A COUNTEREXAMPLE TO THE BOUNDED ORBIT CONJECTURE

By

Stephanie M. Boyles

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF THE
UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1980

To My Mother and Father

ACKNOWLEDGEMENTS

I would like to express my appreciation to several people who have contributed in various ways toward the completion of this dissertation.

To my mentor, Professor Gerhard X. Ritter, I extend my deepest thanks. By suggesting that I solve such an old mathematical problem, his confidence in my ability has helped me both professionally and personally. His friendship, support, and constant willingness to listen and advise will never be forgotten. I am especially indepted to him and Professor Serge Zarantonello for sharing with me the benefit of their experiences when I was selecting a career in mathematics.

To my fellow graduate students in mathematics I offer a toast. Your attitude of comradery and festive natures have made these last four years most enjoyable. I also owe many thanks to the mathematics professors at the University of Florida for their love of mathematics and their interest in the students.

Most of all I wish to thank my parents, Marion and Charles Boyles, for their unending encouragement, faith, and love. Were it not for them and their great respect for education, I would never have reached this goal. I owe them more than I can ever repay and dedicate this work to them.

TABLE OF CONTENTS

Abstract of Dissertation presented to the Graduate Council
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

A COUNTEREXAMPLE TO THE BOUNDED ORBIT CONJECTURE

By

Stephanie M. Boyles

June 1980

Chairman: Gerhard X. Ritter
Major Department: Mathematics

A long outstanding problem in the topology of Euclidean spaces is the Bounded Orbit Conjecture, which states that every homeomorphism of the plane onto itself, with the property that the orbit of every point is bounded, must have a fixed point. It is well known that the conjecture is true for orientation preserving homeomorphisms. We provide a counterexample to the conjecture by constructing a fixed point free orientation reversing homeomorphism which satisfies the hypothesis of the conjecture.

Chapter I contains a summary of definitions and notation used in the text, as well as a historical perspective of the Bounded Orbit Conjecture. In Chapter II, we present the counterexample to the Bounded Orbit Conjecture. This homeomorphism can be outlined as follows. On the complement of the strip $|x| < 1$, h is a reflection across the y-axis. In the strip itself, h is defined on the images of an arc $A = \{(x,y) : |x| < 1 \text{ and } y = 0\}$ and extended in a piecewise fashion to the remainder of the strip. With this construction, every

point on A has a bounded orbit even though the orbit of A itself is unbounded. Consequences of the counterexample in Chapter II are revealed in the first part of Chapter III; in the final section of Chapter III, we furnish two problems in planar fixed point theory which remain unsolved.

CHAPTER I
PRELIMINARIES

The first section of this chapter introduces notation and definitions used in the sequel. For a more detailed approach the reader is refered to (11). In the second section we provide a historical perspective of the Bounded Orbit Conjecture. This summary begins with Brouwer's original Translation Theorem and mentions results obtained by other researchers since that time.

1. Notation and definitions. The letters p, q, v are used to denote points in E^2. Line segments and open arcs are designated by S and A, respectively. The letter t always stands for a real number between 0 and 1; and i, j, k, m, n take only integer values. We let x and y correspond to the x and y coordinates of a point in E^2.

A homeomorphism of the plane is a bijective continuous transformation of 2-dimensional Euclidean space E^2. Homeomorphisms of the plane fall into two categories, orientation preserving and orientation reversing. The intuitive idea is simple. A homeomorphism of the plane preserves orientation if the image of every clockwise oriented simple closed curve is again clockwise oriented; it reverses orientation if the image of every clockwise oriented simple closed curve is counter-clockwise oriented. Note that both a translation and a rotation are orientation preserving, while a reflection is an orientation reversing homeomorphism.

The rigorous definition of this concept is given in terms of homology theory as follows. Let h be a homeomorphism from E^n onto itself, and let h' denote the unique extension of h to $S^n = E^n \cup \{\infty\}$, defined by $h'(\infty) = \infty$ and h' restricted to E^n equals h. The homeomorphism h' is said to be <u>orientation preserving</u> or <u>orientation reversing</u> according as the induced isomorphism $h'_*: H_n(S^n) \to H_n(S^n)$ is or is not the identity. The map h is <u>orientation preserving</u> if h' is orientation preserving; it is <u>orientation reversing</u> if h' is orientation reversing. We note that the composition of an odd (even) number of orientation preserving homeomorphisms is an orientation reversing (preserving) homeomorphism.

The <u>orbit</u> of a point p is defined as the set of all $h^k(p)$, where k ranges over the integers. Note that under a rotation, as well as a reflection, the orbit of every point is bounded. Whereas, if h is a translation, the orbit of every point is unbounded.

A set of n points in E^k is said to be in <u>general position</u> if no m+2 of them lie in an m-plane. Let $v_1, v_2, \ldots, v_n$ be n points in general position in E^k. Denote the closed <u>convex hull</u> of $\{v_1, v_2, \ldots, v_n\}$ (i.e. the smallest convex set containing $\{v_1, v_2, \ldots, v_n\}$) by $\langle v_1, v_2, \ldots, v_n \rangle$. By a <u>linear map</u> we shall mean a map

$$h : \langle v_1, v_2, \ldots, v_n \rangle \to \langle h(v_1), h(v_2), \ldots, h(v_n) \rangle$$

defined by

$$\Sigma_{i=1}^n t_i v_i \to \Sigma_{i=1}^n t_i h(v_i), \quad \text{where } \Sigma_{i=1}^n t_i \leq 1.$$

2. <u>History of the Bounded Orbit Conjecture</u>. In 1912, L. E. J. Brouwer proved his famous translation theorem (9) which states that if h is an orientation preserving homeomorphism of E^2 onto itself having no fixed points, then h is a translation. By a translation, Brouwer meant that for each point p in E^2, $h^n(p) \to \infty$ as $n \to \pm\infty$; that is, the orbit of every point is unbounded. Thus, if h is an orientation preserving homeomorphism of E^2 onto itself such that the orbit of some point is bounded, then h must have a fixed point. Obviously, Brouwer's theorem cannot be true for orientation reversing homeomorphisms. For instance, let h be a reflection across the y-axis followed by the upward shift $f(x,y) = (-x\ ,\ -y-|x|+1)$ in the strip determined by $|x| < 1$ (figure 1). Then every point (x,y) with $|x| \geq 1$ has a bounded orbit, even though h is fixed point free.

The question arose as to whether or not any homeomorphism of E^2 onto itself with the property that the orbit of every point is bounded must have a fixed point. This eventually became known as the bounded orbit problem, and a considerable amount of research relating to this problem has accumulated in the literature (1), (2), (3), (4), (6), (7), (8), (10), (12), (13), (15), (16), (17), and (18). Since the Bounded Orbit Conjecture is true for orientation preserving homeomorphisms (1), any counterexample to the conjecture must be orinetation reversing. However, if a homeomorphism h is orientation reversing with bounded orbits, then h^2 is orientation preserving with bounded orbits and, according to Brouwer's theorem (9), must have a fixed point. Thus, one considers the question: does every homeomorphism, whose square has a fixed point, leave some point fixed itself? In 1974, Gordon Johnson (12) answered this question in the negative by giving an example of a fixed point free homeomorphism of E^2 onto itself with the property that each of its iterates has a fixed point. We note that Johnson's example is not a counterexample to the Bounded Orbit Conjecture since the orbits of points are not bounded.

In a 1975 paper Brechner and Mauldin (7) showed that if there exists a fixed point free homeomorphism h of E^2 onto itself such that the orbits of bounded sets are bounded, then there is a compact continuum M in E^2 which does not separate the plane and which is invariant under h. The next year Bell (3) announced that every homeomorphism of the plane onto itself leaving a non-separating continuum M invariant has a fixed point in M. Thus, any counterexample to the Bounded Orbit Conjecture must be orientation reversing and must contain a compact set whose orbit is unbounded. Initially, the concept of a compact set whose orbit is unbounded, with every point on that compact set having bounded orbit, seems contradictory. However, by modifying an example in (14), Brechner and Mauldin (7) used the standard stereographic projection in order to obtain a homeomorphism of the plane in which the orbit of an arc is unbounded, even though the orbit of every point on that arc is bounded. We note that the example mentioned above is not fixed point free.

The objective of this dissertation is to give a counterexample to the Bounded Orbit Conjecture; that is, to construct a fixed point free orientation reversing homeomorphism of E^2 onto itself with the property that the orbit of every point is bounded. The counterexample constructed in Chapter II has the property that any arc connecting a point p with its image h(p) has unbounded orbit. To see that every counterexample to the Bounded Orbit Conjecture must exhibit this property, let f be a homeomorphism of the plane onto itself, and suppose that A is an arc connecting a point p with its image f(p). If the orbit of A is bounded, then the closure of the orbit of A, together with the bounded components contained in this closure, is a non-separating continuum which is invariant under f. Thus, by Bell's theorem (3), if the orbit of A is bounded, then f must leave some point fixed and cannot be a counter-example to the Bounded Orbit Conjecture.

CHAPTER II
A FIXED POINT FREE HOMEOMORPHISM WITH BOUNDED ORBITS

In this chapter we present the counterexample to the Bounded Orbit Conjecture. The definition of the homeomorphism h, which will serve as the counterexample, is given in section one. Section two explains why the homeomorphism is fixed point free. In section three we prove that the orbit of every point is bounded. We conclude this chapter by showing that h is indeed a homeomorphism.

1. Definition of h. Let B denote the open strip $\{(x,y) : |x| < 1\}$. If p is not in B, define $h(p) = h(x,y) \equiv (-x,y)$, where $\equiv$ indicates definition. Thus, on the complement of B the map is a reflection across the y-axis.

To describe the homeomorphism on B, we first define h on a countable set of points which will be the vertices of convex polygons in B. For all $m \geq 0$ and $k \geq 1$, let

$$v_{\pm m,0} = (\pm m/(m+1) , 0), \text{ and}$$

$$v_{\pm m,k} = (\pm m/(m+1) , \Sigma_{i=1}^{k} 1/(m+i)).$$

For all j and $k \geq 0$, define

$$h(v_{j,k}) \equiv v_{(-1)^{k+1}-j,k+1} .$$

Denote $\langle v_{j-1,k} , v_{j,k} \rangle$ by $S_{j,k}$. Extend h linearly on $S_{j,k}$ by defining

$$h(S_{j,k}) = h(<v_{j-1,k} , v_{j,k}>) \equiv <h(v_{j-1,k}) , h(v_{j,k})>, \text{ for } j, k \geq 0.$$

Let $A = \{(x,y) : |x| < 1 \text{ and } y = 0\}$. Thus, $h^k(A) = \bigcup_{j=-\infty}^{\infty} S_{j,k}$ (figure 2). Note that the positive orbit of A forms an unbounded sequence of arcs contained in the half of B having non-negative y-coordinate, and that the intersection of any two of these arcs is empty. One may view the homeomorphism h on $h^k(A)$, $k \geq 0$, as the following composition: $h = \pi \cdot r \cdot s$, where s is a shift of every segment $S_{j,k}$ either "left" one segment if k is odd (i.e. for all j, $S_{j,k} \to S_{j-1,k}$), or "right" one segment if k is even (i.e. for all j, $S_{j,k} \to S_{j+1,k}$), r is a reflection through the y-axis, and π is an upward projection to $h^{k+1}(A)$.

To describe h on the regions between $h^{k-1}(A)$ and $h^k(A)$, $k > 0$, consider the following two sequences of points on each vertical segment $<v_{j,k-1} , v_{j,k}>$:

$$v_{j,k}^{1/n} \equiv v_{j,k-1} + (v_{j,k} - v_{j,k-1})/n, \text{ where } n \geq 3, \text{ and}$$

$$v_{j,k}^{(n-1)/n} \equiv v_{j,k-1} + (v_{j,k} - v_{j,k-1})(n-1)/n, \text{ where } n \geq 2.$$

Note that $\{v_{j,k}^{1/n}\}_{n\geq 3}$ and $\{v_{j,k}^{(n-1)/n}\}_{n\geq 2}$ are sequences converging to $v_{j,k-1}$ and $v_{j,k}$, respectively. Let

$$S_{j,k}^t = <v_{j-1,k}^t, v_{j,k}^t>, \text{ and}$$

$$A_{k,t} = \bigcup_{j=-\infty}^{\infty} S^{t}_{j,k},$$

where $t = 1/n$ for $n \geq 3$, or $t = (n-1)/n$ for $n \geq 2$. Thus, for every positive integer k $\{A_{k,1/n}\}_{n\geq 3}$ and $\{A_{k,(n-1)/n}\}_{n\geq 2}$ are disjoint sequences of open arcs between $h^{k-1}(A)$ and $h^{k}(A)$ limiting to $h^{k-1}(A)$ and $h^{k}(A)$, respectively. For $k \geq 1$, and for all j, define

$$h(v^{1/n}_{j,k}) \equiv v^{1/(n-2)}_{(-1)^k-j,k+1}\ , \text{ for } n \geq 4,$$

$$h(v^{1/3}_{j,k}) \equiv v^{2/3}_{-j,k+1}\ , \text{ and}$$

$$h(v^{(n-1)/n}_{j,k}) \equiv v^{(n+1)/(n+2)}_{(-1)^{k+1}-j,k+1}\ , \text{ for } n \geq 2.$$

For all j and $k \geq 1$, extend h linearly to a homeomorphism on the segments $S^{(n-1)/n}_{j,k}$, $n \geq 2$, and $S^{1/n}_{j,k}$, $n \geq 3$, by defining

$$h(S^{(n-1)/n}_{j,k}) = h(<v^{(n-1)/n}_{j-1,k}\ ,\ v^{(n-1)/n}_{j,k}>)$$

$$<h(v^{(n-1)/n}_{j-1,k})\ ,\ h(v^{(n-1)/n}_{j,k})>\ , \text{ and}$$

$$h(S^{1/n}_{j,k}) = h(<v^{1/n}_{j-1,k}\ ,\ v^{1/n}_{j,k}>)$$

$$<h(v^{1/n}_{j-1,k})\ ,\ h(v^{1/n}_{j,k})>\ .$$

We define h on the region between $A_{k,(n-1)/n}$ and $A_{k,n/(n+1)}$, where $n \geq 2$, by mapping each vertical segment $<p,q>$ joining $A_{k,(n-1)/n}$ to $A_{k,1/n}$. Refer to figure 3 for k odd.

We map the region between $A_{k,1/4}$ and $A_{k,1/2}$ to the region between $A_{k+1,1/2}$ and $A_{k+1,3/4}$ as follows: for all j, $k \geq 1$, and n = 3,4, we define h as a linear map so that

$$h(\langle v^{1/(n-1)}_{j-1,k}, v^{1/(n-1)}_{j,k}, v^{1/n}_{j+(-1+(-1)^k)/2,k}\rangle)$$

$$\equiv \langle h(v^{1/(n-1)}_{j-1,k}), h(v^{1/(n-1)}_{j,k}), h(v^{1/n}_{j+(-1+(-1)^k)/2,k})\rangle , \text{ and}$$

$$h(\langle v^{1/n}_{j-1,k}, v^{1/n}_{j,k}, v^{1/(n-1)}_{j-(1+(-1)^k)/2,k}\rangle)$$

$$\equiv \langle h(v^{1/n}_{j-1,k}), h(v^{1/n}_{j,k}), h(v^{1/(n-1)}_{j-(1+(-1)^k)/2,k})\rangle .$$

For odd k refer to figure 4. Observe that, for $n \geq 2$, $h(A_{k,(n-1)/n}) = A_{k+1,(n+1)/(n+2)}$. Note also that the image under h of a vertical segment joining $A_{k,1/2}$ to $h^k(A)$ is a vertical segment joining $A_{k+1,3/4}$ to $h^{k+1}(A)$. Similarly, a vertical segment joining $h^{k-1}(A)$ to $A_{k,1/4}$ is mapped under h to a vertical segment joining $h^k(A)$ to $A_{k+1,1/2}$. On the region between $A_{k,1/4}$ and $A_{k,1/2}$, h may also be viewed as the composition of three maps: a shift s, a reflection r through the y-axis, and an upward projection π to the region between $A_{k+1,1/2}$ and $A_{k+1,3/4}$. Note that when k is odd $A_{k,1/4}$ shifts "right" and $A_{k,1/2}$ shifts "left," the reverse occurs when k is even; s is the identity on $A_{k,1/3}$ whether k is odd or even (figure 4).

For a point p in B on or below the reflection of h(A) through the x-axis, define $h(p) = -h^{-1}(-p)$. We extend h linearly on the region between $h^{-1}(A)$ and A by sending every vertical segment <p,q> joining $h^{-1}(A)$ to A to the segment <h(p),h(q)> .

2. <u>h is fixed point free</u>. Clearly no point in the complement of B remains fixed under h. For every k, h is fixed point free on $h^k(A)$ since the intersection of $h^k(A)$ and $h^{k+1}(A)$ is empty. If p is a point between $h^{k-1}(A)$ and $h^k(A)$, then h(p) is between $h^k(A)$ and $h^{k+1}(A)$; thus, $p \neq h(p)$.

3. <u>Every point has bounded orbit</u>. If p is in the complement of B, then the orbit of p is the two point set p and h(p).

Observe that

$$h^n(v_{k,0}) = h^n(\ k/(k+1)\ ,\ 0\ )$$

$$\equiv (\ (-1)^n(n+k)/(n+k+1)\ ,\ \sum_{i=n+k+1}^{2n+k} 1/i\ ),\ \text{for } n,k \geq 1, \text{ and}$$

$$h^n(v_{-k,0}) = h^n(\ -k/(k+1)\ ,\ 0\ )$$

$$\equiv (\ (-1)^n(n-k)/(n-k+1)\ ,\ \sum_{i=n-k+1}^{2n-k} 1/i\ ),\ \text{for } k \geq 1,\ n \geq k.$$

Since $(\sum_{i=1}^{n} 1/i - \log n) \to \gamma$ as $n \to \infty$, where γ is the Euler constant, one can show that, for fixed k,

$$\lim_{n\to\infty} \sum_{i=n+k+1}^{2n+k} 1/i = \lim_{n\to\infty} \sum_{i=n-k+1}^{2n-k} 1/i = \log 2.$$

Hence, for every j, the positive orbit of $v_{j,0}$ is bounded. If p is in $\langle v_{j-1,0}, v_{j,0}\rangle$, then $h^n(p)$ is in $\langle h^n(v_{j-1,0}), h^n(v_{j,0})\rangle$ implies that the positive orbit of every point on A is bounded. Observe that if p is on $h^k(A)$, then $h^{-k}(p)$ is on A; thus, the positive orbit of every point on $h^k(A)$ is bounded for every $k \geq 0$.

To check points between $h^{k-1}(A)$ and $h^k(A)$, for $k \geq 1$, we first consider a point p in the region above $A_{k,1/2}$ and below $h^k(A)$. Let q be the point on $h^k(A)$ which is directly above p (i.e. q is the intersection of $h^k(A)$ with the vertical line containing p). Since q has bounded positive orbit and $h^n(p)$ is directly below $h^n(q)$, for all $n > 0$, the positive orbit of p must also be bounded. Next suppose p is between $A_{k,1/(n+1)}$ and $A_{k,1/n}$, where $n \geq 2$. By construction there is a positive integer m such that $h^m(p)$ is between $A_{k+m,1/2}$ and $h^{k+m}(A)$. Thus, the above argument shows that p has a bounded positive orbit. In fact, since a point between $h^{-k}(A)$ and $h^{-k+1}(A)$, $k > 0$, is between A and h(A) after k applications of h, all points in B have bounded positive orbits.

One can see that the negative orbits of points in B are bounded by recalling that $h^{-1}(p) = -h(-p)$.

4. <u>h is a homeomorphism</u>. It follows from the construction that h is bijective, and that h is continuous at every point (x,y) with $|x| \neq 1$. To show that h is continuous on the lines $x = \pm 1$ we first consider a point (-1,y), where $y \geq 0$. Since h is a reflection on the left side of the line $x = -1$, it suffices to show that if $(x_m, y_m) \to (-1,y)$, then $h(x_m,y_m) \to (1,y)$, where $\{(x_m,y_m)\}$ is contained in B.

Observe that B is partitioned into convex polygons bounded by the lines $x = \pm n/(n+1)$ and the arcs $h^k(A)$, where n and k are integers. Thus, for m sufficiently large (x_m, y_m) is contained in a quadrilateral bounded by the lines $x = -n_m/(n_m+1)$, $x = -(n_m-1)/n_m$, and the arcs $h^{k_m-1}(A)$ and $h^{k_m}(A)$, where $n_m \geq 4$. Hence,

$$-n_m/(n_m+1) \leq x_m \leq -(n_m-1)/n_m \quad \text{and} \quad \sum_{i=n_m+1}^{n_m+k_m+1} 1/i \leq y_m \leq \sum_{i=n_m}^{n_m+k_m+1} 1/i.$$

Let $h(x_m, y_m) = (x_m', y_m')$. By viewing the image of the quadrilateral one can see that

$$(n_m-2)/(n_m-1) \leq x_m' \leq (n_m+1)/(n_m+2) \quad \text{and} \quad \sum_{i=n_m+2}^{n_m+k_m+3} 1/i \leq y_m' \leq \sum_{i=n_m-1}^{n_m+k_m+1} 1/i.$$

Since $x_m \to -1$, n_m must increase without bound; thus, $x_m' \to 1$. To see that $y_m' \to y$ observe that

$$\sum_{i=n_m+1}^{n_m+k_m+1} 1/i - \sum_{i=n_m-1}^{n_m+k_m+1} 1/i \leq y_m - y_m' \leq \sum_{i=n_m}^{n_m+k_m+1} 1/i - \sum_{i=n_m+2}^{n_m+k_m+3} 1/i.$$

The differences of the sums on each side of the bound decrease to 0 as $m \to \infty$; hence, $y_m - y_m' \to 0$ implies that $y_m' \to y$. The arguments for the remaining points of form $(\pm 1, \pm y)$ are analogous to the argument given above.

CHAPTER III
CONCLUDING REMARKS

1. Consequences. By modifying an example of Bing's (5), Brechner and Mauldin (7) gave an example of a fixed point free orientation preserving homeomorphism of E^3 onto itself with the property that the orbit of every point is bounded. Examples of fixed point free orientation preserving and orientation reversing homeomorphisms of E^3 onto itself, with the property that the orbit of every point is bounded, follow as easy corollaries from the example we constructed in Chapter II. The homeomorphism f defined by $f(x,y,z) = (h(x,y),z)$, where h is the homeomorphism constructed above, verifies the corollary for the orientation reversing case. For the orientation preserving case simply define $g(x,y,z) = (h(x,y),-z)$. Observe that the first homeomorphism can be viewed as a reflection across the yz-plane, followed by an orientation preserving map; while the second homeomorphism is the composition of a reflection across the yz-plane, an orientation preserving homeomorphism, and a reflection across the xy-plane.

2. Related questions. Research in the area of planar fixed point theory continues to be quite active. We state two unsolved problems which are closely related to the counterexample presented in Chapter II. Let h be an orientation reversing homeomorphism of E^2 onto itself having the property that the orbit of every point is bounded. Define h to be

<u>bounded at p</u> if there exists an open set U containing p such that the orbit of U is bounded. Let D be the set of all points p such that h is bounded at p. Observe that D is open in E^2. To see that D is dense in E^2, let $F = E^2 - D$, and suppose that F contains some open set U. Define

$$F_n \equiv \{p \in F : \text{the orbit of } p \text{ is contained in } B_n\},$$

where $B_n = \{p \in E^2 : |p| \leq n\}$. Since each F_n is closed and $F = \bigcup_{n=1}^{\infty} F_n$, the Baire Category Theorem implies that some F_n contains an open set W. Therefore, since h is bounded at every point in W, we obtain the contradiction that W is in D as well as F.

In the example we give,D has infinitely many components. The question as to whether D can have finitely many components without h having a fixed point remains unsolved.[1] Even when D has just two components, the answer to this question is not known.

Another question of a similar nature can be phrased as follows: if h is a continuous function on E^2 with the property that the positive orbit of every bounded set is bounded, then does h have a fixed point? As mentioned in Section 2 of Chapter I, when h is a homeomorphism, this question can be answered in the affirmative.

[1]This problem was communicated to me by Professor R. Dan Mauldin.

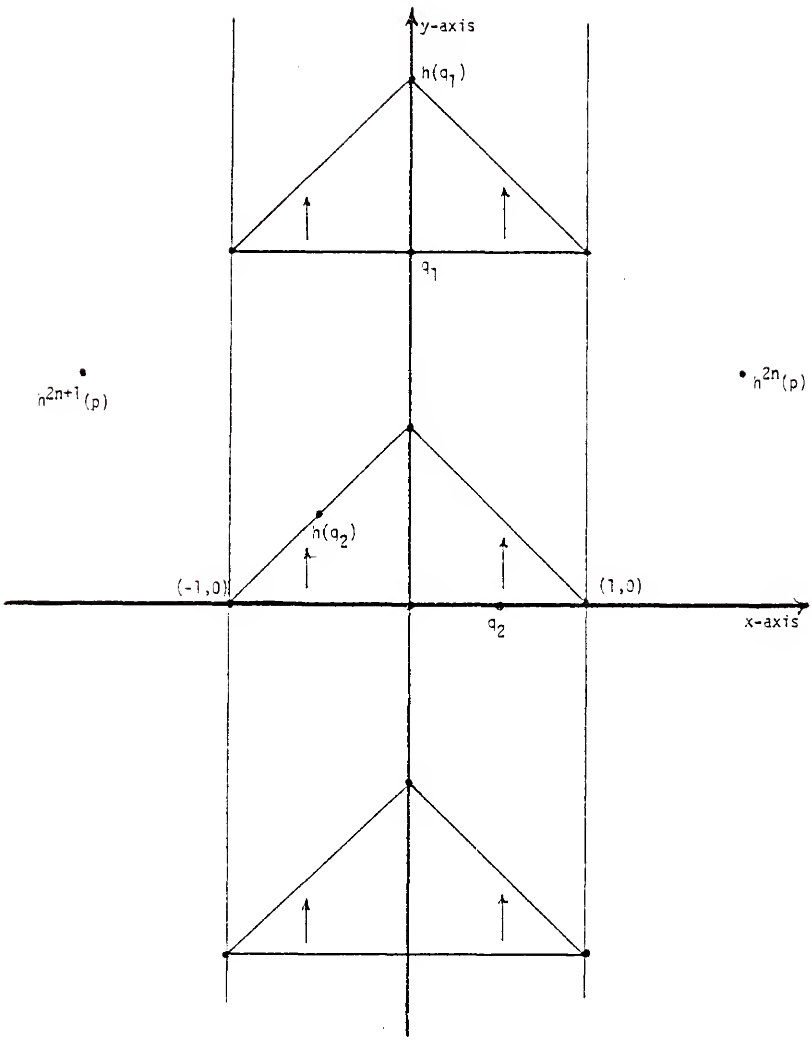


FIGURE 1
A fixed point free homeomorphism with some bounded orbits

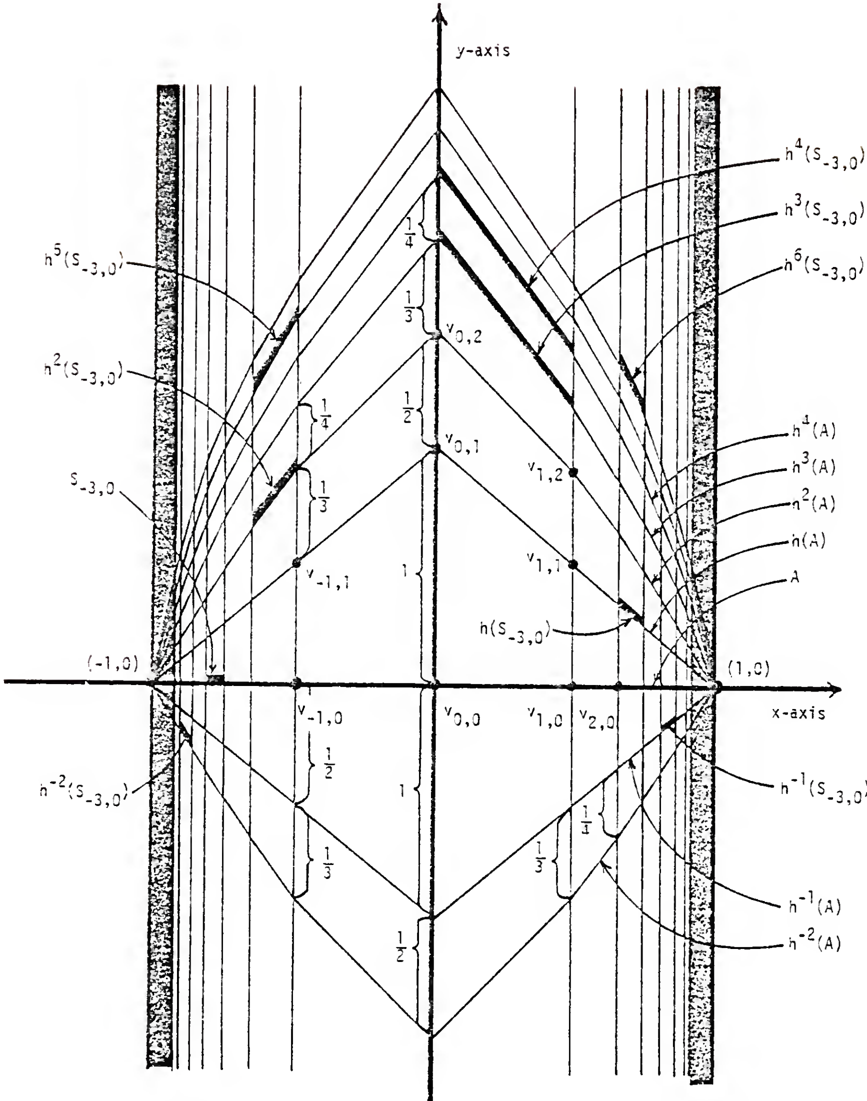
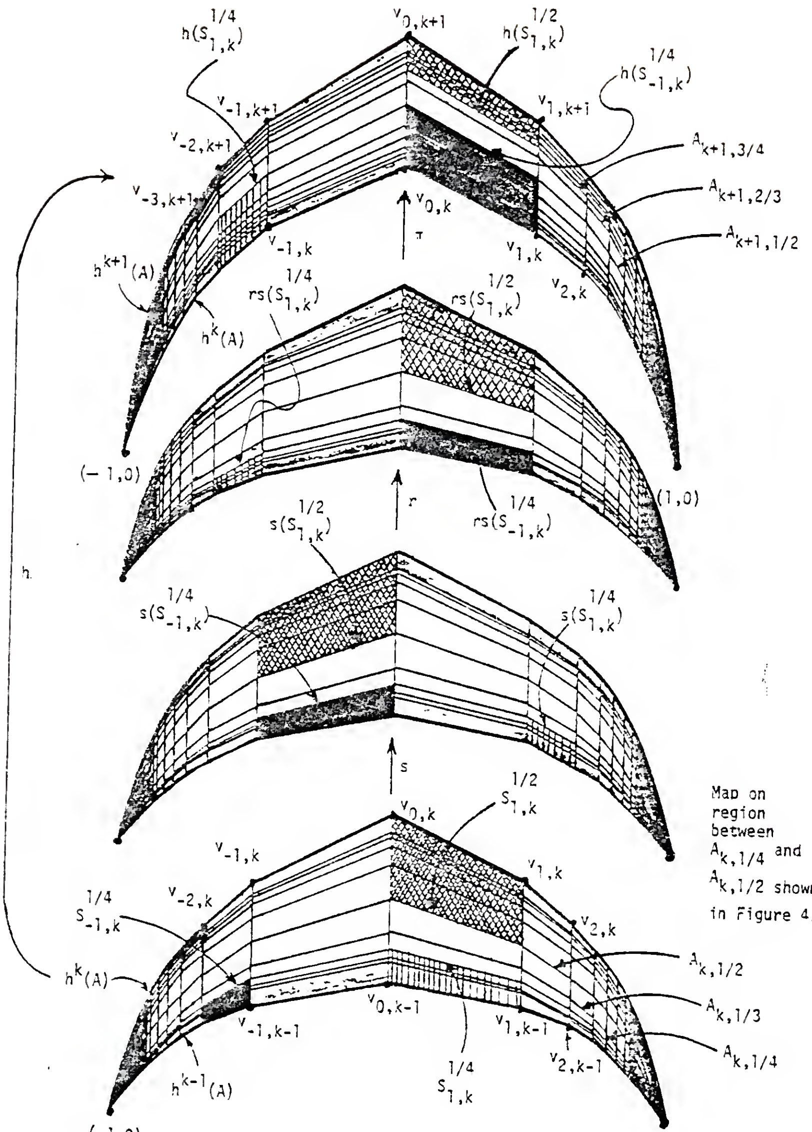


FIGURE 2
A counterexample to the Bounded Orbit Conjecture



Map on region between $A_{k,1/4}$ and $A_{k,1/2}$ shown in Figure 4.

FIGURE 3
Map of h on region between $h^{k-1}(A)$ and $h^k(A)$

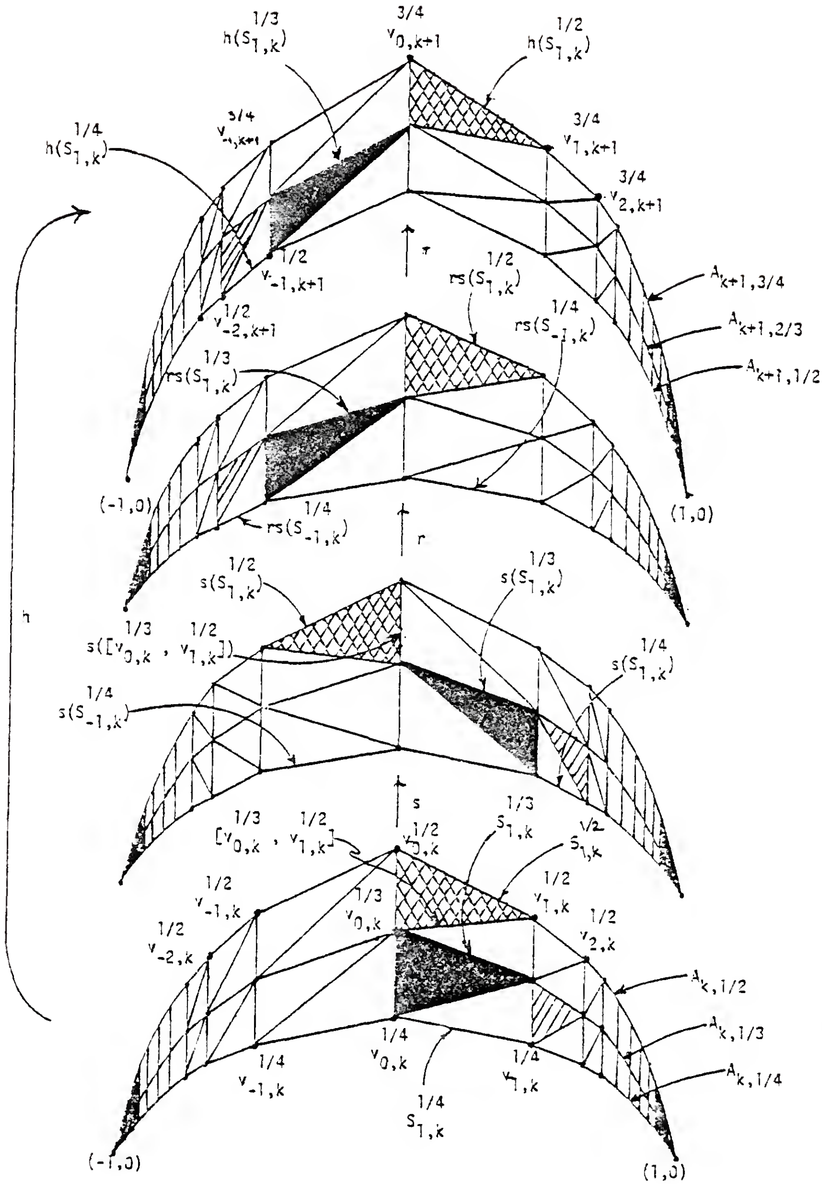


FIGURE 4
Map of h on region between $A_{k,1/4}$ and $A_{k,1/2}$

BIBLIOGRAPHY

1. Stephen Andrea, On homeomorphisms of the plane which have no fixed points, Abh. Math. Sem. Univ. Hamburg. 30 (1967), 61-74.

2. __________, The plane is not compactly generated by a free mapping, TAMS, 151 (1970), 481-498. MR 42 #2445.

3. Harold Bell, A fixed point theorem for plane homeomorphisms, BAMS, 82 (1976), 778-780.

4. __________, A fixed point theorem for plane homeomorphisms, Fund. Math., 100 (1978), 119-128.

5. R. H. Bing, The elusive fixed point property, Amer. Math. Monthly, 76 (1969), 119-132. MR 38 #5201.

6. S. Boyles, A Counterexample to the Bounded Orbit Conjecture, TAMS (to appear).

7. B. L. Brechner and R. D. Mauldin, Homeomorphisms of the plane, Pac. J. of Math., 59 (1975), 375-381. MR 52 #9199.

8. __________, EC^+ Homeomorphisms of Euclidean spaces, Top. Proc., 1 (1976), 335-343.

9. L. E. J. Brouwer, Beweis des ebenen Translationesatzes, Math. Ann., 72 (1912), 39-54.

10. M. L. Cartwright and J. E. Littlewood, Some fixed point theorems, Ann. of Math., 54 (2) (1951), 1-37.

11. J. G. Hocking and G. S. Young, Topology, Addison-Wesley Pub. Co., Reading, Mass., 1961.

12. Gordon Johnson, An example in fixed point theory, PAMS, 44 (1974), 511-514.

13. B. v. Kerekjarto, On a geometric theory of continous groups, Ann. Math., 59 (1925), 105-117.

14. W. K. Mason, Fixed points of pointwise almost periodic homeomorphisms on the 2-sphere, preprint.

15. Deane Montgomery, Pointwise periodic homeomorphisms, Amer. J. Math., 59 (1937), 118-120.

16. E. v. Sperner, Uber die fixpunktfreien Abbilfungen der Ebene, Abh. Math. Sem. Hamburg, 10 (1934), 1-47.

17. H. Terasaka, On quasi translations in E^n, Proc. Japan Acad., 30 (1954), 80-84.

18. S. M. Ulam, Problems in Modern Mathematics, Wiley and Sons, Inc., New York, 1960. MR 22 #10884.

BIOGRAPHICAL SKETCH

Stephanie Marion Boyles was born in Atlanta, Georgia, on December 29, 1954. In 1974, she obtained her Bachelor of Science degree in mathematics from the University of Georgia. Two years later she received her Master of Science degree in applied mathematics from Michigan State University. She then entered the University of Florida and in 1980, attained the degree of Doctor of Philosophy in mathematics. She is a member of the American Mathematical Society, the Mathematical Association of America, and Phi Kappa Phi.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Gerhard X. Ritter, Chairman
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Elroy J. Bolduc
Professor of Subject Specialization
Teacher Education

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

James Edgar Keesling
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Patrick Joseph McKenna
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Sergio E. Zarantonello
Assistant Professor of Mathematics

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate Council and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June, 1980

Dean, Graduate School