TENSOR PRODUCTS OF SPACES OF MEASURES AND VECTOR INTEGRATION IN TENSOR PRODUCT SPACES

by

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To my Parents

Whose faith in me never waivered.

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This dissertation investigates the concepts of measure and integration within the framework of the topological tensor product of two Banach spaces. In Chapter I, basic existence theorems are given for the tensor product of two vector measures. The topological tensor product of certain spaces of measures is studied in Chapter II, where the space of all measures with the Radon-Nikodym Property and the space of all measures with relatively norm compact range are identified in terms of tensor products. Chapter III discusses the theory of integration of vector valued functions with respect to a vector measure; the value of the integral is in the inductive tensor product of the range spaces, and the integral is a generalization of B. J. Pettis' weak integral. Normed Pettis and Bochner-Lebesgue spaces are considered and the Vitali and Lebesgue Dominated Convergence theorems are proved. Finally, in Chapter IV, the integration theory of Chapter III is used with the product measures discussed in Chapter I and II, to prove some Fubini theorems for product integration.

INTRODUCTION

This dissertation concerns the topological tensor product of certain spaces of measures, and integration theory for vector valued functions with respect to a vector-valued measure.

The tensor product of spaces of scalar measures with arbitrary Banach spaces has been studied by Gil de Lamadrid [13] and most recently by D. R. Lewis [16]; however, with the introduction of the notions of the inductive product measure in 1967 by Duchon and Kluvánek [11], and the projective product measure in 1969 by Duchon [9], it is possible to study the topological tensor product of two spaces of measures. The existence of the inductive and projective product measures was shown in [9] and [11] by essentially two different methods. In Chapter I, we generalize a lemma of Duchon and Kluvánek from which we obtain those product measures directly. In Chapter II, we then study various tensor products of spaces of measures; using tensor products we obtain various isometric embeddings into natural spaces of vector measures. Characterizations of certain spaces of vector measures are obtained as a consequence of this study; for example, we identify in Chapter II the space of all X-valued measures, where X is a Banach space, on which the vector form of the Radon-Nikodym theorem is valid. Aside from its own intrinsic value, the study of tensor products of spaces of measures can be used to attack the very important problems of establishing criteria

for weak and norm compactness of sets of vector measures; this method is exemplified by Lewis' paper on weak compactness [16].

For X and Y Banach spaces, probably the most natural integration theory for X-valued functions with respect to a Y-valued measure is developed in Chapter III; in this chapter, we define the strong, the weak, and the "Pettis" integrals, which are successively inclusive. Each of these integrals takes its values in the inductive tensor product space $X \hat{\otimes}_\varepsilon Y$. On the space of strongly integrable functions, a norm is defined which makes it into a Banach space and integral convergence is characterized by norm convergence; the strong integral reduces to the Bochner integral when the measure is scalar valued, and is a special case of the Brooks-Dinculeanu integral defined in [5]. The weak integral is proven to be a particular case of Bartle's bilinear integral [1]. In his general theory, Bartle defines an integral which is Z-valued, where Z is a Banach space, where he presupposes the existence of a fixed bilinear map from $X \times Y$ into Z. In our context, the bilinear map is the canonical one from $X \times Y$ into $X \hat{\otimes}_\varepsilon Y$, and we obtain Bartle's theory; however, more can be said. A norm can be defined on the space of weakly integrable functions which characterizes integral convergence, and the Lebesgue Dominated Convergence theorem is obtained as well as the Vitali Convergence theorem. Finally, we define the Pettis integral for weakly measurable X-valued functions with respect to a Y-valued measure. In case the measure is scalar valued, the Pettis

integral is precisely Pettis' weak integral defined in [17], and for strongly measurable functions, reduces to the weak integral.

In Chapter IV, the notion of tensor product measure as discussed in Chapters I and II, and the integration theory of Chapter III, are combined to obtain vector forms of the Fubini theorem. In order to obtain the main result (Theorem IV.3.6) it was necessary to assume that one of the two measures has the Beppo Levi Property, a property analogous to the Beppo Levi theorem. This property seems essential in proving a general Fubini theorem, and it avoids making the even stronger assumption that both measures have finite variation.

Throughout the dissertation, some related topics in Operator Theory are discussed.

CHAPTER I
TENSOR PRODUCTS OF VECTOR MEASURES

1. Basic Notions.

We shall begin by establishing notation and basic concepts used throughout this dissertation.

X, Y, and Z will always denote abstract Banach spaces over the same scalar field (real or complex). The norm of a vector $x \in X$ is the number $|x|$. X^* is the continuous dual of X and X_1^* denotes the unit sphere of X^*, that is, $X_1^* = \{x^* \in X^* : |x^*|=1\}$. If $x \in X$ and $x^* \in X^*$, then the action of x^* on x is denoted by $x^*(x)$, $\langle x^*,x\rangle$, or $\langle x,x^*\rangle$. The scalar field is denoted by Φ, unless otherwise specified. R is the set of real numbers, R^+ the nonnegative real numbers, $R\# = R^+ \cup \{\infty\}$, and ω is the collection of all natural numbers.

An algebra A of subsets of a pointset S is a family of subsets of S closed under finite unions and complements. Ω is a σ-algebra of subsets of S if Ω is an algebra of subsets of S and is closed under countable unions. The ordered pair (S,Ω) consisting of a pointset S and a σ-algebra of subsets of S form a measurable space. Any function $\mu: A \to X$ is called a set function on A. A set function μ is countably additive (σ-additive) if for every disjoint sequence $(A_i) \subseteq A$, with $\cup_i A_i \in A$ implies

$$\mu(\cup_i A_i) = \Sigma_i \mu(A_i),$$

where the convergence of the infinite series is unconditional.

The set function μ is finitely additive if the above equality holds for every finite disjoint family $(A_i)_{i=1}^n \subseteq \mathcal{A}$. A set function $\mu:\mathcal{A} \to X$ is a measure if the algebra $\mathcal{A}$ is a σ-algebra and μ is σ-additive. A measure μ will sometimes be referred to as a vector valued measure, an X-valued measure, or simply a vector measure. A measure which takes its values in the scalar field Φ is called a scalar measure; if the range of a measure is R^+, it is a positive measure.

For $A \in \mathcal{A}$, let $\Pi(A)$ denote the collection of all measurable partitions of A, that is, the collection of all finite disjoint families $(A_i)_{i=1}^n \subseteq \mathcal{A}$ such that $A = \bigcup_{i=1}^n A_i$.

The set function μ has a variety of associated $R^\#$-valued set functions: the semivariation of μ, the quasivariation of μ, and the total variation of μ. They are defined for $A \in \mathcal{A}$ as follows:

(1) Semivariation:

$$\|\mu\|(A) = \sup\{|\textstyle\sum_{i=1}^n \alpha_i \mu(A_i)| : \alpha_i \in \Phi, |\alpha_i| \leq 1, (A_i)_{i=1}^n \in \Pi(A)\}$$

(2) Quasivariation:

$$\overline{\mu}(A) = \sup\{|\mu(B)| : B \in \mathcal{A}, \ B \subseteq A\}$$

(3) Total variation:

$$|\mu|(A) = \sup\{\textstyle\sum_{i=1}^n |\mu(A_i)| : (A_i)_{i=1}^n \in \Pi(A)\}.$$

Sometimes it is convenient to extend the definition of the semivariation of μ from the algebra $\mathcal{A}$ to the power set of S as follows: for $E \subseteq S$,

$$\|\mu\|(E) = \inf\{\|\mu\|(A) : A \in \mathcal{A}, \ E \subseteq A\}.$$

We remark that if μ is an X-valued set function on the algebra A, and $x^* \in X^*$, then we can define a scalar set function $x^*\mu$ by $x^*\mu(A) = \langle x^*, \mu(A)\rangle$, $A \in A$. We now state a proposition which will help establish a relationship between the three variation set functions and which is of vital importance throughout this dissertation.

1.1 Proposition. (Dinculeanu [7, p.55]) For $A \in A$,

$$\|\mu\|(A) = \sup_{x^* \in X_1^*} |x^*\mu|(A).$$

It is well known that if λ is a scalar set function on A, then $\overline{\lambda}(A) \leq |\lambda|(A) \leq 4\overline{\lambda}(A)$ for all $A \in A$, from this we see that $\overline{x^*\mu}(A) \leq |x^*\mu|(A) \leq 4\overline{x^*\mu}(A)$, for all $x^* \in X^*$. Taking the supremum over X_1^* we get

$$\overline{\mu}(A) \leq \|\mu\|(A) \leq 4\overline{\mu}(A), \quad A \in A.$$

Thus, the semivariation and the quasivariation are equivalent in the sense that $\overline{\mu}(A) = 0$ if and only if $\|\mu\|(A) = 0$. If A is a σ-algebra and μ is a measure, then μ has bounded semivariation: $\sup_{A \in A} \|\mu\|(A) < +\infty$. In the same situation, we may still have $|\mu|(S) = +\infty$; it is for this reason that the semivariation of a vector measure is used as a "control" set function. If $|\mu|(S) < +\infty$, μ is said to have bounded variation or finite total variation. If μ is scalar set function, then $|\mu|(A) = \|\mu\|(A)$ for all $A \in A$.

Let $\mu: A \to X$ and $\lambda: A \to R^+$ be set functions. We write $\mu \ll \lambda$ and say μ is absolutely continuous with respect to λ if

$$\lim_{\lambda(A) \to 0} \mu(A) = 0, \quad A \in A,$$

that is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $A \in \mathcal{A}$ such that $\lambda(A) < \delta$ we have $|\mu(A)| < \varepsilon$.

If $(\mu_\alpha)_{\alpha\in\Delta}$ is a family of set functions on $\mathcal{A}$, then (μ_α) is uniformly bounded provided

$$\sup\{|\mu_\alpha(A)| : A\in\mathcal{A}, \alpha \in \Delta\} < +\infty.$$

The family (μ_α) is pointwise bounded if for each $A \in \mathcal{A}$

$$\sup\{|\mu_\alpha(A)| : \alpha\in\Delta\} < +\infty.$$

It is a result of Nikodym's [12,p. 309] that if (μ_α) is a pointwise bounded family of scalar measures, then (μ_α) is uniformly bounded.

We say that the scalar measure $\lambda:\Omega \to R^+$ is a control measure for the vector measure $\mu:\Omega \to X$, where Ω is a σ-algebra, if $\mu \ll \lambda$ and $\lambda(A) \leq \overline{\mu}(A)$ for all $A \in \Omega$; consequently, we have $\mu(A) \to 0$ if and only if $\lambda(A) \to 0$. We state the following theorem taken from Dunford and Schwartz [12].

1.2 Theorem. Let $\mu:\Omega \to X$ be a vector measure. Then

(1) There exists a control measure λ for μ;

(2) There exists a sequence $(x_n^*) \subseteq X_1^*$ such that $|x_n^*\mu|(A) = 0$ for every $n \in \omega$ if and only if $\|\mu\|(A) = 0$, where $A \in \Omega$.

Proof. Part (1) follows from Corollary IV.9.3 and Lemma IV.10.5 of [12]. Part (2) is derived from the proof of Theorem IV.9.2 of [12]. □

Finally, the end of a proof and the end of a numbered remark will be denoted by □.

2. Tensor Products.

This dissertation is mainly interested in the topological tensor products of Banach spaces, and the tensor product of vector measures. We shall state the definitions of the former concept, and give basic existence theorems for the latter. A standard reference for topological tensor products is Treves [18].

We state here in the form of a theorem, the definition of the algebraic tensor product of X and Y.

2.1 Theorem. A tensor product of X and Y is a pair (M,ϕ) consisting of a vector space M and a bilinear mapping ϕ of $X \times Y$ into M such that the following conditions be satisfied.

(1) The image of $X \times Y$ spans the whole of M;

(2) X and Y are ϕ-linearly disjoint, that is, if $\{x_i\}_{i=1}^n \subseteq X$ and $\{y_i\}_{i=1}^n \subseteq Y$ such that $\sum_{i=1}^n \phi(x_i,y_i) = 0$, then the linear independence of one set of vectors implies that each member of the other set is the zero vector.

There are many equivalent definitions for the tensor product of two spaces as well as constructions available. The map ϕ is called cannonical, and the space M is unique up to vector space isomorphism. This follows from the universal mapping property of M; namely, if G is a vector space and $b: X \times Y \to G$ is a bilinear map, then there exists a unique linear map $\overline{b}: M \to G$ such that $b = \overline{b} \circ \phi$. The space M is usually denoted by $X \otimes Y$, and the elements of the cannonical image of $X \times Y$ by $\phi(x,y) = x \otimes y$; consequently, any element may be written in the form $\sum_{i=1}^n x_i \otimes y_i$ for $x_i \in X$ and $y_i \in Y$.

$X \otimes_\varepsilon Y$ is the tensor product of X and Y endowed with the ε-norm (least crossnorm): for $\theta = \sum_{i=1}^{n} x_i \otimes y_i$,

$$|\theta|_\varepsilon = \sup\{|\sum_{i=1}^{n} \langle x^*, x_i\rangle\langle y^*, y_i\rangle| : (x^*, y^*) \in X_1^* \times Y_1^*\}.$$

The completion of the normed linear space $X \otimes_\varepsilon Y$ is the Banach space $X \hat{\otimes}_\varepsilon Y$ and is called the inductive (or weak) tensor product of X and Y. $X \otimes_\pi Y$ is $X \otimes Y$ equipped with the π-norm (greatest crossnorm): for $\theta \in X \otimes_\pi Y$,

$$|\theta|_\pi = \inf\{\sum_{i=1}^{n} |x_i| \cdot |y_i| : \theta = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

The space $X \hat{\otimes}_\pi Y$ is the completion of $X \otimes_\pi Y$ and is called the projective (or strong) tensor product of X and Y. Obviously, $|\theta|_\varepsilon \leq |\theta|_\pi$, $\theta \in X \otimes Y$.

Let (S,Ω) and (T,Λ) be two measurable spaces, and $\mu:\Omega \to X$ and $\nu:\Lambda \to Y$ measures. $\Omega \otimes \Lambda$ will denote the algebra of finite disjoint unions of measurable rectangles of the set $S \times T$; $\Omega \otimes_\sigma \Lambda$ is the σ-algebra generated by $\Omega \otimes \Lambda$ and is called the product σ-algebra. We are concerned with the existence of "product" measures, $\mu \otimes \nu$, on $\Omega \otimes_\sigma \Lambda$ with values in $X \hat{\otimes}_\varepsilon Y$ or $X \hat{\otimes}_\pi Y$ subject to the identity $\mu \otimes \nu(E \times F) = \mu(E) \otimes \nu(F)$, for $E \in \Omega$ and $F \in \Lambda$.

We begin by making the following definition.

The semivariation of μ with respect to γ (for $\gamma = \varepsilon$ or π) and Y is the $R^\#$-valued set function $\|\mu\|_\gamma^Y$ on Ω defined by

$$\|\mu\|_\gamma^Y(A) = \sup\{|\sum_{i=1}^{n} \mu(A_i) \otimes y_i|_\gamma : y_i \in Y, |y_i| \leq 1, (A_i)_{i=1}^{n} \in \pi(A)\}$$

2.2 Lemma. For each $A \in \Omega$, $\|\mu\|_\varepsilon^Y(A) = \|\mu\|(A)$.

Proof. $\|\mu\|_\varepsilon^Y(A) = \sup\{|\sum_{i=1}^n \mu(A_i)\otimes y_i|_\varepsilon : (A_i)_{i=1}^n \in \Pi(A), y_i \in Y, |y_i| \le 1\}$

$= \sup\{|\sum_{i=1}^n x^*\mu(A_i)\cdot y_i| : (A_i)\in\Pi(A), |y_i|\le 1, x^*\in X_1^*\}$

$\le \sup\{\sum_{i=1}^n |x^*\mu(A_i)| : (A_i)\in\Pi(A), x^*\in X_1^*\}$

$= \sup\{|x^*\mu|(A) : x^*\in X_1^*\}$

$= \|\mu\|(A).$

The last equality is due to Proposition 1.1. Thus $\|\mu\|_\varepsilon^Y(A) \le \|\mu\|(A)$.

Now let $\varepsilon > 0$ be given, there exists $(A_i)_{i=1}^n \in \Pi(A)$ and scalars (α_i) with $|\alpha_i| \le 1$ such that

$$\|\mu\|(A) = \varepsilon + |\sum_{i=1}^n \alpha_i \mu(A_i)|.$$

Choose $y \in Y$ with $|y| = 1$ and $y^* \in Y_1^*$ such that $y^*(y) = 1$; this is possible by the Hahn-Banach theorem. Then if $y_i = \alpha_i y$ for $1 \le i \le n$, then $y^* y_i = \alpha_i$ and $|y_i| \le 1$. Thus,

$\|\mu\|(A) \le \varepsilon + |\sum_{i=1}^n y^*(y_i)\mu(A_i)|$

$\le \varepsilon + |\sum_{i=1}^n y_i \otimes \mu(A_i)|_\varepsilon$

$\le \varepsilon + \|\mu\|_\varepsilon^Y(A).$

Since $\varepsilon > 0$ was arbitrary, it follows that $\|\mu\|(A) \le \|\mu\|_\varepsilon^Y(A)$. Consequently, $\|\mu\|(A) = \|\mu\|_\varepsilon^Y(A)$. □

We now state a generalization of a lemma due to Duchon and Kluvánek which appears in [11].

(S,Ω) and (T,Λ) are measurable spaces. For vector measures $\mu:\Omega \to X$ and $\nu:\Lambda \to Y$, define

$\mu\otimes\nu:\Omega\otimes\Lambda \to X\otimes Y$ by $\mu\otimes\nu(\cup_i E_i \times F_i) = \Sigma_i \mu(E_i)\otimes\nu(F_i)$, where

$\cup_i E_i \times F_i$ is a finite disjoint union, $E_i \in \Omega$ and $F_i \in \Lambda$. Then $\mu \otimes \nu$ is a finitely additive set function on the algebra $\Omega \otimes \Lambda$.

2.3 Lemma. (Duchon and Kluvánek) Let $\gamma = \varepsilon$ or π and suppose

(1) (μ_α) is a family of X-valued measures on Ω and (ν_β) is a family on Y-valued measures on Λ;

(2) $\sup_\alpha \|\mu_\alpha\|_\gamma^Y(S) < +\infty$ and $\sup_\beta \|\nu_\beta\|(T) < +\infty$;

(3) $\lambda:\Omega \to R^+$ and $\phi:\Lambda \to R^+$ are positive measures such that $\|\mu_\alpha\|_\gamma^Y(\cdot) \ll \lambda$ uniformly in α and

$$\nu_\beta \ll \phi \text{ uniformly in } \beta.$$

Then for the family $(\mu_\alpha \otimes \nu_\beta)$ of $X \hat{\otimes}_\gamma Y$-valued finitely additive set functions on $\Omega \otimes \Lambda$, we have $\mu_\alpha \otimes \nu_\beta \ll \lambda \times \phi$ uniformly in α and β on $\Omega \otimes \Lambda$, where $\lambda \times \phi : \Omega \otimes_\sigma \Lambda \to R^+$ is the usual product measure of λ and ϕ.

Proof. We must show to every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $G \in \Omega \otimes \Lambda$ and $\lambda \times \phi(G) < \delta$, we have

$$|\mu_\alpha \otimes \nu_\beta(G)|_\gamma < \varepsilon, \text{ for all } \alpha \text{ and } \beta.$$

To that end, let $\varepsilon > 0$ be given, there exists $\delta > 0$ such that $\lambda(E) < \delta$ implies $\|\mu_\alpha\|_\gamma^Y(E) < \varepsilon$ uniformly in α, and $\phi(F) < \delta$ implies $|\nu_\beta(F)| < \varepsilon$ uniformly in β.

Suppose $G = \bigcup_{i=1}^k E_i \times F_i \in \Omega \otimes \Lambda$ and $\lambda \times \phi(G) < \delta^2$ where $(E_i) \subseteq \Omega$ is disjoint and $(F_i) \subseteq \Lambda$.

Recall that for $s \in S$, the s-section of G is

$$G^s = \{t \in T : (s,t) \in G\}.$$

Write $D = \{s \in \bigcup_{i=1}^k E_i : \phi(G^s) < \delta\}$.

We then have

$$\delta^2 > \lambda \times \phi(G) = \int_S \phi(G^s) d\lambda(s)$$

$$= \int_{\cup_i E_i} \phi(G^s)\,d\lambda(s) \geq \int_{\cup_i E_i - D} \phi(G^s)\,d\lambda(s)$$

$$\geq \delta\ \lambda(\cup_i E_i - D).$$

From this we obtain $\lambda(\cup_i E_i - D) < \delta$ and so

$$\|\mu_\alpha\|_\gamma^Y(\cup_i E_i - D) < \varepsilon \text{ for all } \alpha. \qquad (\#)$$

We may suppose $\phi(F_i) < \delta$ for $i = 1,2,\ldots,p$, hence

$$|\nu_\beta(F_i)| < \varepsilon \text{ for all } \beta \text{ and } 1 \leq i \leq p,$$

that is, $\dfrac{|\nu_\beta(F_i)|}{\varepsilon} < 1$ for all β and $1 \leq i \leq p$.

Therefore, for $i = p+1,\ldots,k$, we have $\phi(F_i) \geq \delta$ and so $D = \bigcup_{i=1}^{p} E_i$. To see this, suppose $s \in D$, then $\phi(\bigcup_{i=1}^{k} E_i \times F_i)^s < \delta$. If $s \in E_j$ we must have $(\cup_i E_i \times F_i)^s = F_j$ and so $\phi(F_j) < \delta$ but then $1 \leq j \leq p$. Conversely, if $s \in E_j$ for some j, $1 \leq j \leq p$, then $\phi(F_j) < \delta$ which implies $s \in D$ since $F_j = (\cup_i E_i \times F_i)^s$. Note that $\|\mu_\alpha\|_\gamma^Y(\bigcup_{i=p+1}^{k} E_i) < \varepsilon$ uniformly in α because of (#).

By assumptions (2), there exists a positive number N such that $\|\mu_\alpha\|_\gamma^Y(S) \leq N$ and $\|\nu_\beta\|(T) \leq N$ for all α and β, it follows that $\dfrac{|\nu_\beta(F)|}{N} \leq 1$ for all β and $F \in \Omega$.

Then,

$$|\mu_\alpha \otimes \nu_\beta(G)|_\nu = \Big|\sum_{i=1}^{k} \mu_\alpha(E_i) \otimes \nu_\beta(F_i)\Big|_\gamma$$

$$\leq \Big|\sum_{i=1}^{p} \mu_\alpha(E_i) \otimes \nu_\beta(F_i)\Big|_\gamma + \Big|\sum_{i=p+1}^{k} \mu_\alpha(E_i) \otimes \nu_\beta(F_i)\Big|_\gamma$$

$$= \varepsilon \Big|\sum_{i=1}^{p} \mu_\alpha(E_i) \otimes \frac{\nu_\beta(F_i)}{\varepsilon}\Big|_\gamma + N \cdot \Big|\sum_{i=p+1}^{k} \mu_\alpha(E_i) \otimes \frac{\nu_\beta(F_i)}{N}\Big|_\gamma$$

$$\leq \varepsilon \cdot \|\mu_\alpha\|_\gamma^Y(\bigcup_{i=1}^{p} E_i) + N \cdot \|\mu_\alpha\|_\gamma^Y(\bigcup_{i=p+1}^{k} E_i)$$

$$\leq \varepsilon \cdot N + N \cdot \varepsilon = 2\varepsilon N.$$

We have $|\mu_\alpha \otimes \nu_\beta(G)|_\gamma < 2\varepsilon N$ regardless of α or β whenever $\lambda\times\phi(G) < \delta^2$, that is, $\mu_\alpha \otimes \nu_\beta \ll \lambda\times\phi$ uniformly. □

As a corollary of Lemma 2.3, we prove the existence theorem of Duchon and Kluvánek [11].

2.4 Theorem. Let $\mu:\Omega \to X$ and $\nu:\Lambda \to Y$ be measures. Then the set function $\mu\otimes\nu:\Omega\otimes\Lambda \to X\otimes_\varepsilon Y$ can be extended uniquely to a measure $\mu\otimes_\varepsilon\nu:\Omega\otimes_\sigma\Lambda \to X\hat{\otimes}_\varepsilon Y$ and

$$\mu\otimes_\varepsilon\nu(E\times F) = \mu(E)\otimes\nu(F), \qquad E \in \Omega,\ F \in \Lambda.$$

Proof. There exists control measures λ and ϕ of μ and ν, respectively, by Theorem 1.2. We then have $\mu \ll \lambda$ and $\nu \ll \phi$. Since vector measures are bounded we have $\|\mu\|(S) < +\infty$ and $\|\nu\|(T) < +\infty$. Regarding Lemma 2.2, $\|\mu\|(E) = \|\mu\|_\varepsilon^Y(E)$ so that $\|\mu\|_\varepsilon^Y(S) < +\infty$ and $\|\mu\|_\varepsilon^Y(\cdot) \ll \lambda$. Thus the hypothesis of Lemma 2.3 is satisfied for the singleton families (μ) and (ν). So we have $\mu\otimes\nu \ll \lambda\times\phi$ on $\Omega\otimes\Lambda$ when $X\otimes Y$ is endowed with its ε-norm.

Because $\mu\otimes\nu \ll \lambda\times\phi$ on $\Omega \otimes \Lambda$ and $\lambda \times \phi$ is a positive measure on $\Omega \otimes_\sigma \Lambda$, we may extend $\mu \otimes \nu$ uniquely to a measure $\mu \otimes_\varepsilon \nu$ defined on $\Omega \otimes_\sigma \Lambda$ with values in $X \hat{\otimes}_\varepsilon Y$ by [7], p. 507. □

Lemma 2.3 suggests the following definition. A vector measure $\mu:\Omega \to X$ is dominated (with respect to Y) if there exists a positive measure λ on Ω such that $\|\mu\|_\pi^Y(E) \to 0$ whenever $\lambda(E) \to 0$, that is, $\|\mu\|_\pi^Y(\cdot) \ll \lambda$.

2.5 Theorem. Let $\mu:\Omega \to X$ and $\nu:\Lambda \to Y$ be vector measures, and suppose μ is dominated (with respect to Y) by a positive

measure λ. Then there exists a unique measure $\mu\otimes_\pi\nu:\Omega\otimes_\sigma\Lambda \to X\hat{\otimes}_\pi Y$ which extends $\mu \otimes \nu$; consequently, $\mu\otimes_\pi\nu(E\times F) = \mu(E)\otimes\nu(F)$ for $E \in \Omega$ and $F \in \Lambda$.

Proof. Choose a control measure ϕ of ν. Then $\nu \ll \phi$ and $\|\mu\|_\pi^Y(\cdot) \ll \lambda$. According to Lemma 2.3, $\mu\otimes\nu \ll \lambda\times\phi$ on $\Omega \otimes \Lambda$ provided $\|\mu\|_\pi^Y(S) < +\infty$. If this is shown, the theorem is proven because the extension is guaranteed by [7], p. 507.

To show $\|\mu\|_\pi^Y(S) < +\infty$, there exists $\delta > 0$ such that $\lambda(E) < \delta$ implies $\|\mu\|_\pi^Y(E) < 1$. By Saks lemma ([12], IV.9.7), there exists $E_1,E_2,\ldots,E_n \in \Omega$ disjoint such that $S = \cup_i E_i$ and each E_i is either an atom or $\lambda(E_i) < \delta$. Since $\|\mu\|_\pi^Y(S) \le \sum_{i=1}^{n}\|\mu\|_\pi^Y(E_i)$ and $\|\mu\|_\pi^Y(E_i) < 1$ for all those i for which $\lambda(E_i) < \delta$, to show $\|\mu\|_\pi^Y(S) < +\infty$, it suffices to prove that if E is an atom, then $\|\mu\|_\pi^Y(E) < +\infty$.

Let $E \in \Omega$ be an atom of λ, that is, if $G \subset E$ and $G \in \Omega$ then $\lambda(G) = 0$ or $\lambda(G) = \lambda(E)$. Because μ is σ-additive, it is bounded, so we can find a number N such that $|\mu(A)| \le N\cdot\lambda(E)$, for all $A \in \Omega$. Now for $G \subset E$, $G \in \Omega$, either $\lambda(G) = 0$ (in which case $\|\mu\|_\pi^Y(G) = 0$, hence $|\mu(G)| = 0$) or $\lambda(G) = \lambda(E)$; in either case, we have $|\mu(G)| \le N\lambda(G)$.

Now

$$\begin{aligned}\|\mu\|_\pi^Y(E) &= \sup\{|\textstyle\sum_{i=1}^{k}\mu(G_i)\otimes y_i|_\pi : y_i\in Y, |y_i| \le 1, (G_i)\in\Pi(E)\}\\ &\le \sup\{\textstyle\sum_{i=1}^{k}|\mu(G_i)| : (G_i)\in\Pi(E)\}\\ &\le N\textstyle\sum_{i=1}^{k}\lambda(G_i)\\ &= N\cdot\lambda(E) < +\infty,\end{aligned}$$

where $\Pi(E)$ is the collection of all measurable partitions (G_i) of E. □

Theorem 2.5 was first proved by M. Duchon in [9]. The measures $\mu \otimes_\varepsilon \nu$ and $\mu \otimes_\pi \nu$ are called the inductive and projective tensor products of μ and ν, respectively.

2.6 Corollary. If either μ or ν have finite variation, then $\mu \otimes_\pi \nu$ exists.

Proof. If μ or ν has finite variation, say μ, then $\|\mu\|_\pi^Y(A) \leq |\mu|(A)$. But then μ is dominated by the positive peasure $|\mu|$, by Theorem 2.5, $\mu \otimes_\pi \nu$ exists. □

In the next chapter, we shall study various tensor products of spaces of measures and give some structure theorems.

CHAPTER II
TENSOR PRODUCTS OF SPACES OF MEASURES

1. Algebraic Tensor Products of Spaces of Measures.

M. Duchon seemed to have developed the theory of product measures primarily for the study of Borel and Baire measures on locally compact Hausdorff spaces [8] and for the study of convolutions of Borel measures defined on a compact Hausdorff topological semigroup with values in a Banach algebra. Here, however, we develope the study of tensor products of abstract spaces of measures.

Throughout this chapter, (S,Ω) and (T,Λ) will denote fixed but arbitrary measurable spaces; X and Y are Banach spaces.

The space $ca(S,\Omega;X)$, or simply $ca(\Omega;X)$, is the space of all measures $\mu:\Omega \to X$. $ca(\Omega;X)$ is a Banach space when equipped with the semivariation norm $\|\cdot\|(S)$. When $X = \Phi$, we write $ca(\Omega)$ instead of $ca(\Omega;\Phi)$. In this case, the semivariation norm is identical with the total variation norm $|\cdot|(S)$.

When various subspaces of $ca(\Omega;X)$ are under consideration, descriptive letters are placed in juxtaposition with "ca," for example: $cabv(\Omega;X)$ is the subspace of $ca(\Omega;X)$ consisting of all those measures with finite total variation, $Ccabv(\Omega;X)$ is the subspace of all measures of finite total variation and with relatively norm compact range. Any subspace of

$ca(\Omega;X)$ consisting of measures with finite total variation will have as its norm, the total variation norm $|\cdot|(S)$ rather than the semivariation norm. Since $\|\cdot\|(S) \leq |\ |(S)$, the total vairation norm defines on this subspace a topology which is, ingeneral, strictly finer than the topology induced by the semivariation norm.

Recall that from the universal mapping property of tensor products, any bilinear map from the Cartesian product of two Banach spaces into a third Banach space induces a unique linear map from the algebraic tensor product of the first two spaces into the third (see the remarks following Theorem I.2.1). The following theorem establishes the basic algebraic structure in which we shall be working throughout this chapter.

1.1 Theorem. (a) The bilinear map $\phi_\varepsilon:(\mu,\nu) \to \mu\otimes_\varepsilon\nu$ induces an algebraic isomorphism which embeds $ca(\Omega;X) \otimes ca(\Lambda;Y)$ into $ca(S\times T,\Omega\otimes_\sigma\Lambda;X\hat{\otimes}_\varepsilon Y)$.

(b) The bilinear map $\phi_\pi:(\mu,\nu) \to \mu\otimes_\pi\nu$ induces an algebraic isomorphism which embeds $cabv(\Omega;S) \otimes ca(\Lambda;Y)$ into $ca(S\times T, \Omega\otimes_\sigma\Lambda;X\hat{\otimes}_\pi Y)$.

Proof. Since $\mu\otimes_\varepsilon\nu$ always exists whenever μ and ν are measures, the map ϕ_ε is defined on $ca(\Omega;X) \times ca(\Lambda;Y)$ and takes its values in $ca(\Omega\otimes_\varepsilon\Omega;X\hat{\otimes}_\pi Y);\mu\otimes_\pi\nu$ exists whenever μ has finite total variation, so that ϕ_π is defined on $cabv(\Omega;X) \times ca(\Lambda;Y)$ and has its range in $ca(\Omega\otimes_\sigma\Lambda;X\otimes_\pi Y)$. It is not difficult to see that ϕ_ε and ϕ_π are bilinear.

In order to prove that the unique linear maps induced by ϕ_ε and ϕ_π are isomorphisms, it suffices, according to

Theorem I.2.1 (b) to prove that the coordinate spaces of $\phi_\omega, \omega = \varepsilon$ and π, are ϕ_ω-linearly disjoint. To this end, let $\omega = \varepsilon$ or π be fixed. Suppose $\{\mu_1,\mu_2,\ldots,\mu_n\} \subseteq ca(\Omega;X)$ is a linearly independent set ($\mu_i, 1\le i\le n$, is assumed to have bounded variation if $\omega = \pi$), and $\{\nu_1,\nu_2,\ldots,\nu_n\} \subseteq ca(\Lambda;Y)$ such that $\sum_{i=1}^{n}\phi_\omega(\mu_i,\nu_i) = 0$, that is, $\sum_{i=1}^{n}\mu_i\otimes_\omega\nu_i = 0$. We want to show $\nu_1 = \nu_2 = \ldots = \nu_n = 0$.

$\sum_{i=1}^{n}\mu_i\otimes_\omega\nu_i = 0$ means $\sum_{i=1}^{n}\mu_i\otimes_\omega\nu_i(G) = 0$ for all $G \in \Omega\otimes_\sigma\Lambda$, in particular

$$0 = \sum_{i=1}^{n}\mu_i\otimes_\omega\nu_i(E\times F) = \sum_{i=1}^{n}\mu_i(E)\otimes\nu_i(F), \qquad (1)$$

for all $E \in \Omega$ and $F \in \Lambda$.

Fix $F \in \Lambda$ and choose $x^* \in X_1^*$ and $y^* \in Y_1^*$ arbitrarily, and apply the functional $x^*\otimes_\omega y^*$, to both sides of equation (1):

$$\begin{aligned} 0 = x^*\otimes_\omega y^*(0) &= \langle x^*\otimes_\omega y^*, \sum_{i=1}^{n}\mu_i(E)\otimes\nu_i(F)\rangle \\ &= \sum_{i=1}^{n}x^*\mu_i(E)\cdot y^*\nu_i(F) \\ &= \langle x^*, \sum_{i=1}^{n}\mu_i(E)\cdot y^*\nu_i(F)\rangle . \end{aligned}$$

$x^* \in X_1^*$ arbitrary implies $\sum_{i=1}^{n}\mu_i(E)\cdot y^*\nu_i(F) = 0$ for all $E \in \Omega$. But $y^*\nu_i(F)$ are scalar quanities which appear in linear combination with the measures $\mu_1,\mu_2,\ldots,\mu_n$, and since they form an independent set and $\sum_{i=1}^{n}y^*\nu_i(F)\cdot\mu_i(\cdot) \equiv 0$ this implies $y^*\nu_i(F) = 0$, $i = 1,2,\ldots,n$. $y^* \in Y_1^*$ was choosen arbitrarily also so that $\nu_i(F) = 0$ for $i = 1,2,\ldots,n$; this then implies $\nu_i \equiv 0$ for all i.

This only proves half the condition for being ϕ_ω-linearly disjoint, we must also prove that if $\{\nu_1,\nu_2,\ldots,\nu_n\} \subseteq ca(\Lambda;Y)$

forms a linearly independent set and $\{\mu_1,\mu_2,\ldots,\mu_n\} \subseteq ca(\Omega;X)$ such that $\sum_{i=1}^{n}\mu_i\otimes_\omega\nu_i = 0$, then $\mu_1 = \mu_2 = \ldots = \mu_n = 0$. The proof of this is analogous to the above proof.

Thus the linear maps induced by ϕ_ε and ϕ_π are isomorphisms, which proves (a) and (b). □

1.2 Corollary. $cabv(\Omega;X) \otimes cabv(\Lambda;Y) \subseteq cabv(\Omega\otimes_\sigma\Lambda;X\hat{\otimes}_\pi Y)$ algebraically.

Proof. In view of Theorem 1.1, we have

$$cabv(\Omega;X) \otimes cabv(\Lambda;Y) \subseteq cabv(\Omega;X) \otimes ca(\Lambda;Y)$$
$$\subseteq ca(\Omega\otimes_\sigma\Lambda;X\hat{\otimes}_\pi Y).$$

It suffices therefore to prove that all measures in the space on the left have finite variation. Let $\mu \in cabv(\Omega;X)$ and $\nu \in cabv(\Lambda;Y)$ and take disjoint sets $G_n = \bigcup_{i=1}^{k_n} E_i^n\times F_i^n$ in $\Omega\otimes\Lambda$, $n = 1,2,\ldots,p$. Then

$$\begin{aligned}
\sum_{n=1}^{p}|\mu\otimes_\pi\nu(G_n)|_\pi &\le \sum_{n=1}^{p}\sum_{i=1}^{k_n}|\mu(E_i^n)\otimes\nu(F_i^n)|_\pi \\
&= \sum_{n=1}^{p}\sum_{i=1}^{k_n}|\mu(E_i^n)|\cdot|\nu(F_i^n)| \\
&\le \sum_{n=1}^{p}\sum_{i=1}^{k_n}|\mu|(E_i^n)\cdot|\nu|(F_i^n) \\
&= \sum_{n=1}^{p}\sum_{i=1}^{k_n}|\mu|\times|\nu|(E_i^n\times F_i^n) \\
&= |\mu|\times|\nu|(\bigcup_{n=1}^{p}G_n).
\end{aligned}$$

It follows that for any $G \in \Omega\otimes\Lambda$ we have $|\mu\otimes_\pi\nu|(G) \le |\mu|\times|\nu|(G)$, hence for all $G \in \Omega\otimes_\sigma\Lambda$.

Thus $|\mu\otimes_\pi\nu|(S\times T) \le |\mu|\times|\nu|(S\times T) < +\infty$, so that $cabv(\Omega;X) \otimes cabv(\Lambda;Y)$ consists of measures with finite variation and therefore lies in $cabv(\Omega\otimes_\sigma\Lambda;X\hat{\otimes}_\pi Y)$. □

1.3 Remark. Duchon [9] has shown that $|\mu\otimes_\pi\nu|(G) = |\mu|x|\nu|(G)$ for all $G \in \Omega\otimes_\sigma\Lambda$ whenever both μ and ν have finite variation. □

Topological embeddings of (a) in Theorem 1.1 will be considered later in this chapter; first, however, we prove the following theorem.

1.4 Theorem. (a) The bilinear map $\psi_\varepsilon:(\mu,x) \to x\mu$ induces an isometric algebraic isomorphism on $ca(\Omega)\otimes_\varepsilon X$ into $ca(\Omega;X)$.

(b) The bilinear map $\psi_\pi:(\mu,x) \to x\mu$ induces an isometric algebraic isomorphism on $ca(\Omega)\otimes_\pi X$ into $cabv(\Omega;X)$.

Thus $ca(\Omega)\hat{\otimes}_\varepsilon X \subseteq ca(\Omega;X)$ and $ca(\Omega)\hat{\otimes}_\pi X \subseteq cabv(\Omega;X)$ isometrically.

Proof. The proof of that $ca(\Omega)\otimes_\pi X \subseteq cabv(\Omega;X)$ isometrically will be postponed until Theorem 2.3 infra, where we shall characterize this space; we state (b) now only for completeness.

It is clear that $ca(\Omega)\otimes X \subseteq ca(\Omega;X)$ by considering the bilinear map $(\mu,x) \to x\mu$, where $x\mu \in ca(\Omega;X)$ is defined by $(x\mu)(E) = x\cdot\mu(E)$. Consequently, $ca(\Omega)\otimes X$ consists of all X-valued "step-measures" on Ω of the form $\sum_{i=1}^{n} x_i\mu_i(\cdot)$ for $x_i \in x$ and $\mu_i \in ca(\Omega)$.

The step measure $\sum_{i=1}^{n} x_i\mu_i$ has finite total variation since μ_i has finite variation for each $i = 1,2,\ldots,n$. We can therefore consider $ca(\Omega)\otimes X$ as an algebraic subspace of $cabv(\Omega;X)$. Part (a) claims that when we consider $ca(\Omega)\otimes X$ as a subspace of $ca(\Omega;X)$, the ε-norm is exactly the norm induced on $ca(\Omega)\otimes X$ by $ca(\Omega;X)$, namely, the semivariation norm. Part (b) claims that the π-norm is the total variation norm. Here we prove the isometry of part (a).

To that end, let $\sum_{i=1}^{n} x_i \mu_i \in ca(\Omega) \otimes X$. From the general theory, the ε-norm can be defined as the norm of $\Sigma_i x_i \mu_i$ when it is considered as a linear map from X^* into $ca(\Omega)$ defined by $\langle \Sigma_i x_i \mu_i, x^* \rangle = \Sigma_i x^*(x_i)\mu_i$. Thus

$$|\Sigma_i x_i \mu_i|_\varepsilon = \sup_{x^* \in X_1^*} |\Sigma_i x^*(x_i)\mu_i|(S)$$

$$= \sup_{x^* \in X_1^*} |x^*(\Sigma_i x_i \mu_i)|(S) \qquad (1)$$

$$= ||\Sigma_i x_i \mu_i||(S) \qquad (2)$$

In going from (1) to (2), we have invoked the Dinculeanu result, Proposition I.1.1. Thus the ε-norm is equal to the semivariation norm. □

We now prove two technical lemmas followed by a theorem which gives insight into the algebraic structure of vector measures defined on product σ-algebras and taking their values in a Banach space X, that is, measures of the form $\lambda : \Omega \otimes_\sigma \Lambda \to X$. This situation is, of course, a bit more general than measures λ of the form $\mu \otimes_\omega \nu$, $\omega = \varepsilon$ or π, which take their values in the tensor product of two Banach spaces.

1,5 Lemma. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n linearly independent scalar measures defined on Ω. Then there exists sets $E_1, E_2, \ldots, E_n$ in Ω such that the determinant of the $n \times n$ matrix $(\lambda_i(E_j))_{n \times n}$ is non-zero. We write

$$\Phi(\lambda_1, \lambda_2, \ldots, \lambda_n; E_1, E_2, \ldots, E_n) = \det(\lambda_i(E_j))_{n \times n} \neq 0.$$

Proof. The proof is by induction on n.

Case n=2. Suppose λ_1, λ_2 form a linearly independent set of scalar measures on Ω such that $\Phi(\lambda_1, \lambda_2; E_1, E_2) = 0$ for all

choices of $E_1, E_2 \in \Omega$. This means that

$$\lambda_1(E_1)\cdot\lambda_2(E_2) - \lambda_1(E_2)\lambda_2(E_1) = 0$$

for all $E_1, E_2 \in \Omega$. Fix $E_2 \in \Omega$ and let E_1 vary over Ω. λ_1 and λ_2 independent implies $\lambda_2(E_2) = 0$ and $\lambda_1(E_2) = 0$. Since E_2 was arbitrary we have $\lambda_1 = \lambda_2 = 0$, a contradiction.

Case n = k+1. Suppose the lemma is true whenever $n \leq k$, and that $\{\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}\}$ is a linearly independent set of $k + 1$ scalar measures such that for all choices of $E_1, E_2, \ldots E_{k+1} \in \Omega$

$$\Phi(\lambda_1, \lambda_2, \ldots, \lambda_k, \lambda_{k+1}; E_1, E_2, \ldots, E_k, E_{k+1}) = 0. \qquad (1)$$

Writing the determinent in (1) in terms of its first row expansion:

$$\sum_{i=1}^{k+1}(-1)^{i+1}\lambda_i(E_1)\cdot\Phi(\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \lambda_{k+1}; E_2, E_3, \ldots, E_k, E_{k+1}) = 0, \qquad (2)$$

where $\hat{\lambda}_i$ means that λ_i is deleted from the list of entries.

Since the measures $\{\lambda_i\}_{i=1}^{k+1}$ are linearly independent and (2) is valid as E_1 varies over Ω, we obtain

$$\Phi(\lambda_1, \ldots, \hat{\lambda}_i, \ldots, \lambda_{k+1}; E_2, E_3, \ldots, E_k, E_{k+1}) = 0 \qquad (3)$$

for any i and any choice of $(E_j)_{j=2}^{k+1} \subseteq \Omega$. (3) is a contradiction of our induction hypothesis since we are back to the case $n = k$. □

By Theorem 1.1, the spaces $cabv(\Omega;X) \otimes ca(\Lambda)$ and $ca(\Omega) \otimes cabv(\Lambda;X)$ lie algebraically in $ca(\Omega \otimes_0 \Lambda; X)$ and, in fact, lie in $cabv(\Omega \otimes_0 \Lambda; X)$; consequently, we may consider the set-theoretic intersection of these two subspaces:

$$I(\Omega, \Lambda; X) = cabv(\Omega;X) \otimes ca(\Lambda) \cap ca(\Omega) \otimes cabv(\Lambda;X)$$

1.6 Lemma. If $\theta \in I(\Omega,\Lambda;X)$, then there exists an integer $n \geq 1$ and vectors $x_1,x_2,\ldots,x_n$, scalar measures $\mu_1,\mu_2,\ldots,\mu_n \in ca(\Omega)$ and $\nu_1,\nu_2,\ldots,\nu_n \in ca(\Lambda)$ such that $\theta = \sum_{i=1}^{n} x_i(\mu_i \times \nu_i)$.

Proof. Without loss of generality, we may assume $\theta \neq 0$.

$\theta \in cabv(\Omega;X)\otimes ca(\Lambda)$ implies $\theta = \sum_{i=1}^{p} \bar{\mu}_i \otimes \nu_i$

for $\bar{\mu}_i \in cabv(\Omega;X)$ and $\nu_i \in ca(\Omega)$. $\theta \in ca(\Omega)\otimes cabv(\Lambda;X)$ implies $\theta = \sum_{j=1}^{n} \mu_j \otimes \bar{\nu}_j$ for $\mu_j \in ca(\Omega)$ and $\bar{\nu}_j \in cabv(\Lambda;X)$.

We assume henceforth that $p \leq n$, and that each family $\{\bar{\mu}_i\}$, $\{\nu_i\}$, $\{\mu_j\}$, $\{\bar{\nu}_j\}$ is a linearly independent family of measures.

By Lemma 1.5, there exists sets $F_1,F_2,\ldots,F_p \in \Lambda$ such that $\Phi(\nu_1,\nu_2,\ldots,\nu_p;F_1,F_2,\ldots,F_p) \neq 0$. Write as simply $\Phi \neq 0$.

With this observation, we use Cramer's rule to solve the system

$$\nu_1(F_1)\bar{\mu}_1(\cdot)+\nu_2(F_1)\bar{\mu}_2(\cdot)+\ldots+\nu_p(F_1)\bar{\mu}_p(\cdot) = \sum_{j=1}^{n} \bar{\nu}_j(F_1)\mu_j(\cdot)$$

$$\nu_1(F_2)\bar{\mu}_1(\cdot)+\nu_2(F_2)\bar{\mu}_2(\cdot)+\ldots+\nu_p(F_2)\bar{\mu}_p(\cdot) = \sum_{j=1}^{n} \bar{\nu}_j(F_2)\mu_j(\cdot)$$

$$\nu_1(F_p)\bar{\mu}_1(\cdot)+\nu_2(F_p)\bar{\mu}_2(F_p)\bar{\mu}_2(\cdot)+\ldots+\nu_p(F_p)\bar{\mu}_p(\cdot) = \sum_{j=1}^{n} \bar{\nu}_j(F_p)\mu_j(\cdot).$$

Define for $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,n$ the vector

$$x_i^j = \sum_{\sigma\in S_p} \frac{\cdots\nu_{i-1}(F_{\sigma(i-1)})\bar{\nu}_j(F_{\sigma(i)})\nu_{i+1}(F_{\sigma(i+1)})\cdots}{\Phi},$$

where S_p is the symmetric group on p-letters; obviously, $x_i^j \in X$ for all i and j.

Thus by Cramer's rule $\bar{\mu}_i(\cdot) = \sum_{j=1}^{n} x_i^j \mu_j(\cdot)$ for $i = 1,2,\ldots,p$.

Substituting this into $\theta = \sum_{i=1}^{p} \bar{\mu}_i \otimes \nu_i$ we get

$$\theta = \sum_{i=1}^{p} \sum_{j=1}^{n} x_i^j (\mu_j \times \nu_i);$$

the lemma is proved upon re-indexing this representation. □

1.7 Theorem. $I(\Omega,\Lambda;X) = X \otimes ca(\Omega) \otimes ca(\Lambda)$.

Proof. From Lemma 1.6 we have $I(\Omega,\Lambda;X) \subseteq X \otimes ca(\Omega) \otimes ca(\Lambda)$ since any member of $I(\Omega,\Lambda;X)$ can be represented in the form $\Sigma_i x_i(\mu_i \times \nu_i)$ where $x_i \in X$, $\mu_i \in ca(\Omega)$ and $\nu_i \in ca(\Lambda)$, which clearly puts it in $X \otimes ca(\Omega) \otimes ca(\Lambda)$.

Conversely, if $\theta \in X \otimes ca(\Omega) \otimes ca(\Lambda)$, then we can write θ as $\theta = \Sigma_i x_i m_i$ where $x_i \in X$ and $m_i \in ca(\Omega) \otimes ca(\Lambda)$. Let i be fixed, we can write m_i in the form $m_i = \Sigma_j \mu_j^i \times \nu_j^i$ where $\mu_j^i \in ca(\Omega)$ and $\nu_j^i \in ca(\Lambda)$. Now we have that $x_i(\mu_j^i \times \nu_j^i) = (x_i \mu_j^i) \otimes \nu_j^i \in cabv(\Omega;X) \otimes ca(\Lambda)$, but $x_i(\mu_j^i \times \nu_j^i) = \mu_j^i \otimes (x_i \nu_j^i) \in ca(\Omega) \otimes cabv(\Lambda;X)$. Thus $x_i(\mu_j^i \times \nu_j^i) \in I(\Omega,\Lambda;X)$ and therefore $x_i m_i = \Sigma_j x_i(\mu_j^i \times \nu_j^i) \in I(\Omega,\Lambda;X)$, and in turn $\theta = \Sigma_i x_i m_i \in I(\Omega,\Lambda;X)$. □

2. The Radon-Nikodym Property

We now introduce a notion which has not appeared in the literature -- that of the Radon-Nikodym property of a measure.

A vector valued measure $\tau:\Omega \to X$ which has finite total variation is said to have the Radon-Nikodym property, or simply the R-N property, if whenever $\lambda:\Omega \to R^+$ is a positive measure such that $\tau \ll \lambda$, then there exists a Bochner integrable function $f:S \to X$ such that

$$\tau(E) = \int_E f \, d\lambda$$

for all $E \in \Omega$. We say that f is the Radon-Nikodym derivative of τ with respect to λ and write $f = \frac{d\mu}{d\lambda}$ or $d\mu = f \, d\lambda$.

Recall that a Banach space X has the Radon-Nikodym property if for every measurable space (S,Ω) and any vector measure $\tau:\Omega \to X$ of finite variation, τ can be written as an indefinite Bochner integral with respect to any positive measure λ on Ω for which $\tau \ll \lambda$. Thus, the Banach space X has the Radon-Nikodym property if and only if every vector measure that takes its values in X has the Radon-Nikodym property. The R-N property of a Banach space is a global property whereas the R-N property of a measure is a local property.

The R-N property of a measure is important in classifying certain tensor products of spaces of measures. In preparation for this, we establish an important lemma.

2.1 Lemma. Suppose $\tau:\Omega \to X$ is a vector measure of bounded variation such that $\tau \ll \lambda \ll \nu$, where λ and ν are two positive measures on Ω. If τ has a Radon-Nikodym derivative with respect to ν, then it has a derivative with respect to λ.

Proof. By the Lebesgue Decomposition Theorem, write $\nu = \mu + i$ where $\mu \ll \lambda$ and $i \perp \lambda$. Since $i \perp \lambda$, there exists $E_o \in \Omega$ such that $i(E_o) = 0$ but $\lambda(S-E_o) = 0$. From $\mu \ll \lambda$, there exists $h \in L_1^+(S,\Omega,\lambda)$ such that $\mu(E) = \int_E h \, d\lambda$.

Thus

$$\tau(E) = \int_E f \, d\nu = \int_E f \, d\mu + \int_E f \, di = \int_E fh \, d\lambda + \int_E f \, di$$

Assert that $\int_E f\, di = 0$ for all $E \in \Omega$.

Case I. $E \subset E_o$. Since $i(E_o) = 0$, we must have $i(E) = 0$ and so $\int_E f\, di = 0$

Case II. $E \subset S-E_o$. Since $\lambda(S-E_o) = 0$, we have $\lambda(E) = 0$ and so $\int_E fh\, d\lambda = 0$; furthermore, $\tau \ll \lambda$ and $\lambda(E) = 0$ implies $\tau(E) = 0$.

Thus

$$0 = \tau(E) = \int_E fh\, d\lambda + \int_E f\, di = 0 + \int_E f\, di$$

or $\int_E f\, di = 0$.

Cases I and II are sufficient to conclude $\int_E f\, di = 0$ since $E = (E \cap E_o) \cup E \cap (S-E_o)$ and the integral $\int f\, di$ is additive.

Thus $\tau(E) = \int_E fh\, d\lambda$, that is $fh = \frac{d\tau}{d\lambda}$ and the lemma is proved. □

2.2 Theorem. Let $(\tau_k) \subseteq cabv(\Omega;X)$ such that $\sum_{k=1}^{\infty} |\tau_k|(S) < +\infty$. If τ_k has the R-N property for each $k \in \omega$, then so does the measure $\tau = \sum_{k=1}^{\infty} \tau_k$.

Proof. We remark first that the infinite series $\Sigma_k\, \tau_k$ does define the measure because the series $\Sigma_k\, \tau_k(E)$ converges absolutely for each $E \in \Omega$:

$$\sum_{k=1}^{\infty} |\tau_k(E)| \leq \sum_{k=1}^{\infty} |\tau_k|(E) \leq \sum_{k=1}^{\infty} |\tau_k|(S) < +\infty \text{ by hypothesis.}$$

To show τ has the R-N property, begin by supposing $\tau \ll \lambda$, where λ is a positive measure on Ω.

Note that $\sum_{k=1}^{\infty} |\tau_k|(E)$ converges and consequently defines a σ-additive measure on Ω such that $\tau_n \ll \sum_{k=1}^{\infty} |\tau_k|$ for each $n \in \omega$.

Write $\nu = \lambda + \sum_{k=1}^{\infty} |\tau_k|$, then ν is a positive measure on Ω such that $\lambda \ll \nu$; consequently, $\tau \ll \lambda \ll \nu$. We intend to show τ has a Radon-Nikodym derivative with respect to ν, and then use Lemma 2.1 to prove the theorem.

Indeed, for each $n \in \omega$ we have also that $\tau_n \ll \nu$. τ_n has by assumption the R-N property; hence $\tau_n(E) = \int_E f_n d\nu$ for some $f_n \in B_X(S,\Omega,\nu)$, where $B_X(S,\Omega,\nu)$ is the space of Bochner integrable X-valued functions.

Write $|f_n|_1 = \int_S |f_n| d\nu$ and note $|\tau_n|(S) = |f_n|_1$. $|f_n|_1$ is the norm of f_n in $B_X(S,\Omega,\nu)$.

Since $\sum_{n=1}^{\infty} |f_n|_1 = \sum_{n=1}^{\infty} |\tau_n|(S) < +\infty$ and $B_X(S,\Omega,\nu)$ is a Banach space, $\sum_{n=1}^{\infty} f_n$ converges in norm to a function $f \in B_X(S,\Omega,\nu)$, that is, $f = \sum_{n=1}^{\infty} f_n$.

But then

$$\tau(E) = \sum_{n=1}^{\infty} \tau_n(S) = \sum_{n=1}^{\infty} \int_E f_n \, d\nu = \int_E \sum_{n=1}^{\infty} f_n \, d\nu = \int_E f \, d\nu.$$

That is, $f = \frac{d\tau}{d\nu}$. By Lemma 2.1 then, $\frac{d\tau}{d\lambda}$ exists, which means, since λ was arbitrary, τ has the R-N property. □

We now prove a theorem which identifies the space $ca(\Omega)\hat{\otimes}_\pi X$. This is a generalization of a theorem of Gil de Lamadrid [13], where he identifies $C^*(H)\hat{\otimes}_\pi X$, $C^*(H)$ is the dual of the Banach space of all continuous functions on a compact Hausdorff space H. $C^*(H)$ is of course the space of all regular Radon measures on H. Our setting is based on an abstract measurable space (S,Ω). Lamadrid's identification was that $C^*(H)\hat{\otimes}_\pi X$ was the class of all regular X-valued Radon measures of bounded variation which can be represented as an absolutely series

of "step measures." Theorem 2.2 implies that such a representation does have the R-N property. Our approach is quite different than his and the result was independently obtained.

2.3 Theorem. Let (S,Ω) be a measurable space and X a Banach space. Then $ca(\Omega)\hat{\otimes}_\pi X$ is isometrically embedded in $Ccabv(\Omega;X)$, the space of all X-valued measures with bounded variation and relatively norm compact range.

Furthermore, $ca(\Omega)\hat{\otimes}_\pi X$ is the Banach space of all X-valued measures on Ω with the R-N property. Symbolically,

$$ca(\Omega)\hat{\otimes}_\pi X = RNca(\Omega;X).$$

Proof. By Theorem 1.4, $ca(\Omega)\otimes X \subset cabv(\Omega;X)$. It is clear that any measure $\theta = \sum_{i=1}^{n} x_i\lambda_i \in ca(\Omega)\otimes X$ has relative norm compact range since each λ_i does. To show the initial assertion, if suffices to prove that on $ca(\Omega)\otimes X$, the π-topology is identical to the bounded variation norm. Indeed, if the π-norm on $ca(\Omega)\otimes X$ is the variation norm then since $Ccabv(\Omega;X)$ is a Banach space, the completion $ca(\Omega)\hat{\otimes}_\pi X$ of $ca(\Omega)\otimes X$ is just the closure of $ca(\Omega)\otimes X$ in $Ccabv(\Omega;X)$, hence $ca(\Omega)\hat{\otimes}_\pi X \subseteq Ccabv(\Omega;X)$.

Take $\theta = \sum_{i=1}^{n} x_i\mu_i$ where $x_i \in X$ and $\mu_i \in ca(\Omega)$. Then

$$|\theta|(S) = |\sum_{i=1}^{n} x_i\mu_i|(S) \leq \sum_{i=1}^{n} |x_i|\cdot|\mu|(S).$$

If we take the infimum on the right hand sice over all representations of θ in the form $\Sigma_i x_i\mu_i$ we obtain $|\theta|(S) \leq |\theta|_\pi$.

Suppose again $\theta = \sum_{i=1}^{n} x_i\mu_i \in ca(\Omega)\otimes X$, and put $\lambda = \sum_{i=1}^{n} |\mu_i|$, then $\mu_i \ll \lambda$ for each i. Write $f_i = \frac{d\mu_i}{d\lambda}$ which exists by the classical Radon-Nikodym theorem. $f_i \in L^1(\lambda)$ and

$f = \sum_{i=1}^{n} x_i f_i \in B_X(\lambda)$. Note that $f = \frac{d\theta}{d\lambda}$, that $|\mu_i|(S) = |f_i|_1$, where $|f_i|_1$ is the norm of f_i in $L^1(\lambda)$, and $|\theta|(S) = |f|_1$, where $|f|_1$ is the norm of f in $B_X(\lambda)$.

Define $B \subseteq B_X(\lambda)$ and $M \subseteq ca(\Omega)\otimes X$ as follows

$$B = \{\sum_{j=1}^{k} \bar{x}_i g_i : \bar{x}_i \in X, g_i \in L^1(\lambda), \text{ and } f = \sum_{i=1}^{k} \bar{x}_i g_i \ \lambda\text{-a.e.}\}$$

$$M = \{\sum_{j=1}^{k} \bar{x}_i \nu_i : x_i \in X, \nu_i \in ca(\Omega) \text{ and } \theta = \sum_{i=1}^{k} \bar{x}_i \nu_i\}.$$

There exists an injection $\psi: B \to M$ defined by $\psi(\sum_{i=1}^{k} \bar{x}_i g_i) = \sum_{i=1}^{k} \bar{x}_i \int_{(\cdot)} g \, d\lambda$; furthermore if $\sum_{i=1}^{k} \bar{x}_i g_i$ and $\sum_{i=1}^{k} \bar{x}_i \nu_i$ are in correspondence, then

$$\sum_{i=1}^{k} |\bar{x}_i| \cdot |g_i|_1 = \sum_{i=1}^{k} |\bar{x}_i| \, |\nu_i|(S).$$

We conclude that

$$\inf_B \sum_{i=1}^{k} |\bar{x}_i| \cdot |g_i|_1 \geq \inf_M \sum_{j=1}^{p} |\bar{x}_j| \cdot |\nu_j|(S) \qquad (\#)$$

for we have argued that for each number from the left side, there is a number from the right side which is at least as small.

It is well known that $B_X(\lambda) = X\hat{\otimes}_\pi L^1(\lambda)$ isometrically (see Treves [18]). Since $f \in B_X(\lambda)$ we have that $|f|_1 = |f|_\pi$, but $|f|_\pi = \inf_B \sum_{i=1}^{k} |\bar{x}_i| \cdot |g_i|_1$ so that $|\theta|(S) = |f|_1 = |f|_\pi$.

On the other hand, $|\theta|_\pi = \inf_M \sum_{i=1}^{k} |\bar{x}_i| \cdot |\nu_i|(S)$.

Thus from (#), $|\theta|(S) = |f|_\pi \geq |\theta|_\pi$. We already have $|\theta|(S) \leq |\theta|_\pi$ so that $|\theta|(S) = |\theta|_\pi$, which proves the first assertion.

We now prove $ca(\Omega)\hat{\otimes}_\pi X = RNca(\Omega; X)$.

If $\tau \in RNca(\Omega;X)$, then τ necessarily has finite total variation; put $\lambda = |\tau|$. Then $\tau \ll \lambda$ and since τ has the R-N property, there exists $f \in B_X(\lambda)$ such that $\tau(E) = \int_E f\, d\lambda$ for all $E \in \Omega$.

Since f is Bochner integrable, we may write f in the form $f(s) = \sum_{n=1}^{\infty} x_n \xi_{E_n}(s)$ λ-a.e., where $x_n \in X$ and $E_n \in \Omega$ (the family (E_n) is not in general pairwise disjoint), and possessing the property that $\sum_{n=1}^{\infty} |x_n| \lambda(E_n) < +\infty$. This is a well-known result which can be derived from Theorem III.5.5 infra, or see Brooks [3].

Define $\tau_n : \Omega \to X$ for each $n \in \omega$ by $\tau_n(E) = x_n \lambda(E \cap E_n)$. τ_n is easily seen to have the R-N property and $\tau_n \in ca(\Omega) \otimes X$, also $\sum_{n=1}^{\infty} |\tau_n|(S) = \sum_{n=1}^{\infty} |x_n| \cdot \lambda(E_n) < +\infty$. (1)

So we have

$$\tau(E) = \int_E f\, d\lambda = \int_E \sum_{n=1}^{\infty} x_n \xi_{E_n}\, d\lambda = \sum_{n=1}^{\infty} x_n \lambda(E \cap E_n),$$

or $\tau(E) = \sum_{n=1}^{\infty} \tau_n(E)$ for $E \in \Omega$. (2)

As remarked above $\tau_n \in X \hat{\otimes}_\pi ca(\Omega)$, hence $\sum_{k=1}^{n} \tau_k \in X \hat{\otimes}_\pi ca(\Omega)$. Note that (1) implies the sequence $\{\sum_{k=1}^{n} \tau_k\}_{n=1}^{\infty}$ is Cauchy in $X \hat{\otimes}_\pi ca(\Omega)$. From the first half of the proof, the π-norm is equal to the variation norm, so for $n < m$ positive integers

$$\left| \sum_{k=n}^{m} \tau_k \right|_\pi = \left| \sum_{k=n}^{m} \tau_k \right|(S) \leq \sum_{k=n}^{m} |\tau_k|(S) \longrightarrow 0$$

as n and m approach infinity because of (1). Regarding (2), $\sum_{k=1}^{\infty} \tau_k$ must converge in variation to τ since it converges to τ setwise. Therefore $\tau \in X \hat{\otimes}_\pi ca(\Omega)$ since it is the sum of a sequence (τ_n) in $X \hat{\otimes}_\pi ca(\Omega)$. Thus we have $RNca(\Omega;X) \subseteq X \hat{\otimes}_\pi ca(\Omega)$.

Conversely, if $\tau \in X\hat{\otimes}_\pi ca(\Omega)$, then from the general theory of projective products (see Treves [18], Cp. 45), there exists $x_n \in X$ and $\lambda_n \in ca(\Omega)$ such that $\sum_{n=1}^{\infty} |x_n| \cdot |\lambda_n|(S) < +\infty$ and such that $\tau(E) = \sum_{n=1}^{\infty} x_n \lambda_n(E)$, where the series will converge absolutely in X. Write $\tau_n = x_n \lambda_n$; τ_n clearly has the R-N property for each $n \in \omega$, also $\tau = \sum_{n=1}^{\infty} \tau_n$ and $\sum_{n=1}^{\infty} |\tau_n|(S) < +\infty$. We conclude from Theorem 2.2 that τ has the R-N property and so $\tau \in RNca(\Omega;X)$. Therefore $X\hat{\otimes}_\pi ca(\Omega) \subseteq RNca(\Omega;X)$, hence we have equality. □

2.4 Corollary. A measure $\mu:\Omega \to X$ with bounded variation has the R-N property if and only if μ is expressible as an indefinte Bochner integral with respect to some measure $\lambda:\Omega \to R^+$.

Proof. If f has the R-N property, then μ is expressible as an indefinite Bochner integral with respect to any measure with which μ is absolutely continuous.

Conversely, if $\mu(E) = \int_E f\, d\lambda$ for some positive measure λ, then there exists a sequence of simple functions (f_n) converging to f λ-a.e. such that $|f_n - f|_1 = \int_S |f_n - f| d\lambda \to 0$.

Write $\mu_n(E) = \int_S f_n\, d\lambda$; consequently, $\mu_n \in ca(\Omega)\otimes_\pi X$. We will show that $\mu \in ca(\Omega)\hat{\otimes}_\pi X$ so that by Theorem 2.3, μ will have the R-N property. Because $ca(\Omega)\otimes_\pi X$ is isometrically embedded in $cabv(\Omega;X)$, it suffices to show that the sequence $\mu_n \in ca(\Omega)\otimes_\pi X$ converges in variation to μ. This is indeed the case because $|\mu_n - \mu|(S) = |f_n - f|_1$ and $\lim_{n\to+\infty} \|f_n - f\|_1 = 0$. □

2.5 Corollary. A Banach space X is a Radon-Nikodym space if and only if $ca(S,\Omega)\hat{\otimes}_\pi X = cabv(S,\Omega;X)$ for every measurable space (S,Ω).

Proof. One always has $ca(S,\Omega)\hat{\otimes}_\pi X \subseteq cabv(S,\Omega;X)$ to begin with. If X is a Radon-Nikodym space, then that means any X-valued measure on Ω has the R-N property, that is, we have containment in the other direction, hence equality.

Conversely, if $ca(S,\Omega)\hat{\otimes}_\pi X = bvca(S,\Omega;X)$ for every measurable space, then, regarding Theorem 2.3, this means every X-valued measure of bounded variation has the R-N property regardless of the measurable space (S,Ω). This is the definition of X being a Radon-Nikodym space. □

2.6 Remark. In particular $ca(S,\Omega)\hat{\otimes}_\pi X = bvca(S,\Omega;X)$ if X is a reflexive Banach space of if X is a separable dual space. □

We have shown by Theorem 2.3 that $ca(\Omega)\hat{\otimes}_\pi X$ lies isometrically isomorphically in $Cbvca(\Omega;X)$. The question is raised whether this isomorphism is onto. The answer is no in general as demonstrated by the following example.

2.7 Example. This is an example of a vector valued measure with bounded bariation and relative norm compact range which does not have the R-N property. This is an example due to Yosida [19].

Let $S = [0,1]$, $\Omega = \mathcal{B}_o$ the σ-field of Baire sets of $[0,1]$ and $\lambda:\mathcal{B}_o \to R^+$ the Lebesgue measure. Denote by $m[1/3,2/3]$ the Banach space of real-valued functions $\xi = \xi(\theta)$ defined on $[1/3,2/3]$ and normed by $\|\xi\| = \sup_\theta |\xi(\theta)|$.

Define an $m[1/3,2/3]$-valued function $x(s) = \xi(\theta;s)$ on $[0,1]$ by :

$$x(s)(\theta) = \xi(\theta;s) = \begin{cases} \frac{s}{\theta} \text{ if } o \le s \le \theta; \\ \frac{s-1}{\theta-1} \text{ if } \theta \le s \le 1. \end{cases}$$

Yosida has shown that $x(s)$ satisfies the Lipschitz condition: $|x(s)-x(s')| \leq 3|s-s'|$ for all $s,s' \in [0,1]$.

Define a set function on the class of intervals of $[0,1]$ by $x(I) = x(s)-x(s')$ where s is the right end point of I and s' is the left end point. The set function x has its values in $m[1/3,2/3]$, and extends to the class of Baire sets on $[0,1]$ as a set function with values in $m[1/3,2/3]$. Because of the Lipschitz condition, $|x(B)| \leq 3\lambda(B)$ for $B \in \mathcal{B}_o$, it follows x is σ-additive, λ-continuous, and of finite total variation. Yosida has shown in [19], that x cannot be expressed as a Bochner integral with respect to Lebesgue measure λ even though $x \ll \lambda$. We shall show that x has relatively norm compact range.

According to Dunford-Schwartz [12], IV.5.6, a bounded set K in $m[1/3,2/3]$ is relatively compact if and only if for every $\varepsilon > 0$, there exists a finite collection $\{E_1,E_2,\ldots,E_n\}$ of disjoints sets with union $[1/3,2/3]$ and points $\theta_k \in E_k$ such that $\sup_{\theta \in E_k} |f(\theta)-f(\theta_k)| < \varepsilon$, for all $f \in K$ and $k = 1,2,3,\ldots,n$.

It shall be shown that $\{x(I):I \in \mathcal{I}\}$, where $\mathcal{I}$ is the algebra of unions of disjoint intervals of $[0,1]$, is relatively norm compact in $m[1/3,2/3]$.

Because x is σ-additive, its range is bounded.

Let $\varepsilon > 0$ be given. Choose n so large that $1/n < \varepsilon$. For $k = 0,1,2,\ldots,9n-1$, define $E_k = (\frac{1}{3} + \frac{k}{27n}, \frac{1}{3} + \frac{k+1}{27n})$. E_k has length $1/27n$. Write θ_k as the midpoint of E_k.

Let $0 \leq k \leq 9n-1$ and $I \in \mathcal{I}$ be fixed. We claim for any $\theta \in E_k$ that $|x(I)(\theta)-x(I)(\theta_k)| < \varepsilon$.

Suppose to begin with that $\theta \in E_k$ and $\theta_k < \theta$. We may write $x(I)(\theta) = \sum_{i=1}^{m}(-1)^{i+1}x(s_i)(\theta)$, where $s_1 > s_2 > s_3 > \ldots > s_m$.

Let q be the largest integer such that $s_q \geq \theta_k$ and p be the greatest integer such that $s_p \geq \theta$. Since $\theta_k < \theta$, we have $p \leq q$.

Thus,

$$x(I)(\theta) = \sum_{i=1}^{n}(-1)^{i+1}\xi(\theta,s_i) = \sum_{i=1}^{p}(-1)^{i+1}\frac{s_i-1}{\theta-1} + \sum_{i=p+1}^{m}(-1)^{i+1}\frac{s_i}{\theta}$$

and

$$x(I)(\theta_k) = \sum_{i=1}^{q}(-1)^{i+1}\frac{s_i-1}{\theta_k-1} + \sum_{i=q+1}^{m}(-1)^{i+1}\frac{s_i}{\theta_k}.$$

So that

$$|x(I)(\theta) - x(I)(\theta_k)| = \left|\sum_{i=1}^{p}(-1)^{i+1}\left(\frac{s_i-1}{\theta-1} - \frac{s_i-1}{\theta_k-1}\right) + \sum_{i=p+1}^{q}(-1)^{i+1}\left(\frac{s_i}{\theta} - \frac{s_i-1}{\theta_k-1}\right) + \sum_{i=q+1}^{m}(-1)^{i+1}\left(\frac{s_i}{\theta} - \frac{s_i}{\theta_k}\right)\right|$$

Simplifying, we get

$$|x(I)(\theta) - x(I)(\theta_k)| \leq \left|\sum_{i=1}^{p}(-1)^{i+1}(s_i-1)\frac{\theta_k-\theta}{(\theta_k-1)(\theta-1)}\right| + \left|\sum_{i=p+1}^{q}(-1)^{i+1}s_i\frac{\theta_k-\theta}{\theta(\theta_i-1)}\right| + \left|\sum_{i=p+1}^{q}(-1)^{i+1}\frac{\theta-s_i}{\theta(\theta_k-1)}\right| + \left|\sum_{i=q+1}^{m}(-1)^{i+1}s_i\frac{\theta_k-\theta}{\theta\cdot\theta_k}\right|$$

Write $|x(I)(\theta) - x(I)(\theta_k)| \leq Q_1 + Q_2 + Q_3 + Q_4$.

(1) $Q_1 = \left|\frac{\theta_k-\theta}{(\theta_k-1)(\theta-1)}\right| \cdot \left|\sum_{i=1}^{p}(-1)^{i+1}(s_i-1)\right| \leq 2\left|\frac{\theta_k-\theta}{(\theta_k-1)(\theta-1)}\right|$

$\leq 18|\theta_k-\theta| \leq \frac{18}{54n}$ since $\theta,\theta_k \in E_k$ which has half width of $1/54n$.

So $Q_1 \leq \frac{1}{3n}$.

(2) $Q_2 = \left|\frac{\theta_k-\theta}{\theta(\theta_k-1)}\right| \cdot \left|\sum_{i=p+1}^{q}(-1)^{i+1}s_i\right| \le \left|\frac{\theta_k-\theta}{\theta(\theta_k-1)}\right|$

$\le 9|\theta_k-\theta| = \frac{9}{54n} = \frac{1}{6n}$

(3) $Q_3 = \frac{1}{|\theta(\theta_k-1)|}\left|\sum_{i=p+1}^{q}(-1)^{i+1}(\theta-s_i)\right| \le 9\cdot\left|\sum_{i=p+1}^{q}(-1)^{i+1}(\theta-s_i)\right|$

Note that for $p+1 \le i \le q$, we have $\theta_k \le s_i \le \theta$, so that

$\left|\sum_{i=p+1}^{q}(-1)^{i+1}(\theta-s_k)\right| \le |\theta-\theta_k| \le \frac{1}{54n}$

Thus $Q_3 \le \frac{1}{6n}$.

(4) $Q_4 = \left|\frac{\theta_k-\theta}{\theta\cdot\theta_k}\right| \cdot \left|\sum_{i=q+1}^{m}(-1)^{i+1}s_i\right| \le \frac{|\theta_k-\theta|}{\theta\cdot\theta_k} \le 9|\theta_k-\theta| \le \frac{9}{54n}$

So $Q_4 \le \frac{1}{6n}$.

As a result of (1)-(4),

$|x(I)(\theta)-x(I)(\theta_k)| \le \frac{1}{3n} + \frac{1}{6n} + \frac{1}{6n} + \frac{1}{6n} = \frac{5}{6n} \le \frac{1}{n} < \varepsilon.$

Thus $\sup_{\theta\ge\theta_k} |x(I)(\theta)-x(I)(\theta_k)| < \varepsilon$ where $\theta \in E_k$. This was shown regardless of the set $I \in \mathcal{I}$ and of k, $0 \le k \le 9n-1$.

In a similar manner, we can show that $\sup_{\theta\le\theta_k} |x(I)(\theta)-x(I)(\theta_k)<\varepsilon$ where $\theta \in E_k$, whence it follows that $\sup_{\theta\in E_k} |x(I)(\theta)-x(I)(\theta_k)|< \varepsilon$ for each k, $o \le k \le 9n-1$, and each $K \in \mathcal{I}$. We have just shown that $x(\mathcal{I})$ is a relatively norm compact subset of $m[1/2,2/3]$. Because $B_o = \sigma(\mathcal{I})$ and $x(B_o) \subset cl[x(\mathcal{I})]$, we conclude that x has relatively norm compact range. □

3. The Space of Measures with Relatively Norm Compact Range.

We shall devote this section to identifying the space $ca(\Omega)\hat{\otimes}_\varepsilon X$. In Theorem 3.3, it is shown that this is the space of all X-valued measures on Ω with relatively norm compact range. In the next chapter, we consider an important class of measures with relatively norm compact range, namely, the class of indefinite integrals of "weakly" integrable functions.

Throughout this section $\mu:\Omega \to X$ is a measure, and $\lambda:\Omega \to R^+$ is a control measure for μ. We shall denote by Π the collection of all measurable partitions of S; that is, $\pi \in \Pi$ if and only if $\pi = \{F_1,F_2,\ldots,F_n\}$ where $F_i \in \Omega$ are pairwise disjoint, $\lambda(F_i) > 0$ and $S = \bigcup_{i=1}^{n} F_i$. Partially order Π as follows: for $\pi,\pi' \in \Pi$, write $\pi \geq \pi'$ if and only if every member of π lies in some member of π'.

For each $\pi \in \Pi$, define $\mu_\pi:\Omega \to X$ by

$$\mu_\pi(E) = \sum_{F\in\pi} \frac{\mu(F)}{\lambda(F)}\lambda_F(E), \text{ for } E \in \Omega,$$

and where $\lambda_F(E) = \lambda(E\cap F)$. Since λ is a control measure we have that $\mu_\pi \ll \mu$ and in fact $\lambda_F \ll \mu$; also, observe $\mu_\pi(F) = \mu(F)$ for each $F \in \pi$.

It is clear that for each $\pi \in \Pi$, $\mu_\pi \in ca(\Omega)\hat{\otimes}_\varepsilon X$ and with the partial ordering of Π, (μ_π) is a net (or generalized sequence) of measures in $ca(\Omega)\hat{\otimes}_\varepsilon X$ (See Dunford and Schwartz [12], Section I.7). We shall show that if μ has relatively norm compact range, then $\lim_{\pi} \mu_\pi = \mu$ is semivariation.

3.1 Lemma. If the Banach space X is the scalar field, then

$$\lim_{\pi} |\mu_\pi-\mu|(S) = 0$$

Proof. Because $\mu \ll \lambda$ and μ is scalar valued, there exists a λ-integrable function f such that $\mu(E) = \int_E f\, d\lambda$ for each $E \in \Omega$, by the Radon-Nikodym Theorem.

For $\pi \in \Pi$, define

$$f_\pi(s) = \sum_{F\in\pi} \frac{1}{\lambda(F)}\int_F f\, d\mu\ \xi_F(S),\ s \in S;$$

consequently, we have

$$\mu_\pi(E) = \int_E f_\pi\, d\mu.$$

From [12] (IV.8.18), $\lim_{\pi} f_\pi = f$ is $L^1(\lambda)$. Because the $L^1(\lambda)$-norm of a function is the total variation of its indefinite integral, we have $\lim_{\pi}|\mu-\mu_\pi|(S) = 0$. □

We use this lemma to prove the same result for the general case of X being an arbitrary Banach space and μ a measure with relatively norm compact range. We remark that Theorem 3.2 was proven by Lewis [16]; here we present a more direct proof of the theorem.

3.2 Theorem. If $\mu:\Omega \to X$ is a vector measure with relatively norm compact range, then $\lim_{\pi}\|\mu-\mu_\pi\|(S) = 0$, and consequently, $\mu \in ca(\Omega)\hat{\otimes}_\varepsilon X$.

Proof. Let λ, Π and (μ_π) be as above, and define

$$ca(\Omega,\lambda) = \{\phi \in ca(\Omega) : \phi \ll \lambda\}.$$

Define on $ca(\Omega,\lambda)$, for each $\pi \in \Pi$, the linear operator U_π by $U_\pi\phi = \phi_\pi$, where $\phi \in ca(\Omega,\lambda)$. By Lemma 3.1, $\lim_{\pi} U_\pi\phi = \phi$ in $ca(\Omega,\lambda)$; consequently, by the Phillips' Lemma [12,IV.5.4], $\lim_{\pi} U_\pi\phi = \phi$ uniformly on compact subsets of $ca(\Omega,\lambda)$.

For each $x^* \in X_1^*$, we have $x^*\mu \in ca(\Omega,\lambda)$ so that $\lim_{\pi}|U_\pi(x^*\mu)-x^*\mu|(S) = \lim_{\pi}|x^*\mu_\pi-x^*\mu|(S) = 0$. Since $\sup_{x^*\in X_1^*}|x^*\mu_\pi-x^*\mu|(S) = \|\mu_\pi-\mu\|(S)$, in order to show $\lim_{\pi}\|\mu_\pi-\mu\|(S) = 0$, it suffices to show, therefore, that $\lim_{\pi}|x^*\mu_\pi-x^*\mu|(S) = 0$ uniformly for $x^* \in X_1^*$, that is, $\lim_{\pi} U_\pi(x^*\mu) = x^*\mu$ uniformly for $x^* \in X_1^*$. From the above discussion, we need only show that $\Gamma = \{x^*\mu : x^* \in X_1^*\}$ is a compact subset of $ca(\Omega,\lambda)$.

To this end, let $(x_n^*\mu) \subseteq \Gamma$; we shall show that there is a subsequence which converges in variation. Since Γ is weakly sequentially compact by [12,IV.10.4], there exists $x_0^* \in X_1^*$ and a subsequence $(x_{n_i}^*)$ of (x_n^*) such that $x_{n_i}^*\mu \to x_0^*\mu$ weakly in $ca(\Omega,\lambda)$, that is $\lim_i x_{n_i}^*\mu(E) = x_0^*\mu(E)$ for each $E \in \Omega$. Since μ has relatively norm compact range, the set $R_\mu = \{\mu(E):E\in\Omega\}$ is relatively compact in X; but $x_{n_i}^* \to x_0^*$ pointwise on R_μ implies, by the Banach-Steinhaus theorem, $x_{n_i}^* \to x_0^*$ uniformly on R_μ. Thus, $\lim_i \sup_{E\in\Omega} |x_{n_i}^*\mu(E)-x_0^*\mu(E)| = 0$. But then

$$0 = \lim_i \sup_{E\in\Omega} |x_{n_i}^*\mu(E)-x_0^*\mu(E)| \geq \frac{1}{4} \lim_i |x_{n_i}^*\mu-x_0^*\mu|(S)$$

(see the remarks following Proposition I.1.1).

It follows then that $x_{n_i}^*\mu \to x_0^*\mu$ in variation, and that Γ is compact. Thus $\lim_\pi \|\mu_\pi-\mu\|(S) = 0$.

Finally, $\mu \in ca(\Omega)\hat{\otimes}_\varepsilon X$ since it is the limit in norm of a net (μ_π) of elements from $ca(\Omega)\hat{\otimes}_\varepsilon X$. □

3.3 Theorem. $ca(\Omega)\hat{\otimes}_\varepsilon X = Cca(\Omega;X)$ isometrically, where $Cca(\Omega;X)$ is the Banach space of X-valued measures with relatively norm compact range.

Proof. It is clear that any of the step measures which comprise the space $ca(\Omega)\otimes X$ have relatively norm compact range because they are linear combinations of elements of X with bounded scalar measures; consequently, each step measure is bounded with range in a finite dimensional subspace of X, hence has relatively norm compact range.

By Theorem 1.3, $ca(\Omega)\otimes_\varepsilon X$ is isometrically embedded in $ca(\Omega;X)$ and consequently in $Cca(\Omega;X)$. The closure of $ca(\Omega)\otimes_\varepsilon X$

in $Cca(\Omega;X)$ is $ca(\Omega)\hat{\otimes}_\varepsilon X$ and $ca(\Omega)\hat{\otimes}_\varepsilon X \subseteq Cca(\Omega;X)$. By Theorem 3.2, we have reverse inclusion. □

3.4 Corollary. If (S,Ω) and (T,Λ) are measurable spaces with X and Y Banach spaces, then

$$Cca(S\times T,\Omega\otimes_\sigma\Lambda;X\hat{\otimes}_\varepsilon Y) = Cca(S,\Omega;X)\hat{\otimes}_\varepsilon Cca(T,\Lambda;Y).$$

Proof. This follows from Theorem 3.3 and the associativity of the inductive tensor product of four Banach spaces. □

3.5 Corollary. $ca(S\times T,\Omega\otimes_\sigma\Lambda) = ca(S,\Omega)\hat{\otimes}_\varepsilon ca(T,\Lambda)$.

Proof. Any scalar measure has a bounded range, hence a relatively norm compact range. □

4. The Space $ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y)$

In this section (S,Ω) and (T,Λ) are measurable spaces with X and Y Banach spaces. We shall prove that the ε-norm on $ca(\Omega;X)\otimes ca(\Lambda;Y)$ is the semivariation norm, and that $ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y)$ can be isometrically embedded in a certain space of separately continuous bilinear maps.

Recall that $X^*\otimes Y^*$ is a vector subspace of $(X\hat{\otimes}_\varepsilon Y)^*$ since each $x^* \in X^*$ and $y^* \in Y^*$ defines a linear functional $x^*\otimes_\varepsilon y^* \in (X\hat{\otimes}_\varepsilon Y)^*$ such that $\langle x^*\otimes_\varepsilon y^*, x\otimes y\rangle = x^*(x)\cdot y^*(y)$ and $|x^*\otimes_\varepsilon y^*| = |x^*|\cdot|y^*|$. Observing the definition of the ε-norm, we see that the set

$$\Gamma = \{x^*\otimes_\varepsilon y^* : x^*\in X_1^*,\ y^*\in Y_1^*\}$$

is a norming family for $X\hat{\otimes}_\varepsilon Y$. Also, for each $\mu \in ca(\Omega;X)$, $\nu \in ca(\Lambda;Y)$, $x^* \in X^*$ and $y^* \in Y^*$, the linear functional $x^*\otimes_\varepsilon y^*$ acting on the vector measure $\mu\otimes_\varepsilon\nu$ yields scalar measure defined

by:

$$\langle x^* \otimes_\varepsilon y^*, \mu \otimes_\varepsilon \nu \rangle (G) = x^*\mu \times y^*\nu(G),$$

where $G \in \Omega \otimes_\sigma \Lambda$.

Thus, $x^* \otimes_\varepsilon y^*$ can be thought of as a linear map from $ca(\Omega;X) \otimes ca(\Lambda;Y)$ into $ca(\Omega \otimes_\sigma \Lambda)$; furthermore, $x^* \otimes_\varepsilon y^*$ is continuous when the former space has on it the ε-norm and the latter space is supplied with the total variation norm. We prove this in the next lemma.

4.1 Lemma. Let $x^* \in X^*$ and $y^* \in Y^*$. $x^* \otimes_\varepsilon y^*$ when considered as a linear map from $ca(\Omega;X) \hat{\otimes}_\varepsilon ca(\Lambda;Y)$ into $ca(\Omega \otimes_\sigma \Lambda)$ defined by $\langle x^* \otimes_\varepsilon y^*, \mu \otimes_\varepsilon \nu \rangle = x^*\mu \times y^*\nu$ is a continuous linear map. Moreover, $|x^* \otimes_\varepsilon y^*| = |x^*| \cdot |y^*|$.

Proof. The maps $\langle x^*, \mu \rangle = x^*\mu$ and $\langle y^*, \nu \rangle = y^*\nu$ are defined from $ca(\Omega;X)$ (resp. $ca(\Lambda;Y)$) into $ca(\Omega)$ (resp. $ca(\Lambda)$), are both clearly linear and they are both continuous since by Proposition I.1.1,

$$|x^*\mu|(S) \leq |x^*| \cdot \|\mu\|(S) \text{ and } |y^*\nu|(T) \leq |y^*| \cdot \|\nu\|(T).$$

From the general theory of tensor products, the map $x^* \otimes_\varepsilon y^*$ is continuous from $ca(\Omega;X) \otimes_\varepsilon ca(\Lambda;Y)$ into $ca(\Omega) \otimes_\varepsilon ca(\Lambda)$ (see Treves [18], Theorem 43.6), and $|x^* \otimes_\varepsilon y^*| = |x^*| \cdot |y^*|$.

The map $x^* \otimes_\varepsilon y^*$ can be extended to $ca(\Omega;X) \hat{\otimes}_\varepsilon ca(\Lambda;Y)$ with values in $ca(\Omega) \hat{\otimes}_\varepsilon ca(\Lambda)$. By Corollary 3.5, $ca(\Omega \otimes_\sigma \Lambda) = ca(\Omega) \hat{\otimes}_\varepsilon ca(\Lambda)$. □

From this lemma, we observe the next proposition which shall be used in this chapter and in Chapter IV.

4.2 Proposition. Let $\sum_{i=1}^{n}\mu_i\otimes_\varepsilon\nu_i \in ca(\Omega;X)\otimes ca(\Lambda;Y)$. Then the semivariation of this measure is given by

$$\|\sum_{i=1}^{n}\mu_i\otimes_\varepsilon\nu_i\|(S\times T) = \sup\,|\sum_{i=1}^{n}x^*\mu_i\times y^*\nu_i|(S\times T),$$

where the supremum is taken over $x^* \in X_1^*$ and $y^* \in Y_1^*$.

Proof. The collection $\Gamma = \{x^*\otimes_\varepsilon y^* : x^*\in X_1^* \text{ and } y^*\in Y_1^*\}$ is a norming family for $X\hat{\otimes}_\varepsilon Y$; therefore by Proposition I.1.1,

$$\|\sum_{i=1}^{n}\mu_i\otimes_\varepsilon\nu_i\|(S\times T) = \sup_{(x^*,y^*)\in\Gamma}|\langle x^*\otimes_\varepsilon y^*, \sum_{i=1}^{n}\mu_i\otimes_\varepsilon\nu_i\rangle|(S\times T).$$

The proposition follows then from this equality and Lemma 4.1. □

If we now endow $ca(\Omega;X)\otimes ca(\Lambda;Y)$ with the semivariation norm it is easy to see from the above proposition that this is a cross norm:

$$\|\mu\otimes_\varepsilon\nu\|(S\times T) = \sup|x^*\mu\times y^*\nu|(S\times T) = \sup|x^*\mu|(S)\cdot|y^*\nu|(T)$$
$$= \|\mu\|(S)\cdot\|\nu\|(T),$$

where all supremums are taken over $X_1^*\times Y_1^*$.

Also from Lemma 4.1, we have $\|\theta\|(S\times T) \le |\theta|_\varepsilon$ for any $\theta \in ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y)$; indeed, for $x^* \in X_1^*$ any $y^* \in Y_1^*$, the function $x^*\otimes_\varepsilon y^*$ is continuous and $|x^*\otimes_\varepsilon y^*| = |x^*|\cdot|y^*| = 1$ so that $|x^*\otimes_\varepsilon y^*\theta|(S\times T) \le |\theta|_\varepsilon$. Now taking the supremum over $X_1^*\times Y_1^*$ we get by Proposition 4.2, $\|\theta\|(S\times T) \le |\theta|_\varepsilon$. We shall see that in fact, equality reigns.

4.3 Theorem. For any $\theta \in ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y)$, we have $\|\theta\|(S\times T) = |\theta|_\varepsilon$, that is,

$ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y) \subseteq ca(\Omega\otimes_\sigma\Lambda; X\hat{\otimes}_\varepsilon Y)$ isometrically.

Proof. Let $\mu^* \in ca(\Omega;X)^*$ and $\nu^* \in ca(\Lambda;Y)^*$ and consider $\mu^* \otimes \nu^* \in ca(\Omega;X)^* \otimes ca(\Lambda;Y)^*$. The norm of $\mu^* \otimes \nu^*$ associated with the semivariation norm is defined by

$$|\mu^* \otimes \nu^*| = \sup \left| \sum_{i=1}^{n} \mu^*(\mu_i) \cdot \nu^*(\nu_i) \right|, \tag{1}$$

where the supremum is taken over all elements $p = \sum_{i=1}^{n} \mu_i \otimes_\varepsilon \nu_i$ such that $\|p\|(S \times T) \leq 1$. We claim that this norm is a crossnorm: $|\mu^* \otimes \nu^*| = |\mu^*| \cdot |\nu^*|$.

It is clear that $|\mu^*| \cdot |\nu^*| \leq |\mu^* \otimes \nu^*|$ by considering the supremum in (1) as being over a smaller class, namely, over all $p = \mu \otimes_\varepsilon \nu$ such that $\mu \in ca(\Omega;X)$, $\nu \in ca(\Lambda;Y)$ and $\|p\|(S \times T) \leq 1$.

Now let $p = \Sigma_i \mu_i \otimes_\varepsilon \nu_i$ be arbitrary with $\|p\|(S \times T) \leq 1$.

Since μ^* is a linear functional on $ca(\Omega;X)$ and $\Sigma_i \nu^*(\nu_i)\mu_i \in ca(\Omega;X)$ we have

$$|\Sigma_i \mu^*(\mu_i) \cdot \nu^*(\nu_i)| \leq |\mu^*| \cdot \|\Sigma_i \nu^*(\nu_i)\mu_i\|(S). \tag{2}$$

Choose $(E_j) \subseteq \Omega$, a finite collection of pairwise disjoint sets and scalars $(\alpha_j) \subseteq \Phi$ with $|\alpha_j| \leq 1$ such that

$$\begin{aligned} |\mu^*| \cdot \|\Sigma_i \nu^*(\nu_i)\mu_i\|(S) &< \frac{\varepsilon}{2} + |\mu^*| \cdot |\Sigma_j \alpha_j \Sigma_i \nu^*(\nu_i)\mu_i(E_j)| \\ &\leq \frac{\varepsilon}{2} + |\mu^*| \cdot |\nu^*| \cdot \|\Sigma_j \alpha_j \Sigma_i \mu_i(E_j)\nu_i\|(T). \end{aligned}$$

Again choosing sets $(F_k) \subseteq \Lambda$ pairwise disjoint and scalars $(\beta_k) \subseteq \Phi$ with $|\beta_k| \leq 1$ such that

$$|\mu^*| \cdot |\nu^*| \cdot \|\Sigma_j \alpha_j \Sigma_i \mu_i(E_j)\nu_i\|(T)$$

$$\leq \frac{\varepsilon}{2} + |\mu^*| \cdot |\nu^*| \cdot |\Sigma_k \beta_k \Sigma_j \alpha_j \Sigma_i \mu_i(E_j) \otimes \nu_i(F_k)|$$

Combining these inequalities with (2) we get

$$|\Sigma_i \mu^*(\mu_i) \cdot \nu^*(\nu_i)| \leq \varepsilon + |\mu^*| \cdot |\nu^*| \cdot |\Sigma_j \Sigma_k \alpha_j \beta_k \Sigma_i \mu_i \otimes_\varepsilon \nu_i(E_j \times F_k)| \tag{3}$$

Now since the family $\{E_j \times F_k\}_{j,k}$ is pairwise disjoint and covers $S \times T$, and $|\alpha_j \beta_k| \leq 1$ for all j and k, we see that the quantity on the right hand side of (3) is one of the numbers overwhich the supremum is taken in the definition of the semivariation of the measure $\Sigma_i \mu_i \otimes_\varepsilon \nu_i$.

Thus,

$$|\Sigma_i \mu^*(\mu_i) \cdot \nu^*(\nu_i)| \leq \varepsilon + |\mu^*| \cdot |\nu^*| \cdot \|\Sigma_i \mu_i \otimes_\varepsilon \nu_i\|(S \times T).$$

But now since $p = \Sigma_i \mu_i \otimes_\varepsilon \nu_i$ was arbitrary with $\|p\|(S \times T) \leq 1$, taking the supremum over all such p, we get by definition

$$|\mu^* \otimes \nu^*| \leq \varepsilon + |\mu^*| \cdot |\nu^*|.$$

Since $\varepsilon > 0$ was arbitrary, we get $|\mu^* \otimes \nu^*| \leq |\mu^*| \cdot |\nu^*|$ and the assertion that $|\mu^* \otimes \nu^*| = |\mu^*| \cdot |\nu^*|$ is proved.

Finally in order to prove $|\theta|_\varepsilon = \|\theta\|(S \times T)$ for any $\theta \in ca(\Omega;X) \hat{\otimes}_\varepsilon ca(\Lambda;Y)$, if suffices to prove this for θ of the form $\sum_{i=1}^{n} \mu_i \otimes \nu_i$. We have shown that $|\mu^* \otimes \nu^*| = |\mu^*| \cdot |\nu^*|$, this means

$$|\mu^* \otimes \nu^*(\theta)| \leq |\mu^* \otimes \nu^*| \cdot \|\theta\|(S \times T) = |\mu^*| \cdot |\nu^*| \cdot \|\theta\|(S \times T).$$

So that

$$\begin{aligned} |\theta|_\varepsilon &= \sup |\Sigma_i \mu^*(\mu_i) \cdot \nu^*(\nu_i)| \\ &= \sup |\langle \mu^* \otimes \nu^*, \Sigma_i \mu_i \otimes_\varepsilon \nu_i \rangle| \\ &\leq \sup |\mu^*| \cdot |\nu^*| \cdot \|\theta\|(S \times T) \\ &= \|\theta\|(S \times T), \end{aligned}$$

where the supremum is taken over $|\mu^*| = 1$ and $|\nu^*| = 1$.

Thus, $|\theta|_\varepsilon \leq \|\theta\|(S \times T)$. Since we have already observed the reverse inequality, the theorem is proved. □

4.4 Corollary. $ca(\Omega;X)\hat{\otimes}_\varepsilon Y \subseteq ca(\Omega;X\hat{\otimes}_\varepsilon Y)$ isometrically.

Proof. Let $T = \{0\}$ and $\Lambda = \{T,\phi\}$, the power set of T, then $ca(T,\Lambda;Y) = Y$ isometrically. Apply Theorem 4.3. □

Let X,Y, and Z be Banach spaces. Then $B(X,Y;Z)$ will denote the vector space of all separately continuous bilinear maps from $X \times Y$ into Z. Separately continuous bilinear maps need not be bounded; however, they are bounded whenever each factor of the product space on which they are defined is a dual space. For this reason, the space $B(X^*,Y^*;Z)$ can be normed by $|b| = \sup|b(x^*,y^*)|$ where the supremum is taken over $X_1^* \times Y_1^*$. This topology on $B(X^*,Y^*,Z)$ is the topology of uniform convergence on equicontinuous (simply bounded) subsets of $X^* \times Y^*$ of the form $A \times B$. $B(X^*,Y^*;Z)$ equipped with this norm topology is denoted by $B_\varepsilon(X^*,Y^*;Z)$. It is not difficult to see that $B_\varepsilon(X^*,Y^*;Z)$ is a Banach space.

Analogous to the usual embedding of $ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y)$ into $B_\varepsilon(ca(\Omega;X)^*,ca(\Omega;Y)^*;\Phi)$, from which the definition of the ε-topology was derived to begin with, we have the following theorem.

4.5 Theorem. There exists an isometric isomorphism from $ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y)$ into $B_\varepsilon(X^*,Y^*;ca(\Omega\otimes_\sigma\Lambda))$.

Proof. Define $\Theta : ca(\Omega;X) \times ca(\Lambda;Y) \to B_\varepsilon(X^*,Y^*;ca(\Omega\otimes_\sigma\Lambda))$ by $\Theta(\mu,\nu)(x^*,y^*) = x^*\mu \times y^*\nu$. Θ is a bilinear map; using once again the universal mapping property of tensor products, there exists a unique linear map $\tilde{\Theta} : ca(\Omega;X) \otimes ca(\Lambda;Y) \to B_\varepsilon(X^*,Y^*;ca(\Omega\otimes_\sigma\Lambda))$ such that $\tilde{\Theta}(\mu\otimes_\varepsilon\nu)(x^*,y^*) = x^*\mu \times y^*\nu$.

To prove that Θ is a one-to-one and an isometry, it suffices to show it is an isometry.

Let $\theta = \sum_{i=1}^{n} \mu_i \otimes_\varepsilon \nu_i$, and prove $|\tilde{\Theta}(\theta)| = |\theta|_\varepsilon$.

$|\tilde{\Theta}(\theta)| = \sup|\tilde{\Theta}(\theta)(x^*,y^*)| = \sup\left|\sum_{i=1}^{n} x^*\mu \times y^*\nu\right|(S\times T)$

$= \|\theta\|(S\times T)$ where the supremums are taken over $X_1^* \times Y_1^*$. By Theorem 4.3, $|\theta|_\varepsilon = \|\theta\|(S\times T)$.

Thus $|\Theta(\theta)| = |\theta|_\varepsilon$. □

There are a few advantages as well as disadvantages to embedding $ca(\Omega;X)\hat{\otimes}_\varepsilon ca(\Lambda;Y)$ in $B_\varepsilon(X^*,Y^*;ca(\Omega\otimes_\sigma\Lambda))$ rather than $B_\varepsilon(ca(\Omega;X)^*,ca(\Lambda;Y)^*;\Phi)$. Because we know very little of the structure of the duals of $ca(\Omega;X)$ and $ca(\Lambda;Y)$, it may be advantageous to use the embedding $B_\varepsilon(X^*,Y^*;ca(\Omega\otimes_\sigma\Lambda))$, the structure of the Banach spaces X^* and Y^* may be well-known or more easily worked with. The range space of the bilinear maps of $B_\varepsilon(X^*,Y^*;ca(\Omega\otimes_\sigma\Lambda))$ is more complicated than the scalar bilinear maps of the other embedding, though quite a lot is known of the structure of $ca(\Omega\otimes_\sigma\Lambda)$. At any rate, both embeddings induce the ε-norm on $ca(\Omega;X) \otimes ca(\Lambda;Y)$.

CHAPTER III
PETTIS AND LEBESGUE TYPE SPACES
AND VECTOR INTEGRATION

1. Measure Theory

Throughout this chapter, (S,Ω) is a measureable space, X and Y are Banach spaces, and $\mu:\Omega \to Y$ is a vector measure.

A set $A \subseteq S$ is μ-null if there exists a set $E \in \Omega$ such that $A \subseteq E$ and $\|\mu\|(E) = 0$. The phrase "μ-almost everywhere," or μ-a.e., refers to μ-null sets.

An X-valued Ω-simple function is a function of the form $f(s) = \sum_{i=1}^{n} x_i \zeta_{E_i}(s)$, where $x_i \in X$, $(E_i) \subseteq \Omega$ is pairwise disjoint, and $\zeta_{E_i}(s)$ is the characteristic function of E_i. The sets E_i are called the characteristic sets of f. The vector space of all such simple functions will be denoted by $S_X(\Omega)$, and when $X = \Phi$, by $S(\Omega)$. A function $f:S \to X$ is μ-measurable if there exists a sequence of simple functions from $S_X(\Omega)$ converging to f pointwise μ-a.e. The same function is weakly μ-measurable if for each $x^* \in X^*$, the scalar function x^*f is μ-measurable. Obviously, any μ-measurable function is weakly μ-measurable; the two concepts coincide if X is separable, by a theorem due to Pettis [17]. A scalar function $f:\Omega \to \Phi$ is Ω-measurable provided $f^{-1}(B) \in \Omega$ for every Borel set B. Any Ω-measurable function is μ-measurable, and any μ-measurable function is equal μ-a.e. to a Ω-measurable function.

A sequence (f_n) of μ-measurable functions converges in μ-measure to a function f means

$$\lim_n \|\mu\|([|f_n - f| > \varepsilon]) = 0$$

for each $\varepsilon > 0$. In this case, f is μ-measurable and there exists a subsequence (f_{n_i}) which converges pointwise to f μ-a.e., this is the theorem of F. Riesz. The Riesz theorem and the Egorov theorem are valid for vector measures because we can choose a control measure λ for μ. The measures μ and λ have the same null sets, and therefore the same measurable functions; convergence in μ-measurable is equivalent to convergence in λ-measure. Since these two theorems are valid for λ, they are valid for μ as well. Consequently, any sequence of functions converging μ-a.e. also converges in μ-measure. The phrases "in μ-measure" and "μ-a.e." are virtually interchangeable.

2. Normed Spaces of μ-measurable Functions.

If f is weakly μ-measurable, we can consider a number of scalar integrals associated with f in order to define a variety of seminorms on the space of X-valued weakly μ-measurable functions.

Define the two seminorms N and N* on the space of weakly μ-measurable X-valued functions as follows:

$$(1) \quad N(f) = \sup_{y^* \in Y_1^*} \int_S |f| \, d|y^*\mu|;$$

$$(2) \quad N^*(f) = \sup_{(x^*, y^*) \in X_1^* \times Y_1^*} \int_S |x^* f| \, d|y^*\mu|.$$

We remark that N and N* are indeed seminorms because each is the supremum of seminorms. Since $|x^* f| \leq |f|$ pointwise for $x^* \in X_1^*$, we have immediately that $0 \leq N^*(f) \leq N(f) \leq +\infty$.

The N-seminorm, which is a Lebesque-Bochner type, was introduced by Brooks and Dinculeanu [5]; this seminorm will sometimes be referred to as the strong seminorm. The N*-seminorm, which is a Pettis type seminorm, will be called the weak seminorm. These seminorms, of course, depend on many parameters such as the measure μ, and the Banach spaces X and Y; it will be clear from the context which parameters are being considered.

If f is X-valued, then $|f|$ is scalar valued, and we shall write $N(f) = N^*(|f|)$. Note that it is always the case that $N^*(f) = \sup_{x^* \in X_1^*} N(|x^*f|)$.

We list some properties of these seminorms

2.1 Proposition. (1) N and N* are subadditive and homogeneous;

(2) $N^*(f) = N(f)$ for f scalar valued;

(3) $N^*(f) = \sup_{A \in \Omega} N^*(f\zeta_A)$, $N(f) = \sup_{A \in \Omega} N(f\zeta_A)$;

(4) $N(\sup_n f_n) = \sup_n N(f_n)$ whenever (f_n) is increasing and positive;

(5) $N(\Sigma_n f_n) \leq \Sigma_n N(f_n)$ for every sequence of positive functions (f_n);

(6) $N(\lim_n \inf f_n) \leq \lim_n \inf N(f_n)$;

(7) $N(f) < +\infty$ implies f is finite μ-a.e. for f $R^\#$-valued.

Proof. Numbers (1), (2), and (3) are clear from the definitions.

(4): $$\sup_n N(f_n) = \sup_n \sup_{y^* \in Y_1^*} \int_S |f_n| \, d|y^*\mu|$$

$$= \sup_{y^* \in Y_1^*} \sup_n \int_S f_n \, d|y^*\mu|$$

$$= \sup_{y^* \in Y_1^*} \int_S \sup_n f_n \, d|y^*\mu| = N(\sup_n f_n).$$

(5): $N(\Sigma_n f_n) = N(\sup_k \sum_{n=1}^{k} f_n) = \sup_k N(\sum_{n=1}^{k} f_n)$

$$\leq \sup_k \sum_{n=1}^{k} N(f_n) = \Sigma_n N(f_n).$$

(6): From Fatou's lemma,

$$\int_S \lim_n \inf |f_n| d|y^*\mu| \leq \lim_n \inf \int_S |f_n| d|y^*\mu|.$$

So for $y^* \in Y_1^*$,

$$\int_S \lim_n \inf |f_n| d|y^*\mu| \leq \sup_{y^* \in Y_1^*} \lim_n \inf \int_S |f_n| d|y^*\mu|$$

$$= \lim_n \inf \sup_{y^* \in Y_1^*} \int_S |f_n| d|y^*\mu|$$

$$= \lim_n \inf N(f_n).$$

Finally, (6) is obtained by taking the supremum of the left-hand inequality over Y_1^*.

(7): If f is $R^{\#}$-valued, and $N(f) < +\infty$, then for each $y^* \in Y_1^*$, f is finite $|y^*\mu|$-a.e., from the classical theory. By Theorem I.1.2, μ-null sets are determined by only a countable family of $\{|y^*\mu|\}$, that is, there exists $(y_n^*) \subseteq Y_1^*$ such that a subset $A \subseteq S$ is μ-null if and only if A is $|y_n^*\mu|$-null for each $n \in \omega$. As a result, f is finite μ-a.e. □

The set $F_X(S,\Omega,\mu;Y)$ of functions $f:S \to X$ which are μ-measurable and satisfy $N(f) < +\infty$ is a vector space with seminorm N. When no confusion will arise, we write $F_X(\mu)$ for $F_X(S,\Omega,\mu;Y)$. The set $W_X(S,\Omega,\mu;Y)$, or simply $W_X(\mu)$, is the set of all functions f which are X-valued and μ-measurable that satisfy $N^*(f) < +\infty$. $W_X(\mu)$ is also a vector space with seminorn N^*. It is clear that $F_X(\mu) \subseteq W_X(\mu)$ and the topology induced on $F_X(\mu)$ by the seminorm N^* is weaker than the N-norm topology of $F_X(\mu)$ since $N^*(f) \leq N(f)$.

Brooks and Dinculeanu [5] have shown, and it follows from (5) in Proposition 2.1, that the system $(F_X(\mu),N)$ is a Banach space if functions equal μ-a.e. are identified. $(W_X(\mu),N^*)$ need not be a Banach space however, since it may not be complete, even if functions equal μ-a.e. are identified.

We can make $W_X(\mu)$ into a complete metric space by considering the metric:

$$d(f,g) = N^*(f-g) + \inf_{\alpha>0}\{\alpha+\|\mu\|([|f-g|>\alpha])\}, \quad f,g \in W_X(\mu).$$

Recall that the second term in the definition is itself a metric equivalent to convergence in μ-measure (see Dunford and Schwartz [12], p. 102).

2.2 Proposition. The semimetric space $(W_X(\mu),d)$ is complete.

Proof. Suppose $(f_n) \subseteq W_X(\mu)$ is d-Cauchy, then (f_n) is Cauchy in μ-measure; consequently, there exists a function f from S into X which is μ-measurable and to which (f_n) converges in μ-measure, that is, $\lim_n \|\mu\|([|f_n-f|>\varepsilon]) = 0$, for each $\varepsilon > 0$. To show $\lim_n d(f_n,f) = 0$, it suffices to show $\lim_n N^*(f_n-f) = 0$.

Let $x^* \in X_1^*$ be fixed. Since $|x^*f_n-x^*f| \leq |f_n-f|$ pointwise, we must have $x^*f_n \to x^*f$ in μ-measure too. Now for each $y^* \in Y_1^*$, $\int_S |x^*f_n-x^*f_m|d|y^*\mu| \leq N^*(f_n-f_m)$, so (x^*f_n) is Cauchy in $L_1(y^*\mu)$, the classical Lebesgue space. But $x^*f_n \to x^*f$ in μ-measure implies $x^*f_n \to x^*f$ in $y^*\mu$-measure, so therefore $x^*f_n \to x^*f$ in $L_1(y^*\mu)$.

Let $\varepsilon > 0$ be given, choose $K \in \omega$ such that whenever $m,n \geq K$, $N^*(f_n-f_m) < \varepsilon$.

But $\int_S |x^*f_n-x^*f_m|d|y^*\mu| \leq N^*(f_n-f_m) < \varepsilon$, for every $(x^*,y^*) \in X_1^* \times Y_1^*$ and $m,n \geq K$.

Because $x^*f_n \to x^*f$ in $L_1(y^*\mu)$ for each $(x^*,y^*) \in X_1^* \times Y_1^*$ we have $\lim_m \int_S |x^*f_n - x^*f_m| d|y^*\mu| = \int_S |x^*f_n - x^*f| d|y^*\mu|$.

Therefore, for $n \geq K$,

$$\int_S |x^*f_n - x^*f| d|y^*\mu| = \lim_m \int_S |x^*f_n - x^*f_m| d|y^*\mu| \leq \varepsilon.$$

Taking the supremum over $X_1^* \times Y_1^*$, we get $N^*(f_n - f) \leq \varepsilon$ for all $n \geq K$. □

This semimetric topology of $W_X(\mu)$ is the topology where a sequence $(f_n) \subseteq W_X(\mu)$ converges to a function f in $W_X(\mu)$ if and only if $N^*(f_n - f) \to 0$ and $f_n \to f$ in μ-measure. It is possible, though we shall not do so here, to consider a slightly more general space, the space of weakly μ-measurable functions with finite N^*-seminorm.

We next prove that N^* is a norm on $W_X(\mu)$, if we agree to identify two functions which disagree only on a μ-null set.

2.3 Proposition. If $f \in W_X(\mu)$ and $f = g$ μ-a.e. for some X-valued function g on S, then $g \in W_X(\mu)$ and $N^*(f-g) = 0$, in particular, $N^*(f) = N^*(g)$.

Conversely, if $N^*(f) = 0$, then $f = 0$ μ-a.e.

Proof. It is clear that g is μ-measurable since it is equal μ-a.e. to a μ-measurable function.

Now for each $x^* \in X_1^*$ and $y^* \in Y_1^*$, $x^*f = x^*g$ μ-a.e., and therefore $|y^*\mu|$-a.e. since $|y^*\mu| \leq \|\mu\|$ by Proposition I.1.1. This being the case, from the Lebesgue theory of integration we have $\int_S |x^*f - x^*g| d|y^*\mu| = 0$. $N^*(f-g) = 0$ is obtained by taking the supremum over $X_1^* \times Y_1^*$.

Conversely, $N^*(f) = 0$ implies $\sup_{x^* \in X_1^*} \int_S |x^*f| d|y^*\mu| = 0$, for each $y^* \in Y_1^*$. This supremum is the Pettis norm of f

with respect to the measure $|y^*\mu|$; it follows then from Pettis [17] that $f = 0$ $|y^*\mu|$-a.e. By Theorem I.2.1, we have $f = 0$ μ-a.e. □

Thus the space $(W_X(\mu),d)$ is complete metric space; in fact, it is a Frechét space. To see this, it suffices to show that $\lim_{\alpha\to 0} d(\alpha f,0) = 0$, where $\alpha \in \Phi$ and $f \in W_X(\mu)$. This fact was proven in Dunford and Schwartz [12], p. 329.

Notice that $S_X(\Omega)$ is a vector subspace of both $F_X(\mu)$ and $W_X(\mu)$. We shall denote by $B_X(\mu)$, the closure of $S_X(\Omega)$ in $F_X(\mu)$ and remark that $B_X(\mu)$ is a Banach space with norm N. $P_X(\mu)$ will denote the closure of $S_X(\Omega)$ in the metric topology of $W_X(\mu)$; $P_X(\mu)$ is a Frechét space.

As a result we have

(1) $f \in B_X(\mu)$ if there exists a sequence $(f_n) \subseteq S_X(\Omega)$ converging μ-a.e. to f such that $\lim_n N(f_n-f) = 0$.

(2) $f \in P_X(\mu)$ if $\lim_n N^*(f_n-f) = 0$ for some sequence $(f_n) \subseteq S_X(\Omega)$ converging μ-a.e. to f.

For a simple function g we have $N^*(g) \leq N(g)$, therefore, $B_X(\mu) \subseteq P_X(\mu)$. If $X = \Phi$, we write $B_X(\mu) = B(\mu)$ and $P_X(\mu) = P(\mu)$.

2.4 Proposition. $B(\mu) = P(\mu)$ and $N(f) = N^*(f)$ for all $f \in B(\mu)$.

Proof. By Proposition 2.1 (2), $N(f) = N^*(f)$ whenever f is scalar valued and μ-measurable. Thus $N \equiv N^*$ on $S(\Omega)$ so $B(\mu) = P(\mu)$. □

2.5 Proposition. Any bounded, μ-measurable, X-valued function on S is in $B_X(\mu)$.

Proof. Let $g:S \to X$ be bounded and μ-measurable and write $K = \sup_{s\in S}|g(s)|$. There exists a sequence $(g_n) \subseteq S_X(\Omega)$ converging in μ-measurable to g and uniformly bounded by $2K$.

For given $\varepsilon > 0$, there exists $M \in \omega$ such that $n \geq M$ implies $\|\mu\|([|g-g_n|>\varepsilon]) < \varepsilon$. Write $E_n = [|g-g_n|>\varepsilon]$.

When $n \geq M$,

$$N(g-g_n) = \sup_{y^*\in Y_1^*} \int_S |g-g_n|\,d|y^*\mu| \qquad .$$

$$\leq \sup_{y^*\in Y_1^*} \int_{E_n} |g-g_n|\,d|y^*\mu| + \sup_{y^*\in Y_1^*} \int_{S-E_n} |g-g_n|\,d|y^*\mu|$$

$$\leq 3K\cdot\|\mu\|(E_n) + \varepsilon\cdot\|\mu\|(S-E_n)$$

$$\leq 3K\varepsilon + \varepsilon\cdot\|\mu\|(S) = \varepsilon(3K+\|\mu\|(S)).$$

Therefore, $\lim_n N(g-g_n) = 0$ and $g \in B_X(\mu)$. □

Because $B_X(\mu) \subseteq W_X(\mu)$, it follows that $W_X(\mu)$ contains the bounded, μ-measurable functions too.

2.6 Proposition. A function $f \in W_X(\mu)$ is in $P_X(\mu)$ if and only if $\lim_n N^*(f\zeta_{A_n}) = 0$ for every sequence $(A_n) \subseteq \Omega$ with $\bigcap_{n=1}^{\infty} A_n = \phi$.

Proof. ($\rightarrow$) Suppose $f \in P_X(\mu)$, then letting $\varepsilon > 0$ be arbitrary, choose a simple function g such that $N^*(f-g) < \frac{\varepsilon}{2}$.

Now if $(A_n) \subseteq \Omega$ and $\bigcap_{n=1}^{\infty} A_n = \phi$, and $K = \sup_{s\in S}|g(s)|$, then there exits $M \in \omega$ such that $n \geq M$ implies $\|\mu\|(A_n) < \frac{\varepsilon}{2K}$.

For $n \geq M$,

$$N^*(f\zeta_{A_n}) \leq N^*(f-g) + N^*(g\zeta_{A_n}) < \frac{\varepsilon}{2} + K\cdot\|\mu\|(A_n) < \varepsilon.$$

($\leftarrow$) Assume $N^*(f\zeta_{A_n}) \to 0$ whenever $\bigcap_{n=1}^{\infty} A_n = \phi$.

Put $B_n = [|f| \leq n]$ and $A_n = S-B_n$; obviously $\bigcap_{n=1}^{\infty} A_n = \phi$. $f\zeta_{B_n}$ is bounded, so it is in $P_X(\mu)$ by Proposition 2.5. Since $\lim_n N^*(f-f\zeta_{B_n}) = \lim_n N^*(f\zeta_{A_n}) = 0$ by assumption, $f \in P_X(\mu)$. □

2.7 Proposition. For $f \in W_X(\mu)$, $f \in P_X(\mu)$ if and only if $N^*(f\zeta_{(\cdot)}) \ll \mu$.

Proof. The condition $N^*(f\zeta_{(\cdot)}) \ll \mu$ means for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|\mu\|(A) < \delta$, then $N^*(f\zeta_A) < \varepsilon$.

($\rightarrow$) Let $\varepsilon > 0$ be given, choose a simple function g so that $N^*(f-g) < \frac{\varepsilon}{2}$. Put $K = \sup_{s\in S}|g(s)|$, then for $\|\mu\|(A) < \frac{\varepsilon}{2K}$ we have

$$\begin{aligned} N^*(f\zeta_A) &\le N^*(f-g) + N^*(g\zeta_A) \\ &< \frac{\varepsilon}{2} + K\cdot N^*(\zeta_A) = \frac{\varepsilon}{2} + K\cdot\|\mu\|(A) \\ &< \frac{\varepsilon}{2} + K\,\frac{\varepsilon}{2K} = \varepsilon. \end{aligned}$$

Thus for $\|\mu\|(A) < \frac{\varepsilon}{2K}$, we have $N^*(f\zeta_A) < \varepsilon$.

($\leftarrow$) Let $\bigcap_{n=1}^{\infty} A_n = \phi$. Then $\|\mu\|(A_n) \rightarrow 0$ so by assumption $\lim_n N^*(f\zeta_{A_n}) = 0$. This implies by Proposition 2.6 that $f \in P_X(\mu)$. □

2.8 Remark. In Propositions 2.6 and 2.7, the properties of N^* that distinguish it from N where not used; consequently, 2.6 and 2.7 remain valid when $W_X(\mu)$, $P_X(\mu)$ and N^* are replaced with $F_X(\mu)$, $B_X(\mu)$ and N, respectively.

3. Convergence Theorems.

We now consider convergence properties of the N^*-norm in order to obtain criteria for a function to be in $P_X(\mu)$ given that it is the pointwise limit of a sequence of functions in $P_X(\mu)$.

3.1 Theorem. (Vitali Convergence Theorem) Let $(f_n) \subseteq P_X(\mu)$ and $f:S \rightarrow X$. Suppose (1) $f_n \rightarrow f$ in μ-measure;

(2) $N^*(f_n\zeta_{(\cdot)}) \ll \mu$ uniformly in n.

Then $f \in P_X(\mu)$ and $\lim_n N^*(f-f_n) = 0$.

Conversely, if $f_n \to f$ in $P_X(\mu)$, that is, in the d-metric, then (1) and (2) hold.

Proof. We first show that (f_n) is d-Cauchy. Since by (1), (f_n) is Cauchy in μ-measure, it suffices to show this sequence is N*-Cauchy.

For $\varepsilon > 0$ given, there exists, by assumption (2), a $\delta > 0$ such that $\|\mu\|(A) < \delta$ implies $N^*(f_n\zeta_A) < \frac{\varepsilon}{3}$ for all $n \in \omega$.

$f_n \to f$ in μ-measure implies the existence of $M \in \omega$ such that $\|\mu\|([|f_n - f_m| > \frac{\varepsilon}{3\cdot\|\mu\|(S)}]) < \delta$, when $n,m \geq M$.

Fix $m,n \geq M$ and write $B = [|f_n - f_m| > \frac{\varepsilon}{3\|\mu\|(S)}]$.

$$N^*(f_n - f_m) \leq N^*(f_n - f_m\zeta_B) + N^*(f_n - f_m\zeta_{S-B})$$
$$\leq N^*(f_n\zeta_B) + N^*(f_m\zeta_B) + N^*(f_n - f_m\zeta_{S-B})$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3\cdot\|\mu\|(S)}\, N^*(\zeta_{S-B}) \leq \varepsilon.$$

So (f_n) is N*-Cauchy as well as in μ-measure, thus (f_n) is d-Cauchy. Since $P_X(\mu)$ is complete and $f_n \to f$ in μ-measure, we see that $\lim_n N^*(f_n - f) = 0$ and $f \in P_X(\mu)$.

Conversely, suppose $d(f_n, f) \to 0$; then $f_n \to f$ in μ-measure and $N^*(f_n - f) \to 0$.

Let $\varepsilon > 0$ be given, there exists $M \in \omega$ such that $n \geq M$ implies $N^*(f_n\zeta_A) < N^*(f\zeta_A) + \frac{\varepsilon}{2}$ for all $A \in \Omega$. This follows from the inequality

$$N^*(f_n\zeta_A) - N^*(f\zeta_A) \leq N^*(f_n - f).$$

Since $f \in P_X(\mu)$, we have $N^*(f\zeta_{(\cdot)}) \ll \mu$. Thus there exists a $\delta > 0$ so that $N^*(f\zeta_A) < \frac{\varepsilon}{2}$ whenever $\|\mu\|(A) < \delta$. For $n \geq M$, $N^*(f_n\zeta_A) < N^*(f\zeta_A) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ whenever $\|\mu\|(A) < \delta$.

Because $f_1, f_2, \ldots, f_M \in P_X(\mu)$, it must be true that $N^*(f_k\zeta_{(\cdot)}) \ll \mu$, for $k = 1,2,\ldots,M$. There are only finitely many, so we may find a single $\delta_1 > 0$ such that $\|\mu\|(A) < \delta_1$ implies $N^*(f_k\zeta_A) < \varepsilon$, $1 \leq k \leq M$. Finally, putting $\delta_2 = \min\{\delta, \delta_1\}$, we see that $N^*(f_n\zeta_A) < \varepsilon$ for all $n \in \omega$ whenever $\|\mu\|(A) < \delta_2$. This is condition (2); condition (1) follows from the assumption that $d(f_n, f) \to 0$. □

3.2 Theorem. (Lebesgue Dominated Convergence) Let $(f_n) \subseteq P_X(\mu)$. Assume $g \in P(\mu)$ and $f:S \to X$ such that

(1) $f_n \to f$ in μ-measure;

(2) $|f_n| \leq |g|$ pointwise μ-a.e. for every $n \in \omega$.

Then $f \in P_X(\mu)$ and $N^*(f_n - f) \to 0$.

Proof. Note that $|x^*f_n| \leq |f_n| \leq |g|$ pointwise μ-a.e. for every $n \in \omega$ and $x^* \in X_1^*$. Let $A \in \Omega$.

$$N^*(f_n\zeta_A) = \sup_{x^* \in X_1^*} N(x^*f_n\zeta_A) \leq N(g\zeta_A).$$

$g \in P(\mu)$, and $P(\mu) = B(\mu)$, so $N(g\zeta_{(\cdot)}) \ll \mu$

by Proposition 2.7. By the above inequality we then have $N^*(f_n\zeta_{(\cdot)}) \ll \mu$ uniformly in $n \in \omega$.

By the Vitali Theorem 3.1, $f \in P_X(\mu)$ and $N^*(f_n - f) \to 0$. □

3.3 Corollary. (Bounded Convergence) If $|f_n| \leq M$ pointwise μ-a.e., for every n, where M is some positive constant, then $f \in P_X(\mu)$ and $N^*(f_n - f) \to 0$.

Proof. Put $g(s) = M$ for $s \in S$. g is a constant function so $g \in P(\mu)$ by Proposition 2.5. Apply Theorem 3.2. □

3.4 Remark. No crucial role was played by the N*-norm; consequently, the Vitali and Lebesgue Dominated Convergence theorems are valid when N* and $P_X(\mu)$ is replaced by N and $B_X(\mu)$, with only minor changes in the proof.

Brooks and Dinculeanu [5] have studied the space $B_X(\mu)$ in more detail and generality. Under suitable conditions, the space $B_X(\mu)$ is weakly sequentially complete, a workable dual space has been identified, and sufficient conditions have been given for subsets of $B_X(\mu)$ to be weakly compact. The space $P_X(\mu)$ is more difficult to work with because it is not a Banach space but a Frechét space.

4. Integration.

Let $\mu:\Omega \to Y$ be countably additive and X a Banach space. In this section we develop an integration theory for functions in $P_X(\mu)$ and $B_X(\mu)$.

Suppose $f(s) = \sum_{i=1}^{n} x_i \zeta_{E_i}(s)$ where $x_i \in X$ and $(E_i) \subseteq \Omega$ forms a measurable partition of S, define for any $E \in \Omega$,

$$\int_E f \otimes_\varepsilon d\mu = \sum_{i=1}^{n} x_i \otimes \mu(E \cap E_i).$$

We note that the value of the integral of a simple function lies in the space $X \hat{\otimes}_\varepsilon Y$.

4.1 Proposition. (1) The integral of a simple function is well defined.

(2) The integral is linear, homogeneous, and countably additive.

Proof. Suppose $f = \sum_{i=1}^{n} x_i \zeta_{E_i}$ and $g = \sum_{j=1}^{n} \bar{x}_j \zeta_{A_j}$. Then $f-g = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - \bar{x}_j) \zeta_{E_i A_j}$ and

$$\int_E f \otimes_\varepsilon d\mu - \int_E g \otimes_\varepsilon d\mu = \sum_{i=1}^{n} x_i \otimes \mu(E\cap E_i) - \sum_{j=1}^{m} \bar{x}_j \otimes \mu(E\cap A_j)$$

$$= \sum_{i=1}^{n} x_i \otimes (\sum_{j=1}^{m} \mu(E\cap E_i \cap A_j)) - \sum_{j=1}^{m} \bar{x}_j \otimes (\sum_{i=1}^{n} \mu(E\cap E_i \cap A_j))$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} x_j \otimes \mu(E\cap E_i \cap A_j) - \sum_{j=1}^{m} \sum_{i=1}^{n} \bar{x}_j \otimes \mu(E\cap E_i \cap A_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - \bar{x}_j) \otimes \mu(E\cap E_i \cap A_j) = \int_E f-g \otimes_\varepsilon d\mu .$$

This proves the linearity and uniqueness; indeed, if $f = g$ μ-a.e. then whenever $\|\mu\|(E_i \cap A_j) \neq 0$, we have $x_i - \bar{x}_j = 0$. It is always true then that $\|\mu\|(E\cap E_i \cap A_j) = 0$ or $x_i - \bar{x}_j = 0$; therefore,

$$\int_E f \otimes_\varepsilon d\mu - \int_E f \otimes_\varepsilon d\mu = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i - \bar{x}_j) \otimes \mu(E\cap E_i \cap A_j) = 0.$$

This proves the integral is well defined.

In order to show the countable additivity of the indefinite integral of a simple function, it suffices to consider a simple function of the form $f(s) = x\zeta_E(s)$, for some $x \in X$ and $E \in \Omega$. indeed, if $(A_n) \subseteq \Omega$ is disjoint and $A = \bigcup_{n=1}^{\infty} A_n$, then since μ is σ-additive, $\mu(E\cap A) = \sum_{n=1}^{\infty} \mu(E\cap A_n)$ converges unconditionally in Y.

It is clear that $\sum_{n=1}^{\infty} x\otimes\mu(E\cap A_n)$ converges unconditionally in $X\hat{\otimes}_\varepsilon Y$ and

$$\sum_{n=1}^{\infty} x\otimes\mu(E\cap A_n) = x\otimes \sum_{n=1}^{\infty} \mu(E\cap A_n) = x\otimes\mu(E\cap A).$$

But this is equivalent to

$$\int_A f\otimes_\varepsilon d\mu = \sum_{n=1}^{\infty} \int_{A_n} f\otimes_\varepsilon d\mu. \quad \square$$

4.2 Propositon. Let $f \in S_X(\Omega)$. Then for every $(x^*,y^*) \in X^* \times Y^*$,

$$(x^*\otimes_\varepsilon y^*) \int_E f\otimes_\varepsilon d\mu = \int_E x^* f d(y^*\mu).$$

Furthermore,

$$\left|\int_E f \otimes_\varepsilon d\mu\right|_\varepsilon \leq N^*(f\zeta_E).$$

Proof. The first assertion follows immediately upon writing the integral as a finite sum determined by the simple function f. The cannonical image of $X_1^* \times Y_1^*$ in $(X \hat{\otimes}_\varepsilon Y)^*$ is norming for $X \hat{\otimes}_\varepsilon Y$ so that

$$\begin{aligned}\left|\int_E f \otimes_\varepsilon d\mu\right|_\varepsilon &= \sup \left|(x^* \otimes_\varepsilon y^*)\int_E f \otimes_\varepsilon d\mu\right| \\ &= \sup \left|\int_S x^*(f\zeta_E)\, d(y^*\mu)\right| \\ &\leq N^*(f\zeta_A),\end{aligned}$$

where the supremum is over $(x^*,y^*) \in X_1^* \times Y_1^*$. □

4.3 Corollary. Let $f \in S_X(\Omega)$ and $\tau(E) = \int_E f \otimes_\varepsilon d\mu$, for $E \in \Omega$. Then $\tau:\Omega \to X \hat{\otimes}_\varepsilon Y$ is a vector measure and $\tau \ll \mu$.

Proof. The fact that τ is a vector measure follows from Proposition 4.1. Observing Proposition 4.2, $|\tau(E)| \leq N^*(f\zeta_E)$. $N^*(f\zeta_{(\cdot)}) \ll \mu$ by Proposition 2.6, so $\tau \ll \mu$. □

5. The Weak and Strong Integrals.

Consider a function $f \in P_X(\mu)$. There exists a sequence $(f_n) \subseteq S_X(\Omega)$ which converges to f in μ-measure and $\lim_n N^*(f_n - f) = 0$. The sequence (f_n) is said to determine f. Now, for each $E \in \Omega$, the sequence $\{\int_E f_n \otimes_\varepsilon d\mu\}$ is Cauchy in the norm of $X \hat{\otimes}_\varepsilon Y$. Indeed, by Proposition 4.2,

$$\left|\int_E f_n \otimes_\varepsilon d\mu - \int_E f_m \otimes_\varepsilon d\mu\right|_\varepsilon \leq N^*(f_n - f_m).$$

Define $\int_E f \otimes_\varepsilon d\mu = \lim_n \int_E f_n \otimes_\varepsilon d\mu$, $E \in \Omega$.

This integral takes its values in $X \hat{\otimes}_\varepsilon Y$ and is called the weak integral of f over E with respect to μ.

The weak integral in unambigously defined for if (f_n) and (g_n) both determined f, then $\lim_n N^*(f - f_n) = 0$ and

$\lim_n N^*(f-g_n) = 0.$

$$\left|\int_E f_n \otimes_\varepsilon d\mu - \int_E g_n \otimes_\varepsilon d\mu\right|_\varepsilon \le N^*(f_n-g_n)$$

$$\le N^*(f_n-f) + N^*(f-g_n).$$

From this we see that

$$\lim_n \int_E f_n \otimes_\varepsilon d\mu = \lim_n \int_E g_n \otimes_\varepsilon d\mu.$$

Let now $f \in B_X(\mu)$ and $(f_n) \subseteq S_X(\Omega)$ a determining sequence for f in $B_X(\mu)$, that is, $\lim_n N(f-f_n) = 0$.

Because

$$\left|\int_E f_n \otimes_\varepsilon d\mu - \int_E f_m \otimes_\varepsilon d\mu\right|_\varepsilon \le N(f_n-f_m),$$

we see that the sequence $\{\int_E f_n \otimes_\varepsilon d\mu\}$ is Cauchy in the norm of $X \hat{\otimes}_\varepsilon Y$; hence, it tends to a limit in that space.

Define the strong integral of f by

$$\int_E f \otimes_\varepsilon d\mu = \lim_n \int_E f_n \otimes_\varepsilon d\mu, \text{ for } E \in \Omega.$$

Since $B_X(\mu) \subseteq P_X(\mu)$ and the norm $N \le N^*$, it follows if $f \in B_X(\mu)$, then any sequence in $S_X(\Omega)$ that determines f in $B_X(\mu)$ also determines f in $P_X(\mu)$; consequently, a determining sequence for f in $B_X(\mu)$ defines the same value in $X\hat{\otimes}_\varepsilon Y$ for both the weak and strong integrals. In this way we see that the strong integral is well defined and can be unambiguously denoted in the same way as the weak integral.

We next consider the countable additivity of these integrals as well as a decomposition theorem for weakly integrable functions.

5.1 Propositon. The indefinite integral of a weakly (resp. strongly) integrable function is countably additive.

Proof. The countable additivity of the strong integral will follow from the countable additivity of the weak. To that end, let $f \in P_X(\mu)$ and write $\tau(E) = \int_E f \otimes_\varepsilon d\mu$, where $E \in \Omega$. There exists a sequence (f_n) of simple functions that determines f in $P_X(\mu)$.

If $\tau_n(E) = \int_E f_n \otimes_\varepsilon d\mu$, then by definition $\lim_{n\to\infty} \tau_n(E) = \tau(E)$ for every $E \in \Omega$.

By Proposition 4.1, each τ_n is countably additive, and by Corollary 4.3, $\tau_n \ll \mu$ for each $n \in \omega$; therefore, by the vector form of the Nikodym theorem ([12], IV.10.6), τ is countably additive. □

5.2 Proposition. If $f \in P_X(\mu)$, then for each $x^* \in X_1^*$ and $y^* \in Y_1^*$, we have $x^*f \in L^1(y^*\mu)$, the classical Lebesgue space.

Furthermore, $\int_E x^*f \, d(y^*\mu) = (x^* \otimes_\varepsilon y^*) \int_E f \otimes_\varepsilon d\mu$.

Proof. The first assertion follows from the inequality $\int_S |x^*f| d|y^*\mu| \leq N^*(f) < +\infty$. The left-side of this inequality is the L^1-norm of x^*f in $L^1(y^*\mu)$, that is $\|x^*f\|_1 < +\infty$. The second assertion follows by considering a determining sequence for f, Proposition 4.2, and the continuity of $x^* \otimes_\varepsilon y^*$. □

5.3 Proposition. If $f \in P_X(\mu)$, then $\left|\int_E f \otimes_\varepsilon d\mu\right|_\varepsilon \leq N^*(f\zeta_E)$. If $f \in B_X(\mu)$, then $\left|\int_E f \otimes_\varepsilon d\mu\right|_\varepsilon \leq N(f\zeta_E)$.

Proof. For $f \in P_X(\mu)$,

$$\left|\int_E f \otimes_\varepsilon d\mu\right|_\varepsilon = \sup \left|(x^* \otimes_\varepsilon y^*) \int_E f \otimes_\varepsilon d\mu\right|$$

$$= \sup \left|\int_S x^*(f\zeta_E) \, d(y^*\mu)\right|$$

$$\leq \sup \int_S |x^*(f\zeta_E)|\, d|y^*\mu|$$

$$= N^*(f\zeta_E).$$

The second assertion follows because $B_X(\mu) \subset P_X(\mu)$ and $N^*(f) \leq N(f)$. □

5.4 Theorem. If $h = \sum_{i=1}^{\infty} x_i \zeta_{E_i} \in P_X(\mu)$, where $x_i \in X$ and the family $(E_i) \subseteq \Omega$ is pariwise disjoint, then for each $E \in \Omega$ we have

$\int_E h \otimes_\varepsilon d\mu = \sum_{i=1}^{\infty} x_i \otimes \mu(E \cap E_i)$, and the series converges unconditionally in $X \hat{\otimes}_\varepsilon Y$.

Proof. Define $\tau(E) = \int_E h \otimes_\varepsilon d\mu$.

Since τ is σ-additive, Proposition 5.1, and $S = \bigcup_{i=1}^{\infty} E_i$, we have $\tau(E) = \tau(\bigcup_{i=1}^{\infty} E \cap E_i) = \sum_{i=1}^{\infty} \tau(E \cap E_i)$ where the last series converges unconditionally in $X \hat{\otimes}_\varepsilon Y$.

Now for each i, $\tau(E \cap E_i) = \int_{E \cap E_i} h \otimes_\varepsilon d\mu$, and for $(x^*, y^*) \in X_1^* \times Y_1^*$, we have

$$x^* \otimes_\varepsilon y^* \tau(E \cap E_i) = \int_{E \cap E_i} x^* h \, d(y^*\mu) = x^*(x_i) \cdot y^*\mu(E \cap E_i)$$

$$= x^* \otimes_\varepsilon y^*(x_i \otimes \mu(E \cap E_i)).$$

But this implies $\tau(E \cap E_i) = x_i \otimes \mu(E \cap E_i)$ because $\{x^* \otimes_\varepsilon y^* \mid (x^*, y^*) \in X_1^* \times Y_1^*\}$ is norming for $X \hat{\otimes}_\varepsilon Y$.

Thus $\int_E h \otimes_\varepsilon d\mu = \tau(E) = \sum_{i=1}^{\infty} \tau(E \cap E_i) = \sum_{i=1}^{\infty} x_i \otimes \mu(E \cap E_i)$ converges unconditionally in $X \hat{\otimes}_\varepsilon Y$. □

5.5 Theorem. (Decomposition Theorem) Suppose $f \in P_X(\mu)$, then f can be written in the form $f = g + h$ μ-a.e., where

(1) g is bounded (hence $g \in B_X(\mu)$);

(2) $h = \sum_{i=1}^{\infty} x_i \zeta_{E_i}$ where $x_i \in X$ and $E_i \in \Omega$ are disjoint.

Furthermore,

(#) $\int_E f \otimes_\varepsilon d\mu = \int_E g \otimes_\varepsilon d\mu + \sum_{i=1}^{+\infty} x_i \otimes \mu(E \cap E_i)$ where the last series converges unconditionally for each $E \in \Omega$.

Proof. Since f is μ-measurable, it has an almost separable range, so that we assume from the beginning that the range of f is separable.

Let $\alpha_n \downarrow 0$ be summable. Define, for each n, S(n,f(s)) to be the sphere of radius α_n about f(s). For each n, $f(S) \subseteq \bigcup_{s \in S} S(n,f(s))$. The range of f is separable metric space -- hence it is Lindelöf; consequently, there exists a sequence (s_i^n) in S such that $f(S) \subseteq \bigcup_{i=1}^{+\infty} S(n,f(s_i^n))$.

Pettis has proved in [17] that the function $s \to |f(s)-f(s_i^n)|$ is μ-measurable, hence $A_i^n = |f-f(s_i^n)|^{-1}([0,\alpha_n)) \in \Omega$ is μ-measurable.

Define $E_i^n = A_i^n - \bigcup_{k=1}^{i-1} A_k^n$ and write $f_n(s) = \sum_{i=1}^{+\infty} f(s_i^n)\zeta_{E_i^n}(s)$. Note that for each n, $S = \bigcup_{i=1}^{+\infty} E_i^n$ and $E_i^n \cap E_j^n = \phi$ for $i \neq j$. Obviously, for any n, and $s \in S$, we must have $s \in E_i^n$ for some i; but this implies that $|f(s)-f(s_i^n)| < \alpha_n$ and, therefore, $|f(s) - f_n(s)| < \alpha_n$. This means $f_n \to f$ uniformly on S.

Write $g(s) = \sum_{n=1}^{+\infty} (f_{n+1}(s)-f_n(s))$, then g is measurable, and bounded since $|g(s)| \leq \sum_{n=1}^{+\infty} |f_{n+1}(s)-f_n(s)| \leq 2\sum_{n=1}^{\infty} \alpha_n$.

Finally, define $E_i = E_i^1$, $x_i = f(s_i^1)$ and $h(s) = f_1(s) = \sum_{i=1}^{\infty} f(s_i^1)\zeta_{E_i^1}(s) = \sum_{i=1}^{+\infty} x_i \zeta_{E_i}(s)$.

We clearly have $f(s) = g(s) + h(s)$, for all $s \in S$, because $f(s) = \lim_{n \to +\infty} f_n(s) = f_1(s) + \sum_{i=1}^{+\infty} (f_{n+1}(s)-f_n(s)) = h(s) + g(s)$.

g is bounded so that $g \in P_X(\mu)$ (Proposition 2.5); consequently $h \in P_X(\mu)$, since $h = f - g$ and $f,g \in P_X(\mu)$.

(#) follows from Theorem 5.4. □

The Decomposition theorem is similar to the one published by Brooks in [3].

We now turn to a deeper study of the weak and strong integrals by comparison with well known, more familiar integrals.

6. The Weak Integral and its Relationships to Other Integrals.

The purpose of this section is to explore the various relationships between the weak integral, the general bilinear integral of Bartle [1], and a Pettis-type integral which will be introduced below.

Let us first consider the Bartle general bilinear integral. Bartle considers μ-measurable functions $f:S \to X$ and a measure $\mu:\Omega \to Y$ with a bilinear map b from $X \times Y$ into a third space Z. In our context, $b:X\times Y \to X\hat{\otimes}_\varepsilon Y$ is the canonical bilinear map defined by $b(x,y) = x\otimes y$. Note that $|b(x,y)|_\varepsilon = |x|\cdot|y|$. Bartle requires the "control" set function for the measure μ to be $\|\mu\|_\varepsilon^X:\Omega \to R^+$, the semivariation of μ with respect to X (and ε), defined in Chapter I. From Lemma I.2.2, we have that $\|\mu\|_\varepsilon^Y(A) = \|\mu\|(A)$ for every $A \in \Omega$. Consequently, Bartle's control set function turns out to be the "usual" one. It is important to note, that the measure then has the *-property (see Bartle [1]).

In order for a function f from S into X to be integrable in the sense of Bartle, there must exist a sequence $(f_n) \subseteq S_X(\Omega)$

converging μ-a.e. such that the sequence $\{\int_E f_n d\mu\}$ is Cauchy is the norm of $X \hat{\otimes}_\varepsilon Y$ for each $E \in \Omega$, where the integral of a simple function is defined in the usual fashion. In this case, one defines

$$(B)\int_E f d\mu = \lim_n \int_E f_n d\mu.$$

We say the sequence (f_n) determines the Bartle integral of f.

We now consider the relationship between the weak integral, defined for functions in $P_X(\mu)$, and the Bartle integral.

6.1 Theorem. A function $f:S \to X$ is Bartle integrable if and only if $f \in P_X(\mu)$. Moreover,

(1) a sequence $(f_n) \subseteq S_X(\Omega)$ determines the Bartle integral of f if and only if $\lim_n N^*(f_n - f) = 0$;

(2) $\int_E f \otimes_\varepsilon d\mu = (B)\int_E f\, d\mu$ for every $E \in \Omega$.

Proof. If $f \in P_X(\mu)$, then there exist a determining sequence $(f_n) \subseteq S_X(\Omega)$ for f in $P_X(\mu)$, that is, $\lim_n N^*(f - f_n) = 0$. From our observations preceding the definition of the weak integral in section 5, the sequence $\{\int_E f_n \otimes_\varepsilon d\mu\}$ is Cauchy in $X \hat{\otimes}_\varepsilon Y$ for every $E \in \Omega$. Since the Bartle and weak integrals of simple functions obviously coincide, (f_n) determines the Bartle integral of f, f is Bartle integrable, and

$$\int_E f \otimes_\varepsilon d\mu = (B)\int_E f\, d\mu.$$

Conversely, suppose $f:S \to X$ is Bartle integrable, that is, there exists a sequence $(f_n) \subseteq S_X(\Omega)$ converging to f μ-a.e. such that $\lim_n \int_E f_n\, d\mu$ exists in $X \hat{\otimes}_\varepsilon Y$ for every $E \in \Omega$, the limit being $(B)\int_E f_n\, d\mu$. We wish to show $f \in P_X(\mu)$, to do so, it suffices to show that $\lim_n N^*(f - f_n) = 0$, this will also prove (1).

Write

$\tau_n(E) = (B)\int_E f_n \, d\mu$ and $\tau_0(E) = (B)\int_E f \, d\mu$, $E \in \Omega$.

By the definition of the Bartle integral, $\lim_n \tau_n(E) = \tau_0(E)$ for each $E \in \Omega$. The measures τ_n are integrals of simple functions so they are σ-additive and $\tau_n \ll \mu$ for each $n \in \omega$; consequently, by the vector form of the Vitali-Hahn-Saks theorem (see [12,III.7.2]), we have $\tau_n \ll \mu$ uniformly for $n \in \omega$.

Because $|x^* \otimes_\varepsilon y^* \tau_n| \leq \|\tau_n\|$ for $x^* \in X_1^*$ and $y^* \in Y_1^*$, we have $|x^* \otimes_\varepsilon y^* \tau_n| \ll \mu$ uniformly for $n \in \omega$, $x^* \in X_1^*$ and $y^* \in Y_1^*$. Note that $|x^* \otimes_\varepsilon y^* \tau_n|(E) = \int_E |x^* f_n| \, d|y^* \mu|$ for $E \in \Omega$. Taking the supremum over $X_1^* \times Y_1^*$, we have $N^*(f_n \zeta_{(\cdot)}) \ll \mu$ uniformly in $n \in \omega$. But $f_n \to f$ μ-a.e., and $N^*(f_n \zeta_{(\cdot)}) \ll \mu$ uniformly implies by the Vitali Theorem 3.1, that $f \in P_X(\mu)$ and $\lim_n N^*(f_n - f) = 0$. The validity of (2) follows because (f_n) determines the weak and Bartle integrals of f. □

In [12], Dunford and Schwartz developed a theory of integration of scalar valued functions with respect to a vector valued measure. This theory is that of Bartle's for $X = \Phi$, in this case we shall say a scalar function is Bartle-Dunford-Schwartz integrable, or B-D-S integrable. We have the following corollary to Theorem 6.1.

6.2 Corollary. A scalar valued, μ-mearurable function f is B-D-S integrable if and only if $f \in P(\mu)$.

6.3 Remark. By Proposition 2.4, $P(\mu) = B(\mu)$; we shall denote this space by $D(\mu)$. From Corollary 6.2, $D(\mu)$ is the Banach space of all scalar functions which are B-D-S integrable with respect to μ.

7. A Pettis-type Integral.

In this section, we will introduce an integral which is more general than the weak integral in the sense that more functions are integrable. The definition of this integral is reminiscent of B. J. Pettis' integral introduced in [17], and it will be shown that for strongly measurable functions, $P_X(\mu)$ is exactly the class of all functions integrable in the new sense. Again, $\mu:\Omega \to Y$ is σ-additive.

A function $f:S \to X$ is $X\hat{\otimes}_\varepsilon Y$-integrable, or Pettis-integrable, on a set $E \in \Omega$, if there exists an element $\theta_E \in X\hat{\otimes}_\varepsilon Y$ such that for all $x^* \in X^*$ and $y^* \in Y^*$ we have $x^*\otimes_\varepsilon y^*(\theta_E) = \int_E x^* f d(y^*\mu)$. We shall denote the element θ_E by $(P)\int_E f\, d\mu$. A function of this type is $X\hat{\otimes}_\varepsilon Y$-integrable if it is $X\hat{\otimes}_\varepsilon Y$-integrable over every set $E \in \Omega$. Any $X\hat{\otimes}_\varepsilon Y$-integrable function is weakly μ-measurable.

Because $X^* \otimes Y^*$ is a subspace of $(X\hat{\otimes}_\varepsilon Y)^*$ which is norming for $X\hat{\otimes}_\varepsilon Y$, the Pettis integral as defined above is single valued, linear, and finitely additive. Note that if Y is the scalar field, then this integral is Pettis' "weak" integral.

7.1 Theorem. If f is $X\hat{\otimes}_\varepsilon Y$-integrable, then the range of the indefinite integral of f is bounded.

Proof. Put $\tau(E) = (P)\int_E f\, d\mu$ and consider the family
$K = \{x^*\otimes_\varepsilon y^*\tau : x^* \in X_1^* \text{ and } y^* \in Y_1^*\}$.
We have $K \subseteq ca(S,\Omega)$.

For $E \in \Omega$, $|x^*\otimes_\varepsilon y^*\tau(E)| \leq |x^*|\cdot|y^*|\cdot|\tau(E)|_\varepsilon = |\tau(E)|_\varepsilon$. This shows the set K to be pointwise bounded; by a result of Nikodým ([12], IV.9.8), the set K is uniformly bounded.

that is, there exists a number M such that $|x^* \otimes_\varepsilon y^* \tau(E)| < M$ for all $x^* \in X_1^*$ and $y^* \in Y_1^*$. Taking the supremum over $X_1^* \times Y_1^*$ we get $|\tau(E)|_\varepsilon \leq M$ for all $E \in \Omega$, that is, τ is bounded. □

7.2 Corollary. If f is an $X \hat{\otimes}_\varepsilon Y$-integrable, then $N^*(f) < +\infty$. Thus if f is μ-measurable, $f \in W_X(\mu)$.

Proof. By Theorem 7.1, $\sup_{E \in \Omega} |(P)\int_E f d\mu| |_\varepsilon \leq M$, for some number M.

Consequently, for $x^* \in X_1^*$ and $y^* \in Y_1^*$ we have

$$\begin{aligned} \int_S |x^* f| d|y^* \mu| &\leq 4 \sup_{E \in \Omega} |\int_E x^* f d(y^* \mu)| \\ &= 4 \sup_{E \in \Omega} |x^* \otimes_\varepsilon y^* (P) \int_E f \, d\mu| \\ &\leq 4 \sup_{E \in \Omega} |(P) \int_E f \, d\mu| \\ &\leq 4M < +\infty. \end{aligned}$$

If f is μ-measurable and $N^*(f) < +\infty$, then by definition, $f \in W_X(\mu)$. □

We now prove the countable additivity of the indefinite integral of a Pettis-integrable function. Pettis proved that the weak integral of [17] was countably additive by showing weak countable additivity implies countable additivily; in our context, it is not clear that we have weak countable additivity.

7.3 Remark. The dual of $X \hat{\otimes}_\varepsilon Y$ is J(X,Y), the space of integral forms on $X \times Y$. For any functional $z \in (X \hat{\otimes}_\varepsilon Y)^*$, there exists closed and bounded subsets $P \subseteq X^*$ and $Q \subseteq Y^*$ and a positive

Radon measure ν on the w*-compact set $P \times Q$ with total variation ≤ 1, such that for all $\theta \in X\hat{\otimes}_\varepsilon Y$

$$z(\theta) = \int_{P\times Q} \theta(x^*,y^*)\, d\nu(x^*,y^*).$$

Here we consider θ as a bounded bilinear map on $X^* \times Y^*$ restricted to $P \times Q$; the integral is the ordinary Lebesgue integral.

7.4 Theorem. Let f be $X\hat{\otimes}_\varepsilon Y$-integrable. Then the indefinite integral of f is countably additive.

Proof. Suppose $(E_i) \subseteq \Omega$ is a disjoint family and $E_o = \bigcup_{i=1}^{\infty} E_i$. Write $\tau(A) = (P)\int_A f\, d\mu$ for $A \in \Omega$. We want to show $\sum_{i=1}^{\infty} \tau(E_i)$ converges unconditionally in $X \hat{\otimes}_\varepsilon Y$ and converges to $\tau(E_o)$.

Because of the Pettis lemma ([12]IV.10.1), it suffices to prove that τ is weakly countably additive, that is, $\langle z,\tau(E_o)\rangle = \sum_{i=1}^{\infty} \langle z,\tau(E_i)\rangle$ for each $z \in (X\hat{\otimes}_\varepsilon Y)^*$ with $|z| \leq 1$. Let z be fixed.

Indeed, z is an integral form on $X \times Y$, regarding Remark 7.3, there exists closed and bounded subsets $P \subseteq X^*$ and $Q \subseteq Y^*$, and a Radon measure ν on $P \times Q$ such that

$$z(\theta) = \int_{P\times Q} \theta(x^*,y^*)\, d\nu(x^*,y^*), \text{ for } \theta \in X\hat{\otimes}_\varepsilon Y.$$

Define $T_i(x^*,y^*) = \int_{E_i} x^* f\, d(y^*\mu)$ for $i = o,1,2,\ldots$.

Note that $T_o(x^*,y^*) = \sum_{i=1}^{\infty} T_i(x^*,y^*)$ for $x^* \in P$ and $y^* \in Q$, and that $T_i \in C(P\times Q)$ for $i = 0,1,2,\ldots$.

Write $K = \sup\{|x^*|\cdot|y^*| : x^*\in P \text{ and } y^*\in Q\} < +\infty$, and since τ is bounded by Theorem 7.1,

$$M = \sup_{A\in\Omega} |\tau(A)|_\varepsilon < +\infty.$$

Finally, define $F_n = \bigcup_{i=1}^{n} E_i$, then for $x^* \in P$ and $y^* \in Q$ we have

$$\begin{aligned}
\left|\sum_{i=1}^{n} T_i(x^*,y^*)\right| &= \left|\sum_{i=1}^{n}\int_{E_i} x^*f\, d(y^*\mu)\right| \\
&= \left|\int_{F_n} x^*f\, d(y^*\mu)\right| \\
&= \left|\langle x^*\otimes_\varepsilon y^*,\ (P)\int_{F_n} f\, d\mu\rangle\right| \\
&\le |x^*|\cdot|y^*|\cdot\left|(P)\int_{F_n} f\, d\mu\right|_\varepsilon \\
&\le K\cdot M < +\infty.
\end{aligned}$$

Thus the sequence $\{\sum_{i=1}^{n} T_i\}_{n=1}^{\infty}$ is pointwise dominated by $K\cdot M$ on $P \times Q$. By the Lebesgue Dominated Convergence Theorem

$$\int_{P\times Q} T_o(x^*,y^*)\,d\nu(x^*,y^*) = \sum_{i=1}^{\infty}\int_{P\times Q} T_i(x^*,y^*)\,d\nu(x^*,y^*).$$

But this says

$$\begin{aligned}
\langle z,\tau(E_o)\rangle &= \int_{P\times Q}\langle \tau(E_o),\ (x^*,y^*)\rangle d\nu(x^*,y^*) \\
&= \int_{P\times Q}\langle (P)\int_{E_o} f\, d\mu,\ (x^*,y^*)\rangle d\nu(x^*,y^*) \\
&= \int_{P\times Q}\int_{E_o} x^*f\, d(y^*\mu)\ d\nu(x^*,y^*) \\
&= \int_{P\times Q} T_o(x^*,y^*)\ d\nu(x^*,y^*) \\
&= \sum_{i=1}^{\infty}\int_{P\times Q} T_i(x^*,y^*)\ d\nu(x^*,y^*) \\
&= \sum_{i=1}^{\infty}\int_{P\times Q}\int_{E_i} x^*f\, d(y^*\mu)\ d\nu(x^*,y^*) \\
&= \sum_{i=1}^{\infty}\int_{P\times Q}\langle (P)\int_{E_i} f\, d\mu, (x^*,y^*)\rangle d\nu(x^*,y^*) \\
&= \sum_{i=1}^{\infty}\langle z,\tau(E_i)\rangle.
\end{aligned}$$

That is, $\langle z,\tau(E_o)\rangle = \sum_{i=1}^{\infty}\langle z,\tau(E_i)\rangle$, and the theorem is proved. □

7.5 Theorem. If f is $X\hat{\otimes}_\varepsilon Y$-integrable, then the indefinite Pettis integral of f is absolutely continuous with respect to μ.

Proof. Write $\tau(A) = (P)\int_A f\, d\mu$; τ is an $X\hat{\otimes}_\varepsilon Y$-valued measure on Ω by Theorem 7.4; consider $K \subseteq ca(S,\Omega)$ defined by $K = \{x^*\otimes_\varepsilon y^*\tau : x^* \in X_1^* \text{ and } y^* \in Y_1^*\}$.

It is clear that $K \ll \mu$, that is, $x^*\otimes_\varepsilon y^*\tau \ll \mu$ for each $x^*\otimes_\varepsilon y^*\tau \in K$. Furthermore, the family K is uniformly strongly additive; this means that for any disjoint sequence $(E_i) \subseteq \Omega$ we have $\lim_i x^*\otimes_\varepsilon y^*\tau(E_i) = 0$ uniformly for $x^*\otimes_\varepsilon y^*\tau \in K$. To see this, by Theorem 7.4, the series $\sum_{i=1}^{\infty}\tau(E_i)$ converges unconditionally in $X\ \hat{\otimes}_\varepsilon\ Y$, from the Orlicz-Pettis lemma [17], we have $\lim_i |\tau(E_i)|_\varepsilon = 0$. Since $|x^*\otimes_\varepsilon y^*\tau(E_i)| \le |\tau(E_i)|$ for $x^* \in X_1^*$ and $y^* \in Y_1^*$, we must have $\lim_i |x^*\otimes_\varepsilon y^*\tau(E_i)| = 0$ uniformly for $x^* \in X_1^*$ and $y^* \in Y_1^*$.

In [2], Brooks has shown that $K \ll \mu$ and K uniformly strongly additive together imply $K \ll \mu$ uniformly, that is, $x^*\otimes_\varepsilon y^*\tau \ll \mu$ uniformly for $x^* \in X_1^*$ and $y^* \in Y_1^*$.

Given $\varepsilon > 0$, there exists $\delta > 0$ such that when $E \in \Omega$ and $\|\mu\|(E) < \delta$, then $|x^*\otimes_\varepsilon y^*\tau(E)| < \varepsilon$. Taking the supremum over $X_1^* \times Y_1^*$, we have $|\tau(E)|_\varepsilon < \varepsilon$ whenever $\|\mu\|(E) < \delta$, that is, $\tau \ll \mu$. □

7.6 Corollary. If f is $X\hat{\otimes}_\varepsilon Y$-integrable, then $N^*(f\zeta_{(\cdot)}) \ll \mu$.

Proof. Let $\tau(A) = (P)\int_A f\, d\mu$. By Theorem 7.4, $\tau \ll \mu$ which implies $x^*\otimes_\varepsilon y^*\tau \ll \mu$ uniformly for $x^* \in X_1^*$ and $y^* \in Y_1^*$. But this means for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $E \in \Omega$ and $\|\mu\|(E) < \delta$, then

$$|x^*\otimes_\varepsilon y^*\tau|(E) = \int_E |x^*f|\,d|y^*\mu| < \varepsilon$$

uniformly for $x^* \in X_1^*$ and $y^* \in Y_1^*$. Taking supremum over $X_1^* \times Y_1^*$ we get $N^*(f\zeta_E) \leq \varepsilon$ whenever $\|\mu\|(E) < \delta$. □

We now prove that for μ-measurable functions, to be in $P_X(\mu)$ and to be $X\hat{\otimes}_\varepsilon Y$-integrable are equivalent notions and the weak and Pettis integrals coincide.

7.7 Theorem. A μ-measurable function f is $X\hat{\otimes}_\varepsilon Y$-integrable if and only if $f \in P_X(\mu)$. In this case,

$$\int_E f\otimes_\varepsilon d\mu = (P)\int_E f\, d\mu \text{ for every } E \in \Omega.$$

Proof. Suppose f is μ-measurable, by Theorem 7.1, $N^*(f) < +\infty$, hence $f \in W_X(\mu)$; from Corollary 7.6, $N^*(f\zeta_{(\cdot)}) \ll \mu$, which is sufficient, by Proposition 2.7, to imply that $f \in P_X(\mu)$.

Conversely, if $f \in P_X(\mu)$, then by Proposition 5.2, we have $(x^*\otimes_\varepsilon y^*)\int_E f\otimes_\varepsilon d\mu = \int_E x^* f\, d(y^*\mu)$ for each $x^* \in X_1^*$, $y^* \in Y_1^*$ and $E \in \Omega$. Thus it is true that for each $E \in \Omega$, there exists a vector $\theta_E \in X\hat{\otimes}_\varepsilon Y$, namely $\theta_E = \int_E f\otimes_\varepsilon d\mu$, such that $x^*\otimes_\varepsilon y^*(\theta_E) = \int_E x^* f d(y^*\mu)$; this is the definition of the Pettis integral. Therefore, f is $X\hat{\otimes}_\varepsilon Y$-integrable and $(P)\int_E f\, d\mu = \int_E f\otimes_\varepsilon d\mu$. □

7.8 Remark. Using a Pettis definition, we can define an integral for functions in $W_X(\mu)$ in such a way that this integral, for functions in $P_X(\mu)$, is the weak integral defined in section 5. For $f \in W_X(\mu)$, define a linear map on $(X\hat{\otimes}_\varepsilon Y)^*$ as follows:

$$z \longrightarrow \int_{A\times B}\int_E x^* f\, d(y^*\mu)\, d\nu(x^*,y^*),$$

where $z \in (X\otimes_\varepsilon Y)^*$ and $\int_{A\times B}(\cdot)d\nu(x^*,y^*)$ is the representation for z as an integral form -- see Remark 7.3. We designate

this linear map by $(P)\int_E f\,d\mu$ and is defined on $(X\hat{\otimes}_\varepsilon Y)^*$. Since $N^*(f) < +\infty$, it is easy to see that $(P)\int_E f\,d\mu$ is a continuous linear functional on $(X\hat{\otimes}_\varepsilon Y)^*$, hence $(P)\int_E f d\mu \in (X\otimes_\varepsilon Y)^{**}$. Indeed, we can choose, for each $|z| \leq 1$, positive Radon measures ν_z on $X_1^* \times Y_1^*$ such that z represented as an integral form is

$$\theta \to \int_{X_1^* \times Y_1^*} \theta(x^*,y^*)\,d\nu_z(x^*,y^*), \theta \in X\hat{\otimes}_\varepsilon Y.$$

For $|z| \leq 1$, we have

$$\begin{aligned} |\langle z, (P)\textstyle\int_E f d\mu\rangle| &= \left|\int_{X_1^* \times Y_1^*} \int_E x^* f\, d(y^*\mu)\, d\nu_z(x^*,y^*)\right| \\ &\leq \int_{X_1^* \times Y_1^*} \int_E |x^* f|\, d|y^*\mu|\, d\nu_z(x^*,y^*) \\ &\leq N^*(f\zeta_E) < +\infty. \end{aligned}$$

Thus $|(P)\int_E f d\mu| \leq N^*(f\zeta_E)$, that is, $(P)\int_E f d\mu \in (X\otimes_\varepsilon Y)^{**}$. Theorem 7.7 states that for $f \in P_X(\mu)$ we have $\int_E f\otimes_\varepsilon d\mu = (P)\int_E f d\mu \in X\hat{\otimes}_\varepsilon Y$.

7.9 Remarks. We have defined a general integral which takes its values in the inductive tensor product of two Banach spaces. In the most general case, the integral is that of Bartle's bilinear integral with the improvement that convergence of integrals is characterized by norm convergence. Bartle's integration theory yielded a Vitali convergence theorem; however, in our context, Lebesgue's Dominated convergence theorem is obtained as well.

A generalization of the Pettis integral is given. If the range space of the measure is the scalar field, our definition is exactly the Pettis definition and $N^*(\cdot) = ((\cdot))_1$. It was shown that the weak and Pettis integral are equivalent for strongly measurable functions.

If the range space of the functions is the scalar field, then the definition of the weak integral is equivalent to the Bartle-Dunford-Schwartz integral.

Finally, if $X = Y = \Phi$, then the weak integral is the Lebesgue integral for scalar functions.

8. The Strong Integral.

The strong integral was defined for the functions in $B_X(\mu)$, and had its values in $X \hat{\otimes}_\varepsilon Y$. We now show that the strong integral is included in the integral of Brooks and Dinculeanu introduced in [5]. They defined a Lebesgue space of integrable functions with respect to an operator valued measure. We shall outline very briefly the basic development of the theory in [5].

Let E and F be Banach spaces and consider an operator valued measure $m:\Omega \to L(E,F)$, where $L(E,F)$ is the space of bounded linear operators from E into F. For each $z \in F^*$, define $m_z:\Omega \to E^*$ by $m_z(A)x = \langle m(E)x,z\rangle$; m_z is countably additive and of finite total variation. For an m-measurable function $f:S \to E$, define $N_1(f) = \sup_{z\in F_1^*}\int_S |f|d|m_z|$. Letting $F_E(N_1)$ be the collection of all m-measurable functions f with $N_1(f) < +\infty$, it follows that $F_E(N_1)$ is a Banach space which contains $S_E(\Omega)$. Finally, $L_E(N_1)$ is the closure in $F_E(N_1)$ of $S_E(\Omega)$.

An integral can be defined for functions $f \in F_E(N_1)$ as follows: $\int_S f dm$ is the vector in F** defined by

$$\langle z, \int_S f dm\rangle = \int_S f dm_z, \text{ where } z \in F^*.$$

If $f \in L_E(N_1)$, then $\int_S f\, dm \in F$; indeed, the integral of

simple functions is F-valued, the mapping $f \to \int_S f\, dm$ from $L_E(N_1)$ to F** is continuous, and the simple functions $S_E(\Omega)$ are dense in $L_E(N_1)$.

The following theorem shows that the strong integral is included in the integral of Brooks and Dinculeanu.

8.1 Theorem. Let $\mu:\Omega \to Y$ and define $m:\Omega \to L(X, X\hat{\otimes}_\varepsilon Y)$ by $m(A)x = x\otimes\mu(A)$. Then

(1) f is μ-measurable if and only if f is m-measurable;

(2) $N(f) = N_1(f)$ for all μ-measurable $f:S \to X$;

(3) $L_X(N_1) = B_X(\mu)$;

(4) $\int_S f\, dm = \int_S f \otimes_\varepsilon d\mu$, $f \in B_X(\mu)$.

The proof of this theorem is contained in the following sequence of lemmas.

8.2 Lemma. f is μ-measurable if and only if f is m-measurable.

Proof. It suffices to prove μ and m have the same null sets. For $E \in \Omega$, each of the following are pairwise equivalent:

E is μ-null; $\sup_{A\in\Omega} |\mu(E\cap A)| = 0$;

$$\sup_{A\in\Omega} \sup_{x^*\in X_1^*} |x\otimes\mu(E\cap A)|_\varepsilon = 0;$$

$$\sup_{A\in\Omega} \sup_{x^*\in X_1^*} |m(E\cap A)X|_\varepsilon = 0;$$

$$\sup_{A\in\Omega} |m(E\cap A)| = 0;$$

E is m-null. □

8.3 Lemma. Let $\Gamma = \{x^*\otimes_\varepsilon y^* : x^*\in X_1^*, y^*\in Y_1^*\}$. Then

$$N_1(f) = \sup_{\omega\in\Gamma} \int_S |f|\, d|m_\omega|, \quad f\in S(\Omega).$$

Proof. Write $f(s) = \sum_{i=1}^{n} \alpha_i \zeta_{E_i}(s)$ where $(\alpha_i) \subseteq \Phi$ and $(E_i) \subseteq \Omega$ are disjoint. Put $J(f) = \sup_{\omega \in \Gamma} \int_S |f| d|m_\omega|$, then $J(f) \leq N_1(f)$ since $\Gamma \subseteq (X \hat{\otimes}_\varepsilon Y)_1^*$.

Let $\varepsilon > 0$ be given, there exists $z \in (X \hat{\otimes}_\varepsilon Y)_1^*$ such that

$$N_1(f) \leq \varepsilon + \int_X |f| d|m_z| = \varepsilon + \sum_{i=1}^{n} |\alpha_i| |m_z|(E_i).$$

There exists an integer $p \in \omega$ and set $A_{ij} \in \Omega$, $1 \leq j \leq p$ and $1 \leq j \leq n$, such that

$$|m_z|(E_i) \leq \frac{\varepsilon}{\Sigma_i |\alpha_i|} + \sum_{j=1}^{p} |m_z(A_{ij})|.$$

Substituting this into the above inequality:

$$N_1(f) \leq \varepsilon + \sum_{i=1}^{n} |\alpha_i| \{\frac{\varepsilon}{\Sigma_i |\alpha_i|} + \sum_{j=1}^{p} |m_z(A_{ij})|\}$$

$$= 2\varepsilon + \Sigma_i \Sigma_j |\alpha_i| |m_z(A_{ij})|.$$

Choose for each pair (i,j), $\overline{x}_{ij} \in X$ with $|x_{ij}| \leq 1$ and

$$|m_z(A_{ij})| \leq \frac{\varepsilon}{p\Sigma_i |\alpha_i|} + |m_z(A_{ij}) \overline{x}_{ij}|.$$

Then

$$N_1(f) \leq 2\varepsilon + \Sigma_i \Sigma_j |\alpha_i| \{\frac{\varepsilon}{p\Sigma_i |x_i|} + |m_z(A_{ij}) \overline{x}_{ij}|\}$$

$$= 3\varepsilon + \Sigma_i \Sigma_j |\alpha_i| |\langle x_{ij} \otimes \mu(A_{ij}), z \rangle|.$$

Choose complex numbers θ_{ij} with $|\theta_{ij}| \leq 1$ and

$$|\langle \overline{x}_{ij} \otimes \mu(A_{ij}), z \rangle| = \theta_{ij} \langle \overline{x}_{ij} \otimes \mu(A_{ij}), z \rangle$$

$$= \langle \theta_{ij} \overline{x}_{ij} \otimes \mu(A_{ij}), z \rangle.$$

Substituting once again,

$$N_1(f) \leq 3\varepsilon + \Sigma_i \Sigma_j |\alpha_i| \langle \theta_{ij} \overline{x}_{ij} \otimes \mu(A_{ij}), z \rangle$$

$$= 3\varepsilon + \langle z, \Sigma_i \Sigma_j |\alpha_i| \theta_{ij} \bar{x}_{ij} \otimes \mu(A_{ij}) \rangle$$

$$\leq 3\varepsilon + |\Sigma_i \Sigma_j |\alpha_i| \theta_{ij} \bar{x}_{ij} \otimes \mu(A_{ij})|_\varepsilon$$

Since Γ is norming for $X \hat{\otimes}_\varepsilon Y$, there exists $x^* \otimes_\varepsilon y^* \in \Gamma$ such that if $\omega = x^* \otimes_\varepsilon y^*$, then

$$N_1(f) \leq 3\varepsilon + \varepsilon + |\Sigma_i \Sigma_j |\alpha_i| \theta_{ij} \langle x^* \otimes_\varepsilon y^*, \bar{x}_{ij} \otimes \mu(A_{ij}) \rangle|$$

$$\leq 4\varepsilon + \Sigma_i \Sigma_j |\alpha_i| \cdot |m_\omega(A_{ij}) \bar{x}_{ij}|$$

$$\leq 4\varepsilon + \Sigma_i |\alpha_i| \Sigma_j |m_\omega|(A_{ij})$$

$$= 4\varepsilon + \Sigma_i |\alpha_i| \cdot |m_\omega|(E_i)$$

$$= 4\varepsilon + \int_S |f| d|m_\omega|$$

$$\leq 4\varepsilon + J(f).$$

Thus $N_1(f) \leq 4\varepsilon + J(f)$ which is sufficient to conclude $N_1(f) \leq J(f)$, and in turn; $N_1(f) = J(f)$. □

8.4 Lemma. For all $f: S \to X$ μ-measurable we have

$$N_1(f) = J(f) \text{ where } J(f) = \sup_{\omega \in \Gamma} \int_S |f| d|m_\omega|.$$

Proof. Case I: $N_1(f) = +\infty$.

Let $n \in \omega$ be arbitrary, there exists $z \in (X \hat{\otimes}_\varepsilon Y)^*_1$ such that $n < \int_S |f| d|m_z|$. We can choose a simple function $g \in S(\Omega)$ with $0 \leq g(s) \leq |f(s)|$ for all $s \in S$ and

$$n < \int_S g \, d|m_z|.$$

By Lemma 8.3, $N_1(g) = J(g)$, since $g \leq |f|$ we also have $J(g) \leq J(f)$; finally,

$$n < \int_S g \, d|m_z| \leq N_1(g) = J(g) \leq N_1(f).$$

This implies that $N_1(f) = +\infty$ so that for this case $N_1(f) = J(f)$.

Case II: $N_1(f) < +\infty$.

Letting $\varepsilon > 0$, we can choose $z \in (X\hat{\otimes}_\varepsilon Y)_1^*$ such that $N_1(f) < \frac{\varepsilon}{2} + \int_S |f|d|m_z|$

Choose $g \geq 0$ simple such that $g \leq |f|$ and

$$\int_S |f|d|m_z| < \frac{\varepsilon}{2} + \int_S g\, d|m_z|.$$

Thus,

$$\begin{aligned} N_1(f) &< \varepsilon + \int_S g d|m_z| < \varepsilon + N_1(g) \\ &= \varepsilon + J(g) \\ &\leq \varepsilon + J(f). \end{aligned}$$

We deduce $N_1(f) \leq \varepsilon + J(f)$, that $N_1(f) \leq J(f)$ and $N_1(f) = J(f)$. □

8.5 Lemma. Given $\omega = x^* \otimes_\varepsilon y^* \in \Gamma$, $|m_\omega|(E) \leq |y^*\mu|(E)$ for every $E \in \Omega$. Conversely, for each $y^* \in Y_1^*$, there exists $x^* \in X_1^*$ such that $|y^*\mu|(E) \leq |m_\omega|(E)$, where $\omega = x^* \otimes_\varepsilon y^*$ and $E \in \Omega$.

Proof. Given $\omega = x^* \otimes_\varepsilon y^* \in \Gamma$, $E \in \Omega$ and $\varepsilon > 0$, we can choose $(A_i) \subseteq \Omega$ a partition of E and vectors $(x_i) \subseteq X$ with $|x_i| \leq 1$ so that

$$\begin{aligned} |m_\omega|(E) &< \varepsilon + \Sigma_i |m_\omega(A_i)x_i| \\ &= \varepsilon + \Sigma_i |x^*(x_i)| \cdot |y^*\mu(A_i)| \\ &\leq \varepsilon + \Sigma_i |y^*\mu|(A_i) \leq \varepsilon + |y^*\mu|(E). \end{aligned}$$

In this way we obtain the inequality $|m_\omega|(E) \leq |y^*\mu|(E)$.

Let $(A_i) \subseteq \Omega$ be an arbitrary partition of $E \in \Omega$. Choose $x \in X$ with $|x| \leq 1$ and $x^* \in X_1^*$ such that $x^*(x) = 1$. Then if we put $\omega = x^* \otimes_\varepsilon y^*$, we have

$$\Sigma_i |y^*\mu(A_i)| = \Sigma_i |x^*(x_i)| \cdot |y^*\mu(A_i)| = \Sigma_i |m_\omega(A_i)x|$$

$$\leq \Sigma_i |m_\omega(A_i)| \leq |m_\omega|(E).$$

From this we obtain $|y^*\mu|(E) \leq |m_\omega|(E)$. □

8.6 Lemma. $N(F) = N_1(f)$ for all μ-measurable $f:S \to X$.

Proof. By Lemma 8.4, it suffices to show $N(f) = J(f)$, where $N(f) = \sup_{y^* \in Y_1^*} \int_S |f|d|y^*\mu|$ and $J(f) = \sup_{\omega \in \Gamma} \int_S |f|d|m_\omega|$.

The equality follows from 8.5. Since $|y^*\mu|(\cdot) \leq |m_\omega|(\cdot)$ for some $\omega \in \Gamma$ given $y^* \in Y_1^*$, we have

$$\int_S |f|d|y^*\mu| \leq \int_S |f|d|m_\omega|$$

and therefore $N(f) \leq J(f)$.

Also, given $\omega \in \Gamma$, $|m_\omega|(\cdot) \leq |y^*\mu|(\cdot)$ for some $y^* \in Y_1^*$, which implies in much the same way as above that $J(f) \leq N(f)$.

Putting these two together: $N(f) = J(f)$. □

8.7 Lemma. $L_X(N_1) = B_X(\mu)$ and $\int_S f\, dm = \int_S f \otimes_\varepsilon d\mu$, $f \in B_X(\mu)$.

Proof. By Lemma 8.6, $N(f) = N_1(f)$ for all μ-measurable f, and since the spaces $L_X(N_1)$ and $B_X(\mu)$ are the closure of $S_X(\Omega)$ with respect to norms N_1 and N, respectively, we must have $L_X(N_1) = B_X(\mu)$.

To prove the second assertion, it suffices to prove it for simple functions; this is because $S_X(\Omega)$ is dense in $B_X(\mu)$ ($= L_X(N_1)$) and the linear maps

$$f \to \int_S f \otimes_\varepsilon d\mu \text{ and } f \to \int_S f\, dm$$

are continuous from $B_X(\mu)$ into $X \hat{\otimes}_\varepsilon Y$. To show $\int_S f\, dm = \int_S f \otimes_\varepsilon d\mu$ for $f \in S_X(\mu)$, we need only show $\int_S f\, dm_z = \langle \int_S f \otimes_\varepsilon d\mu, z \rangle$, where

$z \in (X\hat{\otimes}_\varepsilon Y)_1^*$. Writing $f(s) = \Sigma_i x_i \zeta_{E_i}(s)$ where $x_i \in X$ and $(E_i) \subseteq \Omega$ are disjoint, we have

$$\int_S f\, dm_z = \Sigma_i m_z(E_i)x_i = \Sigma_i \langle x_i \otimes \mu(E_i), z\rangle$$

$$= \langle \Sigma_i x_i \otimes \mu(E_i), Z\rangle = \langle \int_S f \otimes_\varepsilon d\mu, z\rangle. \quad \square$$

This ends the proof of Theorem 8.1. $\square$

8.8 Remark. For the elementary cases, the strong integral reduces to some well-known integrals. For $X = \Phi$, we have the Bartle-Dunford-Schwartz integral as proven in section 6. If $Y = \Phi$, that is, if μ is a scalar measure, the strong integral takes its values in X, the range space of the functions integrated; in this case, we clearly have the Bochner integral with N the Bochner norm.

9. The Spaces $P_X^\infty(\mu)$ and $B_X^\infty(\mu)$.

In preparation to proving some theorems concerning the topological properties of the weak and strong integrals, as well as the classification of certain natural linear operators, it is necessary to make a few remarks on essentially bounded measurable functions. As throughout this chapter $\mu:\Omega \to Y$ is a vector measure.

Let $f:S \to X$ be weakly μ-measurable. Define the following functions:

(1) $N_\infty(f) = \mu\text{-ess}\sup_{s\in S}|f(s)| = \inf_H \sup_{s\in S-H}|f(s)|$,

where the infimum is taken over all μ-null sets $H \subseteq S$.

(2) $N^*_\infty(f) = \sup_{x^*\in X_1^*} (\mu\text{-ess}\sup_{s\in S}|x^*f(s)|)$.

It is clear that $N^*_\infty(f) = \sup_{x^* \in X_1^*} N_\infty(X^*f)$ and that $0 \le N^*_\infty(f) \le N_\infty(f) \le +\infty$.

The magnitude of $N^*_\infty(f)$ and $N_\infty(f)$ depends on the collection of μ-null sets. Sometimes this dependence will be denoted by $N^*_\infty(f;\mu)$ and $N_\infty(f;\mu)$. For example, if $\nu:\Omega \to R^+$ is a positive measure such that $\nu \ll \mu$, then the collection of all ν-null sets contains the μ-null sets; consequently, any μ-measurable function is also ν-measurable. It is always true, therefore, that $N^*_\infty(f;\nu) \le N^*_\infty(f;\mu)$ and $N_\infty(f;\nu) \le N_\infty(f;\mu)$ for any μ-measurable function f, because the infimum in $N^*_\infty(f;\nu)$ and $N_\infty(f;\nu)$ is taken a larger collection of null sets.

A weakly μ-measurable function f is weakly μ-essentially bounded if $N^*_\infty(f) < +\infty$, and is (strongly) μ-essentially bounded if $N_\infty(f) < +\infty$; obviously, if it is μ-essentially bounded, then it is weakly μ-essentially bounded.

Define the following spaces:

(1) $P_X^\infty(\mu)$ is the space of all weakly μ-measurable functions $f:S \to X$ which are weakly μ-essentially bounded.

(2) $B_X^\infty(\mu)$ is the space of all μ-measurable, μ-essentially bounded functions $f:S \to X$.

We have $B_X^\infty(\mu) \subseteq P_X^\infty(\mu)$, and the inclusion is in general strict. If X is separable, then $B_X^\infty(\mu) = P_X^\infty(\mu)$ (see Pettis [17]). The space $B_X^\infty(\mu)$ with the norm $N_\infty(\cdot)$ is a Banach space if we identify functions equal μ-a.e., $P_X^\infty(\mu)$ is a normed linear space with norm $N^*_\infty(\cdot)$. For the case $X = \Phi$, the spaces $B_X^\infty(\mu) = P_X^\infty(\mu)$ and we shall denote this space by $L^\infty(\mu)$, the classic Lebesgue space of μ-essentially bounded scalar

functions with norm

$$\|\phi\|_\infty = \|\phi\,;\mu\|_\infty = \mu - \operatorname{ess\,sup}_{s\in S} |\phi(s)|, \quad \phi \in L^\infty(\mu).$$

9.1 Proposition. If $f \in B_X^\infty(\mu)$, then $f \in B_X(\mu)$ and

(#) $\quad N(f\zeta_E) \le N_\infty(f)\cdot\|\mu\|(E)$

for $E \in \Omega$.

Proof. Since $f \in B_X^\infty(\mu)$, f is μ-measurable. For each $y^* \in Y_1^*$, we have $y^*\mu \ll \mu$ and consequently

$$N_\infty(f;y^*\mu) \le N_\infty(f;\mu) < +\infty.$$

Thus $f \in L^\infty(y^*\mu) \subseteq L^1(y^*\mu)$ and

$$\int_E |f|\,d|y^*\mu| \le N_\infty(f,y^*\mu)\cdot|y^*\mu|(E)$$

$$\le N_\infty(f;\mu)\cdot\|\mu\|(E).$$

Taking the supremum over X_1^* of the left hand side,

$$N(f\zeta_E) \le N_\infty(f;\mu)\cdot\|\mu\|(E) < +\infty$$

This proves part of the assertion and shows $f \in F_X(\mu)$ since $N(f) < +\infty$. Also, $N(f\zeta_{(\cdot)}) \ll \mu$, by Proposition 2.7 and Remark 2.8, $f \in B_X(\mu)$. □

9.2 Proposition. If $f \in B_X^\infty(\mu)$ and $\phi \in L^\infty(\mu)$, then $f \in B_X^\infty(\mu)$ and $N(f\,\zeta_E) \le N_\infty(f)\cdot\|\phi\|_\infty\cdot\|\mu\|(E)$, for $E \in \Omega$.

Proof. Obviously $N_\infty(f\phi) \le N_\infty(f)\cdot\|\phi\|_\infty$.

The result then follows from Proposition 9.1:

$$N(f\phi\zeta_E) \le N_\infty(f\phi)\cdot\|\mu\|(E) \le N_\infty(f)\cdot\|\phi\|_\infty\cdot\|\mu\|(E). \quad \square$$

9.3 Proposition. If $f\phi \in P_X(\mu)$ and $\phi \in L^\infty(\mu)$, then $f \in P_X(\mu)$ and

$$N^*(f\phi) \le \|\phi\|_\infty N^*(f).$$

Proof. Let $x^* \in X_1^*$ and $y^* \in Y_1^*$ be arbitrary, then $x^*f \in L^1(y^*\mu)$ by Proposition 5.2 and $\phi \in L^\infty(y^*\mu)$. From Hölder's inequality, we have for $E \in \Omega$

$$\int_E |\phi x^* f| d|y^*\mu| \leq \|\phi\|_\infty \int_E |x^* f| d|y^*\mu|$$

$$\leq \|\phi\|_\infty N^*(f\zeta_E)$$

Taking the supremum over $X_1^* \times Y_1^*$ of the left-hand side:

$$N^*(f\phi\zeta_E) \leq \|\phi\|_\infty N^*(f\zeta_E). \qquad (\#)$$

$f \in P_X(\mu)$ implies $N^*(f\zeta_{(\cdot)}) \ll \mu$ by Proposition 2.7; this fact and (#) combine together to imply that $N^*(f\phi\zeta_{(\cdot)}) \ll \mu$, but then $f\phi \in P_X(\mu)$ by Propositon 2.7 again. The second assertion is (#) for $E = S$. □

10. Compact Operators.

Let X and Y be Banach spaces and V a bounded linear map from X to Y. V is said to be a compact operator if V maps bounded sets in X onto relatively compact subsets of Y. Compact operators (and weakly compact operators) have been studied by many people in connection with integral representations of operators on spaces of continuous functions; see Dunford and Schwartz [12] for a discussion of the known results in compact operators.

In this section, we classify a certain natural linear operation as being compact provided the measure $\mu:\Omega \to Y$ has a relatively norm compact range.

10.1 Lemma. Let $f \in S_X(\Omega)$. Define a map $V:L^\infty(\mu) \to X\hat{\otimes}_\varepsilon Y$ by

$$V(\phi) = \int_S \phi f \otimes_\varepsilon d\mu.$$

V is linear and continuous. Furthermore, V is a compact operation if μ has a relatively norm compact range.

Proof. Let $f \in S_X(\Omega)$ and $\phi \in L^\infty(\mu)$. Then by Proposition 9.2, $f\phi \in B_X^\infty(\mu)$ and therefore $f\phi \in B_X(\mu)$ by Proposition 9.1. The map $V(\phi) = \int_S f\phi \otimes_\varepsilon d\mu$ is then well-defined since the integral exists, and maps $L^\infty(\mu)$ into $X \hat{\otimes}_\varepsilon Y$. V is clearly linear; it is also bounded:

$$\begin{aligned} |V| &= \sup_{\|\phi\|_\infty = 1} \left| \int_S f\phi \otimes_\varepsilon d\mu \right|_\varepsilon \\ &\leq \sup_{\|\phi\|_\infty = 1} N_\infty(f) \cdot \|\phi\|_\infty \cdot \|\mu\|(S) \\ &= N_\infty(f) \cdot \|\mu\|(S) < +\infty. \end{aligned}$$

Suppose μ has relatively norm compact range; by Theorem II.3.2, $\mu \in ca(\Omega)\hat{\otimes}_\varepsilon Y$; consequently, there exists a sequence $(\mu_k) \subseteq ca(\Omega) \otimes Y$ of step measures such that $\|\mu - \mu_k\|(S) \to 0$. We may assume, according to Theorem II.3.2, that $\mu_k \ll \mu$ for each $k \in \omega$; in fact, we may take $\mu_k = \sum_{i=1}^{n_k} y_i^k \nu_i^k$ where $y_i^k \in Y$, $\nu_i^k \in ca(\Omega)$ and $\nu_i^k \ll \mu$.

For each $k \in \omega$, define $V_k(\phi) = \int_S f\phi \otimes_\varepsilon d\mu_k$. Then $V_k : L^\infty(\mu) \to X \hat{\otimes}_\varepsilon Y$, this is because $f\phi \in B_X^\infty(\mu)$ and $\mu_k \ll \mu$ implies $f\phi \in B_X^\infty(\mu_k) \subseteq B_X(\mu_k)$ which makes V_k well-defined. V_k is bounded:

$$\begin{aligned} |V_k(\phi)| &\leq N_\infty(f;\mu_k) \cdot \|\phi;\mu_k\|_\infty \cdot \|\mu\|(S) \\ &\leq N_\infty(f;\mu) \cdot \|\phi;\mu\|_\infty \cdot \|\mu\|(S). \end{aligned}$$

We have used here the fact that $\mu_k \ll \mu$ implies $N_\infty(f;\mu_k) \leq N_\infty(f;\mu)$ and $\|\phi;\mu_k\|_\infty \leq \|\phi;\mu\|_\infty$ (see the remarks in section 9). We have thus shown that $|V| \leq N_\infty(f;\mu) \cdot \|\mu\|(S) < +\infty$.

Assert that the operators V_k are compact for each $k \in \omega$.

Proof of assertion. Let $k \in \omega$ be fixed and write $\mu_k = \sum_{i=1}^{n} y_i \nu_i$ where $y_i \in Y$ and $\nu_i \in ca(\Omega)$ with $\nu_i \ll \mu$. We again deduce that $\phi \in L^\infty(\nu_i)$, $f \in B_X^\infty(\nu_i)$ and so $f\phi \in B_X^\infty(\nu_i)$ for $i = 1,2,\ldots,n$. Since $\nu_i \ll \mu$ we have $L^\infty(\mu) \subseteq L^\infty(\nu_i)$ for each i. Define $V_k^i : L^\infty(\nu_i) \to X$ by $V_k^i(\psi) = \int_S f\psi \, d\nu_i$. This is well-defined since $f\psi \in B_X(\nu_i)$ for each $\psi \in L^\infty(\nu_i)$ by Proposition 9.2. Since f is a simple function and ν_i is a scalar measure, we can apply a lemma of Pettis [17], Lemma 6.11, to conclude V_k^i is a compact operator on $L^\infty(\nu_i)$. Because $\nu_i \ll \mu$, we always have $\|\psi;\nu_i\|_\infty \leq \|\psi;\mu\|_\infty$ for all $\psi \in L^\infty(\nu_i)$, thus any set in $L^\infty(\mu)$ which is bounded in the $\|\cdot;\mu\|_\infty$-norm, is bounded in the $\|\cdot;\nu_i\|_\infty$-norm and so V_k^i will map this set into a relatively compact subset of X -- this means then that V_k^i restricted to $L^\infty(\mu)$ is a compact operator.

It is easy to see that $V_k(\phi) = \sum_{i=1}^{n} V_k^i(\phi) \otimes y_i$, for $\phi \in L^\infty(\mu)$; since each V_k^i is compact, and V_k is a finite sum of compact operators, V_k is compact too. This proves the assertion.

Now for $\phi \in L^\infty(\mu)$ with $\|\phi\|_\infty = 1$,

$$
\begin{aligned}
|(V-V_k)\phi| &= \left| \int_S f\phi \otimes_\varepsilon d(\mu-\mu_k) \right|_\varepsilon \\
&\leq \sup_{y^* \in Y_1^*} \int_S |f\phi| \, d|y^*(\mu-\mu_k)| \\
&\leq N_\infty(f) \cdot \|\phi\|_\infty \cdot \|\mu-\mu_k\|(S) \\
&= N_\infty(f) \cdot \|\mu-\mu_k\|(S).
\end{aligned}
$$

Thus $|V-V_k| \leq N_\infty(f) \cdot \|\mu-\mu_k\|(S)$. But since $\|\mu-\mu_k\|(S) \to 0$ we also have $|V-V_k| \to 0$; hence $V_k \to V$ in the uniform operator

topology of $L(L^\infty(\mu), X\hat{\otimes}_\varepsilon Y)$. By Lemma VI.5.3 of [12], V is a compact operator since it is the limit in the uniform operator topology of compact operators. □

10.2 Theorem. Suppose $\mu:\Omega \to Y$ has relatively norm compact range. Then for each $f \in P_X(\mu)$, the map

$$V(\phi) = \int_S f\phi\otimes_\varepsilon d\mu,$$

is a compact operator from $L^\infty(\mu)$ into $X \hat{\otimes}_\varepsilon Y$.

Proof. By Proposition 9.3, $f\phi \in P_X(\mu)$ so V is well-defined. Since $f \in P_X(\mu)$, there exists a sequence $(f_n) \subseteq S_X(\Omega)$ which determines f in $P_X(\mu)$, that is, $f_n \to f$ in μ-measure and $N^*(f-f_n) \to 0$.

Define $V_n(\phi) = \int_S f_n\phi\otimes_\varepsilon d\mu$. By Lemma 10.2, $V_n:L^\infty(\mu) \to X\hat{\otimes}_\varepsilon Y$ is a compact operator. Let $\phi \in L^\infty(\mu)$ with $\|\phi\|_\infty = 1$.

$$\begin{aligned} |(V-V_n)\phi| &= \left|\int_S (f-f_n)\phi\otimes_\varepsilon d\mu\right|_\varepsilon \\ &\le N^*((f-f_n)\phi) \\ &\le \|\phi\|_\infty N^*(f-f_n) \\ &= N^*(f-f_n). \end{aligned}$$

Thus $|V-V_n| \le N^*(f-f_n)$; since $N^*(f-f_n) \to 0$ we have $V_n \to V$ in the uniform operator topology so that V is compact also. □

10.3 Corollary. If the range of μ is relatively norm compact, then the indefinite integral of any function in $P_X(\mu)$ has a relatively norm compact range too.

Proof. Let $B = \{\zeta_E : E\in\Omega\}$. Then $B \subseteq L^\infty(\mu)$ is bounded. For any $f \in P_X(\mu)$, the map $V(\phi) = \int_S f\phi\otimes_\varepsilon d\mu$ is a compact operator,

therefore, sends the set B onto a relatively norm compact subset of $X \hat{\otimes}_\varepsilon Y$. But $V(B) = \{V(\zeta_E) : E \in \Omega\} = \{\int_E f \otimes_\varepsilon d\mu : E \in \Omega\}$, that is, $V(B)$ is the range of the indefinite integral $\int f \otimes_\varepsilon d\mu$. $V(B)$ is relatively norm compact in $X \hat{\otimes}_\varepsilon Y$. □

10.4 Corollary. The indefinite integral of functions in $B_X(\mu)$ has a relatively norm compact range if μ does.

Proof. Recall $B_X(\mu) \subseteq P_X(\mu)$ and apply Corollary 10.3. □

10.5 Corollary. Let μ have relative norm compact range. Then

$$P_X(\mu) \subseteq ca(\Omega)\hat{\otimes}_\varepsilon X \hat{\otimes}_\varepsilon Y$$

isometrically.

Proof. The space $ca(\Omega)\hat{\otimes}_\varepsilon X\hat{\otimes}_\varepsilon Y$ is the space of all $X\hat{\otimes}_\varepsilon Y$-valued measures on Ω with relatively norm compact range by Theorem II.3.3.

Define $T: P_X(\mu) \to ca(\Omega)\hat{\otimes}_\varepsilon X\hat{\otimes}_\varepsilon Y$ by

$$T(f) = \int_{(\cdot)} f \otimes_\varepsilon d\mu, \quad f \in P_X(\mu).$$

By Theorem 10.2, the indefinite integral $T(f)$ has relatively norm compact range so that $T(f) \in ca(\Omega)\hat{\otimes}_\varepsilon X\hat{\otimes}_\varepsilon Y$. T is clearly linear, it suffices to show $N^*(f) = \|T(f)\|(S)$ for T to be an isometry since the norm on $ca(\Omega)\hat{\otimes}_\varepsilon X\hat{\otimes}_\varepsilon Y$ is the semivariation norm by Theorem II.3.3. But this is obvious, from Proposition I.1.1 we have

$$\|T(f)\|(S) = \sup^*_{(x^*,y^*)\in X_1^* \times Y_1^*} |x^* \otimes_\varepsilon y^* T(f)|(S)$$

So that

$$\|T(f)\|(S) = \sup_{(x^*,y^*)\in X_1^* \times Y_1^*} \int_S |x^* f| d|y^* \mu|$$

$$\|T(f)\|(S) = N^*(f). \quad \square$$

CHAPTER IV
THE FUBINI THEOREM

1. Preliminaries.

Throughout this chapter, (S,Ω) and (T,Λ) are measurable spaces; X and Y are Banach spaces; $\mu:\Omega \to X$ and $\nu:\Lambda \to Y$ are vector measures. The symbol $\Omega\otimes\Lambda$ denotes the algebra of rectangles of Ω and Λ while $\Omega\otimes_\sigma\Lambda$ is the σ-algebra generated by $\Omega\otimes\Lambda$.

We shall consider three Fubini type theorems for integrals of scalar functions with respect to the inductive product measure $\mu\otimes_\varepsilon\nu$. In section 2, we prove the multiplicative property of product integration; it is obtained in the most general form possible. The classic Fubini theorem is proven in section 3 with only minimal restrictions placed in the hypothesis. The existing vector valued Fubini theorems place severe restrictions on the measures by requiring both measures to have finite total variation (see [10] and [14]); we require that only one of the measures, μ or ν, have the Beppo Levi Property. Finally, in section 4, we derive a Fubini theorem for continuous function.

We use the integration theory developed in Chapter 2 throughout this chapter. Recall that for a vector valued measure $\lambda:\Omega \to X$, the spaces $P_\Phi(\lambda)$ and $B_\Phi(\lambda)$ coincide (Proposition III.2.4), and they define the Banach space of all scalar functions integrable in the Dunford-Schwartz sense with respect to λ

(Corollary III.6.2). As in Remark III.6.3, we use the notation $D(S,\Omega,\lambda;X)$ or simply $D(\lambda)$ for this space. Consequently, we shall write $D(\mu)$ for $D(S,\Omega,\mu;X)$, $D(\nu)$ for $D(T,\Lambda,\nu;Y)$, and $D(\mu\otimes_\varepsilon\nu)$ for $D(S\times T,\Omega\otimes_\sigma\Lambda,\mu\otimes_\varepsilon\nu;X\hat{\otimes}_\varepsilon Y)$.

In this chapter, the variables of integration will sometimes be written in for clarity; for example:

(1) $\int_S f\,d\mu$ will be written $\int_S f(s)\,d\mu(s)$, $f\in D(\mu)$;

(2) $\int_{S\times T}h\,d(\mu\otimes_\varepsilon\nu)$ will be written $\int_{S\times T}h(s,t)\,d(\mu\otimes_\varepsilon\nu)(s,t)$, for $h\in D(\mu\otimes_\varepsilon\nu)$.

2. The Product Theorem.

In this section, all functions are measurable; f and g, with or without subscripts, will always denote functions defined on S and T, respectively.

2.1 Proposition. Suppose f and g are scalar simple functions. The function $(fg)(s,t) = f(s)g(t)$ is a scalar simple function on $S\times T$ and for each $E\in\Omega$ and $F\in\Lambda$

$$\int_{E\times F}fg\,d(\mu\otimes_\varepsilon\nu) = \int_E f\,d\mu\otimes\int_F g\,d\nu.$$

Proof. Suppose $f(s) = \sum_{i=1}^{n}a_i\zeta_{E_i}(s)$ and $g(t) = \sum_{j=1}^{m}b_j\zeta_{F_j}(t)$.

$$\int_{E\times F}fg\,d(\mu\otimes_\alpha\nu) = \int_{E\times F}\sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_j\zeta_{E_i\times F_j}\,d(\mu\otimes_\varepsilon\nu)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_j(\mu\otimes_\alpha\nu)(EE_i\times FF_j) = \sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_j\mu(EE_i)\otimes\nu(FF_j)$$

$$= \left(\sum_{i=1}^{n}a_i\mu(EE_i)\right)\otimes\left(\sum_{j=1}^{m}b_j\nu(FF_j)\right)$$

$$= \int_E f\,d\mu\otimes\int_F g\,d\nu. \quad \square$$

2.2 Lemma. Let f and g be scalar simple functions.

Define $\tau(E) = \int_E f\, d\mu$, for $E \in \Omega$,

and $\rho(F) = \int_F g\, d\nu$, for $F \in \Delta$.

Then $\tau \otimes_\varepsilon \rho(G) = \int_G fg\, d(\mu \otimes_\varepsilon \nu)$ for each $G \in \Omega \otimes_\sigma \Delta$.

Proof. Both τ and ρ are σ-additive taking their values in X and Y, respectively. The inductive product measure always exists and agrees with the indefinite integral $\int fg\, d(\mu \otimes_\varepsilon \nu)$ on the algebra of rectangles:

$$\begin{aligned} \tau \otimes_\varepsilon \rho(E \times F) &= \tau(E) \otimes \rho(E) \\ &= \int_E f\, d\mu \otimes \int_F g\, d\nu \\ &= \int_{E \times F} fg\, d(\mu \otimes_\varepsilon \nu), \end{aligned}$$

by Proposition 2.1.

$\tau \otimes_\varepsilon \rho$ agrees with $\int fg\, d(\mu \otimes_\varepsilon \nu)$ on $\Omega \otimes \Lambda$, so they must agree on $\Omega \otimes_\sigma \Lambda$ because both measures have unique extensions from the algebra to the σ-algebra. □

We now prove the main theorem of this section.

2.3 Theorem. Suppose $f \in D(\mu)$ and $g \in D(\nu)$. Then $fg \in D(\mu \otimes_\varepsilon \nu)$ and $\int_{E \times F} fg\, d(\mu \otimes_\varepsilon \nu) = \int_E f\, d\mu \otimes \int_F g\, d\nu$, for each $E \in \Omega$ and $F \in \Lambda$.

Proof. Let λ and ϕ be control measures for μ and ν, respectively. Let (f_n) and (g_n) be two sequences of simple functions which determines the integrals of f and g, respectively.

Define $\tau_n(E) = \int_E f_n d\mu$ and $\rho_n(F) = \int_F g_n d\nu$ for $n = 1,2,3,\ldots$, and $E \in \Omega$, $F \in \Lambda$. By the Vitali-Hahn-Saks Theorem, $\tau_n \ll \lambda$ $\rho_n \ll \phi$ uniformly in n.

Now by Lemma I.2.3, we have $\tau_n \otimes_\varepsilon \rho_n \ll \lambda \times \phi$ uniformly on $\Omega \otimes \Lambda$. Write $\gamma_n = \tau_n \otimes_\varepsilon \rho_n$.

Note that $\gamma_n(G) = \int_G f_n g_n d(\lambda \otimes_\varepsilon \nu)$, for $G \in \Omega \otimes_\sigma \Lambda$, by Lemma 2.2.

Assert that (γ_n) converges on $\Omega \otimes \Lambda$. To see this, let $A = \bigcup_{i=1}^{k} E_i \times F_i$ be a disjoint union, $E_i \in \Omega$, $F_i \in \Lambda$.

For $n,m \in \omega$, we have

$$|\gamma_n(A) - \gamma_m(A)| = |\sum_{i=1}^{k} \tau_n \otimes_\varepsilon \rho_n(E_i \times F_i) - \tau_m \otimes_\varepsilon \rho_m(E_i \times F_i)|$$

$$= |\sum_{i=1}^{k} \tau_n(E_i) \otimes \rho_n(F_i) - \tau_m(E_i) \otimes \rho_m(F_i)|$$

$$\leq \sum_{i=1}^{k} |\tau_n(E_i) - \tau_m(E_i)| \cdot |\rho_n(F_i)|$$

$$+ \sum_{i=1}^{k} |\tau_m(E_i)| \cdot |\rho_n(F_i) - \rho_m(F_i)|.$$

The sequences $\{\tau_n(E)\}$ and $\{\rho_n(F)\}$ are Cauchy in X and Y, respectively since

$$\lim_n \tau_n(E) = \int_E f \, d\mu \quad \text{and} \quad \lim_n \rho_n(F) = \int_F f \, d\nu.$$

Therefore, $\lim_{n,m} |\tau_n(E_i) - \tau_m(E_i)| = 0$, $1 \leq i \leq k$,

and $\lim_{n,m} |\rho_n(F_i) - \rho_m(F_i)| = 0$, $1 \leq i \leq k$.

From this and the above inequalities we have that

$$\lim_{n,m} |\gamma_n(A) - \gamma_m(A)| = 0;$$

that is, $\{\gamma_n(A)\}$ is Cauchy in $X \hat{\otimes}_\varepsilon Y$ and therefore converges in $X \hat{\otimes}_\varepsilon Y$.

The measures (γ_n) converging on $\Omega \otimes \Lambda$ and $\gamma_n \ll \lambda \times \phi$ uniformly on $\Omega \otimes \Lambda$ are sufficient to imply that the sequence (γ_n) converges on $\Omega \otimes_\sigma \Lambda$ ([4], Corollary 4).

Thus, $\lim_n \gamma_n(G) = \lim_n \int_G f_n g_n d(\mu \otimes_\varepsilon \nu)$ exists for each $G \in \Omega \otimes_\sigma \Lambda$, and $f_n g_n \to fg$ pointwise $\mu \otimes_\varepsilon \nu$-a.e.; this implies by Theorem III.6.1, $fg \in D(\mu \otimes_\varepsilon \nu)$ and the sequence $(f_n g_n)$ of simple functions determines the integral of fg.

Consequently, since the theorem is true for simple functions, we have,

$$\begin{aligned} \int_{E\times F} fg \; d(\mu \otimes_\varepsilon \nu) &= \lim_n \int_{E\times F} f_n g_n \; d(\mu \otimes_\varepsilon \nu) \\ &= \lim_n \int_E f_n \, d\mu \otimes \int_F g_n \, d\mu \\ &= \int_E f \, d\mu \otimes \int_F g \, d\mu. \quad \square \end{aligned}$$

2.4 Remark. The crucial point in the proof of Theorem 2.3 was invoking Lemma I.2.3 to conclude $\tau_n \otimes_\varepsilon \rho_n \ll \lambda \times \phi$ uniformly on $\Omega \otimes \Lambda$; this is because for the inductive product $\|\tau_n\|_\varepsilon^\gamma(\cdot) = \|\tau_n\|(\cdot)$ so that condition (3) of that lemma is fulfilled ($\tau_n \ll \lambda$ uniformly if and only if $\|\tau_n\| \ll \lambda$ uniformly). For the projective product measure, $\mu \otimes_\pi \nu$, the proof of Theorem 2.3 will not work since $\tau_n \ll \lambda$ uniformly and $\rho_n \ll \phi$ uniformly need not imply $\tau_n \otimes_\varepsilon \rho_n \ll \lambda \times \phi$ uniformly on $\Omega \otimes \Lambda$; consequently, further hypothesis may be required for a result analogous to Theorem 2.3 for $\mu \otimes_\pi \nu$. If μ and ν both have finite variation the Theorem 2.3 is true for $\mu \otimes_\pi \nu$. $\square$

2.5 Corollary. Let $f \in D(\mu)$ and $g \in D(\nu)$. Then for each $E \in \Omega$ and $F \in \Lambda$,

(1) the function $s \to \int_F f(s)g(t)d\nu(t)$ is a member of $B_Y(\mu)$;

(2) the function $t \to \int_E f(s)g(t)d\mu(s)$ is a member of $B_X(\nu)$;

(3) $$\begin{aligned} \int_{E\times F} f(s)g(t)d(\mu \otimes_\varepsilon \nu)(s,t) &= \int_E \int_F f(s)g(t)d\nu(t) \otimes_\varepsilon d\mu(s) \\ &= \int_F \int_E f(s)g(t)d\mu(s) \otimes_\varepsilon d\nu(t). \end{aligned}$$

Proof. Fix $E \in \Omega$ and $F \in \Lambda$ and write

$$x = \int_E f\,d\mu \quad \text{and} \quad y = \int_F g\,d\nu$$

where $x \in X$ and $y \in Y$.

Obviously, $y \cdot f(\cdot) \in B_Y(\mu)$ and $x \cdot g(\cdot) \in B_X(\nu)$; indeed, if $(f_n) \subseteq S(\Omega)$ determines f in $D(\mu)$, that is, $N(f-f_n) \to 0$, then the sequence $(yf_n) \subseteq S_Y(\Omega)$ determines yf:

$$\begin{aligned} N(yf_n - yf) &= \sup_{x^* \in X_1^*} \int_S |yf_n - yf|\,d|x^*\mu| \\ &= |y| \sup_{x^* \in X_1^*} \int_S |f_n - f|\,d|x^*\mu| \\ &= |y| \cdot N(f_n - f). \end{aligned}$$

Thus $\lim_n N(yf_n - yf) = |y| \lim_n N(f_n - f) = 0$, and $yf \in B_Y(\mu)$. Similarly, $xg \in B_X(\nu)$.

Because $\int_E yf_n \otimes_\varepsilon d\mu = (\int_E f_n d\mu) \otimes y$, we must have $\int_E yf \otimes_\varepsilon d\mu = (\int_E f\,d\mu) \otimes y$ too.

Finally,

$$\int_F f(s)g(t)\,d\nu(t) = f(s) \int_F g(t)\,d\nu = yf(s)$$

so that (1) is just the function $s \to yf(s)$ which we have shown to be in $B_Y(\mu)$, and using Theorem 2.3 we have

$$\begin{aligned} \int_{E\times F} f(s)g(t)\,d(\mu \otimes_\varepsilon \nu)(s,t) &= \int_E f(s)\,d\mu(s) \otimes \int_F g(t)\,d\nu(t) \\ &= \int_E f(s)\,d\mu(s) \otimes y \\ &= \int_E yf(s) \otimes_\varepsilon d\mu(s) \\ &= \int_E \int_F g(t)\,d\nu(t) f(s) \otimes_\varepsilon d\mu(s) \\ &= \int_E \int_F f(s)g(t)\,d\nu(t) \otimes_\varepsilon d\mu(s). \end{aligned}$$

Condition (2) and the second equality of condition (3) are proved similarly. □

3. The Classic Fubini Theorem.

In this section, we shall use the following notation for the norms of the spaces $D(\mu\otimes_\varepsilon\nu)$, $D(\mu)$, and $B_X(\nu)$ which was introduced in Chapter III.

(1) For $h \in D(\mu\otimes_\varepsilon\nu)$,

$$N(h) = \sup_{(x^*,y^*)\in X_1^*\times Y_1^*} \int_{S\times T} |h(s,t)|\, d|x^*\mu\times y^*\nu|(s,t),$$

(see Remark 3.4 infra);

(2) for $f \in D(\mu)$

$$N_1(f) = \sup_{x^*\in X_1^*} \int_S |f(s)|\, d|x^*\mu|(s);$$

(3) for $g \in B_X(\nu)$,

$$N_2(g) = \sup_{y^*\in Y_1^*} \int_T |g(t)|\, d|y^*\nu|(t).$$

Let $A \in \Omega\otimes_\sigma\Lambda$. For $s \in S$ and $t \in T$, the s-section and the t-section of A are, respectively,

$$A^s = \{t\in T: (s,t)\in A\},$$

and $A^t = \{s\in S: (s,t)\in A\}$.

From the classical theory of Lebesgue integration, we know that $A^s \in \Lambda$ and $A^t \in \Omega$.

3.1 Theorem. Let $A \in \Omega\otimes_\sigma\Lambda$ then

(1) the map $t \to \mu(A^t)$ from T into X is in $B_X(\nu)$;

(2) the map $s \to \nu(A^s)$ from S into Y is in $B_Y(\mu)$;

(3) $\mu\otimes_\varepsilon\nu(A) = \int_T (A^t)\otimes_\varepsilon d\nu(t) = \int_S (A^s)\otimes_\varepsilon d\mu(s)$.

Proof. Let H be the class of all sets $A \in \Omega\otimes_\sigma\Lambda$ for which the conclusions (1), (2), and (3) hold. We shall show that H is a monotone class containing the algebra of rectangles.

H contains the class of rectangles. If $A = \bigcup_{i=1}^{k} E_i \times F_i$, where (E_i) are disjoint, then

$$\nu(A^s) = \sum_{i=1}^{k} \nu(F_i)\zeta_{E_i}(s),$$

which is a Y-valued simple function, clearly in $B_Y(\mu)$.

Also,

$$\int_S \nu(A^s) \otimes_\varepsilon d\mu(s) = \sum_{i=1}^{k} \int_S \nu(F_i)\xi_{E_i}(s) \otimes_\varepsilon d\mu(s)$$

$$= \sum_{i=1}^{k} \mu(E_i) \otimes \nu(F_i) = \sum_{i=1}^{k} \mu \otimes_\varepsilon \nu(E_i \times F_i)$$

$$= \mu \otimes_\varepsilon \nu(A).$$

This proves (2) and half of (3).

We can write $A = \bigcup_{j=1}^{p} E_j' \times F_j'$ where now the sets (F_j') are pairwise disjoint and undergo a similar analysis to obtain (1) and (3).

Thus if $A \in \Omega \otimes \Lambda$, then A satisfies (1), (2) and (3) and therefore, $A \in H$. Finally we conclude $\Omega \otimes \Lambda \subseteq H$.

We now demonstrate that H is a monotone class.

Suppose $(A_n) \subseteq H$ is a monotone sequence and $A = \lim_n A_n$, then $\mu \otimes_\varepsilon \nu(A) = \lim_n \mu \otimes_\varepsilon \nu(A_n)$ and $\mu(A^t) = \lim_n \mu(A_n^t)$ and $\nu(A^s) = \lim_n \nu(A_n^s)$ for each $s \in S$ and $t \in T$.

The functions $\mu(A_n^t) \in B_X(\nu)$ and $\nu(A_n^s) \in B_Y(\mu)$ since $A_n \in H$ and (1) and (2) hold. Now because vector measures are bounded we see that there exists constants P and Q such that $|\mu(A_n^t)| \leq P$ and $|\nu(A_n^s)| \leq Q$ for all $n \in \omega$, $s \in S$, and $t \in T$. By the Bounded Convergence Theorem (Corollary III.3.3), $\mu(A^t) \in B_X(\nu)$, $\nu(A^s) \in B_Y(\mu)$, $\mu(A_n^t) \to \mu(A^t)$ in $B_X(\nu)$ and $\nu(A_n^s) \to \nu(A^s)$ in $B_Y(\mu)$.

Consequently

$$\mu\otimes_\varepsilon\nu(A) = \lim_n \mu\otimes_\varepsilon\nu(A_n) = \lim_n \int_T \mu(A_n^t)\otimes_\varepsilon d\nu(t)$$

$$= \int_T \mu(A^t)\otimes_\varepsilon d\nu(t).$$

Similarly,

$$\mu\otimes_\varepsilon\nu(A) = \int_S \nu(A^s)\otimes_\varepsilon d\mu(s).$$

This proves (1), (2) and (3) so $A \in H$, and H is a monotone class.

H then is a monotone class containing the algebra $\Omega \otimes \Lambda$ so it must necessarily be the σ-algebra $\Omega\otimes_\sigma\Lambda$; that is $H = \Omega\otimes_\sigma\Lambda$ and the theorem holds for all $A \in \Omega\otimes_\sigma\Lambda$. □

3.2 Corollary. Let $h \in D(\mu\otimes_\varepsilon\nu)$ be an $\Omega\otimes_\sigma\Lambda$-simple function. Then

(1) the function $s \to h(s,t)$ is in $D(\mu)$ for each $t \in T$;

(2) the function $t \to \int_S h(s,t)\, d\mu$ is in $B_X(\nu)$;

(3) $\int_{S\times T} h(s,t)\, d(\mu\otimes_\varepsilon\nu)(s,t) = \int_T \int_S h(s,t)\, d\mu(s) \otimes_\varepsilon d\nu(t)$.

Proof. Suppose $h(s,t) = \sum_{i=1}^n a_i \zeta_{G_i}(s,t)$ where $a_i \in \Phi$ and $(G_i) \subseteq \Omega\otimes_\sigma\Lambda$ is disjoint.

Fix $t \in T$, the function in (1) is $h^t(s) = h(s,t)$ and is an Ω-simple function since

$$h^t(s) = h(s,t) = \sum_{i=1}^n a_i \zeta_{G_i}(s,t) = \sum_{i=1}^n a_i \zeta_{G_i^t}(s).$$

So $h^t \in D(\mu)$ as claimed in (1).

$$\int_S h^t(s)d\mu(s) = \sum_{i=1}^n a_i \mu(G_i^t).$$

Since $\mu(G_i^t) \in B_X(\nu)$ by Theorem 3.1, we have the function $g(t) = \int_S h^t(s)d\mu(s)$ in $B_X(\nu)$.

Also, by Theorem 3.1.

$$\int_{S\times T} h(s,t)d(\mu\otimes_\varepsilon\nu)(s,t) = \sum_{i=1}^n a_i(\mu\otimes_\varepsilon\nu)(G_i)$$

$$= \sum_{i=1}^{n} a_i \int_T \mu(G_i^t) \otimes_\varepsilon d\nu(t)$$

$$= \int_T \sum_{i=1}^{n} a_i \mu(G_i^t) \otimes_\varepsilon d\nu(t)$$

$$= \int_T \int_S h^t(s)\, d\mu(s) \otimes_\varepsilon d\nu(t)$$

$$= \int_T \int_S h(s,t)\, d\mu(s) \otimes_\varepsilon d\nu(t). \quad \square$$

3.3 Proposition. Let $\Gamma \subset X_1^*$ be norming for X. Then for each $f \in D(\mu)$ we have $N_1(f) = \sup_{x^* \in \Gamma} \int_S |f|\, d|x^*\mu|$.

Proof. It suffices to show equality for f a simple function because simple functions are dense in $D(\mu)$.

Since $\Gamma \subset X_1^*$, we have $N_1(f) \geq \sup_{x^* \in \Gamma} \int_S |f|\, d|x^*\mu|$, we need to prove the reverse inequality.

Let $\varepsilon > 0$ be fixed, and suppose $f(s) = \sum_{i=1}^{n} \alpha_i \zeta_{E_i}(s)$, where $E_i \in \Omega$ are disjoint. Finally, let $x_o^* \in X_1^*$ be arbitrary.

$$\int_S |f|\, d|x_o^*\mu| = \sum_{i=1}^{n} |\alpha_i|\, |x_o^*\mu|(E_i).$$

Choose for each i, sets $A_{i_j} \in \Omega$, $1 \leq j \leq p$, pairwise disjoint such that $E_i = \sum_{j=1}^{p} A_{i_j}$ and

$$|x^*\mu|(E_i) < \frac{\varepsilon}{2} + \sum_{j=1}^{p} |x^*\mu(A_{i_j})|.$$

Also choose complex numbers θ_{i_j}, $1 \leq i \leq n$, $1 \leq j \leq p$, With $|\theta_{i_j}| \leq 1$ such that $|x^*\mu(A_{i_j})| = \theta_{i_j} x^*\mu(A_{i_j})$.

Thus,

$$\int_S |f|\, d|x_o^*\mu| \leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} |\alpha_i| \sum_{j=1}^{p} \theta_{i_j} x_o^*\mu(A_{i_j})$$

$$\leq \frac{\varepsilon}{2} + \left| \left\langle x_o^*, \sum_{i=1}^{n} |\alpha_i| \sum_{j=1}^{p} \theta_{i_j} \mu(A_{i_j}) \right\rangle \right|$$

$$\leq \frac{\varepsilon}{2} + \left| \sum_{i=1}^{n} |\alpha_i| \sum_{j=1}^{p} \theta_{i_j} \mu(A_{i_j}) \right|.$$

Since Γ is norming, there exists $x^* \in \Gamma$ such that

$$\left|\sum_{i=1}^{n}|\alpha_i|\sum_{j=1}^{p}\theta_{i_j}\mu(A_{i_j})\right| < \frac{\varepsilon}{2} + \left|\langle x^*, \sum_{i=1}^{n}|\alpha_i|\sum_{j=1}^{p}\theta_{i_j}\mu(A_{i_j})\rangle\right|$$

Therefore,

$$\begin{aligned}\int_S |f|d|x_o^*\mu| &< \varepsilon + \left|\sum_{i=1}^{n}|\alpha_i|\sum_{j=1}^{p}\theta_{i_j}x^*\mu(A_{i_j})\right| \\ &\le \varepsilon + \sum_{i=1}^{n}|\alpha_i|\sum_{j=1}^{p}|x^*\mu|(A_{i_j}\cdot) \\ &= \varepsilon + \sum_{i=1}^{n}|\alpha_i|\cdot|x^*\mu|(E_i) \\ &= \varepsilon + \int_S |f|d|x^*\mu|.\end{aligned}$$

We conclude from this that $\int_S |f|d|x_o^*\mu| \le \sup_{x^*\in\Gamma}\int_S |f|d|x^*\mu|$, and since $x_o \in X_1^*$ was arbitrary, $N_1(f) \le \sup_{x^*\in\Gamma}\int_S |f|d|x^*\mu|$. □

3.4 Remark. Proposition 3.3 shows that the definition of the N-norm for $D(\mu\otimes_\varepsilon\nu)$ in the beginning of this section is identical to the definition of the N-norm given in Chapter III. Since the set $\Gamma = \{x^*\otimes_\varepsilon y^* : x^*\in X_1^*, y^*\in Y_1^*\}$ is norming for $X\hat{\otimes}_\varepsilon Y$, we have

$$\begin{aligned}&\sup\{\int_S |h|d|x^*\mu\times h^*\nu| : x^*\otimes_\varepsilon y^*\in\Gamma\} \\ &\quad = \sup\{\int_S |h|d|z(\mu\otimes_\varepsilon\nu)| : z\in(X\hat{\otimes}_\varepsilon Y)_1^*\},\end{aligned}$$

where the right-hand supremum is the definition of the N-norm as in Chapter III, that is, the supremum is taken over the entire unit ball of the dual. Obviously, when working with products measures, the supremum over Γ is much more convenient and useful. □

3.5 Proposition. Let Z be a Banach space and $\lambda:\Omega \to Z$ a vector measure. Suppose A is a sub sigma algebra of Ω, $\overline{\lambda}$ is the restriction of λ to A, and X is a Banach subspace of Z which contains the range of $\overline{\lambda}$. Then

(1) $D(S,A,\overline{\lambda};X) \subseteq D(S,\Omega,\lambda;Z)$ isometrically;

(2) $\int_E f\, d\overline{\lambda} = \int_E f\, d\lambda$ for all $E \in A$ and $f \in D(S,A,\overline{\lambda};X)$.

Proof. Since $A \subseteq \Omega$, we have $S(A) \subseteq S(\Omega)$. To prove (1) it suffices to prove $S(A) \subseteq S(\Omega)$ isometrically; that is, for $h \in S(A)$, we shall prove

$$\sup_{x^* \in X_1^*} \int_S |h|\,d|x^*\overline{\lambda}| = \sup_{z^* \in Z_1^*} \int_S |h|\,d|z^*\lambda|,$$

the left hand number is the norm of h in $D(S,A,\overline{\lambda};X)$ and the right hand number is the norm of h in $D(S,\Omega,\lambda;Z)$.

Write $h(s) = \sum_{i=1}^{n} a_i \zeta_{E_i}(s)$ where $(E_i) \subseteq A$ is a pairwise disjoint family. Since $\overline{\lambda}$ is the restriction of λ to A we have for $E \in A$

$$\int_E f\, d\overline{\lambda} = \Sigma_i a_i \overline{\lambda}(E \cap E_i) = \Sigma_i a_i \lambda(E \cap E_i) = \int_E h\, d\lambda,$$

which proves (2) for simple functions.

If we restrict each member of Z_1^* to X, then each member $z^* \in Z_1^*$ when considered as a functional on X has norm ≤ 1 and furthermore, the family Z_1^* restricted to X is norming family for X. By Proposition 3.3 we have that the norm on $D(S,A,\overline{\lambda};X)$ is

$$\sup_{z^* \in Z_1^*} \int_S |f|\,d|z^*\overline{\lambda}| = \sup_{x^* \in X_1^*} \int_S |f|\,d|x^*\overline{\lambda}|,$$

where each z^* is restricted to X. But for $h \in S(A)$ we have

$$\sup_{x^* \in X_1^*} \int_S |h|\,d|x^*\overline{\lambda}| = \sup_{z^* \in Z_1^*} \int_S |h|\,d|z^*\overline{\lambda}|$$

$$= \sup_{z^* \in Z_1^*} \int_S |h|\,d|z^*\lambda|.$$

This proves $S(A) \subseteq S(\Omega)$ isometrically. Taking the closures of these spaces we get

$$D(S,A,\overline{\lambda};X) \subseteq D(S,\Omega,\lambda;Z) \text{ isometrically,}$$

which proves (1). Finally (2) follows from (1) and the fact (2) is true for A-simple functions. □

We now define a condition on the measure $\mu:\Omega \to X$ which seems crucial in proving the Fubini theorem for vector valued measures.

The measure μ has the Beppo Levi Property (BLP) if every increasing sequence (f_n) of positive Ω-simple functions with $\sup_n N_1(f_n) < +\infty$, is a Cauchy sequence in $D(\mu)$; consequently we have $\sup_n f_n \in D(\mu)$.

A detailed study of the Beppo Levi Property appears in [5] where a sufficient condition under which μ has the BLP is given: the condition is that the range space X of μ is weakly sequentially complete.

Finally, recall that a σ-algebra is separable it if is the σ-algebra generated by some countable subfamily of its members.

3.6 Theorem. (Fubini) Suppose $\mu:\Omega \to X$ has the Beppo Levi Property; in particular, this condition is satisfied if X is weakly sequentially complete. Let h be a non negative function in $D(S\times T;\Omega\otimes_\sigma \Lambda;\mu\otimes_\varepsilon \nu;X\hat{\otimes}_\varepsilon Y)$. Then

(1) for ν-almost every $t \in T$, $h^t \in D(\mu)$ where $h^t(s) = h(s,t)$;

(2) the function defined ν-a.e. by $t \to \int_S f^t(s)d\mu(s)$ is in $B_X(\nu)$;

(3) $\int_{S\times T} h(s,t)d(\mu\otimes_\varepsilon \nu)(s,t) = \int_T \int_S h(s,t)d\mu(s)\otimes_\varepsilon d\nu(t)$.

Proof. The norms N, N_1, N_2 which appear in the proof will be those defined in the introduction of this section.

We begin by assuming that h is $\Omega\otimes_\sigma\Lambda$-measurable, that is $h^{-1}(B) \in \Omega\otimes_\sigma\Lambda$ for all Borel subsets $B \subseteq \Phi$. In this case we can get a sequence (h_n) of increasing positive $\Omega\otimes_\sigma\Lambda$-simple functions converging pointwise everywhere to h. By the Lebesgue Dominated Convergence Theorem (Theorem III.3.2), (h_n) determines h in $D(\mu\otimes_\varepsilon\nu)$.

Let A be the σ-algebra generated by the collection of all characteristic sets of the functions h_n, and $n \in \omega$. Since there are countably many such characteristic sets, A is a separable σ-algebra. Each characteristic set lies in a σ-algebra generated by countably many retangles; consequently, there exists separable σ-algebras $\Omega' \subseteq \Omega$ and $\Lambda' \subseteq \Lambda$ such that $A \subseteq \Omega'\otimes_\sigma\Lambda'$. Let $\bar{\mu}$ and $\bar{\nu}$ be the restrictions of μ(resp. ν) to Ω' (resp. Λ'); finally, let X' and Y' be the closed subspaces generated by the ranges of $\bar{\mu}$ and $\bar{\nu}$, respectively. X' and Y' are separable Banach spaces, and $\bar{\mu}\otimes_\varepsilon\bar{\nu}:\Omega'\otimes_\sigma\Lambda' \to X'\otimes_\varepsilon Y'$ is the restriction of $\mu\otimes_\varepsilon\nu$ to $\Omega' \otimes_\sigma \Lambda'$. By Proposition 3.5, $D(S\times T,\Omega'\otimes_\sigma\Lambda',\bar{\mu}\otimes_\varepsilon\bar{\nu};X'\hat{\otimes}_\varepsilon Y') \subseteq D(S\times T,\Omega\otimes_\sigma\Lambda,\mu\otimes_\varepsilon\nu;X\hat{\otimes}_\varepsilon Y)$ isometrically and $\int_{S\times T} h\, d(\bar{\mu}\otimes_\varepsilon\bar{\nu}) = \int_{S\times T} h\, d(\mu\otimes_\varepsilon\nu)$ for all $h \in D(\bar{\mu}\otimes_\varepsilon\bar{\nu})$.

Now since each h_n is A-measurable it is $\Omega'\otimes_\sigma\Lambda'$-measurable, hence $(h_n) \subseteq S(\Omega'\otimes_\sigma\Lambda')$. Also, (h_n) is Cauchy in $D(\mu\otimes_\varepsilon\nu)$ and since $(h_n) \subseteq D(\bar{\mu}\otimes_\varepsilon\bar{\nu})$, (h_n) is Cauchy in $D(\bar{\mu}\otimes_\varepsilon\bar{\nu})$ because the norms of these spaces agree. Consequently, $h \in D(\bar{\mu}\otimes_\varepsilon\bar{\nu})$ and (h_n) determines h in $D(\bar{\mu}\otimes_\varepsilon\bar{\nu})$.

From this discussion then, we can assume that Ω and Λ are separable σ-algebras, and X and Y are separable Banach spaces to begin with.

For each $t \in T$, the t-section $h^t(s) = h(s,t)$ is μ-measurable since it is a pointwise limit of Ω-simple functions (h_n^t).

Since X is separable, we can find a countable set $(x_n^*) \subseteq X_1^*$ which is norming for X. By Proposition 3.3 the N_1-norm on $D(\mu)$ is

$$N_1(f) = \sup_n \int_S f(s)\, d|x_n^*\mu|(s), \quad f \in D(\mu).$$

For each $n \in \omega$, define

$$g_n(t) = \int_S h^t(s)\, d|x_n^*\mu|(s).$$

Then g_n is defined on T with values in the extended real number system $R\#$. Since for each t, h^t is Ω-measurable, from the classic theory, g_n is Λ-measurable. We now define $g(t) = \sup_n g_n(t)$ and note that g is Λ-measurable since the supremum of a countable family of measurable functions is measurable.

Note that $g(t) = N_1(h^t)$, and

$$N_2(g) = \sup_{y^* \in Y_1^*} \int_T g(t)\, d|y^*\nu|(t) = \sup_{y^* \in Y_1^*} \int_T \sup_n \int_S h(s,t)\, d|x_n^*\mu|\, d|y^*\nu|$$

$$= \sup_{y^* \in Y_1^*} \sup_n \int_T \int_S h(s,t)\, d|x_n^*\mu|\, d|y^*\nu|$$

$$= \sup_{y^* \in Y_1^*} \sup_n \int_{S \times T} h(s,t)\, d|x_n^*\mu \times y^*\nu|(s,t)$$

$$= N(h).$$

Thus $N_2(g) = N(h) < +\infty$, which implies g is finite ν-a.e., or $N_1(h^t) < +\infty$ for almost all $t[\nu]$.

For each $t \in T$, we have (h_n^t) is a sequence of simple functions on S which increase to h^t. Also, $\sup_n N_1(h_n^t) \leq N_1(h^t) < +\infty$

for ν-almost every $t \in T$. Because μ has the Beppo Levi property, we conclude that $h^t \in D(\mu)$ and $\lim_{n\to\infty} N_1(h^t - h_n^t) = 0$ for ν-almost every $t \in T$. This proves assertion (1).

As a result, if we define

$$H_n(t) = \int_S h_n^t \, d\mu(s) \text{ and } H(t) = \int_S h^t(s) \, d\mu(s),$$

then H_n maps T into X is defined everywhere on T, while H is defined only ν-a.e. on T; furthermore $\lim_{n\to\infty} H_n(t) = H(t)$ ν-a.e.

Since H_n is a μ-integral of t-sections of the simple function h_n, by Corollary 3.2 (2), we have $H_n \in B_X(\nu)$. But this implies H_n is ν-measurable, and since $H_n \to H$ ν-a.e. we conclude H is ν-measurable. Actually, we shall show (H_n) is Cauchy in $B_X(\nu)$.

Now for each $n, m \in \omega$, we have

$$N_2(H_n - H_m) = \sup_{y^* \in Y_1^*} \int_T |H_n(t) - H_n(t)| \, d|y^*\nu|(t)$$

$$\leq \sup_{(x^*,y^*) \in X_1^* \times Y_1^*} \int_T \int_S |h_n(s,t) - h_m(s,t)| \, d|x^*\mu|(s) \, d|y^*\nu|(t)$$

$$= N(h_n - h_m).$$

But $\lim_{n,m} N(h_n - h_m) = 0$, so $\lim_{n,m} N_2(H_n - H_m) = 0$. Thus we have $(H_n) \subset B_X(\mu)$ is Cauchy in $B_X(\mu)$ and $H_n \to H$ ν-a.e.; therefore, since $B_X(\nu)$ is complete, we have $H \in B_X(\nu)$ and $\lim_n N_2(H - H_n) = 0$. This then implies $\lim_n \int_T H_n(t) \otimes_\varepsilon d\nu(t) = \int_T H(t) \otimes_\varepsilon d\nu(t)$.

Finally,

$$\int_{S\times T} h(s,t) \, d(\mu \otimes_\varepsilon \nu)(s,t) = \lim_n \int_{S\times T} h_n(s,t) \, d(\mu \otimes_\varepsilon \nu)(s,t)$$

$$= \lim_n \int_T \int_S h_n(s,t) \, d\mu(s) \otimes_\varepsilon d\nu(t) = \lim_n \int_T H_n(t) \otimes_\varepsilon d\nu(t)$$

$$= \int_T H(t) \otimes_\varepsilon d\nu(t) = \int_T \int_S h(s,t) \, d\mu(s) \otimes_\varepsilon d\nu(t).$$

In this series of equalities, we used the fact that this theorem is true for the simple functions h_n, by Corollary 3.2.

The theorem is proven if h is $\Omega \otimes_\sigma \Lambda$-measurable.

Suppose now $h \in D(\mu \otimes_\varepsilon \nu)$ hence h is $\mu \otimes_\varepsilon \nu$-measurable.

We can get a sequence $(\bar{h}_n) \subseteq S(\Omega \otimes_\sigma \Lambda)$ of non negative functions which increases $\mu \otimes_\varepsilon \nu$-a.e. to h. Define $\bar{h}(s,t) = \lim_n \bar{h}_n(s,t)$, then $\bar{h} = h$ $\mu \otimes_\varepsilon \nu$-a.e. and $\bar{h}$ is $\Omega \otimes_\sigma \Lambda$-measurable. Thus the theorem holds for $\bar{h}$.

Assert that for ν-almost every $t \in T$ we have

$$\bar{h}^t(s) = h^t(s) \quad \mu\text{-a.e.} \qquad (\#)$$

Indeed, let λ and ϕ be control measures for μ and ν, respectively. Then $\lambda \times \phi$ is a control measure for $\mu \otimes_\varepsilon \nu$; consequently h is $\lambda \times \phi$-measurable and $\bar{h} = h$ $\lambda \times \phi$-a.e. From the classical theory, for ϕ-almost every $t \in T$, we have $\bar{h}^t(s) = h^t(s)$ λ-a.e. But this is exactly (#) since μ and λ have the same null sets as does ν and ϕ.

We have shown $\bar{h}^t \in D(\mu)$ for almost every $t \in T$; because $h^t = \bar{h}^t$ μ-a.e., we have $h^t \in D(\mu)$ and

$$\int_S h^t(s)\,d\mu(s) = \int_S \bar{h}^t(s)\,d\mu(s) \quad \nu\text{-a.e.}$$

But the function $t \to \int_S \bar{h}^t(s)\,d\mu(s)$ is in $B_X(\nu)$ by the first part of the proof, therefore so is the function $t \to \int_S h^t(s)\,d\mu(s)$, and

$$\int_T \int_S h^t(s)\,d\mu(s) \otimes_\varepsilon d\nu(t) = \int_T \int_S \bar{h}^t(s)\,d\mu(s) \otimes_\varepsilon d\nu(t).$$

Thus

$$\int_{S\times T} h(s,t)\,d(\mu \otimes_\varepsilon \nu)(s,t) = \int_{S\times T} \bar{h}(s,t)\,d(\mu \otimes_\varepsilon \nu)(s,t)$$

$$= \int_T \int_S \bar{h}(s,t)\, d\mu(s) \otimes_\varepsilon d\nu(t)$$

$$= \int_T \int_S h(s,t)\, d\mu(s) \otimes_\varepsilon d\nu(t). \quad \square$$

4. Fubini Theorem on $C(S\times T)$.

In this section, the pointsets S and T will be compact Hausdorff spaces. The space $C(S)$ is the Banach space of continuous scalar valued functions on S supplied with its standard uniform norm $|\cdot|_u$. For the case of continuous functions on $S \times T$, one obtains a Fubini theorem without the Beppo Levi property.

The class of Borel sets of S, denoted by $\mathcal{B}(S)$, is the σ-algebra generated by the family of all compact subsets of S. Let X be a Banach space; then μ is an X-valued Borel measure on S provided μ is a countably additive set function defined on $\mathcal{B}(S)$ with values in X. The measure μ is regular if for any $E \in \mathcal{B}(S)$ and any $\varepsilon > 0$, there exists a compact set C on an open set U with $C \subset E \subset U$ such that $|\mu(H)| < \varepsilon$ for every $H \in \mathcal{B}(S)$ with $H \subset U - C$.

Miloslav Duchon has shown in [8] that if $\mu:\mathcal{B}(S) \to X$ and $\nu:\mathcal{B}(T) \to Y$ are regular Borel measures, then there exists a unique regular Borel measure ρ on $S \times T$ with values in $X \hat{\otimes}_\varepsilon Y$ which extends $\mu \otimes_\varepsilon \nu$. Recall that $\mu \otimes_\varepsilon \nu$ is defined on $\mathcal{B}(S) \otimes_\sigma \mathcal{B}(T)$ while ρ is defined on the Borel sets of $S \times T$, that is, on $\mathcal{B}(S\times T)$; in general $\mathcal{B}(S)\otimes_\sigma\mathcal{B}(T) \subset \mathcal{B}(S\times T)$ properly.

We use freely in the next two theorems the well-known fact that $C(S\times T) = C(S)\hat{\otimes}_\varepsilon C(T)$ isometrically.

4.1 Theorem. Suppose $\mu:\mathcal{B}(S) \to X$ and $\nu:\mathcal{B}(T) \to Y$ are Borel measures. If $h \in C(S\times T)$, then

$$\int_{S\times T} h(s,t)\,d(\mu\otimes_\varepsilon\nu)(s,t) = \int_S \int_T h(s,t)\,d\nu(t)\otimes_\varepsilon d\mu(s). \qquad (\#)$$

If, in addition, μ and ν are regular Borel measures, then

$$\int_{S\times T} h(s,t)\,d\rho(s,t) = \int_{S\times T} h(s,t)\,d(\mu\otimes_\varepsilon\nu)(s,t),$$

where ρ is the unique Borel measure obtained by extending $\mu\otimes_\varepsilon\nu$ form $\mathcal{B}(S)\otimes_\sigma\mathcal{B}(T)$ to $\mathcal{B}(S\times T)$.

Proof. Suppose μ and ν are regular. Since $h \in C(S\times T)$, it is bounded and $\mathcal{B}(S\times T)$-measurable, so by Proposition III.2.5, $h \in D(S\times T,\mathcal{B}(S\times T),\rho;X\otimes_\varepsilon Y)$. Also, since $C(S\times T) = C(S)\hat{\otimes}_\varepsilon C(T)$, h is $\mathcal{B}(S)\otimes_\sigma\mathcal{B}(T)$-measurable, hence $h \in D(S\times T;\mathcal{B}(S)\otimes_\sigma\mathcal{B}(T), \mu\otimes_\varepsilon\nu;X\otimes_\varepsilon Y)$. $\mathcal{B}(S)\otimes_\sigma\mathcal{B}(T)$ is a sub σ-algebra of $\mathcal{B}(S\times T)$ and $\mu\otimes_\varepsilon\nu$ is the restriction of ρ to $\mathcal{B}(S)\otimes_\sigma\mathcal{B}(T)$, so by Proposition 3.5, $D(\mu\otimes_\varepsilon\nu) \subseteq D(\rho)$ isometrically and

$$\int_{S\times T} h(s,t)\,d(\mu\otimes_\varepsilon\nu)(s,t) = \int_{S\times T} h(s,t)\,d\rho(s,t),$$

this proves the second assertion.

To prove (#), we divide the proof into two cases.

Case I. Suppose $h \in C(S)\otimes C(T)$, so that $h(s,t) = \sum_{i=1}^{n} f_i(s)g_i(t)$ for some $f_i \in C(S)$ and $g_i \in C(T)$. In this case, (#) is a direct consequence of Corollary 2.5.

Case II. Let $h \in C(S\times T)$ be arbitrary. Because $C(S\times T) = C(S)\hat{\otimes}_\varepsilon C(T)$, there exists a sequence $(h_n) \subseteq C(S)\otimes C(T)$ such that $\lim_n |h_n - h|_u = 0$, that is $\lim_n h_n(s) = h(s)$ uniformly for $s \in S$. By Proposition III.9.1 we have

$$N(h_n-h) \le N_\infty(h_n-h)\cdot\|\mu\|(S) = |h_n-h|_u\cdot\|\mu\|(S),$$

where N is the norm in the space $D(\mu\otimes_\varepsilon\nu)$. Thus $h_n \to h$ in $D(\mu\otimes_\varepsilon\nu)$.

The function $h^s(t) = h(s,t)$ is continuous on T for each $s \in S$, hence $h^s \in D(\nu)$. Define

$$f_n(s) = \int_T h_n^s(t)d\nu(t) \text{ and } f(s) = \int_T h^t(t)d\nu(t).$$

Then

$$|f(s)-f_n(s)| \le \sup_{y^*\in Y_1^*}\int_T |h^s(t)-h_n^s(t)|\,d|y^*\nu|(t)$$

$$\le |h-h_n|_u\cdot\|\nu\|(T),$$

by Proposition III.9.1 again. This shows that $f_n \to f$ uniformly on S; since f_n is obviously continuous, f is continuous and so f_n, $f \in B_Y(\mu)$ for all $n \in \omega$.

But $f_n \to f$ uniformly implies as before that $f_n \to f$ in $B_Y(\mu)$; thus

$$\int_S f(s)\otimes_\varepsilon d\mu(s) = \lim_n \int_S f_n(s)\otimes_\varepsilon d\mu(s).$$

Finally, regard the definition of f and f_n, and apply Case I:

$$\begin{aligned}\int_S \int_T h(s,t)d\nu(t)\otimes_\varepsilon d\mu(s) &= \int_S f(s)\otimes_\varepsilon d\mu(s)\\ &= \lim_n \int_S f_n(s)\otimes_\varepsilon d\mu(s)\\ &= \lim_n \int_S\int_T h_n(s,t)d\nu(t)\otimes_\varepsilon d\mu(s)\\ &= \lim_n \int_{S\times T} h_n(s,t)d(\mu\otimes_\varepsilon\nu)(s,t)\\ &= \int_{S\times T} h(s,t)d(\mu\otimes_\varepsilon\nu)(s,t). \quad \square\end{aligned}$$

4.2 Remark. Unlike Theorem 3.6, the order of integration in (#) of Theorem 4.1 is not important; thus

$$\int_S\int_T h(s,t)d\nu(t)\otimes_\varepsilon d\mu(s) = \int_T\int_S h(s,t)d\mu(s)\otimes_\varepsilon d\nu(t). \quad \square$$

We may know apply this theorem to operator theory. An operator $U:C(S) \to X$ is weakly compact if U sends bounded sets onto relatively weakly compact subsets of X. A necessary and sufficient condition [12,IV.7.3] for the operator U to be weakly compact is the existence of a regular Borel measure $\mu:\mathcal{B}(S) \to X$ such that

$$U(f) = \int_S f(s)d\mu(s), \quad f \in C(S).$$

Any compact operator (see definition in Section 4, Chapter III) is weakly compact. The weakly compact operator U is a compact operator if and only if μ has relatively norm compact range [12,IV.7.7].

The measure μ is called the measure associated with U.

Suppose now that $P:C(S) \to X$ and $Q:C(T) \to Y$ are operators, then the map $P\otimes Q$ defined on $C(S) \otimes_\varepsilon C(T)$ into $X \hat{\otimes}_\varepsilon Y$ by $P\otimes Q(\sum_{i=1}^{n} f_i g_i) = \sum_{i=1}^{n} P(f_i)\otimes Q(g_i)$, where $\Sigma_i f_i g_i$ is a typical element of $C(S) \otimes C(T)$, is a linear operator which extends to domain of definition to $C(S) \hat{\otimes}_\varepsilon C(T)$ and is denoted by $P \otimes_\varepsilon Q$; moreover $|P\otimes_\varepsilon Q| = |P|\cdot|Q|$ (see [18], Chapter 46). Since $C(S) \hat{\otimes}_\varepsilon C(T) = C(S\times T)$, $P \otimes_\varepsilon Q$ is an operator on $C(S\times T)$.

4.3 Theorem. Suppose $P:C(S) \to X$ and $Q:C(T) \to Y$ are weakly compact (compact) operators. Then $P\otimes_\varepsilon Q:C(S\times T) \to X\hat{\otimes}_\varepsilon Y$ is a weakly compact (compact) operator.

Moreover, if μ is the Borel measure associated with P and ν is the Borel measure associated with Q, then the $X\hat{\otimes}_\varepsilon Y$-valued Borel measure associated with $P\hat{\otimes}_\varepsilon Q$ is ρ, the unique Borel extension of $\mu\otimes_\varepsilon\nu$. Thus,

$$P\otimes_\varepsilon Q(h) = \int_{S\times T} h(s,t)d\rho(s,t) = \int_{S\times T} h(s,t)d(\mu\otimes_\varepsilon\nu)(s,t) \quad (\#)$$

for every $h \in C(S\times T)$.

Proof. Suppose $h \in C(S) \otimes C(T)$ and $h = \sum_{i=1}^{n} f_i g_i$ for $f_i \in C(S)$ and $g_i \in C(T)$. Then

$$P\otimes_\varepsilon Q(h) = \Sigma_i P(f_i)\otimes Q(g_i) = \Sigma_i \int_S f_i d\mu \otimes \int_T g_i d\nu$$

$$= \Sigma_i \int_{S\times T} f_i g_i d(\mu\otimes_\varepsilon\nu) = \int_{S\times T} h\, d(\mu\otimes_\varepsilon\nu).$$

By Theorem 4.1, $\int_{S\times T} h\, d(\mu\otimes_\varepsilon\nu) = \int_{S\times T} h\, d\rho$.

Thus $P\otimes_\varepsilon Q(h) = \int_{S\times T} h\, d\rho$, $h \in C(S) \otimes C(T)$, where ρ is the unique Borel extension of $\mu\otimes_\varepsilon\nu$. Since this equality is valid on a dense subset of $C(S\times T)$, and the linear maps $P\otimes_\varepsilon Q$ and $h \to \int_{S\times T} h\, d\rho$ are continuous on $C(S\times T)$, we have

$$P\otimes_\varepsilon Q(h) = \int_{S\times T} h\, d\rho$$

for all $h \in C(S\times T)$. To prove (#), we now apply Theorem 4.1 once again. This also shows that $P\otimes_\varepsilon Q$ is a weakly compact operator since it has an integral representation with respect to the Borel measure $\rho: \mathcal{B}(S\times T) \to X \hat{\otimes}_\varepsilon Y$.

To prove that $P \otimes_\varepsilon Q$ is a compact operator if P and Q are compact, it is necessary to show that $P \otimes_\varepsilon Q$ maps bounded subsets of $C(S\times T)$ onto relatively compact subsets of $X \hat{\otimes}_\varepsilon Y$. Because $C(S\times T) \subseteq L^\infty(\mu\otimes_\varepsilon\nu)$ isometrically and

$$P\otimes_\varepsilon Q(h) = \int_{S\times T} h\, d(\mu\otimes_\varepsilon\nu),$$

it suffices, by Theorem III.9.2, to show $\mu\otimes_\varepsilon\nu$ has relatively norm compact range.

To this end, since P is compact, μ has relatively norm compact range; by Theorem III.3.2, there exists a sequence $(\mu_n) \subseteq X \otimes ca(\Omega)$ such that $\lim_n \|\mu_n - \mu\|(S) = 0$.

Note that

$$\begin{aligned}\lim_n \|(\mu_n \otimes_\varepsilon \nu) - (\mu \otimes_\varepsilon \nu)\|(S \times T) &= \lim_n \|(\mu_n - \mu) \otimes_\varepsilon \nu\|(S \times T) \\ &= \lim_n \|\mu_n - \mu\|(S) \cdot \|\nu\|(T) \\ &= 0.\end{aligned}$$

To show $\mu \otimes_\varepsilon \nu$ has relatively norm compact range we need only have to show that, for each $n \in \omega$, $\mu_n \otimes_\varepsilon \nu$ has relatively norm compact range, since, in this case $\mu \otimes_\varepsilon \nu$ will be the limit, in semivariation, of measures in $Cca(\Omega \otimes_\sigma \Lambda; X \hat{\otimes}_\varepsilon Y)$, hence would belong to this space and have relatively norm compact range.

Let $n \in \omega$ be fixed, and write $\mu_n = \sum_{i=1}^{k} x_i \lambda_i$, where $x_i \in X$, and λ_i is a non-negative measure on Ω.

Q is a compact operator so ν has relatively norm compact range. Let J be a balanced, convex, compact subset of Y which contains the range of ν. Put $\alpha = \max_i \lambda_i(S)$. Obviously the set $C = \sum_{i=1}^{k} x_i \otimes \alpha J$ is a compact subset of $X \hat{\otimes}_\varepsilon Y$. Claim C contains the range of $\mu_n \otimes_\varepsilon \nu$. To see this, let $G = \cup_j E_j \times F_j$, where the sets $E_j \in \Omega$ are pairwise disjoint and $F_j \in \Lambda$. Then

$$\begin{aligned}\mu_n \otimes_\varepsilon \nu(G) &= \Sigma_j \mu_n(E_j) \otimes \nu(F_j) \\ &= \Sigma_j \Sigma_i x_i \lambda_i(E_j) \otimes \nu(F_j) \\ &= \Sigma_i \Sigma_j x_i \otimes \lambda_i(E_j) \nu(F_i) \\ &= \Sigma_i x_i \otimes \Sigma_j \lambda_i(E_j) \nu(F_j).\end{aligned}$$

For each i, $\Sigma_j \lambda_i(E_j) \leq \lambda(S) \leq \alpha$, and because $\nu(F_j) \in J$, and J is balanced and convex, we see $\Sigma_j \lambda_i(E_j)\nu(F_j) \in \alpha J$. Thus $\mu_n \otimes_\varepsilon \nu(G) \in \Sigma_i x_i \otimes \alpha J = C$, where $G \in \Omega \otimes \Lambda$. Because the range of $\mu_n \otimes_\varepsilon \nu$ restricted to the algebra $\Omega \otimes \Lambda$ lies in C, we conclude the range of $\mu_n \otimes_\varepsilon \nu$ over $\Omega \otimes_\sigma \Lambda$ also lies in C, and so $\mu_n \otimes \nu$ has relatively norm compact range. □

BIBLIOGRAPHY

[1] R. G. Bartle, "A General Bilinear Vector Integral," Studia Math. 15, 337-352 (1956).

[2] J. K. Brooks, "Eqicontinuous Sets of Measures and Applications to Vitali's Integral Convergence Theorem and Control Measures," Adv. in Math. 9, 1-7 (1972).

[3] J. K. Brooks, "Representations of Weak and Strong Integrals in Banach Spaces," Proc. Nat. Acad. Sci. 63, 266-270 (1969).

[4] J. K. Brooks, "Interchange of Limit Theorems for Finitely Additive Measures," Revue Romaine (to appear).

[5] J. K. Brooks and N. Dinculeanu, "Lebesgue-Type Spaces for Vector Integration, Linear Operators, Weak Completeness and Weak Compactness," J. Math. Anal. Appl. (to appear).

[6] N. Dinculeanu, Vector Measures, Pergamon Press, Oxford, 1967.

[7] N. Dinculeanu and I. Kluvánek, "On Vector Measures," Proc. London Math. Soc. 17, 505-512 (1967).

[8] M. Duchon, "On Tensor Product of Vector Measures in Locally Compact Spaces," Mat. casop. 19, 324-329 (1969).

[9] M. Duchon, "On the Projective Tensor Product of Vector Valued Measures II," Mat. casop. 19, 228-234 (1969).

[10] M. Duchon, "The Fubini Theorem and Convolution of Vector Valued Measures," Mat. casop. 23, 170-178 (1973).

[11] M. Duchon and I. Kluvánek, "Inductive Tensor Products of Vector Valued Measures," Mat. casop. 17, 108-112 (1967).

[12] N. Dunford and R. Schwartz, Linear Operators, Part I, Interscience, New York, 1958.

[13] J. Gil de Lamadrid, "Measures and Tensors," Trans. Amer. Math. Soc. 114, 98-121 (1965).

[14] J. E. Huneycutt, "Products and Convolutions of Vector Valued Set Function," preprint.

[15] I. Kluvánek, "An Example Concerning the Projective Tensor Product of Vector Measures," Mat. casop. 20, 81-83 (1970).

[16] D. R. Lewis, Conditional Weak Compactness in Certain Inductive Tensor Products," Math. Ann. 201, 201-209 (1973).

[17] B. J. Pettis, "On Integration in Vector Spaces," Trans. Amer. Math. Soc. 44, 277-304 (1938).

[18] F. Treves, Topological Vector Spaces, Distributions and Kernals, Academic Press, New York, 1967.

[19] K. Yosida, Functional Analysis, Springer-Verlag, New York, 1968.

BIOGRAPHICAL SKETCH

Donald P. Story was born on December 17, 1946, in El Paso, Texas. Son of a career air force officer, he attended various schools throughout the United States, Britain, and France. Upon graduating from Chowtawhatchee High School, Shalimar, Florida, in 1965, he entered Okaloosa-Walton Junior College, Niceville, Florida. In 1967, he enrolled in the University of Florida as an undergraduate, and after receiving a B.A. in Mathematics in 1969, continued on as a graduate student at the University of Florida. He became a member of the American Mathematical Society in 1972.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is full adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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