A SIGNAL DETECTION FROM NOISE UTILIZING ZERO-CROSSING INFORMATION

By

WOON CHEON YEO

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Regretful to leave the University of Florida
Where knowledge is delivered with such warmth 'n
Where friendly hearts always surround you.

Supported and given the opportunity to delve into the
Meaning of knowledge,
Grateful to Dr. Jack R. Smith as
I owe all these to his generosity.

Admire and take pride in all my committee as
They embed confidence in your strength.

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Woon Cheon Yeo

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A method of detecting signals from noise utilizing the zero-crossing information is analyzed.

When the noise is stationary white Gaussian, a correlation detector is considered to be optimal. But, if the process of signal and noise is complex, then a nonlinear detector may give a superior result in overall performance.

In a situation where the signals and noise have nonstationary statistics, a detector which measures the zero-crossing intervals of the process has been found to be very efficient in detecting signals from noise. The operating characteristic of the zero-crossing measuring detector is analyzed in a model with signals of sine wave form imbedded in a stationary bandlimited white Gaussian noise. Then, to

observe the operating characteristic of the detector in a complex situation, the detector is analyzed under deviated conditions of signals and noise from the assumptions. The performance of a correlation detector is also analyzed under identical conditions, as a comparative reference to the performance of the zero-crossing measuring detector.

The operating characteristic of the zero-crossing measuring detector is shown to be effective in detecting phasic events from a sleep electroencephalogram, where the process of the signals and noise is very complex.

CHAPTER I
INTRODUCTION

An optimum detector of a signal from noise may well be described as "a detector which best satisfies given criteria under a given set of assumptions" (Whalen).

The first problem in designing an optimum detector is, then, making proper assumptions on parameters that completely describe the process of signal and noise, and making proper selections of parameters that are relevant to discriminating signals from noise. If either the real process is different from that described by the assumptions on parameters, or any of the relevant parameters are neglected, then the detector may be only an approximation to the optimum.

In the case when the process is complex, we may not be able to make assumptions on parameters, or even if we could define the process by the parameters, an optimum detector based on all the relevant parameters becomes too complicated to implement. Thus we need to select among the definable parameters the most efficient features from the process as the discriminating parameters so that a detector based on these parameters should give an acceptable performance in the complex situation. The efficiency of the parameters is judged by the given criteria under the assumptions on the parameters. The performance of the detector based on these efficient parameters, however, may not be optimum if the environment of signal and noise changes so as to include other efficient definable parameters.

In a stationary Gaussian noise, a correlation detector is considered to be optimal (Wainstein). If the noise process is complex so that it could not be simply definable as stationary Gaussian, then a superior detector may require a nonlinear characteristic, and the implementation of the detector may be complicated and costly. Especially if the process is not stationary, then the adaptation of the detector becomes a serious problem, and it may leave the detector complicated to operate. Thus, in general, the design of a detector should include considerations such as performance, cost of implementation, simplicity of implementation, and simplicity of operation. The priority of the considerations may vary depending on the circumstances of the particular application of the detector.

In designing a system to detect phasic events from sleep electroencephalograms(EEGs), one encounters the following complexities in the process of the EEG activities: The phasic events, which are designated by arrows in Figure 23, are distinguished from the irregular background EEG by their resemblance to sinusoidal waves. Sigma spindles, for example, last from 6 to 20 cycles with a frequency between 12 and 15 Hz depending on the individual. The amplitude of spindles and background EEG activities, which is considered as noise when we desire to detect sigma spindles, also varies widely depending on the individual and the electrode resistance. The a priori probability densities of the frequency and amplitude distribution are not usually known. Even though the strength of the signal (sigma spindles) and noise varies, the signal-to-noise ratio remains relatively constant; that is, a subject with high background EEG activity also has high amplitude in the spindle activity. The strength and the power spectra of the background EEG undergo a continuous change as a sleep progresses.

In practical applications of the phasic event detector, it is desirable that the detector be able to perform equally well an any subject without a priori information on the frequency and the strength of the EEG activity. To accomplish this a detector could be designed which adapts to the varying frequency and intensity of the EEG activity, but it might be too costly to implement.

Smith et al.(43,44) designed a phasic event detector which employs the zero-crossing interval of the EEG as the discriminating parameter, and they obtained rather superior performance from the detector. The detection rate was high enough to meet the requirement, while the false alarm rate was extremely low for any kind of noise situation. The detector could be applicable to a wide variety of subjects without any adjustment of thresholds to accommodate each subject. It was simple to implement, and its overall performance was superior to a correlation detector based on weak assumptions on the signal and noise. And thus a mathematical analysis of the zero-crossing measuring detector was suggested and carried out in this study to provide a better understanding on the efficiency of the feature of zero-crossing interval as a discriminating parameter of signals from noise.

Since both the signal(phasic events) and the noise(background EEG activity) processes in sleep EEG are too complicated to be explicitly defined, an analysis of any phasic event detector from sleep EEG is necessarily based on improper assumptions. Also any attempt to make the model close to the reality would leave the analysis untenable. Thus the analysis of the detector is carried on with an idealized model of signal and noise. The signal is modeled by sinusoidal waves and the noise is assumed to be bandlimited stationary Gaussian. Then, to observe the performance of the detector in a nonideal situation, the operating characteristic of the detector is analyzed under conditions of signal and noise different from the assumptions. In each case the performance of the correlation detector is also analyzed as a comparative reference to the zero-crossing measuring detector.

To compute the detection and false alarm probabilities of the zero-crossing measuring detector, we need the zero-crossing interval density functions of the processes with noise alone and signal plus noise. The distribution functions for the successive zero-crossing intervals of the bandlimited Gaussian noise have been developed by Rice(36,37), McFadden (26,27) and Longuet-Higgins(23,24). Experimental works on the zero-crossing interval distribution of Gaussian noise have been presented by Rainal(32) and Blotekjaer(6). Rice's zero-crossing interval density function agrees well with experimental data for small values of zero-crossing interval. The zero-crossing interval density function derived by McFadden gives good approximations to Gaussian processes having arbitrary power spectra. For the particular Gaussian process which has an ideal low pass spectrum, the approximation by Longuet-Higgins fits closely to the experimental data given by Blotekjaer. Longuet-Higgins derived a probability density function of the spacing between the ith zero and the (i+m+1)th zero of a stationary random function. The density function is expressed as a rapidly convergent series, and, when the process of the random function is Gaussian, the first two terms of the series may be expressed in terms of known functions. This Longuet-Higgins approximation is used here to compute the zero-crossing interval distribution of a bandlimited Gaussian process.

In the signal-plus-noise case, the zero-crossing interval density function is approximated by the first term of the Longuet-Higgins series. This approximation is also derived by Rice(38) for Gaussian processes. Cobb(10) gave a treatment on this approximation for a process of sine wave plus bandlimited Gaussian noise. He reports in the same paper about the zero-crossing interval measuring principle in distinguishing two signals separated in frequency. Yet his paper does not provide details of the detector and any comparative result to other kinds of detecting methods. The computation of the zero-crossing interval density for sine wave plus noise in this study is based on the treatment of Cobb.

Derivations of the zero-crossing interval density function for Gaussian noise and Gaussian noise plus signal are presented in Chapter II. An expression of the first two terms of the Longuet-Higgins' series in terms of the autocorrelation function of the Gaussian noise is derived. The computed density function is plotted along with the experimental data as a comparison. The derivation and the computed data of the zero-crossing interval density for signal plus noise are also presented in Chapter II.

In Chapter III, the schematic of the zero-crossing measuring detector is described and the method of maximizing the detection probability for given false alarm rate is discussed. The performance of the detector is compared to that of the conventional correlation detector under the identical situation of signal and noise.

The advantages and disadvantages of the zero-crossing measuring detector compared to the correlation detector are further discussed in Chapter IV, with practical considerations in detecting phasic events from sleep EEGs.

All the programs to compute various functions were written in FORTRAN and assembly language, and they were processed through a minicomputer PDP-8/e. Without an extended arithmetic unit, the speed of the PDP-8/e was very slow to compute some numerical integrations. Thus, to save the computing time, we employed data interpolations in the computation of the zero-crossing interval density functions and evaluated the statistical functions from their series expansions.

CHAPTER II
ZERO-CROSSING INTERVAL DENSITY FUNCTIONS

Derivation of Zero-crossing Interval Density Function

For simplicity, we will state that a random function $f(t)$ has a zero or a crossing at $t=t_1$ if $f(t_1)=0$. Similarly we use up-crossing or down-crossing to indicate a crossing with $f'(t_1)>0$ or $f'(t_1)<0$ respectively.

Consider the instance when a random function $f(t)$ has up-crossings in the infinitesimal intervals (t_i, t_i+dt_i) $(i=1,\ldots,n)$, and let us denote the joint probability of this happening by $W(+,+,\ldots,+)dt_1 dt_2 \ldots dt_n$. To denote the joint probability that the random function has down-crossings in some intervals, we change the plus sign to minus at the corresponding positions in W. Thus $W(+,-,+)dt_1 dt_2 dt_3$, for instance, would represent the joint probability that a random function $f(t)$ has an up-crossing in the interval (t_1, t_1+dt_1), down-crossing in (t_2, t_2+dt_2), and an up-crossing in (t_3, t_3+dt_3).

We next introduce $p(m;\tau)$ to denote the probability density of the interval between the ith and the (i+m+1)th zeros of $f(t)$. This probability density is denoted by $P_m(\tau)$ by other authors, but, since in this paper a probability is represented by an uppercase P and a probability density is written in a lowercase p, the symbol $p(m;\tau)$ is used here. Then $p(0;\tau)d\tau$ is the probability that the interval between two successive zeros has length τ, and $p(0;\tau)d\tau W(+)dt_1$ expresses the probability that $f(t)$ has an up-crossing at $t=t_1$ and a down-crossing after an interval τ without having any other crossings in between the two crossings. Likewise $p(2n;\tau)d\tau W(+)dt_1$ represents the probability of $f(t)$ having an up-crossing at $t=t_1$ and a down-crossing after an interval τ, containing 2n crossings anywhere in between the two crossings.

If $f(t)$ has an up-crossing at $t=t_1$ and a down-crossing at $t=t_2$, then $f(t)$ may contain 2n $(n=0,1,2,\dots)$ crossings between $t=t_1$ and $t=t_2$. $W(+,-)dt_1dt_2$, the probability of $f(t)$ having an up-crossing at $t=t_1$ and a down-crossing at $t=t_2$, is then the sum of the probabilities of all possible cases when $f(t)$ has an up-crossing at $t=t_1$ and a down-crossing after an interval $\tau=t_2-t_1$, including 2n $(n=0,1,2,\dots)$ crossings within the interval τ. That is

$$W(+,-)dt_1dt_2=[p(0;\tau)+p(2;\tau)+p(4;\tau)+\dots]d\tau W(+)dt_1, \quad (1.1)$$

or, since $d\tau=dt_2$,

$$W(+,-)/W(+)=p(0;\tau)+p(2;\tau)+p(4;\tau)+\dots, \quad (1.2)$$

(McFadden 1958).

In a similar fashion we can derive an expression related to $W(+,-,-)$. Let us consider the situation when $f(t)$ has an up-crossing at $t=t_1$ and a down-crossing at $t=t_3$, containing a down-crossing at $t=t_2$ anywhere in between t_1 and t_3. Since the crossings at $t=t_1$ and $t=t_3$ are an up-crossing and a down-crossing respectively, f(t) can only have an even number of crossings in the interval (t_1,t_3), and one of the down-crossings among these crossings should pass zero at $t=t_2$. Suppose we have 2n crossings in (t_1,t_3) where $t_3-t_1=\tau$. Then the down-crossing at $t=t_2$ could be one of those n down-crossings, and, therefore, there are n different possibilities of f(t) having a down-crossing at $t=t_2$. Since the down-crossing at $t=t_2$ could be anywhere in (t_1,t_3), if we integrate the probability of each possibility with respect to t_2 from t_1 to t_3 we get $p(2n;\tau)d\tau W(+)dt_1$, and we obtain for the n possibilities a total probability of $np(2n;\tau)d\tau W(+)dt_1$. And the sum of the probabilities for all possible values of n,

$$\left[p(2;\tau)+2p(4;\tau)+\ldots+np(2n;\tau)+\ldots\right]d\tau W(+)dt_1, \qquad (1.3)$$

is then the probability of f(t) having an up-crossing at $t=t_1$ and a down-crossing at $t=t_1+\tau$ containing at least one down-crossing at $t=t_2$ anywhere in between t_1 and t_3.

Now if we integrate $W(+,-,-)dt_1dt_2dt_3$ with respect to t_2 over the interval (t_1,t_3), we obtain the probability of f(t) having an up-crossing at $t=t_1$ and a down-crossing at $t=t_3$ containing at least one down-crossing at $t=t_2$ anywhere in between t_1 and t_3. This probability should be equal to that expressed by Equation (1.3). Therefore we obtain (Longuet-Higgins),

$$\int_{t_1<t_2<t_3} dt_2\left[W(+,-,-)/W(+)\right]= p(2;\tau)+2p(4;\tau)+\ldots+np(2n;\tau)+\ldots \qquad (1.4)$$

By rearranging Equations (1.2) and (1.4),

$$p(0;\tau)=W(+,-)/W(+)-[p(2;\tau)+p(4;\tau)+\ldots], \tag{1.5}$$

$$p(2;\tau)=\int_{t_1<t_2<t_3} dt_2[W(+,-,-)/W(+)] - [2p(4;\tau)+3p(6;\tau)+\ldots]. \tag{1.6}$$

Substituting Equation (1.6) into Equation (1.5),

$$p(0;\tau)=W(+,-)/W(+) - \int_{t_1<t_2<t_3} dt_2[W(+,-,-)/W(+)] + p(4;\tau)+2p(6;\tau)+3p(8;\tau)+\ldots. \tag{1.7}$$

If we neglect the terms $[p(4;\tau)+2p(6;\tau)+3p(8;\tau)+\ldots]$ in the above equation, we get an approximate expression for the zero-crossing interval density (Longuet-Higgins):

$$p(0;\tau)=W(+,-)/W(+)-\int_{t_1<t_2<t_3} dt_2[W(+,-,-)/W(+)]. \tag{1.8}$$

By neglecting the remaining terms in Equation (1.7) we are ignoring the probabilities of $f(t)$ having 4 or more zeros within the interval of length τ.

Evaluation of W for Gaussian Process

Let us consider the probability $W(+,+,\dots,+)dt_1 dt_2 \dots dt_n$ that $f(t)$ has an up-crossing with an arbitrary positive slope $f'(t)$ in each of the small intervals (t_i, t_i+dt_i) $(i=1,\dots,n)$. For convenience write

$$f(t_i)=f_i,$$

$$f'(t_i)=g_i,$$

and let $p(f_1,\dots,f_n,g_1,\dots,g_n)$ denote the joint probability density of the f_i and g_i $(i=1,\dots,n)$. Thus $p(f_1,\dots,f_n, g_1,\dots,g_n)df_1\dots df_n dg_1\dots dg_n$ is the probability that the f_i and g_i lie in the intervals (f_i,f_i+df_i), (g_i,g_i+dg_i) respectively for each i.

If $f(t)$ has a zero in (t_i,t_i+dt_i) with a gradient g_i, then $f(t_i)$ must lie in a small range of value of extent $|g_i|\,dt_i$. Especially if $f(t)$ has an up-crossing in the interval (t_i,t_i+dt_i), then the range of the gradient is $0<g_i<\infty$, and the range of f_i is $-g_i dt_i<f_i<0$. Thus the integral of $p(f_1,\dots,f_n,g_1,\dots,g_n)$ over the ranges $(0<g_i<\infty)$, $(-g_i dt_i<f_i<0)$ $(i=1,\dots,n)$ is the probability of $f(t)$ having up-crossings at the intervals (t_i,t_i+dt_i) $(i=1,\dots,n)$. That is,

$$W(+,\dots,+)dt_1\dots dt_n$$
$$=\int_0^\infty dg_1\dots\int_0^\infty dg_n\int_{-g_1dt_1}^{0} df_1\dots\int_{-g_ndt_n}^{0} df_n p(f_1,\dots,f_n,g_1,\dots,g_n). \tag{2.1}$$

Since the range of f_i $(-g_i dt_i, 0)$ is small for each i, we may consider $p(f_1,\ldots,f_n,g_1,\ldots,g_n)$ to be constant with respect to f_i within each interval (t_i, t_i+dt_i). Then the integrals in Equation (2.1) could be approximated as

$$W(+,\ldots,+)dt_1\ldots dt_n$$

$$=\int_0^\infty dg_1\ldots\int_0^\infty dg_n p(f_1,\ldots,f_n,g_1,\ldots,g_n)\int_{-g_1 dt_1}^{0} df_1\ldots\int_{-g_n dt_n}^{0} df_n,$$

which becomes

$$W(+,\ldots,+)dt_1\ldots dt_n$$

$$=\int_0^\infty dg_1\ldots\int_0^\infty dg_n g_1\ldots g_n p(0,\ldots,0,g_1,\ldots,g_n)dt_1\ldots dt_n,$$

and thus,

$$W(+,\ldots,+)$$

$$=\int_0^\infty dg_1\ldots\int_0^\infty dg_n g_1\ldots g_n p(0,\ldots,0,g_1,\ldots,g_n). \tag{2.2}$$

Now we try to find the expression of $p(f_1,\ldots,f_n,$ $g_1,\ldots,g_n)$ in terms of the autocorrelation function of the Gaussian process f(t). The covariance matrix of the 2n variables $f_1,\ldots,f_n,g_1,\ldots,g_n$ is

$$M=\begin{pmatrix} Ef_1f_1,\ldots,Ef_1f_n,Ef_1g_1,\ldots,Ef_1g_n \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ Ef_nf_1,\ldots,Ef_nf_n,Ef_ng_1,\ldots,Ef_ng_n \\ Eg_1f_1,\ldots,Eg_1f_n,Eg_1g_1,\ldots,Eg_1g_n \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ Eg_nf_1,\ldots,Eg_nf_n,Eg_ng_1,\ldots,Eg_ng_n \end{pmatrix}, \tag{2.3}$$

where Ef_ig_j represents $E[f_ig_j]$, an expectation of f_ig_j.

Let R_{ij} denote the autocorrelation function of f(t) such as

$$R_{ij}=R_{ji}=R(t_j-t_i)=R(\tau)=E[f_i f_j], \tag{2.4}$$

where $\tau=t_j-t_i$. Then,

$$E[f_i g_j]=E[f(t_i)\frac{d}{dt}f(t)\Big|_{t=t_j}]$$

$$=E[f(t_i)\frac{\delta}{\delta t}f(t+\tau)\Big|_{t=t_i}]$$

$$=E[f(t_i)\frac{\delta}{\delta\tau}f(t_i+\tau)]$$

$$=\frac{\delta}{\delta\tau}E[f(t_i)f(t_i+\tau)]$$

$$=\frac{\delta}{\delta\tau}R(\tau)=R_{ij}'(\tau), \tag{2.5}$$

where the prime denotes differentiation with respect to τ, and δ denotes partial differentiation.

$$E[g_i f_j]=E[\frac{d}{dt}f(t)\Big|_{t=t_i}f(t_j)]$$

$$=E[\frac{\delta}{\delta t}f(t-\tau)\Big|_{t=t_j}f(t_j)]$$

$$=E[\frac{\delta}{\delta(-\tau)}f(t_j-\tau)f(t_j)]$$

$$=-\frac{\delta}{\delta\tau}E[f(t_j-\tau)f(t_j)]$$

$$=-\frac{\delta}{\delta\tau}R(-\tau)=-\frac{\delta}{\delta\tau}R(\tau)=-R_{ij}'(\tau), \tag{2.6}$$

$$E[g_i g_j]=E[\frac{d}{dt}f(t)\Big|_{t=t_i}\ \frac{d}{dt}f(t)\Big|_{t=t_j}]$$

$$=E\left[\frac{d}{dt}f(t)\Big|_{t=t_i}\frac{\delta}{\delta t}f(t+\tau)\Big|_{t=t_i}\right]$$

$$=E\left[\frac{d}{dt}f(t)\Big|_{t=t_i}\frac{\delta}{\delta\tau}f(t_i+\tau)\right]$$

$$=\frac{\delta}{\delta\tau}E\left[\frac{d}{dt}f(t)\Big|_{t=t_i}f(t_i+\tau)\right]$$

$$=\frac{\delta}{\delta\tau}\left[-\frac{\delta}{\delta\tau}E[f(t_i)f(t_i+\tau)]\right]$$

$$=-\frac{\delta^2}{\delta\tau^2}R(\tau)=-R_{ij}''(\tau). \qquad (2.7)$$

Thus the covariance matrix in terms of $R(\tau)$ is

$$M=\begin{bmatrix} R_{11},\ldots,R_{1n}, & R_{11}',\ldots, R_{1n}' \\ \vdots & \vdots \\ R_{n1},\ldots,R_{nn}, & R_{n1}',\ldots, R_{nn}' \\ -R_{11}',\ldots,-R_{1n}', & -R_{11}'',\ldots,-R_{1n}'' \\ \vdots & \vdots \\ -R_{n1}',\ldots,-R_{nn}', & -R_{n1}'',\ldots,-R_{nn}'' \end{bmatrix}. \qquad (2.8)$$

By the Gaussian hypothesis we have

$$p(f_1,\ldots,f_n,g_1,\ldots,g_n)$$

$$=(2\pi)^{-n}D^{-\frac{1}{2}}\exp(-\tfrac{1}{2}X^TM^{-1}X)$$

$$=(2\pi)^{-n}D^{-\frac{1}{2}}\exp\left(-\tfrac{1}{2}\sum_{i,j=1}^{2n}q_{ij}f_if_j\right), \qquad (2.9)$$

where X and X^T are column and row vectors of $2n$ variables $(f_1,\ldots,f_n,g_1,\ldots,g_n)$ respectively, $D=|M|$, $f_{n+i}=g_i$, and q_{ij} is the element of the ith row and jth column of the matrix $Q=M^{-1}$. We will denote a matrix with elements q_{ij} by (q_{ij}).

Substituting Equation (2.9) into Equation (2.2) we get

$$W(+,\dots,+)$$
$$=(2\pi)^{-n}D^{-\frac{1}{2}}\int_0^\infty dg_1\dots\int_0^\infty dg_n g\ g_2\dots g_n \exp(-\tfrac{1}{2}\sum_{i,j=1}^{n} q_{n+i,n+j}g_i g_j). \tag{2.10}$$

The summation in Equation (2.10) involves only the last n rows and columns of (q_{ij}). We let (h_{ij}) denote the inverse matrix of the matrix composed of the last n rows and columns of (q_{ij}) such as

$$(h_{ij}) = \begin{bmatrix} q_{n+1,n+1},\dots,q_{n+1,2n} \\ \vdots \qquad\qquad \vdots \\ q_{2n,n+1},\dots,q_{2n,2n} \end{bmatrix}^{-1}, \tag{2.11}$$

and modify Equation (2.10) using this matrix.

$$W(+,\dots,+)$$
$$=(2\pi)^{-\frac{1}{2}n}D^{-\frac{1}{2}}|(h_{ij})|^{\frac{1}{2}}\int_0^\infty dg_1\dots\int_0^\infty dg_n g_1 g_2\dots g_n[(2\pi)^{-\frac{1}{2}n}$$
$$|(h_{ij})|^{-\frac{1}{2}}\exp(-\tfrac{1}{2}\sum_{i,j=1}^{n} q_{n+i,n+j}g_i g_j)]$$
$$=(2\pi)^{-\frac{1}{2}n}D^{-\frac{1}{2}}|(h_{ij})|^{\frac{1}{2}}\int_0^\infty dg_1\dots\int_0^\infty dg_n g_1 g_2\dots g_n Z(g,h), \tag{2.12}$$

where

$$Z(g,h)=(2\pi)^{-\frac{1}{2}n}|(h_{ij})|^{-\frac{1}{2}}\exp(-\tfrac{1}{2}\sum_{i,j=1}^{n} q_{n+i,n+j}g_i g_j). \tag{2.13}$$

Now $Z(g,h)$ is an ordinary normal distribution of n variables $(g_1,\dots,g_n)$ with the covariance matrix (h_{ij}). For convenience, we normalize the covariance matrix (h_{ij}) such that the (i,j)th element of the new covariance matrix is

$$v_{ij}=h_{ij}/(h_{ii}h_{jj})^{\frac{1}{2}},$$

and (v_{ij}) is the covariance matrix of the new variables $w_i=g_i/h_{ii}^{\frac{1}{2}}$. Then, with the above change of variables, Equation (2.12) becomes (Longuet-Higgins)

$$W(+,\ldots,+)=(2\pi)^{-\frac{1}{2}n}D^{-\frac{1}{2}}\left|(h_{ij})\right|^{\frac{1}{2}}(h_{11}h_{22}\ldots h_{nn})^{\frac{1}{2}}J_n, \quad (2.14)$$

where

$$J_n=\int_0^\infty dw_1\ldots\int_0^\infty dw_n w_1 w_2\ldots w_n Z(w,v), \quad (2.15)$$

and $Z(w,v)$ is a normal probability density of n variables $(w_1,\ldots,w_n)$ with the normalized covariance matrix (v_{ij}).

By Jacobi's theorem (Appendix A), the determinant and (i,j)th element of the matrix (h_{ij}) are computed as

$$\left|(h_{ij})\right|=D/D_1, \quad (2.16)$$

$$h_{ij}=D_1^{-1}\begin{vmatrix} R_{11},\ldots, & R_{1n}, & R_{1j}' \\ \vdots & & \vdots \\ R_{n1},\ldots, & R_{nn}, & R_{nj}' \\ -R_{i1}',\ldots, & -R_{in}', & -R_{ij}'' \end{vmatrix}, \quad (2.17)$$

where

$$D_1=\begin{vmatrix} R_{11},\ldots,R_{1n} \\ \vdots \qquad \vdots \\ R_{n1},\ldots,R_{nn} \end{vmatrix}. \quad (2.18)$$

Then, by substitution of Equation (2.16) into Equation (2.14),

$$W(+,\ldots,+)=(2\pi)^{-\frac{1}{2}n}D_1^{-\frac{1}{2}}(h_{11}h_{22}\ldots h_{nn})^{\frac{1}{2}}J_n. \quad (2.19)$$

We now compute W for the special cases when n is 1, 2 or 3. If n=1, then Z(w,v) is a normal distribution of a single variate, and we have

$$J_1=\int_0^\infty dw(2\pi)^{-\frac{1}{2}}w\exp(-\tfrac{1}{2}w^2)=(2\pi)^{-\frac{1}{2}},$$

$$D_1=|R_{11}|=R(0),$$

and

$$h_{11}=D_1^{-\frac{1}{2}}\begin{vmatrix} R_{11}, & R_{11}' \\ -R_{11}', & -R_{11}'' \end{vmatrix}=R(0)^{-1}\begin{vmatrix} R(0), & R(0)' \\ -R(0)', & -R(0)'' \end{vmatrix}$$

$$=-R(0)''.$$

Thus Equation (2.19) for the case n=1 becomes

$$W(+)=(2\pi)^{-\frac{1}{2}}D_1^{-\frac{1}{2}}h_{11}^{\frac{1}{2}}J_1$$

$$=(2\pi)^{-1}[-R(0)''/R(0)]^{\frac{1}{2}}. \tag{2.20}$$

When n=2 or 3, we may use the results of Nabeya(30) and Kamat(19) to compute J_2 and J_3:

$$J_2=(2\pi)^{-1}[(1-v_{12}^2)^{\frac{1}{2}}+v_{12}\cos^{-1}(-v_{12})], \tag{2.21}$$

$$J_3=(2\pi)^{-3/2}[|(v_{ij})|^{\frac{1}{2}}+(s_1b_1+s_2b_2+s_3b_3)], \tag{2.22}$$

where

$$s_1=\cos^{-1}[(v_{31}v_{12}-v_{23})(1-v_{31}^2)^{-\frac{1}{2}}(1-v_{12}^2)^{-\frac{1}{2}}],$$

$$b_1=v_{31}v_{12}+v_{23}.$$

The angles of arc cosine are to be taken in the range $(0,\pi)$, and s_2, s_3, b_2 and b_3 are obtained by cyclic permutation of the v_{ij}. This gives

$$W(+,+)=(2\pi)^{-2}(R_{11}R_{22}-R_{12}R_{21})^{-\frac{1}{2}}(h_{11}h_{22})^{\frac{1}{2}}[(1-v_{12}{}^2)^{\frac{1}{2}}+ v_{12}\cos^{-1}(-v_{12})], \tag{2.23}$$

and

$$W(+,+,+) =(2\pi)^{-3}D_1{}^{-\frac{1}{2}}(h_{11}h_{22}h_{33})^{\frac{1}{2}}[\,|(v_{ij})|^{\frac{1}{2}}+(s_1b_1+s_2b_2+s_3b_3)]. \tag{2.24}$$

Since, for our purpose, we need to compute $W(+,-)$ and $W(+,-,-)$, we consider the case when there are down-crossings in W. Suppose the kth zero-crossing is down-crossing. Then in calculating the corresponding probability density W we need only to take the range of integration of g_i from $-\infty$ to 0 instead of 0 to ∞. Equivalently, we may simply reverse the sign of the (n+k)th row and column of M^{-1}, and hence the kth row and column of (h_{ij}) and (v_{ij}). Thus by changing sign of v_{12} in Equation (2.23) we obtain (Longuet-Higgins)

$$W(+,-) =(2\pi)^{-2}(R_{11}R_{22}-R_{12}R_{21})^{-\frac{1}{2}}(h_{11}h_{22})[(1-v_{12}^2)^{\frac{1}{2}}-v_{12}\cos^{-1}v_{12}], \tag{2.25}$$

and also by changing corresponding signs of v_{ij} in s and b in Equation (2.23), we get

$$W(+,-,-)=(2\pi)^{-3}D_i{}^{-\frac{1}{2}}(h_{11}h_{22}h_{33})^{\frac{1}{2}}[\,|(v_{ij})|^{\frac{1}{2}}+s_1b_1+ (s_2-\pi)b_2+(s_3-\pi)b_3]. \tag{2.26}$$

Zero-crossing Interval Density Function for a Bandlimited White Gaussian Noise

In this study the noise is assumed to be bandlimited white Gaussian with a power spectral density

$$S(w)=\begin{cases}\pi, & \text{for } |w|\leqslant 1,\\ 0, & \text{for } |w|>1,\end{cases} \qquad (2.27)$$

and autocorrelation function

$$R(\tau)=\tau^{-1}\sin\tau. \qquad (2.28)$$

The power spectral density and the autocorrelation function of this noise have the shapes as shown in Figure 1.

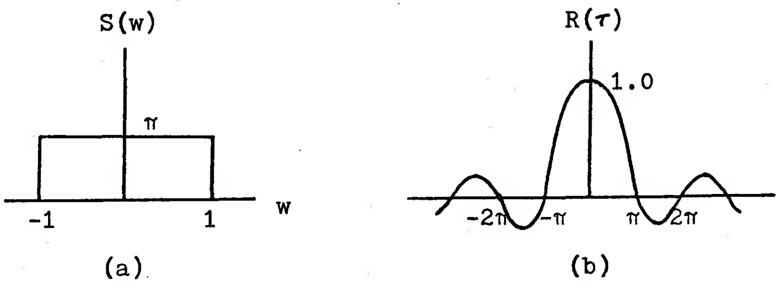
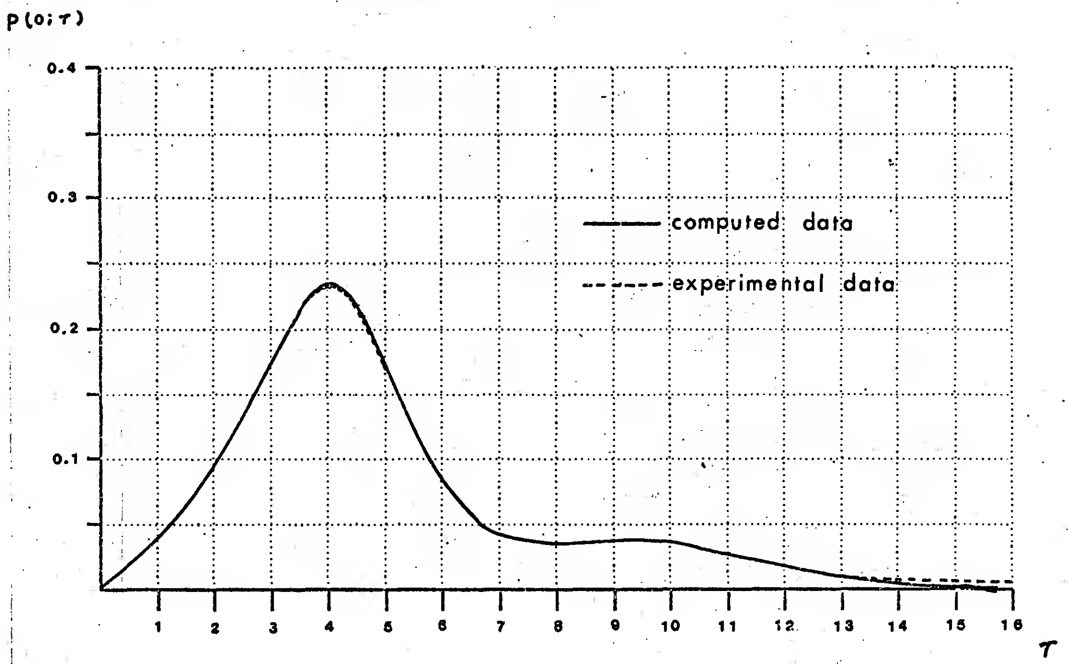


Figure 1.

The zero-crossing interval density function of the noise is computed according to Equation (1.8)

$$p(0;\tau)=W(+,-)/W(+)-\int_{t_1<t_2<t_3} dt_2[W(+,-,-)/W(+)]$$

and plotted in Figure 2. Experimental data by Blotekjaer(6)



p(o;τ)
0.4
0.3
0.2
0.1
computed data
experimental data
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
τ

Figure 2.

are also included for a comparison. The programs used in the computation are listed under LONGO, LONG1, and MATIN. For given τ, W(+) and W(+,-) are computed by LONGO according to the Equations (2.20) and (2.25) respectively with the autocorrelation function of noise

$$R(\tau)=\tau^{-1}\sin\tau.$$

The program LONGO calls the subroutine LONG1 to compute the term

$$\int_{t_1<t_2<t_3} dt_2[W(+,-,-)/W(+)],$$

where $\tau=t_3-t_1$.

The computation repeats with different value of τ to cover the range $0<\tau<15$. MATIN is a subroutine called by LONGO and LONG1 to compute the determinants of matrices which are needed in evaluating Equations (2.17), (2.18) and $|(v_{ij})|$ in Equation (2.26).

Evaluation of W for Sine Wave Plus Noise

For the zero-crossing interval density function of sine wave plus noise, we will use the approximation

$$p(0;\tau)=W(+,-)/W(+). \tag{3.1}$$

The assumption used here is that the signal-to-noise ratio is fairly high so that the zero-crossing intervals are mainly determined by the signal frequency, and the probability of having two down-crossings in the interval τ is negligible.

Let us denote the signal plus noise by f(t). Thus

$$f(t)=s(t)+n(t), \tag{3.2}$$

where n(t) is stationary Gaussian random noise with an autocorrelation function of

$$R(\tau)=\tau^{-1}\sin\tau,$$

and where

$s(t)=a\cos(qt+\theta_0)$, with

a=ratio of sine wave amplitude to rms noise,

q=radian frequency of the signal, and

θ_0=initial phase angle which is unknown.

The signal s(t) is assumed to be statistically independent to the noise n(t).

To compute W(+) we consider, as before, the joint probability that f(t) should pass zero with a slope g_1 in the time interval (t_1,t_1+dt_1), that is, the probability of

$$\left.\begin{aligned} &f(t)=s(t)+n(t)=0,\\ &f'(t)=s'(t)+n'(t)=g_1,\end{aligned}\right\}\ (t_1<t<t_1+dt_1), \tag{3.3}$$

or, equivalently,

$$\left.\begin{aligned} &n(t)=-s(t),\\ &n'(t)=g_1-s'(t),\end{aligned}\right\}\ (t_1<t<t_1+dt_1). \tag{3.4}$$

Since the signal and noise are statistically independent, the probability that the conditions in Equation (3.3) are satisfied is determined by the statistics of the noise alone; that is, it is equal to the probability of the noise satisfying the conditions in Equation (3.4). Thus

$$p(f_1=0,f_1'=g_1)=p(n_1=-s_1,n_1'=g_1-s_1')$$

$$=(2\pi)^{-1}|M|^{-\frac{1}{2}}\exp(-\tfrac{1}{2}X^TM^{-1}X), \tag{3.5}$$

where

$$X=\begin{bmatrix}-s_1\\ g_1-s_1'\end{bmatrix}=\begin{bmatrix}-a\cos\theta\\ g_1+qa\sin\theta\end{bmatrix},$$

$$M=\begin{bmatrix}R_{11}, & R_{11}'\\ R_{11}', & -R_{11}''\end{bmatrix}=\begin{bmatrix}1, & 0\\ 0, & -R_0''\end{bmatrix},$$

θ=the instantaneous phase angle of the sine wave given by $\theta=qt+\theta_0$,

R=the autocorrelation function of the noise.

Since θ_0 is unknown, we may with a sufficiently large sampling time consider θ as a random variable which can take all values from $-\pi$ to π with equall probability. Hence from Equation (2.2)

$$W(+)= (2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\int_{0}^{\infty}dg_1 g_1 p(n_1,n_1')$$

$$=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\int_{0}^{\infty}dg_1 g_1(2\pi)^{-1}|M|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}[a^2\cos^2\theta+ (-R_0")^{-1}(g_1+qa\sin\theta)^2]]$$

$$=(2\pi)^{-2}(-R_0")^{-\frac{1}{2}}\int_{-\pi}^{\pi}d\theta\exp(-\tfrac{1}{2}a^2\cos^2\theta)\int_{0}^{\infty}dg_1 g_1 \exp[-\tfrac{1}{2}(-R_0")^{-1}(g_1+qa\sin\theta)^2].$$

By a change of variable

$$g=(-R_0")^{-\frac{1}{2}}(g_1+qa\sin\theta),$$

$$W(+)=(2\pi)^{-2}(-R_0")^{-\frac{1}{2}}\int_{-\pi}^{\pi}d\theta\exp(-\tfrac{1}{2}a^2\cos^2\theta)\int_{c}^{\infty}dg(-R_0")^{\frac{1}{2}}[(-R_0")^{\frac{1}{2}}g- qa\sin\theta]\exp(-\tfrac{1}{2}g^2),$$

where

$$c=(-R_0")^{-\frac{1}{2}}qa\sin\theta.$$

Letting $b=(-R_0")^{-\frac{1}{2}}qa$,

$$W(+)=(2\pi)^{-3/2}(-R_0")^{\frac{1}{2}}\int_{-\pi}^{\pi}d\theta\exp(-\tfrac{1}{2}a^2\cos^2\theta)[(2\pi)^{-\frac{1}{2}}\int_{b\sin\theta}^{\infty}dgg\exp(-\tfrac{1}{2}g^2)-b\sin\theta(2\pi)^{-\frac{1}{2}}\int_{b\sin\theta}^{\infty}dg\exp(-\tfrac{1}{2}g^2)]$$

$$=(2\pi)^{-3/2}(-R_0")^{\frac{1}{2}}\int_{-\pi}^{\pi}d\theta\exp(-\tfrac{1}{2}a^2\cos^2\theta)[(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}b^2\sin^2\theta) -b\sin\theta[\tfrac{1}{2}-(2\pi)^{-\frac{1}{2}}\int_{0}^{b\sin\theta}dg\exp(-\tfrac{1}{2}g^2)]].$$

Since $\mathrm{erf}(x)=2\pi^{-\frac{1}{2}}\int_{0}^{x}dt\exp(-t^2)$ and the average of $\sin\theta$ from $-\pi$ to π is zero,

$$W(+)=(2\pi)^{-3/2}(-R_0")^{\frac{1}{2}}\int_{-\pi}^{\pi}d\theta\exp(-\tfrac{1}{2}a^2\cos^2\theta)\ [$$

$$(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}b^2\sin^2\theta)+\tfrac{1}{2}b\sin\theta\,\mathrm{erf}(2^{-\frac{1}{2}}b\sin\theta)]. \tag{3.6}$$

This result was given by Rice(38).

To compute W(+,-) we need to consider the instance when

$$\left.\begin{array}{l} f(t)=s(t)+n(t)=0, \\ f'(t)=s'(t)+n'(t)=g_i, \end{array}\right\} (t_i<t<t_i+dt_i) \quad (i=1,2), \tag{3.7}$$

or

$$\left.\begin{array}{l} n(t)=-s(t), \\ n'(t)=g_i-s'(t), \end{array}\right\} (t_i<t<t_i+dt_i) \quad (i=1,2). \tag{3.8}$$

The probability density for this instance is

$$p(n_1,n_2,n_1',n_2')=(2\pi)^{-2}|M|^{-\frac{1}{2}}\exp(-\tfrac{1}{2}X^TM^{-1}X), \tag{3.9}$$

where

$$X=\begin{bmatrix} -s_1 \\ -s_2 \\ g_1-s_1' \\ g_2-s_2' \end{bmatrix} = \begin{bmatrix} -a\cos\theta \\ -a\cos(\theta+q\tau) \\ g_1+qa\sin\theta \\ g_2+qa\sin(\theta+q\tau) \end{bmatrix}$$

$$M=\begin{bmatrix} R_{11}, & R_{12}, & R'_{11}, & R'_{12} \\ R_{21}, & R_{22}, & R'_{21}, & R'_{22} \\ -R'_{11}, & -R'_{12}, & -R''_{11}, & -R''_{12} \\ -R'_{21}, & -R'_{22}, & -R''_{21}, & -R''_{22} \end{bmatrix} = \begin{bmatrix} 1, & R, & 0, & R' \\ R, & 1, & R', & 0 \\ 0, & -R', & -R''_0, & -R'' \\ -R', & 0, & -R'', & -R''_0 \end{bmatrix}.$$

Again we consider θ as a random variable distributed evenly from $-\pi$ to π. And recognizing that the range of g_2 is from $-\infty$ to 0, we write from Equation (2.2)

$$W(+,-)=(2\pi)^{-1}\int_{-\pi}^{\pi}d\theta\int_{0}^{\infty}dg_1\int_{-\infty}^{0}dg_2\,g_1(-g_2)p(n_1,n_2,n_1',n_2')$$

$$=(2\pi)^{-3}|M|^{-\frac{1}{2}}\int_{-\pi}^{\pi}d\theta\int_{0}^{\infty}dg_1\int_{0}^{\infty}dg_2 g_1 g_2 \exp(-\tfrac{1}{2}U^T M^{-1} U), \qquad (3.10)$$

where

$$U=\begin{bmatrix} -a\cos\theta \\ -a\cos(\theta+q\tau) \\ g_1+qa\sin\theta \\ -g_2+qa\sin(\theta+q\tau) \end{bmatrix}.$$

The exponent in Equation (3.10) may be written as

$$-\tfrac{1}{2}U^T M^{-1} U$$

$$=-(2|M|)^{-1}\Big\{2a^2M_{11}[\cos^2\theta+\cos^2(\theta+q\tau)]+2a^2M_{14}\cos\theta\cos(\theta+q\tau)$$

$$-2aM_{12}[(g_1+qa\sin\theta)\cos\theta+[g_2-qa\sin(\theta+q\tau)]\cos(\theta+q\tau)]$$

$$+2aM_{13}[[g_2-qa\sin(\theta+q\tau)]\cos\theta+(g_1+qa\sin\theta)\cos(\theta+q\tau)]$$

$$+M_{22}[(g_1+qa\sin\theta)^2+[g_2-qa\sin(\theta+q\tau)]^2]$$

$$-2M_{23}(g_1+qa\sin\theta)[g_2-qa\sin(\theta+q\tau)]\Big\}, \qquad (3.11)$$

where

$$M_{11}=[(-R_0'')^2-(-R'')^2]-(-R_0'')(R')^2,$$

$$M_{12}=R'[-(-R_0'')R+(-R'')],$$

$$M_{13}=-R'[(-R_0'')-R(-R'')]+(R')^3,$$

$$M_{14}=-R[(-R_0'')^2-(-R'')^2]+(-R'')(R')^2,$$

$$M_{22}=(-R_0'')(1-R^2)-(R')^2,$$

$$M_{23}=-(-R'')(1-R^2)+R(R')^2,$$

$$|M|=[(1-R)[(-R_0'')+(-R'')]-(R')^2][(1+R)[(-R_0'')-(-R'')]-(R')^2].$$

We make a change of variables

$$(2|M|)^{-\frac{1}{2}}g_1=r\cos(\lambda+\tfrac{1}{4}\pi),$$

$$(2|M|)^{-\frac{1}{2}}g_2=r\sin(\lambda+\tfrac{1}{4}\pi),$$

which yields, after some rearrangement of terms,

$$W(+,-)$$
$$=\tfrac{1}{4}\pi^{-3}|M|^{3/2}\int_{-\pi}^{\pi}d\theta\int_{0}^{\infty}dr\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi}d\lambda\, r^3\cos(2\lambda)\exp[-(D_1r^2+2D_2r+D_3)], \tag{3.12}$$

where

$$D_1=M_{22}-M_{23}\cos(2\lambda)$$
$$=(M_{22}-M_{23})\cos^2\lambda+(M_{22}+M_{23})\sin^2\lambda,$$

$$D_2=a|M|^{-\frac{1}{2}}[-[(M_{12}-M_{13})\cos(\tfrac{1}{2}q\tau)+q(M_{22}-M_{23})\sin(\tfrac{1}{2}q\tau)]$$
$$\cos(\theta+\tfrac{1}{2}q\tau)\cos\lambda$$
$$+[(M_{12}+M_{13})\sin(\tfrac{1}{2}q\tau)-q(M_{22}+M_{23})\cos(\tfrac{1}{2}q\tau)]\sin(\theta+\tfrac{1}{2}q\tau)\sin\lambda],$$

$$D_3=a^2|M|^{-1}[[(M_{11}+M_{14})\cos^2(\tfrac{1}{2}q\tau)+2q(M_{12}-M_{13})\sin(\tfrac{1}{2}q\tau)\cos(\tfrac{1}{2}q\tau)$$
$$+q^2(M_{22}-M_{23})\sin^2(\tfrac{1}{2}q\tau)]\cos^2(\theta+\tfrac{1}{2}q\tau)$$
$$+[(M_{11}-M_{14})\sin^2(\tfrac{1}{2}q\tau)-2q(M_{12}+M_{13})\sin(\tfrac{1}{2}q\tau)\cos(\tfrac{1}{2}q\tau)$$
$$+q^2(M_{22}+M_{23})\cos^2(\tfrac{1}{2}q\tau)]\sin^2(\theta+\tfrac{1}{2}q\tau)].$$

We simplify the above expressions by writing

$$C_0=R',$$
$$C_1=1-R,$$
$$C_2=1+R,$$
$$C_3=-R_0''+R'',$$

$$C_4=-R_0''-R'',$$

$$C_5=(1-R)(-R_0''-R'')-(R')^2=C_1C_4-C_0{}^2,$$

$$C_6=(1+R)(-R_0''+R'')-(R')^2=C_2C_3-C_0{}^2,$$

$$C_7=C_0\cos(\tfrac{1}{2}q\tau)+qC_2\sin(\tfrac{1}{2}q\tau),$$

$$C_8=C_0\sin(\tfrac{1}{2}q\tau)+qC_1\cos(\tfrac{1}{2}q\tau),$$

and replacing $(\theta+\frac{1}{2}q\tau)$ by θ to obtain

$$D_1=C_2C_5\cos^2\lambda+C_1C_6\sin^2\lambda,$$

$$D_2=-a(C_5C_6)^{-\frac{1}{2}}(C_5C_7\cos\theta\cos\lambda+C_6C_8\sin\theta\sin\lambda),$$

$$D_3=a^2(C_5C_6)^{-1}[C_2{}^{-1}C_5(C_7{}^2+C_6\cos^2(\tfrac{1}{2}q\tau)\cos^2\theta$$
$$+C_1{}^{-1}C_6(C_8{}^2+C_5\sin^2(\tfrac{1}{2}q\tau))\sin^2\theta],$$

$$|M|=C_5C_6.$$

By integrating the Equation (3.12) with respect to r we finally write (Cobb)

$$W(+,-)=(2\pi)^{-3}|M|^{3/2}\int_{-\pi}^{\pi}d\theta\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi}d\lambda D_1{}^{-2}\cos(2\lambda)[(1+D_4{}^2)\exp(-D_3)$$
$$-\pi^{\frac{1}{2}}D_4(3/2+D_4{}^2)(1-\mathrm{erf}D_4)\exp(-D_5)], \qquad (3.13)$$

where

$$D_4=D_2/D_1{}^{\frac{1}{2}},$$

$$D_5=D_3-D_4{}^2.$$

Zero-crossing Interval Density Function for Sine Wave Plus Noise

The zero-crossing interval density function for sine wave plus noise is computed according to the Equation (3.1)

$$p(0;\tau)=W(+,-)/W(+)$$

with the same noise characteristics as before. Several of these density functions are plotted in Figures 3-8. STOSN, SINO, and COREK are the programs used to compute $p(0;\tau)$. The computation of $W(+,-)$ is carried out by SINO according to the Equation (3.13). The program COREK computes Equation (3.6) to evaluate $W(+)$ for the signal-plus-noise case. The data computed by SINO were stored in the computer operating system by the program STOSN, and later retrieved by COREK to complete the computation.

The second term in Equation (1.8)

$$\int_{t_1<t_2<t_3} dt_2[W(+,-,-)/W(+)] \qquad (3.14)$$

represents the probability density of there being at least one down-crossing between an up-crossing and a down-crossing separated by an interval $\tau=t_3-t_1$. When the amplitude of the signal is high, the zero-crossing intervals of the signal plus noise are close to the half period of the signal wave, $T_h=\pi/q$, and the probability that the zero-crossing intervals are much longer than T_h is very low. In other words, if τ is larger than T_h, the probability of having at least one down-crossing within the interval τ is very high.

The zero-crossing interval density function in Equation

(3.1) therefore contains, with the omission of the second term in Equation (1.8), erroneous distribution near $\tau=3T_h$. When the signal radian frequency q is 0.6 or higher, the erroneous distribution falls within the interval (0,15) of τ. These erroneous data are discarded, and the main distribution curves are extended smoothly to vanish at $\tau=15$ by the program DPOL.

The zero-crossing interval density function (3.1) is a good approximation if the ratio of sine wave amplitude to rms noise is large. In the cases when the ratio a=1 or 2, the error in the approximation is fairly high. Instead of computing the density functions for these cases directly from the Equation (3.1), we interpolated the data by the program SAPOL from the density functions for a=0,3,4 and 5 using a spline interpolating function (Appendix D). The zero-crossing interval density function for a=0 is just the density function of noise alone.

The program INPOL expands the data points of the density function by locally interpolating with a spline function. The area under the computed zero-crossing interval density function usually has a slight deviation (±5%) from 1, and this is corrected by normalization.

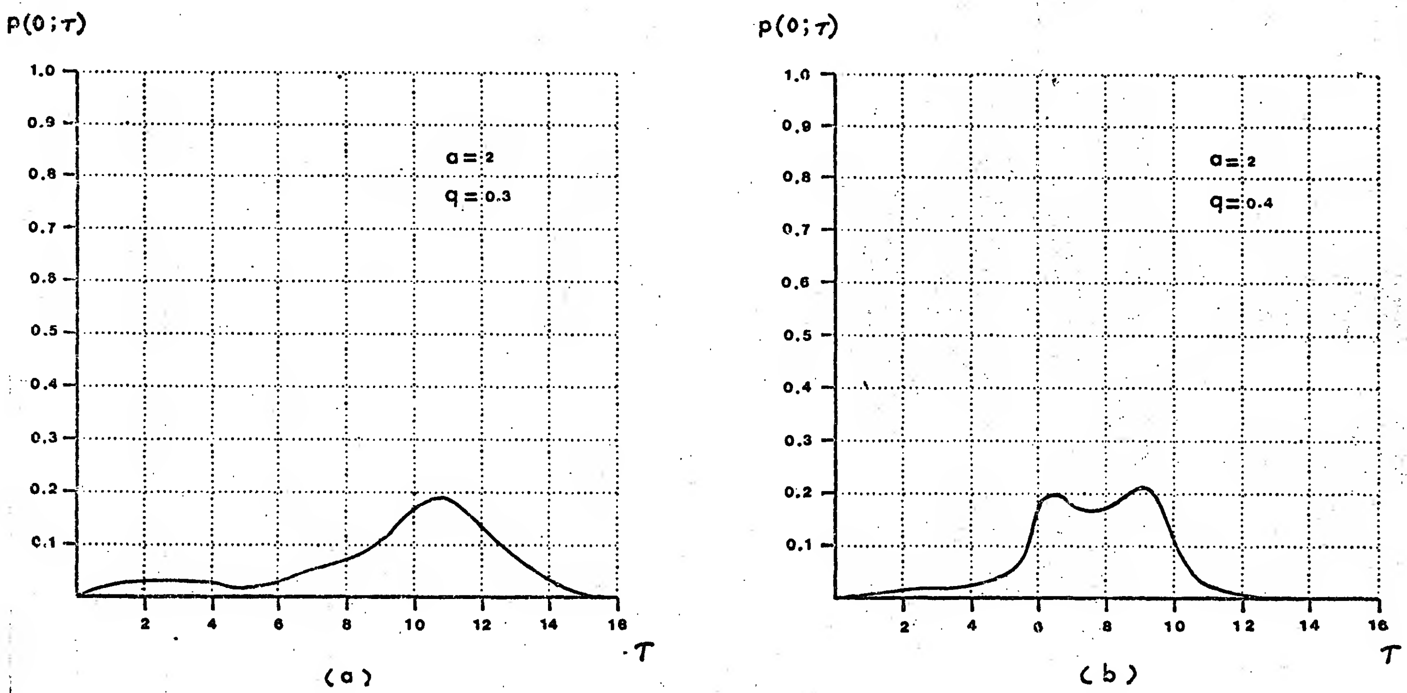

p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a = 2
q = 0.3
2
4
6
8
10
12
14
16
τ
(a)
p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a = 2
q = 0.4
2
4
6
8
10
12
14
16
τ
(b)

Figure 5.

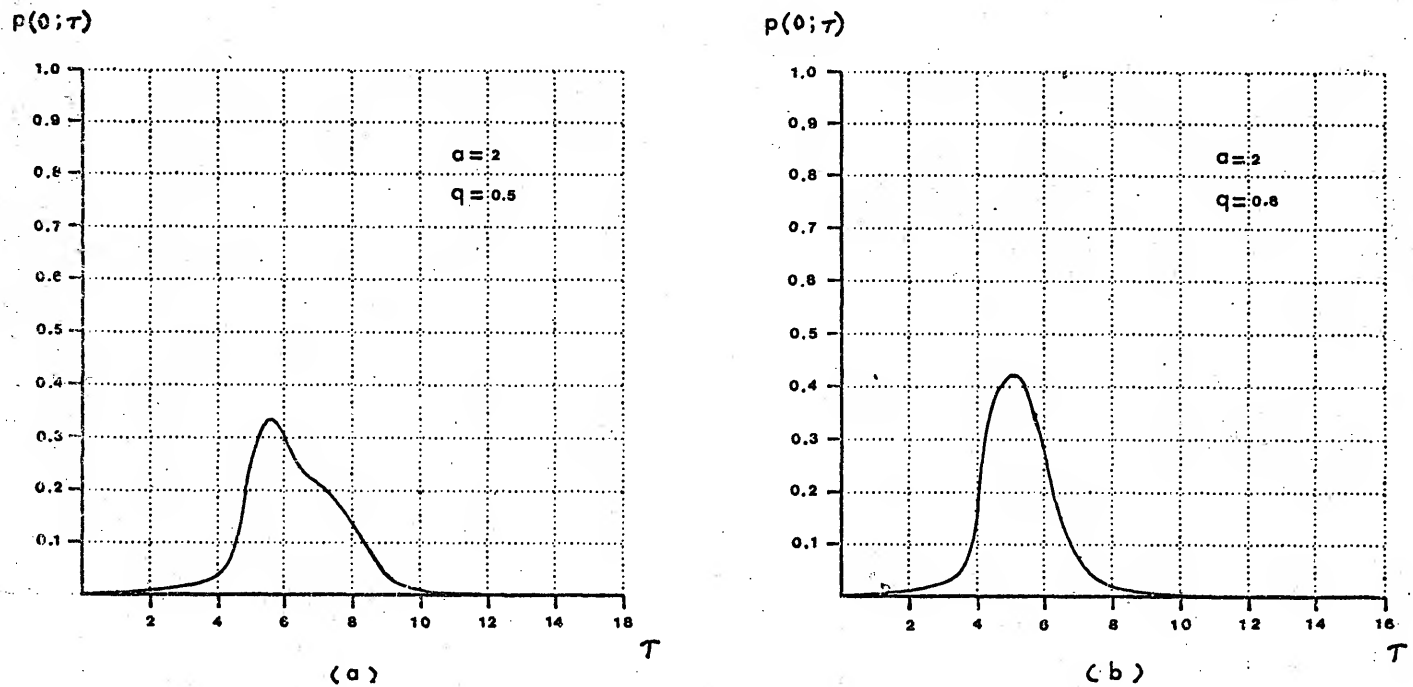


Figure 6.

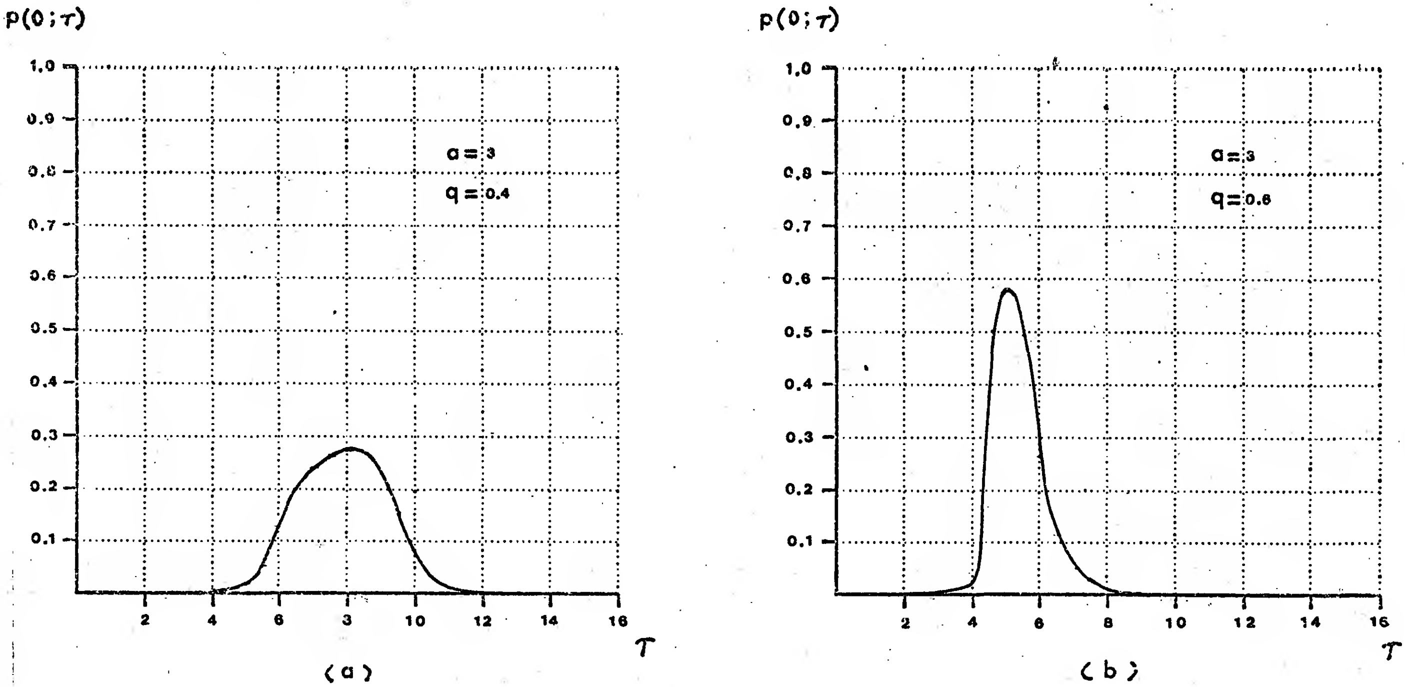

p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
3
10
12
14
16
τ
a=3
q=0.4
(a)
p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
8
10
12
14
16
τ
a=3
q=0.6
(b)

Figure 7.

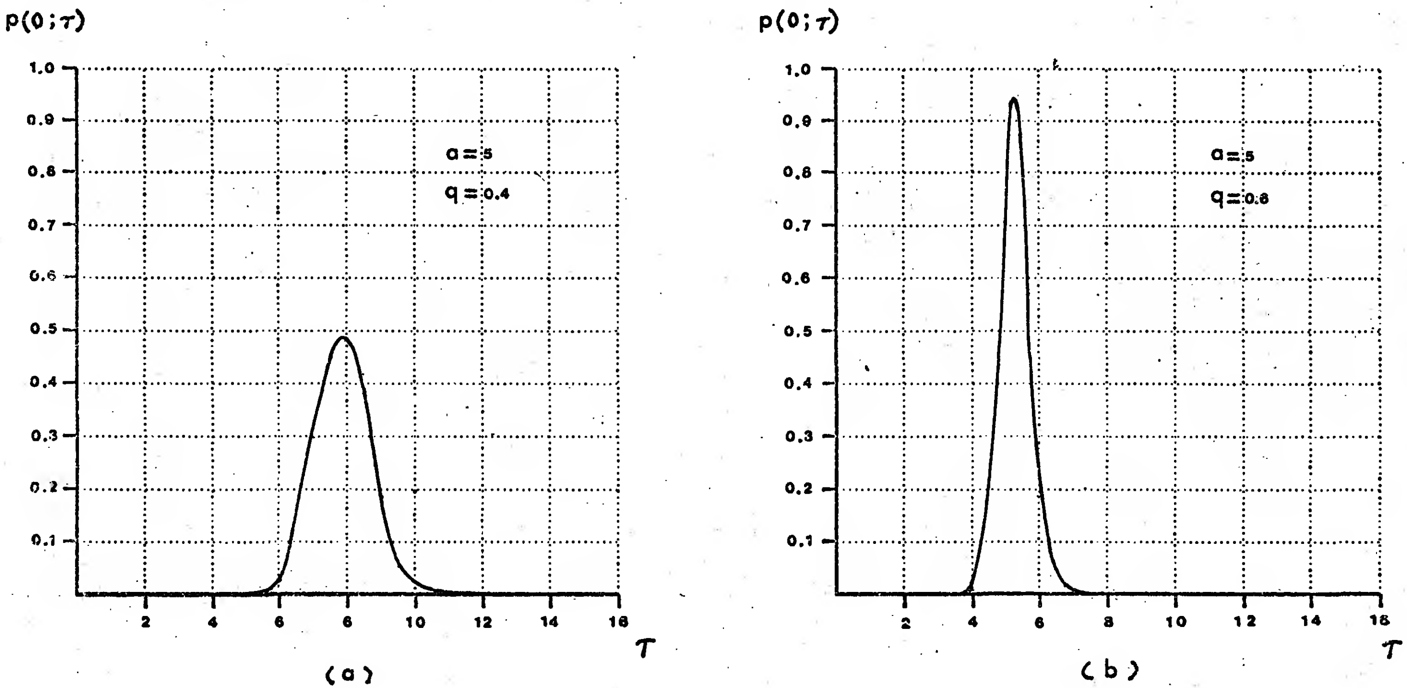

p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
8
10
12
14
16
τ
a = 5
q = 0.4
(a)
p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
8
10
12
14
16
τ
a = 5
q = 0.8
(b)

Figure 8.

CHAPTER III
IMPLEMENTATION AND EVALUATION OF THE DETECTOR

Implementation of the Detector

In implementing the detector it is assumed that the signal-to-noise ratio and the length of a signal are known to us. We also assume that the signal-to-noise ratio is fairly large so that the additive noise on a signal only disturbs the zero-crossing interval determined by the signal frequency and does not add more zero-crossings in the process. Then we know the number of zero-crossings needed to observe the signal in full length.

The schematic of the detector is shown in Figure 9. Zero-crossings of $f(t)$, both up-crossings and down-crossings, are first detected and the output pulse of the zero-crossing detector is used to strobe the system. The intervals between zero-crossings are measured and compared to predetermined thresholds. The output Z_i of this comparator has values such as

$$Z_i = \begin{cases} 1, & \text{if } T_0 < T_i < T_1, \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

where

T_0=the lower limit of the threshold of the zero-crossing interval comparator,

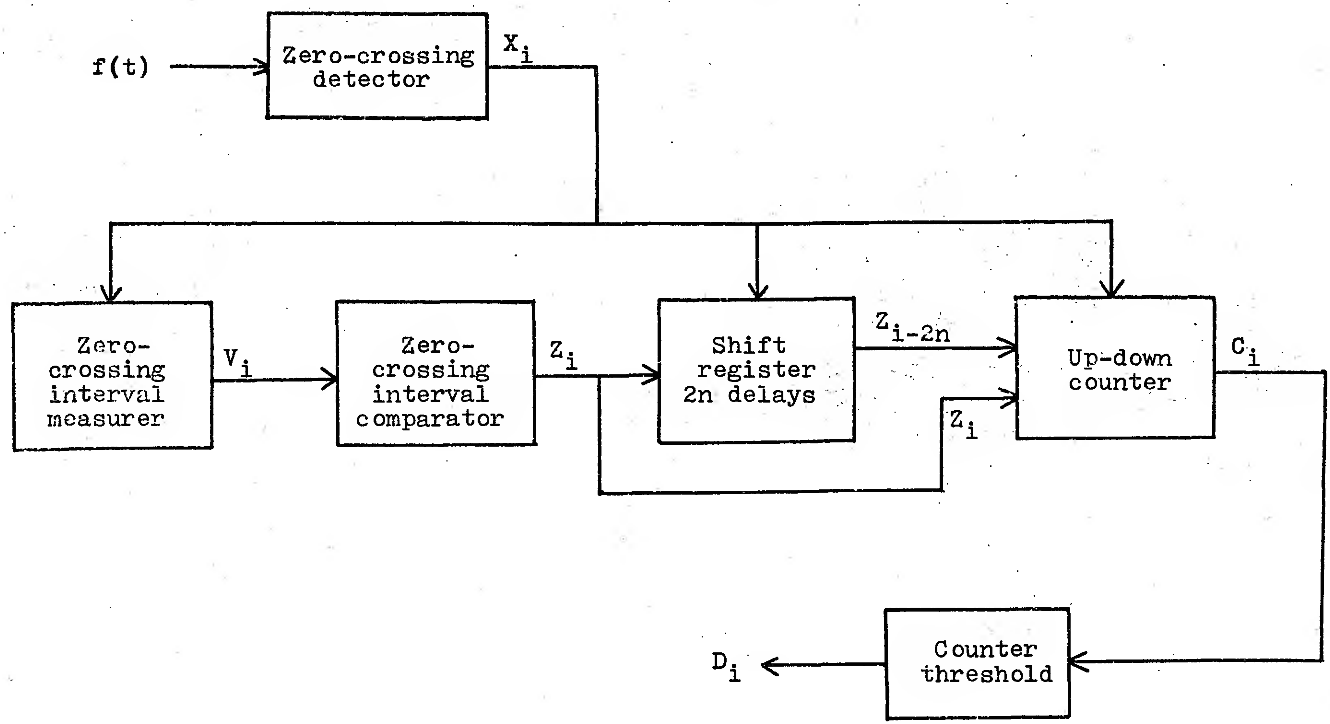

f(t)
Zero-crossing detector
X_i
Zero-crossing interval measurer
V_i
Zero-crossing interval comparator
Z_i
Shift register 2n delays
Z_{i-2n}
Z_i
Up-down counter
C_i
Counter threshold
D_i

Figure 9.

T_1=the higher limit of the threshold of the zero-crossing interval comparator.

The outputs of the zero-crossing interval comparator are stored in a shift register and the content of the shift register is counted by an up-down counter. If the signal s(t) is composed of n cycles of sinusoidal waves, then there are 2n zero-crossings generated by the signal. To observe the signal in full length, the counter sums up the content of the shift register in such a way that

$$C_i = \begin{cases} 0, & \text{if } i=0, \\ C_{i-1}, & \text{if } Z_i = Z_{i-2n}, \\ C_{i-1}+1, & \text{if } Z_i=1 \text{ and } Z_{i-2n}=0, \\ C_{i-1}-1, & \text{if } Z_i=0 \text{ and } Z_{i-2n}=1, \end{cases} \tag{4.2}$$

where $Z_i=0$ for $i \leqslant 0$. Therefore the value of C_i at any moment is

$$C_i = \sum_{k=i-2n}^{i} Z_k \tag{4.3}$$

and this is compared to a threshold to determine the presence of a signal. Thus if we set the threshold to m, then the output of the comparator is

$$D_i = \begin{cases} 1, & \text{if } m \leqslant C_i \leqslant 2n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Let us denote that

H_1=the hypothesis that a signal is present,

H_0=the hypothesis that a signal is absent,

and for given a and q of the signal,

$P_s=P(Z_i=1|H_1)$=the probability that when a signal is present T_i lies within the interval (T_0,T_1),

$Q_s=P(Z_i=0|H_1)$=the probability that when a signal is present T_i lies outside the interval (T_0,T_1),

$P_n=P(Z_i=1|H_0)$=the probability that when a signal is absent T_i lies within the interval (T_0,T_1),

$Q_n=P(Z_i=0|H_0)$=the probability that when a signal is absent T_i lies outside the interval (T_0,T_1).

Then

$$P_s+Q_s=1,$$

$$P_n+Q_n=1,$$

and P_s corresponds to the area under the curve bounded by $T=T_0$ and T_1 in Figure 10(b), and P_n corresponds to the area under the curve bounded by the same limits in Figure 10(a).

To simplify the analysis, the detection probability of a signal is approximated by the detection probability determined by comparing threshold at the end of the signal. This is the lower bound of the detection probability and is given by

$$P_d=P(D_i=1|H_1)=P(C_i\geq m|H_1)$$

$$=\sum_{k=m}^{2n}\binom{2n}{k}P_s^{\,k}Q_s^{\,2n-k}. \tag{4.5}$$

The corresponding false alarm rate is

$$P_f=P(D_i=1|H_0)=P(C_i\geq m|H_0)$$

$$=\sum_{k=m}^{2n}\binom{2n}{k}P_n^{\,k}Q_n^{\,2n-k}. \tag{4.6}$$

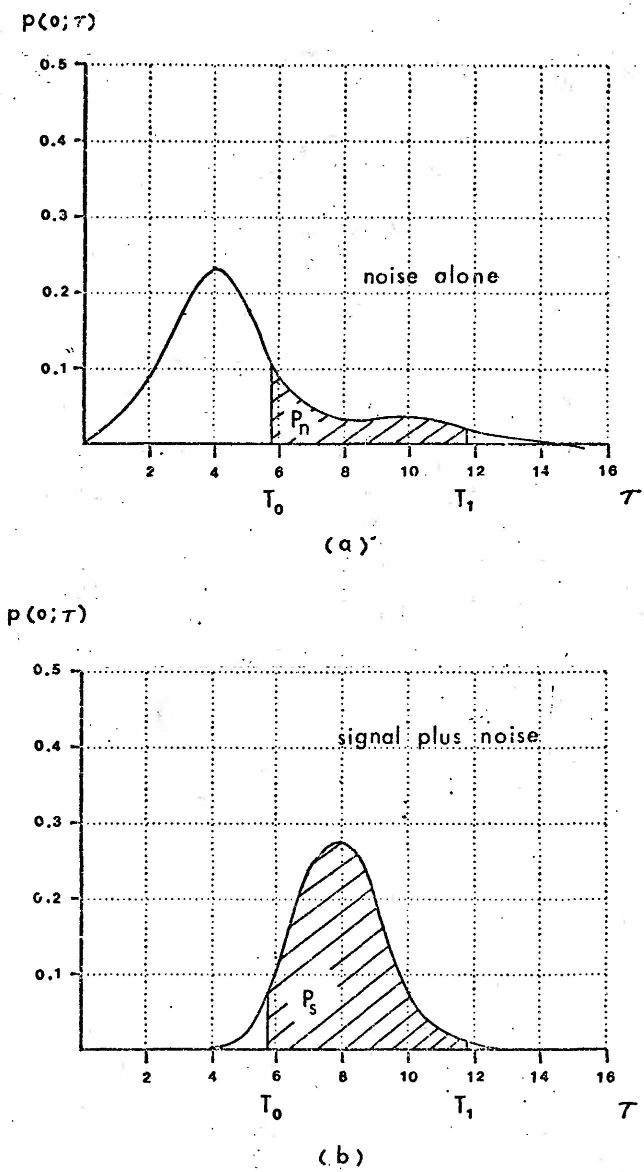

p(o;τ)
0.5
0.4
0.3
0.2
0.1
noise alone
P_n
2
4
6
8
10
12
14
16
T_0
T_1
τ
(a)
p(o;τ)
0.5
0.4
0.3
0.2
0.1
signal plus noise
P_s
2
4
6
8
10
12
14
16
T_0
T_1
τ
(b)

Figure 10.

Now the objective is to maximize the detection probability P_d for a given false alarm probability P_f. This strategy is known as the Neyman-Pearson criterion of decision making, which does not involve either a priori probability or cost estimates (Whalen).

Let $r=P_s/P_n$. Then

$$P_d=\sum_{k=m}^{2n}\binom{2n}{k}(rP_n)^k(1-rP_n)^{2n-k}. \tag{4.7}$$

The binomial expansion

$$f(n,m;x)=\sum_{k=m}^{n}\binom{n}{k}x^k(1-x)^{n-k} \tag{4.8}$$

is related to the incomplete beta function by

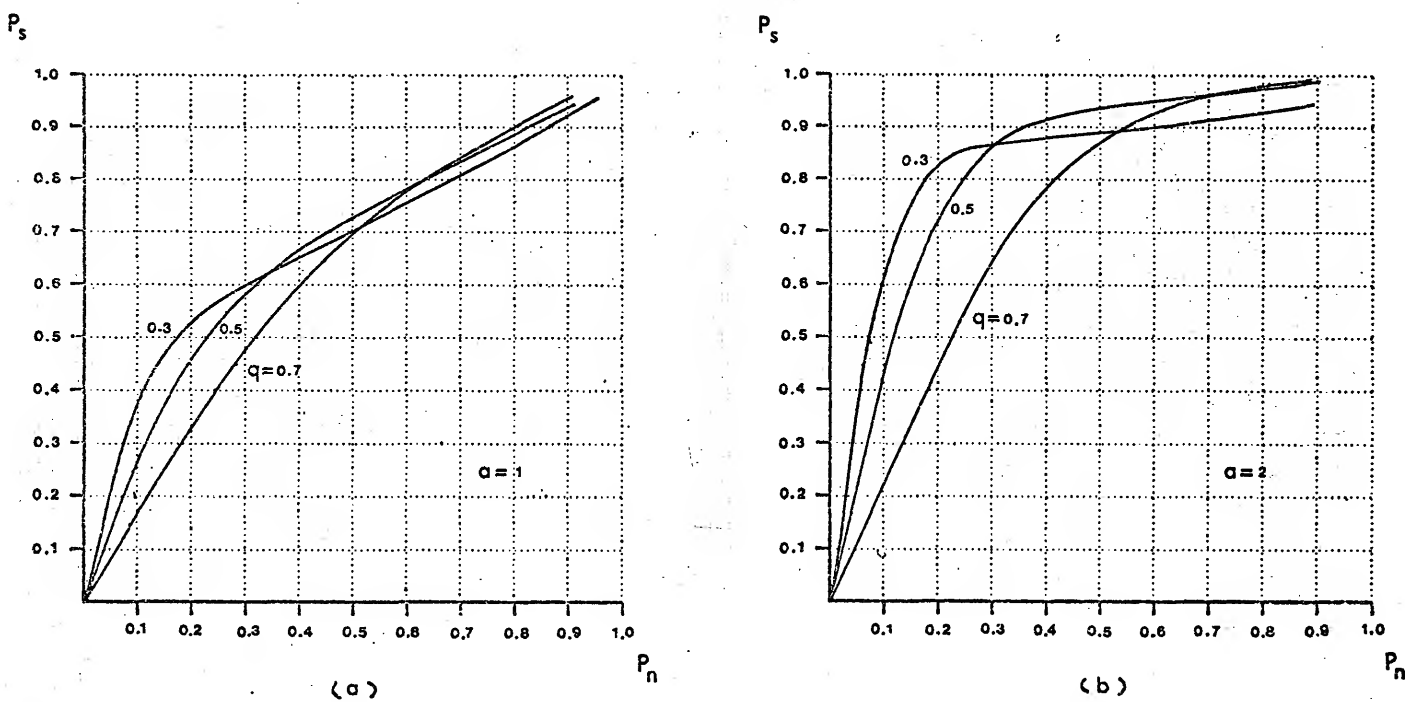
$$f(n,m;x)=\left[\int_0^x dt\,t^{m-1}(1-t)^{n-m}\right]/\left[\int_0^1 dt\,t^{m-1}(1-t)^{n-m}\right] \tag{4.9}$$

(Abramowitz). Since the right side of the above equation increases monotonically with respect to x for given values of n and m, the binomial expansion on the left side also monotonically increases with respect to x. Then we see that, for a signal with parameters a, q and n, P_d in Equation (4.7) is maximized by maximizing the ratio r for given m and P_n, and thus P_f. The maximization of r is done by the program PCOM3.

PCOM3 integrates the zero-crossing interval density function of noise from $\tau=0$ until the integrated area equals P_n, and the zero-crossing interval density function of signal plus noise is also integrated for the same interval to obtain the corresponding P_s. This integration procedure is repeated with an incremented lower limit of integration until the range of τ is exhausted. Among the P_s's obtained by such integrations, the largest value is picked out as the maximized P_s for a given P_n. The integration interval corresponding to the maximized P_s is then the optimum threshold for zero-crossing interval as in Equation (4.1).

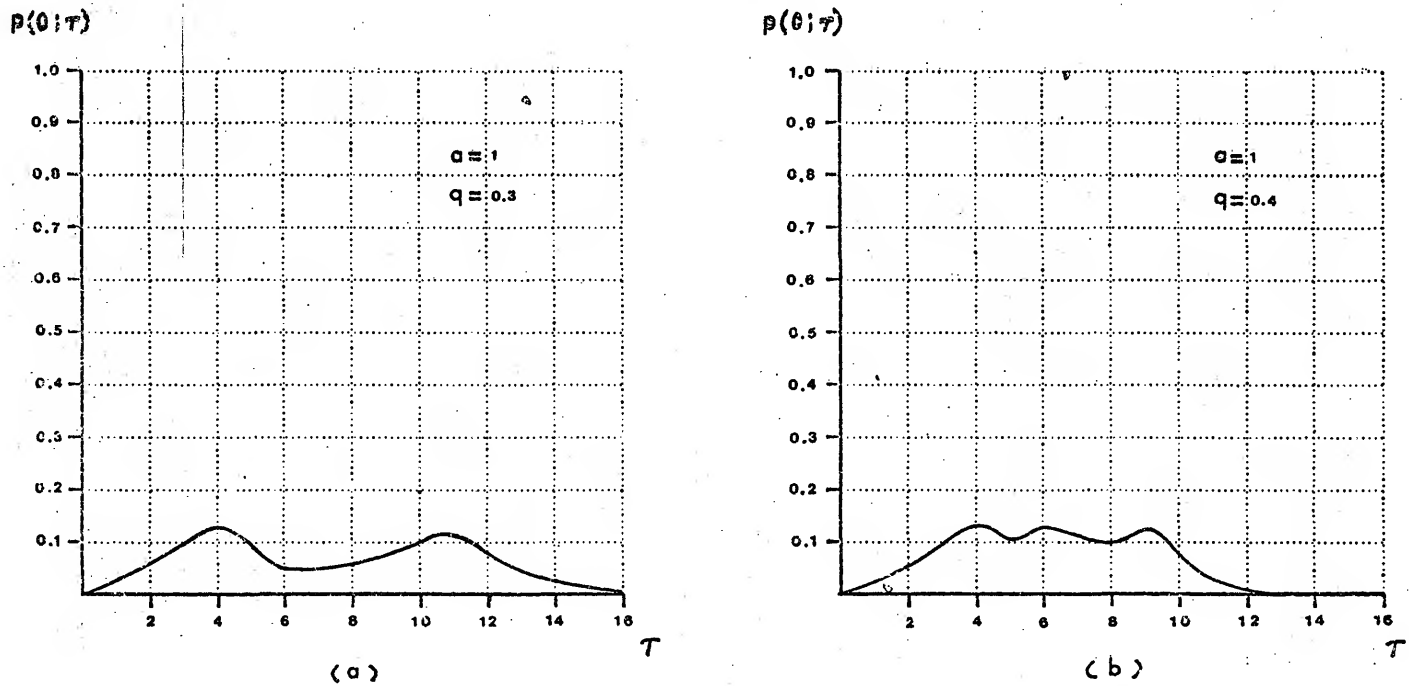
Figures 11 through 14 show the relations between P_n and the maximized P_s for different values of parameters. In the Figures 13 and 14, P_n is plotted against q with fixed parameters P_s and a. From these graphs we observe that q=0.4 in general gives a large value to the ratio $r=P_s/P_n$.

The final step of maximizing the detection probability P_d for given P_f is the determination of the threshold m. The program RCOM computes P_d and P_f for different values of P_n and m according to the Equations (4.7) and (4.6) respectively as shown in Figures 15-19. From these graphs the optimum value for m could be determined to give the maximum P_d for given P_f of a signal with given parameters. For instance, the optimum value of m for a signal with parameters a=2, q=0.4 and n=4 to be detected with maximum probability for a given false alarm probability $P_f=10^{-2}$ is 6, while 7 is the optimum for $P_f=10^{-4}$.



P_s
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.3
0.5
q=0.7
a=1
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
P_n
(a)
P_s
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.3
0.5
q=0.7
a=2
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
P_n
(b)

Figure 11.



p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a = 1
q = 0.3
2
4
6
8
10
12
14
16
τ
(a)
p(0;τ)
a = 1
q = 0.4
(b)

Figure 3.

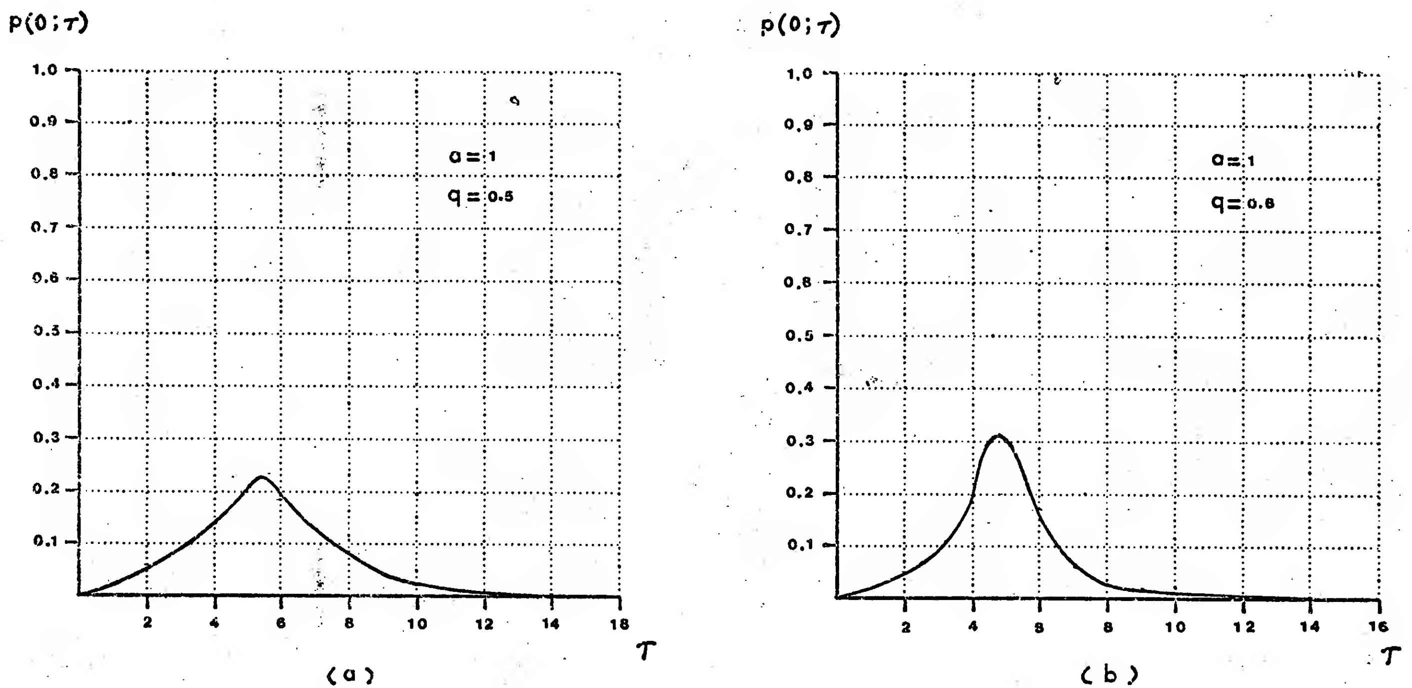
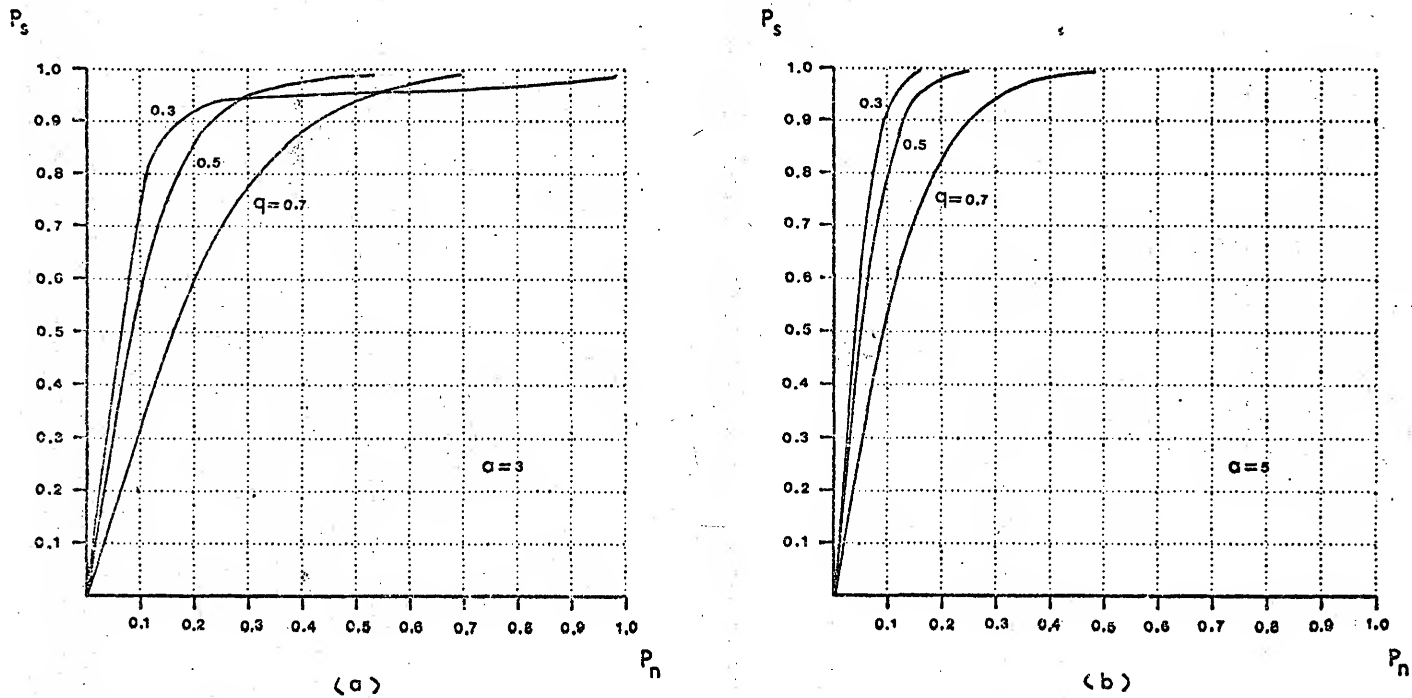

p(0;τ)
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
2
4
6
8
10
12
14
18
τ
a=1
q=0.5
(a)
p(0;τ)
a=1
q=0.8
16
τ
(b)

Figure 4.



P_s
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.3
0.5
q=0.7
a=3
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
P_n
(a)
P_s
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.3
0.5
q=0.7
a=5
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
P_n
(b)

Figure 12.

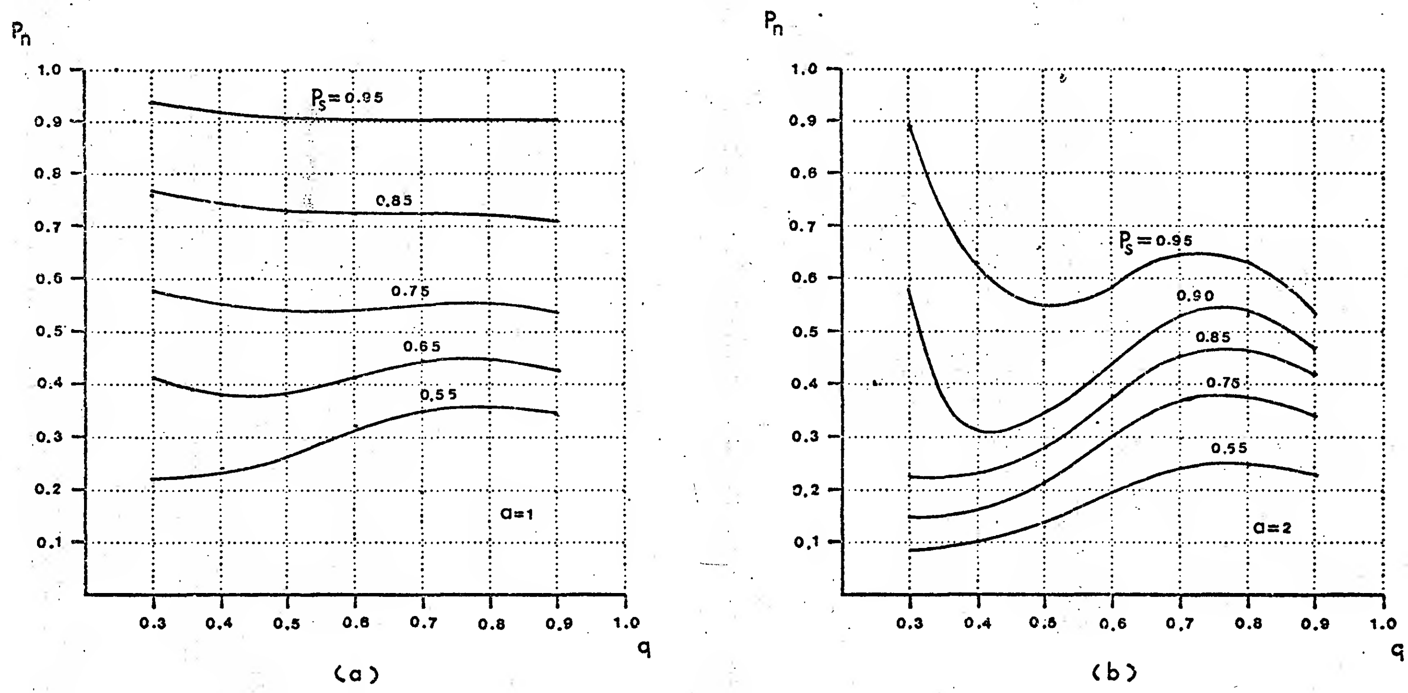
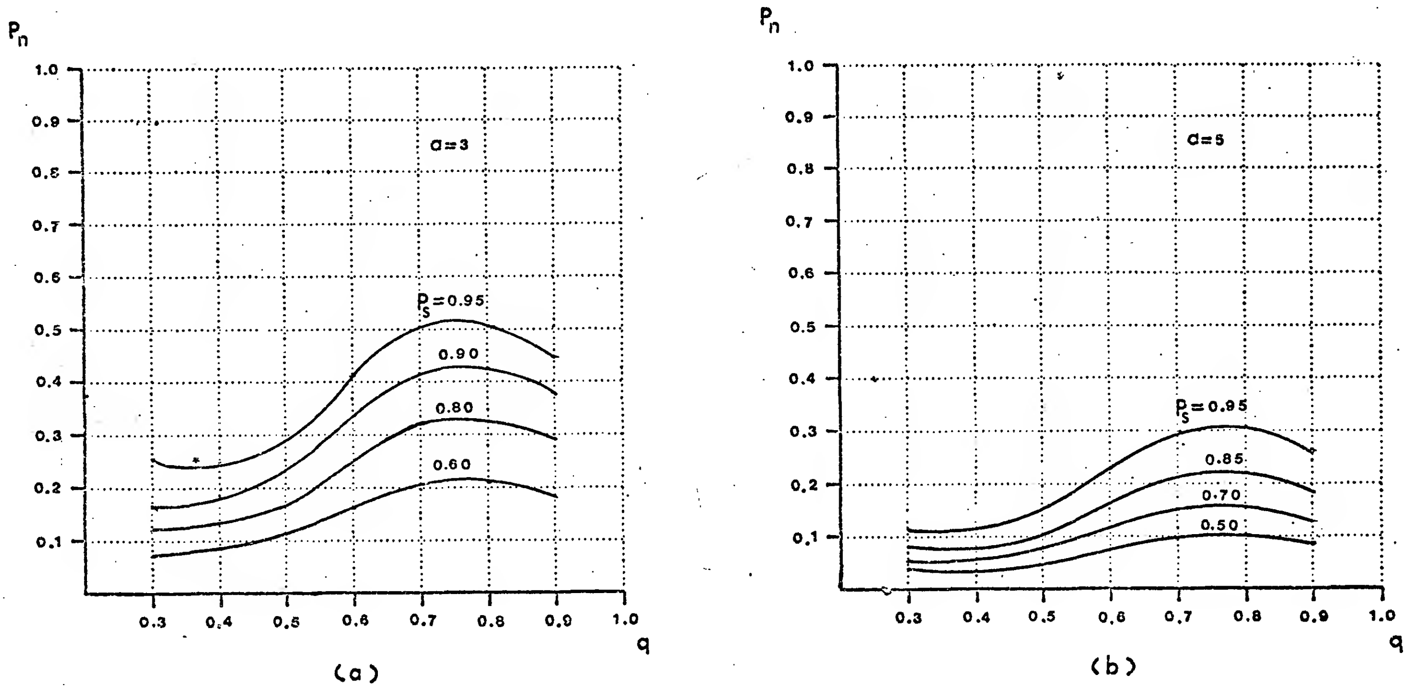
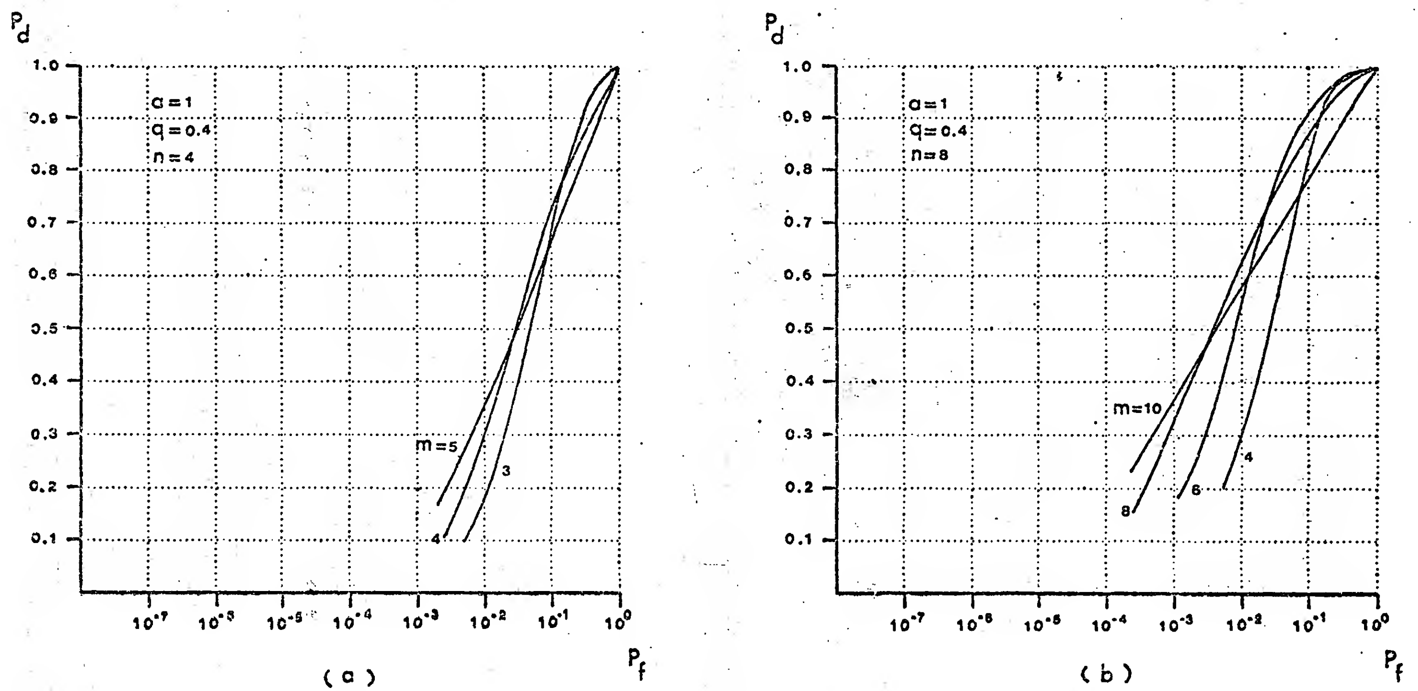

P_n
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
q
$P_s = 0.95$
0.85
0.75
0.65
0.55
a=1
(a)
P_n
$P_s = 0.95$
0.90
0.85
0.75
0.55
a=2
(b)

Figure 13.



P_n
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a=3
P_s=0.95
0.90
0.80
0.60
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
q
(a)
P_n
a=5
P_s=0.95
0.85
0.70
0.50
(b)

Fibure 14.



Pd
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a=1
q=0.4
n=4
m=5
3
4
10^-7 10^-6 10^-5 10^-4 10^-3 10^-2 10^-1 10^0
Pf
(a)
Pd
a=1
q=0.4
n=8
m=10
4
6
8
(b)
Pf

Figure 15.

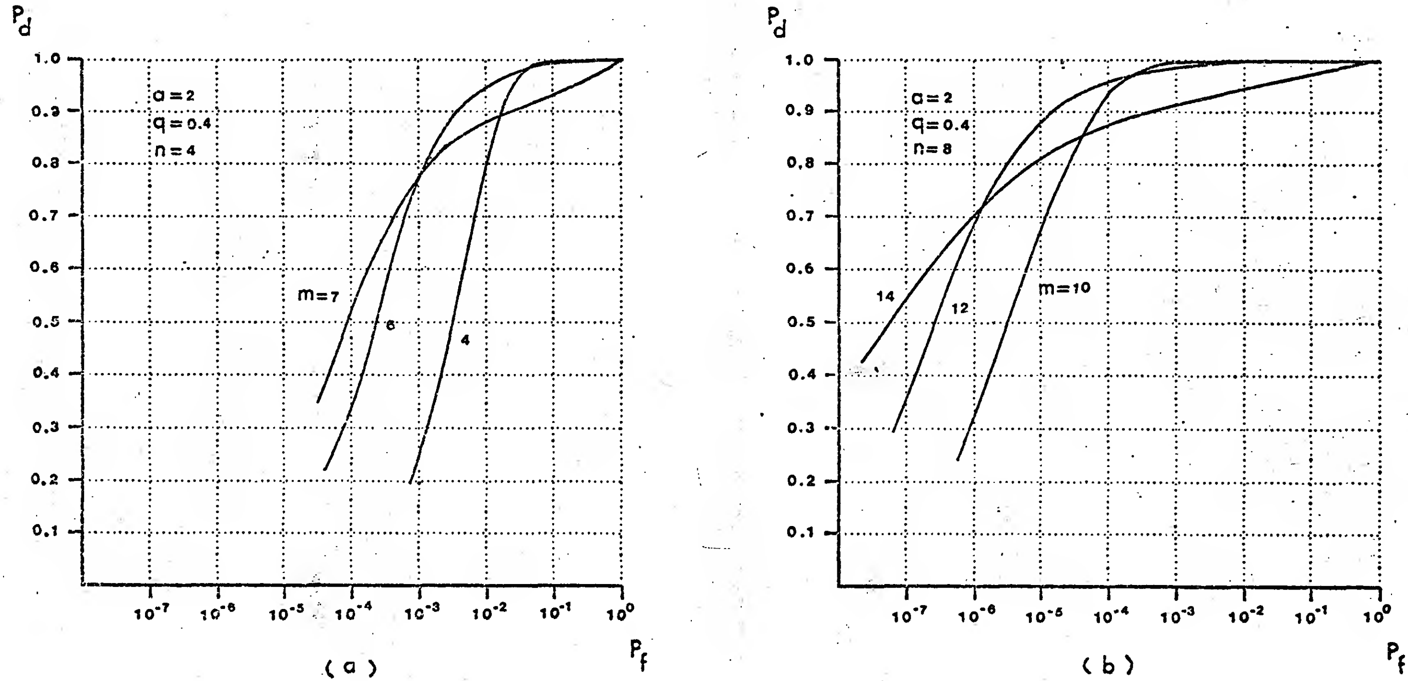
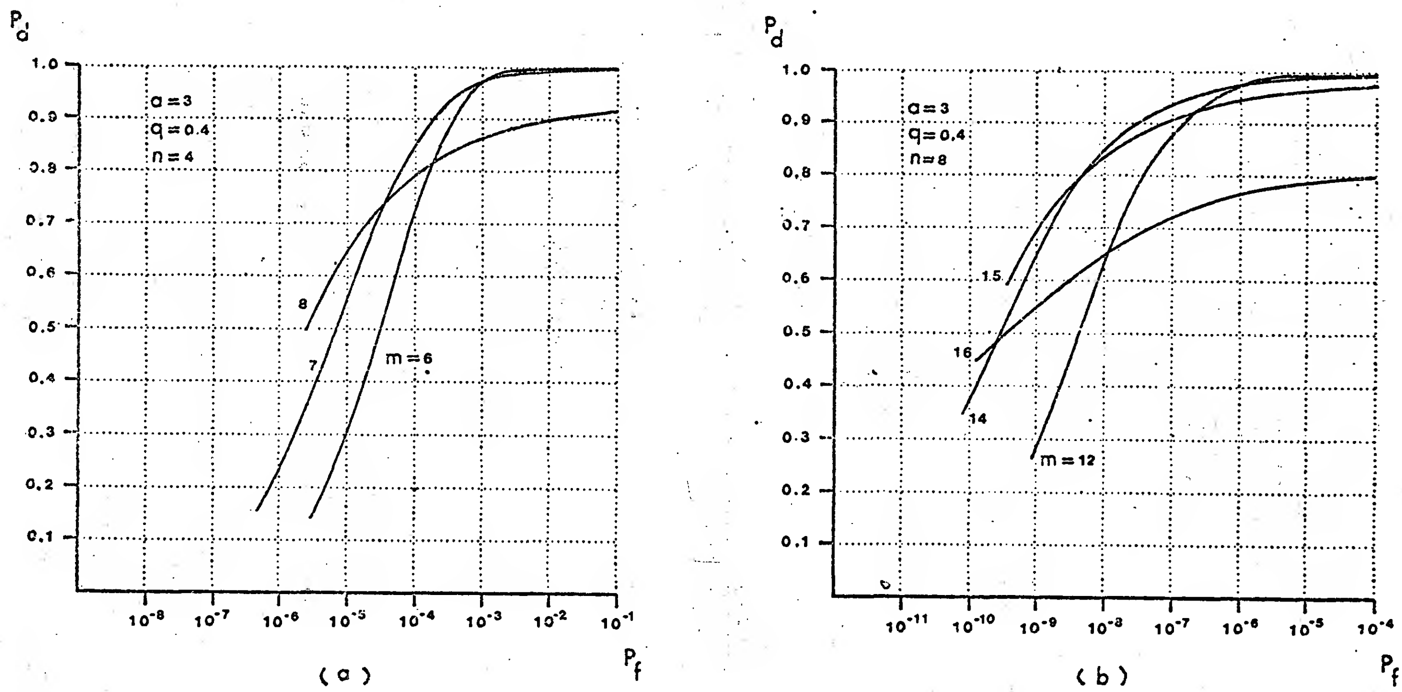
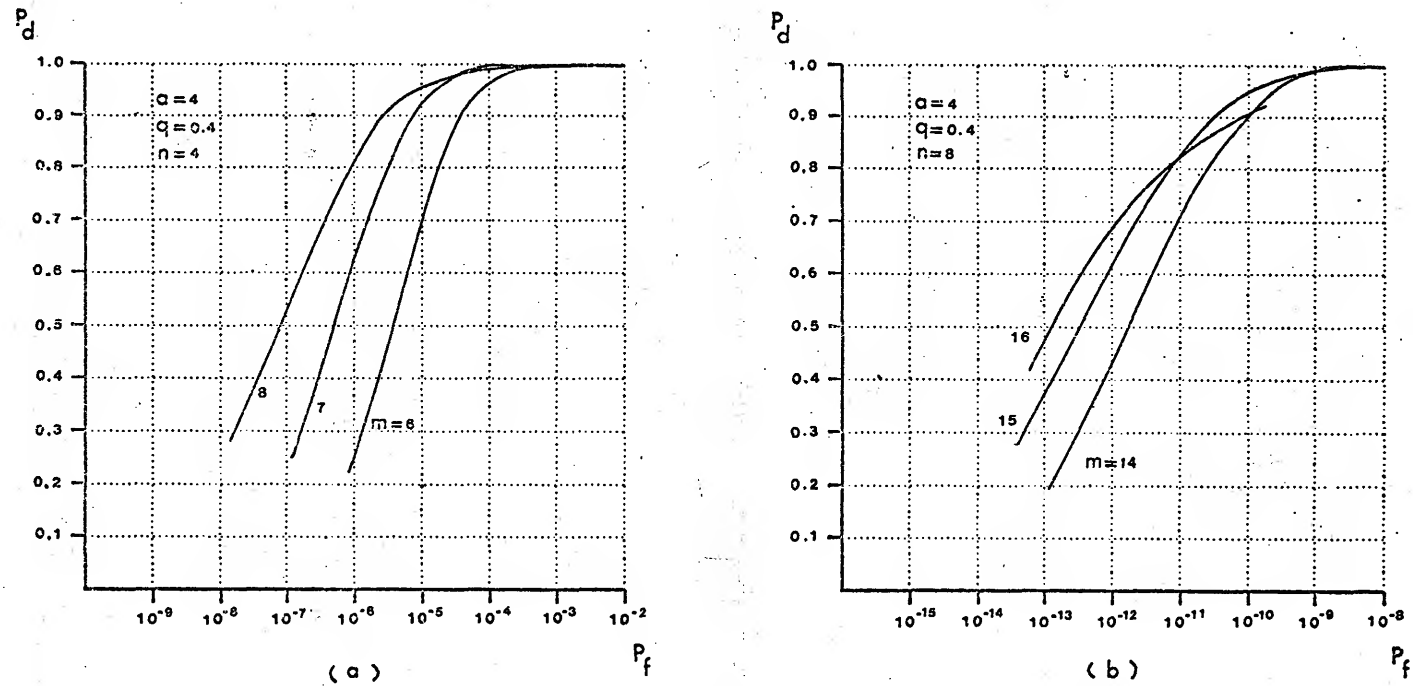

(a)
P_d
P_f
a = 2
q = 0.4
n = 4
m=7
6
4
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^{0}
(b)
P_d
P_f
a = 2
q = 0.4
n = 8
m=10
12
14
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 10^{0}

Figure 16.



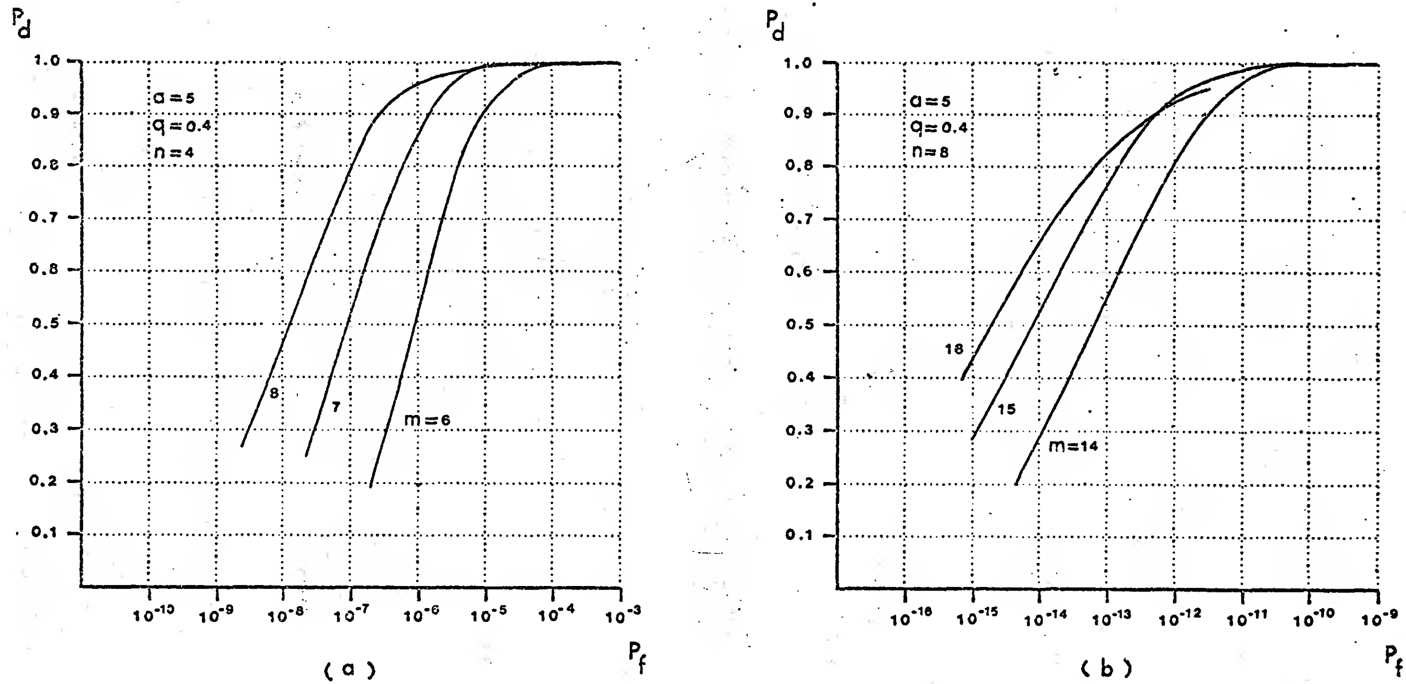
P_d
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a = 3
q = 0.4
n = 4
8
7
m = 6
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1
P_f
(a)
P_d
a = 3
q = 0.4
n = 8
15
16
14
m = 12
10^-11
10^-10
10^-9
10^-8
10^-7
10^-6
10^-5
10^-4
P_f
(b)

Figure 17.



P_d
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a=4
q=0.4
n=4
8
7
m=6
10^-9
10^-8
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
P_f
(a)
P_d
a=4
q=0.4
n=8
16
15
m=14
10^-15
10^-14
10^-13
10^-12
10^-11
10^-10
10^-9
10^-8
P_f
(b)

Figure 18.



(a)
(b)
P_d
P_f
a=5
q=0.4
n=4
n=8
m=6
7
8
m=14
15
18
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
10^{-10}
10^{-9}
10^{-8}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-16}
10^{-15}
10^{-14}
10^{-13}
10^{-12}
10^{-11}

Figure 19.

Comparison to Correlation Detector

The performance of the detector is compared to the conventional correlation detector. Suppose that the parameters of the signal s(t) are completely known to us, and let H_1 and H_0 denote the hypotheses that the signal is present and absent respectively. The correlation detector chooses H_1 if

$$\int_0^T f(t)s(t)dt \geqslant V, \tag{5.1}$$

where the interval (0,T) is the duration in which the signals are received and V is a threshold. The implementation of the correlation detector is shown in Figure 20.

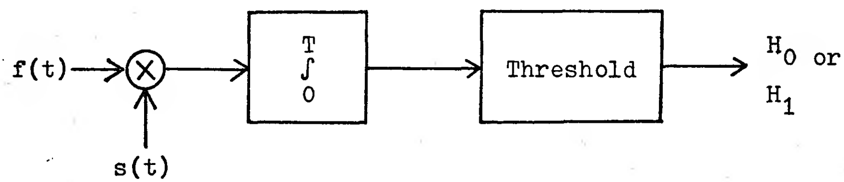


Figure 20.

A correlation detector is equivalent to a matched filter followed by a threshold comparator (Whalen). This equivalent detector which implements the operation of the correlation detector using a matched filter with impulse response $h(t)$ is shown in Figure 21.

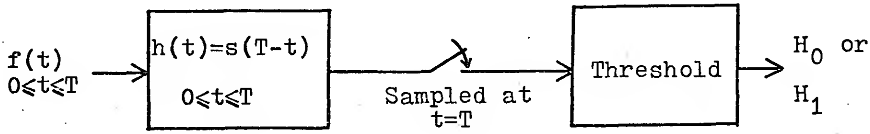


Figure 21.

The decision rule in Equation (5.1) stands on the assumption that the noise is white Gaussian. The threshold V is determined to meet an employed detection criterion for the particular application. To make the environment similar to our previous analysis in analyzing the correlation detector, we assume that the noise is bandlimited white Gaussian with power spectral density

$$S(w)=\begin{cases} \pi, & \text{for } |w|\leqslant 1, \\ 0, & \text{otherwise} \end{cases} \tag{5.2}$$

and autocorrelation function

$$R(\tau)=\tau^{-1}\sin\tau . \tag{5.3}$$

The Neyman-Pearson criterion, which involves neither a priori probabilities nor cost estimates, is also employed here. The strategy of this criterion is to maximize the detection probability for a given false alarm probability, and this could be accomplished by using a likelihood ratio test (Whalen). We approach the analysis with the case where the signal is sampled at m discrete times.

Let us denote the m sampled values by

$$f_k = s_{ik} + n_k, \quad (k=1,\ldots,m) \quad (i=0,1), \tag{5.4}$$

where

$$s_{0k}=0, \quad s_{1k}=s(t_k), \quad n_k=n(t_k), \quad f_k=f(t_k),$$

and t_k $(k=1,\ldots,m)$ are sampling times.

If a noise is sampled at m discrete points, the covariance matrix $M=(m_{ij})$ of the noise has the elements

$$m_{ij}=E[n_i n_j]=R(t_j-t_i) \quad (i,j=1,\ldots,m). \tag{5.5}$$

Since the noise is Gaussian, the joint probability density of the m sampled values of noise is

$$\begin{aligned} p(n) &= p(n_1,n_2,\ldots,n_m) \\ &= (2\pi)^{-\frac{1}{2}m}|M|^{-\frac{1}{2}}\exp(-\tfrac{1}{2}X^T M^{-1} X), \end{aligned} \tag{5.6}$$

where X is the column vector of sampled values. By writing

$$Q=M^{-1}=(q_{ij}) \quad (i,j=1,\ldots,m),$$

the Equation (5.6) becomes

$$p(n)=(2\pi)^{-\frac{1}{2}m}|M|^{-\frac{1}{2}}\exp\Big(-\tfrac{1}{2}\sum_{i,j=1}^{m} q_{ij} n_i n_j\Big). \tag{5.7}$$

Since $n_k=f_k-s_{hk}$ $(h=0,1)$, the above equation could be written as

$$p(n)=p(f-s_h)$$
$$=(2\pi)^{-\frac{1}{2}m}|M|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}\sum_{i,j=1}^{m}q_{ij}(f_i-s_{hi})(f_j-s_{hj})].$$

Then, when the signal is absent in $f(t)$, $h=0$ and the joint probability density of m sampled values of (f_k-s_{0k}) $(k=1,\ldots,m)$ is

$$p_0(f)=p(f-s_0)=p(n)$$
$$=(2\pi)^{-\frac{1}{2}m}|M|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}\sum_{i,j=1}^{m}q_{ij}(f_i-s_{0i})(f_j-s_{0j})]. \tag{5.8}$$

If the signal is present in $f(t)$, then $h=1$ and the corresponding joint probability density of the m samples (f_k-s_{1k}) $(k=1,\ldots,m)$ is

$$p_1(f)=p(f-s_1)=p(n)$$
$$=(2\pi)^{-\frac{1}{2}m}|M|^{-\frac{1}{2}}\exp[-\tfrac{1}{2}\sum_{i,j=1}^{m}q_{ij}(f_i-s_{1i})(f_j-s_{1j})]. \tag{5.9}$$

The likelihood function for the hypothesis H_0 is the joint probability density function $p_0(f)=p_0(f_1,\ldots,f_m)$. Likewise $p_1(f_1,\ldots,f_m)$ represents the likelihood function for the hypothesis H_1. In terms of these likelihood functions, a likelihood ratio r is defined by

$$r=p_1(f_1,\ldots,f_m)/p_0(f_1,\ldots,f_m)$$
$$=\exp[-\tfrac{1}{2}\sum_{i,j=1}^{m}q_{ij}(f_i-s_{1i})(f_j-s_{1j})]/$$
$$\exp[-\tfrac{1}{2}\sum_{i,j=1}^{m}q_{ij}(f_i-s_{0i})(f_j-s_{0j})]$$

$$=\exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{m} q_{ij}(2f_i s_{0j}-2f_i s_{1j}-s_{0i}s_{0j}+s_{1i}s_{1j})\right], \tag{5.10}$$

and the likelihood ratio test is to choose

$$\begin{cases} H_1, & \text{if } r \geqslant r_0, \\ H_0, & \text{if } r < r_0, \end{cases} \tag{5.11}$$

where r_0 is the decision threshold.

If the noise is not white, the evaluation of the likelihood ratio r in Equation (5.10) involves a matrix inversion from the covariance matrix M. When the signal is sampled at a large number of points, this matrix inversion is not practical in considerations of speed and simplicity of the system. This is also the case with our bandlimited white noise, and we simplify the analysis by sampling f(t) at the time intervals for which the sampled noises are uncorrelated.

Since the autocorrelation function of the noise has zeros at $\tau=k\pi$ $(k=\pm1,\pm2,\ldots)$, if we sample f(t) at the intervals $\delta t=\pi$, the samples are uncorrelated; that is

$$E[n_i n_j]=R(t_j-t_i)=\begin{cases} R(0), & \text{if } i=j \\ 0, & \text{otherwise.} \end{cases} \tag{5.12}$$

Then, since, from Equation (5.3), R(0)=1, the elements of the matrix M in Equation (5.5) become

$$m_{ij}=E[n_i n_j]=\begin{cases} 1, & \text{if } i=j, \\ 0, & \text{otherwise,} \end{cases} \tag{5.13}$$

and thus,

$$M=I,$$

and

$$q_{ij}=\begin{cases}1, & \text{if } i=j,\\ 0, & \text{otherwise.}\end{cases} \tag{5.14}$$

Then the likelihood functions in Equation (5.8) and (5.9) simply become

$$p_0(f)=(2\pi)^{-\frac{1}{2}m}\exp\left[-\tfrac{1}{2}\sum_{k=1}^{m}(f_k-s_{0k})^2\right] \tag{5.15}$$

and

$$p_1(f)=(2\pi)^{-\frac{1}{2}m}\exp\left[-\tfrac{1}{2}\sum_{k=1}^{m}(f_k-s_{1k})^2\right]. \tag{5.16}$$

The likelihood ratio r in this case is

$$\begin{aligned} r(f)&=p_1(f)/p_0(f)\\ &=\exp\left[-\tfrac{1}{2}\sum_{k=1}^{m}(f_k-s_{1k})^2\right]/\exp\left[-\tfrac{1}{2}\sum_{k=1}^{m}(f_k-s_{0k})^2\right], \end{aligned} \tag{5.17}$$

or, upon combining terms,

$$r(f)=\exp\left[-\tfrac{1}{2}\sum_{k=1}^{m}(2f_ks_{0k}-2f_ks_{1k}-s_{0k}^{\ 2}+s_{1k}^{\ 2})\right]. \tag{5.18}$$

The decision rule is to choose H_1 if $r\geqslant r_0$, or equivalently, by taking natural logarithm, choose H_1 if

$$\sum_{k=1}^{m}(f_ks_{1k}-f_ks_{0k}+\tfrac{1}{2}s_{0k}^{\ 2}-\tfrac{1}{2}s_{1k}^{\ 2})\geqslant\ln(r_0). \tag{5.19}$$

Let

$$G=\sum_{k=1}^{m}(f_ks_{1k}-f_ks_{0k}+\tfrac{1}{2}s_{0k}^{\ 2}-\tfrac{1}{2}s_{1k}^{\ 2}). \tag{5.20}$$

To evaluate the performance of this detector we need to know the distribution of G under each hypothesis. When H_0 is true,

$f_k=n_k$ and the expectation of G is

$$E_0[G]=E[\sum_{k=1}^{m}(n_k s_{1k}-n_k s_{0k}+\tfrac{1}{2}s_{0k}^2-\tfrac{1}{2}s_{1k}^2)]. \qquad (5.21)$$

Since $s_{0k}=0$,

$$E_0[G]=-\tfrac{1}{2}\sum_{k=1}^{m}s_{1k}^2. \qquad (5.22)$$

The variance of G under H_0 is

$$V_0[G]=E[(G-E_0[G])^2]. \qquad (5.23)$$

But

$$G-E_0[G]=\sum_{k=1}^{m}n_k s_{1k},$$

so

$$V_0[G]=E[\sum_{k=1}^{m}\sum_{j=1}^{m}n_k n_j s_{1k}s_{1j}]$$

$$=\sum_{k=1}^{m}\sum_{j=1}^{m}E[n_k n_j]s_{1k}s_{1j}.$$

Since

$$E[n_k n_j]=\begin{cases}R(0)=1, & \text{if } k=j,\\ 0, & \text{otherwise,}\end{cases}$$

it becomes

$$V_0[G]=\sum_{k=1}^{m}s_{1k}^2. \qquad (5.24)$$

Define

$$S=\sum_{k=1}^{m}s_{1k}^2. \qquad (5.25)$$

Then,

$$E_0[G]=-\tfrac{1}{2}S,$$

$$V_0[G]=S.$$

Therefore the density function of G is

$$p_0(G)=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\exp[-\tfrac{1}{2}(G+\tfrac{1}{2}S)^2/S]. \tag{5.26}$$

In a similar fashion we obtain the density function of G under the hypothesis H_1. If the hypothesis H_1 is true, then $f_k=s_{1k}+n_k$ and

$$E_1[G]=E[\sum_{k=1}^{m}(s_{1k}+n_k)s_{1k}+\tfrac{1}{2}\sum_{k=1}^{m}(-s_{1k}^2)]$$

$$=\sum_{k=1}^{m}s_{1k}^2-\tfrac{1}{2}\sum_{k=1}^{m}s_{1k}^2$$

$$=\tfrac{1}{2}\sum_{k=1}^{m}s_{1k}^2. \tag{5.27}$$

$$G-E_1[G]=\sum_{k=1}^{m}(s_{1k}+n_k)s_{1k}+\tfrac{1}{2}\sum_{k=1}^{m}(-s_{1k}^2)-\tfrac{1}{2}\sum_{k=1}^{m}s_{1k}^2$$

$$=\sum_{k=1}^{m}n_k s_{1k}.$$

$$V_1[G]=E[(G-E_1[G])^2]$$

$$=E[\sum_{k=1}^{m}\sum_{j=1}^{m}n_k n_j s_{1k}s_{1j}]$$

$$=\sum_{k=1}^{m}\sum_{j=1}^{m}E[n_k n_j]s_{1k}s_{1j}$$

$$=\sum_{k=1}^{m}s_{1k}^2. \tag{5.28}$$

Thus

$$p_1(G)=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\exp[-\tfrac{1}{2}(G-\tfrac{1}{2}S)^2/S]. \tag{5.29}$$

Since we choose H_1 if $G \geq \ln(r_0)$, by writing $c=\ln(r_0)$, the false alarm probability is

$$P_f=P(D_1|H_0)=\int_c^\infty dG p_0(G)$$

$$=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\int_c^\infty dG\exp[-\tfrac{1}{2}(G+\tfrac{1}{2}S)^2/S].$$

By a change of variable $z=S^{-\frac{1}{2}}(G+\frac{1}{2}S)$,

$$P_f=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\int_{(c+\frac{1}{2}S)/S^{\frac{1}{2}}}^\infty dzS^{\frac{1}{2}}\exp(-\tfrac{1}{2}z^2)$$

$$=(2\pi)^{-\frac{1}{2}}\int_{(c+\frac{1}{2}S)/S^{\frac{1}{2}}}^\infty dz\exp(-\tfrac{1}{2}z^2). \qquad (5.30)$$

The probability of detection is

$$P_d=P(D_1|H_1)=\int_c^\infty dG p_1(G)$$

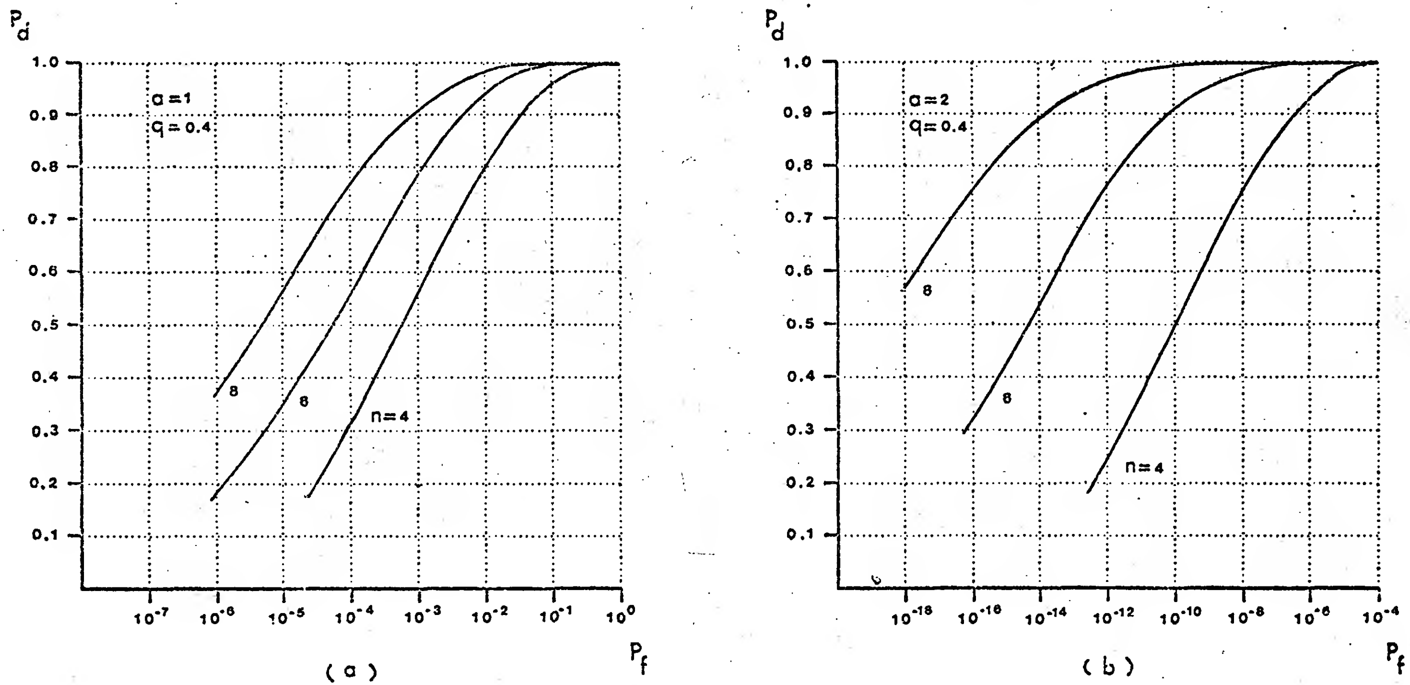
$$=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\int_c^\infty dG\exp[-\tfrac{1}{2}(G-\tfrac{1}{2}S)^2/S]$$

$$=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\int_{(c-\frac{1}{2}S)/S^{\frac{1}{2}}}^\infty dzS^{\frac{1}{2}}\exp(-\tfrac{1}{2}z^2)$$

$$=(2\pi)^{-\frac{1}{2}}\int_{(c-\frac{1}{2}S)/S^{\frac{1}{2}}}^\infty dz\exp(-\tfrac{1}{2}z^2) \qquad (5.31)$$

The performance of the correlation detector is computed by the program PLIN and shown in Figure 22. We note that the performance of the correlation detector is superior to the zero-crossing measuring detector in the assumed circumstances of the signal and background noise. For instance when a=2, q=0.4 and n=4, the false alarm probability of the correlation detector at $P_d=0.9$ is $4x10^{-7}$ while the false alarm probability of the zero-crossing measuring detector at the same detection rate is $3x10^{-3}$.

This superior performance of the correlation detector to the zero-crossing measuring detector is based on the following assumptions about the signal and noise: the frequency and amplitude of the signal are completely known and fixed; the noise is stationary in Gaussian distribution. If the practical situation deviates from the assumptions, then the performances of the two detectors will also deviate from the previous results. The merits and operating characteristics of the two detectors are further discussed in the next chapter under such practical considerations in detecting signals from sleep electroencephalograms(EEGs).



P_d
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
a=1
q=0.4
8
8
n=4
10^-7
10^-6
10^-5
10^-4
10^-3
10^-2
10^-1
10^0
P_f
(a)
P_d
a=2
q=0.4
8
8
n=4
10^-18
10^-16
10^-14
10^-12
10^-10
10^-8
10^-6
10^-4
P_f
(b)

Figure 22.

CHAPTER IV
FURTHER DISCUSSIONS AND CONCLUSION

Operating Characteristics in Practical Situation

The zero-crossing measuring detector has been applied to the detection of phasic events in sleep electroencephalograms(EEGs). Examples of EEG activities from a cat are shown in Figure 23. The arrows indicate phasic events in a sleep EEG. Among various phasic events, alpha, sigma, and beta waves could be modeled as signals of sine wave form imbedded in the noise of background EEG.

The physiological origins of the phasic events as well as the irregular activities of EEG have not been completely understood as yet and neither the statistical characteristics between the phasic events nor the irregular activities have been studied. Even if we can assume for practical purposes that the phasic events are independent processes superimposed on the background EEG, the nonstationarity and the lack of Gaussian property in the sleep EEG activities make it difficult to apply the previous analysis to the quantitative determination of the advantages of the two methods in detecting phasic events from sleep EEG. Hence in the following we attempt only to make qualitative observations on the merits of the zero-crossing measuring detector.

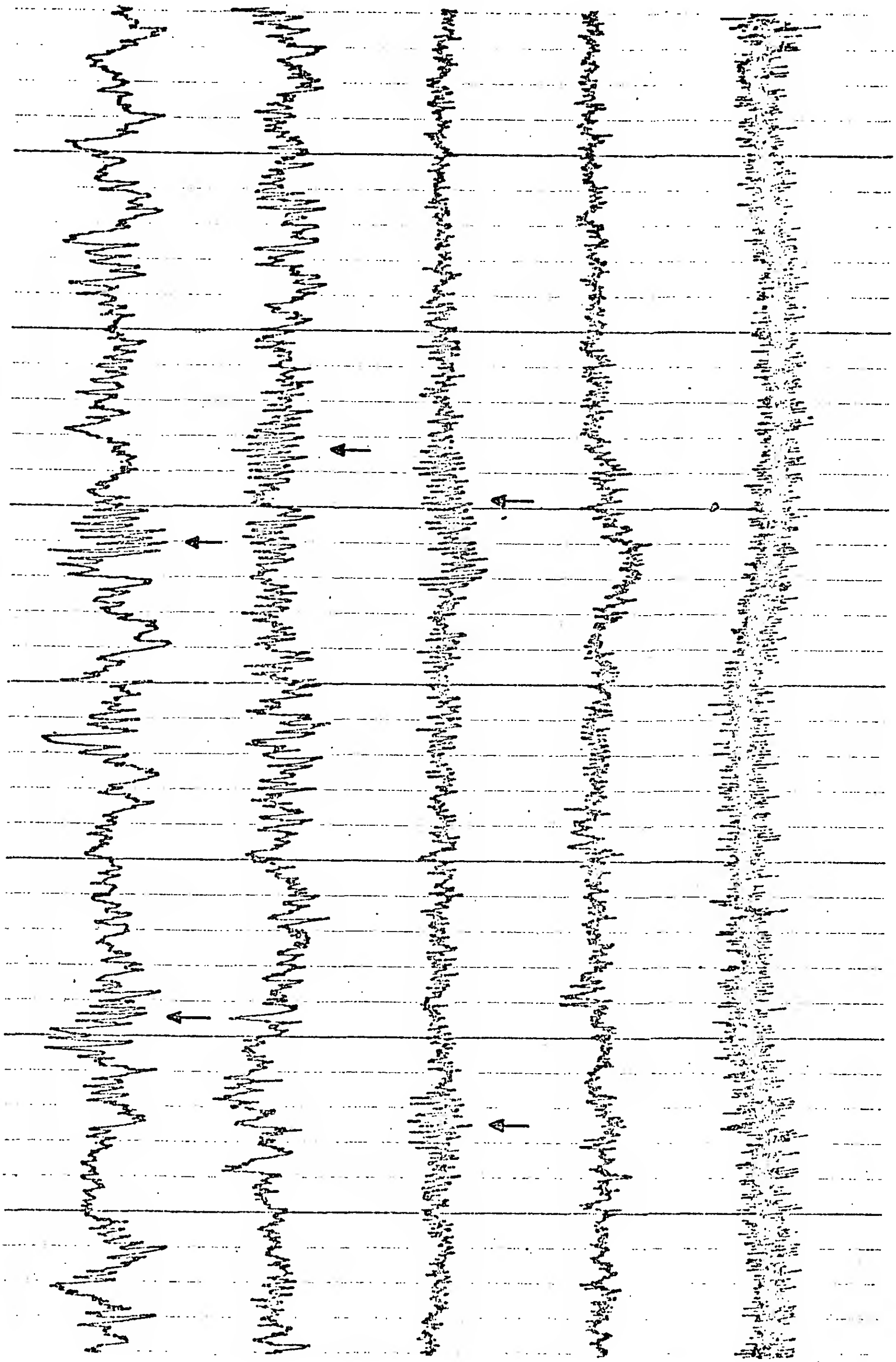


Figure 23.

When we attempt to detect phasic events from a sleep EEG we encounter the the following situations: 1). The frequency of the phasic events varies among individuals. The frequency of alpha activity, for instance, ranges between 8 and 12 Hz, sigma activity between 12 and 15, and the a priori probability density of the frequency distribution is unknown. 2). The amplitude of phasic events varies due to the electrode resistance and individual differences in age and physical parameters. 3). Background EEG activity level varies and artifacts are introduced by movements. The background EEG activity varies also due to the differences between individuals and, within an individual, the variation of activity inherent to a sleep process. Examples of movement artifacts recorded from a cat are shown in Figure 26.

To qualitatively observe the operating characteristics of the two detectors in the described situations, we do the following analyses:

1). To see the effect of the varied frequency of the signal on the performance, we compute the detection probabilities of the detectors to signals of different frequency. Suppose the correlation detector is tuned to signal s_1, and we receive a signal s_2 with a same amplitude but different frequency from s_1. Then, since $f_k=s_{2k}+n_k$, G in Equation (5.20) becomes

$$G=\sum_{k=1}^{m}(s_{2k}+n_k)s_{1k}-\tfrac{1}{2}\sum_{k=1}^{m}s_{1k}^{2}. \tag{6.1}$$

The expectation of G is

$$E_2[G]=\sum_{k=1}^{m} s_{2k}s_{1k}-\tfrac{1}{2}\sum_{k=1}^{m} s_{1k}{}^2. \tag{6.2}$$

The variance of G is

$$V_2[G]=E[(G-E_2[G])^2]=E[\sum_{k=1}^{m}\sum_{j=1}^{m} n_k n_j s_{1k}s_{1j}]$$

$$=\sum_{k=1}^{m} s_{1k}{}^2=S. \tag{6.3}$$

Thus the density function for G is

$$p_2(G)=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\exp[-\tfrac{1}{2}(G-E_2)^2/S], \tag{6.4}$$

where $E_2=E_2[G]$.

Since the detector is tuned to the signal s_1, the threshold remains the same as for s_1, and, hence, the detection probability for s_2 is

$$P_d=P(D_1|H_2)=\int_c^\infty dGp_2(G)$$

$$=(2\pi)^{-\frac{1}{2}}S^{-\frac{1}{2}}\int_c^\infty dG\exp[-\tfrac{1}{2}(G-E_2)^2/S]$$

$$=(2\pi)^{-\frac{1}{2}}\int_{(c-E_2)/S^{\frac{1}{2}}}^\infty dz\exp(-\tfrac{1}{2}z^2), \tag{6.5}$$

where H_2 is the hypothesis that signal s_2 is present, and c is the detector threshold set for signal s_1.

The detection probability of the correlation detector for signals with a different frequency is computed by the program PSEG with the following parameters:

a=2 for both s_1 and s_2,

n=6,

c=threshold set to detect s_1 with a probability of 0.95, that is, set for $P(D_1|H_1)=0.95$,

where a is the ratio of signal amplitude to rms noise, and n is the length of signal in number of cycles. The result is shown in Figure 24(a). The detection probability for a signal with a different frequency, $P(D_1|H_2)$, sharply drops off as the frequency of the signal slightly departs from the tuned frequency of the detector.

The detection probability $P(D_1|H_2)$ for the zero-crossing measuring detector is computed in the following way. Since the detector is tuned to s_1, the thresholds m in Equation (4.4) and T_0 and T_1 in Equation (4.1) are already determined. From the Figures 16(a) and (b), we note that the optimum threshold for m is 6 for a signal with $a=2$ and $n=4$, and $m=12$ when $a=2$ and $n=8$. So with the present signal parameters of $a=2$ and $n=6$, we assume that the threshold for m has been determined at 9. As before we want $P(D_1|H_1)=0.95$, which consequently determines the optimum thresholds (T_0,T_1) for the zero-crossing interval.

The probability of detecting s_2 is then

$$P_d=P(D_1|H_2)$$

$$=\sum_{k=m}^{2n} \binom{2n}{k} P_2^{\,k}(1-P_2)^{2n-k}, \tag{6.6}$$

where P_2 is the area under the zero-crossing interval density curve of s_2 plus noise bounded by the interval (T_0,T_1). The computation is carried out by the program PCOM4 and RCOM, and the result is plotted in Figure 24(b).

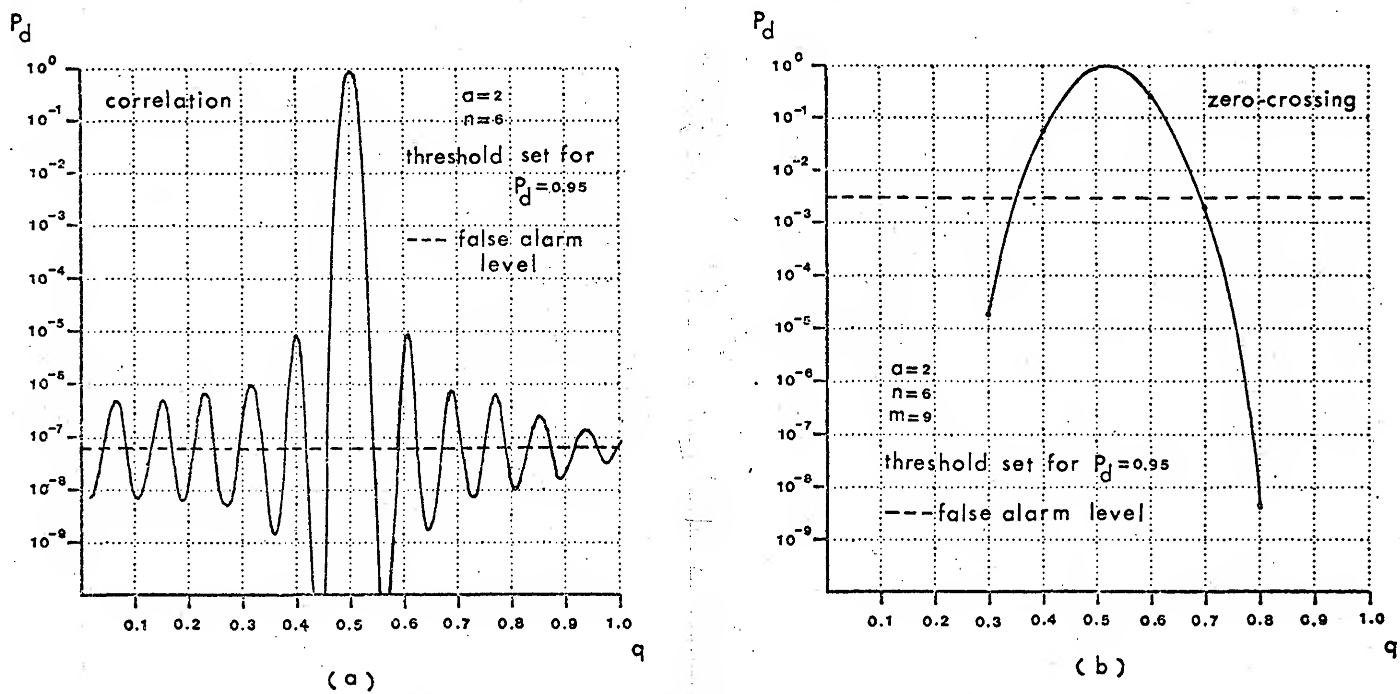

P_d
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
correlation
a=2
n=6
threshold set for
P_d=0.95
--- false alarm
level
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
q
(a)
P_d
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
zero-crossing
a=2
n=6
m=9
threshold set for P_d=0.95
---false alarm level
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
q
(b)

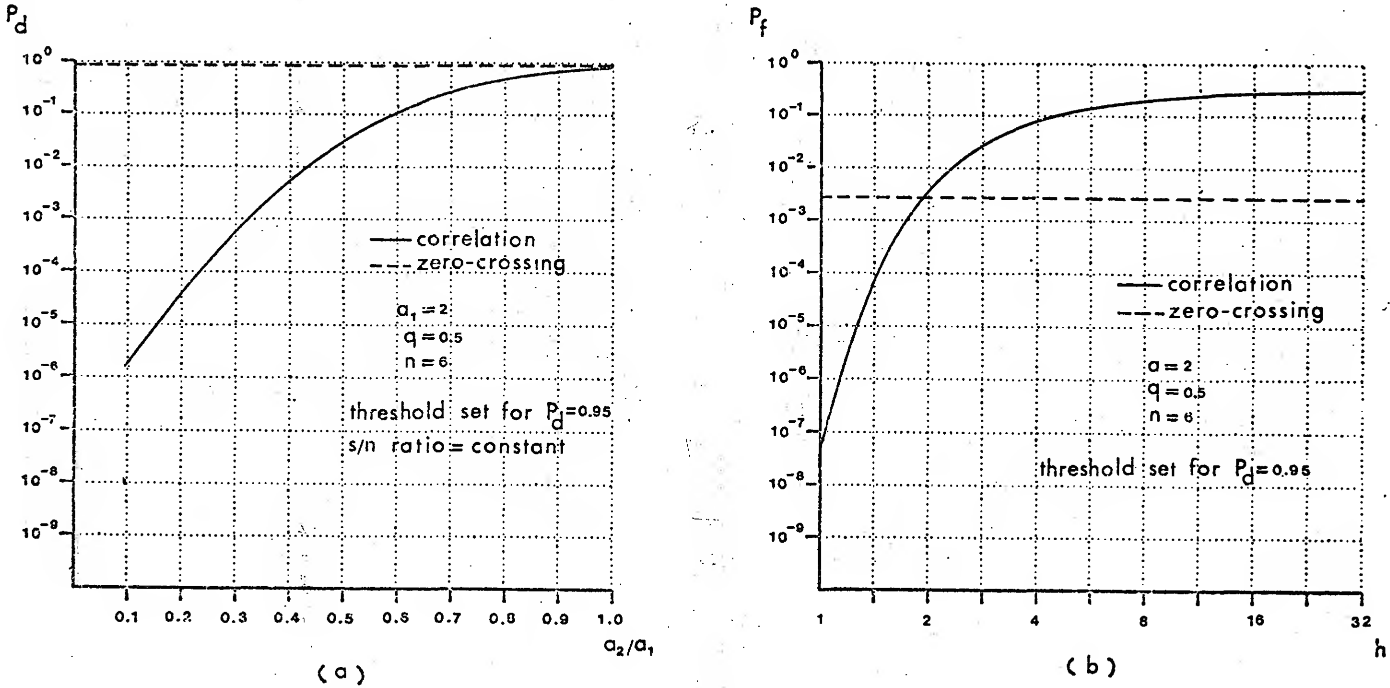
Figure 24.

We note from the two curves that the operating characteristic of the zero-crossing measuring detector is more suitable to cover the broad range of signal frequencies. To cover the same range with the correlation detector without affecting the detection and false alarm performance, we need several channels of detectors in parallel, which is not very practical.

2). The zero-crossing interval distribution is not dependent upon the amplitude of the signal and noise so long as the signal to noise power ratio is maintained constant, which is the usual case with sleep EEGs. And, therefore, the operating characteristic of the zero-crossing measuring detector is not affected by the varied strength of signal and noise.

However, a reduced amplitude of a signal greatly decreases the detection probability of the correlation detector. The detection probability of the correlation detector for a reduced signal can be computed in the same way as done for varied frequency. In this case the first term in Equation (6.2) becomes a function of amplitudes of two signals instead of frequencies, and the detection probability for the reduced signal is expressed again by the Equation (6.5) which could be computed by the program PSEG with a few modifications.

The changes in performance of the two detectors are shown in Figure 25(a). The horizontal coordinate is the amplitude ratio of the reduced signal s_2 to the tuned signal s_1. The frequency and the length of s_2 are assumed to be the same



P_d
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^-8
10^-9
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
a_2/a_1
correlation
zero-crossing
a_1 = 2
q = 0.5
n = 6
threshold set for P_d=0.95
s/n ratio= constant
(a)
P_f
1 2 4 8 16 32
h
a = 2
q = 0.5
n = 6
threshold set for P_d=0.95
(b)

Figure 25.

as s_1 and the signal-to-noise ratio is assumed to be constant. The detection probability of the correlation detector for signals of reduced amplitude drops off rapidly as the ratio decreases while that of the zero-crossing measuring detector remains constant.

3). The increased noise level, assuming that the noise characteristics remain the same, does not increase the false alarm rate of the zero-crossing measuring detector. Also, since the signal to noise power ratio usually remains constant, the detection probability of the signal is not affected.

However, if the noise level is increased in the correlation detector, the false alarm rate also increases to a considerable degree. Suppose the power level of the noise has been changed by a factor of h^2. Then

$$E[n_k n_j]=\begin{cases} h^2, & \text{if } k=j \\ 0, & \text{otherwise,} \end{cases} \tag{6.7}$$

and, since $f_k=n_k$,

$$E[G]=-\tfrac{1}{2}\sum_{k=1}^{m} s_{1k}{}^2=-\tfrac{1}{2}S, \tag{6.8}$$

$$V[G]=E[(G-E[G])^2]=E[\sum_{k=1}^{m}\sum_{j=1}^{m} n_k n_j s_{1k} s_{1j}]$$

$$=h^2\sum_{k=1}^{m} s_{1k}{}^2=h^2S. \tag{6.9}$$

Thus

$$p_0(G)=(2\pi)^{-\frac{1}{2}}h^{-1}S^{-\frac{1}{2}}\exp[-\tfrac{1}{2}(G+\tfrac{1}{2}S)^2/(h^2S)], \tag{6.10}$$

and the false alarm probability is

$$P_f=P(D_1|H_0)=\int_c^\infty dGp_0(G)$$

$$=(2\pi)^{-\frac{1}{2}}h^{-1}S^{-\frac{1}{2}}\cdot\int_c^\infty dG\exp[-\tfrac{1}{2}(G+\tfrac{1}{2}S)^2/(h^2S)]$$

$$=(2\pi)^{-\frac{1}{2}}\int_{(c+\frac{1}{2}S)/(hS^{\frac{1}{2}})}^{\infty} dz\exp(-\tfrac{1}{2}z^2), \tag{6.11}$$

where c is the threshold set for the desired detection probability of the signal. The effect of the increased noise level on the false alarm probability of the correlation detector is computed by the program PNOI according to the Equation (6.11), and plotted in Figure 25(b). The false alarm probability of the correlation detector increases rapidly to 0.1 and gradually approaches to 0.5 as the noise level increases. The false alarm probability of the zero-crossing measuring detector remains constant regardless of the increased noise level.

Since the false alarm probability of the correlation detector is susceptible to the noise level, the detection threshold of the correlation detector could not be lowered arbitrarily to increase the detection probability of weak signals. Consequently, the optimum threshold of the correlation detector for detecting phasic events in EEG should be determined according to the individual strength of the EEG activity.

On the other hand, the zero-crossing measuring detector does not need any threshold adjustments even if the strength of the EEG activity varies over a wide range due to individual differences or electrode resistance.

Typical examples of the detection performance of the zero-crossing measuring detector are shown in Figure 27. A square pulse is the output of the detector indicating the presence of a signal. The immunity of the detector to high power artifacts is demonstrated in Figure 26.

Conclusion

The performance of the zero-crossing measuring detector in an idealized environment of signal and noise was inferior to that of the correlation detector. But when the conditions of the signal and noise deviated from the assumptions, the zero-crossing measuring detector maintained the quality of performance while the performance of the correlation detector was susceptible to the change of environment. The sleep EEG is a very complex process; both the signal and noise have a wide range of variation. When the zero-crossing measuring technique is employed to detect phasic events from sleep EEGs, it should provide a detector which is excellent in performance and applicable to a broad range of subjects with a simplicity of implementation.

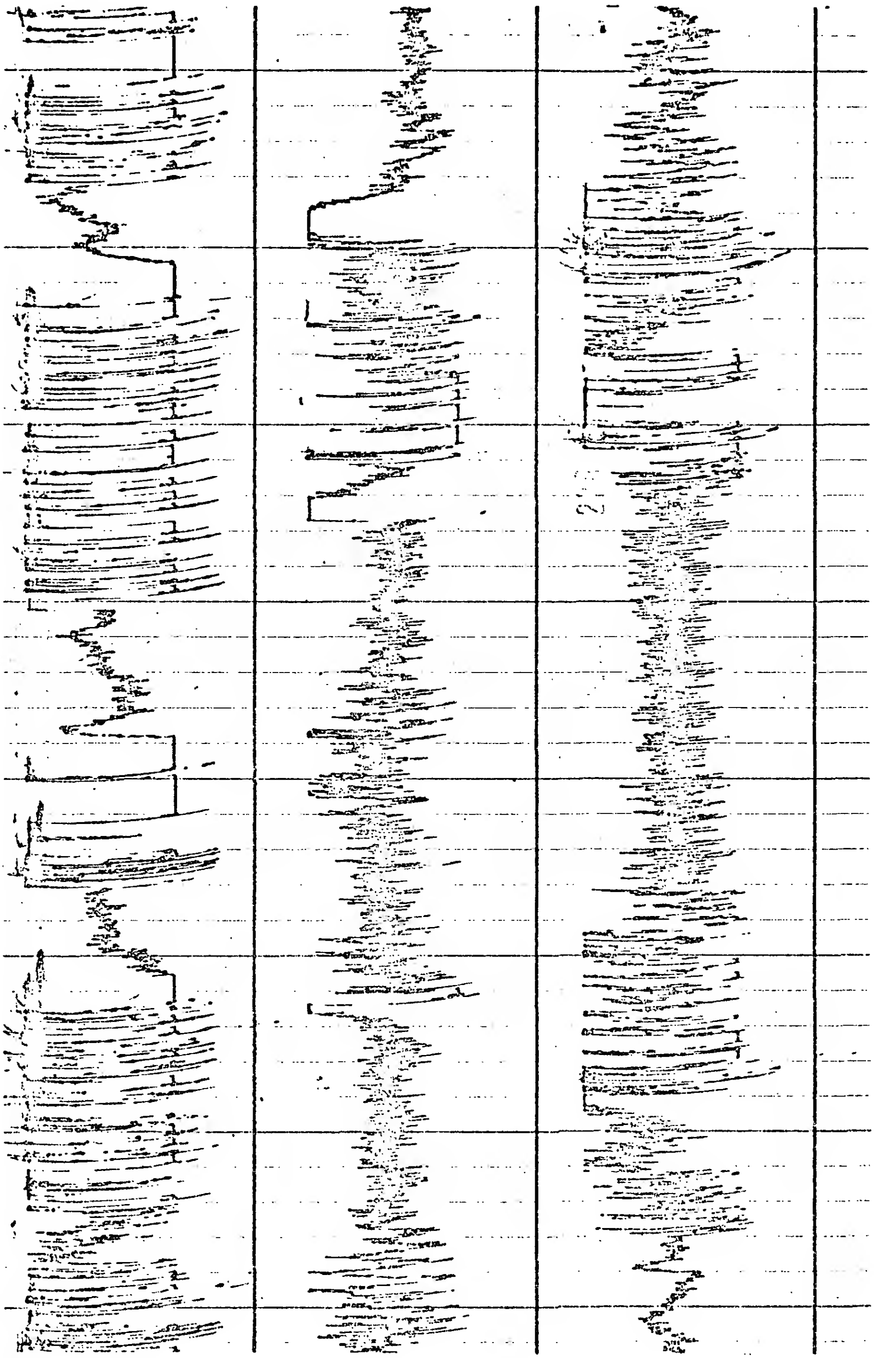


Figure 26.

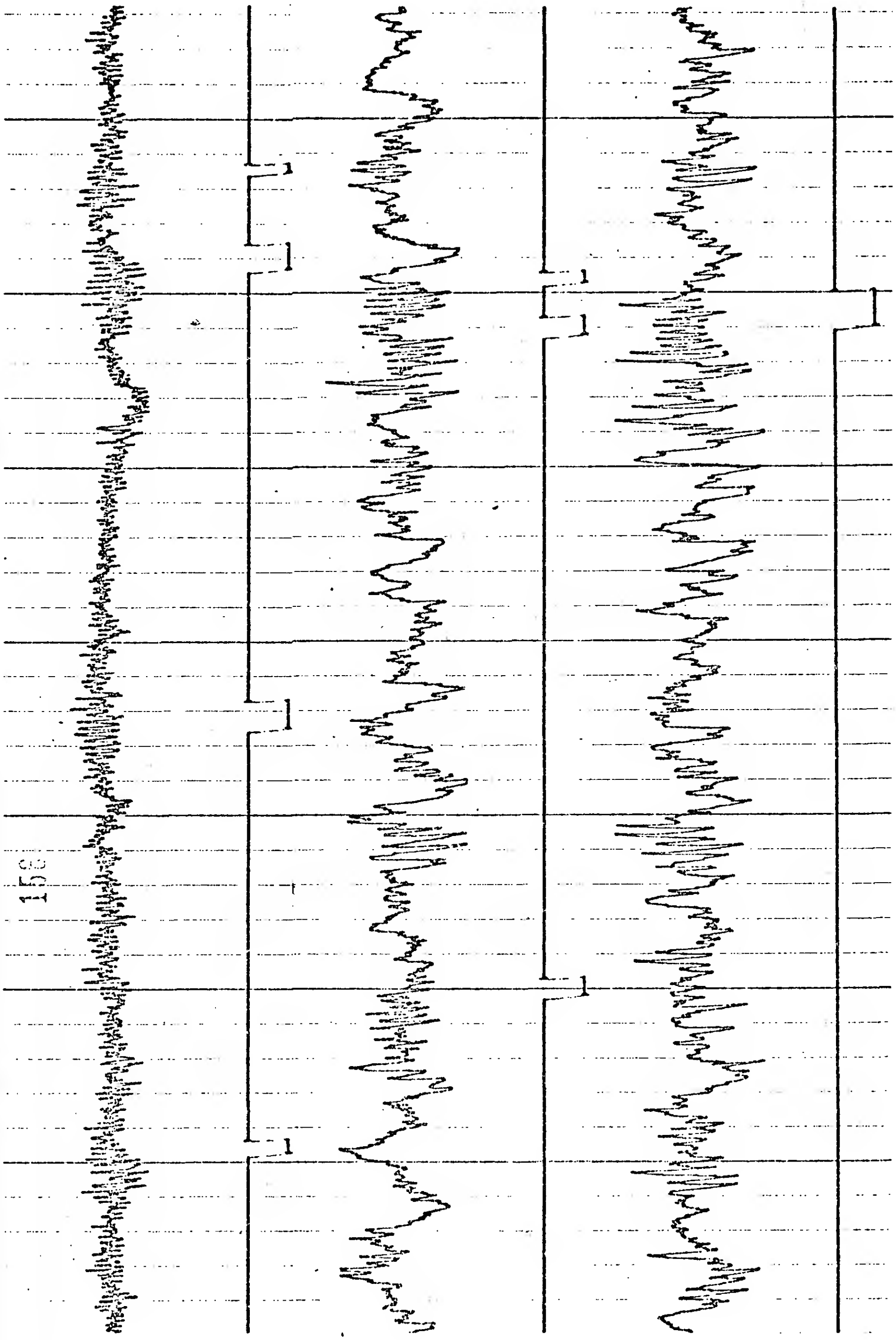


Figure 27.

APPENDICES

APPENDIX A
JACOBI'S THEOREM

In evaluating h_{ij} in Equation (2.17) we used Jacobi's theorem which is stated as

$$(A^{-1})^{(k)}=|A|^{-1}\mathrm{adj}^{(k)}A. \tag{7.1}$$

The derivation of the theorem is quite lengthy, and so we will add here only a brief explanation on the notions in Equation (7.1) and the application of the theorem in evaluating h_{ij}. A reader interested in the development of the theorem could refer to Aitken(2).

Let A be a matrix of order nxn. The determinant obtained by suppressing m rows and m columns of $|A|$ is called a minor of $|A|$ of order n-m.

The kth compound $A^{(k)}$ of A is obtained by forming its elements with minors of $|A|$ of order k in the following way: let all minors which come from the same group of k rows (or columns) of A be placed in the same row (or column) of this derived matrix, and let the priority of elements in rows or columns of this matrix be decided on the principle by which words are ordered in a dictionary or lexicon. For example, minors from rows 1,2,4 of A will appear in earlier rows than those from 1,2,5 or 1,3,4 or 2,3,4; and similarly for columns.

Consider a minor $|B|$ taken from rows $i_1, i_2, \ldots, i_m$ and columns $j_1, j_2, \ldots, j_m$ of A. Then the complementary minor formed by the remaining n-m rows and n-m columns of A, with the sign factor $(-1)^{i_1+\ldots+i_m+j_1+\ldots+j_m}$ multiplied in front, is called a cofactor of minor $|B|$.

Then the kth adjugate compound of A which is denoted by $\mathrm{adj}^{(k)}A$ is obtained in the following way: take the kth compound $A^{(k)}$, replace every element in it by its cofactor in $|A|$, and transpose the resulting matrix.

Now let the matrix M in Equation (2.8) be equal to A^{-1} in Equation (7.1). Then

$$M^{(k)}=|M^{-1}|^{-1}\mathrm{adj}^{(k)}M^{-1}, \tag{7.2}$$

and the determinant

$$D_1=\begin{vmatrix} R_{11},\ldots,R_{1n} \\ \vdots \qquad \vdots \\ R_{n1},\ldots,R_{nn} \end{vmatrix} \tag{7.3}$$

is the element in the 1st row and 1st column of the nth compound matrix of M. This element, if divided by $|M|$, should be by the Jacobi theorem equal to the element in the 1st row and 1st column of the nth adjugate compound of M^{-1}, which is

$$D_2=\begin{vmatrix} m_{n+1,n+1},\ldots,m_{n+1,2n} \\ \vdots \qquad \vdots \\ m_{2n,n+1},\ldots,m_{2n,2n} \end{vmatrix} \tag{7.4}$$

where the matrix $(m_{ij})=M^{-1}$. Thus

$$D_1=|M|D_2=DD_2. \tag{7.5}$$

Since

$$|(h_{ij})|=D_2^{-1}, \tag{7.6}$$

by substituting Equation (7.5) into Equation (7.6) we obtain the relation in Equation (2.16)

$$|(h_{ij})|=D/D_1.$$

Now let

$$P=(p_{ij})=\begin{bmatrix} m_{n+1,n+1},\ldots,m_{n+1,2n} \\ \vdots \qquad\qquad \vdots \\ m_{2n,n+1},\ldots,m_{2n,2n} \end{bmatrix}. \tag{7.7}$$

Then, from Equation (2.11), the (i,j)th element of (h_{ij}) is

$$h_{ij}=D_2^{-1}|P_{ji}| , \tag{7.8}$$

where $|P_{ji}|$ is the cofactor of the element p_{ji}, which is obtained by deleting the jth row and ith column in $|P|$ and multiplying it by the sign factor $(-1)^{i+j}$.

But this cofactor could be also obtained by suppressing n+1 rows and n+1 columns from the 2nx2n matrix M^{-1}. That is, we suppress from M^{-1} the first m rows and m columns, the (n+j)th row and (n+i)th column, and multiply the sign factor $(-1)^{i+j}$. By the Jacobi theorem, however, we can evaluate this cofactor by evaluating the corresponding minor in M. From Equation (7.2) we note that the corresponding minor consists of the first m rows and m columns, the (n+i)th row and (n+j)th column in M.

Thus

$$\left|P_{ji}\right| = \left|M^{-1}\right| \begin{vmatrix} R_{11}, \ldots, & R_{1n}, & R_{1j}' \\ \vdots & \vdots & \vdots \\ R_{n1}, \ldots, & R_{nn}, & R_{nj}' \\ -R_{i1}', \ldots, & -R_{in}', & -R_{ij}'' \end{vmatrix} \tag{7.9}$$

and, substituting the above equation into (7.8), we finally obtain as in Equation (2.17)

$$\begin{aligned} h_{ij} &= D_2^{-1}\left|P_{ji}\right| \\ &= D_1^{-1} \begin{vmatrix} R_{11}, \ldots, & R_{1n}, & R_{1j}' \\ \vdots & \vdots & \vdots \\ R_{n1}, \ldots, & R_{nn}, & R_{nj}' \\ -R_{i1}', \ldots, & -R_{in}', & -R_{ij}'' \end{vmatrix} \end{aligned} \tag{7.10}$$

APPENDIX B
EVALUATION OF A DETERMINANT

Since the OS-8 operating system for the PDP-8/e computer did not have in its library a program to compute a determinant of a matrix, a subroutine is provided to invert a matrix or evaluate a determinant of a matrix. The Gauss-Jordan pivotal elimination method is employed here.

In the evaluation of a determinant by the pivotal elimination method, two rules of transformation for determinants are involved:

1). If a matrix C equals the product of matrices A and B, then $|C|=|A||B|$. (7.11)

2). If a matrix B is formed by interchanging row i and row k of A, then $|B|=-|A|$. (7.12)

If a non-singular matrix A can be reduced to the identity matrix I by premultiplying A by a matrix B

$$BA=I, \tag{7.13}$$

then, by postmultiplying both sides by A^{-1},

$$BAA^{-1}=IA^{-1},$$

or

$$B=A^{-1}, \tag{7.14}$$

that is, we see that B is the inverse matrix of A.

The operation in Equation (7.13) could be carried out in n transformation steps

$$E_nE_{n-1}\cdots E_2E_1A \tag{7.15}$$

successively diagonalyzing one column at each step. For example, at the first stage the resultant matrix is

$$E_1A=\begin{bmatrix} 1/a_{11},0,\dots\dots\dots,0 \\ -a_{21}/a_{11},1,0,\dots\dots,0 \\ -a_{31}/a_{11},0,1,0,\dots,0 \\ \vdots \qquad\qquad\qquad \vdots \\ -a_{n1}/a_{11},0,\dots\dots\dots,1 \end{bmatrix} \begin{bmatrix} a_{11},a_{12},\dots,a_{1n} \\ \vdots \qquad\qquad \vdots \\ a_{n1},a_{n2},\dots,a_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} 1, \quad a_{12}/a_{11},\dots, \quad a_{1n}/a_{11} \\ 0,a_{22}-a_{21}a_{12}/a_{11},\dots,a_{2n}-a_{21}a_{1n}/a_{11} \\ \vdots \qquad\qquad\qquad \vdots \\ 0,a_{n2}-a_{n1}a_{12}/a_{11},\dots,a_{nn}-a_{n1}a_{1n}/a_{11} \end{bmatrix}=A^{(1)} \quad (a_{11}\neq 0) \; . \tag{7.16}$$

In general, after k steps, we want

$$E_kE_{k-1}\cdots E_1A$$

$$=\begin{bmatrix} 1,\dots\dots\dots,0,a^k_{1,k+1},\dots,a^k_{1,n} \\ 0,1,0,\dots,0,a^k_{2,k+1},\dots,a^k_{2,n} \\ \vdots \qquad\qquad\qquad \vdots \\ 0,0,0,\dots,1,a^k_{k,k+1},\dots,a^k_{k,n} \\ \vdots \qquad\qquad\qquad \vdots \\ 0,0,0,\dots,0,a^k_{n,k+1},\dots,a^k_{n,n} \end{bmatrix}=A^{(k)} \; , \tag{7.17}$$

where the superscript k indicates that the elements belong to $A^{(k)}$. Then, after n steps

$$E_nE_{n-1}\cdots E_1A=I. \tag{7.18}$$

In order to produce $A^{(k)}$, the matrix E_k should have the form

$$E_k = \begin{bmatrix} 1,0,0,\ldots,-a_{1k}^{k-1}/a_{kk}^{k-1},\ldots,0 \\ 0,1,0,\ldots,-a_{2k}^{k-1}/a_{kk}^{k-1},\ldots,0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ 0,0,0,\ldots, \quad 1/a_{kk}^{k-1},\ldots,0 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ 0,0,0,\ldots,-a_{nk}^{k-1}/a_{kk}^{k-1},\ldots,1 \end{bmatrix} . \qquad (a_{kk}^{k-1} \neq 0) \qquad (7.19)$$

We note

$$|E_k| = 1/a_{kk}^{k-1}. \qquad (7.20)$$

The operation in Equation (7.18) could be expressed with recursion formulas which calculate the elements of $A^{(k)}$ from the elements of $A^{(k-1)}$:

$$\left.\begin{aligned} a_{kj}^{k} &= a_{kj}^{k-1}/a_{kk}^{k-1}, \quad (a_{kk}^{k-1} \neq 0) \\ a_{ij}^{k} &= a_{ij}^{k-1} - a_{ik}^{k-1} a_{kj}^{k}, \quad (i \neq k) \end{aligned}\right\} (j=1,\ldots,n), \qquad (7.21)$$

for $k=1,\ldots,n$, provided we define $a_{ij} = a_{ij}^{0}$.

The procedure to obtain the inverse matrix is then

$$E_n E_{n-1} \ldots E_1 I = A^{-1}. \qquad (7.22)$$

Let us denote

$$B^{(k)} = E_k E_{k-1} \ldots E_1 I,$$

then the elements of $B^{(k)}$ are calculated from the elements of $B^{(k-1)}$ by the recursion formulas

$$\left.\begin{aligned} b^k_{kj} &= b^{k-1}_{kj}/a^{k-1}_{kk} \\ b^k_{ij} &= b^{k-1}_{ij} - a^{k-1}_{ik} b^k_{kj}, \quad (i \neq k) \end{aligned}\right\} \quad (j=1,\ldots,n) \tag{7.23}$$

Since, from Equation (7.18),

$$1=|I|=|E_n E_{n-1}\ldots E_1 A|,$$

using the rule in Equation (7.11) and the result in Equation (7.20), we obtain

$$\begin{aligned} 1 &= |E_n||E_{n-1}|\ldots|E_1|\,|A| \\ &= (a^{n-1}_{nn} a^{n-2}_{n-1,n-1}\ldots a^2_{33} a^1_{22} a_{11})^{-1}|A|, \end{aligned}$$

or

$$|A|=a_{11}a^1_{22}\ldots a^{n-1}_{nn}. \tag{7.24}$$

When the computations are carried out by a digital computer, the roundoff error occurs in each stage of the elimination process and propagates to the next stage. An analysis of the error expressions corresponding to the recursion formulas for the elimination process reveals that this roundoff error can be minimized by proper choice of the ratio a_{ik}/a_{kk} (McCalla). To reduce the error, the numerically largest element is selected as the element a_{kk}, which is called a pivot element, from the elements a_{ij} $(i,j \geqslant k)$. In order to bring the largest element to the pivotal position, row and column exchanges are necessary and the corresponding sign change in the determinant should be applied by the rule in Equation (7.12).

APPENDIX C
NUMERICAL INTEGRATIONS

1). The error function in Equation (3.13)

$$\mathrm{erf}(x)=2\pi^{-\frac{1}{2}}\int_0^x dt\exp(-t^2) \tag{7.25}$$

is evaluated by the rational approximation (Abramowitz)

$$\mathrm{erf}(x)=1-(a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5)\exp(-x^2)+r(x), \quad (0\leqslant x<\infty) \tag{7.26}$$

where

$t=(1+px)^{-1}$,

$p=0.3275911$,

$a_1=0.2548296$,

$a_2=-0.2844967$,

$a_3=1.4214137$,

$a_4=-1.453152$,

$a_5=1.061405$.

The error bound in this expansion is

$$|r(x)|\leqslant 1.5\mathrm{x}10^{-7}.$$

2). To evaluate the statistical function

$$Q(x)=(2\pi)^{-\frac{1}{2}}\int_x^\infty dt\exp(-\tfrac{1}{2}t^2), \tag{7.27}$$

the following expansion is used (Abramowitz):

$$Q(x)=Z(x)(b_1t+b_2t^2+b_3t^3+b_4t^4+b_5t^5)+r(x), \qquad (0\leqslant x<\infty) \tag{7.28}$$

where

$Z(x)=(2\pi)^{-\frac{1}{2}}\exp(-\frac{1}{2}x^2)$,

$t=(1+px)^{-1}$,

$p=0.2316419$,

$b_1=0.3193815$,

$b_2=-0.3565638$,

$b_3=1.7814779$,

$b_4=-1.821256$,

$b_5=1.330274$.

The error bound is

$$|r(x)| \leqslant 7.5\text{x}10^{-8}.$$

For a large value of x, the approximation of Q(x) by continued fractions

$$Q(x)=Z(x)[1/(x+1/(x+2/(x+3/(x+...))))], \quad (x>0) \tag{7.29}$$

would be a better evaluation. But, since the difference between the two approximations was not significant enough to show up on the plotted graphs, the expansion in Equation (7.28) is used in the computation throughout the range of x.

3). For other numerical integrations, the trapezoidal rule is employed.

Suppose f(x) is defined at equally spaced points x_i ($i=0,1,...,n$), then the area under the curve of f(x) between x_0 and x_n is computed by the trapezoidal rule

$$T=\sum_{i=0}^{n-1} \tfrac{1}{2}h[f(x_i)+f(x_i+h)], \tag{7.30}$$

where h is the distance between two points of x

$$h=x_{i+1}-x_i, \quad (i=0,1,\ldots,n-1). \tag{7.31}$$

Writing the above formula in expanded form and substituting the relation $x_{i+1}=x_i+h$, we obtain the classical form of the trapezoidal rule

$$T=\tfrac{1}{2}h[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)]. \tag{7.32}$$

APPENDIX D
INTERPOLATION BY SPLINE FUNCTION

When an interpolation of data is needed, the spline function is used to approximate the intermediate values of the data. The spline function is a numerical representation of the curve of a flexible strip or spline used in drafting to draw a smooth line through a set of points (Carnahan).

Suppose we have n functional values $(x_i, f(x_i))$ $(i=1,\ldots,n)$, and want to interpolate the data over the interval (x_1, x_n). The restraints that define the spline function over interval (x_1, x_n) are then as follows:

1). In each interval the spline function is a cubic polynomial, that is

$$p_{3,i}(x), \quad (i=1,2,\ldots,n-1) \tag{7.33}$$

where the subscript in p_3 indicate that p is a polynomial of degree 3.

2). Each cubic polynomial $p_{3,i}(x)$ should match the functional values at its two ends x_i and x_{i+1}:

$$\left.\begin{aligned} p_{3,i}(x_i) &= f(x_i), \\ p_{3,i}(x_{i+1}) &= f(x_{i+1}), \end{aligned}\right\} (i=1,2,\ldots,n-1) \tag{7.34}$$

3). The first and second derivatives of successive polynomials must match each other in value at the intermediate points:

$$\left.\begin{aligned} p'_{3,i}(x_i)&=p'_{3,i-1}(x_i), \\ p''_{3,i}(x_i)&=p''_{3,i-1}(x_i), \end{aligned}\right\} (i=2,3,\ldots,n-1), \tag{7.35}$$

and the second derivatives vanish at two end points:

$$\begin{aligned} p''_{3,1}(x_1)&=0, \\ p''_{3,n-1}(x_n)&=0. \end{aligned} \tag{7.36}$$

Equations (7.34) through (7.36) amount to $4(n-1)$ relations, which enables us to determine the $4(n-1)$ coefficients needed to completely describe the spline function in Equation (7.33).

For convenience, define

$$\begin{aligned} f_i&=f(x_i), \\ h_i&=x_{i+1}-x_i, \\ g_i&=p''_{3,i-1}(x_i)=p''_{3,i}(x_i). \end{aligned} \tag{7.37}$$

Since each $p''_{3,i}(x)$ is a linear function of x, it can be obtained on the interval (x_i,x_{i+1}) by linear interpolation of the values g_i and g_{i+1} at each end of the interval:

$$p''_{3,i}(x)=g_i(x_{i+1}-x)/h_i+g_{i+1}(x-x_i)/h_i. \tag{7.38}$$

By integrating twice and imposing the conditions in Equation (7.34), we find that

$$\begin{aligned} p_{3,i}(x)=g_i(x_{i+1}-x)^3/(6h_i)+g_{i+1}(x-x_i)^3/(6h_i)+ \\ (f_{i+1}/h_i-g_{i+1}h_i/6)(x-x_i)+ \end{aligned} \tag{7.39}$$

$$(f_i/h_i-g_ih_i/6)(x_{i+1}-x), \quad (i=1,2,\ldots,n-1).$$

By differentiating Equation (7.39) and imposing the conditions in Equation (7.35), we obtain the following system of linear equations in the g_i:

$$g_{i-1}h_{i-1}/h_i+2g_i(1+h_{i-1}/h_i)+g_{i+1}$$
$$=6[(f_{i+1}-f_i)/h_i-(f_i-f_{i-1})/h_{i-1}]/h_i, \qquad (7.40)$$
$$(i=2,3,\ldots,n-1),$$

and

$$g_1=0,$$
$$g_n=0, \qquad (7.41)$$

which follow from Equation (7.36).

Note that Equations (7.40) and (7.41) amount to n simultaneous linear equations in the n unknowns $g_1,\ldots,g_n$. Thus the values of $g_1,\ldots,g_n$ can be determined, and the substitution of the obtained g_i $(i=1,2,\ldots,n)$ into Equation (7.39) yields the individual cubic spline polynomials for use over successive intervals.

APPENDIX E
LIST OF PROGRAMS

```
C       PROGRAM LONGO

        COMMON P,A,U,V,ALP,ES,MUG,T,PI,WP,SIGMA
        DIMENSION P(6,6),A(6,6),U(6,6),V(6,6),ALP(3),ES(3),MUG(100)
        DIMENSION TAU(200),DATA(200)

        PI=3.1415926536
        DELT=0.1
        DO 10 I=1,100
10      MUG(I)=-1376

        DO 100 MAXT=1,150

        T=DELT*FLOAT(MAXT)
        S=SIN(T)
        C=COS(T)
        R=S/T
        DUM0=R-C
        RD=-DUM0/T
        RDD=2.0*DUM0/T**2-R
        R0=1.0
        RD0=0.0
        RDD0=-1.0/3.0

        P(1,1)=R0
        P(1,2)=R
        P(1,3)=RD0
        P(1,4)=RD
        P(2,2)=R0
        P(2,3)=-RD
        P(2,4)=RD0
        P(3,3)=-RDD0
        P(3,4)=-RDD
        P(4,4)=-RDD0

        DO 200 I=2,4
        MUD=I-1
        DO 200 J=1,MUD
200     P(I,J)=P(J,I)

        SUD1=1.0-R**2

        DO 250 I=1,2
        DO 250 J=1,2
        DO 235 I1=1,4
        DO 235 J1=1,4
235     A(I1,J1)=P(I1,J1)
        DO 240 L=1,4
240     A(3,L)=A(2+I,L)
        DO 245 L=1,4
245     A(L,3)=A(L,2+J)
        KA1=3
        CALL MATIN (A,MAT,D)
250     U(I,J)=D/SUD1
```

```
      DO 270 I=1,2
      DO 270 J=1,2
270   V(I,J)=U(I,J)/(U(I,I)*U(J,J))**0.5

      DUM0=(U(1,1)*U(2,2))**0.5/4.0/PI**2/(1.0-R**2)**0.5
      DUM1=(1.0-V(1,2)**2)**0.5/V(1,2)
      ARC=ATAN(DUM1)
      IF(DUM1)275,280,280
275   ARC=PI+ARC

280   WPM=DUM0*((1.0-V(1,2)**2)**0.5-V(1,2)*ARC)
      WP=SQRT(-RDD0)/2.0/PI


      P(1,1)=R0
      P(1,3)=R
      P(1,4)=RD0
      P(1,6)=RD
      P(2,2)=R0
      P(2,5)=RD0
      P(3,3)=R0
      P(3,4)=-RD
      P(3,6)=RD0
      P(4,4)=-RDD0
      P(4,6)=-RDD
      P(5,5)=-RDD0
      P(6,6)=-RDD0

      CALL LONG1

      POT=WPM/WP-SIGMA

      TAU(MAXT)=T
100   DATA(MAXT)=POT

      CALL OOPEN('SYS','ZXNO')
      WRITE(4,700) MAXT,(TAU(I),DATA(I),I=1,MAXT)
700   FORMAT(A2,400A6)
      CALL OCLOSE

      END
```

```
      SUBROUTINE LONG1

      COMMON P,A,U,V,ALP,ES,MUG,T,PI,UP,SIGMA
      DIMENSION P(6,6),A(6,6),U(6,6),V(6,6),ALP(3),ES(3),MUG(100)

      DSIG=0.0
      SIGMA=0.0
      DDELT=0.5
      DTF=0.9*DDELT
      G=1.0

300   TF=G*DDELT
      IF(T-TF-DTF)200,200,210
200   SIGMA=SIGMA+DSIG
      GO TO 100
210   IF(G-2.01)220,220,230
220   SIGMA=SIGMA+DSIG
      GO TO 305
230   SIGMA=SIGMA+2.0*DSIG

305   TS=T-TF
      S=SIN(TF)
      C=COS(TF)
      RF=S/TF
      DUMB=RF-C
      RFD=-DUMB/TF
      RFDD=2.0*DUMB/TF**2-RF
      S=SIN(TS)
      C=COS(TS)
      RS=S/TS
      DUMB=RS-C
      RSD=-DUMB/TS
      RSDD=2.0*DUMB/TS**2-RS

      P(1,2)=RF
      P(1,5)=RFD
      P(2,3)=RS
      P(2,4)=-RFD
      P(2,6)=RSD
      P(3,5)=-RSD
      P(4,5)=-RFDD
      P(5,6)=-RSDD

      DO 310 I=2,6
      MUD=I-1
      DO 310 J=1,MUD
310   P(I,J)=P(J,I)

      DO 320 I=1,3
      DO 320 J=1,3
320   A(I,J)=P(I,J)

      MAT=3
      CALL MATIN(A,MAT,D)
      SUD2=D
```

```
      DO 350 I=1,3
      DO 350 J=1,3
      DO 335 I1=1,6
      DO 335 J1=1,6
335   A(I1,J1)=P(I1,J1)
      DO 340 L=1,6
340   A(4,L)=A(3+I,L)
      DO 345 L=1,6
345   A(L,4)=A(L,3+J)

      MAT=4
      CALL MATIN(A,MAT,D)
350   U(I,J)=D/SUD2

      DO 370 I=1,3
      DO 370 J=1,3
370   V(I,J)=U(I,J)/(U(I,I)*U(J,J))**0.5

      DO 387 I0=1,3
      KUD=I0+3
      I1=IREM(KUD/3)+1
      MUD=I0+4
      I2=IREM(MUD/3)+1

      DUM0=V(I2,I0)
      DUM0=ABS(DUM0)
      IF(DUM0-0.99)381,383,383
381   DUM0=V(I0,I1)
      DUM0=ABS(DUM0)
      IF(DUM0-0.99)382,383,383
382   ALP(I0)=V(I2,I0)*V(I0,I1)+V(I1,I2)
      DUM0=V(I2,I0)*V(I0,I1)-V(I1,I2)
      DUM0=DUM0/((1.0-V(I2,I0)**2)*(1.0-V(I0,I1)**2))**0.5
      DUM1=ABS(DUM0)
      IF(DUM1-1.0)384,386,386
384   DUM0=(1.0-DUM0**2)**0.5/DUM0
      ARC=ATAN(DUM0)
      IF(DUM0)380,385,385
380   ARC=PI+ARC
385   ES(I0)=ARC
      GO TO 387
383   ALP(I0)=0.0
386   ES(I0)=0.0
387   CONTINUE

      MAT=3
      CALL MATIN(V,MAT,D)
      D=ABS(D)

      DUM0=(U(1,1)*U(2,2)*U(3,3))**0.5
      DUM1=D**0.5+ES(1)*ALP(1)+(ES(2)-PI)*ALP(2)+(ES(3)-PI)*ALP(3)
```

```
      VPMM=DUN0*DUN1/8.0/PI**3/SUD2**0.5

      DSIG=VPMN/VP*DDELT/2.0

      G=G+1.0
      GO TO 300

100   RETURN
      END
```

```
      SUBROUTINE MATIN(A,N,D)
C     PROGRAM MATIN

C     THIS PROGRAM INVERTS A MATRIX A(N,N). THE DETERMINANT D
C     OF THE MATRIX A(N,N) IS ALSO CALCULATED.

      DIMENSION A(6,6),IP(6),JP(6)

      K=1
      D=1.0
10    DUM0=0.0
      DO 20 I=K,N
      DO 20 J=K,N
      DUM1=A(I,J)
      DUM1=ABS(DUM1)
      IF(DUM1-DUM0)20,20,25
25    IP(K)=I
      JP(K)=J
      DUM0=DUM1
20    CONTINUE
      IF(DUM0)920,910,45
45    IF(JP(K)-K)920,52,47
47    DO 50 I=1,N
      DUM0=-A(I,K)
      A(I,K)=A(I,JP(K))
50    A(I,JP(K))=DUM0
52    IF(IP(K)-K)920,65,54
54    DO 55 J=1,N
      DUM0=-A(K,J)
      A(K,J)=A(IP(K),J)
55    A(IP(K),J)=DUM0
65    PIV=A(K,K)
      D=D*PIV
      DO 70 I=1,N
      IF(I-K)75,70,75
75    A(I,K)=-A(I,K)/PIV
      DO 70 J=1,N
      IF(J-K)80,70,80
80    A(I,J)=A(I,J)+A(I,K)*A(K,J)
70    CONTINUE
85    DO 90 J=1,N
90    A(K,J)=A(K,J)/PIV
      A(K,K)=1.0/PIV
      K=K+1
      IF(K-N)10,10,115
115   NN=N-1
      DO 130 M=1,NN
      I=N-M
      IF(IP(I)-I)920,127,125
```

```
125     DO 128 J=1,N
        DUMG=A(J,I)
        A(J,I)=-A(J,IP(I))
        A(J,IP(I))=DUMB
128     CONTINUE
127     IF(JP(I)-I)920,130,135
135     DO 130 J=1,N
        DUMG=A(I,J)
        A(I,J)=-A(JP(I),J)
        A(JP(I),J)=DUMB
130     CONTINUE
        GO TO 999
910     WRITE(1,911)
911     FORMAT(' MATRIX IS SINGULAR.')
        GO TO 999
920     WRITE(1,921)
921     FORMAT(' HARDWARE ERROR.')
999     RETURN
        END
```

```
C         PROGRAM STOSN

          COMMON NDA,A,Q,TAU,DATA
          DIMENSION TAU(200),DATA(200),SN(50)

          MIN1=1
          MAX1=5
          INC1=1
          MIN2=1
          MAX2=9
          INC2=1
          SCAL1=1.0
          SCAL2=0.1
          DA=0.0

          DO 100 LP1=MIN1,MAX1,INC1
          A=SCAL1*FLOAT(LP1)

          DO 100 LP2=MIN2,MAX2,INC2
          Q=SCAL2*FLOAT(LP2)

          CALL SINO

S         CLA CLL
S         TAD \LP1
S         TAD (60
S         RTL;RTL;RTL
S         TAD (56
S         DCA \DA
S         CLL
S         TAD \LP2
S         TAD (60
S         RTL;RTL;RTL
S         DCA \DA#

          CALL OOPEN('SYS',DA)
          WRITE(4,200) NDA,A,Q,(TAU(I),DATA(I),I=1,NDA)
200       FORMAT(A2,402A6)
          CALL OCLOSE

100       CONTINUE

          END
```

```
      SUBROUTINE SINO
C     PROGRAM SINO

      COMMON NDA,A,Q,TAU,DATA
      DIMENSION TAU(200),DATA(200)

      E1=0.25482959
      E2=-0.28449674
      E3=1.4214137
      E4=-1.4531520
      E5=1.0614054
      E6=0.3275911
      NDPI=20
      PI=3.1415926
      PI1=PI/FLOAT(NDPI)
      PI2=PI/4.0
      PI3=PI**0.5
      F0=Q/2.0/PI
      KLP2=2*NDPI+1
      KLP3=NDPI/2+1
      NDA=0

      DO 200 LP1=5,150,5

      T=0.1*FLOAT(LP1)
      T2=T**2
      DUMQ=Q*T/2.0
      CQ=COS(DUMQ)
      SQ=SIN(DUMQ)

      R=SIN(T)/T
      DUMO=R-COS(T)
      RD=-DUMO/T
      RDD=2.0*DUMO/T2-R
      RDD0=-1.0/3.0

      C0=RD
      C1=1.0-R
      C2=1.0+R
      C3=-RDD0+RDD
      C4=-RDD0-RDD
      C5=C1*C4-C0**2
      C6=C2*C3-C0**2
      C7=C0*C0+Q*C2*SQ
      C8=C0*SQ+Q*C1*CQ
      DET=C5*C6
      X0=C1*C6
      X1=C2*C5
      X2=(A*C7)**2/C2/C6
      X3=(A*C8)**2/C1/C5
      X4=(A*CQ/C2)**2
      X5=(A*SQ/C1)**2
      X6=X2+X4
      X7=X3+X5
      X8=(C5*C6)**0.5
```

```
      SUM1=0.0

      DO 100 LP2=1,NLP2

      NUD0=LP2-1
      DUM0=FLOAT(NUD0)
      THE=DUM0*PI1-PI
      CT=COS(THE)
      ST=SIN(THE)

      D3=X6*CT**2+X7*ST**2
      X9=C5*C7*CT
      X10=C6*C8*ST
      IF(D3-73.0)46,47,47
46    EXD3=EXP(-D3)
      GO TO 49
47    EXD3=0.0

49    SUM2=0.0

      DO 150 LP3=1,NLP3

      NUD0=LP3-1
      DUM0=FLOAT(NUD0)
      PHI=DUM0*PI1-PI2
      CP=COS(PHI)
      SP=SIN(PHI)
      DUM0=2.0*PHI
      C2P=COS(DUM0)

      D1=X1*CP**2+X8*SP**2
      D2=-A*(X9*CP+X10*SP)/X8
      D4=D2/D1**0.5
      D8=D4**2
      D5=D3-D8

      DUM1=ABS(D4)
      E0=1.0/(1.0+E6*DUM1)
      DUM0=E1*E0+E2*E0**2+E3*E0**3+E4*E0**4+E5*E0**5
      IF(D8-73.0)50,51,51
50    ERF=1.0-DUM0*EXP(-D8)
      IF(D4)54,55,55
54    ERF=-ERF
      GO TO 55
51    ERF=1.0

55    DUM0=(1.0+D8)*EXD3
60    IF(D5-73.0)61,62,62
61    DUM1=PI3*D4*(1.5+D8)*(1.0-ERF)*EXP(-D5)
      GO TO 65
62    DUM1=0.0
65    SUM=(DUM0-DUM1)*C2P/D1**2
```

```
      IF(LP3-1)70,70,72
70    SUM2=SUM2+SUM
      GO TO 150
72    IF(LP3-MLP3)74,70,70
74    SUM2=SUM2+2.0*SUM
150   CONTINUE

      IF(LP2-1)80,80,82
80    SUM1=SUM1+SUM2
      GO TO 100
82    IF(LP2-MLP2)84,80,80
84    SUM1=SUM1+2.0*SUM2
100   CONTINUE

      WPM=DET**1.5*SUM1*PI1**2/PI**3/32.0
      WT=WPM/FQ

      NDA=NDA+1
      TAU(NDA)=0.1*FLOAT(LP1)
200   DATA(NDA)=WT

      RETURN
      END
```

```
C       PROGRAM COREK

        COMMON NDA,A,Q,TAU,DATA
        DIMENSION TAU(200),DATA(200),MUG(100)

        MIN1=1
        MAX1=5
        INC1=1
        MIN2=1
        MAX2=9
        INC2=1
        SCAL1=1.0
        SCAL2=0.1
        DA=0.0
        DTAU=0.5
        PI=3.1415927
        CRDD=(1.0/3.0)**0.5
        POSE=CRDD/PI
        CON1=1.0/(2.0*PI)**0.5
        DELT=PI/20.0
        E1=0.25482959
        E2=-0.28449674
        E3=1.4214137
        E4=-1.4531520
        E5=1.0614054
        E6=0.3275911

        DO 50 I=1,100
50      MUG(I)=-1376

        DO 100 LP1=MIN1,MAX1,INC1

        DO 100 LP2=MIN2,MAX2,INC2

S       CLA CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

        CALL IOPEN('SYS',DA)
        READ(4,200) NDA,A,Q,(TAU(I),DATA(I),I=1,NDA)
200     FORMAT(A2,402A6)
```

```
      WRITE(3,300) A,Q
300   FORMAT('1'10X,'A='F3.0,10X,'Q='F4.1///)

      DO 320 IN=1,NDA
      DUM0=100.0*DATA(IN)
      MUD0=IFIX(DUM0)
320   WRITE(3,321) TAU(IN),DATA(IN),(MUG(J),J=1,MUD0)
321   FORMAT(F7.2,E20.7,5X,100A1)
      WRITE(3,323)
323   FORMAT(/////)

      B=A*Q/CRDD
      SUM1=0.0

      DO 400 LP5=1,21

      MUD0=LP5-1
      THE=PI*FLOAT(MUD0)/20.0
      S=SIN(THE)
      C=COS(THE)

      VAL0=B*S/2.0**0.5
      DUM1=ABS(VAL0)
      E0=1.0/(1.0+E6*DUM1)
      DUM0=E1*E0+E2*E0**2+E3*E0**3+E4*E0**4+E5*E0**5
      DUM1=VAL0**2
      IF(DUM1-73.0)450,451,451
450   ERF=1.0-DUM0*EXP(-DUM1)
      IF(VAL0)454,455,455
454   ERF=-ERF
      GO TO 455
451   ERF=1.0

455   DUM0=-(A*C)**2/2.0
      VAL0=CON1*EXP(DUM0)
      DUM0=-(B*S)**2/2.0
      VAL1=CON1*EXP(DUM0)
      DSUM1=VAL0*(VAL1+B*S/2.0*ERF)

      IF(LP5-1)461,461,462
461   SUM1=SUM1+DSUM1
      GO TO 400
462   IF(LP5-21)463,461,461
463   SUM1=SUM1+2.0*DSUM1

400   CONTINUE

      UP=POSE*SUM1*DELT/2.0
      SZX=Q/2.0/PI
```

```
      SUM2=0.0

      DO 520 I=1,NDA

      DATA(I)=DATA(I)*SZX/WP
      DUM1=100.0*DATA(I)
      MUDO=IFIX(DUM1)
      IF(MUDO-90)590,590,591
591   MUDO=90
590   CONTINUE

      IF(I-1)510,510,512
510   SUM2=SUM2+DATA(I)
      GO TO 520
512   IF(I-NDA)514,510,510
514   SUM2=SUM2+2.0*DATA(I)

520   WRITE(3,521) TAU(I),DATA(I),(MUG(J),J=1,MUDO)
521   FORMAT(F7.2,E20.7,5X,100A1)

      PROB=0.5*SUM2*DTAU
      WRITE(3,530) SZX,WP,PROB
530   FORMAT(////10X'Q/(2*PI)='F7.4,10X'WP='F7.4,10X
     1 'PROBABILITY='F5.2)

100   CONTINUE

      END
```

```
C         PROGRAM RET1

          DIMENSION TAU(200),DATA(200),MUG(100)

          CALL IOPEN('SYS','ZXHO')
          READ(4,20) NDA,(TAU(I),DATA(I),I=1,NDA)
20        FORMAT(A2,400A6)

          DO 30 I=1,100
30        MUG(I)=-1376

          DO 40 I=1,NDA
          DUMO=200.0*DATA(I)
          MUDO=IFIX(DUMO)
40        WRITE(3,41) TAU(I),DATA(I),(MUG(J),J=1,MUDO)
41        FORMAT(F10.2,E20.7,5X,100(A1))

          END
```

```
C       PROGRAM RET2

        COMMON NDA,A,Q,TAU,DATA
        DIMENSION TAU(200),DATA(200),MUG(100)

        MIN1=1
        MAX1=5
        INC1=1
        MIN2=3
        MAX2=9
        INC2=1
        SCAL1=1.0
        SCAL2=0.1
        DA=0.0

        DO 50 I=1,100
50      MUG(I)=-1376

        DO 100 LP1=MIN1,MAX1,INC1

        DO 100 LP2=MIN2,MAX2,INC2

S       CLA CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

        CALL IOPEN('SYS',DA)
        READ(4,200) NDA,A,Q,(TAU(I),DATA(I),I=1,NDA)
200     FORMAT(A2,402A6)

        WRITE(3,300) A,Q
300     FORMAT('1'10X,'A='F3.0,10X,'Q='F4.1///)

        SUM=0.0
        DO 320 IN=1,NDA
        DUM0=100.0*DATA(IN)
        MUD0=IFIX(DUM0)
        IF(MUD0-90)305,305,306
306     MUD0=90
305     CONTINUE
        SUM=SUM+DATA(IN)
320     WRITE(3,321) TAU(IN),DATA(IN),(MUG(J),J=1,MUD0)
321     FORMAT(F7.2,E20.7,5X,100(A1))
        PROB=SUM*0.1
        WRITE(3,350) PROB
350     FORMAT(///10X,'PROBABILITY='F5.3)

100     CONTINUE
        END
```

```
C       PROGRAM DPOL

        DIMENSION TAU(30),DATA(30),MUG(100),DAPO(31),TAPO(31),PAD(31)

        DO 50 I=1,100
50      MUG(I)=-1376

        DO 100 LP1=3,5

        DO 100 LP2=3,9

S       CLA CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

199     CALL IOPEN('SYS',DA)
        READ(4,200) NDA,A,Q,(TAU(I),DATA(I),I=1,NDA)
200     FORMAT(A2,62A6)

        WRITE(3,300) A,Q
300     FORMAT('1'10X,'A='F3.0,10X,'Q='F4.1///)

304     WRITE(1,305) A,Q
305     FORMAT(2F10.1,10X'MAXIMUM TAU=')
        READ(1,310) TH
310     FORMAT(F20.2)
        DUMB=TH/0.5+0.01
        NP=IFIX(DUMB)

        IND2=0
        IND1=NP-2
        DO 400 LP3=IND1,NP
        IND2=IND2+1
        TAPO(IND2)=TAU(LP3)
400     DAPO(IND2)=DATA(LP3)
        TAPO(4)=15.0
        DAPO(4)=0.0

        H3=30.0-FLOAT(NP)
        C1=4.0
        C2=1.0
        C3=1.0/H3
        C4=2.0*(1.0+1.0/H3)
        F1=6.0*(DAPO(3)-DAPO(2)-DAPO(2)+DAPO(1))
        F2=6.0/H3*((DAPO(4)-DAPO(3))/H3-DAPO(3)+DAPO(2))
        P1=0.0
        P2=(F1*C4-F2)/(C1*C4-C3)
        P3=F1-C1*P2
        P4=0.0
```

```
699     HOLD=100.0
        DO 600 LP4=NP,30
        POS=FLOAT(LP4)
        DUM1=P3/6.0/H3*(30.0-POS)**3
        DUM2=(DAPO(3)/H3-H3*P3/6.0)*(30.0-POS)
        DUM0=DUM1+DUM2
        IF(DUM0)610,620,620
620     IF(HOLD-DUM0)610,630,630
610     P3=0.95*P3
        GO TO 699
630     HOLD=DUM0
600     DATA(LP4)=DUM0

        SUM=0.0
        DO 500 LP5=1,30

        DUM0=100.0*DATA(LP5)
        MUD0=IFIX(DUM0)
        IF(MUD0-90)505,505,506
506     MUD0=90
505     SUM=SUM+DATA(LP5)
500     WRITE(3,510) TAU(LP5),DATA(LP5),(MU5(I),I=1,MUD0)
510     FORMAT(F7.2,E20.7,5X,100A1)

        PROB=SUM*0.5
        WRITE(3,515) PROB
515     FORMAT(///10X'PROBABILITY='F7.4////)

        WRITE(1,700)
700     FORMAT('APPROVE? THEN TYPE IN 1.')
        READ(1,705) IN
705     FORMAT(I1)
        IF(IN-1)199,720,199
720     CALL OOPEN('SYS',DA)
        WRITE(4,730) NDA,A,O,(TAU(I),DATA(I),I=1,NDA)
730     FORMAT(A2,62A6)
        CALL OCLOSE

100     CONTINUE
        END
```

```
C       PROGRAM SAPOL

        COMMON TAUB,DATAB,TAU,DATA
        DIMENSION TAUB(150),DATAB(150),MUG(100),TAU(8,30)
        DIMENSION DATA(8,30),APO(6),DAPO(6),LOPO(6),PAD(6)

        KPO=3
        NPO=KPO-1
        DO 50 I=1,100
50      MUG(I)=-1376

        CALL IOPEN('SYS','ZXHO')
        READ(4,100) NDA,(TAUB(I),DATAB(I),I=1,NDA)
100     FORMAT(A2,400A6)

        DO 120 I=1,30
        MUDB=5*I
        TAU(1,I)=TAUB(MUDB)
120     DATA(1,I)=DATAB(MUDB)

        DO 200 LP1=1,9

        DO 300 LP2=1,5

        IND=LP2+1
S       CLA CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

        CALL IOPEN('SYS',DA)
300     READ(4,310) NDA,A,B,(TAU(IND,J),DATA(IND,J),J=1,NDA)
310     FORMAT(A2,62A6)

        DO 400 LP3=1,30

        DAPO(1)=DATA(1,LP3)
        APO(1)=0.0
        KP=1
```

```
      DO 500 LP4=KPO,5

      KP=KP+1
      IND=LP4+1
      DAPO(KP)=DATA(IND,LP3)
500   APO(KP)=LP4

      C1=8.0
      C2=1.0
      C3=1.0
      C4=4.0
      F1=6.0*(DAPO(3)-DAPO(2)-(DAPO(2)-DAPO(1))/3.0)
      F2=6.0*(DAPO(4)-DAPO(3)-DAPO(3)+DAPO(2))
      P1=0.0
      P2=(F1*C4-F2)/(C1*C4-C3)
      P3=F1-C1*P2
      P4=0.0

      DO 600 LP5=1,NPO
      IND=LP5+6
      POS=FLOAT(LP5)
      DUM0=P2/18.0*POS**3+(DAPO(2)/3.0-P2/2.0)*POS
     1 +(DAPO(1)/3.0)*(3.0-POS)
      IF(DUM0)610,600,600
610   DUM0=0.0
600   DATA(IND,LP3)=DUM0

400   CONTINUE

      DO 700 LP6=1,NPO

      WRITE(3,702) Q
702   FORMAT('1',10X'Q='F5.1////)

      DO 700 LP7=2,2

      LAG=5*(LP7-1)
      SUM=0.0

      DO 750 LP8=1,30

      IND2=LP6+1+LAG
      DUM1=DATA(IND2,LP8)
      DUM0=100.0*DUM1
      MUD0=IFIX(DUM0)
      IF(MUD0-90)705,705,706
706   MUD0=90
705   DUM0=0.5*FLOAT(LP8)

      IF(LP8-1)755,755,756
755   SUM=SUM+DUM1
      GO TO 750
756   IF(LP8-30)757,755,755
757   SUM=SUM+2.0*DUM1
```

```
750     WRITE(3,710) DUM0,DUM1,(MUG(J),J=1,NUD0)
710     FORMAT(F7.2,E20.7,5X,100A1)

        PROB=0.5*SUM*0.5
        WRITE(3,720) PROB
720     FORMAT(///10X'PROBABILITY='F7.4////)

S       CLA CLL
S       TAD \LP6
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLA
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

        A=FLOAT(LP6)
        CALL OOPEN('SYS',DA)
        WRITE(4,791) NDA,A,0,(TAU(1,J),DATA(IND2,J),J=1,NDA)
791     FORMAT(A2,62A6)
        CALL OCLOSE

700     CONTINUE
200     CONTINUE
        END
```

```
C       PROGRAM INPOL

        DIMENSION TAU(31),DATA(31),MUG(100),PTA(150),PDA(150),C(4,4)
        DIMENSION F(4),DAPO(6),P(6)

        DO 50 I=1,100
50      MUG(I)=-1376

        C(1,1)=4.0
        C(1,2)=1.0
        C(1,3)=0.0
        C(1,4)=0.0
        C(2,1)=1.0
        C(2,2)=4.0
        C(2,3)=1.0
        C(2,4)=0.0
        C(3,1)=0.0
        C(3,2)=1.0
        C(3,3)=4.0
        C(3,4)=1.0
        C(4,1)=0.0
        C(4,2)=0.0
        C(4,3)=1.0
        C(4,4)=4.0
        D=0.0
        M=4
        CALL MATIN(C,M,D)

        DO 100 LP1=1,5

        DO 100 LP2=3,9

S       CLA CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

199     CALL IOPEN('SYS',DA)
        READ(4,180) NDA,A,Q,(TAU(I),DATA(I),I=2,31)
180     FORMAT(A2,62A6)

        TAU(1)=0.0
        DATA(1)=0.0
        KEEP=0
```

```
      DO 200 LP3=1,26

      IND=LP3
      DO 300 LP4=1,6
      DAPO(LP4)=DATA(IND)
300   IND=IND+1

      DO 320 I=1,4
320   F(I)=6.0*(DAPO(I+2)-2.*DAPO(I+1)+DAPO(I))

      IND=1
      DO 350 I=1,4
      IND=IND+1
      P(IND)=0.0
      DO 350 J=1,4
350   P(IND)=P(IND)+C(I,J)*F(J)
      P(1)=0.0
      P(6)=0.0

      IF(LP3-1)410,410,400
400   IF(LP3-26)420,430,430
410   LOC=0
      GO TO 450
430   LOC=2
      GO TO 450

420   IND1=KEEP
      DO 421 LP5=1,5
      IND1=IND1+1
      DTAU=0.2*FLOAT(LP5)
      PTA(IND1)=0.1*FLOAT(IND1)
      PDA(IND1)=P(3)*(1.0-DTAU)**3/6.0+P(4)*DTAU**3/6.0
     1 +(DAPO(4)-P(4)/6.0)*DTAU+(DAPO(3)-P(3)/6.0)*(1.0-DTAU)
      IF(PDA(IND1))425,425,421
425   P(3)=0.9*P(3)
      P(4)=0.9*P(3)
      GO TO 420
421   CONTINUE
      KEEP=IND1
      GO TO 200

450   IND1=KEEP
      DO 451 LP6=1,3
      DO 451 LP7=1,5
      IND1=IND1+1
      DTAU=0.2*FLOAT(LP7)
      PTA(IND1)=0.1*FLOAT(IND1)
      K=LP6+LOC
      PDA(IND1)=P(K)*(1.0-DTAU)**3/6.0+P(K+1)*DTAU**3/6.0
     1 +(DAPO(K+1)-P(K+1)/6.0)*DTAU+(DAPO(K)-P(K)/6.0)*(1.0-DTAU)
451   CONTINUE
      KEEP=IND1
      GO TO 200

200   CONTINUE
```

```
      SUM=0.0
      DO 600 I=1,150
600   SUM=SUM+PDA(I)
      PROB=SUM*0.1
      DO 620 I=1,150
620   PDA(I)=PDA(I)/PROB
      NDA=150

      CALL OOPEN('SYS',DA)
      WRITE(4,630) NDA,A,O,(PTA(I),PDA(I),I=1,NDA)
630   FORMAT(A2,302A6)
      CALL OCLOSE

100   CONTINUE

      END
```

```
C       PROGRAM PCON3

        COMMON TAU,DA0,DA1,MUG
        DIMENSION PDET(20),TL(20),TH(20),TINV(20),SMAL(20),RAT(20)
        DIMENSION TAU(150),DA0(150),DA1(150),MUG(100)

        DO 50 I=1,100
50      MUG(I)=-1376

        CALL IOPEN('SYS','ZXNO')
        READ(4,60) NDA,(TAU(I),DA0(I),I=1,NDA)
60      FORMAT(A2,302A6)

        DO 100 LP1=1,5

        DO 100 LP2=3,9

S       CLA CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

199     CALL IOPEN('SYS',DA)
        READ(4,200) NDA,A,Q,(TAU(I),DA1(I),I=1,NDA)
200     FORMAT(A2,302A6)

        GO TO 220
        WRITE(3,210) A,Q
210     FORMAT('1'10X,'A='F3.0,10X,'Q='F4.1///)

220     J=0
        DO 400 LP3=1,20

        J=J+1
        PDET(J)=1.0-0.005*FLOAT(LP3)
        TP=10.0*PDET(J)
        HOLD=DA1(1)
        LOL=1
        LOH=1
        IND1=1
        SMAL(J)=1.0
```

```
      DO 380 LP4=1,149

      HOLD=HOLD-0.5*(DA1(LOL)+DA1(LP4))
      LOL=LP4
      SUM1=HOLD

315   IF(SUM1-TP)320,330,350
320   IND1=IND1+1
      IF(IND1-150)322,322,380
322   HOLD=SUM1
      SUM1=SUM1+0.5*(DA1(LOH)+DA1(IND1))
      LOH=IND1
      GO TO 315

330   SUM0=0.0
      DO 335 I=LOL,LOH
      IF(I-LOL)336,336,337
337   IF(I-LOH)338,336,336
336   SUM0=SUM0+0.5*DA0(I)
      GO TO 335
338   SUM0=SUM0+DA0(I)
335   CONTINUE
      PROB=SUM0*0.1
      TINT=0.0
      GO TO 360

350   DUM2=SUM1-HOLD
      DUM1=TP-HOLD
      TINT=DUM1/DUM2

      SUM0=0.0
      LOH=LOH-1
      DO 355 I=LOL,LOH
      IF(I-LOL)356,356,357
357   IF(I-LOH)358,356,356
356   SUM0=SUM0+0.5*DA0(I)
      GO TO 355
358   SUM0=SUM0+DA0(I)
355   CONTINUE

      HOLD0=SUM0
      SUM0=SUM0+0.5*(DA0(LOH)+DA0(IND1))
      PROB=(HOLD0+TINT*(SUM0-HOLD0))*0.1
      IND1=LOH
360   IF(PROB-SMAL(J))370,380,380
370   SMAL(J)=PROB
      TL(J)=TAU(LOL)
      TH(J)=TAU(LOH)+TINT*0.1
      TINU(J)=TH(J)-TAU(LOL)

380   CONTINUE
```

```
      RAT(J)=PDET(J)/SMAL(J)
      GO TO 400
390   WRITE(3,392) PDET(J),TL(J),TH(J),TINU(J),SMAL(J),RAT(J)
392   FORMAT(4F8.3,2E15.7)

400   CONTINUE

      NDA=J
S     CLA CLL
S     TAD \LP1
S     TAD (60
S     RTL;RTL;RTL
S     TAD (46
S     DCA \DA
S     CLL
S     TAD \LP2
S     TAD (60
S     RTL;RTL;RTL
S     DCA \DA%

      CALL OOPEN('SYS',DA)
      WRITE(4,410) NDA,
     1(PDET(I),TL(I),TH(I),TINU(I),SMAL(I),RAT(I),I=1,NDA)
410   FORMAT(A2,200A6)
      CALL OCLOSE

100   CONTINUE
      END
```

```
C         PROGRAM RCON

          DIMENSION PD(20),TL(20),TH(20),TI(20),PF(20),RA(20),F(21)

          F(1)=1.0
          F(2)=1.0
          F(3)=2.0
          F(4)=6.0
          F(5)=24.0
          F(6)=120.0
          F(7)=720.0
          F(8)=5040.0
          F(9)=40320.0
          F(10)=3.6288*10.0**5
          F(11)=3.6288*10.0**6
          F(12)=3.99168*10.0**7
          F(13)=4.79*10.0**8
          F(14)=6.227*10.0**9
          F(15)=8.71783*10.0**10
          F(16)=1.30767*10.0**12
          F(17)=2.09228*10.0**13
          F(18)=3.55687*10.0**14
          F(19)=6.40237*10.0**15
          F(20)=1.21645*10.0**17
          F(21)=2.43290*10.0**18

          DO 100 LP1=4,5

          DO 100 LP2=4,4

S         CLA CLL
S         TAD \LP1
S         TAD (60
S         RTL;RTL;RTL
S         TAD (46
S         DCA \DA
S         CLL
S         TAD \LP2
S         TAD (60
S         RTL;RTL;RTL
S         DCA \DA#

          CALL IOPEN('SYS',DA)
          READ(4,110) NDA,
        1 (PD(I),TL(I),TH(I),TI(I),PF(I),RA(I),I=1,NDA)
110       FORMAT(A2,200A6)

          Q=0.1*FLOAT(LP2)
          WRITE(3,120) LP1,Q
120       FORMAT('1'10X'A='I2,10X'Q='F4.1///)
```

```
      DO 100 LP3=4,8,2
      N=2*LP3

      DO 100 LP4=10,10
      AL=0.05*FLOAT(LP4)

      DO 100 LP5=LP3,N
      WRITE(3,130) LP3,AL,LP5
130   FORMAT(1X,I10,F10.2,I10)

      DO 300 LP8=1,18
      SUM1=0.0
      KM=N-LP5+1

      DO 200 LP6=1,KM
      C=F(N+1)/F(LP6)/F(N-LP6+2)
      DUM0=PF(LP8)
      MUD1=N-LP6+1
      MUD2=LP6-1
200   SUM1=SUM1+C*DUM0**MUD1*(1.0-DUM0)**MUD2

      GO TO 220
      IF(SUM1-0.005)100,210,210
210   IF(SUM1-AL)220,220,300
220   SUM2=0.0

      DO 250 LP7=1,KM
      C=F(N+1)/F(LP7)/F(N-LP7+2)
      DUM0=PD(LP8)
      MUD1=N-LP7+1
      MUD2=LP7-1
250   SUM2=SUM2+C*DUM0**MUD1*(1.0-DUM0)**MUD2

      DUM0=SUM2/SUM1
      WRITE(3,270) LP8,PD(LP8),PF(LP8),SUM1,SUM2,DUM0
270   FORMAT(10X,I3,5E15.7)

300   CONTINUE
100   CONTINUE

      END
```

```
C         PROGRAM PLIN

          PI=3.1415927
          A0=0.2316419
          A1=0.3193815
          A2=-0.3565638
          A3=1.781478
          A4=-1.821256
          A5=1.330274
          LAV=10

          DO 100 LP1=1,3
          A=FLOAT(LP1)

          DO 100 LP2=4,4
          Q=0.1*FLOAT(LP2)
          WRITE(3,110) LP1,Q
110       FORMAT('1'10X'A='I2,10X'Q='F4.1///)

          DO 100 LP3=4,8,2
          T=2.0*PI*FLOAT(LP3)/Q
          WRITE(3,120) LP3
120       FORMAT(/5X'N='I2)

          SUM2=0.0
          DO 200 LP4=1,LAV

          MUD=LP4-1
          THE=FLOAT(MUD)*PI/FLOAT(LAV)
          SUM1=0.0
          K=-1
220       K=K+1
          TK=THE+FLOAT(K)*PI
          IF(TK-T)230,230,200
230       DUM0=Q*TK
          S=SIN(DUM0)
          SUM1=SUM1+S**2
          GO TO 220
200       SUM2=SUM2+SUM1
          EN=0.5*SUM2*A**2/FLOAT(LAV)

          DO 100 LP5=1,26
          MUD=LP5-6
          DUM0=1.0*FLOAT(MUD)
          VEL=DUM0*ABS(DUM0)/PI
```

```
      X0=(EN+VEL)/(2.0*EN)**0.5
      DUM0=-X0**2/2.0
      Z=EXP(DUM0)/(2.0*PI)**0.5
      DUM1=ABS(X0)
      Y=1.0/(1.0+A0*DUM1)
      P0=Z*(A1*Y+A2*Y**2+A3*Y**3+A4*Y**4+A5*Y**5)
      IF(X0)250,250,251
250   P0=1.0-P0

251   X1=(-EN+VEL)/(2.0*EN)**0.5
      DUM0=-X1**2/2.0
      Z=EXP(DUM0)/(2.0*PI)**0.5
      DUM1=ABS(X1)
      Y=1.0/(1.0+A0*DUM1)
      P1=Z*(A1*Y+A2*Y**2+A3*Y**3+A4*Y**4+A5*Y**5)
      IF(X1)260,260,261
260   P1=1.0-P1

261   RAT=P1/P0
      WRITE(3,300) LP5,VEL,X0,X1,P0,P1,RAT
300   FORMAT(10X,I3,6E15.7)

100   CONTINUE
      END
```

```
C       PROGRAM PSEG

        PI=3.1415927
        A0=0.2316419
        A1=0.3193815
        A2=-0.3565638
        A3=1.781478
        A4=-1.821256
        A5=1.330274
        LAV=10
        DO 100 LP8=2,100,2
        QN=0.01*FLOAT(LP8)

        DO 100 LP1=1,2
        A=FLOAT(LP1)

        DO 100 LP7=1,11
        MUD=LP7-1
        AN=A*FLOAT(MUD)

        DO 100 LP2=4,4
        Q=0.1*FLOAT(LP2)
        WRITE(3,110) A,Q,AN,QN
110     FORMAT(/////////10X'A='F5.2,10X'Q='F5.2,10X'AN='F5.2
     1  ,10X'QN='F5.2///)

        DO 100 LP3=4,6,2
        T=2.0*PI*FLOAT(LP3)/Q
        WRITE(3,120) LP3
120     FORMAT(/5X'N='I2)

        SUM2=0.0
        TSUMN=0.0
        DO 200 LP4=1,LAV

        MUD=LP4-1
        THE=FLOAT(MUD)*PI/FLOAT(LAV)
        SUM1=0.0
        SUMN=0.0
        K=-1
220     K=K+1
        TK=THE+FLOAT(K)*PI
        IF(TK-T)230,230,210
230     DUMO=Q*TK
        S=SIN(DUMO)
        SUM1=SUM1+S**2
        DUMO=QN*TK
        SN=SIN(DUMO)
        COR=S*SN
        SUMN=SUMN+COR
        GO TO 220
210     SUM2=SUM2+SUM1
200     TSUMN=TSUMN+SUMN
        EN=0.5*SUM2*A**2/FLOAT(LAV)
        EN2=TSUMN*A*AN/FLOAT(LAV)-EN
```

```
      DO 100 LP5=1,15
      MUD=LP5-6
      DUM0=1.0*FLOAT(MUD)
      VEL=DUM0*ABS(DUM0)/PI

      X0=(-EN2+VEL)/(2.0*EN)**0.5
      DUM0=-X0**2/2.0
      Z=EXP(DUM0)/(2.0*PI)**0.5
      DUM1=ABS(X0)
      Y=1.0/(1.0+A0*DUM1)
      P0=Z*(A1*Y+A2*Y**2+A3*Y**3+A4*Y**4+A5*Y**5)
      IF(X0)250,250,251
250   P0=1.0-P0

251   X1=(-EN+VEL)/(2.0*EN)**0.5
      DUM0=-X1**2/2.0
      Z=EXP(DUM0)/(2.0*PI)**0.5
      DUM1=ABS(X1)
      Y=1.0/(1.0+A0*DUM1)
      P1=Z*(A1*Y+A2*Y**2+A3*Y**3+A4*Y**4+A5*Y**5)
      IF(X1)260,260,261
260   P1=1.0-P1

261   RAT=P1/P0
      WRITE(3,300) LP5,VEL,X0,X1,P0,P1,RAT
300   FORMAT(10X,I3,6E15.7)

100   CONTINUE
      END
```

```
C       PROGRAM PCOM4

        COMMON TAU,DA0,DA1,MUS
        DIMENSION PDET(20),TL(20),TH(20),TINU(20),SMAL(20),RAT(20)
        DIMENSION TAU(150),DA0(150),DA1(150),MUS(100)

        DO 50 I=1,100
50      MUS(I)=-1376

        WRITE(1,57)
57      FORMAT(1X'TYPE IN THE REFERENCE DATA.')
        READ(1,59) DAS
59      FORMAT(A6)
        CALL IOPEN('SYS',DAS)
        READ(4,60) NDA,AS,QS,(TAU(I),DA1(I),I=1,NDA)
60      FORMAT(A2,302A6)

        DO 100 LP1=2,2

        DO 100 LP2=3,9

S       CLA CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       TAD (56
S       DCA \DA
S       CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

199     CALL IOPEN('SYS',DA)
        READ(4,200) NDA,AN,QN,(TAU(I),DA0(I),I=1,NDA)
200     FORMAT(A2,302A6)

        WRITE(3,210) AS,QS,AN,QN
210     FORMAT('1'10X,'AS='F3.0,10X,'QS='F4.1,10X'AN='F3.0,10X
     1  'QN='F4.1///)

220     J=0
        DO 400 LP3=1,20

        J=J+1
        PDET(J)=1.0-0.05*FLOAT(LP3)
        TP=10.0*PDET(J)
        HOLD=DA1(1)
        LOL=1
        LOH=1
        IND1=1
        SMAL(J)=1.0
```

```
      DO 300 LP4=1,149

      HOLD=HOLD-0.5*(DA1(LOL)+DA1(LP4))
      LOL=LP4
      SUM1=HOLD

315   IF(SUM1-TP)320,330,350
320   IND1=IND1+1
      IF(IND1-150)322,322,300
322   HOLD=SUM1
      SUM1=SUM1+0.5*(DA1(LOH)+DA1(IND1))
      LOH=IND1
      GO TO 315

330   SUM0=0.0
      DO 335 I=LOL,LOH
      IF(I-LOL)336,336,337
337   IF(I-LOH)338,336,336
336   SUM0=SUM0+0.5*DA0(I)
      GO TO 335
338   SUM0=SUM0+DA0(I)
335   CONTINUE
      PROB=SUM0*0.1
      TINT=0.0
      GO TO 360

350   DUM2=SUM1-HOLD
      DUM1=TP-HOLD
      TINT=DUM1/DUM2

      SUM0=0.0
      LOH=LOH-1
      DO 355 I=LOL,LOH
      IF(I-LOL)356,356,357
357   IF(I-LOH)358,356,356
356   SUM0=SUM0+0.5*DA0(I)
      GO TO 355
358   SUM0=SUM0+DA0(I)
355   CONTINUE

      HOLD0=SUM0
      SUM0=SUM0+0.5*(DA0(LOH)+DA0(IND1))
      PROB=(HOLD0+TINT*(SUM0-HOLD0))*0.1
      IND1=LOH
360   IF(PROB-SMAL(J))370,300,300
370   SMAL(J)=PROB
      TL(J)=TAU(LOL)
      TH(J)=TAU(LOH)+TINT*0.1
      TINV(J)=TH(J)-TAU(LOL)

300   CONTINUE
```

```
        RAT(J)=PDET(J)/SMAL(J)
390     WRITE(3,392) PDET(J),TL(J),TH(J),TINW(J),SMAL(J),RAT(J)
392     FORMAT(4F8.3,2E15.7)

400     CONTINUE

        NDA=J
S       CLA CLL
S       TAD \LP1
S       TAD (60
S       RTL;RTL;RTL
S       TAD (46
S       DCA \DA
S       CLL
S       TAD \LP2
S       TAD (60
S       RTL;RTL;RTL
S       DCA \DA#

        CALL OOPEN('SYS',DA)
        WRITE(4,410) NDA,
     1  (PDET(I),TL(I),TH(I),TINW(I),SMAL(I),RAT(I),I=1,NDA)
410     FORMAT(A2,200A6)
        CALL OCLOSE

100     CONTINUE
        END
```

```
C       PROGRAM PNOI

        PI=3.1415927
        A0=0.2316419
        A1=0.3193815
        A2=-0.3565638
        A3=1.781478
        A4=-1.821256
        A5=1.330274
        LAV=10

        DO 100 LP1=1,3
        A=FLOAT(LP1)

        DO 100 LP7=2,10,2
        CNOI=FLOAT(LP7)

        DO 100 LP2=3,9,2
        Q=0.1*FLOAT(LP2)
        WRITE(3,110) A,Q,CNOI
110     FORMAT('1'10X'A='F5.2,10X'Q='F4.1,10X'NOISE INC='F5.2///)

        DO 100 LP3=4,8,2
        T=2.0*PI*FLOAT(LP3)/Q
        WRITE(3,120) LP3
120     FORMAT(/5X'N='I2)

        SUM2=0.0
        DO 200 LP4=1,LAV

        MUD=LP4-1
        THE=FLOAT(MUD)*PI/FLOAT(LAV)
        SUM1=0.0
        K=-1
220     K=K+1
        TK=THE+FLOAT(K)*PI
        IF(TK-T)230,230,200
230     DUM0=Q*TK
        S=SIN(DUM0)
        SUM1=SUM1+S**2
        GO TO 220
200     SUM2=SUM2+SUM1
        EN=0.5*SUM2*A**2/FLOAT(LAV)

        DO 100 LP5=1,10
        MUD=LP5-1
        DUM0=1.0*FLOAT(MUD)
        VEL=DUM0**2/PI
```

```
      X0=(EN+VEL)/(2.0*EN)**0.5/CHOI
      DUM0=-X0**2/2.0
      Z=EXP(DUM0)/(2.0*PI)**0.5
      DUM1=ABS(X0)
      Y=1.0/(1.0+A0*DUM1)
      P0=Z*(A1*Y+A2*Y**2+A3*Y**3+A4*Y**4+A5*Y**5)
      IF(X0)250,250,251
250   P0=1.0-P0

251   X1=(-EN+VEL)/(2.0*EN)**0.5
      DUM0=-X1**2/2.0
      Z=EXP(DUM0)/(2.0*PI)**0.5
      DUM1=ABS(X1)
      Y=1.0/(1.0+A0*DUM1)
      P1=Z*(A1*Y+A2*Y**2+A3*Y**3+A4*Y**4+A5*Y**5)
      IF(X1)260,260,261
260   P1=1.0-P1

261   RAT=P1/P0
      WRITE(3,300) LP5,VEL,X0,X1,P0,P1,RAT
300   FORMAT(10X,I3,6E15.7)

100   CONTINUE
      END
```

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BIOGRAPHICAL SKETCH

Born March 14, 1944, Woon Cheon Yeo was raised and educated in the city of Seoul, Korea, until his graduation in 1969 from the Seoul National University. To continue his education in engineering and life sciences, he started in the master's program in the field of Biomedical Engineering at the University of Florida. He received the degree of Master of Science in Electrical Engineering from this University in 1970. While pursuing the Master's and Ph.D. degrees, Mr. Yeo was engaged in the research of automated analysis of sleep electroencephalograms in the Department of Electrical Engineering. Mr. Yeo is a member of Eta Kappa Nu and the Institute of Electrical and Electronics Engineers.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Jack R. Smith, Chairman
Professor of Electrical Engineering

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Donald G. Childers
Professor of Electrical Engineering

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Arnold Paige
Associate Professor of Speech and Electrical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Dong S. Kim
Assistant Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Michael E. Warren
Assistant Professor of Electrical Engineering

This dissertation was submitted to the Graduate Faculty of the College of Engineering and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December, 1975

Dean, College of Engineering

Dean, Graduate School