

GEOMETRY OF LINK INVARIANTS

By

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$k$ -Quasi-isotopy is an equivalence relation on piecewise-linear (or smooth) links in the Euclidean 3-space, whose definition closely resembles the first  $k$  stages of the construction of a Casson handle. It generalizes and refines both  $k$ -cobordism of Cochran and Orr and  $k$ -linking of S. Eilenberg, as modified by N. Smythe and K. Kobayashi.  $k$ -Quasi-isotopy turned out to be helpful for understanding the geometry of the Alexander polynomial and Milnor's  $\bar{\mu}$ -invariants, leading, in particular, to the following results.

- There exists a *direct* multi-variable analogue of the Conway polynomial  $\nabla_L(z)$ , i.e., a (finite) polynomial  $\hat{\nabla}_L(z_1, \dots, z_n)$  with integer coefficients, satisfying  $\hat{\nabla}_{L_+} - \hat{\nabla}_{L_-} = z_i \hat{\nabla}_{L_0}$  for any Conway triple  $(L_+, L_-, L_0)$  whose disagreement is within the components of the  $i$ -th color. Moreover, for 2-component links all coefficients of the 2-variable  $\hat{\nabla}_L$  are integer liftings of Milnor's invariants  $\bar{\mu}(1 \dots 12 \dots 2)$  of even length.
- Each Cochran's derived invariant  $\beta^i$  (defined originally when the linking number vanishes) extends to a Vassiliev invariant of 2-component links.
- If  $K_i$  are the components of a link  $L$ , each coefficient of the power series  $\nabla_L / \prod \nabla_{K_i}$  is invariant under sufficiently close  $C^0$ -approximation, and can be (uniquely) extended, preserving this property, to all topological links in  $\mathbb{R}^3$ . The same holds for the power series  $\hat{\nabla}_L(z_1, \dots, z_n) / \prod \nabla_{K_i}(z_i)$ , as well as  $H_L / \prod H_{K_i}$  and  $F_L / \prod F_{K_i}$ , where  $H_L$  and  $F_L$  are certain exponential parameterizations of the two-variable HOMFLY and Kauffman polynomials.

- No difference between PL isotopy and TOP isotopy (as equivalence relations on PL links in  $\mathbb{R}^3$ ) can be detected by Vassiliev invariants.

## CHAPTER 1

### INTRODUCTION

Recall that *PL isotopy* of links in  $S^3$  can be viewed as the equivalence relation generated by ambient isotopy and insertion of local knots. (Thus any knot in  $S^3$  is PL isotopic to the unknot.) The “links modulo knots” version of the Vassiliev Conjecture for knots [37] could be as follows.

**Problem 1.1.** *Are PL isotopy classes of links in  $S^3$  separated by finite type invariants that are well-defined up to PL isotopy?*

It is to be noted that both questions depend heavily on the topology of  $S^3$  (or  $\mathbb{R}^3$ ) as opposed to other 3-manifolds. Indeed, the answer to 1.1 is negative for links (in fact, even knots) in any contractible open 3-manifold  $W$  other than  $\mathbb{R}^3$  ([25] and Theorem 2.2 below), whereas the original Vassiliev Conjecture fails for knots in any such  $W$  embeddable in  $\mathbb{R}^3$  (e.g., in the original Whitehead manifold) [7, 8]. Similarly, if  $M$  is a homotopy sphere, then  $M \cong S^3$  given either a positive solution to 1.1 for knots in  $M$  ([25] and Theorem 2.2 below), or a proof of the Vassiliev Conjecture for knots in  $M \# M$  [7, 8].

Some further geometry behind Problem 1.1 is revealed by

**Theorem 1.2.** *The answer to 1.1 is affirmative if and only if the following four statements (on PL links in  $S^3$ ) hold simultaneously:*

- (i) *finite type invariants, well-defined up to PL isotopy, are not weaker than finite KL-type invariants (cf. §2), well-defined up to PL isotopy;*
- (ii) *indistinguishability by finite KL-type invariants, well-defined up to PL isotopy, implies  $k$ -quasi-isotopy for all  $k \in \mathbb{N}$ ;*
- (iii)  *$k$ -quasi-isotopy for all  $k \in \mathbb{N}$  implies TOP isotopy in the sense of Milnor (i.e., homotopy within the class of topological embeddings);*
- (iv) *TOP isotopy implies PL isotopy.*

The assertion follows since the converse to each of (i)–(iv) holds with no proviso: to (ii) by Theorem 2.2 below, to (iii) by Theorem 1.3(b) below, to (i) and (iv) by definitions.

We recall that PL links  $L, L': mS^1 \hookrightarrow S^3$  (where  $mS^1 = S_1^1 \sqcup \cdots \sqcup S_m^1$ ) are called *k-quasi-isotopic* [25], if they are joined by a generic PL homotopy, all whose singular links are *k-quasi-embeddings*. A PL map  $f: mS^1 \rightarrow S^3$  with precisely one double point  $f(p) = f(q)$  is called a *k-quasi-embedding*,  $k \in \mathbb{N}$ , if, in addition to the singleton  $P_0 := \{f(p)\}$ , there exist compact subpolyhedra  $P_1, \dots, P_k \subset S^3$  and arcs  $J_0, \dots, J_k \subset mS^1$  such that  $f^{-1}(P_j) \subset J_j$  for each  $j \leq k$  and  $P_j \cup f(J_j) \subset P_{j+1}$  for each  $j < k$ , where the latter inclusion is null-homotopic for each  $j < k$ .

This definition is discussed in detail and illustrated by examples in Melikhov and Repoš [25] and [26]. The only result we need from these papers is the following obvious

**Theorem 1.3.** [25] (a) *For each  $k \in \mathbb{N}$ ,  $k$ -quasi-isotopy classes of all sufficiently close PL approximations to a topological link coincide.*

(b) *TOP isotopic PL links are  $k$ -quasi-isotopic for all  $k \in \mathbb{N}$ .*

The paper is organized as follows. §2 contains a little bit of general geometric theory of finite KL-type invariants, which we interpret in the context of colored links. (We do not attempt at extending any of the standard algebraic results of the theory of finite type invariants.) The rest of the paper is devoted to specific examples. Except for invariants of genuine finite type — from the Conway polynomial (§3), the multi-variable Alexander polynomial (Theorems 4.2 and 4.3) and the HOMFLY and Kauffman polynomials (§5), — we need to work just to produce interesting examples (Theorems 4.4 and 4.6). The payoff is a further clarification of the relationship between the Alexander polynomial and geometrically more transparent invariants of Milnor [28] and Cochran [5] (Theorem 4.8). We are also able to extract additional geometric information (Theorem 3.4 and Corollary 4.10b), which was the initial motivation for this paper (compare Problem 1.5 in Melikhov and Repovš [25]).

## CHAPTER 2

### COLORED FINITE TYPE INVARIANTS AND $k$ -QUASI-ISOTOPY

We introduce finite type invariants of colored links, as a straightforward generalization of the Kirk–Livingston setting of finite type invariants of links [16]. Kirk–Livingston type  $k$  invariants are recovered as type  $k$  invariants of colored links whose components have pairwise distinct colors, whereas the usual type  $k$  invariants of links (as defined by Goussarov [9] and Stanford [33]) coincide with type  $k$  invariants of monochromatic links. Although normally everything boils down to one of these two extreme cases, the general setting, apart from its unifying appeal, may be useful in proofs (cf. proof of Proposition 3.1) and in dealing with colored link polynomials (cf. Theorem 4.6). Refer to the following bracketed citations for a background on finite type invariants: [37, 10, 31, 4, 33, 3].

An  $m$ -component classical *colored link map* corresponding to a given *coloring*  
 $c: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$  is a continuous map  $L: mS^1 \rightarrow S^3$  such that  $L(S_i^1) \cap L(S_j^1) = \emptyset$  whenever  $c(i) \neq c(j)$ . Thus usual link maps correspond to the identity coloring and arbitrary maps to the constant coloring. Let  $LM_{1, \dots, 1}^3(c)$  be the space of all PL  $c$ -link maps  $mS^1 \rightarrow S^3$ , and let  $\mathcal{LM}(c)$  denote its subspace consisting of maps whose only singularities are transversal double points (the integer  $m$  will be fixed throughout this section). Note that  $\mathcal{LM}(c)$  is disconnected unless  $c$  is constant. Let  $\mathcal{LM}_n(c)$  (resp.  $\mathcal{LM}_{\geq n}(c)$ ) denote the subspace of  $\mathcal{LM}(c)$  consisting of maps with precisely (resp. at least)  $n$  singularities. Note that  $\mathcal{LM}_0(c)$  does not depend on  $c$ .

Given any ambient isotopy invariant  $\chi : \mathcal{LM}_0(c) \rightarrow G$  with values in a finitely generated abelian group  $G$ , it can be extended to  $\mathcal{LM}(c)$  inductively by the formula

$$\chi(L_s) = \chi(L_+) - \chi(L_-),$$

where  $L_+, L_- \in \mathcal{LM}_n(c)$  differ by a single positive crossing of components of the same color, and  $L_s \in \mathcal{LM}_{n+1}(c)$  denotes the intermediate singular link with the additional double point. If thus obtained extension  $\bar{\chi}$  vanishes on  $\mathcal{LM}_{\geq r+1}(c)$ , i.e., on all  $c$ -link maps

with at least  $r + 1$  transversal double points, for some finite  $r$ , then  $\chi$  is called a *finite c-type* invariant, namely of *c-type*  $r$ . Note that in our notation any *c-type*  $r$  invariant is also of *c-type*  $r + 1$ . Sometimes we will identify a finite *c-type* invariant  $\chi: \mathcal{LM}_0(c) \rightarrow G$  with its extension  $\bar{\chi}: \mathcal{LM}(c) \rightarrow G$ .

For  $m = 1$  there is only one coloring  $c$ , and finite *c-type* invariants, normalized by  $\chi(\text{unknot}) = 0$ , coincide with the usual Goussarov–Vassiliev invariants of knots. If  $c$  is a constant map, we write “type” instead of “*c-type*”, and if  $c$  is the identity, we write “KL-type” instead of “*c-type*”, and  $\mathcal{LM}_*$  instead of  $\mathcal{LM}_*(c)$ . If the coloring  $c$  is a composition  $c = de$ , then any *d-type*  $k$  invariant is also a *c-type*  $k$  invariant. In particular, any type  $k$  invariant is a KL-type  $k$  invariant, but not vice versa. Indeed, the linking number  $\text{lk}(L)$  and the generalized Sato-Levine invariant (see §3) are of KL-types 0 and 1, respectively, but of types exactly 1 and 3.

It is arguable that triple and higher  $\bar{\mu}$ -invariants with distinct indices should be regarded as having KL-type 0 (as they assume values in different cyclic groups depending on link homotopy classes of proper sublinks, we must either agree to imagine these groups as subgroups, say, of  $\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ , or extend the definition of finite KL-type invariants to include the possibility where each path-connected component of  $\mathcal{LM}$  maps to its own range). In contrast, there are almost no finite type invariants of link homotopy [24], although this can be remedied by consideration of string links (see references in Hughes [12] or Masbaum and Vaintrob [23]) or partially defined finite type invariants [9, 12]. We show in §4 that certain  $\bar{\mu}$ -invariants of 2-component links have integer liftings of finite KL-type, at least some of which are not of finite type due to

**Proposition 2.1.** *The KL-type 0 invariant  $v(L) = (-1)^{\text{lk}(L)}$  is not of finite type.*

*Proof.* By induction,  $v(L_s) = \pm 2^k$  for any singular link  $L_s$  with  $k$  transversal intersections between distinct components and no self-intersections.  $\square$

It is clear that KL-type  $r$  invariants  $\chi: \mathcal{LM}_0 \rightarrow G$  form an abelian group, which we denote by  $G_r$  (or  $G_r^{(m)}$  when the number of components needs to be specified). Clearly,  $G_0$  is the direct sum of  $|\pi_0(\mathcal{LM})|$  copies of  $G$ . Let  $G_r(\lambda)$  denote the subgroup of  $G_r$  consisting of invariants vanishing on all links whose link homotopy class is not  $\lambda \in \pi_0(\mathcal{LM})$ , and

on a fixed link  $B_\lambda \in \lambda$  with unknotted components. The latter can always be achieved by adding a KL-type 0 invariant, so the quotient map  $q: G_r \rightarrow G_r/G_0$  takes  $G_r(\lambda)$  isomorphically onto a subgroup, independent of the choice of  $B_\lambda$ ; moreover  $G_r/G_0 = \bigoplus_\lambda q(G_r(\lambda))$ . For any  $\text{lk} \in \mathbb{Z}$ , it was proved in Kirk and Livingston [16] that  $\mathbb{Z}_1^{(2)}(\text{lk}) \simeq \mathbb{Z}$  (generated by the generalized Sato–Levine invariant) and conjectured that  $\mathbb{Z}_r^{(2)}(\text{lk})$  is not finitely generated for  $r > 1$ . It is well-known [33] that the group of monochromatic  $G$ -valued type  $r$  invariants of  $m$ -component links is finitely generated; in particular, so is  $G_r^{(1)}$ . Let  $\tilde{G}_r$  denote the subgroup of  $G_r$  consisting of those invariants which remain unchanged under tying local knots, i.e., under PL isotopy. Notice that the well-known proof that type 1 invariants of knots are trivial works also to show that  $\tilde{G}_1 = G_1$ . In the following remarks we address the difference between  $G_r$  and  $\tilde{G}_r$  for  $r > 1$ .

*Remarks.* (i). By evaluating any type  $r$  invariant of knots on the components of a link  $L \in \lambda$  we obtain a monomorphism

$$\Lambda: \bigoplus_{i=1}^m G_r^{(1)} \hookrightarrow G_r(\lambda).$$

We claim that its image (whose elements we regard here as the “least interesting” among all) is a direct summand, with complement containing  $\tilde{G}_r(\lambda)$ . Indeed, for any  $\chi \in G_r(\lambda)$  define a knot invariant  $\chi_i$  by  $\chi_i(K) = \chi(K_i)$ , where  $K_i$  is the link, obtained by tying the knot  $K$  locally on the  $i^{\text{th}}$  component of  $B_\lambda$ . Then  $\chi_i$  is a type  $r$  knot invariant, since the local knot on the  $i^{\text{th}}$  component of  $K_i$ , viewed as a knotted ball pair  $(B^3, B^1)$ , can be chosen to look precisely the same as  $K$  outside a small ball. Define an endomorphism  $\Phi$  of  $G_r(\lambda)$  by  $\Phi(\chi)(L) = \chi(L) - \sum_{i=1}^m \chi_i(L^i)$ , where  $L^i$  denotes the  $i^{\text{th}}$  component of  $L$ . Then  $\Phi$  takes any  $\chi \in \text{im } \Lambda$  to zero and any  $\chi \in \tilde{G}_r(\lambda)$  to itself, and consequently defines a splitting (depending, in general, on the choice of  $B_\lambda$ ) of the quotient map  $G_r(\lambda) \rightarrow \text{coker } \Lambda$ , with image containing  $\tilde{G}_r(\lambda)$ .

(ii). We claim that an invariant  $\chi$  in the complement to  $\text{im } \Lambda$  is invariant under tying local knots iff the restriction of  $\bar{\chi}$  to  $\mathcal{LM}_1$  (which is a KL-type  $r - 1$  invariant of singular links) is. Indeed, the “only if” implication is trivial. Conversely, it suffices to find one link  $L \in \lambda$

such that  $\chi(L)$  is unchanged when any local knot is added to  $L$ . Since  $\chi = \Phi(\chi)$ , clearly  $B_\lambda$  is such a link.

In particular, since KL-type 1 invariants of singular links are invariant under tying local knots due to the one-term (framing independence) relation, it follows that

$$G_2(\lambda) = \tilde{G}_2(\lambda) \oplus \text{im } \Lambda.$$

**Theorem 2.2.** *Let  $\chi$  be a KL-type  $k$  invariant. If  $\chi$  is invariant under PL isotopy, then it is invariant under  $k$ -quasi-isotopy.*

*Proof.* For  $k = 0$  there is nothing to prove, so assume  $k \geq 1$ . It suffices to show that any  $\chi \in \tilde{G}_k$  vanishes on any  $k$ -quasi-embedding  $f: mS^1 \rightarrow S^3$  with precisely one double point  $f(p) = f(q)$ . Let  $P_0, \dots, P_k$  and  $J_0, \dots, J_{k-1}$  be as in the definition of  $k$ -quasi-embedding, and let  $\tilde{J}_0$  denote the subarc of  $J_0$  with  $\partial \tilde{J}_0 = \{p, q\}$ . In order to have enough room for general position we assume that  $P_i$ 's are compact 3-manifolds (with boundary) and  $\tilde{J}_0 \subset \text{Int } J_0$  (this can be achieved by taking small regular neighborhoods).

Since the inclusion  $P_0 \cup f(J_0) \hookrightarrow P_1$  is null-homotopic, there exists a generic PL homotopy  $f'_t: f(mS^1) \rightarrow S^3$  such that the homotopy  $f_t := f'_t f: mS^1 \rightarrow S^3$  satisfies  $f_0 = f$ ,  $f_t(J_0) \subset P_1$ ,  $f_t = f$  outside  $J_0$ , and  $f_1(\tilde{J}_0)$  is a small circle, bounding an embedded disk in the complement to  $f_1(mS^1)$ . Using the one-term (framing independence) relation, we see that any finite KL-type invariant vanishes on  $f_1$ . Hence any KL-type 1 vanishes on  $f_0 = f$ , which completes the proof for  $k = 1$ .  $\square$

Before proceeding to the general case, we state the following generalization of the one-term (framing independence) relation.

**Lemma 2.3.** *If  $f \in \mathcal{LM}_n$  has all its double points inside a ball  $B$  such that  $f^{-1}(B)$  is an arc, then for any  $\chi \in \tilde{G}_n$ ,  $\chi(f) = 0$ .*

*Proof.* By induction on  $n$ .  $\square$

*Proof of 2.2 (continued).* Let us study the jump of our invariant  $\chi \in \tilde{G}_k$  on the link maps  $f_t$  occurring at singular moments of the homotopy  $f'_t$ . Let  $f_{t_1} \in \mathcal{LM}_2$  be one. Using

the definition of  $k$ -quasi-isotopy,  $k \geq 2$ , we will now construct a generic PL homotopy  $f'_{t_1,t}: f_{t_1}(mS^1) \rightarrow S^3$  such that the homotopy  $f_{t_1,t} := f'_{t_1,t}f: mS^1 \rightarrow S^3$  satisfies  $f_{t_1,0} = f_{t_1}$ ,  $f_{t_1,t}(J_1) \subset P_2$ ,  $f_{t_1,t} = f_{t_1}$  outside  $J_1$ , and  $f_{t_1,1}$  takes  $J_1$  into a ball  $B_1$  in the complement to  $f_{t_1,1}(mS^1 \setminus J_1)$ .

Indeed, since  $f_{t_1}(J_1) \subset P_1 \cup f(J_1)$  is null-homotopic in  $P_2$ , we can contract the 1-dimensional polyhedron  $f_{t_1}(J_1 \setminus U)$ , where  $U$  is a regular neighborhood of  $\partial J_1$  in  $J_1$ , embedded by  $f_{t_1}$ , into a small ball  $B'_1 \subset S^3 \setminus f_{t_1}(mS^1 \setminus J_1)$ , by a homotopy  $f_{t_1,t}|_{J_1}$  as required above. Joining the endpoints of  $f_{t_1}(mS^1 \setminus \partial J_1)$  to those of  $f_{t_1,t}(J_1 \setminus U)$  by two embedded arcs  $f_{t_1,t}(U)$  in an arbitrary way (continuously depending on  $t$ ), we obtain the required homotopy. Taking a small regular neighborhood of  $B'_1 \cup f_{t_1,1}(J_1)$  relative to  $f_{t_1,1}(\partial J_1)$  we obtain the required ball  $B_1$ .<sup>1</sup>

By Lemma 2.3, any finite KL-type invariant vanishes on  $f_{t_1,1}$ . The homotopy  $f_{t_1,t}$  as well as the analogously constructed  $f_{t_i,t}$  for each  $f_{t_i} \in \mathcal{LM}_2$  does not change any invariant of KL-type 2, so for  $k = 2$  we are done. The proof in the general case must be transparent now. For completeness, we state  $\square$

**2.4. The  $n^{\text{th}}$  inductive step.** *For each critical level  $f_{t_{i_1}, \dots, t_{i_n}} \in \mathcal{LM}_{n+1}$  of any of the homotopies constructed in the previous step, there exists a generic PL homotopy*

$f'_{t_{i_1}, \dots, t_{i_n}, t}: f_{t_{i_1}, \dots, t_{i_n}}(mS^1) \rightarrow S^3$  such that the homotopy  $f_{t_{i_1}, \dots, t_{i_n}, t} := f'_{t_{i_1}, \dots, t_{i_n}, t}f_{t_{i_1}, \dots, t_{i_n}}$  satisfies:

- (i)  $f_{t_{i_1}, \dots, t_{i_n}, t} = f_{t_{i_1}, \dots, t_{i_n}}$  for  $t = 0$  and outside  $J_n$ ;
- (ii)  $f_{t_{i_1}, \dots, t_{i_n}, t}(J_n) \subset P_{n+1}$  for each  $t$ ;
- (iii)  $f_{t_{i_1}, \dots, t_{i_n}, 1}^{-1}(B_{n-1}) = J_n$  for some PL 3-ball  $B_{n-1}$ .

*Remark.* In fact, the proof of Theorem 2.2 works under a weaker assumption and with a stronger conclusion. Namely,  $k$ -quasi-isotopy can be replaced with virtual  $k$ -quasi-isotopy (see definition in Melikhov and Repovš [25]), whereas indistinguishability by KL-type  $k$

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<sup>1</sup> Using the notation from Melikhov and Repovš [25], the homotopy  $f_{t_1,t}$  can be visualized as shifting the arcs  $f(I'_j)$  onto the arcs  $F(I_j)$  and then taking the image of  $J_1$  into a ball along the track of the null-homotopy  $F$ .

invariants (which can be thought of as a formal generalization of  $k$ -equivalence in the sense of Goussarov [10], Habiro and Stanford) can be replaced with *geometric  $k$ -equivalence*, defined as follows. For each  $n > 0$ , let  $\mathcal{LM}_{n;0}$  denote the subspace of  $\mathcal{LM}_n$  consisting of the link maps  $l$  such that all singularities of  $l$  are contained in a ball  $B$  such that  $l^{-1}(B)$  is an arc. We call link maps  $l, l' \in \mathcal{LM}_n$  *geometrically  $k$ -equivalent* if they are homotopic in the space  $\mathcal{LM}_n \cup \mathcal{LM}_{n+1;k}$ , where  $\mathcal{LM}_{i;k}$  for  $k > 0, i > 0$  consists of those link maps with  $i$  singularities which are geometrically  $(k-1)$ -equivalent to a link map in  $\mathcal{LM}_{i;0}$ .

To see why geometric  $k$ -equivalence may differ from  $k$ -quasi-isotopy, we employ the notion of weak  $k$ -quasi-isotopy, which is defined similarly to  $k$ -quasi-isotopy, but with null-homotopies replaced by null-homologies, cf. [26].

The difference between geometric  $k$ -equivalence and weak  $k$ -quasi-isotopy is analogous to the difference between the lower central series and the derived series, and for  $k \geq 2$  can be visualized as follows. Let  $M_k$  denote the  $k$ -th Milnor link (cf., e.g., Fig. 1 in Melikhov and Repovš [26]), and let  $M_k^W, k \geq 2$ , be  $M_k$  where the “long” component is replaced by its Whitehead double. Unclasping the clasp of the doubled component is a 1-quasi-isotopy  $h_t$  from  $M_k^W$  to the unlink. This  $h_t$  is not a 2-quasi-isotopy (even not a weak 2-quasi-isotopy), since  $M_k$  is not a boundary link, as detected by Cochran’s invariants (cf., e.g., [26]). It would thus be in the spirit of Conjecture 1.1 in Melikhov and Repovš [26] to conjecture that there exists no 2-quasi-isotopy, even weak, from  $M_k^W$  to the unlink. However, since  $M_k$  is  $(k-1)$ -quasi-isotopic to the unlink,  $h_t$  realizes geometric  $k$ -equivalence between  $M_k^W$  and the unlink.

For  $k = 1$  no such example exists, since geometric 1-equivalence clearly implies weak 1-quasi-isotopy. For 2-component links the converse also holds (cf. Fig. 2 in Milnor [27]), amplifying the analogy with the lower central series and the derived series, whose respective initial terms coincide. Thus the difference between 1-quasi-isotopy and geometric 1-equivalence for 2-component links reduces to the difference between 1-quasi-isotopy and weak 1-quasi-isotopy (see Fig. 2(d) in Melikhov and Repovš [25]).

Existence of difference between geometric  $k$ -equivalence and “indistinguishability by KL-type  $k$  invariants, well-defined up to PL isotopy,” already for  $k = 1$ , would follow

from the (expected) positive solution to Problem 1.5 in Melikhov and Repovš [25] for weak 1-quasi-isotopy.

## CHAPTER 3

### THE CONWAY POLYNOMIAL

Recall that the *Conway polynomial* of a link  $L$  is the unique polynomial  $\nabla_L(z)$  satisfying  $\nabla_{\text{unknot}} = 1$  and the crossing change formula

$$\nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z), \quad (1)$$

where  $L_+$  and  $L_-$  differ by a positive crossing change, and  $L_0$  is obtained by oriented smoothing of the self-intersection of the intermediate singular link  $L_s$ . Note that  $L_0$  has one more (respectively less) component than  $L_+$  and  $L_-$  if the intersection is a self-intersection of some component (resp. an intersection between distinct components).

The skein relation (1) shows that the coefficient at  $z^k$  in  $\nabla_L$  is a type  $k$  invariant. The generalized one-term relation for finite type invariants specializes here to the equation  $\nabla_L = 0$  for any split link  $L$  (i.e., a link whose components can be split in two nonempty parts by an embedded sphere). Using (1) it is now easy to see that the Conway polynomial of an  $m$ -component link  $L$  is necessarily of the form

$$\nabla_L(z) = z^{m-1}(c_0 + c_1 z^2 + \cdots + c_n z^{2n})$$

for some  $n$ . By (1),  $c_0(K) = 1$  for any knot  $K$ , which can be used to recursively evaluate  $c_0$  on any link  $L$ . For example, it is immediate that  $c_0(L)$  is the linking number of  $L$  for  $m = 2$ , and  $c_0(L) = ab + bc + ca$  for  $m = 3$ , where  $a, b, c$  are the linking numbers of the 2-component sublinks (cf. [22]). For arbitrary  $m$ , it is easy to see that  $c_0(L)$  is a symmetric polynomial in the pairwise linking numbers (and thus a KL-type 0 invariant); see [18, 23] for an explicit formula.

**Proposition 3.1.**  $c_k(L)$  is of KL-type  $2k$ .

*Proof.* Let  $L$  be a  $c$ -colored link, and let  $|c|$  denote the number of colors used, i.e., the cardinality of the image of  $c$ . Then the coefficient of  $\nabla_L$  at  $z^{|c|-1}$  is either  $c_0(L)$  or 0

according as  $c$  is onto or not; in either case, it is a  $c$ -type 0 invariant. An induction on  $i$  using the skein relation (1) shows that the coefficient of  $\nabla_L$  at  $z^{|c|-1+i}$  is of  $c$ -type  $i$ . In the case where  $c$  is the identity, this is our assertion.  $\square$

In order to compute  $\nabla_K$  for a knot  $K$ , we could consider a sequence of crossing changes in a plane diagram of  $K$ , turning  $K$  into the unknot (cf. [31]). Then  $\nabla_K = \nabla_{\text{unknot}} + z \sum \varepsilon_i \nabla_{L_i}$ , where  $L_i$  are the 2-component links obtained by smoothing the crossings and  $\varepsilon_i = \pm 1$  are the signs of the crossing changes. We could further consider a sequence of crossing changes in the diagram of each  $L_i$ , involving only crossings of distinct components and turning  $L_i$  into a split link. This yields  $\nabla_{L_i} = z \sum \varepsilon_{ij} \nabla_{K_{ij}}$ , where  $K_{ij}$  are the knots obtained by smoothing the crossings and  $\varepsilon_{ij} = \pm 1$  are the signs of the crossing changes. Since the diagram of each  $K_{ij}$  has fewer crossings than that of  $K$ , we can express  $\nabla_K$ , iterating this procedure, as  $\sum_k z^{2k} (\sum_i \epsilon_{ki} \nabla_{\text{unknot}})$ , where the signs  $\epsilon_{ki} = \pm 1$  are determined by the above construction. Since  $\nabla_{\text{unknot}} = 1$ , plainly  $\nabla_K = \sum_k z^{2k} \sum_i \epsilon_{ki}$ . Note that this procedure shows that  $\nabla_K$  is indeed a polynomial (rather than a power series) for any knot  $K$ ; a similar argument works for links.

Now let  $L$  be a link, and suppose that  $L'$  is obtained from  $L$  by tying a knot  $K$  locally on one of the components. We can echo the above construction, expressing  $\nabla_{L'}$  as  $\sum_k z^{2k} \sum_i \epsilon_{ki} \nabla_L$ , where the signs  $\epsilon_{ki}$  are same as above. Thus<sup>1</sup>  $\nabla_{L'} = \nabla_L \nabla_K$ . It follows that the power series

$$\nabla_L^*(z) := \frac{\nabla_L(z)}{\nabla_{K_1}(z) \cdots \nabla_{K_m}(z)},$$

where  $K_1, \dots, K_m$  are the components of  $L$ , is invariant under PL isotopy of  $L$ . Note that the above formula can be rewritten as  $\nabla_L^* = \nabla_L - (\nabla_{K_1} \cdots \nabla_{K_m} - 1) \nabla_L^*$ , meanwhile  $\nabla_{K_1} \cdots \nabla_{K_m}$  is of the form  $1 + b_1 z^2 + \cdots + b_n z^{2n}$  for some  $n$ , where  $b_i(L) = \sum_{i=i_1+\dots+i_k} c_{i_1}(K_1) \cdots c_{i_m}(K_m)$ . We find that  $\nabla_L^*$  is of the form

$$\nabla_L^*(z) = z^{m-1} (\alpha_0 + \alpha_1 z^2 + \alpha_2 z^4 + \dots),$$

---

<sup>1</sup> The conclusion is, of course, well-known, cf. [20], but we need the argument in order to set up notation for use in the proof of Theorem 3.4.

where  $\alpha_i = c_i - (\alpha_{i-1}b_1 + \cdots + \alpha_0 b_i)$ . Hence  $\nabla_L^* \in \mathbb{Z}[[z]]$  (rather than just  $\mathbb{Q}[[z^{\pm 1}]]$ ), and  $c_k \equiv \alpha_k \pmod{\gcd(c_0, \dots, c_{k-1})}$ . In particular, for  $m = 2$  we see that  $\alpha_0(L) = c_0(L)$  is the linking number, and  $\alpha_1(L) = c_1(L) - c_0(L)(c_1(K_1) + c_1(K_2))$  is (cf. [22]) the generalized Sato–Levine invariant!

Under the *generalized Sato–Levine invariant* we mean the invariant that emerged in the work of Polyak–Viro (see [2]), Kirk–Livingston [16, 22], Akhmetiev (see [1]) and Nakanishi–Ohyama [30]; see also [25].

*Remark.* For  $m = 2$  we can also obtain  $\alpha_1(L)$  from  $c_1(L)$  by applying the projection  $\Phi$  from Remark (i) in §2, with a certain choice of  $B_\lambda$  (cf. [16]). This is not surprising, because  $c_1$  has KL-type 2 by Proposition 3.1, hence  $\Phi(c_1)$  is invariant under PL-isotopy by Remark (ii) in §2.

*Remark.* Clearly, the power series  $\nabla_L^*$  is actually a polynomial if the components of  $L$  are unknotted or, more generally, have no non-local knots. Due to a splitting of the multi-variable Alexander polynomial (see §4),  $\nabla_L^*$  splits into a product of a polynomial (which is a quotient of the original  $\nabla_L$ ) and a power series, both invariant under PL isotopy.

**Theorem 3.2.** (i) For each  $L: mS^1 \hookrightarrow S^3$  and any  $n \in \mathbb{N}$  there exists an  $\varepsilon_n > 0$  such that if  $L': mS^1 \hookrightarrow S^3$  is  $C^0$   $\varepsilon_n$ -close to  $L$ ,

$$\nabla_{L'}^*(z) \equiv \nabla_L^*(z) \pmod{(z^n)}.$$

- (ii)  $\nabla_L^*$  can be uniquely extended to all topological links in  $S^3$ , preserving (i).
- (iii) The extended  $\nabla_L^*$  is invariant under TOP isotopy of  $L$ .

Of course, the extended  $\nabla_L^*$  need not be a rational power series for some wild links (which therefore will not be TOP isotopic to any PL links).

*Proof.* The coefficient  $d_k$  of  $\nabla_L^*$  at  $z^k$  is of (monochromatic) type  $k$  since it is a polynomial in the coefficients of the Conway polynomials of  $L$  and its components, homogeneous of degree  $k$  with respect to the degrees (in  $z$ ) of the corresponding terms. Since  $d_k$  is invariant under PL isotopy, by Theorem 2.2 it is invariant under  $k$ -quasi-isotopy. The

assertions (i) and (ii) now follow from Theorem 1.3(a), and (iii) either from [25] or, alternatively, from (ii) and compactness of the unit interval.  $\square$

Proposition 3.1 implies that each  $\alpha_k(L)$  is of KL-type  $2k$ . It is easy to check that  $\alpha_1(L) = c_1(L) - c_0(L)(\sum_i c_1(K_i))$  is, in fact, of KL-type 1 and that

$$\alpha_2(L) = c_2(L) - \alpha_1(L) \left( \sum_i c_1(K_i) \right) - c_0(L) \left( \sum_i c_2(K_i) + \sum_{i \neq j} c_1(K_i)c_1(K_j) \right)$$

is of KL-type 3.

**Proposition 3.3.**  $\alpha_2(L)$  is not of KL-type 2 for  $m = 2$ .

*Proof.* Consider the case  $\text{lk}(L) = 0$ , then  $\alpha_2(L) = c_2(L) - c_1(L)(c_1(K_1) + c_1(K_2))$ , which can be also written as  $a_5(L) - a_3(L)(a_2(K_1) + a_2(K_2))$ , where  $a_i(L)$  denotes the coefficient of  $\nabla_L$  at  $z^i$ . The “third derivative” of  $\alpha_2(L)$ , i.e., restriction of  $\bar{\alpha}_2(L)$  to  $\mathcal{LM}_3^2$  can be found using the Leibniz rule  $(\chi\psi)(L_s) = \chi(L_+)\psi(L_s) + \chi(L_s)\psi(L_-)$ . In the case where the 3 singular points are all on the same component  $K_{sss}$ , it is given by

$$\bar{\alpha}_2(L_{sss}) = a_2(L_{000}) - a_2(L_{++0})a_0(K_{00-}) - a_2(L_{+0+})a_0(K_{0-0}) - a_2(L_{0++})a_0(K_{-00})$$

where, as usual, + and – stand for the overpass and the underpass, and 0 for the smoothing of the crossing  $s$ . Assuming that each  $L_{***}$  on the right hand side has three components and each  $K_{***}$  only one component, we can simplify this as  $c_0(L_{000}) - c_0(L_{++0}) - c_0(L_{+0+}) - c_0(L_{0++})$ .

Let  $\varphi: S^1 \looparrowright S^2$  be a generic  $C^1$ -approximation with 3 double points of the clockwise double cover  $S^1 \rightarrow S^1 \subset \mathbb{R}^2$ , and let  $A, B, C, D$  denote the 4 bounded components of  $\mathbb{R}^2 \setminus \varphi(S^1)$  such that  $D$  contains the origin. Let  $K_{sss} \in \mathcal{LM}_3^1$  be the composition of  $\varphi$  and the inclusion  $\mathbb{R}^2 \subset S^2 \subset S^3$ , and let  $K: S^1 \hookrightarrow S^3 \setminus K_{sss}(S^1)$  be a knot in the complement of  $K_{sss}$  linking the clockwise oriented boundaries of  $A, B, C, D$  with linking numbers  $a, b, c, d$  such that  $a + b + c + 2d = 0$ . Finally, define  $L_{sss} \in \mathcal{LM}_3^2$  to be the union of  $K$  and  $K_{sss}$ , then  $\text{lk}(L_{sss}) = a + b + c + 2d = 0$  and we find that

$$\bar{\alpha}_2(L_{sss}) = (a + b + c + d)d - (a + c + d)(b + d) - (a + b + d)(c + d) - (b + c + d)(a + d).$$

This expression is nonzero, e.g., for  $a = 1, b = 2, c = -3, d = 0$ .  $\square$

By Proposition 3.1 and Theorem 2.2,  $\alpha_n(L)$  is invariant under  $2n$ -quasi-isotopy.

However, according to Proposition 3.3, the following strengthening of this assertion cannot be obtained by means of Theorem 2.2.

**Theorem 3.4.**  $\alpha_n(L)$  is invariant under  $n$ -quasi-isotopy.

The proof makes use of the following notion. We define *colored link homotopy* to be the equivalence relation on the set of links colored with  $m$  colors, generated by intersections between components of the same color (including self-intersections) and addition of trivial component of any color, separated from the link by an embedded sphere. Thus an  $m$ -component colored link  $L$  is colored link homotopic with an  $(m+k)$ -component colored link  $L'$  iff  $L'$  is homotopic to  $L \sqcup T_k$  through colored link maps, where  $T_k$  is the  $k$ -component unlink, split from  $L$  by an embedded sphere and colored in some way. Such a homotopy will be called a colored link homotopy between  $L$  and  $L'$ .

*Proof.* Let us start by considering the above procedure for computing the Conway polynomial of a knot  $K$  in more detail. One step of this procedure yields  $c_k(K) = c_k(\text{unknot}) + \sum \varepsilon_i c_k(L_i)$  and  $c_k(L_i) = \sum \varepsilon_{ij} c_{k-1}(K_{ij})$ . Since  $c_0 = 1$  for every knot,  $c_k(K)$  can be computed in  $k$  steps, regardless of the number of crossings in the diagram of  $K$ . Moreover, if one is only interested in finding  $c_k(K)$  for a given  $k$  (which would not allow, e.g., to conclude that  $\nabla_K$  is a polynomial not just a power series), the computation could be based on arbitrary generic PL homotopies rather than those suggested by the diagram of  $K$ . In particular, we allow (using that  $c_0 = 0$  for every link with  $> 1$  components and each  $c_i = 0$  for a trivial link with  $> 1$  components) self-intersections of components in the homotopies from  $L_i$ 's to split links, so that  $K_{ij}$ 's may have three and, inductively, any odd number of components. For such an  $n$  step procedure, the equality  $\nabla_K = \sum_{k \leq n} z^{2k} \sum_i \epsilon_{ki}$ , where the signs  $\epsilon_{ki}$  are determined by the homotopies, holds up to degree  $2n$ .

Now let  $L$  be an  $m$ -component link, and suppose that  $L'$  is obtained from  $L$  by a generic PL link homotopy  $H$  with a single singular level  $L_s$ . Color the components of  $L$  with distinct colors, then the  $(m+1)$ -component smoothing  $L_0$  of  $L_s$  is naturally colored with  $m$  colors. Suppose inductively that  $L_{0i_2 \dots i_{k+1}}$  is one of the smoothed singular links

in a generic PL colored link homotopy  $H_{0i_2\dots i_k}$  between the link  $L_{0i_2\dots i_k}$  and some link  $L'_{0i_2\dots i_k}$  which either has one less component than  $L_{0i_2\dots i_k}$  or coincides (geometrically) with  $L$ . If there exist such homotopies  $H_{0i_2\dots i_l}$  for  $l = 1, \dots, 2k$ , we say that  $L_s$  is a  $+(-)$ -*skein k-quasi-embedding*, provided that the self-intersection in  $H$  is positive (negative). A *skein k-quasi-isotopy* is a link homotopy where every singular level is a  $+$  or  $--$ -skein  $k$ -quasi-embedding, depending on the sign of the self-intersection.

To see that  $\alpha_n$  is invariant under skein  $n$ -quasi-isotopy <sup>2</sup>, we return to the original link homotopy  $H$  and denote by  $K$  and  $K'$  the components of  $L$  and  $L'$  each

concolor with two components of  $L_0$ . As in the above argument for knots, we have

$$\nabla_{K'} = \sum_{k \leq n} z^{2k} (\sum_i \epsilon_{ki} \nabla_K) + z^{2n+1} P \text{ and } z^{1-m} \nabla_{L'} = \sum_{k \leq n} z^{2k} (\sum_i \epsilon_{ki} z^{1-m} \nabla_L) + z^{2n+1} Q,$$

where  $P(z)$  and  $Q(z)$  are some polynomials, and the signs  $\epsilon_{ki}$  are determined by the homotopies  $H_{0i_2\dots i_k}$ . Then

$$\frac{z^{1-m} \nabla_{L'}}{\nabla_{K'}} = \frac{z^{1-m} \nabla_L R + z^{2n+1} P}{\nabla_K R + z^{2n+1} Q} = \frac{z^{1-m} \nabla_L}{\nabla_K} + z^{2n+1} S,$$

where  $R(z) = \sum_{k \leq n} z^{2k} \sum_i \epsilon_{ki}$  and  $S(z)$  is some power series, and the assertion follows.

To complete the proof, we show that  $n$ -quasi-isotopy implies skein  $n$ -quasi-isotopy. It suffices to consider a crossing change on a component  $K$  of  $L$ , satisfying the definition of  $n$ -quasi-isotopy. Let  $f, J_0, \dots, J_n$  and  $P_0, \dots, P_n$  be as in the definition of  $n$ -quasi-embedding; we can assume that  $P_1$  contains a regular neighborhood of  $f(J_0)$  containing  $L(J_0)$ . We associate to every link  $L_{0*}$ , such that  $L \setminus K$  is a geometric sublink of  $L_{0*}$ , but  $L$  itself is not, the collection of positive integers  $d(L_{0*}) = (d_0, \dots, d_m)$ , where  $d_i$  for  $i > 0$  (resp.  $i = 0$ ) is the minimal number such that the  $i^{\text{th}}$  component of  $L_{0*}$  not in  $L \setminus K$  is null-homotopic in  $P_{d_i}$  (resp. is homotopic to  $K$  with support in  $P_{d_i}$ ). It is easy to see that

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<sup>2</sup> By the proof of Theorem 4.9 below, and since  $\alpha_{0,2n}$  and  $\alpha_{2n,0}$  in Theorem 4.8 depend only on the linking number, for  $n > 0$  each  $\alpha_n$  is actually invariant under skein  $(n - \frac{1}{2})$ -quasi-isotopy.

$d(L_{0i_2 \dots i_k})$  is obtained from  $d(L_{0i_2 \dots i_k})$  by one of the following two operations:

$$(d_0, \dots, d_i, \dots, d_m) \xrightarrow{\alpha_i} (d_0, \dots, d_i + 1, d_i + 1, \dots, d_m);$$

$$(d_0, \dots, d_i, \dots, d_j, \dots, d_m) \xrightarrow{\beta_{ij}} (d_0, \dots, \max(d_i, d_j), \dots, \hat{d}_j, \dots, d_m).$$

Conversely, if for some  $i$  the operation  $\alpha_i$  and all operations  $\beta_{ij}$ 's lead to collections of integers not exceeding  $n$ , one can construct a homotopy  $H_{0i_2 \dots i_k}$  between  $L_{0i_2 \dots i_k}$  and some  $L'_{0i_2 \dots i_k}$  which either has one less component than  $L_{0i_2 \dots i_k}$  or coincides with  $L$ .

Suppose that none of the integers  $d(L_{0i_2 \dots i_k})$  exceeds  $l + 1$ , and at least  $r$  of them do not exceed  $l$ . If  $r \geq 1$ , let  $d_i$  be one of these  $r$ , then none of the integers  $\gamma(d(L_{0i_2 \dots i_k}))$ , where  $\gamma$  is  $\alpha_i$  or  $\beta_{ij}$ , exceeds  $l + 1$ , and at least  $r - 1$  of them do not exceed  $l$ . Thus if at least two components of  $L_{0i_2 \dots i_k}$  are not in  $L \setminus K$  and none of the integers  $d(L_{0i_2 \dots i_k})$  exceeds  $l$ , we can construct a homotopy  $H_{0i_2 \dots i_k}$  as above, and for each singular link  $L_{0i_2 \dots i_{k+1}}$  in this homotopy also a homotopy  $H_{0i_2 \dots i_{k+1}}$  as above, so that for each singular link  $L_{0i_2 \dots i_{k+2}}$  in this homotopy, none of the integers  $d(L_{0i_2 \dots i_{k+2}})$  exceeds  $l + 1$ . But it is indeed the case for  $k = 1$ , hence for any odd  $k$ , that at least two components of  $L_{0i_2 \dots i_k}$  are not in  $L \setminus K$ .  $\square$

*Remark.* Note that the above argument does not work for geometric  $k$ -equivalence (see end of §2) in place of  $k$ -quasi-isotopy.

Since  $c_k \equiv \alpha_k \pmod{\gcd(c_0, \dots, c_{k-1})}$ , Theorem 3.4 implies the case  $l = 1$ , and also the 2-component case of

**Theorem 3.5.** [26] Set  $\lambda = \lceil \frac{(l-1)(m-1)}{2} \rceil$ . The residue class of  $c_{\lambda+k}$  modulo  $\gcd$  of  $c_\lambda, \dots, c_{\lambda+k-1}$  and all  $\bar{\mu}$ -invariants of length  $\leq l$  is invariant under  $(\lfloor \frac{\lambda}{m-1} \rfloor + k)$ -quasi-isotopy.

(Here  $\lceil x \rceil = n$  if  $x \in [n - \frac{1}{2}, n + \frac{1}{2})$ , and  $\lfloor x \rfloor = n$  if  $x \in (n - \frac{1}{2}, n + \frac{1}{2}]$  for  $n \in \mathbb{Z}$ .)

One special case not covered by Theorem 3.4 asserts that for 3-component links the residue class of  $c_k$  modulo the greatest common divisor  $\Delta_k$  of all  $\bar{\mu}$ -invariants of length  $\leq k + 1$  is invariant under  $\lfloor \frac{k}{2} \rfloor$ -quasi-isotopy. Naturally, one could wonder whether an integer invariant of  $\lfloor \frac{k}{2} \rfloor$ -quasi-isotopy of 3-component links, congruent to  $c_k \pmod{\Delta_k}$ , can

be found among coefficients of rational functions in  $\nabla^*$  of the link and its two-component sublinks. It turns out that this is not the case already for  $k = 1$ . Indeed, we would get an integer link homotopy invariant of (monochromatic) finite type (specifically, of type 4), which is not a function of the pairwise linking numbers (since  $c_1 \equiv \mu(123)^2 \pmod{\Delta_1}$ , cf. [18, 23]). But this is impossible for 3-component links [24].

Alternatively, one can argue directly as follows. Consider the invariant  $\gamma(L) := \alpha_1(L) - \sum \alpha_0(L_0)\alpha_1(L_1)$ , where the summation is over all ordered pairs  $(L_0, L_1)$  of distinct 2-component sublinks of  $L$ . (Recall that in the 2-component case  $\alpha_0(L)$  and  $\alpha_1(L)$  coincide with the linking number and the generalized Sato-Levine invariant, respectively.) It can be easily verified that  $\gamma(L)$  jumps by

$$\pm\mu(12)(\mu(1, 3^+)\mu(2, 3^-) + \mu(1, 3^-)\mu(2, 3^+))$$

on any singular link  $L_s = K_1 \cup K_2 \cup K_s$  with smoothing  $K_1 \cup K_2 \cup K_{3^+} \cup K_{3^-}$ . One can check that this jump cannot be cancelled by the jump of any polynomial expression in  $c_0(L)$  and the coefficients of the Conway polynomials of the sublinks of  $L$ , homogeneous of degree 4.

*Remark.* An integer invariant of link homotopy of 3-component links is given by  $q\Delta_1 + r$ , where  $q$  is any polynomial in  $\frac{\mu(ij)}{\Delta_1}$  and  $r \equiv \bar{\mu}(123)^2 \pmod{\Delta_1}$ ,  $0 \leq r < \Delta_1$ .

## CHAPTER 4

### THE MULTI-VARIABLE ALEXANDER POLYNOMIAL

The Conway polynomial is equivalent to the monochromatic case of the *Conway potential function*  $\Omega_L$  of the colored link  $L$ , namely  $\nabla_L(x - x^{-1}) = (x - x^{-1})\Omega_L(x)$  for monochromatic  $L$ . For a link  $L$  colored with  $n$  colors,  $\Omega_L \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$  (the ring of Laurent polynomials) if  $L$  has more than one component; otherwise  $\Omega_L$  belongs to the fractional ideal  $(x - x^{-1})^{-1}\mathbb{Z}[x^{\pm 1}]$  in the field of fractions of  $\mathbb{Z}[x^{\pm 1}]$ .

$\Omega_L(x_1, \dots, x_n)$  is a normalized version of the sign-refined Alexander polynomial  $\Delta_L(t_1, \dots, t_n)$ , which is well-defined up to multiplication by monomials  $t_1^{i_1} \dots t_n^{i_n}$ . If  $K$  is a knot,  $(x - x^{-1})\Omega_K(x) = x^\lambda \Delta_K(x^2)$ , whereas for a link  $L$  with  $m > 1$  components,  $\Omega_L(x_1, \dots, x_n) = x_1^{\lambda_1} \dots x_n^{\lambda_n} \Delta_L(x_{c(1)}^2, \dots, x_{c(m)}^2)$ , where  $c: \{1, \dots, m\} \rightarrow \{1, \dots, n\}$  is the coloring, and the integers  $\lambda, \lambda_1, \dots, \lambda_n$  are uniquely determined by the symmetry relation  $\Omega_L(x_1, \dots, x_n) = (-1)^m \Omega_L(x_1^{-1}, \dots, x_n^{-1})$ . We refer to Hartley [11], Traldi [35] and Turaev [36] for definition of the sign-refined Alexander polynomial and discussion of the Conway potential function.

*Remark (not used in the sequel).* The asymmetry of the Alexander polynomial, which forces one to work with its symmetrized version  $\Omega_L$ , goes back to the asymmetry of presentations of the Alexander module or, equivalently, of the Fox differential calculus. The Leibniz rule for the Fox derivative (restricted to the group elements)

$$D(fg) = D(f) + fD(g)$$

arises geometrically from considering lifts of the generators of the fundamental group of the link complement to the universal abelian cover, *starting* at a fixed lift  $\tilde{p}$  of the basepoint. If we wish to base the whole theory on the lifts *ending* at  $\tilde{p}$ , we will have to change the Leibniz rule to

$$D(fg) = D(f)g^{-1} + D(g).$$

The “symmetric” Leibniz rule

$$D(fg) = D(f)g^{-1} + fD(g) + (f^{-1}g^{-1} - g^{-1}f^{-1})$$

does not seem to have a clear geometric meaning. (Note the analogy with the three first-order finite differences.) However, it does correspond to a “symmetric” version of the Magnus expansion, which will be behind the scenes in this section. Indeed, its abelian version gives rise, in Theorem 4.2 below, to a power series  $\mathcal{U}_L$ , in the same way as the abelian version of the usual Magnus expansion leads to Traldi’s parametrization of  $\Omega_L$ , discussed in the proof of Theorem 4.8 below.

**Lemma 4.1.** (compare [15]) (a)  $\Omega_L(x_1, \dots, x_n) = \Omega_L(-x_1^{-1}, \dots, -x_n^{-1})$ .

(b) If  $L$  has  $m > 1$  components, the total degree of every nonzero term of  $\Omega_L$  is congruent to  $m \pmod{2}$ .

(c) If  $L$  has  $> 1$  components, the  $x_i$ -degree of every nonzero term of  $\Omega_L$  is congruent mod 2 to the number  $k_i$  of components of color  $i$  plus  $l_i := \sum \text{lk}(K, K')$ , where  $K$  runs over all components of color  $i$  and  $K'$  over those of other colors.

*Proof.* It is well-known that, if all components have distinct colors, the integer  $\lambda_i$  from the above formula relating  $\Omega_L$  to the Alexander polynomial is not congruent to  $l_i \pmod{2}$  (cf. [35]). This proves (c), which implies (b). For links with  $> 1$  components (a) follows from (b) and from the symmetry relation in the definition of  $\Omega_L$ ; for knots – from the relation of  $\Omega_L(z)$  with the Conway polynomial.  $\square$

Computation of  $\Omega_L$  is much harder than that of  $\nabla_L$ , for the skein relation (1) is no longer valid for arbitrary crossing changes. It survives, in the form

$$\Omega_{L_+} - \Omega_{L_-} = (x_i - x_i^{-1})\Omega_{L_0}, \quad (2)$$

in the case where both strands involved in the intersection are of color  $i$ . There are other formulas for potential functions of links related by local moves, which suffice for evaluation of  $\Omega$  on all links; see references in Murakami [29].

Dynnikov [6] and H. Murakami [29] noticed that the coefficients of the power series  $\Omega_L(e^{h_1/2}, \dots, e^{h_n/2})$  are (monochromatic) finite type invariants of  $L$ .

**Theorem 4.2.** *For a link  $L$  colored with  $n$  colors there exists a unique power series  $\mathfrak{U}_L \in \mathbb{Q}[[z_1, \dots, z_n]]$  (unless  $L$  is a knot, in which case  $\mathfrak{U}_L \in z^{-1}\mathbb{Z}[z]$ ) such that*

$$\mathfrak{U}_L(x_1 - x_1^{-1}, \dots, x_n - x_n^{-1}) = \Omega_L(x_1, \dots, x_n).$$

*The coefficient of  $\mathfrak{U}_L$  at a term of total degree  $n$  is of type  $n + 1$ ; moreover,  $\mathfrak{U}_L$  satisfies the skein relation*

$$\mathfrak{U}_{L_+} - \mathfrak{U}_{L_-} = z_i \mathfrak{U}_{L_0}$$

*for intersections between components of color  $i$ . Furthermore, the total degree of every nonzero term of  $\mathfrak{U}_L$  is congruent mod 2 to the number of components of  $L$ , and  $\mathfrak{U}_L(4y_1, \dots, 4y_n) \in \mathbb{Z}[[y_1, \dots, y_n]]$  (unless  $L$  is a knot).*

For  $n = 1$  we see that  $z\mathfrak{U}_L(z)$  coincides with the (finite) polynomial  $\nabla_L(z)$ .

*Proof.* Let  $x(z)$  denote the power series in  $z$ , obtained by expanding the radical in either of  $x_{\pm}(z) = \pm\sqrt{1+z^2/4} + z/2$  by the formula  $(1+t)^r = 1 + rt + \frac{r(r-1)}{2}t^2 + \dots$ . The Galois group of the quadratic equation  $z = x - x^{-1}$  acts on its roots by  $x \mapsto -x^{-1}$ , so Lemma 4.1(a) implies that the coefficients of the power series

$$\mathfrak{U}_L(z_1, \dots, z_n) := \Omega_L(x(z_1), \dots, x(z_n))$$

are independent of the choice of the root  $x_{\pm}$ .

The second assertion follows by Murakami's proof [29] of the result mentioned above, and the third from the skein relation (2). Next, since  $x^{-1} - x = -(x - x^{-1})$ ,

$$\begin{aligned} \mathfrak{U}_L(-z_1, \dots, -z_n) &= \Omega_L(x(z_1)^{-1}, \dots, x(z_n)^{-1}) \\ &= (-1)^m \Omega_L(x(z_1), \dots, x(z_n)) = (-1)^m \mathfrak{U}_L(z_1, \dots, z_n), \end{aligned}$$

which proves the assertion on total degrees of nonzero terms.

The final assertion may be not obvious (since  $(k+1) \nmid \binom{2k-1}{k}$  for  $k = 1, 3, 7$ ) from

$$(1 + 4y)^{1/2} = 1 + 4y \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \binom{2k-1}{k} y^k.$$

However, this power series does have integer coefficients, since this is the case for

$$(1 + 4y)^{-1/2} = 1 + 2 \sum_{k=1}^{\infty} (-1)^k \binom{2k-1}{k} y^k.$$

(In fact, we see that  $(k+1) \mid 2 \binom{2k-1}{k}$ , and  $x_{\pm}(y) \equiv 1 \pmod{2}$ .)  $\square$

Let us consider the case of two colors in more detail. Let  $c_{ij}$  denote the coefficient of  $\mathcal{U}_L(z_1, z_2)$  at  $z_1^i z_2^j$ . Since  $\mathcal{U}(z, z) = z^{-1} \nabla(z)$ ,  $c_{00}$  coincides with the linking number  $c_0$  in the 2-component case and is zero otherwise. The skein relation (2) implies that for a 3-component link  $K_1 \cup K'_1 \cup K_2$ , colored as indicated by subscript,  $c_{10}$  coincides with  $\text{lk}(K_1, K'_1)(\text{lk}(K_1, K_2) + \text{lk}(K'_1, K_2))$  up to a type 0 invariant of the colored link. The latter is identically zero by the connected sum<sup>1</sup> formula

$$\Omega_{L \#_i L'} = (x_i - x_i^{-1}) \Omega_L \Omega_{L'}, \quad (3)$$

which follows from the definition of  $\Omega_L$  in Hartley [11]. Since  $c_{01} = c_0 - c_{10}$  or by an analogous argument,  $c_{01} = \text{lk}(K_1, K_2) \text{lk}(K'_1, K_2)$ . It follows that for 2-component links  $c_{11}$  coincides with the generalized Sato–Levine invariant  $\alpha_1$  up to a KL-type 0 invariant. Since  $\alpha_1 - c_{11}$  is of type 3, it has to be a degree 3 polynomial in the linking number, which turns out to be  $\frac{1}{12}(\text{lk}^3 - \text{lk})$ . (Since  $\Omega_{\text{unlink}} = 0$  and  $\Omega_{\text{Hopf link}} = 1$ , the second Conway identity [11, 15] can be used to evaluate  $\Omega_L$  on a series of links  $\mathcal{H}_n$  with  $\text{lk } \mathcal{H}_n = n$ .) The half-integer  $c_{11}$  is thus the *unoriented generalized Sato–Levine invariant*, which is implicit in the second paragraph of Kirk and Livingston [16]. Specifically,  $2c_{11}/\text{lk}^2$  is (unless  $\text{lk} = 0$ ) the Casson–Walker invariant of the  $\mathbb{Q}$ -homology sphere obtained by 0-surgery on the components of the link; in fact,  $c_{11}(L) = \frac{\alpha_1(L) + \alpha_1(L')}{2}$ , where  $L'$  denotes the link obtained from  $L$  by reversing the orientation of one of the components [16].

The preceding paragraph implies the first part of the following

<sup>1</sup> The multi-valued operation of Hashizume connected sum  $L \#_i L'$  is defined as follows. Let  $L'(mS^1)$  be split from  $L(mS^1)$  by an embedded  $S^2$ , and  $b: I \times I \hookrightarrow S^3$  be a band meeting the embedded  $S^2$  in an arc, and  $L(mS^1)$  (resp.  $L'(mS^1)$ ) in an arc of color  $i$ , identified with  $b(I \times \{0\})$  (resp.  $b(I \times \{1\})$ ). Then  $L \#_i L'(mS^1) = ((L(mS^1) \cup L'(mS^1)) \setminus b(I \times \partial I)) \cup b(\partial I \times I)$ .

**Theorem 4.3.** Consider the power series

$$\mathcal{U}_L^*(z_1, \dots, z_n) := \frac{\mathcal{U}_L(z_1, \dots, z_n)}{\nabla_{K_1}(z_{c(1)}) \cdots \nabla_{K_m}(z_{c(m)})},$$

where  $K_1, \dots, K_m$  denote the components of  $L$ . Let  $\alpha_{ij}$  denote the coefficient of  $\mathcal{U}_L^*(z_1, z_2)$  at  $z_1^i z_2^j$ . For 2-component links

- (i)  $\alpha_{00} = \text{lk}$ , and  $\alpha_{11}$  is the unoriented generalized Sato–Levine invariant, which assumes all half-integer values;
- (ii)  $\alpha_{1,2k-1}$  and  $\alpha_{2k-1,1}$  coincide, when multiplied by  $(-1)^{k+1}$ , with Cochran’s [5] derived invariants  $\beta_\pm^k$ , whenever the latter are defined (i.e.,  $\text{lk} = 0$ );
- (iii)  $\alpha_{ij}$  is of type  $i + j$ ; when  $i + j$  is odd,  $\alpha_{ij} = 0$ .

The last part is an immediate corollary of Theorem 4.2.

*Proof of (ii).* It follows from [5] that  $\sum_{i=1}^{\infty} \beta_L^i (-z^2)^i = \eta_L(y^2)$ , where  $\eta_L$  is Kojima’s  $\eta$ -function and  $z = y - y^{-1}$ . On the other hand, it follows from [13] that

$$\eta_L(y^2) = \pm \frac{(1-y^2)(1-y^{-2})\Omega'_L(1,y)}{\nabla_{K_2}(y-y^{-1})},$$

where  $\Omega'_L$  is given by  $\Omega_L(x_1, x_2) = (x_1 - x_1^{-1})(x_2 - x_2^{-1})\Omega'_L(x_1, x_2)$  for 2-component links with  $\text{lk} = 0$ . It is shown in Jin [13] (see also proof of Theorem 4.8(ii) below) that under this assumption  $\Omega'_L$  is a Laurent polynomial (rather than just a rational function). Define  $\mathcal{U}'_L$  by  $\mathcal{U}_L(z_1, z_2) = z_1 z_2 \mathcal{U}'_L(z_1, z_2)$  for 2-component links with  $\text{lk} = 0$ , then

$$\sum_{i=1}^{\infty} (-1)^i \beta_L^i z^{2i} = \mp \frac{z^2 \mathcal{U}'_L(0, z)}{\nabla_{K_2}(z)},$$

and consequently

$$\sum_{i=0}^{\infty} (-1)^i \beta_L^{i+1} z^{2i} = \pm \frac{\mathcal{U}'_L(0, z)}{\nabla_{K_1}(0) \nabla_{K_2}(z)}.$$

According to (i), the sign must be positive.  $\square$

It turns out that each coefficient of the power series  $\mathcal{U}$  can be canonically split into a  $\mathbb{Q}$ -linear combination of the coefficients of certain  $2^{n-1}$  polynomials.

**Theorem 4.4.** *The Conway potential function of any colored link  $L$  with  $> 1$  components can be uniquely written in the form*

$$\Omega_L(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} \left\{ \frac{x_{i_1}}{x_{i_2}} \cdots \frac{x_{i_{2k-1}}}{x_{i_{2k}}} \right\} P_{i_1 \dots i_{2k}}(\{x_1\}, \dots, \{x_n\}) \quad (4)$$

for some  $P_{i_1 \dots i_{2k}} \in \frac{1}{2}\mathbb{Z}[z_1, \dots, z_n]$ , where  $\{f(x_1, \dots, x_n)\}$  denotes  $f(x_1, \dots, x_n) + f(-x_1^{-1}, \dots, -x_n^{-1})$  for any function  $f(x_1, \dots, x_n)$ . Moreover, if  $n = 2l$ , the coefficients of  $P_{12 \dots n}$  are integer.

Here is the case  $n = 2$  in more detail:

$$\Omega(x, y) = 2P(x - x^{-1}, y - y^{-1}) + (xy^{-1} + x^{-1}y)P_{12}(x - x^{-1}, y - y^{-1}).$$

This case was essentially known in 1986; indeed it is equivalent (see formula (\*) below) to Kidwell's decomposition [15]:

$$\Omega(x, y) = K_1(x - x^{-1}, y - y^{-1}) + (xy + x^{-1}y^{-1})K_2(x - x^{-1}, y - y^{-1}).$$

However, the proof of Theorem 4.8(iii) below (on  $\bar{\mu}$ -invariants) breaks down for  $K_1$  and  $K_2$  in place of  $2P$  and  $P_{12}$ . Moreover, the assertion on integrality of  $2P_{i_1 \dots i_{2k}}$  in Theorem 4.4 will not hold already for  $n = 3$  (resp.  $n = 5$ ) if  $\{\frac{x_{i_1}}{x_{i_2}} \cdots \frac{x_{i_{2k-1}}}{x_{i_{2k}}}\}$  is replaced with  $\{x_{i_1} \cdots x_{i_{2k}}\}$  (resp. with  $\{\frac{x_{i_1} \cdots x_{i_k}}{x_{i_{k+1}} \cdots x_{i_{2k}}}\}$ ) in (4), at least for some Laurent polynomial  $\Omega$  satisfying the conclusion of Lemma 4.1(a). (Lemma 4.1(a) is the only property of  $\Omega_L$  used in the proof of Theorem 4.4.)

*Proof.* By Lemma 4.1(a),  $\Omega_L$  includes together with every term  $Ax_1^{p_1} \cdots x_n^{p_n}$  the term  $(-1)^{p_1 + \dots + p_n} Ax_1^{-p_1} \cdots x_n^{-p_n}$ , and so can be written as a  $\mathbb{Z}$ -linear combination of the L-polynomials  $\{x_1^{p_1} \cdots x_n^{p_n}\}$ . The formula

$$\{x_i M\} - \{x_i^{-1} M\} = \{x_i\} \{M\}, \quad (**)$$

where  $M$  is any monomial in  $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ , can be verified directly (separately for  $M$  of odd and even total degrees), and allows to express each  $\{x_1^{p_1} \cdots x_n^{p_n}\}$  in the form

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} \{x_{i_1} x_{i_2}^{-1} x_{i_3} x_{i_4}^{-1} \cdots (x_{i_{k-1}} x_{i_k}^{-1})^{(-1)^k}\} P'_{i_1 \dots i_k}(\{x_1\}, \dots, \{x_n\})$$

for some  $P'_{i_1 \dots i_k} \in \mathbb{Z}[z_1, \dots, z_n]$ ,  $k > 0$ , and some  $P' \in \frac{1}{2}\mathbb{Z}[z_1, \dots, z_n]$ . The summands corresponding to  $k = 1$  can be included in  $P'$ , and one can get rid of the summands corresponding to odd  $k \geq 3$  by repeated use of the formula

$$2\{x_i x_j^{-1} M\} = \{x_i\} \{x_j^{-1} M\} + \{x_j^{-1}\} \{x_i M\} + \{x_i x_j\} \{M\}, \quad (**)$$

combined with (\*). Note that (\*\*) follows immediately from (\*) and its analogue

$$\{x_i x_j M\} + \{x_i^{-1} x_j^{-1} M\} = \{x_i x_j\} \{M\}.$$

But it is not clear from this approach that the resulting polynomials have half-integer coefficients. To see this, represent  $2\{x_{i_1} x_{i_2}^{-1} x_{i_3} \dots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}$  as

$$\begin{aligned} & \{x_{i_1} x_{i_2}^{-1} x_{i_3} \dots x_{i_{2k}}^{-1} x_{i_{2k+1}}\} - \{x_{i_1}^{-1} x_{i_2} x_{i_3}^{-1} \dots x_{i_{2k}} x_{i_{2k+1}}^{-1}\} \\ & + (1-1) \sum_{j=1}^{2k} \{x_{i_1}^{-1} x_{i_2} x_{i_3}^{-1} \dots (x_{i_{j-2}} x_{i_{j-1}}^{-1} x_{i_j} x_{i_{j+1}} x_{i_{j+2}}^{-1} x_{i_{j+3}})^{(-1)^j} \dots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}. \end{aligned}$$

Then formula (\*) yields:  $2\{x_{i_1} x_{i_2}^{-1} x_{i_3} \dots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}$

$$= \sum_{j=1}^{2k+1} (-1)^{j+1} \{x_{i_j}\} \{x_{i_1}^{-1} x_{i_2} x_{i_3}^{-1} \dots (x_{i_{j-2}} x_{i_{j-1}}^{-1} x_{i_{j+1}} x_{i_{j+2}}^{-1} x_{i_{j+3}})^{(-1)^j} \dots x_{i_{2k}}^{-1} x_{i_{2k+1}}\}.$$

Thus each  $P'_{i_1 \dots i_{2k+1}}$  can be included in the polynomials  $\frac{1}{2}P'_{j_1 \dots j_{2k}}$ .

It remains to verify uniqueness of (4). Suppose, by way of contradiction, that a nontrivial expression  $Q(x_1, \dots, x_n)$  in the form of the right hand side of (4) is identically zero. Then so is  $Q(x_1, \dots, x_{n-1}, x_n) - Q(x_1, \dots, x_{n-1}, -x_n^{-1})$ , which can be rewritten as

$$[x_n] \sum_{1 \leq i_1 < \dots < i_{2k-1} \leq n-1} [x_{i_1} x_{i_2}^{-1} x_{i_3} \dots x_{i_{2k-2}}^{-1} x_{i_{2k-1}}] P_{i_1 \dots i_{2k-1}, n}(\{x_1\}, \dots, \{x_n\}) = 0,$$

where  $[f(x_1, \dots, x_n)]$  denotes  $f(x_1, \dots, x_n) - f(-x_1^{-1}, \dots, -x_n^{-1})$ . Denote the left hand side by  $[x_n]R(x_1, \dots, x_n)$ , then  $R(x_1, \dots, x_n)$  is identically zero. Hence so is  $R(x_1, \dots, x_{n-2}, x_{n-1}, x_n) - R(x_1, \dots, x_{n-2}, -x_{n-1}^{-1}, x_n)$ , which can be rewritten as

$$[x_{n-1}] \sum_{1 \leq i_1 < \dots < i_{2k} \leq n-2} \{x_{i_1} x_{i_2}^{-1} \dots x_{i_{2k-1}} x_{i_{2k}}^{-1}\} P_{i_1 \dots i_{2k}, n-1, n}(\{x_1\}, \dots, \{x_n\}) = 0.$$

Repeating this two-step procedure  $\lfloor \frac{n}{2} \rfloor$  times, we will end up with

$$[x_1]\{1\}P_{1\dots n}(\{x_1\}, \dots, \{x_n\}) = 0 \quad \text{or} \quad [x_2]\{1\}P_{2\dots n}(\{x_1\}, \dots, \{x_n\}) = 0$$

according as  $n$  is even or odd. Consider, for example, the case of odd  $n$ . By symmetry,  $P_{1\dots i\dots n} = 0$  for each  $i$ . Returning to the previous stage

$$[x_4] \sum_{1 \leq i_1 < \dots < i_{2k} \leq 3} \{x_{i_1}x_{i_2}^{-1}\dots x_{i_{2k-1}}x_{i_{2k}}^{-1}\} P_{i_1\dots i_{2k}, 4\dots n}(\{x_1\}, \dots, \{x_n\}) = 0,$$

we can now substitute zeroes for  $P_{23,4\dots n}$ ,  $P_{13,4\dots n}$ ,  $P_{12,4\dots n}$ , and so we get  $P_{4\dots n} = 0$ .

Continuing to the earlier stages, we will similarly verify that each  $P_{i_1\dots i_{2k}} = 0$ .  $\square$

Since every term of, say,  $x - x^{-1}$  has an odd degree in  $x$ , Lemma 4.1 implies

**Lemma 4.5.** *For every nonzero term of  $P_{i_1\dots i_k}$*

- (i) *the total degree is congruent mod 2 to the number of components;*
- (ii) *the  $z_i$ -degree is congruent to  $k_i + l_i \pmod{2}$  iff  $i \notin \{i_1, \dots, i_k\}$ .*

Let us turn again to the case of two-component links colored with two colors. Let  $d'_{ij}$  (resp.  $d''_{ij}$ ) denote the coefficient of  $P(z_1, z_2)$  (resp.  $P_{12}(z_1, z_2)$ ) at  $z_1^i z_2^j$ . Substituting  $x(z_i)$  for  $x_i$  as in the proof of Theorem 4.2, we get

$$x_1 x_2^{-1} + x_1^{-1} x_2 = 2 - \frac{z_1 z_2}{2} + \frac{z_1^2}{4} + \frac{z_2^2}{4} + \text{terms of total degree } \geq 4.$$

Thus  $c_{00} = 2(d'_{00} + d''_{00})$  and

$$c_{11} = 2(d'_{11} + d''_{11}) - \frac{d''_{00}}{2}, \quad c_{20} = 2(d'_{20} + d''_{20}) + \frac{d''_{00}}{4}, \quad c_{02} = 2(d'_{02} + d''_{02}) + \frac{d''_{00}}{4}.$$

By Lemma 4.5(ii),  $d'_{2i,2j} = d''_{2i+1,2j+1} = 0$  if the linking number is even; otherwise  $d'_{2i+1,2j+1} = d''_{2i,2j} = 0$ . On the other hand,  $c_{00} = \text{lk}$  and  $c_{11} = \alpha_1 - (\text{lk}^3 - \text{lk})/12$  by Theorem 4.3(i). Thus  $2d''_{00}$  is  $\text{lk}$  or 0, and  $2d''_{11}$  is 0 or the integer  $\alpha_1 - (\text{lk}^3 - \text{lk})/12$ , according as  $\text{lk}$  is even or odd. Similarly,  $2d'_{00}$  is 0 or  $\text{lk}$ , and  $2d'_{11}$  is the integer  $\alpha_1 - (\text{lk}^3 - 4\text{lk})/12$  or 0, according as  $\text{lk}$  is even or odd. On the other hand, it follows from Proposition 2.1 that the KL-type 0 invariant  $2d'_{00} = ((-1)^{\text{lk}} + 1)\text{lk}/2$  is not of finite type. Since the linking number  $c_{00}$  and the half-integer  $c_{11}$  are of finite type, the integer  $d_{11} := 2(d'_{11} + d''_{11})$  is also not of finite type.

*Remark.* The decomposition (4) in the case  $n = 2$  can be modified, using the identity

$$xy^{-1} + x^{-1}y = \frac{(x-y)^2}{xy} + 2,$$

to the form

$$\Omega_L(x, y) = 2(P(\{x\}, \{y\}) + P_{12}(\{x\}, \{y\})) + \frac{(x-y)^2}{xy}P_{12}(\{x\}, \{y\}).$$

Lemma 4.5(ii) implies that each coefficient of  $P + P_{12}$  coincides with the corresponding coefficient of either  $P$  or  $P_{12}$ , depending on the parity of  $\text{lk}(L)$ . Since  $\text{lk}(L)$  is the constant term of  $2(P + P_{12})$ , the remainder  $(xy)^{-1}(x-y)^2P_{12}$  is redundant, in the sense that it contains no additional information with respect to  $2(P + P_{12})$ .

More generally, we have

**Theorem 4.6.** *Let  $P_{i_1, \dots, i_{2k}}$  be the polynomials of the colored link  $L$ , defined in (4). The polynomial  $\nabla_L$ , defined by*

$$\nabla_L(z_1, \dots, z_n) := 2 \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} P_{i_1 \dots i_{2k}}(z_1, \dots, z_n) \in \mathbb{Z}[z_1, \dots, z_n]$$

*if  $L$  has at least two components, and by  $\nabla_L(z) = z^{-1}\nabla_L(z) \in z^{-1}\mathbb{Z}[z]$  if  $n = 1$ , determines, and is determined by, the Conway potential function  $\Omega_L(x_1, \dots, x_n)$ . The coefficient of  $\nabla_L$  at a term of total degree  $d$  is of colored type  $d + 1$ , and vanishes unless  $d$  is congruent mod 2 to the number of components of  $L$ .*

*Proof.* By Theorem 4.4,  $\Omega_L$  determines each  $P_{i_1 \dots i_{2k}}$ , hence  $\nabla_L$ . Conversely, Lemma 4.5(ii) implies that the coefficients of each  $P_{i_1 \dots i_{2k}}$ , hence of  $\Omega_L$ , are determined by  $\nabla_L$ . The last two assertions follow from Lemma 4.5(i) and the skein relation (2), which can be rewritten in the form

$$\nabla_{L_+} - \nabla_{L_-} = z_i \nabla_{L_0}$$

for intersections between components of color  $i$ . □

Let  $K_1, \dots, K_m$  denote the components of a colored link  $L$ , where  $m > 1$ . It follows from Theorem 4.4 that the power series

$$\Omega_L^*(x_1, \dots, x_n) := \frac{\Omega_L(x_1, \dots, x_n)}{\nabla_{K_1}(x_{c(1)} - x_{c(1)}^{-1}) \cdots \nabla_{K_m}(x_{c(m)} - x_{c(m)}^{-1})} \in \mathbb{Z}[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]]$$

can be expressed in the form

$$\Omega_L^*(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_{2k} \leq n} \left\{ \frac{x_{i_1}}{x_{i_2}} \cdots \frac{x_{i_{2k-1}}}{x_{i_{2k}}} \right\} P_{i_1 \dots i_{2k}}^*(\{x_1\}, \dots, \{x_n\}),$$

where each  $P_{i_1 \dots i_{2k}}^* \in \mathbb{Z}[[z_1, \dots, z_n]]$ , and the proof of Theorem 4.4 shows that such an expression is unique. The doubled sum of all these polynomials is, of course, nothing but

$$\nabla_L^*(z_1, \dots, z_n) := \frac{\nabla_L(z_1, \dots, z_n)}{\nabla_{K_1}(z_{c(1)}) \cdots \nabla_{K_m}(z_{c(m)})},$$

which contains the same information as  $\Omega_L^*$  by the proof of Theorem 4.6.

**Corollary 4.7.** (i) For each  $L: mS^1 \hookrightarrow S^3$  and any  $k \in \mathbb{N}$  there exists an  $\varepsilon_k > 0$  such that if  $L': mS^1 \hookrightarrow S^3$  is  $C^0$   $\varepsilon_k$ -close to  $L$ ,

$$\nabla_{L'}^* = \nabla_L^* + \text{terms of total degree } \geq k,$$

$$\mathcal{U}_{L'}^* = \mathcal{U}_L^* + \text{terms of total degree } \geq k.$$

(ii)  $\nabla_L^*$  and  $\mathcal{U}_L^*$  can be uniquely extended to all TOP links in  $S^3$ , preserving (i).

(iii) The extended  $\nabla_L^*$  and  $\mathcal{U}_L^*$  are invariant under TOP isotopy of  $L$ .

*Proof.* As in the above argument for  $\nabla_L^*$ , it is easy to see that  $\nabla_L^*$ , as well as  $\mathcal{U}_L^*$ , are invariant under PL isotopy (cf. [35]). By Theorems 4.2 and 4.6, the coefficients of  $\nabla_L^*$  and  $\mathcal{U}_L^*$  at terms of total degree  $k$  are of colored type  $k+1$ , hence by Theorem 2.2 they are invariant under  $(k+1)$ -quasi-isotopy. The rest of the proof is as in Theorem 3.2.  $\square$

*Remark.* If the components of  $L$  are unknotted, or more generally have no non-local knots,  $\nabla_L^*$  is a (finite) polynomial. Actually,  $\nabla_L^*$  splits into a product of  $m$  one-variable power series and a polynomial (namely, the product of the irreducible factors of  $\nabla_L$ , involving more than one variable), so that each of them is individually invariant under PL isotopy;

compare [35, 32]. There is, of course, such a splitting for every  $P_{i_1 \dots i_{2n}}^*$ , producing a plethora of PL isotopy invariants.

Lack of an analogue of Lemma 4.5(ii) for the power series  $\mathcal{U}_L$  hinders establishing a simple relation between the coefficients of  $\mathcal{U}_L$  and those of Traldi's expansion of  $\Omega_L$  [35], which prevents one from expressing the finite type rationals  $\alpha_{ij}$  (discussed in Theorem 4.3) as liftings of  $\bar{\mu}$ -invariants. However, each  $\alpha_{ij}$  can be split into a linear combination (whose coefficients depend on the linking number) of the integer coefficients of the power series  $\nabla_L^*$ , which do admit such an expression.

**Theorem 4.8.** *Let  $\delta_{ij}$  denote the coefficient of  $\nabla_L^*$  at  $z_1^i z_2^j$ . For 2-component links*

- (i)  $\delta_{00} = \text{lk}$ , and  $\delta_{11}$  is an integer lifting of the Sato–Levine invariant  $\bar{\mu}(1122)$ , but not a finite type invariant;
- (ii)  $\delta_{1,2k-1}$  and  $\delta_{2k-1,1}$  are Cochran's [5] derived invariants  $(-1)^{k+1} \beta_\pm^k$ , whenever the latter are defined (i.e.,  $\text{lk} = 0$ );
- (iii)  $\delta_{ij}$  is an integer lifting of Milnor's [28] invariant  $(-1)^j \bar{\mu}(\underbrace{1 \dots 1}_{i+1} \underbrace{2 \dots 2}_{j+1})$ , provided that  $i + j$  is even;
- (iv)  $\delta_{ij}$  is of KL-type  $i + j$ ;
- (v) when  $i + j$  is odd,  $\delta_{ij} = 0$ ;
- (vi) when  $i + 1, j + 1$  and  $\text{lk}$  are all even or all odd,  $\delta_{ij}$  is even;
- (vii) for a given  $L$ , there are only finitely many pairs  $(i, j)$  such that  $\delta_{ij} \not\equiv 0$  modulo the greatest common divisor  $\Delta_{ij}$  of all  $\delta_{kl}$ 's with  $k \leq i, l \leq j$  and  $k + l < i + j$ ; the congruence can be replaced by equality if the components of  $L$  are unknotted.

The last four parts are immediate consequences of Theorem 4.6; Lemma 4.5(i); Lemma 4.5(ii) and integrality of the coefficients of  $P_{12}^*$ ; the definition of  $\nabla_L^*$  as a rational power series. The first part follows from the discussion after Lemma 4.5.

*Remark.* The geometry of the relationship between  $\bar{\mu}$ -invariants and the multi-variable Alexander polynomial is now better understood [17]. If  $L$  is the link closure of a string link  $\ell: \{1, \dots, m\} \times I \hookrightarrow \mathbb{R}^2 \times I$ ,

$$\Delta_L = \Delta_\ell \Gamma_\ell,$$

where  $\Delta_L$  denotes the usual Alexander polynomial,  $\Delta_\ell$  the Alexander polynomial of the string link [19], i.e., the Reidemeister torsion of the based chain complex of the pair  $(X, X \cap \mathbb{R}^2 \times \{0\})$ , where  $X = \mathbb{R}^2 \times I \setminus \text{im } \ell$ , and  $\Gamma_\ell$  a certain rational power series, determined by the  $\mu$ -invariants of  $\ell$ .

*Proof of (iii).* Since we are only interested in the residue class of every  $\delta_{ij}$  mod  $\Delta_{ij}$ , we may consider the polynomial  $\nabla_L = 2(P + P_{12})$  in place of the power series  $\nabla_L^* = 2(P^* + P_{12}^*)$ . By Theorem 4.4 and Lemma 4.5(ii),  $\Omega_L$  is uniquely expressible in the form

$$(\{x_1\}\{x_2\})^\lambda \Omega_L(x_1, x_2) = Q_1(\{x_1\}^2, \{x_2\}^2) + \{x_1 x_2^{-1}\}\{x_1\}\{x_2\} Q_2(\{x_1\}^2, \{x_2\}^2)$$

for some  $Q_1, Q_2 \in \mathbb{Z}[z_1, z_2]$ , where  $\lambda = 0$  or  $1$  according as  $\text{lk}(L)$  is odd or even. Set  $y_i = x_i^2 - 1$ , then  $x_i^{-2} = 1 - y_i + y_i^2 - y_i^3 + \dots$ . The identities

$$\begin{aligned} \{x_i\}^2 &= x_i^{-2}(x_i^2 - 1)^2 = (1 - y_i + y_i^2 - \dots)y_i^2; \\ \{x_1 x_2^{-1}\}\{x_1\}\{x_2\} &= (x_1^{-2} + x_2^{-2})(x_1^2 - 1)(x_2^2 - 1) = (2 - y_1 - y_2 + y_1^2 + y_2^2 - \dots)y_1 y_2 \end{aligned}$$

allow to express  $(x_1 x_2)^{-\lambda} \Omega_L(x_1, x_2)$  as a power series  $T_L(y_1, y_2)$  with integer coefficients.

Let us study this substitution more carefully. Let  $d'_{ij}, d''_{ij}$  denote the coefficients at  $z_1^i z_2^j$  in  $P$  and  $P_{12}$ , and let us write  $(k, l) \leqslant (i, j)$  if  $k \leqslant i$  and  $l \leqslant j$ . Then the coefficient  $e''_{ij}$  at  $y_1^i y_2^j$  in the power series  $R_2$ , defined by the equality  $(y_1 y_2)^{\lambda-1} R_2(y_1, y_2) = (z_1 z_2)^{\lambda-1} P_{12}(z_1, z_2)$ , is given by

$$e''_{ij} = \sum_{(k,l) \leqslant (i,j)} (-1)^{(i-k)+(j-l)} d''_{kl} = (-1)^{i+j} \sum_{(k,l) \leqslant (i,j)} d''_{kl}$$

(the latter equality uses that  $d''_{kl} = 0$  if  $k + l$  is odd), and similarly for the coefficients  $e'_{ij}$  of the power series  $R_1$ , defined by  $(y_1 y_2)^\lambda R_1(y_1, y_2) = (z_1 z_2)^\lambda P(z_1, z_2)$ . Now the coefficients  $e_{ij}$  of  $T_L(y_1, y_2)$  are given by

$$\begin{aligned} e_{ij} &= 2e'_{ij} + 2e''_{ij} + \sum_{k < i} (-1)^{i-k} e''_{kj} + \sum_{l < j} (-1)^{j-l} e''_{il} \\ &= 2e'_{ij} + 2e''_{ij} + (-1)^{i+j} \sum_{(k,l) < (i,j)} ((i-k) + (j-l)) d''_{kl}. \end{aligned}$$

The key observation here is that all coefficients on the right hand side are even if  $i + j$  is even.

Let us consider the case  $\lambda = 1$ . Then by Lemma 4.5(ii),  $d''_{ij} = 0$  unless both  $i$  and  $j$  are even, and  $d'_{ij} = 0$  unless both  $i$  and  $j$  are odd. Hence  $e''_{ij} \equiv 0$  modulo  $\gcd\{e''_{kl} \mid (k, l) < (i, j)\}$ , unless both  $i$  and  $j$  are even, and similarly for  $e'_{ij}$ . Now it follows by induction that  $e_{ij} \equiv 2e''_{ij} \equiv 2d''_{ij}$  or  $e_{ij} \equiv 2e'_{ij} \equiv 2d'_{ij}$  modulo  $E_{ij} := \gcd\{e_{kl} \mid (k, l) < (i, j)\}$  according as  $i$  and  $j$  are both even or both odd. Thus  $e_{ij} \equiv 2(d'_{ij} + d''_{ij}) \pmod{E_{ij}}$  if  $i + j$  is even. Clearly, the latter assertion holds in the case  $\lambda = 0$  as well, which can be proved by the same argument.

Finally, since  $-\lambda \equiv \text{lk} - 1 \pmod{2}$ , and  $x_i^{\pm 2}$  are expressible as power series in  $y_i$  with integer coefficients and constant term 1, the coefficients  $\tilde{e}_{ij}$  of Traldi's power series  $\tilde{T}_L(y_1, y_2) = (x_1 x_2)^{\text{lk}-1} \Omega_L$  are related to  $e_{ij}$  by congruence mod  $E_{ij}$ . This completes the proof, since by Traldi [35], each  $\tilde{e}_{ij}$  is an integer lifting of  $(-1)^{j+1} \bar{\mu}(\underbrace{1 \dots 1}_{i+1} \underbrace{2 \dots 2}_{j+1})$ .

□

*Remark.* The above argument yields a new proof of another result due to Traldi:

$$2\bar{\mu}(\underbrace{1 \dots 1}_{i+1} \underbrace{2 \dots 2}_{j+1}) = 0 \text{ when } i + j \text{ is odd [34].}$$

*Proof of (ii).* We recall that  $\bar{\mu}(1 \dots 12)$  and  $\bar{\mu}(12 \dots 2)$  identically vanish, with the exception of  $\bar{\mu}(12) = \text{lk}$  [28]. Hence by (iii),  $\delta_{0,2k} \equiv \delta_{2k,0} \equiv 0 \pmod{\text{lk}}$  for each  $k$ . So if  $\text{lk} = 0$ , every nonzero term of either  $P^*$  or  $P_{12}^*$  involves both  $z_1$  and  $z_2$ . (Alternatively, this follows from Jin's lemma mentioned in the proof of Theorem 4.3(ii).) By Lemma 4.5(ii), every nonzero term of  $P_{12}^*$  has to further include each of them once again, i.e.,  $P_{12}^*$  is divisible by  $z_1^2 z_2^2$ . Hence, firstly,  $\delta_{2k-1,1}$  coincides with the coefficient of  $2P^*(z_1, z_2)$  at  $z_1^{2k-1} z_2$ , and, secondly, this coefficient is not affected by adding  $(x(z_1)x(z_2)^{-1} + x(z_2)x(z_1)^{-1})P_{12}^*(z_1, z_2)$  to  $2P^*(z_1, z_2)$ , where  $x(z)$  is as in the proof of Theorem 4.2. □

**Theorem 4.9.** *The coefficient of  $\nabla_L^*$  at  $z_1^{i_1} \dots z_n^{i_n}$  is invariant under  $k$ -quasi-isotopy if  $\max(i_1, \dots, i_n) \leq 2k$ .*

*Proof.* This is analogous to Theorem 3.4. One only needs to show that the coefficient in question is invariant under *restricted* skein  $k$ -quasi-isotopy, where a *restricted*  $\pm$ -skein

$k$ -quasi-embedding is defined as before, but with additional restriction that each  $L_{0i_2\dots i_k}$  includes  $L \setminus K$  and each  $H_{0i_2\dots i_k}$  is fixed on  $L \setminus K$ . This is done by the same argument, replacing every occurrence of the polynomial  $z^{1-m}\nabla_L$  with  $\nabla_L$ .  $\square$

**Corollary 4.10.** (a) [26] Cochran's invariants  $\beta_\pm^k$  are invariant under  $k$ -quasi-isotopy;  
 (b) Milnor's invariants  $\bar{\mu}(1\dots 12\dots 2)$  of even length are invariant under  $k$ -quasi-isotopy, if each index occurs at most  $2k+1$  times.

Part (b) covers (and largely improves) the corresponding case of

**Theorem 4.11.** [26] All  $\bar{\mu}$ -invariants of length  $\leq 2k+3$  are invariant under  $k$ -quasi-isotopy.

## CHAPTER 5

### THE HOMFLY AND KAUFFMAN POLYNOMIALS

We recall that the *HOMFLY(PT) polynomial* and the Dubrovnik version of the *Kauffman polynomial* are the unique Laurent polynomials  $H_L, F_L \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$  satisfying  $H_{\text{unknot}} = F_{\text{unknot}} = 1$  and

$$\begin{aligned} xH_{L_+} - x^{-1}H_{L_-} &= yH_{L_0}, \\ xF_{L_+} - x^{-1}F_{L_-} &= y(F_{L_0} - x^{w(L_\infty)-w(L_0)}F_{L_\infty}), \end{aligned}$$

where  $L_+, L_-, L_0$  and  $L_s$  are as in the definition of the Conway polynomial (cf. §3),  $L_\infty$  is obtained by changing the orientation of the “right” of the two loops in  $L_s$  (corresponding to either the two intersecting components or the two lobes of the singular component) and oriented smoothing of the crossing of the obtained singular link  $L'_s$ , and  $w(L)$  denotes the *writhe* of the diagram of  $L$ , i.e., the number of positive crossings minus the number of negative crossings (so that  $w(L_+) - 1 = w(L_-) + 1 = w(L_0)$ ). The versions of  $H_L$  and  $F_L$  in Lickorish [20] are obtained as  $H_L(-ia, iz)$  and  $(-1)^{m-1}F_L(-ia, iz)$ .

**Theorem 5.1.** *Let  $e^t$  denote the (formal) power series  $\sum \frac{t^n}{n!}$ , and consider the power series*

$$H_L^* := \frac{H_L}{H_{K_1} \cdots H_{K_m}} \quad \text{and} \quad F_L^* := \frac{F_L}{F_{K_1} \cdots F_{K_m}},$$

where  $K_1, \dots, K_m$  denote the components of the link  $L$ .

(i) For each  $L: mS^1 \hookrightarrow S^3$  and any  $n \in \mathbb{N}$  there exists an  $\varepsilon_n > 0$  such that if  $L': mS^1 \hookrightarrow S^3$  is  $C^0$   $\varepsilon_n$ -close to  $L$ ,

$$\begin{aligned} H_{L'}^*(e^{ch/2}, e^{h/2} - e^{-h/2}) &\equiv H_L^*(e^{ch/2}, e^{h/2} - e^{-h/2}) \pmod{(h^n)}, \\ F_{L'}^*(e^{(c-1)h/2}, e^{h/2} - e^{-h/2}) &\equiv F_L^*(e^{(c-1)h/2}, e^{h/2} - e^{-h/2}) \pmod{(h^n)}. \end{aligned}$$

(ii)  $H_L^*$  and  $F_L^*$  can be uniquely extended to all TOP links in  $S^3$ , preserving (i).

(iii) The extended  $H_L^*$  and  $F_L^*$  are invariant under TOP isotopy of  $L$ .

*Proof.* The connected sum formulae for  $H_L$  and  $F_L$  [20] imply that  $H^*$  and  $F^*$  are invariant under PL isotopy. (Note that the connected sum in Lickorish [20] is Hashizume's, not the componentwise connected sum of Melikhov and Repovš [25].) On the other hand, it was noticed in Lieberum [21] (compare [9, 4, 33]) that the coefficients of the power series

$$H_L(e^{ch/2}, e^{h/2} - e^{-h/2}) = \sum_{k=0}^{\infty} \sum_{i=0}^{k+m-1} p_{ki} c^i h^k \in \mathbb{Q}[c][[h]]$$

$$F_L(e^{(c-1)h/2}, e^{h/2} - e^{-h/2}) = \sum_{k=0}^{\infty} \sum_{i=0}^{k+m-1} q_{ki} c^i h^k \in \mathbb{Q}[c][[h]]$$

are (monochromatic) finite type invariants of  $L$ . Specifically, each  $p_{ki}$  and each  $q_{ki}$  is of type  $k$ , moreover  $p_{0i} = q_{0i} = \delta_{m-1,i}$  (the Kronecker delta). (The argument in Lieberum [21] was for  $H_L H_{T_2}$  and  $F_L F_{T_2}$ , where  $T_2$  denotes the trivial 2-component link, but it works as well for  $H_L$  and  $F_L$ , compare [3].) The rest of the proof repeats that of Theorem 3.2.  $\square$

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