would usually imply that the final replenishment interval for, say, item $i$ has a length different from $T_i$, and the ending inventory should possibly also be valued using different costs than $p_i$ and $h_i$. This is in principle easy to do by appropriately modifying the distribution of $D_i$ in the final replenishment cycle. In fact, in a similar way we could also handle replenishment interval lengths that are varying over the planning horizon. However, for ease of notation and exposition we have omitted this generalization. Finally, the optimal cost over the entire planning horizon is then equal to $G_1(I_1)$, where $I_1$ represents the initial inventory levels.

## 5.3 Solution Approaches

### 5.3.1 Equal Replenishment Intervals

Before we discuss heuristic solution approaches to the dynamic programming recursion (5.1) for the general case of unequal replenishment intervals, we will first study the case of equal replenishment intervals in more detail. In particular, we will determine the optimal policy for this case under a somewhat milder set of assumptions on the demand distributions than has been considered in the literature to date. In this case, $T_i$ is identical for all items, and $\tau_j = (j-1)T_1 + 1$ and $R_j = \{1, \ldots, \ell\}$ for $j = 1, \ldots, m$. The dynamic programming recursion (5.1) then simplifies to

$$G_j(I_j) = \min_{Q_j : \sum_{i=1}^{\ell} (I_{i,j} + Q_{i,j})^+ \leq V} \left\{ g_j(I_j, Q_j) + E\left[G_{j+1}(I_{j+1})\right] \right\}.$$

The following lemma shows that, without loss of optimality, we may assume that $I_{i,j} + Q_{i,j} \geq 0$, so that the $^+$ superscript can be ignored in the feasible region of this dynamic programming recursion if these nonnegativity constraints are added.

**Lemma 5.3.1** *The dynamic programming recursion is equivalent to*

$$G_j(I_j) = \min_{Q_j \geq (-I_j)^+ : \sum_{i=1}^{\ell} (I_{i,j} + Q_{i,j}) \leq V} \left\{ g_j(I_j, Q_j) + E\left[G_{j+1}(I_{j+1})\right] \right\}.$$

**Proof:** Consider a replenishment period $j$ such that, for some $i$, the inventory level just after replenishment is negative, i.e., $I_{i,j} + Q_{i,j} < 0$. Then consider the following alternative set of order quantities at replenishment period $j$ that ensure that the inventory level just after replenishment is always nonnegative:

$$\tilde{Q}_{i,j} = \begin{cases} Q_{i,j} & \text{if } I_{i,j} + Q_{i,j} \geq 0 \\ -I_{i,j} & \text{otherwise.} \end{cases}$$

Clearly, $\tilde{Q}_{i,j} \geq Q_{i,j}$ for all $i$. Nevertheless, it is easy to see that these order quantities satisfy the capacity constraint. If $I_{i,j}+Q_{i,j} < 0$, we have that $E[(I_{i,j}+Q_{i,j}-D_i)^+] = 0$ and $E[(I_{i,j}+\tilde{Q}_{i,j}-D_i)^+] = E[(0-D_i)^+] = 0$, as well as $E[(D_i-I_{i,j}-Q_{i,j})^+] > E[D_i]$. This immediately implies that

$$g_j(I_j, \tilde{Q}_j) \leq g_j(I_j, Q_j).$$

Now note that the initial inventories at the start of replenishment time $j+1$ satisfy

$$\tilde{I}_{i,j+1} = I_{i,j+1} \qquad \text{if } I_{i,j} + Q_{i,j} \geq 0$$

and

$$0 \geq \tilde{I}_{i,j+1} > I_{i,j+1} \qquad \text{otherwise}$$

for any realization of demands. This then implies that

$$E[G_{j+1}(\tilde{I}_{i,j+1})] \leq E[G_{j+1}(I_{i,j+1})]$$

since the total future costs are nonincreasing in $I_{i,j+1}$ when $I_{i,j+1} < 0$. This proves the desired result. $\square$

Ignall and Veinott [33] and Beyer et al. [10] show that, in a stochastic, multi-item inventory model with limited warehouse capacity and where all items are replenished simultaneously, a myopic policy is optimal if the product demands are stationary and independent, the cost functions are separable, and the inventory after any replenishment is always nonnegative. Choi [14] used this result to derive an explicit form of this myopic policy for our inventory system in the case where the demand distributions are absolutely continuous. The following theorem characterizes the optimal replenishment policy for general demand distributions. Before we state the theorem, we first introduce some additional notation and definitions that are used in the characterization of the optimal policy. Note that if $F$ is the cumulative distribution function of a random variable it is continuous from the right. We will denote a related function that is continuous from the left through

$$F^{-}(y) = \lim_{h \downarrow 0} F(y - h).$$

Furthermore, we will define the generalized inverse of $F$ through

$$F^{\leftarrow}(p) = \min\{y : F(y) \geq p\}.$$

**Theorem 5.3.2** *The optimal replenishment quantity for item $i$ in period $j$ in the capacitated multi-item inventory model with equal replenishment intervals is given by*

$$Q^*_{i,j} = S^*_i - I_{i,j}$$

*where the vector of order-up-to levels $(S^*_1, \ldots, S^*_\ell)$ is the optimal solution to the following optimization problem:*

$$\textit{minimize } \mu$$

*subject to*

$$\begin{aligned}
\sum_{i=1}^{\ell} S_i &= \min\left\{V, \sum_{i=1}^{\ell} F_i^{\leftarrow}\left(\frac{p_i}{h_i + p_i}\right)\right\} \\
F_i(S_i) &\geq \frac{p_i - \mu}{h_i + p_i} \qquad \textit{for } i = 1, \ldots, \ell \\
F_i^{-}(S_i) &\leq \frac{p_i - \mu}{h_i + p_i} \qquad \textit{for } i = 1, \ldots, \ell \\
\mu &\geq 0.
\end{aligned}$$

**Proof:** As mentioned above, Ignall and Veinott [33] and Beyer et al. [10] showed that the optimal policy is a myopic policy. Therefore, we can determine the optimal inventory policy for each replenishment interval separately by solving the following optimization problem:

$$\text{minimize} \sum_{i=1}^{\ell} E[h_i(I_{i,j} + Q_{i,j} - D_i)^+ + p_i(D_i - I_{i,j} - Q_{i,j})^+]$$

subject to

$$\begin{aligned} \sum_{i=1}^{\ell}(I_{i,j} + Q_{i,j}) &\leq V \\ I_{i,j} + Q_{i,j} &\geq 0 \qquad i = 1, \ldots, \ell. \end{aligned}$$

By making the substitution $S_{i,j} = I_{i,j} + Q_{i,j}$, we see that in fact an order-up-to policy is optimal, and its parameters can be found by solving the following optimization problem (where the subscripts $j$ are now omitted for ease of notation):

$$\text{minimize} \sum_{i=1}^{\ell} E[h_i(S_i - D_i)^+ + p_i(D_i - S_i)^+]$$

subject to

$$\begin{aligned} \sum_{i=1}^{\ell} S_i &\leq V \\ S_i &\geq 0 \qquad i = 1, \ldots, \ell. \end{aligned}$$

In the following, we will ignore the nonnegativity constraints, and verify that the optimal solution thus found indeed is nonnegative.

Defining

$$\gamma_i(S_i) = E[h_i(S_i - D_i)^+ + p_i(D_i - S_i)^+]$$

and introducing a nonnegative Lagrange multiplier $\mu$ with the capacity constraint, the KKT conditions for the optimization problem are given by:

$$\begin{aligned} \sum_{i=1}^{\ell} \partial\gamma_i(S_i) + \mu &\ni 0 && (5.2) \\ \mu\left(\sum_{i=1}^{\ell} S_i - V\right) &= 0 && (5.3) \\ \sum_{i=1}^{\ell} S_i &\leq V \\ \mu &\geq 0 \end{aligned}$$

(See Hiriart-Urruty and Lemarechal [32]). Now note that

$$\partial\gamma_i(S_i) = \left[(h_i + p_i)F_i^{-}(S_i) - p_i, (h_i + p_i)F_i(S_i) - p_i\right]$$

so that condition (5.2) can be written as

$$\begin{aligned} (h_i + p_i)F_i^{-}(S_i) - p_i + \mu &\leq 0 \\ (h_i + p_i)F_i(S_i) - p_i + \mu &\geq 0 \end{aligned}$$

Now noting that condition (5.3) says that the capacity constraint can only be non-binding if $\mu = 0$, we obtain the desired result. $\square$

The following corollary shows how the above result simplifies when the demand distribution of each item is absolutely continuous:

**Corollary 5.3.3** *When all demand distributions are absolutely continuous, the optimal replenishment quantity for item $i$ in period $j$ in the capacitated multi-item inventory model with equal replenishment intervals is given by*

$$Q_{i,j}^* = F_i^{\leftarrow}\left(\frac{p_i - \mu^*}{h_i + p_i}\right) - I_{i,j}$$

*where* $\mu^*$ *satisfies*

$$\mu^* = \arg \min_{\mu \geq 0} \left[ \sum_{i=1}^{\ell} F_i^{\leftarrow} \left( \frac{p_i - \mu}{h_i + p_i} \right) \leq V \right].$$

**Proof:** If the distribution $F_i$ is absolutely continuous, then the function $\gamma_i$ in the proof of Theorem 5.3.2 is continuous. The result then follows immediately from the fact that condition (5.2) can be written as

$$(h_i + p_i)F_i(S_i) - p_i + \mu = 0.$$

□

We will now draw insights from the structure of this optimal policy to develop heuristics for our general problem. These heuristics were first proposed by Choi [14], but only work with discrete demand models. We generalize the results and make them work for both discrete and continuous demand models.

### 5.3.2 Heuristics for a Two-item Case

For ease of exposition in presenting our heuristic solution approaches to the inventory problem with unequal replenishment intervals, we first consider a two-item inventory system in which the first item is replenished in every odd period and the second item is replenished in every even period until the end of the time horizon (see Figure 5–2).

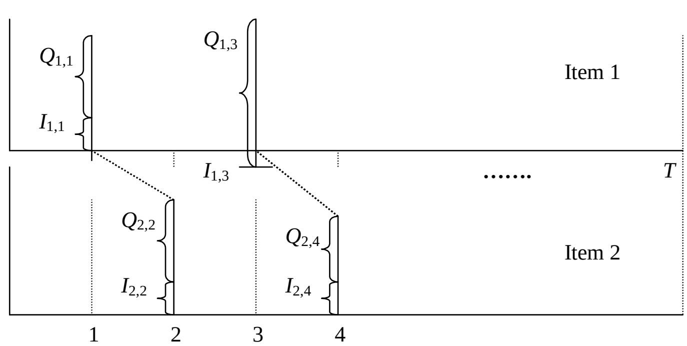


Figure 5–2: An inventory system with unequal replenishment intervals.

### A Nonintrusive Heuristic

The first heuristic is directly motivated by the result of Theorem 5.3.2 which considered the problem under equal replenishment intervals. In particular, in this heuristic we determine a policy by ignoring the fact that the two items are not replenished simultaneously, and simply compute order-up-to levels as in the case with equal replenishment intervals. We will denote these order-up-to levels by $S_1^N$ and $S_2^N$. Note that these levels can be interpreted as individual item capacities, i.e., this policy acts as if the warehouse is partitioned into dedicated sections for the two items. Clearly, if neither item violates its individual capacity, the joint capacity constraint is also satisfied, regardless of the timing of the item replenishments. We therefore call this policy a *nonintrusive* one, since the value of the demand (in particular, an exceptionally small demand) for a particular given item in the period following its replenishment will not impose any additional constraint on the replenishment amount for the other item. Note that (in the absence of positive lowerbounds on the demands), any order-up-to policy with levels $S_1', S_2'$ such that $S_1' + S_2' > V$ does not enjoy this property! The actual policy given by this heuristic becomes:

$$Q_{i,j}^N = V_i^N - I_{i,j}$$

(see Figure 5–3 for an illustration of this policy).

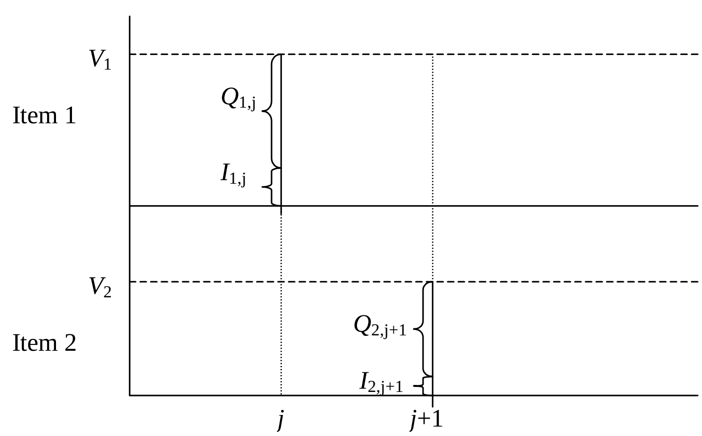


Figure 5–3: An inventory policy for the nonintrusive heuristic.

Note that, if $\mu^* = 0$ and thus the capacity constraint is not binding, $S_1^N$ and $S_2^N$ are the unconstrained optimal order-up-to quantities for each item individually. Therefore, it is easy to see that the nonintrusive heuristic enjoys the desirable property that it finds the *optimal* (i.e., unconstrained) solution whenever the capacity constraint is not binding.

On the other hand, the obvious drawback to this heuristic is that, when the capacity constraint *is* binding, at the time of replenishment of, say, item 1 there will usually be available capacity in the warehouse that will remain unused until the replenishment of item 2. This intuitively appears to be a waste of resources since at least some of that available capacity could be used for item 1, since demand for item 1 will likely free up this space before it is needed by item 2.

**A Greedy Heuristic**

Our second heuristic can be viewed as a *greedy* heuristic. In this heuristic, we compute the optimal replenishment quantity for the item to be replenished, taking into account the *total* available capacity at the warehouse at that time. Referring to Figure 5–4, item $i$ ($i = 1, 2$) is replenished up to its unconstrained optimal inventory level or the total available capacity at the warehouse, whichever is smaller. That is, the order-up-to level is

$$S_{i,j}^G = \min\left(F_i^{\leftarrow}\left(\frac{p_i}{h_i + p_i}\right), V - I_{3-i,j}^+\right)$$

or, equivalently, the order quantity is

$$Q_{i,j}^G = \min\left(F_i^{\leftarrow}\left(\frac{p_i}{h_i + p_i}\right), V - I_{3-i,j}^+\right) - I_{i,j}.$$

Note that if the capacity constraint is not binding, it is easy to see that this heuristic also enjoys the desirable property that it finds the *optimal* (i.e., the unconstrained) solution.

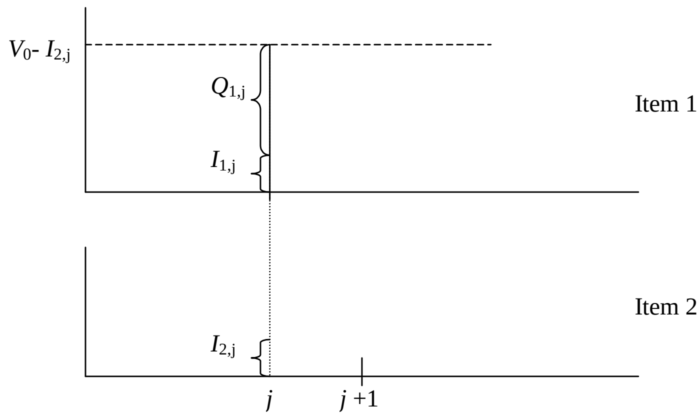


Figure 5–4: An inventory policy for the greedy heuristic.

On the other hand, the obvious drawback to this heuristic is its greedy nature. At the time of replenishment of say item 1, potentially all available capacity is used for replenishing that item, possibly leaving too little space for item 2 at its replenishment time to be able to achieve an adequate cost (and customer service) level.

**A Sharing Heuristic**

Note that, in some sense, the two heuristics proposed above can be viewed as being on opposite sides of the spectrum. The first heuristic *never* uses any of the storage space that is reserved for the other item, thereby wasting valuable storage space. On the other hand, the second heuristic almost *always* uses storage space that is intended for the other item, thereby potentially severely limiting replenishment of the other item. It seems that the best policy would be a compromise way between these two extremes, where some of the warehouse capacity is shared between the two items.

The sharing heuristic we propose is similar in spirit to the first heuristic in that we determine, a priori, individual pseudo-capacities for each of the two items, which we will denote by $V_1'$ and $V_2'$. The sum of these individual capacities however will, in general, exceed the total warehouse capacity $V$, reflecting the fact that either item can temporarily use some of the storage space intended for the other, thereby counting on this space to be freed up by demand before the other item actually needs it. As noted above, this cannot be guaranteed. We deal with this situation in a similar manner as in the greedy heuristic, by limiting the order-up-to quantity to the actual available space in the warehouse at the time of replenishment. The policy then becomes:

$$Q_{i,j}^S = \min\left(V_i' - I_{i,j}, Q_{i,j}^G\right).$$

The critical remaining issue is how to determine good values for the individual pseudo-capacities $V_1'$ and $V_2'$. To account for the different demand rates of the two different items, we will choose

$$V_i' = S_i^N + \alpha E[d_i] \qquad i = 1, 2$$

where $S_i^N$ is the optimal capacity used in the nonintrusive heuristic. It is easy to see that $\alpha$ should be nonnegative, and that with respect to each individual item we would like to maximize the value of $\alpha$. In the remainder, we will derive some suitable candidate values for $\alpha$. Assuming that capacity is very tight, and we therefore usually order up to $V_i'$, the inventory level of item 2 when item 1 is replenished will on average be equal to $V_2' - E[d_2] = S_2^N - (1-\alpha)E[d_2]$. To make sure that the actual available capacity at the warehouse is not constraining *on average*, we impose the constraint

$$V_1' + (V_2' - E[d_2])^+ \leq V$$

which is equivalent to

$$S_1^N + \alpha E[d_1] + (S_2^N - (1-\alpha)E[d_2])^+ \leq V$$

or

$$S_1^N + \alpha E[d_1] + (S_2^N - E[d_2] + \alpha E[d_2])^+ \leq V.$$

Since the left-hand-side of this inequality is increasing in $\alpha$, the maximum value of $\alpha$ that satisfies it, say $\bar{\alpha}^1$, can easily be found using binary search. In a similar way, we can find the value of $\bar{\alpha}^2$. We can then ensure that, on average, the capacity constraint is satisfied both at times when item 1 is replenished and when item 2 is replenished by choosing

$$\alpha = \alpha_1 \equiv \min(\bar{\alpha}^1, \bar{\alpha}^2).$$

Alternatively, we may want to ensure that the capacity constraint is satisfied only on average over all replenishment times. In that case, we should choose

$$\alpha = \alpha_2 \equiv \frac{1}{2}(\bar{\alpha}^1 + \bar{\alpha}^2).$$

As a third alternative, we propose a version of the sharing heuristic where the value of $\alpha$ depends on the item being replenished. In particular, we propose using an expected demand-weighted average of the two values that motivated $\alpha_1$ and $\alpha_2$, i.e.,

$$\alpha = \alpha_3^{(1)} \equiv \bar{\alpha}^1$$

when item 1 is replenished, and

$$\alpha = \alpha_3^{(2)} \equiv \bar{\alpha}^2$$

when item 2 is replenished. Whatever the choice of $\alpha$ is, note that the pseudo-capacities are larger than the nonintrusive order-up-to levels. However, simply ordering up to the pseudo-capacity may be either capacity infeasible or cost inefficient. We therefore propose to restrict the order-up-to level in the sharing heuristic to never exceed the greedy level $S_i^G$. In other words, we define the order-up-to level for item $i$ at time $j$ to be equal to

$$S_i^S = \min\left[V_i', S_i^G\right].$$

See Figure 5–5 for an illustration of this heuristic.

### Summary of Heuristics

We next provide a brief summary of the three proposed heuristics.

**Nonintrusive heuristic**

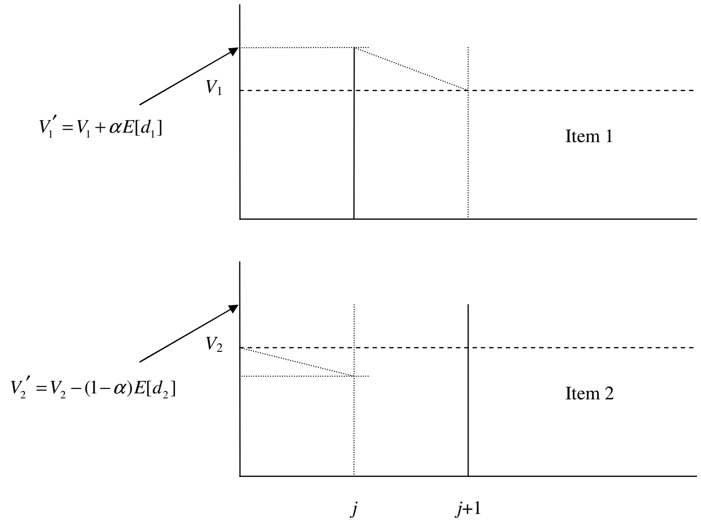


Figure 5–5: An inventory policy for the sharing heuristic.

- Compute the separate capacity of each item by solving the following:

$$\text{minimize } \mu$$

subject to

$$\begin{aligned} S_1 + S_2 &= \min\left\{V, F_1^{\leftarrow}\left(\frac{p_1}{h_1+p_1}\right) + F_2^{\leftarrow}\left(\frac{p_2}{h_2+p_2}\right)\right\} \\ F_i(S_i) &\geq \frac{p_i - \mu}{h_i + p_i} \qquad i = 1, 2 \\ F_i^{-}(S_i) &\leq \frac{p_i - \mu}{h_i + p_i} \qquad i = 1, 2 \\ \mu &\geq 0 \end{aligned}$$

and let $(\mu^*, S_1^N, S_2^N)$ be an optimal solution to this problem.

- The order-up-to levels are then equal to

$$S_i^N \qquad i = 1, 2.$$

**Greedy heuristic**

- If item $i$ is replenished in period $j$, determine the available capacity, $V - I_{3-i,j}$ for item $i$.

- The order-up-to level for item $i$ is then equal to

$$S_i^G = \min\left\{F_i^{\leftarrow}\left(\frac{p_i}{h_i + p_i}\right), V - I_{3-i,j}^+\right\}.$$

**Sharing heuristic**

- Compute the individual item capacities $S_1^N, S_2^N$ as for the nonintrusive heuristic.
- If item $i$ is replenished, determine its adjusted capacity in one of the following three ways:
  (i) $V_i' = S_i^N + \alpha_1 E[d_i]$
  (ii) $V_i' = S_i^N + \alpha_2 E[d_i]$
  (iii) $V_i' = S_i^N + \alpha_3^{(i)} E[d_i]$
- The order-up-to level for item $i$ is then equal to

$$S_i^S = \min\left\{S_i^G, V_i'\right\}.$$

### 5.3.3 Heuristics for the General Multi-item Case

The heuristics that we have developed in Section 5.3.2 are limited to the two-item case, in which products are replenished at different points in time. In real warehouse systems, however, it is more likely that more than two items are replenished, and that replenishments of some items may take place at the same time. Based on the results from the two-item case, we can extend the proposed heuristics to multi-item cases. Let $\ell \geq 2$ be the number of items in the system and let $b$ be the least common multiple of the individual interval lengths (or the length of one *common cycle*). If $T_i$ represents the replenishment interval length of item $i$ as defined earlier in this section, $b_i = b/T_i$ represents the number of replenishments of item $i$ in one common cycle. In the following three sections, we will generalize the three heuristics to this general case.

### The Nonintrusive Heuristic

Recall that the idea behind the *nonintrusive heuristic* is to allocate *individual* capacities for all $\ell$ items. The best nonintrusive order-up-to levels can be obtained by solving the following problem

$$\text{minimize } \mu$$

subject to

$$\begin{aligned}
\sum_{i=1}^{\ell} S_i &= \min\left\{V, \sum_{i=1}^{\ell} F_i^{\leftarrow}\left(\frac{p_i}{h_i+p_i}\right)\right\} \\
F_i(S_i) &\geq \frac{b_i p_i - \mu}{b_i(h_i+p_i)} \qquad i=1,\ldots,\ell \\
F_i^{-}(S_i) &\leq \frac{b_i p_i - \mu}{b_i(h_i+p_i)} \qquad i=1,\ldots,\ell \\
\mu &\geq 0
\end{aligned}$$

and letting $(\mu^*, S_1^N, \ldots, S_\ell^N)$ denote an optimal solution, where the values $S_i^N$ are the order-up-to levels. Note that for this heuristic, the process to determine individual capacities in the multi-item case is the same regardless of whether certain items are replenished simultaneously. This follows directly from the fact that, in the nonintrusive heuristic, we determine the individual capacity of each item *under the assumption that all items are replenished at the same time.* The individual capacities are therefore not affected by whether items are replenished simultaneously in the actual replenishment schedule or not. And we see that this heuristic is time-independent, and not affected by the actual inventory levels at the time of replenishments.

Note that, if $\mu^* = 0$ and thus the capacity constraint is not binding, $S_1^N$ and $S_2^N$ are the unconstrained optimal order-up-to quantities for each item individually. Therefore, it is easy to see that the nonintrusive heuristic enjoys the desirable property that it finds the *optimal* (i.e., unconstrained) solution whenever the capacity constraint is not binding.

**The Greedy Heuristic**

In the two-item case without simultaneous replenishments, the greedy replenishment policy used the total available capacity at the warehouse as a constraint on the replenishment quantity for the single item under consideration. We extend this approach to the multi-item case. However, because of the possibility of simultaneous replenishments, we need to allocate the total available capacity at replenishment time $j$, namely $V - \sum_{i \notin R_j} I^+_{i,j}$, to each of the individual items that are replenished at that time. To this end, we determine individual order-up-to quantities for each of these items, in a similar way as for the nonintrusive heuristic. The best greedy order-up-to levels for all items $i \in R_j$ can then be found by solving the following optimization problem:

$$\text{minimize } \mu$$

subject to

$$\begin{aligned}
\sum_{i \in R_j} S_i &= \min\left\{V, \sum_{i \in R_j} F_i^{\leftarrow}\left(\frac{p_i}{h_i + p_i}\right)\right\} - \sum_{i \notin R_j} I^+_{i,j} \\
F_i(S_i) &\geq \frac{b_i p_i - \mu}{b_i(h_i + p_i)} \qquad i \in R_j \\
F_i(S_i-) &\leq \frac{b_i p_i - \mu}{b_i(h_i + p_i)} \qquad i \in R_j \\
\mu &\geq 0.
\end{aligned}$$

Let $(\mu^*; S_i^G, \ i \in R_j)$ be an optimal solution for this problem. The order-up-to levels are then equal to $V_i^G$, $i \in R_j$. Different from the nonintrusive heuristic, this greedy heuristic will result in different order-up-to levels at different times for the same item, and it also depends on the inventory levels of the items that are not replenished at the time, in order to figure out the total available capacity that can be distributed.

**The Sharing Heuristic**

Consider the $j^{\text{th}}$ replenishment stage, and define $e_{i,j}$ as the number of time periods that have elapsed for item $i$, $i = 1, \ldots, \ell$, since its last replenishment. If item

$i$ is replenished in the current period, we let $e_{i,j} = 0$. For example, in Figure 5–2, we have $e_{1,2} = 1$, $e_{2,3} = 1$, $e_{1,4} = 1$, and so on. Recall that the set of items that are replenished in the $j^{\text{th}}$ replenishment period is given by $R_j$. Then, similarly to the two-item case, we define the pseudo-capacity for items $i \in R_j$ by

$$V_i' = S_i^N + \alpha T_i E[d_i] \qquad i \in R_j$$

where again $S_i^N$ denotes the best individual capacity for item $i$ in the nonintrusive heuristic. At replenishment period $j$, the average inventory level for all items that are not replenished will be equal to

$$S_i^N + (\alpha T_i - e_{i,j}) E[d_i] \qquad i \notin R_j.$$

We impose the following to ensure that the actual available capacity at the warehouse is not constraining:

$$\sum_{i \in R_j} V_i' + \sum_{i \notin R_j} (V_i' - e_{i,j} E[d_i])^+ \quad \leq \quad V$$

which is equivalent to

$$\sum_{i \in R_j} (S_i^N + \alpha T_i E[d_i]) + \sum_{i \notin R_j} (S_i^N + (\alpha T_i - e_{i,j}) E[d_i])^+ \quad \leq \quad V.$$

As in the two-item case, we may find the largest value of $\alpha$ satisfying this inequality, say $\bar{\alpha}^{(j)}$, by binary search. Similarly to the two-item case, we use these values to obtain three variants of the heuristic:

$$\begin{aligned} \alpha_1 &= \min_{j=1,\ldots,m} \bar{\alpha}^{(j)}, \\ \alpha_2 &= \frac{1}{m} \sum_{j=1}^{m} \bar{\alpha}^{(j)} \end{aligned}$$

which are both independent of the replenishment time, or, at replenishment time $j$,

$$\alpha_3^{(j)} = \bar{\alpha}^{(j)}.$$

As in the two-item case, we will restrict the order-up-to level in the sharing heuristic to the greedy level:

$$S_i^S = \min\left\{V_i', S_i^G\right\}.$$

### 5.3.4 Proof of Optimality of Heuristics for Simultaneous Replenishment Case

As we have mentioned above, it is easy to see that all proposed heuristics provide the optimal solution to the inventory problem in the absence of a warehouse capacity. However, it can be shown that the heuristic solutions are also optimal in the presence of a warehouse capacity when the replenishments of all items take place simultaneously.

**Theorem 5.3.4** *In the equal replenishment intervals case, all three proposed heuristics provide the optimal solution to the replenishment problem.*

**Proof:** If all products are replenished at the same time, then at any replenishment time $j$, $R_j = \{1, \ldots, \ell\}$. Moreover, all $T_i$ values are equal, and thus $b_i = 1$ for all $i = 1, \ldots, \ell$. The optimality of the nonintrusive heuristic for the simultaneous replenishment case now follows directly from Theorem 5.3.2. Furthermore, if all items are replenished at the same time, then greedy heuristic will be identical to the nonintrusive heuristic since all items $i$ belong to $R_j$ at an replenishment time. Finally, in the sharing heuristic recall that $V_i' \geq S_i^N$ by construction, regardless of the value of $\alpha$. Since in the equal replenishment case we have $S_i^G = S_i^N$, we obtain that

$$S_i^S = \min\left\{V_i', S_i^G\right\} = \min\left\{V_i', S_i^N\right\} = S_i^N$$

and is therefore identical to the optimal nonintrusive solution as well. □

## 5.4 Numerical Results

This section presents numerical results illustrating the performance of the heuristics described in the previous section as compared to the optimal solution found by the stochastic dynamic programming formulation developed in Section 5.2. We have tested both two-item and three-item models, with planning horizons of $T = 60$ and $T = 12$, respectively (in order to allow us to compute the optimal policies).

For the two-item models, we have generated problem instances with different relative levels of the unit holding and penalty cost parameters: $h_i = 2$ and $p_i \in \{3, 10\}$. The demand per period for each item is assumed to follow a discrete uniform distribution with a lower bound of 0 and an upper bound $U_1 \in \{5, 10, 15, 20\}$ and $U_2 = 40 - U_1 \in \{20, 25, 30, 35\}$, in order to study the effect of (non-)homogeneities in the demand distributions of the items. The initial inventory levels for all items are set to 0, which should not have a major impact on the replenishment policy in the long run. For each instance, we first determined the storage capacity used in the unconstrained optimal solution. We then considered problem instances in which the capacity is equal to 90%, 80%, 60%, and 40% of this value. Finally, we varied the replenishment cycle lengths (2 or 3) as well as the timings of the replenishments.

We solved all problem instances using the stochastic dynamic programming recursion derived in this chapter. We then simulated the system for all proposed heuristics and compared the expected cost of each heuristic policy to the expected cost of the optimal policy. Tables 5–1 - 5–3 show measures of the relative error of the heuristic solutions as compared to the optimal solutions. In particular, each entry of the tables gives the average error obtained from all instances with the specified parameters. The tables also show the average time required for solving the problem to optimality. The time required for the heuristics is negligible.

The tables clearly show that, in general, the performance of all heuristics tends to improve with more homogeneity in the demand distributions or with higher capacity levels. The latter observation should not be surprising since we showed that all heuristics provide the optimal solution when the capacity constraint is not binding. Furthermore, in almost all cases variant 2 of the sharing heuristic seems to outperform the other heuristics, with variant 3 of that heuristic a reasonable alternative. However, due to the fact that variant 2 is somewhat easier to implement in practice there seems to be little reason to apply that variant. A somewhat surprising result is that the greedy heuristic seems to outperform variant 1 of the sharing heuristic for most of the tested instances.

Tables 5–4 - 5–6 focus only on instances where the length of the replenishment intervals is equal to 2 for both items and the replenishments are staggered. In these instances, note that the nonintrusive heuristic in fact provides the *optimal* solution corresponding to the case where the items are replenished at the same time. The error associated with the nonintrusive heuristic therefore also measures the cost associated with coordinating the replenishment of the items to take place in the same periods. These results show that the benefit of staggering replenishments is largest when capacity is very limited and demands are homogeneous. An unrelated but nevertheless interesting observation that can be made by comparing the results in Tables 5–4 - 5–6 to the results in Tables 5–1 - 5–3 is that the time required by the stochastic dynamic programming method is quite reasonable when only one item is replenished in each replenishment period, but increases dramatically when multiple items are replenished simultaneously in some replenishment periods. This follows immediately from the fact that the state space of the stochastic dynamic program increases rapidly in the number of items that can be replenished in any period.

For the three-item models, we have again considered different relative levels of the unit holding and penalty cost parameters: $h_i = 2$ and $p_i \in \{3, 10\}$, different capacity levels, and different replenishment schedules. However, in the interest of time we have focused on problem instances with only a single demand pattern, i.e., $U_1 = U_2 = 10$ and $U_3 = 15$. Tables 5–7 and 5–8 show the results of these tests. The performance of the heuristics for the three-item cases seems to follow the same pattern as for the two-item cases. Although in general the solution errors are larger for three-item instances, the solutions found by variant 2 of the sharing heuristic are

Table 5–1: Error (in %) of the solution obtained by the heuristics as compared to the optimal solution, as a function of the tightness of the storage capacity (2 items).

| Capacity (% of unconstrained) | Time (opt.) (h:mm:ss) | Nonintrusive | Greedy | Sharing | | |
|---|---|---|---|---|---|---|
| | | | | 1 | 2 | 3 |
| 90 | 3:45:24 | 2.17 | 0.26 | 1.40 | 0.24 | 0.26 |
| 80 | 2:50:51 | 6.57 | 1.56 | 4.17 | 0.86 | 1.22 |
| 60 | 1:28:43 | 12.15 | 5.36 | 7.97 | 1.35 | 2.51 |
| 40 | 0:36:08 | 10.91 | 6.29 | 7.66 | 2.08 | 2.82 |

Table 5–2: Error (in %) of the solution obtained by the heuristics as compared to the optimal solution, as a function of demand variability between items (2 items).

| Mean demands | Time (opt.) (h:mm:ss) | Nonintrusive | Greedy | Sharing | | |
|---|---|---|---|---|---|---|
| | | | | 1 | 2 | 3 |
| 20,20 | 2:44:31 | 10.23 | 3.80 | 6.56 | 1.54 | 1.87 |
| 15,25 | 2:42:12 | 9.53 | 4.35 | 6.16 | 1.27 | 2.00 |
| 10,30 | 2:05:13 | 7.64 | 3.40 | 5.30 | 1.16 | 1.84 |
| 5,35 | 1:09:09 | 4.40 | 1.92 | 3.18 | 0.56 | 1.11 |

Table 5–3: Error (in %) of the solution obtained by the heuristics as compared to the optimal solution, as a function of the underage penalty costs of the items (2 items).

| Penalty costs | Time (opt.) (h:mm:ss) | Nonintrusive | Greedy | Sharing | | |
|---|---|---|---|---|---|---|
| | | | | 1 | 2 | 3 |
| 3,3 | 1:37:01 | 7.51 | 1.87 | 4.69 | 0.83 | 1.48 |
| 10,10 | 2:44:04 | 10.98 | 4.15 | 6.95 | 1.24 | 2.46 |
| 3,10 | 2:22:09 | 5.71 | 5.01 | 4.26 | 1.26 | 1.09 |
| 10,3 | 1:57:50 | 7.59 | 2.44 | 5.30 | 1.20 | 1.80 |

Table 5–4: Relative cost associated with coordinating replenishments for different items to occur at the same time, as a function of the tightness of the storage capacity.

| Capacity (% of unconstrained) | Time (opt.) (h:mm:ss) | Nonintrusive |
|---|---|---|
| 90 | 0:09:23 | 2.57 |
| 80 | 0:07:51 | 8.19 |
| 60 | 0:04:56 | 15.76 |
| 40 | 0:02:32 | 14.77 |

still quite acceptable, especially considering the time required to find the optimal solution.

## 5.5 Summary

In this chapter, we generalize a stochastic multi-item, periodic-review inventory model, by relaxing the assumption of identical replenishment schedules for items. We construct three heuristics to determine replenishment quantities for the case of unequal replenishment intervals. These heuristics all use relatively simple and intuitively attractive decision rules. Numerical testing of these heuristics on an extensive set of test problems with various cost and demand parameters suggests that an excellent solution can be found in very limited time.

The nonintrusive heuristic, which uses a separate capacity for each product for replenishment, is very easy to implement, but is not able to deal as effectively with the scarce resource as the other two heuristics for most problems. This behavior is not unexpected because this heuristic is more likely to retain unused capacity for some products while others may suffer from the lack of the resource. The greedy heuristic uses a replenishment policy that replenishes products up to the total available capacity of the system at the time of replenishment. This heuristic outperforms the nonintrusive heuristic in most cases, but sometimes still leads to poor performance due to the fact that it is too aggressive in replenishing products in a given period so that the other items suffer from the lack of resource in the following replenishment periods, causing high warehouse shortage penalty costs. The sharing heuristic, however, attempts to combine the positive qualities of both other heuristics by defining individual capacities for each item, but allows the sum of the individual capacities to exceed the total warehouse capacity – to reflect the possibility of sharing some of the warehouse space among items due to the different replenishment schedules. In particular, the second variant of this heuristic seems to enjoy a very good performance over all instances studied.

Table 5–5: Relative cost associated with coordinating replenishments for different items to occur at the same time, as a function of demand variability between items (2 items).

| Mean demands | Time (opt.) (h:mm:ss) | Nonintrusive |
|---|---|---|
| 20,20 | 0:08:00 | 13.47 |
| 15,25 | 2:42:12 | 12.49 |
| 10,30 | 2:05:13 | 9.81 |
| 5,35 | 1:09:09 | 5.52 |

Table 5–6: Relative cost associated with coordinating replenishments for different items to occur at the same time, as a function of the underage penalty costs of the items.

| Penalty costs | Time (opt.) (h:mm:ss) | Nonintrusive |
|---|---|---|
| 3,3 | 0:04:40 | 10.01 |
| 10,10 | 0:07:44 | 14.64 |
| 3,10 | 0:06:47 | 7.47 |
| 10,3 | 0:05:32 | 9.18 |

Table 5–7: Error (in %) of the solution obtained by the heuristics as compared to the optimal solution, as a function of the tightness of the storage capacity (3 items).

| Capacity (% of unconstrained) | Time (opt.) (h:mm:ss) | Nonintrusive | Greedy | Sharing | | |
|---|---|---|---|---|---|---|
| | | | | 1 | 2 | 3 |
| 90 | 35:02:18 | 16.27 | 3.11 | 8.91 | 2.34 | 4.92 |
| 80 | 17:30:36 | 20.88 | 4.22 | 14.91 | 3.77 | 7.08 |
| 65 | 9:45:32 | 23.60 | 4.93 | 18.60 | 4.95 | 9.27 |
| 50 | 5:04:38 | 22.40 | 4.73 | 17.66 | 6.07 | 8.71 |

Table 5–8: Error (in %) of the solution obtained by the heuristics as compared to the optimal solution, as a function of the underage penalty costs of the items (3 items).

| Penalty costs | Time (opt.) (h:mm:ss) | Nonintrusive | Greedy | Sharing | | |
|---|---|---|---|---|---|---|
| | | | | 1 | 2 | 3 |
| 3, 3 ,3 | 10:47:29 | 17.02 | 2.73 | 10.11 | 1.97 | 5.85 |
| 10,10,10 | 22:51:43 | 23.46 | 4.81 | 18.91 | 3.91 | 7.81 |
| 3, 3 ,10 | 15:22:36 | 17.45 | 3.25 | 15.17 | 6.22 | 6.87 |

We conclude this chapter by pointing out possible extensions of the stochastic models considered in this study. The current optimization model is established for a fixed warehouse capacity over time. However, the approach used in this chapter can be used to determine how much one should be willing to pay for extra warehouse capacity by simulating different values of this capacity. One interesting extension worth pursuing is to actually incorporate a variable warehouse capacity into the problem, for situations where extra warehouse capacity can be leased to accommodate higher inventory levels. Another future research effort may consider a relaxation of the assumption of zero delivery leadtimes between manufacturers and the warehouse. Finally, one may consider the optimization of the replenishment schedule (which we have assumed given), and even extend consideration to cases with nonzero fixed ordering costs.

The major contribution of this study is that we extend the aforementioned multi-item, periodic-review inventory model by relaxing the assumption of identical replenishment schedules for the different items. To our knowledge, no one has considered the case in which products from different manufacturers have distinct replenishment schedules or unequal replenishment interval lengths (we call this the case of *unequal replenishment intervals*). Unfortunately, except for relatively small problems, it is difficult to determine an optimal replenishment policy in this case. In this chapter, we therefore develop three efficient and effective heuristics to determine replenishment quantities under unequal replenishment intervals. We show that each of these heuristics provides the optimal inventory ordering quantities for the case where the replenishment intervals for the different items coincide. Extensive numerical tests are employed that compare the performance of the heuristics to the optimal policies. These results not only show that high-quality solutions can be obtained in very limited time, but also suggests guidelines on which heuristic to use for various classes of instances.

## REFERENCES


[1] V. Anantharam. The optimal buffer allocation problem. *IEEE Transactions on Information Theory*, 35:721–725, 1989.

[2] S. Anily. Multi-item replenishment and storage problem: heuristics and bounds. *Operations Research*, 39:233–243, 1991.

[3] K.J. Arrow, T. Harris, and J. Marschak. Optimal inventory policy. *Econometrica*, 19:250–272, 1951.

[4] K.S. Azoury. Bayes solution to dynamic inventory models under unknown demand distribution. *Management Science*, 31:1150–1160, 1985.

[5] R.H. Ballou. *Business Logistics Management.* Prentice Hall, Upper Saddle River, New Jersey, 4th edition, 1999.

[6] M. Beckmann. An inventory model for arbitrary interval and quantity distributions of demands. *Management Science*, 8:35–57, 1961.

[7] R. Bellman, I. Glicksberg, and O. Gross. On the optimal inventory equation. *Management Science*, 2:83–104, 1955.

[8] D.P. Bertsekas. *Dynamic Programming: Deterministic and Stochastic Models.* Prentice Hall, Englewood Cliffs, New Jersey, 1988.

[9] D. Beyer, S.P. Sethi, and R. Sridhar. *Decision and Control in Management Sciences in honor of Professor Alain Haurie*, chapter Average-cost optimality of a base-stock policy for a multi-product inventory model with limited storage, pages 241–260. Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[10] D. Beyer, S.P. Sethi, and R. Sridhar. Stochastic multi-product inventory models with limited storage. *Journal of Optimization Theory and Applications*, 111:553–588, 2001.

[11] S.A. Carr, A.R. Güllü, P.L. Jackson, and J. Muckstadt. Exact analysis of the no b/c stock policy. Technical Report, Cornell University, Ithaca, New York, 1993.

[12] X. Chen and D. Simchi-Levi. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: the finite horizon case. *Operations Research*, 52:887–896, 2004.

[13] X. Chen and D. Simchi-Levi. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: the infinite horizon case. *Mathematics of Operations Research*, 29:698–723, 2004.

[14] J. Choi. *Stochastic Production and Inventory Models with Limited Resources.* PhD dissertation, University of Florida, Gainesville, Florida, 2001.

[15] F.W. Ciarallo, R. Akella, and T.E. Morton. A periodic review production planning model with uncertain capacity and uncertain demand. *Management Science*, 40:320–332, 1994.

[16] G.A. DeCroix and A. Arreola-Risa. Optimal production and inventory policy for multiple products under resource constraints. *Management Science*, 44:950–961, 1998.

[17] G. Dobson. The economic lot scheduling problem: a resolution to feasibility using time varying lot sizes. *Operations Research*, 35:764–771, 1987.

[18] R.V. Evans. Inventory control of a multiproduct system with a limited production resource. *Naval Research Logistics Quarterly*, 14:173–184, 1967.

[19] A. Federgruen and A. Heching. Combined pricing and inventory control under uncertainty. *Operations Research*, 47:454–474, 1999.

[20] A. Federgruen and Y. Zheng. An efficient algorithm for computing an optimal $(r, q)$ policy in continuous review stochastic inventory systems. *Operations Research*, 40:808–813, 1992.

[21] A. Federgruen and P. Zipkin. An inventory model with limited production capacity and uncertain demands i: the average-cost criterion. *Mathematics of Operations Research*, 11:193–207, 1986.

[22] A. Federgruen and P. Zipkin. An inventory model with limited production capacity and uncertain demands ii: the discounted-cost criterion. *Mathematics of Operations Research*, 11:208–215, 1986.

[23] Y. Feng and F.Y. Chen. Joint pricing and inventory control with setup costs and demand uncertainty. Working paper, Chinese University of Hong Kong, Hong Kong, China, 2003.

[24] M. Florian and M. Klein. Deterministic production planning with concave costs and capacity constraints. *Management Science*, 18:12–20, 1971.

[25] G. Gallego. New bounds and heuristics for $(q, r)$ policies. *Management Science*, 44:219–233, 1998.

[26] G. Gallego, M. Queyranne, and D. Simchi-Levi. Single resource multi-item inventory systems. *Operations Research*, 44:580–595, 1996.

[27] P. Glasserman. Allocating production capacity among multiple products. *Operations Research*, 44:724–734, 1996.

[28] S.K. Goyal. A note on "multi-product inventory situation with one restriction". *Journal of the Operational Research Society*, 29:269–271, 1978.

[29] G. Hadley and T.M. Whitin. *Analysis of Inventory Systems*. Prentice Hall, Englewood Cliffs, New Jersey, 1963.

[30] R. Hartley and L.C. Thomas. The deterministic, two-product, inventory system with capacity constraint. *Journal of the Operational Research Society*, 33:1013–1020, 1982.

[31] D.P. Heyman and M.J. Sobel. *Stochastic Models in Operations Research, Volume II: Stochastic Optimization*. McGraw-Hill, 1984.

[32] J.-B. Hiriart-Urruty and C. Lemarechal. *Convex Analysis and Minimization Algorithms I: Fundamentals*. Springer-Verlag, Berlin, Germany, 1996.

[33] E. Ignall and A.F. Veinott. Optimality of myopic inventory policies for several substitute products. *Management Science*, 15:284–304, 1969.

[34] P.C. Jones and R.R. Inman. When is the economic lot scheduling problem easy? *IIE Transactions*, 21:11–20, 1989.

[35] R. Kapuscinski and S. Tayur. A capacitated production-inventory model with periodic demand. *Operations Research*, 46:899–911, 1998.

[36] S. Karlin. Dynamic inventory policy with varying stochastic demands. *Management Science*, 6:231–258, 1960.

[37] A. De Kok, H. Tijms, and F. Van der Duyn Schouten. Approximations for the single-product production-inventory problem with compound poisson demand and service level constraints. *Advances in Applied Probability*, 16:378–402, 1984.

[38] A.A. Kurawarwala and H. Matsuo. Forecasting and inventory management of short life-cycle products. *Operations Research*, 44:131–150, 1996.

[39] H. Lau and A.H. Lau. The newsstand problem under price-dependent demand distribution. *IIE Transactions*, 20:168–175, 1988.

[40] H. Lau and A.H. Lau. The newsstand problem: a capacitated multiple-product single-period inventory problem. *European Journal of Operational Research*, 94:29–42, 1996.

[41] H.L. Lee and S. Nahmias. *Handbooks in Operations Research and Management Science: Logistics of Production and Inventory*, chapter Single product, single-location models, pages 1–55. Elsevier Science, Amsterdam, The Netherlands, 1993.

[42] L. Li. A stochastic theory of the firm. *Mathematics of Operations Research*, 13:447–466, 1988.

[43] W.S. Lovejoy. Myopic policies for some inventory models with uncertain demand distributions. *Management Science*, 36:724–738, 1990.

[44] W.S. Lovejoy. Stopped myopic policies in some inventory models with generalized demand processes. *Management Science*, 38:688–707, 1992.

[45] W.S. Lovejoy. Suboptimal policies, with bounds, for parameter adaptive decision processes. *Operations Research*, 41:583–599, 1993.

[46] S. Nahmias and C. Schmidt. An efficient heuristic for the multi-item newsboy problem with a single constraint. *Naval Research Logistics Quarterly*, 31:463–474, 1984.

[47] N.C. Petruzzi and M. Dada. Pricing and the newsvendor problem: A review with extensions. *Operations Research*, 47:183–194, 1999.

[48] M. Rosenblatt and U. Rothblum. On the single resource capacity problem for multi-item inventory systems. *Operations Research*, 38:686–693, 1990.

[49] S.M. Ross. *Stochastic Processes*. Wiley, New York, New York, 2nd edition, 1996.

[50] H.E. Scarf. Bayes solution of the statistical inventory problem. *Annals of Mathematical Statistics*, 30:490–508, 1959.

[51] H.E. Scarf. Some remarks on bayes solution to the inventory problem. *Naval Research Logistics Quarterly*, 7:591–596, 1960.

[52] J. Song and P. Zipkin. Inventory control in a fluctuating demand environment. *Operations Research*, 41:351–370, 1993.

[53] J. Song and P. Zipkin. Managing inventory with the prospect of obsolescence. *Operations Research*, 44:215–222, 1996.

[54] L.J. Thomas. Price and production decisions with random demand. *Operations Research*, 22:513–518, 1974.

[55] J.T. Treharne and C.R. Sox. Adaptive inventory control for nonstationary demand and partial information. *Management Science*, 48:607–624, 2002.

[56] A.F. Veinott. Optimal policy for a multi-product, dynamic, nonstationary inventory problem. *Management Science*, 12:206–222, 1965.

[57] Y. Wang and Y. Gerchak. Continuous review inventory control when capacity is variable. *International Journal of Production Economics*, 45:381–388, 1996.

[58] Y. Wang and Y. Gerchak. Periodic review production models with variable capacity, random yield and uncertain demand. *Management Science*, 42:130–137, 1996.

[59] Y.Z. Wang, L. Jiang, and Z.-J. Shen. Channel performance under consignment contract with revenue sharing. *Management Science*, 50:34–47, 2004.

[60] L.M. Wein. Capacity allocation in generalized jackson networks. *Operations Research Letters*, 8:143–146, 1989.

[61] T.M. Whitin. Inventory control and price theory. *Management Science*, 2:61–68, 1955.

[62] Y. Zheng and P. Zipkin. A queueing model to analyze the value of centralized inventory information. *Operations Research*, 38:296–307, 1990.

[63] P. Zipkin. *Foundations of Inventory Management.* McGraw-Hill, New York, New York, 2000.

# BIOGRAPHICAL SKETCH

Jie Cao was born in Yidu, Hubei Province, the People's Republic of China. He earned his bachelor's degree from Department of Control Science and Engineering, Huazhong University of Science and Technology, majoring in automatic control, and master's degree from the Institute of Systems Engineering, Huazhong University of Science and Technology, majoring in decision support systems. In 2001, he joined the Industrial and Systems Engineering department at the University of Florida. He expects to get his doctorate degree from the Industrial and Systems Engineering department, the University of Florida, in August 2005.