EFFECT OF VAPOR DYNAMICS ON INTERFACIAL INSTABILITIES

By

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This doctoral thesis is dedicated to the teachers all around the world who dedicated their lives to future generations.
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Instabilities at fluid-fluid interfaces are of interest to researchers on account of their role in many industrial and natural processes. These instabilities are often accompanied by phase-change between liquids and their vapors, and in some instances are caused by it. Examples of processes involving such instabilities are film boiling and drying processes. An everyday occurrence is seen in the pattern formation caused by the drying of coffee drops on a kitchen counter.

It turns out that the instabilities are accompanied by fluid flow in both phases but many past studies of the phenomena have been done by considering only the liquid fluid dynamics while the role of the vapor phase has been either left out of the study or relegated to heat transfer only while its fluid dynamics has been ignored.

In this study, evaporative instabilities are analyzed theoretically and experimentally at the interface between a liquid and its vapor where both the liquid and the vapor are fluid dynamically active in order to investigate and understand the role of vapor dynamics on evaporative and interfacial instabilities. In the theoretical model, the Navier
Stokes and energy equations in the liquid and vapor domains were using linear and nonlinear perturbation methods with the assumption that both fluids are incompressible using the Boussinesq approximation. The linear theory tells us the conditions at the onset of instability and the critical point at which the interface becomes unstable. The weak nonlinear method tells us the behavior of the system slightly beyond the onset of the instability. The linear and weak nonlinear theories are of importance for the experimental reasons as they provide us with the information on how and when the instability will occur.

In a separate study of non-evaporative systems, experiments were performed in a liquid-gas bilayer system in a circular container with the intention of studying instabilities in the absence of evaporation. The depth of the gas layer was changed to observe the effect of the fluid dynamics in the gas layer on the onset flow pattern and conditions. These experiments were then compared with a corresponding theoretical model and it was concluded that the gas dynamics played an ever-increasing role as the gas layer height was increased. Past workers had assumed the vapor to be a conductive, fluid dynamically passive medium. However, the experiments show us that deep enough vapor layers can cause the bilayer system to convect earlier than expected by thermally coupling with the liquid at the interface.

In summary, from theoretical studies it was found out that active vapor layers play an important role on the evaporative instabilities even in the absence of natural convection. Moreover, it is discovered that the fluid dynamically active vapor plays a very strong stabilizing role on the evaporation problem. In addition to that, from the experimental studies in the absence of phase-change it was revealed that the vapor layers
can and do affect the onset flow pattern and conditions. Both theoretical and experimental studies also indicate that the commonly used infinitely deep vapor layer assumption is not always true and can lead to errors.
Evaporation and condensation involve the transformation of a substance from the liquid to the vapor state or the other way around by the addition or depletion of energy. It turns out that these processes can lead to an interfacial instability. This instability is typically manifested by the sudden undulation of an erstwhile flat surface between a liquid and a vapor as a control variable, such as the temperature drop across the bilayer, exceeds a critical value. There is a need to understand the phenomenon of the instability in the light of the application of evaporation to many technologies and this is the focus of our work. Evaporation arises in a variety of industrial and natural processes, such as in heat pipe technology, the drying of oxide films, the drying of lakebeds and in dry-eye syndrome.

However, evaporation comes accompanied by fluid flow and natural convection due to buoyancy and interfacial tension gradients. In order to have an understanding of the various phenomena associated with evaporation and the first order effects that promote instability during evaporation we advance some “picture arguments,” first for the pure phase-change problem and later for connected instabilities such as convection and gravitational instability.

Physics of Various Instabilities Connected to Vapor-Liquid Phase Change

Evaporative Instability

Evaporative instability is best understood by considering an evaporating liquid in contact with its own vapor. To get an idea of the physical model the reader’s attention is
drawn to Figure 1-1, which depicts a liquid-vapor bilayer where a temperature difference is imposed across it. The input variables are the fluid depths, the temperature difference and the pressure of the vapor at the upper plate. As a result of fixing these variables the evaporation rate, the fluid behavior and the interface morphology can be determined. It may be noted that evaporation can be obtained in the absence of a temperature difference by manipulating the pressure at the upper plate. However, in this work we shall imagine that a temperature difference is always imposed.

The interface is originally thought to be flat and it is the stability of this state that is in question. The liquid and vapor depths may be fixed by suitably adjusting the liquid feed and vapor removal rates. In fact, given a temperature drop the upper plate pressure may be adjusted in such a way that the evaporation or condensation rate is zero. We will consider much of the analysis in this thesis for a case where initially there is no mass transfer across the interface as it simplifies the calculations and is sufficient to reveal the physics behind this problem. We now proceed in a stepwise manner to consider evaporative instability and return to Figure 1-1.

Figure 1-1. Physical model of the phase-change problem

First, imagine that the base flow rate is zero and the fluid dynamics is left out. By imagining that the fluid mechanics is omitted, we are also pretending that pressure
perturbations do not arise. This will obviously change when the perturbed fluid motion is taken into account. Interesting conclusions can be drawn by simply looking at the temperature profiles in the base state. The schematics in Figures 1-2a and 1-2b depict an evaporation problem at a zero evaporation rate. Figure 1-2a shows the heating arrangement where the heat needed for evaporation is supplied from the liquid side while Figure 1-2b shows heat input from the vapor side. The arrows in the figures show the direction of the front motion if the base state were to have some evaporation though this is not central to the physical arguments presented here.

![Figure 1-2a. Heat input from the liquid side](image)

In both figures the dotted wave represents a perturbed interface while the dotted lines represent the temperature profiles in the perturbed state at the interface. Note that the temperature at the perturbed interface is the same as the unperturbed state because the temperature is assumed to be at equilibrium with the pressure at the interface, an assumption that is justified at low evaporation rates. When the fluid dynamics is ignored the pressure is not perturbed and consequently also the temperature is unperturbed at the interface (see Appendix A for a description of the thermodynamic equilibrium relation at the interface and its development upon the imposition of an infinitesimal perturbation). When heated from the liquid side, a crest is further away from the heat source and also closer to the heat sink. Therefore the temperature gradient on the vapor side gets sharper
whereas the temperature gradient on the liquid side becomes less sharp (i.e., less heat is transferred to that point and more heat is taken away). The rate of evaporation from the crest is thus decreased which makes the interface unstable. On the other hand, when heated from the vapor side, the trough gets closer to the heat sink and is further away from the heat source. Consequently, less heat is transferred to the front and more heat is taken away from it. The speed of the trough is thus decreased and as a result the front is stabilized. Note that the picture argument presented here suggests that when the interface is unstable, it is unstable for disturbances of all periodicities or wavenumbers and when it is stable, it is stable for all wavenumbers.

![Diagram](image)

Figure 1-2b. Heat input from the vapor side

Now include the fluid dynamics in both phases in the thought experiment, but do not consider any other convective effect. Figure 1-3 is similar to Figure 1-2.a except that fluid dynamics is now taken into account in both phases. Observe that a deflected surface does not have the same temperature as an erstwhile flat interface for a trough is expected to be at a higher temperature than a crest as it is closer to the heat source. Remember also that we started out with a base state where there was thermodynamic equilibrium at the interface and no phase-change was taking place. Now that the trough is at a temperature higher than the saturation temperature there will be evaporation taking place there and consequently, when perturbed, the vapor will condense into its own liquid at the crest.
where the temperature is colder than the saturation temperature. There will be upward flow at the troughs and downward flow at the crests as indicated in Figure 1-3. Again, instability occurs at every wavenumber.

![Figure 1-3. Physics with fluid dynamics](image)

Now surface tension acts on pressure differences in fluid dynamics problems via the surface curvature and in many phase-change instability problems it is the surface tension that comes in to stabilize the situation. Its effect is felt most strongly at small wavelengths and weakly on lazy perturbations. But, surface tension comes attended by the complication that it depends upon temperature and we will take this into account as we continue to study the problem in stages. There is also another complication and this time it is due to gravity. If the vapor were at the bottom as in pool boiling then gravity would play the destabilizing role of a heavy fluid on top of a light fluid. The destabilization would take place in that case when gravitational potential energy overwhelms surface potential energy. This is the familiar Rayleigh-Taylor instability problem and we give a brief discussion of this instability later in this chapter. However, even if the vapor were on the top and the liquid were heated, gravity still plays a role via buoyancy and to understand the nature of natural convection due to buoyancy and interfacial tension gradient we turn to more picture arguments.
Before we move on to a discussion of these other effects, it might be noted that evaporative instability can take place even if the interface were assumed to be flat. In this case, the instability would be manifested by fluid motion. This case will not be of interest to us.

**Rayleigh Convection or Buoyancy-Driven Convection**

Buoyancy-driven convection, often termed natural convection or Rayleigh convection, occurs when a fluid is subject to a temperature gradient in a gravitational field and when there is a variation of density with respect to temperature. Imagine a layer of liquid bounded vertically by two horizontal surfaces, with the lower surface at a temperature greater than the upper one as in Figure 1-4.

![Figure 1-4. Rayleigh convection](image)

As density typically decreases with an increase in temperature, the fluid near the top surface is heavier than the fluid at the bottom plate, creating a top-heavy arrangement, one that is gravitationally unstable. For sufficiently small temperature differences, the fluid simply conducts heat from the lower surface to the upper one, creating a linear temperature field across the fluid. Now imagine a mechanical perturbation being imparted to an element of fluid so that it is projected downward. As the density of the “fluid packet” is greater than its environment it will proceed downward. Due to mass conservation, fluid from below will be displaced and will move upward. This motion
will continue unabated unless the fluid’s kinematic viscosity and thermal diffusivity are large enough that the mechanical perturbations and thermal perturbations die out quickly and the original mechanically quiescent state is re-established. From this thought experiment it is immediately clear that unlike many classical problems of fluid mechanics, the problem of the onset of convection can not be considered for fluids in the limit of vanishing viscosity or thermal diffusivity. In other words, disturbances to the fluid are dampened by the kinematic viscosity and thermal diffusivity of the fluid and enhanced by the buoyancy forces. Under steady conditions there is a balance that is established between these effects, represented by the dimensionless Rayleigh number, a group that arises from the scaling of the modeling equations. It is given by

\[ Ra = \frac{\alpha g \Delta T}{\nu \kappa} d^3 \]  

Here, \( \alpha \) is the negative thermal expansion coefficient and is usually positive, \( g \) is the magnitude of gravity, \( \Delta T \) is the vertical temperature difference across the fluid layer and is taken positive when the fluid is heated from below, \( d \) is the depth of the fluid, \( \nu \) is its kinematic viscosity, and \( \kappa \) is its thermal diffusivity. If the temperature difference is increased beyond what will be called the critical temperature difference, then the gravitational instability overcomes the viscous and thermal damping effects and the fluid is set into motion, causing buoyancy-driven convection. (See Appendix B for an elementary derivation of the critical condition for the onset of buoyancy driven convection.) If the upper surface adjoins a vapor phase then convection can initiate in either phase. If it begins in the lower phase, it will propagate to the upper phase by way of viscous dragging. If convection is simultaneous in both phases they can become
thermally coupled with the interface becoming an isotherm. If it initiates in the upper phase, it cannot propagate to the lower phase except by way of generating interfacial tension gradients, which in turn will generate weak flow in the lower liquid. For that reason, we turn to interfacial tension gradient-driven or Marangoni convection in the following section.

**Pure Marangoni Convection or Interfacial Tension Gradient-Driven Convection**

Interfacial tension gradient-driven convection or Marangoni convection is unlike buoyancy-driven convection and can occur in a fluid even in the absence of a gravitational field. Imagine a layer of fluid which is bounded below by a rigid plate and whose upper surface is in contact with a passive gas as depicted in Figure 1-5.

![Figure 1-5. Marangoni convection](image)

A passive gas is not one without density or viscosity. It is merely a gas wherein one tacitly assumes that thermal or mechanical perturbations cannot occur. Pretend that above the passive gas, there is another rigid plate. For the sake of consistency with the earlier argument, let the lower plate be at a temperature higher than the upper plate's temperature. Now, imagine that the interface between the lower liquid and the passive gas is momentarily disturbed. The regions of the interface that are pushed up experience a cooler temperature. Likewise, the regions of the interface that are pushed down
increase in temperature. Typically, interfacial tension decreases with an increase in
temperature. Therefore, the regions of the interface, which are pushed up increase in
interfacial tension, which pulls on the interface, while the regions of the interface, which
are pushed down, decrease in surface tension. When the fluid is pulled along the
interface, warmer fluid from the bulk replaces the fluid at the interface near ‘x’ enhancing
the interfacial tension gradient-induced flow. If the temperature difference across the
liquid is sufficiently small, then the thermal diffusivity of the fluid will conduct away the
heat faster than the disturbance can amplify. The dynamic viscosity will also resist the
flow causing the interface to become flat again and the interfacial tension to become
constant. As in the case of buoyancy-driven convection, there exists a critical
temperature difference when the surface tension gradient-driven flow is not dampened by
the thermal diffusivity or viscosity, and the fluid is set into motion. Scaling of the
equations that model the interfacial momentum balance leads to a dimensionless group
that exhibits the balance between dissipative effects of the viscosity and thermal
diffusivity and the promoting effects of interfacial tension gradient. This dimensionless
group is called the Marangoni number and is given by

\[ Ma = \frac{\Delta T}{\gamma T \mu \kappa} d \]

Here \( \gamma T \) is the change in the interfacial tension with respect to the temperature, and
\( \mu \) is the dynamic viscosity of the fluid. It is noteworthy that while the above argument
assumes the interface to be deflectable it is not necessary for the interface to undulate in
order for Marangoni convection to take place. To see this a little more clearly, look at
Figure 1-5, but pretend that the interface is flat. Points ‘x’ and ‘y’ will therefore be on
the same horizontal level. Imagine an element of fluid projected downward from the interface at point ‘y’. Fluid nearby must replace the lost fluid at ‘y’ and this fluid comes from the region near ‘x’. Consequently, hot fluid from below moves to ‘x’. If the interfacial tension decreases with temperature, as is the case in most substances, then fluid flows from warm surface regions to cold surface regions. Therefore, fluid moves from ‘x’ to ‘y’ and the convection continues unless large thermal diffusivity and viscosity dampen out the disturbances. This means, theoretically speaking, that Marangoni convection can be analyzed assuming a flat free surface, i.e., a surface with an infinite surface tension. This fact was observed by Pearson (1958) who gave the first theoretical analysis of Marangoni convection using a methodology similar to that used by Rayleigh for the case of buoyancy driven convection.

The physical argument presented above can easily be extended to a bilayer system where the gas layer is fluid dynamically active. In this case, the gas layer will also convect due to surface tension gradients and will bring in colder fluid from the cold plate to ‘x’ opposing the destabilizing nature of the convection in the liquid; however, the effect of the convection in the gas is less because of its high kinematic viscosity.

The extent of either Rayleigh convection or Marangoni convection is primarily a function of the fluid depths. By examining equations 1.1 and 1.2, we notice that Rayleigh convection is proportional to the cube of the fluid depth and that Marangoni convection is directly proportional to the fluid depth. From these scaling arguments, we can conclude that for deep fluids, buoyancy-driven convection is more prevalent, whereas for shallow depths, interfacial tension gradient-driven convection is more prevalent, and at intermediate fluid depths, both Rayleigh and Marangoni convection can occur. If the
interface is allowed to deflect one might anticipate that in the case of pure Marangoni convection, warm fluid flows towards a trough while in the case of Rayleigh convection cold fluid flows away from a trough leading us to believe that Marangoni motion can counter Rayleigh driven motion. This counter effect was demonstrated computationally by Sarma (1987) and could not be seen in the study by Nield (1964) wherein flat surfaces were assumed.

When a layer of fluid is heated from above, it creates stable density stratification. Therefore the buoyancy force does not cause convection. Marangoni convection, though, may still occur in fluids being heated from above. For example, consider a single layer of fluid superimposed by a passive gas and heated from above. One might imagine that purely steady Marangoni convection would not occur. For instance, suppose a random perturbation causes some part of the surface to become warmer than the rest of the surface. The interfacial tension will decrease in this region and the higher interfacial tension elsewhere will pull fluid away from this hot spot. By continuity, fluid lying below the hot spot will move up to replace the displaced fluid. As the lower fluid is cooler than the surface, this region now cools off and the interfacial tension increases, thereby either stabilizing the region or causing an overshoot in temperature of that spot and generating oscillatory motion. Only calculations can tell precisely which case will occur. In real systems the upper fluid is never truly passive. Given the same scenario, fluid movement along the interface will also drag warmer fluid from above and cooler fluid from below. Depending upon the fluid properties and depth ratios the instability will or will not be reinforced.
Oscillatory convection can occur if the fluid is heated from above and buoyancy effects are included. This was shown by Rednikov et al. (1998) for the case when the upper fluid is passive. The explanation is as follows. If a small volume of fluid is displaced within the bulk of the fluid, the density stratification acts as a restoring force. The Marangoni force acts similarly and the two effects reinforce each other causing an overshoot and sustained oscillations. Indeed, as was demonstrated in their paper, this only occurs in certain fluids, at certain depths, where the buoyancy and interfacial tension forces are comparable.

There is another phenomenon associated with surface tension gradient-driven convection, often called the long wavelength Marangoni instability. This instability typically occurs when either the surface tension or the depth of the fluid layer is very small. The initiation of this instability is similar to the description given above. However, in the long wavelength scenario, the convection cells are much larger than the regular, or short wavelength, Marangoni convection. As the convection propagates, it causes large-scale deformation in the interface, which can actually cause the interface to rupture; that is, the interface deforms to such an extent that it comes in contact with the lower plate. This phenomenon occurs in the drying of films and in coating processes (Schatz).

**Rayleigh-Taylor Instability**

So far we have talked about the liquid and gas systems where the liquid in consideration underlies the gas phase. There are however situations, such as film boiling, where the opposite arrangement is encountered. When the vapor phase underlies a liquid phase, this is a top-heavy arrangement called Rayleigh-Taylor instability and it is potentially unstable. The instability occurs when the gravitational potential energy
exceeds the stabilizing effect of surface potential energy. In the classical treatment of the problem, it is assumed that there is no heat and mass transfer, and it is well known that the system is unstable to disturbances with a wavenumber smaller than the critical wavenumber given by

\[ \omega_c = \left( \frac{(\rho - \rho^*) g}{\gamma} \right)^{1/2} \]

where \( \gamma \) is the surface tension, \( \rho \) and \( \rho^* \) are the liquid and vapor densities, respectively.

A derivation for the classical Rayleigh-Taylor critical wavenumber from the book by Chandrasekhar (1960) for infinitely deep fluid layers is given in Appendix D. This instability is also discussed in connection to the phase-change problem in Chapter 5.

What makes the Rayleigh-Taylor problem of special interest to us is the unavoidable existence of both the heat and mass transfer in film boiling.

**Plan of the Thesis**

The rest of the thesis contains the modeling equations, their scaling and the method of solution to determine the onset conditions of the instabilities. This is followed by a discussion of the results of calculations for the onset of instability in the evaporation problem, the Rayleigh-Taylor problem and the pure convection problem. In each of these discussions, the role of vapor dynamics is stressed: a role that has not been emphasized up to this point in this chapter. The vapor is a source of heat transfer in all of these problems. To assume that the vapor plays a passive role is then ordinarily incorrect.

When the vapor moves it does so on account of several mechanisms. It may move because pressure perturbations are imparted to it or because the thermal profiles generate phase-change at least upon perturbation, or it can move on account of viscous dragging.
Whatever the mechanism for its motion, it must change the mode of heat transport from conduction to convection. This change in the transport must feed back to the interface and either assist the undulations or hinder them. Our goal is to understand the role of the feedback, the modeling and the physics of instability mechanisms.

This thesis is different from the traditional type of dissertations because of the absence of a chapter on the literature search. I have decided to refer to the literature in the context of the work presented.
CHAPTER 2
THE MATHEMATICAL MODEL

This chapter includes all of the equations used to analyze the problems described in Chapter 1. The equations are divided into the field equations that are meant to describe the behavior of the state domain variables and the interfacial equations that describe the relation of the domain variables with surface variables such as surface speed and surface curvature which, in turn, depend upon surface position. The domain equations and interfacial conditions are written in a general manner without regard for distinction between the various problems. We shall draw the attention of the reader to the differences as we go along and thereby indicate how each problem belongs to a special case of the general equations.

What may not have been apparent from the discussion in the earlier chapter is that the instabilities are related to a nonlinear property of the modeling equations. The nonlinearities arise on account of more than one reason. One of the reasons to expect nonlinear characteristics in the modeling equations is that the interface position is coupled to the transport behavior and the two depend on each other. Another reason is that the domain variables are coupled. For example, the interface temperature distribution affects the pressure distribution, which in turn affects the flow profiles. The flow profiles then affect the temperature profiles, either reinforcing them or causing the disturbances to subside. We now turn to the nonlinear equations and shall point out their characteristics as we go along.
Nonlinear Equations

The various problems we will deal with are all fluid flow problems; hence, the
equations that model the fluid physics in unscaled form are given by the Navier Stokes
equations in each phase, i.e., by

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \rho(T) \mathbf{g} + \mu \nabla^2 \mathbf{v} \quad 2.1 \]

and

\[ \rho^* \left( \frac{\partial \mathbf{v}^*}{\partial t} + \mathbf{v}^* \cdot \nabla \mathbf{v}^* \right) = -\nabla P^* + \rho^*(T^*) \mathbf{g}^* + \mu^* \nabla^2 \mathbf{v}^* \quad 2.2 \]

Each of the problems is connected with temperature gradients. These affect the
energy transport. Therefore the energy equations in each phase are needed. They are
given by

\[ \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T = \kappa \nabla^2 T \quad 2.3 \]

and

\[ \frac{\partial T^*}{\partial t} + \mathbf{v}^* \cdot \nabla T^* = \kappa^* \nabla^2 T^* \quad 2.4 \]

Mass conservation in each phase requires the continuity equations. They are

\[ \nabla \cdot \mathbf{v} = 0 \quad 2.5 \]

and

\[ \nabla \cdot \mathbf{v}^* = 0 \quad 2.6 \]

In the equations above, \( \mathbf{v} \), \( P \) and \( T \) are velocity, pressure and temperature fields,
respectively, and the asterisk denotes the vapor phase. In the equations, \( \rho \), \( \mu \), and \( \kappa \) are
the density, viscosity and thermal diffusivity, respectively.

A major assumption that has tacitly been made is that the densities of the fluids are
treated to be constant in all of the terms save those associated with gravitational
acceleration. This assumption is fair provided the temperature drop is not very large or at least as long as the total density change divided by its mean value in each phase is a small fraction. This is the celebrated Boussinesq approximation. It can be justified by a perturbation series in a scaled thermal expansion coefficient. This justification, given by Mihaljan (1962), has proven to be invaluable in studies in thermal convection greatly simplifying the calculations. More on this approximation is given in Appendix E.

We continue with the modeling equations and assume that the bottom and top plate temperatures are kept constant and therefore \( T(-d) = T_{\text{bottom}} \) and \( T^*(d^*) = T_{\text{top}} \) hold.

The top and bottom walls are impermeable, therefore, \( \nabla \cdot \mathbf{n} = 0 \) and
\[
\nabla^* \cdot \mathbf{n} \bigg|_{x=d^*} = 0
\]
hold. The no-slip condition applies along the top and bottom plates, and gives rise to \( \nabla \cdot \mathbf{i} \bigg|_{x=d} = 0 \) and \( \nabla^* \cdot \mathbf{i} \bigg|_{x=d^*} = 0 \).

At the interface, the mass balance equation is given by
\[
\rho \left( \mathbf{v} - \mathbf{\bar{u}} \right) \cdot \mathbf{n} = \rho^* \left( \mathbf{v}^* - \mathbf{\bar{u}} \right) \cdot \mathbf{n}
\]
where \( \mathbf{n} = \frac{-Z \mathbf{i} + \mathbf{k}}{\left( 1 + \left( \frac{\partial Z}{\partial x} \right)^2 \right)^{\frac{1}{2}}} \) is the unit outward normal (Appendix F) and
\[
\mathbf{\bar{u}} \cdot \mathbf{n} = \frac{\partial Z}{\partial t} \left( \frac{\left( \frac{\partial Z}{\partial x} \right)^2}{1 + \left( \frac{\partial Z}{\partial x} \right)^2} \right)^{\frac{1}{2}}
\]
is the interface speed (Appendix G). When there is no phase-change at the interface, we set the interfacial mass balance equal to zero, and we obtain two interface conditions from the interfacial mass balance equation:
\[
\rho \left( \mathbf{v} - \mathbf{\bar{u}} \right) \cdot \mathbf{n} = 0
\]
\[ \rho^* \left( \bar{v}^* - \bar{u} \right) \cdot \hat{n} = 0 \]

At the interface, the temperatures and the tangential components of velocities of both fluids are equal to each other, so \( T = T^* \) and \( \bar{v} \cdot \hat{t} = \bar{v}^* \cdot \hat{t} \) hold where

\[
\hat{t} = \frac{\vec{t} + \frac{\partial Z}{\partial x} \vec{k}}{\left( 1 + \left( \frac{\partial Z}{\partial x} \right)^2 \right)^{\frac{1}{2}}} \text{ is the unit tangent vector.}
\]

The interfacial tension and its gradient with respect to the temperature at the interface come into the picture through the force balance. By taking the dot product of the stress balance with the unit normal and unit tangent vectors separately, we get the normal and tangential stress balance equations. The force balance is given by

\[
\left( \rho \bar{v} \left( \bar{v} - \bar{u} \right) - \bar{T} \right) \cdot \hat{n} + 2H \bar{n} + \nabla \gamma \bar{v} \cdot \hat{t} = \left( \rho^* \bar{v}^* \left( \bar{v}^* - \bar{u} \right) - \bar{T}^* \right) \cdot \hat{n}
\]

where \( \bar{T} = -\bar{P} \bar{l} + \bar{S} \) is the total stress, \( \gamma \) is the interfacial tension and

\[
2H = \frac{\partial^2 Z}{\partial x^2} \left( 1 + \left( \frac{\partial Z}{\partial x} \right)^2 \right)^{\frac{1}{2}} \text{ is the surface mean curvature.}
\]

In addition we need the interfacial energy balance, viz.

\[
\left( \rho \left( \dot{H} + \frac{1}{2} (\bar{v} - \bar{u})^2 \right) (\bar{v} - \bar{u}) + \bar{q} - \bar{S} \cdot (\bar{v} - \bar{u}) \right) \cdot \hat{n} = \left( \rho^* \left( \dot{H}^* + \frac{1}{2} (\bar{v}^* - \bar{u})^2 \right) (\bar{v}^* - \bar{u}) + \bar{q}^* - \bar{S}^* \cdot (\bar{v}^* - \bar{u}) \right) \cdot \hat{n}
\]

Here \( \dot{H} \) is the enthalpy per unit mass, \( \bar{S} \) is the extra stress, and \( \bar{q} \) is the heat flux. Now, if there is phase-change across the interface, we are short of one equation as the
interfacial mass balance (Equation 2.7) gives us only one relation. To get this additional equation we observe that when the evaporation rate is very small we may assume that there is thermodynamic equilibrium at the phase-change boundary, that is

\[ P^* - P_{base}^* = \rho^* h \ln \left( \frac{T^*}{T_{base}^*} \right) \]

where \( P_{base}^* \) and \( T_{base}^* \) are the base state pressure and temperature of the fluid, and \( h \) is the latent heat of evaporation per unit mass.

**Dimensionless Nonlinear Equations**

The nonlinear equations in dimensionless form, if necessary, are given below. The following scales are used: length, \( d \); velocity, \( \nu = \frac{\kappa}{d} \); time, \( \frac{d}{\nu} \); pressure, \( \bar{P} = \frac{\mu \nu}{d} \) and temperature, \( \bar{T} = \frac{T - T_{cold}}{\Delta T} \) where \( \Delta T = T_{hot} - T_{cold} \). The instability in these problems is caused by the temperature gradients, so it is reasonable to scale the velocities with respect to the thermal diffusivity since it is the dissipation of the temperature perturbations that is important. The temperature is scaled with respect to the total temperature difference across the bilayer system and the reference temperature is the cold temperature, which in many systems we cover will be the top plate. This will prove to be of convenience when we move to higher order expansions of the problem. Throughout the course of our study we will keep the top and bottom plate temperatures constant and by scaling the temperature with respect to our choice of reference temperature and the total temperature difference, we will significantly simplify the top and bottom temperature boundary conditions. Hereafter, we will drop the tilde on the scaled temperature. Using the expansion for the density given in Appendix E, the momentum equations in each phase become
\[
\frac{1}{\Pr} \left( \frac{\partial \bar{v}}{\partial t} + \bar{v} \cdot \nabla \bar{v} \right) = -\nabla P - G \bar{\alpha}_x + \text{Ra} \, \bar{\alpha}_x + \nabla^2 \bar{v} \tag{2.11}
\]

and

\[
\frac{\nu}{\nu^*} \frac{1}{\Pr} \left( \frac{\partial \bar{v}^*}{\partial t} + \bar{v}^* \cdot \nabla \bar{v}^* \right) = -\frac{\mu}{\mu^*} \nabla P^* - \frac{\nu}{\nu^*} G \bar{\alpha}_x^* + \text{Ra}^* \, \bar{\alpha}_x^* + \nabla^2 \bar{v}^* \tag{2.12}
\]

where \( \Pr = \frac{\nu}{\kappa} \) and \( G = \frac{gd^3}{v \kappa} \) are the Prandtl and Gravity numbers, respectively.

The energy equations in each domain are given by

\[
\nabla^2 T = \left( \frac{\partial T}{\partial t} + \bar{v} \cdot \nabla T \right) \tag{2.13}
\]

and

\[
\nabla^2 T^* = \frac{\kappa}{\kappa^*} \left( \frac{\partial T^*}{\partial t} + \bar{v}^* \cdot \nabla T^* \right) \tag{2.14}
\]

The mass balance at the interface, \( z = Z(x,t) \), now becomes

\[
(\bar{v} - \bar{u}) \cdot \bar{n} = \frac{\rho}{\rho^*} (\bar{v}^* - \bar{u}) \cdot \bar{n} \tag{2.15}
\]

The normal stress balance in dimensionless form is given by

\[
\text{Ca} \left( \frac{1}{\Pr} \left( \bar{v} - \bar{u} \right) - \bar{T} \right) : \bar{n} \bar{n} + 2H = \text{Ca} \left( \frac{\rho}{\rho^*} \frac{1}{\Pr} \bar{v}^* \left( \bar{v}^* - \bar{u} \right) - \frac{\mu}{\mu^*} \bar{T}^* \right) : \bar{n} \bar{n} \tag{2.16}
\]

where \( \text{Ca} = \frac{\mu K}{\gamma d} \) is the Capillary number.

The tangential stress balance becomes

\[
\bar{S}^* : \bar{n} \bar{t} - \bar{S} : \bar{n} \bar{t} + \text{Ma} \bar{V}_s T = 0 \tag{2.17}
\]

In addition, the interfacial energy balance becomes
Here, $K_{pc} = \frac{k^2}{h}$, $E = \frac{C_p \Delta T}{h}$ and $V_{pc} = \frac{\nu \kappa}{h d^2}$. $E$ is the Evaporation number.

At the interface, the temperatures of both fluids are equal to each other, and the thermodynamic equilibrium at the phase-change boundary holds, that is

$$\Pi_{ke} \left( p^* - p_{base} \right) = \ln \left( \frac{T^*}{T_{base}} \right)$$

where $\Pi_{ke} = \frac{\rho \nu \kappa}{\rho^* h d^2}$.

Some Observations

We can make several observations on the above equations. First, as noted, the interfacial mass balance is simplified when there is no phase-change and the thermodynamic equilibrium relationship is then ignored. It is also possible to consider phase-change with a zero temperature drop across the bilayer. However, in this case a non-zero evaporation base state is required. This case is not considered in this study.

Second, when phase-change takes place and the thermodynamic equilibrium is taken into account, the effect of curvature on latent heat also needs to be taken into account. However, the effect is weak and this fact is proven in Appendix A. Third, by imposing a restriction of a flat undulated interface and infinite interfacial tension, the equations simplify considerably. The normal component of the stress balance is then used only to determine the value of the curvature times the interfacial tension but not to close the problem. Fourth, if the Marangoni effect were excluded, then the tangential stress
balance at the interface simplifies by rendering the interfacial tension gradient to zero but no singularity arises on account of this. Finally, the interfacial balances are all frame invariant. This is seen in the force balance if the mass balance is taken into account and it is seen in the energy balance in the form that it is presented. The energy balance is derived for the reader’s benefit in Appendix H.

The Linear Model

To see the equations that determine the onset of instability requires us to linearize the equations about a base state. First, a word about the base state: The base state is always a solution to the nonlinear equations and often it might seem defeating to look for a base state if it means solving a full set of intricate coupled nonlinear equations. But in practice for a large class of problems the base state is seen almost by inspection or by guessing it. For example, in the problems considered in this thesis an obvious base state is the quiescent state with a flat interface. It is not in global thermodynamic equilibrium as a temperature gradient is imposed but it is a state whose stability is called into question.

To study the stability of the interface, arbitrary disturbances of infinitesimal amplitude are applied and for a given set of input variables, the growth and decay time constants of these disturbances or equivalently the critical temperature difference across both layers that result in marginal stability are determined. The reader might wonder why the disturbances are taken to be small and what this restriction implies. If a state is unstable to infinitesimal disturbances it must be unstable to all disturbances. The same is untrue of finite amplitude disturbances. Thus, stability to small disturbances usually means very little but instability implies a lot. Using small disturbances also allows the dropping of terms with quadratic interactions and higher degree interactions. Therefore,
the plan is to linearize the above equations about a known base state and to find the onset of the interfacial instability from the perturbed model.

**Perturbation Equations**

Hereon, the variables of the base state are denoted by the subscript 0, and the variables of the perturbed state by the subscript 1. Thus, the temperature, when perturbed, is described by

\[ T = T_0 + \epsilon \left( T_1 + \frac{dT_0}{dz} z_1 \right) + O(\epsilon^2) \]

where \( \epsilon \) is a small perturbation parameter representing the deviation from the base state and \( z_1 \) is the mapping of the perturbed configuration onto the reference configuration. Its meaning is explained in the Appendix I and, at the interface, it is simply the perturbation of the surface deflection to first order, \( Z_1 \), a variable to be determined during the course of the calculation. We can further expand \( T_1 \) and other subscript ‘one’ variables using a normal mode expansion. For example,

\[ T_1 = \hat{T}_1(z) e^{\sigma t} e^{i\omega x}. \]

Here, \( \sigma \) is the inverse time constant while \( \omega \) is a wavenumber associated with the given perturbation. A wavenumber arises because the system is infinite in lateral extent. However, the wavelength, which is proportional to reciprocal of the wavenumber, may be related to the spacing between the sidewalls in containers with periodic boundary conditions.

The same expansion as above is used for both components of velocity and pressure as well as temperature in both phases. It also holds for the interface deflection variable...
Z₁. The first priority is to get the base state, as this is the state whose stability is in question and about which the expansions must be done.

The Base State Solution and the Perturbed Equations

In the base state there is no flow in either the liquid or the gas. For the phase-change problem, assume that there is no phase-change. In other words, assume that the upper plate pressure is fixed so that the phase-change rate in the base state is zero. This of course does not mean that the phase-change rate will be zero upon perturbation. Consequently, both fluids are in a conductive state, thus \( \bar{v}_0 = \bar{v}'_0 = \bar{0} \). Hereon, we will drop the overbar of the scaled temperature. The temperature profiles in the liquid and its vapor in dimensionless form become

\[
T = -\frac{k'}{k \delta + k^*} z + \frac{k \delta'}{k \delta + k^*}
\]

and

\[
T^* = -\frac{k}{k \delta + k^*} z + \frac{k \delta'}{k \delta + k^*}
\]

where \( \delta = \frac{d^*}{d} \).

After we perturb the domain and boundary equations in the manner given in the previous section, we arrive at the following equations in the domain in dimensionless form:

\[
\left( \frac{d^2}{dz^2} - w^2 \right) v_{x1} - i w P_1 = \frac{1}{Pr} v_{x1} \quad 2.20
\]

\[
\left( \frac{d^2}{dz^2} - w^2 \right) v_{z1} + Ra T_1 - \frac{\partial P_1}{\partial z} = \frac{1}{Pr} v_{z1} \quad 2.21
\]
\[
\left( \frac{d^2}{dz^2} - w^2 \right) T_i - \frac{dT_0}{dz} v_{z1} = \sigma T_i
\]

2.22

and

\[
iw v_{x1} + \frac{dv_{z1}}{dz} = 0
\]

2.23

for the lower phase, and

\[
\left( \frac{d^2}{dz^2} - w^2 \right) v^*_{x1} - iw \frac{\mu}{\mu} P_1^* = \sigma \frac{1}{Pr} \frac{\nu}{\nu} v_{x1}^*
\]

2.24

\[
\left( \frac{d^2}{dz^2} - w^2 \right) v_{z1}^* + \frac{\alpha^*}{\alpha} \frac{\nu}{\nu} Ra T_i^* - \frac{\mu}{\mu} \frac{\partial}{\partial z} P_1^* = \sigma \frac{1}{Pr} \frac{\nu}{\nu} v_{z1}^*
\]

2.25

\[
\left( \frac{d^2}{dz^2} - w^2 \right) T_i^* - \frac{\kappa}{\kappa} \frac{dT_0^*}{dz} v_{z1}^* = \sigma \frac{\kappa}{\kappa} T_i^*
\]

2.26

and

\[
iw v_{x1}^* + \frac{dv_{z1}^*}{dz} = 0
\]

2.27

for the upper phase.

At the lower plate, \( z = -1 \), the liquid is subject to \( v_{z1} = 0, \ v_{x1} = 0 \) and \( T_i = 0 \)

while the conditions at the top plate become \( v_{z1}^* = 0, \ v_{x1}^* = 0 \) and \( T_i^* = 0 \).

At the interface the mass balance turns into

\[
v_{z1} - \frac{\rho^*}{\rho} v_{z1}^* = \sigma \left( 1 - \frac{\rho^*}{\rho} \right) Z_1
\]

2.28

which, in the absence of phase-change, becomes

\[
v_{z1} = \sigma Z_1
\]

and

\[
v_{z1}^* = \sigma Z_1.
\]

The continuity of temperature and the no-slip conditions become
\[ T_i + \frac{dT_0}{dz} Z_i = T_i^* + \frac{dT_0^*}{dz} Z_i \]  

and

\[ v_{x1} = v_{x1}^* \]  

The normal and tangential stress balances assume the following dimensionless form:

\[
Ca \left( P_1 - P_1^* \right) - 2 \text{Ca} \left( \frac{dv_{x1}}{dz} - \mu \frac{dv_{x1}^*}{dz} \right) - \left( Bo + w^2 \right) Z_i = 0
\]  

and

\[
\frac{\mu^*}{\mu} \left( \frac{dv_{x1}^*}{dz} + iw v_{x1}^* \right) - \left( \frac{dv_{x1}}{dz} + iw v_{x1} \right) - iw \text{Ma} \left( T_i + \frac{dT_0}{dz} Z_i \right) = 0
\]  

where \( Bo = \left( \frac{\rho - \rho^*}{\gamma} \right) g d^2 \), and again \( Ca = \frac{\mu \kappa}{\gamma d} \) and \( Ma = \frac{\gamma_1 \Delta T}{\mu \kappa} \) are the Bond, Capillary and Marangoni numbers, respectively. The interfacial energy balance is

\[
v_{x1} - E \left( \frac{k^* dT_i^*}{k dz} - \frac{dT_i}{dz} \right) = \sigma Z_i
\]  

where \( E \) stands for the Evaporation number and is given by \( E = \frac{C_p \Delta T}{h} \). \( C_p \) is the liquid heat capacity. Again, one has to be careful in order to avoid a mistake here. The interfacial energy balance in the absence of phase-change simply becomes

\[
\frac{k^* dT_i^*}{k dz} = \frac{dT_i}{dz}
\]

The equilibrium condition at the interface, when there is phase-change across the interface, is

\[
\Pi_{KE} P_i^* - \Pi_{PE} Z_i = \frac{T_i^* + \frac{dT_0}{dz} Z_i}{T_{base}}
\]
where $\Pi_{PE} = \frac{g \, d}{h}$ is a dimensionless parameter from the linearized Clapeyron equation, and $T^*_\text{base}$ is the scaled base state temperature of the vapor at the flat interface and it is equal to $T^*_0$.

There are several observations to be made in the perturbed equations. First, the growth constant $\sigma$ appears in the domain as well as interfacial equations. Dropping them in the domain equations lead to an interfacial mode, however dropping them may not always be justifiable but when the growth constant is dropped from the domain its calculation becomes straightforward. It must then also be real as it is given by the ratio of two determinants. Second, the growth constant is ordinarily an eigenvalue for a sensible control parameter such as $E$. Third, it is conceivable that solutions exist such that $Z_1$ is zero but they may not be the first to become unstable as a control parameter, such as $E$, changes. Fourth, when we pay closer attention to some of the dimensionless numbers we recognize that their values are very small. A typical value for Capillary number is of the order of $10^{-8}$ and for $\Pi_{KE}$ is of the order of $10^{-10}$. This indicates that for systems where there is a high surface tension between the two fluids it may not be a bad idea to set $Ca$ equal to zero, which consequently will imply that the surface deflection $Z_1$ is equal to zero. That is to say that we can assume the interface to be flat at all times and the instability will still set in because the temperature and velocity will still affect each other through the domain equations. The same goes also for the $\Pi_{KE}$ term.

In the next chapter the perturbation problem is solved for the special case of pure evaporation in the absence of natural convection, and from the calculations a physical interpretation of the problem is sought.
Endnote: Challenges in phase-change problems: Interface conditions

There is a crucial difference between the classical Rayleigh-Marangoni-Bénard problem and the evaporation problem. The difference lies in the interfacial conditions. In the Rayleigh-Marangoni-Bénard, the interfacial mass balance gives us 2 interface conditions as a result of the no mass transfer condition at the interface. However, in the evaporation problem, we only get one condition from the interfacial mass balance, and consequently, we are 1 interface condition short. The interface conditions are completed with our choice of equilibrium conditions for the temperature and pressure at the interface. The choice of conditions at the interface between the liquid and the vapor poses a challenge in phase-change problems. Ordinarily, the continuity of temperature is used along with some other thermodynamic equilibrium condition (Huang and Joseph, 1992) or a relation derived from the kinetic theory (Oron et al., 1997) is used instead. Ward and Stanga (2001) have investigated the assumption of continuity of temperature and they have verified the existence of a temperature discontinuity at the interface. Margerit et al. (2003) studied the role of such interfacial nonequilibrium effects on Bénard-Marangoni instability. However, Ward and Stanga report that the magnitude of the temperature discontinuity increases with the evaporation flux and note that when equilibrium exists between the liquid and vapor phases, the temperature is the same in each phase. Consequently, when the evaporation rate is very small we may assume that there is thermodynamic equilibrium at the phase-change boundary and the temperature of both fluids are equal to each other at the interface, that is

\[ P^*=f(T^*) \text{ and } T=T^*. \]
It is no problem to consider non-zero evaporation in the base state. The equilibrium relation is replaced by one that involves a phenomenological coefficient, which may, in fact, depend upon the wavelength of the disturbance.
CHAPTER 3
INSTABILITY DUE TO EVAPORATION – THE LINEAR MODEL

In this chapter we shall concentrate on the solution of the linearized equations given in the last chapter to find the effect of active vapor layers on the evaporative instability without and with convection. The calculations are presented first for the phase-change problem without the complications of Rayleigh and Marangoni instabilities. The effects of Rayleigh and Marangoni convection are added later and each step towards the complete phase-change problem is discussed. The results for different cases are presented in the form of plots and physical explanations are given.

The Pure Phase-Change Problem

To understand the physics of the pure phase-change problem, we first display the results of calculations using the full set of phase-change problem equations as mentioned in Chapter 2. The only terms left out of the equations are the terms that have gravity and surface tension-gradient in them. The first goal of the calculations is to determine the conditions at the onset of instability. In order to do this, we set $\sigma$, the dimensionless inverse time constant, to zero. This implies that we seek only the conditions for steady bifurcation from the base state. Another set of calculations, where we have calculated the values of $\sigma$, was also performed for the case of fluids heated from the liquid side where it was discovered the leading $\sigma$ to be always real. Upon setting $\sigma$ to zero, the unscaled eigenvalue in the linearized problem is the temperature drop. The Chebyshev spectral tau method (Johnson, 1996) is used to solve the resulting eigenvalue problem that determines the critical temperature difference. The interested readers are referred to Appendix J for
more details on the method. The calculations are done assuming the physical properties
of water and water vapor at the saturation temperature under 1 atm, i.e., at 100 °C as
given by Rubinstein et. al. (2002). The properties used in the calculations are given in
Table 3.1.

Table 3-1. Physical properties of water and water vapor at 100°C

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$960 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>$\rho^*$</td>
<td>$0.6 \text{ kg/m}^3$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$2.9 \cdot 10^{-7} \text{ kg/m sec}$</td>
</tr>
<tr>
<td>$\mu^*$</td>
<td>$1.3 \cdot 10^{-5} \text{ kg/m sec}$</td>
</tr>
<tr>
<td>$k$</td>
<td>$6.8 \cdot 10^{-7} \text{ J/m sec °C}$</td>
</tr>
<tr>
<td>$k^*$</td>
<td>$2.5 \cdot 10^{-5} \text{ J/m sec °C}$</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$1.7 \cdot 10^{-7} \text{ m}^2/\text{sec}$</td>
</tr>
<tr>
<td>$\kappa^*$</td>
<td>$2.0 \cdot 10^{-5} \text{ m}^2/\text{sec}$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$6.0 \cdot 10^{-5} \text{ °C}$</td>
</tr>
<tr>
<td>$\alpha^*$</td>
<td>$6.0 \cdot 10^{-3} \text{ °C}$</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>$2.3 \cdot 10^6 \text{ J/kg}$</td>
</tr>
<tr>
<td>$\gamma_T$</td>
<td>$2.0 \cdot 10^{-4} \text{ N/m°C}$</td>
</tr>
</tbody>
</table>

To make a proper comparison we consider different physical problems and present
graphs of the ratio of the critical temperature differences for the different cases against
the wavenumber. The definitions of these ratios are shown in Table 3.2 where the
subscripts pc, Ma, g, and Ra stand for phase-change, Marangoni convection, gravity and
buoyancy-driven convection, respectively.

Table 3-2. Definitions of $R^1$, $R^2$ and $R^3$

<table>
<thead>
<tr>
<th>$R^1$</th>
<th>$\frac{\Delta T_{pc-Ma}}{\Delta T_{Ma}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>$\frac{\Delta T_{pc-Ma-g}}{\Delta T_{pc-Ma}}$</td>
</tr>
<tr>
<td>$R^3$</td>
<td>$\frac{\Delta T_{pc-Ma-Ra}}{\Delta T_{pc-Ma}}$</td>
</tr>
</tbody>
</table>
For example, $\Delta T_{pc-Ma}$ denotes the critical temperature difference for the case where there is phase-change and Marangoni convection, and $R^1$ therefore is the ratio of $\Delta T_{pc-Ma}$ to the critical temperature difference for the case of pure Marangoni convection, $\Delta T_{Ma}$. 

First, we present the results for the case where the critical temperature difference for a pure phase-change problem is compared to the critical temperature difference for a case where the Marangoni effect is added to the phase-change problem. This comparison is given in Table 3-3 and the readers can see for themselves that the addition of the Marangoni effect does little to change the threshold of evaporative instability.

<table>
<thead>
<tr>
<th>Wavenumber, $\omega$</th>
<th>$\Delta T_{pc}$ ($^\circ$C)</th>
<th>$\Delta T_{pc-Ma}$ ($^\circ$C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>0.2</td>
<td>0.47</td>
<td>0.47</td>
</tr>
<tr>
<td>0.3</td>
<td>1.78</td>
<td>1.78</td>
</tr>
<tr>
<td>1.1</td>
<td>6.70</td>
<td>6.69</td>
</tr>
<tr>
<td>1.2</td>
<td>6.96</td>
<td>6.95</td>
</tr>
<tr>
<td>1.3</td>
<td>7.25</td>
<td>7.24</td>
</tr>
<tr>
<td>2.1</td>
<td>10.64</td>
<td>10.63</td>
</tr>
<tr>
<td>2.2</td>
<td>11.24</td>
<td>11.23</td>
</tr>
<tr>
<td>2.3</td>
<td>11.89</td>
<td>11.88</td>
</tr>
<tr>
<td>3.1</td>
<td>19.21</td>
<td>19.19</td>
</tr>
<tr>
<td>3.2</td>
<td>20.44</td>
<td>20.42</td>
</tr>
<tr>
<td>3.3</td>
<td>21.75</td>
<td>21.73</td>
</tr>
</tbody>
</table>

Marangoni convection is driven by the temperature variations along the interface. In order to understand the effect of phase-change on temperature gradients at a deflected interface, let us first consider such an interface in the absence of gravity and observe that the deflection must make the temperature vary along the interface. As discussed earlier, the temperature at a trough is expected to be higher than that at a crest as it is closer to the heat source. When perturbed, the liquid starts to evaporate into its own vapor at a trough since the temperature at the trough is now higher than the base state temperature.
Likewise, condensation takes place at the crests. When the liquid evaporates at the trough, the otherwise hot trough temperature goes down since part of the energy input is now used for phase-change whereas at the crest the heat is released, and the otherwise low crest temperature goes up. In other words this phase-change action reduces the temperature perturbations along the interface in phase-change problems, which, in turn, reduces the surface tension gradient-driven convection. Marangoni convection therefore ultimately never plays a significant role on the stability of the phase-change problem.

We can also predict the reduction of the temperature perturbations at the interface by considering the linearized thermodynamic equilibrium relation at the interface. They are seen from the $\Pi$ terms in the equation 2.34 in Chapter 2, which are very small for most fluids, thereby making the temperature perturbations very small.

![Figure 3-1](image)

**Figure 3-1.** $R^1$ vs. wavenumber.

In Figure 3-1, we observe that at large wavenumbers the pure Marangoni problem is considerably more unstable than the phase-change problem with Marangoni
convection. At low wave numbers the reverse is true. Before we go on and explain these observations let us make clear the restrictions behind the calculations that generated Figure 3-1.

In producing Figure 3-1 we have performed calculations by taking two similar systems where a liquid underlies its own vapor at zero gravity, and in each system heat is supplied from the liquid side. In one case, we pretend that there is no phase-change across the interface between the liquid and its vapor. In this case, the flow can only be caused by surface-tension gradients along the interface. In the other case, we allow phase-change across the interface. To make a fair comparison, as there is initially no flow in the pure Marangoni problem, we adjust the pressure on the vapor side in the second case such that the phase-change rate is zero. As a result, phase-change is only possible through perturbations at the interface. Accordingly, we have two systems with no-flow in their base states. They are two similar systems with only one difference; one has the possibility of phase-change upon perturbation while the other has no such possibility at all. We then plot the ratio, $R^1$, versus wavenumber, $\omega$, in Figure 3.1. At small wavenumbers, not at or very near the zero wavenumber, the pure Marangoni problem is more stable than the phase-change problem with Marangoni convection for a water and water vapor system. Small wavenumbers imply that there are reduced transverse thermal gradients thereby stabilizing the Marangoni effect, yet allowing the phase change instability to continue unabated. At large wavenumbers, we observe the strong action of thermal and momentum diffusion.

In many physical situations and laboratory experiments, gravity plays an important role. Gravity comes equipped with the burden of buoyancy-driven convection.
However, the effects of Rayleigh convection can be avoided in applications that involve very thin liquid and vapor layers even though gravity continues to play a role. Gravity simply pulls the perturbed interface back to its original position. In other words, gravity plays a stabilizing effect at all wavenumbers. Even so, its effect on the stability of the interface is more obvious at small wavenumbers, as small wavenumbers imply weaker surface curvature, and that explains why the critical temperature difference is higher at small wavenumbers when gravity is present. This is depicted by the curve labeled $R^2$ in Figure 3-2.

Working with very thin fluid layers is difficult in a laboratory environment. Thus, eventually we have to account for the effect of buoyancy on the problem. For that reason, Rayleigh convection is now introduced into the problem. It is well known from the Rayleigh problem that, in the absence of phase-change, buoyancy-driven convection can easily be stabilized at very small and very large wavenumbers. Consequently, in Figure 3-2 we observe that when $R^2$ and $R^3$ are plotted against the wavenumber, both plots understandably merge at small wavenumbers. At large enough wavenumbers, in the absence of buoyancy-driven convection, the stability offered by gravity is swept away by the stabilizing effect of the phase-change mechanism and the curve, depicted by $R^2$, merges with the unity line. This occurs because the surface curvature and, in turn, the phase-change becomes more important compared to gravity at large wavenumbers. In contrast, $R^3$ diverges to become less stable because of the destabilizing character of buoyancy-driven convection. Yet, at even larger wavenumbers $R^3$ ultimately merges with the unity line, not shown in Figure 3-2.
In order to demonstrate the crucial role of the active vapor layer in the stability of a bilayer phase-change problem, we vary the depth of the vapor layer for a constant liquid depth and calculate the critical temperature difference for a phase-change problem with Marangoni and Rayleigh convection. The results are plotted in Figure 3-3, where the ratio of the critical temperature difference for systems with different values of $\delta$ is graphed against wavenumber. The ratios are defined in Table 3-4.

Table 3-4. Definitions of $R^4$, $R^5$ and $R^6$

<table>
<thead>
<tr>
<th>$R^4$</th>
<th>$\frac{\Delta T_{\text{critical}} \text{ of a system with } \delta = 0.2}{\Delta T_{\text{critical}} \text{ of a system with } \delta = 0.1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^5$</td>
<td>$\frac{\Delta T_{\text{critical}} \text{ of a system with } \delta = 0.3}{\Delta T_{\text{critical}} \text{ of a system with } \delta = 0.1}$</td>
</tr>
<tr>
<td>$R^6$</td>
<td>$\frac{\Delta T_{\text{critical}} \text{ of a system with } \delta = 0.4}{\Delta T_{\text{critical}} \text{ of a system with } \delta = 0.1}$</td>
</tr>
</tbody>
</table>

Figure 3-3 shows us that increasing the vapor depth stabilizes the interface. The depth of the vapor layer plays a dual role when we account for the Rayleigh convection in our problem. The deeper the vapor layer, the more unstable it becomes due to Rayleigh convection which is scaled as the cube of the depth; however, the important stabilizing
role of the increased vapor depth on the interfacial stability due to phase-change is still present. The stabilization comes about because of the increase in the vapor flow; meanwhile the destabilizing effect of heat transfer is also weakened. To understand the stability mechanism in Figure 3-3, let us first look at the problem from a heat transfer point of view.

The heat that is transported to the interface is used for phase-change and kinetic energy with the balance being conducted away from the interface through the vapor. In our model, the interface temperature in the base state is always equal to the saturation temperature. Its value depends on the pressure at the top plate and the top plate pressure is set to make the evaporation rate in the base state equal to zero. Hence, a change in fluid depths must influence the pressure required at the top plate to keep the phase-change rate equal to zero and hence the saturation temperature at the interface.

Therefore, the proximity of the top plate to the interface affects the temperature gradients at the interface in both phases. The further away the top plate, or in other words, the greater the ratio of the vapor depth to the liquid depth, the smaller the temperature gradients in the base state. We learned from our preliminary analysis of the phase-change problem in the absence of fluid dynamics that the heat transfer mechanism destabilizes the interface. Thus, from a heat transfer point of view, the deeper the vapor depth the less the destabilizing effect of the heat transfer. From a fluid dynamic point of view, the deeper the vapor layer the less the top plate impedes the stabilizing flow in the vapor. Hence, the interface becomes more stable for these two reasons. It is this stabilizing role of the vapor flow that makes the passive vapor assumption questionable.
Figure 3-3. R^4 through R^6 vs. wavenumber.

**Stabilizing Effect of the Vapor Flow**

In order to understand how the vapor flow stabilizes the system, it is best to look at the velocity profiles at the onset of instability of a pure phase-change problem in the absence of gravity and Marangoni effects.

In Figure 3-4 the velocity profiles are shown for a case where the vapor depth to the liquid depth ratio is taken to be equal to 0.4. The position, z_0 = 0, represents the liquid-vapor interface in the base state. To determine the direction of the flow at a trough (or at a crest) we calculate the value of the vertical component of the velocity in either phase at the interface relative to the computed value of Z_1. This is done using the eigenvector that corresponds to the critical temperature difference. Then, we find the sign of \( \frac{v_{z_l}}{Z_1} \) (or \( \frac{v_{z_l}^*}{Z_1} \)) where \( v_{z_l} \) (or \( v_{z_l}^* \)) is the liquid (or vapor) velocity at the interface and \( Z_1 \) is the surface deflection. This ratio turns out to be negative, implying that when the deflection
is negative, the flow is in the positive direction. In other words, there is up flow at the
trough and down-flow at the crest. The fluid flows upset the temperature gradients in
both phases at the interface and more so in the vapor because of the stronger flow at the
interface in the vapor. Once the phase-change takes place upon perturbation, the flow in
the liquid will bring the warmer liquid from below attempting to make the trough hotter
whereas the stronger flow in the vapor tries to convect this heat away from the trough.
Observe the difference in the magnitudes of the velocities and the nature of the profiles
between the two phases. The horizontal component of the velocity in the vapor is very
strong compared to the liquid velocity. The vapor brings the warm fluid from the trough
to the colder crest, making its temperature go back up again. In other words, the liquid
flow convects heat to the interface and destabilizes the interface while the heat
conduction from the bottom plate to the interface also encourages instability. The vapor
flow offers stability though the heat conduction in the vapor encourages instability. The
velocity profiles of the perturbed flow tell us that the liquid flow plays a destabilizing
role whereas the vapor flow plays a stabilizing role.

To give further credibility to the statement that the vapor flow is stabilizing
whereas the liquid flow is destabilizing, we can perform a little test. By deliberately
changing the viscosity of the vapor we can strengthen or weaken the vapor flow. Thus, if
we increase the vapor viscosity by a small amount, the vapor flow should be impeded and
the system will become unstable. Indeed, an increase in the vapor viscosity destabilizes
the interface at all wavenumbers. Accordingly, an increase in the liquid viscosity
stabilizes the interface because the destabilizing liquid flow is impeded. It is useful to
remember that in both phases the flow is caused by the temperature perturbations at the
interface. This is obvious from the energy equation at the interface wherein one sees that
perturbed temperature gradients must result in perturbed flows at the interface. This flow
at the interface generates pressure perturbations, which in turn feedback to the
temperature through the thermodynamic equilibrium at the interface. To summarize, the
stability of a phase-change problem is determined by the competitive nature of heat
conduction and heat convection, where the relative values of physical properties of both
the liquid and the vapor affect the magnitude of the heat transfer mechanisms. The
phase-change problem in the absence of gravity becomes unstable at all wavenumbers
given that a certain critical temperature difference is applied across the fluid layers.

Figure 3-4. The velocity vectors in both phases at the onset of instability. $z = 0$
represents the interface position in the base state.

Thus, if we were to conduct experiments in space, we would have to cut-off the
small wavenumbers by limiting the size of the container so that the system is not unstable
from the beginning of the experiment. However, when gravity is present, there exists a
critical wavenumber at which the interface becomes unstable, as depicted in Figure 3-5.
Figure 3-5. The critical temperature difference versus the dimensionless wavenumber for water-water vapor system where $d = 3 \text{mm}$ and $d' = 1.5 \text{mm}$.

This figure shows the critical temperature drop in °C vs. the dimensionless wavenumber, $\omega$. We immediately observe that the critical $\Delta T$ is within an experimentally realizable range.

In this chapter, we have attempted to explain the physics of the phase-change problem at the onset of instability in a comprehensive manner using a straightforward linear stability calculation. We conclude that the consideration of the vapor fluid dynamics is vital for the study of stability of the phase-change problem and leaving it out is an invalid assumption. The linear calculations tell us what the flow and temperature profiles are at the onset of instability. This information is enough for us to understand the physics of the instability; however, one needs to support the theory by conducting experiments. In order to know what to expect during the course of an experiment we need to turn to the nonlinear analysis. This analysis will give us the information on the type of the instability; in other words, it will tell us whether the instability is subcritical or
supercritical. In Chapter 2, we defined the input variables to be the fluid depths, the pressure at the top plate and the overall temperature drop between the hot and cold plates. The critical conditions for the onset of the instability are determined along with the critical wavenumber. Advancing the temperature drop beyond the critical value will not only reveal the magnitudes of the flow and temperature profiles, but will also give the liquid feed rate and the vapor removal rate, as well as the enhancement in heat transport. In order to find the answer to the behavior of the two-phase problem beyond the onset of instability, we turn to the nonlinear problem in the next chapter.

Endnotes

The Exchange of Instability when Heated from the Vapor Side

Another set of calculations was performed for the evaporation problem when heated from the vapor side. It is not necessary for the onset mode to be steady. Sometimes the onset modes can also be oscillatory and one should always check the values of the inverse time constants with the leading real part. When the bilayer system is heated from the vapor side and only gravity effect, but not buoyancy-driven and surface tension gradient-driven convection, is taken into account it turns out that onset of instability is oscillatory.

The Assumption of Flat Surface in the Stability Analysis of the Phase-Change Problem

The phase-change problem can also be studied under the assumption that the interface does not deflect with the onset of instability, i.e., the onset of instability is represented by the transition from the conductive state to the convective state. If the interface were artificially constrained to be flat then surface curvature would be zero and interfacial tension would be infinity making the product a determinate value only upon
introduction of the solved variables into the normal component of the force balance at the interface. Under the flat interface restriction the pure phase-change problem exhibits a minimum as seen in Figure 3-6. Weak transverse gradients stabilize on the low end, as one needs these gradients in order to have local evaporative regions and adjoining condensation regions. In the absence of transverse gradients we would not have local evaporation and condensation. Both are needed to have an unstable base state. At the high end of wave number stabilization is once again due to viscous and thermal diffusive effects.

![Figure 3-6. Evaporation number vs. dimensionless wavenumber for two water-water vapor systems. Curve (a) represents the results for calculations where the interface was assumed to be flat whereas curve (b) represents calculations with a deflected interface.](image-url)
CHAPTER 4
WEAKLY NONLINEAR ANALYSIS

The Pure Phase-Change Problem

In Chapter 3, linear stability methods were used to analyze the phase-change between a liquid and its own vapor where the vapor, in addition to the liquid, was assumed to be fluid dynamically active. We concluded there that the active vapor layer plays a significant role in determining the stability of the phase-change problem, and offered explanations for the physics behind this stability. In this chapter, we will examine what happens once we pass the onset of instability. There are several reasons to study the post onset region. One reason is to construct the steady solution beyond the onset of the instability and to find the magnitude of the state variables at the given conditions as this is of experimental importance. In a laboratory experiment the onset of instability can be detected by different means, such as IR-imaging, flow tracers or measuring the change in heat transport with respect to a change in the control variable once the critical state is passed. The use of tracers to observe the onset of the instability may cause problems since it could affect the surface tension at the interface between the liquid and the vapor. Moreover, tracer accumulation at the bottom of the liquid or at the liquid-vapor interface may present another experimental difficulty. IR imaging, though proven to be efficient in flow visualization of bilayer convection by Johnson & Narayanan (1996), may be difficult because of the likelihood of the vapor condensing on the cold top plate when the bilayer is heated from the liquid side; thereby obstructing the imaging.
The change in the heat transfer at the bottom plate is one way to detect the onset of instability in many convective instability problems (Chandrasekhar, 1961), provided that the change is measurable. Therefore, another reason to study the post-onset region via nonlinear calculations is to find the magnitude of the change in the heat transfer. However, there are other advantages of doing the nonlinear calculations. Knowing the magnitude of the perturbed domain variables will also tell us how the flows in both domains change when we change any one of the input variables. The linear calculations only reveal the velocities of both fluid phases relative to each other for a specific set of input variables, whereas the nonlinear calculations give us the chance to compare different bilayer systems with each other. This way, we can better understand the effect of each input variable or physical property on the instability.

The physical model is the same as described in Figure 1.1. To get an idea on the nonlinear features, we will investigate the phase-change problem in the absence of gravity and Marangoni convection in this chapter.

The problem is studied in two spatial dimensions, x and z. The nonlinear equations in dimensionless form for the pure phase-change problem are repeated below for the sake of convenience. The momentum equations in each phase are

\[
\frac{1}{Pr} \left( \frac{\partial \tilde{v}}{\partial t} + \tilde{v} \cdot \nabla \tilde{v} \right) = -\nabla P + \nabla^2 \tilde{v} \tag{4.1}
\]

and

\[
\nu \frac{1}{Pr} \left( \frac{\partial \tilde{v}^*}{\partial t} + \tilde{v}^* \cdot \nabla \tilde{v}^* \right) = -\frac{\mu}{\nu^*} \nabla P^* + \nabla^2 \tilde{v}^* \tag{4.2}
\]

Recall that the energy equations in each domain are given by
\[ \nabla^2 T = \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) \] 

4.3

and

\[ \nabla^2 T^* = \frac{\kappa}{\kappa^*} \left( \frac{\partial T^*}{\partial t} + \mathbf{v}^* \cdot \nabla T^* \right) \] 

4.4

Also remember that the continuity equations in each phase hold assuming that the fluids are incompressible in the sense of the Boussinesq approximation.

We will complete the mathematical model by repeating the interfacial conditions at the phase-change boundary. The mass balance at the interface, \( z = Z(x,t) \), is given by

\[ (\mathbf{v} - \bar{u}) \cdot \mathbf{n} = \rho^* \left( \mathbf{v}^* - \bar{u} \right) \cdot \mathbf{n} \] 

4.5

At the interface, the tangential components of velocities of both fluids are equal to each other, so \( \mathbf{v} \cdot \mathbf{t} = \mathbf{v}^* \cdot \mathbf{t} \) holds.

The interfacial tension comes into the picture through the force balance. This balance equation has normal and tangential components and is given by

\[ \text{Ca} \left( \frac{1}{\text{Pr}} (\mathbf{v} - \bar{u}) - \mathbf{T} \right) \cdot \mathbf{n} + 2H \mathbf{n} = \text{Ca} \left( \rho^* \frac{1}{\rho} \mathbf{v}^* (\mathbf{v}^* - \bar{u}) - \frac{\mu^*}{\mu} \mathbf{T}^* \right) \cdot \mathbf{n} \] 

4.6

In addition, we need the interfacial energy balance, viz.

\[ \left( -1 + K_{\text{pc}} \left( \frac{1}{2} (\mathbf{v} - \bar{u})^2 - \frac{1}{2} (\mathbf{v}^* - \bar{u})^2 \right) \right) (\mathbf{v} - \bar{u}) \cdot \mathbf{n} - E \left( \nabla T \cdot \mathbf{n} - \frac{k^*}{k} \nabla T^* \cdot \mathbf{n} \right) \] 

\[ - V_{\text{pc}} \left( \frac{\tilde{S}}{\mu} (\mathbf{v} - \bar{u}) - \frac{\mu^*}{\mu} \tilde{S}^* (\mathbf{v}^* - \bar{u}) \right) \cdot \mathbf{n} = 0 \] 

4.7
At the interface, the temperatures of both fluids are equal to each other and the thermodynamic equilibrium at the phase-change boundary is

\[
\Pi_{KE} \left( P^* - P_{BASE} \right) = \ln \left( \frac{T^*}{T_{BASE}} \right)
\]

4.8

In the nonlinear analysis one goes beyond the critical state by advancing a control variable such as the temperature drop across the plates. The scaled temperature drop, however, appears as \( E \), the Evaporation number, therefore it is \( E \) that will be advanced from its critical value. Once \( E \) is increased all of the state variables will respond and it is their response that interests us. Whether the response is linearly proportional to the change or not is also of interest and will be determined. To this end we introduce perturbation expansions about the base state for each response state variable.

**Perturbation Expansions**

The domain variables and their derivatives are expanded along the mapping as follows:

\[
\begin{align*}
\mathbf{u} &= \mathbf{u}_0 + (\lambda - \lambda_c)^{\alpha} \left( \mathbf{u}_1 + z_1 \frac{\partial \mathbf{u}}{\partial z_0} \right) + \frac{1}{2} (\lambda - \lambda_c)^{2\alpha} \left( \mathbf{u}_2 + 2z_1 \frac{\partial \mathbf{u}_1}{\partial z_0} + z_1^2 \frac{\partial^2 \mathbf{u}}{\partial z_0^2} + z_2 \frac{\partial \mathbf{u}}{\partial z_0} \right) \\
&\quad + \frac{1}{6} (\lambda - \lambda_c)^{3\alpha} \left( \mathbf{u}_3 + 3z_1 \frac{\partial \mathbf{u}_2}{\partial z_0} + 3z_2 \frac{\partial \mathbf{u}_1}{\partial z_0} + 3z_1^2 \frac{\partial^2 \mathbf{u}_1}{\partial z_0^2} + z_1^3 \frac{\partial^3 \mathbf{u}}{\partial z_0^3} + z_3 \frac{\partial \mathbf{u}}{\partial z_0} \right) + \ldots
\end{align*}
\]

and

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial \mathbf{z}} &= \frac{\partial \mathbf{u}_0}{\partial \mathbf{z}_0} + (\lambda - \lambda_c)^{\alpha} \left( \frac{\partial \mathbf{u}_1}{\partial \mathbf{z}_0} + z_1 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{z}_0^2} \right) + \frac{1}{2} (\lambda - \lambda_c)^{2\alpha} \left( \frac{\partial \mathbf{u}_2}{\partial \mathbf{z}_0} + 2z_1 \frac{\partial^2 \mathbf{u}_1}{\partial \mathbf{z}_0^2} + z_1^2 \frac{\partial^3 \mathbf{u}}{\partial \mathbf{z}_0^3} + z_2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{z}_0^2} \right) \\
&\quad + \frac{1}{6} (\lambda - \lambda_c)^{3\alpha} \left( \frac{\partial \mathbf{u}_3}{\partial \mathbf{z}_0} + 3z_1 \frac{\partial^2 \mathbf{u}_2}{\partial \mathbf{z}_0^2} + 3z_2 \frac{\partial^2 \mathbf{u}_1}{\partial \mathbf{z}_0^2} + 3z_1^2 \frac{\partial^3 \mathbf{u}_1}{\partial \mathbf{z}_0^3} + z_1^3 \frac{\partial^3 \mathbf{u}}{\partial \mathbf{z}_0^3} + z_3 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{z}_0^2} \right) + \ldots
\end{align*}
\]

In the above expansions, \( z_1, z_2 \) and \( z_3 \) specify the mappings from the current state in the \((x,z)\)-domain to the reference quiescent state in the \((x_0,z_0)\)-domain at first, second
and third orders, respectively. Higher derivatives with respect to \( z_0 \) and derivatives in other directions follow in a like manner. It has been shown by Johns and Narayanan (2002) that the derivatives of the spatial mappings \( z_1, z_2, z_3 \) etc. disappear due to algebraic cancellations and never need be of any concern. In the above expansions, \( \lambda \) is the dimensionless control variable, and \( \lambda_c \) is its value at which the system becomes unstable, namely the critical value. In our problem \( \lambda \) will be replaced by the parameter \( E \). We go on and set the \( \frac{\partial}{\partial t} \) terms equal to zero and investigate the nature of the steady-state solutions to the problem once we cross the threshold value. To do this, we will go beyond the critical point by an amount \( \varepsilon \), i.e., \( \lambda = \lambda_c + \varepsilon \). The control variable is now defined as \( E = E_c + \varepsilon \). We will substitute the expansions of the domain variables into the nonlinear equations and find the values of \( \alpha \), along with the amplitude of the perturbations that excited the instability, using the method of dominant balance at every order of the calculation as given by Grindrod (1996). The equations are very long and the process of balancing the orders is tedious; thus, in order to assist the reader, we give an overview of what we are going to do in the calculation.

**Overview**

Suppose that the steady nonlinear problem is in the form of \( Nu = f \), where \( N \) is a nonlinear operator containing the control variable \( \lambda \), \( u \) is the solution vector and \( b \) is an inhomogeneity vector that drives the problem. The vector \( u \) contains the velocity components, the pressure and the temperature, in addition to the surface position, \( Z \), which is the last element of the vector \( u \). The base problem must satisfy the nonlinear equation and its solution is trivially known. In our application, the base state is a pure
conductive state, so in each fluid the velocity is zero and the temperature field is linear. The first order problem is required to be an eigenvalue problem and this means that $\alpha$ cannot be determined at this order. The first order problem is written as:

$$Lu_1 = 0.$$ 

Here, the operator $L$ derived from the $O(\varepsilon^\alpha)$ equations by linearization of $N$ and contains the control variable $\lambda$. The eigenvector $u_1$ is, of course, known only up to an arbitrary multiplicative constant, $A$, which is the amplitude of the disturbances that cause the instability. At first order we can determine $u_1$ and the critical value of $\lambda$, but not $\alpha$ and $A$. It is our aim to determine what $A$, as well as $\alpha$, is through the dominant balance method, and therefore we proceed to the next order. The second order problem is in the form

$$Lu_2 = f_1(u_1,u_1).$$

The solvability condition is applied to this system of equations. But, in our problem the operator $L$ is not self-adjoint. Hence, to make the algebra simpler, the elements of the vector $u_2$ are solved in terms of its last element, i.e., $Z_2$ and $Z_1^2$, where $Z_2$ is the second order correction to the surface position $Z$. Solvability is then applied to the differential equation for $Z_2$, which is actually the normal stress balance at the liquid-vapor interface as the differential operator for $Z$ at every order is self-adjoint. The normal stress balances in the first and second orders look as follows;

$$\frac{d^2Z_1}{dx_0^2} - aZ_1 = 0$$

and
\[ \frac{d^2Z_2}{dx_0^2} - aZ_2 = b_2 \left( Z_1^2 \right). \]

We find in this problem that, because \( Z_1 \) is a multiple of \( \cos(\omega x) \), the solvability condition is automatically satisfied at the second order; thus we move on to the third order problem. In our problem, at third order, we find that a balance can be struck between the various terms for a specific value of \( \alpha \). We then introduce the specified value of \( \alpha \) into the equations and apply the solvability condition on the normal stress balance, which is now in the form of

\[ \frac{d^2Z_3}{dx_0^2} - aZ_3 = b_3 \left( Z_1^3, Z_1Z_2 \right). \]

By doing so, we get an equation that determines the value of \( A^2 \). Hence, in the dominant balance method, we find the value of \( \alpha \) and \( A \) at the same order. The value of \( \alpha \) tells us what type of bifurcation we should expect to see once the instability sets in. For example, if \( \alpha \) is equal to one the bifurcation is trans-critical; and if \( \alpha \) is equal to \( \frac{1}{2} \) it is a pitchfork. To determine whether the pitchfork is forward or backward we focus on the sign of \( A^2 \). If it is positive the pitchfork is forward, else it is backward. In the latter case, \( E \) must be expanded as \( E = E_C - \varepsilon \). The work showing how to determine \( \alpha \) is shown in detail as an endnote to this chapter.

**Perturbed Equations**

Using the method outlined above, we expand the nonlinear domain equations given earlier up to the third order. Thus far we have pretended that our physical system is infinite in lateral extent, but the pure phase-change problem is unlike the standard Bénard problem, which produces a nonzero critical wavenumber. For that reason, in this chapter...
we will choose an arbitrary wavenumber, and will proceed with the calculations using	hat wavenumber. As a result, we have tacitly assumed that the fluids are confined within
a laterally finite system with slippery and insulating walls. An analysis for long
wavelength modes assuming a passive vapor has been considered by Oron et. al. (1997).

Returning to the expansions of the governing equations, we first write the
momentum equations in both phases as

\[ (E - E_c)^\alpha \left( -\nabla_0 P_1 + \nabla_0^2 \tilde{v}_1 \right) + \frac{1}{2} (E - E_c)^{2\alpha} \left( -\nabla_0 P_2 + \nabla_0^2 \tilde{v}_2 - \frac{2}{Pr} \tilde{v}_1 \cdot \nabla_0 \tilde{v}_1 \right) \]
\[ + \frac{1}{6} (E - E_c)^{3\alpha} \left( -\nabla_0 P_3 + \nabla_0^2 \tilde{v}_3 - \frac{3}{Pr} \left( \tilde{v}_1 \cdot \nabla_0 \tilde{v}_2 + \tilde{v}_2 \cdot \nabla_0 \tilde{v}_1 \right) \right) = 0 \]

and

\[ (E - E_c)^\alpha \left( -\frac{\mu}{\mu} \nabla_0 P_1^* + \nabla_0^2 \tilde{v}_1^* \right) + \frac{1}{2} (E - E_c)^{2\alpha} \left( -\frac{\mu}{\mu} \nabla_0 P_2^* + \nabla_0^2 \tilde{v}_2^* - \frac{\nu}{\nu} \frac{2}{Pr} \tilde{v}_1^* \cdot \nabla_0 \tilde{v}_1^* \right) \]
\[ + \frac{1}{6} (E - E_c)^{3\alpha} \left( -\frac{\mu}{\mu} \nabla_0 P_3^* + \nabla_0^2 \tilde{v}_3^* - \frac{\nu}{\nu} \frac{3}{Pr} \left( \tilde{v}_1^* \cdot \nabla_0 \tilde{v}_2^* + \tilde{v}_2^* \cdot \nabla_0 \tilde{v}_1^* \right) \right) = 0 \]

The energy equations in each phase are

\[ \nabla_0^2 T_0 + (E - E_c)^\alpha \left( \nabla_0^2 T_1 - \nabla_1 \cdot \nabla_0 T_0 \right) + \frac{1}{2} (E - E_c)^{2\alpha} \left( \nabla_0^2 T_2 - \nabla_2 \cdot \nabla_0 T_0 - 2\nabla_1 \cdot \nabla_0 T_1 \right) \]
\[ + \frac{1}{6} (E - E_c)^{3\alpha} \left( \nabla_0^2 T_3 - \nabla_3 \cdot \nabla_0 T_0 - 3 \left( \nabla_2 \cdot \nabla_0 T_1 + \nabla_1 \cdot \nabla_0 T_2 \right) \right) = 0 \]

and

\[ \nabla_0^2 T_0^* + (E - E_c)^\alpha \left( \nabla_0^2 T_1^* - \frac{K}{K} \tilde{v}_1^* \cdot \nabla_0 T_0^* \right) \]
\[ + \frac{1}{2} (E - E_c)^{2\alpha} \left( \nabla_0^2 T_2^* - \frac{K}{K} \tilde{v}_2^* \cdot \nabla_0 T_0^* - 2 \frac{K}{K} \tilde{v}_1^* \cdot \nabla_0 T_1^* \right) \]
\[ + \frac{1}{6} (E - E_c)^{3\alpha} \left( \nabla_0^2 T_3^* - \frac{K}{K} \tilde{v}_3^* \cdot \nabla_0 T_0^* - 3 \frac{K}{K} \left( \tilde{v}_2^* \cdot \nabla_0 T_1^* + \tilde{v}_1^* \cdot \nabla_0 T_2^* \right) \right) = 0 \]
The continuity equations become

\begin{equation}
(E - E_c)^\alpha \nabla_0 \cdot \tilde{v}_1 + \frac{1}{2} (E - E_c)^{2\alpha} \nabla_0 \cdot \tilde{v}_2 + \frac{1}{6} (E - E_c)^{3\alpha} \nabla_0 \cdot \tilde{v}_3 = 0
\end{equation}

4.13

and

\begin{equation}
(E - E_c)^\alpha \nabla_0 \cdot \tilde{v}_1^* + \frac{1}{2} (E - E_c)^{2\alpha} \nabla_0 \cdot \tilde{v}_2^* + \frac{1}{6} (E - E_c)^{3\alpha} \nabla_0 \cdot \tilde{v}_3^* = 0
\end{equation}

4.14

The unit normal and the unit tangential vectors, upon perturbation up to the third order, become

\begin{equation}
\tilde{n} = \tilde{k} + (E - E_c)^\alpha \left( -\frac{\partial Z_1}{\partial x_0} \tilde{i} \right) + \frac{1}{2} (E - E_c)^{2\alpha} \left( -\frac{\partial Z_2}{\partial x_0} \tilde{i} - \left( \frac{\partial Z_1}{\partial x_0} \right)^2 \tilde{k} \right)

+ \frac{1}{6} (E - E_c)^{3\alpha} \left( \frac{3}{\partial x_0} \frac{\partial Z_1}{\partial x_0} \tilde{i} - 3 \frac{\partial Z_1}{\partial x_0} \frac{\partial Z_2}{\partial x_0} \tilde{k} \right)
\end{equation}

4.15

and

\begin{equation}
\tilde{t} = \tilde{i} + (E - E_c)^\alpha \frac{\partial Z_1}{\partial x_0} \tilde{k} + \frac{1}{2} (E - E_c)^{2\alpha} \left( -\left( \frac{\partial Z_1}{\partial x_0} \right)^2 \tilde{i} + \frac{\partial Z_2}{\partial x_0} \tilde{k} \right)

+ \frac{1}{6} (E - E_c)^{3\alpha} \left( -3 \frac{\partial Z_1}{\partial x_0} \frac{\partial Z_2}{\partial x_0} \tilde{i} + \frac{\partial Z_3}{\partial x_0} - 3 \left( \frac{\partial Z_1}{\partial x_0} \right)^3 \tilde{k} \right)
\end{equation}

4.16

At the lower plate, \( z = -1 \), the boundary conditions \( \tilde{v} \cdot \tilde{n} = 0 \), \( \tilde{v} \cdot \tilde{t} = 0 \), and \( T = 1 \) are expanded as

\begin{equation}
(E - E_c)^\alpha v_{z1} + \frac{1}{2} (E - E_c)^{2\alpha} v_{z2} + \frac{1}{6} (E - E_c)^{3\alpha} v_{z3} = 0
\end{equation}

4.17

\begin{equation}
(E - E_c)^\alpha v_{x1} + \frac{1}{2} (E - E_c)^{2\alpha} v_{x2} + \frac{1}{6} (E - E_c)^{3\alpha} v_{x3} = 0
\end{equation}

4.18

and
\[ T_0 + (E - E_c)^\alpha T_1 + \frac{1}{2} (E - E_c)^{2\alpha} T_2 + \frac{1}{6} (E - E_c)^{3\alpha} T_3 = 1 \]  \hspace{1cm} (4.19)\\

while the conditions at the top plate \( \vec{v}^* \cdot \vec{n} = 0 \), \( \vec{v}^* \cdot \vec{t} = 0 \), and \( T^* = 0 \) become

\[ (E - E_c)^\alpha v_{z1}^* + \frac{1}{2} (E - E_c)^{2\alpha} v_{z2}^* + \frac{1}{6} (E - E_c)^{3\alpha} v_{z3}^* = 0 \]  \hspace{1cm} (4.20)\\

\[ (E - E_c)^\alpha v_{x1}^* + \frac{1}{2} (E - E_c)^{2\alpha} v_{x2}^* + \frac{1}{6} (E - E_c)^{3\alpha} v_{x3}^* = 0 \]  \hspace{1cm} (4.21)

and

\[ T_0^* + (E - E_c)^\alpha T_1^* + \frac{1}{2} (E - E_c)^{2\alpha} T_2^* + \frac{1}{6} (E - E_c)^{3\alpha} T_3^* = 0 \]  \hspace{1cm} (4.22)

For reasons of brevity, we do not provide all of the expansions of all interfacial conditions save three that are crucial to the analysis of the problem. One of those three conditions is the interfacial energy balance and, after simplifications, it assumes the form
\[-E \left( \frac{\partial T_0}{\partial z_0} - \frac{k^*}{k} \frac{\partial T_0^*}{\partial z_0} \right) + (E - E_C)^2 \left( -v_{z1} + E \left( \frac{\partial T_1}{\partial z_0} - \frac{k^*}{k} \frac{\partial T_1^*}{\partial z_0} \right) \right) \]

\[+ \frac{1}{2} (E - E_C)^2 \left[ \begin{array}{c}
\left( v_{z1} + 2 \frac{\partial v_{z1}}{\partial z_0} Z_1 - 2v_{z1} \frac{\partial Z_1}{\partial x_0} \right)
\end{array} \right] \]

\[-E \left( \frac{\partial T_2}{\partial z_0} + 2 \frac{\partial^2 T_1}{\partial x_0^2} Z_1 - 2 \frac{\partial T_1}{\partial z_0} \frac{\partial Z_1}{\partial x_0} \right) \]

\[-\frac{k^*}{k} \left( \frac{\partial T_2^*}{\partial z_0} + 2 \frac{\partial^2 T_1^*}{\partial x_0^2} Z_1 - 2 \frac{\partial T_1^*}{\partial z_0} \frac{\partial Z_1}{\partial x_0} \right) \]

\[-2V_{PC} \left( \frac{\partial v_{z1}}{\partial z_0} - \frac{\mu^*}{\mu} \frac{\partial v_{z1}^*}{\partial z_0} \right) \]

\[+ 3K_{PC} v_{z1} \left( v_1^2 + v_{x1}^2 - v_{z1}^2 - v_{z1}^2 \right) \]

\[+ \frac{1}{6} (E - E_C)^3 \left( \begin{array}{c}
\frac{\partial T_3}{\partial z_0} + 3 \frac{\partial^2 T_1}{\partial x_0^2} Z_1 + 3 \frac{\partial^2 T_1}{\partial x_0^2} Z_2 + 3 \frac{\partial^2 T_1}{\partial x_0^2} Z_1
\end{array} \right) \]

\[-E \left( -3 \frac{\partial T_1}{\partial x_0} \frac{\partial Z_2}{\partial x_0} - 3 \frac{\partial T_2}{\partial x_0} \frac{\partial Z_1}{\partial x_0} - 6 \frac{\partial^2 T_1}{\partial x_0^2} Z_1 \frac{\partial Z_1}{\partial x_0} \right) \]

\[-\frac{k^*}{k} \left( \frac{\partial T_2^*}{\partial z_0} + 3 \frac{\partial^2 T_1^*}{\partial x_0^2} Z_1 + 3 \frac{\partial^2 T_1^*}{\partial x_0^2} Z_2 + 3 \frac{\partial^2 T_1^*}{\partial x_0^2} Z_1 \right) \]

\[-3 \frac{\partial T_1^*}{\partial x_0} \frac{\partial Z_2^*}{\partial x_0} - 3 \frac{\partial T_2^*}{\partial x_0} \frac{\partial Z_1^*}{\partial x_0} - 6 \frac{\partial^2 T_1^*}{\partial x_0^2} Z_1 \frac{\partial Z_1^*}{\partial x_0} \right) \]

\[= 0 \]

\[\left( \begin{array}{c}
6v_{z1} \left( \frac{\partial v_{z1}}{\partial z_0} - 2 \left( \frac{\partial v_{z1}}{\partial z_0} + \frac{\partial v_{z1}}{\partial z_0} \right) \frac{\partial Z_1}{\partial x_0} \right) + 2 \frac{\partial^2 v_{z1}}{\partial x_0^2} Z_1
\end{array} \right) \]

\[-V_{PC} \left( 6v_{z1} \left( \frac{\partial v_{z1}}{\partial z_0} - 2 \left( \frac{\partial v_{z1}}{\partial z_0} + \frac{\partial v_{z1}}{\partial z_0} \right) \frac{\partial Z_1}{\partial x_0} \right) + 2 \frac{\partial^2 v_{z1}}{\partial x_0^2} Z_1 \right) \]

\[-\frac{\mu^*}{\mu} \left( 6v_{z1} \left( \frac{\partial v_{z1}}{\partial z_0} - 2 \left( \frac{\partial v_{z1}}{\partial z_0} + \frac{\partial v_{z1}}{\partial z_0} \right) \frac{\partial Z_1}{\partial x_0} \right) - 6 \frac{\partial v_{z1}}{\partial z_0} \frac{\partial v_{z1}}{\partial z_0} \frac{\partial v_{z1}}{\partial z_0} \right) \]

\[-12v_{x1} \frac{\partial v_{z1}}{\partial z_0} + 6 \frac{\partial v_{z1}}{\partial z_0} \left( v_{z1}^2 + 2 \frac{\partial v_{z1}}{\partial z_0} Z_1 \right) \]

\[= 0 \]
Another one is the thermodynamic equilibrium relation between the vapor pressure and the vapor temperature;

\[
\Pi_{KE} \left( p^*_0 - p_{sat} \right) - \ln \left( \frac{T^*_0}{T_{sat}} \right) + \left( E - E_c \right)^\alpha \left( \Pi_{KE} p^*_1 - \frac{1}{T_{sat}} \left( T^*_1 + \frac{dT^*_0}{dz_0} Z_1 \right) \right) \\
+ \frac{1}{2} (E - E_c)^{2\alpha} \left( \Pi_{KE} \left( p^*_2 + 2 \frac{\partial p^*_1}{\partial z_0} Z_1 \right) - \frac{1}{T_{sat}} \left( T^*_2 + 2 \frac{\partial T^*_1}{\partial z_0} Z_1 + \frac{dT^*_0}{dz_0} Z_2 \right) + \left( \frac{1}{T_{sat}} \right)^2 \left( T^*_1 + \frac{dT^*_0}{dz_0} Z_1 \right) \right) \\
+ \frac{1}{6} (E - E_c)^{3\alpha} \left( \Pi_{KE} \left( p^*_3 + 3 \frac{\partial p^*_2}{\partial z_0} Z_1 + 3 \frac{\partial p^*_1}{\partial z_0} Z_2 + 3 \frac{\partial^2 p^*_1}{\partial z_0^2} Z_1^2 \right) - \frac{1}{T_{sat}} \left( T^*_3 + 3 \frac{\partial T^*_2}{\partial z_0} Z_1 + 3 \frac{\partial T^*_1}{\partial z_0} Z_2 + 3 \frac{\partial^2 T^*_1}{\partial z_0^2} Z_1^2 + \frac{dT^*_0}{dz_0} Z_3 \right) \\
+ 3 \left( \frac{1}{T_{sat}} \right)^2 \left( T^*_2 + 2 \frac{\partial T^*_1}{\partial z_0} Z_1 + \frac{dT^*_0}{dz_0} Z_2 \right) \left( T^*_1 + \frac{dT^*_0}{dz_0} Z_1 \right) \right) = 0
\]

We also give the third order normal stress balance equation because we will use it to find the values of \( \alpha \) and \( A \). The third order normal stress balance is given by
\[
Ca\left( P_0 - P^*_0 \right) + (E - E_C)^a \left( Ca \left( P_1 - P^*_1 - 2 \frac{\partial v_{z1}}{\partial z_0} + 2 \frac{\mu^*}{\mu} \frac{\partial v_{z1}}{\partial x_0} \right) + \frac{\partial^2 Z_1}{\partial x_0^2} \right) \\
+ \frac{1}{2} (E - E_C)^{2a} \left( Ca \left( P_2 + 2 \frac{\partial P_1}{\partial z_0} Z_1 - P^*_2 - 2 \frac{\partial P_1}{\partial z_0} Z_1 \right) \\
- 2 \left( \frac{\partial v_{z2}}{\partial z_0} + 2 \frac{\partial^2 v_{z1}}{\partial z_0} Z_1 \right) + \frac{2 \mu^*}{\mu} \left( \frac{\partial v_{z2}}{\partial z_0} + 2 \frac{\partial^2 v_{z1}}{\partial z_0^2} Z_1 \right) \right) \\
+ \frac{1}{6} (E - E_C)^{3a} \left( Ca \left( P_3 + 3 \frac{\partial P_2}{\partial z_0} Z_1 + 3 \frac{\partial P_1}{\partial z_0} Z_1 + 3 \frac{\partial^2 P_1}{\partial z_0^2} Z_1^2 \right) \\
- P^*_3 - 3 \frac{\partial P^*_2}{\partial z_0} Z_1 - 3 \frac{\partial P^*_1}{\partial z_0} Z_2 - 3 \frac{\partial^2 P^*_1}{\partial z_0^2} Z_1^2 \right) \right) \\
+ 2 \frac{Ca}{Pr} v_{z1} \left( v_{z1} - v_{z1}^* \right) + \frac{\partial^2 Z_1}{\partial x_0^2} \\
+ \frac{1}{6} (E - E_C)^{3a} \left( Ca \left( \frac{\partial v_{z3}}{\partial z_0} + 3 \frac{\partial^2 v_{z2}}{\partial z_0^2} Z_1 + 3 \frac{\partial^2 v_{z1}}{\partial z_0^2} Z_2 + 3 \frac{\partial^3 v_{z1}}{\partial z_0^3} Z_1^2 + 12 \frac{\partial v_{z1}}{\partial z_0} \left( \frac{\partial Z_1}{\partial x_0} \right)^2 \right) \right) \\
+ \frac{3}{Pr} \left( v_{z2} + 2 \frac{\partial v_{z1}}{\partial z_0} Z_1 - 2 v_{x1} \frac{\partial Z_1}{\partial x_0} \right) \left( v_{z1} - v_{z1}^* \right) \\
+ \frac{3}{Pr} \left( v_{z2} + 2 \frac{\partial v_{z1}}{\partial z_0} Z_1 - 2 v_{x1} \frac{\partial Z_1}{\partial x_0} - v_{z2} + 2 \frac{\partial v_{z2}}{\partial z_0} Z_1 + 2 v_{x1} \frac{\partial Z_1}{\partial x_0} \right) v_{z1} \\
+ \frac{\partial^2 Z_3}{\partial x_0^2} - 9 \frac{\partial^2 Z_1}{\partial x_0^2} \left( \frac{\partial Z_1}{\partial x_0} \right)^2 \right) = 0
\]

Remember that in all of the expansion formulas, E will be replaced by \( E_{C^+} \). The first order problem is an eigenvalue problem and it was solved in an earlier chapter. The solution to the first order problem is used at every subsequent order along with the solvability condition. We learn from the first order equations that the eigenvector \( u_1 \) in terms of \( Z_1 \) is
\[ u_1 = A \hat{u}_1(Z) \cos \omega x_0. \]

Recall that \( \omega \) is an input and it is determined by the width of the compartment in the lateral direction. We cannot determine the value of \( \alpha \) either from the first or the second order equations. As noted in the earlier section, the solvability condition is automatically satisfied at the second order. The solution to the second order, \( u_2 \), will go as

\[ u_2 = B u_{2a}(Z) \cos \omega x_0 + A^2 u_{2b}(Z) + A^2 u_{2c}(Z) \cos 2\omega x_0. \]

It is at the third order that we learn that \( \alpha \) is equal to \( \frac{1}{2} \). At this order, there exists a value for \( A \) other than zero. \( A \) appears as \( A^2 \), and the value of \( A^2 \) is obtained using the solvability condition on the third order normal stress balance. The third order solution, \( u_3 \), depends upon the form of \( u_1, u_2 \) as well as \( Z_3 \), i.e.,

\[ u_3 = C u_{3a}(Z) \cos \omega x_0 + A^3 u_{3b}(Z) \cos \omega x_0 \]
\[ + A^3 u_{3c}(Z) \cos 3\omega x_0 + A u_{3d}(Z) \cos \omega x_0 + B A u_{3e}(Z) \cos 2\omega x_0 + 1 \]

**Results of Calculations**

The Chebyshev spectral tau method is used to solve the eigenvalue problem that determines the critical temperature difference and the value of the unknown multiplicative constant \( A \). Once we know what \( A \) is, we can calculate the actual change in the heat transfer rate at the bottom plate when we advance \( \Delta T \) a little beyond the critical point. Our system is finite in the lateral extent with periodic conditions. The heat transfer up to the second order is calculated as
\[
\int_0^\lambda \tilde{q} \cdot \tilde{n} = -k \int_0^\lambda \left( \frac{dT_0}{dz_0} + \varepsilon \frac{d\hat{T}_1}{dz_0} \cos \omega x_0 + \frac{1}{2} \varepsilon \left( \frac{d\hat{T}_2}{dz_0} + \frac{d\hat{T}_1}{dz_0} \cos 2\omega x_0 \right) \right) dx_0
\]

Clearly, \( \cos \omega x_0 \) and \( \cos 2\omega x_0 \), when integrated over the width, give zero. We are more interested in the percentage of change of the heat transfer, so we have plotted \( \frac{(\tilde{q} - \tilde{q}_0) \cdot \tilde{n}}{\tilde{q}_0 \cdot \tilde{n}_0} \) against \( \frac{E}{E_C} \) in Figure 4-1 for a system where the liquid and vapor depths are 4mm and 0.4mm, and the wavenumber is \( \omega = 0.3 \). In nonlinear calculations the wavenumber of the disturbance is fixed. Accordingly, one can find the width of the box using the wavelength of the disturbance. The wavenumber, \( \omega = 0.3 \), corresponds to a cell width of about 84mm.

We learn that once the interface is unstable, more heat is transferred through the bottom plate to the system. Thus, if we wanted to know when the system becomes unstable, we only have to measure the amount of heat that is needed to keep the bottom plate at a constant temperature. The heat necessary changes with the onset of instability. For a typical problem we can observe a 2\% increase for a supercritical change of about 10\%.
Figure 4-1. Heat transfer across the bottom plate.

The nonlinear analysis results do more than giving the heat transfer enhancement. They can also be of use in helping us to better understand the physics of evaporative instability. To see how, let us first observe from the linear calculations that the deeper vapor layers make the interface more stable. Then apply the nonlinear calculations to two cases with the same base state interface pressure and temperature where in one of the systems $\delta = 0.1$ and in the other $\delta = 0.3$, and compare the magnitudes of the pressure perturbations in the vapor at the interface. Because $P^* = P_0^* + \varepsilon \sqrt{E} P_1^*$ at the interface and because we wish to compare the problems with each other for the same percentage of departure from the critical state, we will express $P^*$ as

$$P^* = P_0^* + \left( \frac{E - E_c}{E_c} \right)^{1/2} E_c^{1/2} \varepsilon \sqrt{E} P_1^*. $$
In Figures 4-2 and 4-3, we have plotted the magnitudes of pressure perturbations in the vapor, $E_c^{1/2} P_1^*$, for the cases where $\delta = 0.1$ and $\delta = 0.3$. We learn from the calculations that even though the nature of the profiles of the vapor pressure perturbations are the same for both cases, the magnitudes are not. The vapor pressure perturbation at the interface for the case with the shallow vapor layer ($\delta = 0.1$) is about three times more than the pressure perturbation at the interface for the case with $\delta = 0.3$, resulting in a more unstable configuration. In Chapter 3, during our discussion of the physics of the pure phase-change problem we had deliberately changed the value of the vapor viscosity to see its effect on the instability. The system, where the vapor viscosity was increased, proved to be more unstable.

Figure 4-2. The scaled pressure perturbations in the vapor domain for the case where $d = 1\text{mm}$, $\delta = 0.1$ and $\omega = 1$. $z = 0$ represents the interface position in the base state.
Figure 4-3. The scaled pressure perturbations in the vapor domain for the case where \( d = 1mm \), \( \delta = 0.3 \) and \( \omega = 1 \). \( z_0 = 0 \) represents the interface position in the base state.

As shown in Figure 4-4, our nonlinear calculations return a larger pressure perturbation value at the interface for the system with the more viscous vapor. Thus, the nonlinear calculations support the argument that the vapor flow dynamics affect the stability of the phase-change problem.
Endnote: The Determination of $\alpha$

There is a lot of analytical work that goes into determining the value of $\alpha$. For that reason, we will demonstrate this cumbersome procedure on a simpler problem.

Assume the following domain equation

$$\frac{d^3u}{dx^3} + \lambda f(u) = 0 \tag{4.27}$$

where $f(0) = 0$, $f'(0) = \frac{df}{dx}(0) > 0$ and $f''(0) = \frac{d^2f}{dx^2}(0) > 0$ are given, and the domain equation is subjected to Dirichlet condition at $x = 0$ and $x = 1$. To determine the behavior of the system around the critical point, in this case $\lambda_c$, we will go a little beyond the critical point and replace $\lambda$ by $\lambda_c + \epsilon$. When we expand the domain variable $u$ around a base state $u_0$ as a power series of $\epsilon^\alpha$, the domain equation can be written as
\[ \frac{d^2}{dx^2} \left( \varepsilon^\alpha u_1 + \frac{1}{2} \varepsilon^{2\alpha} u_2 + \frac{1}{6} \varepsilon^{3\alpha} u_3 \right) + \lambda_c \left( f'(0) \left( \varepsilon^\alpha u_1 + \frac{1}{2} \varepsilon^{2\alpha} u_2 + \frac{1}{6} \varepsilon^{3\alpha} u_3 \right) + \frac{1}{2} f''(0) \left( \varepsilon^\alpha u_1 + \frac{1}{2} \varepsilon^{2\alpha} u_2 + \frac{1}{6} \varepsilon^{3\alpha} u_3 \right)^2 + \ldots \right) + \varepsilon \left( f'(0) \left( \varepsilon^\alpha u_1 + \frac{1}{2} \varepsilon^{2\alpha} u_2 + \frac{1}{6} \varepsilon^{3\alpha} u_3 \right) + \frac{1}{2} f''(0) \left( \varepsilon^\alpha u_1 + \frac{1}{2} \varepsilon^{2\alpha} u_2 + \frac{1}{6} \varepsilon^{3\alpha} u_3 \right)^2 + \ldots \right) = 0 \]

The \( \varepsilon^\alpha \) order problem is then

\[ \frac{d^2 u_1}{dx^2} + \lambda_c f'(0) u_1 = 0 \quad 4.28 \]

subject to \( u_1(0) = 0 \) and \( u_1(0) = 0 \). When we solve for \( u_1 \), we find that \( u_1 = A \sin(\pi x) \) and \( \lambda_c = \frac{\pi^2}{f'(0)} \). Now we will solve for \( u_2 \) and \( u_3 \) and so on. To do that we will move to the next order of equations, which is unknown since \( \alpha \) is yet unknown and needs to be determined. There exist 3 possibilities:

1. \( 1 + \alpha < 2\alpha \) is one of the possibilities, and this implies that \( f'(0) u_1 = 0 \).

Now, this can only be true if only if \( u_1 = 0 \), so obviously this is not a plausible choice. Another possibility is that \( 1 + \alpha = 2\alpha \). When we set \( \alpha = 1 \), we get the following equation as the next order equation:

\[ \frac{1}{2} \frac{d^2 u_2}{dx^2} + \frac{1}{2} \lambda_c f'(0) u_2 + \frac{1}{2} \lambda_c f''(0) u_2^2 + f'(0) u_1 = 0 \quad 4.29 \]

If we multiply equation 4.28 by \( u_2 \) and equation 4.29 by \( u_1 \), then integrate both equations over the domain, and subtract them from each other we get
Using the relation above, we can determine a value for $A$ which is non-zero, and it is

$$A = -\frac{3}{4\pi} \left( \frac{f'(0)}{f''(0)} \right)^2$$

So far, one of the three possibilities gives us a non-zero value for $A$. The next possibility is that $1 + \alpha > 2\alpha$. Now this gives us

$$\frac{1}{2} \lambda \int_0^1 u_i^3 \, dx = 0$$

This is not possible since $\int_0^1 u_i^3 \, dx = \frac{4}{3\pi}$. Thus, we conclude that $\alpha = 1$. Note that the value of $A$ is determined at the same order where we determined the value of $\alpha$.

**A different approach to the weak nonlinear analysis**

There is more information one can actually get from the weak nonlinear analysis. To do that, we will have to modify our physical model. This time we will again set the evaporation rate in the base state equal to zero by adjusting the vapor pressure, however, now we will leave the feed and removal valves open, and as we increase the temperature difference applied across the bilayer system we will adjust the vapor pressure accordingly such that the evaporation rate is equal to zero in the base state even if we cross the critical point. Intuitively, one expects the net evaporation rate to change once we cross the critical point. Beyond the critical point the net evaporation rate, $U$, can be expanded as

$$U = U_{\text{base}} + U_{\text{extra}}$$
where $U_{\text{extra}}$ is

$$U_{\text{extra}} = U_{\text{flat}} + \delta$$  \hspace{1cm} 4.31

In equation 4.31, $U_{\text{flat}}$ is the change in the net evaporation rate if the instability had not occurred, and $\delta$ is the correction to it because of the onset of instability. In this analysis, we will adjust the vapor pressure as we increase the temperature difference applied accordingly, $U_{\text{flat}}$ is always zero even after we cross the critical point. Thus, the only new unknown in the problem is $\delta$. The sign of $\delta$ tells us what will happen to the net evaporation rate beyond the critical point. If it turns out to be positive, then with the onset of instability the system will suck in liquid through the bottom valve and remove the excess vapor through the top valve. One does need to solve the complete weak nonlinear problem to get this information. Moreover, one can even determine the sign $\delta$ by analytical calculations. From our earlier analysis we know that the solvability condition was automatically satisfied at the second order. That means that we can go ahead and solve the second order problem. By doing that we will find $\delta$ in terms of the amplitude of the deflection. The momentum equations in each phase at the second order are

$$-\nabla_0^2 P_2 + \nabla_0^2 \tilde{v}_2 = \frac{2}{\Pr} \tilde{v}_1 \cdot \nabla_0 \tilde{v}_1$$ \hspace{1cm} 4.32

and

$$-\frac{\mu}{\mu^*} \nabla_0^2 P_2^* + \nabla_0^2 \tilde{v}_2^* = \frac{\nu}{\Pr} \tilde{v}_1^* \cdot \nabla_0 \tilde{v}_1^*$$ \hspace{1cm} 4.33

The energy equations in each phase are
\[ \nabla^2 T_2 - \tilde{v}_2 \cdot \nabla T_0 = 2 \tilde{v}_1 \cdot \nabla T_1 \] \hspace{1cm} 4.34

and

\[ \nabla^2 T^*_2 - \frac{\kappa}{\kappa^*} \tilde{v}^*_2 \cdot \nabla T^*_0 = 2 \frac{\kappa}{\kappa^*} \tilde{v}^*_1 \cdot \nabla T^*_1 \] \hspace{1cm} 4.35

The continuity equations become

\[ \nabla \cdot \tilde{v}_2 = 0 \] \hspace{1cm} 4.36

and

\[ \nabla \cdot \tilde{v}^*_2 = 0 \] \hspace{1cm} 4.37

At the lower plate, \( z = -1 \), the boundary conditions \( \tilde{v} \cdot \tilde{n} = U \), \( \tilde{v} \cdot \tilde{t} = 0 \), and \( T_1 = 0 \) are expanded as

\[ v_{z2} = U_2 \]
\[ v_{x2} = 0 \]

and

\[ T_2 = 0 \]

while the conditions at the top plate \( \tilde{v}^* \cdot \tilde{n} = U^*_2 \), \( \tilde{v}^* \cdot \tilde{t} = 0 \), and \( T^* = 0 \) become

\[ v^*_{z2} = U^*_2 \]
\[ v^*_{x2} = 0 \]

and

\[ T^*_2 = 0 \]
For reasons of brevity, we do not provide all of the expansions of all interfacial conditions save three that are crucial to the analysis of the problem. One of those three conditions is the interfacial energy balance and, after simplifications, it assumes the form

\[-v_{z1} - E \left( \frac{\partial T_z}{\partial z_0} - \frac{k^*}{k} \frac{\partial T^*_i}{\partial z_0} \right) = -2 \frac{\partial v_{z1}}{\partial z_0} Z_1 + 2 v_{z1} \frac{\partial Z^*_i}{\partial x_0} \]

\[-E \left( 2 \frac{\partial^2 T_i}{\partial z_0^2} Z_1 - 2 \frac{\partial T_i^*}{\partial x_0} \frac{\partial Z^*_i}{\partial x_0} \right) - \frac{k^*}{k} \left( 2 \frac{\partial^2 T_i^*}{\partial z_0^2} Z_1 - 2 \frac{\partial T_i^*}{\partial x_0} \frac{\partial Z^*_i}{\partial x_0} \right) + 2 V_{pc} \left( \frac{\partial v_{z1}}{\partial z_0} - \frac{\mu^*}{\mu} \frac{\partial v_{z1}^*}{\partial z_0} \right) \]

Another one is the thermodynamic equilibrium relation between the vapor pressure and the vapor temperature;

\[\Pi_{KE} \left( P^*_2 + 2 \frac{\partial P_i^*}{\partial z_0} Z_1 \right) - \frac{1}{T^*_{base}} \left( T^*_2 + \frac{d T^*_2}{dz_0} Z_2 \right) = \frac{1}{T^*_{base}} \left( 2 \frac{\partial T_i^*}{\partial z_0} Z_1 \right) - \left( \frac{1}{T^*_{base}} \right)^2 \left( T^*_1 + \frac{d T^*_0}{dz_0} Z_1 \right) \]

The normal stress balance assumes the following form

\[Ca \left( P_2 - P^*_2 - 2 \frac{\partial v_{z1}}{\partial z_0} + 2 \frac{\mu^*}{\mu} \frac{\partial v_{z1}^*}{\partial z_0} \right) + \frac{\partial^2 Z_2}{\partial x_0^2} \]

\[= Ca \left( 2 \frac{\partial P_i^*}{\partial z_0} Z_1 - 2 \frac{\partial P_i}{\partial z_0} Z_1 + 4 \frac{\partial^2 v_{z1}}{\partial z_0^2} Z_1 - 4 \frac{\mu^*}{\mu} \frac{\partial^2 v_{z1}^*}{\partial z_0^2} Z_1 \right) - \frac{2Ca}{Pr} v_{z1} \left( v_{z1} - v_{z1}^* \right) \]

When one solves the first order problem, one determines that the domain variables are actually trigonometric functions in the \(x_0\)-direction. They are, in the liquid,

\[v_{z1} = \hat{v}_{z1} (z_0) \sin \omega x_0 \]

\[v_{z1} = \hat{v}_{z1} (z_0) \cos \omega x_0 \]

\[T_i = \hat{T}_i (z_0) \cos \omega x_0 \]

and
\[ P_1 = \hat{P}_1(\omega x_0) \cos \omega x_0 \]

The vapor domain variables assume similar forms. We plug them into the second order domain equation and find the form of the second order domain variables. What follows is the demonstration of how we find the forms of the second order domain variables. The second order energy equation is

\[ \nabla^2 T_2 - \hat{v}_2 \cdot \nabla T_0 = 2\hat{v}_1 \cdot \nabla T_1 \]

After substituting the first order domain variables we get

\[ \nabla^2 T_2 - v_{z2} \frac{dT_0}{dz_0} = 2 \left( v_{x1} \frac{\partial T_1}{\partial x_0} + v_{z1} \frac{\partial T_1}{\partial z_0} \right) \]

which translates to

\[ \nabla^2 T_2 - v_{z2} \frac{dT_0}{dz_0} = 2 \left( -\omega \hat{v}_{z1}(z_0) \hat{T}_1(z_0) \sin^2 \omega x_0 + \hat{v}_{z1}(z_0) \frac{d\hat{T}_1}{dz_0}(z_0) \cos^2 \omega x_0 \right) \]

Using trigonometric manipulations we get

\[ \nabla^2 T_2 - v_{z2} \frac{dT_0}{dz_0} = \left( \hat{v}_{z1}(z_0) \frac{d\hat{T}_1}{dz_0}(z_0) + \hat{T}_1(z_0) \frac{d\hat{v}_{z1}}{dz_0}(z_0) \right) \]

\[ + \left( \hat{v}_{z1}(z_0) \frac{d\hat{T}_1}{dz_0}(z_0) - \hat{T}_1(z_0) \frac{d\hat{v}_{z1}}{dz_0}(z_0) \right) \cos 2\omega x_0 \]

From the manipulations above, one learns that

\[ T_2 = \hat{T}_2(z_0) + \hat{T}_2(z_0) \cos 2\omega x_0 \]

Following the same method, one obtains

\[ v_{x2} = \hat{v}_{x2}(z_0) + \hat{v}_{x2}(z_0) \sin 2\omega x_0 \]

\[ v_{z2} = \hat{v}_{z2}(z_0) + \hat{v}_{z2}(z_0) \cos 2\omega x_0 \]

and
\[ P_2 = \hat{P}_2(z_0) + \hat{\tilde{P}}_2(z_0) \cos 2\omega x_0 \]

We are trying to find the sign of \( \delta \). At the second order, \( \delta \) is actually equal to the \( x \) independent part of the correction to the net evaporation \( U \), which is \( \hat{U}_2 \). Thus, we only need to solve for the \( x \)-independent part of the particular solution for \( U_2 \). The domain equations in the liquid for \( x \)-independent part of the second order problem are

\[
\frac{d^2 \hat{\nu}_{z2}}{dz_0^2} = 0
\]

\[-\frac{d\hat{P}_2}{dz_0} + \frac{d^2 \hat{\nu}_{z2}}{dz_0^2} = \frac{2}{\text{Pr}} \hat{\nu}_{z1} \frac{d\hat{\nu}_{z1}}{dz_0} \]

\[
\frac{d\hat{T}_2}{dz_0} - \hat{\nu}_{z2} \frac{dT_0}{dz_0} = \left( \hat{\nu}_{z1} \frac{dT_1}{dz_0} + \hat{T}_1 \frac{d\hat{\nu}_{z1}}{dz_0} \right)
\]

and

\[
\frac{d\hat{\nu}_{z2}}{dz_0} = 0
\]

In the vapor domain, they are

\[
\frac{d^2 \hat{\nu}_{z2}^*}{dz_0^2} = 0
\]

\[-\frac{\mu}{\mu} \frac{d\hat{P}_2^*}{dz_0} + \frac{d^2 \hat{\nu}_{z2}^*}{dz_0^2} = \frac{2}{\text{Pr}^*} \frac{\nu^*}{\nu} \frac{d\hat{\nu}_{z1}^*}{dz_0} \]

\[
\frac{d\hat{T}_2^*}{dz_0} - \frac{\kappa}{\kappa^*} \frac{d\hat{T}_0^*}{dz_0} = \frac{\kappa}{\kappa^*} \left( \hat{\nu}_{z1}^* \frac{dT_1^*}{dz_0} + \hat{T}_1^* \frac{d\hat{\nu}_{z1}^*}{dz_0} \right)
\]

and

\[
\frac{d\hat{\nu}_{z2}^*}{dz_0} = 0
\]
At the bottom plate, the boundary conditions are

\[ \hat{v}_{x2} = 0 \]
\[ \hat{v}_{z2} = \hat{U}_2 \]
\[ \hat{T}_2 = 0 \]

At the top plate,

\[ \hat{v}_{x2} = 0 \]
\[ \hat{v}_{z2} = \hat{U}_2^* \]
\[ \hat{T}_2^* = 0 \]
\[ \hat{P}_2^* = 0 \]

At the interface, we have

\[ \hat{v}_{x2} = \frac{\rho^*}{\rho} \hat{v}_{x2} \]
\[ \hat{v}_{z2} = \hat{v}_{z2}^* \]
\[ \left( \hat{T}_2 + \frac{dT_0}{dz_0} \hat{Z}_2 \right) - \left( \hat{T}_2^* + \frac{dT_0^*}{dz_0} \hat{Z}_2 \right) = \left( \frac{dT_1^*}{dz_0} - \frac{dT_1}{dz_0} \right) \hat{Z}_1 \]

\[ \frac{d\hat{v}_{x2}}{dz_0} = \mu^* \frac{d\hat{v}_{x2}^*}{dz_0} \]

\[ \frac{1}{T_{\text{base}}^*} \left( \hat{T}_2^* + \frac{dT_0^*}{dz_0} \hat{Z}_2 \right) - \Pi_{\text{KE}} \hat{P}_2^* \]

\[ = - \frac{1}{T_{\text{base}}^*} \frac{d\hat{T}_1^*}{dz_0} \hat{Z}_1 + \left( \frac{1}{T_{\text{base}}^*} \right)^2 \left( \hat{T}_1^* + \frac{dT_0^*}{dz_0} \hat{Z}_1 \right)^2 + \Pi_{\text{KE}} \frac{d\hat{P}_1^*}{dz_0} \hat{Z}_1 \]
\[-\dot{v}_{x2} = E \left( \frac{d\hat{T}_2}{dz_0} - \frac{k^*}{k} \frac{dT}{dz_0} \right) + \phi \frac{d\hat{T}_0}{dz_0} \left( \frac{d\hat{v}_{x2}}{dz_0} - \frac{\kappa^*}{\kappa} \frac{\hat{v}_{z1}}{\hat{v}_{z1}} \right) \hat{Z}_1 + 2V_{pc} \left( 1 - \frac{\nu^*}{\nu} \right) \dot{\hat{v}}_{z1} \frac{d\hat{v}_{z1}}{dz_0} \]

\[4.46\]

\[\text{Ca} \left( \hat{P}_2 - \hat{P}_2^* - 2 \frac{d\hat{v}_{x2}}{dz_0} + 2 \hat{\mu} \frac{d\hat{v}_{x2}}{dz_0} \right) + \hat{\nu} \frac{d\hat{v}_{x2}}{dz_0} = \text{Ca} \frac{\rho^*}{\text{Pr}} \left( 1 - \frac{\rho^*}{\rho} \right) \hat{v}_{z1} \]

\[4.47\]

These equations allow an analytical solution for the problem. First, we learn that everywhere in each domain

\[\ddot{v}_{x2}(z_0) = 0\]

\[\ddot{v}_{x2}(z_0) = 0\]

\[\ddot{v}_{z2}(z_0) = \hat{U}_2\]

\[\ddot{v}_{z2}(z_0) = \frac{\rho}{\rho} \hat{U}_2\]

We also learn that

\[\hat{P}_2(z_0) = -\frac{1}{\text{Pr}} \ddot{v}_{z1}\]

\[\hat{P}_2^*(z_0) = -\frac{\rho^*}{\rho} \frac{1}{\text{Pr}} \ddot{v}_{z1}\]

For reasons of brevity, I will omit giving the analytical solution for \(\hat{T}_2(z_0)\) and \(\hat{T}_2^*(z_0)\).

In conclusion, we find the correction to the net evaporation, \(\hat{U}_2\), and it turns out to be positive at each wavenumber. This information is useful if we were to do experiments since now we have an additional indicator, other than measuring the heat transfer at the bottom wall, to determine whether the instability has set in or not.

Opening the valves and allowing the bilayer system such in liquid through the bottom
wall with the onset of instability is a preferable method since it is a visual method, unlike measuring the change in the heat transfer.
CHAPTER 5
THE RAYLEIGH-TAYLOR PROBLEM WITH PHASE-CHANGE

Introduction

As we discussed earlier in Chapter 1, a vapor underlying a liquid results in a configuration that is potentially unstable. This problem, known as the Rayleigh-Taylor problem, is well known and an account of its underlying physics is given by Chandrasekhar (1961). It may be recalled from Chandrasekhar that the interface is unstable for wavenumbers smaller than a dimensionless critical wavenumber given by

\[ w_{RT} = \left( \frac{(\rho^l - \rho^g)gd^2}{\gamma} \right)^{\frac{1}{2}} \]

Here, \( \rho^l \) and \( \rho^g \) are the liquid and vapor densities, respectively, \( \gamma \) is the interfacial tension, \( d \) is the horizontal length scale and \( g \) is the coefficient of gravitational acceleration. The derivation of this result is also given in Appendix D. The classical Rayleigh-Taylor instability problem is ordinarily studied under the assumption that there is no mass transfer across the interface. The problem was reformulated by Hsieh (1972) who considered the Rayleigh-Taylor problem with mass and heat transfer in inviscid fluids of finite vertical depths. He solved both the velocity and the temperature fields in both fluids and derived an analytical dispersion relation i.e., a relation between \( \sigma \) and \( \omega \). Later, Hsieh (1978) studied the same problem again for inviscid fluids but introduced a phenomenological coefficient, \( \alpha \), which allowed him to decouple the velocity and temperature fields for both infinitely deep fluids as well as for fluid layers with finite
depths. Using the same idea of a phenomenological coefficient, Ho (1980) analytically obtained a result for viscous fluids with finite depths. In the same paper, Ho also derived the critical dispersion relation for viscous fluids with equal kinematic viscosities for infinitely deep fluid layers. As an application to pool boiling, Badratinova et al. (1996) have found an analytical expression for the critical wavenumber for a system where a vapor with finite depth underlies its own infinitely deep liquid, solving for both the velocity and the temperature fields. However, we have seen in Chapter 3 that the assumption of large depth fluids may lead to conclusions that do not quite represent the physics. We might also like to know what the phenomenological coefficient used by Ho (1980) and other workers such as Adham-Khodaparast et al. (1995) really means. These workers found a critical dispersion relation for infinitely deep fluid layers with equal kinematic viscosities to be given by \( w^2 + 4 \frac{\alpha \nu^l}{\gamma} w - w_{RT}^2 = 0 \). Here, \( \nu^l \) is the kinematic viscosity of both the liquid and the gas phases. Note that upon setting the phenomenological coefficient equal to zero in the above relation, one recovers the classical Rayleigh-Taylor instability result.

In this chapter we therefore investigate the Rayleigh-Taylor problem with interfacial mass and heat transfer and present the results of our calculations for viscous fluids where both fluid layers have finite depths. In addition, we derive an analytical expression for the phenomenological coefficient, \( \alpha \), albeit for infinitely deep viscous fluid layers and show that this coefficient is actually a function of the wavenumber.

**Physical and Mathematical Model**

Unlike the earlier chapters this time the physical model is a system where a vapor of depth, \( d^g \), underlies its own liquid of depth, \( d^l \), and is heated from below (Figure 5-1).
As before the base state is one where there is a flat interface between the liquid and its underlying vapor, and the stability of this base state is in question. Also as before the liquid and vapor depths are assumed constant by suitably adjusting the vapor removal and liquid feed rates. The phase-change rate at the interface can be controlled by adjusting the vapor pressure at the vapor-side wall. The phase-change rate is not constant as it can change upon perturbation. Consequently, in this problem as in the pure evaporation problem the input variables are the liquid and vapor depths, the temperature difference between the bottom and top plates, and the pressure at the lower plate. Furthermore, the pressure at the lower plate can be adjusted such that the phase-change rate at the interface is equal to zero. In the analysis of the problem, we assume as usual that the liquid and its vapor are in thermodynamic equilibrium because we will only consider either zero or very small phase-change rates at the interface in the base state.

The equations that model the physics are given by the Navier-Stokes, energy and continuity equations in each phase assuming that the fluids are incompressible. It is assumed that the bottom and top plate temperatures are kept constant. When we assume that the liquid and its vapor are in equilibrium and that no phase change is taking place across the interface, we need not feed and remove fluid through the bottom and top plates. The no-slip condition applies along the top and bottom plates. Hereon, $v_x$, $v_z$, 

Figure 5-1. The physical model
and \( T \) are taken to be the x and z-components of velocity, and temperature fields, respectively, while the asterisk denotes the vapor phase.

The equations are brought into dimensionless form using the following scales:

- length, \( d = \frac{\mu^l \kappa^l}{\gamma} \);
- velocity, \( \bar{v} = \frac{\kappa^l}{d} \);
- time, \( \frac{d}{\bar{v}} \) where \( \mu^l \) and \( \kappa^l \) are the liquid viscosity and the liquid thermal diffusivity, respectively. The temperature is scaled as \( \bar{T} = \frac{T}{dT_0^g dz_0} \).

In the temperature scale, \( \frac{dT_0^g}{dz_0} \) is base state temperature gradient in the vapor at the interface. Linearized stability is performed in the standard way by expanding the state variables in the form of \( u = \hat{u}(z_0)e^{\sigma z_0} e^{iwz} \) leading to the following dimensionless equations of motion

\[
\left( \frac{d^2}{dz^2} - w^2 \right) \left( \frac{d^2}{dz^2} - w^2 - \frac{\sigma}{\Pr} \right) v_{z1}^l = 0 \quad 5.1
\]

and

\[
\left( \frac{d^2}{dz^2} - w^2 \right) \left( \frac{d^2}{dz^2} - w^2 - \frac{\sigma}{\Pr} \frac{1}{\nu} \right) v_{z1}^v = 0 \quad 5.2
\]

where \( \sigma \) and \( \Pr \) are the inverse time constant and liquid Prandtl number, respectively, \( \nu \) is the ratio of the vapor kinematic viscosity to the liquid kinematic viscosity and \( w \) is the dimensionless wavenumber of the perturbation.

The perturbed energy equations in dimensionless form become

\[
\left( \frac{d^2}{dz^2} - w^2 - \sigma \right) T_{z1}^l = k v_{z1}^l \quad 5.3
\]
In the equations above, $k$ is the ratio of the vapor thermal conductivity to the liquid thermal conductivity, and $\kappa$ here is the ratio of the vapor thermal diffusivity to the liquid thermal diffusivity. At the lower plate, the no-flow, no-slip and constant temperature imply that the vapor is subject to $v_{z_1}^g = 0$, $\frac{dv_{z_1}^g}{dz_0} = 0$ and $T_{z_1}^g = 0$ while similar conditions at the top plate become $v_{z_1}^l = 0$, $\frac{dv_{z_1}^l}{dz_0} = 0$ and $T_{z_1}^l = 0$.

At the interface the mass balance is given by

$$v_{z_1}^l - \rho v_{z_1}^g = \sigma \left(1 - \rho\right) Z_1$$

where $Z_1$ is the perturbed surface deflection, and $\rho$ is the ratio of vapor density to liquid density.

The continuity of temperature and the no-slip conditions at the interface become

$$T_{z_1}^l + k Z_1 = T_{z_1}^g + Z_1 \text{ and } \frac{dv_{z_1}^g}{dz_0} = \frac{dv_{z_1}^l}{dz_0}.$$
\[ \mu \left( \frac{d^2 v^g_{z1}}{dz_0^2} + w^2 v^g_{z1} \right) - \left( \frac{d^2 v^l_{z1}}{dz_0^2} + w^2 v^l_{z1} \right) = 0 \quad 5.7 \]

where \( \mu \) is the ratio of the vapor viscosity to the liquid viscosity.

The interfacial energy balance is

\[ v^l_{z1} - \sigma Z_i + E \left( \frac{dT^l_i}{dz_0} - k \frac{dT^g_i}{dz_0} \right) = 0 \quad 5.8 \]

where again \( E \) stands for the Evaporation number and is given by \( E = \frac{C_p \Delta T}{h} \). Here, \( C_p \)

and \( h \) are the liquid heat capacity and the latent heat of evaporation per unit mass, respectively. The equilibrium condition at the interface becomes

\[ \Pi_{KE} \left( \frac{d^3 v^g_{z1}}{dz_0^3} - \left( w^2 + \frac{\sigma}{Pr \nu} \right) \frac{dv^g_{z1}}{dz_0} \right) - w^2 \Pi_{PE} Z_i = w^2 \eta \left( T^g_i + Z_i \right) \quad 5.9 \]

where again \( \Pi_{KE} = \frac{v^g \kappa}{h d^2} \) and \( \Pi_{PE} = \frac{g d}{h} \) are two dimensionless parameters from the linearized Clapeyron equation, and \( \eta \) is the inverse of the scaled saturation temperature at the flat interface.

**Analytical Results for Infinitely Deep Fluids**

Ordinarily the fluid layers are assumed to be of finite depths but to provide the reader with an expression for the phenomenological coefficient, \( \alpha \) by way of an analytical procedure and avoid a numerical calculation we assume that they are infinitely deep.

The Rayleigh-Taylor problem is essentially an interfacial instability problem, so to find the leading inverse time constant, \( \sigma \), one can omit the \( \sigma \)’s coming from the domain equations and domain variables, and keep the \( \sigma \)’s that appear only along with the surface
deflection. Doing this makes it easier to find an analytical dispersion relation for $\alpha$.

When we subject the vapor and liquid domain equations to the conditions at $z_0 = -\infty$ and $z_0 = \infty$, respectively, we get the following expressions for the velocity and temperature fields.

\[
v^{l}_{zl} = A^{l} e^{-wz_0} + B^{l} z_0 e^{-wz_0}
\]

\[
v^{g}_{zl} = A^{g} e^{wz_0} + B^{g} z_0 e^{wz_0}
\]

\[
T^{l}_{i} = C^{l} e^{-wz_0} - \frac{k}{2w} \left( z_0 e^{-wz_0} + \frac{1}{2} k B^{l} z_0^2 e^{-wz_0} \right)
\]

\[
T^{g}_{i} = C^{g} e^{wz_0} + \frac{1}{2w} \kappa \left( z_0 e^{wz_0} + \frac{1}{2} k B^{g} z_0^2 e^{wz_0} \right)
\]

The steps taken to get the analytical result are given below. Using the no slip condition and the tangential stress balance one gets

\[-A^{l} w + B^{l} = A^{g} w + B^{g}\]

and

\[A^{l} w - B^{l} = \mu \left( A^{g} w + B^{g} \right).
\]

We deduce from these two equations that

\[A^{l} w = B^{l}\]

and

\[A^{g} w = -B^{g}.
\]
Now we can write two of the unknowns, $B^l$ and $B^g$, in terms of $A^l$ and of $A^g$. We then plug in the expressions for $\nu_{z1}^l$ and $\nu_{z1}^g$ into the normal stress balance and we get

$$2A^l w + 2\mu A^g w = \left(w_{RT}^2 - w^2\right)Z_1.$$

The mass balance gives us

$$A^l = \rho A^g + (1 - \rho)\sigma Z_1.$$

We can use the relation obtained from the interfacial mass balance to eliminate $A^l$ from the normal stress balance and we get $A^g$ in terms of the first order perturbation of the surface deflection, $Z_1$, which is

$$A^g = \frac{w_{RT}^2 - w^2 - 2w(1 - \rho)\sigma}{2w(\mu + \rho)} Z_1.$$

When we plug the expressions for $T_{1}^l$ and $T_{1}^g$ into the continuity of temperature and the thermodynamic equilibrium relation we get

$$C^g + Z_1 = C^l + k Z_1$$

and

$$\eta \left( C^g + Z_1 \right) = -2\Pi_{KE} w A^g - \Pi_{PE} Z_1.$$

Now we are in a position to write $C^l$ and $C^g$ in terms of $Z_1$. We still have not used the interfacial energy balance. When we plug everything into the energy balance at the interface we get

$$(...) Z_1 = 0.$$

For this equality to hold either $Z_1$ or the term in the parenthesis must be equal to zero. $Z_1$ equal to zero is not an option since then there would be no solution to the
problem solved. Setting the terms in the parenthesis equal to zero we get the following expression as the dispersion relation:

$$
\sigma = \frac{F + G}{H}
$$

where

$$
F = \frac{(w_{RT}^2 - w^2)}{2w(\mu + \rho)} \left[ \rho + E \left( \frac{2 \Pi_{KE} w^2 (1 + k)}{\eta} - \frac{3k}{4w} \left( \rho + \frac{1}{k} \right) \right) \right]
$$

$$
G = w E \left( \frac{\Pi_{PE} (1 + k)}{\eta} + 2k \right)
$$

and

$$
H = \rho + \frac{3k}{4w} E (1 - \rho) + \frac{(1 - \rho)}{\mu + \rho} \left[ \rho + E \left( \frac{2 \Pi_{KE} w^2 (1 + k)}{\eta} - \frac{3k}{4w} \left( \rho + \frac{1}{k} \right) \right) \right]
$$

When we set \( \sigma = 0 \), we get an expression for the critical wavenumber in terms of the vapor temperature gradient at the interface and the physical properties of both fluids.

We mentioned earlier that Ho (1980) and Adham-Khodaparast et al. (1995) derived a dispersion relation for the phenomenological coefficient, \( \alpha \), by decoupling the temperature and velocity fields. Once \( \alpha \) is introduced only the velocity fields in both fluids were obtained. To obtain the result of Adham-Khodaparast et al., one only needs to drop all the energy equations from our model and replace the interfacial energy balance by the following balance equation in dimensional form:

$$
v_{\alpha i}^1 - \sigma Z_i = \alpha Z_i
$$
Simply put, the phenomenological coefficient is introduced in the interfacial energy balance where it replaces the heat conduction terms. The dispersion relation of Adham-Khodaparast et al. using dimensionless equations becomes

\[ w^2 + 2 \frac{\alpha (\nu^l + \nu^g)}{\gamma} d w - w_{RT}^2 = 0 \]

When we assume that the kinematic viscosities of both fluids are equal to each other, we get the familiar result of Adham-Khodaparast et al. in dimensionless form. It is

\[ w^2 + 4 \frac{\alpha \nu^l d}{\gamma} w - w_{RT}^2 = 0 \]

Going back to analytical dispersion relation, when we rearrange the equation for the critical wavenumber, we get the expression for the phenomenological coefficient, \( \alpha \), at the onset of instability. It is

\[
\alpha = -\frac{\rho^g k^l}{d^2} \left[ G \left[ \rho + E \left( \frac{2 \Pi_{KE} w^2 (1+k)}{\eta} - \frac{3k \left( \rho + \frac{1}{k} \right)}{4w} \right) \right] \right]
\]

We see that \( \alpha \), at the onset of instability, itself is a function of the wavenumber as well as the physical properties of the fluids.

In Figure 5-2, the dimensionless phenomenological coefficient, \( \alpha \), is plotted against the dimensionless wavenumber to see the shape of the curve and investigate whether there is any range of \( w \) in which \( \alpha \) remains roughly unchanged. The calculations are done assuming the physical properties of water and water vapor at its
saturation temperature under 1 atm, i.e., at 100°C. The physical properties used in the calculations are given in Table 3-1.

Figure 5-2 shows the strong dependence of the phenomenological coefficient on the wavenumber in the plotted wavenumber range. The classical Rayleigh-Taylor dimensionless critical wavenumber for the same system is $3.4e^{-7}$. We observe that at exactly the classical Rayleigh-Taylor critical wavenumber the value of the phenomenological coefficient is equal to zero. When we look at the relation derived by Adham-Khodaparast et al., it is easier to see why $\alpha = 0$ when $w = w_{RT}$.

![Graph showing dependence of phenomenological coefficient on wavenumber](image)

**Figure 5-2.** The dependence of the phenomenological coefficient on the wavenumber.

The graph therefore tells us that the use of a phenomenological coefficient to decouple the temperature and velocity fields is not a good assumption as the phenomenological coefficient itself depends strongly upon the wavenumber. In fact, the coefficient even turns negative for large wave numbers.
Numerical Results for Finite Depth Fluids

To see whether the assumption that one or both fluids are infinitely deep is a valid assumption, we have plotted the critical temperature gradient in the vapor against the wavenumber in Figure 5-3 for different systems. In one of the cases, the fluids are assumed to be infinitely deep whereas in the others both fluid layers have finite depths. Positive temperature gradients mean that the bilayer system is heated from the liquid side, namely from the top.

We know from the classical Rayleigh-Taylor problem that the system is unstable for wavenumbers smaller than the critical wavenumber, however, when there is phase-change taking place at the liquid-vapor interface, the interface can be stabilized by heating from the liquid side even at small wavenumbers. In the plots, the region under the critical curve is the unstable region. As we increase the liquid and vapor depths, the critical values of the vapor temperature gradients approach the values for the infinitely deep layers. We observe that for large wavenumbers the curves overlap whereas for small wavenumbers the critical temperature gradients in the vapor at the interface diverge from the values for the infinitely deep layers. From Figure 5-3 we can tell that the assumption of infinitely deep layers is also not a good one for all wavenumbers. For a water-water vapor system, the finite layers, where \( d^l = d^g = 0.1 \) m. deep, give the same results as the infinitely deep layers at all wavenumbers, however, a 0.1-meter depth for a vapor layer is a relatively large fluid depth under many practical circumstances.
Figure 5-3. Comparison of models a) Infinitely deep fluid layers  
b) $d_l = 0.05 \text{ m and } d_v = 0.05 \text{ m} \quad c) d_l = 0.01 \text{ m and } d_v = 0.01 \text{ m}$

In many applications of the Rayleigh-Taylor instability, the vapor layer is shallower than the liquid layer. For that reason, in Figure 5-4 we plotted the dimensionless critical temperature gradient against the wavenumber for two different cases to see the effect of the vapor layer depth on the instability. Observe that when realistic fluid depth ratios are considered the effect of the fluid depths is considerably more important. The larger the ratio of the liquid depth to the vapor depth is, the easier it is to stabilize the bilayer system by heating from the liquid side. That is, the minimum temperature gradient required to stabilize the system is less. For shallow vapor layers, the flow in the vapor is impeded by the bottom wall and the flow in the liquid starts to play a role on the stability. The liquid flow brings warmer liquid to the troughs encouraging evaporation, thus preventing them from growing deeper. More thorough explanations of the stabilization mechanism through the flow generated by phase-change are given in chapter 3. Thus far, in all the cases we considered, the instability was delayed by heating from the liquid side. Again, when we turn to industrial applications,
we recognize that the system is mostly heated from the vapor side, namely heated from below.

Figure 5-5 gives a plot of the critical temperature difference across the bilayer system against the wavenumber where both of the fluids have finite depths and the system is heated from below, namely the vapor side. The liquid and the vapor layers are taken to be 1mm and 0.1mm deep, respectively. The region above the curve is the stable region. So, this calculation also pertains to a physically realizable experiment. Observe that the curve shows an increase of the critical temperature difference with wavenumber at low wavenumbers, a maximum and then a decrease. This shows the dual effect of wavenumber. The waviness of the disturbance finds itself participating in transverse diffusion of heat and momentum as well as in stabilization on account of surface tension. At low wave numbers, surface tension plays a minimal role. Transverse diffusion itself plays two roles. In the domain it offers a dissipative effect and therefore plays a stabilizing role but at the interface it encourages the movement of the interface as hot locations move toward cold regions and so forth. This is the destabilizing effect. At high wave numbers the stabilizing effect of surface tension and the critical temperature difference for the realization of stability is decreased.
Figure 5-4. Effect of shallow fluid layers.
   a) $d^l = 50e^{-3}m$ and $d^v = 50e^{-3}m$      b) $d^l = 50e^{-3}m$ and $d^v = 5e^{-3}m$

Note that heating from below can stabilize an arrangement where a vapor underlies its own liquid at a wavenumber at which the classical Rayleigh-Taylor analysis indicates otherwise.

In summary, we conclude in this chapter that the assumption of infinitely deep fluid layers give results that are very different than for the fluid layers with finite depths as one can conclude that a system that is unstable for a given temperature gradient in the vapor based on the infinitely deep layers assumption may actually give stability for fluid layers with finite depths.
Another result of this chapter is that the use of a phenomenological coefficient is questionable given the observation fact that such a coefficient itself is a function of the wavenumber.
CHAPTER 6
EFFECT OF VAPOR DYNAMICS ON THE RAYLEIGH-MARANGONI PROBLEM

Introduction

In keeping with the general goals of this thesis as stated in Chapter 1 we move on to the discussion of the role of vapor dynamics in Rayleigh Marangoni convection without phase change. In this regard we first discuss the physics of the pure Rayleigh-Marangoni problem. The physical arguments made are then supported by the calculations done using nothing more than a linear stability analysis. The results of calculations are followed by a discussion of experiments that were done to demonstrate the calculations.

When we have a liquid layer underlying a gas layer, say air, and heat the bilayer from below, there are several mechanisms through which the convection will be initiated. To better understand the possible convection mechanisms that can occur, let us consider Figure 6-1.

![Figure 6-1. Different types of convection mechanisms](image)

a  

b  

c
When the bottom layer, i.e., the liquid layer, is deep compared to the top layer (case a), it is more likely for the convection to start in the liquid layer due to buoyancy. The convection in the liquid layer will cause the top layer to convect through viscous coupling at the interface. For certain depths in both fluid layers (case b), the interface will be an isotherm and the bilayer system will act as two separate one-layer fluid systems. The convection in each layer will be due to buoyancy and we say that the layers are coupled thermally. Another mechanism, one that is of importance to us, is when the gas layer is deep compared to the liquid layer (case c). In the latter case, convection will start in the gas layer due to buoyancy. In this case, the convection in the air cannot cause the liquid to convect through viscous coupling because of the low ratio of the gas viscosity to the liquid viscosity. However, the convection in the upper layer will cause perturbations in the temperature at the interface, which in turn will cause Marangoni convection along the interface and consequently in the liquid layer. This last case is important to us because it demonstrates the role of an active vapor layer on the onset of convective instability. Note that if the gas layer were assumed to be passive, we would not expect to predict the sort of convection depicted in case c.

Several experiments have been done as part of this doctoral study and the results of these experiments have been compared to the calculations to verify the theory. In the following section, we will briefly describe the experimental setup. This setup was originally designed and constructed by Johnson (1997). Fine details on the experiment such as the data acquisition system and control algorithms are given in the thesis of Johnson (1997).
Experimental Apparatus and Procedure

The objectives of the experiments were to study the effect of the air height on the flow pattern of the lower liquid. The experimental apparatus was designed so that various air heights, as well as various liquid aspect ratios could be used. A schematic of the test section is given in Figure 6-2 and the picture of the experimental setup is in Figure 6-3.

The lower heating element consisted of a heating plate under a hollowed cylinder made of lucite, \(\frac{3}{4}\)" thick, called the lower bath. The top and bottom of the cylinder were capped with thin copper plates, \(\frac{1}{4}\)" thick, and the interior of the cylinder was filled with water. A magnetic stir bar was placed in the water and the entire lower bath sat on top of a magnetic stirrer. The stirred water helped to stabilize any temperature fluctuations from the heating block, ensuring a uniform temperature up to ± 0.05°C.

![Schematic of the experimental setup](image)

Figure 6-2. Schematic of the experimental setup

The test section itself consisted of four pieces: a liquid insert, an air insert, a zinc selenide IR transparent window and a clamp for structural integrity. The oil insert, the air
insert and the clamp are made of lucite. To ensure that the liquid-gas interface was flat, a pinning edge was used in the liquid insert. The liquid insert could be made of different radii and heights, to achieve the desired liquid aspect ratio. The radius of the air insert was always the same as the radius of the liquid insert. However, different air heights could be used for the same liquid insert. The lucite clamp fastened the liquid and air inserts down, preventing silicone oil from leaking and ensuring structural integrity. The outer radius of the oil inserts was 1 1/16”. The outer radius of the air inserts were 2 ¼”. However to reiterate, all of the inserts had the same inner radius. The zinc selenide window had a diameter of 2” and was 5 mm thick. The accuracy of each piece was machined to within 0.1 mm.

An Inframetrics, model 760, infrared camera was used to visualize the flow patterns. This particular model of infrared camera is capable of measuring in the 3 to 5 µm range or the 8 to 12 µm range. However, only the 8 to 12 µm range was used. The infrared camera was placed directly above the test section and measured the infrared radiation being emitted by the silicone oil. As silicone oil readily absorbs infrared radiation, only the radiation from the first few angstroms of the silicone oil interface could be detected. An effective emissivity could be calibrated, and programmed into the infrared camera to find the temperature of the interface. The IR camera was never used to measure the temperature rather it was used to detect the variation in interfacial temperature from which the flow structure could be deduced. Zinc selenide is 60% transparent to infrared radiation in the 8 to 12 µm range. Additionally, an anti-reflective infrared polymer was coated on the zinc selenide window by II-VI Inc. This coating was
necessary to eliminate any false images generated by reflected, ambient infrared radiation.

Figure 6-3. Experimental setup

To control the temperature at the top of the test section, an infrared transparent medium was needed. The medium of choice was air. A lucite box enclosed the test section, the lower bath, the magnetic stirrer, the infrared camera and the heating control elements. A heater, a fan, and a car radiator were used to control the temperature of the ambient air in the lucite box. To increase the ambient temperature, the heater would turn on. Cool water, pumped through the radiator, continuously removed heat from the air. Proper mixing was ensured by a blowing fan. The overall temperature control across the top of the zinc selenide and the bottom of the top copper plate of the lower bath was never more than 2.4% of the setpoint. When the smoothing of the temperature fluctuations due to the highly conductive plates (copper and zinc selenide) are taken into consideration, the actual control across the two fluid layers was probably better. The temperature of the top and bottom plate, and the temperature of the cooling water were
measured. All of the temperatures were recorded and controlled using a computerized data acquisition system. The temperature control was later improved to ±0.05°C, with no substantial effect on the experimental results.

The infrared images were sent to a VCR and displayed on a TV. As some of the experiments lasted several days, the VCR was wired into the computer so that the control program would periodically record and stop recording the images.

Each experiment was conducted in the same systematic manner. First the silicone oil was placed into the liquid insert. The corresponding air insert was chosen and placed on top of the liquid insert. The lucite clamp was placed on top of the air insert and screwed into the lower bath. The level of the oil-air interface could be checked by both a standard bubble level and the infrared camera. As it turns out, the oil-air interface acts as a lens to infrared radiation, concentrating infrared radiation if the surface is depressed in the center or diffusing infrared radiation if the interface is elevated in the center. If there is an insufficient amount of silicone oil in the insert, the liquid-air interface will be depressed and the temperature will appear to be higher in the center, even though the temperature is uniform. If there is too much oil in the insert, the liquid-air interface will be elevated in the center and the temperature will appear to be lower in the center. Checking the oil level with the infrared camera was more accurate than the bubble level.

Once the test section was secured, a temperature difference was applied across the liquid-gas bilayer. The initial temperature difference was less than the critical temperature difference necessary to initiate convection in either layer. When a temperature difference was applied, it was held constant for several time constants. The longest time constant in these experiments is the horizontal thermal diffusion time
constant. Here the thermal diffusivity of silicone oil is $\kappa = 1.1 \cdot 10^{-7} \, \text{m}^2/\text{sec}$ and the typical diameter is about 25 mm to give a horizontal time constant of

$$\frac{d^2}{\kappa} = \left(25 \cdot 10^{-3} \, m\right)^2 = 1.6 \text{hrs}.$$  The temperature difference for these experiments was held constant for four hours. However, steady state was usually reached well within the four-hour period. After the temperature difference was held constant for four hours, the temperature difference was increased to a new set point and held constant for another four hours. This procedure was repeated until the fluid began to flow. Once the fluid began to flow, the temperature difference and the flow pattern were noted.

**Experimental Results**

We have compared the experimental results with two types of calculations. The results of the linear calculations are in good qualitative agreement with the experiments. Calculations were done to support the physical arguments given above using the physical properties in Table 6-1. The first set of calculations was done using a code written by the author assuming that the two fluid layers are bounded vertically by two rigid plates of infinite lateral extent. The physical model and the equations are given in Chapter 1 and 2. Let us start out by verifying the physical arguments we made using the first set of calculations.

The fluid properties used in the calculations are those that correspond to fluids used in the experiments, viz. silicone oil and air. The calculations were done assuming the liquid and air depths to be 5 mm and 3 mm, respectively.
Table 6-1. Physical properties of silicone oil and air at 20°C

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>968 kg/m³</td>
<td>( \rho^* )</td>
<td>1.2 kg/m³</td>
</tr>
<tr>
<td>( \mu )</td>
<td>96.8 \times 10^{-3} kg/(m·sec)</td>
<td>( \mu^* )</td>
<td>1.9 \times 10^{-5} kg/(m·sec)</td>
</tr>
<tr>
<td>( k )</td>
<td>1.6 \times 10^{-1} J/(m·sec·°C)</td>
<td>( k^* )</td>
<td>2.6 \times 10^{-2} J/(m·sec·°C)</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>1.1 \times 10^{-7} m²/sec</td>
<td>( \kappa^* )</td>
<td>2.1 \times 10^{-5} m²/sec</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>9.6 \times 10^{-4} 1/°C</td>
<td>( \alpha^* )</td>
<td>32.4 \times 10^{-4} 1/°C</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>20.9 \times 10^{-3} N/m</td>
<td>( \gamma^* )</td>
<td>0.5 \times 10^{-4} N/m</td>
</tr>
</tbody>
</table>

In Figure 6-4, \( z = 0 \) represents the liquid-gas interface. It is evident that at the onset of convection, for this is what the calculations predict, at any point in the \( x \)-direction, the \( z \)-components of velocity in both fluids have opposite signs, that is, the convection rolls in each phase are in the opposite direction of the other phase. If at any given ‘\( x \)’ the liquid moves upward, the air layer at that point in the direction of ‘\( x \)’ moves downward. The flow in the liquid layer is stronger than the flow in the gas layer as convection initially starts due to buoyancy in the liquid layer. The calculations therefore represent the scenario depicted by case (a).

Next, the calculations are depicted in Figure 6.5 for a silicone oil-air system where the liquid and air depths are 5 mm and 5mm, respectively. We will see that the calculations correspond to a case similar to case (b). We observe that there is strong flow in both fluid layers due to buoyancy. There is a small roll in the vapor near the interface, which arises because of the no-slip condition at the interface. To better visualize the flow pattern in the system, let us turn to Figure 6-6. In Figure 6-6, \( z = 0 \) represents the interface between silicone oil and air. It is more obvious from this plot, that the
convection in the gas near the interface is very weak in comparison. The convection occurs in both domains due to buoyancy. A further adjustment in the gas depth could result in a more accurate representation of case (b) than the above calculations.

Figure 6-4. z-component of velocity in a silicone oil-air system at the onset of instability where the silicone oil and air depths are 5mm and 3mm, respectively.

Figure 6-5. z-component of velocity in a silicone oil-air system at the onset of instability where the silicone oil and air depths are 5mm and 5mm, respectively.
Now let us consider the case c where the air layer is deeper than the silicone oil layer. The convection starts initially due to buoyancy and the flow in the vapor causes convection in the liquid through surface tension gradients at the interface.

Figure 6-6. Flow profiles in a silicone oil-air system at the onset of convection where the silicone oil and air depths are 5mm and 3mm, respectively.

Figure 6-7 depicts a silicone oil-air bilayer system where the silicone oil and air depths are 5mm and 9mm, respectively. In Figure 6-7, the convection in the air layer is much stronger than the convection in the liquid layer since the convection in the liquid is actually caused by surface tension gradients at the interface, and Marangoni forces are not strong enough to cause flow as strong as the buoyancy forces that initiated the convection in the vapor.

The second set of calculations was done using a code supplied by Dauby. In the second set of calculations, a single liquid layer of silicone oil superposed by a conductive, active gas in a closed cylindrical container is modeled. The interface between the liquid and the gas is assumed to be flat. All sides of the cylindrical container have no-slip
boundaries. The lower and upper vertical boundaries are rigid, no-slip, conducting plates. The lateral sidewalls could be either perfect conductors or perfect insulators.

Figure 6-7. z-component of velocity in a silicone oil-air system at the onset of instability where the silicone oil and air depths are 5mm and 9mm, respectively.

The calculations are able to predict the critical temperature difference (or Rayleigh number or Marangoni number) for different azimuthal modes over a range of the cylinder’s aspect ratio. Figure 6-8 shows the predicted critical temperature difference for a range of the air heights for a fixed liquid depth and radius from the calculations as well as the results of experiments performed at several air heights. In Figure 6-8 there are three curves for each air height. Each of these curves stands for a different azimuthal flow pattern. To better explain the convection patterns of those modes, let us turn to Figure 6-9. \( m = 0 \) is an axisymmetric flow pattern, meaning that the flow will go up in the middle and go down on the sides. In Picture 4, one can observe the lighter color indicating the upflow in the middle. \( m = 1 \), on the other hand, is symmetric around an imaginary diameter cutting the cylinder in the middle. That is, the flow will go up on one side and go down the other, which is very clear in Picture 3.
Figure 6-8. The critical temperature difference vs. air height for a bilayer system where the silicone oil depth is 8mm and the aspect ratio is 2.5.

The results of experiments are marked with an ‘x’ on the same plot, and some of the results are also numbered. The pictures of the experiment at the given numbered points are below.

Figure 6-9. Pictures of experiments taken by IR camera at the onset of instability

We observe that the experiments and calculations agree perfectly for small and large air depths whereas for intermediate air heights the onset of convection is observed before the expected critical temperature difference is reached. When the air depths are very small and very large, the convection is due to buoyancy and one can show mathematically that as the Rayleigh convection problem is self-adjoint, the instability is supercritical. For intermediate air depths, Marangoni convection starts to play a role on the stability of the problem, and Marangoni problem can be subcritical for \( m=0 \). And in
this case, it turns out to be subcritical. That is why we believe that we can observe the convection set in before the critical point is reached.

In summary, we conclude that even in the absence of phase-change the fluid flow in the gas layer plays an active role in determining the stability of a bilayer system indicating it is a mistake to leave the fluid dynamics of gas layers in convective instability problems. Our conclusions have been backed by experiments. Moreover we have been able to interpret the results of these experiments in light of bifurcation theory.
CHAPTER 7
CONCLUDING REMARKS AND FUTURE SCOPE

In this chapter, the main discoveries of each chapter are re-evaluated and the future scope for further research is suggested based on the conclusions of this doctoral work. In order to do this, the discussion is divided into three parts.

**Active Vapor Layers**

A theoretical model was written to describe the physics of the interfacial instability arising from evaporation. The numerical calculations arising form this model showed without any doubt that it is inappropriate to assume that the vapor layer is only a conductive heat transfer medium. In fact, it was found that the fluid dynamics of the vapor layer has an important role on stabilizing the evaporation problem. As past workers have assumed that the vapor can be taken to be infinitely deep and fluid mechanically passive, this assumption has been shown to not always be true. In a separate experimental study using non-evaporative liquids in the presence of air, it was again shown that gas layers are fluid mechanically active, thus have as strong effect on the onset point as well as the pattern at the onset of instability. These experiments agreed well with companion numerical models. As a result of this work it is suggested that one should numerically investigate the role of vapor layers on the onset flow patterns during phase-change in cylindrical containers. The companion experiments might, however, prove to be harder than corresponding problem in the absence of phase-change. One of the main reasons for the difficulty is the fact that a temperature gradient has to be applied
across the bilayer system and the vapor will condense on the cold plate obscuring the IR-
imaging of the liquid-vapor interface. There are several ways to get around this
hindrance. One of them is to heat the system from the vapor side. The calculations
suggest that the interface will become unstable even when heated from above within the
range of temperature differences that are suitable for laboratory experiments, i.e., at low
values of temperature gradients. Another way is to not apply a temperature difference
across the system and to let the temperature gradient occur through evaporation at the
interface. This can be done by lowering the vapor pressure at the top plate and allowing a
small phase-change rate at the interface, which in turn will lower the temperature of the
interface. In that case, the liquid will still be potentially unstable for Rayleigh convection
whereas the vapor layer will be Rayleigh stable. By using shallow liquid depths, one can
delay the buoyancy-driven instability in the liquid and investigate the role of active vapor
layer, which can then only convect due to surface tension gradients or the phase-change.
Another necessary calculation to do would be the nonlinear analysis of the problem in its
complete form, i.e., including the buoyancy- and surface tension gradient-driven
convection. Several patterns are expected to emerge with the onset of instability and
nonlinear calculations will provide information on the stability of these possible patterns
and tell us which one of them will prevail after the onset of instability.

In the last problem investigated in this work, Chapter 5, the role of an active vapor
layer was investigated in relation to a film-boiling configuration. There is plenty of data
in that line of research and the results of this work will definitely shed more light on the
Rayleigh-Taylor instability problem in the presence of phase-change including a finite
base state evaporation rate.
Conditions at the Interface of an Already Evaporating Liquid

In this doctoral work, all calculations were done assuming a zero phase-change rate in the base state. Naturally, the thermodynamic equilibrium assumption at the liquid-vapor interface was sufficient to complete the model for the interface conditions. On the other hand, in practical systems this is not always applicable as very high evaporation rates are common in film boiling. In that case, continuum thermostatics is not adequate to describe what is going on at the interface. One can investigate the possibility of a nonequilibrium relation or an accommodation coefficient and a range of possibilities where that coefficient may replace the interface conditions used in the theoretical model specified in this study. There is, of course, the complication that such an accommodation coefficient would depend upon the wave number.

Coupling of Evaporative Instabilities with Other Phase-Change Phenomenon

There are plenty of examples where the evaporation at a liquid-vapor interface is accompanied by the solidification at a liquid-solid interface, such as in coating processes and in the drying of car paints. Evaporative instability and the depth of the vapor layer can be used to stabilize an otherwise unstable solidification front. For that reason, it would be interesting to investigate the coupling of evaporative instability with the solidification problem. Experiments can be performed to investigate the effect of evaporating liquid layers and active vapor layers on the morphology of a solidification front. In other words, there is plenty of scope to take the results of this thesis toward other problems with equally interesting physics and applications.
APPENDIX A
THERMODYNAMIC EQUILIBRIUM RELATION AT THE INTERFACE

Consider an interface between an evaporating liquid and its vapor. For two phases of a pure species coexisting at equilibrium,

$$\hat{G}^L = \hat{G}^G$$

where $\hat{G}^L$ and $\hat{G}^G$ are the liquid and gas specific Gibbs free energies, respectively.

This relation holds regardless of the shape of the interface. We will make use of this relation to get information about the effect of curvature on the pressure. So, we write

$$\hat{G}^L_{\text{curved}} = \hat{G}^G_{\text{curved}}$$

and

$$\hat{G}^G_{\text{flat}} = \hat{G}^L_{\text{flat}}$$

Combining all these equations we can write

$$\hat{G}^L_{\text{curved}} - \hat{G}^G_{\text{curved}} - \hat{G}^L_{\text{flat}} + \hat{G}^G_{\text{flat}} = 0$$

or

$$\left( \hat{G}^L_{\text{curved}} - \hat{G}^L_{\text{flat}} \right) - \left( \hat{G}^G_{\text{curved}} - \hat{G}^G_{\text{flat}} \right) = 0$$

The changes in $\hat{G}$ due to changes in $T$ and $P$ are evaluated using

$$\text{d}\hat{G} = -\hat{S}\text{d}T + \hat{V}\text{d}P$$

where $\hat{S}$ and $\hat{V} = \frac{1}{\rho}$ are the specific entropy and volume, respectively.

Assuming that the specific entropy and volume do not change much on account of small changes in $T$ and $P$, we get
\[- S^L dT + V^L dP^L + S^G dT - V^G dP^G = 0 \]

or

\[ (S^G - S^L) dT + V^L dP^L - V^G dP^G = 0 \]

The above equation can be rearranged and rewritten assuming \( \frac{1}{\rho^G} dP^G \) and \( \frac{1}{\rho^L} dP^L \)

to

\[ (S^G - S^L) dT^G - \frac{1}{\rho^G} dP^G = 0 \]

Assuming an ideal gas, we then have

\[ (S^G - S^L) dT + V^L dP^L - \frac{RT}{P^G} dP^G = 0 \]

This becomes

\[ \frac{H^G - H^L}{T} dT + V^L dP^L - \frac{RT}{P^G} dP^G = 0 \]

But, \( H^G - H^L \) is equal to the latent heat of evaporation, \( \lambda \). Therefore,

\[ \frac{\dot{\lambda}}{T} dT + V^L dP^L - \frac{RT}{P^G} dP^G = 0 \]

or

\[ \frac{\dot{\lambda}}{T^2} dT + \frac{V^L}{T} dP^L - \frac{R}{P^G} dP^G = 0 \]

We can integrate both sides to get a relation between the temperature and pressure at the interface. The integration limits are between the flat and curved states. Hence

\[ \int_{T_y}^{T_c} \frac{\dot{\lambda}}{T^2} dT + \int_{P_y^L}^{P_c^L} \frac{V^L}{T} dP^L - \int_{P_y^G}^{P_c^G} \frac{R}{P^G} dP^G = 0 \]
This equation can be worked on further to get the first order terms, namely the perturbed equation. It is

\[
\frac{\lambda}{T_0^2} \left( T_1 + Z_1 \frac{dT_0}{dz} \right) = \frac{R}{P_0^G} \left( P_1^G + Z_1 \frac{dP_0^G}{dz} \right) + \frac{V^1}{T_0} \gamma \frac{\partial^2 Z_1}{\partial x^2}
\]
APPENDIX B
A SIMPLE DERIVATION TO OBTAIN THE CRITICAL CONDITION FOR ONSET OF CONVECTION IN A FLUID LAYER

Consider an infinite horizontal layer of fluid of a finite depth, d, where a constant temperature gradient, \( \frac{dT_0}{dz_0} = \frac{\Delta T}{d} \), is maintained by heating from below. Here, \( \Delta T = T_{\text{hot}} - T_{\text{cold}} \). Making use of the continuity equation, the dimensionless perturbed Navier-Stokes equations are brought into the following form:

\[
\begin{align*}
\frac{d^2}{dz^2} - \omega^2 &\left( \frac{d^2}{dz^2} - \omega^2 - \frac{1}{\text{Pr} \sigma} \right) v_{z1} = \omega^2 \text{Ra} T_1 \\
T_1 &\left( \frac{d^2}{dz^2} - \omega^2 \right) = -v_{z1}
\end{align*}
\]

When we set \( \sigma = 0 \) to determine the critical Rayleigh number and eliminate \( T_1 \) between the two equations, we get

\[
\left( \frac{d^2}{dz^2} - \omega^2 \right)^3 v_{z1} = -\omega^2 \text{Ra} v_{z1}
\]

The temperatures at the top and bottom plates are kept constant. Using the domain energy equation we turn the constant temperature conditions at \( z_0 = 0 \) and \( z_0 = 1 \) into

\[
\left( \frac{d^2}{dz^2} - \omega^2 \right)^2 v_{z1} = 0
\]

The fluid is confined between the top and the bottom plates, that is \( v_{z1} = 0 \) at \( z_0 = 0 \) and \( z_0 = 1 \). Two more boundary conditions are needed. For a simpler algebra, let
us require that both surfaces are stress free and the boundary conditions at $z_0 = 0$ and $z_0 = 1$ are given by

$$\frac{d^2 v_{z1}}{dz^2} = 0 \quad \text{B.5}$$

Using the no mass transfer and stress free boundary conditions it is easy to convert the boundary condition obtained from the temperature boundary conditions into

$$\frac{d^4 v_{z1}}{dz_0^4} = 0 \quad \text{B.6}$$

From equations A.3 through A.6, we determine that $\frac{d^6 v_{z1}}{dz_0^6} = 0$. By differentiating the equation A.3, we find that $\frac{d^8 v_{z1}}{dz_0^8} = 0$. By further differentiations of A.3, we conclude that all even derivatives of $v_{z1}$ vanish on the boundaries. Consequently,

$$\frac{d^{2m} v_{z1}}{dz_0^{2m}} = 0 \text{ at } z_0 = 0 \text{ and } z_0 = 1 \text{ for } m = 1, 2, \ldots$$

At this point, it is not hard to guess that the solution is in the form

$$v_{z1} = A \sin n \pi z_0 \quad (n = 1, 2, \ldots)$$

where $A$ is an unknown constant, and $n$ is an integer. After substituting this solution into A.3, we get the following relation:

$$Ra = \left( \frac{n^2 \pi^2 + \omega^2}{\omega^2} \right)^3$$

For any given positive value of $\omega^2$, the lowest value of $Ra$ occurs when $n = 1$, accordingly,

$$Ra = \left( \frac{\pi^2 + \omega^2}{\omega^2} \right)^3.$$
Our aim is to determine the critical wavenumber that corresponds to the lowest value of Ra. In order to determine the critical Rayleigh number along with its corresponding wavenumber, let us take the first derivative of Ra with respect to $\omega^2$ and set it equal to zero. It is

$$\frac{d Ra}{d \omega^2} = 3\left(\frac{\pi^2 + \omega^2}{\omega^2}\right)^2 - \left(\frac{\pi^2 + \omega^2}{\omega^4}\right)^3 = 0$$

from which we find that $\omega_c^2 = \frac{\pi^2}{2}$.

The corresponding value of Ra is

$$Ra_c = \frac{27}{4} \pi^4 = 657.5.$$

Below is the plot of the Rayleigh number against the wavenumber.

![Figure B–1. Critical Rayleigh number vs. dimensionless wavenumber](image-url)
APPENDIX C
A SIMPLE DERIVATION FOR THE RAYLEIGH TAYLOR INSTABILITY

When a vapor underlies a liquid, the resulting configuration is potentially unstable. This problem, known as the Rayleigh-Taylor problem, is well known and a nice account of this instability is given by Chandrasekhar (1961). Here I will give a derivation of the problem for infinitely deep fluid layers and find the critical wavenumber at the onset of instability.

Linearized stability is performed in the standard way leading to the following dimensionless equations of motion

\[
\left( \frac{d^2}{dz^2} - w^2 \right)^2 v_{z_1}^{l} = 0
\]

and

\[
\left( \frac{d^2}{dz^2} - w^2 \right)^2 v_{z_1}^{g} = 0.
\]

When we subject the vapor and liquid domain equations to the conditions at \( z_0 = -\infty \) and \( z_0 = \infty \), respectively, we get the following expressions for the velocity and temperature fields.

\[
v_{z_l}^{l} = A^l e^{-wz_0} + B^l z_0 e^{-wz_0}
\]

\[
v_{z_l}^{g} = A^g e^{wz_0} + B^g z_0 e^{wz_0}
\]

At the interface, \( z_0 = 0 \), the mass balance turns into
\( v_{z1} = 0 \)

and

\( v^*_{z1} = 0 \).

The no-slip conditions becomes

\[
\frac{dv_{z1}}{dz_0} = \frac{dv^*_{z1}}{dz_0}.
\]

The tangential and normal stress balances assume the following dimensionless form:

\[
\mu \left( \frac{d^2v^g_{z1}}{dz_0^2} + w^2 v^g_{z1} \right) - \left( \frac{d^2v^l_{z1}}{dz_0^2} + w^2 v^l_{z1} \right) = 0
\]

and

\[
\mu \left( \frac{d^3v^g_{z1}}{dz_0^3} - 3w^2 \frac{dv^g_{z1}}{dz_0} \right) - \left( \frac{d^3v^l_{z1}}{dz_0^3} - 3w^2 \frac{dv^l_{z1}}{dz_0} \right) + w^2 \left( \frac{(\rho^l - \rho^g)gd^2}{\gamma} - w^2 \right) Z_1 = 0
\]

where \( \mu \) is the ratio of the vapor viscosity to the liquid viscosity.

Using the interfacial conditions we arrive at

\[
w_c = \left( \frac{(\rho^l - \rho^g)g}{\gamma} \right)^{\frac{1}{2}} d.
\]
APPENDIX D
THE BOUSSINESQ APPROXIMATION

The Rayleigh problem is a stability problem because of the nonlinearity introduced to the problem through the convective heat equation term in the domain energy equation. After we linearize the domain equations we observe that the temperature and velocity are still dependent on each other through the energy equation. However, that relation is not the only reason for the Rayleigh convection. The assumption that the fluid density is a linear function of temperature makes the fluid dynamics problem dependent on the heat transfer problem. Clearly, the analysis of the Rayleigh problem may be very complicated. For that reason, the assumption of Boussinesq approximation is employed. It is based on the assumption that

\[ \frac{1}{\rho} \frac{\partial \rho}{\partial T} \left. \right|_{T=T_0} \Delta T \approx 1 \]

which is basically \( \frac{\Delta \rho}{\rho} \approx 1 \) where \( \Delta \rho \) is the change in the fluid density for the temperature difference applied. The density can be expressed by the first term of a Taylor series expansion about the base state as

\[ \rho = \rho_0 + \left. \frac{\partial \rho}{\partial T} \right|_{T=T_0} (T - T_0) \]

Let us now look at the perturbed equation of motion and the energy equation, and perform a dimensional analysis. The perturbed energy equation is

\[ \frac{\partial T_1}{\partial t_0} + \bar{v}_1 \cdot \nabla_0 T_0 = \kappa \nabla_0^2 T_1 \]

and we learn from a dimensional analysis that the prevailing velocities would be of order \( \frac{\kappa}{d} \), then plugging that velocity scale into the motion equation...
\[
\frac{\rho \partial \vec{v}}{\partial t} = -\nabla \cdot \vec{P} + \rho \vec{g} + \mu \nabla^2 \vec{v}
\]

we determine that the term where the perturbed density multiplies gravitational acceleration has to be of order $10^{-4}$ or $10^{-5}$ for most fluids for a depth of 1 cm.

\[-\frac{1}{\rho} \frac{\partial \rho}{\partial T} \bigg|_{T=0}\]

itself is of the order of $10^{-4}$ or $10^{-5}$ before it even multiplies the gravitational acceleration and a temperature difference. A simple dimensional analysis of the motion equation tells us that the density, $\rho$, can be replaced by its base state value, $\rho_0$, in all the terms except in the term that multiplies the gravitational acceleration.

During the course of my study, Prof. Narayanan and I had an interesting observation on Boussinesq approximation.

**A Remark on the Boussinesq Approximation**

Consider a fluid in a rigid container. The energy balance for the fluid is given by

\[
\frac{d}{dt} \rho \left( \hat{U} + \frac{1}{2} |\vec{v}|^2 + \Phi \right) = -\nabla \cdot \vec{q} + \nabla \cdot (\vec{\tau} \cdot \vec{v})
\]

where $\vec{\tau} = -\vec{P}^\parallel + \vec{S}^\parallel$. $\vec{S}$ is the viscous stress tensor and for Newtonian fluids it is $\vec{S} = 2\mu \vec{D}$.

Now let us take the volume integral of the equation above and rearrange it. We get

\[
\int \rho \frac{d}{dt} \left( \hat{U} + \frac{1}{2} |\vec{v}|^2 + \Phi \right) = -\int \nabla \cdot \vec{q} + \int (\vec{\tau} \cdot \vec{v}) \cdot \vec{n}
\]

At steady-state all terms except the heat conduction term are equal to zero. The last integral is zero because of the rigid walls and the no slip at the boundary which require $\vec{v} \cdot \vec{n} = 0$ and $\vec{v} \cdot \vec{t} = 0$. So, we have

\[
\int \nabla \cdot \vec{q} = 0
\]
Now let us look at another form of the energy equation for a fluid, which can be brought into the same form as the one given in the beginning of this Appendix. It is

$$\frac{d}{dt} \int \rho \dot{U} + \int \rho \dot{U} (\vec{v} - \vec{u}) \cdot \vec{n} = -\int \nabla \cdot q - \int P(\nabla \cdot \vec{v}) + \int 2 \mu \vec{D} : \vec{D}$$

At steady-state and rigid boundaries, the integrals on the left hand side are equal to zero. We have shown that $\int \nabla \cdot q = 0$, and then the equation reduces to

$$\int P(\nabla \cdot \vec{v}) = \int 2 \mu \vec{D} : \vec{D}$$

The continuity equation in the domain in our analysis is

$$\nabla \cdot \vec{v} = 0$$

Thus, we are only left with one term and the equation further reduces to

$$\int 2 \mu \vec{D} : \vec{D} = 0$$

When there is convection in our system, we know that

$$\int 2 \mu \vec{D} : \vec{D} > 0.$$  

This poses no problem in the linear stability analysis because this term disappears in the first order perturbation of the equation and is automatically equal to zero, however, it seems that nonlinear analysis there is a violation when one uses Boussinesq approximation.
APPENDIX E
THE UNIT NORMAL OF A SURFACE

In this Appendix, the unit normal vector in 2D system where the normal vector is on a surface on the z-axis. Let a surface be denoted by

\[ f = z - Z(x, t) = 0 \]

Then, \( f \) is positive on one side of \( f = 0 \), negative on the other, and the normal pointing into the region where \( f \) is positive is given by

\[ \vec{n} = \frac{\nabla f}{|\nabla f|}. \]

Here,

\[ \nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial z} k. \]

Then, the equation for the normal stress balance is given by

\[ \begin{align*}
\ddot{\vec{n}} &= \left( -\frac{\partial Z}{\partial x} i + k \right) \\
&= \frac{-\frac{\partial Z}{\partial x} i + k}{\left( \left( \frac{\partial Z}{\partial x} \right)^2 + 1 \right)^{\frac{1}{2}}}. \\
\end{align*} \]

The derivation of the unit tangent vector is straightforward using the definition \( \vec{n} \cdot \vec{t} = 0 \) from which we get

\[ \vec{t} = \frac{i + \frac{\partial Z}{\partial x} k}{\left( \left( \frac{\partial Z}{\partial x} \right)^2 + 1 \right)^{\frac{1}{2}}}. \]
APPENDIX F
DERIVATION OF THE SURFACE SPEED

Let a surface be denoted by

\[ f(\vec{r}, t) = 0 \]

where \( \vec{r} = (x, y, z) \) and the unit normal of the surface is given by

\[ \vec{n} = \frac{\nabla f}{|\nabla f|}. \]

Let the surface move a small distance \( \Delta s \) along its normal in time \( \Delta t \). Then,

\[ f(\vec{r} \pm \Delta s \vec{n}, t + \Delta t) \]

is given by

\[ f(\vec{r} \pm \Delta s \vec{n}, t + \Delta t) = f(\vec{r}, t) \pm \Delta s \vec{n} \cdot \nabla f(\vec{r}, t) + \Delta t \frac{\partial f(\vec{r}, t)}{\partial t} + ... \]

whence \( f(\vec{r} \pm \Delta s \vec{n}, t + \Delta t) = 0 = f(\vec{r}, t) \) requires

\[ \pm \Delta s \vec{n} \cdot \nabla f(\vec{r}, t) = -\Delta t \frac{\partial f(\vec{r}, t)}{\partial t} \]

The normal speed of the surface, \( u \), is then given by

\[ u = \pm \frac{\Delta s}{\Delta t} = - \frac{\partial f(\vec{r}, t)}{\vec{n} \cdot \nabla f(\vec{r}, t)} \]

Now, using the definition of the unit normal given earlier we get

\[ u = - \frac{\partial f}{\partial t} \frac{\vec{n}}{|\nabla f|}. \]

In our problems, the definition of \( u \) becomes
\[
u = \frac{\partial Z}{\partial t} \left( \left( \frac{\partial Z}{\partial x} \right)^2 + 1 \right)^{-\frac{1}{2}}.
\]
APPENDIX G
THE INTERFACIAL ENERGY BALANCE

There are three difference equations of interest to us in thermal problems and they are

\[ \rho (\bar{v} - \bar{u}) \cdot \hat{n} = \rho^* (\bar{v}^* - \bar{u}) \cdot \hat{n} \]
\[ \left( \rho \bar{v} (\bar{v} - \bar{u}) - \bar{T} \right) \cdot \hat{n} + \gamma 2H \bar{n} + \nabla \cdot \bar{v} = \left( \rho^* \bar{v}^* (\bar{v}^* - \bar{u}) - \bar{T}^* \right) \cdot \hat{n} \]
\[ \left( \rho \left( \hat{U} + \frac{1}{2} \bar{v}^2 \right) (\bar{v} - \bar{u}) + \bar{q} - \bar{T} \cdot \bar{v} \right) \cdot \hat{n} + \gamma 2H \bar{n} \cdot \bar{u} \]
\[ = \left( \rho^* \left( \hat{U}^* + \frac{1}{2} \bar{v}^{*2} \right) (\bar{v}^* - \bar{u}) + \bar{q}^* - \bar{T}^* \cdot \bar{v}^* \right) \cdot \hat{n} \]

where \( \bar{u} = u \cdot \hat{n} \).

Notice that \( \hat{n} \) points out of the non-asterisk phase into asterisk phase and \( 2H \) is taken to be negative at a crest, looking along \( \hat{n} \).

The first of these is form invariant under a change of observer; the second and third are not. The second and third equations are taken to hold in a laboratory frame and our aim is to use the second equation to rewrite the mechanical terms in the third equation, thereby producing a form invariant difference equation that accounts for the latent heat released as a phase-change front moves into the asterisk fluid.

To do this, work only on the left hand-side, take the dot product of the second equation 2 with \( \bar{u} \), subtract the result from the third equation to obtain
\[
\left( \rho \left( \bar{v} - \bar{u} \right) \hat{U} + \rho \left( \bar{v} - \bar{u} \right) \left( \frac{1}{2} \bar{v}^2 - \bar{v} \cdot \bar{u} \right) + \bar{q} - \bar{T} \cdot \left( \bar{v} - \bar{u} \right) \right) \cdot \bar{n} + 2H \cdot \bar{n} \cdot \bar{u} = \left( \rho^* \left( \bar{v}^* - \bar{u} \right) \hat{U}^* + \rho^* \left( \bar{v}^* - \bar{u} \right) \left( \frac{1}{2} \bar{v}^{*2} - \bar{v}^* \cdot \bar{u} \right) + \bar{q}^* - \bar{T}^* \cdot \left( \bar{v}^* - \bar{u} \right) \right) \cdot \bar{n}
\]

and use \( \bar{T} = -\bar{P} \bar{T} + \bar{S} \) and \( \hat{H} = \hat{U} + \frac{P}{\rho} \) to get

\[
\left( \rho \left( \hat{H} + \frac{1}{2} \left( \bar{v} - \bar{u} \right)^2 - \frac{1}{2} \bar{u}^2 \right) \left( \bar{v} - \bar{u} \right) + \bar{q} - \bar{S} \cdot \left( \bar{v} - \bar{u} \right) \right) \cdot \bar{n} = \left( \rho^* \left( \hat{H}^* + \frac{1}{2} \left( \bar{v}^* - \bar{u} \right)^2 - \frac{1}{2} \bar{u}^2 \right) \left( \bar{v}^* - \bar{u} \right) + \bar{q}^* - \bar{S}^* \cdot \left( \bar{v}^* - \bar{u} \right) \right) \cdot \bar{n}
\]

The resulting difference equation is then

\[
\left( \rho \left( \hat{H} + \frac{1}{2} \left( \bar{v} - \bar{u} \right)^2 \right) \left( \bar{v} - \bar{u} \right) + \bar{q} - \bar{S} \cdot \left( \bar{v} - \bar{u} \right) \right) \cdot \bar{n} = \left( \rho^* \left( \hat{H}^* + \frac{1}{2} \left( \bar{v}^* - \bar{u} \right)^2 \right) \left( \bar{v}^* - \bar{u} \right) + \bar{q}^* - \bar{S}^* \cdot \left( \bar{v}^* - \bar{u} \right) \right) \cdot \bar{n}
\]

and observe that it is form invariant.
APPENDIX H
THE EXPANSION OF A DOMAIN VARIABLE AND ITS DERIVATIVES ALONG THE MAPPING

In this appendix, the perturbation equations used in the theoretical work are explained. This is done for a one-dimensional domain and the reader is referred to Johns and Narayanan (2002) for details.

Let ‘u’ denote the solution of a problem in an irregular domain D where D is not specified and must be determined as part of the solution. Imagine that D lies in the vicinity of a reference domain $D_0$ and can be expressed in terms of the reference domain via a parameter $\varepsilon$. Let $u$ be a function of a spatial coordinate ‘$z$’. Then ‘$u$’ must be a function of ‘$\varepsilon$’ directly because it lies on D and also because it is a function of ‘$z$’. By an irregular domain it is meant a domain that is not convenient. Now to solve for $u$ and obtain D simultaneously we can solve a series of companion problems defined on the nearby regular or convenient domain, $D_0$, which we called the reference domain. What needs to be done is to discover how to determine $u$ in terms of the solutions to problems defined on $D_0$. The points of $D_0$ will be denoted by the coordinate $z_0$ and those of D by the coordinate $z$.

Imagine a family of domains $D_\varepsilon$ growing out of the reference domain $D_0$. $u$ must be determined on each of these. The point $z$ of the domain $D_\varepsilon$ is then determined in terms of the point $z_0$ of the reference domain $D_0$ by the mapping

$$z = f(z_0, \varepsilon).$$
The equation above is a little more general than necessary to explain the mapping for our purposes. In the problems of this thesis, only one part of the domain is irregular, namely the interface.

Now, to get going let us expand the function \( f \) in powers of \( \varepsilon \) as

\[
f (z_0, \varepsilon) = f (z_0, \varepsilon = 0) + \varepsilon \frac{\partial f (z_0, \varepsilon = 0)}{\partial \varepsilon} + \frac{1}{2} \varepsilon^2 \frac{\partial^2 f (z_0, \varepsilon = 0)}{\partial \varepsilon^2} + ...\]

where \( f (z_0, \varepsilon = 0) = z_0 \) and the derivatives of \( f \) are evaluated holding \( z_0 \) fixed. Then in terms of the notation

\[
z_1 (z_0, \varepsilon) = \frac{\partial f (z_0, \varepsilon = 0)}{\partial \varepsilon},
\]

\[
z_2 (z_0, \varepsilon) = \frac{\partial^2 f (z_0, \varepsilon = 0)}{\partial \varepsilon^2}
\]

etc.

the mapping can be written as

\[
z = z_0 + \varepsilon z_1 (z_0, \varepsilon = 0) + \frac{1}{2} \varepsilon^2 z_2 (z_0, \varepsilon = 0) + ...
\]

Also, the boundary of the reference domain must be carried into the boundary of the present domain by the same mapping. The function, \( Z \), which describes the boundary of the new domain, inherits its expansion in powers of \( \varepsilon \) from the mapping given in and can be written as

\[
Z = Z_0 + \varepsilon Z_1 (Z_0, \varepsilon = 0) + \frac{1}{2} \varepsilon^2 Z_2 (Z_0, \varepsilon = 0) + ...
\]

It is the \( Z_i \)'s that need to be determined to specify the domain \( D_\varepsilon \) in terms of the domain \( D_0 \).
Accordingly, $u(z, \epsilon)$ can be expanded in powers of $\epsilon$ along the mapping as

$$u(z, \epsilon) = u(z = z_0, \epsilon = 0) + \epsilon \frac{du}{d\epsilon}(z = z_0, \epsilon = 0) + \frac{1}{2} \epsilon^2 \frac{d^2u}{d\epsilon^2}(z = z_0, \epsilon = 0) + \ldots$$

where $\frac{du}{d\epsilon}$ denotes the derivative of the function $u$ depending on $z$ and $\epsilon$ taken along the mapping. To obtain a formula for $\frac{du}{d\epsilon}(z = z_0, \epsilon = 0)$, differentiate $u$ along the mapping taking $z$ to depend on $\epsilon$, holding $z_0$ fixed. Using the chain rule, this gives

$$\frac{du(z, \epsilon)}{d\epsilon} = \frac{\partial u(z, \epsilon)}{\partial \epsilon} + \frac{\partial u(z, \epsilon)}{\partial z} \frac{\partial f(z_0, \epsilon)}{\partial \epsilon}.$$

Now, set $\epsilon$ to zero in the above equation to get

$$\frac{du(z = z_0, \epsilon = 0)}{d\epsilon} = \frac{\partial u(z_0, \epsilon = 0)}{\partial \epsilon} + \frac{\partial u(z_0, \epsilon = 0)}{\partial z} \frac{\partial f(z_0, \epsilon)}{\partial \epsilon}.$$

Then, introduce the definition of $u_1$ via

$$u_1(z_0) = \frac{\partial u(z_0, \epsilon = 0)}{\partial \epsilon}$$

and observe that

$$\frac{\partial u(z_0, \epsilon = 0)}{\partial z} = \frac{\partial u_0(z_0)}{\partial z_0}$$

to get

$$\frac{du(z = z_0, \epsilon = 0)}{d\epsilon} = u_1(z_0) + \frac{\partial u_0(z_0)}{\partial z_0} z_1(z_0, \epsilon).$$

All the other orders of the derivatives of $u$ can be determined the same way. Finally, if a domain variable needs to be specified at the boundary it is written as
\[
\frac{du(z = Z_0, \varepsilon = 0)}{d\varepsilon} = u_1(Z_0) + \frac{\partial u_0}{\partial z_0} Z_1(Z_0, \varepsilon). 
\]

When additional derivatives are obtained and plugged into the expansion of \( u \), it becomes

\[
u(z, \varepsilon) = u_0 + \varepsilon \left( u_1 + \frac{\partial u_0}{\partial z_0} z_1 \right) + \frac{1}{2} \varepsilon^2 \left( u_2 + 2 \frac{\partial u_1}{\partial z_0} z_1 + \frac{\partial^2 u_0}{\partial z_0^2} z_1^2 + \frac{\partial u_0}{\partial z_0} z_2 \right) + \ldots
\]

The careful reader will notice that the mapping \( z_1 \) does not appear in the domain equations given in the thesis. To show why, let us work out an example and so let \( u \) satisfy an equation

\[
\frac{\partial u}{\partial z} = 0
\]

in the new domain. Our reference domain, however, is in the coordinate system, \( z_0 \).

Thus,

\[
\frac{\partial u}{\partial z} = \frac{\partial u}{\partial z_0} \frac{\partial z_0}{\partial z}.
\]

Hence, we must differentiate the right hand side of the expansion of \( u \) with respect to \( z_0 \), holding \( \varepsilon \) fixed and then multiply it by \( \frac{\partial z_0}{\partial z} = 1 - \varepsilon \frac{\partial z_1}{\partial z} - \frac{1}{2} \varepsilon^2 \frac{\partial z_2}{\partial z} - \ldots \)

This gives us to the first order

\[
\frac{\partial u}{\partial z} = \frac{\partial u_0}{\partial z_0} + \varepsilon \left( \frac{\partial u_1}{\partial z_0} + \frac{\partial^2 u_0}{\partial z_0^2} z_1 \right) + \ldots
\]

See how the derivatives of \( z_1 \) and \( z_2 \) are lost during the step where we multiplied the derivative of \( u \) with respect to \( z_0 \) by \( \frac{\partial z_0}{\partial z} \). Now going back to our example and plugging this in, we get
\[
\frac{\partial u}{\partial z} = \frac{\partial u_0}{\partial z_0} + \varepsilon \left( \frac{\partial u_1}{\partial z_0} + \frac{\partial^2 u_0}{\partial z_0^2} z_1 \right) + \ldots = 0
\]

That is to say that in the domain

\[
\frac{\partial u_0}{\partial z_0} = 0
\]

and accordingly, \( \frac{\partial^2 u_0}{\partial z_0^2} = 0 \), which gives

\[
\frac{\partial u_1}{\partial z_0} = 0.
\]

The mapping is lost from domain equations. If surface variables were considered the mapping would not be lost. What we show in this appendix is mostly sufficient for us to get on with the work of this thesis.
APPENDIX I
NUMERICAL METHOD

The Chebyshev spectral tau method was used to solve the resulting eigenvalue problem. Numerical methods using this method are often considerably faster with greater accuracy than other standard techniques such as finite differencing. The tau method easily incorporates complicated boundary conditions. A great explanation of the Chebyshev spectral tau method is given by Johnson with nice examples. In this section, the Chebyshev spectral tau method is described briefly, the reader is referred to Canuto et al. and Gottlieb and Orszag.

Spectral method is discretization scheme developed from the method of weighted residuals (Finlayson and Scriven). Here follow the details of the method:

Consider the following problem:

\[ \frac{\partial u}{\partial t} + Lu = \lambda Au \]

with boundary conditions represented by

\[ Bu = 0 \]

where L and A are linear operators, B is a linear boundary operator, and \( \lambda \) is the eigenvalue. The solution vector \( u \) can be expressed in terms of infinite series of trial functions. The trial functions chosen are the Chebyshev polynomials, then \( u \) is written as

\[ u(x, t) = \sum_{n=0}^{\infty} a_n(t) T_n(x) \]
The solution vector \( u \) is then approximated by the truncated series up to some finite value of \( N \) and is given by

\[
\begin{align*}
  u(x, t) &\cong u_N(x, t) = \sum_{n=0}^{N} a_n(t)T_n(x)
\end{align*}
\]

Now, \( u_N(x, t) \) must explicitly satisfy the boundary conditions. Since this is not an easy task, \( k \) more terms are added to the polynomial, so the expression for \( u_N(x, t) \) becomes

\[
\begin{align*}
  u_N(x, t) &\cong \sum_{n=0}^{N+k} a_n(t)T_n(x)
\end{align*}
\]

Here, \( N \) is the dimension in the domain and \( k \) is the number of boundary conditions. Now there are \( N+k \) unknowns and \( N \) equations. The \( k \) additional equations obtained from the \( k \) boundary equations.

\[
\sum_{n=0}^{N+k} a_n(t)T_n(x) = 0
\]

The Chebyshev polynomials are orthogonal on the interval \((-1,1)\) with respect to the weight function \( w(x) = \left(1 - x^2\right)^{-\frac{1}{2}} \). These polynomials satisfy the following:

\[
\langle T_m(x), T_n(x) \rangle = \int_{-1}^{1} w(x)T_m(x)T_n(x)dx = \begin{cases} 
  0 & m \neq n \\
  \pi & m = n = 0 \\
  \frac{1}{2}\pi & m = n \neq 0 
\end{cases}
\]

Now choose a test function \( T_m(x) \) and take the inner product with the problem to be solved. It gives

\[
\left\langle \sum_{m=0}^{N} T_m(x), \sum_{n=0}^{N} T_n(x) \right\rangle \frac{da_n(t)}{dt} = a_n(t) \left\langle \sum_{m=0}^{N} T_m(x), \sum_{n=0}^{N} L T_n(x) \right\rangle + \lambda a_n(t) \left\langle \sum_{m=0}^{N} T_m(x), \sum_{n=0}^{N} A T_n(x) \right\rangle
\]
which reduces to

$$\frac{da_n(t)}{dt} = a_n(t)\left(\sum_{m=0}^{N} T_m(x), \sum_{n=0}^{N} L T_n(x)\right) + \lambda a_n(t)\left(\sum_{m=0}^{N} T_m(x), \sum_{n=0}^{N} A T_n(x)\right)$$

The equation above can be solved now using the definition of Chebyshev polynomials.
LIST OF REFERENCES


Rayleigh, Lord, “On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side,” Phil. Mag., 32(6), 529 (1916).


BIOGRAPHICAL SKETCH

Ozgur Ozen was born in Turkey. He graduated from Bogazici University in Istanbul, Turkey, receiving a B.S. degree in Chemical Engineering in 1999. He then attended the University of Florida for graduate studies under the supervision of Prof. Ranga Narayanan. In 2004, he graduated from the University of Florida with a Ph.D. in chemical engineering.