THE USE OF COMPLEX VARIABLES FOR SOLVING CERTAIN ELASTICITY PROBLEMS INVOLVING INTERSECTING BOUNDARIES

By

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CHAPTER I

INTRODUCTION

Elasticity

In the study of elastic properties of materials forming regions with greater than microscopic dimensions, the molecular nature of the material is disregarded and replaced by a continuous mathematical model. The problem does not deal with movements in the body as a whole, but in relative movement of points within the body, which movements are called strain.

The forces that produce strain are referred to as stresses and are divided into two classes. Normal stresses tending to elongate or shorten a segment are called tensile or compressive stresses, respectively. A stress tending to deform an angle within the body is called a shear stress.

To describe the stresses associated with rectangular coordinates, three symbols, σ_x, σ_y, and σ_z, are necessary for the normal streses; and three symbols, τ_{xy}, τ_{xz}, and τ_{yz}, for the shearing stresses. Similarly, for the strains, we use ϵ_x, ϵ_y, ϵ_z, γ_{xy}, γ_{xz}, and γ_{yz}.

Moreover, it may be shown that the stress coordinates are covariant tensors of the second kind, as are also the strain coordinates (6, p. 275). This fact simplifies the changing of spatial coordinates.

To the theory Hooke (8, p. 1) contributed his hypothesis that there exists a linear relationship between stresses and strains when they are of infinitesimal order. Below are the relations of Hooke's Law as they are regularly now given.

$$
\begin{aligned}
&\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right], && \gamma_{xy} = \frac{1}{G}\tau_{xy}\,, \\
(1.1)\qquad &\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right], && \gamma_{xz} = \frac{1}{G}\tau_{xz}\,, \\
&\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right], && \gamma_{yz} = \frac{1}{G}\tau_{yz}\,,
\end{aligned}
$$

where E, G, and ν are elastic constants pertaining to the material in question, and are called the modulus of elasticity in tension, the modulus of elasticity in shear, and Poisson's ratio, respectively.

Equilibrium within the elastic body demands that the following equations, called the differential equations of equilibrium (9, p. 229), be satisfied:

$$
\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0\,, \\
(1.2)\qquad &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0\,, \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0\,,
\end{aligned}
$$

where X, Y, and Z are components of any body forces present.

Moreover, relations between displacements and strains require that

$$
\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}\,,
$$

$$
\frac{\partial^2 \epsilon_x}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z}\,,
$$

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z},$$

$$(1.3) \qquad \begin{aligned} 2\frac{\partial^2 \epsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \\ 2\frac{\partial^2 \epsilon_y}{\partial x \partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \\ 2\frac{\partial^2 \epsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right), \end{aligned}$$

which equations are called the conditions of compatibility (8, p. 28).

To solve a general problem in three dimensions when a body is submitted to the action of given forces, it is necessary to solve equation (1.2), subject to boundary conditions, and to obtain a set of stresses which produce a corresponding set of strains that satisfy equation (1.3). Anyone familiar with such problems will realize that this is a tremenduously complicated and difficult task, except in the case of the simplest stress distributions. One must also realize that it is here assumed that the material has the same elastic properties in each direction. This means that a completely general solution in elasticity is difficult to the point of impossibility. It is, therefore, customary to introduce certain simplifying assumptions that remove many conditions on the general solution and still yield results that approximate many situations closely enough to be very useful. In the remainder of this work, the subject of plane-elasticity will be followed.

Let it be assumed that the dimension of the body in the z direction is small relative to the other two. Suppose, also, that the loading on the edges of such a body is distributed uniformly along the thickness and that its faces are not loaded. A large plate of steel, loaded on the edges, would be an example of this problem. With this loading pattern, σ_z, τ_{xz}, and τ_{yz} are zero on the boundary, and it is also assumed that they are zero inside the body. Likewise, we assume that the remaining stresses are functions of x and y, but not of z. Such a situation is called plane stress.

Under these assumptions, it is seen that the equations (1.1), (1.2), and (1.3) reduce to

(1.1a)
$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad , \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy} \, ,$$
$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \quad ;$$

(1.2a)
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \, ,$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \, ;$$

(1.3a)
$$\frac{\partial^2 \epsilon_y}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \, .$$

Another similar simple case called plain strain is derived by assuming that the z-direction is very large with respect to the other two. Consider the case when forces load the body perpendicular to the z-direction and are uniformly distributed along its length. Then, it is assumed that there is no displacement in the z-direction; hence, there are no strains in this direction. As before, the stresses and strains will depend upon x and y only. Since ϵ_z, γ_{xz}, and γ_{yz} are zero, then (1.1) reduces to

(1.1b)
$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] ,$$
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] ,$$
$$0 = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] ,$$
$$\gamma_{xy} = \frac{1}{G}\tau_{xy} ,$$

showing that σ_z may be expressed in terms of σ_x and σ_y. Equations (1.1b) then become

$$\sigma_z = \nu(\sigma_x + \sigma_y) ;$$

(1.1c)
$$\epsilon_x = \frac{1}{E}[(1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y] ,$$
$$\epsilon_y = \frac{1}{E}[(1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x] ,$$
$$\gamma_{xy} = \frac{1}{G}\tau_{xy} .$$

Equations (1.2) and (1.3) reduce to precisely the same form as (1.2a) and (1.3a) above.

It can here be seen that plane stress and plane strain are identical mathematically, and differ only in the elastic constants employed in equations (1.1a) and (1.1c). These two can then be grouped together under the subject of two-dimensional problems.

Two-Dimensional Problems

As has been seen, in order to solve a two-dimensional problem, we must satisfy at least three sets of conditions. These are the equilibrium equations (1.2a), the conditions of compatibility (1.3a), and the boundary conditions.

G. B. Airy (9, p. 26) first introduced the stress function ϕ such that, when there are no body forces,

$$(1.4) \qquad \frac{\partial^2\phi}{\partial y^2} = \sigma_x , \qquad \frac{\partial^2\phi}{\partial x^2} = \sigma_y , \qquad \frac{\partial^2\phi}{\partial x\partial y} = -\tau_{xy} .$$

This function satisfies the equilibrium conditions.

J. C. Maxwell (7, p. 105) found that if the function ϕ satisfies the bi-harmonic equation, then the condition of compatibility is also satisfied. The problem now resolves into finding a ϕ satisfying

$$(1.5) \qquad \frac{\partial^4\phi}{\partial y^4} + 2\frac{\partial^4\phi}{\partial y^2\partial x^2} + \frac{\partial^4\phi}{\partial x^4} = 0 ,$$

and also satisfying the boundary conditions.

It is found that a general solution of the bi-harmonic equations is

$$\phi = R\left\{x f_1(x-iy) + f_2(x-iy) + x f_3(x+iy) + f_4(x+iy)\right\} .$$

This may be written

$$(1.6) \qquad \phi = R\left\{\bar{z}\psi(z) + \chi(z)\right\} ,$$

where $z = x + iy$ and ψ and χ are complex functions of z, analytic in the region under consideration. Some elementary manipulations lead us to two relations that are used to evaluate the stresses, avoiding the necessity of finding the real part of ϕ. They are

$$\sigma_y + \sigma_x = 2(\psi' + \overline{\psi'}), \tag{1.7}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2(\bar{z}\psi'' + \chi''). \tag{1.8}$$

It is also found that the resultant force acting along a boundary, where X and Y are the resultants in the x and y directions, respectively, and A and B are the ends of the boundary, is (7, p. 114)

$$X + iY = -i\left[\psi + z\overline{\psi'} + \overline{\chi'}\right]_A^B. \tag{1.9}$$

It follows that, along an unloaded boundary, this quantity must be constant. Written in the way that we shall repeatedly use, it is

$$\left[\psi + z\overline{\psi'} + \overline{\chi'}\right] = H, \tag{1.10}$$

where H is a complex constant.

With the foregoing relations, (1.7) through (1.10), certain intersecting boundary problems will be attacked. It is most important, however, to discuss the functions ψ and χ, the complex potentials, before proceeding.

The Complex Potentials

Several rather general statements may be made about ψ and χ. It has been noted that they must be analytic in the region under consideration. They may have singularities in deleted regions, and therefore, may be expanded in a Laurent series about these regions, except for any terms that might produce multi-valued stresses or displacements. It is to be noted that the imaginary part of the function

$\left[\bar{z}\psi + \chi\right]$ may be multi-valued. Muskhelishvili (7, p. 121) shows that ψ and χ must have the following forms:

$$\psi = \sum_{K=1}^{m} A_K \operatorname{Log}(z - z_K) + \psi^*, \tag{1.11}$$

$$\chi = \sum_{K=1}^{m} B_K \operatorname{Log}(z - z_K) + z\sum_{K=1}^{m} C_K \operatorname{Log}(z - z_K) + \chi^*, \tag{1.12}$$

where A_k, B_k, and C_k are complex constants, and ψ^* and χ^* are holomorphic functions. Z_k is a point in the k^{th} deleted neighborhood.

Since the stresses do not depend on χ but on χ' and higher derivatives, the χ' form is here inserted.

$$\chi' = \sum_{K=1}^{m} C_K \operatorname{Log}(z - z_K) + \chi_1^*, \tag{1.13}$$

where χ_1^* is holomorphic in the region of the problem.

Since ψ^* and χ_1^* are holomorphic, they may be freely expanded in Laurent expansions about points in deleted neighborhoods.

CHAPTER II

THE MOTIVATION FOR THE METHOD

As can be readily seen, the complex functions ψ and χ yield results that are sometimes readily handled by the method of undetermined coefficients. The usual procedure is to insert the boundary conditions in an appropriate relation, such as (1.10), and to use the fact that two expressions are identical if coefficients of like powers of z are equal. This gives a series of equations which are to be solved for the unknown coefficients.

In the past, attempts to solve certain problems by the method of undetermined coefficients have failed to yield results. One of these cases, the semi-circular notch problem, which has been solved using the methods of real variables (5), and later generalized by the same methods (3), will now be discussed to show where the difficulty arises. As shall be seen, inconsistent coefficient equations result from the presence of a multiplicity of boundaries.

The notch in question has been oriented, as at the right, so that it becomes, by appropriate choice of axes and scale, the half of the unit circle for which y is positive. This portion of the unit circle will be called the half unit circle.

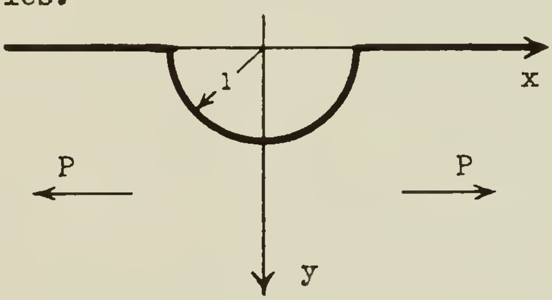

Figure 1

The x-axis is the edge of our semi-infinite sheet. Since this can be thought of as an infinite sheet with a circular hole coupled with some distributed force superimposed on the x-axis, we can expect the complex potentials to have singularities only within the unit circle. Then ψ and χ' will assume the following forms:

$$\psi = A\,\mathrm{Log}\,z + \sum_{n=-\infty}^{\infty} a_n z^{+n},$$

$$\chi' = B\,\mathrm{Log}\,z + \sum_{n=-\infty}^{\infty} b_n z^{+n}.$$

Moreover, since the stresses are bounded at infinity, we can immediately eliminate all positive powers of z except the first. Then, also deleting the constant terms not affecting stresses, we have

$$\psi = A\,\mathrm{Log}\,z + az + \sum_{n=1}^{\infty} a_n z^{-n},$$

$$\chi' = B\,\mathrm{Log}\,z + bz + \sum_{n=1}^{\infty} b_n z^{-n},$$

or, in a better form for our purposes,

(2.1) $$\psi = A\,\mathrm{Log}\,z + (A' + A''i)z + \sum_{n=1}^{\infty} a_n z^{-n},$$

(2.2) $$\chi' = B\,\mathrm{Log}\,z + (B' + B'')z + \sum_{n=1}^{\infty} b_n z^{-n}.$$

The uniform tension P in the x-direction fixes the coefficients of z, for from (1.5)

$$(\sigma_y + \sigma_x)\Big|_{z=\infty} = 2(\psi' + \overline{\psi'})\Big|_{\infty} = 4A' = P,$$

since $\sigma_x\big|_{\infty} = P$ and $\sigma_y\big|_{\infty} = 0$. Then $A' = \frac{P}{4}$.

Also, from (1.6)

$$\sigma_y - \sigma_x + 2i\tau_{xy}\Big|_{\infty} = 2(\bar{z}\psi'' + \chi'') = 2(B' + iB'') = -P \; ,$$

so that $B' = -\frac{P}{2}$, $B'' = 0$, since $\tau_{xy}\Big|_{\infty} = 0$. A" is set equal to zero, since it does not affect the stresses and since the movement about the origin also vanishes.

The final form of ψ and χ' to be used in succeeding work is

(2.1a) $$\psi = A\,\mathrm{Log}\,z + \frac{P}{4}z + \sum_{n=1}^{\infty} a_n z^{-n} \, ,$$

(2.2a) $$\chi' = B\,\mathrm{Log}\,z - \frac{P}{2}z + \sum_{n=1}^{\infty} b_n z^{-n} \, .$$

Since no force acts on the x-axis, then (1.10) holds and

(1.10a) $$\left[\psi + z\overline{\psi'} + \overline{\chi'}\right]\Big|_{z=x} = H \, .$$

Substituting from (2.1a) and (2.2a) yields

(2.3) $$A\,\mathrm{Log}\,z + \frac{P}{4}z + \sum_{n=1}^{\infty} a_n z^{-n} + z\left(\frac{\bar{A}}{\bar{z}} + \frac{P}{4} - \sum_{n=1}^{\infty} n\bar{a}_n \bar{z}^{-n-1}\right) + \bar{B}\,\mathrm{Log}\,\bar{z} - \frac{P}{2\bar{z}^{-1}} + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^{-n} = H \, .$$

The above expression must be equal to a constant along the line z = x. However, on this line, $\bar{z}$ = z = x, and the foregoing becomes

$$(A + \bar{B})\,\mathrm{Log}\,x + \bar{A} + \sum_{n=1}^{\infty} (a_n - n\bar{a}_n + \bar{b}_n) x^{-n} = H \, .$$

This is satisfied if $\bar{B} = -A$ and $\bar{b}_n = n\bar{a}_n - a_n$, or

(2.4) $$B = -\bar{A} \, ,$$
$$b_n = n a_n - \bar{a}_n \, ,$$

by taking conjugates of both sides. Then (2.1a) and (2.2a) simplify to

$$\psi = A\,\mathrm{Log}\,z + \frac{p}{4}z + \sum_{n=1}^{\infty} a_n z^{-n}, \tag{2.1b}$$

$$\chi' = -\bar{A}\,\mathrm{Log}\,z - \frac{p}{2}z + \sum_{n=1}^{\infty}(na_n - \bar{a}_n)z^{-n}. \tag{2.2b}$$

Now, considering the unit circle, it is seen that $z = e^{i\theta}$ along this curve, since the modulus R is equal to one. For simplicity, $e^{i\theta}$ will be called σ. $\bar{\sigma} = e^{-i\theta} = \sigma^{-1}$, so that $\bar{\sigma} = \sigma^{-1}$. Inserting this boundary in (2.3) leads to

$$H = A\,\mathrm{Log}\,\sigma - A\,\mathrm{Log}\,\sigma^{-1} + \frac{p}{2}\sigma - \frac{p}{2}\sigma^{-1} + A\sigma^2$$

$$+\sum_{n=1} a_n\sigma^{-n} - \sum_{n} n\bar{a}_n\sigma^{n+2} + \sum \bar{b}_n\sigma^n,$$

$$= (A+A)\,\mathrm{Log}\,\sigma + \left(\frac{p}{2} + \bar{b}_1\right)\sigma + \left(-\frac{p}{2} + a_1\right)\sigma^{-1}$$

$$+(A + \bar{b}_2)\sigma^2 + \sum_{n=2}^{\infty} a_n\sigma^{-n} + \sum_{n=3}^{\infty}\left[\bar{b}_n - (n-2)\bar{a}_{n-2}\right]\sigma^{+n}.$$

The usual attack would yield the following equations when coefficients are set equal:

(a) $A + A = 0$ (d) $\bar{A} + b_2 = 0$

(b) $\frac{p}{4} + b_1 = 0$ (e) $a_n = 0, \quad n > 1$

(c) $a_1 - \frac{p}{2} = 0$ (f) $b_n - (n-z)a_{n-2} = 0, \quad n > 2$

Careful scrutiny of these equations shows them to be inconsistent when considered with equations (2.4). Note that $b_n = na_n - \bar{a}_n$ coupled with (f) above shows a contradiction, in that, from the former $b_3 = 0$ since it depends on a_3, whereas, the latter yields b_3 as dependent on a_1 and therefore $b_3 \neq 0$. There is agreement at this point, however, that A must vanish.

The usual method has therefore failed to produce fruitful results. This, however, poses a more far-reaching problem: that is, the existence of the problem, at least theoretically, would show that the very functions, which have led into this inconsistency, must in some way give a solution. This would indicate that, somewhere in the process, there must have been some step taken that is based on the grounds of sufficiency but not necessity. That step is discussed in the next chapter.

In summary, there is need to investigate the methods commonly used because they lead to no solution, where it can demonstrated that a solution exists. This investigation will begin with a look at identities on the half unit circle.

CHAPTER III

IDENTITIES ON THE HALF UNIT CIRCLE

The half unit circle will, as has already been noted, in the following context, refer to the upper half of a unit circle centered at the origin, that is the part of the unit circle with positive ordinate. If z is expressed as $Re^{i\theta}$, then on this portion of the unit circle

$$R = 1 \quad ,$$

$$0 < \theta < \pi \, .$$

But Euler's familiar formula states that

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

or, by rearrangements,

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \, , \tag{3.1}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \, .$$

Previously, the symbol $\sigma = e^{i\theta}$ was introduced, and it was there noted that $e^{-i\theta} = \bar{\sigma} = \sigma^{-1}$. Also $e^{ni\theta} = \sigma^n$. Then

$$\sin(m\theta) = \frac{\sigma^m - \sigma^{-m}}{2i} \, , \tag{3.2}$$

$$\cos(m\theta) = \frac{\sigma^m + \sigma^{-m}}{2} \, .$$

In the region under consideration sin $m\theta$ and cos $m\theta$, where m is a positive integer, can be expanded by an ordinary Fourier series into an expression involving only cos $n\theta$ and sin $n\theta$, respectively. This is not true of the whole circle where uniqueness demands that the expansions for sin $m\theta$ and cos $m\theta$ be precisely sin $m\theta$ and cos $m\theta$, respectively.

By appropriate Fourier theorems (1, pp. 57, 70), we obtain expansions of the form

$$\frac{\sigma^m - \bar{\sigma}^m}{2i} = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \frac{\sigma^n + \bar{\sigma}^n}{2}, \qquad 0<\theta<\pi,$$

where

$$C_0 = \frac{2}{\pi}\int_0^{\pi} \sin(m\theta)\, d\theta \quad ,$$

$$C_n = \frac{2}{\pi}\int_0^{\pi} \sin(m\theta)\cos(n\theta)\, d\theta, \quad n=1,2,3,\cdots,$$

and

$$\frac{\sigma^m + \bar{\sigma}^m}{2} = \sum_{n=1}^{\infty} D_n \frac{\sigma^n - \bar{\sigma}^n}{2i}, \qquad 0<\theta<\pi,$$

where

$$D_n = \frac{2}{\pi}\int_0^{\pi} \cos(m\theta)\sin(n\theta)\, d\theta, \quad n=1,2,3,\cdots.$$

It is found by simple integration that, when $m = 2k - 1$, then

$$C_{2n} = \frac{2}{\pi}\left[\frac{2(2K-1)}{(2K-1)^2-(2n)^2}\right], \quad n=0,1,2,3,\cdots,$$

$$C_{2n-1} = 0 \quad ,$$

and, when $m = 2k$, then

$$C_{2n} = 0, \qquad n = 0, 1, 2, 3, \cdots,$$

$$C_{2n-1} = \frac{2}{\pi}\left[\frac{2(2K)}{(2K)^2 - (2n-1)^2}\right], \quad n = 1, 2, 3, \cdots.$$

Also if $m = 2k - 1$, then

$$D_{2n-1} = 0, \qquad n = 1, 2, 3, \cdots,$$

$$D_{2n} = \frac{2}{\pi}\left[\frac{2(2n)}{(2n)^2 - (2K-1)^2}\right], \quad n = 1, 2, 3, \cdots,$$

and, when $m = 2k$, then

$$D_{2n} = 0, \qquad n = 1, 2, 3, \cdots,$$

$$D_{2n-1} = \frac{2}{\pi}\left[\frac{2(2n-1)}{(2n-1)^2 - (2K)^2}\right], \quad n = 1, 2, 3, \cdots.$$

These lead to the following relations:

$$\frac{\sigma^{2K-1} - \bar{\sigma}^{2K-1}}{2i} = \frac{2}{\pi(2K-1)} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2(2K-1)}{(2K-1)^2 - (2n)^2}\right]\frac{\sigma^{2n} + \bar{\sigma}^{2n}}{2},$$

$$\frac{\sigma^{2K} - \bar{\sigma}^{2K}}{2i} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2(2K)}{(2K)^2 - (2n-1)^2}\right]\frac{\sigma^{2n-1} + \bar{\sigma}^{2n-1}}{2},$$

$$\frac{\sigma^{2K-1} + \bar{\sigma}^{2K-1}}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2(2n)}{(2n)^2 - (2K-1)^2}\right]\frac{\sigma^{2n} - \bar{\sigma}^{2n}}{2i},$$

$$\frac{\sigma^{2K} + \bar{\sigma}^{2K}}{2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2(2n-1)}{(2n-1)^2 - (2K)^2}\right]\frac{\sigma^{2n-1} - \bar{\sigma}^{2n-1}}{2i},$$

where $0 < \theta < \pi$.

By elementary manipulation, we may obtain

$$
(3.3)\quad
\begin{aligned}
&(a)\quad \sigma^{2K-1}-\bar{\sigma}^{2K-1} = \frac{4i}{\pi}\left\{\frac{1}{2K-1} + \sum_{n=1}^{\infty}\left[\frac{(2K-1)}{(2K-1)^2-(2n)^2}\right](\sigma^{2n}+\bar{\sigma}^{2n})\right\},\\
&(b)\quad \sigma^{2K}-\bar{\sigma}^{2K} = \frac{4i}{\pi}\sum_{n=1}^{\infty}\frac{2K}{(2K)^2-(2n-1)^2}(\sigma^{2n-1}+\bar{\sigma}^{2n-1}),\\
&(c)\quad \sigma^{2K-1}+\bar{\sigma}^{2K-1} = \frac{4i}{\pi}\sum_{n=1}^{\infty}\frac{2n}{(2K-1)^2-(2n)^2}(\sigma^{2n}-\bar{\sigma}^{2n}),\\
&(d)\quad \sigma^{2K}+\bar{\sigma}^{2K} = \frac{4i}{\pi}\sum_{n=1}^{\infty}\frac{2n-1}{(2K)^2-(2n-1)^2}(\sigma^{2n-1}-\bar{\sigma}^{2n-1}).
\end{aligned}
$$

Properties of infinite series would allow series to be added, with the assurance that the resulting series will also converge in the same region. If we add (a) and (c) from above, we obtain

$$
\begin{aligned}
\sigma^{2K-1} &= \frac{2i}{\pi}\left\{\frac{1}{2K-1} + \sum_{n=1}^{\infty}\left[\frac{2K-1+2N}{(2K-1)^2-(2n)^2}\sigma^{2n} + \frac{2K-1-2n}{(2K-1)^2-(2n)^2}\bar{\sigma}^{2n}\right]\right\},\\
&= \frac{2i}{\pi}\left\{\frac{1}{2K-1} + \sum_{n=1}^{\infty}\left[\frac{\sigma^{2n}}{2K-1-2n} + \frac{\bar{\sigma}^{2n}}{2K-1+2n}\right]\right\}.
\end{aligned}
$$

Consider the series $\sum_{n=1}^{\infty}\frac{1}{(2K-1)-(2n)}z^{2n}$. By Abel's test (4, p. 80), this series will converge at all points on the unit circle except possibly at $z = 1$, since the coefficients of the negative of this series form, after the $2k^{th}$ term, a decreasing sequence of positive terms approaching zero as n increases without limit. A similar statement may be made about the second series given in (3.2).

Since both series converge on the unit circle where $0 < \theta < \pi$, the terms may be separated as long as use of the series is confined to this region. Then

$$
\sigma^{2K-1} = \frac{2i}{\pi}\left\{\frac{1}{2K-1} + \sum_{n=1}^{\infty}\frac{\sigma^{2n}}{2K-1-2n} + \sum_{n=1}^{\infty}\frac{\bar{\sigma}^{2n}}{2K-1+2n}\right\},
$$

$$0 < \theta < \pi$$

If we use $\bar{\sigma} = \sigma^{-1}$ and substitute -n for n in the second summation, we have

$$\sigma^{2K-1} = \frac{2i}{\pi}\left\{\frac{1}{2K-1} + \sum_{n=1}^{\infty}\frac{\sigma^{2n}}{2K-1-2n} + \sum_{n=-1}^{-\infty}\frac{\sigma^{2n}}{2K-1-2n}\right\}.$$

Since $\frac{1}{2K-1} = \left.\frac{\sigma^{2n}}{(2K-1)-(2n)}\right|_{n=0}$, summations may be combined giving

(3.4a) $$\sigma^{2K-1} = \sum_{n=-\infty}^{\infty}\frac{\sigma^{2n}}{(2K-1)-2n}, \qquad 0 < \theta < \pi .$$

Now, adding (b) and (d) of (3.1) produces

$$\sigma^{2K} = \frac{2i}{\pi}\sum_{n=1}^{\infty}\left[\frac{2K+(2n-1)}{(2K)^2-(2n-1)^2}\,\sigma^{2n-1} + \frac{2K-(2n-1)}{(2K)^2-(2n-1)^2}\,\bar{\sigma}^{2n-1}\right]$$

$$= \frac{2i}{\pi}\left[\sum_{n=1}^{\infty}\left[\frac{\sigma^{2n-1}}{2K-(2n-1)} + \frac{\bar{\sigma}^{2n-1}}{2K+(2n-1)}\right]\right].$$

As before, the two series, made up of the first and second terms of each term of this series, each converge in the region $0 < \theta < \pi$. Then the same rearrangement yields

$$\sigma^{2K} = \frac{2i}{\pi}\left[\sum_{n=1}^{\infty}\frac{\sigma^{2n-1}}{2K-(2n-1)} + \sum_{n=1}^{\infty}\frac{\bar{\sigma}^{2n-1}}{2K+(2n-1)}\right].$$

Substituting in the second summation -(2n - 1) for 2n - 1 gives

$$\sigma^{2K} = \frac{2i}{\pi}\left[\sum_{n=1}^{\infty}\frac{\sigma^{2n-1}}{2K-(2n-1)} + \sum_{n=0}^{-\infty}\frac{\sigma^{2n-1}}{2K-(2n-1)}\right],$$

and combining summations obtains

(3.5a) $$\sigma = \frac{2i}{\pi}\sum_{n=-\infty}^{\infty}\frac{\sigma^{2n-1}}{2K-(2n-1)} .$$

From these equations, (3.4a) and (3.5a), it is seen that there exists a relation between power terms in σ on the unit circle.

Since this is the case, then there might be several ways for a function to vanish on the half unit circle. It might vanish because the coefficient of each power vanishes, by certain infinite sums of coefficients vanishing through the relation (3.4a) and (3.5a), or through some other relation between one power of σ and others in an isolated region. The second possibility will be used in the sequel.

CHAPTER IV

THE METHOD

On the half unit circle, it has been shown that there exists a relationship between powers of σ. These relations are

$$(3.4a) \qquad \sigma^{2K-1} = \frac{2i}{\pi}\sum_{n=-\infty}^{\infty} \frac{\sigma^{2n}}{(2K-1)-2n},$$

$$(3.5a) \qquad \sigma^{2K} = \frac{2i}{\pi}\sum_{n=-\infty}^{\infty} \frac{\sigma^{2n-1}}{(2K)-(2n-1)}, \quad 0 < \theta < \pi.$$

The form of the above equations would imply that the imaginary part of the coefficient of odd powers affects the real part of the coefficients of even powers, and, similarly, that the imaginary part of the coefficient of even powers affects the real part of the coefficients of odd powers. This would suggest dividing the expressions which are to be evaluated into three parts, i.e., powers of σ with real coefficients, even powers of σ with imaginary coefficients, and odd powers of σ with imaginary coefficients. Then (3.4a) and (3.5a) can be applied, the imaginary unit therein causing all coefficients involved to become real. Gathering the coefficients of like powers will then produce real coefficient expressions which are usually infinite series. Then, setting each such expression equal to zero, or to corresponding boundary conditions, will remove all stresses, or otherwise match given boundary conditions.

To illustrate the use of (3.4a) and (3.5a), application will be made to the circular notch problem which was started in the second chapter of this work. Since this problem has been solved, this application will serve to corroborate the method, as well as to increase the evidence of the accuracy of the first solution.

This method is applicable to any two-boundary problem in which one boundary is mapped onto the half unit circle. If the boundaries are disjoint, then application becomes very difficult. The easiest and most straightforward application is to boundaries which intersect, such that one can be mapped on the semi-circle and the other on the x-axis. Such is the problem of Chapter VII, which heretofore has been unsolved.

It should be here added that, although the results are usually infinite series in undetermined coefficients, they tend to assume simple forms which possibly could be treated generally to get exact results, or, at least, to corroborate hypotheses concerning general solutions.

CHAPTER V

THE SEMI-CIRCULAR NOTCH

The method just described will now be applied to the circular notch problem which was begun in Chapter II. The complex potentials were restricted in form, by the vanishing of the logarithmic term and the tension at infinity, to

$$\psi = \frac{p}{4} z + \sum_{n=1}^{\infty} a_n z^{-n},$$
$$(5.1) \qquad \chi' = -\frac{p}{2} z + \sum_{n=1}^{\infty} b_n z^{-n}.$$

To remove stress on the x-axis, it was necessary that

$$(2.4) \qquad b = n a_n - \bar{a}_n .$$

The boundary conditions must now be satisfied on the semi-circular arc. First, substituting (2.4) in (5.1) yields

$$\psi = \frac{p}{4} z + \sum_{n=1}^{\infty} a_n z^{-n},$$
$$(5.1a) \qquad \chi' = -\frac{p}{2} z + \sum_{n=1}^{\infty} (n a_n - \bar{a}_n) z^{-n}.$$

Then

$$(1.10) \qquad \psi + z\overline{\psi'} + \overline{\chi'}\Big|_{z=\sigma} = H ,$$

where H is a complex constant.

Inserting (5.1a) in (1.10) gives

$$H = \frac{p}{4} z + \sum_{n=1}^{\infty} a_n z^{-n} + z\left[\frac{p}{4} - \sum_{n=1}^{\infty} n\bar{a}_n \bar{z}^{-n-1}\right]$$

$$- \frac{p}{2}\bar{z} + \sum_{n=1}^{\infty}(n\bar{a}_n - a_n) z^{-n}\Big|_{z=\sigma}$$

$$= \frac{p}{4}\sigma + \sum_{n=1}^{\infty} a_n \sigma^{-n} + \frac{p}{4}\sigma - \sum_{n=1}^{\infty} n a_n \sigma^{n+2}$$

$$- \frac{p}{2}\sigma^{-1} + \sum_{n=1}^{\infty}(n\bar{a}_n - a_n)\sigma^{n} .$$

Now, if we let $a_n = c_n + id_n$, then

$$H = \frac{p}{2}\sigma - \frac{p}{2}\sigma^{-1} + \sum_{n=1}^{\infty}(c_n + id_n)\sigma^{-n} - \sum_{n=1}^{\infty} n(c_n - id_n)\sigma^{n+2}$$

$$+ \sum_{n=1}^{\infty}\left[(n-1)c_n - (n+1)id_n\right]\sigma^{n}$$

$$= \frac{p}{2}\sigma - \frac{p}{2}\sigma^{-1} + \sum_{n=1}^{\infty} c_n \sigma^{-n} - \sum_{n=1}^{\infty} n c_n \sigma^{n+2} + \sum_{n=1}^{\infty}(n+1)c_n \sigma^{n}$$

$$+ \sum_{n=1}^{\infty} id_n \sigma^{-n} + \sum_{n=1}^{\infty} id_n \sigma^{n+2} - \sum_{n=1}^{\infty}(n+1)\sigma^{n}.$$

This separation is permissible since the convergence of a complex series insures the convergence of the series composed of its real and imaginary coefficients, respectively.

In the equation at the top of the next page, the coefficients of some of the powers of σ are gathered, and some of the summations are altered and combined.

$$H = \frac{p}{2}\sigma + \left[-\frac{p}{2} + C_1\right]\sigma^{-1} + C_2\sigma^2 + \sum_{n=2}^{\infty} C_n\sigma^{-n} + \sum_{n=3}^{\infty}\left[(n-1)C_n - (n-2)C_{n-2}\right]\sigma^n$$

$$+\sum_{n=1}^{\infty} i d\,\sigma^{-n} + \sum_{n=3}^{\infty} i\left[-(n+1)d_n + (n-2)d_{n-2}\right]\sigma^n - 2id_1\sigma - 3id_2\sigma^2.$$

Letting the part of the above equation with real coefficient be called $F(\sigma)$, and dividing the imaginary coefficient part into even and odd powers of σ gives

$$H = F(\sigma) + \sum_{n=1}^{\infty} i d_{2n}\sigma^{-2n} + \sum_{n=2}^{\infty} i\left[-(2n+1)d_{2n} + (2n-2)d_{2n-2}\right]\sigma^{2n} - 3id_2\sigma^2$$

$$+\sum_{n=1}^{\infty} i d_{2n-1}\sigma^{-2n+1} + \sum_{n=2}^{\infty} i\left[-2nd_{2n-1} + (2n-3)d_{2n-3}\right]\sigma^{2n-1} - 2id_1\sigma.$$

The preceeding expression is now in convenient form to apply formulas (3.4a) and (3.5a) to obtain

$$H = F(\sigma) + \sum_{n=1}^{\infty} i d_{2n}\sum_{K=-\infty}^{\infty}\left(\frac{2i}{\pi}\right)\frac{\sigma^{2K-1}}{-2n-(2K-1)} - 3id_2\left(\frac{2i}{\pi}\right)\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{2-(2K-1)}$$

$$+\sum_{n=2}^{\infty} i\left[-(2n+1)d_{2n} + (2n-2)d_{2n-2}\right]\left(\frac{2i}{\pi}\right)\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{2n-(2K-1)}$$

$$+\sum_{n=1}^{\infty} i d_{2n-1}\left(\frac{2i}{\pi}\right)\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{-(2n-1)-2K} - 2id_1\left(\frac{2i}{\pi}\right)\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{1-2K}$$

$$+\sum_{n=2}^{\infty} i\left[-2nd_{2n-1} + (2n-3)d_{2n-3}\right]\left(\frac{2i}{\pi}\right)\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{2n-1-2K}$$

$$= F(\sigma) + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty} d_{2n}\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{2n+2K-1} + 3d_2\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{2-(2K-1)}\right.$$

$$+\sum_{n=2}^{\infty}\left[(2n-1)d_{2n} - (2n-2)d_{2n-2}\right]\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{2n-(2K-1)}$$

$$+\sum_{n=1}^{\infty} d_{2n-1}\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{(2n-1)+2K} + 2d_1\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{1-2K}$$

$$\left.+\sum_{n=1}^{\infty}\left[2nd_{2n-1} - (2n-3)d_{2n-3}\right]\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{2n-1-2K}\right\}.$$

The expression now has all real coefficients. By referring back to the value of $F(\sigma)$,

$$F(\sigma)=\frac{p}{2}\sigma+\left[-\frac{p}{2}+C_1\right]\sigma^{-1}+C_2\sigma^2+\sum_{n=2}^{\infty}C_n\sigma^{-n}+\sum_{n=3}^{\infty}\left[(n-1)C_n-(n-2)C_{n-2}\right]\sigma^n,$$

it is possible to pick out coefficients of various powers of σ, and to set each one equal to zero, since the left member of the equation, H, is a complex constant.

The coefficient of the first power of σ is

$$\frac{p}{2}+\frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{d_{2n}}{2n+1}+\sum_{n=2}^{\infty}\left[(2n+1)d_{2n}-(2n-2)d_{2n-2}\right]\frac{1}{2n-1}\right.$$

$$\left.+\frac{3d_2}{2-1}\right\}$$

$$=\frac{p}{2}+\frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{d_{2n}}{2n+1}+\sum_{n=2}^{\infty}\frac{(2n+1)d_{2n}}{2n-1}-\sum_{n=1}^{\infty}\frac{(2n)d_{2n}}{2n+2-1}\right.$$

$$\left.+\frac{3d_2}{2-1}\right\}$$

$$=\frac{p}{2}+\frac{2}{\pi}\left\{\sum_{n=2}^{\infty}\left[\frac{1}{2n+1}+\frac{2n+1}{2n-1}-\frac{2n}{2n+2-1}\right]d_{2n}\right.$$

$$\left.+\left[\frac{3}{1}+\frac{1}{3}-\frac{2}{3}\right]d_2\right\}$$

$$=\frac{p}{2}+\frac{2}{\pi}\left\{\frac{8}{3}d_2+\sum_{n=2}^{\infty}\frac{4(2n)}{(2n)^2-1}d_{2n}\right\}$$

$$(5.2)\qquad =\frac{p}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4(2n)}{(2n)^2-1}d_{2n}$$

$$=0\,.$$

The coefficients of σ^3 and all higher odd powers take the form for the $(2k - 1)^{th}$ power:

$$0 = (2K-2)C_{2K-1} - (2K-3)C_{2K-3} + \frac{2}{\pi}\left\{\frac{3d_2}{2-(2K-1)} + \sum_{n=1}^{\infty}\frac{d_{2n}}{2n+(2K-1)}\right.$$

$$\left. + \sum_{n=2}^{\infty}\left[\frac{(2n+1)d_{2n} - (2n-2)d_{2n-2}}{2n-(2K-1)}\right]\right.$$

$$= (2K-2)C_{2K-1} - (2K-3)C_{2K-3} + \frac{2}{\pi}\left\{\left[\frac{3}{2-(2K-1)} + \frac{1}{2+(2K-1)}\right]d_2\right.$$

$$\left. + \sum_{n=2}^{\infty}\left[\frac{1}{2n+2K-1} - \frac{2n-1}{2n-(2K-1)}\right]d_{2n} - \sum_{n=1}^{\infty}\frac{2n\, d_{2n}}{2n+2-(2K-1)}\right\}$$

$$= (2K-2)C_{2K-1} - (2K-3)C_{2K-3} + \frac{2}{\pi}\left\{\frac{24\, d_2}{[4-(2K-1)^2][4-(2K-1)]}\right.$$

$$\left. + \sum_{n=2}^{\infty}\frac{4(2n)(2n+1)d_{2n}}{[(2n)^2-(2K-1)^2][2n+2-(2K-1)]}\right.$$

(5.3)
$$= (2K-2)C_{2K-1} - (2K-3)C_{2K-3} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4(2n)(2n+1)d_{2n}}{[(2n)^2-(2K-1)^2][2n+2-(2K-1)]},$$

$$K = 2, 3, 4, \cdots.$$

The coefficient of σ^{-1} is

$$0 = -\frac{P}{2} + C_1 + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{d_{2n}}{2n-1} + \sum_{n=2}^{\infty}\frac{(2n-1)d_{2n} - (2n-2)d_{2n-2}}{2n+1} + \frac{3d_2}{2+1}\right\}$$

$$= -\frac{P}{2} + C_1 + \frac{2}{\pi}\left\{\sum_{n=2}^{\infty}\left[\frac{1}{2n-1} + \frac{2n+1}{2n+1}\right]d_{2n} - \sum_{n=1}^{\infty}\frac{2n\, d_{2n}}{2n+2+1} + \left[\frac{3}{3} + \frac{1}{1}\right]d_2\right\}$$

$$= -\frac{P}{2} + C_1 + \frac{2}{\pi}\left\{\frac{8d_2}{5} + \sum_{n=2}^{\infty}\left[\frac{1}{2n-1} + 1 - \frac{2n}{2n+3}\right]d_{2n}\right\}$$

$$= -\frac{P}{2} + C_1 + \frac{2}{\pi}\left\{\frac{8d_2}{5} + \sum_{n=2}^{\infty}\frac{4(2n)d_{2n}}{(2n-1)(2n-3)}\right\}$$

(5.4)
$$= -\frac{P}{2} + C_1 + \frac{2}{\pi}\left\{\sum_{n=2}^{\infty}\frac{4(2n)d_{2n}}{(2n-1)(2n-3)}\right\}.$$

The coefficient of σ^{2K-1}, when $k > 1$, is

$$0 = C_{2K-1} + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{d_{2n}}{2n-(2K-1)} + \sum_{n=2}^{\infty}\frac{(2n+1)d_{2n}-(2n-2)d_{2n-2}}{2n+2K-1} + \frac{3d_2}{2+2K-1}\right\}$$

$$= C_{2K-1} + \frac{2}{\pi}\left\{\left[\frac{3}{2+(2K-1)} + \frac{1}{2-(2K-1)} - \frac{2}{4+(2K-1)}\right]d_2 + \sum_{n=2}^{\infty}\left[\frac{1}{2n-(2K-1)} - \frac{2n+1}{2n+(2K-1)}\right]d_{2n} - \sum_{n=2}^{\infty}\frac{2n\,d_{2n}}{2n+2+(2K-1)}\right\}$$

$$= C_{2K-1} + \frac{2}{\pi}\left\{\frac{24d_2}{[4-(2K-1)^2][4+2K-1]} + \sum_{n=2}^{\infty}\frac{4(2n)(2n+1)d_{2n}}{[(2n)^2-(2K-1)][2n+2+2K-1]}\right\}$$

(5.5) $$= C_{2K-1} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{4(2n)(2n+1)d_{2n}}{[(2n)^2-(2K-1)^2][2n+2+2K-1]}, \quad K = 2, 3, 4, \cdots.$$

It is seen that the equations (5.2) through (5.5) contain only even numbered d's and odd numbered c's. The coefficient equations obtained from the even powers of σ contain only odd numbered d's and even numbered c's. Furthermore, these equations contain no constants, and therefore, form a homogeneous system of equations. On this basis, we may solve all that system merely be letting all odd d and even c coefficients be equal to zero. The solutions thus obtained agrees with the solution of this problem by Maunsell (5) as is shown in Chapter VII.

A system of forty equations in forty of these coefficients was composed in the following order:

1st and 2nd equations from (5.4) and (5.2) respectively,

3rd, 5th, 7th,···, 39th from equation (5.5),

4th, 6th, 8th,···, 40th from equation (5.3) .

This order produces a system with the coefficient matrix diagonal that is large compared to the rest of the matrix. Its simplicity of form allows rapid machine computation, and the heavy diagonal greatly improves accuracy. This system was solved to eight significant digits on an IBM 650 calculator in the Statistical Laboratory of the University of Florida. Below are the first thirty rounded coefficients of ψ and χ'.

TABLE I

COEFFICIENTS OF ψ AND χ'

n	ψ	χ'
1........	+0.7924848	+0.0000000
2........	- .3314121i	- .9942363i
3........	- .2173459	- .4346918
4........	+ .0847689i	+ .4238446i
5........	+ .0088164	+ .0352655
6........	+ .0095928i	+ .0671493i
7........	+ .0033255	+ .0199532
8........	+ .0022085i	+ .0198764i
9........	+ .0015117	+ .0120935
10........	+ .0006217i	+ .0068387i
11........	+ .0007765	+ .0077650
12........	+ .0001772i	+ .0023031i
13........	+ .0004341	+ .0052094
14........	+ .0000637i	+ .0009553i
15........	+ .0003296	+ .0046145
16........	- .0000726i	- .0012347i
17........	+ .0001624	+ .0025988
18........	- .0000550i	- .0010443i
19........	+ .0001077	+ .0019383
20........	- .0000525i	- .0011027i
21........	+ .0000694	+ .0013876
22........	- .0000435i	- .0009998i
23........	+ .0000493	+ .0010836
24........	- .0000383i	- .0009577i
25........	+ .0000349	+ .0008364
26........	- .0000333i	- .0008997i
27........	+ .0000251	+ .0006535
28........	- .0000292i	- .0008456i
29........	+ .0000184	+ .0005160
30........	- .0000259i	- .0008029i

Figuring the stresses at $z = i$ with these values yields

$$\sigma_x = +3.073,$$

$$\sigma_y = -0.0083,$$

$$\tau_{xy} = 0.$$

These results agree quite well with those of Maunsell (5) and Isibasi (3). They also indicate that Frocht (2) may have been correct in that his results indicate that σ_x at $z = i$ slightly exceeds three.

There might be some doubt as to the use of equation (1.10) in removing stresses from the semi-circle, since the actual stresses depend on higher derivatives of ψ and χ. To satisfy such question, formulas (3.4a) and (3.5a) will now be applied directly to the stresses on the notch.

Here it is most convenient to change to polar coordinates; since, on the notch, the radial stress is σ_r and the tangential stress is $\tau_{r\theta}$. The relation used at this point is (7, p. 138)

$$\sigma_r - i\tau_{r\theta} = \psi' + \overline{\psi'} - e^{2i\theta}\left[\bar{z}\psi'' + \chi''\right]. \tag{5.6}$$

On the curve in question, $z = \sigma = e^{i\theta}$, $e^{2i\theta} = \sigma^2$, $\sigma_r = 0$, and $\tau_{r\theta} = 0$. Then equation (5.6) becomes

$$0 = \psi' + \psi' - \sigma^2\left[\sigma^{-1}\psi'' + \chi''\right]. \tag{5.7}$$

where ψ and χ' are given by (5.1). Substituting (5.1) in (5.7) gives

$$0 = \frac{p}{4} - \sum_{n=1}^{\infty} n a_n \sigma^{-n-1} + \frac{p}{4} - \sum_{n=1}^{\infty} n \bar{a}_n \sigma^{n+1}$$

$$- \sigma^2 \left[\sigma^{-1} \sum_{n=1}^{\infty} a_n (n)(n+1) \sigma^{-n-2} - \frac{p}{2} - \sum_{n=1}^{\infty} n(n a_n - \bar{a}_n) \sigma^{-n-1} \right]$$

$$= \frac{p}{2} + \frac{p}{2}\sigma^2 - \sum_{n=1}^{\infty} n a_n \sigma^{-n-1} - \sum_{n=1}^{\infty} n \bar{a}_n \sigma^{n+1}$$

$$- \sum_{n=1}^{\infty} n(n+1) a_n \sigma^{-n-1} + \sum_{n=1}^{\infty} n(n a_n - \bar{a}_n) \sigma^{-n+1}$$

$$= \frac{p}{2} + \frac{p}{2}\sigma^2 - \sum_{n=1}^{\infty} n a_n \sigma^{n+1} - \sum_{n=1}^{\infty} \left[n a_n + n(n+1) a_n \right] \sigma^{-n-1}$$

$$+ \sum_{n=-1}^{\infty} (n+2) \left[(n+2) a_{n+2} - \bar{a}_{n+2} \right] \sigma^{-n-1}$$

$$= \frac{p}{2} + \frac{p}{2}\sigma^2 - \sum_{n=1}^{\infty} \left[n a_n + n(n+1) a_n - (n+2)^2 a_{n+2} + (n+2) \bar{a}_{n+2} \right] \sigma^{-n-1}$$

$$+ (a_1 - \bar{a}_1) + 2(2a_2 - \bar{a}_2)\sigma^{-1}.$$

Setting $a_n = c_n + id_n$, and separating real and imaginary coefficients, we have

$$0 = \left\{ \frac{p}{2} + \frac{p}{2}\sigma^2 - \sum_{n=1}^{\infty} n c_n \sigma^{n+1} + 2c_2 \sigma^{-1} \right.$$

$$\left. - \sum_{n=1}^{\infty} \left[n(n+2) c_n + (n+2)(-n-1) c_{n+2} \right] \sigma^{-n-1} \right\}$$

$$+ \left\{ 2id_1 + 6id_2 \sigma^{-1} + \sum_{n=1}^{\infty} n i d_n \sigma^{n+1} \right.$$

$$\left. - \sum_{n=1}^{\infty} \left[n(n+2) i d_n - (n+2)(n+3) i d_{n+2} \right] \sigma^{-n-1} \right\}.$$

The real coefficient part above will be denoted by $G(\sigma)$, and the imaginary coefficients will be separated into even and odd powers of σ. Then

$$0 = G(\sigma) + i\sum_{n=1}^{\infty}(2n-1)d_{2n-1}\sigma^{2n} - i(2d_1)$$
$$-i\sum_{n=1}^{\infty}\left[(2n-1)(2n+1)d_{2n-1} - (2n+1)(2n+2)d_{2n+1}\right]\sigma^{-2n}$$
$$+i\sum_{n=1}^{\infty}2nd_{2n}\sigma^{2n+1} + 6id_2\sigma^{-1}$$
$$-i\sum_{n=1}^{\infty}\left[2n(2n+2)d_{2n} - (2n+2)(2n+3)d_{2n+2}\right]\sigma^{-2n-1}.$$

Now formulas (3.4a) and (3.5a) may be applied to give

$$0 = G(\sigma) + \frac{2}{\pi}\left\{-\sum_{n=1}^{\infty}(2n-1)d_{2n-1}\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{2n-(2K-1)} - 2d_1\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{-(2K-1)}\right.$$
$$-\sum_{n=1}^{\infty}\left[(2n-1)(2n+1)d_{2n-1} - (2n+1)(2n+2)d_{2n+1}\right]\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K-1}}{2n+2K-1}$$
$$-\sum_{n=1}^{\infty}2nd_{2n}\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{2n+1-2K} + 6d_2\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{1+2K}$$
$$\left.-\sum_{n=1}^{\infty}\left[2n(2n-2)d_{2n} - (2n+2)(2n+3)d_{2n+2}\right]\sum_{K=-\infty}^{\infty}\frac{\sigma^{2K}}{2n+1+2K}\right\}.$$

Satisfying this equation removes stresses on the notch.

Next we equate the coefficients of the various powers to zero, where

$$G(\sigma) = \frac{p}{2} + \frac{p}{2}\sigma^2 - \sum_{n=1}^{\infty}nc_n\sigma^{n+1} + 2c_2\sigma^{-1}$$
$$-\sum_{n=1}^{\infty}\left[n(n+2)c_n + (n+2)(-n-1)C_{n+2}\right]\sigma^{-n-1}.$$

For the constant term, we get

$$0 = \frac{p}{2} + \sum_{n=1}^{\infty} 2n d_{2n} \left(\frac{2}{\pi}\right) \frac{-1}{2n+1} \quad + \frac{2}{\pi} 6 d_2$$

$$- \frac{2}{\pi} \sum_{n=1}^{\infty} \left[2n(2n+2) d_{2n} - (2n+2)(2n+3) d_{2n+2}\right] \frac{1}{2n+1}$$

$$= \frac{p}{2} + \frac{2}{\pi} \left\{ 6d_2 + \sum_{n=1}^{\infty} \left[\frac{-2n}{2n+1} - \frac{2n(2n+2)}{2n+1}\right] d_{2n} + \sum_{n=2}^{\infty} \frac{2n(2n+1)}{2n-1} d_{2n} \right\}$$

$$= \frac{p}{2} + \frac{2}{\pi} \left\{ \frac{8}{3} d_2 + \sum_{n=2}^{\infty} \frac{4(2n)}{(2n)^2 - 1} d_{2n} \right\}$$

$$= \frac{p}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{4(2n)}{(2n)^2 - 1} d_{2n} \;,$$

which is exactly equation (5.2).

The coefficient of the second power gives

$$0 = \frac{p}{2} - C_1 + \frac{2}{\pi} \left\{ 2d_2 - \sum_{n=1}^{\infty} \frac{2n d_{2n}}{2n-1} \right.$$

$$\left. - \sum_{n=1}^{\infty} \left[2n(2n+2) d_{2n} - (2n+2)(2n+3) d_{2n+2}\right] \frac{1}{2n+3} \right\}$$

$$= \frac{p}{2} - C_1 + \frac{2}{\pi} \left\{ 2d_2 + \sum_{n=1}^{\infty} \left[\frac{-2n}{2n-1} + \frac{-2n(2n+2)}{2n+3}\right] d_{2n} \right.$$

$$\left. + \sum_{n=2}^{\infty} \frac{(2n)(2n+1)}{2n+1} d_{2n} \right\}$$

$$= \frac{p}{2} - C_1 + \frac{2}{\pi} \left\{ -\frac{8}{5} d_2 - \sum_{n=2}^{\infty} \frac{4(2n) d_{2n}}{(2n-1)(2n+3)} \right\}$$

$$= -\frac{p}{2} + C_1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{4(2n) d_{2n}}{(2n-1)(2n+3)} \;.$$

The preceeding equation is exactly equation (5.4). Likewise, the other coefficients yield the same equations. Therefore, the formulas (3.4a) and (3.5a) may be used either to remove resultant force, or to remove directly stresses on a boundary with identical results.

We next apply these formulas to a problem which has not been solved heretofore.

CHAPTER VI

THE SEMI-ELLIPTICAL NOTCH

The Mapping

One of the advantages of the use of complex variables is that regions which are difficult to handle may be mapped conformally onto regions which are easily handled. To solve the problem of a semi-infinite sheet with a semi-elliptical notch under uniform tension, the notch will be mapped on the half unit circle. The mapping function which will be used is

$$z = c\zeta + \frac{e}{\zeta}, \tag{6.1}$$

where $\zeta = \xi + i\eta$, $c = \frac{a+b}{2}$, and $e = \frac{a-b}{2}$.

The curve $\zeta = \sigma$ is the unit circle in the ζ-plane. Then

$$z = \frac{a+b}{2}\sigma + \frac{a-b}{2}\sigma^{-1}$$

$$= a\frac{\sigma+\sigma^{-1}}{2} + b\frac{\sigma-\sigma^{-1}}{2}$$

$$= a\cos\theta + bi\sin\theta\,.$$

This is an ellipse with major axis 2a and minor axis 2b situated as in Figure 2 on the next page. The points $\zeta = \pm 1$ give $z = \pm a$, so that the semi-ellipse maps onto the semi-circle. Moreover, $\zeta = +i$

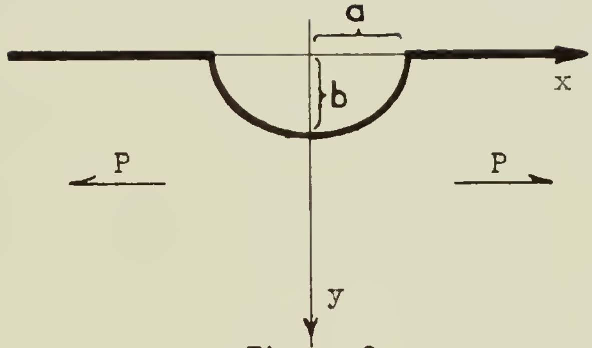

Figure 2

gives $z = ib$, showing the half of the ellipse with positive η maps onto the part of the circle with positive y. Then this mapping transforms the semi-ellipse exactly as desired.

The following theorem as stated by Townsend is useful in analyzing the mapping function $\omega(\zeta)$.

> Given a multiple-valued analytic function $w = f(z)$ whose inverse function $z = \phi(w)$ is single-valued. If the derived function $\phi'(w)$ has a zero of order k, or a pole of order $k + 2$ at w_0, then $f(z)$ has a branch point of order k at the corresponding point z_0. Also if $w = \infty$ is a pole of order k of $\phi'(w)$, then the corresponding point of the z-plane is a branch point of $f(z)$ of order k. (10, p. 335)

Now $z = c\zeta + e\zeta^{-1}$ is single-valued, and its derived function $\omega' = c - e\zeta^{-2}$ has a pole of order 2 at $\zeta = 0$ and zeros of order one at $\zeta = \pm\sqrt{e/c}$. Therefore, by the theorem, $\zeta = f(z)$ has branch points at the z values corresponding to $\zeta = \pm\sqrt{e/c}$. To find these points, the ζ values are substituted in (6.1).

$$z = c\sqrt{\frac{e}{c}} + e\sqrt{\frac{c}{e}}$$

$$= 2\sqrt{ce}$$

$$= 2\sqrt{\frac{a^2 - b^2}{4}}$$

$$= 2\sqrt{a^2 - b^2}\,.$$

Likewise, $\zeta = -\sqrt{\frac{e}{c}}$ yields the value $z = -\sqrt{a^2 - b^2}$. These are the well-known values for the foci. They may be joined by a branch cut, leaving the function holomorphic in the plane. These branch points lie outside the region of the body in question, and, therefore, cause no difficulty in this problem.

Before proceeding, the x-axis image must be found in the ζ-plane. To do this, equation (6.1) is solved for corresponding values in the ζ-plane to $z = x + i0$.

$$x = c(\xi + i\eta) + \frac{e}{\xi + i\eta}$$

$$= c\xi + \frac{e\xi}{\xi^2 + \eta^2} + \left(c - \frac{e}{\xi^2 + \eta^2}\right) i\eta .$$

Equating real and imaginary coefficients gives

$$c\xi + \frac{e\xi}{\xi^2 + \eta^2} = x ,$$

$$\left(c - \frac{e}{\xi^2 + \eta^2}\right)\eta = 0 .$$

The above equation is satisfied if $\eta = 0$. Then

$$c\xi + e\xi^{-1} = x ,$$

and x can take on any value numerically greater than $\sqrt{\frac{e}{c}}$ so that $\eta = 0$ obtains all values of x outside the foci of the ellipse.

If $c - e/\xi^2 + \eta^2 = 0$, the relations (6.2) are also satisfied. Then

$$c\xi^2 + c\eta^2 - e = 0 ,$$

$$\xi^2 + \eta^2 = \frac{e}{c} .$$

This evidently yields a circle of radius $\sqrt{\frac{e}{c}}$ in the ζ-plane. The corresponding value of x would be

$$x = \left(c + \frac{e}{e/c}\right)\xi$$

$$= 2c\xi .$$

But

$$|\xi| \leq \sqrt{\frac{e}{c}} ,$$

so that

$$|x| \leq \sqrt{ce}\,(2) = \sqrt{a^2 - b^2} .$$

The above values of x fall between the foci of the ellipse and out of the region of consideration. Then the part of the x-axis on which the boundary lies is given by the ξ-axis in the ζ-plane.

The mapping is now ready to be used, but it is helpful to consider, before proceeding, the form that equation (1.10) takes in a mapped region.

Equation (1.10) in the ζ-Plane

The formula (1.10) will be used as before, although its form will be somewhat altered by the mapping. In the z-plane it takes the form below in which subscripts have been added to simplify succeeding work.

(1.10a) $$\psi_1(z) + z\overline{\psi_1'(z)} + \overline{\chi_1'(z)} = H ,$$

where H is a complex constant.

Now $z = \omega(\zeta)$ so that (1.10a) becomes

$$\psi_1(\omega(\zeta)) + \omega(\zeta)\,\overline{\frac{d\psi_1(\omega(\zeta))}{dz}} + \overline{\chi_1'(\omega(\zeta))} = H\,,$$

$$\psi(\zeta) + \omega(\zeta)\,\overline{\psi'(\zeta)}\,\overline{\frac{d\omega(\zeta)}{dz}} + \overline{\chi'(\zeta)} = H\,,$$

(6.2)
$$\psi(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\psi'(\zeta)} + \overline{\chi'(\zeta)} = H\,.$$

This form will be useful after the mapping has been applied.

The Complex Potentials in the ζ-Plane

Since the tension at infinity and the conditions on the x-axis are the same as were taken for the circular notch, the complex potentials take the form in the z-plane

(2.1b)
$$\psi_1 = A \operatorname{Log} z + \frac{P}{4} z + \sum_{n=1}^{\infty} a_n z^{-n}\,,$$

(2.2b)
$$\chi_1' = -\bar{A} \operatorname{Log} z - \frac{P}{2} z + \sum_{n=1}^{\infty} (n a_n - \bar{a}_n) z^{-n}\,.$$

Since the boundary of the problem is symmetrical about the y-axis and (1.10a) holds, then the logarithmic term must vanish as it did in the circular notch. Then the equations (2.1b) and (2.2b) simplify to

$$\psi_1 = \frac{P}{4} z + \sum_{n=1}^{\infty} a_n z^{-n}\,,$$

$$\chi_1' = -\frac{P}{2} z + \sum_{n=1}^{\infty} (n a_n - \bar{a}_n) z^{-n}\,.$$

After mapping the functions, we have

(6.3)
$$\psi = \frac{Pc}{4}\zeta + \sum_{n=1}^{\infty} a_n \zeta^{-n}\,,$$

$$\chi' = -\frac{Pc}{2}\zeta + \sum_{n=1}^{\infty} b_n \zeta^{-n}\,.$$

The first degree terms arise from the first degree terms of the preceeding forms. All other positive powers vanish since in the ζ-plane the stresses are bounded at infinity. The other coefficients are difficult to ascertain in terms of the original z-plane coefficients, so the solution will be carried out in the ζ-plane. To find the stresses in such a solution, the ζ value corresponding to a desired z value is found, and this is used in the ζ-plane solution.

The Problem

Now the preceeding work will be applied to the elliptical notch. It has been shown that the ξ-axis corresponds to the x-axis. Stresses will first be removed from this axis. In the application of (6.2), use must be made of

$$\left.\frac{\overline{\omega(\zeta)}}{\overline{\omega'(\zeta)}}\right|_{\xi} = \frac{c\xi + \frac{e}{\xi}}{c - \frac{e}{\xi^2}}$$

$$= \frac{c\xi^3 + e\xi}{c\xi^2 - e}$$

$$= \xi + 2\lambda\xi^{-1} + 2\lambda\xi^{-3} + 2\lambda\xi^{-4} + \cdots$$

$$= \xi \sum_{n=0}^{\infty} D_n \xi^{-n}, \tag{6.4}$$

where

$$\begin{cases} D_0 = 1, & \\ D_{2n} = 2\lambda, & \lambda = \frac{e}{c}, \\ D_{2n-1} = 0. & \end{cases}$$

Substituting (6.3) and (6.4) in (6.2) and setting $\zeta = \xi$ gives

$$H = \frac{p_c}{4}\xi + \sum_{n=1}^{\infty} a_n \xi^{-n} + \left[\xi \sum_{n=0}^{\infty} D_n \xi^{-n}\right]\left[\frac{p_c}{4} - \sum_{n=1}^{\infty} n\bar{a}_n \xi^{-n-1}\right]$$
$$- \frac{p_c}{2}\xi + \sum_{n=1}^{\infty} \bar{b}_n \xi^{-n}.$$

If we take the series product by the Cauchy method and combine terms, we have

$$H = -\frac{p_c}{4}\xi + \sum_{n=1}^{\infty}(a_n + \bar{b}_n)\xi^{-n} + \sum_{n=0}^{\infty} D_n \frac{p_c}{4}\xi^{-n+1}$$
$$-\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} k a_k D_{n-k}\right]\xi^{-n}.$$

where H is a complex constant. Simplifying yields

$$H = -\frac{p_c}{4}\xi + \sum_{n=1}^{\infty}\left[a_n + \bar{b}_n + D_{n+1}\left(\frac{p_c}{4}\right) - \sum_{k=1}^{n} k\bar{a}_k D_{n-k}\right]\xi^{-n} + \frac{p_c}{4}D_0\xi$$
$$= \sum_{n=1}^{\infty}\left[a_n + \bar{b}_n + D_{n+1}\left(\frac{p_c}{4}\right) - \sum_{k=1}^{n} k\bar{a}_k D_{n-k}\right]\xi^{-n}.$$

There is no constant term so that H is zero as before. Equating the coefficient of each power to zero gives

$$0 = a_n + \bar{b}_n + D_{n+1}\left(\frac{p_c}{4}\right) - \sum_{k=1}^{n} k\bar{a}_k D_{n-k}, \quad n = 1, 2, 3, \cdots.$$

If the coefficients conform to the above relation, then the ξ-axis will be free of stress. Thus stresses are removed from the ξ-axis if

$$b_n = \sum_{k=1}^{n} k a_k D_{n-k} - \bar{a}_n - D_{n+1}\left(\frac{p_c}{4}\right), \quad n = 1, 2, 3, \cdots. \tag{6.5}$$

Consideration must be made of the curve $\zeta = \sigma$. Along this curve, it will be necessary to use

$$\left.\frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\right|_{\sigma} = \frac{c\sigma + e\sigma^{-1}}{c + e\bar{\sigma}^{-2}}$$

$$= \frac{e\sigma^{-1} + c\sigma}{c - e\sigma^{2}}$$

$$= \lambda\sigma^{-1} + (1+\lambda^2)\sigma + \lambda(1+\lambda^2)\sigma^3 + \cdots$$

(6.6) $$= \frac{1}{\sigma}\sum_{n=0}^{\infty} B_n\sigma^n,$$

where

$$\begin{cases} B_0 = \lambda, & \lambda = \frac{e}{c}, \\ B_{2n} = \lambda^{n-1}(1+\lambda^2), & n > 0, \\ B_{2n-1} = 0, & n > 0. \end{cases}$$

Setting $\zeta = \sigma$ and substituting (6.3) and (6.6) in (6.2) leads to

$$0 = \frac{pc}{4}\sigma + \sum_{n=1}^{\infty} a_n\sigma^{-n} + \left[\frac{1}{\sigma}\sum_{n=0}^{\infty} B_n\sigma^n\right]\left[\frac{pc}{4} - \sum n\bar{a}_n\sigma^{n+1}\right]$$

$$- \frac{pc}{2}\sigma^{-1} + \sum_{n=1}^{\infty} \bar{b}_n\sigma^n$$

$$= \frac{pc}{4}\sigma - \frac{pc}{2}\sigma^{-1} + \sum_{n=1}^{\infty} a_n\sigma^{-n} + \frac{pc}{4}\sum_{n=0}^{\infty} B_n\sigma^{n-1}$$

$$- \left[\sum_{n=0}^{\infty} B_n\sigma^n\right]\left[\sum_{n=1}^{\infty} n\bar{a}_n\sigma^n\right] + \sum_{n=1}^{\infty} \bar{b}_n\sigma^n.$$

Again taking the Cauchy product and simplifying gives

$$0 = \frac{Pc}{4}\sigma - \frac{Pc}{2}\sigma^{-1} + \sum_{n=1}^{\infty} a_n \sigma^{-n} + \sum_{n=1}^{\infty}[\bar{b}_n + (\tfrac{Pc}{4})B_{n+1}]\sigma^n$$

$$+ \frac{Pc\lambda}{4}\sigma^{-1} - \sum_{n=1}^{\infty}[\sum_{k=1}^{n} k\bar{a}_k B_{n-k}]\sigma^n$$

$$= \frac{Pc}{4}\sigma + [\frac{Pc\lambda}{4} - \frac{Pc}{2}]\sigma^{-1} + \sum_{n=1}^{\infty} a_n \sigma^{-n}$$

$$+ \sum_{n=1}^{\infty}[\bar{b} + \frac{Pc}{4}B_{n+1} - \sum_{k=1}^{n} k\bar{a}_k B_{n-k}]\sigma^n \quad ,$$

and substituting equation (7.4) into this equation gives

$$0 = \frac{Pc}{4}\sigma + [\frac{Pc\lambda}{4} - \frac{Pc}{2}]\sigma^{-1} + \sum_{n=1}^{\infty} a_n \sigma^{-n}$$

$$+ \sum_{n=1}^{\infty}[\sum_{k=1}^{n} k\bar{a}_k(D_{n-k} - B_{n-k}) - a_n + \frac{Pc}{4}(B_{n+1} - D_{n+1})]\sigma^n .$$

Into the above expression is substituted $a_n = c_n + id_n$ and the part with the real coefficients is written first to get

$$0 = \frac{Pc}{4}\sigma + [\frac{Pc\lambda}{4} - \frac{Pc}{2}]\sigma^{-1} + \sum_{n=1}^{\infty} c_n \sigma^{-n}$$

$$+ \sum_{n=1}^{\infty}[\sum_{k=1}^{n} kc_k(D_{n-k} - B_{n-k}) - c_n + \frac{Pc}{4}(B_{n+1} - D_{n+1})]\sigma^n$$

$$+ \sum_{n=1}^{\infty} id_n \sigma^{-n} + \sum_{n=1}^{\infty} i[\sum_{k=1}^{n}(-kd_k)(D_{n-k} - B_{n-k}) - d_n]\sigma^n .$$

Let the part with real coefficients be called $g(\sigma)$ and let the even and odd powers on the remaining part be separated. Then

$$0 = g(\sigma) + \sum_{n=1}^{\infty} i d_{2n} \sigma^{-2n} + \sum_{n=1}^{\infty} i \left[\sum_{k=1}^{2n} (-k d_k)(D_{2n-k} - B_{2n-k}) - d_{2n}\right] \sigma^{2n}$$
$$+ \sum_{n=1}^{\infty} i d_{2n-1} \sigma^{-2n+1} + \sum_{n=1}^{\infty} i \left[\sum_{k=1}^{2n-1} (-k d_k)(D_{2n-1-k} - B_{2n-1-k}) - d_{2n-1}\right] \sigma^{2n-1}.$$

The equation is now in proper form for the application of formulas (3.4a) and (3.5a).

$$(6.7) \quad \begin{aligned} 0 = \; & g(\sigma) + \frac{2}{\pi} \left\{ \sum_{n=1}^{\infty} d_{2n} \sum_{m=-\infty}^{\infty} \frac{\sigma^{2m-1}}{2n + 2m - 1} \right. \\ & + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{2n} k d_k (D_{2n-k} - B_{2n-k}) + d_{2n}\right] \sum_{m=-\infty}^{\infty} \frac{\sigma^{2m-1}}{2n - (2m-1)} \\ & + \sum_{n=1}^{\infty} d_{2n-1} \sum_{m=-\infty}^{\infty} \frac{\sigma^{2m}}{2n - 1 + 2m} \\ & \left. + \sum_{n=1}^{\infty} \left[\sum_{k=1}^{2n-1} k d_k (D_{2n-1-k} - B_{2n-1-k}) + d_{2n-1}\right] \sum_{k=-\infty}^{\infty} \frac{\sigma^{2m}}{2n - 1 - 2m} \right\}, \end{aligned}$$

where

$$g(\sigma) = \frac{p_c}{4}\sigma + \left[\frac{p_c \lambda}{4} - \frac{p_c}{2}\right]\sigma^{-1} + \sum_{n=1}^{\infty} c_n \sigma^{-n}$$
$$+ \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} k c_k (D_{n-k} - B_{n-k}) - c_n + \frac{p_c}{4}(B_{n+1} - D_{n+1})\right] \sigma^{n}.$$

From (6.7) the coefficients may be taken to form the final equations. As in the circular notch, the coefficients of even powers yield a homogeneous system in odd d's and even c's. Moreover, these numbered c's and d's do not occur in the coefficients of odd powers which have only even d's and odd c's. Therefore, all even c's and odd d's are set equal to zero, and that much of the system is determined.

In simplifying the coefficient equations, the following information, taken from (6.4) and (6.6), is needed:

$$D_m - B_m = \begin{cases} 0\,, & m = 1,3,5,\cdots, \\ 1-\lambda\,, & m = 0, \\ -\lambda^{\frac{m}{2}-1}(1-\lambda)^2\,, & m = 2,4,6,\cdots. \end{cases}$$

Setting the coefficients of the odd powers equal to zero gives the following equations:

$$1^{st} \quad 0 = \frac{p_c}{4} + \frac{p_c}{4}(B_2 - D_2) - c_1 + c_1(D_0 - B_0) + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty} \frac{d_{2n}}{2n+1} + \sum_{n=1}^{\infty}\left[\sum_{k=1}^{2n} k d_k (D_{2n-k} - B_{2n-k}) + d_{2n}\right]\frac{1}{2n-1}\right\},$$

$$(6.8)\qquad 2m-1^{th} \quad 0 = \sum_{k=1}^{2m-1} k c_k (D_{2m-1-k} - B_{2m-1-k}) - c_{2m-1} + \frac{p_c}{4}(B_{2m} - D_{2m}) + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\left[\frac{2(2n)d_{2n}}{(2n)^2-(2m-1)^2} + \sum_{k=1}^{2n} k d_k (D_{2n-k} - B_{2n-k})\frac{1}{2n-(2m-1)}\right]\right\},$$

$$-1^{st} \quad 0 = -\frac{p_c}{2} + \frac{p_c\lambda}{4} + c_1 + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{2(2n)d_{2n}}{(2n)^2-1} + \sum_{n=1}^{\infty}\left[\sum_{k=1}^{2n} k d_k (D_{2n-k} - B_{2n-k})\right]\frac{1}{2n+1}\right\},$$

$$-(2m-1)^{th} \quad 0 = c_{2m-1} + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\left[\frac{2(2n)d_{2n}}{(2n)^2-(2m-1)^2} + \sum_{k=1}^{2n} k d_k (D_{2n-k} - B_{2n-k})\frac{1}{2n+2m-1}\right]\right\},$$

$$m = 2,3,4,\cdots.$$

These equations produce a system of equations which give the desired result. To show that they are compatible with the semi-circle equations, λ will be set equal to zero and the results investigated. Since $\lambda = \frac{a-b}{a+b}$, $\lambda = 0$ gives the result when a = b, i.e., a circle. With this substitution, the equation for the first power coefficient becomes

$$0 = \frac{p_c}{4} + \frac{p_c}{4} - C_1 + C_1 + \frac{2}{\pi}\Bigg\{\sum_{n=1}^{\infty} \frac{d_{2n}}{2n+1}$$

$$+\sum_{n=1}^{\infty}\Big[\sum_{k=1}^{2n} k d_k \left(D_{2n-k} - B_{2n-k}\right) + d_{2n}\Big]\frac{1}{2n-1}\Bigg\}$$

$$= \frac{p_c}{2} + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty} \frac{2(2n)d_{2n}}{(2n)^2-1} + \sum_{n=1}^{\infty} \frac{2n d_{2n} - (2n-2)d_{2n-2}}{2n-1}\right\}$$

$$= \frac{p_c}{2} + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\left[\frac{2(2n)d_{2n}}{(2n)^2-1} + \frac{2n d_{2n}}{2n-1}\right] - \sum_{n=2}^{\infty} \frac{(2n-2)d_{2n-2}}{2n-1}\right\}$$

$$= \frac{p_c}{2} + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty} \frac{3(2n)+(2n)^2}{(2n)^2-1} d_{2n} - \sum_{n=1}^{\infty} \frac{2n d_{2n}}{2n+1}\right\}$$

$$= \frac{p_c}{2} + \frac{2}{\pi}\left\{\sum_{n=1}^{\infty} \frac{4(2n)d_{2n}}{(2n)^2-1}\right\}.$$

This is identically equation (5.2) of the semi-circle solution when c = 1. Thus, this solution of the elliptical notch is compatible with the solution in Chapter V.

The equations (6.8) may be written in a form that is better for computation. The altered equations are

$$1^{st} \quad 0 = \frac{p_c}{4}\left(2-2\lambda+\lambda^2\right) - C_1\lambda + \frac{2}{\pi}\Bigg\{\sum_{n=1}^{\infty}\Big[\frac{2(2n)}{(2n)^2-1} + \frac{2n(1-\lambda)}{2n-1}$$

$$-\sum_{k=1}^{\infty} \frac{2n\lambda^{k-1}(1-\lambda)^2}{2k+2n-1}\Big] d_{2n}\Bigg\},$$

$$2m\text{-}1^{th} \quad 0 = +\frac{p_c}{4}\lambda^{m-1}(1-\lambda)^2 - \sum_{k=1}^{m-1}(2k-1)C_{2k-1}\lambda^{m-k-1}(1-\lambda)^2$$

$$+\left[2m-2-(2m-1)\lambda\right]C_{2m-1}$$

$$(6.8a) \qquad +\frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\left[\frac{2(2n)}{(2n)^2-(2m-1)^2}+\frac{2n(1-\lambda)}{2n-(2m-1)}-\sum_{k=1}^{\infty}\frac{2n\lambda^{k-1}(1-\lambda)^2}{2k+2n-(2m-1)}\right]d_{2n}\right\},$$

$$m = 2,3,4,\cdots,$$

$$-1^{st} \quad 0 = -\frac{p_c}{2}+\frac{p_c\lambda}{4}+C_1$$

$$+\frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\left[\frac{2(2n)}{(2n)^2-1}+\frac{2n(1-\lambda)}{2n+1}-\sum_{k=1}^{\infty}\frac{2n\lambda^{k-1}(1-\lambda)^2}{2k+2n+1}\right]d_{2n}\right\},$$

$$-(2m-1)^{th} \quad 0 = C_{2m-1}$$

$$+\frac{2}{\pi}\left\{\sum_{n=1}^{\infty}\left[\frac{2(2n)}{(2n)^2-(2m-1)^2}+\frac{2n(1-\lambda)}{2n+2m-1}-\sum_{k=1}^{\infty}\frac{2n\lambda^{k-1}(1-\lambda)^2}{2k+2n+2m-1}\right]d_{2n}\right\},$$

$$m = 2,3,4,\cdots.$$

These equations compose the exact solution to the elliptic notch. The infinite series in the coefficient causes no question since it converges for all values of λ.

To find solutions of any particular ellipse, a finite system of dimensions corresponding to accuracy desired is taken from this infinite set. Suppose an $\alpha \times \alpha$ system has been chosen. Then n is summed from 1 to $\frac{\alpha}{2}$ in each equation and k is summed from 1 to $\frac{\alpha}{2} - n$ in each d coefficient. The first equation is taken from the -1 power coefficient of (6.a), the second from the $+1$ power coefficient. The odd numbered equations from 3 to $\alpha - 1$ are taken from the $-(2m - 1)$ power coefficients, and the even numbered equations above two from the $(2m - 1)$ power coefficients.

When taken in this way, the equations tend to have the greatest coefficients on the main diagonal of the coefficient matrix, as did the circular notch. This increases the value of the associated determinant so that the accuracy of the solution is improved.

CHAPTER VII

CONCLUSION

Previous Solution of the Semi-circular Notch

The solution of the semi-circular notch problem by Maunsell (5) has been mentioned. It is interesting to consider the relationship of his solution to the one developed in this paper.

Maunsell started with the real stress function ϕ which placed uniform tension P in the x-direction. To this he added certain stress functions which did not alter the stresses on the x-axis, but did give him something with which to remove stresses on the notch. The functions he added were even numbered derivatives with respect to x of $R\theta\cos\theta$ and odd numbered derivatives with respect to x of $R\theta\sin\theta$. These are, respectively,

$$(7.1) \qquad \frac{\partial^{2n}(R\theta\cos\theta)}{\partial x^{2n}} = \frac{M}{R^{2n-1}}\Big[(2n-1)\sin(2n+1)\theta - (2n+1)\sin(2n-1)\theta\Big], \; m \geq 1,$$

$$(7.2) \qquad \frac{\partial^{2n-1}(R\theta\sin\theta)}{\partial x^{2n-1}} = \frac{M_1}{R^{2n}}\Big[\cos 2n\theta - \cos(2n+2)\theta\Big], \; m \geq 0.$$

The numerical factor is combined and called M and M_1 for simplicity. Attached to each of these was an undetermined coefficient which he later evaluated by taking a system of equations from an infinite set.

Analyzing this solution by use of complex variables, we find that Maunsell's original stress function is

$$(7.3) \qquad \phi = R\left\{A\bar{z}z - Az^2\right\}, \qquad \text{where } A = \frac{P}{4}.$$

Now the functions that he added, (7.1) and (7.2), can be expressed in complex form as

$$(7.1a) \qquad R\left\{M i\left[\frac{2K+1}{z^{2K-1}} - \frac{(2K-1)\bar{z}}{z^{2K}}\right]\right\}, \qquad K \geq 1,$$

$$(7.2a) \qquad R\left\{M_1\left[\frac{1}{z^{2K-2}} - \frac{\bar{z}}{z^{2K-1}}\right]\right\}, \qquad K \geq 1.$$

Adding these forms to (7.3), attaching a real coefficient to each, and allowing these coefficients to absorb the M factors gives

$$\phi = R\left\{A\bar{z}z - Az^2 + a_K i\left[\frac{2K+1}{z^{2K-1}} - \frac{(2K-1)\bar{z}}{z^{2K}}\right] + b_K\left[\frac{1}{z^{2K-2}} - \frac{\bar{z}}{z^{2K-1}}\right]\right\}$$

$$= R\left\{\bar{z}\left[Az - \frac{(2K-1)a_K i}{z^{2K}} - \frac{b_K}{z^{2K-1}}\right] + \left[-Az^2 + \frac{(2K+1)a_K i}{z^{2K-1}} + \frac{b_K}{z^{2K-2}}\right]\right\}.$$

This separation suggests the use of the bracketed terms for ψ and χ as complex potentials. It is now seen that if we take the first bracket for ψ, we have real coefficients on odd powers and pure imaginary coefficients on the even powers of z. The more general attack of this paper shows that this must be the case. Taking this function for ψ associates the b_k above with the c_{2k-1} of Chapter V. Also $(2k-1)a_k$ is associated with d_{2k}. These solutions show five significant digit correlation for the first two terms and decreasing, but good, correlation thereafter. This is shown by the samples from the two solutions in the table at the top of the next page.

TABLE II

COMPARISON OF COEFFICIENTS

Maunsell	Chapter V
$b_1 = 0.7924661$	$c_1 = 0.79248481$
$a_1 = .3314167$	$d_2 = .33141212$
$b_7 = .00041$	$c_{13} = .00043$
$11a_6 = .00017$	$d_{12} = .00018$

The fact that the results of this paper seem to be more accurate than those of Maunsell, judging from the lower σ_y at $z = i$ and more coefficients being figured, also tends to lend evidence to the photoelastic results of Frocht. Maunsell achieved a value of 3.05 P for σ_x at $z = i$, and predicted a value of 3 P as the exact answer (5). The improved accuracy of this paper shows a value of 3.07 which agrees with Frocht's prediction that the stress slightly exceeds 3 P (2). Isibasi's figures show 3.06 P (3).

The Way Ahead

This work suggests that there may be many such expansions which could be useful in solving multiple boundary problems. At first glance, these methods may seem rather complicated, but they have the advantage of giving equations in forms that may be easily manipulated. By allowing the use of conformal mapping, these methods render possible the solution of problems not practically feasible by other methods. Also, if other such solutions follow the general trend of the two shown in this paper, they will lend themselves readily to electronic calculator solutions. With the increasing availability of such computers in research and industry, solutions as these may well soon be used in many practical applications.

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BIOGRAPHICAL SKETCH

Paul Henry Hutcheson was born December 7, 1928, in Nashville, Tennessee. In June, 1947, he was graduated from David Lipscomb High School of that city. In August, 1950, he received the degree of Bachelor of Arts cum laude from David Lipscomb College. Studying with the aid of a Jesse Jones Scholarship, he received the degree of Master of Arts from George Peabody College for Teachers in August, 1951. From 1951 until 1954, Mr. Hutcheson taught mathematics and chemistry in Florida Christian College, Tampa, Florida. From 1954 until 1958, he was an Assistant Professor of Mathematics at the University of Tampa. In September, 1958, upon the receipt of a Southern Fellowships Fund grant, he began full-time study in the Graduate School of the University of Florida. He served as an Instructor in the Department of Mathematics there during the summer of 1959, and from September, 1959 until June, 1960, he was a Graduate Assistant. The receipt of a second Southern Fellowships Fund grant enabled the completion of this dissertation during the summer of 1960.

Paul Henry Hutcheson is married to the former Nancy Dennison, and they have three children. He is a past member of the National Council of Teachers of Mathematics. In September, 1960, he will join the staff of Middle Tennessee State College.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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