Design and Implementation of a 6 DOF Parallel Manipulator With Passive Force Control

By

Bo Zhang

A Dissertation Presented to the Graduate School Of The University of Florida In Partial Fulfillment Of the Requirements For the Degree of Doctor of Philosophy

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Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

DESIGN AND IMPLEMENTATION OF A 6 DOF PARALLEL MANIPULATOR WITH PASSIVE FORCE CONTROL

By

Bo Zhang

August 2005

Chair: Carl D. Crane III
Major Department: Mechanical and Aerospace Engineering

Parallel mechanism has been studied for several decades. It has various advantages such as high stiffness, high accuracy and high payload capacity compared to the commonly used serial mechanism. This work presents the design, analysis, and control strategy for a parallel compliance coupler for force control (PCCFC) based on a parallel platform design. The device is installed on the distal end of an industrial manipulator to regulate the contact wrench experienced when the manipulator comes into contact with objects in its environment. The parallel mechanism is comprised of a top platform and a base platform that are connected by six instrumented compliant leg connectors. The pose of the top platform relative to the base as well as the external wrench applied to the top platform is determined by measuring the displacements of the individual leg connectors. The serial robot then moves the PCCFC in order to achieve the desired contact wrench at the distal end.
A mechanical system has been fabricated with an emphasis placed on minimizing the system size and minimizing friction at the joints. Each leg has been calibrated individually off-line to determine connector properties, i.e., spring constant and free length.

The spatial compliance matrix of the PCCFC has been studied to better understand the compliant property of the passive manipulator. The forward analysis for the special 6-6 parallel platform as well as the kinematic control is also studied. The outcome of this research will advance the contact force/torque and pose regulation for in-contact operations.
CHAPTER 1
MOTIVATION AND INTRODUCTION

Motivation

With the development of robotics and control technologies, many tasks can now be done with higher efficiency than ever before. Robotic systems can perform extremely high precision operations in automatic assembly lines as well as perform operations in environments that are too dangerous for humans to work in directly. Simple operations can be completed by position control alone and manipulators are capable of high repeatability which makes these operations easy to control. More complex operations require the robot to be controlled as it manipulates a work piece which comes into contact with other objects in its environment.

The control of forces during in-contact operations is a more difficult problem than that which occurs during non-contact operations. For example, during an in-contact operation it is necessary to monitor the loads applied externally to the end effector to ensure that the forces and torques applied to the object being manipulated and the objects in the environment do not exceed allowable specified values. The manipulator should provide the appropriate loads to the object to function properly without exceeding the external load limits of the object. As an example, the manipulator may need to implant a prism-shaped part into a corresponding hole of a precision instrument. During the assembly operation, the part should be carefully fitted with minimal collision so as not to damage the component or the instrument.
Industrial manipulators have been in use since the mid 1970’s. Traditional serial industrial robots can be positioned and oriented very accurately and moved along a desired trajectory. However, without some form of force control, any positional misalignment of the manipulator could cause unexpected yielding interaction with other parts if the manipulated end-effector is rigidly positioned.

To satisfy these kinds of requirements, one solution is to integrate a force control scheme into the manipulator controller where measured forces are fed back in a closed-loop approach. Load cells that are commercially available are capable of measuring a spatial wrench. These devices, however, are quite stiff and the response of the manipulator may not be fast enough to prevent high contact forces. The focus of this dissertation is on the creation and analysis of a more compliant spatial device that can be incorporated into the control loop of a manipulator in order to successfully accomplish the in-contact force control task.

Compliance control must be integrated in the robotic control when the manipulator is handling some fragile or dangerous objects with force regulation. A properly designed compliance control system has following advantages [Hua98]:

1. Avoiding collision and damage of the object and the environment.
2. Regulating the loads and wrench applied on the manipulator to meet any special external load requirements.
3. Compensating for the inevitable positional inaccuracies that results form rigid position/trajectory control.

A simple example is shown in Figure 1-1. A serial pair of actuated prismatic joints supports a wheel via a two-spring system. The actuated prismatic joints are controlled so that the wheel could maintain a desired contact force with a rigid wall [Gri91A, Duf96].
The objective of this simple application is to control the contact force between the wheel and the rigid wall when the wheel is rolling along the surface. The serial pair of actuated prismatic joints supports the body containing points B₁ and B₂. This body is connected to the wheel by two compliant connectors ($B_iC$ and $B_2C$). The actuator drives the end effector body with pure motion in the i and j directions to sustain the desired contact force between the wheel and the environment and to move the wheel along the surface as specified. This can be accomplished given the compliant propriety of the two connectors (spring constants and free lengths) and the geometrical values of the
mechanism ($\theta_1$ and $\theta_2$). The relationship between a change in the actuator positions ($\delta d_1$ and $\delta d_2$) and the change in contact force can be analytically determined.

**Introduction**

Most industrial manipulators are serial in structure. The serial structure is an open-ended structure consisting of several links connected one after another as shown in Figure 1-2. The human arm is a good example of a serial manipulator. The kinematic diagrams of most industrial manipulators look similar to that of the G.E. P-60 and PUMA 560 industrial robots in that most consist of seven links (including ground) interconnected by six rotational joints using a special geometry (such as having three consecutive joint axes being parallel or intersecting). These structures are well constructed, highly developed, and are widely used in the industrial applications. Serial manipulators do not have closed kinematical loops and are actuated at each joint along the serial linkage.

![Figure 1-2. Serial structure](image)

Accordingly, each actuator can significantly contribute to the net force/torque that is applied to the end effector link, either to accelerate this link to cause it to move or to influence the contact force and torque if the motion is restricted by the environment. Since motions are provided serially, the effects of control and actuation errors are
compounded. Further, with serially connected links, the stiffness of the whole structure may be low and it may be very difficult to realize very fast and highly accurate motions with high stiffness.

Compared to the serial manipulator, the parallel manipulator is a closed-loop mechanism in which the end-effector (mobile platform) is connected to the base by at least two independent kinematic chains (see for example Figure 1-3). With the multiple closed loops, it can improve the stiffness of the manipulator because all the leg connectors sustain the payload in a distributive manner as long as the device is far from a singular configuration. The problem of end-point positioning error is also reduced due to no accumulation of errors.

![Figure 1-3. In-parallel platform](image)

Hence this type of manipulator enjoys the advantages of compact, high speed, high accuracy, high loading capacity, and high stiffness, compared to serial manipulators. Disadvantages, however, include a limited work space due to connector actuation limits and interference.
Therefore, the in-parallel mechanism is perceived to be a good counterpart and
necessary complement to that of the serial manipulator. It is very useful to combine these
two types of structures to utilize both advantages and decrease the weaknesses. One of
these applications is force control of serial manipulators with an in-parallel manipulator,
whose leg connectors are compliant and instrumented, is mounted at the end of the serial
arm. This is the approach that will be taken here in this dissertation; i.e., a commercially
available serial industrial manipulator will be augmented by attaching a compliant in-
parallel platform to the end effector so that contact forces can be effectively controlled.

Compared to parallel manipulators, the direct forward position analysis of serial
manipulators is quite simple and straightforward and the reverse position analysis is very
complex and often requires the solution of multiple non-linear equations to obtain
multiple solution sets. For the parallel manipulator, the opposite is the case in that the
reverse position analysis is straightforward while the forward position analysis is
complicated. This kind of phenomenon is usually referred to as serial-parallel duality.
There are similar duality properties between serial robots and fully parallel manipulator
with regards to instantaneous kinematics and statics.

The first parallel spatial industrial robot is credited to Pollard’s five degree-of-
freedom parallel spray painting manipulator [Pol40] that had three branches. This
manipulator was never actually built. In late 1950’s, Dr. Eric Gough invented the first
well-known octahedral hexapod with six struts symmetrically forming an octahedron
called the universal tire-testing machine to respond to the problem of aero-landing loads
[Gou62]. Then in 1965, Stewart published his paper of designing a parallel-actuated
mechanisms as a six-DOF flight simulator, which is different from the octahedral
hexapod and widely referred to as the “Stewart platform” [Ste65]. Stewart’s paper gained much attention and has had a great impact on the subsequent development of parallel mechanisms. Since then, much work has been done in the field of parallel geometry and kinematics such as geometric analysis [Lee00, Hun98, Tsa99], kinematics and statics [Das98A], and parallel dynamics and controls [Gri91B]. Relative works will be discussed in a more detailed manner in the literature review section.

In addition to theoretical studies, experimental works on prototypes of the Stewart platform have also been conducted for studying its property and performance. The fine positioning and orienting capability of the Stewart platform make it very suitable for use in control applications as dexterous wrists and various constructions of Stewart platform based wrists.

Because there are different nomenclatures and definitions used by different researchers in the field of parallel mechanisms, it is quite necessary to define the nomenclature [Mer00] used in current research. The following terms are defined:

1. Parallel mechanism: A closed-loop mechanism in which the end-effector (mobile platform) is connected to the base by at least two independent kinematic chains. It is also called Parallel Platform or a Parallel Kinematic Mechanism (PKM).

2. In-parallel mechanism: A 6-DOF parallel mechanism with two rigid bodies connected through six identical leg connectors, such as for example six extensible legs each with spherical joints at both ends or with a spherical joint at one end and a universal joint at the other.

3. Base Platform: The immovable plate of the parallel platform. It is also called the fixed platform.

4. Top Platform: The moving body connected to the base platform via extensible legs. It is also called moving platform or mobile platform.

5. Legs: The independent kinematic chains in parallel connecting the top platform and the base platform. Also called a connector.

6. Joint: Kinematic connection between two rigid bodies providing relative motion.
7. PCCFC: Parallel Compliance Coupler for Force Control. A parallel mechanism whose leg connectors are compliant and instrumented.

8. Platform Configuration: The combined positions and orientations of all leg connectors and the top platform.

9. Direct Analysis: Given the kinematic properties of the legs, determine the position and orientation of the top platform. Also called the forward analysis.

10. Reverse Analysis: Given the position and orientation of the top platform, determine the kinematic properties of the leg actuators.

11. M-N PKM: A PKM with M joint connection points on the base platform and N joint connection points on the top platform.

12. DOF: An abbreviating for degree of freedom.

A generalized parallel platform is shown in Figure 1-4 [Rid04]. In this example the base platform and top platform are connected by six extensible legs. The joints connecting the legs to the top platform and base platform are spherical joints, which would introduce an additional trivial rotation of legs about their axes. These additional freedoms will not affect the overall system performance and could be eliminated by replacing the spherical joint on one end of each connector with a universal joint.

The above parallel platform has six legs connecting the base platform with the top platform and there are six separate joint points on the base platform and top platform respectively. This geometric arrangement it is called a 6-6 PKM or “6-6 platform” according to the above nomenclature. The six joint points on the base platform are not necessarily located on one plane, but it is often the case that all joint points lie on a single plane and are arranged in some symmetric pattern.
Besides the general 6-6 parallel platform, there are some other configurations. One common configuration of a parallel mechanism is the 3-3 parallel manipulator as shown in Figure 1-5.
The 3-3 parallel platform also has six connector legs, but each leg shares one joint point with another leg on the base platform and similarly on the top platform. The three shared joint points form a triangle on both the planar top platform and the planar base platform. A literature review of these types of devices will be presented in the following chapter.
CHAPTER 2
BACKGROUND AND LITERATURE REVIEW

Although it was Dr. Eric Gough [Gou62] who invented the first variable-length-strut octahedral hexapod in England in the 1950’s, his parallel mechanism, also called the universal tire-testing machine, did not draw much public attention. Then in 1965, Stewart presented his paper on the design of a flight simulator based upon a 6 DOF parallel platform [Ste65]. This work had a great impact on subsequent developments in the field of parallel mechanisms. Since then, many researchers have done much work on parallel mechanisms and both theoretical analyses and practical applications have been studied.

Kinematics Analysis

The forward kinematic analysis of parallel mechanisms was one of the central research interests in this field in the 1980s and 1990s. While the reverse kinematics analysis, which is to calculate connector properties based on the top platform position and orientation information, is quite straightforward, the forward kinematic analysis is comparatively difficult to solve. The problem is to determine the position and orientation information of the top platform (mobile platform) based on the connector properties, i.e., typically the connector lengths. Usually the legs are composed of actuated prismatic joints that are connected to the top platform and base platform by a spherical joint at one end and either a spherical or universal joint at the other. The connector properties are generally referred to as the connector length or leg length as this parameter is usually measured as part of a closed-loop feedback scheme to control the prismatic actuator.
It has been observed [Das00] that the closed-form solution for the forward kinematics problem could be simplified by regulating the connector joint locations. The simplest case is the 3-3 platform, which is degenerated by grouping the connector joints into three pairs for both the top platform and base platform; each pair of connectors shares one joint point. Although it is difficult to design a physical octahedral structure with the concentric double-spherical joints that would have sufficient loading capacity and adequate workspace due to collisions of the leg connectors themselves, the forward analysis is the simplest for this class [Hun98]. The forward analysis of this device was first solved by Griffis and Duffy [Gri89] who showed how the position and orientation of the top platform can be determined with respect to the base when given the lengths of the six connectors as well as the geometry of the connection points on the top and base. This solution is a closed-form solution and is based on the analysis of the input/output relationship of a series of three spherical four bar mechanisms. The resulting solution yielded a single eighth degree polynomial in the square of one variable which resulted in a total of sixteen distinct solution poses for a given set of connector lengths. Eight solutions were reflected about the plane formed by the base connector points. The solution technique was extended to solve other special configurations such as the 4-4 and 4-5 platforms with extended methods [Lin90, Lin94]. Other methods [Inn93, Mer92] used constraints to solve the problem. First, part of the structure is ignored so that the locus curves of the connector joints could be determined according to the connector and joint properties. Then the removed part together with the angular and distance constraints is added back for further determinations of the solution for the forward kinematics analysis.
Some theoretical analyses [Rag93, Wen94] found that there are as many as 40 solutions for the forward kinematic equations. Hunt and Primrose used a geometrical method to determine the maximum number of the assembly [Hun93]. In 1996, Wampler [Wam96], found that the maximum number of possible configurations for a general Stewart platform is forty by using a continuation method.

**Singularity Analysis**

A singular configuration is some special configuration in which the parallel mechanism gains some uncontrollable freedom. For parallel manipulators, there are different types of singularity conditions based on the analysis of the Jacobian matrix that is formed from the lines of action of the six leg connectors [Zha04].

Both Tsai [Tsa99] and Merlet [Mer00] point out in their books that there are 3 types of singularities based on the Jacobian matrix analysis: the inverse kinematic singularities where the manipulator loses one or more degrees of freedom and can resist external loads in some directions; the direct kinematic singularities where the top platform gains additional degree(s) of freedom and the parallel platform cannot sustain external loads in certain direction (or just uncontrollable); and the combined singularities which could exhibit the features of both the direct kinematic singularities and reverse kinematic singularities.

Other researchers [Dan95, Ma91] found an interesting phenomenon called architecture singularities, which means that some parallel manipulators with particular configurations exhibit continuous motion capabilities in a relatively large portion of the whole workspace with all actuators fixed. This kind of singularity should be avoided in the early stage of design.
One of the reasons to study singularities is to avoid or restrict any singularities from occurring within the effective workspace. Dasgupta [Das98B] developed a method to plan paths with good performance in the workspace of the manipulator for some special cases although there is not solid evidence to prove that the method is applicable to all begin-pose-to-end-pose path planning.

**Statics and Compliance Analysis**

The forward static analysis is very straightforward as the end effector (top platform) force can be directly mapped from the connector forces, as described by [Fic86]. Duffy [Duf96] performed “stiffness mapping” analyses when considering the relationship between the twist (instantaneous motion of the top platform) in response to a change in the externally applied wrench. The mapping matrix (compliance matrix) is related to the stiffness properties of the connectors as well as the geometric dimension values.

The static equilibrium equation for a parallel mechanism with six leg connectors can be written as

\[
\mathbf{\dot{w}} = \{\mathbf{f} ; \mathbf{m}\} = \{\mathbf{f}_1 ; \mathbf{m}_{01}\} + \{\mathbf{f}_2 ; \mathbf{m}_{02}\} + \ldots + \{\mathbf{f}_6 ; \mathbf{m}_{06}\}
\]  

(2-1)

where \(\mathbf{\dot{w}}\) is the applied external wrench applied to the top platform which is comprised of a force \(\mathbf{f}\) applied through the origin point of the reference coordinate system and a moment \(\mathbf{m}\) (dynamic interpretation). The terms \(\mathbf{f}_i\) and \(\mathbf{m}_i\), \(i = 1..6\), represent the pure force that is applied along each of the leg connectors, i.e., \(\mathbf{f}_i\) represents the force applied along connector \(i\) and \(\mathbf{m}_{0i}\) represents the moment of the force along connector \(i\) measured with respect to the origin of the reference frame.

This equation may also be written as

\[
\mathbf{\dot{w}} = \mathbf{f}\{\mathbf{S} ; \mathbf{S}_0\} = \mathbf{f}_1\{\mathbf{S}_1 ; \mathbf{S}_{01}\} + \mathbf{f}_2\{\mathbf{S}_2 ; \mathbf{S}_{02}\} + \ldots + \mathbf{f}_6\{\mathbf{S}_6 ; \mathbf{S}_{06}\}
\]  

(2-2)
where \( f \) is the magnitude of the external wrench, \( \{ S; S_0 \} \) are the coordinates of the screw associated with the wrench, \( f_i \) represents the magnitude of the force along connector \( i \), and \( \{ S_i; S_{0Li} \} \) are the Plücker line coordinates of the line along connector \( i \) [Cra01].

This equation can also be written in matrix format as

\[
\begin{bmatrix}
S \\
S_0
\end{bmatrix}
f =
\begin{bmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
S_{01} & S_{02} & S_{03} & S_{04} & S_{05} & S_{06}
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
f_5 \\
f_6
\end{bmatrix}
\]

where the \( 6 \times 6 \) matrix formed by the Plücker coordinates of the lines along the six leg connectors comprises the Jacobian matrix that relates the force magnitudes along each leg to the externally applied wrench.

A derivative of the above equation will yield a relationship between the changes in individual leg forces to the change in the externally applied wrench. Griffis [Gri91A] extended this analysis to show how the change in the externally applied wrench could be mapped to the instantaneous motion of the top platform.

According to the static force analysis presented above, it is straightforward to design a compliant parallel platform to detect forces and torques. Knowledge of the spring constant and free length of each connector and measurement of the current leg lengths allows for the computation of the external wrench, once a forward position analysis is completed to determine the current pose. The first sensor of this type was developed by Gaillet [Gai83] based on the 3-3 parallel octahedral structure. Griffis and Duffy [Gri91B] then introduced the kinestatic control theory to simultaneously control force and displacement for a certain constrained manipulator based on the general spatial
stiffness of a compliant coupling. In their theoretical analysis of the compliant control strategy, a model of a passive Stewart platform with compliant legs was utilized to describe the spatial stiffness of the parallel platform based wrench sensor.

Compared to the open loop Remote Center Compliance Modules (RCC) that are commercially available (see Figure 2-1), the Stewart-platform-based force-torque sensor can provide additional information about external wrench to assist force/position control. Dwarakanath and Crane [Dwa00], and Nguyen and Antrazi [Ngu91] studied a Stewart platform based force sensor, using a LVDT to measure the compliant deflection for wrench detection. Some researchers [Das94, Svi95] also considered the optimality of the condition number of the force transformation matrix for isotropy and stability problems.

It is important that any wrench sensing device be far from a singularity configuration. Lee [Lee94] defined the problem of “closeness” to a singularity measure by defining what is known as quality index (QI) for planar in-parallel devices. Lee et al. [Lee98] extended the definition of quality index to spatial 3-3 in-parallel devices. The quality index is the ratio of the determinant of the matrix formed by the Plücker line coordinates of the six connector legs of the platform in some arbitrary position to the maximum value of the determinant that is possible for the in-parallel mechanism. The index ranges from 0 to 1 and the maximum value of 1 refers to the optimal configuration. Lee [Lee00] provides a detailed quality index presentation on several typical in-parallel structures.
The theory of screws is a very powerful tool to investigate the compliance or stiffness characteristic of a compliant device. Ball [Bal00] first introduced the theory and used it to describe the general motion of a rigid body. He presented the idea of the principal screw of a rigid body, where a wrench applied along a principal screw of inertia generates a twist on the same screw. The principal screw can be found by solving the spatial eigenvector problem. Dimentberg [Dim65] analyzed the static and dynamic properties of a spring suspended system using screw theory.

Patterson and Lipkin [Pat90] studied the necessary and sufficient conditions for the existence of compliant axes and showed a new classification of general compliance matrix based on the numbers of the compliant axes produced.
Loncaric [Lon87] also did research on unloaded compliant systems for the spatial stiffness matrix and normal form analysis by using Lie groups. Then Loncaric [Lon91A] used a geometric approach to define and analyze the elastic systems of rigid bodies and described how to choose coordinates to simplify the interpretation of the compliance and stiffness matrix.

Lipkin and Duffy [Lip88] presented a hybrid control theory for twist and wrench control of a rigid body, and showed that it is based on the metric of elliptic geometry and is noninvariant with the change of Euclidean unit length and change of basis.

Duffy [Duf96] analyzed planer parallel spring systems theoretically. Griffis and Duffy [Gri91B] studied the non-symmetric stiffness behavior for a special octahedron parallel platform with 6 springs as the connectors, and the stiffness mapping could be represented by a $6 \times 6$ matrix $[K]$. Patterson and Lipkin [Pat92] also found that a coordinate transformation changes the stiffness matrix into an asymmetric matrix but the eigenvectors and eigenvalues are invariant under the coordinate transformation, and the eigenvalue problems for compliance matrix and stiffness matrix are shown to be equivalent.

Lipkin [Lip92A] introduce a geometric decomposition to diagonalize the $6 \times 6$ compliance matrix and the diagonal elements are the linear and rotational compliance and stiffness values. It is also proved that the decomposition always exists for all cases. Another paper [Lip92B] presents several geometric results of the $6 \times 6$ compliance matrix via screw theory.

Huang and Schimmels [Hua00] also studied the decomposition of the stiffness matrix by evaluating the rank-1 matrices that compose a spatial stiffness matrix by using
the stiffness-coupling index. The decomposition, which is also called eigenscrew decomposition, is shown to be invariant in coordinate transformation with the concept of screw springs.

Roberts [Rob02] points out that the normal stiffness matrix with a preload wrench is asymmetric. However when only a small twist from the equilibrium pose is considered, the spatial stiffness matrix is a symmetric positive semi-defined matrix and the normal form could be derived from it.

Ciblak and Lipkin [Cib94] have shown that for a preload system with relative large deflection from its original position, the stiffness matrix is no longer symmetric. In a more recent study, Ciblak and Lipkin [Cib99] present a systematic approach to the synthesis of Cartesian stiffness by springs using screw theory, and shown that a stiffness matrix with rank n can be synthesized by at least n springs. Researchers [Cho02, Hua98, Lon91B, Rob00] also work on synthesis for a predefined spatial stiffness matrix.
The parallel mechanism offers a straightforward method to determine the external wrench applied to the top platform based on measured leg lengths and knowledge of the system geometry (location of joint points in top and base platforms), connector spring constants, and connector spring free lengths. Because of this, the Passive Compliant Coupler for Force Control (PCCFC) can be utilized in force feedback applications for serial robots by employing it as a wrist element that is between the distal end of the manipulator and the environment. In this chapter, a prototype for the compliant parallel platform has been designed and tested in order to study the position/wrench control with the PKM device.

**Design Specification**

The design process could roughly be composed by 3 stages [Tsa99]:

1. Product specification and planning stage,
2. Conceptual design stage,
3. Physical design stage.

During the first stage, the functionality, dimension, and other requirements are specified for the development of the product. In the second stage, several rough conceptual design alternatives are developed based on the product specification and the one with best overall performance is chosen to be developed. In the last stage the dimensional and functional design, analysis and optimization, assembly simulation, material selection, and engineering documentation are completed. Although the third
stage usually is the most time-consuming stage, the first two stages have great impact on design results and cost control.

In the PCCFC specification and planning stage, the product’s geometrical and functional requirements are specified as follows:

SP1: The mechanism should be compliant in nature to measure contact loads;
SP2: The mechanism should have 6 degrees of freedom;
SP3: The mechanism should be a spatial parallel structure;
SP4: The mechanism should be ability to measure loads without large errors;
SP5: the structure should have a relatively compact size.

As mentioned in the previous chapter, the Stewart platform is a promising structure for such spatial compliant devices. The Stewart platform structure satisfies the SP2 and SP3 criteria. With each connector being composed of a prismatic joint and compliant component, the SP1 and SP4 criteria could also be satisfied. One of the main advantages of the Stewart platform is that it can withstand a large payload (compared to serial geometries) with relative compact size. As a result, the dimensional requirement - SP5 could be met during the conceptual design and physical design stage. The detailed design specifications are presented in Table 3-1. Note that in this table, the direction of the z axis that is referred to is perpendicular to the plane of the base platform.

Table 3-1. Design objective specification

<table>
<thead>
<tr>
<th>Specification</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Individual Connector Deflection: Magnitude</td>
<td>±5mm</td>
</tr>
<tr>
<td>Maximum perpendicular load: Magnitude</td>
<td>50N</td>
</tr>
<tr>
<td>Motion range in Z direction: Magnitude</td>
<td>±4mm</td>
</tr>
<tr>
<td>Motion range in X,Y direction: Magnitude</td>
<td>±3.5mm</td>
</tr>
</tbody>
</table>
Table 3-1. Continued

<table>
<thead>
<tr>
<th>Specification</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rotation range about Z axis: Magnitude</td>
<td>±6°</td>
</tr>
<tr>
<td>Rotation range about X,Y axis: Magnitude</td>
<td>±4°</td>
</tr>
<tr>
<td>Position Measuring Resolution:</td>
<td>0.01mm</td>
</tr>
</tbody>
</table>

**Conceptual Design**

In the conceptual design stage, a “best” conceptual design is selected from among alternatives. The 3-3 octahedral structure was the first to be considered since it is the simplest structure of its kind.

The mobility of general spatial mechanisms can be calculated from the Grübler Criterion [Tsa99]:

\[
M = \lambda (n - 1) - \sum_{i=1}^{j}(\lambda - f_i),
\]

(3-1)

where:

- \(M\): Mobility, or number of degree of freedom of the system.
- \(\lambda\): Degrees of freedom of the space in which a mechanism is intended to function, for the spatial case, \(\lambda = 6\)
- \(n\): Number of links in the mechanism, including the fixed link
- \(j\): Numbers of joints in a mechanism, assuming that all the joints are binary.
- \(f_i\): Degrees of relative motion permitted by joint \(i\).

The octahedral 3-3 parallel platform is shown in Figure 3-1. It is the simplest geometry for a spatial parallel platform. For the 3-3 parallel platform, the octahedron has a top platform and a base platform, 6 connectors, and 6 concentric spherical joints, three on the base platform and three on the top platform. Each connector is treated as 2 bodies
connected by a prismatic joint. Thus there are a total of 14 bodies; 2 for each of the 6 legs, the top platform, and the base platform. Each spherical joint has 3 relative degrees of freedom and thus there are a total of 12 joints that each has 3 degrees of relative freedom.

![Figure 3-1. Plan view of octahedral 3-3 parallel platform](image)

For this case, the mobility equation becomes

$$M = 6(14 - 1) - \sum_{i=1}^{12} (6 - 3) - \sum_{i=1}^{6} (6 - 1) = 78 - 36 - 30 = 12$$  \hspace{1cm} (3-2)

Six out of the 12 degrees of freedom are trivial in that each leg connector can rotate about its own axis. Eliminating these from consideration results in a 6 degree of freedom device.

The forward kinematic analysis was first performed by Griffis and Duffy [Gri89]. They discovered that a maximum of sixteen pose configurations can exist for this device (eight above the base platform and eight more reflections below the base platform) when given a set of leg connector lengths. Hunt does a comprehensive study of the octahedral
3-3 parallel platform [Hun98]. Lee [Lee94] analyzed the singular configurations of this device and defined the problem of “closeness” to a singularity by defining what is known as the quality index (QI) for planer in-parallel devices. Lee et al. [Lee98] extended the definition of quality index to the spatial 3-3 in-parallel device. The quality index is the ratio of the determinant of the matrix formed by the Plücker line coordinates of the six connector legs of the platform in some arbitrary position to the maximum value of determinant that is possible for the in-parallel mechanism. This index is a dimensionless value:

\[ \lambda = \frac{|J|}{|J|_{\text{max}}}, \]  

(3-3)

where \( J \) is the matrix of the line coordinates. The quality index has a maximum value of 1 at an optimal configuration that is shown to correspond to the maximum value of the determinant. As the top platform departs from this configuration relative to the base, the determinant would decrease, and at some positions and orientations it can reach zero which indicates an uncontrollable singularity state.

As previously stated, the mechanism has been designed such that the determinant of the matrix \( J \) will be near the maximum at the unloaded home position. Lee [Lee00] showed that for a 3-3 platform with a top triangle of side length \( a \) and a bottom triangle of side length \( b \), that the maximum value of the determinant of \( J \) would occur when the top platform was parallel to the base and was rotated as depicted by the plan view shown in Figure 3-1. The determinant of the matrix \( J \) was shown to be given by

\[ |J| = \frac{3 \sqrt{3} a^3 b^3 h^3}{4 \left\{ \frac{a^2 - ab + b^2}{3} + h^2 \right\}^3} \]  

(3-4)
where $h$ is the separation distance or height of the top platform with respect to the base. Lee proved that the maximum value for the determinant of $J$ would occur when

$$h = \frac{1}{\sqrt{3}} (a^2 - ab + b^2) .$$

(3-5)

and

$$b = 2a .$$

(3-6)

From (3-5) and (3-6), it is easy to obtain

$$b = 2a = 2h .$$

(3-7)

Although the geometric configuration of the 3-3 parallel platform has the simplest geometry with regards to the forward analysis, it is difficult to design and implement in practice due to the pair of concentric spherical joints or other type of universal joints that connect each leg to the base and top platform. These kind of concentric joint pairs are not only very difficult to manufacture but also very hard to have sufficient payload capacity. This configuration also induces unwanted interference between moving legs.

One other alternative is the 6-3 platform, which has six points of connection on the base platform and three points of connection on the top platform. While this 6-3 configuration has less concentric joints pair in the base platform, it still has the same difficulty and complicated arrangement of joint connections on the top platform. In order to completely eliminate the need for coincident connections, the 6-6 parallel platform was studied. The forward analysis for a general 6-6 platform, however, is very complicated requiring lengthy computation and resulting in a maximum of forty configurations.

Griffis and Duffy [Gri93] developed a special 6-6 platform geometry, however, where the forward analysis was comparable in difficulty to that of the 3-3 platform. As
opposed to a general 6-6 parallel platform, the special 6-6 platform has six leg connector points in a plane on the top and base platforms. The arrangement of these connecting points is such that three connecting points of the platform define a triangle as the apices and the other three connecting points lie on the sides of this triangle.

There are a few configurations of the leg connection: one configuration, known as the “apex to apex”, has three of the legs attaching to the corners or apices of the triangle of the top platform and the base platform, and the remaining legs are attached to the sides or edges of the platform. The other preferred configuration is known as “apex to midline” as shown in Figure 3-2.

Figure 3-2. Special 6-6 parallel platform.
In order to eliminate the six additional rotational freedoms, the six spherical joints connecting the top platform and leg connectors are replaced by universal joints. Equation (3-1) is utilized here to calculate the mobility of the special 6-6 parallel platform

\[ M = 6(14 - 1) - \sum_{i=1}^{6} (6 - 3) - \sum_{i=1}^{6} (6 - 2) - \sum_{i=1}^{6} (6 - 1) = 78 - 18 - 24 - 30 = 6 \quad (3-8) \]

The legs 1-6 are arranged in such a way that each of them is connecting to a corner connecting point on either the top or base platform while the other connecting point is in the midline of the triangle of the opposing platform. For example, leg 6 is connecting the top platform 19 on the corner connecting point 18 with the base platform 20 on the midline connecting point 12. Griffis noted that this particular configuration is singular.

Figure 3-3. 3-D model for special 6-6 parallel mechanism

If the midline connecting points are not exactly on the middle of the triangle side and for each side of the triangle, and the points are distributed symmetrically, there are two different configurations which are usually called “clockwise” configuration and
“anti-clockwise” (the rotation direction of the top platform when pressed) configuration based on the plan view of the structure. There is no significant analytical difference between the two configurations for the kinematic analysis and control, so only one configuration is used to build the prototype. One computer generated model of these configurations is shown in Figure 3-3.

Prototype Design

After choosing the desired configuration of the prototype, more detailed and specific information about the parallel platform should be studied in order to build the physical mechanism. One plan view of the configuration of the parallel platform is shown in Figure 3-4.

The dimensional relationship between the top/base platform and the free length of the legs was studied by Lee [Lee00] based on the optimal dimensional restriction for the special 6-6 parallel platform stated in the previous section. The free lengths of the connectors are calculated by the following equations:

\[
\begin{align*}
l_{\text{long}} &= \sqrt{\frac{1}{3} \left( 3h^2 + a^2(3\rho^2 - 3\rho + 1) - (\cos \theta + \sqrt{3}(2\rho - 1)\sin \theta)ab + b^2 \right)} \\
l_{\text{short}} &= \sqrt{\frac{1}{3} \left( 3h^2 + a^2 - (\cos \theta - \sqrt{3}(2\rho - 1)\sin \theta)ab + b^2(3\rho^2 - 3\rho + 1) \right)}
\end{align*}
\] (3-9)

As shown in Figure 3-4:

l: free length of the legs, divided into two groups based on the fact that there are two different values for free length of the legs.

a: refers to the functional triangle edge length of the top form

b: refers to the functional triangle edge length of the base form.

\( \theta \): refers to the rotation angle of the top platform about the z-axis.
$\rho_a, \rho_b$: refer to the distance between adjacent joint points on the base platform and top platform.

Figure 3-4. Plan view of special 6-6 platform

The design process is shown in the Figure 3-5 below
Consider the requirements of feasibility with regards to dimensional limits for manufacturing parts, encoders available in the market, and the need for a continuous working space with no interference of moving parts, a 3-D modeling and assembly simulation resulted in the following choices for the size of the base and top platforms and the lengths of the two sets of connectors at the unloaded home position:
a = 60 mm, 
b = 120 mm, 
\( \rho = \frac{28}{120} = 0.2333, \)
h = a = 60 mm, 
\( L_{\text{short}} = 68.021 \text{ mm}, \)
\( L_{\text{long}} = 80.969 \text{ mm}. \)

The elastic component is a crucial element for the passive compliant parallel manipulator prototype. Precision linear springs were selected for use as the elastic component to provide elongation and compression of the leg connectors. After carefully studying and comparing different springs, a group of cylindrical precision springs with different spring constants and the same high linear coefficient (± 5%) were selected to use as the elastic component in the legs. The spring rates were respectively 8.755 N/mm, 2.627 N/mm and 0.8755 N/mm (or 50 lb/in, 15 lb/in, and 5 lb/in). By using different groups of springs, the stiffness matrix of the parallel platform could be changed by replacing the spring and thus the device can be applied to different applications where the expected force ranges are different.

The free length variation of each connector is measured by a linear optical encoder. Each leg connector is comprised of two parts that translate with respect to each other and are interconnected by the spring. The encoder read head is attached to one of the leg parts and the encoder linear scale is attached to the other. The two leg parts are constrained by a ball spline to ensure that they translate relative to one another and to help maintain alignment of the encoder read head and linear scale.
The initial free length of each connector is measured by micro-photography with a specially designed clamp apparatus. Since the six connecting points on the base platform are in the same plane, the centers of the pseudo-spherical joints are also in the same virtual plane. The free length of the connectors is defined as the distance between the two centers of the joints, which connect the connector with the top platform and the base platform.

Figure 3-6 shows renderings of the prototype device. Figure 3-7 presents a photograph of the prototype.

Figure 3-6. Assembled model for special 6-6 parallel platform. A) Solid model, B) Frame model.
Figure 3-7. Photo of the assembled parallel platform
CHAPTER 4
SINGULARITY ANALYSIS

In this chapter, a singularity analysis is presented and a special singularity configuration for a special 6-6 parallel platform is analyzed.

It is important singularity configurations, which are identified as the case when the Plücker line coordinates of the six leg connectors become linearly dependent, be avoided throughout the effective workspace of the device. Singularities can occur in both planar and spatial mechanisms. As an example, a planar structure is shown in Figure 4-1.

The structure is in a singular configuration when point S is at location (0, 0). At this instant, considering the displacement of point S as the input and the displacement of point D as the output, it is not possible to obtain a user desired velocity for point D, no matter what the velocity of the input is. This condition where point S is at (0, 0) is called a singular point as distinct from other simple points, which are also called regular points [Hun78]. For this group of mechanism, a singularity only occurs when the mechanism comes to some discrete special configuration while for all other points or configurations the mechanism is normal or controllable. Particularly for parallel structures, it could be said that singularity configurations exist in the workspace at those configurations where the parallel structure gains some uncontrollable freedom.
For spatial parallel manipulators, there are different types of singularity conditions based on the analysis of the Jacobian matrix that is formed from the lines of action of the six leg connectors. For example, a parallel manipulator may be able to resist external loads in some directions with zero actuator forces at an inverse kinematical singular configuration. Some manipulators can also have a different kind of singularity condition, in which with all the actuators are locked, the top platform could still move infinitesimally in some direction. Further, there are combined singularities that occur for some special kinematic architecture when both conditions mentioned above occur [Tsa99].

All the singularity conditions described above are temporary and conditional. This means that only when the special geometrical configuration occurs, the mechanism will function differently from the norm. The special 6-6 parallel platform is considered here and the joint arrangement is shown in Figure 4-2.

Lee [Lee00] showed that when the dimensional ratio $p=q=1/2$, where $p$ and $q$ represent the offset percentage of the joints in the base and top platform respectively, the determinant of the Jacobian matrix is zero and the platform is in a singularity. He also
identified the singularity case that occurs when the top platform rotates 90° around the Z-axis.

Certain geometries exist, however, that are always singular. One planar example is the parallelogram. A parallelogram with four revolute joints is normally unstable i.e., it could not sustain external loads with internal angles unchanged. Compared to the very stable triangle, such special geometries are often referred to as unstable as opposed to being described as being in a singular state [Tsa 99].

Examples of unstable spatial mechanisms can also be found. Figure 4-3 shows the particular case that was identified in this research. This special geometry is always unstable, or in a singularity condition. It will be shown that the top platform will have continuous mobility when the lengths of the six leg connectors are fixed. For this
mechanism, both the top and the base platforms are equilateral triangles. The following notation is used for this special singularity configuration:

- \( a \): the length of the side of the top equilateral triangle,
- \( b \): the length of the side of the base equilateral triangle,
- \( q, p \): a dimensionless number in the range of 0 to 0.5 that defines the offset distance of the connection points (for example, the distance between points \( E_{c1} \) and \( E_{c2} \) is \( pb \) and the distance between points \( B_1 \) and \( B_2 \) is \( qa \)),
- \( h \): the vertical distance from the geometric center of the base plate to the center of top plate.

Figure 4-3. Plan view of the special singularity configuration

In order to simplify the process, here the structure will be analyzed only in an initial configuration where the two plates are parallel and the relative rotation angle is zero. The analysis process is the same for other positions and orientations. The matrix of the line coordinates along the connectors for this configuration is written as
The determinant of the Jacobian matrix is

\[
\begin{bmatrix}
\frac{(q-1)a+b}{2} & \frac{-a-(2p-1)b}{2} & \frac{(1-2q)a-b}{2} & \frac{a-(1-p)b}{2} & \frac{qa}{2} & \frac{pb}{2} \\
\frac{\sqrt{3(a(3q-1)+h)}}{2} & \frac{\sqrt{3(b-a)}}{2} & \frac{\sqrt{3(b-a)}}{2} & \frac{\sqrt{3[-a-(3p-1)b]}}{2} & \frac{\sqrt{3[a(2-3q)-2h]}}{2} & \frac{\sqrt{3[2a-(2-3p)b]}}{2} \\
6 & h & h & h & h & h \\
6 & h & h & h & h & h \\
6 & h & h & h & h & h \\
\frac{\sqrt{3}bh}{6} & \frac{-\sqrt{3}bh}{6} & \frac{-\sqrt{3}bh}{6} & \frac{\sqrt{3}(3p-1)b}{6} & \frac{2\sqrt{3}bh}{6} & \frac{\sqrt{3}(2-3p)b}{6} \\
6 & h & h & h & h & h \\
\frac{\sqrt{3}qab}{6} & \frac{-\sqrt{3}qab}{6} & \frac{-\sqrt{3}qab}{6} & \frac{-\sqrt{3}qab}{6} & \frac{-\sqrt{3}qab}{6} & \frac{-\sqrt{3}qab}{6} \\
6 & 6 & 6 & 6 & 6 & 6 \\
\end{bmatrix}
\]  

(4-1)

The determinant of the Jacobian matrix is

\[
Det(J) = \frac{\sqrt{3}}{8} (abh)^3 (9q^3 p - 9q^3 p^2 - 3q^3 - 9qp^3 + 9q^2 p^3 + 3 p^3) 
\]

\[
= \frac{3\sqrt{3}}{8} (abq)^3 (p-q) \left[(p-q)^2 + 3pq(1-p)(1-q) \right]
\]  

(4-2)

It was shown by Lee [Lee00] when analyzing the mechanism depicted in Figure 4-2 that when p=q=0.5, the determinant of the Jacobian matrix equals zero and the structure is unstable. For all other values for p and q in the range of 0 to 0.5, the structure is stable.

For the structure shown in Figure 4-3, the result is quite different. Clearly the determinant of J is zero when p=q. Under the constraints 0\(\leq\)q, p\(\leq\)0.5, the term

\[
\left[(p-q)^2 + 3pq(1-p)(1-q) \right]
\]

is clearly nonnegative and is zero precisely when p=q=0.

Consequently, we have that under the natural physical constraints 0\(\leq\)q, p\(\leq\)0.5, the determinant of J is zero if and only if p=q, regardless of the values for a, b, and h. This means that the matrix is degenerate and the structure is singular and has gained some uncontrollable freedom. Since the structure now could not sustain any external load, the gravity of the top plate would change the configuration of the structure and would move it downwards in Z direction with a counterclockwise rotation along the Z axis.
A photo of the platform model is shown in Figure 4-4. From the picture it can be seen that the structure has collapsed due to the gravity effect until leg interference occurs. The structure could sustain the gravity of the top plate in the shown state only because of collisions of adjacent leg connectors which stops it from moving downwards further. The top plate could easily be moved upwards with little upwards force applied.

Figure 4-4. Photo of the parallel platform in singularity

Although singular conditions are typically something to be avoided, there is one application where this particular geometry would be of use. Platforms that incorporate tensegrity must be in a state that when no external load is applied to the top platform, the lines of action of the six leg connectors must be linearly dependent. Tensegrity based platforms incorporate compliance such as springs to pre-stress certain leg connectors in tension which will place the other connectors in compression. The internal forces in the
leg connectors can only sum to zero when the lines of action are linearly dependent. Thus, the platform geometry discussed here could have particular application when tensegrity is incorporated in the platform.
CHAPTER 5
STIFFNESS ANALYSIS OF COMPLIANT DEVICE

For a compliant component, such as a spring, the basic property of it is its stiffness. Theoretically, no object is infinitely stiff; the only difference between a so called “stiff component” and “compliant device” is that comparing with the “compliant” component, the “stiff” component has a relatively much higher rigidity or stiffness. The spring is the most commonly used compliant component. The stiffness property of a spring is usually quantified by a ratio called the spring constant, also known as the elastic coefficient or stiffness coefficient and is used to describe compliant devices.

In this chapter, simple analyses for planar compliant devices are introduced together with an example of a compliant 2-D contact force control system. Then the stiffness analysis of a 3-D system is studied using Screw theory. The stiffness matrix for the special 6-6 parallel manipulator is developed and a numerical example is provided for the specific design.

**Simple Planar Case Stiffness Analysis**

Hooke’s Law describes the fundamental relationship between an external force and the compliant displacement in static equilibrium

\[ F_s = -k \Delta l = -k(l - l_0) \]  

(5-1)

where \( F_s \) is the force exerted on the spring (or more generally, the compliant component), \( k \) is the spring constant, \( l \) is the current length of the spring, \( l_0 \) is the free length, and \( \Delta l \) is the displacement from the equilibrium position.
Consider two springs that are connected in-parallel, with spring constants $k_1$ and $k_2$ respectively. An external force is applied on the right end of the two springs and the force direction is along the axes of the springs.

The force in each spring can be calculated as

\[ f_{s1} = -k_1 \Delta l_1 \]
\[ f_{s2} = -k_2 \Delta l_2 \]

where in this case $\Delta l_1 = \Delta l_2 = \Delta l$. The total force applied to the across the two springs can be written as

\[ f_s = f_{s1} + f_{s2} \quad (5-2) \]

The equivalent mapping of stiffness of the model is derived by:

\[ f_s = -(k_1 \Delta l_1 + k_2 \Delta l_2) = -(k_1 + k_2) \Delta l 
= -k_c \Delta l \quad (5-3) \]

where $k_c$ is the equivalent stiffness/spring constant of the in-parallel connected springs. It is the sum of the individual spring constants. This shows that when compliant components are connected in-parallel, the overall equivalent stiffness is much higher than the stiffness of any individual compliant component.
Figure 5-2. Planar serially connected springs system

Now consider two springs connected in series, as shown in Figure 5-2 above. The axes of the two springs are collinear and they are connected end to end. The direction of the external force applied on them is also along the axes of the springs. The derivation is quite simple, so only the result is provided here as

\[
f_s = -k_s \Delta l = -k_s (\Delta l_1 + \Delta l_2)
\]

(5-4)

where \( \frac{1}{k_s} = \frac{1}{k_1} + \frac{1}{k_2} \)

The term \( k_s \) is the equivalent stiffness/spring constant of the serially connected springs. Its reciprocal value is the sum of the reciprocal values of the individual spring constants. This means that when compliant components are connected serially, the overall equivalent stiffness is less than the stiffness of individual compliant component.

If \( k_1 = k_2 = k \) then \( k_e = \frac{k}{2} \) and if \( k_1 \neq k_2 \), then \( k_e \approx k_1 \). This shows that serially connected components with widely different spring constants, have an equivalent stiffness, which is more dependent on the component with the smaller spring constant.

According to this result, if one compliant component is connected to a relatively stiff bar, the equivalent stiffness of this two-component system is very close to the stiffness of the compliant component. The systems discussed above have one degree-of-freedom (DOF). A more general planar case would have two or more DOF.
Figure 5-3. Planar 2 DOF spring

Figure 5-3 shows a planar 2 DOF spring case. In this spring system, two translational springs are connected at one end P and grounded separately at pivot points A and B respectively. Here translational springs behave like prismatic joints in revolute-prismatic-revolute (RPR) serial chains. In the X-Y plane, two such springs form the simple compliant coupling as shown in Figure 5-3.

The two-spring compliant coupling system is equivalent to a planar two-dimensional spring. The spring is two-dimensional because two independent forces act in the translational spring, and it is planar since the forces remain in a plane.

The external force applied at point P is in static equilibrium with the forces acting in the springs. The two-dimensional spring remains in quasi static equilibrium as the point P move gradually.

In order to analyze the two-dimensional force/displacement relationship, or mapping of stiffness, it is necessary to decompose both the external force and displacements into standard Cartesian coordinate vectors such that $\delta \mathbf{f} = \delta \mathbf{f}_x \hat{x} + \delta \mathbf{f}_y \hat{y}$. 
The locations of points A, B, and P, the initial and current lengths of AP, BP, and
the angles \( \theta_1, \theta_2 \) are all known. The free length of AP is \( l_{01} \) and the free length of BP
is \( l_{02} \). The spring constants are \( k_1 \) and \( k_2 \). The current length of AP and BP are \( l_1 \) and
\( l_2 \) respectively. To simplify the equations, dimensionless parameters \( \rho_1 = l_{01}/l_1 \) and
\( \rho_2 = l_{02}/l_2 \) are introduced. By definition, these two scalar values are always positive, and
no negative spring lengths are allowed. When \( \rho > 1 \), the corresponding spring is
elongated, and if \( \rho_i \) is less than 1, then it is compressed.

The external force applied on the spring system is given by:

\[
\begin{bmatrix}
    f_x \\
    f_y
\end{bmatrix}
= \begin{bmatrix}
    c_1 & c_2 \\
    s_1 & s_2
\end{bmatrix}
\begin{bmatrix}
    k_1(1-\rho_1)l_1 \\
    k_2(1-\rho_2)l_2
\end{bmatrix}
\rho_1
\]

where \( c_i = \cos(\theta_i) \) and \( s_i = \sin(\theta_i) \). Differentiating the above equation will result in the
following equation

\[
\begin{bmatrix}
    \delta f_x \\
    \delta f_y
\end{bmatrix} = [k]
\begin{bmatrix}
    \delta x \\
    \delta y
\end{bmatrix}
\]

where \([k]\) is the mapping of the stiffness of the system which can be written as [Gri91A]

\[
[k] = \begin{bmatrix}
    k_{11} & k_{12} \\
    k_{21} & k_{22}
\end{bmatrix}
= \begin{bmatrix}
    c_1 & c_2 \\
    s_1 & s_2
\end{bmatrix}
\begin{bmatrix}
    k_1 & 0 \\
    0 & k_2
\end{bmatrix}
\begin{bmatrix}
    c_1 & s_1 \\
    c_2 & s_2
\end{bmatrix}
\rho_1
\]

where \([j]\) is the static Jacobian matrix of the system, \([\delta j]\) is the differential matrix with
respect to \( \theta_1 \) and \( \theta_2 \), and \([k_i],[k_i(1-\rho_i)]\) are 2 × 2 diagonal matrices. In general \( \theta_1 \neq \theta_2 \),
as if the two angles are equal, then the spring matrix expression is singular and meaningless. For that case the two springs are also parallel and equation (5-3) can be applied instead.

### Planar Displacement and Force Representation

Screw theory is utilized in the analysis of more complex compliant systems, such as the structure shown in Figure 1-1. In screw theory [Bal00, Duf96], a line in the XY plane is written using Plücker coordinates as a triple of real numbers \{L; M; R\}. L and M represent the direction of the line in the XY plane and as such are dimensionless. R represents the moment of the line about the Z axis and has units of length. For this analysis the direction of the line will be represented by a unit vector and thus

\[
(L^2 + M^2)^{1/2} = 1.
\]

The Plücker coordinates of a line are often written as

\[
\hat{s} = \{S; S_0\} \tag{5-9}
\]

where \(S = (L_i + M_j)\) and \(S_0 = (R_k)\). Although Plücker coordinates are homogeneous, i.e., \(\{\lambda S; \lambda S_0\}\) describe the same line for all nonzero values for \(\lambda\), it will be assumed throughout this analysis that \(S\) is a unit vector. The subscript 0 is introduced to indicate that the moment of the line about the Z axis will change with a translation of the reference coordinate system.

A force is represented by a line multiplied by a force magnitude. A twist is represented by a line multiplied by an angular velocity. It is interesting to note that a pure moment of magnitude \(m\) is written as \(\{0, 0; m\}\) and a pure translation of magnitude \(v\) is written as \(\{0, 0; v\}\) which represent lines at infinity that are multiplied by the moment magnitude and the velocity magnitude respectively.
The resultant of an arbitrary set of planar forces $f_i \{S_i; S_{0i}\}, i=1..n$, can always be represented by a wrench which in the planar case reduces to a particular line-bound force which may be calculated as

$$w = f \{S; S_0\} = \sum_{i=1}^{n} f_i \{S_i; S_{0i}\} \quad (5-10)$$

This resultant is often written as

$$\hat{w} = \{f; m_0\} \quad (5-11)$$

where $f = f S$ and $m_0 = f S_0$.

Now, consider that three forces in the XY plane of magnitude $f_1, f_2, f_3$ are applied to a rigid body. The equivalent resultant force can be determined by the sum of the force vectors as

$$\hat{w} = \begin{bmatrix} f_1 \\ m_1 \end{bmatrix} + \begin{bmatrix} f_2 \\ m_2 \end{bmatrix} + \begin{bmatrix} f_3 \\ m_3 \end{bmatrix} = f_1 \hat{s}_1 + f_2 \hat{s}_2 + f_3 \hat{s}_3 \quad (5-12)$$

Equation (5-12) can be further transformed with the following expression

$$\hat{w} = \begin{bmatrix} c_1 \\ s_1 \\ p_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ s_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} c_3 \\ s_3 \\ p_3 \end{bmatrix} \quad (5-13)$$

where $s_i$ and $c_i$ represent the sine and cosine of an orientation angle of line $i$ measured with respect to the X axis and $p_i$ represent the moment of line $i$.

Equation (5-13) may be written in matrix form as

$$\hat{w} = j\lambda, \quad (5-14)$$

where
Given the magnitude of the external forces and their coordinates of each line of action, the coordinates of the resultant force (both magnitude and the line of action) can be determined by the above equation. For static equilibrium [Duf96], there must be one corresponding external force also applied on the rigid body with equal magnitude and opposite direction.

**Stiffness Mapping for a Planar System**

The general mapping of stiffness is a one-to-one correspondence that associates a twist describing the relative displacement between the bodies with the corresponding resultant wrench which interacts between them. Considering a simple 2-D planar system shown in Figure 5-4, the rigid moving body is connected to ground by three RPR compliant serial chains. A linear spring is located inside the prismatic joint and axial spring forces are applied both on the top moving stiff body (to simplify, it is also called top platform) and the ground. According to equation (5-13), the resultant force can be calculated. In order to maintain equilibrium, an external force with the exact same magnitude \( f \) and opposite coordinates \( \hat{\mathbf{w}} \) much be applied on the top platform.
Figure 5-4. Planar compliant system

\[ \dot{\omega} = f_1 \dot{s}_1 + f_2 \dot{s}_2 + f_3 \dot{s}_3 = \]
\[ k_1 (l_1 - l_{01}) \dot{s}_1 + k_2 (l_2 - l_{02}) \dot{s}_2 + k_3 (l_3 - l_{03}) \dot{s}_3 \]  
(5-15)

where twist \( \dot{\omega} \) contains the magnitude and geometrical information of the resultant force, \( f_1, f_2 \) and \( f_3 \) are the magnitudes of the spring forces correspondingly, \( \dot{s}_1, \dot{s}_2 \) and \( \dot{s}_3 \) are the line coordinate of the axis of the connectors, \( k_1, k_2 \) and \( k_3 \) are the spring constants and \( (l_i - l_{0i}), (i = 1, 2, 3) \) is the change in length between the current connector length and the initial length correspondingly.

The detailed derivation process can be found in [Duf96], and only the result of the stiffness mapping is provided here as

\[ \delta \dot{\omega} = [K] \delta \dot{D} \]  
(5-16)
where the stiffness matrix $[K]$ is

$$[K] = j[k]j^T + [B][k(1 - \rho)][C]^T$$

(5-17)

where $[k] = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}$, $[k(1 - \rho)] = \begin{bmatrix} k_1(1 - \rho_1) & 0 & 0 \\ 0 & k_2(1 - \rho_2) & 0 \\ 0 & 0 & k_3(1 - \rho_3) \end{bmatrix}$ and

$$\rho_i = \frac{l_{ii}}{l_i} \quad j = \begin{bmatrix} \hat{s}_1 & \hat{s}_2 & \hat{s}_3 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \\ p_1 & p_2 & p_3 \end{bmatrix}, \quad [B] = \begin{bmatrix} \hat{s}_{iB} & \hat{s}_{iB} & \hat{s}_{iB} \end{bmatrix} = \begin{bmatrix} -s_1 & -s_2 & -s_3 \\ c_1 & c_2 & c_3 \\ q_{iB} & q_{iB} & q_{iB} \end{bmatrix}$$

and

$$[C] = \begin{bmatrix} \hat{s}_{iC} & \hat{s}_{iC} & \hat{s}_{iC} \end{bmatrix} = \begin{bmatrix} -s_1 & -s_2 & -s_3 \\ c_1 & c_2 & c_3 \\ q_{iC} & q_{iC} & q_{iC} \end{bmatrix}, \quad \hat{s}_{iB} = ds_i/d\theta_i$$

are the coordinates of the line that is perpendicular to $\hat{s}_i$ that passes through a fixed pivot $B_i$, $\hat{s}_{iC}$ are the coordinates of the line $S_{iC}$ that is parallel to $S_{iB}$ and through pivot $C_i$, as shown in Figure 5-5.

Figure 5-5. Stiffness mapping for a planar compliant system
Stiffness Matrix for Spatial Compliant Structures

The analysis of stiffness matrix for the planar structure was presented in the previous sections. Now the spatial stiffness property will be explored. In this section, necessary concepts of projective geometry for point and line-vector are provided briefly, followed by the analysis of a spatial 3-3 octahedron and the desired special 6-6 spatial parallel manipulator.

In order to perform the analysis, it is essential to expand some of the concepts from the 2-D plane to 3-D space. In the planar case, the Cartesian coordinate for a point is defined by two dimensionless scalar values: X and Y, which are also dependent on the reference point of the selected coordinate system. In Screw theory [Cra01], homogeneous coordinates \((w; x, y, z)\) are used to describe the location of a point and

\[
\mathbf{r} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{w}
\]

refers to the position vector from the reference point \(O\) to the point \(A\) with Cartesian coordinate \((x_i, y_i, z_i)\). Usually it is assumed that \(w=1\) so that the ratios of \(x/w, y/w, z/w\) are equal to \(x, y\) and \(z\). This homogeneous coordinate system also allows for a point at infinity. When \(|w|=0\), the point \(A\) is at infinity in the direction parallel to \((x_1 + y_1 + z_1)\). For points not at infinity, \(w\) is always a non-zero value. To simplify the expressions in this work, the coordinate of a point will also be expressed as \((x, y, z)\).

It is clear that in the planar case, the equation of a line can be expressed by three numbers \(L, M,\) and \(R\), which are first introduced by Plücker, and are called Plücker line coordinates. In 3-D space, the ray coordinate of a line is defined by any two distinct points on the line. The coordinate of point 1 and point 2 are \(\mathbf{r}_1(x_1, y_1, z_1)\) and
\( \mathbf{r}_2(x_2, y_2, z_2) \) respectively. The vector \( \mathbf{S} \) whose direction is parallel to the line could be written as:

\[
\mathbf{S} = (\mathbf{r}_2 - \mathbf{r}_1),
\]

or

\[
\mathbf{S} = Li + Mj + Nk
\]  

(5-18)

(5-19)

So vector \( \mathbf{S} \) provides directional information for the line segment composed by the two points. Now consider another vector \( \mathbf{r} \) from the reference origin to any general point \( (x_1, y_1, z_1) \) on the line, the cross product of the two vectors defines a new vector which is perpendicular to both of the vectors. This is written as

\[
\mathbf{S}_{OL} = \mathbf{r} \times \mathbf{S}.
\]  

(5-20)

The vector \( \mathbf{S}_{OL} \) is the moment of the line about the origin O and is clearly origin dependent, while the vector \( \mathbf{S} \) provides the directional information for the line. Thus the coordinate of a line is written as \( \{\mathbf{S}; \mathbf{S}_{OL}\} \) and this is also called the Plücker coordinates of the line. The semi-colon in the coordinates indicates that the dimensions of \( \mathbf{S} \) and \( \mathbf{S}_{OL} \) are different, i.e., \( \mathbf{S} \) is dimensionless while \( \mathbf{S}_{OL} \) has units of length.

From equation (5-20), the moment vector \( \mathbf{S}_{OL} \) is:

\[
\mathbf{S}_{OL} = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ L & M & N \end{vmatrix} = Pi + Qj + Rk,
\]  

(5-21)

where

\[
P = y_1N - z_1L
\]

\[
Q = z_1L - x_1N
\]

\[
R = x_1M - y_1L
\]  

(5-22)

The Plücker coordinates for the line joining two points with coordinates \((1; x_1, y_1, z_1)\) and \((1; x_2, y_2, z_2)\) was expressed by Grassmann by the six 2×2 determinants of the array:
as
\[
\begin{bmatrix}
1 & x_1 & y_1 & z_1 \\
1 & x_2 & y_2 & z_2
\end{bmatrix}
\]
\(\text{(5-23)}\)

The Plücker line coordinates \(\{S; S_{ol}\} = \{L, M, N; P, Q, R\}\) are homogeneous since non-zero scalar multiples of all the coordinates still determine the same line [Cra01]. The semicolon inside the coordinate separates the \((L, M, N)\) and \((P, Q, R)\) because their dimensions are different. From equation \((5-19)\), the direction ratios \((L, M, N)\) are related to the distance \(|S|\) by
\[
L^2 + M^2 + N^2 = |S|^2
\]
\(\text{(5-24)}\)

It is useful to unitize these directional values of these homogeneous coordinates to simplify the application;
\[
L = \frac{x_2 - x_1}{|S|}, \quad M = \frac{y_2 - y_1}{|S|}, \quad N = \frac{z_2 - z_1}{|S|}
\]
\(\text{(5-25)}\)

which are also called direction cosines of the lines or unit direction ratios. Therefore, the new set of \(L, M,\) and \(N\) has the new restriction
\[
L^2 + M^2 + N^2 = 1 .
\]
\(\text{(5-26)}\)

Using the Plücker line coordinates, a force \(\mathbf{f}\) is expressed as a scalar multiple \(fS\).

The reference point is selected in such a way that the moment of the force \(\mathbf{f}\) about this reference point, \(\mathbf{m}_0\), can be expressed as a scalar multiple \(fS_{ol}\) where \(S_{ol}\) is the moment vector of the line. So the action of the force applied on the body can be expressed as a scalar multiple of the standard Plucker line coordinates and the coordinates of the force are:
\[
\hat{w} = f \cdot \hat{S} = f \{S; S_{OL}\} = \{f, m_o\}, 
\]  
(5-27)

where \( \{S; S_{OL}\} \) is the Plucker line coordinates and \(|S| = 1\). It is clear that \( f \) is a line bound vector and coordinate independent, while the moment \( m_o \) is origin dependent.

For spatial parallel mechanisms, the forward static analysis consists of computing the resultant wrench \( \hat{w} = \{f; m\} \) due to linearly independent forces generated in the legs acting upon the moving platform. The resultant wrench could be simply expressed:

\[
\hat{w} = \{f; m\} = \{f_1; m_{o1}\} + \{f_2; m_{o2}\} + \{f_3; m_{o3}\} + ... \\
= f_1\{S_1; S_{o1}\} + f_2\{S_2; S_{o2}\} + f_3\{S_3; S_{o3}\} + ... . 
\]  
(5-28)

Equation (5-28) may be written as

\[
\hat{w} = j\lambda, 
\]  
(5-29)

where \( j \) is the Jacobian matrix of the structure. The columns of the matrix are composed of the Plücker coordinates of the lines of the leg connector forces as

\[
j = \begin{bmatrix}
S_1 & S_2 & S_3 & ... \\
S_{o1} & S_{o2} & S_{o3} & ... 
\end{bmatrix}. 
\]  
(5-30)

\( \lambda \) is the column vector with the scalar of the forces as its elements.

\[
\lambda = \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
... 
\end{bmatrix}. 
\]  
(5-31)
Figure 5-6. 3-D Model for special 6-6 parallel mechanism

Figure 5-7. Top view of the special 6-6 configuration
Now consider the proposed special 6-6 parallel passive structure, its 3-D model and plan view are shown in Figures 5-6 and 5-7.

This special 6-6 spatial parallel manipulator has a moveable top platform connected to the fixed platform (ground) by six translational springs acting in-parallel. Each leg has a conventional spring as the compliant component and as acting in the prismatic joint of spherical-prismatic-spherical serial chain. The parallel manipulator with six such compliant legs as well as one top platform and one base platform thus has six degrees of freedom.

The top platform is connected to the base platform by the initially unloaded compliant coupling which restricts any relative spatial motion between the bodies. The corresponding stiffness mapping is thus a one-to-one correspondence that associates a twist describing the relative displacement between the top and base platform with the corresponding resultant wrench provided by the springs between them.

The basic shapes of the top platform and the base platform of this special configuration are equilateral triangles with the top and base triangle side lengths being a and b respectively while pa and pb are the joints separation distance between the pivots. The distance h between the geometrical center of the top platform and the geometrical center of the base platform is selected in such a way that it is qualify the following expression based on the analysis in the previous chapters:

\[ h = a = \frac{b}{2}. \]  

(5-32)

The top platform has 6 spherical pivot points A1, B1, C1, D1, E1, F1 located along its triangle sides and the base platform has 6 spherical pivot points A, B, C, D, E, F along its triangle sides. The six springs are connected respectively and in order to simplify the
process of the analysis, it is necessary to establish the geometry of the system. The following are position vectors that define the directions of the six legs and their magnitudes define the lengths of the six legs:

\[
\begin{align*}
    l_1 &= \|AA\| = \|OA_i - OA\|, \\
    l_2 &= \|BB_i\| = \|OB_i - OB\|, \\
    l_3 &= \|CC_i\| = \|OC_i - OC\|, \\
    l_4 &= \|DD_i\| = \|OD_i - OD\|, \\
    l_5 &= \|EE_i\| = \|OE_i - OE\|, \\
    l_6 &= \|FF_i\| = \|OF_i - OF\|.
\end{align*}
\]

(5-33)

One coordinate system is attached to the base platform, which is fixed to ground, with the z axis pointing upwards to the top platform. The origin of the coordinate system is located right on the geometrical center of the base triangle platform.

Now considering some external wrench is applied on the top platform and the whole system is in static equilibrium with the six spring forces as the moving top platform twists relative to the fixed base platform. Here a wrench \( \hat{\omega} = [f;0] \) can be thought of as a force \([f;0]\) acting though the origin together with a general couple \([0;m]\), adding these sets of coordinates together reproduces the original wrench coordinates.

Similar to the wrench, a twist (written in axial coordinates) can be thought of as a rotation \([0,\phi]\) about a line through the origin together with a general translation \([X,0]\), adding these sets of coordinates together reproduces the original six twist coordinates:

\[
\hat{D} = [X,\phi], \text{[Gri91A]}
\]

In order to maintain the system equilibrium, the top platform moves as the external wrench changes. The incremental change of wrench may be written as \(\delta\hat{\omega} = [\delta f;\delta m]\), and the form of the desired stiffness mapping of the compliant structure may be written as
\[ \varepsilon w = [K] \delta \dot{D}, \quad (5-34) \]

where \([K]\) is the 6x6 stiffness matrix, which relates the incremental twist \((\delta \dot{D} = [\delta X, \delta \phi])\) of the top platform relative to the fixed base platform/ground to the incremental change of the wrench.

As it is specified in the previous chapters, the geometrical properties of the platform are known: the leg lengths are measured, while the dimensional values of the top and base triangles are invariant and pre-defined. The individual spring constants of the springs are also known.

In order to determine the mapping of (5-34), it is necessary to first differentiate (5-29) [Pig98], which could also be written as

\[
\begin{bmatrix}
\lambda f \\
j m
\end{bmatrix}
= \dot{w} = \hat{j} \hat{\kappa} = j,
\]

\[ (5-35) \]

where \(k_i\) is the individual spring constant of the \(i^{th}\) spring, and \(l_i, l_{0i}\) are the current length and free length of the \(i^{th}\) spring respectively, so that \(k_i(l_i - l_{0i})\) is the force in the \(i^{th}\) leg due to the extension of the \(i^{th}\) spring. The Jacobian matrix \(j\) is:

\[
j = \begin{bmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
OA \times S_1 & OB \times S_2 & OC \times S_3 & OD \times S_4 & OE \times S_5 & OF \times S_6
\end{bmatrix},
\]

\[ (5-36) \]

where, \(S_i\) is the direction cosine of the axial line of the \(i^{th}\) leg. These direction cosines could be calculated and unitized via (5-25):
The lower three rows of the $6 \times 6$ matrix in (5-36) are the moments of the six lines relative to the reference point O. Because $\mathbf{OA}$, $\mathbf{OB}$, $\mathbf{OC}$, $\mathbf{OD}$, $\mathbf{OE}$ and $\mathbf{OF}$ are all fixed line vectors on the base platform triangle, and are constant, this matrix is solely a function of the direction cosines of the six legs.

The differential of equation (5-35) is expressed as

$$
\begin{pmatrix}
\delta f \\
\delta m_0
\end{pmatrix} =
\begin{bmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
OA \times S_1 & OB \times S_2 & OC \times S_3 & OD \times S_4 & OE \times S_5 & OF \times S_6
\end{bmatrix}
\begin{bmatrix}
k_1 \delta l_1 \\
k_2 \delta l_2 \\
k_3 \delta l_3 \\
k_4 \delta l_4 \\
k_5 \delta l_5 \\
k_6 \delta l_6
\end{bmatrix}

+ \begin{bmatrix}
\delta S_1 & \delta S_2 & \delta S_3 & \delta S_4 & \delta S_5 & \delta S_6 \\
OA \times \delta S_1 & OB \times \delta S_2 & OC \times \delta S_3 & OD \times \delta S_4 & OE \times \delta S_5 & OF \times \delta S_6
\end{bmatrix}
\begin{bmatrix}
k_1 (l_1 - l_{01}) \\
k_2 (l_2 - l_{02}) \\
k_3 (l_3 - l_{03}) \\
k_4 (l_4 - l_{04}) \\
k_5 (l_5 - l_{05}) \\
k_6 (l_6 - l_{06})
\end{bmatrix}
$$

(5-37)

Here $\delta l_i$ is related to $\delta D$ by the following expression:

$$
\delta l = [\mathbf{j}]^T \delta \hat{D}
$$

(5-38)

where $[\mathbf{j}]$ is the Jacobian matrix of the parallel structure. The column of this $6 \times 6$ matrix is the line coordinates of the $i^{th}$ leg [Gri91A].

Griffis demonstrated that in order to relate $\delta S_i$ with $\delta \hat{D}$, each leg needs to have two derivatives, which are perpendicular to the axis of that leg at its base point, to describe the $\delta S_i$. Because each leg is individually connected to the base platform and top platform via a ball joint and a hook joint, it only possess two degrees of freedom: two
rotations whose rotation axis are mutual perpendicular. Griffis also provides the stiffness
mapping analysis for a 3-3 octahedral spatial platform: the analysis is based on Screw
theory and utilizes some geometrical restrictions and configurations of the octahedral
structure. Three examples include that two legs sharing one concentric ball joint at one
fixed pivot, four legs sharing one triangle side and the pivots are located on the vertexes
of the triangles. Although the analysis result is exclusively for the 3-3 octahedral
mechanism, the conceptual approach is applicable to the general spatial stiffness analysis.
The stiffness mapping of the special 6-6 parallel manipulator is determined by following
the same procedure.

The general expression of the stiffness matrix for the special 6-6 parallel platform
is presented as

\[
[K] = [j][k_i][j]^T + [\delta_i^0][k_i(1-\rho_i)][\delta_i^0]^T + [\delta_i^a][k_i(1-\rho_i)][\delta_i^a]^T + [\delta_i^0][k_i(1-\rho_i)][V_\theta]^T + [\delta_i^a][k_i(1-\rho_i)][V_a]^T,
\]  
(5-39)

where 
\[
[j] = \begin{bmatrix}
S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\
OA \times S_1 & OB \times S_2 & OC \times S_3 & OD \times S_4 & OE \times S_5 & OF \times S_6
\end{bmatrix}
\]  
(5-40)

\[
[k_i] = \begin{bmatrix}
k_1 & 0 & 0 & 0 & 0 & 0 \\
0 & k_2 & 0 & 0 & 0 & 0 \\
0 & 0 & k_3 & 0 & 0 & 0 \\
0 & 0 & 0 & k_4 & 0 & 0 \\
0 & 0 & 0 & 0 & k_5 & 0 \\
0 & 0 & 0 & 0 & 0 & k_6
\end{bmatrix}
\]

\[
[k_i(1-\rho_i)] = \begin{bmatrix}
k_1(1-\rho_i) & 0 & 0 & 0 & 0 & 0 \\
0 & k_2(1-\rho_2) & 0 & 0 & 0 & 0 \\
0 & 0 & k_3(1-\rho_3) & 0 & 0 & 0 \\
0 & 0 & 0 & k_4(1-\rho_4) & 0 & 0 \\
0 & 0 & 0 & 0 & k_5(1-\rho_5) & 0 \\
0 & 0 & 0 & 0 & 0 & k_6(1-\rho_6)
\end{bmatrix}
\]  
(5-41)
and where $\rho_i = l_{0i}/l_i$ are dimensionless ratios incorporated to simplify the expressions.

Further,

$$
[\delta \mathbf{j}_b] = \begin{bmatrix}
\delta S_1^\theta & \delta S_2^\theta & \delta S_3^\theta & \delta S_4^\theta & \delta S_5^\theta & \delta S_6^\theta \\
\mathbf{OA} \times \delta S_1^\theta & \mathbf{OB} \times \delta S_2^\theta & \mathbf{OC} \times \delta S_3^\theta & \mathbf{OD} \times \delta S_4^\theta & \mathbf{OE} \times \delta S_5^\theta & \mathbf{OF} \times \delta S_6^\theta 
\end{bmatrix} \quad (5-42)
$$

and

$$
[\delta \mathbf{j}_a] = \begin{bmatrix}
\delta S_1^\alpha & \delta S_2^\alpha & \delta S_3^\alpha & \delta S_4^\alpha & \delta S_5^\alpha & \delta S_6^\alpha \\
\mathbf{OA} \times \delta S_1^\alpha & \mathbf{OB} \times \delta S_2^\alpha & \mathbf{OC} \times \delta S_3^\alpha & \mathbf{OD} \times \delta S_4^\alpha & \mathbf{OE} \times \delta S_5^\alpha & \mathbf{OF} \times \delta S_6^\alpha 
\end{bmatrix}, \quad (5-43)
$$

where $\delta S_i^\theta$, $\delta S_i^\alpha$ are all unitized direction cosine vectors for the derivatives of the $S_i$ vector; the three vectors are mutually perpendicular at the pivot point.

Next, six unit vectors are defined as intermediate variables describing the directional information of the base platform triangle sides as

$$
u_1 = \frac{-\mathbf{EA}}{||\mathbf{EA}||}, \quad u_2 = \frac{\mathbf{AC}}{||\mathbf{AC}||}, \quad u_3 = \frac{-\mathbf{AC}}{||\mathbf{AC}||}, \quad u_4 = \frac{-\mathbf{EC}}{||\mathbf{EC}||}, \quad u_5 = \frac{\mathbf{EC}}{||\mathbf{EC}||}, \quad u_6 = \frac{\mathbf{EA}}{||\mathbf{EA}||} \quad (5-44)
$$

$\mathbf{V}_i$ is given by the follow expression:

$$
\mathbf{V}_i = \frac{\mathbf{u}_i \times S_i}{||\mathbf{u}_i \times S_i||} \quad (5-45)
$$

$$
\delta S_i^\prime = \mathbf{V}_i \quad \text{and} \quad \delta S_i^{\prime\prime} = \mathbf{V}_i \times S_i \quad (5-46)
$$

$$
[\mathbf{V}_a] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
l_1 \cdot \mathbf{V}_1 & l_2 \cdot \mathbf{V}_2 & l_3 \cdot \mathbf{V}_3 & l_4 \cdot \mathbf{V}_4 & l_5 \cdot \mathbf{V}_5 & l_6 \cdot \mathbf{V}_6 
\end{bmatrix} \quad (5-47)
$$

$$
[\mathbf{V}_b] = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
l_1 \cdot S_1 \times \mathbf{V}_1 & l_2 \cdot S_2 \times \mathbf{V}_2 & l_3 \cdot S_3 \times \mathbf{V}_3 & l_4 \cdot S_4 \times \mathbf{V}_4 & l_5 \cdot S_5 \times \mathbf{V}_5 & l_6 \cdot S_6 \times \mathbf{V}_6 
\end{bmatrix} \quad (5-48)
$$

where $\mathbf{0}$ is a zero vector.

Substituting all the necessary components in equation (5-45) yields the global stiffness matrix of the special 6-6 parallel passive mechanism via the stiffness mapping analysis.
It is clear that the stiffness matrix can be written as the sum of five different matrices, the first three matrices are symmetric, while the last two matrices are asymmetric, so the overall stiffness matrix is asymmetric. It is obvious that the stiffness matrix is dependent on the selection of the coordinate system, so by using different coordinate systems, it might be possible to have a shorter or longer expression, but the change of coordinate system does not change the stiffness properties of the system, such as the rank of the matrix, or the eigenvectors and eigenvalues of the stiffness matrix (which are also called corresponding eigen-screw of stiffness and eigen-stiffness) [Gri91B, Sel02]. It is also intuitive that the stiffness property of a given compliant manipulator should not change just by using a different coordinate system, and neither could it be changed to be a symmetric matrix by choosing a different coordinate system.

In the five matrices constituting the global stiffness matrix, only the first matrix does not have the diagonal matrix of \( [k_i (1 - \rho_i)] \). When the parallel compliant manipulator sustains a relatively small deflection (and a correspondingly small twist), and each leg also endures a relatively small amount of axial force, therefore each leg length is close to its initial length.

Then because \( \rho_i = \frac{l_{0i}}{l_i} \approx \frac{l_{0i}}{l_{0i}} = 1 \), all the other four matrices’ components who contain the diagonal matrix of \( [k_i (1 - \rho_i)] \) are so close to the zero matrix that their effects can be neglected and only the first part of the global stiffness matrix is left. With these simplifications, the stiffness matrix is a positive definite symmetric matrix.

Consider an example parallel passive platform with the special 6-6 configuration shown in Figure 5-7) that has the following specifications:
a = 60 mm, b = 120 mm,
p=0.233,

\[ k_i = 20 \text{ lb/ in} = 3.5003 \text{ N/mm}. \]

The stiffness matrix of the system will be calculated for the case when it is sustaining a small external wrench in static equilibrium and the top platform is slightly deflected from its original position. The corresponding Jacobian matrix of the structure is calculated as

\[
\begin{bmatrix}
-0.65457 & -0.23529 & 0.19761 & -0.23529 & 0.45696 & 0.47059 \\
0.14974 & 0.40754 & -0.64174 & -0.40754 & 0.492 & 0 \\
0.74102 & 0.88235 & 0.74102 & 0.88235 & 0.74102 & 0.88235 \\
-44.461 & -40.588 & 0 & 12.353 & 44.461 & 28.235 \\
\end{bmatrix}
\]

where each column is the Plücker line coordinates of the axis for each leg, the upper three numbers are unitless, the lower three numbers have the unit of mm.

The corresponding stiffness matrix is calculated as

\[
\begin{bmatrix}
3.5301 & 0 & 0 & -22.256 & 242.75 & 0 \\
0 & 3.5301 & 0 & -242.75 & -22.256 & 0 \\
0 & 0 & 13.942 & 0 & 0 & 44.511 \\
-22.256 & -242.75 & 0 & 22930 & 0 & 0 \\
242.75 & -22.256 & 0 & 0 & 22930 & 0 \\
0 & 0 & 44.511 & 0 & 0 & 4758.7 \\
\end{bmatrix}
\]

In this matrix, the units are also different. The four 3×3 sub-matrices have the following units

\[
\begin{bmatrix}
\frac{N}{mm} & N \\
N & \frac{N}{mm} \\
\end{bmatrix}
\]

Introduce the equation (5-38) into the stiffness mapping (5-34):
\[ \delta \hat{w} = [K] \delta \hat{D} = [K][j]^T \delta l \]  \hspace{1cm} (5-51)

For a given set of minute deflections of the legs, the above equation can be used to calculate the applied external wrench.

\[
\hat{w} = \begin{bmatrix}
-2.2912 & -0.8236 & 0.69168 & -0.8236 & 1.5995 & 1.6472 \\
0.52414 & 1.4265 & -2.2463 & -1.4265 & 1.7222 & 0 \\
2.5938 & 3.0885 & 2.5938 & 3.0885 & 2.5938 & 3.0885 \\
-89.852 & -32.097 & 179.7 & 139.09 & 89.852 & -106.99 \\
-155.63 & -142.07 & 0 & 43.239 & 155.63 & 98.832 \\
-47.921 & 57.061 & -47.921 & 57.061 & -47.921 & 57.061
\end{bmatrix} \begin{bmatrix} \Delta l_1 \\ \Delta l_2 \\ \Delta l_3 \\ \Delta l_4 \\ \Delta l_5 \\ \Delta l_6 \end{bmatrix} \hspace{1cm} (5-52)

If the top platform of the special 6-6 parallel passive platform is deflected much from its unladed home position, then the compliance matrix can be calculated from equation (5-39).
CHAPTER 6
FORWARD ANALYSIS FOR SPECIAL 6-6 PARALLEL PLATFORM

The kinematic forward analysis is very important to control applications. In this chapter, the forward kinematic analysis for parallel 3-3 manipulator is introduced. Then a geometrical method, patented by Duffy and Griffis [Gri93], of determining the equivalent 3-3 parallel structure for a special 6-6 configuration mechanism is described.

**Forward Kinematic Analysis for a 3-3 Platform**

It is necessary to determine the position and orientation of the end effector of the manipulator. Given the geometrical properties of the parallel platform, including leg length, connector pivots locations and etc., the forward kinematic analysis determines the relation between the end effector pose and its geometrical properties. As stated in the previous chapters, the forward kinematic analysis is complicated compared to the reverse kinematic analysis. This problem is also geometrically equivalent to the problem of finding out ways to place a rigid body such that six of its given points lie on six given spheres.

During the late 80’s and early 90’s of last century, many researchers had worked on the forward kinematic analysis of parallel manipulators. Many different approaches such as closed-form solutions of special cases, numerical schemes, and analytical approaches were considered for both special cases and general 6-6 configurations. Hunt [Hun98] studied the geometry and mobility of the 3-3 in-parallel manipulator. Wen and Liang [Wen94] used an analytical approach and solved the problem for the 6-6 Stewart platform with a planar base and platform by reducing the kinematic equations into a uni-variate
polynomial and concluded that the upper bound of the solutions for forward kinematic problems is 40 for this class. Merlet [Mer92, Mer00] had studied direct kinematic solutions for general 6-6 parallel platform and special cases with additional sensors.

Various researchers have stated that for a general 6-6 in-parallel platform and with a given set of fixed leg lengths, it is possible to assemble it in 40 different configurations [Das00, Hun98]. It is not likely that all of the 40 configurations are real and applicable. Due to the difficulty of the forward kinematic analysis for the general 6-6 in-parallel platform, it is worthwhile to consider some special cases with reduced complexity. The simplest form of this class is first analyzed.

Although there are different configurations and designs for a simple spatial parallel platform, the symmetric 3-3 parallel platform shown in Figure 6-1 is among the most important and widely analyzed structures.

This 3-3 parallel platform is also called an “octahedron”. It has eight triangular faces, six vertices and twelve edges. Four edges are concurrent at each vertex. Every vertex is contained in four faces. The device has all six degrees of freedom, ignoring the trivial freedom of each leg rotating about its own axis.

The forward kinematic analysis for the 3-3 in-parallel manipulator was first solved by Griffis and Duffy [Gri89] who showed that the position and orientation of the top platform can be determined with respect to the base when given the lengths of the six connectors as well as the geometry of the connection points on the top and base. The solution is a closed-form solution and is based on the analysis of the input/output relationship of three spherical four bar mechanisms. The three spherical four bar mechanisms (see Figure 6-3) result in three equations in the tan-half angle of the three
unknowns $\theta_x$, $\theta_y$, and $\theta_z$. These tan-half angles are simply named $x$, $y$, and $z$. Elimination of the variables $y$ and $z$ results in the following polynomial which contains the variable $x$ as its only unknown [Gri89].

Figure 6-1. 3-D drawing of the 3-3 in-parallel platform

\[ E = \alpha^2 - 4\beta \rho_1 \rho_2, \]  

(6-1)

where
\[
\alpha = 2A_3B_2a_2c_1^2c_2 - 4A_3B_3c_1^2b_2^2 + 2A_3C_3a_1c_1c_2^2 - 4A_3C_3a_1c_2^2b_1^2 \\
- 2A_3E_3a_4c_1c_2 + 4A_3E_3a_5c_2^2 - 4A_3E_3b_1c_3c_2 \\
- 8A_3E_3b_1^2b_2^2 - 2A_3D_3c_1b_1b_2 - 2B_3C_3a_1c_2c_3 \\
+ 4B_3C_3a_2c_2c_1 - 8B_3C_3b_2^2 + 2B_3E_3c_2c_1c_3 - 4B_3E_3c_2^2b_1^2 - 2B_3D_3a_2b_2c_1b_2 \\
+ 2C_3E_3a_1^2c_2c_2 - 2C_3D_3a_1c_2b_2c_1b_2 \\
- 2D_3E_3a_2b_1b_2 - D_3^2c_1a_2c_2 - A_3^2c_1^2c_2^2 \\
- B_3^2c_1^2a_2^2 - C_3^2a_1^2c_2^2 - E_3^2a_1^2a_2^2
\]

\[
\beta = 4A_3E_3b_1b_2 + A_3D_3c_1c_2 + D_3E_3a_1a_2 \\
- 4B_3C_3b_1b_2 - B_3D_3c_1a_2 - C_3D_3a_1c_2
\]

\[
\rho_1 = b_1^2 - a_1c_1 \\
\rho_2 = b_2^2 - a_2c_2
\]

\[
a_1 = A_1x_1^2 + C_1, \quad b_1 = 0.5D_1x, \quad c_1 = B_1x^2 + E_1 \\
a_2 = A_2x_2^2 + B_2, \quad b_2 = 0.5D_2x, \quad c_2 = C_2x^2 + E_2
\]

The coefficients \(A_i, \ldots, E_i, i = 1, 2, 3\) are expressed in terms of known quantities. Their values for the generic spherical four-bar mechanism shown in Figure 6-2 may be written as

\[
A = s_{12}c_{41}s_{34} - s_{12}s_{41}c_{34} - s_{41}s_{34}c_{12} + c_{23} - c_{12}c_{41}c_{34} \\
B = -s_{12}c_{41}s_{34} - s_{12}s_{41}c_{34} + s_{41}s_{34}c_{12} + c_{23} - c_{12}c_{41}c_{34} \\
C = -s_{12}c_{41}s_{34} + s_{12}s_{41}c_{34} - s_{41}s_{34}c_{12} + c_{23} - c_{12}c_{41}c_{34} \\
D = -4s_{12}s_{34} \\
E = s_{12}c_{41}s_{34} + s_{12}s_{41}c_{34} + s_{41}s_{34}c_{12} + c_{23} - c_{12}c_{41}c_{34}
\]

where \(s_{ij} = \sin(\alpha_{ij})\) and \(c_i = \cos(\alpha_i)\). For this problem, the coefficients are obtained by replacing the vertex as the origin of the three spherical four-bar linkages (shown in Figure 6-3) with points o, p, and q respectively. The corresponding mapping of the angles of the spherical four-bar linkages is shown in table 6-1.
Table 6-1. Mappings of angles of spherical four bar mechanism.

<table>
<thead>
<tr>
<th>Origin</th>
<th>o</th>
<th>p</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output: $a_{12}$</td>
<td>$\angle qo!r$</td>
<td>$\angle ops$</td>
<td>$\angle pqr$</td>
</tr>
<tr>
<td>Coupler: $a_{23}$</td>
<td>$\angle ros$</td>
<td>$\angle spt$</td>
<td>$\angle tqr$</td>
</tr>
<tr>
<td>Input: $a_{34}$</td>
<td>$\angle sop$</td>
<td>$\angle tpq$</td>
<td>$\angle rqo$</td>
</tr>
<tr>
<td>Ground: $a_{41}$</td>
<td>$\angle poq$</td>
<td>$\angle pqo$</td>
<td>$\angle qqp$</td>
</tr>
</tbody>
</table>

The resulting solution shown above is an eighth degree polynomial in the square of one variable which yields a maximum of sixteen solution poses for any given set of connector lengths and base-top triangles lengths. Eight solutions were reflected about the
plane formed by the base connector points. There can be up to 16 real solutions and therefore, there can be 0, 2, 4, 6 or 8 pairs of real, reflected solutions.

Figure 6-3. 3-3 in-parallel mechanism with three spherical four-bar linkages, Griffis [Gri89]

**Forward Kinematic Analysis for Special 6-6 In-Parallel Platform**

The configuration of the special 6-6 in-parallel platform is carefully chosen such that the forward kinematic analysis can use a method similar to the method for the 3-3 in-parallel platform. This section will show the geometric relationship between the special 6-6 in-parallel platform and an equivalent 3-3 platform and how to solve the direct kinematic analysis problem by using this conversion. The relationship between a Special 6-6 platform and its equivalent 3-3 platform was discovered by Griffis and Duffy [Gri93] and is presented here for completeness as the forward analysis is an important part of the control algorithm for the force control device developed in this dissertation.
A Special 6-6 platform is defined as one which is geometrically reducible to an
equivalent 3-3 platform. Figure 6-4 depicts a perspective and plan view of a 6-6 platform
where the leg connector points R₀, S₀, and T₀ lie along the edges of the triangle defined
by points O₀, P₀, and Q₀ and the leg connector points O₁, P₁, and Q₁ lie along the edges of
the triangle defined by the points R₁, S₁, and T₁. The objective here is to determine the
distance between the pairs of points O₀ - R₁, O₀ - S₁, P₀ - S₁, P₀ - T₁, Q₀ - T₁, and Q₀ - R₁.
These distances represent the leg connector lengths for an equivalent 3-3 platform.

Throughout this analysis, the notation \( m_i n_j \) will be used to represent the distance
between the two points \( M_i \) and \( N_j \) and the notation \( m_i n_j \) will represent the vector from
point \( M_i \) to \( N_j \). Using this notation, the problem statement can be written as:

given:  \( o_0o_1, p_0p_1, q_0q_1, r_0r_1, s_0s_1, t_0t_1 \)  \hspace{1cm} \text{connector lengths for Special 6-6 platform}
\( o_0p_0, p_0q_0, q_0o_0, o_0s_0, p_0t_0, q_0r_0 \)  \hspace{1cm} \text{base triangle parameters}
\( r_1s_1, s_1t_1, t_1r_1, r_1o_1, s_1p_1, t_1q_1 \)  \hspace{1cm} \text{top triangle parameters}

find:  \( o_0r_1, o_0s_1, p_0s_1, p_0t_1, q_0t_1, q_0r_1 \)  \hspace{1cm} \text{connector lengths for equivalent 3-3 platform}

Obviously it is the case that

\[
\begin{align*}
s_0p_0 &= o_0p_0 - o_0s_0 \quad ; \quad t_0q_0 = p_0q_0 - p_0t_0 \quad ; \quad r_0o_0 = q_0o_0 - q_0r_0 \\
o_1s_1 &= r_1s_1 - r_1o_1 \quad ; \quad p_1t_1 = s_1t_1 - s_1p_1 \quad ; \quad q_1r_1 = t_1r_1 - t_1q_1 .
\end{align*}
\]  \hspace{1cm} (6-2)

The six leg connectors of the equivalent 3-3 platform define an octahedron, i.e.,
there are eight triangular faces; the top and bottom platform triangles and six faces
defined by two intersecting connectors and an edge of either the base or top platform.
The solution begins by first defining angles \( \beta \) and \( \gamma \) which, for each of the six faces
defined by intersecting connectors, defines the angle in the plane between a connector of
the Special 6-6 platform and an edge of either the top or base platform as appropriate.

Figure 6-4. Special 6-6 platform and equivalent 3-3 platform, A) Perspective view, B) Plan view

Figure 6-5 shows the angles $\gamma_1$ and $\beta_1$. The angle $\gamma_2$ is defined as the angle between
$r_0r_1$ and $r_0o_0$ and the angle $\gamma_3$ is defined as the angle between $s_0s_1$ and $s_0o_0$. Similarly, the
angle $\beta_2$ is defined as the angle between $q_1t_1$ and $q_1q_0$ and the angle $\beta_3$ is defined as the angle between $o_1s_1$ and $o_1o_0$.

Figure 6-5. Definition of angles $\beta$ and $\gamma$

Figure 6-6 shows a planar triangle that is defined by the vertex points $G_1$, $G_2$, and $G_3$. A point $G_0$ is defined as a point on the line defined by $G_1$ and $G_2$ and the angle $\varphi$ is shown as the angle between $g_0g_2$ and $g_0g_3$. The cosine law for the triangle defined by points $G_0$, $G_2$, and $G_3$ yields

$$\cos \varphi = \frac{b^2 + c^2 - \ell_b^2}{2bc} \quad (6-3)$$

Next consider that the triangle lies in a plane defined by $u$ and $v$ coordinate axes. The coordinates of any point $G_i$ may then be written as $(u_i, v_i)$. The distance $\ell_a$ may then be expressed as
\[ \ell_a^2 = (u_3 - u_1)^2 + (v_3 - v_1)^2 \]
\[ = (a + c \cos \phi)^2 + (c \sin \phi)^2 \]
\[ = c^2 (\sin^2 \phi + \cos^2 \phi) + a^2 + 2ac \cos \phi \cdot \]
\[ = a^2 + c^2 + 2ac \cos \phi \quad (6-4) \]

Figure 6-6. Planar triangle

Substituting (6-3) into (6-4) and simplifying gives

\[ \ell_a^2 = a^2 + c^2 + \frac{a}{b} (b^2 + c^2 - \ell_b^2) \]
\[ = \frac{1}{b} (ba^2 + b c^2 + ab^2 + a c^2 - a \ell_b^2) \quad (6-5) \]

Regrouping this equation gives

\[ b \ell_a^2 + a \ell_b^2 = (a + b)c^2 + ab^2 + a^2b = (a + b)(c^2 + ab) \quad (6-6) \]

The result of equation (6-6) can be applied to each of the six side faces of the octahedron defined by the leg connectors of the equivalent 3-3 platform. The parameter substitutions are shown in Table 6-2.
Table 6-2. Parameter substitutions

<table>
<thead>
<tr>
<th>$\Phi$</th>
<th>$G_0$</th>
<th>$G_1$</th>
<th>$G_2$</th>
<th>$G_3$</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>$\ell_a$</th>
<th>$\ell_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>$P_1$</td>
<td>$T_1$</td>
<td>$S_1$</td>
<td>$P_0$</td>
<td>$p_1t_1$</td>
<td>$p_1s_1$</td>
<td>$p_0p_1$</td>
<td>$p_0t_1$</td>
<td>$p_0s_1$</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>$Q_1$</td>
<td>$R_1$</td>
<td>$T_1$</td>
<td>$Q_0$</td>
<td>$q_1r_1$</td>
<td>$t_1q_1$</td>
<td>$q_0q_1$</td>
<td>$q_0r_1$</td>
<td>$q_0l_1$</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>$O_1$</td>
<td>$S_1$</td>
<td>$R_1$</td>
<td>$O_0$</td>
<td>$s_1o_1$</td>
<td>$o_1r_1$</td>
<td>$o_0o_1$</td>
<td>$o_0s_1$</td>
<td>$o_0r_1$</td>
</tr>
<tr>
<td>$\gamma_1$</td>
<td>$T_0$</td>
<td>$P_0$</td>
<td>$Q_0$</td>
<td>$T_1$</td>
<td>$p_0t_0$</td>
<td>$t_0q_0$</td>
<td>$t_0t_1$</td>
<td>$p_0t_1$</td>
<td>$q_0t_1$</td>
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<tr>
<td>$\gamma_2$</td>
<td>$R_0$</td>
<td>$Q_0$</td>
<td>$O_0$</td>
<td>$R_1$</td>
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<td>$r_0r_1$</td>
<td>$q_0r_1$</td>
<td>$o_0r_1$</td>
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<tr>
<td>$\gamma_3$</td>
<td>$S_0$</td>
<td>$O_0$</td>
<td>$P_0$</td>
<td>$S_1$</td>
<td>$o_0s_0$</td>
<td>$s_0p_0$</td>
<td>$s_0s_1$</td>
<td>$o_0s_1$</td>
<td>$p_0s_1$</td>
</tr>
</tbody>
</table>

Using Table 6-2 to perform appropriate parameter substitutions in (A-5) will result in six equations that can be written in matrix form as

$$ A q = M, $$

where

$$ A = \begin{bmatrix}
p_1s_1 & p_1t_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t_1q_1 & q_1r_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & o_1r_1 & s_1o_1 & 0 & 0 \\
t_0q_0 & 0 & 0 & p_0t_0 & 0 & 0 & 0 & 0 \\
0 & 0 & r_0o_0 & 0 & 0 & q_0r_0 & 0 & 0 \\
0 & o_0s_0 & 0 & 0 & s_0p_0 & 0 & 0 & 0
\end{bmatrix} \quad q = \begin{bmatrix} p_0t_1^2 \\
p_0s_1^2 \\
q_0r_1^2 \\
q_0t_1^2 \\
o_0s_1^2 \\
o_0r_1^2 \end{bmatrix} \quad M = \begin{bmatrix} M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5 \\
M_6 \end{bmatrix} $$

where

- $M_1 = (p_1t_1 + p_1s_1)(p_0p_12 + p_1t_1 + p_1s_1)$,
- $M_2 = (q_1r_1 + t_1q_1)(q_0q_12 + q_1r_1t_1q_1)$,
- $M_3 = (s_1o_1 + o_1r_1)(o_0o_12 + s_1o_1 + o_1r_1)$,
- $M_4 = (p_0t_0 + t_0q_0)(t_0t_12 + p_0t_0 + t_0q_0)$,
- $M_5 = (q_0r_0 + r_0o_0)(r_0r_12 + q_0r_0 + r_0o_0)$,
- $M_6 = (o_0s_0 + o_0p_0)(s_0s_12 + o_0s_0 + o_0p_0)$.  

(6-9)
Then these equations are manipulated such that dividing the first rows of \( A \) and \( M \) by \( p_1 t_1 + p_1 s_1 = s_1 t_1 \), the second rows by \( q_1 r_1 + t_1 q_1 = r_1 t_1 \), and so on to obtain

\[
A'q = M',
\]

where

\[
A' = \begin{bmatrix}
p & 1-p & 0 & 0 & 0 & 0 \\
0 & 0 & q & 1-q & 0 & 0 \\
0 & 0 & 0 & 0 & w & 1-w \\
1-t & 0 & 0 & t & 0 & 0 \\
0 & 0 & 1-r & 0 & 0 & r \\
0 & s & 0 & 0 & 1-s & 0
\end{bmatrix},
\]

\[
M' = \begin{bmatrix}
p_0 p_1^2 + p_1 t_1 p_1 s_1 \\
q_0 q_1^2 + q_1 r_1 t_1 q_1 \\
o_0 o_1^2 + s_1 o_1 o_1 r_1 \\
t_0 t_1^2 + p_0 t_0 t_0 q_0 \\
r_0 r_1^2 + q_0 r_0 r_0 o_0 \\
s_0 s_1^2 + o_0 s_0 s_0 p_0
\end{bmatrix},
\]

with

\[
p = \frac{p_1 s_1}{p_1 s_1 + p_1 t_1} = \frac{p_1 s_1}{s_1 t_1},
\]

\[
t = \frac{t_0 p_0}{t_0 p_0 + t_0 q_0} = \frac{t_0 p_0}{p_0 q_0},
\]

\[
q = \frac{q_1 t_1}{q_1 t_1 + q_1 r_1} = \frac{q_1 t_1}{r_1 t_1},
\]

\[
r = \frac{r_0 q_0}{r_0 q_0 + r_0 o_0} = \frac{r_0 q_0}{q_0 o_0},
\]

\[
w = \frac{o_1 r_1}{o_1 r_1 + o_1 s_1} = \frac{o_1 r_1}{r_1 s_1},
\]

\[
s = \frac{s_0 o_0}{s_0 o_0 + s_0 p_0} = \frac{s_0 o_0}{o_0 p_0}.
\]

Note that \( 0 \leq p, q, w, t, r, s \leq 1 \).

The matrix \( A' \) is dimensionless and depends only on the ratios of where the connections occur along the sides of the upper and lower bases. Matrix \( A' \) is also independent of the shapes of the upper and lower triangles. Now all the terms in matrix \( A' \) are known as they are dimensionless ratios expressed in terms of given distances of the top and bottom platform. All the terms in vector \( M' \) are also known as they are expressed in terms of given dimensions as well as the square of the lengths of the six leg connectors of the Special 6-6 platform. Thus the square of the lengths of the connectors for the equivalent 3-3 platform may be determined as:
\[ q = A^{-1}M' \]  

Further more, the determinant of \( A' \) is:

\[
\det(A) = (1 - p)(1 - q)(1 - w)(1 - t)(1 - r)(1 - s) - pqwtrs
\]  

It is interesting to note that the matrix \( A' \) is singular if and only if the following condition is satisfied:

\[
\begin{align*}
 s_0p_0 & \quad t_0q_0 & \quad o_0r_0 \\
 s_1p_1 & \quad t_1q_1 & \quad o_1r_1
\end{align*}
\]

\[
= \begin{align*}
 q_0r_0 & \quad o_0s_0 & \quad p_0t_0 \\
 q_1r_1 & \quad o_1s_1 & \quad p_1t_1
\end{align*}
\]  

\[ (6-13) \]

One case where this could happen would be if all the middle connector pointes \( S_0, T_0, R_0, O_1, P_1, \) and \( Q_1 \) are located at the midpoints of the sides of the upper and lower triangles. We can also conclude that for \( A' \) to be singular, if some of the parameters \( p, q, w, t, r, s \) are less than \( \frac{1}{2} \) then some of the other parameters must be greater than \( \frac{1}{2} \). Thus we immediately conclude that if all of the parameters are less than \( \frac{1}{2} \) (as is the case in Figure A-1), then \( A' \) is non-singular. The same statement of course is true if all the parameters are greater than \( \frac{1}{2} \).

Once the leg length dimensions of the equivalent 3-3 platform are determined, the forward analysis of the 3-3 device as described in Section 6.1 is used to determine the position and orientation of the top platform relative to the base.
In this chapter, the individual components of the designed manipulator are tested to validate the machine design and theoretical analysis presented in the previous chapters. In order to calibrate each leg’s stiffness, first the force sensor is calibrated. An optical encoder measures the compliant displacement of the leg and the load cell records the applied force. The force-displacement relationship is provided for each leg and the results are analyzed. The parallel platform is then assembled for the 6 DOF wrench measuring testing. Several experiments are presented and the results are analyzed. At the end of this chapter, a summary of the work is followed by conclusions and future work suggestions.

**Calibration Experiment for the Force Sensor**

In order to get the force/displacement relationship of the compliant connectors, the detected force signal and the corresponding displacement signal should be recorded simultaneously during the individual leg calibration experiment.

A load cell is used to measure the axial force applied on the individual leg. The rated capacity of the load cell is 5 lb, (22.246 N) and the output of the load cell is an analog voltage signal, ranging from -5--+5 Vdc. The rated output is 2mV/V ± 20%. It is necessary to calibrate the load cell first to ensure the validity and accuracy of the following experiments’ results.
Figure 7-1. Load cell calibration experiment

Figure 7-2. Load cell calibration experiment - opposite direction
The load cell output analog signal is connected to an A/D port of a multi-functional I/O board and the value of the voltage is recorded with the corresponding known force, which is provided by a set of standard weights for accurate force calibration. The mapping of the force/voltage relationship is shown in Figure 7-1 and Figure 7-2.

Five additional sets of experimental data were analyzed and the plots are very similar, so only one plot is shown here. The linear equation that relates the applied force to the measured voltage can be written as

\[ y = k_1 x, \quad (7-1) \]

where the coefficient \( k_1 \) is determined to equal 0.00225 Volt/gram. The linear regression statistics are shown in Table 7-1.

<table>
<thead>
<tr>
<th>Regression Statistics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiple R</td>
<td>0.99915584</td>
</tr>
<tr>
<td>R Square</td>
<td>0.998231949</td>
</tr>
<tr>
<td>Adjusted R Square</td>
<td>0.966981949</td>
</tr>
<tr>
<td>Standard Error</td>
<td>0.059572297(V)</td>
</tr>
</tbody>
</table>

From the table and the figure above, the load cell shows very high linearity and is adequate to be used in the following experiments.

**Individual Leg Calibration Experiment**

After calibrating the load cell, it is necessary to calibrate the compliant connectors and identify the stiffness property for each of them. During the experiment, the compliant connector is fixed vertically and is attached to the load cell in such a way that the applied external force causing either elongation or compression can be detected properly. The two physical quantities, force and displacement, are detected by the load
cell and the optical encoder. The optical encoder is attached on the compliant leg both for the calibration experiment and for the designed operation.

Axial force is applied on the top of the compliant leg with variable magnitude and direction in order to get the stiffness mapping of the leg for both compression and elongation. The resolution of the optical encoder is 1000 counts/inch.

The individual plot of the stiffness mapping for each of the six leg connectors is shown in Figures 7-3 through 7-8.

Figure 7-3. Calibration plot for leg-1
Figure 7-4. Calibration plot for leg-2

Figure 7-5. Calibration plot for leg-3
Figure 7-6. Calibration plot for leg-4

Figure 7-7. Calibration plot for leg-5
In the calibration plots, the displacement is measured in mm and force measured in Newtons. In the original data file, the displacement at each sample time is recorded as encoder counts and the force is recorded as digitized voltage value. Then the units are converted by the load cell calibration experiment results together with the following conversion equations:

\[
\begin{align*}
1 \text{ in} &= 25.4 \text{ mm} \\
1 \text{ lb} &= 453.6 \text{ gram} = 4.448 \text{ N}
\end{align*}
\] (7-2)

Then, the data in the plots is used to build six individual look-up tables of displacement vs. force for each leg. During operation, the relative displacement is measured and the corresponding force value is obtained from the look-up table. These calibration tables can also be replaced by linear regression of the data for less-accurate operations, and thus a “linear spring constant” is assigned to each leg.
Parallel Platform Force/Wrench Testing Experiment

The force/torque sensor has been factory calibrated, so only the resolution is listed here. The resolutions for force and torque are shown in the table below.

Table 7-2. Force-torque Sensor Resolution

<table>
<thead>
<tr>
<th>Fx</th>
<th>Fy</th>
<th>Fz</th>
<th>Tx</th>
<th>Ty</th>
<th>Tz</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1 (N)</td>
<td>0.1 (N)</td>
<td>0.2 (N)</td>
<td>5 (N-mm)</td>
<td>5 (N-mm)</td>
<td>5 (N-mm)</td>
</tr>
</tbody>
</table>

The parallel platform is attached on the top of the force/torque sensor as shown in Figure 7-9 and its mass center is carefully balanced such that its weight is directly over the origin of the sensor’s reference coordinate system. This coordinate system is based on the force/torque sensor, so there is no moment introduced. This z-direction bias force is easily eliminated in the sampling program, so it will not be discussed further.
The home position refers to the position of the platform when it is attached with the force/torque sensor and placed horizontally, with no other force applied on the top platform or any leg. The home position is also called the initial position/orientation, and it can be specified by its relative displacement from the encoder’s indexed position. Before use, it is necessary to home the device. This is done by applying some small force/wrench to the top platform, so each leg pass through its index position. Thus after homing, the current position and orientation can be accurately measured even when it is not in its home position.

**Repeatability test**

It is important to know the repeatability of the manipulator. The previous prototype, built by Tyler [Dwa00], did not have good repeatability. Significant friction in the spherical joints was noted when an external force was applied and then removed. This friction impacted the ability of the top platform to return repeatedly to the same home position when external loading was removed.

In order to test its repeatability, two sets of experiment are performed.

1. Apply random force/wrench to the top plate and then remove it, measure the unloaded position/orientation via the leg lengths measurements
2. Apply a weight of 500 grams on the top plate several times, and compare the measured leg lengths.

<table>
<thead>
<tr>
<th>Test Number</th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>#2</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>#3</td>
<td>-6</td>
<td>0</td>
<td>-4</td>
<td>-6</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>#4</td>
<td>-5</td>
<td>2</td>
<td>0</td>
<td>-4</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>#5</td>
<td>-3</td>
<td>-3</td>
<td>-1</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>#6</td>
<td>-2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>-2.83333</td>
<td>-0.16667</td>
<td>-1.16667</td>
<td>-1.66667</td>
<td>1</td>
<td>-0.83333</td>
</tr>
<tr>
<td><strong>STD</strong></td>
<td>2.316607</td>
<td>1.602082</td>
<td>1.47196</td>
<td>2.73252</td>
<td>1.788854</td>
<td>1.602082</td>
</tr>
</tbody>
</table>
The experiment results are shown in Tables 7-3 and 7-4. The numbers in the column shows the encoder counts when there is no external load applied (for method 1) and when the external load is applied on the platform (for method 2). The overall repeatability for the parallel platform is reasonable.

**Forward analysis verification**

The forward kinematic analysis was presented in Chapter 5. An experiment is shown here to validate the analysis result and also to provide for the static analysis process.

The parallel platform has some external wrench applied on the top plate. The counts of the relative displacement for each leg is then recorded. Table 7-5 shows the lengths of the six leg connectors, both in units of encoder counts from the unloaded position and in units of mm, with the external wrench applied.

The table also shows the calculated lengths of the six connectors of the equivalent 3-3 platform. After performing the forward analysis of the equivalent 3-3 platform, a reverse analysis of the special 6-6 platform is conducted to calculate the leg lengths and to verify the forward analysis result.
Table 7-5. Forward analysis

<table>
<thead>
<tr>
<th>Encoder Counts</th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-15</td>
<td>-10</td>
<td>-2</td>
<td>-13</td>
<td>-26</td>
<td>-19</td>
</tr>
<tr>
<td>Leg length for special 6-6 (measured) (mm)</td>
<td>80.588</td>
<td>67.746</td>
<td>80.918</td>
<td>67.669</td>
<td>80.308</td>
<td>67.517</td>
</tr>
<tr>
<td>Leg length for the equivalent 3-3 platform (mm)</td>
<td>84.595</td>
<td>84.829</td>
<td>84.722</td>
<td>84.145</td>
<td>84.469</td>
<td>80.588</td>
</tr>
<tr>
<td>Leg length for the 6-6 platform (reverse analysis) (mm)</td>
<td>80.587</td>
<td>67.746</td>
<td>80.918</td>
<td>67.669</td>
<td>80.309</td>
<td>67.517</td>
</tr>
</tbody>
</table>

From the table, it is clear that the forward analysis is very accurate, the leg lengths calculated from the reverse analysis, which are also based on the results of the forward kinematic analysis, are very close to the leg length values measured by the optical encoder.

The coordinate systems of the parallel platform are shown in Figure 7-10. The base points are $O_0$, $P_0$, and $Q_0$, the origin of the base coordinate system is at point $O_0$, $P_0$ is on the X axis, and $Q_0$ is on the XY plane. This coordinate system is fixed in the base. The top plate corner points are $R_i$, $S_i$, $T_i$, the origin of the top platform coordinate system is at point $R_i$, $S_i$ is on the x axis, and $T_i$ is in the xy plane.

After finding the leg lengths of the equivalent 3-3 platform by the geometrical conversion, the position and orientation of the platform can be calculated by the forward kinematic analysis. The detail of this analysis is presented in Chapter 6, so only the results are presented here.
For the measured leg lengths listed in Table 7-5, the forward analysis has only 8 solutions.\(^1\) Four are above the base and four are reflected through the base. The four configurations above the base are discussed here. Each solution can be represented by a \(4 \times 4\) transformation matrix. The first three elements in the fourth column have units of length (mm) and all other elements are dimensionless. From now on, the units for the element in the transformation matrices are omitted. The four calculated transformation matrices that relate the top and base coordinate systems are

\[
\begin{bmatrix}
0.5105 & -0.1095 & 0.8529 & 29.2029 \\
-0.8598 & -0.0718 & 0.5055 & 52.4117 \\
0.0059 & -0.9914 & -0.1308 & 59.4408 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
0.5021 & 0.8648 & 0.0007 & 29.7107 \\
-0.8648 & 0.5020 & 0.0007 & 52.1185 \\
0.0058 & -0.0042 & 0.9999 & 59.4469 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
0.4987 & 0.8668 & 0.0061 & 29.9149 \\
0.1211 & -0.0627 & -0.9907 & 52.0006 \\
-0.8583 & 0.4947 & -0.1363 & 59.4478 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
-0.3526 & 0.3757 & -0.8570 & 80.9928 \\
-0.3670 & 0.7869 & 0.4960 & 22.5108 \\
0.8608 & 0.4894 & -0.1396 & 8.1510 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

Although all four solutions are real and satisfy the geometrical conditions, there is only one solution that corresponds to the actual current configuration. With these transformation matrices, the coordinates of all points of the 3-3 structure can be determined. Then drawings of these configurations (or the numerical determination of

\(^1\) In general, the forward analysis of the special 6-6 platform will yield 16 possible solutions, 8 above the base and 8 reflections through the base. However for the measured set of leg lengths, only 8 real solutions exist, 4 above the base and 4 reflections through the base.
the configuration closest to the most previous pose) can be used to choose the correct solution set.

Figure 7-10. Coordinate systems of the parallel platform

Here the geometrical meanings of the $4 \times 4$ transformation matrices are used to make the selection. By using homogeneous coordinates, the $4 \times 4$ transformation matrix can be represented as

$$
\begin{bmatrix}
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
A & B \\
A & B \\
A & B \\
A & B \\
\end{bmatrix}
\begin{bmatrix}
P & R \\
P & R \\
P & R \\
P & R \\
\end{bmatrix}
\begin{bmatrix}
\mathbf{P}_B \\
\mathbf{P}_B \\
\mathbf{P}_B \\
\mathbf{P}_B \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
\end{bmatrix},
$$

where $\mathbf{P}_B$ is the location of the origin of the B coordinate system measured with respect to the A coordinate system, $\mathbf{R} = \begin{bmatrix}
\mathbf{X}_B & \mathbf{Y}_B & \mathbf{Z}_B \\
\mathbf{X}_B & \mathbf{Y}_B & \mathbf{Z}_B \\
\mathbf{X}_B & \mathbf{Y}_B & \mathbf{Z}_B \\
\mathbf{X}_B & \mathbf{Y}_B & \mathbf{Z}_B \\
\end{bmatrix}$, its columns vectors are the orientation of the B coordinate system measured with respect to the A coordinate system.
So by comparing the columns of the transformation matrices, the right solution can be properly selected.

The transformation matrix for the home position is given below as

$$
\begin{bmatrix}
0.5056 & 0.8628 & 0 & 29.6651 \\
-0.8628 & 0.5056 & 0 & 52.1547 \\
0 & 0 & 1 & 59.9987 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

(7-5)

Clearly, the second solution is the correct one since physically the Z axis of the top coordinate system remained relatively parallel to the Z axis of the base coordinate system after the loading was applied. For all other solutions, the coordinate systems have too much rotation relative to the base coordinates system.

**Wrench and force testing.**

After choosing the correct transformation matrix, the coordinates of all points are known and the Jacobian matrices of the structure at any instant be calculated. Using the equilibrium condition, the following equation can be used to calculate the applied force/wrench as stated in the previous chapters.

$$\hat{w} = j\lambda .$$

(7-6)

**Numerical example:**

In order to verify the accuracy of the PCCFC, several numerical experiments have been done. For each set of experiment data, the external wrench applied on the top platform of the PCCFC is measured by the force/torque sensor on which the device is mounted for this experiment and compared to the calculated wrench based on the sensed leg lengths. The force/torque sensor data is reported in terms of the sensor’s coordinate system and the calculated wrench is calculated in terms of the base coordinate system. The wrenches are compared by converting the calculated wrench to the sensor coordinate
system. It is important to note that the force/torque sensor has a left-handed coordinate system. That is its Z-axis and X-axis are parallel to the corresponding axes of the base coordinate system of the PCCFC, but the Y-axes are anti-parallel.

The sensor coordinate system is used as the common reference system and the coordinates of the external wrench detected by the PCCFC are transformed by the following equation:

\[
\{^2S; ^2S_0\} = \{^1S; ^1S_o - ^1p \times ^1S\}
\]

where superscript 1 and 2 refer to the Screw coordinates in the former coordinate system and new coordinate system respectively, \(^1p\) refers the coordinates of origin of the second coordinate system in terms of the first coordinate system. This transformation will determine the calculated wrench in terms of a right-handed coordinate system whose origin is at the origin of the sensor coordinate system. The x and z axes will be parallel to the x and z axes of the sensor coordinate system, but the y axis will be anti-parallel to that of the sensor coordinate system.

For each set of experiment data, \(S, S_o\) will be different, but \(^1p\) will remain the same as \(^1p = [60, 20, 3, 19]\) mm. Several numerical experiments are conducted here. The spring constants are shown in Table 7-6.

<table>
<thead>
<tr>
<th>Table 7-6. Spring constants</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Spring Constant(N/mm)</td>
</tr>
</tbody>
</table>

**Numerical experiments:**

Six different loads were applied to the top platform. The encoder counts and the corresponding leg lengths for the six cases are shown in Tables 7-7 through 7-12.
Table 7-7. Numerical experiment 1- encoder counts and leg lengths

<table>
<thead>
<tr>
<th></th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counts</td>
<td>-15</td>
<td>-15</td>
<td>-12</td>
<td>-73</td>
<td>-61</td>
<td>8</td>
</tr>
<tr>
<td>Leg length(mm)</td>
<td>80.59</td>
<td>67.62</td>
<td>80.66</td>
<td>66.15</td>
<td>79.42</td>
<td>68.20</td>
</tr>
</tbody>
</table>

Table 7-8. Numerical experiment 2- encoder counts and leg lengths

<table>
<thead>
<tr>
<th></th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counts</td>
<td>-76</td>
<td>13</td>
<td>-13</td>
<td>-4</td>
<td>-5</td>
<td>-87</td>
</tr>
<tr>
<td>Leg length(mm)</td>
<td>79.04</td>
<td>68.33</td>
<td>80.64</td>
<td>67.90</td>
<td>80.84</td>
<td>65.79</td>
</tr>
</tbody>
</table>

Table 7-9. Numerical experiment 3- encoder counts and leg lengths

<table>
<thead>
<tr>
<th></th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counts</td>
<td>-6</td>
<td>-80</td>
<td>-63</td>
<td>-6</td>
<td>-20</td>
<td>2</td>
</tr>
<tr>
<td>Leg length(mm)</td>
<td>80.82</td>
<td>65.97</td>
<td>79.37</td>
<td>67.85</td>
<td>80.46</td>
<td>68.05</td>
</tr>
</tbody>
</table>

Table 7-10. Numerical experiment 4- encoder counts and leg lengths

<table>
<thead>
<tr>
<th></th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counts</td>
<td>74</td>
<td>-30</td>
<td>-37</td>
<td>-54</td>
<td>-21</td>
<td>-83</td>
</tr>
<tr>
<td>Leg length(mm)</td>
<td>82.85</td>
<td>67.24</td>
<td>80.03</td>
<td>66.63</td>
<td>80.44</td>
<td>65.89</td>
</tr>
</tbody>
</table>

Table 7-11. Numerical experiment 5- encoder counts and leg lengths

<table>
<thead>
<tr>
<th></th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counts</td>
<td>46</td>
<td>-31</td>
<td>-37</td>
<td>-47</td>
<td>-22</td>
<td>-67</td>
</tr>
<tr>
<td>Leg length(mm)</td>
<td>82.14</td>
<td>67.21</td>
<td>80.03</td>
<td>66.81</td>
<td>80.41</td>
<td>66.30</td>
</tr>
</tbody>
</table>

Table 7-12. Numerical experiment 6- encoder counts and leg lengths

<table>
<thead>
<tr>
<th></th>
<th>Leg 1</th>
<th>Leg 2</th>
<th>Leg 3</th>
<th>Leg 4</th>
<th>Leg 5</th>
<th>Leg 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leg length(mm)</td>
<td>82.06</td>
<td>65.43</td>
<td>80.28</td>
<td>66.32</td>
<td>82.47</td>
<td>67.11</td>
</tr>
</tbody>
</table>
The corresponding wrench information for each of the six cases is shown in Tables 7-13 through 7-18. The sensor data are shown in the first row, the calculated data from PCCFC in terms of its base coordinate system are shown in the second row.

<table>
<thead>
<tr>
<th>Table 7-13. Numerical experiment 1 - wrench</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>PCCFC (base coord system)</td>
</tr>
<tr>
<td>PCCFC (Transformed)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 7-14. Numerical experiment 2 - wrench</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>PCCFC (base coord system)</td>
</tr>
<tr>
<td>PCCFC (Transformed)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 7-15. Numerical experiment 3 - wrench</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>PCCFC (base coord system)</td>
</tr>
<tr>
<td>PCCFC (Transformed)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 7-16. Numerical experiment 4 - wrench</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>PCCFC (base coord system)</td>
</tr>
<tr>
<td>PCCFC (Transformed)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 7-17. Numerical experiment 5 - wrench</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>F/T sensor</td>
</tr>
<tr>
<td>PCCFC (base coord system)</td>
</tr>
<tr>
<td>PCCFC (Transformed)</td>
</tr>
</tbody>
</table>
Table 7-18. Numerical experiment 6 - wrench

<table>
<thead>
<tr>
<th></th>
<th>Fx(N)</th>
<th>Fy(N)</th>
<th>Fz(N)</th>
<th>Tx(Nmm)</th>
<th>Ty(Nmm)</th>
<th>Tz(Nmm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>F/T sensor</td>
<td>1.8</td>
<td>1.9</td>
<td>-9.4</td>
<td>420.8</td>
<td>-254.2</td>
<td>-276.8</td>
</tr>
<tr>
<td>PCCFC (base coord system)</td>
<td>2.0</td>
<td>-2.3</td>
<td>-9.9</td>
<td>29.6</td>
<td>794.8</td>
<td>-540.9</td>
</tr>
<tr>
<td>PCCFC (Transformed)</td>
<td>2.0</td>
<td>-2.3</td>
<td>-9.9</td>
<td>417.9</td>
<td>239.0</td>
<td>-327.8</td>
</tr>
</tbody>
</table>

The different signs between Fy and Ty of the two set of data are caused by the different coordinate systems. Considering the resolution of the sensor and environmental noise, the results are very good.

**Determination of Stiffness Matrix at a Loaded Position**

In this section the stiffness matrix for the prototype device when it is at a loaded position will be determined in two ways i.e., analytically and experimentally, and the two results will be compared. Figure 7-11 shows a model of the compliant platform with coordinate system 0 attached to the base and coordinate system 1 attached to the top platform. The constant platform dimensions are given as

\[
\begin{align*}
L_{O_0\rightarrow p_b} &= L_{p_b\rightarrow Q_0} = L_{Q_0\rightarrow O_b} = 120 \text{ mm}, \\
L_{O_0\rightarrow s_b} &= L_{p_b\rightarrow t_b} = L_{Q_0\rightarrow R_b} = 28 \text{ mm}, \\
L_{R_1\rightarrow S_b} &= L_{S_b\rightarrow T_1} = L_{T_1\rightarrow R_1} = 60 \text{ mm}, \\
L_{R_1\rightarrow O_1} &= L_{S_b\rightarrow T_1} = L_{T_1\rightarrow Q_1} = 14 \text{ mm}
\end{align*}
\]  

(7-8)

where the notation \( L_{i\rightarrow j} \) is used to represent the distance between points \( i \) and \( j \).

A load was applied to the top platform and the leg lengths were measured as

\[
\begin{align*}
L_{O_0\rightarrow O_1} &= 80.1054 \text{ mm}, \\
L_{p_b\rightarrow p_1} &= 80.8928 \text{ mm}, \\
L_{Q_0\rightarrow Q_1} &= 80.8928 \text{ mm}, \\
L_{R_0\rightarrow R_1} &= 66.8570 \text{ mm}, \\
L_{S_b\rightarrow S_b} &= 68.2032 \text{ mm}, \\
L_{T_0\rightarrow T_1} &= 67.9492 \text{ mm}.
\end{align*}
\]  

(7-9)
Figure 7-11. Compliant Platform Model

At this position, a forward displacement analysis was conducted and four possible real positions and orientations of the top platform were determined. During the experiment, the z axis of the top platform remained close to parallel to the z axis of the bottom platform and the solution which most closely satisfied this constraint was selected. The $4 \times 4$ transformation matrix that relates the top and base coordinate systems was determined to be

$$
^{0}\mathbf{T} = \begin{bmatrix}
0.5017 & 0.8647 & -0.0243 & 30.1091 \\
-0.8647 & 0.5021 & 0.0154 & 51.6518 \\
0.0256 & 0.0133 & 0.9996 & 58.5839 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

(7-10)

the units of the terms in the first three columns are dimensionless and the units of the top three terms of the fourth column are millimeters.

At this position, the force in each of the six leg connectors was determined. The displacement of each of the legs from the unloaded home position measured in terms of encoder counts was as follows:
\[ \Delta L_{O_h \rightarrow O_i} = -34, \]
\[ \Delta L_{P_2 \rightarrow P_1} = -3, \]
\[ \Delta L_{Q_2 \rightarrow Q_1} = -3, \]
\[ \Delta L_{R_2 \rightarrow R_1} = -45, \]
\[ \Delta L_{S_2 \rightarrow S_1} = 8, \]
\[ \Delta L_{T_2 \rightarrow T_1} = -2. \quad (7-11) \]

The resolution of the encoder is 1000 counts/inch (39 counts/mm). The spring constant of each leg was previously determined experimentally as

\[ k_{O_h \rightarrow O_i} = 3.0914 \frac{N}{mm}, \]
\[ k_{P_2 \rightarrow P_1} = 3.1003 \frac{N}{mm}, \]
\[ k_{Q_2 \rightarrow Q_1} = 2.9757 \frac{N}{mm}, \]
\[ k_{R_2 \rightarrow R_1} = 3.0380 \frac{N}{mm}, \]
\[ k_{S_2 \rightarrow S_1} = 3.0469 \frac{N}{mm}, \]
\[ k_{T_2 \rightarrow T_1} = 3.1047 \frac{N}{mm}. \quad (7-12) \]

The force in each of the legs is thus calculated as

\[ f_{O_h \rightarrow O_i} = -2.6697 \, N, \]
\[ f_{P_2 \rightarrow P_1} = -0.2362 \, N, \]
\[ f_{Q_2 \rightarrow Q_1} = -0.2267 \, N, \]
\[ f_{R_2 \rightarrow R_1} = -3.4724 \, N, \]
\[ f_{S_2 \rightarrow S_1} = 0.6191 \, N, \]
\[ f_{T_2 \rightarrow T_1} = -0.1577 \, N. \quad (7-13) \]

The external wrench applied at position 1, \( \mathbf{\hat{w}}_1 \), is calculated as the sum of the individual forces as
where the first three terms have units of Newtons and the last three terms have units of Newton-millimeters. The stiffness matrix for the device at this position will now be computed analytically and experimentally.

**Analytical Determination of Stiffness Matrix**

In chapter 5, the spatial stiffness matrix for PCCFC was studied and presented. The spatial stiffness matrix can be calculated analytically based on the geometrical dimension values, spring constants of the compliant devices and the current transformation matrix. Equation (5-39) is used to determine the spatial stiffness matrix.

For the specified configuration discussed above, its stiffness matrix can be calculated as

\[
\begin{bmatrix}
3.0505 & 0.0426 & -0.0239 & -21.406 & 211.79 & -101.96 \\
0.0426 & 3.0365 & 0.0265 & -205.78 & -21.241 & 180.95 \\
-0.024 & 0.0265 & 12.101 & 412.15 & -732.38 & 42.647 \\
216.73 & -21.241 & -732.36 & -24898 & 64519 & -11169 \\
-101.78 & 180.93 & 42.647 & -10226 & -10906 & 18475 \\
\end{bmatrix}
\]

where \( \alpha \) stands for analytically determined. As stated in Chapter 5, the four 3×3 sub-matrices have the following units

\[
\begin{bmatrix}
\frac{N}{mm} & N \\
N & \frac{N}{mm}
\end{bmatrix}
\]
Experimental Determination of Stiffness Matrix

An additional external load was used to displace the platform slightly from the position defined by (7-10) (to be referred to as position 1) and the leg lengths were recorded. This procedure was repeated for a total of six times and these six cases will be identified by the letters A through F. For each of these cases it was necessary to determine the change in the applied wrench (to be written in ray coordinates) and the twist that describes the motion of the platform from position 1 to the new position (to be written in axis coordinates). The wrench and twist will both be evaluated in terms of the coordinate system attached to the base platform.

As previously stated, the stiffness matrix $K$ relates the change in the applied wrench to the instantaneous motion of the top platform as

$$\delta \mathbf{w} = [K] \mathbf{D}$$  \hspace{1cm} (7-16)

By evaluating the change in the wrench and the twist six times, (7-16) can be written in matrix format as

$$[\delta \mathbf{w}_{6\times6}] = [K] [\mathbf{D}_{6\times6}]$$  \hspace{1cm} (7-17)

where $[\delta \mathbf{w}_{6\times6}]$ and $[\mathbf{D}_{6\times6}]$ are $6\times6$ matrices defined as

$$[\delta \mathbf{w}_{6\times6}] = [\delta \mathbf{w}_{1\rightarrow A} \; \delta \mathbf{w}_{1\rightarrow B} \; \delta \mathbf{w}_{1\rightarrow C} \; \delta \mathbf{w}_{1\rightarrow D} \; \delta \mathbf{w}_{1\rightarrow E} \; \delta \mathbf{w}_{1\rightarrow F}],$$  \hspace{1cm} (7-18)

$$[\mathbf{D}_{6\times6}] = [\mathbf{D}_{1\rightarrow A} \; \mathbf{D}_{1\rightarrow B} \; \mathbf{D}_{1\rightarrow C} \; \mathbf{D}_{1\rightarrow D} \; \mathbf{D}_{1\rightarrow E} \; \mathbf{D}_{1\rightarrow F}].$$  \hspace{1cm} (7-19)

The columns of $[\delta \mathbf{w}_{6\times6}]$ are the changes in the external wrench as the platform moved from position 1 to position A, position 1 to position B, etc. The columns of $[\mathbf{D}_{6\times6}]$ are the twists that represent the motion of the platform from position 1 to position
A, from position 1 to position B, etc. Once the matrices $\hat{\delta \mathbf{w}}_{6 \times 6}$ and $\hat{\mathbf{D}}_{6 \times 6}$ are determined, the stiffness matrix can be calculated as

$$[\mathbf{K}] = [\hat{\delta \mathbf{w}}_{6 \times 6}][\hat{\mathbf{D}}_{6 \times 6}]^{-1}. \quad (7-20)$$

Prior to calculating $[\hat{\delta \mathbf{w}}_{6 \times 6}]$ and $[\hat{\mathbf{D}}_{6 \times 6}]$ it is necessary to document the measured leg lengths at the six new poses of the top platform. Table 7-19 presents this information.

**Table 7-19. Measured leg connector lengths at positions A through F (all units mm)**

<table>
<thead>
<tr>
<th></th>
<th>$L_{O_1 \rightarrow O_1}$</th>
<th>$L_{P_1 \rightarrow P_1}$</th>
<th>$L_{Q_1 \rightarrow Q_1}$</th>
<th>$L_{R_1 \rightarrow R_1}$</th>
<th>$L_{S_1 \rightarrow S_1}$</th>
<th>$L_{T_1 \rightarrow T_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>80.5118</td>
<td>79.3688</td>
<td>79.9530</td>
<td>66.9586</td>
<td>69.5748</td>
<td>68.3048</td>
</tr>
<tr>
<td>B</td>
<td>79.9530</td>
<td>80.2578</td>
<td>80.9182</td>
<td>66.9840</td>
<td>67.3142</td>
<td>68.0508</td>
</tr>
<tr>
<td>C</td>
<td>79.9784</td>
<td>80.7912</td>
<td>80.5372</td>
<td>66.9840</td>
<td>68.2032</td>
<td>67.1364</td>
</tr>
<tr>
<td>D</td>
<td>80.4102</td>
<td>81.3246</td>
<td>81.3500</td>
<td>66.4760</td>
<td>67.5428</td>
<td>67.7206</td>
</tr>
<tr>
<td>E</td>
<td>81.2738</td>
<td>79.6228</td>
<td>80.6134</td>
<td>66.5014</td>
<td>69.2192</td>
<td>67.7206</td>
</tr>
<tr>
<td>F</td>
<td>79.3434</td>
<td>80.4356</td>
<td>82.2390</td>
<td>67.4158</td>
<td>68.0762</td>
<td>67.4412</td>
</tr>
</tbody>
</table>

The position and orientation of the top platform was determined via a forward displacement analysis. The transformation matrices were determined as

$$^0\mathbf{T}_A = \begin{bmatrix} 0.5391 & 0.8420 & -0.0191 & 30.8315 \\ -0.8419 & 0.5394 & 0.0164 & 51.0055 \\ 0.0241 & 0.0072 & 0.9997 & 58.5788 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ^0\mathbf{T}_B = \begin{bmatrix} 0.5041 & 0.8633 & -0.0237 & 30.0674 \\ -0.8637 & 0.5040 & -0.0068 & 51.5888 \\ 0.0061 & 0.0239 & 0.9997 & 58.6874 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$^0\mathbf{T}_C = \begin{bmatrix} 0.5019 & 0.8648 & -0.0099 & 30.0147 \\ -0.8645 & 0.5020 & 0.0242 & 51.4415 \\ 0.0259 & -0.0036 & 0.9997 & 58.6024 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ^0\mathbf{T}_D = \begin{bmatrix} 0.4778 & 0.8781 & -0.0265 & 30.4274 \\ -0.8782 & 0.4782 & 0.0109 & 52.4777 \\ 0.0222 & 0.0181 & 0.9996 & 58.6251 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$^0\mathbf{T}_E = \begin{bmatrix} 0.5097 & 0.8602 & -0.0165 & 31.7774 \\ -0.8601 & 0.5099 & 0.0154 & 52.0331 \\ 0.0217 & 0.0064 & 0.9997 & 58.7887 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad ^0\mathbf{T}_F = \begin{bmatrix} 0.4993 & 0.8659 & -0.0305 & 30.4078 \\ -0.8661 & 0.4998 & 0.0084 & 50.0301 \\ 0.0225 & 0.0222 & 0.9995 & 58.5063 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7-21)$$

The matrix $[\hat{\delta \mathbf{w}}_{6 \times 6}]$ is relatively simple to calculate. For example, the wrench at position 1, $\hat{\mathbf{w}}_1$, is calculated as the sum of the forces along the leg connector lines at
position 1. Similarly, the wrench at position A, \( \hat{\mathbf{w}}_A \), is also calculated as the sum of the forces along the leg connector lines at position A. The change in the wrench, \( \delta\hat{\mathbf{w}}_{1\rightarrow A} \) is simply calculated as the difference between these two wrenches as

\[
\delta\hat{\mathbf{w}}_{1\rightarrow A} = \hat{\mathbf{w}}_A - \hat{\mathbf{w}}_1. \tag{7-22}
\]

Similar calculations are performed for the other five legs. The displacements of each leg measured in encoder counts from the unloaded home position are listed in the table below.

Table 7-20. Displacement of leg connectors from unloaded home position at positions A through F (all units are encoder counts)

<table>
<thead>
<tr>
<th></th>
<th>( L_{O_0\rightarrow O_1} )</th>
<th>( L_{P_1\rightarrow P} )</th>
<th>( L_{O_0\rightarrow O_1} )</th>
<th>( L_{R_1\rightarrow R_1} )</th>
<th>( L_{S_0\rightarrow S} )</th>
<th>( L_{T_0\rightarrow T_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-18</td>
<td>-63</td>
<td>-40</td>
<td>-41</td>
<td>62</td>
<td>12</td>
</tr>
<tr>
<td>B</td>
<td>-40</td>
<td>-28</td>
<td>-2</td>
<td>-40</td>
<td>-27</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>-39</td>
<td>-34</td>
<td>-17</td>
<td>-40</td>
<td>8</td>
<td>-17</td>
</tr>
<tr>
<td>D</td>
<td>-22</td>
<td>14</td>
<td>15</td>
<td>-60</td>
<td>-18</td>
<td>-11</td>
</tr>
<tr>
<td>E</td>
<td>12</td>
<td>-53</td>
<td>-14</td>
<td>-59</td>
<td>48</td>
<td>-11</td>
</tr>
<tr>
<td>F</td>
<td>-64</td>
<td>-21</td>
<td>50</td>
<td>-23</td>
<td>3</td>
<td>-22</td>
</tr>
</tbody>
</table>

The calculated wrenches at each of the six positions are presented as

\[
\hat{\mathbf{w}}_A = \begin{bmatrix}
4.7730 \\
2.2219 \\
-4.8295 \\
-433.9854 \\
508.2140 \\
138.8671
\end{bmatrix}, \quad \hat{\mathbf{w}}_B = \begin{bmatrix}
-0.3449 \\
-0.4166 \\
-8.4476 \\
-223.6900 \\
363.1519 \\
-21.4402
\end{bmatrix}, \quad \hat{\mathbf{w}}_C = \begin{bmatrix}
0.3539 \\
-0.5456 \\
-8.1272 \\
-370.3918 \\
465.1178 \\
-63.1008
\end{bmatrix}, \quad \hat{\mathbf{w}}_D = \begin{bmatrix}
-0.6523 \\
0.1190 \\
-5.7095 \\
-257.3332 \\
156.5356 \\
-88.9454
\end{bmatrix}, \quad \hat{\mathbf{w}}_E = \begin{bmatrix}
5.8796 \\
2.0262 \\
-4.8078 \\
-420.4532 \\
600.3909 \\
-47.7374
\end{bmatrix}, \quad \hat{\mathbf{w}}_F = \begin{bmatrix}
0.3890 \\
-4.9762 \\
-5.0937 \\
125.7469 \\
212.4282 \\
-329.0032
\end{bmatrix}. \tag{7-23}
\]
where the first three components of these vectors have units of Newtons and the last three components have units of Newton-millimeters. From (7-21) and (7-14) the matrix \( [\hat{\omega}_{6:6}] \) is calculated as

\[
[\hat{\omega}_{6:6}] = \begin{bmatrix}
4.7467 & -0.3712 & 0.3277 & -0.6785 & 5.8533 & 0.3628 \\
2.0391 & -0.5994 & -0.7285 & -0.0639 & 1.8434 & -5.1591 \\
0.1141 & -3.5041 & -3.1836 & -0.7659 & 0.1358 & -0.1501 \\
337.65 & 192.59 & 294.56 & -14.025 & 429.82 & 41.868 \\
\end{bmatrix}.
\] (7-24)

The determination of the twists as the top platform moves from position 1 to each of the six other positions is slightly more complicated. The determination of the twist \( \hat{D}_{1 \rightarrow A} \) will be presented.

The twist \( \hat{D}_{1 \rightarrow A} \) is comprised of six components. When written in axis coordinates, the last three represent the instantaneous angular velocity of the moving body and the first three represent the instantaneous linear velocity of a point in the moving body that is coincident with the origin of the reference frame. In this case, both of these instantaneous quantities must be approximated from the finite motion of the platform as it moves between pose 1 and pose A.

The angular velocity was approximated by calculating the axis and angle of rotation that would rotate the top coordinate system from its orientation at pose 1 to its orientation at pose A. This was accomplished by first evaluating the net rotation matrix that relates the A pose to the 1 pose as

\[
^1_A R = \left[ ^0_A R \right]^T \left[ ^0_A R \right].
\] (7-25)
where the $3 \times 3$ rotation matrices $^0\mathbf{R}_i$ and $^0\mathbf{R}_A$ are obtained as the upper left $3 \times 3$ elements of the transformation matrices $^0\mathbf{T}_i$ and $^0\mathbf{T}_A$ respectively. For this case the rotation matrix $^A\mathbf{R}_i$ was evaluated as

$$
^A\mathbf{R}_i = 
\begin{bmatrix}
0.999 & -0.0438 & 0.0018 \\
0.0437 & 0.999 & 0.0050 \\
-0.0020 & -0.0049 & 1.0000
\end{bmatrix}.
$$  \tag{7-26}

From [Cra98] the angle of rotation and axis of rotation that would rotate coordinate system 1 to be aligned with coordinate system A are calculated as $\theta = 2.526$ degrees and $^1\mathbf{m} = [-0.1124, 0.0431, 0.9927]^T$. The superscript associated with the axis vector $\mathbf{m}$ is used to indicate that the rotation axis that was determined from (7-26) was expressed in terms of the coordinate system attached to the top platform at position 1. The direction of the rotation axis can be expressed in the ground, 0, coordinate system via

$$
^0\mathbf{m} = ^0\mathbf{R}_i^A \mathbf{m}.
$$  \tag{7-27}

Evaluating the axis of rotation in the 0 coordinate system yielded $^0\mathbf{m} = [-0.0433, 0.1342, 0.9900]^T$. Finally, the last three components of $\mathbf{D}_{i\rightarrow A}$ are approximated by $(^0\mathbf{m})$ and will be dimensionless (radians).

As previously stated, the first three components of $\mathbf{D}_{i\rightarrow A}$ represent the instantaneous linear velocity of a point in the moving body that is coincident with the origin of the reference frame. This quantity is estimated by first determining the coordinates of the origin point of the ground, 0, coordinate system in the 1 coordinate system, i.e., $^1\mathbf{P}_{0\text{orig}}$. This quantity is determined as

$$
^1\mathbf{P}_{0\text{orig}} = -[^0\mathbf{R}_i^T ]^0\mathbf{P}_{0\text{orig}}
$$  \tag{7-28}
where \( \mathbf{P}_{0 \text{orig}} \) is obtained as the top three terms of the fourth column of \( \mathbf{T}_0 \) which was numerically determined in (7-10). For infinitesimal motion, the coordinates of the origin of the ground coordinate system should be the same when measured in the 1 or the A coordinate systems. Thus it is assumed that \( \mathbf{P}_{0 \text{orig}} = \mathbf{P}_{0 \text{orig}} \) where the notation \( 0' \text{orig} \) is introduced. This point is then transformed to the 1 coordinate system as

\[
\mathbf{1}_{0' \text{orig}} = \mathbf{T}_A \mathbf{0}_{0' \text{orig}}
\]

where

\[
\mathbf{1}_{A} = \left[ \mathbf{T}_0 \right]^{-1} \mathbf{A}_{0} \mathbf{T}_0 .
\]

The translation of the origin point of coordinate system 0 as seen from coordinate system 1 can be written as the net displacement of the point as

\[
\mathbf{1}_{0 \text{orig}} = \mathbf{1}_{0' \text{orig}} - \mathbf{1}_{0 \text{orig}}.
\]

Lastly, this translation vector can be evaluated in the 0 coordinate system as

\[
\mathbf{0}_{0} = \mathbf{R}_{0' \text{orig}} \mathbf{1}_{0 \text{orig}} .
\]

For this particular case, the translation of the point in the top platform that is coincident with the origin of the reference coordinate system, for the case where the top platform moves from pose 1 to pose A, was evaluated as

\[
\mathbf{0}_{0} = \begin{bmatrix} 0.7223 \\ -0.6462 \\ -0.0052 \end{bmatrix} \text{ mm}.
\]

Thus for the case where the top platform moves from position 1 to position A, the instantaneous twist written in axis coordinates is estimated as
\[
\mathbf{\hat{D}}_{1\rightarrow A} = \begin{bmatrix}
\begin{bmatrix}
\theta^0_v
\theta^0_m
\end{bmatrix}
&= \\
0.7223 \\
-0.6462 \\
-0.0052 \\
-0.00191 \\
0.00592 \\
0.04364
\end{bmatrix}
\] (7-34)

where the first three components have units of mm and the last three components have units of rad.

The process can be repeated to determine the twists associated with the motion from pose 1 to pose B, pose 1 to pose C, …, pose 1 to pose F. The resulting twists were determined as

\[
\mathbf{\hat{D}}_{1\rightarrow B} = \begin{bmatrix}
-0.04179 \\
-0.6291 \\
0.1035 \\
0.0222 \\
0.006 \\
0.0027
\end{bmatrix}, \quad \mathbf{\hat{D}}_{1\rightarrow C} = \begin{bmatrix}
-0.0944 \\
-0.2102 \\
0.0185 \\
-0.0088 \\
0.0144 \\
-0.0001
\end{bmatrix}, \quad \mathbf{\hat{D}}_{1\rightarrow D} = \begin{bmatrix}
0.3183 \\
0.8259 \\
0.0411 \\
0.0053 \\
-0.0025 \\
-0.0273
\end{bmatrix}
\] (7-35)

\[
\mathbf{\hat{D}}_{1\rightarrow E} = \begin{bmatrix}
1.6683 \\
0.3813 \\
0.2048 \\
-0.0001 \\
0.0079 \\
0.0091
\end{bmatrix}, \quad \mathbf{\hat{D}}_{1\rightarrow F} = \begin{bmatrix}
0.2987 \\
1.6217 \\
-0.0776 \\
0.0071 \\
-0.0062 \\
-0.0025
\end{bmatrix}
\]

Lastly, substituting these values into (7-19) and then substituting (7-19) and (7-24) into (7-17) yields the stiffness matrix for the device at pose 1. This matrix was determined as
\[
[K] = \begin{bmatrix}
2.9045 & -0.0499 & 1.7812 & 67.167 & 136.53 & -97.152 \\
-0.0239 & 2.9637 & 0.8046 & -167.05 & -56.028 & 181.38 \\
0.2085 & 0.0308 & 11.182 & 364.17 & -693.53 & 32.81 \\
-8.5556 & -205.44 & 326.45 & 29390 & -21122 & -10796 \\
189.53 & -29.633 & -524.65 & -14459 & 55733 & -10174 \\
-98.136 & 179.71 & 18.616 & -11395 & -9997.3 & 18204
\end{bmatrix}.
\tag{7-36}
\]

**Comparison of analytical result and experimental result**

Comparing the stiffness matrix determined analytically with the stiffness matrix calculated experimentally, it is clear that both matrices are non-symmetric and they are very close to each other. This result also validates the previous stiffness analysis as well as the force/ twist approximation.

**Future Research**

The passive parallel platform has been designed, fabricated, and tested. Future experiments should be applied to test its dynamic properties such as the effects of vibration, velocity, and acceleration. The performance of kinematic control depends not only on the stiffness matrix of the compliant component but also on the whole robot system. The system stiffness is dependent on the configuration of the robot system, including the serial industrial robot, passive parallel platform, a gripper or tool, and any permanently attached compliant devices. The ability of accurately modeling the system stiffness ensures the control performance for displacement and force.

There are ways to improve the design of the PCCFC. The theoretical analysis presented in this work can be applied directly to the future project. A non-contact displacement laser sensor can be used to replace the optical encoder to greatly reduce the size of the structure. Also a plastic or rubber cover can be designed to protect the connectors for use of the device in a far ranging set of applications.
APPENDIX
MECHANICAL DRAWINGS OF THE PARTS FOR PCCFC
Figure A-1. Strip holder
Figure A-2. Lower leg part.
Figure A-3. Upper leg part
Figure A-4. Leg
Each triangle side is identical to other 2 sides on every dim (including size dms and on location dms).
Figure A-6. Base plate.
LIST OF REFERENCES


BIOGRAPHICAL SKETCH

Bo Zhang was born in Xi’an, China, 1975. He received his Bachelor of Science degree in mechanical engineering from XiDian University in his home town in 1997. Then he went to Beijing to continue his study and received his Master of Science degree in mechanical engineering from Tsinghua University in 2000. He found his strong interest in robotics and joined the Department of Mechanical and Aerospace Engineering, University of Florida, for his PhD degree in mechanical engineering. Since then, he has continued his studies and worked as a research assistant with Dr. Carl D. Crane III at the Center for Intelligent Machines and Robotics.