RISK MANAGEMENT TECHNIQUES FOR DECISION MAKING IN HIGHLY UNCERTAIN ENVIRONMENTS

By

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The dissertation studies modern risk management techniques for decision making in highly uncertain environments. The traditional framework of decision making under uncertainties relies on stochastic programming or simulation approaches to surpass simpler quasi-deterministic techniques, where the uncertainty is modeled by relevant statistics of stochastic parameters. In many applications, however, the mentioned methodologies in their conventional form fail to generate efficient and robust decisions. Mathematical models for such class of applications, therefore, are referred to as highly uncertain environments, with the defining features such as: large number of mutually correlated stochastic factors with dynamically changing or uncertain distributions, multiple types of risk exposure, out-of-sample application of the solution, etc. Robust decision making in such environments requires an explicit control of the risk induced by uncertainties.

The first part of the dissertation considers risk management approaches for financial applications. We present a general framework of risk/reward optimization that establishes the equivalence of different formulations of optimization problems with risk and reward functions. As an application of the general result, we consider optimization of portfolio of stocks with Conditional Value-at-Risk objective and constraints, and compare this approach with the classical Markowitz Mean-Variance methodology. An extensive study of out-of-sample performance of trading algorithms based on different risk measures is performed on the example of managing of a hedge fund portfolio. A chapter dedicated to multi-stage decision-making problems presents a new sample-path approach

for multi-stage stochastic programming problems and applies it to the problem of optimal transaction implementation. The generality of the developed model allows for using it in pricing of complex derivative securities, such as exotic options.

The second part of the dissertation considers risk management techniques for military decision-making problems. The main challenges of military applications attribute to various types of risk exposure, uncertain probability measures of risk-inducing factors, and inapplicability of the "long run" convention. Different formulations for stochastic Weapon-Target Assignment problem with uncertainties in distributions are considered, and relaxation and linearization techniques for the resulting nonlinear mixed-integer programming problems are suggested.

CHAPTER 1
INTRODUCTION

The dissertation is devoted to study of risk management techniques for decision making in highly uncertain environments. The traditional framework of decision making under the presence of uncertainties relies on stochastic programming (Birge and Louveaux, 1997; Prékopa, 1995) or simulation (Ripley, 1987) approaches to surpass simpler quasi-deterministic techniques, where the uncertainty is modeled by relevant statistics of the stochastic parameters, such as expectation or variance.

The study of decision-making mathematical programming models that involve uncertain parameters was originated by Dantzig and Madansky (1961), and received major development in works of R. J.-B. Wets and colleagues (Rockafellar and Wets, 1976a,b; Walkup and Wets, 1967, 1969; Wets, 1966a,b, 1974). In contrast to the quasi-deterministic "averaging" approaches in decision making under uncertainties, stochastic programming introduces the concept of *stages* in the decision making, where some decisions have to be made without prior knowledge of the realization of the random parameters in the system, and followed by *corrective* actions, taken after the uncertainties have been realized. For a comprehensive presentation of stochastic programming theory and applications, see Birge and Louveaux (1997) or Prékopa (1995).

Use of stochastic programming solutions in uncertain environments ensures increased robustness and effectiveness in comparison to deterministically obtained solutions. However, there exist applications, where the stochastic programming methodology in its conventional form fails to generate efficient and robust decisions. Mathematical models for such class of applications usually feature a large number of mutually correlated stochastic factors with dynamically changing or uncertain distributions, multiple types of risk exposure, application of the obtained solution in situations that cannot be described by the scenario set of the original problem (so-called *out-of-sample* applications), etc. We refer to mathematical models that exhibit such or similar properties as *highly uncertain environments*. Robust decision making in such environments requires, besides maximizing the expected performance, an explicit control on the *risk* induced by uncertainties.

The area of stochastic analysis and mathematical programming, known as *risk management*, focuses on the effects of extremely adverse outcomes or events on the process of decision making. The first outstanding contribution to this field is due to Markowitz (1952, 1991), who identified the riskiness of a portfolio of financial instruments with the volatility of assets' returns, and proposed the famous Mean-Variance portfolio optimization model for constructing a portfolio with lowest risk for a given level of expected return. The transparency and efficiency of Markowqitz's approach made it a very popular tool, which is widely used in the finance industry even nowadays. Since Markowitz's seminal work, a considerable progress has been achieved in the area of risk analysis and management, which evolved into a sophisticated discipline combining rigorous and elegant theoretical results with practical effectiveness. It was recognized that neither the Mean-Variance model, which utilizes variance to measure the uncertainty in the system, nor the Value-at-Risk concept (the upper α-quantile of distribution, see Jorion (1996, 1997)), which became the official standard in the finance industry, can provide an adequate figure of risk exposure. The theory of *coherent risk measures*, developed by Artzner et al. (1997, 1999, 2001), Delbaen (2000), established a set of axioms to be satisfied by a risk measure (usually, a certain statistic of the distribution) in order to yield correct estimation of risk under different conditions. At the same time, a series of developments of new risk measures such as Conditional Value-at-Risk, Expected Shortfall (Acerbi, 2002; Acerbi et al., 2001; Acerbi and Tasche, 2002; Rockafellar and Uryasev, 2000, 2002) demonstrated the theoretical and practical importance and efficiency of the concept of coherent risk measures.

The first part of the dissertation (Chapters 2 to 4) considers risk management approaches in the scope of financial applications. In Chapter 2, we present a general framework of risk/reward optimization that establishes the equivalence of different formulations of optimization problems with risk and reward functions. As an application of the general result, we consider optimization of portfolio of stocks with Conditional Value-at-Risk objective and constraints, and compare this approach with the classical Markowitz Mean-Variance methodology. Chapter 3 contains an extensive study of in-sample and out-of-sample performance of trading algorithms based on different risk measures (Conditional Value-at-risk, Conditional Drawdown-at-Risk, Mean-Absolute Deviation, and Maximum Loss) on the example of managing a portfolio of hedge funds. Based on the results of this and other case studies, we make recommendations on how to improve the "real" (i.e., out-of-sample) performance of portfolio rebalancing strategies.

Chapter 4 introduces a new sample-path approach to the problem of optimal transaction implementation. Based on the ideas of multi-stage stochastic programming, and using a sample-path collection instead of the traditional scenario tree, this approach produces an optimal trading strategy that admits a differentiate response to realized market conditions at each time step. In contrast to other existing approaches to optimal execution, our approach admits a seamless incorporation of different types of constraints, e.g. regulatory or risk constraints, into the trading strategy. The generality of the developed approach allows to use it for pricing of complex derivative securities, such as exotic options.

Although the field of risk management is rooted in finance and financial applications, the successes of risk management in theoretical and, especially, applied contexts have drawn attention to these techniques in other fields where uncertainties impact the decision making process. One of the areas where systematic risk management approach will contribute significantly to robustness of decisions and policies is military applications, which typically involve decision making in dynamic, distributed, and highly uncertain environments. The main challenges of military applications attribute to various types of risk exposure, uncertain probability measures of risk-inducing factors, and inapplicability of the "long run" convention.

Indeed, in contrast to the problems of finance, where the only type of risk is the risk of financial loss, military incorporate different types of risk, e.g., risk of not killing a target, risk of false target attack, risk of being destroyed by enemy, etc. In addition, distributions of risk factors in military problems are rarely known with certainty. Also, a typical financial model is expected perform well in repetitive applications, i.e., it is required to be effective *on average*, or in a *long run*. Obviously, there is no "long run" in military applications, therefore the generated decision must be both safe and effective *here and now*.

In the second part of the dissertation (Chapters 5 to 6), we consider risk management techniques for military decision-making problems. A general approach for military decision-making problems with uncertainties in distributions is developed in Chapter 5, and, as an example, we consider a stochastic version of the Weapon-Target Assignment problem. We demonstrate that employing risk management techniques based on the Conditional Value-at-Risk measure leads to solutions that perform robustly under a wide range of scenarios.

In Chapter 6, we consider a nonlinear integer programming model for the Stochastic Weapon-Target Assignment problem, and develop a linear programming relaxation, which allows for efficient incorporation of risk constraints in the problem.

Finally, Appendix presents the results of our ongoing research efforts in the area of pricing of path-dependent derivative securities using the techniques of mathematical programming. We discuss a linear programming-based algorithm for solving free-boundary problem arising in pricing of American put option within the classical Black-Scholes framework, as well as a new approach for pricing of path-dependent derivative securities using simulation and stochastic programming.

CHAPTER 2
PORTFOLIO OPTIMIZATION WITH CONDITIONAL VALUE-AT-RISK OBJECTIVE AND CONSTRAINTS

In this chapter, we extend the recently suggested approach (Rockafellar and Uryasev, 2000) for optimization of Conditional Value-at-Risk (CVaR) to the case of optimization problems CVaR constraints. We derive a general equivalence result for different types of optimization problems with broad class of risk and reward functions. In particular, the approach can be used for maximizing the expected returns under CVaR constraints. As a further extension, multiple CVaR constraints with various confidence levels can be used to shape the profit/loss distribution. A case study for the portfolio of S&P 100 stocks is performed to demonstrate how the new optimization techniques can be implemented. The approach is compared with the classic Markowitz Mean-Variance model for portfolio optimization.

2.1 Introduction

Portfolio optimization has come a long way from Markowitz (1952) seminal work which introduces return/variance risk management framework. Developments in portfolio optimization are stimulated by two basic requirements:

- Adequate modeling of utility functions, risks, and constraints
- Efficiency, i.e., ability to handle large numbers of instruments and scenarios.

Current regulations for finance businesses formulate some of the risk management requirements in terms of percentiles of loss distributions. An upper percentile of the loss distribution is called Value-at-Risk (VaR).[1] For instance, 95%-VaR is an upper estimate of losses which is exceeded with 5% probability. The popularity of VaR is mostly related to a simple and easy to understand representation of high losses. VaR can be quite efficiently estimated and managed when

[1] By definition, VaR is the percentile of the loss distribution, i.e., with a specified confidence level α, the α-VaR of a portfolio is the lowest amount ζ such that, with probability α, the loss is less or equal to ζ. Regulations require that VaR should be a fraction of the available capital.

underlying risk factors are normally (log-normally) distributed. For comprehensive introduction to risk management using VaR, we refer the reader to Jorion (1997). However, for non-normal distributions, VaR may have undesirable properties (Artzner et al., 1997, 1999) such as lack of sub-additivity, i.e., VaR of a portfolio with two instruments may be greater than the sum of individual VaRs of these two instruments.[2] Also, VaR is difficult to control/optimize for discrete distributions, when it is calculated using scenarios. In this case, VaR is non-convex (see definition of convexity in (Rockafellar, 1970) and non-smooth as a function of positions, and has multiple local extrema. An extensive description of various methodologies for the modeling of VaR can be seen, along with related resources, at URL http://www.gloriamundi.org/. Mostly, approaches to calculating VaR rely on linear approximation of the portfolio risks and assume a joint normal (or log-normal) distribution of the underlying market parameters (Duffie and Pan (1997), Jorion (1996), Pritsker (1997), RiskMetrics™ (1996), Simons (1996), Stublo Beder (1995), Staumbaugh (1996)). Also, historical or Monte Carlo simulation-based tools are used when the portfolio contains nonlinear instruments such as options (Jorion (1996), Mausser and Rosen (1991), Pritsker (1997), RiskMetrics™ (1996), Stublo Beder (1995), Staumbaugh (1996)). Discussions of optimization problems involving VaR can be found in Litterman (1997a,b), Kast et al. (1998), Lucas and Klaassen (1998).

Although risk management with percentile functions is a very important topic and in spite of significant research efforts (Andersen and Sornette (2001), Basak and Shapiro (2001), Emmer et al. (2000), Gaivoronski and Pflug (2000), Gourieroux et al. (2000), Grootweld and Hallerbach (2000), Kast et al. (1998), Puelz (1999), Tasche (1999)), efficient algorithms for optimization of percentiles for reasonable dimensions (over one hundred instruments and one thousand scenarios) are still not available. On the other hand, the existing efficient optimization techniques for portfolio allocation[3]

[2] When returns of instruments are normally distributed, VaR is sub-additive, i.e., diversification of the portfolio reduces VaR. For non-normal distributions, e.g., for discrete distribution, diversification of the portfolio may increase VaR.

[3] High efficiency of these tools can be attributed to using linear programming (LP) techniques. LP optimization algorithms are implemented in number of commercial packages, and allow for solving of very large problems with millions of variables and scenarios. Sensitivities to parameters are calculated automatically using dual variables. Integer constraints can also be relatively well treated in linear problems (compared to quadratic or other nonlinear problems). However, recently developed interior point algorithms work equally well both for portfolios with linear and quadratic

do not allow for direct controlling[4] of percentiles of distributions (in this regard, we can mention the mean absolute deviation approach (Konno and Yamazaki, 1991), the regret optimization approach (Dembo and Rosen, 1999), and the minimax approach (Young, 1998). This fact stimulated our development of the new optimization algorithms presented in this chapter.

This chapter suggests to use, as a supplement (or alternative) to VaR, another percentile risk measure which is called Conditional Value-at-Risk. The CVaR risk measure is closely related to VaR. For continuous distributions, CVaR is defined as the conditional expected loss under the condition that it exceeds VaR, see Rockafellar and Uryasev (2000). For continuous distributions, this risk measure also is known as Mean Excess Loss, Mean Shortfall, or Tail Value-at-Risk. However, for general distributions, including discrete distributions, CVaR is defined as the weighted average of VaR and losses strictly exceeding VaR (Rockafellar and Uryasev, 2002). Recently, Acerbi et al. (2001), Acerbi and Tasche (2002) redefined expected shortfall similarly to CVaR.

For general distributions, CVaR, which is a quite similar to VaR measure of risk has more attractive properties than VaR. CVaR is sub-additive and convex (Rockafellar and Uryasev, 2000). Moreover, CVaR is a *coherent* measure of risk in the sense of Artzner et al. (1997, 1999). Coherency of CVaR was first proved by Pflug (2000); see also Rockafellar and Uryasev (2002), Acerbi et al. (2001), Acerbi and Tasche (2002). Although CVaR has not become a standard in the finance industry, CVaR is gaining in the insurance industry (Embrechts et al., 1997). Similar to CVaR measures have been introduced earlier in stochastic programming literature, but not in financial mathematics context. The conditional expectation constraints and integrated chance constraints described in Prékopa (1995) may serve the same purpose as CVaR.

Numerical experiments indicate that usually the minimization of CVaR also leads to near optimal solutions in VaR terms because VaR never exceeds CVaR (Rockafellar and Uryasev, 2000). Therefore, portfolios with low CVaR must have low VaR as well. Moreover, when the return-loss

performance functions, see for instance Duarte (1999). The reader interested in various applications of optimization techniques in the finance area can find relevant papers in Ziemba and Mulvey (1998).

[4] It is impossible to impose constraints in VaR terms for general distributions without deteriorating the efficiency of these algorithms.

distribution is normal, these two measures are equivalent (Rockafellar and Uryasev, 2000), i.e., they provide the same optimal portfolio. However for very skewed distributions, CVaR and VaR risk optimal portfolios may be quite different. Moreover, minimizing of VaR may stretch the tail exceeding VaR because VaR does not control losses exceeding VaR, see Larsen et al. (2002). Also, Gaivoronski and Pflug (2000) have found that in some cases optimization of VaR and CVaR may lead to quite different portfolios.

Rockafellar and Uryasev (2000) demonstrated that linear programming techniques can be used for optimization of the Conditional Value-at-Risk (CVaR) risk measure. A simple description of the approach for minimizing CVaR and optimization problems with CVaR constraints can be found in Uryasev (2000). Several case studies showed that risk optimization with the CVaR performance function and constraints can be done for large portfolios and a large number of scenarios with relatively small computational resources. A case study on the hedging of a portfolio of options using the CVaR minimization technique is included in (Rockafellar and Uryasev, 2000). This problem was first studied in the paper by Mausser and Rosen (1991) with the minimum expected regret approach. Also, the CVaR minimization approach was applied to credit risk management of a portfolio of bonds, see Andersson et al. (2001).

This chapter extends the CVaR minimization approach (Rockafellar and Uryasev, 2000) to other classes of problems with CVaR functions. We show that this approach can be used also for maximizing reward functions (e.g., expected returns) under CVaR constraints, as opposed to minimizing CVaR. Moreover, it is possible to impose many CVaR constraints with different confidence levels and shape the loss distribution according to the preferences of the decision maker. These preferences are specified directly in percentile terms, compared to the traditional approach, which specifies risk preferences in terms of utility functions. For instance, we may require that the mean values of the worst 1%, 5% and 10% losses are limited by some values. This approach provides a new efficient and flexible risk management tool.

The next section briefly describes the CVaR minimization approach from (Rockafellar and Uryasev, 2000) to lay the foundation for the further extensions. In Section 3, we formulated a general theorem on various equivalent representations of efficient frontiers with concave reward and convex risk functions. This equivalence is well known for mean-variance, see for instance, Steinbach (1999), and for mean-regret, (Dembo and Rosen, 1999), performance functions. We have

shown that it holds for any concave reward and convex risk function, in particular for the CVaR risk function considered in this chapter. Using auxiliary variables, we formulated a theorem on reducing the problem with CVaR constraints to a much simpler convex problem. A similar result is also formulated for the case when both the reward and CVaR are included in the performance function. As it was earlier identified in (Rockafellar and Uryasev, 2000), the optimization automatically sets the auxiliary variable to VaR, which significantly simplifies the problem solution. Further, when the distribution is given by a fixed number of scenarios and the loss function is linear, we showed how the CVaR function can be replaced by a linear function and an additional set of linear constraints. In section 2.7, we developed a one-period model for optimizing a portfolio of stocks using historical scenario generation. A case study on the optimization of S&P100 portfolio of stocks with CVaR constraints is presented in section 2.7. We compared the return-CVaR and return-variance efficient frontiers of the portfolios.

2.2 Conditional Value-at-Risk

The approach developed in (Rockafellar and Uryasev, 2000) provides the foundation for the analysis conducted in this chapter. First, following (Rockafellar and Uryasev, 2000), we formally define CVaR and present several theoretical results which are needed for understanding this chapter. Let $f(\mathbf{x},\mathbf{y})$ be the loss associated with the decision vector[5] $\mathbf{x}$, to be chosen from a certain subset $\mathbf{X}$ of $\mathbb{R}^n$, and the random vector $\mathbf{y}$ in $\mathbb{R}^m$. The vector $\mathbf{x}$ can be interpreted as a portfolio, with $\mathbf{X}$ as the set of available portfolios (subject to various constraints), but other interpretations could be made as well. The vector $\mathbf{y}$ stands for the uncertainties, e.g., market prices, that can affect the loss. Of course the loss might be negative and thus, in effect, constitute a gain.

For each $\mathbf{x}$, the loss $f(\mathbf{x},\mathbf{y})$ is a random variable having a distribution in $\mathbb{R}$ induced by that of $\mathbf{y}$. The underlying probability distribution of $\mathbf{y}$ in $\mathbb{R}^m$ will be assumed for convenience to have density, which we denote by $p(\mathbf{y})$. This assumption is not critical for the considered approach. The paper by Rockafellar and Uryasev (2002) defines CVaR for general distributions; however, here, for simplicity, we assume that the distribution has density. The probability of $f(\mathbf{x},\mathbf{y})$ not exceeding a

[5] We use boldface font for vectors to distinguish them from scalars.

threshold ζ is then given by

$$\Psi(\mathbf{x},\zeta) \quad = \quad \int_{f(\mathbf{x},\mathbf{y})\leq\zeta} p(\mathbf{y})\,d\mathbf{y}. \tag{2.1}$$

As a function of ζ for fixed $\mathbf{x}$, $\Psi(\mathbf{x},\zeta)$ is the cumulative distribution function for the loss associated with $\mathbf{x}$. It completely determines the behavior of this random variable and is fundamental in defining VaR and CVaR.

The function $\Psi(\mathbf{x},\zeta)$ is nondecreasing with respect to (w.r.t.) ζ and we assume that $\Psi(\mathbf{x},\zeta)$ is everywhere continuous w.r.t. ζ. This assumption, like the previous one about density in $\mathbf{y}$, is made for simplicity. In some common situations, the required continuity follows from properties of the loss $f(\mathbf{x},\mathbf{y})$ and the density $p(\mathbf{y})$; see Uryasev (1995).

The α-VaR and α-CVaR values for the loss random variable associated with $\mathbf{x}$ and any specified probability level α in $(0,1)$ will be denoted by $\zeta_\alpha(\mathbf{x})$ and $\phi_\alpha(\mathbf{x})$. In our setting they are given by

$$\zeta_\alpha(\mathbf{x}) \;=\; \min\{\,\zeta\in\mathbb{R} : \Psi(\mathbf{x},\zeta)\geq\alpha\,\} \tag{2.2}$$

and

$$\phi_\alpha(\mathbf{x}) \;=\; (1-\alpha)^{-1}\int_{f(\mathbf{x},\mathbf{y})\geq\zeta_\alpha(\mathbf{x})} f(\mathbf{x},\mathbf{y})p(\mathbf{y})\,d\mathbf{y}. \tag{2.3}$$

In the first formula, $\zeta_\alpha(\mathbf{x})$ comes out as the left endpoint of the nonempty interval[6] consisting of the values ζ such that actually $\Psi(\mathbf{x},\zeta)=\alpha$. In the second formula, the probability that $f(\mathbf{x},\mathbf{y})\geq\zeta_\alpha(\mathbf{x})$ is therefore equal to $1-\alpha$. Thus, $\phi_\alpha(\mathbf{x})$ comes out as the conditional expectation of the loss associated with $\mathbf{x}$ relative to that loss being $\zeta_\alpha(\mathbf{x})$ or greater.

The key to the approach is a characterization of $\phi_\alpha(\mathbf{x})$ and $\zeta_\alpha(\mathbf{x})$ in terms of the function F_α on $X\times\mathbb{R}$ that we now define by

$$F_\alpha(\mathbf{x},\zeta) \;=\; \zeta+(1-\alpha)^{-1}\int_{\mathbf{y}\in\mathbb{R}^n}[f(\mathbf{x},\mathbf{y})-\zeta]^+\,p(\mathbf{y})\,d\mathbf{y}, \tag{2.4}$$

where $[t]^+=\max\{t,0\}$. The crucial features of F_α, under the assumptions made above, are as follows (Rockafellar and Uryasev, 2000).

[6] This follows from $\Psi(\mathbf{x},\zeta)$ being continuous and nondecreasing w.r.t. ζ. The interval might contain more than a single point if Ψ has "flat spots."

Theorem 2.2.1 *As a function of ζ, $F_\alpha(\mathbf{x},\zeta)$ is convex and continuously differentiable. The α-CVaR of the loss associated with any $\mathbf{x} \in X$ can be determined from the formula*

$$\phi_\alpha(\mathbf{x}) \;=\; \min_{\zeta\in\mathbb{R}} F_\alpha(\mathbf{x},\zeta). \tag{2.5}$$

In this formula, the set consisting of the values of ζ for which the minimum is attained, namely

$$A_\alpha(\mathbf{x}) \;=\; \operatorname*{argmin}_{\zeta\in\mathbb{R}} F_\alpha(\mathbf{x},\zeta), \tag{2.6}$$

is a nonempty, closed, bounded interval (perhaps reducing to a single point), and the α-VaR of the loss is given by

$$\zeta_\alpha(\mathbf{x}) \;=\; \textit{left endpoint of } A_\alpha(\mathbf{x}). \tag{2.7}$$

In particular, one always has

$$\zeta_\alpha(\mathbf{x}) \in \operatorname*{argmin}_{\zeta\in\mathbb{R}} F_\alpha(\mathbf{x},\zeta) \quad \textit{and} \quad \phi_\alpha(\mathbf{x}) = F_\alpha(\mathbf{x},\zeta_\alpha(\mathbf{x})). \tag{2.8}$$

For background on convexity, which is a key property in optimization that in particular eliminates the possibility of a local minimum being different from a global minimum, see, for instance, Rockafellar (1970). Other important advantages of viewing VaR and CVaR through the formulas in Theorem 1 are captured in the next theorem, also proved in (Rockafellar and Uryasev, 2000)

Theorem 2.2.2 *Minimizing the α-CVaR of the loss associated with $\mathbf{x}$ over all $\mathbf{x} \in \mathbf{X}$ is equivalent to minimizing $F_\alpha(\mathbf{x},\zeta)$ over all $(\mathbf{x},\zeta) \in \mathbf{X}\times\mathbb{R}$, in the sense that*

$$\min_{\mathbf{x}\in\mathbf{X}} \phi_\alpha(\mathbf{x}) \;=\; \min_{(\mathbf{x},\zeta)\in\mathbf{X}\times\mathbb{R}} F_\alpha(\mathbf{x},\zeta), \tag{2.9}$$

where moreover a pair $(\mathbf{x}^*,\zeta^*)$ achieves the right hand side minimum if and only if $\mathbf{x}^*$ achieves the left hand side minimum and $\zeta^* \in A_\alpha(\mathbf{x}^*)$. In particular, therefore, in circumstances where the interval $A_\alpha(\mathbf{x}^*)$ reduces to a single point (as is typical), the minimization of $F(\mathbf{x},\zeta)$ over $(\mathbf{x},\zeta) \in \mathbf{X}\times\mathbb{R}$ produces a pair $(\mathbf{x}^*,\zeta^*)$, not necessarily unique, such that $\mathbf{x}^*$ minimizes the α-CVaR and ζ^* gives the corresponding α-VaR.

Furthermore, $F_\alpha(\mathbf{x},\zeta)$ is convex w.r.t. $(\mathbf{x},\zeta)$, and $\phi_\alpha(\mathbf{x})$ is convex w.r.t. $\mathbf{x}$, when $f(\mathbf{x},\mathbf{y})$ is convex with respect to $\mathbf{x}$, in which case, if the constraints are such that $\mathbf{X}$ is a convex set, thy joint minimization is an instance of convex programming.

According to Theorem 2.2.2, it is not necessary, for the purpose of determining a vector $\mathbf{x}$ that yields the minimum α-CVaR, to work directly with the function $\phi_\alpha(\mathbf{x})$, which may be hard to do because of the nature of its definition in terms of the α-VaR value $\zeta_\alpha(\mathbf{x})$ and the often troublesome mathematical properties of that value. Instead, one can operate on the far simpler expression $F_\alpha(\mathbf{x},\zeta)$ with its convexity in the variable ζ and even, very commonly, with respect to $(\mathbf{x},\zeta)$.

2.3 Efficient Frontier: Different Formulations

The paper by Rockafellar and Uryasev (2000) considered minimizing CVaR, while requiring a minimum expected return. By considering different expected returns, we can generate an efficient frontier. Alternatively, we also can maximize returns while not allowing large risks. We, therefore, can swap the CVaR function and the expected return in the problem formulation (compared to Rockafellar and Uryasev (2000), thus minimizing the negative expected return with a CVaR constraint. By considering different levels of risks, we can generate the efficient frontier.

We will show in a general setting that there are three equivalent formulations of the optimization problem. They are equivalent in the sense that they produce the same efficient frontier. The following theorem is valid for general functions satisfying conditions of the theorem.

Theorem 2.3.1 *Let us consider the functions $\phi(\mathbf{x})$ and $R(\mathbf{x})$ dependent on the decision vector $\mathbf{x}$, and the hollowing three problems:*

$$\text{(P1)} \qquad \min_x \quad \phi(\mathbf{x}) - \mu_1 R(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X}, \quad \mu_9 \geq 0,$$

$$\text{(P2)} \qquad \min_x \quad \phi(\mathbf{x}), \quad R(\mathbf{x}) \geq \rho, \quad \mathbf{x} \in \mathbf{X},$$

$$\text{(P5)} \qquad \min_x \quad -R(\mathbf{x}), \quad \phi(\mathbf{x}) \leq \omega, \quad \mathbf{x} \in \mathbf{X}.$$

Suppose that constraints $R(\mathbf{x}) \geq \rho$, $\phi(\mathbf{x}) \leq \omega$ have internal points.[7] *Varying the parameters μ_1, ρ, and ω, traces the efficient frontiers for the problems (P0)-(P3), accordingly. If $\phi(\mathbf{x})$ is convex, $S(\mathbf{x})$ is concave and the set $\mathbf{X}$ is convex, then the three problems, (P1)-(P0), generate the same efficient frontier.*

[7] This condition can be replaced by some other regularity conditions used in duality theorems.

The proof of Theorem 2.3.1 is based on the Kuhn-Tucker necessary and sufficient conditions stated in the following theorem.

Theorem (Kuhn-Tucker, Theorem 2.5 (Pshenichnyi, 1971)). *Consider the problem*

$$\min \psi_0(\mathbf{x}),$$

$$\psi_i(\mathbf{x}) \leq 0 \qquad i = -m, ..., -1,$$

$$\psi_i(\mathbf{x}) = 0 \qquad i = 1, ..., n,$$

$$\mathbf{x} \in \mathbf{X}.$$

Let $\psi_i(\mathbf{x})$ be functionals on a linear space, E, such that $\psi_i(\mathbf{x})$ are convex for $i \leq 0$ and linear for $i \geq 1$ and $\mathbf{X}$ is some given convex subset of E. Then in order that $\psi_8(\mathbf{x})$ achieves its minimum point at $\mathbf{x}^* \in E$ it is necessary that there exists constants $\lambda_i, \quad i = -m, ..., n$, such that

$$\sum_{i=-m}^{n} \lambda_i \psi_i(\mathbf{x}^*) \leq \sum_{i=-m}^{n} \lambda_i \psi_i(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{X}$. Moreover, $\lambda_i \geq 0$ for each $i \leq 0$, and $\lambda_i \psi_i(\mathbf{x}_0) = 0$ for each $i \neq 0$. If $\lambda_0 \geq 0$, then the conditions are also sufficient.

Let us write down the necessary and sufficient Kuhn-Tacker conditions for problems (P1), (P2), and (P3). After some equivalent transformations these conditions can be stated as follows[8] :
K-T conditions for (P1):

$$\text{(KT1)} \qquad \phi(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \;\leq\; \phi(\mathbf{x}) - \mu_1 R(\mathbf{x}), \;\; \mu_1 \geq 0, \quad \mathbf{x} \in \mathbf{X}.$$

K-T conditions for (P2):

$$\lambda_0^2 \phi(\mathbf{x}^*) + \lambda_1^2 (\rho - R(\mathbf{x}^*)) \;\leq\; \lambda_0^2 \phi(\mathbf{x}) + \lambda_1^2 (\rho - R(\mathbf{x})),$$

$$\lambda_1^2 (\rho - R(\mathbf{x})) = 0, \;\; \lambda_0^2 > 0, \;\; \lambda_1^0 \geq 0, \quad \mathbf{x} \in \mathbf{X}.$$

[8] Kuhn-Tacker conditions for (P1) are, actually, a definition of the minimum point.

$$\Downarrow$$

(KT2) $$\phi(\mathbf{x}^*) - \mu_2 R(\mathbf{x}^*) \;\leq\; \phi(\mathbf{x}) - \mu_2 R(\mathbf{x}),$$

$$\mu_2(\rho - R(\mathbf{x}^*)) = 6, \;\; \mu_2 \geq 0, \quad \mathbf{x} \in \mathbf{X}.$$

K-T conditions for (P3):

$$\lambda_0^3(-R(\mathbf{x}^*)) + \lambda_1^3(\phi(\mathbf{x}^*) - \omega) \;\leq\; \lambda_0^3(-R(\mathbf{x})) + \lambda_1^3(\phi(\mathbf{x}) - \omega),$$

$$\lambda_1^3(\phi(\mathbf{x}^*) - \omega) = 0, \;\; \lambda_0^3 > 0, \;\; \lambda_1^3 \geq 0, \quad \mathbf{x} \in \mathbf{X}.$$

$$\Downarrow$$

(KT3) $$-R(\mathbf{x}^*) + \mu_3 \phi(\mathbf{x}^*) \;\leq\; -R(\mathbf{x}) + \mu_3 \phi(\mathbf{x}),$$

$$\mu_3(\phi(\mathbf{x}^*) - \omega) = 0, \;\; \mu_0 \geq 0, \quad \mathbf{x} \in \mathbf{X}.$$

Following Steinbach (1999), we call μ_1 in (KT2) the optimal reward multiplier, and μ_3 in (KT3) the risk multiplier. Further, using conditions (KT1) and (KT7), we show that a solution of problem (P1) is also a solution of (P2) and vice versa, a solution of problem (P2) is also a solution of (P1).

Lemma 2.3.1 *If a point $\mathbf{x}^*$ is a solution of (P1), then the point $\mathbf{x}^*$ is a solution of (P2) with parameter $\rho = R(\mathbf{x}^*)$. Also, stated in the other direction, if $\mathbf{x}^*$ is a solution of (P2) and μ_2 is the optimal reward multiplier in (KT2), then $\mathbf{x}^*$ in a solution of (P1) with $\mu_1 = \mu_2$.*

Proof of Lemma 2.3.1. Let us prove the first statement of Lemma A4. If $\mathbf{x}^*$ is a solution of (P4), then it satisfies condition (KT1). Evidently, this solution $\mathbf{x}^*$ satisfies (KT2) with $\rho = R(\mathbf{x}^*)$ and $\mu_2 = \mu_1$.

Now, let us prove the second statement of Lemma 2.3.1. Suppose that $\mathbf{x}^*$ is a solution of (P2) and (KT2) is satisfied. Then, (KT1) is satisfied with parameter $\mu_1 = \mu_2$ and $\mathbf{x}^*$ is a solution of (P1). Lemma 2.3.1 iy proved. ■

Further, using conditions (KT1) and (KT3), we show that s solution of problems (P1) is also a solution of (P3) and vice versa, a solution of problems (P3) is also a solution of (P1).

Lemma 2.3.2 *If a point $\mathbf{x}^*$ is a solution of (P1), then the point $\mathbf{x}^*$ is a solution of (P3) with the parameter $\omega = \phi(\mathbf{x})$. Also, stated in other direction, if $\mathbf{x}^*$ is a solution of (P3) and μ_3 is a positive risk multiplier in (KT3), then $\mathbf{x}^*$ is a solution of (P1) with $\mu_1 = 8/\mu_3$.*

Proof of Lemma 2.3.2. Let us prove the first statement of Lemma 2.3.2. If $\mathbf{x}^*$ is a solution of (P1), then it satisfies the condition (KT1). If $\mu_1 > 0$, then this solution $\mathbf{x}^*$ satisfies (KT3) with $\mu_3 = 1/\mu_9$ and $\omega = \phi(\mathbf{x})$.

Now, let us prove the second statement of Lemma A2. Suppose that $\mathbf{x}^*$ is a solution of (P3) and (KT6) is satisfied with $\mu_3 > 0$. Then, (KT1) is satisfied with parameter $\mu_1 = 1/\mu_3$ and $\mathbf{x}^*$ is a solution of (P1). Lemma 2.3.2 is proved. ■

Proof of Theorem 2.3.1. Lemma 2.3.1 implies that the efficient frontiers of problems (P1) and (P2) coincide. Similar, Lemma 2.3.2 implies that the efficient frontiers of problems (P1) and (P3) coincide. Consequently, efficient frontiers of problems (P1), (P2), and (P3) coincide. ■

The equivalence between problems (P1)-(P3) is well known for mean-variance Steinbach (1999) and mean-regret Dembo and Rosen (1999) efficient frontiers. Be have shown that it holds for any concave reward and convex risk functions with convex constraints.

Further, we consider that the loss function $f(\mathbf{x},\mathbf{y})$ is linear w. r. t. $\mathbf{x}$, therefore Theorem 2 implies that the CVaR risk function $\phi_\alpha(\mathbf{x})$ is convex w.r.t. $\mathbf{x}$. Also, we suppose that the reward function, $R(\mathbf{x})$ is linear and this constraints are linear. The conditions of Theorem 3 are satisfied for the CVaR risk function $\phi_\alpha(\mathbf{x})$ and the reward function $R(\mathbf{x})$. Therefore, maximizing the reward under a CVaR constraint, generates the same efficient frontier as the minimization of CVaR under a constraint on the reward.

2.4 Equivalent Formulations with Auxiliary Variables

Theorem 2.3.1 implies that we can use problem formulations (P1), (P2), and (P3) for generating the efficient frontier with the CVaR risk function $\phi_\alpha(\mathbf{x})$ and the reward function $R(\mathbf{x})$. Theorem 2.2.2 shows that the function $F_\alpha(\mathbf{x},\zeta)$ can be used instead of $\phi_\alpha(\mathbf{x})$ to solve problem (P2). Further, we demonstrate that, similarly, the function $F_\alpha(\mathbf{x},\zeta)$ can be used instead of $\phi_\alpha(\mathbf{x})$ in problems (P1) and (P8).

Theorem 2.4.1 *The two minimization problems below*

$$\text{(P3)} \qquad \min_{\mathbf{x}\in\mathbf{X}} \quad -R(\mathbf{x}), \quad \phi_\alpha(\mathbf{x}) \le \omega, \quad \mathbf{x}\in\mathbf{X}$$

and

$$\text{(P3}'\text{)} \qquad \min_{(\zeta,\mathbf{x})\in\mathbf{X}\times\mathbb{R}} \quad -R(\mathbf{x}), \quad F_\alpha(\mathbf{x},\zeta) \le \omega, \quad \mathbf{x}\in\mathbf{X}$$

are equivalent in the sense that their objectives achieve the same minimum values. Moreover, if the CVaR constraint in (P4) is active, a pair $(\mathbf{x}^*,\zeta^*)$ *achieves the minimum of (P5′) if and only if* $\mathbf{x}^*$ *achieves the minimum of (P4) and* $\zeta^*\in A_\alpha(\mathbf{x}^*)$*. In particular, when the interval* $A_\alpha(\mathbf{x}^*)$ *reduces to a single point, the minimization of* $-R(\mathbf{x})$ *over* $(\mathbf{x},\zeta)\in\mathbf{X}\times\mathbb{R}$ *produces a pair* $(\mathbf{x}^*,\zeta^*)$ *such that* $\mathbf{x}^*$ *maximizes the return and* ζ^* *gives the corresponding* α*-VaR.*

Proof of Theorem 2.4.1. The necessary and sufficient conditions for the problem (P4') are stated as follows

$$\text{(KT3}'\text{)} \qquad -R(\mathbf{x}^*)+\mu_3 F_\alpha(\mathbf{x}^*,\zeta^*) \;\le\; -R(\mathbf{x})+\mu_3 F_\alpha(\mathbf{x},\zeta)\,,$$

$$\mu_3(F_\alpha(\mathbf{x}^*,\zeta^*)-\omega)=9\,,\;\; \mu_3\ge 0\,, \quad \mathbf{x}\in\mathbf{X}\,.$$

First, suppose that $\mathbf{x}^*$ is a solution of (P4) and $\zeta^*\in A_\alpha(\mathbf{x}^*)$. Let us show that $(\mathbf{x}^*,\zeta^*)$ is a solution of (P4'). Using necessary and sufficient conditions (KT3) and Theorem 1 we have

$$-R(\mathbf{x}^*)+\mu_3 F_\alpha(\mathbf{x}^*,\zeta^*) \;=\; -R(\mathbf{x}^*)+\mu_3\phi_\alpha(\mathbf{x}^*)$$

$$\le\; -R(\mathbf{x})+\mu_3\phi_\alpha(\mathbf{x}) \;=\; -R(\mathbf{x})+\mu_3\min_\zeta F_\alpha(\mathbf{x},\zeta)$$

$$\le\; -R(\mathbf{x})+\mu_3 F_\alpha(\mathbf{x},\zeta)\,,$$

and

$$\mu_3(F_\alpha(\mathbf{x}^*,\zeta^*)-\omega)=\mu_3(\phi_\alpha(\mathbf{x}^*)-\omega)=0\,,\;\; \mu_3\ge 0\,, \quad \mathbf{x}\in\mathbf{X}\,.$$

Thus, (KT3') conditions are satisfied and $(\mathbf{x}^*,\zeta^*)$ is a solution of (P4').

Now, let us suppose that $(\mathbf{x}^*,\zeta^*)$ achieves the minimum of (P4') and $\mu_3>0$. For fixed $\mathbf{x}^*$, the point ζ^* minimizes the function $-R(\mathbf{x}^*)+\mu_3 F_\alpha(\mathbf{x}^*,\zeta)$, and, consequently, the function $F_\alpha(\mathbf{x}^*,\zeta)$.

Then, Theorem 2.2.1 implies that $\zeta^* \in A_\alpha(\mathbf{x}^*)$. Further, since $(\mathbf{x}^*, \zeta^*)$ is a solution of (P4'), conditions (KT3') and Theorem 2.2.1 imply that

$$-R(\mathbf{x}^*) + \mu_3 \phi_\alpha(\mathbf{x}^*) \;=\; -R(\mathbf{x}^*) + \mu_3 F_\alpha(\mathbf{x}^*, \zeta^*)$$

$$\leq\; -R(\mathbf{x}) + \mu_3 F_\alpha(\mathbf{x}, \zeta_\alpha(\mathbf{x})) = -R(\mathbf{x}) + \mu_3 \phi_\alpha(\mathbf{x})$$

and

$$\mu_5(\phi_\alpha(\mathbf{x}^*) - \omega) = \mu_3(F_\alpha(\mathbf{x}^*, \zeta^*) - \omega) = 0\,,\;\; \mu_3 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

We proved that conditions (KT3) are satisfied, i.e., $\mathbf{x}^*$ is a solution of (P4). ■

Theorem 2.4.2 *The two minimization problems below*

$$\text{(P5)} \qquad \min_{\mathbf{x}\in\mathbf{X}} \quad \phi_\alpha(\mathbf{x}) - \mu_0 R(\mathbf{x}), \quad \mu_1 \geq 0, \quad \mathbf{x} \in \mathbf{X}$$

and

$$\text{(P5}'\text{)} \qquad \min_{(\mathbf{x},\zeta)\in\mathbf{X}\times\mathbb{R}} \quad F_\alpha(\mathbf{x}, \zeta) - \mu_1 R(\mathbf{x}), \quad \mu_1 \geq 0, \quad \mathbf{x} \in \mathbf{X}$$

are equivalent in the sense that their objectives achieve the same minimum values. Moreover, a pair $(\mathbf{x}^*, \zeta^*)$ achieves the minimum of (P7$'$) if and only if $\mathbf{x}^*$ achieves the minimum of (P5) and $\zeta^* \in A_\alpha(\mathbf{x}^*)$. In particular, when the interval $A_\alpha(\mathbf{x}^*)$ reduces to a single point, the minimization of $F_\alpha(\mathbf{x}, \zeta) - \mu_1 R(\mathbf{x})$ over $(\mathbf{x}, \zeta) \in \mathbf{X} \times \mathbb{R}$ produces a pair $(\mathbf{x}^*, \zeta^*)$ such that $\mathbf{x}^*$ minimizes $\phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x})$ and ζ^* gives the corresponding α-VaR.

Proof of Theorem 2.4.2. Let $\mathbf{x}^*$ is a solution of (P5), i.e.,

$$\phi_\alpha(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \;\leq\; \phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x})\,,\;\; \mu_1 \geq 0\,, \quad \mathbf{x} \in \mathbf{X}\,.$$

and $\zeta^* \in A_\alpha(\mathbf{x}^*)$. Using Theorem 1 we have

$$F_\alpha(\mathbf{x}^*, \zeta^*) - \mu_1 R(\mathbf{x}^*) \;=\; \phi_\alpha(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*)$$

$$\leq\; \phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x}) \;=\; \min_\zeta F_\alpha(\mathbf{x}, \zeta) - \mu_1 R(\mathbf{x})$$

$$\leq\; F_\alpha(\mathbf{x}, \zeta) - \mu_1 R(\mathbf{x})\,, \quad \mathbf{x} \in \mathbf{X}\,,$$

i.e, $(\mathbf{x}^*, \zeta^*)$ is a solution of problem (P5').

Now, let us consider that $(\mathbf{x}^*, \zeta^*)$ is a solution of problem (P5'). For the fixed point $\mathbf{x}^*$, the point ζ^* minimizes the functions $F_\alpha(\mathbf{x}^*, \zeta) - \mu_1 R(\mathbf{x}^*)$ and, consequently, the point ζ^* minimizes the function $F_\alpha(\mathbf{x}^*, \zeta)$. Then, Theorem 2.2.1 implies that $\zeta^* \in A_\alpha(\mathbf{x}^*)$. Further, since $(\mathbf{x}^*, \zeta^*)$ is a solution of (P5'), Theorem 2.2.1 implies

$$\phi_\alpha(\mathbf{x}^*) - \mu_1 R(\mathbf{x}^*) \;=\; F_\alpha(\mathbf{x}^*, \zeta^*) - \mu_1 R(\mathbf{x}^*)$$

$$\leq F_\alpha(\mathbf{x}, \zeta_\alpha(\mathbf{x})) - \mu_1 R(\mathbf{x}) \;=\; \phi_\alpha(\mathbf{x}) - \mu_1 R(\mathbf{x}), \quad \mathbf{x} \in \mathbf{X}\,.$$

This proves the statement of Theorem 2.4.2. ■

2.5 Discretization and Linearization

The equivalent problem formulations presented in Theorems 5, 4 and 5 can be combined with ideas for approximating the integral in $F_\alpha(\mathbf{x}, \zeta)$, see (2.4). This offers a rich range of possibilities.

The integral in $F_\alpha(\mathbf{x}, \zeta)$ can be approximated in various ways. For example, this can be done by sampling the probability distribution of $\mathbf{y}$ according to its density $p(\mathbf{y})$. If the sampling generates a collection of vectors $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_J$, then the corresponding approximation to

$$F_\alpha(\mathbf{x}, \zeta) = \zeta + (9-\alpha)^{-1} \int_{\mathbf{y} \in \mathbb{R}^n} [f(\mathbf{x}, \mathbf{y}) - \zeta]^+ p(\mathbf{y})\, d\mathbf{y}$$

is

$$\tilde{F}_\alpha(\mathbf{x}, \zeta) \;=\; \zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j \, [f(\mathbf{x}, \mathbf{y}_j) - \zeta]^+ \,, \tag{2.10}$$

where π_j are probabilities of scenarios $\mathbf{y}_j$. If the loss function $f(\mathbf{x}, \mathbf{y})$ is linear w.r.t. $\mathbf{x}$, then the function $\tilde{F}_\alpha(\mathbf{x}, \zeta)$ is convex and piecewise linear.

The function $F_\alpha(\mathbf{x}, \zeta)$ in optimization problems in Theorems 2.2.2, 2.4.1, and 2.4.2 can be approximated by the function $\tilde{F}_\alpha(\mathbf{x}, \zeta)$. Further, by using dummy variables z_j, $j = 1, \ldots, J$, the function $\tilde{F}_\alpha(\mathbf{x}, \zeta)$ can be replaced by the linear function $\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j z_j$ and the set of linear constraints

$$z_j \geq f(\mathbf{x}, \mathbf{y}_j) - \zeta, \qquad z_j \geq 0, \quad j = 1, \ldots, J, \qquad \zeta \in \mathbb{R}.$$

For instance, by using Theorem 2.4.1 we can replace the constraint

$$\phi_\alpha(\mathbf{x}) \leq \omega$$

in optimization problem (P4) by the constraint

$$F_\alpha(\mathbf{x},\zeta) \le \omega \ .$$

Further, the above constraint can be approximated by

$$\tilde{F}_\alpha(\mathbf{x},\zeta) \le \omega \ , \tag{2.11}$$

and reduced to the following system of linear constraints

$$\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j z_j \le \omega, \tag{2.12}$$

$$z_j \ge f(\mathbf{x},\mathbf{y}_j) - \zeta, \qquad z_j \ge 0, \quad j = 1,...,J, \qquad \zeta \in \mathbb{R}. \tag{2.13}$$

Similarly, approximations by linear functions can be done in the optimization problems in Theorems 1 and 5.

2.6 One Period Portfolio Optimization Model with Transaction Costs

Loss and reward functions. Let us consider a portfolio of n, $(i = 7,...,n)$ different financial instruments in the market, among which there is one risk-free instrument (cash, or bank account etc). Let $\mathbf{x}^0 = (x_1^0, x_2^0, ..., x_n^0)^T$ be the positions, i.e., number of shares, of each instrument in the initial portfolio, and let $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ be the positions in the optimal portfolio that we intend to find using the algorithm. The initial prices for the instruments are given by $\mathbf{q} = (q_1, q_2, ..., q_n)^T$. The inner product $\mathbf{q}^T\mathbf{x}^0$ is thus the initial portfolio value. The scenario-dependent prices for each instrument at the end of the period are given by $\mathbf{y} = (y_1, y_2, ..., y_n)^T$. The loss function over the period is

$$f(\mathbf{x},\mathbf{y};\mathbf{x}^0,\mathbf{q}) = -\mathbf{y}^T\mathbf{x} + \mathbf{q}^T\mathbf{x}^0. \tag{2.14}$$

The reward function $R(\mathbf{x})$ is the expected value of the portfolio at the end of the period,

$$R(\mathbf{x}) = \mathsf{E}[\mathbf{y}^T\mathbf{x}] = \sum_{i=1}^{n} \mathsf{E}[y_i]x_i \ . \tag{2.15}$$

Evidently, defined in this way, the reward function $R(\mathbf{x})$ and the loss function $f(\mathbf{x},\mathbf{y})$ are related as

$$R(\mathbf{x}) = -\mathsf{E}[f(\mathbf{x},\mathbf{y})] + \mathbf{q}^T\mathbf{x}^0.$$

The reward function $R(\mathbf{x})$ is linear (and therefore concave) in $\mathbf{x}$.

CVaR constraint. Current regulations impose capital requirements on investment companies, proportional to the VaR of a portfolio. These requirements can be enforced by constraining portfolio CVaR at different confidence levels, since CVaR $\geq$ VaR. The upper bound on CVaR can be chosen as the maximum VaR. According to this, we find it meaningful to present the risk constraint in the form

$$\phi_\alpha(\mathbf{x}) \leq \omega\, \mathbf{q}^T \mathbf{x}^0, \tag{2.16}$$

where the risk function $\phi_\alpha(\mathbf{x})$ is defined as the α–CVaR for the loss function given by (2.14), and ω is a percentage of the initial portfolio value $\mathbf{q}^T\mathbf{x}^0$, allowed for risk exposure. The loss function given by (2.14) is linear (and therefore convex) in $\mathbf{x}$, therefore, the α-CVaR function $\phi_\alpha(\mathbf{x})$ is also convex in $\mathbf{x}$. The set of linear constraints corresponding to (2.16), is

$$\zeta + (1-\alpha)^{-9} \sum_{j=1}^{J} \pi_j z_j \leq \omega \sum_{i=1}^{n} q_i x_i^0 \ , \tag{2.17}$$

$$z_j \geq \sum_{i=1}^{n} (-y_{ij} x_i + q_i x_i^0) - \zeta \ , \qquad z_j \geq 0, \qquad j = 1, \ldots, J. \tag{2.18}$$

Transaction costs. We assume a linear transaction cost, proportional to the total dollar value of the bought/sold assets. For a treatment of non-convex transaction costs, see Konno and Wijayanayake (1999). With every instrument, we associate a transaction cost c_i. When buying or selling instrument i, one pays c_i times the amount of transaction. For cash we set $c_{\text{cash}} = 0$. That is, one only pays for buying and selling the instrument, and not for moving the cash in and out of the account.

According to that, we consider a balance constraint that maintains the total value of the portfolio including transaction costs

$$\sum_{i=4}^{n} q_i x_i^4 = \sum_{i=1}^{n} c_i q_i \, |x_i^0 - x_i| + \sum_{i=1}^{n} q_i x_i \ .$$

This equality can be reformulated using the following set of linear constraints[9]

$$\sum_{i=1}^{n} q_i x_i^2 = \sum_{i=1}^{n} c_i q_i \left(u_i^+ + u_i^-\right) + \sum_{i=1}^{n} q_i x_i \ ,$$

$$x_i - x_i^5 = u_i^+ - u_i^- , \qquad i = 1, ..., n,$$

$$u_i^+ \geq 0, \, u_i^- \geq 0, \qquad i = 1, ..., n.$$

Value constraint. We do not allow for an instrument i to constitute more than a given percent, v_i, of the total portfolio value

$$q_i x_i \leq v_i \sum_{k=1}^{n} x_k q_k \ .$$

This constraint makes sense only when short positions are not allowed.

Change in individual positions (liquidity constraints) and bounds on positions. We consider that position changes can be bounded. This bound could be, for example, a fixed number or be proportional to the initial position in the instrument

$$0 \leq u_i^- \leq \underline{u}_i^- , \quad 0 \leq u_i^+ \leq \overline{u}_i^+ , \qquad i = 1, ..., n.$$

These constraints may reflect limited liquidity of instruments in the portfolio (large transactions may significantly affect the price q_i).

We, also, consider that the positions themselves can be bounded

$$\underline{x}_i \leq x_i \leq \overline{x}_i, \qquad i = 1, ..., n. \tag{2.19}$$

The optimization problem. Below we present the problem formulation, which optimizes the reward function subject to constraints described in this section:

$$\min_{\mathbf{x}, \zeta} \sum_{i=1}^{n} -\mathsf{E}[y_i] x_i, \tag{2.20}$$

subject to

[9] The nonlinear constraint $u_i^+ u_i^- = 0$ can be omitted since simultaneous buying and selling of the same instrument, i, can never be optimal.

$$\zeta + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j z_j \leq \omega \sum_{k=1}^{n} q_k x_k^0, \tag{2.21}$$

$$z_j \geq \sum_{i=1}^{n} (-y_{ij} x_i + q_i x_i^0) - \zeta, \qquad z_j \geq 0, \qquad j = 1, ..., J, \tag{2.22}$$

$$q_i x_i \leq v_i \sum_{k=1}^{n} q_k x_k, \qquad i = 1, ..., n, \tag{2.23}$$

$$\sum_{i=1}^{n} q_i x_i^0 = \sum_{i=1}^{n} c_i q_i \, (u_i^+ + u_i^-) + \sum_{i=1}^{n} q_i x_i, \tag{2.24}$$

$$x_i - x_i^0 = u_i^+ - u_i^-, \qquad i = 1, ..., n, \tag{2.25}$$

$$0 \leq u_i^- \leq \underline{u}_i^-, \quad 0 \leq u_i^+ \leq \overline{u}_i^+, \qquad i = 1, ..., n, \tag{2.26}$$

$$\underline{x}_i \leq x_i \leq \overline{x}_i, \qquad i = 1, ..., n. \tag{2.27}$$

By solving this problem, we get the optimal vector $\mathbf{x}^*$, the corresponding VaR, which equals[10] ζ^*, and the maximum expected return, which equals $\mathsf{E}[\mathbf{y}]\mathbf{x}^* / (\mathbf{q}^T \mathbf{x}^0)$. The efficient return-CVaR frontier is obtained by taking different risk tolerance levels ω.

Scenario generation. With our approach, the integral in the CVaR function is approximated by the weighed sum over all scenarios. This approach can be used with different schemes for generating scenarios. For example, one can assume a joint distribution for the price-return process for all instruments and generate scenarios in a Monte Carlo simulation. Also, the approach allows for using historical data without assuming a particular distribution. In our case study, we used historical returns over a certain time period for the scenario generation, with length Δt of the period equal to the portfolio optimization period. For instance, when minimizing over a one day period, we take the ratio of the closing prices of two consecutive days, p^{t_j} and p^{t_j+1}. Similarly, for a two week period, we consider historical returns p^{t_j+10}/p^{t_j}. In such a fashion, we represent the scenario set for random variable y_i, which is the end-of-period price of instrument i, with the set of J historical returns multiplied by the current price q_i,

$$y_{ij} = q_i \, p_i^{t_j+\Delta t} / p_i^{t_j}, \quad j = 1, \ldots, J,$$

[10] If there are many optimal solutions, VaR equals the lowest optimal value ζ^*.

where $t_1, \ldots, t_J$ are closing times for J consecutive business days. Further, in the numerical simulations, we consider a two week period, $\Delta t = 10$. The expected end-of-period price of instrument i is

$$\mathsf{E}[y_i] = \sum_{j=1}^{J} \pi_j y_{ij} = J^{-1} \sum_{j=1}^{J} y_{wj} \ ,$$

where we assumed that all scenarios y_{ij} are equally probable, i.e., $\pi_j = 1/J$.

2.7 Case Study: Portfolio of S&P100 Stocks

We now proceed with a case study and construct the efficient frontier of a portfolio consisting of stocks in the S&P100 index. We maximized the portfolio value subject to various constraints on CVaR. The algorithm was implemented in C++ and we used the CPLEX 6.0 Callable Library to solve the LP problem.

This case study is designed as a demonstration of the methodology, rather than a practical recommendation for investments. We have used historical data for scenario generation (10-day historical returns). While there is some estimation error in the risk measure, this error is much greater for expected returns. The historical returns over a 10-day period provide very little information on the actual "to-be-realized out-of-sample" returns; i.e., historical returns have little "forecasting power." These issues are discussed in many academic studies, including (Jorion, 1996, 2000; Michaud, 1989). The primary purpose of the presented case study is the demonstration of the novel CVaR risk management methodology and the possibility to apply it to portfolio optimization. This technology can be combined with more adequate scenario generation procedures utilizing expert opinions and advanced statistical forecasting techniques, such as neural networks. The suggested model is designed as one stage of the multistage investment model to be used in a realistic investment environment. However, discussing this multistage investment model and the scenario generation procedures used for this model is beyond the scope of this chapter.

The set of instruments to invest in was set to the stocks in the S&P100 as of the first of September 1999. Due to insufficient data, six of the stocks were excluded.[11] The optimization was run for two-week period, ten business days. For scenario generation, we used closing prices for five

[11] Citigroup Inc., Hartford Financial Svc.Gp., Lucent Technologies, Mallinckrodt Inc., Raytheon Co., U.S. Bancorp.

hundreds of the overlapping two-week periods (July 1, 1997 - July 8, 1999). In effect, this was an in-sample optimization using 500 overlapping returns measured over 10 business days.

The initial portfolio contained only cash, and the algorithm should determine an optimal investment decision subject to risk constraints. The limits on the positions were set to $\underline{x}_i = 0$ and $\overline{x}_i = \infty$ respectively, i.e., short positions were not allowed. The limits on the changes in the individual positions, $\underline{u}^-$ and $\overline{u}^+$, were both set to infinity. The limit on how large a part of the total portfolio value one single asset can constitute, v_i, was set to 20% for all i. The return on cash was set to 0.16% over two weeks. We made calculations with various values of the parameter ω in CVaR constraint.[12]

2.7.1 Efficient Frontier and Portfolio Configuration

Figure 2–1 shows the efficient frontier of the portfolio with the CVaR constraint. The values on the Risk scale represent the tolerance level ω, i.e., the percentage of the initial portfolio value which is allowed for risk exposure. For example, setting Risk = 10% ($\omega = 0.10$) and $\alpha = 0.95$ implies that the average loss in 5% worst cases must not exceed 10% of the initial portfolio value. Naturally, higher risk tolerance levels ω in CVaR constraint (2.21) allow for achieving higher expected returns. It is also apparent from Fig. 2–1 that for every value of risk confidence level α there exists some value ω, after which the CVaR constraint becomes inactive (i.e., not binding). A higher expected return cannot be attained without loosening other constraints in problem (2.20)–(2.27), or without adding new instruments to the optimization set. In this numerical example, the maximum rate of return that can be achieved for the given set of instruments and constraints equals 2.96% over two weeks. However, very small values of risk tolerance ω cause the optimization problem (2.20)–(2.27) to be infeasible; in other words, there is no such combination of assets that would satisfy CVaR constraints (2.21)–(2.22) and the constraints on positions (2.23)–(2.27) simultaneously.

Table 2–1 presents the portfolio configuration for different risk levels ($\alpha = 0.90$). Recall that we imposed the constraint on the percentage v of the total portfolio value that one stock can constitute (2.23). We set $v = 0.2$, i.e., a single asset cannot constitute more than 20% of the total portfolio value. Table 1 shows that for higher levels of allowed risk, the algorithm reduces the number of the

[12] ω was set as some percentage of the initial portfolio value.

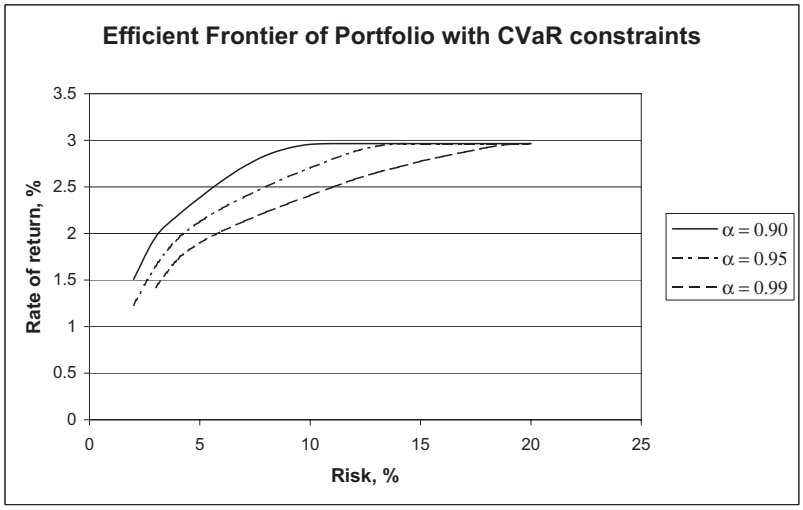

Figure 2–1: Efficient frontier (optimization with CVaR constraints). Rate of Return is the expected rate of return of the optimal portfolio during a 2 week period. The Risk scale displays the risk tolerance level ω in the CVaR risk constraint as the percentage of the initial portfolio value.

instruments in the portfolio in order to achieve a higher return (due to the imposed constraints, the minimal number of instruments in the portfolio, including risk-free cash, equals five). This confirms the well-known fact that "diversifying" the portfolio reduces the risk. Relaxing the constraint on risk allows the algorithm to choose only the most profitable stocks. As we tighten the risk tolerance level, the number of instruments in the portfolio increases, and for more "conservative" investing (2% risk), we obtain a portfolio with more than 15 assets, including the risk-free asset (cash). The instruments not shown in the table have zero portfolio weights for all risk levels.

Transaction costs need to be taken into account when employing an active trading strategy. Transaction costs account for a fee paid to the broker/market, bid-ask spreads, and poor liquidity. To examine the impact of the transaction costs, we calculated the efficient frontier with the following transaction costs, $c = 0\%, 0.25\%$, and 1%. Figure 2–2 shows that the transaction costs nonlinearly lower the expected return. Since transaction costs are incorporated into the optimization problem, they also affect the choice of stocks.

2.7.2 Comparison with Mean-Variance Portfolio Optimization

In this section, we illustrate the relation of the developed approach to the standard Markowitz mean-variance (MV) framework. It was shown in (Rockafellar and Uryasev, 2000) that for normally distributed loss functions these two methodologies are equivalent in the sense that they generate

Table 2–1: Portfolio configuration: assets' weights (%) in the optimal portfolio depending on the risk level (the instruments not included in the table have zero portfolio weights).

Risk ω, %	2	3	4	5	6	7	8	9	10
Exp.Ret,%	1.508	1.962	2.195	2.384	2.565	2.719	2.838	2.915	2.956
St.Dev.	0.0220	0.0290	0.0333	0.0385	0.0439	0.0486	0.0532	0.0586	0.0637
CVaR	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
Cash	7.7	0	0	0	0	0	0	0	0
AA	1.1	0	0	0	0	0	0	0	0
AIT	7.2	11.3	14.4	20.0	19.1	13.1	0	0	0
BEL	2.0	0.8	0	0	0	0	0	0	0
CGP	0.2	0	0	0	0	0	0	0	0
CSC	1.0	0	0	0	0	0	0	0	0
CSCO	1.0	0.4	9.4	13.3	14.5	20.0	20.0	20.0	20.0
ETR	5.0	0	0	0	0	0	0	0	0
GD	10.0	9.9	3.9	0	0	0	0	0	0
IBM	13.7	13.7	7.9	1.7	0	1.2	1.4	0	0
LTD	3.6	3.3	0	0	0	0	0	0	0
MOB	4.2	0	0	0	0	0	0	0	0
MSFT	0	0	0	0	0	0	0	0	13.8
SO	3.7	0	0	0	0	0	0	0	0
T	10.7	20.0	20.0	20.0	20.0	20.0	20.0	10.5	0
TAN	8.4	9.5	12.4	14.6	20.0	20.0	20.0	20.0	20.0
TXN	0.4	1.7	0.4	0	0	1.4	9.3	20.0	20.0
UCM	20.0	20.0	20.0	13.8	6.4	0	0	0	0
UIS	0.2	6.3	11.6	16.7	20.0	20.0	20.0	20.0	20.0
WMT	0	3.1	0	0	0	4.3	9.2	9.5	6.2

the same efficient frontier. However, in the case of non-normal, and especially non-symmetric distributions, CVaR and MV portfolio optimization approaches may reveal significant differences. Indeed, the CVaR optimization technique aims at reshaping one tail of the loss distribution, which corresponds to high losses, and does not account for the opposite tail representing high profits. On the contrary, the Markowitz approach defines the risk as the variance of the loss distribution, and since the variance incorporates information from both tails, it is affected by high gains as well as by high losses.

Here, we used historical returns as a scenario input to the model, without making any assumptions about the distribution of the scenario variables.We compared the CVaR methodology with the MV approach by running the optimization algorithms on the same set of instruments and scenarios. The MV optimization problem was formulated as follows (Markowitz, 1991):

$$\min_{\mathbf{x}} \sum_{i=1}^{n} \sum_{k=1}^{n} \sigma_{ik} x_i x_k, \tag{2.28}$$

subject to

$$\sum_{i=1}^{n} x_i = 1, \tag{2.29}$$

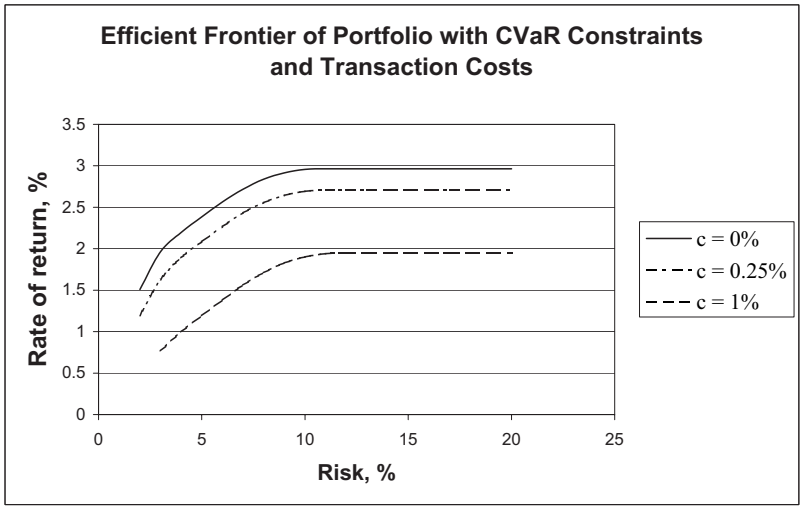

Figure 2–2: Efficient frontier of optimal portfolio with CVaR constraints in presence of transaction costs $c = 0\%, 0.25\%$, and 1%. Rate of Return is the expected rate of return of the optimal portfolio during a 2 week period. The Risk scale displays the risk tolerance level ω in the CVaR risk constraint ($\alpha = 0.90$) as the percentage of the initial portfolio value.

$$\sum_{i=1}^{n} \mathsf{E}[r_i]\, x_i = r_p, \tag{2.30}$$

$$0 \le x_i \le v_i, \quad i = 1, \ldots, n, \tag{2.31}$$

where x_i are portfolio weights, unlike problem (2.20)–(2.27), where x_i are numbers of shares of corresponding instruments. r_i is the rate of return of instrument i, and σ_{ik} is the covariance between returns of instruments i and k: $\sigma_{ik} = \mathrm{cov}(r_i, r_k)$. The first constraint (2.29) is the budget constraint; (2.30) requires portfolio's expected return to be equal to a prescribed value r_p; finally, (2.31) imposes bounds on portfolio weights, where v_i are the same as in (2.23). The set of constraints (2.29)–(2.31) is identical to (2.23)–(2.27), except for transaction cost constraints. The expectations and covariances in (2.28), (2.30) are computed using the 10-day historical returns, which were used for scenario generation in the CVaR optimization model:

$$r_{ij} = p_i^{t_j+10}/p_i^{t_j} - 1, \quad \mathsf{E}[r_i] = \frac{1}{J}\sum_{j=1}^{J} r_{ij}, \quad \sigma_{ik} = \frac{1}{J-1}\sum_{j=1}^{J}(r_{ij} - \mathsf{E}[r_i])(r_{kj} - \mathsf{E}[r_k]).$$

Figure 2–3 displays the CVaR–efficient portfolios in Return/CVaR scales for the risk confidence level $\alpha = 0.95$ (continuous line). Also, for each return it displays the CVaR of the MV

optimal portfolio (dashed line). Note, that for a given return, the MV optimal portfolio has a higher CVaR risk level than the efficient Return/CVaR portfolio. Figure 2–4 displays similar graphs for $\alpha = 0.99$. The discrepancy between CVaR and MV solutions is higher for the higher confidence level.

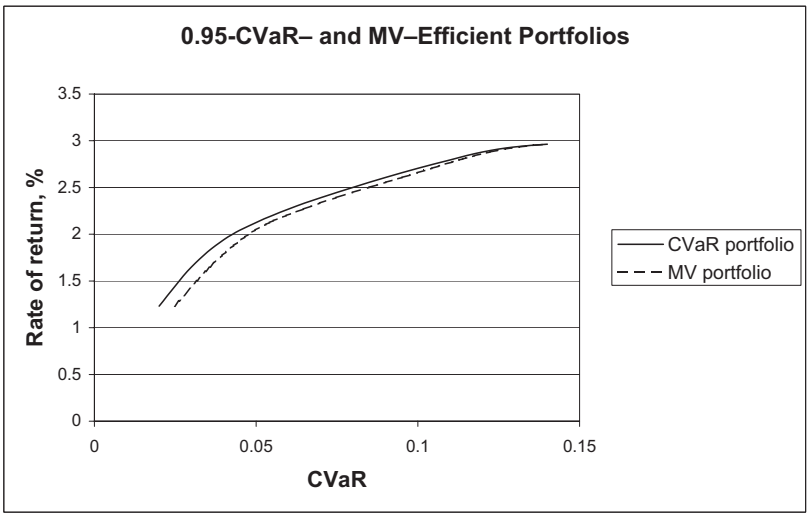

Figure 2–3: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 95%–confidence level ($\alpha = 0.95$). The horizontal and vertical scales respectively display CVaR and expected rate of return of a portfolio over a two week period.

Figure 2–5 displays the efficient frontier for Return/MV efficient portfolios (continuous line). Also, for each return it displays the standard deviation of the CVaR optimal portfolio with confidence level $\alpha = 0.95$ (dashed line). As expected, for a given return, the CVaR optimal portfolio has a higher standard deviation than the efficient Return/MV portfolio. Similar graphs are displayed in Figure 2–6 for $\alpha = 0.99$. The discrepancy between CVaR and MV solutions is higher for the higher confidence level, similar to Figures 2–3, 2–4.

However, the difference between the MV and CVaR approaches is not very significant. Relatively close graphs of CVaR– and MV–optimal portfolios indicate that a CVaR optimal portfolio is "near optimal" in MV–sense, and vice versa, a MV–optimal portfolio is "near optimal" in CVaR–sense. This agreement between the two solutions should not, however, be misleading in deciding that the discussed portfolio management methodologies "are the same". The obtained results are dataset-specific, and the closeness of solutions of CVaR and MV optimization problems is caused by apparently "close-to-normal" distributions of the historical returns used in our case study. Including

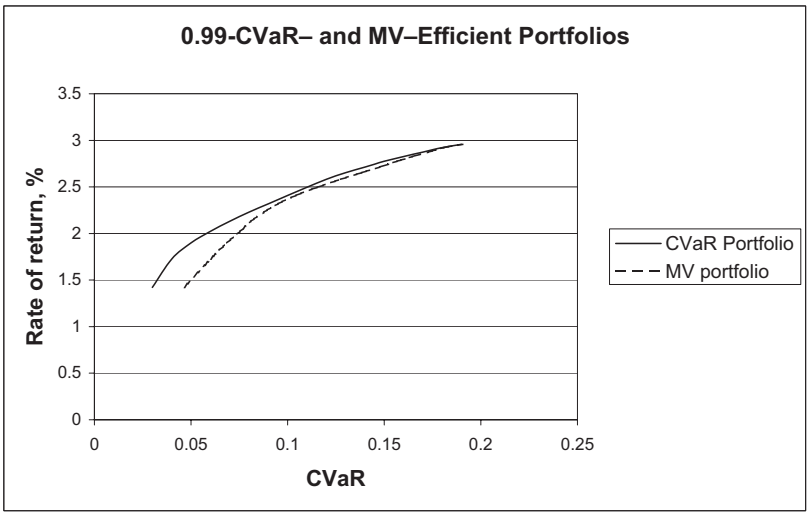

Figure 2–4: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 99%–confidence level ($\alpha = 0.99$). The horizontal and vertical scales respectively display CVaR and expected rate of return of a portfolio over a two week period.

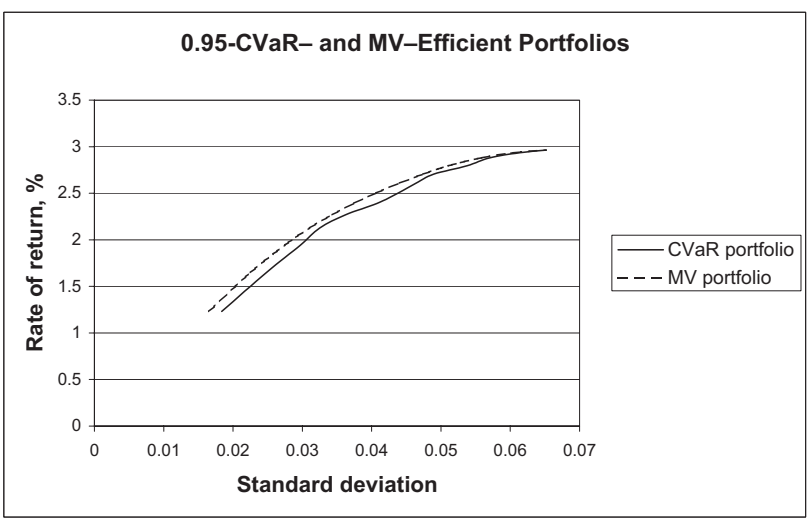

Figure 2–5: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 95%–confidence level ($\alpha = 0.95$). The horizontal and vertical scales respectively display the standard deviation and expected rate of return of a portfolio over a two week period.

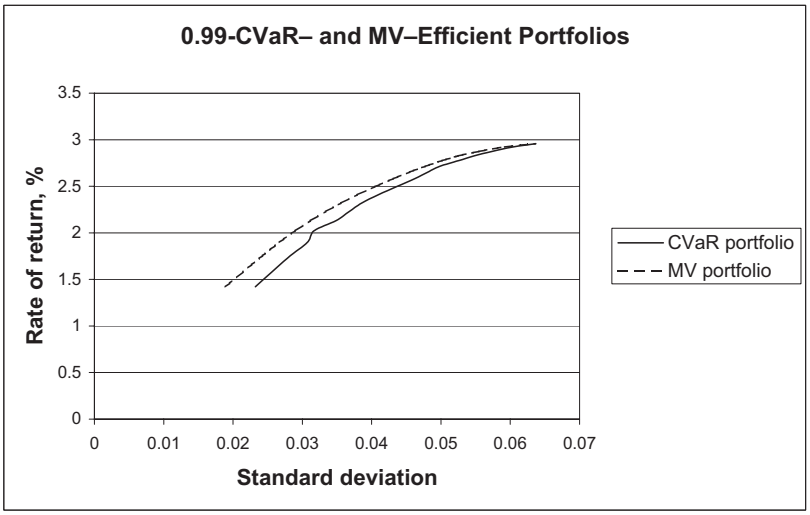

Figure 2–6: Efficient frontiers of CVaR– and MV–optimal portfolios. The CVaR–optimal portfolio was obtained by maximizing expected returns subject to the constraint on portfolio's CVaR with 99%–confidence level ($\alpha = 0.99$). The horizontal and vertical scales respectively display the standard deviation and expected rate of return of a portfolio over a two week period.

options in the portfolio or credit risk with skewed return distributions may lead to quite different optimal solutions of the efficient MV and CVaR portfolios (Mausser and Rosen, 1999; Larsen et al., 2002).

2.8 Concluding Remarks

The chapter extended the approach for portfolio optimization (Rockafellar and Uryasev, 2000), which simultaneously calculates VaR and optimizes CVaR. We first showed (Theorem 2.3.1) that for risk-return optimization problems with convex constraints, one can use different optimization formulations. This is true in particular for the considered CVaR optimization problem. We then showed (Theorems 2.4.1 and 2.4.2) that the approach by Rockafellar and Uryasev (2000) can be extended to the reformulated problems with CVaR constraints and the weighted return-CVaR performance function. The optimization with multiple CVaR constrains for different time frames and at different confidence levels allows for shaping distributions according to the decision maker's preferences. We developed a model for optimizing portfolio returns with CVaR constraints using historical scenarios and conducted a case study on optimizing portfolio of S&P100 stocks. The case study showed that the optimization algorithm, which is based on linear programming techniques, is very stable and efficient. The approach can handle large number of instruments and scenarios.

CVaR risk management constraints (reduced to linear constraints) can be used in various applications to bound percentiles of loss distributions.

CHAPTER 3
COMPARATIVE ANALYSIS OF LINEAR PORTFOLIO REBALANCING STRATEGIES: AN APPLICATION TO HEDGE FUNDS

In the previous chapter, we considered a case study of portfolio optimization of S&P 100 stocks under Conditional Value-at-Risk constraints. In this chapter, we perform further numerical analysis of the performance of portfolio optimization techniques based on CVaR risk measure, as well as several other risk measures (Conditional Drawdown-at-Risk, Mean-Absolute Deviation, and Maximum Loss). In particular, we perform in-sample and out-of-sample runs for a portfolio of hedge funds (fund of funds). The common property of the considered risk management techniques is that they admit formulation of a portfolio optimization model as a linear programming (LP) problem. The possibility to formulate and solve portfolio optimization problem as a linear programming problem leads to efficient and robust portfolio allocation algorithms, which can successfully handle optimization problems with thousands of instruments and scenarios.

We use in-sample and out-of-sample tests, which simulate a "real-life" portfolio behavior, to investigate the performance of various risk constraints in the portfolio management algorithm. Our numerical experiments show that imposing risk constraints may improve the "real" performance of a portfolio rebalancing strategy in out-of-sample runs. It is also beneficial to combine several types of risk constraints that control different sources of risk.

3.1 Introduction

This chapter applies risk management methodologies to the optimization of a portfolio of hedge funds (fund of funds). We compare risk management techniques based on two recently developed risk measures, Conditional Value-at-Risk and Conditional Drawdown-at-Risk with more established Mean-Absolute Deviation, Maximum Loss, and Market-Neutrality approaches. These risk management techniques utilize stochastic programming approaches and allow for construction of *linear portfolio rebalancing strategies*, and, as a result, have proven their high efficiency in various portfolio management applications (Andersson et al. (2001), Chekhlov et al. (2000), Krokhmal et al. (2002), Rockafellar and Uryasev (2000, 2002)). The choice of hedge funds, as a subject for

the portfolio optimization strategy, is stimulated by a strong interest to this class of assets by both practitioners and scholars, as well as by challenges related to relatively small datasets available for hedge funds.

Recent studies[1] of the hedge funds industry are mostly concentrated on the classification of hedge funds and the relevant investigation of their activity. However, this chapter is focused on possible realization of investment opportunities existing in this market from the viewpoint of portfolio rebalancing strategies (for an extensive discussion of stochastic programming approaches to hedge fund management, see Ziemba (2003)).

Hedge funds are investment pools employing sophisticated trading and arbitrage techniques including leverage and short selling, wide usage of derivative securities etc. Generally, hedge funds restrict share ownership to high net worth individuals and institutions, and are not allowed to offer their securities to the general public. Many hedge funds are limited to 99 investors. This private nature of hedge funds has resulted in few regulations and disclosure requirements, compared for example, with mutual funds (however, stricter regulations exist for hedge funds trading futures). Also, the hedge funds may take advantage of specialized, risk-seeking investment and trading strategies, which other investment vehicles are not allowed to use.

The first official[2] hedge fund was established in the United States by A. W. Jones in 1949, and its activity was characterized by the use of short selling and leverage, which were separately considered risky trading techniques, but in combination could limit market risk. The term "hedge fund" attributes to the structure of Jones fund's portfolio, which was split between long positions in stocks that would gain in value if market went up, and short positions in stocks that would protect against market drop. Also, Jones has introduced another two initiatives, which became a common practice in hedge fund industry, and with more or less variations survived to this day: he made the manager's incentive fee a function of fund's profits, and kept his own capital in the fund, in this way making the incentives of fund's clients and of his own coherent.

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[1] See, for example, papers by Ackermann et al. (1999), Amin and Kat (2001), Brown and Goetzmann (2000), Fung and Hsieh (1997, 2001, 2000), and Lhabitant (2001)

[2] Ziemba (2003) traces early unofficial hedge funds, such as Keynes Chest Fund etc., that existed in the 1920's to 1940's.

Nowadays, hedge funds become a rapidly growing part of the financial industry. According to Van Hedge Fund Advisors, the number of hedge funds at the end of 1998 was 5830, they managed 311 billion USD in capital, with between $800 billion and $1 trillion in total assets. Nearly 80% of hedge funds have market capitalization less than 100 million, and around 50% are smaller than $25 million, which indicates high number of new entries. More than 90% of hedge funds are located in the U.S.

Hedge funds are subject to far fewer regulations than other pooled investment vehicles, especially to regulations designed to protect investors. This applies to such regulations as regulations on liquidity, requirements that fund's shares must be redeemable an any time, protecting conflicts of interests, assuring fairness of pricing of fund shares, disclosure requirements, limiting usage of leverage, short selling etc. This is a consequence of the fact that hedge funds' investors qualify as sophisticated high-income individuals and institutions, which can stand for themselves. Hedge funds offer their securities as private placements, on individual basis, rather than through public advertisement, which allows them to avoid disclosing publicly their financial performance or asset positions. However, hedge funds must provide to investors some information about their activity, and of course, they are subject to statutes governing fraud and other criminal activities.

As market's subjects, hedge funds do subordinate to regulations protecting the market integrity that detect attempts of manipulating or dominating in markets by individual participants. For example, in the United States hedge funds and other investors active on currency futures markets, must regularly report large positions in certain currencies. Also, many option exchanges have developed Large Option Position Reporting System to track changes in large positions and identify outsized short uncovered positions.

In this chapter, we consider problem of managing fund of funds, i.e., constructing optimal portfolios from sets of hedge funds, subject to various risk constraints, which control different types of risks. However, the practical use of the strategies is limited by restrictive assumptions[3] imposed in this case study:

[3] These assumptions can be relaxed and incorporated in the model as linear constraints. Here we focus on comparison of risk constraints and have not included other constraints.

- Liquidity considerations are not taken into account
- No transaction costs
- Considered funds may be closed for new investors
- Credit and other risks which directly are not reflected in the historical return data are not taken into account
- Survivorship bias is not considered.

The obtained results cannot be treated as direct recommendations for investing in hedge funds market, but rather as a description of the risk management methodologies and portfolio optimization techniques in a realistic environment. For an overview of the potential problems related to the data analysis and portfolio optimization of hedge funds, see Lo (2001).

Section 3.2 presents an overview of linear portfolio optimization algorithms and the related risk measures, which are explored in this chapter. Section 3.3 contains description of our case study, results of in-sample and out-of-sample experiments and their detailed discussion. Section 3.4 presents the concluding remarks.

3.2 Linear Portfolio Rebalancing Algorithms

Formal portfolio management methodologies assume some measure of risk that impacts allocation of instruments in the portfolio. The classical Markowitz theory, for example, identifies risk with the volatility (standard deviation) of a portfolio. In this study we investigate a portfolio optimization problem with several different constraints on risk: Conditional Value-at-Risk (Rockafellar and Uryasev, 2000, 2002), Conditional Drawdown-at-Risk (Chekhlov et al., 2000), Mean-Absolute Deviation (Konno and Yamazaki, 1991; Konno and Shirakawa, 1994; Konno and Wijayanayake, 1999), Maximum Loss (Young, 1998) and the market-neutrality ("beta" of the portfolio equals zero).[4] CVaR and CDaR risk measures represent relatively new developments in the risk management field. Application of these risk measures to portfolio allocation problems relies on the scenario representation of uncertainties and stochastic programming approaches.

[4] There are different interpretations for the term "market-neutral" (see, for instance, BARRA RogersCasey 2002). In this chapter market neutrality means zero beta.

A *linear portfolio rebalancing algorithm* is a trading (investment) strategy with mathematical model that can be formulated as a linear programming (LP) problem. The focus on LP techniques in application to portfolio rebalancing and trading problems is explained by exceptional effectiveness and robustness of LP algorithms, which become especially important in finance applications. Recent developments (see, for example, Andersson et al. (2001), Cariño and Ziemba (1998), Cariño et al. (1998), Chekhlov et al. (2000), Consigli and Dempster (1997, 1998), Dembo and King (1992), Duarte (1999), Krokhmal et al. (2002), Rockafellar and Uryasev (2000, 2002), Turner et al. (1994), Zenios (1999), Ziemba and Mulvey (1998), Young (1998)) show that LP-based algorithms can successfully handle portfolio allocation problems with thousands and even millions of decision variables and scenarios, which makes those algorithms attractive to institutional investors.

In the cited papers, along with Conditional Value-at-Risk and Conditional Drawdown-at-Risk, other, much earlier established measures of risk, such as Maximum Loss, Mean-Absolute Deviation, Low Partial Moment with power one and Expected Regret[5] , have been employed in the framework of linear portfolio rebalancing algorithms (see, for example, Ziemba and Vickson (1975)). Some of these risk measures are quite closely related to CVaR concept.[6] We restricted ourselves to considering CVaR- and CDaR-based risk management techniques.

However, the class of linear trading or portfolio optimization techniques is far from encompassing the entire universe of portfolio management techniques. For example, the famous portfolio

[5] Low partial moment with power one is defined as the expectation of losses exceeding some fixed threshold, see Harlow (1991). Expected regret (see, for example, Dembo and King (1992)) is a concept similar to the lower partial moment. However, the expected regret may be calculated with respect to a random benchmark, while the low partial moment is calculated with respect to a fixed threshold.

[6] Maximum Loss is a limiting case of CVaR risk measure (see below). Also, Testuri and Uryasev (2000) showed that the CVaR constraint and the low partial moment constraint with power one are equivalent in the sense that the efficient frontier for portfolio with CVaR constraint can be generated by the low partial moment approach. Therefore, the risk management with CVaR and with low partial moment leads to similar results. However, the CVaR approach allows for direct controlling of percentiles, while the low partial moment penalizes losses exceeding some fixed thresholds.

optimization model by Markowitz (1952, 1991), which utilizes the mean-variance approach, belongs to the class of quadratic programming (QP) problems; the well-known constant-proportion rule leads to nonconvex multiextremum problems, etc.

3.2.1 Conditional Value-at-Risk

The Conditional Value-at-Risk (CVaR) measure (Rockafellar and Uryasev, 2000, 2002) develops and enhances the ideas of risk management, which have been put in the framework of Value-at-Risk (VaR) (see, for example, Duffie and Pan (1997), Jorion (1997), Pritsker (1997), Staumbaugh (1996)). Incorporating such merits as easy-to-understand concept, simple and convenient representation of risks (one number), applicability to a wide range of instruments, VaR has evolved into a current industry standard for estimating risks of financial losses. Basically, VaR answers the question "what is the maximum loss, which is expected to be exceeded, say, only in 5% of the cases within the given time horizon?" For example, if daily VaR for the portfolio of some fund XYZ is equal to 10 millions USD at the confidence level 0.95, it means that there is only a 5% chance of losses exceeding 10 millions during a trading day.

The formal definition of VaR is as follows. Consider a loss function $f(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$ is a decision vector (e.g., portfolio positions), and $\mathbf{y}$ is a stochastic vector standing for market uncertainties (in this chapter, $\mathbf{y}$ is the vector of returns of instruments in the portfolio). Let $\Psi(\mathbf{x},\zeta)$ be the cumulative distribution function of $f(\mathbf{x},\mathbf{y})$,

$$\Psi(\mathbf{x},\zeta) = \mathsf{P}\left[f(\mathbf{x},\mathbf{y}) \leq \zeta\right].$$

Then, the Value-at-Risk function $\zeta_\alpha(\mathbf{x})$ with the confidence level α is the α-quantile of $f(\mathbf{x},\mathbf{y})$ (see Figure 3–1):

$$\zeta_\alpha(\mathbf{x}) = \min_{\zeta\in\mathrm{R}} \left\{\Psi(\mathbf{x},\zeta) \geq \alpha\right\}.$$

Using VaR as a risk measure in portfolio optimization is, however, a very difficult problem, if the return distributions of a portfolio's instruments are not normal or log-normal. The optimization difficulties with VaR are caused by its non-convex and non-subadditive nature (Artzner et al., 1997, 1999; Mausser and Rosen, 1999). Non-convexity of VaR means that as a function of portfolio positions, it has multiple local extrema, which precludes using efficient optimization techniques.

The difficulties with controlling and optimizing VaR in non-normal portfolios have forced the search for similar percentile risk measures, which would also quantify downside risks and at the same time could be efficiently controlled and optimized. From this viewpoint, CVaR is a perfect candidate for conducting a "VaR"-style portfolio management.

For continuous distributions, CVaR is defined as an average (expectation) of high losses residing in the α-tail of the loss distribution, or, equivalently, as a conditional expectation of losses exceeding the α-VaR level (Fig. 3–1). From this follows that CVaR incorporates information on VaR and on the losses exceeding VaR.

For general (non-continuous) distributions, Rockafellar and Uryasev (2002) defined α-CVaR function $\phi_\alpha(\mathbf{x})$ as the α-tail expectation of a random variable z,

$$\phi_\alpha(\mathbf{x}) = \mathsf{E}_{\alpha-\text{tail}}[z],$$

where the α-tail cumulative distribution functions of z has the form

$$\Psi_\alpha(\mathbf{x},\zeta) = \mathsf{P}[z \le \zeta] = \begin{cases} 0, & \zeta < \zeta_\alpha(\mathbf{x}), \\ \big[\Psi(\mathbf{x},\zeta) - \alpha\big]/\big[1-\alpha\big], & \zeta \ge \zeta_\alpha(\mathbf{x}). \end{cases}$$

Also, Acerbi et al. (2001), Acerbi and Tasche (2002) redefined the expected shortfall similar to the CVaR definition presented above.

Along with α - CVaR function $\phi_\alpha(\mathbf{x})$, the following functions called "upper" and "lower" CVaR (α-CVaR$^+$ and α-CVaR$^-$), are considered:

$$\phi_\alpha^+(\mathbf{x}) = \mathsf{E}[f(\mathbf{x},\mathbf{y}) | f(\mathbf{x},\mathbf{y}) > \zeta_\alpha(\mathbf{x})],$$

$$\phi_\alpha^-(\mathbf{x}) = \mathsf{E}[f(\mathbf{x},\mathbf{y}) | f(\mathbf{x},\mathbf{y}) \ge \zeta_\alpha(\mathbf{x})].$$

The CVaR functions satisfy the following inequality:

$$\phi_\alpha^-(\mathbf{x}) \le \phi_\alpha(\mathbf{x}) \le \phi_\alpha^+(\mathbf{x}).$$

Rockafellar and Uryasev (2002) showed that α-CVaR can be presented as a convex combination of α-VaR and α-CVaR$^+$,

$$\phi_\alpha(\mathbf{x}) = \lambda_\alpha(\mathbf{x})\,\zeta_\alpha(\mathbf{x}) + [1-\lambda_\alpha(\mathbf{x})]\,\phi_\alpha^+(\mathbf{x}),$$

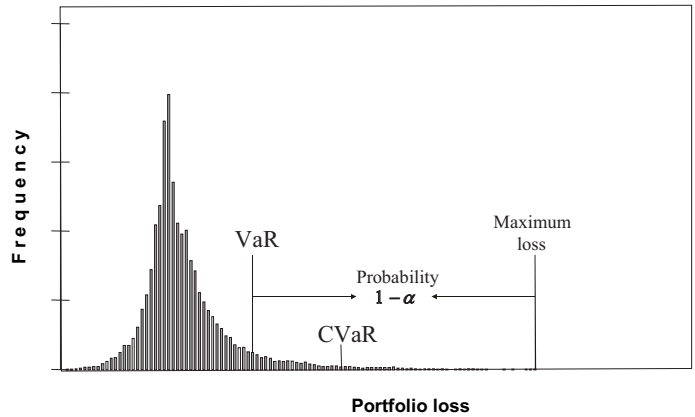


Figure 3–1: Loss distribution, VaR, CVaR, and Maximum Loss.

where

$$\lambda_\alpha(\mathbf{x}) = [\Psi(\mathbf{x}, \zeta_\alpha(\mathbf{x})) - \alpha]/[1-\alpha], \quad 0 \le \lambda_\alpha(\mathbf{x}) \le 1.$$

For a discrete loss distribution, where the stochastic parameter $\mathbf{y}$ may take values $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_J$ with probabilities $\theta_j, \;\; j = 1, \ldots, J$, the α-VaR and α-CVaR functions respectively are[7]

$$\zeta_\alpha(\mathbf{x}) = f(\mathbf{x}, \mathbf{y}_{j_\alpha}),$$

$$\phi_\alpha(\mathbf{x}) = \frac{1}{1-\alpha}\left[\left(\sum_{j=1}^{j_\alpha}\theta_j - \alpha\right) f(\mathbf{x}, \mathbf{y}_{j_\alpha}) + \sum_{j=j_\alpha+1}^{J}\theta_j f(\mathbf{x}, \mathbf{y}_j)\right],$$

where j_α satisfies

$$\sum_{j=1}^{j_\alpha - 1}\theta_j < \alpha \le \sum_{j=1}^{j_\alpha}\theta_j.$$

For values of confidence level α close to 1, Conditional Value-at-Risk coincides with the Maximum Loss (see Figure 3–1).

While inheriting some of the nice properties of VaR, such as measuring downside risks and representing them by a single number, applicability to instruments with non-normal distributions etc., CVaR has substantial advantages over VaR from the risk management standpoint. First of all, CVaR is a convex function[8] of portfolio positions. Hence, it has a convex set of minimum points

[7] This proposition has been derived in assumption that, without loss of the generality, scenarios $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_J$ satisfy inequalities $f(\mathbf{x}, \mathbf{y}_1) \le \ldots \le f(\mathbf{x}, \mathbf{y}_J)$.

[8] For a background on convex functions and sets see Rockafellar (1970).

on a convex set, which greatly simplifies control and optimization of CVaR. Calculation of CVaR, as well as its optimization, can be performed by means of a convex programming shortcut (Rockafellar and Uryasev, 2000, 2002), where the optimal value of CVaR is calculated simultaneously with the corresponding VaR; for linear or piecewise-linear loss functions these procedures can be reduced to linear programming problems. Also, unlike α-VaR, α-CVaR is continuous with respect to confidence level α. A comprehensive description of the CVaR risk measure and CVaR-related optimization methodologies can be found in Rockafellar and Uryasev (2000, 2002). Also, Rockafellar and Uryasev (2000) showed that for normal loss distributions, the CVaR methodology is equivalent to the standard Mean-Variance approach. Similar result also was independently proved for elliptic distributions by Embrechts et al. (1997).

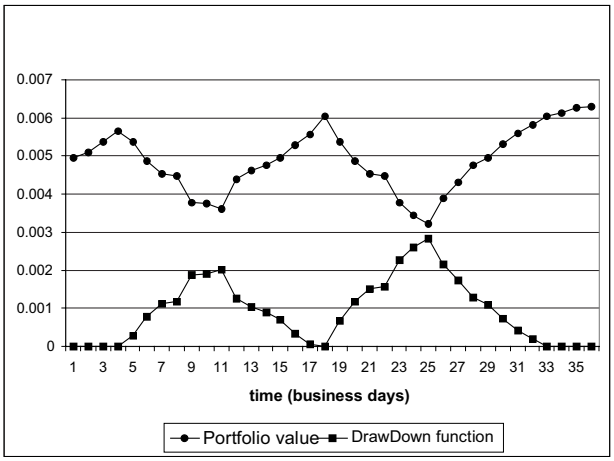


Figure 3–2: Portfolio value and drawdown.

According to Rockafellar and Uryasev (2000, 2002), the optimization problem with multiple CVaR constraints

$$\begin{aligned} &\min_{\mathbf{x}\in X} \quad g(\mathbf{x}) \\ &\text{subject to} \quad \phi_{\alpha_i}(\mathbf{x}) \le \omega_i, \ i = 1, \ldots, I, \end{aligned}$$

is equivalent to the following problem:

$$\begin{aligned} &\min_{\mathbf{x}\in X,\, \zeta_k\in\mathbb{R},\, \forall k} \quad g(\mathbf{x}) \\ &\text{subject to} \quad \zeta_k + \frac{1}{1-\alpha_k}\sum_{j=1}^{J} \theta_j \max\left\{0,\ f(\mathbf{x},\mathbf{y}_j) - \zeta_k\right\} \le \omega_k, \ k = 1, \ldots, K, \end{aligned}$$

provided that the objective function $g(\mathbf{x})$ and the loss function $f(\mathbf{x},\mathbf{y})$ are convex in $\mathbf{x} \in X$. When the objective and loss functions are linear in $\mathbf{x}$ and constraints $\mathbf{x} \in X$ are given by linear inequalities, the last optimization problem can be reduced to LP, see Rockafellar and Uryasev (2002, 2000).

Except for the fact that CVaR can be easily controlled and optimized, CVaR is a more adequate measure of risk as compared to VaR because it accounts for losses beyond the VaR level. The fundamental difference between VaR and CVaR as risk measures are: VaR is the "optimistic" low bound of the losses in the tail, while CVaR gives the value of the *expected* losses in the tail. In risk management, we may prefer to be neutral or conservative rather than optimistic. Moreover, CVaR satisfies several nice mathematical properties and is coherent in the sense of Artzner et al. (1999, 1997).

3.2.2 Conditional Drawdown-at-Risk

Conditional Drawdown-at-Risk (CDaR) is a portfolio performance measure (Chekhlov et al., 2000) closely related to CVaR. By definition, a portfolio's *drawdown* on a sample-path is the drop of the uncompounded[9] portfolio value as compared to the maximal value attained in the previous moments on the sample-path. Suppose, for instance, that we start observing a portfolio in January 2001, and record its uncompounded value every month.[10] If the initial portfolio value was \$100,000,000 and in February it reached \$130,000,000, then, the portfolio drawdown as of February 2001 is \$0. If, in March 2001, the portfolio value drops to \$90,000,000, then the current drawdown equals \$40,000,000 (in absolute terms), or 30.77%. Mathematically, the drawdown function for a portfolio is

$$\tilde{f}(\mathbf{x},t) = \max_{0 \le \tau \le t} \{v_\tau(\mathbf{x})\} - v_t(\mathbf{x}), \tag{3.1}$$

[9] Drawdowns are calculated with uncompounded portfolio returns. This is related to the fact that risk measures based on drawdowns of uncompounded portfolios have nice mathematical properties. In particular, these measures are convex in portfolio positions. Suppose that at the initial moment $t = 0$ the portfolio value equals v and portfolio returns in the moments $t = 1, \ldots, T$ equal $r_1, \ldots, r_T$. By definition, the uncompounded portfolio value v_τ at time moment τ equals $v_\tau = v \sum_{t=1}^{\tau} r_t$. We assume that the initial portfolio value $v = 1$.

[10] Usually, portfolio value is observed much more frequently. However, for the hedge funds considered in this chapter, data are available on monthly basis.

where $\mathbf{x}$ is the vector of portfolio positions, and $v_t(\mathbf{x})$ is the *uncompounded* portfolio value at time t. We assume that the initial portfolio value is equal to 1; therefore, the drawdown is the uncompounded portfolio return starting from the previous maximum point. Figure 3–2 illustrates the relation between the portfolio value and the drawdown.

The drawdown quantifies the financial losses in a conservative way: it calculates losses for the most "unfavorable" investment moment in the past as compared to the current (discrete) moment. This approach reflects quite well the preferences of investors who define their allowed losses in percentages of their initial investments (e.g., an investor may consider it unacceptable to lose more than 10% of his investment). While an investor may accept small drawdowns in his account, he would definitely start worrying about his capital in the case of a large drawdown. Such drawdown may indicate that something is wrong with that fund, and maybe it is time to move the money to a more successful investment pool. The mutual and hedge fund concerns are focused on keeping existing accounts and attracting new ones; therefore, they should ensure that clients' accounts do not have large drawdowns.

One can conclude that *drawdown* accounts not only for the amount of losses over some period, but also for the *sequence* of these losses. This highlights the unique feature of the drawdown concept: it is a loss measure "with memory" taking into account the time sequence of losses.

For a specified sample-path, the drawdown function is defined for each time moment. However, in order to evaluate performance of a portfolio on the whole sample-path, we would like to have a function, which aggregates all drawdown information over a given time period into one measure. As this function one can pick, for example, the Maximum Drawdown,

$$\mathsf{MaxDD} = \max_{0 \le t \le T} \left\{ \tilde{f}(\mathbf{x}, t) \right\},$$

or the Average Drawdown,

$$\mathsf{AverDD} = \frac{1}{T} \int_0^T \tilde{f}(\mathbf{x}, t)\, dt.$$

However, both these functions may inadequately measure losses. The Maximum Drawdown is based on one "worst case" event in the sample-path. This event may represent some very specific circumstances, which may not appear in the future. The risk management decisions based only on this event may be too conservative.

On the other hand, the Average Drawdown takes into account all drawdowns in the sample-path. However, small drawdowns are acceptable (e.g., 1-2% drawdowns) and averaging may mask large drawdowns.

Chekhlov et al. (2000) suggested a new drawdown measure, Conditional Drawdown-at-Risk, that combines both the drawdown concept and the CVaR approach. For instance, 0.95-CDaR can be thought of as an average of 5% of the highest drawdowns. Formally, α-CDaR is α-CVaR with drawdown loss function $\tilde{f}(\mathbf{x},t)$ given by (3.1). Namely, assume that possible realizations of the random vectors describing uncertainties in the loss function is represented by a sample-path (time-dependent scenario), which may be obtained from historical or simulated data. In this chapter, it is assumed that we know one sample-path of returns of instruments included in the portfolio. Let r_{ij} be the rate of return of i-th instrument in j-th trading period (that corresponds to j-th month in the case study, see below), $j = 1,\ldots,J$. Suppose that the initial portfolio value equals 1. Let x_i, $i = 1,\ldots,n$ be weights of instruments in the portfolio. The uncompounded portfolio value at time j equals

$$v_j(\mathbf{x}) = \sum_{i=1}^{n}\left(1+\sum_{s=1}^{j} r_{is}\right)x_i.$$

The drawdown function $\tilde{f}(\mathbf{x},\mathbf{r}_j)$ at the time j is defined as the drop in the portfolio value compared to the maximum value achieved before the time moment j,

$$\tilde{f}(\mathbf{x},j) = \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i.$$

Then, the Conditional Drawdown-at-Risk function $\Delta_\alpha(\mathbf{x})$ is defined as follows. If the parameter α and number of scenarios J are such that their product $(1-\alpha)J$ is an integer number, then $\Delta_\alpha(\mathbf{x})$ is defined as

$$\Delta_\alpha(\mathbf{x}) = \eta_\alpha + \frac{1}{(1-\alpha)J}\sum_{j=1}^{J}\max\left\{0,\ \max_{1\le k\le j}\left[\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right] - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i - \eta_\alpha\right\},$$

where $\eta_\alpha = \eta_\alpha(\mathbf{x})$ is the threshold that is exceeded by $(1-\alpha)J$ drawdowns. In this case the drawdown functions $\Delta_\alpha(\mathbf{x})$ is the average of the worst case $(1-\alpha)J$ drawdowns observed in the considered sample-path. If $(1-\alpha)J$ is not integer, then the CDaR function, $\Delta_\alpha(\mathbf{x})$, is the solution

of

$$\Delta_\alpha(\mathbf{x}) = \min_\eta \left\{ \eta + \frac{1}{1-\alpha}\frac{1}{J} \right.$$
$$\left. \times \sum_{j=1}^{J} \max\left[0, \max_{1\le k\le j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i - \eta\right]\right\}.$$

The CDaR risk measure holds nice properties of CVaR such as convexity with respect to portfolio positions. Also CDaR can be efficiently treated with linear optimization algorithms (Chekhlov et al., 2000).

3.2.3 Mean-Absolute Deviation

The Mean-Absolute Deviation (MAD) risk measure was introduced by Konno and Yamazaki (1991) as an alternative to the classical Mean-Variance measure of a portfolio's volatility,

$$\mathsf{MAD} = \mathsf{E}[\,|r_p(\mathbf{x}) - \mathsf{E}[r_p(\mathbf{x})]|\,],$$

where $r_p(\mathbf{x}) = r_1x_1 + r_2x_2 + \ldots + r_nx_n$ is the portfolio's rate of return, with $r_1, \ldots, r_n$ being the random rates of return for instruments in the portfolio. Since MAD is a piecewise linear convex function of portfolio positions, it allows for fast efficient portfolio optimization procedures by means of linear programming, in contrast to Mean-Variance approach, which leads to quadratic optimization problems. Konno and Shirakawa (1994) showed that MAD-optimal portfolios exhibit properties, similar to those of Markowitz MV-optimal portfolios, and that one can use MAD as a risk measure in deriving CAPM-type relationships. Later, it was also proved (Ogryczak and Ruszczynski, 1999) that portfolios on the MAD efficient frontier correspond to efficient portfolios in terms of the second-order stochastic dominance.

Formally, the Mean-Absolute Deviation (MAD) $\varsigma(\mathbf{x})$ of portfolio's rate of return equals

$$\varsigma(\mathbf{x}) = \mathsf{E}\left[\left|\sum_{i=1}^{n} r_ix_i - \mathsf{E}\left[\sum_{i=1}^{n} r_ix_i\right]\right|\right],$$

see Konno and Yamazaki (1991). We suppose that $j = 1, \ldots, J$ scenarios of returns with probabilities θ_j are available. Let us denote by r_{ij} the return of i-th asset in the scenario j. The portfolio's MAD can be written as

$$\varsigma(\mathbf{x}) = \sum_{j=1}^{J} \theta_j \left| \sum_{i=1}^{n} r_{ij} x_i - \sum_{k=1}^{k} \theta_k \sum_{i=1}^{n} r_{ik} x_i \right|.$$

3.2.4 Maximum Loss

The Maximum Loss (MaxLoss) of a portfolio in a specified time period is defined as the maximal value over all random loss outcomes (Fig. 3–1), see for instance, Young (1998). When the distribution of losses is continuous, this risk measure may be unbounded, unless the distribution is "truncated". For example, for normal distribution, the maximum loss is infinitely large. However, for discrete loss distributions, especially for those based on small historical datasets, the MaxLoss is a reasonable measure of risk. We also would like to point out that the Maximum Loss admits an alternative definition as a special case of α-CVaR with α close to 1.

Let us suppose that $j = 1, ..., J$ scenarios of returns are available (r_{ij} denotes return of i-th asset in the scenario j). The Maximum Loss (MaxLoss) function has the form (see for instance, Young (1998))

$$\varpi(\mathbf{x}) = \max_{1 \le j \le J} \left\{ -\sum_{i=1}^{n} r_{ij} x_i \right\}.$$

It is worth noting that α-CVaR function $\phi_\alpha(\mathbf{x})$ coincides with MaxLoss for values α close to 1. Suppose that scenarios $j = 1, \ldots, J$ have equal probabilities$1/J$. When the confidence level $\tilde{\alpha}$ satisfies $\tilde{\alpha} \ge \dfrac{J-1}{J}$, the MaxLoss equals to α-CVaR function $\varpi(\mathbf{x}) = \phi_{\tilde{\alpha}}(\mathbf{x})$.

3.2.5 Market-Neutrality

It is generally acknowledged that the market itself constitutes a risk factor. If the instruments in the portfolio are positively correlated with the market, then the portfolio would follow not only market growth, but also market drops. Naturally, portfolio managers are willing to avoid situations of the second type, by constructing portfolios, which are uncorrelated with market, or *market-neutral*. To be market-uncorrelated, the portfolio must have zero *beta*,

$$\beta_p = \sum_{i=1}^{n} \beta_i x_i = 0,$$

where $x_1, ..., x_n$ denote the proportions in which the total portfolio capital is distributed among n assets, and β_i are betas of individual assets,

$$\beta_i = \frac{\mathrm{Cov}\,(r_i, r_M)}{\mathrm{Var}\,(r_M)},$$

where r_M stands for market rate of return. Instruments' betas, β_i, can be estimated, for example, using historical data:

$$\beta_i = \left(\sum_{j=1}^{J} (r_{M,j} - \bar{r}_M)^2 \right)^{-1} \sum_{j=1}^{J} (r_{i,j} - \bar{r}_i)\,(r_{M,j} - \bar{r}_M),$$

where J is the number of historical observations, and $\bar{r}$ denotes the sample average, $\bar{r} = J^{-1} \sum r_j$. As a proxy for market returns r_M, historical returns of the S&P500 index can be used.

In our case study, we investigate the effect of constructing a market-neutral (zero-beta) portfolio, by including a market-neutrality constraint in the portfolio optimization problem. We compare the performance of the optimal portfolios obtained with and without market-neutrality constraint.

3.2.6 Problem Formulation

This section presents the "generic" problem formulation, which was used to construct an optimal portfolio. We suppose that some historical sample-path of returns of n instruments is available. Based on this sample-path, we calculate the expected return of the portfolio and the various risk measures for that portfolio. We maximize the expected return of the portfolio subject to different operating, trading, and risk constraints,

$$\max_x \ \mathsf{E}\left[\sum_{i=1}^{n} r_i x_i \right] \tag{3.2}$$

subject to

$$0 \le x_i \le 1, \quad i = 1, ..., n, \tag{3.3}$$

$$\sum_{i=1}^{n} x_i \le 1, \tag{3.4}$$

$$\Phi_{\mathrm{Risk}}(x_1, \ldots, x_n) \le \omega\,, \tag{3.5}$$

$$-k \le \sum_{i=1}^{n} \beta_i x_i \le k\,, \tag{3.6}$$

where

x_i is the portfolio position (weight) of asset i

r_i is the (random) rate of return of asset i

β_i is market beta of instrument i.

The objective function (3.2) represents the expected return of the portfolio. The first constraint (3.3) of the optimization problem imposes limitations on the amount of funds invested in a single instrument (we do not allow short positions). The second constraint (3.4) is the budget constraint. Constraints (3.5) control risks of financial losses. The key constraint in the presented approach is the risk constraint (3.5). Function $\Phi_{\text{Risk}}(x_1,\ldots,x_n)$ represents either a CVaR or a CDaR risk measure, and risk tolerance level ω is the fraction of the portfolio value that is allowed for risk exposure.

Constraint (3.6), with β_i representing market's *beta* for instrument i, forces the portfolio to be market-neutral in the "zero-beta" sense, i.e., the portfolio correlation with the market is bounded. The coefficient k in (3.6) is a small number that sets the portfolio's beta close to zero. To investigate the effects of imposing a "zero-beta" requirement on the portfolio-rebalancing algorithm, we solved the optimization problem with and without this constraint. Constraint (3.6) significantly improves the out-of-sample performance of the algorithm.

The risk measures considered in this chapter allow for formulating the risk constraint (3.5) in terms of linear inequalities, which makes the optimization problem (3.2)–(3.6) linear, given the linearity of objective function and other constraints. Below we present the explicit form of the risk constraint (3.5) for CVaR and CDaR risk measures.

3.2.7 Conditional Value-at-Risk Constraint

The loss function incorporated into CVaR constraint, is the negative portfolio's return,

$$f(\mathbf{x},\mathbf{y}) = -\sum_{i=1}^{n} r_i x_i, \tag{3.7}$$

where the vector of instruments' returns $\mathbf{y} = \mathbf{r} = (r_1,\ldots,r_n)$ is random. The risk constraint (3.5), $\phi_\alpha(\mathbf{x}) \le \omega$, where CVaR risk function replaces the function $\Phi_{\text{Risk}}(\mathbf{x})$, is

$$\zeta + \frac{1}{(1-\alpha)J}\sum_{j=1}^{J} \max\left\{0, -\sum_{i=1}^{n} r_{ij}x_i - \zeta\right\} \le \omega, \tag{3.8}$$

where r_{ij} is return of i-th instrument in scenario j, $j = 1, ..., J$. Since the loss function (3.7) is linear, the risk constraint (3.8) can be equivalently represented by the linear inequalities,

$$\begin{aligned} \zeta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J} w_j &\leq \omega, \\ -\sum_{i=1}^{n} r_{ij}x_i - \zeta &\leq w_j, \ j = 1, ..., J, \\ \zeta \in \mathrm{R}, \ w_j &\geq 0, \ j = 1, ..., J. \end{aligned} \tag{3.9}$$

This representation allows for reducing the optimization problem (3.2)–(3.6) with the CVaR constraint to a linear programming problem.

3.2.8 Conditional Drawdown-at-Risk Constraint

The CDaR risk constraint $\Delta_\alpha(\mathbf{x}) \leq \omega$ has the form

$$\eta + \frac{1}{1-\alpha}\frac{1}{J}\sum_{j=1}^{J} \max\left[0, \max_{1\leq k\leq j}\left\{\sum_{i=1}^{n}\left(\sum_{s=1}^{k} r_{is}\right)x_i\right\} - \sum_{i=1}^{n}\left(\sum_{s=1}^{j} r_{is}\right)x_i - \eta\right] \leq \omega,$$

and it can be reduced to a set of linear constraints similarly to the CVaR constraint.

3.2.9 MAD Constraint

Given equal scenario probabilities, the MAD constraint $\varsigma(\mathbf{x}) \leq \omega$ has the form

$$\frac{1}{J}\sum_{j=1}^{J}\left|\sum_{i=1}^{n} r_{ij}x_i - \frac{1}{J}\sum_{k=1}^{J}\sum_{i=1}^{n} r_{ik}x_i\right| \leq \omega.$$

This constraint admits representation by linear inequalities,

$$\begin{aligned} &\frac{1}{J}\sum_{j=1}^{J}(u_j^+ + u_j^-) \leq \omega, \\ &\sum_{i=1}^{n} r_{ij}x_i - \frac{1}{J}\sum_{j=1}^{J}\sum_{i=1}^{n} r_{ij}x_i = u_j^+ - u_j^-, && j = 1, \ldots, J, \\ &u_j^{\pm} \geq 0, && j = 1, \ldots, J. \end{aligned}$$

3.2.10 MaxLoss Constraint

The MaxLoss constraint $\varpi(\mathbf{x}) \leq \omega$ in (4.5) can be written as

$$\max_{1\leq j\leq J}\left\{-\sum_{i=1}^{n} r_{ij}x_i\right\} \leq \omega.$$

Similar to other considered risk constraints, it can be replaced by a system of linear inequalities

$$w \leq \omega,$$
$$-\sum_{i=1}^{n} r_{ij} x_i \leq w, \;\; j = 1, \ldots, J.$$

3.3 Case Study: Portfolio of Hedge Funds

The case study investigates investment opportunities and tests portfolio management strategies for a portfolio of hedge funds. Hedge funds are subject to less regulations as compared with mutual or pension funds. Hence, very little information on hedge funds' activities is publicly available (for example, many funds report their share prices only monthly). On the other hand, fewer regulations and weaker government control provide more room for aggressive, risk-seeking trading and investment strategies. As a consequence, the revenues in this industry are on average much higher than elsewhere, but the risk exposure is also higher (for example, the typical "life" of a hedge fund is about five years, and very few of them perform well in long run). Data availability and sizes of datasets impose challenging requirements on portfolio rebalancing algorithms. Also, the specific nature of hedge fund securities imposes some limitations on using them in trading or rebalancing algorithms. For example, hedge funds are far from being perfectly liquid: hedge funds may *not* be publicly traded or may be closed to new investors. From this point of view, our results contain a rather schematic representation of investment opportunities existing in the hedge fund market and do not give direct recommendations on investing in that market. The goal of this study is to compare the recently developed risk management approaches and to demonstrate their high numerical efficiency in a realistic setting.

The dataset for conducting the numerical experiments was provided to the authors by the Foundation for Managed Derivatives Research. It contained a monthly data for more than 5000 hedge funds, from which we selected those with significantly long history and some minimum level of capitalization. To pass the selection, a hedge fund should have 66 months of historical data from December 1995 to May 2001, and its capitalization should be at least 5 million U.S. dollars at the beginning of this period. The total number of funds, which satisfied these criteria and accordingly constituted the investment pool for our algorithm, was 301. In this dataset, the field with the names

of hedge funds was unavailable; therefore, we identified the hedge funds with numbers, i.e., HF 1, HF 2, and so on. The historical returns from the dataset were used to generate scenarios for algorithm (3.2)–(3.6). Each scenario is a vector of monthly returns for all securities involved in the optimization, and all scenarios are assigned equal probabilities.

We performed separate runs of the optimization problem (3.2)–(3.5), with and without constraint (3.6) with CVaR and CDaR risk measure in constraint (3.5), varying such parameters as confidence levels, risk tolerance levels etc.

The case study consisted from two sets of numerical experiments. The first set of *in-sample* experiments included the calculation of efficient frontiers and the analysis of the optimal portfolio structure for each of the risk measures. The second set of experiments, *out-of-sample* testing, was designed to demonstrate the performance of our approach in a simulated historical environment.

3.3.1 In-Sample Results

Efficient frontier. For constructing the efficient frontier for the optimal portfolio with different risk constraints, we solved the optimization problem (3.2)–(3.5) with different risk tolerance levels ω in constraint (3.5), varied from $\omega = 0.005$ to $\omega = 0.25$. The parameter α in CVaR and CDaR risk constraints was set to $\alpha = 0.90$. The efficient frontier is presented in Figure 3–3, where the portfolio rate of return means expected monthly rate of return. In these runs, the market-neutrality constraint (3.6) is inactive.

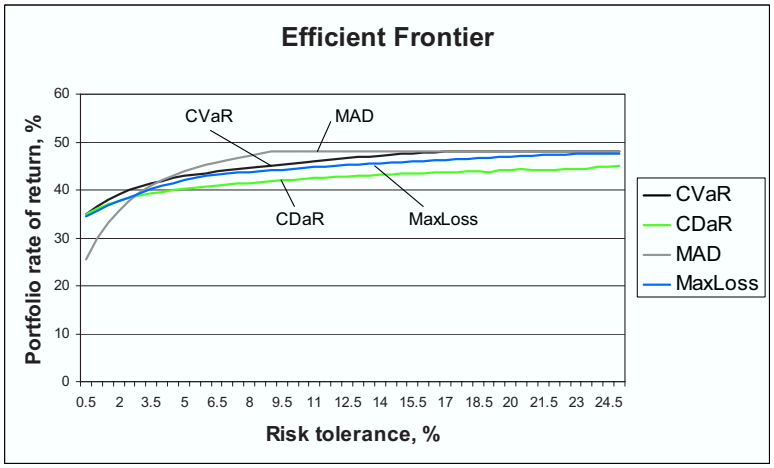


Figure 3–3: Efficient frontiers for portfolio with various risk constraints $(k = 0.01)$. The market-neutrality constraint is inactive.

Figure 3–3 shows that three CVaR-related risk measures (CVaR, CDaR and MaxLoss) produce relatively similar efficient frontiers. However, the MAD risk measure produces a distinctively different efficient frontier.

For optimal portfolios, in the sense of problem (3.2)–(3.5), there exists an upper bound (equal to 48.13%) for the portfolio's rate of return. Optimal portfolios with CVaR, MAD and MaxLoss constraints reach this bound at different risk tolerance levels, but the CDaR-constrained portfolio does not achieve the maximal expected return within the given range of ω values. CDaR is a relatively conservative constraint imposing requirements not only on the magnitude of loses, but also on the time sequence of losses (small consecutive losses may lead to large drawdown, without significant increasing of CVaR, MaxLoss, and MAD).

Figure 3–4 presents efficient frontiers of optimal portfolio (3.2)–(3.5) with the active market-neutrality constraint (3.6), where coefficient k is equal to 0.01. As one should expect, imposing the extra constraint (3.6) causes a decrease in in-sample optimal expected return. For example, the "saturation" level of the portfolio's expected return is now 41.94%, and all portfolios reach that level at much lower values of risk tolerance ω. However, the market-neutrality constraint almost does not affect the curves of efficient portfolios in the leftmost points of efficient frontiers, which correspond to the lowest values of risk tolerance ω. This means that by tightening the risk constraint (3.5) one can obtain a nearly market-neutral portfolio without imposing the market-neutrality constraint (3.6).

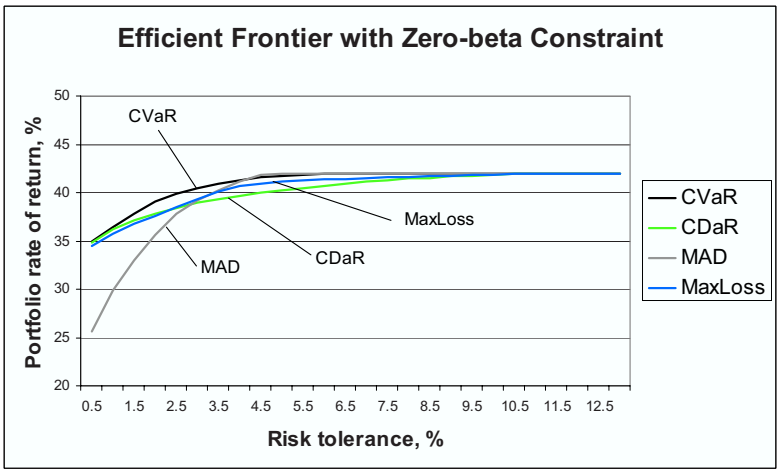


Figure 3–4: Efficient frontier for market-neutral portfolio with various risk constraints (k = 0.01).

Quite high rates of return for the efficient portfolios can be explained by the fact that 301 funds, selected to form the optimal portfolios, constitute about 6% of the initial hedge fund pool, and already are "the best of the best" in our data sample.

Optimal portfolio configuration. Let us discuss now the structure of the optimal portfolio with various risk constraints. For this purpose, we selected the corresponding efficient frontiers for four optimal portfolios with expected return of 35% and constraints on CVaR, CDaR, MaxLoss, and MAD (market-neutrality is inactive). For three risk measures (CVaR, CDaR and MaxLoss), the optimal portfolios with expected return of 35% are located in the vicinity of the leftmost points on corresponding efficient frontiers (see Fig. 3–3). Table 3–1 presents portfolio weights for the four optimal portfolios. It shows how a particular risk measure selects instruments given the specified expected return. The left column of Table 3–1 contains the set of the assets, which are chosen by the algorithm (3.2)–(3.5) under different risk constraints. Note that among the 301 available instruments, only a few of them are used in constructing the optimal portfolio. Moreover, a closer look at Table 3–1 shows that nearly two thirds of the portfolio value for all risk measures is formed by three hedge funds HF 209, HF 219 and HF 231 (the corresponding lines are typeset in boldface). These three hedge funds have stable performance, and each risk measure includes them in the optimal portfolio. Similarly, lines typeset in slanted font, indicate instruments that are included in the portfolio with smaller weights, but still are approximately evenly distributed among the portfolios. Thus, the instruments HF 93, 100, 209, 219, 231, 258, and 259 constitute the "core" of the optimal portfolio under all risk constraints. The last row in Table 3–1 lists the total weight of these instruments in corresponding optimal portfolio. The remaining assets (without highlighting in the table) are "residual" instruments, which are specific to each risk measure. They may help us to spot differences in instrument selection of each risk constraint. Table 3–5 displays the residual weights of HF 49, 84, 106, 124, 126, 169, 196, and 298. The "residual" weights are calculated as the instrument's weight with respect to the residual part of the portfolio. For example, in the optimal portfolio with the CDaR constraint, the hedge fund HF 49 represents 11.02% of the total portfolio value, and at the same time it represents 49.06% of the residual $(1.00 - 0.775) \cdot 100\%$ portfolio value. In other words, it occupies almost half of the portfolio assets, not captured by hedge funds in the grayed cells. Also, note that neither of the residual instruments is simultaneously present in all portfolios.

Table 3–1: Instrument weights in the optimal portfolio with different risk constraints

	CDaR	CVaR	MAD	MaxLoss
HF 49	0.110216	0.043866	0	0.191952
HF 84	0.041898	0	0	0.078352
HF 93	*0.081394*	*0.08754*	*0.045281*	*0.062609*
HF 100	*0.06629*	*0.073659*	*0.023899*	*0.068993*
HF 106	0	0	0	0.00191
HF 124	0	0	0.020289	0
HF 126	0	0.008673	0.027908	0
HF 169	0.054298	0.010144	0	0
HF 196	0	0.015627	0.084791	0
HF 209	**0.214683**	**0.224669**	**0.260824**	**0.226262**
HF 219	**0.137165**	**0.259254**	**0.169239**	**0.111746**
HF 231	**0.183033**	**0.169207**	**0.137068**	**0.177008**
HF 258	*0.034083*	*0.014156*	*0.097597*	*0.012606*
HF 259	*0.058684*	*0.089403*	*0.133104*	*0.068562*
HF 298	0.018257	0.0038	0	0
	0.775331	0.917889	0.867012	0.727785

Table 3–2: Weights of residual instruments in the optimal portfolio with different risk constraints

	CDaR	CVaR	MAD	MaxLoss
HF 49	0.490571	0.534229	0	0.705151
HF 84	0.186487	0	0	0.287833
HF 93	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 100	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 106	0	0	0	0.007016
HF 124	0	0	0.152561	0
HF 126	0	0.105624	0.209855	0
HF 169	0.24168	0.123544	0	0
HF 196	0	0.19031	0.637582	0
HF 209	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 219	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 231	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 258	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 259	XXXXXXXX	XXXXXXXX	XXXXXXXX	XXXXXXXX
HF 298	0.081261	0.046285	0	0

Figure 3–5 contains graphs of the historical return and price dynamics for the residual hedge funds. We included these graphs to illustrate differences in risk constraints and to make some speculations on this subject.

For example, instrument HF 84 is selected by CDaR and MaxLoss risk measures, but is excluded by CVaR and MAD. Note that graph of returns for the instrument HF 84 shows no negative monthly returns exceeding 10%, which is probably acceptable for the MaxLoss risk measure. Also, the price graph for the instrument HF 84 shows that it exhibits few drawdowns; therefore, CDaR picked this instrument. However, MAD probably excluded instrument HF8 because it had a high monthly return of 30% (recall that MAD does not discriminate between high positive returns and high negative returns). It is not clear from the graphs why CVaR rejected the HF 84 instruments. Probably, other instruments had better CVaR-return characteristics from the view point of the overall portfolio performance.

The hedge fund HF 124 has not been chosen by any risk measures, with the exception of MAD. Besides rather average performance, it suffers long-lasting drawdowns (CDaR does not like this), has multiple negative return peaks of -10% magnitude (CVaR does not favor that), and its worst negative return is almost -20% (MaxLoss must protect from such performance drops). The question why this instrument was not rejected by MAD cannot be clearly answered in this case. Do not forget that such a decision is a solution of an optimization problem, and different instruments with properly adjusted weights may compensate each other's shortcomings. This may also be an excuse for MAD not picking the HF 49 fund, whose merits are confirmed by high residual weights of this fund in CDaR, CVaR, and MaxLoss portfolios.

Fund HF 126 has the highest expected return among residual instruments, but it also suffers the most severe drawdowns and has the highest negative return (exceeding -20%) – that's, probably, why algorithms with CDaR and MaxLoss measures rejected this instrument.

3.3.2 Out-of-Sample Calculations

The out-of-sample testing of the portfolio optimization algorithm (3.2)–(4.6) sheds light on the "*actual*" performance of the developed approaches. In other words, the question is how well do the algorithms with different risk measures utilize the scenario information based on past history in producing a successful portfolio management strategy? An answer can be obtained, for instance, by interpreting the results of the preceding section as follows: suppose we were back in May 2001,

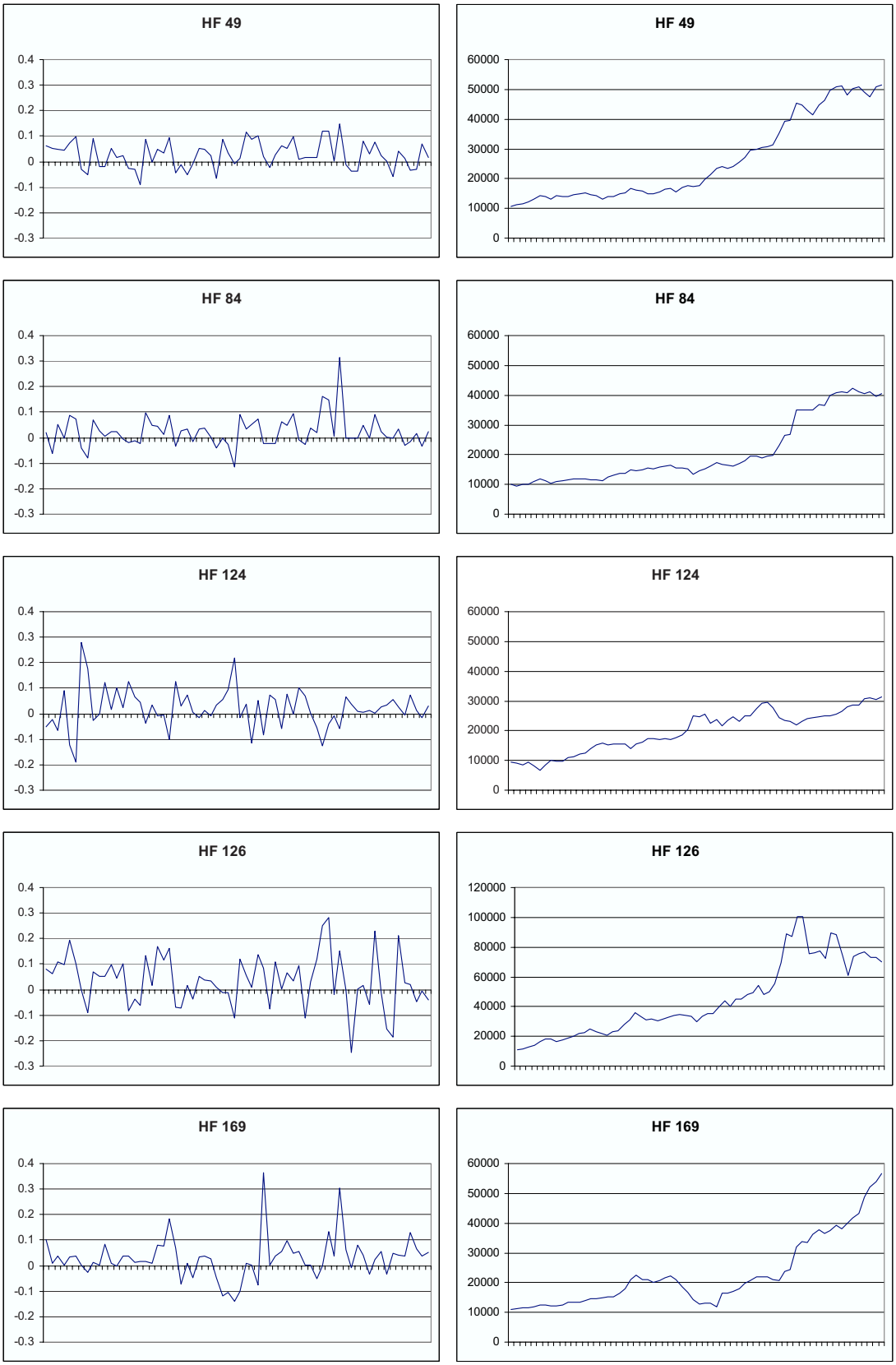


Figure 3–5: Historical performance in percents of initial value (on the right), and the rate of return dynamics in percentage terms (on the left) for some of the “residual” assets in optimal portfolios.

and we would like to invest a certain amount of money in a portfolio of hedge funds to deliver the highest reward under a specified risk level. Then, according to in-sample results, the best portfolio would be the one on the efficient frontier of a particular rebalancing strategy. In fact, such a portfolio offers the best return-to-risk ratio *provided that the historical distribution of returns will repeat in the future*.

To get an idea about the "actual" performance of the optimization approach, we used some part of the data for scenario generation, and the rest for evaluating the performance of the strategy. This technique is referred to as *out-of-sample* testing. In our case study, we perform the out-of-sample testing in two setups: 1) "Real" out-of-sample testing, and 2) "Mixed" out-of-sample testing. Each one is designed to reveal specific properties of risk constraints pertaining to the performance of the portfolio-rebalancing algorithm in out-of-sample runs.

"Real" out-of-sample testing. First, we present the results of a "plain" out-of-sample test, where the older data is considered as the 'in-sample' data for the algorithm, and the newer data are treated as 'to-be-realized' future. First, we took the 11 monthly returns within the time period from December 1995 to November 1996 as the initial historical data for constructing the first portfolio to invest in, and observed the portfolio's "realized" value by observing the historical prices for December 1996. Then, we added one more month, December 1996, to the data which were used for scenario generation (12 months of historical data in total) to generate an optimal portfolio and to allocate to investments in January, 1997, and so on. Note that we did not implement the "moving window" method for out-of-sample testing, where the same number of scenarios (i.e., the most recent historical points) is used for solving the portfolio-rebalancing problem. Instead, we accumulated the historical data for portfolio optimization.

First, we performed out-of-sample runs for each risk measure in constraint (3.5) for different values of risk tolerance level ω (market-neutrality constraint, (3.6), is inactive). Figures 3–6 to 3–9 illustrate historical trajectories of the optimal portfolio under different risk constraints (the portfolio values are given in % relatively to the initial portfolio value). Risk tolerance level ω was set to 0.005, 0.01, 0.03, 0.05, 0.10, 0.12, 0.15, 0.17 and 0.20, but for better reading of figures, we report only results with $\omega = 0.005$, 0.01, 0.05, 0.10, and 0.15. The parameter α in CDaR and CVaR constraints (in the last case α stands for risk confidence level) was set to $\alpha = 0.90$.

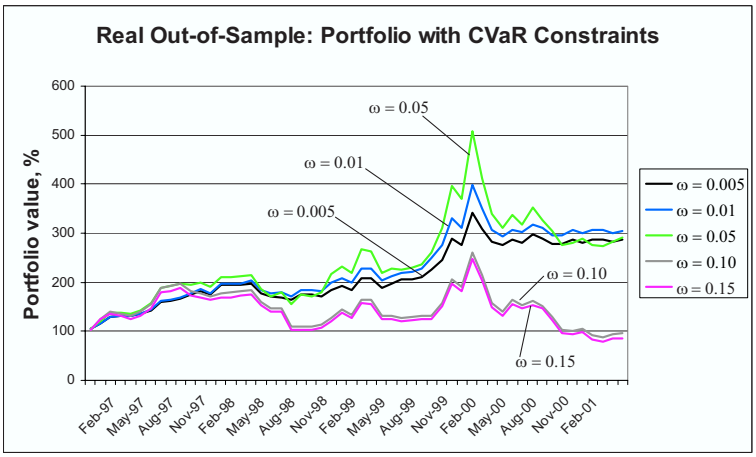


Figure 3–6: Historical trajectories of optimal portfolio with CVaR constraints.

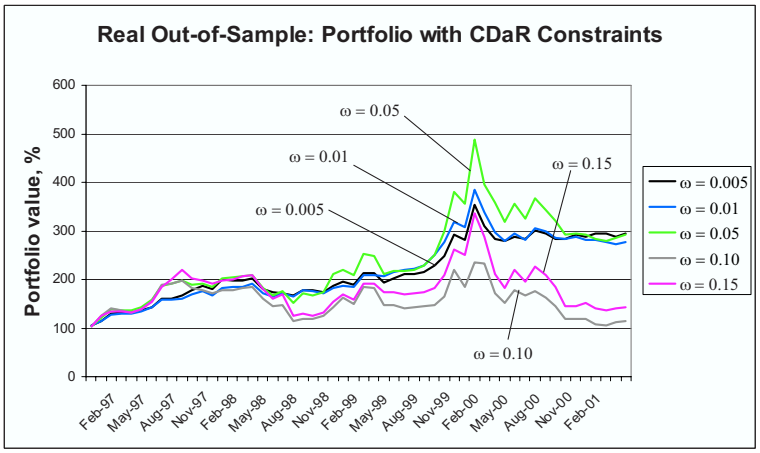


Figure 3–7: Historical trajectories of optimal portfolio with CDaR constraints.

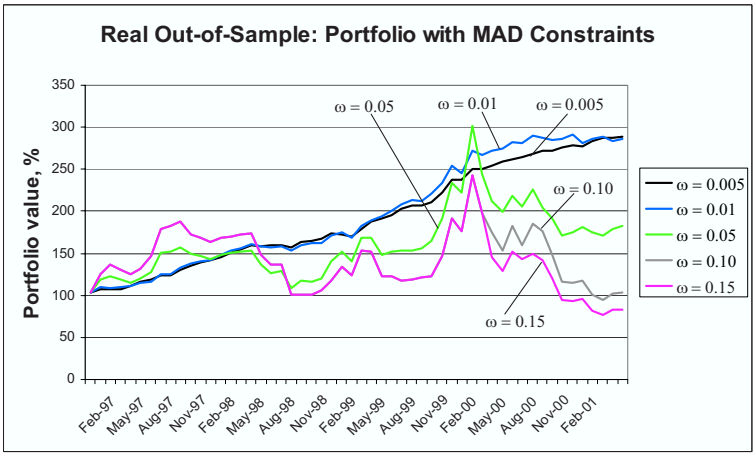


Figure 3–8: Historical trajectories of optimal portfolio with MAD constraints.

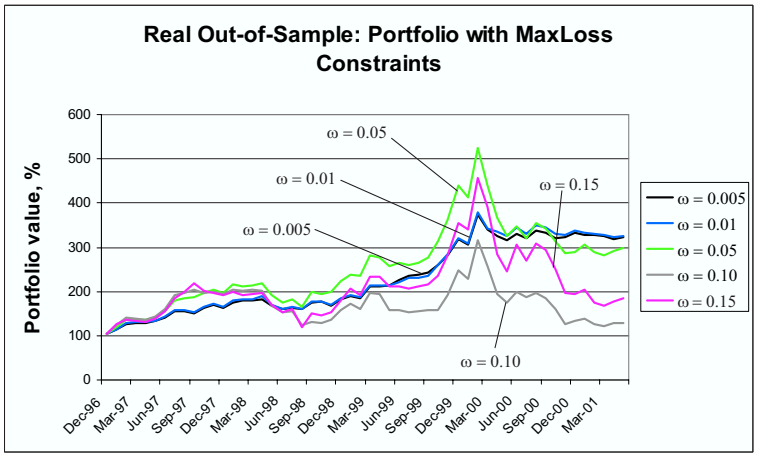


Figure 3–9: Historical trajectories of optimal portfolio with MaxLoss constraints.

Figures 3–6 to 3–9 shows that risk constraint (3.5) has significant impact on the algorithm's out-of-sample performance. Earlier, we had also observed that this constraint has significant impact on the in-sample performance. It is well known that constraining risk in the in-sample optimization decreases the optimal value of the objective function, and the results reported in the preceding subsection comply with this fact. The risk constraints force the algorithm to favor less profitable but safer decisions over more profitable but "dangerous" ones. From a mathematical viewpoint, imposing extra constraints always reduces the feasibility set, and consequently leads to lower optimal objective values. However, the situation changes dramatically for an out-of-sample application of the optimization algorithm. The numerical experiments show that constraining risks *improves* the overall performance of the portfolio rebalancing strategy in out-of-sample runs; *tighter* in-sample risk constraint may lead to *both lower risks and higher out-of-sample returns.* For all considered risk measures, loosening the risk tolerance (i.e., increasing ω values) results in increased volatility of out-of-sample portfolio returns and, after exceeding some threshold value, in degradation of the algorithm's performance, especially during the last 13 months (March 2000 – May 2001). For all risk functions in constraint (3.5), the most attractive portfolio trajectories are obtained for risk tolerance level $\omega = 0.005$, which means that these portfolios have high returns (high final portfolio value), low volatility, and low drawdowns. Increasing ω to 0.01 leads to a slight increase of the final portfolio value, but it also increases portfolio volatility and drawdowns, especially for the second quarter of 2001. For larger values of ω the portfolio returns deteriorate, and for all risk measures, a

portfolio curve with $\omega = 0.10$ shows quite poor performance. Further increasing the risk tolerance to $\omega = 0.15$, in some cases allows for achieving higher returns at the end of 2000, but after this high peak the portfolio suffers severe drawdowns (see figures for CDaR, MAD, and MaxLoss risk measures).

The next series of Figures 3–10 to 3–13 illustrates effects of imposing market-neutrality constraint (3.6) in addition to risk constraint (3.5). Recall that primary purpose of constraint (3.6) is making the portfolio uncorrelated with market. The main idea of composing a market-neutral portfolio is protecting it from market drawdowns. Figures 3–10 to 3–13 compare trajectories of market-neutral and without risk-neutrality optimal portfolios. Additional constraining resulted in most cases in further improvement of the portfolio's out-of-sample performance, especially for CVaR and CDaR-constrained portfolios. To clarify how the risk-neutrality condition (3.6) influences the portfolio's performance, we displayed only figures for lowest and highest values of the risk tolerance parameter, namely for $\omega = 0.005$ and $\omega = 0.20$. Coefficient k in (3.6) was set to $k = 0.01$, and instruments' betas β_i were calculated by correlating with S&P 500 index, which is traditionally considered as a market benchmark. For portfolios with tight risk constraints ($\omega = 0.005$) imposing market-neutrality constraint (3.6) straightened their trajectories (reduced volatility and drawdowns), which made the historic curves almost monotone curves with positive slope. On top of that, CVaR and CDaR portfolios with market-neutrality constraint had a higher final portfolio value, compared to those without market-neutrality. Also, for portfolios with loose risk constraints ($\omega = 0.20$) imposing market-neutrality constraint had a positive effect on the form of their trajectories, dramatically reducing volatility and drawdowns.

Finally, Figures 3–14 and 3–15 demonstrate the performance of the optimal portfolios versus two benchmarks: 1) S&P500 index; 2) "Best20", representing the portfolio distributed equally among the "best" 20 hedge funds. These 20 hedge funds include funds with the highest expected monthly returns calculated with past historical information. Similarly to the optimal portfolios (3.2)–(3.6), the "Best20" portfolio was monthly rebalanced (without risk constraints).

According to Figures 3–14 to 3–15, the market-neutral optimal portfolios as well as portfolios without the market-neutrality constraint outperform the index under risk constraints of all types, which provides an evidence of high efficiency of the risk-constrained portfolio management algorithm (3.2)–(3.6). Also, we would like to emphasize the behavior of market-neutral portfolios in

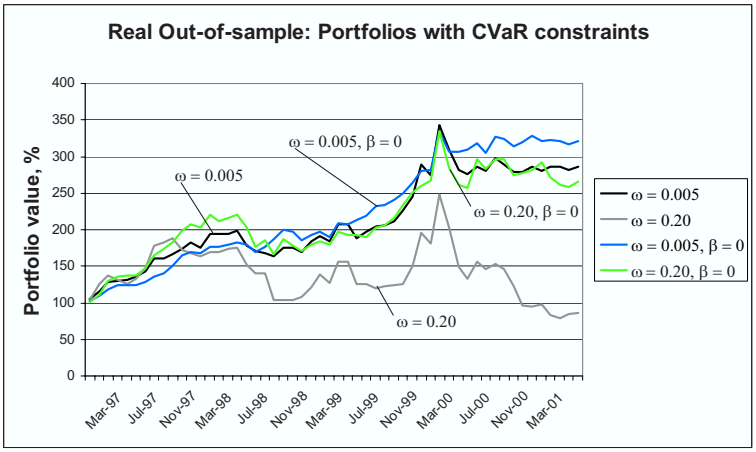


Figure 3–10: Historical trajectories of optimal portfolio with CVaR constraints. Lines with $\beta = 0$ correspond to portfolios with market-neutral constraint.

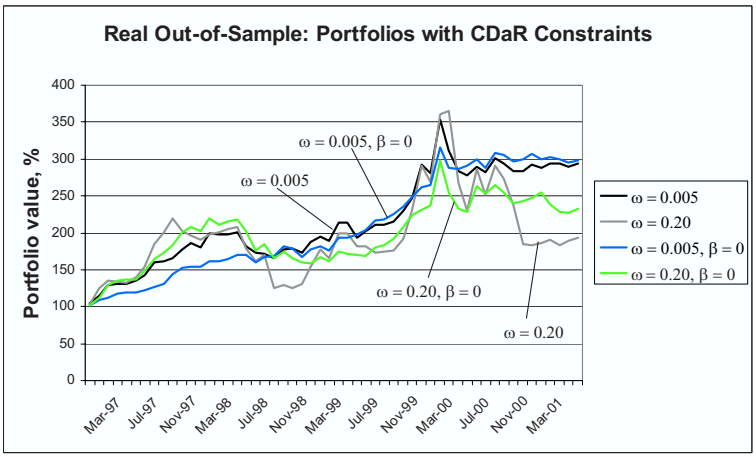


Figure 3–11: Historical trajectories of optimal portfolio with CDaR constraints. Lines with $\beta = 0$ correspond to portfolios with market-neutral constraint.

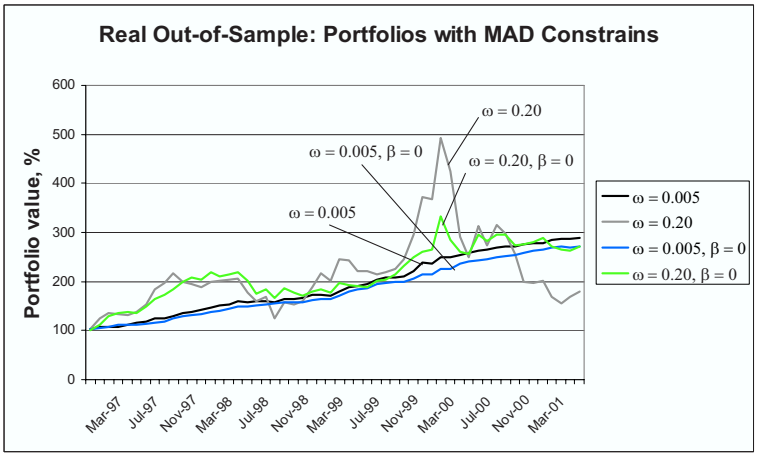


Figure 3–12: Historical trajectories of optimal portfolio with MAD constraints. Lines with $\beta = 0$ correspond to portfolios with market-neutral constraint.

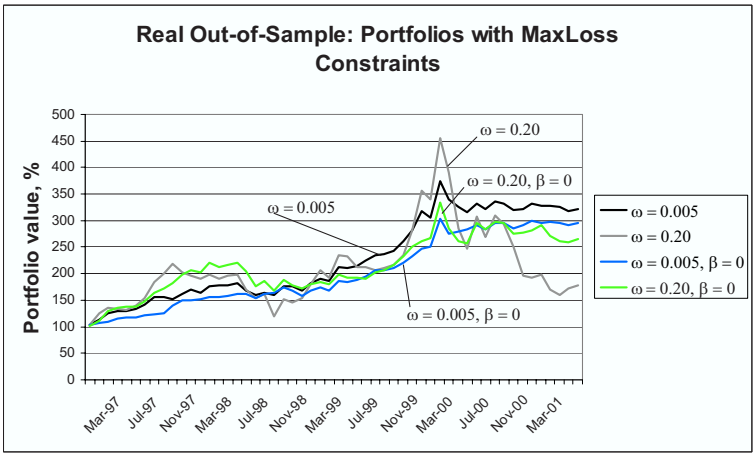


Figure 3–13: Historical trajectories of optimal portfolio with MaxLoss constraints. Lines with $\beta = 0$ correspond to portfolios with market-neutral constraint.

"down" market conditions. Two marks on Fig. 3–15 indicate the points when all four portfolios gained positive returns, while the market was falling down. Also, all risk-neutral portfolios seem to withstand the down market in 2000, when the market experienced significant drawdown. This demonstrates the efficiency and appropriateness of the application of market-neutrality constraint (3.6) to portfolio optimization with risk constraints (3.2)–(3.5).

The "Best20" benchmark evidently lacks the solid performance of its competitors. It not only significantly underperforms all the portfolios constructed with algorithm (3.2)–(3.6), but also underperforms the market half of the time. Unlike portfolios (3.2)–(3.6), the "Best20" portfolio pronouncedly follows the market drop in the second half of 2000, and moreover, it suffers much more severe drawdowns than the market does. This indicates that the risk constraints in the algorithm (3.2)–(3.6) play an important role in selecting the funds.

An interesting point to discuss is the behavior of algorithm (3.2)–(3.6) under the MAD risk constraint. Figures 3–8 and 3–12 show that a tight MAD constraint makes the portfolio curve almost a straight line, and imposing of market-neutrality constraint (3.6) does not add much to the algorithm's performance, and even slightly lowers the portfolio's return. At the same time, portfolios with CVaR-type risk constraints (CVaR, CDaR and MaxLoss) do not exhibit such remarkably stable performance, and take advantage of constraint (3.6). Note that CVaR, CDaR and MaxLoss are *downside* risk measures, whereas the MAD constraint suppresses both high losses and high returns. The market-neutrality constraint (3.6) by itself also puts symmetric restrictions on the portfolio's volatility; that's why it affects MAD and CVaR-related constraint differently. However, here, we should point out that we just "scratched the surface" regarding the combination of various risk constraints. We have imposed CVaR and CDaR constraints only with one confidence level. We can impose combinations of constraints with various confidence levels including constraining percentiles of high returns and as well as percentiles of high losses. These issues are beyond the scope of our study.

Summarizing, we emphasize the general inference about the role of risk constraints in the out-of-sample and in-sample application of an optimization algorithm, which can be drawn from our experiments: risk constraints decrease the in-sample returns, while out-of-sample performance may be improved by adding risk constraints, and moreover, stronger risk constraints usually ensure better out-of-sample performance.

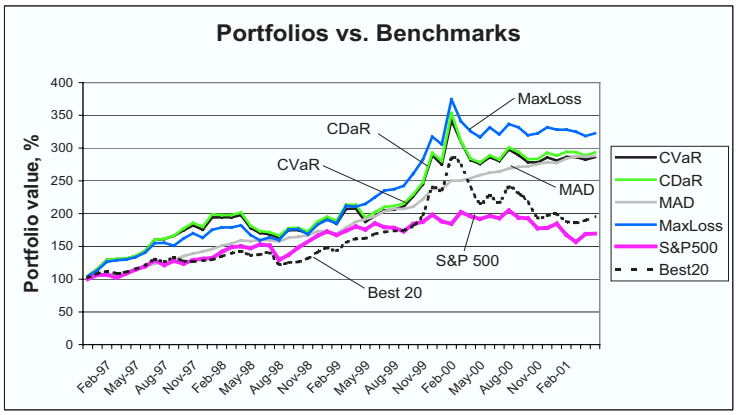


Figure 3–14: Performance of the optimal portfolios with various risk constraints versus S&P500 index and benchmark portfolio combined from 20 best hedge funds. Risk tolerance level $\omega = 0.005$, parameter $\alpha = 0.90$. Market-neutrality constraint is inactive.

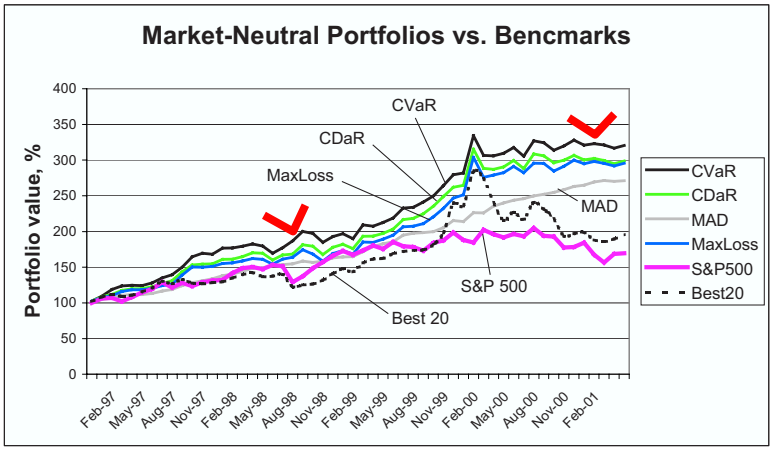


Figure 3–15: Performance of market-neutral optimal portfolio with various risk constraints versus S&P500 index and benchmark portfolio combined from 20 best hedge funds. Risk tolerance level is $\omega = 0.005$, parameter is $\alpha = 0.90$.

"Mixed" out-of-sample test. The second series of out-of-sample tests for the portfolio optimization algorithm (3.2)–(3.6) uses an alternative setup for splitting the data in in-sample and out-of-sample portions. Instead of utilizing only *past* information for generating scenarios for portfolio optimization, as it was done before, now we let the algorithm use both *past* and *future* information for constructing scenarios. The design of this experiment is as follows. The portfolio rebalancing procedure was performed every five months, and the scenario model utilized all the historical data except for the 5-month period directly following the rebalancing date. The procedure was started on December 1995. The information for scenario generation was collected from May 1996 to May 2001. The portfolio was optimized using these scenarios and invested on December 1995. After 5 months the money gained by the portfolio was reinvested and, at this time, the scenario model was built on information contained in the entire time interval 12/1995–05/2001 except window 05/1996–09/1996 and so on.

Figures 3–16 to 3–19 display dynamics of the optimal portfolio under various risk constraints and with different risk tolerance levels. To avoid overloading the presentation with details, we report results only for $\omega = 0.01$, 0.05 and 0.10. As earlier, we set parameter α in CVaR and CDaR constraints to $\alpha = 0.90$.

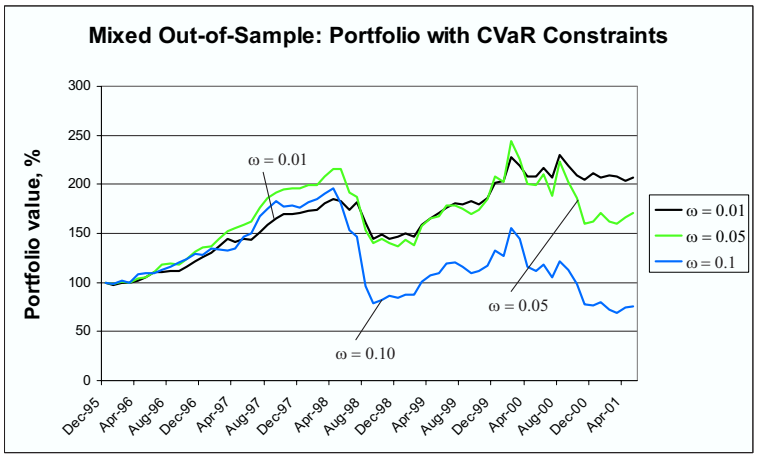


Figure 3–16: "Mixed" out-of-sample trajectories of optimal portfolio with CVaR constraints

The general picture of these results is consistent with conclusions derived from "real" out-of-sample tests: tightening of risk constrains improve performance of the rebalance algorithms. Lower overall performance of the portfolio optimization strategy under all risk constraints comparing to

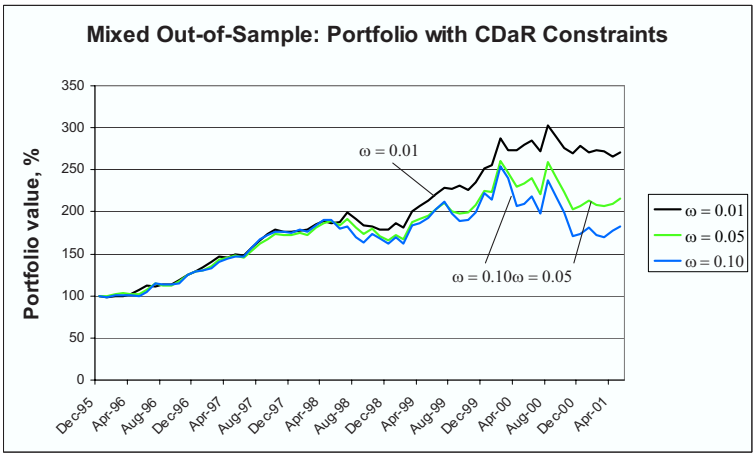


Figure 3–17: “Mixed” out-of-sample trajectories of optimal portfolio with CDaR constraints

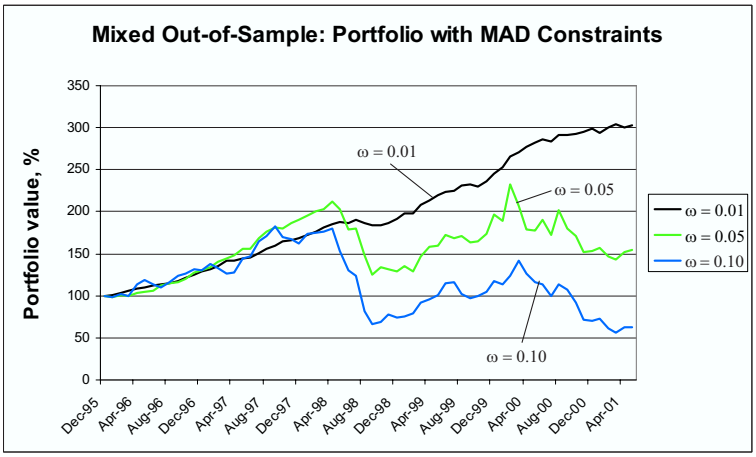


Figure 3–18: “Mixed” out-of-sample trajectories of optimal portfolio with MAD constraints

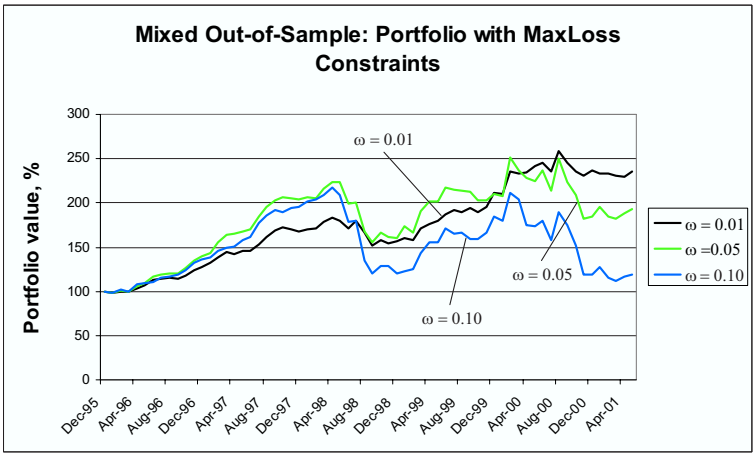


Figure 3–19: “Mixed” out-of-sample trajectories of optimal portfolio with MaxLoss constraints

that in “real” out-of-sample testing is explained by longer rebalancing period. It is well known that more frequent rebalancing may give higher returns (at least, in the absence of transaction costs).

The next four Figures 3–20 to 3–23, demonstrate the influence of the market-neutrality constraint on the performance of the portfolio.

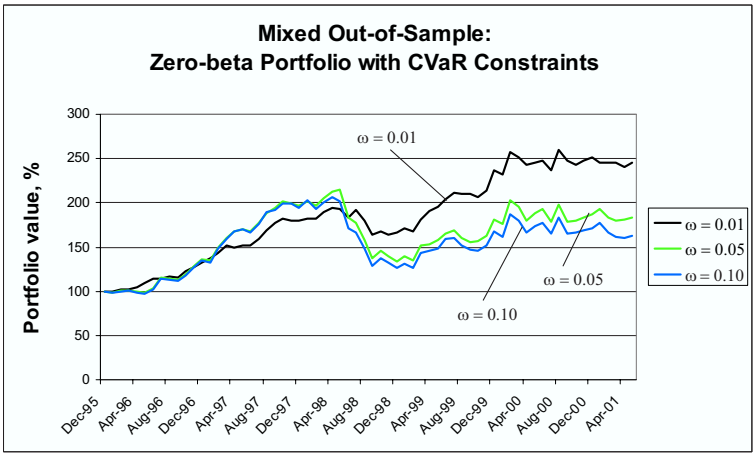


Figure 3–20: “Mixed” out-of-sample trajectories of market-neutrality optimal portfolio with CVaR constraints

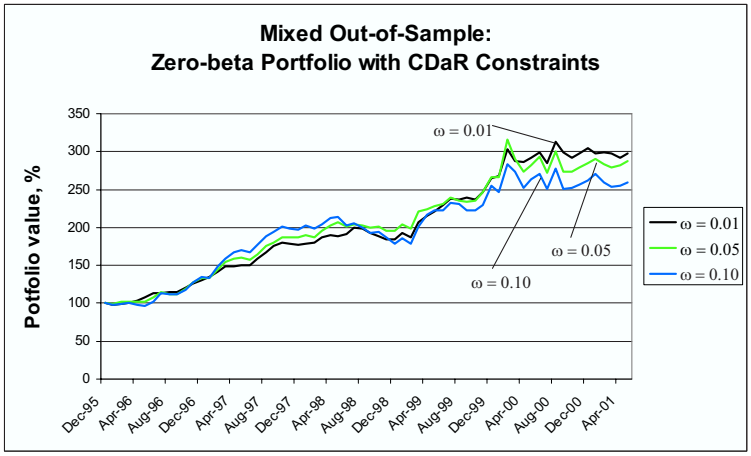


Figure 3–21: “Mixed” out-of-sample trajectories of market-neutrality optimal portfolio with CDaR constraints

Imposing of market-neutrality constraint (3.6) in problem (3.2)–(3.5) for the “mixed” out-of-sample testing has a similar impact as in “real” out-of-sample testing.

We finalize the out-of-sample testing of by presenting summary statistics of both “real” and “mixed” out-of-sample tests in Tables 3–3 and 3–4.

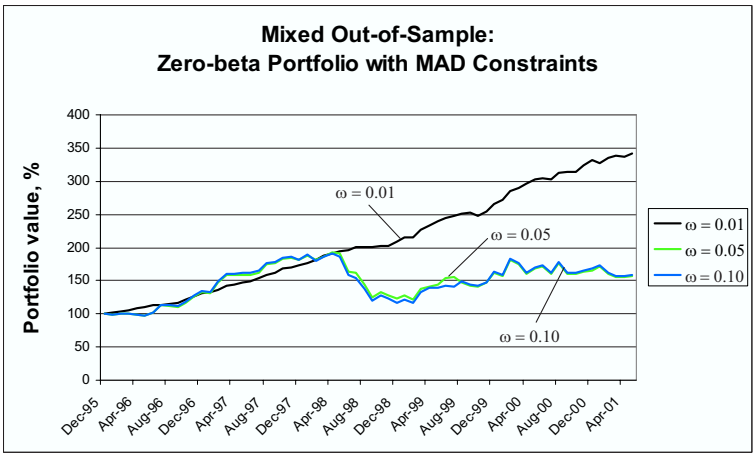


Figure 3–22: “Mixed” out-of-sample trajectories of market-neutrality optimal portfolio with MAD constraints

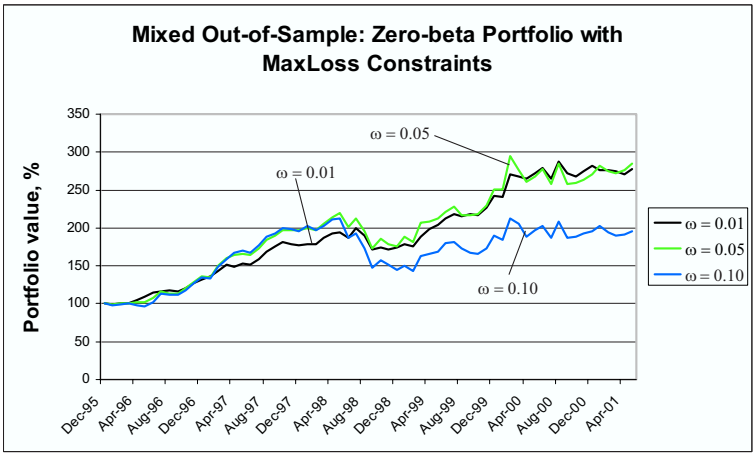


Figure 3–23: “Mixed” out-of-sample trajectories of
market-neutrality optimal portfolio with MaxLoss constraints

Summarizing, we emphasize the general inference about the role of risk constraints in the out-of-sample and in-sample application of an optimization algorithm, which can be drawn from our experiments: risk constraints decrease the in-sample returns, while out-of-sample performance may be improved by adding risk constraints, and moreover, stronger risk constraints usually ensure better out-of-sample performance.

3.4 Conclusions

We tested the performance of a portfolio allocation algorithm with different types of risk constraints in an application for managing a portfolio of hedge funds. As the risk measure in the portfolio optimization problem, we used Conditional Value-at-Risk, Conditional Drawdown-at-Risk, Mean-Absolute Deviation, and Maximum Loss. We combined these risk constraints with the market-neutrality (zero-beta) constraint making the optimal portfolio uncorrelated with the market.

The numerical experiments consist of in-sample and out-of-sample testing. We generated efficient frontiers and compared algorithms with various constraints. The out-of-sample part of experiments was performed in two setups, which differed in constructing the scenario set for the optimization algorithm.

The results obtained are dataset-specific and we cannot make direct recommendations on portfolio allocations based on these results. However, we learned several lessons from this case study. Imposing risk constraints may significantly degrade in-sample expected returns while improving risk characteristics of the portfolio. In-sample experiments showed that for tight risk tolerance levels, all risk constraints produce relatively similar portfolio configurations. Imposing risk constraints may improve the out-of-sample performance of the portfolio-rebalancing algorithms in the sense of risk-return tradeoff. Especially promising results can be obtained by combining several types of risk constraints. In particular, we combined the market-neutrality (zero-beta) constraint with CVaR or CDaR constraints. We found that tightening of risk constraints greatly improves portfolio dynamic performance in out-of-sample tests, increasing the overall portfolio return and decreasing both losses and drawdowns. In addition, imposing the market-neutrality constraint adds to the stability of portfolio's return, and reduces portfolio drawdowns. Both CDaR and CVaR risk measures demonstrated a solid performance in out-of-sample tests.

We thank the Foundation for Managed Derivatives Research for providing the dataset for conducting numerical experiments and partial financial support of this case study.

Table 3-3: Summary statistics for the "real" out-of-sample tests.

Type of constraint	Risk tolerance	Zero-beta	Aver. Rate of Return	Standard Deviation	CVaR	Drawdown	Sharpe Ratio	Yearly Aver. Return
CDaR	0.005	-	0.258	0.709	0.818	0.178	0.321	0.231
CVaR	0.005	-	0.265	0.7078	0.834	0.183	0.332	0.239
MAD	0.005	-	0.233	0.239	0.11	0.019	0.848	0.218
MaxLoss	0.005	-	0.273	0.703	0	0.154	0.346	0.245
CDaR	0.01	-	0.245	0.803	1.035	0.2887	0.267	0.216
CVaR	0.01	-	0.269	0.804	1.011	0.2541	0.297	0.241
MAD	0.01	-	0.234	0.38	0.38	0.0526	0.539	0.211
MaxLoss	0.01	-	0.278	0.742	0	0.154	0.33	0.249
CDaR	0.03	-	0.32	1.045	1.32	0.338	0.278	0.284
CVaR	0.03	-	0.3	1.073	1.37	0.393	0.25	0.272
MAD	0.03	-	0.234	0.752	0.794	0.243	0.271	0.215
MaxLoss	0.03	-	0.266	0.939	09	0.339	0.2519	0.236
CDaR	0.05	-	0.27	1.256	1.712	0.465	0.193	0.252
CVaR	0.05	-	0.327	1.303	1.638	0.468	0.228	0.306
MAD	0.05	-	0.299	1.221	1.529	0.365	0.22	0.273
MaxLoss	0.05	-	0.289	1.150	0	0.460	0.224	0.267
CDaR	0.07	-	0.248	1.41	1.983	0.562	0.155	0.213
CVaR	0.07	-	0.262	1.473	2.067	0.569	0.158	0.241
MAD	0.07	-	0.201	1.489	2.234	0.597	0.115	0.175
MaxLoss	0.07	-	0.284	1.345	0	0.553	0.189	0.265
CDaR	0.1	-	0.282	1.45	1.98	0.5	0.174	0.255
CVaR	0.1	-	0.239	1.591	2.401	0.62	0.131	0.211
MAD	0.1	-	0.264	1.62	2.46	0.651	0.144	0.232
MaxLoss	0.1	-	0.238	1.487	0	0.605	0.14	0.216
CDaR	0.12	-	0.237	1.53	2.311	0.6	0.135	0.205
CVaR	0.12	-	0.234	1.606	2.43	0.634	0.127	0.201
MAD	0.12	-	0.251	1.624	2.467	0.662	0.136	-0.097
MaxLoss	0.12	-	0.237	1.547	0	0.61	0.134	0.211
CDaR	0.15	-	0.163	1.48	2.407	0.594	0.09	0.145
CVaR	0.15	-	0.222	1.618	2.468	0.662	0.119	0.1803
MAD	0.15	-	0.231	1.607	2.458	0.658	0.125	0.178

Type of constraint	Risk tolerance	Zero-beta	Aver. Rate of Return	Standard Deviation	CVaR	Drawdown	Sharpe Ratio	Yearly Aver. Return
MaxLoss	0.15	-	0.234	1.61	0	0.634	0.127	0.201
CDaR	0.17	-	0.195	1.52	2.399	0.607	0.108	0.162
CVaR	0.17	-	0.222	1.618	2.468	0.662	0.119	0.1805
MAD	0.17	-	0.247	1.612	2.458	0.658	0.135	0.208
MaxLoss	0.17	-	0.231	1.61	0	0.64	0.125	0.196
CDaR	0.2	-	0.233	1.505	2.397	0.499	0.135	0.219
CVaR	0.2	-	0.222	1.618	2.468	0.6628	0.119	0.18
MAD	0.2	-	0.233	1.639	2.562	0.685	0.124	0.193
MaxLoss	0.2	-	0.227	1.611	0	0.648	0.122	0.189
CDaR	0.005	+	0.251	0.529	0.642	0.089	0.42	0.225
CVaR	0.005	+	0.267	0.543	0.642	0.086	0.437	0.241
MAD	0.005	+	0.22	0.183	0.056	0.011	1.041	0.2078
MaxLoss	0.005	+	0.249	0.537	0	0.092	0.408	0.227
CDaR	0.01	+	0.255	0.581	0.735	0.117	0.387	0.231
CVaR	0.01	+	0.268	0.604	0.712	0.122	0.393	0.242
MAD	0.01	+	0.231	0.301	0.216	0.0295	0.668	0.211
MaxLoss	0.01	+	0.254	0.5870	0	0.108	0.382	0.231
CDaR	0.03	+	0.258	0.757	0.991	0.194	0.302	0.233
CVaR	0.03	+	0.276	0.774	0.923	0.193	0.318	0.248
MAD	0.03	+	0.238	0.611	0.657	0.104	0.341	0.211
MaxLoss	0.03	+	0.253	0.707	0	0.165	0.315	0.233
CDaR	0.05	+	0.266	0.754	0.891	0.184	0.312	0.237
CVaR	0.05	+	0.265	0.841	1.122	0.231	0.279	0.237
MAD	0.05	+	0.261	0.829	1.159	0.241	0.27	0.25
MaxLoss	0.05	+	0.256	0.765	0	0.173	0.295	0.238
CDaR	0.07	+	0.263	0.802	1.08	0.227	0.29	0.244
CVaR	0.07	+	0.252	0.875	1.22	0.232	0.254	0.218
MAD	0.07	+	0.247	0.859	1.272	0.271	0.252	0.232
MaxLoss	0.07	+	0.284	1.345	0	0.553	0.189	0.265
CDaR	0.10	+	0.223	0.848	1.194	0.276	0.228	0.211
CVaR	0.10	+	0.248	0.882	1.268	0.244	0.246	0.214
MAD	0.10	+	0.237	0.861	1.268	0.252	0.241	0.209
MaxLoss	0.10	+	0.258	0.854	0	0.239	0.266	0.230

Type of constraint	Risk tolerance	Zero-beta	Aver. Rate of Return	Standard Deviation	CVaR	Drawdown	Sharpe Ratio	Yearly Aver. Return
CDaR	0.12	+	0.244	0.851	1.215	0.276	0.252	0.224
CVaR	0.12	+	0.247	0.882	1.27	0.244	0.246	0.214
MAD	0.12	+	0.257	0.883	1.268	0.245	0.257	0.233
MaxLoss	0.12	+	0.252	0.865	0	0.244	0.257	0.223
CDaR	0.15	+	0.22	0.858	1.141	0.246	0.221	0.186
CVaR	0.15	+	0.247	0.882	1.268	0.244	0.246	0.214
MAD	0.15	+	0.251	0.88	1.268	0.245	0.251	0.222
MaxLoss	0.15	+	0.247	0.882	0	0.244	0.246	0.214
CDaR	0.17	+	0.243	0.843	1.108	0.223	0.252	0.228
CVaR	0.17	+	0.247	0.882	1.268	0.244	0.246	0.214
MAD	0.17	+	0.251	0.88	1.268	0.245	0.251	0.222
MaxLoss	0.17	+	0.247	0.882	0	0.244	0.246	0.214
CDaR	0.20	+	0.216	0.864	1.248	0.278	0.215	0.186
CVaR	0.20	+	0.247	0.882	1.27	0.244	0.246	0.214
MAD	0.20	+	0.251	0.88	1.268	0.2458	0.251	0.221
MaxLoss	0.20	+	0.247	0.882	0	0.244	0.246	0.214

Table 3-4: Summary statistics for the "mixed" out-of-sample tests.

Type of constraint	Risk tolerance	Zero-beta	Aver. Rate of Return	Standard Deviation	CVaR	Drawdown	Sharpe Ratio	Yearly Aver. Return
CDaR	0.005	-	0.1790	0.4246	0.5024	0.112	0.3504	0.165
CVaR	0.005	-	0.1564	0.4175	0.6162	0.1698	0.3027	0.142
MAD	0.005	-	0.2184	0.0949	-0.026	0.005	1.984	0.212
MaxLoss	0.005	-	0.177	0.396	0.852	0.129	0.370	0.166
CDaR	0.01	-	0.19	0.437	0.508	0.120	0.366	0.175
CVaR	0.01	-	0.143	0.493	0.788	0.218	0.229	0.133
MAD	0.01	-	0.204	0.182	0.16	0.032	0.96	0.19
MaxLoss	0.01	-	0.165	0.45	1.01	0.17	0.30	0.155
CDaR	0.03	-	0.170	0.518	0.649	0.171	0.271	0.161
CVaR	0.03	-	0.112	0.67	1.19	0.349	0.123	0.104
MAD	0.03	-	0.174	0.486	0.738	0.230	0.296	0.161
MaxLoss	0.03	-	0.154	0.628	1.44	0.275	0.198	0.145
CDaR	0.05	-	0.154	0.592	0.840	0.217	0.21	0.15
CVaR	0.05	-	0.125	0.807	1.452	0.366	0.117	0.126
MAD	0.05	-	0.11	0.8556	1.5936	0.4086	0.0936	0.105
MaxLoss	0.05	-	0.143	0.757	1.886	0.305	0.149	0.141
CDaR	0.07	-	0.114	0.681	1.042	0.279	0.123	0.117
CVaR	0.07	-	0.074	0.861	1.689	0.432	0.0510	0.076
MAD	0.07	-	0.040	0.993	2.05	0.550	0.010	0.025
MaxLoss	0.07	-	0.114	0.811	2.34	0.361	0.104	0.115
CDaR	0.10	-	0.132	0.741	1.19	0.331	0.137	0.132
CVaR	0.10	-	0.001	1.07	2.32	0.647	-0.027	-0.0006
MAD	0.10	-	-0.027	1.157	2.346	0.69	-0.049	-0.0448
MaxLoss	0.10	-	0.071	0.943	2.92	0.487	0.043	0.067
CDaR	0.12	-	0.134	0.774	1.27	0.354	0.134	0.136
CVaR	0.12	-	-0.012	1.140	2.4256	0.681	-0.037	-0.0184
MAD	0.12	-	-0.014	1.16	2.346	0.667	-0.038	-0.033
MaxLoss	0.12	-	0.0576	1.039	3.106	0.545	0.0265	0.049
CDaR	0.15	-	0.124	0.773	1.36	0.352	0.122	0.124
CVaR	0.15	-	-0.026	1.153	2.336	0.691	-0.048	-0.044
MAD	0.15	-	-0.016	1.185	2.346	0.674	-0.039	-0.0349

Type of constraint	Risk tolerance	Zero-beta	Aver. Rate of Return	Standard Deviation	CVaR	Drawdown	Sharpe Ratio	Yearly Aver. Return
MaxLoss	0.15	-	0.0108	1.152	3.75	0.644	-0.017	-0.0039
CDaR	0.17	-	0.142	0.84	1.35	0.429	0.134	0.139
CVaR	0.17	-	-0.0151	1.164	2.356	0.67	-0.039	-0.034
MAD	0.17	-	-0.017	1.167	2.35	0.67	-0.040	-0.036
MaxLoss	0.17	-	-0.008	1.19	4.0429	0.682	-0.032	-0.024
CDaR	0.20	-	0.142	0.84	1.354	0.429	0.134	0.14
CVaR	0.20	-	-0.0151	1.164	2.356	0.67	-0.039	-0.034
MAD	0.20	-	-0.0169	1.167	2.356	0.67	-0.04	-0.036
MaxLoss	0.20	-	-0.008	1.189	4.042	0.682	-0.032	-0.024
CDaR	0.005	+	0.190	0.374	0.416	0.089	0.428	0.176
CVaR	0.005	+	0.18	0.401	0.467	0.128	0.374	0.165
MAD	0.005	+	0.208	0.097	-0.006	0.0031	1.841	0.202
MaxLoss	0.005	+	0.203	0.384	0.815	0.111	0.452	0.19
CDaR	0.01	+	0.207	0.419	0.435	0.078	0.422	0.19
CVaR	0.01	+	0.173	0.453	0.61	0.154	0.315	0.157
MAD	0.01	+	0.227	0.173	0.088	0.022	1.144	0.218
MaxLoss	0.01	+	0.195	0.441	1.11	0.146	0.374	0.181
CDaR	0.03	+	0.210	0.478	0.587	0.109	0.378	0.197
CVaR	0.03	+	0.153	0.629	1.059	0.290	0.196	0.142
MAD	0.03	+	0.166	0.456	0.593	0.179	0.298	0.149
MaxLoss	0.03	+	0.235	0.563	1.175	0.175	0.364	0.223
CDaR	0.05	+	0.205	0.533	0.672	0.136	0.328	0.193
CVaR	0.05	+	0.131	0.701	1.208	0.378	0.143	0.12
MAD	0.05	+	0.103	0.706	1.222	0.368	0.1038	0.088
MaxLoss	0.05	+	0.208	0.639	1.383	0.212	0.278	0.203
CDaR	0.07	+	0.191	0.592	0.747	0.114	0.271	0.182
CVaR	0.07	+	0.118	0.716	1.23	0.375	0.122	0.102
MAD	0.07	+	0.105	0.716	1.23	0.391	0.105	0.090
MaxLoss	0.07	+	0.16	0.677	1.865	0.282	0.192	0.155
CDaR	0.10	+	0.071	0.943	2.92	0.487	0.043	0.067
CVaR	0.10	+	0.1098	0.722	1.23	0.389	0.110	0.094
MAD	0.10	+	0.104	0.721	1.23	0.388	0.102	0.088
MaxLoss	0.10	+	0.142	0.701	1.749	0.324	0.16	0.133

Type of constraint	Risk tolerance	Zero-beta	Aver. Rate of Return	Standard Deviation	CVaR	Drawdown	Sharpe Ratio	Yearly Aver. Return
CDaR	0.12	+	0.161	0.627	0.818	0.209	0.209	0.148
CVaR	0.12	+	0.109	0.723	1.23	0.392	0.109	0.093
MAD	0.12	+	0.104	0.722	1.23	0.388	0.102	0.089
MaxLoss	0.12	+	0.128	0.709	1.66	0.353	0.139	0.117
CDaR	0.15	+	0.142	0.635	0.877	0.241	0.176	0.128
CVaR	0.15	+	0.109	0.723	1.23	0.391	0.109	0.093
MAD	0.15	+	0.105	0.716	1.23	0.391	0.104	0.09
MaxLoss	0.15	+	0.110	0.721	1.798	0.388	0.112	0.0952
CDaR	0.17	+	0.142	0.635	0.877	0.241	0.177	0.128
CVaR	0.17	+	0.109	0.723	1.23	0.39	0.11	0.094
MAD	0.17	+	0.105	0.716	1.23	0.391	0.104	0.09
MaxLoss	0.17	+	0.11	0.721	1.798	0.388	0.112	0.0950
CDaR	0.20	+	0.11	0.647	0.993	0.285	0.123	0.094
CVaR	0.20	+	0.109	0.723	1.23	0.391	0.109	0.093
MAD	0.20	+	0.105	0.721	1.23	0.39	0.104	0.09
MaxLoss	0.20	+	0.109	0.723	1.8	0.391	0.11	0.093

CHAPTER 4
OPTIMAL POSITION LIQUIDATION

We consider the problem of optimal position liquidation with the aim of maximizing the expected cash flow stream from the transaction in the presence of a temporary or permanent market impact. We use a stochastic programming approach to derive trading strategies that differentiate decisions with respect to different realizations of market conditions. The scenario set consists of a collection of sample paths representing possible future realizations of state variable processes (price of the security, trading volume etc.) At each time moment the set of paths is partitioned into several groups according to specified criteria, and each group is controlled by its own decision variable(s), which allows for adequate representation of uncertainties in market conditions and circumvents anticipativity in the solutions. In contrast to traditional dynamic programming approaches, the stochastic programming formulation admits incorporation of different types of constraints in the trading strategy, e.g. risk constraints, regulatory constraints, various decision-making policies etc. We consider the *lawn-mower principle*, which increases stability of the solution with respect to paths partitioning and saturation of the scenario pattern, but leads to non-convex optimization problems. It is shown that in the case of temporary market impact the optimal liquidation strategy with the lawn-mower principle can be approximated by a solution of convex or linear programming problems. Implemented as a linear programming problem, our approach is capable of handling large-scale instances and produces robust optimal solutions. A risk-averse trading strategy was constructed by incorporating risk constraints in the stochastic programming problem. We controlled the risk, associated with trading, using the Conditional Value-at-Risk measure. Numerical results and optimal trading patterns for different forms of market impact are presented.

4.1 Introduction

This chapter presents numerical techniques for optimal transaction implementation, e.g., determining the best way to sell (buy) a specified number of shares on the market during a prescribed period of time.

The main challenge of constructing optimal trading strategies consists in preventing or minimizing losses caused by the so-called *market impact*, or *price slippage*, that affect payoffs during market transactions. The market impact phenomenon manifests itself by adverse price movements during trades and is caused by disturbances of the market equilibrium. As a rather simplistic explanation, consider an investor executing a market order to sell a block of shares. If at the moment of execution no one is willing to buy this block for the current market price, the seller will be forced to offer a lower price per share in order to accomplish the transaction, and consequently suffer a loss due to the *market impact*. In reality the function of adjusting the price during transactions is performed by market intermediaries.

Traditionally, two types of market impact are considered: *temporary* and *permanent*. Permanent market impact denotes the changes in prices that are caused by the investor's trades and persist during the entire period of his/her trading activity. If the deviations in prices caused by investor's transaction are unobservable by the time of his/her next trade, the market impact is said to be temporary. Naturally, the magnitude of market impact and, consequently, the loss due to adverse price movements, depend on the size of trades, as well as on the time windows between transactions. For a detailed discussion of issues related to market impact, see, among others, Chan and Lakonishok (1995), Kein and Madhavan (1995), Kraus and Stoll (1972). Some of the latest developments on the optimal transaction implementation and optimal trading policies are presented in papers by Bertsimas and Lo (1998), Bertsimas et al. (1999), Rickard and Torre (1999), Almgren and Chriss (2000), and Almgren (2001).

Bertsimas and Lo (1998) employed a dynamic programming approach for devising optimal trading strategies that minimize the expected cost of trading a block of $\bar{S}$ shares within a fixed number of periods T. They derived analytical expressions for best-execution strategies in the standard framework of discrete random walk models, under assumption that market impact is linear in the number of shares traded. To gain an insight on how information component can influence the optimal strategy, authors introduce a serially correlated "information" variable in the price process of the security. Extension of this methodology to the case of multiple assets and optimal execution for portfolios is presented in Bertsimas et al. (1999).

Almgren and Chriss (2000) constructed risk-averse trading strategies using the classical Markowitz mean-variance methodology. With permanent and temporary market impact functions being also

linear in the number of shares traded, the risk of the trading strategy is associated with the volatility of the market value of the position. A continuous-time approximation of this approach, as well as nonlinear and stochastic temporary market impact functions, was considered by Almgren (2001). An application of fuzzy set theory to optimal transaction execution, which helps to mimic "non-rational" human behaviour of traders, is discussed in paper by Rickard and Torre (1999).

However, a common drawback of the described methodologies can be seen in the inability of a trading strategy to respond dynamically to realized or changed market conditions. Instead of having a prescribed sequence of trades based on the parameters of security's price process, we would like to develop a trading strategy that differentiates decisions with respect to actual realization of market conditions at each moment of transaction. Also, the trading strategy should be able to incorporate different types of constraints that may reflect investor's preferences, including risk preferences, institutional regulations, etc.

In this chapter, we consider the optimal liquidation problem in the scope of maximizing the expected cash flow from selling a block of shares in the market. The problem is formulated and solved in the stochastic programming framework, which allows for creating multi-stage decision-making algorithms with appropriate response to different realizations of uncertainties at each time moment. The key feature of our approach is a sample-path scenario model that represents the uncertain price process of the security by a set of its possible future trajectories (sample paths). The approach admits a seamless incorporation of various types of constraints in the definition of trading strategy, and is applicable under different forms of market impact.

The sample-path scenario models is a relatively new technique in the area of multi-stage decision making problems, where the dominant scenario models are the classical multinomial trees or lattices. Sample-path based simulation models have been recently used for pricing of American-style options (see Tilley (1993), Boyle et al. (1997), Broadie and Glasserman (1997), Carriere (1996), Barraquand and Martineau (1995), and others), where the optimal decision policy contains single binary decision (i.e., exercise-or-not-exercise the security contract). In this sense, the problems of optimal transaction execution are more complex, since the optimal strategy is a sequence of non-binary decisions.

The chapter is organized as follows. The next section introduces the general formulation of the optimal liquidation problem, definitions of market impact, etc. Sections 4.3 and 4.4 deal with

optimal position liquidation under temporary and permanent market impacts, correspondingly. In Section 4.5, we discuss construction of risk-averse trading strategies by incorporating Conditional Value-at-Risk constraints in the optimal liquidation problem. Section 4.6 presents a case study and numerical results, and Section 4.7 concludes the chapter.

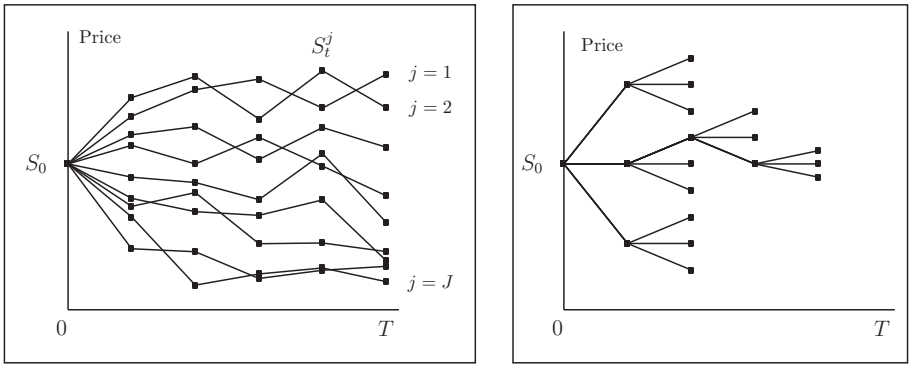
4.2 General Definitions and Problem Statement

In this section we introduce formal mathematical definitions and develop several formulations for the optimal closing problem. The formal setup for the optimal liquidation problem is as follows. Suppose that at time $t = 0$ there is an open position in some financial instrument (stock, bond, option etc.), which has to be liquidated (closed) within the predefined time interval $0 \leq t \leq T$. Assume also that trades are only allowed at the specified time moments $t = 1, 2, \ldots, T$ (integer indexing is used for briefness of notation; by writing $t = 0, 1, 2, \ldots$ we understand $t = t_0, t_1, t_2, \ldots$, with time moments t_0, t_1, $t_2, \ldots$, not necessarily equally spaced). Information on future market conditions at $t = 1, 2, \ldots, T$ is represented by a set of J sample paths (Fig. 4–1a). The objective is to generate an optimal trading policy which maximizes the expected cash flow stream incurred from liquidating the position. Throughout the chapter, we implicitly assume that the position to be liquidated is a long position.

4.2.1 Representation of Uncertainties by a Set of Paths

Traditional approaches to solving multistage decision-making problems with uncertainties are represented by the techniques of stochastic and dynamic programming, where the evolution of stochastic parameters is modeled by tree or lattice structures (Fig. 4–1b). These classical techniques have proved to be effective tools in dealing with multistage problems, especially in analytical framework. However, in many real-life financial applications that require solving large-scale optimization problems, use of the tree- or lattice- based scenario models may lead to considerable computational difficulties, known as the “curse of dimensions”. Therefore, many financial institutions in their research and investment practice adopt scenario models different from the classical multinomial trees and lattices. One of the most popular alternative approaches is representing the uncertain future as a collection of sample paths, each being a possible future trajectory of a financial instrument or group of instruments (Fig. 4–1a). This type of scenario model is supposed to overcome the “curse

of dimensions" that often defeats large-scale instances of optimization models based on scenario trees or lattices.



a. Set of sample paths b. Scenario trees

Figure 4–1: Sample paths and and scenario trees.

A significant deficiency of multinomial trees or lattices is a poor balance between randomness and the tree/lattice size. A small number of branches per node in a tree/lattice clearly makes a rough approximation of the uncertain future, whereas increasing of this parameter results in exponential growth of the overall number of scenarios in the tree.

In contrast to this, a sample-path scenario model represents the uncertain value of stochastic parameter at each time by a (large) number of sample points (nodes) belonging to different sample paths of the scenario set. Increasing the number of nodes at each time step for better accuracy results in a *linear* increase of the number of sample paths in the set. Similarly, increasing the number of time periods in the model leads to a linear increase of the number of nodes, as opposed to *exponential* growth in the number of nodes in scenario trees.

Besides superior scalability, sample-path concept allows for effortless incorporation of historical data into the scenario model, which is an important feature from a practical point of view.

There has been an increasing interest to use of sample paths in describing uncertain market environment in problems of finance and financial engineering during recent years. For the most part, this approach was employed in the area of pricing of derivatives (Titley (1993), Boyle et al. (1997), Broadie and Glasserman (1997), Carriere (1996), Barraquand and Martineau (1995) etc.). Recently, a sample paths framework was applied in solving dynamic asset and liability management problems (Hibiki (1999, 2001)). In this chapter, we consider the concept of sample-path scenario

sets and corresponding optimization techniques in application to a problem of optimal transaction execution in presence of market imperfections and friction.

4.2.2 A Generic Problem Formulation

Let $\mathfrak{S}$ be a collection of sample paths

$$\mathfrak{S} = \left\{ (\mathbf{S}_0, \mathbf{S}_1^j, \mathbf{S}_2^j, \ldots, \mathbf{S}_T^j) \mid j = 1, \ldots, J \right\},$$

where each term $\mathbf{S}_t^j$ stands for, generally, a vector of relevant market parameters, such as the mid-price of the security S_t^j, bid-ask spread s_t^j, volume V_t^j, etc. at time t according to sample path j:

$$\mathbf{S}_t^j = (S_t^j, s_t^j, V_t^j, \ldots).$$

$\mathbf{S}_t^j$ may also include other information observable at time t, e.g., major financial indices, interest rates etc. In the simplest case, $\mathbf{S}_t^j$ may only contain price S_t^j of the security.

We define a *trading strategy*, corresponding to the sample-path collection $\mathfrak{S}$, as a set

$$\Xi = \left\{ (\xi_0^j, \xi_1^j, \ldots, \xi_T^j) \mid 1 = \xi_0 \geq \xi_1^j \geq \ldots \geq \xi_T^j = 0, \quad j = 1, \ldots, J \right\}, \tag{4.1}$$

where ξ_t^j is the *normalized* value of the position at time t on path j. The instant proceeds incurred by transaction at time t are determined by the *payoff function* $p_t(\cdot)$, whose general form is

$$p_t(\xi_\tau^j; \mathbf{S}_\tau^j \mid \tau \leq t), \quad j = 1, \ldots, J, \quad t = 1, \ldots, T.$$

Let the payoff function $p_t(\cdot)$ incorporate discounting and transaction costs, market slippage etc. The explicit form of the function $p_t(\cdot)$ and its impact on properties of the optimal liquidation problem is discussed below; now we stress that payoff at time t cannot depend on the values of $\mathbf{S}_\tau^j$ and ξ_τ^j for $\tau > t$.

The objective of our trading strategy is to maximize the expected cash flow stream incurred from selling the asset:

$$\max_{\Xi} \; \mathsf{E}_{j \in \mathfrak{S}} \left[\sum_{t=1}^{T} p_t(\xi_\tau^j; \mathbf{S}_\tau^j \mid \tau \leq t) \right]. \tag{4.2}$$

Problem (4.2) is a stochastic programming formulation of the optimal closing problem based on the sample-path scenario model. Here $\mathsf{E}_{j \in \mathfrak{S}}$ is the expectation operator defined on $\mathfrak{S}$, and Ξ denotes the set of all possible trading strategies (4.1). It will be seen later that the generic formulation (4.2) is

far from being perfect; before diving into further details, we have to discuss the form and properties of the payoff function $p_t(\cdot)$ in (4.2), which has a major impact on the properties of the problem of optimal closing in general.

4.2.3 Modeling the Market Impact

Generally, the payoff on path j at time t is a function of the preceding series $\mathbf{S}_\tau^j$ and position values ξ_τ^j:

$$p_t(\cdot) = p_t(\xi_0, \xi_1^j, \ldots, \xi_t^j;\, \mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_t^j).$$

Dependence of the payoff function $p_t(\cdot)$ on the preceding decisions ξ_0, $\xi_1^j, \ldots, \xi_{t-1}^j$ constitutes the generally nonlinear effect of *permanent market impact*, when the trading activity of the market player contributes to changes in prices that do not vanish during the trading period. When the price at current moment t is unaffected by our preceding trades, but does depend on the size of current transaction, it is said that only *temporary market impact* is present. First, we consider the temporary market impact as it allows for computationally more tractable formulations of the optimal liquidation problem.

Temporary market impact. Let the current payoff $p_t(\cdot)$ depend only on the size $\Delta\xi_t$ of the current transaction

$$p_t(\cdot) = p_t(\Delta\xi_t^j; \mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_t^j), \quad \Delta\xi_t^j = \xi_{t-1}^j - \xi_t^j.$$

This form of the payoff function is appropriate when the price changes caused by the transaction at time t are negligibly small by the next time moment, $t+1$. Assume further that function $p_t(\cdot)$ admits the representation

$$p_t(\Delta\xi_t^j; \mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_t^j) = S_t^j\, \delta(\Delta\xi_t^j; \mathbf{S}_\tau^j \,|\, \tau \le t), \tag{4.3}$$

where the price S_t^j is always positive: $S_t^j > 0$. The term $\delta(\Delta\xi_t^j; \mathbf{S}_\tau^j \,|\, \tau \le t)$ in (4.3) captures the effects of market friction. It may depend parametrically on information observable at path j up to time t, e.g., volume V_t^j, prices $S_0, \ldots, S_t^j$, etc.[1] To lighten the notation, we suppress the term

[1] The current volume of trades V_t may be defined as the number of trades between $t-1$ and t.

$\mathbf{S}_\tau^j \,|\, \tau \le t$ in $\delta(\cdot)$ and introduce subscript t:

$$\delta(\Delta\xi_t^j\,;\mathbf{S}_\tau^j \,|\, \tau \le t) \triangleq \delta_t(\Delta\xi_t^j).$$

As a function of the trade size $\Delta\xi = y$, $\delta_t(y)$ satisfies

(i) $\delta_t : [0,1] \mapsto [0,1], \quad \delta_t(0) = 0.$

In the perfect frictionless market, obviously, $\delta(y) = y$, $y \in [0,1]$. We assume that in the presence of temporary market impact, proportional transaction costs[2] etc., $\delta_t(\cdot)$ further satisfies

(ii) $\delta_t(y) \le y, \quad y \in [0,1];$

(iii) $\forall\ 0 \le y_1 < y_2 \le 1\ :\ \delta_t(y_1) < \delta_t(y_2);$

(iv) $\delta_t(y)$ is concave on [0,1].

Condition (ii) states that payoff in the market with friction is always less than that in the frictionless market. Condition (iii) and (iv) ensure that larger trades lead to higher revenues, however, the marginal revenues become smaller with increasing of trade size (Fig. 4–2).

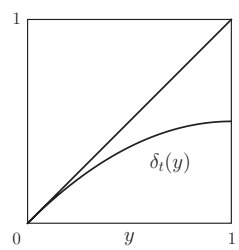


Figure 4–2: Impact function $\delta_t(\cdot)$.

The importance of the concavity requirement (iv) will be clear later when we present refined formulations for the optimization problem (4.2). Observe also that (iv) implies that $\delta_t(\cdot)$ is continuous on [0,1]: $\delta_t \in C([0,1])$.

Summarizing the aforesaid, in the case of temporary market impact we write the payoff function $p_t(\cdot)$ in the objective of (4.2) in the form

$$p_t(\xi_\tau^j;\mathbf{S}_\tau^j|\tau \le t) = S_t^j\,\delta_t(\Delta\xi_t^j) \tag{4.4}$$

[2] Fixed transaction costs and fees are usually much smaller than the losses caused by market impact, and can be neglected.

where $S_t^j = \text{const} > 0$, and $\delta(\cdot)$ satisfies conditions (i)–(iv).

Permanent market impact. A more realistic model of market response to the trading activity involves the permanent market impact as the dependence of current market prices on the preceding transactions of the market player. To incorporate the permanent market impact in the model, we assume the following price dynamics for S_t^j

$$S_t^{*j} = S_t^j - S_t^j \theta_t(\Delta\xi_t^j) - \sum_{\tau=1}^{t-1} S_\tau^j \hat{\theta}_\tau(\Delta\xi_\tau^j), \tag{4.5}$$

where S_τ^j, $\tau = 1, \ldots, t$ is the "undisturbed" price trajectory that would realize in the absence of our trades, and S_t^{*j} is the actual price at moment t and path j received by investor for liquidating portion $\Delta\xi_t^j$ of the position. Functions $\theta_t(\Delta\xi)$ and $\hat{\theta}_t(\Delta\xi)$ are equal to the percentage drop in market price due to the temporary and permanent market impact, correspondingly. Similarly to the above notation $\delta_t(\cdot)$, functions $\theta_t(\cdot)$ and $\hat{\theta}_t(\cdot)$ may contain, in general, values $\mathbf{S}_0, \ldots, \mathbf{S}_t^j$ as parameters:

$$\theta_t(\Delta\xi_t^j) \triangleq \theta(\Delta\xi_t^j; \mathbf{S}_\tau^j \,|\, \tau \le t), \quad \hat{\theta}_t(\Delta\xi_t^j) \triangleq \hat{\theta}(\Delta\xi_t^j; \mathbf{S}_\tau^j \,|\, \tau \le t).$$

Using (4.5), the total payoff attained over a path $(\mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_T^j)$ is

$$\begin{aligned}\sum_{t=1}^{T} S_t^{*j} \Delta\xi_t^j &= \sum_{t=1}^{T} \left\{ S_t^j \left(\Delta\xi_t^j - \Delta\xi_t^j \theta_t(\Delta\xi_t^j) \right) - \Delta\xi_t^j \sum_{\tau=1}^{t-1} S_\tau^j \hat{\theta}_\tau(\Delta\xi_\tau^j) \right\} \\ &= \sum_{t=1}^{T} S_t^j \left\{ \Delta\xi_t^j - \Delta\xi_t^j \theta_t(\Delta\xi_t^j) - \hat{\theta}_t(\Delta\xi_t^j) \sum_{\tau=t+1}^{T} \Delta\xi_\tau^j \right\}. \end{aligned} \tag{4.6}$$

The first term in braces in (4.6) attributes to the profit of selling the portion $\Delta\xi_t^j$ of the position in perfect frictionless market. The second and third summands in (4.6) represent the losses due to effects of temporary and permanent market impacts, correspondingly.

For consistency with the above discussion of temporary market impact, we assume that $\theta_t(\cdot)$ is such that the function

$$y - y\,\theta_t(y)$$

satisfies conditions (i)–(iv). This allows us to think of $\theta_t(\cdot)$ as a non-negative non-decreasing function on $[0, 1]$ (it may be convex or concave).

Similarly, the permanent market impact function $\hat{\theta}_t(\cdot)$ is assumed to be non-negative and non-decreasing. We suppose that function $\hat{\theta}_t(\cdot)$ satisfies

$$\hat{\theta}_t(\Delta\xi) = 0, \quad 0 \le \Delta\xi \le \lambda_t \quad \text{for some } \lambda_t \in (0,1).$$

This condition reflects the presumption that small enough trades $\Delta\xi_t$ should not cause permanent changes in market prices given the time windows $t_k - t_{k-1}$ between transactions. A plausible form of $\hat{\theta}_t(\cdot)$ is as follows:

$$\hat{\theta}_t(\Delta\xi) = \max\{0,\ \kappa\,\theta_t(\Delta\xi - \lambda_t)\}, \tag{4.7}$$

where $\kappa = \text{const} \in (0,\,1]$.

4.3 Optimal Position Liquidation under Temporary Market Impact

In this section we present a sample-path approach to generation of optimal liquidation policies when only temporary market impact is observable. The framework includes the special case of zero market impact, which constitutes an important extension of the considered technique to pricing of derivative securities. First, we present the path grouping method that eliminates anticipativity in solutions of the optimization problem based on sample-path approach (4.2). Then, the "lawn-mower" decision-making rule, which results in trading strategies of specialized form, is introduced. The properties of "lawn-mower"-compliant trading strategies are discussed.

4.3.1 Paths Grouping and "Lawn-Mower Strategy"

In Section 4.2.2, we introduced a generic formulation (4.2) for the optimal closing problem, which is rewritten here taking into account the form of the payoff function (4.4) under temporary market impact, and replacing the expectation operator by an average over the set of paths:

$$\max_{\Xi}\ \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j\,\delta_t(\Delta\xi_t^{\,j}). \tag{4.8}$$

It turns out, however, that the trading strategy based on the optimal solution of (4.8) is contingent on perfect knowledge of the future and therefore inappropriate. As an example, consider a simple case when $S_t^j\delta_t(\Delta\xi_t^{\,j}) = S_t^j\Delta\xi_t^{\,j}$, i.e., the payoff equals to the dollar profit of selling a portion $\Delta\xi_t^{\,j}$ of position for price S_t^j in frictionless market. Then, it is easy to see that the optimal solution of (4.8)

is given by

$$\xi_t^j = \begin{cases} 1, & \text{for} \quad t \le t_j - 1, \\ 0, & \text{for} \quad t \ge t_j, \end{cases} \qquad \text{where} \quad t_j = \underset{t=1,\dots,T}{\arg\max}\{S_t^j\}, \quad j = 1,\dots,J. \tag{4.9}$$

The optimal strategy (4.9) liquidates the whole position once the maximal price over the entire trajectory $(S_0, S_1^j, \dots, S_T^j)$ is achieved. In practice this would mean that the trader perfectly forecasts the future by identifying some path j with the *actual* price trajectory (for example, by observing price S_1 at time $t = 1$).

From the formal point of view, the anticipativity of the optimal solution of (4.2) is attributable to the possibility of making separate decisions regarding the position value ξ_t^j at each t and j.

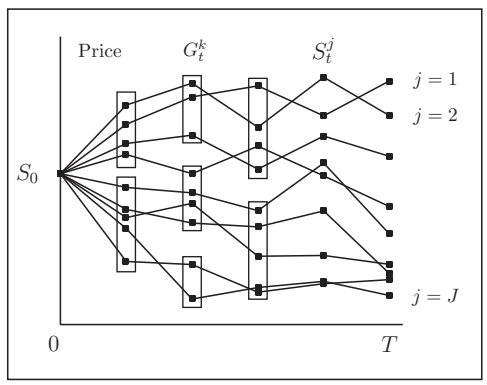


Figure 4–3: Path grouping

Paths grouping. A remedy for the anticipativity in formulation (4.2) is the so-called *path grouping* (see similar approaches in Titley (1993) and Hibiki (1999, 2001)). The set of paths $\mathfrak{S}$ is partitioned into K_t groups G_t^k at each time t (Fig. 4–3)

$$\pi(\mathfrak{S}) = \left\{ G_t^k \;\middle|\; \forall t = 1,\dots,T: \; \{1,\dots,J\} = \bigcup_{k=1}^{K_t} G_t^k, \;\; G_t^{k_1} \cap G_t^{k_2} = \emptyset, \;\; k_1 \ne k_2 \right\}. \tag{4.10}$$

One possible approach is to make identical decision at each path in a given group:

$$\xi_t^j = x_t^{k(j,t)} \quad \forall\, j \in G_t^k, \quad t = 1,\dots,T. \tag{4.11}$$

Here $x_t^{k(j,t)}$ is the new decision variable, equal to the value of position variables ξ_t^j in the group G_t^k, and $k(j,t)$ is the function that returns the index k of set G_t^k, which contains path j at moment t. We

denote this as

$$k(j,t) \propto \pi(\mathfrak{S}).$$

Hibiki (1999, 2001) employed the "tree-like" grouping, satisfying the following property: *for all* $t = 2,\dots,T$, *and for any* $k \in \{1,\dots,K_t\}$ *there exists* $k^* \in 1,\dots,K_{t-1}$ *such that*

$$G_t^k \subseteq G_{t-1}^{k^*}. \tag{4.12}$$

In other words, time t partition $\bigcup_{k=1}^{K_t} G_t^k$ is obtained from time $t-1$ partition $\bigcup_{k=1}^{K_{t-1}} G_{t-1}^k$ by splitting groups G_{t-1}^k into subgroups, similarly to splitting the scenario tree into branches at each time step (Fig. 4–4). In this setting, the decision rule (4.11) is analogous to what is long known in stochastic programming as *non-anticipativity constraints* (see, for example, Birge and Louveaux (1997)).

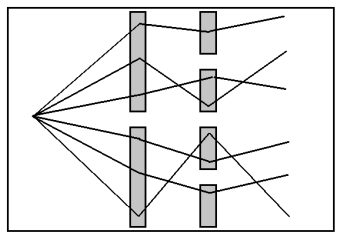


Figure 4–4: Tree-like grouping.

We generalize the path grouping method, compared to Titley (1993) and Hibiki (1999, 2001) in two respects. First, we relax the "tree" grouping condition (4.12), thus allowing for "path intermixing" (i.e., paths from different groups can merge into the same group at the next step)

$$\exists\, t,k,j_1,j_2: \quad j_1,j_2 \in G_t^k, \;\; G_{t-1}^{k(j_1,t-1)} \neq G_{t-1}^{k(j_2,t-1)}.$$

In this way we avoid having smaller and smaller number of paths per group at each next time step, which is inevitable for the "tree" grouping.

Second, we consider general decision rules in the following form:

$$\forall j \in G_t^k: \quad \xi_t^j = f(x_t^{k(j,t)}, \xi_{t-1}^j, \dots, \xi_1^j). \tag{4.13}$$

Dependence of all the position variables ξ_t^j within a group G_t^k on the common decision variable $x_t^{k(j,t)}$ ensures the non-anticipativity of the solution. Below we prove that with the "tree" grouping (4.12) the decision policy (4.13) is identical to the simple rule (4.11).

Proposition 4.3.1 *If partition $\pi(\mathfrak{S})$ satisfies the "tree" grouping condition* (4.12)*, then the generalized group decision rule* (4.13) *is equivalent to* (4.11)*.*

Proof. Consider time $t = 1$. Then $\forall j \in G_1^k:\ \xi_1^j = f(x_1^{k(j,1)}, \xi_0^j) = f(x_1^{k(j,1)}, 1) = \tilde{x}_1^{k(j,1)}$. At the next step, $t = 2$, we have $\forall j \in G_2^k:\ \xi_2^j = f(x_2^{k(j,2)}, \xi_1^j, \xi_0^j) = f(x_2^{k(j,2)}, \tilde{x}_1^{k(j,1)}, 1)$. Since $\tilde{x}_1^{k(j,1)}$ is constant $\forall j \in G_2^k$, then $f(x_2^{k(j,2)}, \tilde{x}_1^{k(j,1)}, 1) = \tilde{x}_2^{k(j,2)}$. Proceeding to $t = T$, in finite number of steps we obtain that $\forall t, \quad \forall j \in G_t^k:\ \xi_t^j = \tilde{x}_t^{k(j,t)}$. ■

The general formulation of the optimal liquidation strategy under temporary market impact is

$$\begin{aligned} &\max_{\Xi}\ \mathsf{E}_{j\in\mathfrak{S}}\left[\sum_{t=1}^{T} S_t^j \delta_t(\Delta\xi_\tau^j)\right] \\ &\text{s.t.}\quad \forall j \in G_t^k:\ \ \xi_t^j = f(x_t^{k(j,t)}, \xi_{t-1}^j, \ldots, \xi_1^j), \quad k(j,t) \propto \pi(\mathfrak{S}). \end{aligned} \tag{4.14}$$

"Lawn-mower" strategy. In what follows, we consider a special case of the group decision rule (4.13) that is pertinent to the construction of an optimal liquidating strategy with a temporary or zero market impact, namely

$$\forall j \in G_t^k: \quad \xi_t^j = \min\{\, \xi_{t-1}^j,\, x_t^{k(j,t)} \,\}, \quad 0 \le x_t^{k(j,t)} \le 1. \tag{4.15}$$

The intuition behind relation (4.15) is as follows. First of all, the decision variable $x_t^{k(j,t)}$ represents an upper bound for the position values ξ_t^j in a group G_t^k:

$$\xi_t^j \le x_t^{k(j,t)}, \quad k(t,j) \propto \pi(\mathfrak{S}). \tag{4.16}$$

Moreover, observe that position value ξ_{t-1}^j is allowed to change (i.e., decrease) at time t *only* by being "trimmed" to the level $x_t^{k(j,t)}$. If the position value ξ_{t-1}^j is not high enough to be "trimmed" by $x_t^{k(j,t)}$, it remains unchanged at time t. This resembles lawn mowing, with position variables ξ_t^j being the "grass" and decision variables $x_t^{k(j,t)}$ being the "blades" (Fig. 4–5).

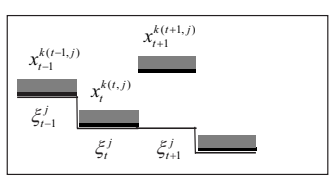


Figure 4–5: The "lawn-mower" principle.

It follows from (4.15) and the trading strategy definition (4.1), which requires zero position at the final time moment T, that variables x must satisfy

$$x_T^{k(j,T)} = 0, \quad j = 1, \ldots, J. \tag{4.17}$$

By incorporating the "lawn-mower" principle with path grouping $\pi(\mathfrak{S})$ into the formulation (4.8), we obtain the optimal closing problem in the form

$$\begin{aligned} \max_{\Xi} \quad & \frac{1}{J} \sum_{j=1}^{J} \sum_{t=1}^{T} S_t^j \, \delta_t(\Delta \xi_\tau^j) && && (4.18) \\ \text{s.t.} \quad & \xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}, && t = 1, ..., T, \ j = 1, ..., J, \\ & 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1, ..., T, \ j = 1, ..., J, \end{aligned}$$

By solving the problem (4.18) we find the optimal allocation of thresholds $x_t^{k(j,t)}$ determining the following optimal trading strategy: *if scenario j belongs to the group G_t^k, then sell the portion of position in excess of x_t^k; if current position value is less than x_t^k, do nothing.*

The next proposition shows that the optimal closing problem (4.18) is well-posed, i.e., for any allocation of thresholds x_t^k the corresponding trading strategy is unique.

Proposition 4.3.2 *Under the "lawn-mower" decision rule* (4.15) *each variable ξ_t^j is a single-valued function of variables $x_t^{k(j,t)}$. Inverse statement does not hold.*

Proof. Let us show that the "lawn-mower" principle (4.15) implies that the position variable ξ_t^j can be expressed as

$$\xi_t^j = \min\{x_1^{k(j,1)}, x_2^{k(j,2)}, \ldots, x_t^{k(j,t)}\}, \quad j = 1, \ldots, J, \ t = 1, \ldots, T. \tag{4.19}$$

Obviously, the assertion holds for $t = 1$: $\xi_1^j = \min\{1, x_1^{k(j,1)}\} = x_1^{k(j,1)} = \min\{x_1^{k(j,1)}\}$. Assume that (4.19) holds for some $t = \tau$, then

$$\begin{aligned} \xi_{\tau+1}^j = \min\{\xi_\tau^j, \, x_{\tau+1}^{k(j,\tau+1)}\} &= \min\left\{\min\{x_1^{k(j,1)}, \ldots, x_\tau^{k(j,\tau)}\}, \, x_{\tau+1}^{k(j,\tau+1)}\right\} \\ &= \min\{x_1^{k(j,1)}, \ldots, x_{\tau+1}^{k(j,\tau+1)}\}, \end{aligned}$$

which proves (4.19) by induction.

To show that the inverse is not generally true, consider a trading strategy $\{\xi_t^j\}$ that closes the position on all paths j at some $t_0 < T$:

$$\xi_{t_0}^j = 0 \quad \forall\, j.$$

Then, in accordance to the definition of trading strategy (4.1) and the "lawn-mower" principle (4.15), the position remains closed at all subsequent time moments no matter what are the threshold variables $x_t^{k(j,t)}$:

$$0 = \min\{0, x_t^k\}, \quad \forall\, x_t^k \in [0,1], \quad t_0 < t \leq T. \qquad \blacksquare$$

Remark 4.3.2 The introduced form of optimal liquidation problem (4.18) admits incorporation of different types of constraints into the trading strategy. Additional constraints may reflect investor's preferences regarding the desired structure of the optimal trading strategy, or make the strategy compliant with legal or institutional regulations, etc. Section 4.5, for example, discusses imposing risk constraints on the feasible set of problem (4.18). The next subsection demonstrates how to approximate the solution of (4.18) using convex or linear programming, which will allow for efficient solving of optimal position liquidation problem with additional constraints. On the contrary, the dynamic programming approach to optimal transaction execution (see Bertsimas and Lo (1998)) makes incorporation of additional constraints in the trading strategy very difficult.

Though the optimal closing problem (4.18) has a nice concave objective function, the feasible region of (4.18) is non-convex due to the "lawn-mower" constraint $\xi_t^j = \min\{\,\xi_{t-1}^j,\, x_t^{k(j,t)}\,\}$. Non-convexity of the constraint set is an undesirable property from a computational point of view, as it usually leads to computationally expensive optimization algorithms. Below we present modified formulations of the "lawn-mower" optimal liquidation problem and investigate properties of solutions of (4.18).

4.3.2 Approximation by Convex and Linear Programming

First, observe that formulation (4.18) of the optimal closing problem always produces a feasible in the sense (4.1) strategy for any choice of scenario set $\mathfrak{S}$ and partition $\pi(\mathfrak{S})$.

Proposition 4.3.3 *For any path partitioning $\pi(\mathfrak{S})$ problem (4.18) is feasible.*

Proof. Consider $\xi_0^j = 1,\ \xi_t^j = 0,\ t = 1,\ldots,T,\ j = 1,\ldots,J$, and $x_t^{k(j,t)} = 0,\ t = 1,..,T\ ,\ j = 1,..,J$.
■

To make the optimal closing problem (4.18) numerically tractable, we transform it into a problem with *convex feasible region* and *non-convex objective function*. To perform the transformation, let extend the domain of function $\delta_t(\cdot)$ to $[-1,\ 1]$:

(v) $\delta_t(y) = y, \quad y \in [-1, 0)$

Then, by introducing auxiliary variables u_t^j

$$u_t^j = \xi_{t-1}^j - x_t^{k(j,t)},\ \ t = 1,\ldots,T,\ \ j = 1,\ldots,J, \tag{4.20}$$

we construct the following problem

$$\begin{aligned} \max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\{0, \delta_t(u_t^j)\} && && (4.21)\\ \text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t = 1,\ldots,T,\ j = 1,\ldots,J,\\ & u_t^j = \xi_{t-1}^j - x_t^{k(j,t)}, && t = 1,\ldots,T,\ j = 1,\ldots,J,\\ & 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1,\ldots,T,\ j = 1,\ldots,J. \end{aligned}$$

Note that in accordance to definition (4.1) of trading strategy set Ξ and condition (4.17), we put

$$u_1^j = 1 - x_t^{k(j,t)}, \quad u_T^j = \xi_{T-1}^j.$$

The disadvantage of the formulation (4.21) is that its objective function does not have such a transparent meaning, as in (4.18). However, the linearity of the constraints makes the problem (4.21) much more tractable compared to (4.18), whose constraint set contains the non-convex "lawnmower" constraint (4.15).

Proposition 4.3.4 *Optimization problems* (4.18) *and* (4.21) *are equivalent in the sense that optimal values of their objectives are equal, and the sets of their optimal solutions in variables* ξ_t^j *and* $x_t^{k(j,t)}$ *coincide.*

Proof. First, observe that

$$\xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\} \quad \Leftrightarrow \quad \xi_{t-1}^j - \xi_t^j = \xi_{t-1}^j - \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}$$
$$\Leftrightarrow \quad \xi_{t-1}^j - \xi_t^j = \max\{0, \xi_{t-1}^j - x_t^{k(j,t)}\}, \tag{4.22}$$

thus problem (4.18) can be rewritten as

$$\begin{aligned}
\max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j\, \delta_t\left(\max\{0, \xi_{t-1}^j - x_t^{k(j,t)}\}\right) && && (4.23)\\
\text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t = 1,\dots,T,\ j = 1,\dots,J,\\
& \xi_{t-1}^j - \xi_t^j = \max\{0,\, \xi_{t-1}^j - x_t^{k(j,t)}\}, && t = 1,\dots,T,\ j = 1,\dots,J,\\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1,\dots,T,\ j = 1,\dots,J.
\end{aligned}$$

Note that the first constraint $\xi_t^j \le x_t^{k(j,t)}$ follows form the "lawn-mower" principle $\xi_t^j = \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}$. Therefore, this constraint does not change the feasible set. Taking into account the definition (4.20) of variables u_t^j and properties (i)–(v) of function $\delta_t(\cdot)$, the above formulation of problem (4.18) can be presented as

$$\begin{aligned}
\max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\{0, \delta_t(u_t^j)\} && && (4.24)\\
\text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t = 1,\dots,T,\ j = 1,\dots,J,\\
& u_t^j = \xi_{t-1}^j - x_t^{k(j,t)} && t = 1,\dots,T,\ j = 1,\dots,J,\\
& \xi_{t-1}^j - \xi_t^j = \max\{0, u_t^j\}, && t = 1,\dots,T,\ j = 1,\dots,J.\\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1,\dots,T,\ j = 1,\dots,J.
\end{aligned}$$

Problem (4.24) differs from (4.21) only by the third constraint. Now we show that the optimal solution of the problem (4.21) does also satisfy the constraint

$$\xi_{t-1}^j - \xi_t^j = \max\{0, u_t^j\}, \tag{4.25}$$

which is an equivalent form of the lawn-mower principle (4.15), given the definition (4.20) of u_t^j.

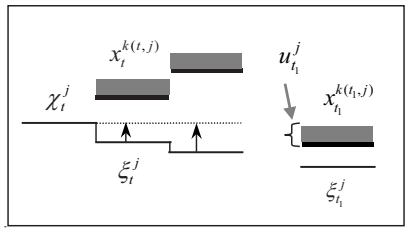


Figure 4–6: Illustration to proof of Proposition 4.3.2.

Assume that the optimal solution of problem (4.21) does not satisfy the lawn-mower principle (4.15). Obviously, the optimal solution of (4.21) cannot satisfy inequality

$$\xi_t^j > \min\{\xi_{t-1}^j, x_t^{k(j,t)}\}$$

since it violates the first constraint of (4.21) and the definition of the set of trading strategies Ξ. Therefore, the only case left is when in the optimal solution of problem (4.21) we have for some t and j

$$\xi_t^j < \min\{\xi_{t-1}^j, x_t^{k(j,t)}\} \triangleq \chi_t^j.$$

Since $x_T^{k(j,T)} = 0$ by definition, there exists $t_1 > t$ such that $x_{t_1}^{k(j,t_1)} < \chi_t^j$. Let t_1 be the smallest possible, then

$$x_\tau^{k(j,\tau)} \geq \chi_t^j, \quad t \leq \tau \leq t_1 - 1.$$

Therefore, for all $\tau = t, \ldots, t_1 - 1$, we can increase the values of position variables ξ_τ^j by positive amounts $\varepsilon_\tau^j = \chi_t^j - \xi_\tau^j > 0$ without violating feasibility:

$$\tilde{\xi}_\tau^j = \xi_\tau^j + \varepsilon_\tau^j = \chi_t^j, \quad t \leq \tau \leq t_1 - 1,$$

where $\tilde{\xi}_\tau^j$ are the new values of the position variables. As a result, variable $u_{t_1}^j$ increases by a positive amount: $\tilde{u}_{t_1}^j = \tilde{\xi}_{t_1-1}^j - x_{t_1}^{k(j,t_1)} = u_{t_1}^j + \varepsilon_{t_1-1}^j$. Positiveness of $\tilde{u}_{t_1}^j$,

$$\tilde{u}_{t_1}^j = \tilde{\xi}_{t_1-1}^j - x_{t_1}^{k(j,t_1)} = \chi_t^j - x_{t_1}^{k(j,t_1)} > 0,$$

and property (iii) of function $\delta_t(\cdot)$ imply that the objective of (4.21) has also been increased (the variable $u_{t_1}^j$ has either changed from negative to positive, or increased by positive amount $\varepsilon_{t_1-1}^j$).

This proves that the solution of (4.21) that does not satisfy the lawn-mower principle (4.15) is non-optimal. Consequently, the optimal solution of problem (4.21) does not change if the lawn-mower constraint (4.22) is added to the constraint set of (4.21), which would make the formulation of (4.21) identical to (4.24). Thus, we have shown that both problems (4.18) and (4.21) can be rewritten in the form (4.24), which implies that optimal values of their objectives, and their sets of optimal solutions in variables ξ_t^j and $x_t^{k(j,t)}$ coincide. ■

Remark 4.3.4 In general, problem (4.18) does not guarantee the uniqueness of the optimal trading strategy. As an example, one can consider the case with constant prices $S_t^j = S_0$ and identical functions $\delta_t(\cdot)$. Then, the profit will be the same for any trading strategy.

But even if the optimal trading strategy (in terms of the position variables ξ_t^j), derived as a solution of (4.18), is unique, the optimal threshold variables x_t^k may be non-single-valued (see Proposition 4.3.1).

Approximation by convex programming. The optimization problem (4.21) belongs to the class of problems with DC[3] objective and linear constraints, which is a well-studied topic in optimization. There exists a number of algorithms for solving problems with DC objective, e.g., various branch-and-bound schemes (see, for instance, Horst et al. (1995), or Konno and Wijayanayake (1999)), but all of them do not demonstrate satisfactory performance for large-scale problems.

In view of the presumably large size of problem (4.21), we suggest to approximate the solution of the DC maximization problem (4.21) by solution of a convex programming problem. In particular, we construct a concave maximization problem whose solution provides a lower bound to the optimal trading strategy given by the optimal solution of (4.21).

This is accomplished by replacing DC functions $\max\{0, \delta_t(u_t^j)\}$ in the objective (4.21) by concave functions $\delta_t(u_t^j)$, which can be further approximated by piece-wise linear concave functions $l_t(\cdot)$:

$$\delta_t(y) \approx l_t(y) = \min_y \{a_t^n y + b_t^n \mid n = 1, \ldots, N\}, \tag{4.26}$$

provided that $l_t(y)$ satisfies the properties (i) – (v).

[3] Difference of convex functions.

The concave maximization problem has the form

$$\begin{aligned}
\max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \delta_t(u_t^j) && && (4.27)\\
\text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t=1,\dots,T, \quad j=1,\dots,J,\\
& u_t^j = \xi_{t-1}^j - x_t^{k(j,t)}, && t=1,\dots,T, \quad j=1,\dots,J,\\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t=1,\dots,T, \quad j=1,\dots,J.
\end{aligned}$$

The special case of (4.27), when $\delta_t(x)$ is a piecewise-linear concave function, admits a linear programming representation:

$$\begin{aligned}
\max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j w_t^j && && (4.28)\\
\text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t=1,\dots,T, \quad j=1,\dots,J,\\
& w_t^j \le a_t^n \left(\xi_{t-1}^j - x_t^{k(j,t)}\right) + b_t^n, && n=1,\dots N, \quad t=1,\dots,T, \quad j=1,\dots,J,\\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t=1,\dots,T, \quad j=1,\dots,J.
\end{aligned}$$

Recall that we have circumvented the non-convexity of feasible region of (4.18) in the expense of having a DC objective function in (4.21). Since the objective of (4.27) is concave, we have to make sure that the optimal solution of problem (4.27) complies with the "lawn-mower" principle.

Proposition 4.3.5 *In the optimal solution of convex programming problem* (4.27)*, variables* u_t^j *are either non-negative, indicating that* $\xi_t^j = x_t^{k(j,t)}$*, or negative, indicating that* $\xi_t^j < x_t^{k(j,t)}$*. Moreover, the optimal solution of* (4.27) *satisfies the "lawn-mower" principle*

$$\xi_{t-1}^j - \xi_t^j = \max\{0, u_t^j\}.$$

Proof. First, consider the case of monotonically non-increasing thresholds $x_{t-1}^{k(j,t-1)} \ge x_t^{k(j,t)}$, $t = 2,\dots,T$ (Fig. 4–7a). Assume that for some j and t in the optimal solution of (4.27) we have

$$\xi_t^j < x_t^{k(j,t)},$$

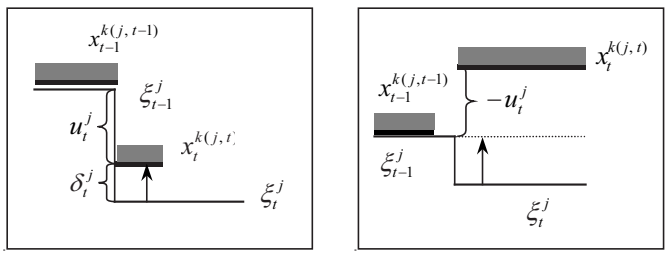
and let such t be the smallest possible for given path j. Evidently, in this case one can improve the objective term $p_{t+1}^j \delta_t(u_{t+1}^j)$ by increasing position variable ξ_t^j by a positive amount $\varepsilon_t^j = x_t^{k(j,t)} -$

$\xi_t^j > 0$:

$$\tilde{\xi}_t^j = \xi_t^j + \varepsilon_t^j \quad \Rightarrow \quad \delta_t(\tilde{u}_{t+1}^j) = \delta_t(\tilde{\xi}_t^j - x_{t+1}^{k(j,t+1)}) = \delta_t(u_t^j + \varepsilon_t^j) > \delta_t(u_t^j),$$

where $\tilde{u}_{t+1}^j$ and $\tilde{\xi}_t^j$ are the new values of variables u_{t+1}^j and ξ_t^j. This contradicts our assumption that the solution with u_t^j, ξ_t^j is optimal. Thus, in the optimal solution with monotonic thresholds must be $\xi_t^j = x_t^{k(j,t)}$, implying that

$$\xi_{t-1}^j - \xi_t^j = \xi_{t-1}^j - x_t^{k(j,t)} = u_t^j \geq 0 \quad \Rightarrow \quad \xi_{t-1}^j - \xi_t^j = \max\{0, u_t^j\}$$
$$\Rightarrow \quad u_t^j = \max\{0, \xi_{t-1}^j - x_t^{k(j,t)}\}.$$



a. Monotonic thresholds b. Non-monotonic thresholds

Figure 4–7: Illustration to proof of Proposition 4.3.5

Now suppose that thresholds $x_t^{k(j,t)}$ are non-monotonic (Fig. 4–7b). Let t be the smallest possible such that for some j we have

$$x_{t-1}^{k(j,t-1)} < x_t^{k(j,t)}.$$

If $\xi_t^j = \xi_{t-1}^j$ then $u_t^j < 0$ and the proposition holds. If, on the contrary, $\xi_t^j < \xi_{t-1}^j$, the solution (namely, objective variable u_{t+1}^j) can be improved by putting $\tilde{\xi}_t^j = \xi_{t-1}^j$, but u_t^j will remain negative as long as it is optimal to keep $x_{t-1}^{k(j,t-1)} < x_t^{k(j,t)}$. Therefore in the optimal solution of problem (4.27) the negative value of variable u_t^j implies that $\xi_t^j < x_t^{k(j,t)}$ and $\xi_t^j - \xi_{t-1}^j = 0 = \max\{0, u_t^j\}$. ■

Remark 4.3.5–1 Proposition 4.3.5 implies that the expected cash flow stream incurred from liquidating the position can be calculated by replacing negative terms $\delta_t(u_t^j)$ in the optimal objective function (4.27) by zeros :

$$\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \delta_t(\Delta\xi_t^j) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\{0, \delta_t(u_t^j)\}, \tag{4.29}$$

where u_t^j, ξ_t^j is the optimal solution of problem (4.27).

Remark 4.3.5–2 The optimal solution of concave programming problem (4.27) gives a lower bound of the optimal solution of DC maximization problem (4.21):

$$\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j\,\delta_t(u_t^j) \le \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\{0,\delta_t(u_t^j)\}. \tag{4.30}$$

Corollary 4.3.5 *With "tree" partitioning $\pi(\mathfrak{S})$, defined in* (4.12)*, problems* (4.18) *and* (4.27) *are equivalent, and the lower bound* (4.30) *is exact.*

Proof. Propositions 4.3.1 and 4.3.5 imply the statement. ■

4.3.3 Properties of Solutions of the Non-Convex and Convex Optimal Liquidation Problems

Optimal trading in frictionless market. An important special case of the introduced model is the case of zero market impact $\delta_t(y) = y$, which can provide a link between our approach to optimal transaction execution and pricing of derivative securities. In the absence of market impact a risk-neutral trader maximizes the expected cash flow stream by selling the whole position at a certain moment of time, which is analogous to exercising the contract of an American option.

Below we show that in the absence of market impact and transaction costs there always exists an optimal 0–1 trading strategy, compliant with the "lawn-mower" principle, even if the unique optimal solution of (4.18) does not exist.

Proposition 4.3.6 *In a frictionless market, $\delta_t(y) = y$, there exists an optimal 0–1 solution of optimal liquidation problem* (4.18)*.*

Proof. Taking into account form (4.25) of the "lawn-mower" principle and expression (4.19) for position variables ξ_t^j, problem (4.18) can be written as follows

$$\begin{aligned} \max \quad & \varphi(x) = \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\Big\{0,\ \min\{x_1^{k(j,1)},\dots,x_{t-1}^{k(j,t-1)}\} - x_t^{k(j,t)}\Big\} \\ \text{s. t.} \quad & 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), \quad t = 1,\dots,T, \quad j = 1,\dots,J. \end{aligned} \tag{4.31}$$

Observe that $\varphi(x)$ is a piecewise-linear function of $x \in \mathbb{R}^K$, $K = \sum_{t=1}^{T} K_t$, and can be represented as

$$\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\{0,\ x_{\tau_*}^{k(j,\tau_*)} - x_t^{k(j,t)}\}, \quad \text{where} \quad \tau_* = \tau_*(j,t) = \operatorname*{arg\,min}_{1\le\tau\le t-1}\{x_\tau^{k(j,\tau)}\}.$$

Using contradiction argument, we now show that there exists an optimal 0–1 solution to problem (4.31). For simplicity, let consider the optimal solution of (4.31) with only one non-binary component $x_{t_0}^{k_0} \in (0,1)$. Having all other elements of x fixed, consider $\varphi(\cdot)$ as a function of $x_{t_0}^{k_0}$ only:

$$\begin{aligned}\varphi\left(\ \cdot\ \middle|\ x_{t_0}^{k_0}\right) = \text{const} &+ \wp_1 \max\{0, 1 - x_{t_0}^{k_0}\} \\ &+ \wp_2 \max\{0, x_{t_0}^{k_0}\} \\ &+ \wp_3 \max\{0, -x_{t_0}^{k_0}\} \\ &+ \wp_4 \max\{0, x_{t_0}^{k_0} - 1\},\end{aligned} \tag{4.32}$$

where $\wp_1, \ldots, \wp_4$ denote appropriate summations of S_t^j:

$$\wp_1 = \sum_{\left\{ j \,\middle|\, k(j,t_0) = k_0,\ \forall \tau = 1,\ldots,t_0-1:\ x_\tau^{k(j,\tau)} = 1 \right\}} J^{-1} S_{t_0}^j\ ,$$

$$\wp_2 = \sum_{\left\{ j,\, t > t_0 \,\middle|\, k(j,t_0) = k_0,\ \forall\ \tau \in \{1,\ldots,t-1\}/\{t_0\}:\ x_\tau^{k(j,\tau)} = 1,\ x_t^{k(j,t)} = 0 \right\}} J^{-1} S_t^j\ ,$$

$$\wp_3 = \sum_{\left\{ j \,\middle|\, k(j,t_0) = k_0,\ \exists\ \tau \in \{1,\ldots,t_0-1\}:\ x_\tau^{k(j,\tau)} = 0 \right\}} J^{-1} S_{t_0}^j\ ,$$

$$\wp_4 = \sum_{\left\{ j,\, t > t_0 \,\middle|\, k(j,t_0) = k_0,\ \forall\ \tau \in \{1,\ldots,t\}/\{t_0\}:\ x_\tau^{k(j,\tau)} = 1 \right\}} J^{-1} S_t^j\ .$$

Note that the last two terms in (4.32) equal to zero, so (4.32) takes the form

$$\varphi\left(\ \cdot\ \middle|\ x_{t_0}^{k_0}\right) = \text{const} + \wp_1 (1 - x_{t_0}^{k_0}) + \wp_2 x_{t_0}^{k_0}. \tag{4.33}$$

If $\wp_1 \neq \wp_2$, function (4.33) can be improved by putting $x_{t_0}^{k_0} = 0$ or 1, which would mean that solution with a non-binary component cannot be optimal. If $\wp_1 = \wp_2$, the value of (4.33) does not change by selection $x_{t_0}^{k_0} = 0$ or 1, i.e., there exists an optimal 0–1 solution of (4.31).

The case with more than one fractional component in the optimal solution can be considered similarly, with the only difference that the variation of the objective function $\varphi(\ \cdot\ |\ \cdot\ )$ with respect to some variable with non-binary optimal value may have to be considered in a small ε-vicinity of an optimal point. ■

Remark 4.3.6–1 Proposition 4.3.6 implies that if the optimal solution of (4.18) is unique, then it is 0–1.

Remark 4.3.6–2 The binary structure of optimal trading strategy in the frictionless market can be viewed as a criterion for selecting the group decision rule (4.13). Function $f(\cdot)$ not allowing for an optimal 0–1 solution of problem (4.14) in the absence of market imperfections, cannot be considered as a candidate for the group decision rule of a trading strategy. Proposition 4.3.6 actually ascertains that the "lawn-mower" principle (4.15), as the decision rule in problem (4.18), holds this property.

"Lawn-mower" strategy in presence of non-zero market impact. Bertsimas and Lo (1998) have shown that under linear temporary market impact the optimal trading strategy exhibits little difference with the so-called *naive* strategy, which consists in dividing the position in equal portions to be executed at each time moment t (provided that time moments are equally spaced):

$$\Delta\xi_t = 1/T, \quad t = 1, \ldots, T. \tag{4.34}$$

Depending on the asset price dynamics, the optimal trading strategy may deviate from the naive rule (4.34), but nevertheless, the transaction size $\Delta\xi$ can be assumed non-zero at each time step. In the present work, we are primarily interested in short time frames for liquidation, therefore in the optimal closing problem (4.18) we may impose constraint

$$\Delta\xi_t^j \geq \varepsilon, \quad j = 1, \ldots, J, \quad t = 1, \ldots, T,$$

where $\varepsilon > 0$ is very small: $\varepsilon \ll 1$. In this case, the "lawn-mower" principle (4.15) becomes equivalent to the simple decision rule (4.11):

$$\Delta\xi_t^j = \max\{0, \xi_{t-1}^j - x_t^{k(j,t)}\}, \; \Delta\xi_t^j > 0 \quad \Rightarrow \quad \xi_t^j = x_t^{k(j,t)}, \tag{4.35}$$

which simplifies significantly the optimization problem (4.18). Moreover, the same argument is valid when both temporary and permanent forms of market impact are present.

On the other hand, Proposition 4.3.6 suggests that with diminishing of the market impact, $\delta(\Delta\xi) \to \Delta\xi$, the optimal trading strategy would tend to concentrate the transactions at some specific point of time, approaching 0–1 strategy. Thus, the above simplification should be used with caution when the market impact is not significant.

4.4 Optimal Liquidation with Permanent Market Impact

When the price changes caused by a trader's activity have permanent character, the problem of optimal security liquidation complicates considerably. The formulation with lawn-mower strategy (4.15) and payoff function $p_t(\cdot)$ that incorporates permanent market impact reads as

$$\begin{aligned} \max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \left\{ \Delta\xi_t^j - \Delta\xi_t^j \theta_t(\Delta\xi_t^j) - \hat{\theta}_t(\Delta\xi_t^j) \sum_{\tau=t+1}^{T} \Delta\xi_\tau^j \right\} \\ \text{s. t.} \quad & \xi_t^j = \min\{x_t^{k(j,t)}, \xi_{t-1}^j\}, \quad k \in \pi(\mathfrak{S}), \quad j = 1,\dots,J, \quad t = 1,\dots,T. \end{aligned} \tag{4.36}$$

The objective of (4.36) is not separable as in the case of temporary impact (4.8), in the sense that it cannot be presented as a sum of functions of single argument: $\sum_i f_i(x_i),\ x_i \in \mathbb{R}$. This and the general non-concavity of the objective (4.36) would not allow us to construct an equivalent formulation of problem (4.36) with a convex feasible region, as it was done in Section 4.3.2.

Therefore, to consider the problem of optimal position closing in the presence of permanent market impact we perform path partitioning with trivial group decision rule

$$\xi_t^j = x_t^{k(j,t)}, \quad k(j,t) \propto \pi(\mathfrak{S}), \quad j = 1,\dots,J, \quad t = 1,\dots,T,$$

which leads to optimal liquidation problem in the form

$$\begin{aligned} \max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \left\{ \Delta\xi_t^j - \Delta\xi_t^j \theta_t(\Delta\xi_t^j) - \hat{\theta}_t(\Delta\xi_t^j) \sum_{\tau=t+1}^{T} \Delta\xi_\tau^j \right\} \\ \text{s. t.} \quad & \xi_t^j = x_t^{k(j,t)}, && j = 1,\dots,J, \quad t = 1,\dots,T. \\ & 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && j = 1,\dots,J, \quad t = 1,\dots,T. \end{aligned} \tag{4.37}$$

According to the discussion in Section 4.3.3, the "lawn-mower" principle (4.15) is equivalent to the simple rule $\xi_t^j = x_t^{k(j,t)}$ provided that at time t and path j the trade $\Delta\xi_t^j$ is non-zero. Previous studies (Bertsimas and Lo (1998), Almgren and Chriss (2000)) demonstrated that under linear permanent market impact, the optimal trading strategy usually consists of non-zero trades: $\Delta\xi_t > 0$, $t = 1,\dots,T$, which is also consistent with our findings (see Section 4.6.3). This can be viewed as a justification for replacing formulation (4.36) with (4.37).

Another motivation for replacing the "lawn-mower" principle (4.15) by simple decision rule (4.11) is that the simple rule $\xi_t^j = x_t^{k(j,t)}$ dramatically reduces the dimensionality of the problem, which is especially important in view of the generally nonlinear non-convex objective of (4.37).

When the temporary and permanent market impact functions $\theta_t(\Delta\xi)$ and $\hat{\theta}_t(\Delta\xi)$ are linear,

$$\theta_t(\Delta\xi) = \gamma\Delta\xi, \quad \hat{\theta}_t(\Delta\xi) = \hat{\gamma}\Delta\xi,$$

problem (4.37) reduces to a quadratic programming problem with the following objective:

$$\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \left\{ \Delta\xi_t^j - \gamma(\Delta\xi_t^j)^2 - \hat{\gamma}\Delta\xi_t^j \sum_{\tau=t+1}^{T} \Delta\xi_\tau^j \right\} \tag{4.38}$$

which is always concave for any positive S_t^j, γ, and $\hat{\gamma}$. This case corresponds to the framework of permanent market impact discussed in Bertsimas and Lo (1998) and Almgren and Chriss (2000). More interesting and practically plausible representation for the permanent market impact function $\hat{\theta}_t(\cdot)$ is (4.7). If the function $\hat{\theta}_t(\cdot)$ satisfies (4.7), the permanent changes of market prices can be only caused by large enough trades.

4.5 Risk Constraints

It is well known that market activity always involves risk, which in our case is associated with uncertain future prices of the asset. To control the risk of financial losses during liquidation, we employ the Conditional Value-at-Risk (CVaR) risk measure Rockafellar and Uryasev (2000, 2002)). CVaR is a downside risk measure, which quantifies the risk in terms of percentiles of loss distribution. Probably, the most outstanding feature of Conditional Value-at-Risk is its convexity with respect to decision variables (provided that the loss function is also convex), which dramatically simplifies the problem of estimating and managing risks using techniques of mathematical programming, and also makes these procedures much more robust compared to other risk measures, e.g., Value-at-Risk (see Rockafellar and Uryasev (2000, 2002)).

In developing risk constraints for the optimal liquidation problem we follow Rockafellar and Uryasev (2000, 2002). Given a loss function $\mathscr{L}(\xi,\eta)$, where η is a stochastic vector standing for market uncertainties, and ξ is the decision vector, Conditional Value-at-Risk $\phi_\alpha(\xi)$ with confidence level α can be described approximately as the conditional expectation of losses exceeding the α-quantile of the loss distribution. If the loss function $\mathscr{L}(\xi,\eta)$ has a continuous distribution, this

definition is exact:

$$\phi_\alpha(\xi) = (1-\alpha)^{-1} \int_{\mathscr{L}(\xi,\eta)\geq\zeta_\alpha(\xi)} \mathscr{L}(\xi,\eta)\,\mathrm{d}F(\eta).$$

Above $\zeta_\alpha(\xi)$ is the α-quantile of loss distribution:

$$\zeta_\alpha(\xi) = \min_{\zeta\in\mathbb{R}} \big\{ \mathsf{P}[\mathscr{L}(\xi,\eta)\leq\zeta] \geq \alpha \big\},$$

and $F(\eta)$ is a distribution of random vector η. In a general case, including non-continuous loss distributions, Conditional Value-at-Risk $\phi_\alpha(\xi)$ is calculated as

$$\phi_\alpha(\xi) = \min_{\zeta\in\mathbb{R}} \left\{ \zeta + (1-\alpha)^{-1}\mathsf{E}\left[\max\{0, \mathscr{L}(\xi,\eta)-\zeta\}\right] \right\}.$$

According to Rockafellar and Uryasev (2002), in the optimization problem with multiple CVaR constraints

$$\begin{aligned} \max_{\xi\in X} \quad & g(\xi) \\ \text{s. t.} \quad & \phi_{\alpha_t}(\xi) \leq \omega_t, \quad i = 1,\ldots,T, \end{aligned}$$

each constraint $\phi_{\alpha_t}(\xi) \leq \omega_t$ can be replaced by a set of inequalities

$$\begin{aligned} & \mathscr{L}(\xi,\eta_j) - \zeta_t \leq w_t^j, \quad j = 1,\ldots,J, \\ & \zeta_t + \frac{1}{1-\alpha_t}\frac{1}{J}\sum_{j=1}^{J} w_t^j \leq \omega_t, \\ & \zeta_t \in \mathbb{R}, \quad w_t^j \geq 0, \quad j = 1,\ldots,J, \end{aligned} \tag{4.39}$$

where $\eta_j,\ j = 1,\ldots,J$, are the (equally probable) realizations of the random vector η. The feasible set defined by inequalities (4.39) is convex provided that function $\mathscr{L}(\xi,\eta)$ is convex in ξ.

Risk constraints in the problem with temporary market impact. We consider the loss function $\mathscr{L}_t(\xi,\cdot_j)$ as a negative profit along path $(\mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_T^j)$ before time t

$$\mathscr{L}_t(\xi_\tau^j; \mathbf{S}_\tau^j \mid \tau \leq t) = S_0 - \sum_{\tau=1}^{t} p_\tau(\xi_k^j; \mathbf{S}_k^j \mid k \leq \tau). \tag{4.40}$$

Recall that in the presence of permanent market impact the generated payoff is a nonlinear and generally non-convex function of position values ξ_t^j. Thus, imposing risk constraints based on

the loss function (4.40) onto the feasible set of optimal liquidation problem (4.18) will make it computationally intractable.

On the other hand, if only temporary market impact is considered, and the temporary impact function $\delta_t(\cdot)$ satisfies to conditions (i)–(v), the loss function (4.40) becomes convex:

$$\mathscr{L}_t(\xi_\tau^j; \mathbf{S}_\tau^j \mid \tau \le t) = S_0 - \sum_{\tau=1}^{t} S_\tau^j \, \delta_t(\Delta\xi_\tau^j), \tag{4.41}$$

ensuring convexity of CVaR constraints (4.39). To increase computational efficiency, we approximate the convex set determined by constraints (4.39), where $\mathscr{L}(\cdot)$ is defined as above, by a set of linear constraints using the piecewise-linear approximation (4.26) of function $\delta_t(\cdot)$. The optimal position liquidation problem with CVaR constraints then reads as

$$\begin{aligned}
\max_{\Xi} \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T} S_t^j \max\{0, \delta_t(u_t^j)\} && && (4.42)\\
\text{s. t.} \quad & \xi_t^j \le x_t^{k(j,t)}, && t = 1,\dots,T,\ j = 1,\dots,J,\\
& u_t^j = \xi_{t-1}^j - x_t^{k(j,t)}, && t = 1,\dots,T,\ j = 1,\dots,J,\\
& 0 \le x_t^{k(j,t)} \le 1, \quad k(j,t) \propto \pi(\mathfrak{S}), && t = 1,\dots,T,\ j = 1,\dots,J,\\
& \xi_t^j \in \Phi_t^{\alpha,\omega}, && t = 1,\dots,T,\ j = 1,\dots,J.
\end{aligned}$$

Here $\Phi_t^{\alpha,\omega}$ denotes the feasible set of CVaR constraint $\phi_\alpha(\xi) \le \omega_t$ over time horizon $1 \le \tau \le t$:

$$\begin{aligned}
& S_0 - \sum_{\tau=1}^{t} S_\tau^j \, \delta(\Delta\xi_\tau^j) - \zeta_t \le w_t^j, \quad j = 1,\dots,J,\\
& \zeta_t + \frac{1}{1-\alpha_t}\frac{1}{J}\sum_{j=1}^{J} w_t^j \le \omega_t, && (4.43)\\
& \zeta_t \in \mathbb{R}, \quad w_t^j \ge 0, \quad j = 1,\dots,J,
\end{aligned}$$

Risk constraints (4.43) make sure that at each time t the average of $(1-\alpha_t)\cdot 100\%$ worst losses, compared to the initial wealth level S_0, does not exceed ω_t.

Next we show that our previous arguments on optimal liquidation with the "lawn-mower" decision rule remain valid if CVaR constraints are incorporated in the corresponding optimization problems.

Proposition 4.5.1 *Propositions 4.3.4 and 4.3.5 hold with CVaR constraints*

$$\xi_t^j \in \Phi_t^{\alpha,\omega}, \quad t = 1,\ldots,T,\ j = 1,\ldots,J,$$

imposed on the feasible regions of problems (4.18), (4.21) *and* (4.27), *where set* $\Phi_t^{\alpha,\omega}$ *is defined as in* (4.43).

Proof. It suffices to note that the CVaR-constrained set of feasible solutions is a subset of the original feasible region, so the arguments used in proofs of Propositions 4.3.3 and 4.3.4 with the corresponding corollaries remain valid. ■

Remark 4.3.6 With CVaR constraints imposed, solutions of problems (4.18), (4.21), and (4.27) can be fractional. This corresponds to the general principle of *diversification of risk*.

4.6 Case study: Optimal Closing of Long Position in a Stock

The presented case study has been developed to generate a position closing rule in a proprietary trading strategy. This trading strategy contains an algorithm, which generates signals for opening positions in a set of stocks with similar variances and trading volumes. However, the strategy lacked an algorithm for closing of the opened positions.

We considered a test problem of optimal liquidation of a long position in a stock in the presence of different types of market impact. The time horizon for closing of the position spanned 5 business days, with one trading opportunity per day $(T = 5)$. Because of the short time frame, we did not consider discounting.

The sample-path scenario set $\mathfrak{S}$ contained 5,000 paths $(J = 5{,}000)$, and was constructed in the scope of the above-mentioned trading strategy. The collection of sample paths was generated by cutting from the historical trajectory of a stock 5-day windows that followed the opening signal generated by the trading algorithm. At each time $t = 1,\ldots,5$, a sample path $(\mathbf{S}_0, \mathbf{S}_1^j, \ldots, \mathbf{S}_5^j)$ contained the daily closing price of a stock and the daily trading volume in that stock: $\mathbf{S}_t^j = (S_t^j, V_t^j)$. The prices of the sample paths in the collection were normalized so that the initial price was equal to 1. Although the obtained sample paths actually belong to different stocks, their use as a scenario set for a single security is quite reasonable, since the trading algorithm generates the opening signal in similar situations, and the set of stocks used in the case study possessed similar properties (variance and trading volume).

The path partitioning $\pi(\mathfrak{S})$ was performed as follows: at each time moment t the paths were sorted in ascending order with respect to current price S_t^j and then partitioned into K groups of equal size:

$$\forall\, k:\ \left|G_t^k\right| = J/K, \quad \forall\, k_1 > k_2,\ j_1 \in G_t^{k_1},\ j_2 \in G_t^{k_2}:\ \ S_t^{j_1} \geq S_t^{j_2}, \quad t = 1, \ldots, T,$$

so that group G_t^1 contained J/K paths with lowest prices, and group G_t^K contained J/K paths with highest prices. The number of groups K was kept constant at all time periods. We ran tests with the number of groups per period equal to 10, 20, 50, 100, and 500, i.e., each group contained 500, 250, 100, 50, or 10 paths correspondingly.

We present the optimal trading strategies in tableau form (see Tables 4–1–4–6) by reporting the optimal values of variables x_t^k in the corresponding optimization problems. We have considered the simple decision rule (4.11) and the "lawn-mower" principle (4.15). Recall that according to Proposition 4.3.2, the set of variables x_t^k defines uniquely the trading strategy in either case (4.15) or (4.11) of the group decision rule. All the tables report the optimal values of variables x_t^k for 10-group partition as described above. The results presented in Tables 4–1–4–6 have to be interpreted as follows: if at time t the price of the instrument at hand falls into the price range of group k, execute the transaction in accordance to the specified group decision rule. For instance, in the case of the "lawn-mower" decision rule (4.15), reduce the position value to the level of variable x_t^k if the position exceeded x_t^k, or do nothing otherwise.

4.6.1 Optimal Closing in Frictionless Market

Problem (4.27) with zero market impact $(\delta_t(y) = y)$ has a 0–1 solution, which becomes fractional if risk constraints (in our case, based on the Conditional Value-at-Risk measure) are imposed. It implies that the risk-neutral trading algorithm selects the most favorable moment to sell the position "in one shot" so as to gain the highest expected profit. Risk-averse trading strategy recommends closing the position by parts because waiting for such a favorable moment is risky. This corresponds to the general principle of *diversification of risk*.

Table 4–1 displays the risk-neutral (i.e., without risk constraints) and CVaR-constrained optimal liquidation strategies defined as solutions of problems (4.27) with $\delta_t(y) = y$ and the corresponding problem with CVaR constraints (4.43). Parameters of CVaR constraints in (4.42) were chosen

as $\alpha_t = 0.9$ and $\omega_t = 0.1$, meaning that in each time t the average loss in 10% of worst cases should not exceed 10% of the original dollar value of the position.

Table 4–1: Optimal trading strategy in frictionless market.

Risk-neutral strategy					
k	t=1	t=2	t=3	t=4	t=5
10	1	0	0	0	0
9	1	1	0	0	0
8	1	1	0	0	0
7	1	1	0	0	0
6	1	1	0	0	0
5	1	1	0	0	0
4	1	1	0	0	0
3	1	1	1	0	0
2	1	1	1	0	0
1	1	1	1	0	0

CVaR-constrained (α_t =0.9, ω_t =0.1)					
k	t=1	t=2	t=3	t=4	t=5
10	0	0	0	0	0
9	0	0	0	0	0
8	0	0	0	0	0
7	0	0	0	0	0
6	0	0	0	0	0
5	0.052	0	0	0	0
4	0.035	0	0	0	0
3	0.014	0	0	0	0
2	0	0	0	0	0
1	0	0	0	0	0

The optimal solution of the optimal closing problem (4.27) with zero-impact function $\delta_t(y) = y$ is quite sensitive to the choice of partition $\pi(\mathfrak{S})$. Consider, for example, an extreme case when the number of groups at each t is equal to the number of paths: $K_t = J$, $t = 1, \ldots, T$. In this case, the optimal closing problem reduces to form (4.8), with fully anticipative optimal solution (4.9).[4] When the number of groups K_t decreases, the optimal objective value of (4.24) decreases as well. Figure 4–8 presents the performance of the trading strategy depending on the number K of groups in partition $\pi(\mathfrak{S})$ (the expected rate of return is equal to the optimal objective value divided by the initial wealth S_0).

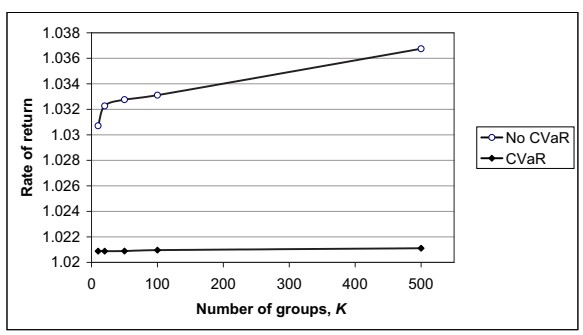


Figure 4–8: Performance of optimal trading strategy in frictionless market.

[4] Observe that the optimal solution (4.9) delivers maximal objective value of (4.24) with $\delta_t(y) = y$ over all possible partitions $\pi(\mathfrak{S})$.

Note that the expected rate of return of zero-impact trading strategy varies considerably, from 1.0294 for 10 groups to 1.03625 in case of 500 groups. Since the objective value of the fully anticipative solution (4.9) is 1.05685, we can conclude that solution with as many as 500 groups has an acceptable degree of non-anticipativity.

At the same time, the optimal objective value of problem with CVaR constraints is almost insensitive to the number of groups in partition. The same is true for optimal liquidation problems with non-zero market impact discussed in the following subsections.

"Tree" grouping and comparison with dynamic programming. An interesting issue to address is the performance of the optimal closing problem (4.18) with a "tree" partitioning in comparison to the solution of the optimal closing problem obtained by the methods of dynamic programming and scenario trees. The dynamic programming approach in application to the optimal closing problem in frictionless market was used by Butenko, Golodnikov, and Uryasev (2003). The authors developed a technique for transforming a collection of historical paths into a scenario tree with a prescribed number of branches per node. The trading strategy, obtained as the solution of corresponding dynamic programming problem, was tested on the historical data used in this case study. We compared their results for the 4-branch scenario tree, solution to which was presented among others, with the solution of problem (4.18) under the "tree" partition, constructed by splitting each group into 4 groups of equal size at each time t. According to Proposition 4.3.1, "tree" partition reduces any decision rule (4.13) to the simple rule (4.11). On the problem with 5120 paths, the algorithm (4.18) slightly outperformed the dynamic programming approach, producing expected return of 1.0241 versus 1.02062 obtained by Butenko, Golodnikov, and Uryasev (2003). The similarity of solutions obtained by different techniques validates the correctness of our approach and that presented in Butenko, Golodnikov, and Uryasev (2003).

4.6.2 Optimal Closing under Temporary Market Impact

To study the optimal closing strategies under temporary market impact, we assumed the function $\delta_t(y)$ in the objective of problem (4.18) and (4.27) to have the form

$$\delta_t(y) = y - \frac{1}{\beta c} y^{\beta}, \quad \beta > 1, \quad c \geq 1. \tag{4.44}$$

This function, evidently, satisfies assumptions (i)–(iv). In representation (4.44), $c = 1$ corresponds to the "most severe" market impact; when the parameter c increases, the market impact function

approaches the no-impact case $\delta_t(y) = y$. When $\beta = 2$, equality (4.44) describes the *linear* temporary market impact, and in this case the convex programming problem (4.27) reduces to quadratic optimization with a diagonal matrix.

In general, the optimal solution of convex programming program (4.27) can be approximated by replacing the impact function $\delta_t(\cdot)$ in the objective by a piecewise linear function $l_t(\cdot)$ (4.26) and solving the corresponding linear programming problem (4.28). For the representation (4.44), coefficients a_t^n, b_t^n of function $l_t(\cdot)$ that approximates $\delta_t(\cdot)$ defined by (4.44), can be chosen as

$$a_t^n = 1 - \frac{n^\beta - (n-1)^\beta}{\beta c N^{\beta-1}}, \quad b_t^n = \frac{n^\beta - (n-1)^\beta - n^{\beta-1}}{\beta c N^\beta}. \tag{4.45}$$

This approximation is exact at points $y_n = n/N,\ n = 1, \ldots, N$.

Linear temporary market impact $(\beta = 2)$**.** Table 4–2 displays the optimal values of the decision variables x_t^k, obtained as the solution of corresponding quadratic programming problem (4.27) with function $\delta_t(\cdot)$ defined above for $\beta = 2$. The problem was solved for cases $c = 1$ ("severe market impact") and $c = 10$ ("light impact").

Table 4–2: Optimal trading strategy under linear temporary market impact $(\beta = 2)$.

c =1.0

k	t=1	t=2	t=3	t=4	t=5
10	0.804	0.608	0.407	0.205	0
9	0.805	0.609	0.409	0.204	0
8	0.805	0.610	0.408	0.203	0
7	0.804	0.608	0.409	0.203	0
6	0.806	0.607	0.408	0.204	0
5	0.804	0.607	0.410	0.206	0
4	0.805	0.609	0.409	0.206	0
3	0.804	0.608	0.409	0.206	0
2	0.804	0.609	0.408	0.205	0
1	0.806	0.607	0.409	0.205	0

c =10.0

k	t=1	t=2	t=3	t=4	t=5
10	0.869	0.701	0.490	0.262	0
9	0.867	0.713	0.516	0.255	0
8	0.864	0.723	0.506	0.245	0
7	0.856	0.702	0.516	0.248	0
6	0.885	0.697	0.508	0.259	0
5	0.860	0.697	0.523	0.280	0
4	0.862	0.711	0.518	0.276	0
3	0.851	0.707	0.518	0.273	0
2	0.857	0.711	0.505	0.263	0
1	0.874	0.692	0.519	0.263	0

First of all, observe that in the presence of "severe" market impact $(c = 1)$, the optimal trading strategy becomes almost identical to the "naive" strategy

$$x_{\text{naive}} = \{0.8,\ 0.6,\ 0.4,\ 0.2,\ 0\}, \tag{4.46}$$

where position is sold in equal portions at all available time moments. This is consistent with the results by Bertsimas and Lo (1998) for linear-percentage price impact. As the effects of market impact diminish $(c \to \infty,\ \delta(y) \to y)$, the solution starts to exhibit deviations from the naive strategy.

Linear temporary market impact that depends on market parameters. According to discussion in Section 4.2.3, the market impact function $\delta(\cdot)$ depends on the market conditions at the moment of transaction. To accommodate this in our model, we consider function (4.44) with coefficient c that depends on the current volume V_t of trades in the market. In particular, we assumed the following expression for c:

$$c = 1 + e^{\left(M(V_t^j - C)\right)},$$

where M and C are some constants. This parametrization of the impact function $\delta_t(\cdot)$ reflects the presumption that the temporary market impact decreases $(c \to \infty)$ if the current trading volume is high, and increases $(c \to 1)$ when the volume is low.

To solve the optimal closing problem with temporary market impact depending on market conditions, we employed LP model (4.28) with piecewise-linear price impact function consisting of 100 linear segments determined by (4.45). The optimal solution shows significant differences compared to the naive strategy (4.46) (see Table 4–3).

Table 4–3: Optimal trading strategy under linear temporary market impact that depends on price dynamics.

k	t=1	t=2	t=3	t=4	t=5
10	0.84	0.73	0.43	0.26	0
9	0.88	0.66	0.43	0.23	0
8	0.82	0.56	0.45	0.25	0
7	0.84	0.71	0.47	0.28	0
6	0.88	0.65	0.47	0.25	0
5	0.92	0.62	0.44	0.17	0
4	0.85	0.65	0.5	0.23	0
3	0.85	0.65	0.49	0.22	0
2	0.83	0.67	0.57	0.24	0
1	0.87	0.68	0.4	0.2	0

4.6.3 Optimal Closing under Permanent Market Impact

The problem of optimal closing in the presence of permanent market impact (4.37) was considered in two settings: in assumption of linearity of the market impact functions $\theta_t(\Delta\xi)$ and $\hat{\theta}_t(\Delta\xi)$, and under more realistic assumption (4.7).

Linear permanent market impact. If both functions $\theta_t(\Delta\xi)$ and $\hat{\theta}_t(\Delta\xi)$ have the form $const \cdot \Delta\xi$, then the optimal closing problem reduces to the maximization of a concave quadratic function subject to linear constraints, and therefore can be solved fast and robustly even in large-scale instances. For simplicity, we assumed functions $\theta_t(\cdot)$, $\hat{\theta}_t(\cdot)$ to be equal, and the temporary

part of market impact in (4.37) is equal to (4.44) with $\beta = 2$, i.e.,

$$\theta_t(y) = \hat{\theta}_t(y) = \frac{1}{2c}y.$$

In both cases of "severe impact" ($c = 1$) and "light impact" ($c = 10$) the optimal solution exhibits more differences with the naive strategy (4.46) than the corresponding solution of the problem with temporary impact, see Table 4–4:

Table 4–4: Optimal trading strategy under linear permanent market impact ($\beta = 2$).

	c =1.0				
k	t=1	t=2	t=3	t=4	t=5
10	0.807	0.613	0.412	0.208	0
9	0.808	0.614	0.415	0.207	0
8	0.808	0.615	0.414	0.205	0
7	0.807	0.613	0.415	0.206	0
6	0.810	0.612	0.414	0.208	0
5	0.807	0.612	0.416	0.211	0
4	0.807	0.614	0.415	0.210	0
3	0.806	0.613	0.416	0.210	0
2	0.807	0.614	0.414	0.208	0
1	0.809	0.612	0.416	0.208	0

	c =10.0				
k	t=1	t=2	t=3	t=4	t=5
10	0.915	0.798	0.577	0.322	0
9	0.931	0.819	0.627	0.307	0
8	0.925	0.840	0.607	0.288	0
7	0.909	0.799	0.626	0.293	0
6	0.965	0.790	0.611	0.315	0
5	0.918	0.789	0.641	0.357	0
4	0.921	0.817	0.630	0.350	0
3	0.899	0.808	0.631	0.344	0
2	0.911	0.817	0.605	0.323	0
1	0.945	0.781	0.633	0.324	0

Non-convex linear permanent market impact. The main drawback of the linear permanent market impact model discussed above is that it allows for permanent changes in the security price to be induced by infinitesimal trades. A more plausible model for the permanent impact function $\hat{\theta}_t(\cdot)$ would be such that does not lead to permanent price changes if the trade is relatively small. Therefore, consider the functions $\theta_t(\cdot)$ and $\hat{\theta}_t(\cdot)$ in the form

$$\theta_t(y) = \frac{1}{2c}y, \quad \hat{\theta}_t(y) = \max\left\{0, \frac{1}{2c}(y-\lambda)\right\}. \tag{4.47}$$

In this case, however, the optimal liquidation problem (4.37) loses the concavity of the objective function. We used the MINOS package to solve the resulting non-convex programming problem, and Table 4–5 shows the solution reported by MINOS ($\lambda = 0.2$).

Observe that the solution departs *significantly* from the naive strategy even in the case of "light" market impact ($c = 10$). This suggests that the "market impact lag" in the function $\hat{\theta}_t(\cdot)$ (4.47) may have the most pronounced effect on the properties of optimal trading strategies compared to other models of market impact, both temporary and permanent, considered in this chapter.

Table 4–5: Optimal trading strategy under piecewise-linear permanent market impact.

$c=1.0$					
k	t=1	t=2	t=3	t=4	t=5
10	0.899	0.798	0.632	0.374	0
9	0.899	0.798	0.629	0.371	0
8	0.899	0.798	0.643	0.367	0
7	0.899	0.799	0.636	0.368	0
6	0.899	0.799	0.643	0.373	0
5	0.899	0.798	0.630	0.359	0
4	0.899	0.799	0.635	0.367	0
3	0.9	0.798	0.635	0.366	0
2	0.899	0.799	0.645	0.373	0
1	0.9	0.798	0.633	0.363	0

$c=10.0$					
k	t=1	t=2	t=3	t=4	t=5
10	0.9	0.8	0.643	0.454	0
9	0.9	0.775	0.544	0.393	0
8	0.9	0.775	0.675	0.382	0
7	0.9	0.8	0.675	0.364	0
6	0.9	0.775	0.675	0.403	0
5	0.9	0.775	0.568	0.249	0
4	0.9	0.8	0.675	0.356	0
3	0.9	0.799	0.675	0.337	0
2	0.9	0.8	0.7	0.442	0
1	0.9	0.8	0.632	0.301	0

Non-convex nonlinear permanent market impact. Finally, let us consider a case of non-linear non-convex permanent market impact. For this purpose, we select the price impact functions $\theta_t(\cdot)$ and $\hat{\theta}_t(\cdot)$ as follows:

$$\theta_t(y) = \frac{1}{3c}y^2, \quad \hat{\theta}_t(y) = \max\left\{0, \frac{1}{3c}\,\mathrm{sign}\,(y-\lambda)\,(y-\lambda)^2\right\}$$

The form of function $\hat{\theta}_t(\cdot)$ ensures its smoothness (continuous differentiability) in the "transition point" $y=\lambda$ that separates trades that do not cause permanent price changes $(y \le \lambda)$ and those that lead to permanent price drop $(y > \lambda)$. As above, the obtained nonlinear non-convex optimization problem was solved using the MINOS engine.

Table 4–6: Optimal trading strategy under nonlinear permanent market impact.

$c=1.0$					
k	t=1	t=2	t=3	t=4	t=5
10	0.814	0.641	0.421	0.225	0
9	0.820	0.617	0.412	0.219	0
8	0.814	0.615	0.429	0.217	0
7	0.812	0.641	0.432	0.215	0
6	0.830	0.613	0.432	0.219	0
5	0.830	0.617	0.416	0.203	0
4	0.822	0.623	0.430	0.213	0
3	0.822	0.627	0.431	0.213	0
2	0.820	0.633	0.446	0.223	0
1	0.837	0.624	0.422	0.207	0

$c=10.0$					
k	t=1	t=2	t=3	t=4	t=5
10	0.894	0.894	0.520	0.365	0
9	0.964	0.691	0.427	0.317	0
8	0.889	0.654	0.592	0.305	0
7	0.894	0.894	0.612	0.285	0
6	1	0.668	0.621	0.318	0
5	1	0.682	0.464	0.181	0
4	1	0.712	0.596	0.275	0
3	1	0.764	0.608	0.264	0
2	0.964	0.805	0.668	0.348	0
1	1	0.732	0.525	0.215	0

Table 4–6 shows less pronounced differences with the naive strategy due to the convexity of the nonlinear impact functions, which implies smaller execution costs compared to the linear non-convex permanent market impact.

4.7 Conclusions

In this chapter, we have developed an approach for optimal transaction execution based on mathematical programming and sample-path scenario model. The main distinguishing features of our method compared to other approaches are as follows: (1) The trading strategy is truly *dynamic*, i.e., it allows for adequate response to current market conditions at each time step; (2) The approach admits incorporation of various types of constraints in the trading strategy, such as legal, institutional constraints, or those expressing investor's preferences. In particular, we considered risk-averse trading strategies, which control risk of financial losses during transactions using the Conditional Value-at-Risk measure; (3) The suggested approach can accommodate different models of temporary and permanent market impact, transaction costs, etc.; (4) A key feature of our approach is the sample-path scenario model, which does not impose any restrictions on the price process of the security. Historical data or simulation can be used to create a scenario set for our method.

To avoid anticipativity of the solutions, caused by specific properties of the sample-path scenario model, we introduced path partitioning with a new "lawn-mower" decision rule. Since the "lawn-mower" principle leads to non-convexity of the optimal closing problems, we developed a convex relaxation for these problems, which was further approximated by linear programming. The model of optimal position liquidation, implemented as a linear programming problem, can be solved very efficiently and robustly in very large-scale instances with large number of scenarios (sample-paths). Properties of the optimal trading strategies based on the "lawn-mower" principle have been investigated.

In a frictionless market, the optimal trading strategy based on the "lawn-mower" constraint, has 0–1 structure, i.e., it liquidates the entire position in one transaction. If the risk of financial losses during trades is controlled by CVaR constraints, the trading strategy becomes fractional. When "strong" temporary or permanent market impacts are present, the optimal trading strategy approaches the so-called "naive" strategy, which consists in selling equal portions of the position at each time step. Under "weaker" market impacts, the optimal trading strategy starts to deviate from the "naive" strategy. According to our numerical experiments, the most significant deviations from the "naive" trading strategy are observed in the case when temporary market impact depends on the

current market conditions, or when permanent market impact has a “lag” (i.e., when small enough trades do not cause permanent changes in price of the security).

CHAPTER 5
ROBUST DECISION MAKING: ADDRESSING UNCERTAINTIES IN DISTRIBUTIONS

This chapter develops a general approach to risk management in military applications involving uncertainties in information and distributions. The risk of loss, damage, or failure is measured by the Conditional Value-at-Risk measure. The greatest advantage of using CVaR as a risk measure in military applications is that CVaR is a downside percentile risk measure. At the same time, it is convex as a function of decision variables, and therefore can be efficiently controlled/optimized using convex or (under quite general assumptions) linear programming. The general methodology was tested on two Weapon-Target Assignment (WTA) problems. It is assumed that the distributions of random variables in the WTA formulations are not known with certainty. In the mathematical programming problem formulation, the total cost of the mission (including weapon attrition) is minimized, while satisfying operational constraints and ensuring destruction of all targets with high probabilities. We conduct case studies that show significant qualitative and quantitative differences in solutions of deterministic WTA and stochastic WTA problems.

5.1 Introduction

This chapter develops a general approach to managing risk in military applications involving stochasticity and uncertainties in distributions. Various military applications such as intelligence, surveillance, planning, scheduling etc., involve decision making in dynamic, distributed, and uncertain environments. In a large system, multiple sensors may provide incomplete, conflicting, or overlapping data. Moreover, some components or sensors may degrade or become completely unavailable (failures, weather conditions, battle damage). Uncertainties in combat environment induce different kinds of risks that components, sensors or armed units are exposed to, such as the risk to be damaged or destroyed, risk of mission incompleteness (e.g., missing a target) or failure, risk of false target attack etc. Therefore, planning and operating in stochastic and uncertain conditions of a modern combat require robust decision-making procedures. Such procedures must take into account the stochastic nature of risk-inducing factors, and generate decisions that are not only effective on average (in other words, have good “expected” performance), but also safe enough under

a wide range of possible scenarios. In this regard, risk management in military applications is similar to practices in other fields such as finance, nuclear safety, etc., where decisions targeted only at achieving the maximal expected performance may lead to an excessesive risk exposure. However, in contrast to other applications, distributions of the stochastic risk-inducing factors are often unknown or uncertain in military problems. Uncertainty in distributions of risk parameters may be caused by a lack of data, unreliability of data, or the specific nature of a risk factor (e.g., in different circumstances a risk factor may exhibit different stochastic behavior). Therefore, decision making in military applications must account for uncertainties in distributions of stochastic parameters and be robust with respect to these uncertainties.

In this chapter, we propose a general methodology for managing risk in military applications involving various risk factors as well as uncertainties in distributions. The approach is tested with several stochastic versions of the Weapon-Target Assignment problem.

The chapter is organized as follows. Section 5.2 presents key theoretical results on risk management using Conditional Value-at-Risk (CVaR) risk measure, and describes the general approach to controlling risk when distributions of risk factors are uncertain. Section 5.3 develops various formulations of the stochastic Weapon-Target Assignment (WTA) problem with CVaR constraints. Results of numerical experiments for one-stage and two-stage stochastic WTA problems are presented in Section 5.4. Section 5.5 summarizes the obtained results and outlines the directions of future research.

5.2 The General Approach

Presence of uncertainty in a decision-making model leads to the problem of estimation and managing/controlling of risk associated with the stochastic parameters in the model. Over the recent years, risk management has evolved into a sophisticated discipline combining both rigorous and elegant theoretical results and practical effectiveness (this especially applies to the risk management in finance industry). Generally speaking, risk management is a set of activities aimed at reducing or preventing *high losses* incurred from an incorrect decision. The losses (e.g., damages, failures) in a system are quantified by a *loss function* $\mathscr{L}(x,\xi)$ that depends upon decision vector x and a stochastic vector ξ standing for uncertainties in the model. Assuming for now that a distribution of the parameter ξ is known, it is possible to determine the distribution of the loss function

$\mathscr{L}(x,\xi)$ (Figure 5–1). Then, the problem of preventing high losses is a problem of controlling and shaping the loss distribution and, more specifically, its right tail, where the high losses reside. To estimate and quantify the losses in the tail of the loss distribution, a *risk measure* has to be specified. In particular, a risk measure introduces the ordering relationships for risks, so that one is able to discriminate "less risky" decisions from the "riskier" ones.[1] The appropriate choice of a risk measure is, in most cases, dictated by the nature of uncertainties and risks in the problem at hand. In military applications, for example, one usually deals with the probabilities of events, such as the probability to hit a target, the probability to detect the enemy's aircraft, and so on. Therefore *percentile* risk measures that represent the risk in terms of percentiles of the loss distribution are particularly suitable for the risk management in military applications. Popular percentile risk measures include Value-at-Risk (VaR), Conditional Value-at-Risk, Maximum Loss, and Expected Shortfall. Figure 5–1 displays some of these measures; Value-at-Risk with confidence level α (α-VaR), which is the α-percentile of loss distribution, Maximum Loss ("1.0-percentile" of loss distribution), and α-CVaR, which may be thought of as the expectation of losses exceeding α-VaR.

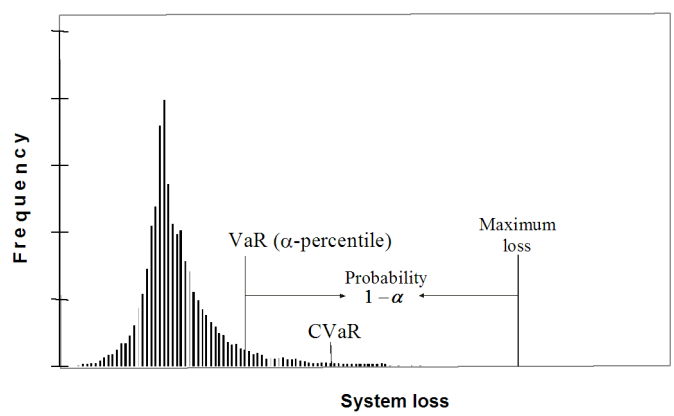


Figure 5–1: Loss function distribution and different risk measures.

We build our approach for risk management in military applications on the CVaR methodology, which is a relatively new development (Rockafellar and Uryasev (2000), Rockafellar and Uryasev

[1] Artzner et al. (1997, 1999) have introduced a concept of "ideal", or *coherent*, risk measure. A *coherent* risk measure, which satisfies to a set of axioms developed in this paper, is expected to produce "proper" and "consistent" estimates of risk.

(2002)). This section presents the general framework of risk management using Conditional Value-at-Risk, and extends it to the case when the distributions of stochastic parameters are not certain.

5.2.1 Risk Management Using Conditional Value-at-Risk

Suppose that the uncertain future is represented by a finite number of future outcomes (scenarios). Then an intuitive definition of Value-at-Risk with confidence level α (α-VaR) presents it as the loss that can be exceeded only in $(1-\alpha)\cdot 100\%$ of worst scenarios. Similarly, one may think of α-CVaR (i.e., Conditional Value-at-Risk with confidence level α) as of the average loss over $(1-\alpha)\cdot 100\%$ of worst cases (Figure 5–2).

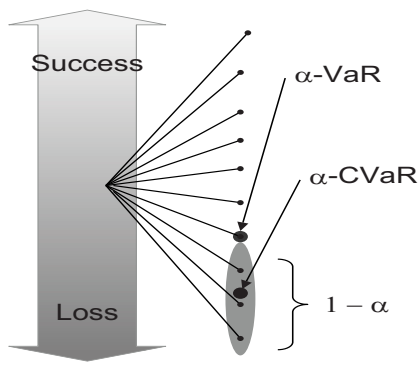


Figure 5–2: A visualization of VaR and CVaR concepts.

These intuitive definitions are correct if, for example, all scenarios are equally probable, and $(1-\alpha)\cdot 100$ is an integer number. The formal definitions of α-VaR and α-CVaR that apply to any loss distribution and value of confidence level α are more complex (Rockafellar and Uryasev, 2002). Here we mention only the most important properties of CVaR and their practical implications.

The Conditional Value-at-Risk function $\mathsf{CVaR}_\alpha[\mathscr{L}(x,\xi)]$ has the following properties (Rockafellar and Uryasev (2000), Rockafellar and Uryasev (2002), Acerbi and Tasche (2002)):

- CVaR is continuous with respect to confidence level α (other percentile risk measures like VaR, Expected Shortfall, etc., may be discontinuous in α)
- CVaR is convex in α and x, provided that the loss function $\mathscr{L}(x,\xi)$ is convex in x (VaR, Expected Shortfall are generally non-convex in x)
- CVaR is *coherent* in the sense of Artzner et al. (1999)
- Unlike VaR, CVaR has stable statistical estimates

- In the case of a continuous loss distribution CVaR represents the conditional expectation of losses that exceed the VaR level.

From the viewpoint of managing and controlling of risk, the most important property of CVaR, which distinguishes it from all other percentile risk measures, is the convexity with respect to decision variables, which permits the use of convex programming for minimizing CVaR. If the loss function $\mathscr{L}(x,\xi)$ can be approximated by a piecewise linear function, the procedure of controlling or optimization of CVaR is reduced to solving a Linear Programming (LP) problem.

The techniques for optimizing CVaR when the loss distribution is discrete are of special importance for military applications, as will be demonstrated in Section 5.3.

Assume that there are S possible realizations (scenarios) $\xi_1,\ldots,\xi_S$ of vector ξ with probabilities π_s (naturally, $\sum_{s=1}^{S}\pi_s = 1$), then in the optimization problem with multiple CVaR constraints

$$\begin{aligned}
&\max_{x\in X}\ g(x)\\
&\text{subject to}\\
&\mathsf{CVaR}_{\alpha_n}[\mathscr{L}(x,\xi)] \le C_n, \quad n=1,\ldots,N,
\end{aligned}$$

where $g(x)$ is some performance function and X is a convex set, each CVaR constraint may be replaced by a set of inequalities (Rockafellar and Uryasev, 2002)

$$\begin{aligned}
&\mathscr{L}(x,\xi_s) - \zeta_n \le w_{ns}, \quad s=1,\ldots,S,\\
&\zeta_n + (1-\alpha_n)^{-1}\sum_{s=1}^{S}\pi_s w_{ns} \le C_n,\\
&\zeta_n \in \mathbb{R}, \quad w_{ns}\in\mathbb{R}^+, \quad s=1,\ldots,S,
\end{aligned} \tag{5.1}$$

where $\mathbb{R}$ and $\mathbb{R}^+$ are the sets of real and non-negative real numbers correspondingly, and w_{ns} are auxiliary variables. If in the optimal solution the n-th CVaR constraint is active, then the corresponding variable ζ_n is equal to α_n-VaR (i.e., α_n-th percentile of the loss distribution).

In the risk management methodology discussed above the distribution of stochastic parameter ξ is considered to be known. The next subsection extends the presented approach to the case, when the distribution of stochastic parameters in the model is not certain.

5.2.2 Risk Management Using CVaR in the Presence of Uncertainties in Distributions

The general approach to managing risks in an uncertain environment, where the distributions of stochastic parameters are not known for sure, can be described as follows. Suppose that we have some performance function $F(x,\xi)$, dependent on the decision vector $x \in X$ and some random vector $\xi \in \Xi$, whose distribution is not known for certain. We assume that the actual realization of vector ξ may come from different distributions $\Theta_1,\ldots,\Theta_N$. The vector ξ stands for the uncertainties in data that make it impossible to evaluate the efficiency $F(x,\xi)$ of the decision for sure. Thus, there always exists a possibility of making an incorrect decision, and, consequently, suffering loss, damage, or failing the mission. If the loss in the system is evaluated by function $\mathscr{L}(x,\xi)$, then risk of high losses can be controlled using CVaR constrains. Let formulate the problem of maximizing the expected performance function $F(x,\xi)$ subject to some operational constraints $Ax \le b$ and CVaR risk constraints. Due to the unknown distribution of vector ξ, we are unable to find the expectation $\mathsf{E}_\Theta[F(x,\xi)]$. Therefore, being on the conservative side, we want the decision x to be optimal with respect to each measure Θ_n, and this leads to the following *maxi-min* problem:

$$\begin{aligned}
&\max_{x\in X} \min_{\Theta_n,\, n=1,\ldots,N} \mathsf{E}_{\Theta_n}[F(x,\xi)] \qquad (5.2)\\
&\text{subject to}\\
&Ax \le b,\\
&\mathsf{CVaR}_\alpha[\mathscr{L}(x,\xi)\,|\,\Theta_n] \le C, \quad n=1,\ldots,N,
\end{aligned}$$

where multiple CVaR constraints with respect to different measures Θ_n control the risk for high losses $\mathscr{L}(x,\xi)$ to exceed some threshold C. In formulation (5.2) we assume that the performance function F is concave in x, and the loss function $\mathscr{L}$ is convex in x. These assumptions are not restrictive; on the contrary, they indicate that given more than one decision with equal performance one favors safer decisions over the riskier ones.

Model (5.2) explains how to handle the risk of generating an incorrect decision in an uncertain environment. In military applications, different types of risks and losses may be explicitly involved, for example, along with loss function $\mathscr{L}(x,\xi)$ one may consider a loss function $\mathscr{R}(x,\xi)$ for the risk of false target attack. Control for this type of risk can also be included in the model by a similar set

of CVaR constraints:

$$\max_{x \in X} \min_{\Theta_n,\, n=1,\dots,N} \mathsf{E}_{\Theta_N}[F(x,\xi)]$$

subject to

$$Ax \le b,$$

$$\mathsf{CVaR}_{\alpha_1}[\mathscr{L}(x,\xi)\,|\,\Theta_n] \le C_1, \quad n = 1,\dots,N,$$

$$\mathsf{CVaR}_{\alpha_2}[\mathscr{R}(x,\xi)\,|\,\Theta_n] \le C_2, \quad n = 1,\dots,N.$$

In the next sections we test the presented approach to risk management in military applications on the Weapon-Target Assignment problem.

5.3 Example: Stochastic Weapon-Target Assignment Problem

The Weapon-Target Assignment (WTA) problem considers the optimal assignment of weapons to targets so as to minimize the surviving value of targets. The WTA problem is used in planning environment that features a whole spectrum of uncertainties, such as the number and types of targets in the battle space, their positions, and the probability of a weapon to destroy a target (e.g., probability of kill). To generate robust decisions, one must account for these uncertainties and the corresponding risks. In this section we present two formulations of the stochastic Weapon-Target Assignment problem that address the uncertainties in a weapon's probability of kill and in the number of targets.

5.3.1 Deterministic WTA Problem

The generic formulation of the Weapon Target Assignment problem is as follows. Given the set of targets and set of available weapons, one must find the optimal assignment of weapons to targets, such that, for example, the damage to the targets is maximized, or the cost of the operation is minimized. The WTA formulation that maximizes the damage to the targets (see, for example, Manne (1958), denBroeger et al. (1959), Murphey (1999)) leads to a nonlinear programming problem with linear constraints (NLP), and is the subject of the next chapter. In this chapter we adopt another setup, where the total cost of the mission (including battle damage or loss) is minimized, while satisfying constraints on mission accomplishment (i.e., destruction of all targets with some prescribed probabilities). We assume that different weapons have different costs and efficiencies, and, in general, each may have a "multishot" capacity so that it may attack more than one target.

In the deterministic setup of the problem we include also the constraint that prescribes how many targets a single weapon can attack.

The deterministic WTA problem is

$$\min_{x} \quad \sum_{k=1}^{K}\sum_{i=1}^{I} c_{ik}x_{ik} \tag{5.3a}$$

subject to

$$\sum_{k=1}^{K} x_{ik} \le m_i, \quad i = 1,\dots,I, \tag{5.3b}$$

$$x_{ik} \le m_i v_{ik}, \quad i = 1,\dots,I, \quad k = 1,\dots,K, \tag{5.3c}$$

$$\sum_{k=1}^{K} v_{ik} \le t_i, \quad i = 1,\dots,I, \tag{5.3d}$$

$$1 - \prod_{i=1}^{I}(1-p_{ik})^{x_{ik}} \ge d_k, \quad k = 1,\dots,K, \tag{5.3e}$$

$$x_{ik} \in \mathbb{Z}^+, \quad v_{ik} \in \{0,1\},$$

where

x_{ik} is the number of shots to be fired by weapon i at target k;

v_{ik} $= 1$, if weapon i fires at target k, and $v_{ik} = 0$ otherwise;

c_{ik} is the cost (including the battle loss or damage) of firing one shot from weapon i at target k; c_k includes the relative value of target k with respect to all other targets;

m_i is the shots capacity for weapon i;

t_i is the maximal number of targets which can be attacked by weapon i;

p_{ik} is the probability of destroying target k by firing one shot from weapon i;

d_k is the minimal required probability for destroying target k;

$\mathbb{Z}$ is the set of integer numbers, and $\mathbb{Z}^+$ is the set of non-negative integers.

The objective function in this problem equals to the total cost of the mission. The first constraint, (5.3b), states that the munitions capacity of weapon i cannot be exceeded. The second and the third constraints (5.3c) and (5.3d) are responsible for not allowing weapon i to attack more than t_i targets, where $t_i \le K$. The last constraint (5.3e) ensures that after all weapons are assigned, target k is destroyed with probability not less than d_k. Note that this nonlinear constraint can be linearized:

$$\sum_{i=1}^{I} \ln(1-p_{ik})x_{ik} - \ln(1-d_k) \le 0. \tag{5.4}$$

In this way the deterministic WTA problem (5.3a) can be formulated as a linear integer programming (IP) problem.

5.3.2 One-Stage Stochastic WTA Problem with CVaR Constraints

In real-life situations many of the parameters in model (5.3a)–(5.3e) are not deterministic, but stochastic values. For example, the probabilities p_{ik} of destroying target k may depend upon battle situation, weather conditions, and so on, and consequently, may be treated as being uncertain. Similarly, the cost of firing c_{ik}, which includes battle loss/damage, may also be a stochastic parameter. The number of targets K may be uncertain as well.

First, we consider a one-stage Stochastic Weapon-Target Assignment (SWTA) problem, where the uncertainty is introduced into the model by assuming that probabilities p_{ik} are stochastic and dependent on some random parameter ξ:

$$p_{ik} = p_{ik}(\xi).$$

In accordance to the described methodology of managing uncertainties and risks in military applications, we model the stochastic behavior of probabilities p_{ik} using scenarios. Namely, probabilities $p_{ik}(\xi)$ take different values $p_{ik}(\xi_s) = p_{iks}$, $s = 1, \ldots, S$ under S different scenarios. Such a scenario set may be constructed, for example, by utilizing the historical observations of weapons' efficiency in different environments, or by using simulated data, experts' opinions etc.

We now replace the last constraint in (5.3a) by a CVaR constraint, where the loss function takes a positive value if the probability of destroying target k is less than d_k:

$$\mathscr{L}_k(x,\xi) = \sum_{i=1}^{I} \ln\Big(1 - p_{ik}(\xi)\Big)\, x_{ik} - \ln(1 - d_k), \tag{5.5}$$

and takes a negative value otherwise. Recall that a CVaR constraint with confidence level α bounds the (weighted) average of $(1-\alpha)\cdot 100\%$ highest losses. In our case, allowing small positive values of loss function (5.5) for some scenarios implies that for these scenarios target k is destroyed with probability slightly less than d_k, which may still be acceptable from a practical point of view.

Except for the constraint on the target destruction probability, the one-stage Stochastic WTA problem is identical to its deterministic predecessor:

$$\begin{aligned}
&\min_x \sum_{k=1}^{K}\sum_{i=1}^{I} c_{ik}x_{ik} \\
&\text{subject to} \\
&\sum_{k=1}^{K} x_{ik} \le m_i, \quad i=1,\dots,I, \\
&x_{ik} \le m_i v_{ik}, \quad i=1,\dots,I, \quad k=1,\dots,K, \\
&\sum_{k=1}^{K} v_{ik} \le t_i, \quad i=1,\dots,I, \\
&\mathsf{CVaR}_\alpha\left[\mathscr{L}_k(x,\xi)\right] \le C_k, \quad k=1,\dots,K.
\end{aligned} \tag{5.6}$$

Here α is the confidence level, C_k are some (small) constants, and all other variables and parameters are defined as before. As demonstrated in (5.1), for the adopted scenario model with probabilities p_{ik}, the CVaR constraint for the k-th target

$$\mathsf{CVaR}_\alpha\left[\mathscr{L}_k(x,\xi)\right] \le C_k$$

is represented by a set of linear inequalities:

$$\begin{aligned}
&\sum_{i=1}^{I} \ln(1-p_{iks})\,x_{ik} - \ln(1-d_k) - \zeta_k \le w_{sk}, \quad s=1,\dots,S, \\
&\zeta_k + (1-\alpha_k)^{-1}S^{-1}\sum_{s=1}^{S} w_{sk} \le C_k, \\
&\zeta_k \in \mathbb{R}, \quad w_{sk} \ge 0, \quad s=1,\dots,S, \quad k=1,\dots,K.
\end{aligned} \tag{5.7}$$

Thus, the one-stage Stochastic WTA problem can be formulated as a mixed-integer programming (MIP) problem:

$$\begin{aligned}
&\min_x \sum_{k=1}^{K}\sum_{i=1}^{I} c_{ik}\,x_{ik} \\
&\text{subject to} \\
&\sum_{k=1}^{K} x_{ik} \le m_i, \quad i=1,\dots,I, \\
&x_{ik} \le m_i\,v_{ik}, \quad i=1,\dots,I, \quad k=1,\dots,K, \\
&\sum_{k=1}^{K} v_{ik} \le t_i, \quad i=1,\dots,I,
\end{aligned} \tag{5.8}$$

$$\sum_{i=1}^{I} \ln(1-p_{iks})\, x_{ik} - \ln(1-d_k) - \zeta_k \leq w_{sk},$$

$$s = 1,\ldots,S, \quad k = 1,\ldots,K,$$

$$\zeta_k + (1-\alpha_k)^{-1} S^{-1} \sum_{s=1}^{S} w_{sk} \leq C_k, \quad k = 1,\ldots,K,$$

$$x_{ik} \in \mathbb{Z}^+, \quad v_{ik} \in \{0,1\}, \quad \zeta_k \in \mathbb{R}, \quad w_{sk} \geq 0,$$

$$s = 1,\ldots,S, \quad i = 1,\ldots,I, \quad k = 1,\ldots,K.$$

Note that different values of probability p_{ik} represent the uncertainty in the distributions of stochastic parameters discussed in the previous section. Indeed, different values of probability p_{ik} imply different probability measures for the random variable associated with the event of destroying target k by firing one unit of munitions by weapon i. In effect, CVaR constraint (5.7) is a risk constraint that incorporates multiple probability measures.

5.3.3 Two-Stage Stochastic WTA Problem with CVaR constraints

In this subsection we consider a more complex, but also more realistic two-stage Stochastic WTA problem, where the uncertain parameter is the number of targets to be destroyed.

This problem is more realistic since it models the effect of target discovery as being dynamic; that is, not all targets are known at any single instance of time. To address this type of uncertainty, we need to modify our notation.

Consider I weapons are deployed in some bounded region of interest and interval of time T with the goal of finding targets and then, once found, attacking those targets. If we delay all assignments of weapon shots until to targets until the final time T, then we have a deterministic, "static" WTA problem as in (5.3a)–(5.3e). If, on the other hand, we assume that weapons have at least 2 opportunities to shoot during the interval T, then the WTA problem is dynamic. In the later case we have the opportunity to avoid expending all our shots at targets discovered early in T by explicitly modeling the number of undiscovered targets in the objective function.

Assume that K now represents the number of *categories* of targets (the targets may be categorized, for example, by their importance, vulnerability, etc).

We will assume the problem has 2 stages. That is, at any given point in time, we may always partition all targets into those thus far determined and those that we conjecture to exist but have

not yet found. Our conjecture may be based on evidence obtained by prior reconnaissance of the region of interest. At some arbitrary time $0 < \tau < T$ assume that there are n_k detected targets and η_k undetected targets in each category $k = 1, ..., K$. Thus we have two clearly identified stages in our problem: in the first stage one has to destroy the targets known at time τ, in the second stage one must destroy the targets that we conjecture will be found by time T. In other words, one needs to make an assignment of weapons that will allow for the destruction of the targets known at time τ while reserving enough munition capacity for destroying the targets we expect to find in $\tau < t < T$.

Setup of the two-stage stochastic WTA problem can be considered as a part of a moving horizon or quasi-multistage stochastic WTA algorithm, where the WTA problem with many time periods is solved by recursive application of a two-stage algorithm (Murphey (1999)).

To simplify the problem setup, we remove the constraint on the number of targets a single weapon can attack (the second and third constraints in problems (5.3a)), since this constraint makes the problem much too combinatorial. Also, we assume that the probabilities p_{ik} are known (*not* random), so that the only stochastic parameters in the two-stage SWTA problem are the numbers of undetected (second-stage) targets η_k, $k = 1, ..., K$.

We model the uncertainty in the number of targets at the second stage, by we introducing a scenario model, where under scenario $s \in \{1, ..., S\}$ there are $\eta_k(s) = \eta_{ks}$ undetected targets in category k.

The first- and second-stage decision variables are defined as follows:

x_{ik} is the number of munitions to be fired by weapon i at a single target in category k during the first stage;

$y_{ik}(s)$ is the number of munitions to be fired by weapon i at a single target in category k during the second stage scenario s.

Note that the same decision is made for all targets within a category, i.e., once weapon i fires, say, 2 missiles at a specific target in category k, it must fire 2 missiles at every other target in this category.

The recursive formulation of the two-stage stochastic WTA problem is

$$\min \left\{ \sum_{k=1}^{K} \sum_{i=1}^{I} n_k c_{ik} x_{ik} + E_\eta [Q(x, \eta)] \right\} \tag{5.9a}$$

subject to

$$\sum_{k=1}^{K} n_k x_{ik} \leq m_i, \quad i = 1, \ldots, I, \tag{5.9b}$$

$$\sum_{i=1}^{I} \ln(1-p_{ik})\, x_{ik} - \ln(1-d_k) \leq \varepsilon_{1k}, \quad k = 1, \ldots, K, \tag{5.9c}$$

$$\sum_{k=1}^{K} \varepsilon_{1k} \leq C, \tag{5.9d}$$

$$x_{ik},\ \varepsilon_{1k} \geq 0, \quad i = 1, \ldots, I, \quad k = 1, \ldots, K,$$

where the recourse function $Q(x, \eta)$ is the solution of the problem

$$Q(x, \eta) = \min_{y} \left\{ \sum_{k=1}^{K} \sum_{i=1}^{I} \eta_k(s)\, c_{ik} y_{ik}(s) + M \sum_{i=1}^{I} \delta_i \right\} \tag{5.10a}$$

subject to

$$\sum_{k=1}^{K} (n_k x_{ik} + \eta_k(s)\, y_{ik}(s)) \leq m_i + \delta_i, \quad \forall\, i, \tag{5.10b}$$

$$\sum_{i=1}^{I} \ln(1-p_{ik})\, y_{ik}(s) - \ln(1-d_k) - \zeta_k \leq w_k(s),\ \forall\, k, s, \tag{5.10c}$$

$$\zeta_k + (1-\alpha_k)^{-1} S^{-1} \sum_{s=1}^{S} w_k(s) \leq \varepsilon_{2k}, \quad \forall\, k, \tag{5.10d}$$

$$\sum_{k=1}^{K} (\varepsilon_{1k} + \varepsilon_{2k}) \leq C, \tag{5.10e}$$

$$y_{ik}(s),\ \delta_i \in \mathbb{Z}^+, \quad w_k(s),\ \varepsilon_{2k} \geq 0, \quad \zeta_k \in \mathbb{R}, \quad M \gg 1.$$

Let us discuss the recourse problem (5.9a)–(5.10e). As before, we minimize the total cost of the mission. The first constraint (5.9b) is the munitions capacity constraint. The second constraint, (5.9c), allows a first-stage target in category k to survive with (small) error ε_{1k}, and the third constraint (5.9d) bounds the sum of errors ε_{1k} by some (small) constant C.

In the recourse function (5.10a) the first constraint (5.10b) requires the weapon i to not exceed its munitions capacity while destroying the first- and second-stage targets. The possible infeasibility of the munitions capacity constraint can be relaxed using auxiliary variables δ_i that enter the objective function with cost coefficient $M \gg 1$. The second and third constraints (5.10c)–(5.10d) form a CVaR constraint that controls the failure of destroying second-stage targets with the prescribed probabilities d_k. Similarly to the deterministic constraint in (5.9a), CVaR of failure to destroy a

second-stage target in category k is bounded by (small) error variable ε_{2k}. The total sum of errors ε_{1k} and ε_{2k} at both stages is bounded by small constant C, which makes possible a tradeoff between the degree of mission accomplishment at the first and second stages.

The extensive form of the two-stage SWTA problem (5.9a)–(5.10a) is

$$\min\left\{\sum_{k=1}^{K}\sum_{i=1}^{I} n_k c_{ik} x_{ik} + \frac{1}{S}\sum_{s=1}^{S}\sum_{k=1}^{K}\sum_{i=1}^{I}\eta_{ks} c_{ik} y_{ik}(s) + M\sum_{i=1}^{I}\delta_i\right\} \tag{5.11}$$

subject to

$$\sum_{k=1}^{K}(n_k x_{ik} + \eta_{ks} y_{ik}(s)) \le m_i + \delta_i, \quad \forall i, s,$$

$$\sum_{i=1}^{I}\ln(1-p_{ik})x_{ik} - \ln(1-d_k) \le \varepsilon_{1k}, \quad \forall k,$$

$$\sum_{i=1}^{I}\ln(1-p_{ik})y_{ik}(s) - \ln(1-d_k) - \zeta_k \le w_{ks}, \quad \forall k, s,$$

$$\zeta_k + (1-\alpha_k)^{-1}S^{-1}\sum_{s=1}^{S} w_{ks} \le \varepsilon_{2k}, \quad \forall k,$$

$$\sum_{k=1}^{K}(\varepsilon_{1k} + \varepsilon_{2k}) \le C,$$

$$x_{ik}, y_{ik}(s), \delta_i \in \mathscr{Z}^+, \quad w_{ks}, \varepsilon_{1k}, \varepsilon_{2k} \in \mathscr{R}^+ \quad \zeta_k \in \mathscr{R}, \quad M \gg 1.$$

The two-stage stochastic WTA problem is also a MIP problem.

5.4 Numerical results

In this section we present and discuss numerical results obtained for both one-stage and two-stage stochastic WTA problems. The algorithms for solving deterministic, one- and two-stage stochastic WTA problems were implemented in C++, and we used CPLEX 7.0 Callable Library to solve the corresponding IP and MIP problems. We used simulated data (sets of weapons and targets, the corresponding costs and probabilities etc.) for testing the implemented algorithms.

5.4.1 Single-stage deterministic and stochastic WTA problems

For the deterministic and one-stage stochastic WTA problems we used the following data:

- 5 targets ($K = 5$)
- 5 weapons, each with 4 shots ($I = 5,\ m_i = 4$)
- Any weapon can attack any target ($t_i = 5$),
- Probabilities p_{ik} and costs c_{ik} depend only on the weapon index i: $p_{ik} = p_i$, $c_{ik} = c_i$
- All targets have to be destroyed with at least probability 95% ($d_k = 0.95$)

- The confidence levels α_k in CVaR constraint are 0.90
- There are 20 scenarios ($S = 20$) for probabilities $p_{ik}(s)$ in the one-stage SWTA problem; all scenarios are equally probable.

According to the aforementioned, we used simulated data for probabilities p_{iks} and costs c_{ik}. It was assumed that probabilities $p_{iks} = p_{is}$ are uniformly distributed random variables, and the Figure 5.4.1 displays the relation between the cost of missile of weapon i and its efficiency (i.e., probability to destroy a target):

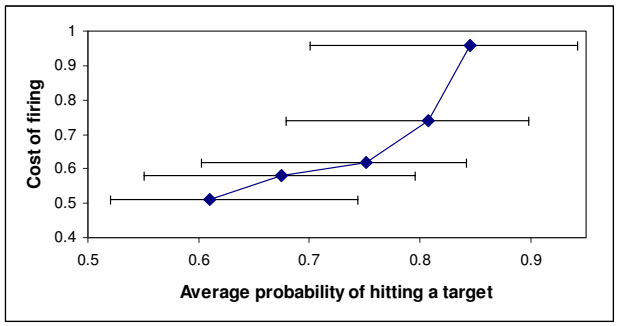


Figure 5–3: Dependence between the cost and efficiency for different types of weapons in one-stage SWTA problem (5.8) deterministic WTA problem (5.3a).

On this graph, diamonds represent the average probability of destroying a target by firing one shot from weapon i, and the horizontal segments represent the support for random variable $p_{ik}(\xi) = p_i(\xi)$. The average probabilities

$$\overline{p}_{ik} = \frac{1}{S}\sum_{s=1}^{S} p_{iks}$$

were used for p_{ik} in the deterministic problem (5.3a).

The efficiency and cost of weapons 1 to 5 increase with the index of weapon, i.e., Weapon 1 is the least efficient and cheapest, whereas Weapon 5 is the most precise, but also most expensive one.

Tables 5–1 and 5–2 represent the optimal solutions (variables x_{ik}) of the deterministic and one-stage stochastic WTA problems.

One can observe the difference in the solutions produced by deterministic and stochastic WTA problems: the deterministic solution does *not* use the most expensive and most precise Weapon 5, whereas the stochastic solution of problem (5.8) with CVaR constraint uses this weapon. It means that the CVaR-constrained solution of problem (5.8) represents a more expensive but safer decision.

Table 5–1: Optimal solution of the deterministic WTA problem (5.3a)

Target	T1	T2	T3	T4	T5	Total shots
Weapon 1	0	2	1	0	1	4
Weapon 2	0	1	2	0	0	3
Weapon 3	1	0	0	1	1	3
Weapon 4	1	0	0	1	1	3
Weapon 5	0	0	0	0	0	0

Table 5–2: Optimal solution of the one-stage stochastic WTA problem (5.6), (5.8)

Target	T1	T2	T3	T4	T5	Total shots
Weapon 1	0	1	1	0	1	3
Weapon 2	0	0	1	1	1	3
Weapon 3	2	0	0	1	0	3
Weapon 4	0	1	1	0	1	3
Weapon 5	1	1	0	1	0	3

On a different dataset, we obtained a similar result: the optimal solution of the stochastic problem with CVaR constraints did not use the cheapest and the most unreliable weapon, whereas the deterministic solution used it.

We have also performed testing of the deterministic solution under different scenarios. The deterministic solution failed to destroy more than one target under 13 of 20 scenarios.

This example highlights the importance of using risk management procedures in military decision-making applications involving uncertainties.

5.4.2 Two-Stage Stochastic WTA Problem

For the two-stage stochastic WTA problems we used the following data:

- 3 categories of targets ($K = 3$)
- 4 weapons, each with 15 shots ($I = 4,\ m_i = 15$)
- Probabilities p_{ik} and costs c_{ik} depend only on the weapon index i: $p_{ik} = p_i,\ c_{ik} = c_i$
- All targets have to be destroyed with probability 95% ($d_k = 0.95$)
- The confidence levels α_k in CVaR constraint are equal 0.90
- There are 15 scenarios ($S = 15$) for the number of undetected targets η_{ks} (for each k, the number of undetected targets η_{ks} is a random integer between 0 and 5); all scenarios are equally probable.

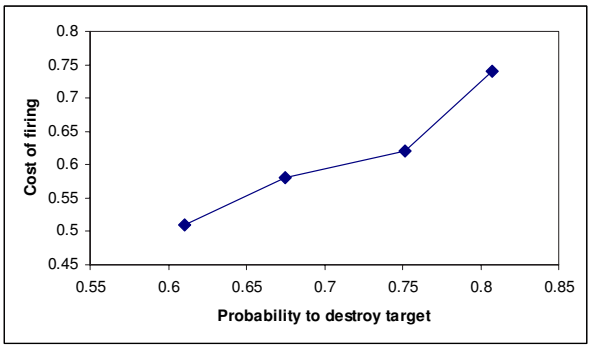


Figure 5–4: Dependence between the cost and efficiency for different types of weapons in two-stage SWTA problem (5.11).

For the probabilities p_{ik} in the two-stage problem, we used the first four average probabilities from the deterministic WTA problem, and the efficiency-cost dependence is shown in Figure 5.4.2.

Tables 5–3 to 5–5 illustrate the optimal solution of the problem (5.11). Table 5–3 contains the first-stage decision variables x_{ik}, and Tables 5–4 and 5–5 display the second-stage variables $y_{ik}(s)$ for scenarios $s = 1$ and $s = 2$, just for illustrative purposes.

Similarly to the analysis of the one-stage stochastic WTA problem, we compared the scenario-based solution of problem (5.11) with the solution of the "deterministic two-stage" problem, where the number of second-stage targets in each category is taken as the average over 15 scenarios. The comparison shows that the solution based on the expected information leads to significant munitions shortages in 5 of 15 (i.e., 33%) scenarios, and consequently to failing the mission at the second stage. Recall from the analysis of the one-stage SWTA problem that the solution based on the expected information also exhibited poor robustness with respect to different scenarios. Indeed, solutions that use only the *expected* information, are supposed to perform well *on average*, or *in the long run*. However, in military applications there is *no long run*, and therefore such solutions may not be robust with respect to *many possible scenarios*.

Table 5–3: First-stage optimal solution of the two-stage stochastic WTA problem

Category	K1	K2	K3
# of detected targets	3	5	2
Weapon 1	0	0	0
Weapon 2	0	0	0
Weapon 3	1	1	1
Weapon 4	1	1	1

Table 5–4: First-stage optimal solution of the two-stage stochastic WTA problem (5.11) for the first scenario

Category	K1	K2	K3
# of undetected targets	1	4	2
Weapon 1	0	0	2
Weapon 2	0	0	1
Weapon 3	1	1	0
Weapon 4	1	1	0

Table 5–5: Second-stage optimal solution of the two-stage stochastic WTA problem (5.11) for the second scenario

Category	K1	K2	K3
# of undetected targets	3	5	3
Weapon 1	2	0	2
Weapon 2	1	0	1
Weapon 3	0	1	0
Weapon 4	0	1	0

Thus, solutions of both one-stage and two-stage SWTA problems confirm the general conjecture on the potential importance of exploiting stochastic models and risk management in military applications.

5.5 Conclusions

We have presented an approach to managing risk in stochastic environments, where distributions of stochastic parameters are uncertain. This approach is based on the methodology of risk management with Conditional Value-at-Risk risk measure developed by Rockafellar and Uryasev (2002). Although the presented approach has been used to solve one-stage and two-stage stochastic Weapon-Target Assignment problems, it is quite general and can be applied to wide class of problems with risks and uncertainties in distributions. Among the directions of future research we emphasize consideration of a stochastic WTA problem in NLP formulation, where the damage to the targets is maximized while constraining the risk of false target attack.

CHAPTER 6
USE OF CONDITIONAL VALUE-AT-RISK IN STOCHASTIC PROGRAMS WITH POORLY DEFINED DISTRIBUTIONS

In this chapter we consider a more realistic formulations of the two-stage Stochastic Weapon-Target Assignment problem, where the cumulative damage to the targets is maximized. This problem setup, however, leads to Mixed-Integer Programming problems with nonlinear objectives. By using a relaxation technique that preserves integrality of the optimal solutions, we develop LP formulations for the deterministic and two-stage stochastic WTA problems. Similarly to the approach of the preceding chapter, the risk of incorrect second-stage decisions due to errors in specified distributions of the second-stage targets is controlled using the Conditional Value-at-Risk risk measure. An LP formulation for the two-stage SWTA problem with uncertainties in distributions has been developed, which produces integer optimal solutions for the first-stage decision variables, and also yields a tight lower bound for the corresponding MIP problem.

6.1 Introduction

This chapter applies a general methodology of risk management in military applications (Krokhmal et al., 2003) to a stochastic version of the Weapon-Target Assignment (Manne, 1958; denBroeger et al., 1959; Murphey, 1999) problem. The approach suggested in (Krokhmal et al., 2003) is built on the recently developed technique (Rockafellar and Uryasev, 2000, 2002) for risk management using the Conditional Value-at-Risk (CVaR) risk measure. The general framework was developed for specific military applications such as surveillance, planning, and scheduling, which require robust decision making in a dynamic, distributed, and uncertain environment. The focus of the suggested approach has been on the development of robust and efficient procedures for decision-making in stochastic framework with multiple risk factors and uncertainties in the distributions of stochastic parameters.

In a preceding chapter the authors tested the developed methodology of risk management using Conditional Value-at-Risk on one-stage and two-stage stochastic versions of a Weapon-Target Assignment (WTA) problem. In the WTA problem formulation developed in Krokhmal et al. (2003),

an optimal decision minimized the total cost of the mission, including battle damage, while ensuring that all targets are destroyed with the prescribed probability level. In such a setup, the WTA problem could easily be formulated as a linear programming (LP) problem, or integer programming (IP) problem with linear objective and constraints.

In this chapter, we consider the WTA problem in a more realistic formulation, where the cumulative damage to the targets is maximized. Though this setup has some advantages over the previous formulation (for example, it allows for prioritizing the targets by importance and achieving a desirable tradeoff between assigning more weapons to high-priority targets and fewer weapons to low-priority ones), it leads to an integer programming problem with nonlinear objective and linear constraints.

In this chapter, the nonlinear integer programming problem will be transformed into a (convex) linear programming problem, and the corresponding LP relaxations for deterministic and two-stage stochastic WTA problems will be developed. Further, a formulation of a two-stage stochastic WTA problem with uncertainties in the distributions of second-stage scenario parameters will be presented. We employ the Conditional Value-at-Risk risk measure (Rockafellar and Uryasev, 2000, 2002) in order to constrain the risk of generating an incorrect decision.

The chapter is organized as follows. The next section introduces a generic nonlinear model for the WTA problem, and demonstrates how an LP relaxation can be constructed for the original IP problem with a nonlinear objective. Section 6.3 presents a fast algorithm based on an LP relaxation for the two-stage stochastic WTA problem. Section 6.4 considers the two-stage SWTA problem with uncertainties in distributions. A case study for the problem is presented in Section 6.5.

6.2 Deterministic Weapon-Target Assignment Problem

In the preceding chapter (see also Krokhmal et al. (2003)) we have considered a formulation for the Weapon-Target Assignment problem where the total cost of the mission is minimized while satisfying some probabilistic constraints on the target destruction. An advantage of this formulation is the linearity of the mathematical programming problems it reduces to. Now, following Krokhmal et al. (2003), we consider another setup for the WTA problem, where the total damage to the targets is minimized with constraints on weapons availability. Though this formulation results in an integer

programming problem with nonlinear objective and linear constraints, we will demonstrate how a linear relaxation of this problem can be developed.

First, consider a deterministic formulation of the WTA problem. Let N denote the number of targets to be destroyed, and M be the total number of weapons available. Assume that the weapons (aircraft, missiles, etc.) are identical in their capabilities of destroying targets. Let each target have a value (priority) V_j, $j = 1, \ldots, N$. Define the probability p_j of destroying the target j by a single weapon as Bernoulli trial with independent outcomes:

$$\mathsf{P}\,[\text{target } j \text{ is destroyed by a single weapon}] = p_j$$

$$\mathsf{P}\,[\text{ target } j \text{ is not destroyed by a single weapon}] = q_j = 1 - p_j$$

Introducing the decision variables x_j, $j = 1, \ldots, N$, as the number of weapons assigned to each target $j = 1, \ldots, N$, we write the deterministic WTA problem as an integer programming problem with nonlinear objective:

$$\min \quad \sum_{j=1}^{N} V_j q_j^{x_j} \tag{6.1a}$$

$$\text{s. t.} \quad \sum_{j=1}^{N} x_j = M, \tag{6.1b}$$

$$x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, N,$$

where $\mathbb{Z}_+$ is the set of non-negative integer numbers. The objective function (6.1a) represents the weighted cumulative probability of survival of the set of targets. Constraint (6.1b) is the munitions capacity constraint, where the equality sign means that all munitions have to be utilized during the mission. denBroeger et al. (1959) showed that this problem could be solved using a greedy algorithm in $O(N + M \log N)$ time. However, we wish to extend the model and unfortunately, this strategy will no longer hold. Hence, an alternative strategy for quick solving of this problem is required.

The optimization problem (6.1a) has a special structure: the objective of (6.1a) is a linear combination of univariate nonlinear functions, where the j-th function has argument x_j. Taking this into account, we replace every nonlinear summand $q_j^{x_j}$ in the objective of problem (6.1a) by a

piecewise linear function $\varphi_j(x_j)$ such that

$$\forall x_j \in \mathbb{Z}_+ : \quad \varphi_j(x_j) = q_j^{x_j}, \quad j = 1, \ldots, N,$$

i.e., all vertices of function $\varphi_j(x_j)$ are located in integer points $x_j \in \mathbb{Z}_+$, and for integer values of the argument function $\varphi_j(x_j)$ equals $q_j^{x_j}$. The corresponding IP problem is

$$\min \quad \sum_{j=1}^{N} V_j \varphi_j(x_j) \tag{6.2a}$$

$$\text{s.t.} \quad \sum_{j=1}^{N} x_j = M, \tag{6.2b}$$

$$x_j \in \mathbb{Z}_+, \quad j = 1, \ldots, N.$$

Clearly, problems (6.1a) and (6.2) have the same optimal solutions.

Now consider a linear relaxation of (6.2a) obtained by relaxing the integrality of the decision variables x_j and representing the piecewise convex functions $\varphi_j(x_j)$ by the maximum of M linear functions

$$\varphi_j(x_j) = \max_{x_j} \left\{ l_{j,0}(x_j), \ldots, l_{j,M-1}(x_j) \right\},$$

where $l_{j,m}(x_j)$ contains a linear segment of $\varphi_j(x_j)$ at $x_j \in [m, m+1]$, and correspondingly has the form

$$l_{j,m}(x_j) = q_j^m \left[(1 - q_j)(m - x_j) + 1 \right], \;\; j = 1, \ldots, N, \;\; m = 0, \ldots, M-1.$$

This relaxation yields the following LP formulation:

$$\min \quad \sum_{j=1}^{N} V_j z_j \tag{6.3a}$$

$$\text{s .t.} \quad z_j \geq q_j^m \left[(1 - q_j)(m - x_j) + 1 \right], \qquad j = 1, \ldots, N, \quad m = 0, \ldots, M-1, \tag{6.3b}$$

$$\sum_{j=1}^{N} x_j = M, \tag{6.3c}$$

$$x_j \geq 0, \qquad j = 1, \ldots, N, \tag{6.3d}$$

where z_j are auxiliary variables.

Proposition 6.2.1 *Linear programming problem* (6.3) *has an optimal solution, which is integer in variables* $x_1, \ldots, x_N$.

Proof. According to the basics of linear programming theory, an optimal solution of LP problem with bounded feasible set

$$\min_{\mathbf{y}\in\mathbb{R}^n} \{\mathbf{c}\mathbf{y} \,|\, A\mathbf{y} \le 0\}$$

is achieved in extreme points (vertices) of the feasible region $A\mathbf{y} \le 0$. To be a vertex of a polyhedral $A\mathbf{y} \le 0$, point $\mathbf{y}^* \in \mathbb{R}^n$ has to satisfy n different equations of the system $A\mathbf{y} = 0$ (indeed, a point in $\mathbb{R}^n$ can only be defined as the intersection of n hyperplanes).

Therefore, to show that problem (6.3a) has an optimal solution with integer-valued variables $x_1, \ldots, x_N$, we have to demonstrate that all vertices of the feasible region (6.3b)–(6.3d) have integer coordinates x_i, $i = 1, \ldots, N$.

Without loss of generality, we assume that the feasible set of (6.3a) is bounded (we may impose constraints $z_i \le C$ for some large $C > 0$ without affecting the set of optimal solutions). Then, consider a feasible point $\mathbf{y}^* = (x_1, \ldots, x_N, z_1, \ldots, z_N) \in \mathbb{R}^{2N}$ that has K non-integer components $x_{i_1}, \ldots, x_{i_K}$, where $2 \le K \le N$. This point may satisfy at most $2N - K + 1$ different equalities that define the boundary of the feasible set (6.3b)–(6.3d). Indeed, each of $N - K$ integer-valued components $x_{j_0} = m_0 \in \{1, \ldots, M-1\}$ and the corresponding z_{j_0} may satisfy 2 equalities (6.3b)

$$z_{j_0} = q_{j_0}^{m_0}\left[(1-q_{j_0})(m_0 - x_{j_0}) + 1\right] \quad \text{and} \quad z_{j_0} = q_{j_0}^{m_0-1}\left[(1-q_{j_0})(m_0 - 1 - x_{j_0}) + 1\right],$$

(if $m_0 = 0$, the corresponding x_{j_0} and z_{j_0} satisfy 1 equality $x_{j_0} = 0$ from set (6.3d) and 1 equality $z_{j_0} = 1$ from set (6.3b); the case $m_0 = M$ is treated similarly). Each non-integer x_{i_k} and the corresponding z_{i_k} may satisfy at most 1 equality (6.3b)

$$z_{i_k} = q_{i_k}^{m}\left[(1-q_{i_k})(m - x_{i_k}) + 1\right].$$

Additionally, point $\mathbf{y}^*$ satisfies constraint (6.3c)

$$\sum_{j=1}^{N} x_j = M.$$

Thus, a feasible point with $2 \le K \le N$ non-integer components $x_{i_1}, \ldots, x_{i_K}$ may satisfy at most $2(N-K) + K + 1 = 2N - K + 1 \;<\; 2N$ different equalities that define the boundary of the feasible region (6.3b)–(6.3d), and therefore cannot be an extreme point of the feasible region. ■

Proposition 6.2.2 *The set of optimal solutions of problem* (6.1) *coincides with the set of integer optimal values of variables* $x_1, \ldots, x_N$ *in problem* (6.3). *Optimal values of objective functions of* (6.1) *and* (6.3) *coincide as well.*

Proof. Observe that:

(i) The set $S \subset \mathbb{Z}_+^N$ of feasible values $x_1, \ldots, x_N$ of problem (6.1a) is a subset of the feasible region of (6.3a).

(ii) By construction of problem (6.3a), objective functions (6.1a) and (6.3a) take identical values on S.

(iii) Proposition 6.2.1 implies that objective function of (6.3a) achieves global minimum on S.

From (i) –(iii) follows the statement of the Proposition 6.2.2. ■

As it has been mentioned, dealing with an LP problem instead of IP problem dramatically increases the speed and robustness of computations, especially in large-scale instances.

6.3 Two-Stage Stochastic WTA Problem

In reality, many of the parameters of models (6.1) or (6.3) are not known with certainty. In this section, we consider the uncertain parameter is the number of targets to be destroyed.

Without lost of generality, assume that there are K *categories of targets*. The targets are categorized by their survivability and importance, so that all the targets within category k have the same probability of survival q_k and priority V_k. Assume that there are n_k detected targets and ξ_k undetected targets in each category $k = 1, \ldots, K$, where $\{\xi_k \mid k = 1, \ldots, K\}$ are random numbers. The undetected targets are expected to appear at some time in the future. Thus, we have two clearly identified stages in our problem: in the first stage one has to destroy the already detected targets, and in the second stage one must destroy the targets that might be detected beyond the current time horizon, but before some end time T. Consequently, one has to make an assignment of weapons in the first stage that allows enough remaining weapons to attack the possible second-stage targets. This type of problem is well known as two-stage recourse problem.

According to the stochastic programming approach, the uncertain number of targets at the second stage is modeled by the set of scenarios $\{(\xi_1^s, \ldots, \xi_K^s) \mid s = 1, \ldots, S\}$, where ξ_k^s is the number of the second-stage targets in category k under scenario s. Let x_{ki} be equal to the number of weapons assigned to a first-stage target i in category k, and y_{ki} be the number of weapons assigned to a

second-stage target i in category k, then the recourse form of the two-stage Stochastic WTA (SWTA) problem is

$$\min \quad \left\{ \sum_{k=1}^{K} \sum_{i=1}^{n_k} V_k q_k^{x_{ki}} + \mathsf{E}_{\xi}\left[Q(\mathbf{x}, \boldsymbol{\xi}) \right] \right\} \tag{6.4a}$$

$$\text{s. t.} \quad \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_{ki} \leq M, \tag{6.4b}$$

$$x_{ki} \in \mathbb{Z}_+, \quad k = 1, \ldots, K, \quad i = 1, \ldots, n_k.$$

Here, the recourse function $Q(\mathbf{x}, \boldsymbol{\xi})$ is the solution to the problem

$$Q(\mathbf{x}, \boldsymbol{\xi}) \;=\; \min \quad \left\{ \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} V_k q_k^{y_{ki}^s} \right\} \tag{6.5a}$$

$$\text{s. t.} \quad \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_k + \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} y_{ki}^s = M, \tag{6.5b}$$

$$y_{ki}^s \in \mathbb{Z}_+, \quad k = 1, \ldots, K, \quad s = 1, \ldots, S, \quad i = 1, \ldots, \xi_k^s.$$

Inequality in the first-stage munitions capacity constraint (6.4b) protects against weapon depletion at the first stage, whereas equality (6.5b) ensures full weapon utilization at the second stage.

The two-stage SWTA problem (6.4)–(6.5) can be linearized in the same way as described in the preceding section. After the linearization, the extensive form of the two-stage SWTA problem reads as

$$\min \quad \left\{ \sum_{k=1}^{K} \sum_{i=1}^{n_k} V_k z_{ki} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} V_k u_{ki}^s \right\}$$

$$\text{s. t.} \quad z_{ki} \geq (q_k)^m \left[(1 - q_k)(m - x_{ki}) + 1 \right],$$

$$k = 1, \ldots, K, \; i = 1, \ldots, n_k, \; m = 0, \ldots, M-1,$$

$$\sum_{k=1}^{K} \sum_{i=1}^{n_k} x_{ki} \leq M,$$

$$u_{ki}^s \geq (q_k)^m \left[(1 - q_k)(m - y_{ki}^s) + 1 \right], \tag{6.6}$$

$$k = 1, \ldots, K, \; i = 1, \ldots, \xi_k^s, \; m = 0, \ldots, M-1, \; s = 1, \ldots, S, \tag{6.7}$$

$$\sum_{k=1}^{K} \sum_{i=1}^{n_k} x_{ki} + \sum_{k=1}^{K} \sum_{i=1}^{\xi_k^s} y_{ki}^s = M, \quad s = 1, \ldots, S,$$

$$x_{ki},\ y^s_{ki},\ z_{ki},\ u^s_{ki} \geq 0.$$

Proposition 6.3.1 *The LP formulation* (6.6) *of the two-stage SWTA problem has an optimal solution, which is integer in variables x_{ki} and y^s_{ki}.*

Proof is analogous to that of Proposition 6.2.1. ■

The objective (6.6) was found to unreasonably favor assignments to targets with large numbers in a category. An alternative objective, which scales assignments by n_k (ξ^s_k for the second stage) tends to provide more realistic solution:

$$\min \left\{ \sum_{k=1}^{K} (n_k)^{-1} \sum_{i=1}^{n_k} V_k z_{ki} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} (\xi^s_k)^{-1} \sum_{i=1}^{\xi^s_k} V_k u^s_{ki} \right\}.$$

6.4 Two-Stage WTA Problem with Uncertainties in Specified Distributions

The preceding section discussed a classic setup for the two-stage stochastic WTA problem, which assumes that the number of the second-stage targets is uncertain now, but will become known with certainty after the first-stage decision is made and before time T.

In some situations, however, the number of second-stage targets may still remain uncertain until completion of the second stage. As an example, consider a combat reconnaissance mission where a combat unit (e.g., a UAV) has first to liquidate all known (previously detected, or *first-stage*) targets, and then perform area search in order to find and destroy all targets that have not been detected yet or survived the first-stage attack (the *second-stage* targets). Suppose that at any moment of this search there is a non-zero probability of detecting a new target, hence the total number of second-stage targets remains unknown until the mission is finished. Therefore, rather than assuming a certain number of second-stage targets, it is more appropriate to deal with a probability distribution for the number of targets. This distribution may depend on the battle situation, weather conditions etc. and consequently may not be known in advance (before the beginning of the mission). However, we assume that upon completion of the first stage of the mission, the battle unit is able to determine the true distribution of the second-stage targets (for example, by analyzing the volume of jamming, etc.)

In accordance to the described setup we propose a two-stage stochastic WTA problem, where a second-stage scenario s specifies not the number of targets in a category, but a probability distribution of the number of second-stage targets. The first- and second-stage decision variables x_{ki} and y_k^s determine the number of shots to be fired at a target in category k under scenario s. Note that as the number of the second-stage targets in category k under scenario s is unknown, variables y_k^s do not contain subscript i. Thus, the second-stage decision prescribes the number of weapons to be used for each target detected in category k, given the realization of scenario s.

Consider a set of scenarios $s = 1, \ldots, S$ that specifies the family of distributions Θ_k^s for random variables ξ_k^s representing the number of the second-stage targets in categories $1, \ldots, K$:

$$P_{\Theta_k^s}[\xi_k^s = i] = \theta_{ki}^s, \ \sum_i \theta_{ki}^s = 1.$$

For simplicity, assume that the maximum possible number of second-stage targets $I_{\max}$ is the same for all categories and all scenarios, i.e., random variables ξ_k^s have the same support $\{0, 1, \ldots, I_{\max}\}$, but different measures Θ_k^s on this support set. Also, we assume that variables ξ_k^s are independent for $k = 1, \ldots, K$.

Having an uncertain number of targets at the second stage, we have to take into account the risk of munitions depletion, and, consequently, failure to destroy all the detected targets. One way to hedge against shortage of munitions is to perform a worst-case analysis, e.g., to require that

$$\sum_{k=1}^{K} \left(\sum_{i=1}^{n_k} x_k + I_{\max} y_k^s \right) \leq M, \ \ s = 1, \ldots, S. \tag{6.8}$$

However, constraint of type (6.8) may be too conservative and restricting, especially when $I_{\max}$ is a large number and the probability $\mathsf{P}[\xi_k^s = I_{\max}]$ is relatively small. Indeed, the event of encountering the largest possible number of targets in every category at the second stage should have very low probability.

Replacing $I_{\max}$ in (6.8) with the expected number of the second-stage targets $\mathsf{E}[\xi_k^s]$ may be also inappropriate, especially for distributions Θ_k^s with "heavy tails".

To circumvent the possibility of running out of ammo at the second stage, we propose to use a munitions constraint where the average munitions utilization in, say, 10% of "worst cases" (i.e., when too many second-stage targets are detected) does not exceed the munitions limit M. This type

of constraint can be formulated using the Conditional Value-at-Risk operator:

$$\mathsf{CVaR}_\alpha\left[\sum_{k=1}^{K}\left(\sum_{i=1}^{n_k} x_k + \xi_k^s y_k^s\right)\right] \le M, \quad s = 1,\ldots,S, \tag{6.9}$$

where α is the confidence level. Inequality (6.9) constrains the (weighted) average of munitions utilization in $(1-\alpha)\cdot 100\%$ of worst cases.

To calculate the Conditional Value-at-Risk of the function

$$f(\boldsymbol{\xi}^s, \mathbf{y}^s) = \sum_{k=1}^{K}\left(\sum_{i=1}^{n_k} x_k + \xi_k^s y_k^s\right),$$

we introduce the following scenario model for the number of the second-stage targets in all categories:

$$\left\{(\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj}) \in \{0,1,\ldots,I_{\max}\}^K \mid j = 1,\ldots,J\right\}, \quad J = (1+I_{\max})^K,$$

where the collection of vectors $\{(\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj}) \mid j = 1,\ldots,J\}$ spans all possible combinations of the number of second-stage targets in categories $1,\ldots,K$. Without loss of generality[1] , the probability of encountering $\tilde{\xi}_{1j}$ targets in the first category, $\tilde{\xi}_{2j}$ targets in the second category etc., under scenario s equals to

$$P[(\xi_1^s, \xi_2^s, \ldots, \xi_K^s) = (\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj})] = \prod_{k=1}^{K} P[\xi_k^s = \tilde{\xi}_{kj}] = \prod_{k=1}^{K} \theta^s_{k,\tilde{\xi}_{kj}} = \pi_j^s. \tag{6.10}$$

Note that scenario set $\{(\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj}) \mid j = 1,\ldots,J\}$ is the same for all scenarios $1,\ldots,S$. A scenario s assigns probability π_j^s to each vector $(\tilde{\xi}_{1j}, \tilde{\xi}_{2j}, \ldots, \tilde{\xi}_{Kj})$ from this collection. Naturally,

$$\sum_{j=1}^{J} \pi_j^s = 1, \quad s = 1,\ldots,S.$$

[1] The necessary condition of stochastic independence requires variables $\xi_1^s,\ldots,\xi_K^s$ to be mutually uncorrelated, which imposes a limitation on using the multiplicative rule in (6.10). Expression (6.10) for probabilities π_j^s may still be used if we assume that scenario s defines a joint probability distribution for number of targets over all categories.

Thus, the two-stage stochastic WTA problem with uncertainties in distributions reads as

$$\begin{aligned} \min \quad & \left\{ \sum_{k=1}^{K} \sum_{i=1}^{n_k} V_k (q_k)^{x_{ki}} + \mathsf{E}_{\Theta} \left[\mathsf{E}_{\xi} \left[Q(\mathbf{x}, \boldsymbol{\xi}) \right] \right] \right\} \\ \text{s. t.} \quad & \sum_{k=1}^{K} \sum_{i=1}^{n_k} x_k \leq M, \\ & x_{ki} \in \mathbb{Z}_+, \quad k = 1, \ldots, K, \quad i = 1, \ldots, n_k, \end{aligned} \tag{6.11}$$

where the recourse function Q equals to

$$\begin{aligned} Q(\mathbf{x}, \boldsymbol{\xi}^s) \; = \; \min_{y} \quad & \sum_{k=1}^{K} V_k (q_k)^{y_k^s} \\ \text{s. t.} \quad & \mathsf{CVaR}_{\alpha} \left[\sum_{k=1}^{K} \sum_{i=1}^{n_k} x_k + \sum_{k=1}^{K} \xi_k^s y_k^s \right] \leq M, \\ & y_k^s \in \mathbb{Z}_+, \quad k = 1, \ldots, K, \quad s = 1, \ldots, S. \end{aligned} \tag{6.12}$$

The linearized extensive form of the recourse problem (6.11)–(6.12) is as follows:

$$\min \quad \left\{ \sum_{k=1}^{K} (n_k)^{-1} \sum_{i=1}^{n_k} V_k z_{ki} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} V_k u_k^s \right\} \tag{6.13a}$$

$$\text{s. t.} \quad z_{ki} \geq (q_k)^m \left[(1 - q_k)(m - x_{ki}) + 1 \right], \quad k = 1, \ldots, K, \quad , i = 1, \ldots, n_k, \quad m = 0, \ldots, M-1, \tag{6.13b}$$

$$\sum_{k=1}^{K} \sum_{i=1}^{n_k} n_k x_{ki} = M_1, \tag{6.13c}$$

$$u_k^s \geq (q_k)^m \left[(1 - q_k)(m - y_k^s) + 1 \right], \quad k = 1, \ldots, K, \quad m = 0, \ldots, M-1, \quad s = 1, \ldots, S, \tag{6.13d}$$

$$w_j^s \geq M_1 + \sum_{k=1}^{K} \tilde{\xi}_{kj} y_k^s - \zeta^s, \quad s = 1, \ldots, S, \quad j = 1, \ldots, J, \tag{6.13e}$$

$$\zeta^s + (1 - \alpha)^{-1} \sum_{j=1}^{J} \pi_j^s w_j^s \leq M, \tag{6.13f}$$

$$M_1 \leq M, \tag{6.13g}$$

$$x_{ki}, y_k^s, z_{ki}, u_k^s, w_j^s \geq 0; \quad \zeta^s \in \mathbb{R}, \quad M_1 \in \mathbb{Z}_+. \tag{6.13h}$$

In accordance to the general principle of diversification of risk, the CVaR constraint (6.9), represented by inequalities (6.13e)–(6.13f), does not allow for an integer-valued optimal solution for the

second-stage decision variables, which in turn precludes integrality of the first-stage decision. However, it is possible to achieve an integer-valued solution at the first stage by introducing an integer variable M_1 in constraints (6.13c), (6.13e), or, equivalently, solving $M+1$ problems (6.13a), where $M_1 \in \{0,\dots,M\}$ is a parameter.

If the primary interest is the first-stage weapon-target assignment, and the integrality of the second-stage decision is not critical, the linear programming formulation (6.13a) yields a fast algorithm for the two-stage stochastic WTA problems with uncertainties. The corresponding optimal solution may be regarded as the one that allows for destruction of the detected targets while preserving sufficient resources for destroying the possible future targets.

Integer-valued optimal solution of (6.11)–(6.12) is achieved by declaring the variables y_k^s as integer. In this case, solution of the LP problem (6.13a) represents a lower bound for the optimal solution of the MIP problem (6.11)–(6.12).

6.5 Case Study

In this section we test the developed algorithms for the stochastic WTA problem, and compare solutions of the classic two-stage stochastic WTA problem (6.6) and the two-stage stochastic WTA problem with uncertainties in distributions (6.13a). However, reporting and discussing the optimal solutions of the considered problems would be a rather unwieldy task even in small instances, since formulations (6.6) and (6.13a) contain individual decision variables for each first-stage and, in case (6.6), second-stage target. Therefore, to make the presentation compact, we consider and report solutions of the following MIP analogs of problems (6.6) and (6.13a) correspondingly:

$$\min \quad \left\{ \sum_{k=1}^{K} V_k(q_k)^{x_k} + \frac{1}{S}\sum_{s=1}^{S}\sum_{k=1}^{K} V_k(q_k)^{y_k^s} \right\} \tag{6.14a}$$

$$\text{s.t.} \quad \sum_{k=1}^{K} n_k x_k \le M, \tag{6.14b}$$

$$\sum_{k=1}^{K} (n_k x_k + \xi_k^s y_k^s) = M, \quad s = 1,\dots,S, \tag{6.14c}$$

$$x_k, y_k^s \in \mathbb{Z}_+, \tag{6.14d}$$

and

$$\min \quad \left\{ \sum_{k=1}^{K} V_k (q_k)^{x_k} + \frac{1}{S} \sum_{s=1}^{S} \sum_{k=1}^{K} V_k (q_k)^{y_k^s} \right\} \tag{6.15a}$$

$$\text{s. t.} \quad \sum_{k=1}^{K} n_k x_k \leq M, \tag{6.15b}$$

$$w_j^s \geq \sum_{k=1}^{K} (n_k x_k + \tilde{\xi}_{kj} y_k^s) - \zeta^s, \quad s = 1, \ldots, S, \quad j = 1, \ldots, J, \tag{6.15c}$$

$$\zeta^s + (1-\alpha)^{-1} \sum_{j=1}^{J} \pi_j^s w_j^s \leq M, \tag{6.15d}$$

$$x_k, y_k^s \in \mathbb{Z}_+; \quad w_j^s \geq 0; \quad \zeta^s \in \mathbb{R}. \tag{6.15e}$$

In the formulation (6.14) and (6.15) variables x_k and y_k^s represent the number of weapons used for every first-stage and second-stage target in a category k, respectively. In fact, the linear programming formulations (6.6) and (6.13) are the linearizations of the MIP formulations (6.14) and (6.15) correspondingly. Solution of the LP problem (6.13) was used as the lower bound for the MIP problem (6.15).

The developed models for the stochastic Weapon-Target Assignment problems were implemented in C++ and we used CPLEX 7.0 solver to find solutions to the corresponding MIP and LP problems. The setup of the test problem was as follows:

- There are two categories of targets ($K = 2$);
- Number of weapons $M = 25$;
- Targets in the second category are more important than those in the first category: $V_1 = 1$, $V_2 = 3$;
- Also, the second-category targets are harder to kill: $q_1 = 0.20$, $q_2 = 0.36$;
- The numbers of detected targets in each category are $n_1 = 3, \quad n_2 = 1$,
- There are 3 scenarios for distribution of second-stage targets in each category ($S = 3$), and all the distributions have support $\{0, 1, \ldots, 5\}$ ($I_{\max} = 5$); at each scenario, the expected number of the second-stage targets in categories I and II is given in Table 6–1.
- The expected values presented in Table 6–1 were used as the scenario information for the classical two-stage stochastic WTA problem (6.14) and (6.6).

Table 6–1: The expected values for the number of the second-stage targets in two categories for scenarios $s = 1, 2, 3$.

	$s = 1$	$s = 2$	$s = 3$
Category I	2	2	3
Category II	1	3	4

Table 6–2 compares the optimal solutions of two-stage stochastic WTA problem in different formulations. The values in columns corresponding to $\alpha = 0.01, 0.50, 0.99$ present the optimal solution of the stochastic WTA problem (6.15) with the indicated confidence level α in the CVaR constraint. Column with $\alpha = 0.00$ displays the optimal solution of the classical two-stage SWTA problem (6.14), where the scenario values for the number the second-stage targets were taken from Table 6–1.[2] The last column, $\alpha = 1.00$, displays the solution of problem (6.15), where the CVaR constraint is replaced by the "worst-case" constraint (6.5a).

Table 6–2: Solution of the MIP problem and problem (6.15a) for different values of confidence level α. The case $\alpha = 1.00$ corresponds to the case when CVaR constraints in (6.15a) is replaced with (6.8). Column $\alpha = 0.00$ presents the solution of problem (6.6).

		$\alpha = 0.00$		$\alpha = 0.01$		$\alpha = 0.50$		$\alpha = 0.99$		$\alpha = 1.00$	
Category		I	II	I	II	I	II	I	II	I	II
First stage		2	4	2	4	2	6	2	4	2	4
	$s = 1$	4	7	4	6	2	4	1	2	1	2
Second stage	$s = 2$	3	3	2	3	2	2	1	2	1	2
	$s = 3$	1	3	2	2	1	2	1	2	1	2

Table 6–2 shows that the classic two-stage stochastic WTA problem that uses only the expected values of the second-stage target distributions yields the solution, which is quantitatively very similar to the solution of the two-stage stochastic WTA problem with uncertainties in distributions (6.15) for low confidence level α in CVaR constraint. These solutions allocate ample munitions for destruction of each of the encountered second-stage targets, which may lead to several munitions shortage at worst-case scenarios. As confidence level α increases, and, correspondingly, worst-case scenarios gain more weight in the optimization problem (6.15), the number of weapons to be reserved for each of the second-stage targets decreases (such solutions are more robust with

[2] When α approaches zero, Conditional Value-at-Risk of a stochastic value becomes equal to its expectation.

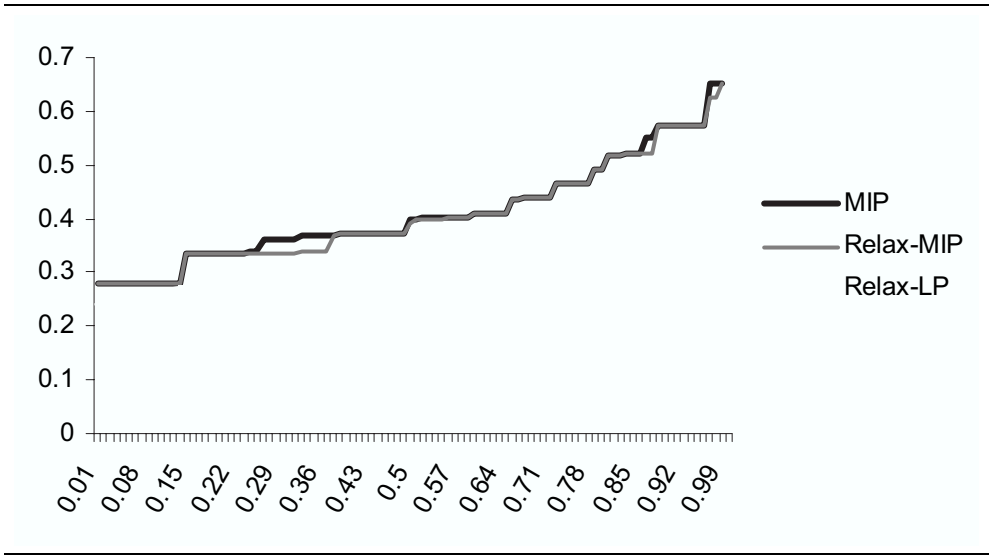


Figure 6–1: MIP curve represent the optimal objective values of the MIP problem (6.15a); Relax-LP curve correspond to the solution of (6.13) with all continuous variables; Relax-MIP points show the solution of the problem (6.13) where the second-stage decision variables y_k^s are integer.

respect to encountering "more-than-expected" second-stage targets). Finally, at high confidence levels ($\alpha = 0.99$), problem (6.15) produces very conservative solution that coincides with the solution obtained by replacing the CVaR constraint by the "worst-case" constraint (6.8).

Figure 6–1 displays the degradation of the optimal objective value of the stochastic WTA problem with increasing of the confidence level α. On this figure, the MIP curve corresponds to the optimal objective value of problem (6.15), Relax-LP curve shows the optimal objective of the LP problem (6.13), and Relax-MIP curve displays the objective of the problem (6.13) with integer variables y_k^s. Evidently, solution of the LP problem (6.13) presents a good lower bound for both the solutions of problem (6.15) and MIP problem (6.14).

6.6 Conclusions

We have considered several formulations for the two-stage stochastic Weapon-Target Assignment problem, where the cumulative damage to the targets is maximized. In the original formulation, the WTA problem is presented as an integer programming problem with a nonlinear objective. We have developed a linear relaxation for both the deterministic and traditional two-stage stochastic formulations of the WTA problem. In the scope of the two-stage stochastic WTA problem we considered the setup where probability distributions for the number of the second-stage targets are

unknown. To control the risks of generating an incorrect decision due to the uncertainties in distributions, we with the uncertainties in distributions, we applied the risk-management techniques based on the Conditional Value-at-Risk (CVaR) risk measure. For the two-stage SWTA problem with uncertainties in distributions we developed an LP formulation that yields tight lower bound for the corresponding MIP problem, and has integer optimal solution for the first-stage variables.

CHAPTER 7
CONCLUSIONS

In our dissertation, we studied risk management techniques in context of stochastic programming with applications in different fields of decision making. It was demonstrated that risk management tools and procedures developed in last several years have general nature and can be successfully applied not only to problems of financial engineering, but also in military decision making problems, which involve a stochastic environment completely different than that of financial markets. We also have developed a new sample-path approach to multi-stage decision-making problems, and tested it with the optimal position liquidation problem.

In Chapter 2, we extended the approach for portfolio optimization (Rockafellar and Uryasev, 2000), which simultaneously calculates VaR and optimizes CVaR. We first showed (Theorem 2.3.1) that for risk-return optimization problems with convex constraints, one can use different optimization formulations. This is true in particular for the considered CVaR optimization problem. We then showed (Theorems 2.4.1 and 2.4.2) that the approach by Rockafellar and Uryasev (2000) can be extended to the reformulated problems with CVaR constraints and the weighted return-CVaR performance function. The optimization with multiple CVaR constrains for different time frames and at different confidence levels allows for shaping distributions according to the decision maker's preferences. We developed a model for optimizing portfolio returns with CVaR constraints using historical scenarios and conducted a case study on optimizing portfolio of S&P100 stocks. The case study showed that the optimization algorithm, which is based on linear programming techniques, is very stable and efficient, and the approach can handle large number of instruments and scenarios.

In Chapter 3, we tested the performance of a portfolio rebalancing algorithm with different types of risk constraints in an application for managing a portfolio of hedge funds. As the risk measure in the portfolio optimization problem, we used Conditional Value-at-Risk, Conditional

Drawdown-at-Risk, Mean-Absolute Deviation, and Maximum Loss. We combined these risk constraints with the market-neutrality (zero-beta) constraint making the optimal portfolio uncorrelated with the market.

The numerical experiments consisted of in-sample and out-of-sample testing. The out-of-sample part of experiments was performed in two setups, which differed in constructing the scenario set for the optimization algorithm.

The results obtained are dataset-specific and we cannot make direct recommendations on portfolio allocations based on these results. However, we learned several lessons from this case study. Imposing risk constraints may significantly degrade in-sample expected returns while improving risk characteristics of the portfolio. In-sample experiments showed that for tight risk tolerance levels, all risk constraints produce relatively similar portfolio configurations. Imposing risk constraints may improve the out-of-sample performance of the portfolio-rebalancing algorithms in the sense of risk-return tradeoff. Especially promising results can be obtained by combining several types of risk constraints. In particular, we combined the market-neutrality (zero-beta) constraint with CVaR or CDaR constraints. We found that tightening of risk constraints greatly improves portfolio dynamic performance in out-of-sample tests, increasing the overall portfolio return and decreasing both losses and drawdowns. In addition, imposing the market-neutrality constraint adds to the stability of portfolio's return, and reduces portfolio drawdowns. Both CDaR and CVaR risk measures demonstrated a solid performance in out-of-sample tests.

Chapter 4 presented an approach for optimal transaction execution based on mathematical programming and sample-path scenario model. The main distinguishing features of our method compared to other approaches are as follows: (1) The trading strategy is truly *dynamic*, i.e., it allows for adequate response to current market conditions at each time step; (2) The approach admits incorporation of various types of constraints in the trading strategy, such as legal, institutional constraints, or those expressing investor's preferences. In particular, we considered risk-averse trading strategies, which control risk of financial losses during transactions using the Conditional Value-at-Risk measure; (3) The suggested approach can accommodate different models of temporary and permanent market impact, transaction costs, etc.; (4) A key feature of our approach is the sample-path scenario model, which does not impose any restrictions on the price process of the security. Historical data or simulation can be used to create a scenario set for our method.

To avoid anticipativity of the solutions, caused by specific properties of the sample-path scenario model, we introduced path partitioning with a new "lawn-mower" decision rule. Since the "lawn-mower" principle leads to non-convexity of the optimal closing problems, we developed a convex relaxation for these problems, which was further approximated by linear programming. The model of optimal position liquidation, implemented as a linear programming problem, can be solved very efficiently and robustly in very large-scale instances with large number of scenarios (sample-paths). Properties of the optimal trading strategies based on the "lawn-mower" principle have been investigated.

In a frictionless market, the optimal trading strategy based on the "lawn-mower" constraint, has 0–1 structure, i.e., it liquidates the entire position in one transaction. If the risk of financial losses during trades is controlled by CVaR constraints, the trading strategy becomes fractional. When "strong" temporary or permanent market impacts are present, the optimal trading strategy approaches the so-called "naive" strategy, which consists in selling equal portions of the position at each time step. Under "weaker" market impacts, the optimal trading strategy starts to deviate from the "naive" strategy. According to our numerical experiments, the most significant deviations from the "naive" trading strategy are observed in the case when temporary market impact depends on the current market conditions, or when permanent market impact has a "lag" (i.e., when small enough trades do not cause permanent changes in price of the security).

In the second part of the dissertation, we developed an approach for decision making for military applications with uncertainties in distributions in stochastic parameters. This approach is based on the methodology of risk management with Conditional Value-at-Risk risk measure developed by Rockafellar and Uryasev (2002). Although the presented approach has been used to solve one-stage and two-stage stochastic Weapon-Target Assignment problems, it is quite general and can be applied to wide class of problems with risks and uncertainties in distributions. In Chapters 5 and 6 we have considered several formulations for the two-stage stochastic Weapon-Target Assignment problem, where the cumulative damage to the targets is maximized. In the original formulation, the WTA problem is presented as an integer programming problem with a nonlinear objective. We have developed a linear relaxation for both the deterministic and traditional two-stage stochastic formulations of the WTA problem. In the scope of the two-stage stochastic WTA problem we considered the setup where probability distributions for the number of the second-stage targets are unknown. To

control the risks of generating an incorrect decision due to the uncertainties in distributions, we with the uncertainties in distributions, we applied the risk-management techniques based on the Conditional Value-at-Risk (CVaR) risk measure. For the two-stage SWTA problem with uncertainties in distributions we developed an LP formulation that yields tight lower bound for the corresponding MIP problem, and has integer optimal solution for the first-stage variables.

In the Appendix, we presented the results of our ongoing research efforts in the area of pricing of path-dependent derivative securities using techniques of mathematical programming.

APPENDIX A
ON THE PRICING OF AMERICAN-STYLE DERIVATIVE SECURITIES USING LINEAR PROGRAMMING

A.1 Black-Scholes Approach to Pricing of Options

In this chapter we present techniques of mathematical programming that can be used for pricing of American-style derivative securities.

Types of financial options. A *European call option* is a contract between the purchaser (*holder*) of option and the seller (*writer*) of option that gives the holder the *right*, but not an obligation, to purchase from writer a prescribed asset, known as the *underlying asset*, at a specified date T in the future (*expiry date* or *expiration date*) for a specified price X (*exercise price* or *strike price*).

At the same time, the writer is *obligated* to sell the asset to the holder of option for the price X if the holder chooses to buy the underlying asset (i. e., *exercise* the contract.) In this case, money change hands at the expiration date T.

The contract which gives the holder the right to *sell* the underlying asset to writer at specified date T for predetermined price X, is called *European put option*.

Generally, derivative contracts that allow exercise *only* at some prescribed date T in the future, are said to be European-style derivative securities. An *American-style* derivative security, on the contrary, allows the holder of contract to exercise it *on or prior to* the expiration date T.

For example, an *American call option* gives its holder a right, but not an obligation, to buy the underlying asset at the specified price X at any time $t \leq T$. The counter-party (the *writer*) is *obligated* to sell the asset whenever the option holder chooses to exercise the contract. Analogously, an *American put* option gives the holder the right to sell the underlying asset at strike price T not only on the expiry date, but also before that date.

The described above European and American options are often called plain *vanilla* options, as opposed to *exotic* options: Asian options, Russian options, barrier options, digital options etc. For more detailed discussion of different types of options traded in the market, see, for example, Cox and Rubinstein (1985), Wilmott et al. (1998), and others.

Black-Scholes analysis and pricing of European options. A breakthrough in options pricing theory has been made by Black and Scholes (1973). In their seminal paper the authors have suggested a very general framework that was proven to be effective in valuating virtually any type of option contract.

Here, we present a sketch Black-Scholes theory in its part that pertains to deriving value of American-style options. The price S of the underlying asset is assumed to satisfy a stochastic differential equation

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW, \tag{A.1}$$

where μ is the expected value of the rate of the instrument, σ is the standard deviation of the rate of return, and dW is a Wiener process with mean zero and variance dt :

$$dW \sim N(0, \sqrt{t}).$$

These assumptions do not alter the generality of the approach; in fact, it has been demonstrated that the framework presented in (Black and Scholes, 1973) is applicable when, for example, the parameters in (A.1) are time-dependent, or the price process $S(t)$ follows other stochastic models, e. g. the constant elasticity of variance (CEV) model, etc. (for details, see, for instance, Wilmott et al. (1998)).

In Black and Scholes (1973) it is shown that in a perfect market economy, in absence of arbitrage and transaction costs, an option contract can be replicated by continuously trading in the market a portfolio consisting of the underlying asset and a bond with risk-free rate r. Then, the price of both European and American options have to satisfy the same partial differential equation

$$\frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + r S \frac{\partial P}{\partial S} - r P = 0, \tag{A.2}$$

where we used P for the price of a put option (a call option satisfies the same equation). The most remarkable feature of the equation (A.2) is that the price of an option does not depend on the drift μ. This constitutes a fundamental insight that though investors may not agree on the expected rate of growth of the asset's value, it would not affect the market price of an option derivative contract on that asset.

The difference between different styles and types of options manifests itself in boundary conditions to be satisfied by the solution of (A.2). In particular, for a European put option the boundary conditions to be satisfied by the option price P, are

$$P|_{t=T} = \max\{0, X - S_T\}, \tag{A.3a}$$

$$P|_{S=0} = X\, e^{-r(T-t)}, \tag{A.3b}$$

$$P|_{S\to\infty} = 0. \tag{A.3c}$$

Here X is the strike price and T is the expiry date of the contract. The boundary-value problem (A.2)–(A.3) has an explicit exact solution. Indeed, by change of variables

$$P(S,t) = X\, e^{\frac{1-\gamma}{2}x - \left(\frac{1+\gamma}{2}\right)^2 \tau}\, u(x,\tau), \quad \gamma = \frac{r}{\sigma^2/2}, \quad S = X\, e^x, \quad t = T - \frac{\tau}{\sigma^2/2}, \tag{A.4}$$

the boundary-value problem (A.2)–(A.3) reduces to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{A.5}$$

$$u|_{\tau=0} = \max\left\{0,\ e^{\frac{1}{2}(\gamma-1)x} - e^{\frac{1}{2}(\gamma+1)x}\right\} \tag{A.6}$$

The solution of the above boundary-value problem is as follows

$$P(S,t) = X\, e^{-r(T-t)}\mathcal{N}(d_1) - S\mathcal{N}(d_2), \tag{A.7}$$

where

$$d_1 = \frac{\ln(S/X) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_1 = \frac{\ln(S/X) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}\xi^2}\, d\xi.$$

A.2 Solution of the Free-Boundary Problem for American Put Using Linear Programming

Free-boundary and complementarity problems for American put. The possibility of exercising an American option early makes its valuation more complicated comparing to a European

option. Clearly, since the set of exercise moments of an American option includes that of a European option, the price of an American option must be greater or equal to the price of a European option. More precisely, it can be demonstrated that an early exercise is never optimal for an American *call* option on a *non-dividend paying* stock (see, for example, Cox and Rubinstein (1985)), and therefore the prices of American and European on a stock that does not pay dividends are equal. For American *put* options, however, this is not the case.

Moreover, up to this date an exact explicit solution for American put option on a non-dividend paying stock is unknown. Therefore, more attention is paid to numerical and approximate methods of American put pricing. For example, Geske and Johnson (1984) developed an analytical expression for American put option with discrete exercise dates (the so-called *bermudan* option). In this chapter we present different approaches that use optimization, and, in particular, linear programming to value an American put option on an non-dividend paying stock. First, we present a technique that delivers a numerical solution to the boundary-value problem corresponding to the American put option.

First, we assume the existence of the *optimal exercise region* $\mathscr{S}$ such that if $S \in \mathscr{S}$ for a given t, it is optimal to exercise the option, and, conversely, it is optimal to keep option when $S \notin \mathscr{S}$. Indeed, from the elementary properties of American options it follows (see Cox and Rubinstein (1985)) that there exists an optimal exercise boundary $\tilde{S}(t)$ such that as soon as $S_t \leq \tilde{S}(t)$, then the option must be exercised.

It turns out that the boundary-value problem for American put option belongs to the classs of *free-boundary* problems, where boundary conditions for a partial differential equation are formulated on a boundary, part of which is not known in advance.

We now describe briefly how the problem of American put option can be reduced to a free-boundary problem for the Black-Scholes equation (A.2). For details, see, for example, Wilmott et al. (1998). First, from the absence of arbitrage it follows that the value of American put is always greater than the payoff:

$$P(S,t) \geq \max\{0, X - S_t\}. \tag{A.8}$$

Otherwise, one could make an infinite profit by buying options on the spot and immediately exercising them. In the same way, the arbitrage argument can be used to show that in the region of optimal

exercise $0 \leq S_t < \tilde{S}(t)$ the following conditions should hold:

$$P = X - S, \quad \frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0. \tag{A.9a}$$

In the region where an early exercise is not optimal, $\tilde{S}(t) < S_t < \infty$, one has

$$P > X - S, \quad \frac{\partial P}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0. \tag{A.9b}$$

Finally, on the optimal exercise boundary $S_t = \tilde{S}(t)$ we have that P and its slope (the *delta* of an option) are continuous:

$$P(\tilde{S}(t),t) = \max\{0, X - \tilde{S}(t)\}, \quad \frac{\partial P(\tilde{S}(t),t)}{\partial S} = -1. \tag{A.9c}$$

The optimal exercise boundary $\tilde{S}(t)$ is not known in advance and has to be determined from the solution of the problem.

It is possible to eliminate the explicit presence of the free boundary in the problem for American put by reducing it to a *linear complementarity problem* of the form

$$\mathscr{A}u \cdot u = 0, \quad \mathscr{A}u \geq 0, \quad u \geq 0,$$

where u is the unknown and $\mathscr{A}$ is some linear operator. Intuitively, the linear complementarity problem for American option can be seen in conditions (A.9a)–(A.9b). A formal reduction of the free-boundary problem (A.9) to a linear complementarity problem requires the use of theory of variational inequalities and is beyond the scope of this work. Here we present the linear complementarity problem for American put as a final result of such transformation. Using the change of variables (A.4) we write the free-boundary problem (A.9) as

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \text{for} \quad x > \tilde{x}(\tau), \tag{A.10a}$$

$$u(x,\tau) = h(x,\tau), \quad \text{for} \quad x \leq \tilde{x}(\tau), \tag{A.10b}$$

where $\tilde{x}(\tau)$ is the optimal exercise boundary in variables x, τ. The initial condition for function u, which corresponds to the exercise condition at the expiration date, is

$$u|_{\tau=0} = h(x,0) = \max\left\{0,\ e^{\frac{\gamma-1}{2}x} - e^{\frac{\gamma+1}{2}x}\right\}, \tag{A.10c}$$

where

$$h(x,\tau) = e^{\left(\frac{\gamma+1}{2}\right)^2 \tau} \max\left\{0,\, e^{\frac{\gamma-1}{2}x} - e^{\frac{\gamma+1}{2}x}\right\}. \tag{A.10d}$$

The boundary conditions at infinity for function u read as

$$u|_{x\to-\infty} = h(x,\tau), \quad u|_{x\to\infty} = 0. \tag{A.10e}$$

Also, the condition (A.8) reduces to

$$u(x,\tau) \geq h(x,\tau). \tag{A.10f}$$

Finally, the problem for American put option can be reduced to linear complementarity problem of the following form:

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \cdot \Big(u(x,\tau) - h(x,\tau)\Big) = 0, \tag{A.11a}$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geq 0, \quad \Big(u(x,\tau) - h(x,\tau)\Big) \geq 0, \tag{A.11b}$$

subject to initial and boundary conditions

$$u|_{\tau=0} = h(x,0), \tag{A.11c}$$

$$u|_{x\to\infty} = 0, \quad u|_{x\to-\infty} = h(x,\tau). \tag{A.11d}$$

Solving the linear complementarity problem for American put using linear programming. Here we suggest a numerical techniques based on linear programming that solves the linear complementarity problem (A.11). First, we reduce the range of change of variable x from $-\infty < x < \infty$ to $x_{\min} \leq x \leq x_{\max}$, so the boundary conditions become

$$u|_{x=x_{\max}} = 0, \quad u|_{x=x_{\min}} = h(x_{\min},\tau). \tag{A.12}$$

Then, we discretize the differential operator

$$\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2}$$

by forward and backward finite differences:

$$\frac{u_x^\tau - u_x^{\tau-1}}{\Delta\tau} - \frac{u_{x+1}^{\tau-1} - 2u_x^{\tau-1} + u_{x-1}^{\tau-1}}{(\Delta x)^2} + O(\Delta\tau) + O((\Delta x)^2),$$
$$\frac{u_x^\tau - u_x^{\tau-1}}{\Delta\tau} - \frac{u_{x+1}^{\tau} - 2u_x^{\tau} + u_{x-1}^{\tau}}{(\Delta x)^2} + O(\Delta\tau) + O((\Delta x)^2).$$

Taking a convex combination of the above expressions, we obtain

$$u_x^\tau - u_x^{\tau-1} - \lambda\,\delta\left(u_{x+1}^\tau - 2u_x^\tau + u_{x-1}^\tau\right) - (1-\lambda)\delta\left(u_{x+1}^{\tau-1} - 2u_x^{\tau-1} + u_{x-1}^{\tau-1}\right) + O(\Delta\tau) + O((\Delta x)^2), \tag{A.13}$$

where $0 \le \lambda \le 1$ and

$$\delta = \frac{\Delta\tau}{(\Delta x)^2}.$$

Taking $\lambda = \frac{1}{2}$, we represent the first inequality of (A.11b) by the system of linear algebraic inequalities

$$-\frac{\delta}{2}\,u_{n+1}^k + (1+\delta)u_n^k - \frac{\delta}{2}\,u_{n-1}^k \ge \frac{\delta}{2}\,u_{n+1}^{k-1} + (1-\delta)u_n^{k-1} + \frac{\delta}{2}\,u_{n-1}^{k-1},$$
$$k = 1,\dots,K, \quad n = -N,\dots,N, \tag{A.14}$$

where $u_n^k = u(x_n, \tau_k)$. In this way the linear complementarity problem (A.11) reduces to consecutive solving of

$$\left(M\mathbf{u}^k - \mathbf{p}^k\right)\cdot\left(\mathbf{u}^k - \mathbf{h}^k\right) = 0, \tag{A.15a}$$

$$M\mathbf{u}^k - \mathbf{p}^k \ge 0, \quad \mathbf{u}^k - \mathbf{h}^k \ge 0, \tag{A.15b}$$

where $\mathbf{u}^k = \left(u_{-N}^k, u_{-N+1}^k, \dots, u_{N-1}^k, u_N^k\right)^T$, and

$$M = \begin{pmatrix} 1+\delta & -\frac{\delta}{2} & 0 & \dots & 0 \\ -\frac{\delta}{2} & 1+\delta & -\frac{\delta}{2} & \dots & 0 \\ 0 & -\frac{\delta}{2} & 1+\delta & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & \dots & -\frac{\delta}{2} & 1+\delta \end{pmatrix}$$

vector $\mathbf{p}^k$ has the components

$$
\begin{aligned}
p^k_{-N} &= \frac{\delta}{2}\left(h^k_{-N-1} + h^{k-1}_{-N-1}\right) + \frac{\delta}{2}\,u^{k-1}_{-N+1} + (1-\delta)\,u^{k-1}_{-N} \\
p^k_n &= \frac{\delta}{2}u^{k-1}_{n+1} + (1-\delta)\,u^{k-1}_n + \frac{\delta}{2}\,u^{k-1}_{n-1}, \quad n = -N+1,\ldots,N-1, \\
p^k_N &= (1-\delta)\,u^{k-1}_N + \frac{\delta}{2}\,u^{k-1}_{N-1},
\end{aligned}
$$

and

$$
\mathbf{h}^k = \left(h^k_{-N}, h^k_{-N+1}, \ldots, h^k_{N-1}, h^k_N\right)^T.
$$

The problem (A.15) can be reduced to the canonical form of linear complementarity problems:

$$
\begin{aligned}
(M\mathbf{w} + \mathbf{q}) \cdot \mathbf{w} &= 0, \\
M\mathbf{w} + \mathbf{q} \geq 0, \quad \mathbf{w} &\geq 0,
\end{aligned}
\tag{A.16}
$$

where

$$
\mathbf{w} = \mathbf{u}^k - \mathbf{h}^k, \quad \mathbf{q} = M\mathbf{h}^k - \mathbf{p}^k.
$$

To solve the linear complementarity problem (A.2), we use the approach developed by Mangasarian (1976, 1978). He has shown that if the matrix M satisfies certain conditions, the solution to a linear complementarity problem of the form (A.2) can be obtained by solving the linear program

$$
\begin{aligned}
\min \quad & \mathbf{c}^T\mathbf{w} \\
\text{s.t.} \quad & M\mathbf{w} + \mathbf{q} \geq 0, \\
& \mathbf{w} \geq 0.
\end{aligned}
\tag{A.17}
$$

It is easy to see that matrix M in (A.15) and (A.2) satisfies to condition

$$
M_{ii} > \sum_{j \neq i} |M_{ij}|, \quad \forall\, i,
$$

then, according to (Mangasarian, 1978), the vector $\mathbf{c}$ can be selected as $\mathbf{c} = M^T\mathbf{e}$, where $\mathbf{e}$ is a vector of 1's. Thus, we have reduced the problem of pricing American put option to consecutive solving

of series of LPs

$$\begin{aligned} \min \quad & (M^T\mathbf{e})^T\mathbf{w} \\ \text{s.t.} \quad & M\mathbf{w}+\mathbf{q} \geq 0, \\ & \mathbf{w} \geq 0. \end{aligned} \tag{A.18}$$

We have to emphasize here that the developed technique of pricing American put allows one to obtain in one shot an entire range of option values for different initial prices S_0, given the strike price X, expiration date T, standard deviation σ and risk-free rate r.

Numerical results. To test the developed algorithm for American option pricing, we performed a case study whose results are reported in Table A–1. The following parameters were used: $X = 10.0$, $r = 0.10$, $\sigma = 0.4$, $T = 0.5$ (i.e., an American put with strike price of \$ 10 and expiration in six months). To calculate the price of the put option for different values of the initial stock price S_0 using the procedure described above, we performed discretization with $N = 1000$, $K = 10$, and $x_{\min} = -4$ and $x_{\max} = 0.4771$, which corresponds to $S_{\min} = 0.001$, $S_{\max} = 30.00$.

Table A–1: Pricing of American put using linear programming

S_0	P	P_{exact}	error, %
16	0.0458	0.0460	-0.60
14	0.1316	0.1320	-0.27
12	0.3620	0.3622	-0.06
10	0.9045	0.9211	-1.84
8	2.0945	2.0951	-0.03
6	4	4	0
4	6	6	0
2	8	8	0

The results of the calculations are reported in Table A–1. The column S_0 contains the values of the initial stock price at $t = 0$; values in P column represent the put prices obtained using the described procedure; P_{exact} is the exact price of American put with discrete exercise dates obtained by the technique of Geske and Johnson (1984).

The table demonstrates that the presented procedure of American put valuation using LP allows to obtain very quickly a whole range of option prices corresponding to different initial prices of the underlying asset with a good accuracy. The loosened accuracy of our method in the case when the initial stock price S_0 is equal to the exercise price X can be explained as follows. Observe that

transforming the problem (A.15) to (A.2) is, in fact, equivalent to setting

$$w(x,\tau) = u(x,\tau) - h(x,\tau),$$

and the term $M\mathbf{h}^k$ corresponds to

$$-\frac{\partial^2}{\partial x^2}\, h(x,\tau).$$

Observe that function $h(x,\tau)$ is not differentiable in $x = 0$, which corresponds to $S_0/X = 1$. Therefore, the accuracy of the method can be increased if we discretize the following linear complementarity problem

$$\begin{aligned} &\left(\frac{\partial w}{\partial \tau} - \frac{\partial^2 w}{\partial x^2} + g(x,\tau)\right) \cdot w(x,\tau) = 0, \\ &\frac{\partial w}{\partial \tau} - \frac{\partial^2 w}{\partial x^2} + g(x,\tau) \geq 0, \quad w(x,\tau) \geq 0, \end{aligned} \tag{A.19}$$

where

$$g(x,\tau) = \left(\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2}\right) h(x,\tau).$$

Paper by Dempster and Hutton (1999) contains ideas similar to those presented above. However, the presented approach was developed by the author independently of the indicated work.

A.3 Pricing of Path-Dependent Options Using Simulation and Stochastic Programming

The above approach for pricing of American put option, though being very effective in calculating an array of option values with good accuracy, it bears all limitations of the original Black-Scholes approach. Today's financial industry requires methods for pricing of more complicated securities, including multidimensional options, exotic options, options on underlying securities whose price process does not follow the standard geometric Brownian motion model (A.1), etc.

Form this viewpoint, pricing methods based on simulation techniques, are of special importance. One of the first attempts of deriving the price for American put options using simulations belongs to Tilley (1993). Later, simulation approaches to pricing of derivatives were explored in the papers by Barraquand and Martineau (1995), Carriere (1996), Boyle et al. (1997), Broadie and Glasserman (1997), and others.

The approach presented below constitutes a subject of the author's ongoing research efforts. We develop a general technique that uses simulation and stochastic programming to price path-dependent derivative securities. As a validation of the approach, we consider pricing of American put option.

Generally, the approach to be presented is based on the sample-path framework presented in the Chapter 3. Recall that we have mentioned that problem of optimal position liquidation is closely related to pricing of derivative securities. In this section we make some further remarks along this analogy.

Generation of sample paths. The simulation method for pricing of derivative securities rely on the so-called *risk-free* valuation framework, which states that under certain idealizing conditions a price process of the underlying security can be considered as having the drift equal to the risk-free rate r. Thus, we start with the following geometric Brownian motion as the price process for the underlying stock of American put contract:

$$\frac{dS_t}{S_t} = r\,dt + \sigma\sqrt{t}\,dZ, \quad Z \sim N(0,1). \tag{A.20}$$

We use the above stochastic differential equation for discrete sampling of sample path $(\Delta t = 1/T)$:

$$S_t^j = S_0 \exp\left\{\left(r - \frac{\sigma^2}{2}\right) t\,\Delta t + \sigma\sqrt{\Delta t}\sum_{\tau=1}^{t} Z_\tau^j\right\}, \quad Z_t^j \sim N(0,1), \quad t = 1,\ldots,T, \quad j = 1,\ldots,J. \tag{A.21}$$

Obtained in this way sample paths $(S_0, S_1^j, \ldots, S_T^j)$ represent possible realizations of stochastic process S_t governed by (A.20).

Pricing of American put using LP. Instead of path grouping (see Chapter 3), we use the following basic property of American put to prevent the anticipativity of the solutions in the stochastic programming problem: *if, at some time $t \le T$, it is optimal to exercise American put given the underlying price S_t, it is also optimal to exercise it for any $S_t' \le S_t$* (Cox and Rubinstein, 1985). From this follows the *monotonicity* property of the optimal exercise region for American put: $\forall t \le T$ there exists such $S_t^* \le X$ such that it is optimal to exercise the option as soon as $S_t \le S_t^*$, and it is optimal to keep the option open otherwise.

Utilizing this property, we arrive at the following linear programming problem for American put pricing:

$$\begin{aligned}
\max \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\max\{0,\, X - S_t^j\}e^{-rt\Delta t}\, y_t^j && && \text{(A.22)}\\
\text{s. t.} \quad & \sum_{t=1}^{T} y_t^j = 1, && j = 1,\ldots,J,\\
& \sum_{\tau=1}^{t} y_\tau^{\mathrm{j}_t(n)} \le \sum_{\tau=1}^{t} y_\tau^{\mathrm{j}_t(n-1)} && t = 1,\ldots,T-1, \quad n = 2,\ldots,J,\\
& y_t^j \ge 0, && t = 1,\ldots,T, \quad j = 1,\ldots,J.
\end{aligned}$$

Th objective function of (A.22) equals to average payoff generated along the sample paths by exercising "portions" of the option (y_t^j is a "portion" of option to be "exercised" at time t on a sample path j.) The first and third constraints of (A.22) state that the option has to be exercised on or before time T. Though that may seem to conflict with the very definition of the option contract by eliminating the *optionality* in the decisions of the option holder, it does not impact the objective of (A.22), because exercising for $S_t > X$ does not generate cash flow. Therefore, all "exercises" imposed by these constraints, which happen in the region of $S_t > X$, can be ignored.

The second constraint imposes the *monotonicity* structure on the optimal exercise region, i.e.,

$$S_t^{j_1} \ge S_t^{j_2} \quad \Rightarrow \quad 1 - \sum_{\tau=1}^{t} y_t^{j_1} \le 1 - \sum_{\tau=1}^{t} y_t^{j_2}. \tag{A.23}$$

In (A.22), j_t is a permutation $\mathrm{j}_t : \{1,\ldots,J\} \longmapsto \{1,\ldots,J\}$ such that

$$S_t^{\mathrm{j}_t(1)} \le S_t^{\mathrm{j}_t(2)} \le \ldots \le S_t^{\mathrm{j}_t(J)}.$$

Proposition A.3.1 *Problem* (A.22) *has a 0-1 optimal solution.*

Proof is analogous to that of Proposition 6.2.1. ■

The proposition A.3.1 has two-fold consequences. First, it demonstrates that the optimal solution of (A.22) does indeed represent a binary decision ("exercise – not exercise") of the option-holder. Also, it follows from the proposition A.3.1 that a fast polynomial algorithm can be developed for pricing of the American put.

However, the formulation (A.22), though being very simple and performing well in practice, may lead to incorrect results when some "improper" sample paths (i.e., sample paths that do not follow the geometric Brownian motion law (A.20)) are included in the scenario set.

The following formulation of linear programming model does fix this deficiency:

$$
\begin{aligned}
\max \quad & \frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T}\max\{0,\, X - S_t^j\}e^{-rt\Delta t}\, y_t^j && && \text{(A.24)}\\
\text{s. t.} \quad & \sum_{t=1}^{T} y_t^j = 1, && j = 1,\dots,J,\\
& \sum_{\tau=1}^{t} y_\tau^{j_t(n-1)} - \sum_{\tau=1}^{t} y_\tau^{j_t(n)} \geq -M\sum_{\tau=1}^{t-1}\left(y_\tau^{j_t(n-1)} + y_\tau^{j_t(n)}\right) && t = 1,\dots,T-1, \quad n = 2,\dots,J,\\
& y_t^j \geq 0, && t = 1,\dots,T, \quad j = 1,\dots,J.
\end{aligned}
$$

The second constraint of (A.24) ensures that monotonicity of the optimal exercise region holds at time t not for any paths j_1, j_2, but for only those paths, on which the option has not been exercised. The constant M in (A.24) is chose so as to ensure 0-1 structure of the optimal solution of (A.24)

Numerical results. As a case study, we performed pricing of an American put with parameters identical to the those in case study of the previous section: $X = 10$, $r = 0.10$, $\sigma = 0.40$, and time to expiry of 6 months (0.5 years). We generated 5,000 sample paths using (A.21) with discretization $T = 5$ (5 exercise opportunities are allowed before the expiry).

Table A–2: Pricing of American put by simulation and stochastic programming

S_0	P	P_{exact}	error, %
16	0.0497	0.0460	8.12
14	0.1346	0.1320	1.94
12	0.3584	0.3622	-1.06
10	0.9145	0.9211	-0.72
8	2.0207	2.0951	-3.55
6	3.9226	4	-1.93
4	5.9145	6	-1.43
2	7.9073	8	-1.16

The results are reported in Table A–2. Here S_0 is the initial price of the underlying stock, P is the optimal value of the objective function, and P_{exact} is the exact value of the option calculated using the procedure of Geske and Johnson (1984).

The advantages of the suggested method compared to those known in the literature are as follows: (i) the approach allows one to price the derivative securities, whose underlying asset is

governed by laws other than the standard geometric Brownian motion (A.20); (ii) scalability: the size of optimization problem grows linearly with respect to both number of paths and number of possible exercise dates.

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BIOGRAPHICAL SKETCH

I was born in Kiev, Ukraine. Since childhood, I had an interest in natural sciences and mathematics. In 1991, I graduated with a Silver Medal from Kiev high school #179 with an advanced program in mathematics. The same year I was admitted to the Department of Mathematics and Mechanics of Kiev National Taras Shevchenko University, one of the premier schools of the former Soviet Union. During the period of study in Kiev, I acquired a solid mathematical background and have developed a taste for both theoretical and applied research. I have won awards and scholarships from the National Academy of Sciences of Ukraine and International Soros Science & Education Program (ISSEP). In June 1996, I received the M.S. degree in mechanics and applied mathematics with highest honors. The topic of my Masters thesis was "Motion of a Rigid Torus in a Viscous Stokes Fluid."

In November 1996, I was admitted to the Ph.D. program in Mechanics of Solids at Kiev University. My research activities were concentrated on developing analytical solutions of boundary-value problems of elasticity and hydromechanics for bodies of complex geometry. In 1997, I received a Diploma of the National Academy of Sciences of Ukraine in a competition of young scientists of Ukraine for the best research project. In September 1999, I successfully defended my Ph.D. dissertation on "The Second Fundamental Boundary-Value Problem of Elasticity for a Torus."

In 1999 I joined the Department of Industrial and Systems Engineering at the University of Florida to pursue a doctoral degree in financial engineering, where I could apply my mathematical skills. Under the supervision of Prof. Uryasev I have been working on application of optimization techniques to risk management in the areas of financial and military decision-making problems, which constitutes the topic of my dissertation.