PROJECTIVE CONVEXITY

By

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PREFACE

In recent years much attention has been given to the study of convexity. The application of this theory to the solutions of systems of linear inequalities has further intensified the research. Klee [3] has studied convexity in the general setting of a topological linear space. It is the purpose of this paper to define a generalization of convexity and to make an analogous study of the generalization in the setting of a topological linear space.

The topology will be as general as possible. That is, no separation axioms will be assumed unless explicitly stated. Thus, the results will hold in spaces whose topologies are even weaker than T_0. (Recall that in a T_0 space X the following property holds: if $x,y \in X$, then there exists a neighborhood U of x such that $y \notin U$, or there exists a neighborhood V of y such that $x \notin V$.)

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This Thesis is Dedicated

to P.M.H. and K.A.C.

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Chapter I

ELEMENTARY ALGEBRAIC
AND TOPOLOGICAL CONSIDERATIONS

Let X be a linear space over the real field R, where R has the usual topology. Suppose, also, that X has a topology superimposed. X is a *topological linear space over R* if and only if (hereafter shortened to iff) vector addition x_1+x_2 and scalar multiplication αx are continuous functions on X x X and R x X to X, respectively.

Let x and y be elements of X. The set of all points $\lambda x+(1-\lambda)y$, $\lambda \in R$, is called the *line through x and y* and is denoted by $\overline{xy}$. Those points for which $0 \leq \lambda \leq 1$ make up the *internal segment* which is denoted by $\overset{\frown}{xy}$. Finally, those points for which $\lambda \leq 0$ or $\lambda \geq 1$ make up the *external segment*, $\overset{\smile}{xy}$.

A subset S of X is *convex* iff, for each pair of points x,y of S, it is the case that $\overset{\frown}{xy} \subset S$.

A subset S of X is *projectively convex* (hereafter shortened to p.c.) iff, for each pair of points x,y of S, either $\overset{\frown}{xy} \subset S$ or $\overset{\smile}{xy} \subset S$.

Thus, we see that projective convexity is a generalization of the notion of convexity. We state this formally as the first theorem.

Theorem 1. A convex set S is p.c.

Proof. By definition, for each pair x,y of S, we have $\overset{\frown}{xy}\subset S$. Thus, S is p.c.

It is obvious that the complement of a convex set need not be convex. However, there is a more pleasant result for p.c. sets.

Theorem 2. The complement of a p.c. set is p.c.

Proof. Let S be a p.c. set and suppose that its complement S' is not p.c. Then there are points x,y of S' such that $\overset{\frown}{xy}\not\subset S'$ and $\overset{\smile}{xy}\not\subset S'$. That is, there are points u,v in S such that

$$u = \lambda_1 x+(1-\lambda_1)y,\ 0<\lambda_1<1,$$

$$v = \lambda_2 x+(1-\lambda_2)y,\ \lambda_2<0 \text{ or } \lambda_2>1.$$

If we solve for x and y in terms of u and v, we have that

$$x = \left[\frac{1-\lambda_2}{\lambda_1-\lambda_2}\right]u+\left[\frac{\lambda_1-1}{\lambda_1-\lambda_2}\right]v,$$

$$y = \left[\frac{-\lambda_2}{\lambda_1-\lambda_2}\right]u+\left[\frac{\lambda_1}{\lambda_1-\lambda_2}\right]v.$$

Setting $\delta_1 = \dfrac{1-\lambda_2}{\lambda_1-\lambda_2}$ and $\delta_2 = \dfrac{-\lambda_2}{\lambda_1-\lambda_2}$, we have that

$$x = \delta_1 u+(1-\delta_1)v,$$

$$y = \delta_2 u+(1-\delta_2)v.$$

We now must consider two cases.

Suppose that $0<\lambda_1<1$ and $\lambda_2<0$. Then $\delta_1>1$ and $0<\delta_2<1$, so we have that $x\epsilon\overset{\smile}{uv}$ and $y\epsilon\overset{\frown}{uv}$ which contradicts the assumption that S is p.c.

If $0<\lambda_1<1$ and $\lambda_2>1$, we have $\delta_2>1$ and $0<\delta_1<1$, making $x\epsilon\overset{\frown}{uv}$ and $y\epsilon\overset{\smile}{uv}$, again a contradiction.

Thus, the assumption that S' is not p.c. is untenable. We conclude that S' is p.c.

Examples of p.c. sets other than convex sets and their complements are furnished by the next theorem. Recall that $Q(x)$ is a quadratic form on X to R iff there exists a bilinear form $B(x,y)$ on $X \times X$ to R such that $Q(x) = B(x,x)$.

<u>Theorem 3</u>. The set of points at which a quadratic form is positive (non-negative, non-positive, negative) is a p.c. set.

<u>Proof</u>. Suppose that x and y are two points for which $Q(x)>0$, $Q(y)>0$. Then

$$Q(\lambda x+(1-\lambda)y) = \lambda^2 Q(x)+(1-\lambda)^2 Q(y)+\lambda(1-\lambda)(B(x,y)+B(y,x)).$$

If $(B(x,y)+B(y,x))\geq 0$, we have that $Q(\lambda x+(1-\lambda)y)>0$ for $0\leq\lambda\leq 1$. On the other hand, if $(B(x,y)+B(y,x))<0$, we have that $Q(\lambda x+(1-\lambda)y)>0$ for $\lambda<0$ or $\lambda>1$.

Thus, if the quadratic form is positive at x and y, it is also positive at either $\overset{\frown}{xy}$ or $\overset{\smile}{xy}$.

We now consider several purely algebraic notions about p.c. sets. First, recall that a <u>linear variety</u> is a translate of a linear subspace of X. That is, a linear variety is a set of the form $x+S$, where x is some element of X and S is a linear subspace of X.

<u>Lemma 1</u>. If V is a linear variety and $y,z\in V$, then $\overline{yz}\subset V$. In particular, a linear variety is convex, and hence p.c.

<u>Proof</u>. Let $V = x+S$, where x is some element of X and S is a linear subspace of X. Then we may uniquely express y and z as $y = x+s_1$ and $z = x+s_2$, where $s_1,s_2\in S$. Now any point on

$\overline{yz}$ has the form

$$\begin{aligned}\lambda y+(1-\lambda)z &= \lambda(x+s_1)+(1-\lambda)(x+s_2)\\ &= x+(\lambda s_1+(1-\lambda)s_2)\\ &= x+s_3,\end{aligned}$$

where $s_3 \in S$. Thus, any point of $\overline{yz}$ is in V, so that $\overline{yz} \subset V$.

It is an elementary theorem in convex set theory that the intersection of a family of convex sets is also convex. However, the intersection of two p.c. sets need not be p.c. To see this, let

$$S = \{(x,y) \mid x^2+y^2 \leq 4\},$$
$$T = \{(x,y) \mid x^2-y^2 \geq 1\}.$$

Both S and T are p.c., but their intersection is not p.c. However, we are able to get a somewhat analogous result, which also serves to characterize linear varieties. First, we need a lemma which gives an alternative characterization of linear varieties.

<u>Lemma 2</u>. A nonvoid subset S of X is a linear variety iff we have $\alpha_1x_1+\alpha_2x_2+\ldots+\alpha_nx_n \in S$ whenever $\{x_1,x_2,\ldots,x_n\} \subset S$ and $\alpha_1+\alpha_2+\ldots+\alpha_n = 1$. The index n is not fixed.

<u>Proof</u>. If S is a linear variety, then S = x+V for a subspace V. Let $\{x_1,x_2,\ldots x_n\} \subset S$ and write $x_i = x+v_i$, $v_i \in V$, $i = 1,2,\ldots,n$. Using this representation we have

$$\begin{aligned}\alpha_1x_1+\alpha_2x_2+\ldots+\alpha_nx_n &= \alpha_1(x+v_1)+\alpha_2(x+v_2)+\ldots+\alpha_n(x+v_n)\\ &= x+\alpha_1x_1+\alpha_2x_2+\ldots+\alpha_nx_n,\end{aligned}$$

which shows that the condition is necessary.

To show sufficiency, let $x \epsilon S$ and define $T = -x+S$. We will show that T is a subspace. Let $t_1, t_2 \epsilon T$ and write

$$t_1 = -x+s_1,$$
$$t_2 = -x+s_2,$$

for $s_1, s_2 \epsilon S$. Now let α, δ be any two real numbers. We have

$$\begin{aligned} \alpha t_1+\delta t_2 &= \alpha(-x+s_1)+\delta(-x+s_2) \\ &= -x+(x-\alpha x+\alpha s_1-\delta x+\delta s_2) \\ &= -x+s_3 \end{aligned}$$

since $(1-\alpha+\alpha-\delta+\delta) = 1$. Thus, T is a linear subspace.

<u>Theorem 4</u>. A set S is a linear variety iff $S \wedge P$ is p.c. for every p.c. set P.

<u>Proof</u>. First we will show the necessity of this condition. Let $x,y \epsilon S \wedge P$. From the definition of projective convexity we know that $\widehat{xy} \subset P$ or $\breve{xy} \subset P$. Both $\widehat{xy}$ and $\breve{xy}$ are contained in S, by Lemma 1, so at least one of the two segments is contained in $S \wedge P$, making this intersection p.c.

Now we consider the somewhat more difficult problem of the sufficiency of the condition. It will be shown that our condition implies the condition of Lemma 2. To do this we will use induction on the number of elements in the subset of S.

For $n = 1$ we have $\{x_1\} \subset S$ and for $\alpha_1 = 1$ we have $\alpha_1 x_1 \epsilon S$ without applying our condition.

Now assume that for $n = k$ $\{x_1, x_2, \ldots, x_k\} \subset S$ and for $\alpha_1+\alpha_2+\ldots+\alpha_k = 1$ we have $\alpha_1 x_1+\alpha_2 x_2+\ldots+\alpha_k x_k \epsilon S$. We will see that this holds for $n = k+1$.

Let $\{y_1, y_2, \ldots, y_{k+1}\} \subset S$, and let $\{\delta_1, \delta_2, \ldots, \delta_{k+1}\}$ be a set of real numbers whose sum is one. If any one of the δ_i's is zero, the condition holds by the inductive assumption. Thus, assume that all are non-zero. We consider three cases.

Suppose that $0 \leq \delta_{k+1} < 1$. We know that

$$y = \frac{\delta_1 y_1 + \ldots + \delta_k y_k}{\delta_1 + \ldots + \delta_k} \epsilon S$$

by the inductive hypothesis. Now $\widehat{yy}_{k+1}$ is a p.c. set and hence $S \wedge \widehat{yy}_{k+1}$ is p.c. Since $\widehat{yy}_{k+1} \subset S$, we have that

$$\sum_{i=1}^{k} \delta_i y + \delta_{k+1} y_{k+1} = \sum_{i=1}^{k+1} \delta_i y_i \epsilon S.$$

Next, suppose that $\delta_{k+1} < 0$ or $\delta_{k+1} > 1$. As in the first case we have $y \epsilon S$ by the inductive hypothesis. $\breve{yy}_{k+1}$ is p.c. so $S \wedge \breve{yy}_{k+1}$ is p.c. Since $\breve{yy}_{k+1} \subset S$, we have that

$$\sum_{i=1}^{k} \delta_i y + \delta_{k+1} y_{k+1} = \sum_{i=1}^{k+1} \delta_i y_i \epsilon S.$$

Finally, suppose that $\delta_{k+1} = 1$. We know that for some j, $\delta_j < 0$. Let $\delta_n' = \delta_n$ for $n \neq k+1$, $n \neq j$, and let $\delta_{k+1}' = \delta_j$, $\delta_j' = \delta_{k+1}$. Make a similar renumbering of the y_i's. We now have the situation of the second case, so the lemma holds.

In all three cases the condition of Lemma 2 is satisfied, so we conclude that S is a linear variety.

A natural question of an algebraic nature is whether or not projective convexity is preserved under the formation of a quotient space or a direct sum of two spaces. That is, if we form a quotient space of X modulo a linear subspace, is the quotient space image of a p.c. set also p.c.? And, if we form a direct sum of two linear spaces, is the direct sum of two p.c. sets also a p.c. subset of the direct sum space?

The answer to the first question is in the affirmative, but the second question receives an affirmative answer only in a very special case. We now state these two results.

Theorem 5. Suppose h is a homomorphism of the space X onto the space Y. Then if S is a p.c. (convex) subset of X, h[S] is a p.c. (convex) subset of Y. In particular, the image under a linear transformation of a p.c. set is p.c.

Proof. Let $u, v \in h[S]$. For at least one pair x,y in S we have that $u = h(x)$, $v = h(y)$. Either $\widehat{xy}$ or $\breve{xy}$ is contained in S, so let us assume that it is the former, and the other possibility will follow in a similar manner.

$$\begin{aligned}\lambda u+(1-\lambda)v &= \lambda h(x)+(1-\lambda)h(y)\\ &= h(\lambda x+(1-\lambda)y)\end{aligned}$$

is an element of h[S] if $0 \leq \lambda \leq 1$. Hence, h[S] is p.c.

To consider the question of direct sums let the spaces X_1 and X_2 be E_1. Let $S_1 = (0,1)$ and $S_2 = (1,\infty) \vee (-\infty,-1)$. Both S_1 and S_2 are p.c. subsets of X_1 and X_2, respectively, but $S_1 \oplus S_2$ is not p.c. in the direct sum space. Thus, if even one of S_1 and S_2 fails to be convex, the direct sum need not be p.c. However, we are able to state the following result.

Theorem 6. Suppose S_1 and S_2 are convex subsets of the spaces X_1 and X_2, respectively. Then $S_1 \oplus S_2$ is convex in $X_1 \oplus X_2$.

Proof. Let (x_1,x_2) and (y_1,y_2) be elements of $S_1 \oplus S_2$. We have that $\lambda(x_1,x_2)+(1-\lambda)(y_1,y_2) = (\lambda x_1+(1-\lambda)y_1, \lambda x_2+(1-\lambda)y_2)$ is an element of $S_1 \oplus S_2$ since $\lambda x_i+(1-\lambda)y_i \in S_i$, $i = 1,2$.

It is a theorem in convex set theory that, if S and T are convex sets and α and δ are non-negative real numbers, then the vector sum set $\alpha S+\delta T$ is convex. For a proof of this theorem see Taylor [5], page 130. This need not be the case for p.c. sets as the following example shows.

Let the space be E_2 and define S and T as follows:

$$S = \{(x,y) \mid x = 0,\ y<-1 \text{ or } y>1\},$$

$$T = \{(x,y) \mid 1<x<2,\ y = 0\}.$$

By direct computation it is seen that

$$S+T = \{(x,y) \mid 1<x<2,\ y<-1 \text{ or } y>1\},$$

which is not p.c.

With a suitable restriction, however, we are able to obtain a theorem.

Theorem 7. If P is p.c. and V is a linear variety, then the vector sum set $\alpha V+\delta P$ is p.c. for any real numbers α and δ.

Proof. Let $V = x+S$, with x some element in X and S a linear subspace of X. Let u,v be members of $\alpha V+\delta P$. We may write

$$u = \alpha(x+s_1)+\delta p_1,$$

$$v = \alpha(x+s_2)+\delta p_2,$$

where $s_i \in S$, $p_i \in P$, $i = 1,2$. We have that

$$\begin{aligned}\lambda u+(1-\lambda)v &= \lambda(\alpha(x+s_1)+\delta p_1)+(1-\lambda)(\alpha(x+s_2)+\delta p_2)\\ &= \alpha(x+(\lambda s_1+(1-\lambda)s_2)+\delta(\lambda p_1+(1-\lambda)p_2).\end{aligned}$$

Now $x+(\lambda s_1+(1-\lambda)s_2 \in V$ for all real λ. For $0\leq\lambda\leq 1$, or for $\lambda\leq 0$

or $\lambda \geq 1$ we have that $\lambda p_1+(1-\lambda)p_2 \epsilon P$. Hence, either $\widehat{uv}$ or $\widecheck{uv}$ is contained in $\alpha V+\delta P$, making this set p.c.

A motivation for the terminology "projective" convexity is given in the next theorem. Recall from projective geometry that if $a,b \epsilon X$ such that $a \epsilon \widehat{xy}$ and $b \epsilon \widecheck{xy}$ for $x,y \epsilon X$, then the pair (a,b) is said to *separate* the pair (x,y). If (a,b) separates (x,y), then (x,y) separates (a,b).

Theorem 8. Let T be a 1-1 transformation on X onto X which maps points into points, lines into lines, and having the property that

(a,b) separates (x,y) iff (Ta,Tb) separates (Tx,Ty).

Then, if S is p.c., so is T[S]. That is, the image of a p.c. set under a projective transformation is p.c.

Proof. Let $Tx,Ty \epsilon T[S]$ and assume that $\widehat{TxTy} \not\subset S$, $\widecheck{TxTy} \not\subset S$. Then there are points Tu,Tv such that $Tu \epsilon \widehat{TxTy}$ and $Tv \epsilon \widecheck{TxTy}$ and $Tu,Tv \epsilon (T[S])' = T[S]'$. Then (Tu,Tv) separates (Tx,Ty), so (u,v) separates (x,y). Thus, $\widehat{xy} \not\subset S$ and $\widecheck{xy} \not\subset S$, a oontradiction to the assumption that S is p.c.

In a linear space it is well-known that there is a unique minimal convex set containing a given set, and this set is called the *convex hull* of the given set. For a proof of this statement see Taylor [5], page 131. However, if our set is simply two distinct points, there are two minimal and non-comparable p.c. sets containing the two points, namely, the two segments determined by the two points. This situation prevails in general as the following theorem states.

Theorem 9. For a subset S of a topological linear space X there exists a minimal p.c. set containing S.

Proof. Let A be the family of all p.c. sets contained in the complement S'. A is partially ordered by set inclusion. For B a chain in A consider $\vee B$. Let $x,y\in\vee B$; there are sets C,D in B such that $x\in C$, $y\in D$. Either $C\subset D$ or $D\subset C$, so let us assume that the former holds. Then $x,y\in D$ and since D is p.c., we have that $\overset{\frown}{xy}$ or $\overset{\smile}{xy}$ is contained in D. This means that $\overset{\frown}{xy}$ or $\overset{\smile}{xy}$ is contained in $\vee B$, making $\vee B$ a p.c. set. Now clearly $\vee B\subset S'$, so we have that each chain in A has an upper bound in A. Thus, by Zorn's Lemma, there is a maximal member F of A. F' is a minimal p.c. set containing S. For, suppose that there is a p.c. set M such that $S\subset M\subset F'$ and $M\neq F'$. Then M' is a p.c. set with $F\neq M'$ and $F\subset M'\subset S'$, contradicting the maximality of F.

Such a minimal p.c. set containing the subset S will be called a p.c. hull of S. A problem that remains open at the present is the determination of the class of sets for which there exists a unique p.c. hull.

Before proceeding with our development, it seems advisable to make an observation about one property in a topological linear space. Since translations are homeomorphisms, the neighborhood system at any point $x\in X$ is obtained by translating the neighborhood system of the origin by the vector x. Thus, if $\mathcal{S}$ is the neighborhood system of the origin, then $\{x+V \mid V\in\mathcal{S}\}$ is the neighborhood system at x. The importance of this observation lies in the fact that the neighborhood system at any point $y \in X$ can be indexed by means of the same index set S. It should also be remarked that S is directed by set inclusion. Both of these facts will be used in the next two theorems.

<u>Theorem 10</u>. The closure of a p.c. set A is p.c.

<u>Proof</u>. Let $x, y \in A^{-}$. For each neighborhood M_V, $V \in S$, of x, there is a point $x_V \in A \wedge M_V$. Likewise, for the corresponding neighborhood N_V of y, there is a point $y_V \in A \wedge N_V$. Now for each V either $\widehat{x_V y_V}$ or $\widecheck{x_V y_V}$ is a member of A, since A is p.c. Thus, for at least one of these two cases there is a subset R of S with the property: if $V \in S$, there exists $U \in R$, with $U \subset V$, such that $\widehat{x_U y_U}$ (or $\widecheck{x_U y_U}$) is contained in A. (Such a subset R of S is said to be a <u>cofinal</u> subset. For a detailed discussion of Moore-Smith Convergence, see Kelley [2], chapter 2.) Suppose that this holds for internal segments. Let p be defined by

$$p = \lambda x + (1-\lambda) y, \quad 0 \le \lambda \le 1.$$

Consider the net $\{p_U; U \in R\} = \{\lambda x_U + (1-\lambda) y_U\}$. It is eventually in each neighborhood of p, so p is a limit. Since all of the values of this net are in A and p is a limit of the net, we have that $p \in A^{-}$. Since p was any member of $\widehat{xy}$, we conclude that $\widehat{xy} \subset A$. The foregoing argument can be modified slightly for the other case.

<u>Theorem 11</u>. The interior of a p.c. set A is p.c.

<u>Proof</u>. From $A^{o} = X - [X-A]^{-}$, the preceding theorem, and the fact that the complement of a p.c. set is p.c., the theorem follows.

We will now digress somewhat to develop some properties of boundedness that will be needed in the sequel.

A subset S of X is <u>bounded</u> iff for each neighborhood of the origin, U, there exists a positive scalar α such that $S \subset \alpha U$.

An equivalent condition is that S is bounded iff: if α_n is a null sequence and $\{x_n\}$ is a sequence in S, $\{\alpha_n x_n\}$ converges to the origin. See Taylor [5], page 128.

<u>Lemma 3</u>. The union of two bounded sets is bounded. (A proof can be found in Bourbaki [1].)

<u>Lemma 4</u>. Any finite set is bounded.

<u>Proof</u>. We will use induction on the number of elements in our set.

For $n = 1$ let $S_1 = \{x\}$. If α_n is a null sequence, then $\{\alpha_n x_n\}$ converges to the origin by continuity of scalar multiplication.

Assume that any set of n elements is bounded. For n+1 let $S_{n+1} = \{x_1, x_2, \ldots, x_n, x_{n+1}\}$. This set can be expressed as the union of a set of one element and a set of n elements, both of which are bounded. Thus, S_{n+1} is bounded by Lemma 3.

<u>Theorem 12</u>. The convex hull of a finite set S is bounded.

<u>Proof</u>. Let $S = \{x_1, x_2, \ldots, x_n\}$. The convex hull of S consists of all points of the form

$$z = \sum_{i=1}^{n} \lambda_i x_i,$$

with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. Let $\{y_m\}$ be a sequence in the convex hull of S. We may write

$$y_m = \sum_{i=1}^{n} \lambda_{im} x_i.$$

Now let $\{\alpha_n\}$ be any null sequence, and consider the sequence $\{\alpha_n y_n\}$. Since $0 \leq \lambda_{in} \leq 1$, $\{\alpha_n \lambda_{in}\}$ is a null sequence for each n. Hence, each term in the expression of y_m approaches the origin as a limit by the continuity of scalar multiplication and thus, by the continuity of vector addition $\{\alpha_n y_n\}$ converges to the origin. Therefore, the convex hull of S is bounded.

The next result, although elementary in appearance, is quite important in our next theorem.

<u>Lemma 5</u>. A line L is an unbounded set.

<u>Proof</u>. Let the line L be defined as the set

$$\{z \mid z = \lambda x + (1-\lambda) y,\ x \neq y\}.$$

Let $\{\alpha_n\} = \{1/n\}$ and $\{\lambda_n\} = \{n\}$. We know that $nx+(1-n)y \epsilon L$ for all positive integers n; thus $\{x_n\} = \{\lambda_n x + (1-\lambda_n) y\} \subset L$. Clearly α_n is a null sequence. Now consider the sequence $\{\alpha_n x_n\} = \{x + (1/n - 1) y\}$. This sequence converges to x-y which is not the origin. Thus, L is not bounded.

<u>Theorem 13</u>. A bounded p.c. set S is convex.

<u>Proof</u>. Since $\{x,y\}$ is bounded, $\widehat{xy}$ is bounded, being the convex hull of a finite set. If $\breve{xy}$ were bounded, $\overline{xy} = \widehat{xy} \vee \breve{xy}$ would be bounded. Hence, $\breve{xy}$ is not bounded. Therefore, if $x, y \epsilon S$, we see that since $\breve{xy} \not\subset S$, we must have $\widehat{xy} \subset S$.

Chapter II

HYPERPLANES AND PROJECTIVELY CONVEX SETS

A <u>hyperplane</u> H is a set of the form $\{x|f(x) = \alpha\}$ for a linear functional f and a scalar α. H is said to <u>separate</u> the set S iff there exist points $x,y \epsilon S$ such that $f(x)<\alpha$ and $f(y)>\alpha$. Clearly, if H separates T and $T \subset S$, then H separates S.

We now give several theorems in which separation by hyperplanes plays an important part.

<u>Lemma 6</u>. Suppose that the hyperplane H fails to separate the set S. If x and y are points of S such that $\overset{\smile}{xy} \subset S$, then either $\overset{\smile}{xy} \subset H$ or $\overset{\smile}{xy} \wedge H = \emptyset$.

<u>Proof</u>. Let H be represented as $\{x|f(x) = \alpha\}$ for a linear functional f and a scalar α. If H does not separate S, either $f[S] \geq \alpha$ or $f[S] \leq \alpha$. Let us assume that the former holds. Also, suppose that $\overset{\smile}{xy} \not\subset H$; then at least one of x and y is not in H in view of Lemma 1. Suppose that it is y. Let $z \epsilon \overset{\smile}{xy} \wedge H$; for $\lambda>1$ consider the point $p = \lambda z+(1-\lambda)y$. We have that

$$\begin{aligned} f(p) &= f(\lambda z+(1-\lambda)y) \\ &= \lambda f(z)+(1-\lambda)f(y) \\ &= \lambda\alpha+(1-\lambda)f(y) \\ &< \lambda\alpha+(1-\lambda)\alpha = \alpha. \end{aligned}$$

This says that H separates $\breve{xy}$ since $f(y)>\alpha$, so we have that H separates S, a contradiction.

Lemma 7. There is no open set contained in a hyperplane.

Proof. Let H be a hyperplane and let x be a vector such that $x \notin H$. If S is open and $S \subset H$, then consider $\widehat{xs}$ for $s \in S$. No point of $\widehat{xs}$ is in S except for s, and s is an accumulation point of S'. To see this, let $\{\lambda_n\}$ be a sequence of real numbers with $0<\lambda_n<1$ and such that the sequence converges to zero. Then $\{\lambda_n x+(1-\lambda_n)s\}$ is a sequence of points in S' which converges to s by the continuity of scalar multiplication and vector addition. S' is closed since S is open, so we have that $s \in S'$, a contradiction to the assumption that $s \in S$.

We may remark that, in view of the preceding lemma, if S is open in Lemma 6, then $S \cap H = \emptyset$ so that $\breve{xy} \cap H = \emptyset$.

Theorem 14. If A is an open or closed p.c. set and if there is a hyperplane which does not separate A, then either A is convex, or A is contained in a hyperplane.

Proof. Suppose that A is a closed p.c. set and H_1 is a hyperplane which does not separate A. Then there exist a linear functional f and a real number α_1 such that $H_1 = \{x \mid f(x) = \alpha_1\}$, and $f(x) \geq \alpha_1$ or $f(x) \leq \alpha_1$ for all $x \in A$. Assume that the former holds. If A is not convex, then there are points $x, y \in A$ such that $\widehat{xy} \not\subset A$. That is, there exists a point $p \in A'$ such that $p = \lambda x+(1-\lambda)y$ with $0<\lambda<1$. Since A is p.c., we know that $\breve{xy} \subset A$. By Lemma 6 either $\overline{xy} \subset H_1$ or $\overline{xy} \cap H_1 = \emptyset$. Thus, in the first case $\overline{xy}$ is contained in H_1 and in the second $\overline{xy} \subset H_2$, where H_2 is the hyperplane defined by $H_2 = \{x | f(x) = \alpha_2\}$ for $\alpha_2 > \alpha_1$. Let us consider the case $\overline{xy} \subset H_2$ and the other follows similarly. If there are no points in $A - \overline{xy}$, the theorem is established. Hence assume that there exists a point $z \epsilon A$ such that $f(z) \neq \alpha_2$. Either $\alpha_1 \leq f(z) < \alpha_2$ or $f(z) > \alpha_2$. Consider the first case; $\breve{yz} \not\subset A$ since for any λ where

$$\lambda > \frac{\alpha_1 - f(y)}{f(z) - f(y)},$$

$\lambda z + (1-\lambda)y$ is a point such that

$$f(\lambda z + (1-\lambda)y) = \lambda f(z) + (1-\lambda)f(y) < \alpha_1.$$

Similarly in the other case $\breve{yz} \not\subset A$. Thus, we must have that $\widehat{yz} \subset A$ in either case.

Now let $q \epsilon \widehat{yz}$; in a manner similar to the foregoing, we can show that $\widehat{xq} \subset A$. That is, the point

$$[\lambda x + (1-\lambda)(\lambda' y + (1-\lambda')z)] \epsilon A$$

for all λ' for which $0 \leq \lambda' < 1$ and λ has the same value as before. Let $\{\lambda_n\}$ be a sequence such that $0 \leq \lambda_n < 1$ and λ_n converges to one. Then

$$\{\lambda x + (1-\lambda)(\lambda_n y + (1-\lambda_n)z)\}$$

is a sequence of points in A which has p as a limit due to the continuity of scalar multiplication and vector addition. Since A is closed, $p \epsilon A$, a contradiction. Hence, if A is not convex, it is contained in a hyperplane.

Let us consider the case in which A is an open p.c. set and H_1 is a hyperplane which does not separate A. We again have that $H_1 = \{x | f(x) = \alpha_1\}$ for a linear functional f and a scalar α_1, and $f(x)<\alpha_1$ or $f(x)>\alpha_1$ for all $x \in A$. If A is not convex, there are points $x', y' \in A$ such that for $0<\lambda<1$

$$p = [\lambda x'+(1-\lambda)y'] \in A.$$

Either $0<\lambda<1/2$, $\lambda = 1/2$, or $1/2<\lambda<1$. If the second case holds, p is the <u>midpoint</u> of $\widehat{xy}$. Suppose that the third case holds; let x be defined to be the point $2\lambda x'+(1-2\lambda)y'$ and y' be called y. Since $1<2\lambda$ and A is p.c., we have $x \in A$. Also, we have that $p = 1/2x+1/2y$. That is, in this case we can find $x,y \in A$ such that the midpoint of $\widehat{xy}$ is not in A. This can also be done in the other case, so for the remainder of the proof we will assume that this has been done. Since A is open $A \wedge H_1 = \emptyset$ so $\overline{xy} \wedge H_1 = \emptyset$; hence, $\overline{xy} \subset H_2 = \{x \mid f(x) = \alpha_2\}$ for $\alpha_1<\alpha_2$ or $\alpha_2<\alpha_1$. Assume that the former holds. If $A \not\subset H_2$, there is a point $z \in A$ such that $f(z) \neq \alpha_2$. Either $\alpha_1<f(z)<\alpha_2$ or $f(z)>\alpha_2$. We will consider in detail the first possibility and the other will follow in a similar manner. As in the first half of the proof we have $\widehat{xz} \subset A$. Let $s = \lambda' p+(1-\lambda')z$, where

$$\lambda' = \frac{\alpha_1-f(z)}{1/2f(x)+1/2f(y)-f(z)} .$$

Also, let $r = \lambda'' y+(1-\lambda'')z$, where λ'' is defined by

$$\lambda'' = \frac{\alpha_1-f(z)}{f(y)-f(z)} .$$

Note that $\lambda' = \lambda''$. The points s and r are, respectively, $\overline{pz} \wedge H_1$ and $\overline{yz} \wedge H_1$. Define the sequence $\{x_n\}$ by

$$x_n = nr+(1-n)s$$

and the sequence $\{y_n\}$ by

$$y_n = \lambda_n p+(1-\lambda_n)x_n,$$

where $\lambda_n = 1-1/\lambda' n$. In a different form we have that

$$y_n = [\lambda_n+(1-\lambda_n)(1-n)\lambda']p+[(1-\lambda_n)n\lambda']y+[(1-\lambda_n)(1-\lambda')]z.$$

$\overset{\frown}{px_n}$ contains points of A so it cannot be contained in A'; thus $\overset{\smile}{px_n} \subset A'$. Since $y_n \epsilon \overset{\smile}{px_n}$, we have that $y_n \epsilon A'$. It is seen that $\{y_n\}$ converges to y; since A' is closed, $y \epsilon A'$. But $y \epsilon A$, so we have reached a contradiction. Thus, if A is not convex, it is contained in a hyperplane.

Corollary. If A is an open p.c. set and if there is a hyperplane not separating A, then A is convex.

Proof. By Theorem 14 A is either convex, or is contained in a hyperplane. By Lemma 6 A cannot be contained in a hyperplane; hence, A is convex.

Chapter III

PROJECTIVE CONVEXITY IN TWO-DIMENSIONAL TOPOLOGICAL LINEAR SPACES

Our attention is now directed toward the study of projective convexity in two-dimensional topological linear spaces which will be denoted by L_2. Such a space will have a basis of two vectors. In particular, a connection will be established between projective convexity and a certain convexity property which is called property P_3. Further information about the latter idea can be found in Valentine [6]. The theorems of this chapter will be quite useful in helping to characterize p.c. sets in n-dimensional topological linear spaces L_n and in spaces of arbitrary dimension.

An _open_ _half-plane_ is a set of the form $\{x|f(x)<\alpha\}$ or $\{x|f(x)>\alpha\}$ for a linear functional f and a scalar α. If less than and greater than are replaced by less than or equal to and greater than or equal to, respectively, the resulting set is called a _closed_ _half-plane_.

An _open_ _strip_ is a subset S of L_2 such that there exist a linear functional f and scalars α and δ with $\alpha<\delta$ for which S is the set $\{x|\alpha<f(x)<\delta\}$. A point x such that $f(x) = \alpha$ or $f(x) = \delta$ is a boundary point of S.

Following the terminology of Euclidean 2-space, a hyperplane in L_2 will be called a <u>line</u>. This is also in agreement with our earlier usage of the term.

Before starting with our main project of this chapter, we will establish a system of terminology and notation which will be very useful in the sequel.

Let p_1, p_2, and p_3 be three non-collinear points of L_2. First, we will let $a_1 = \overline{p_2p_3}$, and a_2 and a_3 will be defined analogously.

We next wish to define seven regions in a manner which agrees with the standard terminology used for the Euclidean plane. We will consider a point of the form

$$\sum_{i=1}^{3} \alpha_i p_i, \text{ where } \sum_{i=1}^{3} \alpha_i = 1.$$

(i) <u>The triangle P</u>.

Here each of the coefficients is non-negative. If one of them is zero, as for example α_1, we have a point on one of the three internal segments, in this case $\widehat{p_2p_3}$. If two of the coefficients are zero, the third is automatically one, so we have one of our three points.

(ii) <u>The three regions A_i</u>.

These regions are defined by requiring that the i^{th} coefficient be non-negative and the others be non-positive.

(iii) <u>The three regions B_i</u>.

These regions have the i^{th} coefficient non-positive and the other two coefficients non-negative.

We first prove a lemma which will be quite useful in establishing our main theorem of the chapter.

Lemma 8. If A is a p.c. subset of L_2 and a line L fails to separate A, then A is either

(i) an open half-plane plus part of its boundary; or

(ii) an open strip plus part of its boundary; or

(iii) contained in a line; or

(iv) convex.

Proof. Suppose that A is not convex; then there are points $x, y \epsilon A$ such that $\widehat{xy} \not\subset A$. We know that $\check{xy} \subset A$, so we have that $\overline{xy}$ is parallel to L. We may represent L as $\{x | f(x) = \alpha_1\}$ for a linear functional f and a scalar α_1. Then $\overline{xy}$ can be represented as $\{x | f(x) = \alpha_2\}$. If $A \not\subset \overline{xy}$, let $z \epsilon A - \overline{xy}$. Then $f(z) \neq \alpha_2$. The line $L' = \{x | f(x) = f(z)\}$ bounds, with $\overline{xy}$, an open strip whose points all belong to A. If we can find a sequence $\{z_n\}$ in A such that the sequence $\{f(z_n)\}$ is strictly monotone increasing and unbounded above, A is the set of case (i). If no such sequence exists, A is the set of case (ii)

It might be remarked that the part common to A and the boundary of the open strip, or the open half-plane, will consist either of a single point, or a finite (open, closed, or semi-open) line segment or the union of two infinite rays (with two, one, or no endpoints included). Any other collection of boundary points would fail to be p.c. and, therefore, A would not be p.c. It, of course, might happen that the entire boundary of the open strip or open half-plane is in A.

A subset S of the topological linear space X is said to have <u>property</u> $\underline{P_3}$ iff for $x,y,z\in S$, at least one of $\widehat{xy}$, $\widehat{yz}$, and $\widehat{xz}$ is contained in S. It is immediate that any convex set has property P_3. Also, the union of two convex sets has this property.

<u>Theorem 15</u>. Suppose that A is a p.c. subset of L_2 which does not have property P_3. Then

(i) A' has property P_3;

(ii) A has at most two components; and

(iii) if A is open or closed, then it is connected.

<u>Proof</u>. Let p_1, p_2, and p_3 be three points of A such that none of their internal segments is contained in A. Since A is p.c., all three external segments are contained in A. We will first determine some general structural properties of A'.

(a) $\underline{A'\wedge A_1}$ <u>is convex</u>. For example, let $x,y\in A'\wedge A_1$; then we have that

$$x = \alpha_1 p_1+\alpha_2 p_2+\alpha_3 p_3,\ \sum_{i=1}^{3}\alpha_i = 1,\ \alpha_1>0,\ \alpha_2<0,\ \alpha_3<0,$$

$$y = \delta_1 p_1+\delta_2 p_2+\delta_3 p_3,\ \sum_{i=1}^{3}\delta_i = 1,\ \delta_1>0,\ \delta_2<0,\ \delta_3<0.$$

Any point on $\widehat{xy}$ has the form $\lambda x+(1-\lambda)y$, $0\leq\lambda\leq 1$. Thus, for $q\in\widehat{xy}$, we have that

$$\begin{aligned} q &= \lambda(\alpha_1 p_1+\alpha_2 p_2+\alpha_3 p_3)+(1-\lambda)(\delta_1 p_1+\delta_2 p_2+\delta_3 p_3) \\ &= (\lambda\alpha_1+(1-\lambda)\delta_1)p_1+(\lambda\alpha_2+(1-\lambda)\delta_2)p_2+(\lambda\alpha_3+(1-\lambda)\delta_3)p_3 \\ &= \pi_1 p_1+\pi_2 p_2+\pi_3 p_3, \end{aligned}$$

where

$$\sum_{i=1}^{3} \pi_i = \sum_{i=1}^{3} \lambda\alpha_i + \sum_{i=1}^{3}(1-\lambda)\delta_i = \lambda+(1-\lambda) = 1,$$

and $\pi_1>0$, $\pi_2<0$, and $\pi_3<0$. Hence, $q\epsilon A_1$ which makes A_1 convex. Likewise A_2 and A_3 are convex.

Since A' is p.c., either $\widehat{xy}$ or $\overset{\smile}{xy}$ is contained in A'. We will see that $\overset{\smile}{xy}\not\subset A'$. If $\alpha_1 = \delta_1$ and $\alpha_2 = \delta_2$, then also $\alpha_3 = \delta_3$ and $x = y$. Thus, let us assume that $\alpha_3>\delta_3$. The same argument applies to the other two pairs of coefficients. Consider the point

$$\lambda x+(1-\lambda)y = \lambda \sum_{i=1}^{3} \alpha_i p_i + (1-\lambda) \sum_{i=1}^{3} \delta_i p_i$$

$$= \sum_{i=1}^{3} [\lambda\alpha_i+(1-\lambda)\delta_i]p_i,$$

with λ defined by

$$\lambda = \frac{-\delta_3}{\alpha_3-\delta_3} .$$

We have that $\lambda>1$, so the point is on $\overset{\smile}{xy}$. If, however, we substitute this value of λ in the above representation of our point, we obtain a point on $\overset{\smile}{p_1p_2}$, which we have already observed to be in A. We have reached a contradiction, so we must have $\widehat{xy}\subset A'$. Hence $\widehat{xy}\subset A'\wedge A_1$, making this intersection convex. A similar argument holds in the other two cases.

(b) *If $A'\wedge A_1 \neq \emptyset$, then A' contains no unbounded subset in either of the B_j's adjacent to A_1.* For example, let $x\epsilon A'\wedge A_1$ and $y\epsilon B_2\wedge A'$. If y lies outside the triangle which is determined by p_1, p_3, and x, then $\overset{\smile}{xy}$ intersects both $\overset{\smile}{p_2p_3}$ and $\overset{\smile}{p_1p_3}$. Regardless of where y lies within its specified region,

$\overset{\frown}{xy}$ intersects $\overset{\smile}{p_1p_2}$, so that A' would not be p.c. Hence, $B_2 \wedge A'$ is contained in a triangle and is thus bounded by Theorem 11. Similarly $B_3 \wedge A'$ is bounded.

(c) <u>A' intersects at most one A_i</u>. For, suppose $x \epsilon A' \wedge A_1$ and $y \epsilon A' \wedge A_2$. $\overset{\frown}{xy}$ contains a point on each of $\overset{\smile}{p_2p_3}$ and $\overset{\smile}{p_1p_3}$, so $\overset{\smile}{xy}$ must be contained in A'. Now $\overline{xy}$ is parallel to $\overline{p_1p_2}$, for otherwise $\overset{\smile}{xy}$ would contain a point of $\overset{\smile}{p_1p_2}$. Thus, $A' \wedge (A_1 \vee A_2)$ is contained in a line parallel to $\overline{p_1p_2}$. Since $\overset{\frown}{p_1p_3} \not\subset A$, there is a point $z \epsilon \overset{\frown}{p_1p_3} \wedge A'$. But since the lines $\overline{yz}$ and $\overline{xy}$ are not identical, we have reached a contradiction.

(d) <u>$A' \wedge (B_1 \vee B_2 \vee B_3 \vee P) = S$ is p.c.</u> Consider the set

$$A \vee (A_1 \vee A_2 \vee A_3),$$

and let x and y be elements of this set. We must take into consideration three cases.

If $x \epsilon A_i$, $y \epsilon A_i$, we have $\overset{\frown}{xy} \subset A_i$ since A_i is convex.

Suppose that $x \epsilon A_i$, $y \epsilon A_j$, with $i \neq j$. For example, let $x \epsilon A_1$, $y \epsilon A_2$. $\overset{\frown}{xy}$ contains points of A, say q_1 and q_2, where $q_1 \epsilon \overline{p_2p_3}$ and $q_2 \epsilon \overline{p_1p_3}$. Now either $\overset{\frown}{q_1q_2} \subset A$, making $\overset{\frown}{xy}$ in our set (since $\overset{\frown}{xq_1} \subset A_1$, $\overset{\frown}{yq_2} \subset A_2$, and $\overset{\frown}{xy} = \overset{\frown}{xq_1} \vee \overset{\frown}{q_1q_2} \vee \overset{\frown}{q_2y}$), or $\overset{\smile}{q_1q_2} \subset A$, making $\overset{\smile}{xy} \subset A \vee (A_1 \vee A_2 \vee A_3)$, since $\overset{\smile}{xy} \subset \overset{\smile}{q_1q_2}$.

Finally, suppose that $x \epsilon A$, $y \epsilon A_i$. If $x \epsilon A_j$ for any j, the case has already been considered. Thus, let $x \epsilon (B_1 \vee B_2 \vee B_3 \vee P)$. We know that $\overset{\frown}{xy}$ contains a point p of A. Either $\overset{\frown}{xp} \subset A$, which makes $\overset{\frown}{xy} \subset A$ (since $\overset{\frown}{yp} \subset A_i$ and $\overset{\frown}{xy} = \overset{\frown}{xp} \vee \overset{\frown}{xy}$), or $\overset{\smile}{xp} \subset A$, which makes $\overset{\smile}{xy} \subset A$, since $\overset{\smile}{xy} \subset \overset{\smile}{xp}$. Thus, this set is p.c. and so is our original set S.

(e) *Let M_i be the family of lines parallel to a_i which intersect B_i, $i = 1,2,3$. Then some member of some M_i does not separate S, where S is the same as in part (d).* Let us first assume that all of the members of M_1 and M_2 separate S. Let $x \epsilon A' \wedge B_1$ outside the closed strip bounded by a_3 and the line through p_3 parallel to a_3. Now suppose that $y \epsilon A' \wedge B_2$ such that y lies outside the closed triangular region determined by $a_2 \wedge a_3$, $a_3 \wedge \overline{xp_3}$, and $a_2 \wedge \overline{xp_3}$. Such a y exists since $A' \wedge B_2$ must be unbounded by the hypothesis about M_2. $\widehat{xy}$ is not in S, since it contains a point of A. By (d) $\breve{xy}$ must be contained in S, so it follows as in our previous work that $\overline{xy}$ is parallel to a_3. Since $\widehat{p_1p_3} \not\subset A$, let $z \epsilon \widehat{p_1p_3} \wedge A'$. There are points on both $\widehat{yz}$ and $\breve{yz}$ which lie in B_2 but not on $\overline{xy}$, since $\overline{xy} \neq \overline{xz}$. One of these segments must be a subset of A'; but we have already observed that $A' \wedge B_2$ must be a subset of $\overline{xy}$. Hence, we have reached a contradiction and it follows that there is no x outside the closed strip and in $A' \wedge B_1$.

(f) *S is convex or is an open strip plus part of its boundary.* By (d) and (e) S satisfies the conditions of Lemma 8. S cannot be an open half-plane plus part of its boundary since there are points from an A_i in every half-plane. It cannot be contained in a line since the three points of S on $\widehat{p_1p_2}$, $\widehat{p_2p_3}$, and $\widehat{p_1p_3}$ cannot be collinear. Hence, the conclusion holds.

Having established these preliminary results, we can now proceed with the proof of our theorem.

<u>A' has property P_3</u>. We must consider two possible cases.

Suppose that A' does not intersect any A_i. By (f) $A' = S$ is convex or is an open strip plus part of its boundary. In either case A' has property P_3.

Suppose that A' intersects one A_i, say A_1. (It cannot intersect more than one by (c).) Then $A' = (A' \wedge A_1) \vee S$. Both $A' \wedge A_i$ and S are convex. The former is convex by (a) and the latter follows from the fact that both $A' \wedge B_2$ and $A' \wedge B_3$ are bounded, by (b), and hence S cannot be an open strip plus part of its boundary. We have remarked earlier that the union of two convex sets has property P_3.

<u>A has at most two components</u>. Let $x \epsilon A \wedge A_1$ and y_2, y_3 be points on a_2, a_3 such that $\overline{y_2 y_3}$ is parallel to a_1 and $x \epsilon \overline{y_2 y_3}$. Either $\widehat{xy_2} \subset A$ or $\breve{xy_2} \subset A$ so that, in either case, x can be joined to the boundary of A_1 by points of A. Thus, $A \wedge A_1$ is connected. From (e) we have that some $A' \wedge B_i$ is bounded. Let it be $A' \wedge B_3$. There is some member of M_3 which does not pass through $A' \wedge B_3$. Then the internal segment K between a_1 and a_2 lies entirely in A. Since any point in $A \wedge A_1$ and $A \wedge A_2$ can be joined segmentally to K, we have that $A \wedge (A_1 \vee A_2)$ is connected. Let $p \epsilon A \wedge B_1$, and consider the line through p parallel to a_1. There are points $q_2 \epsilon a_2$ and $q_3 \epsilon a_3$ on this line. Either $\widehat{pq_2}$ or $\breve{pq_3}$ is contained in A; in either case, p can be joined to one of the two A_j's adjacent to B_i. Finally, if $r \epsilon P \wedge A$, let s be a point of K such that $\overline{rs}$ separates A_3. If $\widehat{rs} \subset A$, r is connected to K by a segment lying in A; if $\breve{rs} \subset A$, then r is

connected to $A \wedge A_3$. Thus, A consists of at most two components, namely, the ones containing $A \wedge A_3$ and $A \wedge (A_1 \vee A_2)$.

<u>If A is either open or closed, it is connected.</u> Suppose that A is open; then we know that A' is closed. The assumption that $A' \wedge B_1$ and $A' \wedge B_2$ are unbounded is equivalent to saying that all members of M_1 and M_2 separate S. We have seen that this requires $A' \wedge (B_1 \vee B_2)$ to be contained in the strip bounded by a_3 and the line through p_3 parallel to a_3. Since $\widehat{p_1 p_2} \not\subset A$, we know that there is a point x such that $x \epsilon A' \wedge \widehat{p_1 p_2}$. Now if both $A' \wedge B_1$ and $A' \wedge B_2$ are unbounded, we can get a sequence of points in A' having p_1 as a limit which says that $p_1 \epsilon A'$. But this is a contradiction to the fact that $p_1 \epsilon A$. Hence, at most one of the sets $A' \wedge B_i$ is unbounded. As before, let us suppose that $A \wedge (A_1 \vee A_2)$ is contained in a component. $A \wedge A_3$ can be joined to $A \wedge A_1$ or $A \wedge A_2$, since either $A' \wedge B_1$ or $A' \wedge B_2$ is bounded, by points of A. Thus, A is connected.

If A is closed, A' is open. We can use a point of $A' \wedge B_3$ in some neighborhood of x to show that at least one of $A' \wedge B_1$ and $A' \wedge B_2$ is bounded. The remaining part of the proof is the same.

<u>Corollary</u>. A p.c. set A in L_2 has at most two components.

<u>Proof</u>. If A does not have property P_3, the preceding theorem says that A has at most two components. If A has property P_3, suppose that x, y, and z are points in three distinct components. All three of $\widehat{xy}$, $\widehat{yz}$, and $\widehat{xz}$ fail to lie in A, contradicting the assumption that A has property P_3.

Chapter IV

APPLICATIONS OF THE RESULTS FOR L_2 TO SPACES OF ARBITRARY DIMENSION

In this chapter we will make use of the results of the previous chapter to establish the analogous results in arbitrary topological linear spaces. The somewhat slow and tedious proofs for the L_2 case will ease the way for shorter proofs in our present considerations.

First we have the analogue of Theorem 15, part (ii).

<u>Theorem 16</u>. A p.c. set A has at most two components.

<u>Proof</u>. Suppose that C_i, $i = 1,2,3$, are three components of A and $x_i \epsilon C_i$, $i = 1,2,3$. First, suppose that the x_i's are collinear. Then one of them is on the internal segment of the other two. Suppose that $x_2 \epsilon \overset{\frown}{x_1 x_3}$. Then

$$x_2 = \lambda x_1 + (1-\lambda) x_3,$$

where $0<\lambda<1$. Now $\overset{\frown}{x_1 x_3} \not\subset A$, for then C_1 and C_3 would be segmentally connected. Thus, since A is p.c., we have that $\overset{\smile}{x_1 x_3} \subset A$. Since $\overset{\frown}{x_1 x_2} \subset \overset{\smile}{x_1 x_3}$, we have that C_1 and C_2 are segmentally connected, once again a contradiction. Thus, in this case, A can have at most two components.

Now suppose that the x_i's are not collinear. Then they generate a two-dimensional linear variety V which is homeomorphi to L_2. It is clear that $V \wedge C_1$, $V \wedge C_2$, and $V \wedge C_3$ are components of $V \wedge A$, a contradiction to the corollary to Theorem 15.

Theorem 17. If an open or closed p.c. set A is not connected, then each component is convex.

Proof. Let C_1 and C_2 be two components of A and suppose that C_1 is not convex; then there exist points $x, y \in C_1$ such that $\overset{\frown}{xy} \not\subset C_1$. Let z be any member of C_2. First, suppose that x, y, and z are collinear. Then one is on the internal segment of the other two; suppose that $y \in \overset{\frown}{xz}$. Now $\overset{\frown}{yz} \not\subset A$ and $\overset{\frown}{xz} \not\subset A$. Hence, $\overset{\smile}{yz} \subset A$ and $\overset{\smile}{xz} \subset A$. Since $\overset{\frown}{xy} \subset \overset{\frown}{yz}$, we have that $\overset{\frown}{xy} \subset A$. Thus, there is a point $p \in C_2$ such that $p \in \overset{\frown}{xy}$. Since $\overset{\frown}{py} \subset \overset{\frown}{xy}$, it follows that p can be segmentally connected with y, which is a contradiction. The case of $x \in \overset{\frown}{yz}$ is similar to the foregoing. The case of $z \in \overset{\frown}{xy}$ is impossible, since z would then be segmentally connected with x from the fact that $\overset{\frown}{xz} \subset \overset{\frown}{xy}$.

Thus, if x, y, and z are collinear, the theorem holds.

Now suppose that x, y, and z are not collinear. Then they generate a two-dimensional linear variety V which is homeomorphic with L_2. Also, $V \wedge C_1$ and $V \wedge C_2$ are components in the open (or closed) p.c. set $V \wedge A$. But then $V \wedge A$ must either have property P_3 or be connected. If it has property P_3, then we have that $\overset{\frown}{xy} \subset A$ since $\overset{\frown}{xz} \not\subset A$ and $\overset{\frown}{yz} \not\subset A$. This is a contradiction. Now $V \wedge A$ cannot be connected since $V \wedge C_1$ and $V \wedge C_2$ are both open and closed in the relative topology and neither is void.

Hence, the theorem is established.

Theorem 18. If A is p.c., then any component of A is also p.c.

Proof. Suppose that we express A as the union of two components C_1 and C_2. If $x, y \in C_1$, we know that $\overset{\frown}{xy} \subset A$ or $\overset{\smile}{xy} \subset A$. Suppose that $\overset{\frown}{xy} \subset A$; then $\overset{\frown}{xy} \subset C_1$ for otherwise A would be connected. A similar argument holds for the case of $\overset{\smile}{xy} \subset A$.

Chapter V

SOME PROPERTIES OF PROJECTIVE CONVEXITY IN EUCLIDEAN SPACE

Several of the results which were desired require a more restrictive topology. It might appear that merely assuming that our topology is T_1 or perhaps T_2 would not be so specialized as restricting our attention to E_n, but a theorem in topological linear space theory tells us the contrary. The theorem states that all n-dimensional topological linear T_1 spaces with the same scalar field are linearly homeomorphic. This theorem may be found in Taylor [5]. Thus, in particular, the study of real n-dimensional topological linear spaces "reduces" to considerations of E_n, since we are only concerned with algebraic and topological properties of p.c. sets.

We will first prove a lemma and then a theorem for the plane. The lemma holds in L_2 and the proof is the same as is given below. The theorem does not hold even in E_3.

<u>Lemma 9</u>. If A is a connected set in E_2 which has property P_3 but is not convex, then there exist points $p,q,r \in A$ with $\widehat{pq}$ and $\widehat{pr}$ contained in A but $\overset{\smile}{qr}$ not contained in A.

<u>Proof</u>. Since A is not convex, there exist points $t,r \in A$ such that $\widehat{tr} \not\subset A$. Since A has property P_3, every point of A can be

joined to t or r (or both) by an internal segment lying entirely within A. If some point can be joined to both, the lemma holds. Hence, suppose that no point can be joined to both. Let T be the set of all points in A which are joined to t and R be the set of points in A which are joined to r. Suppose that T is not convex; then if $x,y \in T$ such that $\widehat{xy} \not\subset T$, there is a point $z \in \widehat{xy} \wedge R$. By property P_3 we must have that $\widehat{xz} \subset A$, $\widehat{rz} \subset A$, and $\widehat{rx} \subset A$. Hence the lemma holds. Thus, let us suppose that both R and T are convex. Since $R \wedge T = \emptyset$ and A is connected, we know that some point $p \in R$ is an accumulation point of T. But this means that p is linearly accessible from T. (See Klee [5]. A point y is <u>linearly accessible from a subset S of X</u> iff there is a point $x \in S - \{y\}$ such that $[x,y) \subset S$.) Thus, there is a point $q \in T$ such that $\widehat{qr} \not\subset A$, but $\widehat{pr} \subset A$ and $\widehat{pq} \subset A$.

<u>Theorem 19</u>. If A and A' are complementary connected p.c. subsets of the plane, then one of them is convex.

<u>Proof</u>. Since either A or A' has property P_3, let us suppose that A does. If A is not convex, by the preceding lemma there exist $x,y,z \in A$, with $\widehat{xz}$ and $\widehat{yz}$ contained in A, but with a point u of A' on $\widehat{xy}$. Then $\smile{xy} \vee \widehat{xz} \vee \widehat{yz}$ lies in A and separates the plane with A' on one side since A' is connected.

Now if A' is not convex, there exist $p,q \in A'$ with $\widehat{pq}$ not in A'; that is, there exists a point r such that $r \in A \wedge \widehat{pq}$. But $\widehat{pu}$ and $\widehat{qu}$ must lie in A', and $\breve{pq} \vee \widehat{pu} \vee \widehat{qu}$ separates r from x and y, contradicting the fact that A is connected. This completes the proof.

That this theorem does not extent even to E_3 can be seen by considering the two p.c. sets bounded by a hyperboloid of one sheet.

For notational purposes we make the following definitions. Suppose that $\{x_1, x_2, \ldots, x_{n+1}\}$ is a set of completely independent points in E_n. The simplex which this set determines will be denoted by $S(x_1, x_2, \ldots, x_{n+1})$ and the hyperplane through the first n of these points will be denoted by $H(x_1, x_2, \ldots, x_n)$. It might be remarked that E_n is the same as $H(x_1, x_2, \ldots, x_{n+1})$.

Lemma 10. Suppose that A is a p.c. subset of E_n which is not contained in a hyperplane. Let $x_1, x_2, \ldots, x_n$ be vertices of a simplex $S(x_1, x_2, \ldots, x_n)$ which is contained in A. Then this simplex contains a simplex $S(y_1, y_2, \ldots, y_n)$ which is a face of a simplex $S(y_1, y_2, \ldots, y_{n+1})$ lying in A. Moreover, for some arbitrary i, we have $x_i = y_i$.

Proof. Let $S(x_1)$ be a simplex in E_1, with $S(x_1) = \{x_1\} \subset A$. Since A is not contained in a hyperplane (a point in this case), there is a point $y \in A$ distinct from x_1. Either $\overset{\frown}{x_1 y}$ or $\overset{\smile}{x_1 y}$ is contained in A. If the former is the case, the lemma holds. In the latter case let $y_2 = 2x_1 - y$. Then we have

$$S(x_1, y_2) = \overset{\frown}{x_1 y_2} \subset \overset{\smile}{x_1 y} \subset A,$$

so that the lemma holds in E_1.

We now make the inductive assumption by assuming that the lemma holds in E_{n-1}.

Suppose that $x_1, x_2, \ldots, x_n \epsilon A$ and $S(x_1, x_2, \ldots, x_n) \subset A$. Since A is not contained in a hyperplane, we know that there is some point $z \epsilon A$ which is not in $H(x_1, x_2, \ldots, x_n)$. Consider the hyperplane $H(x_2, x_3, \ldots, x_n, z) = H_{n-1}$. $S(x_2, x_3, \ldots, x_n)$ is a simplex in $A \wedge H_{n-1}$ and hence there are points $x_3', x_4', \ldots, x_n', z' \epsilon H_{n-1}$ such that $S(x_2, x_3', \ldots, x_n') \subset S(x_2, x_3, \ldots, x_n)$ and $S(x_2, x_3', \ldots, x_n', z')$ is a subset of $A \wedge H_{n-1}$. Now $S(x_1, x_2, x_3', \ldots, x_n')$, which is contained in $S(x_1, x_2, \ldots, x_n)$, and $S(x_2, x_3', \ldots, x_n', z')$ are faces of $S(x_1, x_2, x_3', \ldots, x_n', z')$ which are contained in A. If no point of A' lies in $S(x_1, x_2, x_3', \ldots, z')$, the lemma holds. If some point p of A' lies in this simplex, it can be represented by

$$p = \delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3' + \ldots + \delta_n x_n' + \delta_{n+1} z',$$

where $\sum_{i=1}^{n+1} \delta_i = 1$ and $0<\delta_i<1$, $i = 1, 2, \ldots, n+1$. Now let us consider the region R defined by

$$R = \{x \mid x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3' + \ldots + \alpha_n x_n' + \alpha_{n+1} z',\ \sum_{i=1}^{n+1} \alpha_i = 1,$$

$$\sum_{i=1}^{n} \alpha_i \geq 1,\ 0 \leq \alpha_2,\ 0 \leq \delta_i \alpha_1 - \delta_1 \alpha_i,\ i = 2, \ldots, n\}.$$

If $y \epsilon A' \wedge R$, we have the following considerations. Let p_1 be the point $\lambda y + (1-\lambda)p$, where λ is defined by

$$\lambda = \frac{\delta_{n+1}}{\delta_{n+1} - \alpha_{n+1}}.$$

Since $\alpha_{n+1} \leq 0$, we have that $\delta_{n+1} - \alpha_{n+1} > 0$. Hence $0<\lambda<1$; that is, $p_1 \epsilon \widehat{yp}$. If we write out the expression for p_1, we have that

$$p_1 = \frac{\alpha_1 \delta_{n+1} - \alpha_{n+1} \delta_1}{\delta_{n+1} - \alpha_{n+1}} x_1 + \ldots + \frac{\alpha_n \delta_{n+1} - \alpha_{n+1} \delta_n}{\delta_{n+1} - \alpha_{n+1}} x_n' + \frac{\alpha_{n+1} \delta_{n+1} - \alpha_{n+1} \delta_{n+1}}{\delta_{n+1} - \alpha_{n+1}} z'$$

Each coefficient is non-negative, making $p_1 \epsilon S(x_1, \ldots, x_n')$. That is, $\widehat{yp} \wedge S(x_1, \ldots, x_n') = \{p_1\}$.

Now let p_2 be defined by $p_2 = \lambda y + (1-\lambda)p$, where $\lambda = \frac{\delta_1}{\delta_1 - \alpha_1}$. Since we know that

$$\delta_1 - \alpha_1 = (\delta_1\alpha_2 - \alpha_1\delta_2) + (\delta_1\alpha_3 - \alpha_1\delta_3) + \ldots + (\delta_1\alpha_{n+1} - \alpha_1\delta_{n+1}),$$

we have $\lambda < 0$; thus, $p_2 \epsilon \breve{yp}$. Substituting the value of λ, we have

$$p_2 = \frac{\delta_1\alpha_2 - \alpha_1\delta_2}{\delta_1 - \alpha_1}x + \ldots + \frac{\delta_1\alpha_{n+1} - \alpha_1\delta_{n+1}}{\delta_1 - \alpha_1}z'.$$

Each coefficient is non-negative and less than one, so we have $p_2 \epsilon S(x_2, \ldots, x_n', z')$. Hence, we have that $\breve{yp} \wedge S(x_2, \ldots, x_n', z')$ is not empty.

Thus, we see that no point can be in $A' \wedge R$, making the entire region R lie in A.

For convenience we will let $x_1 = u_1$, $x_2' = u_2, \ldots,$ $z' = u_{n+1}$. Then for $i = 2,3,\ldots,n$ define y_i by

$$y_i = \frac{\delta_1 + \delta_2 + \ldots + \delta_{n-1}}{\delta_1 + \delta_2 + \ldots + \delta_n}u_i + \ldots + \frac{\delta_n}{\delta_1 + \delta_2 + \ldots + \delta_n}u_n$$

and y_{n+1} by

$$y_{n+1} = \frac{2\delta_1}{\delta_1 + \delta_2 + \ldots + \delta_n}u_1 + \ldots + \frac{2\delta_n}{\delta_1 + \delta_2 + \ldots + \delta_n}u_n + (-1)u_{n+1}.$$

Clearly for $i = 2,3,\ldots,n$ we have $y_i \epsilon S(x_1, x_2', \ldots, x_n')$ so that $S(y_1, y_2, \ldots, y_n) \subset S(x_1, x_2', \ldots, x_n')$ and $x_1 = y_1$. Finally, the coefficients of y_{n+1} are such that $y_{n+1} \epsilon R$, but y_{n+1} is not in $S(x_1, x_2', \ldots, x_n')$. It is clear that we could have modified the foregoing argument so that $x_i = y_i$ for any $i = 1,2,\ldots,n$. Clearly $S(y_1, y_2, \ldots, y_{n+1}) \subset A$.

This completes the induction.

Lemma 11. If A is not contained in a hyperplane, then for any $p \epsilon A$ there is a hyperplane through p which is generated by p and n-1 other points of A.

Proof. There are n+1 points $x_1, x_2, \ldots, x_{n+1}$ in A which generate E_n. In particular, we have that

$$p = \delta_1 x_1 + \delta_2 x_2 + \ldots + \delta_{n+1} x_{n+1}, \quad \sum_{i=1}^{n+1} \delta_i = 1.$$

Suppose that $\delta_1 \neq 0$. Then

$$x = \delta_1^{-1}(p - \delta_2 x_2 - \ldots - \delta_{n+1} x_{n+1}).$$

Now $\{p, x_2, \ldots, x_{n+1}\}$ generates E_n, so there is a hyperplane $H(p, x_2, \ldots, x_n)$ through p and n-1 other points of A.

Lemma 12. Suppose that A is a p.c. subset of E_n which is not contained in a hyperplane. Then every point of A is a vertex of a simplex lying in A.

Proof. For n = 1 the proof is trivial. Let us assume the lemma for E_{n-1}. Let $p_1 \epsilon A$; there are points $x_2, x_3, \ldots, x_n \epsilon A$ such that $\{p_1, x_2, \ldots, x_n\}$ generates a hyperplane. In this hyperplane the inductive assumption says that there is a simplex $S(p_1, p_2, \ldots, p_n) \subset A$. By Lemma 10 there are n+1 points $p_1, q_2, \ldots, q_{n+1}$ such that $S(p_1, q_2, \ldots, q_{n+1}) \subset A$.

Theorem 20. If A is a p.c. subset of E_n, then either A is contained in a hyperplane or $A^- = (\text{Int } A)^-$.

Proof. Suppose that A is not contained in a hyperplane. Let $x \epsilon A$; it is a vertex of a simplex whose interior lies in A. Hence, there is a sequence $\{x_n\}$ in (Int A) which converges to x. Thus, $A \subset (\text{Int } A)^-$. If $x \epsilon A^- - A$, there is a sequence $\{x_n'\}$ in A which converges to x. For each n there is a sequence

$\{x_{ni}\}$ in (Int A) which converges to x_n'. Since we have

$$\operatorname*{Lim}_n \left(\operatorname*{Lim}_i x_{ni}\right) = x,$$

we have that

$$\operatorname*{Lim}_i x_{ii} = x,$$

and thus $x \in (\text{Int } A)^-$. Therefore, we conclude that $A \subset (\text{Int } A)^-$.

Since $(\text{Int } A) \subset A$, we know that $(\text{Int } A)^- \subset A^-$, so that $A^- = (\text{Int } A)^-$.

Corollary. If A is a p.c. subset of E_n, then either A is contained in a hyperplane or A has an interior point.

Proof. If A is not contained in a hyperplane, we have already seen that $A \subset (\text{Int } A)^-$. Since $\emptyset^- = \emptyset$, we have $(\text{Int } A) \neq \emptyset$.

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BIOGRAPHICAL SKETCH

William Ray Hare, Jr. was born June 29, 1936, at Murfreesboro, Arkansas. In May, 1953, he was graduated from Delight High School, Delight, Arkansas. His undergraduate studies were completed in June, 1957, at Henderson State Teachers College, Arkadelphia, Arkansas, where he received the degree of Bachelor of Science summa cum laude. The major fields were mathematics and chemistry and the minor was physics. He was awarded a graduate assistantship in the Department of Mathematics of the University of Florida where he enrolled in June, 1957. He received the Master of Science degree in January, 1959. From June, 1959, until graduation he was a Graduate Council Fellow in the Department of Mathematics. He accepted a position in the Department of Mathematics of Duke University beginning in September, 1961.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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