ORTHOMODULAR LATTICES

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2^M, 2^m	1
$\in$	1
$\subset$, $\supset$	1
M-N	1
T.A.E.	1
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CHAPTER 1

DEFINITIONS AND ELEMENTARY RESULTS

1. From Relations to Orthomodular Lattices

1.1.1 <u>Notation</u>. We assume that the reader has a knowledge of the concepts of set theory (including the various forms of the axiom of choice) and cardinal number theory. The cardinal number of a set M is denoted by the symbol $\#M$. The power set of a set M is denoted by 2^M. If $\#M = m$, then 2^M is sometimes written 2^m.

The symbols $\in$, $\subset$, and $\supset$ are used in their usual set-theoretic sense. The complement of N in M is denoted by M-N. Finally, the symbol T.A.E. stands for "the following statements are equivalent."

1.1.2 <u>Definition</u>. Let X be any set. We define $X \times X$ to be the set $\{(x,y):x,y \in X\}$. By a <u>relation</u> on X we mean a subset of $X \times X$. If R is a relation then we call the relation $\{(x,y):(y,x) \in R\}$ the <u>converse</u> of R and write it as R^{-1}. $\{x:(x,y) \in R\}$ is called the <u>domain</u> of R and $\{y:(x,y) \in R\}$ is called the <u>range</u> of R. Note that the domain of R equals the range of R^{-1} and the range of R equals the domain of

R^{-1}. Let $M \subset X$ and $x \in X$, then MR denotes $\{y:(z,y) \in R$ for some $z \in M\}$ and xR denotes $\{x\}R$. (Here, as in the sequel, relations operate on the right. In keeping with this convention, most functions operate on the right; however, in analytically flavored contexts functions may operate on the left.) If R and S are two relations on X, then we define the <u>composition</u> of R and S, denoted by RS, to be the relation $\{(x,y)$:there exists $z \in X$ such that $(x,z) \in R$ and $(z,y) \in S\}$. By the <u>diagonal</u> of $X \times X$, symbolized by Δ_X or simply Δ if there can be no confusion as to what set is under consideration, we mean $\{(x,x):x \in X\}$. If $Y \subset X$, then the <u>diagonal</u> of $X \times X$ <u>restricted to</u> Y, written $\Delta|_Y$, is defined to be $\{(x,x):x \in Y\}$.

Let R be a relation on X. Then R is said to be <u>reflexive</u> if $\Delta \subset R$; R is said to be <u>symmetric</u> if $R^{-1} \subset R$; R is said to be <u>anti-symmetric</u> if $(x,y) \in R$ and $(y,x) \in R$ imply $x = y$; R is said to be <u>transitive</u> if $RR \subset R$. We sometimes write RR as R^2, $(RR)R$ as R^3, etc.

1.1.3 <u>Remark</u>. (1) Any relation of the form $R \cup R^{-1}$ is a symmetric relation.

(2) Δ is a symmetric relation.

(3) The union of two symmetric relations is again a symmetric relation.

(4) Let R_α, α in some index set I, and T be relations

on X. Then $(\bigcup_{\alpha\in I} R_\alpha)T = \bigcup_{\alpha\in I} (R_\alpha T)$ and

$T(\bigcup_{\alpha\in I} R_\alpha) = \bigcup_{\alpha\in I} (TR_\alpha)$.

The proof follows immediately from the definitions.

1.1.4 <u>Definition</u>. If R is a reflexive, symmetric, and transitive relation on X, then R is called an <u>equivalence relation</u> on X. If R is an equivalence relation, then a set of the form xR is called an <u>equivalence class</u>, the set of all such equivalence classes is denoted by X/R, and an element of such an equivalence class is called a <u>representative</u> of the equivalence class. The following statement is an immediate consequence of the above definition: If R is an equivalence relation, then x is a representative of the equivalence class yR if and only if xR = yR.

If R is reflexive, anti-symmetric, and transitive on X, then R is called a <u>partial ordering</u> on X; in this case (X,R) is called a <u>partially ordered set</u> or simply a <u>poset</u>.

If $\leq$ is a relation on the set P such that $(P,\leq)$ is a poset, then we say that $\leq$ <u>partially orders</u> P. If $x,y\in P$ and $x \leq y$, then y is said to <u>dominate</u> x, x is said to <u>be dominated by</u> y. If at least one of $x \leq y$, $y \leq x$ holds,

then x and y are said to be <u>comparable</u>. If x and y are comparable for all x,y in a subset M of P, then M is called a <u>linearly ordered subset</u> of P or a <u>chain</u> in P. If x is dominated by y and $x \neq y$, then we sometimes write $x < y$. Let $M \subset P$; if $u \in P$ is such that $x \in M$ implies that $x \leq u$, then u is called an <u>upper bound</u> for M; if $v \in P$ is such that $x \in M$ implies that $v \leq x$, then v is called a <u>lower bound</u> for M.

The set of all upper bounds for M is denoted by U(M), i.e.,

$$U(M) = \{z \in P : x \leq z \text{ for all } x \in M\}.$$

The set of all lower bounds for M is denoted by T(M), i.e.,

$$T(M) = \{z \in P : z \leq x \text{ for all } x \in M\}.$$

An element $z \in P$ is called the <u>greatest</u> (<u>largest</u>) element of a subset M of P if $z \in M$ and $z \in U(M)$. An element $z \in P$ is called the <u>least</u> (<u>smallest</u>) element of a subset M of P if $z \in M$ and $z \in T(M)$. An immediate consequence of the fact that $\leq$ is anti-symmetric is the fact that the greatest and least elements of a subset M of P are unique (if they exist). An element z of P is called the <u>supremum</u> (<u>join</u>) of the subset M of P, written sup M, if z is an upper bound for M, and, whenever u is an upper bound for M, $z \leq u$. An element z of P is called the <u>infimum</u> (<u>meet</u>) of the subset M of P, written inf M, if z is a lower bound for M, and, whenever v is a lower bound for M, $v \leq z$.

A poset $(P,\leq)$ is called a <u>lattice</u> in case, for each non-empty finite subset M of P, there exist $z,w\in P$ such that $z = \sup M$ and $w = \inf M$. In case $M = \{x,y\}$, then sup M is sometimes written $x\vee y$ and inf M is sometimes written $x\wedge y$. Also if $M = \{x_1, \ldots, x_n\}$, then sup M and inf M are sometimes written $x_1\vee\ldots\vee x_n$ and $x_1\wedge\ldots\wedge x_n$, respectively. It may be shown that sup and inf satisfy the usual (generalized) associative and commutative laws.

The statement that the supremum (resp., infimum) of a subset M of P exists is clearly equivalent to the statement that U(M) has a least element (resp., T(M) has a greatest element). To say that $z = \sup M$ (resp., $w = \inf M$) is to say that $U(M) = U(\{z\})$ (resp., $T(M) = T(\{w\})$).

A poset $(P,\leq)$ is called a <u>complete lattice</u> if, for every $M \subset P$, there exist $z,w\in P$ such that $z = \sup M$ and $w = \inf M$; if such is the case we say simply that sup M and inf M exist.

A subset M of a lattice $(L,\leq)$ is called a <u>sublattice</u> of L in case $(M,\leq|_M)$ is a lattice. A sublattice M of the lattice L is called <u>subcomplete</u> in case the following obtains: if $N \subset M$ and sup N exists as computed in L, then sup N exists as computed in M, the two are equal, and the common value is in M. A sublattice M of a lattice L may be (1) complete but not subcomplete, (2) subcomplete but not complete, (3) neither, or (4) both.

Let L be the lattice of all subspaces of an infinite dimensional Hilbert space H (with set-theoretic inclusion as the partial order), and let M be the sublattice of L consisting of all closed subspaces of H. Then M is an example of (1). Any lattice which is not complete, considered as a sublattice of itself, is an example of (2). The lattice of all finite subsets of an infinite set X, considered as a sublattice of the power set of X, is an example of (3). Any complete lattice, considered as a sublattice of itself, is an example of (4).

An element 0 (resp., 1) of a poset P with the property that $U(0) = P$ (resp., $T(1) = P$) is called the <u>zero</u> (resp., <u>unit</u>) of P. The uniqueness of the zero (resp., unit), if it exists, is an immediate consequence of the fact that the partial ordering is anti-symmetric. The unit is sometimes called the <u>one</u> of the poset. If there exists a zero and a one in P, then P is called a <u>poset with zero and one</u>. In a poset P with zero and one, $P(0,x)$ denotes $\{z \in P : 0 \leq z \leq x\}$ and $P(x,1)$ denotes $\{z \in P : x \leq z \leq 1\}$.

If P contains a zero element 0, then an element x in P is said to be an <u>atom</u> of P in case $0 < x$ and if $0 < y < x$ for some $y \in P$, then $y = 0$ or $y = x$. If P contains a one element 1, then an element x in P is said to be a <u>co-atom</u> of P in case $x < 1$, and if $x \leq y \leq 1$ for some $y \in P$, then $y = x$ or $y = 1$. P is said to be <u>atomic</u> in case every

non-zero element of P dominates an atom of P. If every non-zero element of P is the join of the atoms it dominates, then the atoms of P are said to be <u>join dense</u>.

If x is an element of a poset P with zero and one, then an element y of P is called a <u>complement</u> of x in P in case $x \vee y$ exists, $x \wedge y$ exists, $x \vee y = 1$, and $x \wedge y = 0$. If every element of a poset P has a complement in P, then P is called a <u>complemented poset</u>. If P is a complemented poset and if, moreover, there is a mapping $':P \longrightarrow P$ such that

(1) $x \leq y$ implies $y' \leq x'$,

(2) $x'' = x$ (where, by definition, $x'' = (x')'$), and

(3) x' is a complement for x,

then P is called an <u>orthocomplemented poset</u>. $':P \longrightarrow P$ is said to be an <u>orthocomplementation</u> on P.

1.1.5 <u>Notation</u>. We reserve the right to use any of the symbols P, $(P,\leq)$, or $(P,\leq,')$ to represent the (orthocomplemented) poset P. By judicious utilization of this standard abuse of terminology, we will single out the salient feature of P under discussion, distinguish between two such structures if necessary, and maintain a minimum of notational overweight.

1.1.6 <u>Definition</u>. Because of the symmetry in the definitions of supremum and infimum, zero and one, we have

the following <u>principle of duality for posets with zero and one</u>:

If, in any valid statement which holds for all posets with zero and one, we interchange the symbols for supremum and infimum, interchange zero and one, and reverse all inequalities, then we obtain another valid statement (called the <u>dual</u> of the original statement) which holds for all posets with zero and one.

Similar principles of duality hold for general posets and for orthocomplemented posets.

1.1.7 <u>Lemma</u>. Let P be an orthocomplemented poset, then P satisfies the Generalized DeMorgan Laws, i.e., for $x_\alpha \in P$ $(\alpha \in I)$, (1) if $\sup\{x_\alpha : \alpha \in I\}$ or $\inf\{x_\alpha' : \alpha \in I\}$ exists in P, then they both exist and $(\sup\{x_\alpha : \alpha \in I\})' = \inf\{x_\alpha' : \alpha \in I\}$; and (2) if $\inf\{x_\alpha : \alpha \in I\}$ or $\sup\{x_\alpha' : \alpha \in I\}$ exists in P, then they both exist and $(\inf\{x_\alpha : \alpha \in I\})' = \sup\{x_\alpha' : \alpha \in I\}$.

<u>Proof</u>. If $\sup\{x_\alpha : \alpha \in I\}$ exists and equals x, then $x \geq x_\alpha$ for all $\alpha \in I$, so that $x' \leq x_\alpha'$ for all $\alpha \in I$. If y is any other lower bound for x_α', then $y' \geq x_\alpha$ for all $\alpha \in I$, $y' \geq x$, and finally $y \leq x'$. Hence $x' = \inf\{x_\alpha' : \alpha \in I\}$. Similarly, if $\inf\{x_\alpha' : \alpha \in I\}$ exists and equals x, then $x' = \sup\{x_\alpha : \alpha \in I\}$. Hence (1) is valid. (2) follows from (1) by duality.

1.1.8 <u>Definition</u>. Two elements x,y of an orthocomplemented poset are said to be <u>orthogonal</u>, written $x \perp y$, in case $x \leq y'$. Note that if $x \perp y$ if and only if $y \perp x$. A family $\{x_\alpha : \alpha \in A\}$ of elements of P is said to be an <u>orthogonal family</u> in case $\alpha \neq \beta$ implies that $x_\alpha \perp x_\beta$. An orthocomplemented poset P is called an <u>orthomodular poset</u> in case P satisfies the following two properties:

(1) if $x,y \in P$ are such that $x \perp y$, then $x \vee y$ exists in P, and

(2) for $x,y \in P$, $x \leq y$ implies $y = x \vee (y' \vee x)'$.

The latter condition is called the <u>orthomodular identity</u>, abbreviated OMI.

An orthomodular (resp., orthocomplemented) poset which is a lattice is called an <u>orthomodular</u> (resp., <u>orthocomplemented</u>) <u>lattice</u>. A sublattice L_1 of an orthomodular lattice L is said to be a <u>sub-orthomodular lattice</u> of L, written $L_1 \leq L$, in case the restriction of the orthocomplementation on L to L_1 makes L_1 an orthomodular lattice. In case $L_1 \leq L$ and $L_1 \neq L$, we sometimes write $L_1 < L$.

2. Standard Results in the Theory of Orthomodular Lattices

1.2.1 <u>Theorem</u>. Let L be an orthocomplemented lattice. Then T.A.E.

(1) L is an orthomodular lattice, i.e., L satisfies the OMI,

$e \leq f$ implies $f = e \vee (f \wedge e')$,

for all $e, f \in L$,

(2) L satisfies the dual orthomodular identity, abbreviated DOMI,

$e \leq f$ implies $e = f \wedge (e \vee f')$,

for all $e, f \in L$,

(3) eCf implies fCe, for all $e, f \in L$,

(4) $e \leq f$ and $f \wedge e' = 0$ implies $e = f$, for all $e, f \in L$,

(5) L does not contain a sublattice of the form

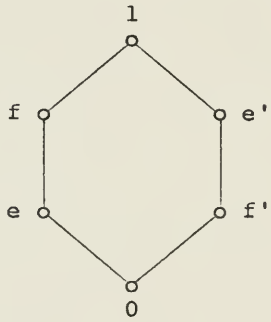


Figure 1

<u>Proof</u>. The equivalence of (1), (2), and (3) is proved in [4, Theorem 1, p. 68]. The equivalence of (4) and (5) is obvious. Moreover, it is clear that (1) implies (4). Hence it suffices to prove that (5) implies (1).

Suppose that (5) is valid but that (1) is invalid. Then there exist $a, f \in L$ such that $a \leq f$ and $f > a \vee (f \wedge a')$. Let $e = a \vee (f \wedge a')$. We claim that $\{0, 1, e, e', f, f'\}$ is a sublattice of L of the type given in

Figure 1. For, $f \geq e$ implies $f \wedge a' \geq e \wedge a'$. Moreover, $e = a \vee (f \wedge a')$ implies $e \geq f \wedge a'$; hence $e \wedge a' \geq (f \wedge a') \wedge a' = f \wedge a'$ and consequently $f \wedge a' = e \wedge a'$. Since $f' \vee e \geq e \geq e \wedge a' = f \wedge a' \geq f \wedge e' = (f' \vee e)'$, it follows that $f' \vee e = 1$, and hence $f \wedge e' = 0$. Therefore $f \vee e' = 1$ and $e \wedge f' = 0$. But $f \wedge f' = e \wedge e' = 0$ and $f \vee f' = e \vee e' = 1$. Hence $\{0,1,e,e',f,f'\}$ is a sublattice of L of the type given in Figure 1 which contradicts the assumption that (5) is valid, i.e., L contains no such sublattice. Consequently (5) implies (1).

1.2.2 <u>Corollary</u>. The atoms of an atomic orthomodular lattice are join dense.

<u>Proof</u>. The result follows from an application of part (4) of Theorem 1.2.1.

1.2.3 <u>Definition</u>. Let L be an orthomodular lattice. Corresponding to every element $f \in L$, we define a mapping $\varphi_f : L \longrightarrow L$ as follows:

$$\text{For } e \in L,\ e\varphi_f = (e \vee f') \wedge f.$$

For $e, f \in L$, we say that e <u>commutes</u> with f if and only if $e\varphi_f = e \wedge f$ in which case we write eCf. Also, for any subset M of L we define

$$C(M) = \{e \in L : eCf \text{ for all } f \in M\}.$$

By $CC(M)$ we mean $C(C(M))$. The set $C(L)$ is called the

<u>center</u> of L. Note that $\{0,1\} \subset C(L) \subset L$ always holds. If $\{0,1\} = C(L)$, then L is said to be <u>irreducible</u>; if $\{0,1\} \subsetneq C(L)$, then L is said to be <u>reducible</u>. If $C(L) = L$, then L is said to be <u>Boolean</u>.

1.2.4 <u>Theorem</u>. Let L be an orthomodular lattice, and let $M,N \subset L$.

(1) $C(M)$ is a subcomplete sub-orthomodular lattice of L.

(2) $M \subset N$ implies $C(N) \subset C(M)$.

(3) $M \subset C(C(M))$.

(4) $C(M) = C(C(C(M)))$.

(5) $M = C(C(M))$ if and only if there exists $N \subset L$ such that $M = C(N)$.

(6) Let M be a sub-orthomodular lattice of L. Then T.A.E.

(a) M is a Boolean lattice,

(b) $M \subset C(M)$,

(c) $C(C(M)) \subset C(M)$.

(7) If $M \subset C(M)$, then $C(C(M))$ is a subcomplete Boolean sub-orthomodular lattice of L.

<u>Proof</u>. (1) is essentially proved in [2, Lemma 3, p. 67]. (2) and (3) follow immediately from Definition 1.2.3; (4), (5), and (6) are immediate consequences of (2) and (3). (7) follows from (1), (4), and (6).

1.2.5 <u>Definition</u>. Let L be an orthomodular lattice, and let $M \subset L$. By a <u>polynomial in M</u>, written Pol(M), is meant any finite combination of suprema and infima of elements of M or primes of such elements, or any finite combination of suprema and infima of such combinations of elements of M or primes of such combinations of elements of M, etc.

It is to be emphasized that, although a polynomial in M may be represented in different ways as combinations of elements of M or the primes of such elements, in any given representation only a finite number of elements of M appear and that the symbols $\vee$, $\wedge$, and ' appear only a finite number of times.

The following proposition contains the elementary properties of polynomials which will be utilized in the sequel.

1.2.6 <u>Proposition</u>. Let L be an orthomodular lattice, and let $M,N \subset L$.

(1) If $M \subset N$, then $\mathrm{Pol}(M) \subset \mathrm{Pol}(N)$.

(2) $\mathrm{Pol}(\mathrm{Pol}(M)) = \mathrm{Pol}(M)$.

(3) If $M \subset \mathrm{Pol}(N)$, then $\mathrm{Pol}(M) \subset \mathrm{Pol}(N)$.

<u>Proof</u>. These results follow immediately from Definition 1.2.5.

1.2.7 <u>Proposition</u>. Let L be an orthomodular lattice, and let $e,f,g \in L$.

(1) T.A.E.

(a) eCf,

(b) fCe,

(c) $e'Cf$.

(2) If $e_\alpha Cf$ for all $\alpha \in I$, then $(\sup \{e_\alpha : \alpha \in I\})Cf$.

(3) If $e \perp f$, then eCf.

(4) $e \in C(L)$ if and only if e has exactly one complement in L.

<u>Proof</u>. (1), (2), and (3) are proved in [2, Lemmata 1, 2, and 3, p. 67]. (4) is proved in [4, Corollary to Theorem 4, p. 71].

1.2.8 <u>Theorem</u> (Foulis-Holland Theorem). Let L be an orthomodular lattice, and let $e,f,g \in L$. If any two of the three relations eCf, eCg, fCg hold, then $(e \vee f) \wedge g = (e \wedge g) \vee (f \wedge g)$ and $(e \wedge f) \vee g = (e \vee g) \wedge (f \vee g)$.

<u>Proof</u>. The proof is found in [2, Theorem 5, p. 68] and in [4, Theorem 3, p. 69].

An immediate consequence of Proposition 1.2.7 and the Foulis-Holland Theorem is the following

1.2.9 <u>Corollary</u>. Let L be an orthomodular lattice, let $M \subset L$ be any family of mutually commuting elements of L,

let $e \in L$ be such that eCf for all $f \in M$, and let $p \in Pol(M)$. Then eCp.

<u>Proof</u>. $p\varphi_e = (p \vee e') \wedge e = (p \wedge e) \vee (e' \wedge e) = p \wedge e$.

1.2.10 <u>Definition</u>. Let L be a lattice, and let $a,b,c \in L$. Then we say that (a,b,c) is a <u>distributive triple</u> and we write $D(a,b,c)$ in case $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$. We say that (b,c) is a <u>modular pair</u> and we write $M(b,c)$ in case $D(c,a,b)$ holds whenever $c \leq b$. L is said to be a <u>distributive</u> (resp., <u>modular</u>) <u>lattice</u> in case $D(a,b,c)$ holds for all $a,b,c \in L$ (resp., $M(b,c)$ holds for all $b,c \in L$).

1.2.11 <u>Remark</u>. An orthomodular lattice is a Boolean lattice if and only if it is a distributive lattice.

<u>Proof</u>. This is an immediate consequence of the Foulis-Holland Theorem.

1.2.12 <u>Lemma</u>. Let L be an orthomodular lattice, and let $M \subset L$ be any family of mutually commuting elements of L. Then $Pol(M)$ is a Boolean sub-orthomodular lattice of L.

<u>Proof</u>. Let $p,q,r \in Pol(M)$. Apply Proposition 1.2.7 to the components of p, q, and r to obtain that p, q, and r commute. Then apply the Foulis-Holland Theorem to obtain $D(p,q,r)$.

1.2.13 <u>Lemma</u>. Let B be a Boolean lattice, let $x \in B$, and let a be an atom of B. Then either $a \leq x$ or $x \leq a'$.

Proof. $a = a\wedge 1 = a\wedge(x\vee x') = (a\wedge x)\vee(a\wedge x')$. Since a is an atom of B, $0 \leq a\wedge x \leq a$ and $0 \leq a\wedge x' \leq a$. If both $a\wedge x = 0$ and $a\wedge x' = 0$, then $a = 0$ which is a contradiction. Hence $a\wedge x = a$ or $a\wedge x' = a$. If both obtain then $a \leq x$ and $a \leq x'$, hence $a = 0$ which is a contradiction. Hence exactly one of $a \leq x$, $x \leq a'$ obtains.

3. Morphisms, Ideals, Filters, and Sections

1.3.1 Definition. Let P and Q be posets. Let $M \subset P$ and $N \subset Q$. Then a mapping f from M into N is said to be an order homomorphism in case $x,y\in M$ and $x \leq y$ imply that $xf \leq yf$. If f has the property that, for all $x,y\in M$, $x\vee y$ exists in M implies $xf\vee yf$ exists in N and $(x\vee y)f = xf\vee yf$, then f is called a join homomorphism. A meet homomorphism is defined dually. If f is both a join homomorphism and a meet homomorphism, then f is said to be a join-meet homomorphism.

If M and N are sub-orthocomplemented posets of P and Q, respectively, then a function $f:M \longrightarrow N$ is said to be an ortho-homomorphism in case $x'f = (xf)'$. If f has the property that, for all $K \subset M$ such that $\sup K$ exists and is in M, $\sup (Kf)$ exists and $(\sup K)f = \sup (Kf)$, then f is said to be a complete join homomorphism. A complete meet homomorphism is defined dually. A homomorphism which is both a complete join homomorphism and a complete meet

homomorphism is said to be a <u>complete join-meet homomorphism</u>.

If f is a homomorphism of any type, then f is said to be a <u>monomorphism</u> from M to N in case $f^{-1}:Mf \longrightarrow M$ is a homomorphism of the same type as f, f is said to be an <u>epimorphism</u> from M to N in case Mf = N, and f is said to be an <u>isomorphism</u> from M to N in case f is both a monomorphism and an epimorphism.

1.3.2 <u>Lemma</u>. Let $(L_1,\leq_1)$ and $(L_2,\leq_2)$ be orthomodular lattices, let S_1 and S_2 be sub-orthomodular lattices of L_1 and L_2, respectively, and let $\theta:S_1\cong S_2$ be an order ortho-isomorphism. Then θ is a join-meet ortho-isomorphism. If, in addition to the above hypotheses, L_1 and L_2 are complete lattices and S_1 and S_2 are subcomplete sublattices, then θ is a complete join-meet ortho-isomorphism.

<u>Proof</u>. We assume that S_1 and S_2 are subcomplete and prove that θ is a complete join-meet ortho-isomorphism. The proof of the other statement is merely a simplification of the proof given.

Let $M \subset S_1$, $m = \sup_1 M$, and $n = \sup_2 (M\theta)$ (where $\sup_i$ indicates the supremum in L_i). Since S_1 and S_2 are subcomplete, $m\in S_1$ and $n\in S_2$; hence there exists $w\in S_1$ such that $w\theta = n$. We must show that $w = m$. From $x \leq_1 m$ for all $x\in M$, it follows that $x\theta \leq_2 m\theta$ for all $x\theta\in M\theta$; hence $\sup_2 (M\theta) = n = w\theta \leq_2 m\theta$, and therefore $w \leq_1 m$. But from

$x\theta \leq n = w\theta$ for all $x\theta \in M\theta$, it follows that $x \leq_1 w$ for all $x \in M$; hence $\sup_1 M = m \leq_1 w$. Therefore $w = m$ and consequently θ is a complete join homomorphism. By a dual argument, θ is a complete meet homomorphism. A similar argument proves that θ^{-1} has these properties; the result follows.

1.3.3 <u>Definition</u>. A subset I of an orthocomplemented poset P is said to be an <u>order ideal</u> of P if and only if $m \in I$ and $n \leq m$ imply $n \in I$. A subset F of P is said to be an <u>order filter</u> of P if and only if $m \in F$ and $m \leq n$ imply $n \in F$.

If I (resp., F) is an order ideal (resp., order filter) on P with $I \neq P$ (resp., $F \neq L$) then I (resp., F) is said to be a <u>proper order ideal</u> (resp., <u>proper order filter</u>) on P. If $I = \{0,1\}$ or $I = P$ (resp., $F = \{0,1\}$ or $F = P$), then I (resp., F) is said to be a <u>trivial order ideal</u> (resp., <u>trivial order filter</u>). An order ideal I (resp., order filter F) is said to be a <u>principal order ideal</u> (resp., <u>principal order filter</u>) in case there exists $x \in P$ such that $I = P(0,x)$ (resp., $F = P(x,1)$).

<u>Note</u>. Let $I \leq L$. I is an order ideal of P if and only if the set $F = \{x \in L : x' \in I\}$ is an order filter of P.

The following terminology is not standard.

1.3.4 <u>Definition</u>. A subset S of the orthocomplemented poset P is said to be a <u>section</u> of P in case $S = I \cup F$ where I is an order ideal of L, F is an order filter of L, and $x \in I$ if and only if $x' \in F$. If $S = L(x,1) \cup L(0,x')$, then S is called a <u>principal section</u> of P and S is denoted by S_x.

1.3.5 <u>Remark</u>. Let P be an orthocomplemented poset. Then every section of P is the union of principal sections of P.

<u>Proof</u>. Let S be a section of P, then $S = I \cup F$ for some order ideal I and some order filter F. Then $S = \cup\{S_x : x \in I\}$.

1.3.6 <u>Lemma</u>. Let P be an orthocomplemented poset, and let S be a section of P. Then S is closed under the orthocomplementation of P.

<u>Proof</u>. By Remark 1.3.5 we need only show that any principal section S_x is closed under the orthocomplementation of P. We may assume $x \neq 0$. If $y \in S_x$, then either $x \leq y$ or $y \leq x'$. If $x \leq y$, then $y' \leq x'$ so that $y' \in S_x$. If $y \leq x'$, then $x \leq y'$ so that $y' \in S_x$. Hence S_x is closed under the orthocomplementation of P.

1.3.7 <u>Remark</u>. If S_x is a principal section of an orthomodular lattice L, then S_x is a sub-orthomodular lattice of L.

<u>Proof</u>. By Lemma 1.1.7 we need only prove that the join (in L) of any two elements in S_x is again in S_x. But this follows immediately from the definition of S_x.

CHAPTER 2

THE PASTE JOB

1. New Lattices from Old

2.1.1 <u>Convention</u>. Throughout this chapter we assume that $(L_1, \leq_1, {}^{\#})$ and $(L_2, \leq_2, {}^{+})$ are two disjoint orthomodular lattices, and that $S_i \subset L_i$ $(i = 1, 2)$ are such that:

(1) $\{0,1\} \subset S_i \neq L_i$ $(i = 1, 2)$;

(2) S_i is closed under the orthocomplementation of L_i $(i = 1, 2)$;

(3) there exists $\theta : S_1 \longrightarrow S_2$ such that θ is an order ortho-isomorphism.

The ambiguity of notation generated by using the symbol S_1 for the principal section and for the subset of L_1 will be resolved by the context in which the symbol appears. Unless otherwise stated S_1 will denote the subset of L_1 given in the above convention.

2.1.2 <u>Definition</u>. (1) Let $L_0 = L_1 \cup L_2$.

(2) Let $P_1 = \{(x,y) \in L_0 \times L_0 : y = x\theta\}$.

(3) Let $\Delta = \{(x,x) : x \in L_0\}$.

(4) Let $P = \Delta \cup P_1 \cup P_1^{-1}$.

2.1.3 <u>Proposition</u>. P is an equivalence relation on L_0.

<u>Proof</u>. (i) P is reflexive since $\Delta \subset P$.

(ii) P is symmetric by Remark 1.1.3.

(iii) To show that P is transitive, we must prove that $PP \subset P$. We first note that $P_1P_1 = \emptyset$ since the range of P_1 is disjoint from the domain of P_1. For a similar reason $P_1^{-1}P_1^{-1} = \emptyset$. We also note that $P_1P_1^{-1} = \Delta|_{S_1}$ and that $P_1^{-1}P_1 = \Delta|_{S_2}$; both identities are immediate from the definition of P_1. We now compute:

$$\begin{aligned}
PP &= (\Delta \cup P_1 \cup P_1^{-1})(\Delta \cup P_1 \cup P_1^{-1}) \\
&= \Delta(\Delta \cup P_1 \cup P_1^{-1}) \cup P_1(\Delta \cup P_1 \cup P_1^{-1}) \cup P_1^{-1}(\Delta \cup P_1 \cup P_1^{-1}) \\
&= \Delta\Delta \cup \Delta P_1 \cup (\Delta P_1^{-1}) \cup P_1\Delta \cup P_1P_1 \cup P_1P_1^{-1} \cup P_1^{-1}\Delta \cup P_1^{-1}P_1 \cup P_1^{-1}P_1^{-1} \\
&= \Delta \cup P_1 \cup P_1^{-1} \cup P_1 \cup P_1P_1 \cup P_1P_1^{-1} \cup P_1^{-1} \cup P_1^{-1}P_1 \cup P_1^{-1}P_1^{-1} \\
&= \Delta \cup P_1 \cup P_1^{-1} \cup P_1P_1 \cup P_1P_1^{-1} \cup P_1^{-1}P_1 \cup P_1^{-1}P_1^{-1} \\
&= (\Delta \cup P_1 \cup P_1^{-1}) \cup \emptyset \cup \Delta|_{S_1} \cup \Delta|_{S_2} \cup \emptyset \\
&= \Delta \cup P_1 \cup P_1^{-1} = P.
\end{aligned}$$

Hence P is transitive.

2.1.4 <u>Definition</u>. (1) Let $L = L_0/P$.

(2) Let $R_1 = \{([x],[y]) \in L \times L$: there exist $x_1 \in [x]$ and $y_1 \in [y]$ such that $x_1 \leq_1 y_1\}$.

(3) Let $R_2 = \{([x],[y]) \in L \times L$: there exist $x_2 \in [x]$ and $y_2 \in [y]$ such that $x_2 \leq_2 y_2\}$.

(4) Let $R = \{([x],[y]) \in L \times L$: there exists $[z] \in L$ such that $([x],[z]) \in R_1 \cup R_2$ and $([z],[y]) \in R_1 \cup R_2\}$.

2.1.5 <u>Remark</u>. $R = (R_1 \cup R_2)^2$.

<u>Proof</u>. This is an immediate consequence of Definition 2.1.4 part (4) and the definition of $(R_1 \cup R_2)^2$.

2.1.6 <u>Lemma</u>. (i) If $y = x\theta$, then $x \in S_1 \subset L_1$ and $y \in S_2 \subset L_2$.

(ii) If $y = x\theta$ and $z = x\theta$, then $y = z$.

(iii) If $y = x\theta$ and $y = z\theta$, then $x = z$.

(iv) If $[x] \in L$, then either $[x]$ is a singleton set whose only element is in $L_1 - S_1$ or in $L_2 - S_2$, or a doubleton set consisting of an element $x_1 \in S_1$ and an element $x_2 \in S_2$ such that $x_1\theta = x_2$.

(v) If $x_1, x_2 \in [x]$, $y_1, y_2 \in [y]$, $x_1\theta = x_2$, and $y_1\theta = y_2$, then the following are equivalent:

(a) $x_1 \leq_1 y_1$,

(b) $x_2 \leq_2 y_2$,

(c) $y_1^{\#} \leq_1 x_1^{\#}$,

(d) $y_2^{+} \leq_2 x_2^{+}$.

<u>Proof</u>. (i) is clear because the domain of θ is contained in S_1 which in turn is a subset of L_1 and the range of θ is contained in S_2 which in turn is a subset of L_2.

(ii) is clear because θ is, among other things, a function.

(iii) is clear because θ is a monomorphism.

(iv) follows immediately from the fact that $P = \Delta \cup P_1 \cup P_1^{-1}$.

In (v), (a) is equivalent to (c) since $^{\#}$ is an orthocomplementation for L_1, and (b) is equivalent to (d) since $^{+}$ is an orthocomplementation for L_2. Moreover (a) is equivalent to (b) since θ is an order isomorphism. Hence all four statements are equivalent.

2.1.7 <u>Notation</u>. If $[z] \in L$ is such that $([x],[z]) \in R_1 \cup R_2$ and $([z],[y]) \in R_1 \cup R_2$, then we say that $[z]$ <u>implements</u> $([x],[y]) \in R$ and we write $[z]:[x]R[y]$.

For $i,j \in \{1,2\}$, if $[z] \in L$ is such that $([x],[z]) \in R_i$ and $([z],[y]) \in R_j$, then we write $[z]:[x]R_iR_j[y]$ and we say that $[z]$ <u>implements</u> $([x],[y]) \in R_iR_j$.

In what follows $[x] = \{x_1\}$, $[x] = \{x_2\}$, and $[x] = \{x_1,x_2\}$ mean, respectively, that $[x] = \{x_1\} \subset L_1 - S_1$, $[x] = \{x_2\} \subset L_2 - S_2$, and $[x] = \{x_1,x_2\}$ where $x_1 \in S_1$, $x_2 \in S_2$, and $x_1\theta = x_2$. We will freely write $[x_1]$ for $[x]$ whenever $x_1 \in [x]$, and $[x_2]$ for $[x]$ whenever $x_2 \in [x]$. A subscript 1 or 2 will always denote the fact that the element in question is in L_1 or in L_2, respectively. The home of unsubscripted elements either will be made explicit or will be left conveniently ambiguous if no confusion is generated by the ambiguity.

2.1.8 <u>Lemma</u>. (1) If $[z]:[x]R[y]$, $[x] = \{x_1\}$, and

$[y] = \{y_2\}$, then $[z] = \{z_1, z_2\}$ and $x_1 \leq_1 z_1$, $z_2 \leq_2 y_2$.

(2) If $[x]R[y]$, $[x] = \{x_1, x_2\}$, and $[y] = \{y_1, y_2\}$, then $[x]:[x]R[y]$ and $[y]:[x]R[y]$.

Proof. Ad (1). One of $[z] = \{z_1\}$, $[z] = \{z_2\}$, or $[z] = \{z_1, z_2\}$ must obtain. If $[z] = \{z_1\}$, then $([z],[y]) \in R_1 \cup R_2$ which is impossible. If $[z] = \{z_2\}$, then $([x],[z]) \in R_1 \cup R_2$ which is impossible. Hence $[z] = \{z_1, z_2\}$, $x_1 \leq_1 z_1$, and $z_2 \leq_2 y_2$.

Ad (2). By the use of the order homomorphism θ, it is easily shown that $x_1 \leq_1 y_1$ and $x_2 \leq_2 y_2$. The result follows.

2.1.9 Proposition. R is a reflexive, anti-symmetric, and transitive relation on L.

Proof. (1) R is reflexive since $[x]:[x]R[x]$.

(2) To prove that R is anti-symmetric, assume that $[z]:[x]R[y]$ and $[w]:[y]R[x]$. We must show that $[x] = [y]$. Note that if $[x]$, $[y]$, $[z]$, and $[w]$ all have representations in L_1, then the following obtains:

(I) $$x_1 \leq_1 z_1 \leq_1 y_1 \leq_1 w_1 \leq_1 x_1.$$

Hence $x_1 = y_1$ and $[x] = [y]$. Note also that if $[x]$, $[y]$, $[z]$, and $[w]$ all have representations in L_2, then the following obtains:

(II) $$x_2 \leq_2 z_2 \leq_2 y_2 \leq_2 w_2 \leq_2 x_2.$$

Hence $x_2 = y_2$ and $[x] = [y]$.

One and only one of the following cases obtains:

Case (1)	$[x] = \{x_1\}$	and	$[y] = \{y_1\}$,
Case (2)	$[x] = \{x_1\}$	and	$[y] = \{y_2\}$,
Case (3)	$[x] = \{x_1\}$	and	$[y] = \{y_1,y_2\}$,
Case (4)	$[x] = \{x_2\}$	and	$[y] = \{y_1\}$,
Case (5)	$[x] = \{x_2\}$	and	$[y] = \{y_2\}$,
Case (6)	$[x] = \{x_2\}$	and	$[y] = \{y_1,y_2\}$,
Case (7)	$[x] = \{x_1,x_2\}$	and	$[y] = \{y_1\}$,
Case (8)	$[x] = \{x_1,x_2\}$	and	$[y] = \{y_2\}$,
Case (9)	$[x] = \{x_1,x_2\}$	and	$[y] = \{y_1,y_2\}$.

Cases (1) and (5), (2) and (4) are symmetric, as are Cases (3), (6), (7), and (8). Accordingly we only consider Cases (1), (2), (3), and (9).

In Case (1), (I) obtains; hence $[x] = [y]$.

In Case (2), by Lemma 2.1.8 part (1), we have $[z] = \{z_1,z_2\}$, $[w] = \{w_1,w_2\}$, and $x_1 \leq_1 z_1$, $z_2 \leq_2 y_2$, $y_2 \leq_2 w_2$, $w_1 \leq_1 x_1$. Hence $w_1 \leq_1 z_1$ and $z_2 \leq_2 w_2$. But, since θ is an order homomorphism, $z_2 \leq_2 w_2 = w_1\theta \leq_2 z_1\theta = z_2$, and so $z_2 = w_2$. Moreover, since θ is a monomorphism, $z_1 = w_1$. Hence $x_1 \leq_1 z_1 = w_1 \leq_1 x_1$ and $y_2 \leq_2 w_2 = z_2 \leq_2 y_2$ imply that $x_1 = z_1$ and $y_2 = z_2$. Consequently $[x] = [x_1] = [z_1] = [z_2] = [y_2] = [y]$ which is the desired result.

In Case (3), (I) also obtains which yields a contradiction since no singleton equivalence class can equal a doubleton equivalence class. Hence Case (3) cannot occur.

In Case (9) we may assume that $[z] = [w] = [x]$ (cf. Lemma 2.1.8 part (2)). Hence (I) occurs and $[x] = [y]$.

Consequently R is anti-symmetric.

(3) To prove that R is transitive, we must show that

(III) $$RR \subset R.$$

But $R = (R_1 \cup R_2)^2$ by Remark 2.1.5, and hence we may compute RR as follows:

$$\begin{aligned}
RR &= R(R_1 \cup R_2)^2 \\
&= R(R_1^2 \cup R_1R_2 \cup R_2R_1 \cup R_2^2) \\
&= RR_1^2 \cup RR_1R_2 \cup RR_2R_1 \cup RR_2^2 \\
&= (R_1 \cup R_2)^2 R_1^2 \cup (R_1 \cup R_2)^2 R_1R_2 \cup (R_1 \cup R_2)^2 R_2R_1 \cup (R_1 \cup R_2)^2 R_2^2 \\
&= (R_1^2 \cup R_1R_2 \cup R_2R_1 \cup R_2^2)R_1^2 \cup (R_1^2 \cup R_1R_2 \cup R_2R_1 \cup R_2^2)R_1R_2 \\
&\quad \cup (R_1^2 \cup R_1R_2 \cup R_2R_1 \cup R_2^2)R_2R_1 \cup (R_1^2 \cup R_1R_2 \cup R_2R_1 \cup R_2^2)R_2^2 \\
&= (R_1^4 \cup R_1R_2R_1^2 \cup R_2R_1^3 \cup R_2^2R_1^2) \cup (R_1^3R_2 \cup (R_1R_2)^2 \cup R_2R_1^2R_2 \cup R_2^2R_1R_2) \\
&\quad \cup (R_1^2R_2R_1 \cup R_1R_2^2R_1 \cup (R_2R_1)^2 \cup R_2^3R_1) \cup (R_1^2R_2^2 \cup R_1R_2^3 \cup R_2R_1R_2^2 \cup R_2^4).
\end{aligned}$$

Hence (III) is equivalent to

(IV) $$\begin{aligned}
&R_1^4 \cup R_1R_2R_1^2 \cup R_2R_1^3 \cup R_2^2R_1^2 \cup R_1^3R_2 \cup (R_1R_2)^2 \cup R_2R_1^2R_2 \cup R_2^2R_1R_2 \\
&\quad \cup R_1^2R_2R_1 \cup R_1R_2^2R_1 \cup (R_2R_1)^2 \cup R_2^3R_1 \cup R_1^2R_2^2 \cup R_1R_2^3 \cup R_2R_1R_2^2 \cup R_2^4 \\
&\qquad \subset R_1^2 \cup R_1R_2 \cup R_2R_1 \cup R_2^2.
\end{aligned}$$

To prove (IV) we consider 16 cases corresponding to the 16 sets exhibited on the left hand side of (IV). In each case we will show that the set in question is contained in one of the sets on the right hand side of (IV).

<u>Case (1)</u>. If $([x],[y]) \in R_1^4$, then there exist

$[p],[q],[r] \in L$ such that

$([x],[p]),([p],[q]),([q],[r]),([r],[y]) \in R_1$. Hence there exist $x_1 \in [x]$, $p_1 \in [p]$, $q_1 \in [q]$, $r_1 \in [r]$, and $y_1 \in [y]$ such that $x_1 \leq_1 p_1 \leq_1 q_1 \leq_1 r_1 \leq_1 y_1$ which implies $x_1 \leq_1 q_1 \leq_1 y_1$. Therefore $([x],[q]),([q],[y]) \in R_1$, $[q]:[x]R_1^2[y]$, and $([x],[y]) \in R_1^2$.

<u>Case (2)</u>. If $([x],[y]) \in R_1R_2R_1^2$, then there exist $[p],[q],[r] \in L$ such that $([x],[p]) \in R_1$, $([p],[q]) \in R_2$, and $([q],[r]),([r],[y]) \in R_1$. By definition of R_1 and R_2, $[p] = \{p_1,p_2\}$, $[q] = \{q_1,q_2\}$, $x_1 \leq_1 p_1$, $p_2 \leq_2 q_2$, and $q_1 \leq_1 r_1 \leq_1 y_1$. Hence (via θ) $p_1 \leq_1 q_1$ and therefore $x_1 \leq_1 p_1 \leq_1 q_1 \leq_1 r_1 \leq_1 y_1$ which implies that $[p]:[x]R_1^2[y]$, i.e., $([x],[y]) \in R_1^2$.

<u>Case (3)</u>. If $([x],[y]) \in R_2R_1^3$, then there exist $[p],[q],[r] \in L$ such that $([x],[p]) \in R_2$ and $([p],[q]),([q],[r]),([r],[y]) \in R_1$. Hence $[p] = \{p_1,p_2\}$, $x_2 \leq_2 p_2$, and $p_1 \leq_1 q_1 \leq_1 r_1 \leq_1 y_1$. Therefore $[p]:[x]R_2R_1[y]$, i.e., $([x],[y]) \in R_2R_1$.

<u>Case (4)</u>. If $([x],[y]) \in R_2^2R_1^2$, then there exist $[p],[q],[r] \in L$ such that $([x],[p]),([p],[q]) \in R_2$ and $([q],[r]),([r],[y]) \in R_1$. Hence $[q] = \{q_1,q_2\}$, $x_2 \leq_2 p_2 \leq_2 q_2$, and $q_1 \leq_1 r_1 \leq_1 y_1$ which implies that $[q]:[x]R_2R_1[y]$.

By arguments similar to those used in Cases (1) - (4) we may conclude the following:

Case (5). If $([x],[y])\in R_1^3R_2$, then there exists $[r]\in L$ such that $[r]:[x]R_1R_2[y]$.

Case (6). If $([x],[y])\in(R_1R_2)^2$, then there exists $[r]\in L$ such that $[r]:[x]R_1R_2[y]$.

Case (7). If $([x],[y])\in R_2R_1^2R_2$, then there exists $[r]\in L$ such that $[r]:[x]R_2^2[y]$.

Case (8). If $([x],[y])\in R_2^2R_1R_2$, then there exists $[r]\in L$ such that $[r]:[x]R_2^2[y]$.

For $n = 1, 2, \ldots, 8$, Case $16-(n-1)$ is obtained from Case n by interchanging the subscripts 1 and 2. Consequently R is transitive.

2.1.10 Notation. Since, by the preceding lemma, R is a partial ordering on L, we write $[x] \leq [y]$ whenever $([x],[y])\in R$. In particular, we will henceforth write $[z]:[x] \leq [y]$ for $[z]:[x]R[y]$.

Moreover, in what follows, when considering a relation such as $[z]:[x] \leq [y]$, the statement "we may assume that there exist $x_1\in[x]$, $z_1\in[z]$ such that $x_1 \leq_1 z_1$" will sometimes be abbreviated to the statement "we may assume $x_1 \leq_1 z_1$"; also in case analyses the abbreviated form "$x_1 \leq_1 z_1$" means "there exist $x_1\in[x]$, $z_1\in[z]$ such that $x_1 \leq_1 z_1$."

2.1.11 Definition. Let 0_i (resp., 1_i) be the zero (resp., unit) element of L_i $(i = 1, 2)$. Define $[0]$ to be $[0_1]$

and $[1]$ to be $[1_1]$. Define $':L \longrightarrow L$ by the following:

$$[x]' = \begin{cases} [x_1^{\#}] \text{ if there exists } x_1 \in L_1 \text{ such that } x_1 \in [x]. \\ [x_2^{+}] \text{ if there exists } x_2 \in L_2 \text{ such that } x_2 \in [x]. \end{cases}$$

2.1.12 <u>Notation</u>. We henceforth write 0_i and 1_i $(i = 1, 2)$ as 0 and 1, respectively. When euphony demands, the notation $\{0,1\}_i$ will be utilized to denote $\{0_i, 1_i\}$. In other occurrences, if a distinction is necessary it will be clear from the context.

2.1.13 <u>Lemma</u>. $[0]$ and $[1]$ are the zero and unit elements, respectively, of L.

<u>Proof</u>. If $[x] \in L$, then $[x] = [x_1]$ or $[x] = [x_2]$. If $[x] = [x_1]$, then $[0] \leq [x]$ since $0 \leq_1 x_1$; if $[x] = [x_2]$. we may similarly conclude that $[0] \leq [x]$. Hence $[0]$ is the zero element of L. Dually, $[1]$ is the unit element of L.

2.1.14 <u>Proposition</u>. $':L \longrightarrow L$ is a well-defined orthocomplementation for the poset L.

<u>Proof</u>. $':L \longrightarrow L$ is well-defined. For, if $[x] = \{x_1\}$, or if $[x] = \{x_2\}$, then there is no ambiguity in the definition of $[x]'$ which is $\{x^{\#}\}$ or $\{x^{+}\}$, respectively. If $[x] = \{x_1, x_2\}$, then, since $x_1^{\#}\theta = x_2^{+}$, $[x]'$ is unambiguous.

Now suppose that $[x] \leq [a]$ and $[x]' \leq [a]$. Assume $[z]:[x] \leq [a]$ and $[w]:[x]' \leq [a]$. We may assume

that $x_1 \leq_1 z_1$ and consider only the following eight possibilities:

Case (1)	$z_1 \leq_1 a_1$,	$x_1^{\#} \leq_1 w_1$,	$w_1 \leq_1 a_1$,
Case (2)	$z_1 \leq_1 a_1$,	$x_1^{\#} \leq_1 w_1$,	$w_2 \leq_2 a_2$,
Case (3)	$z_1 \leq_1 a_1$,	$x_2^{+} \leq_2 w_2$,	$w_1 \leq_1 a_1$,
Case (4)	$z_1 \leq_1 a_1$,	$x_2^{+} \leq_2 w_2$,	$w_2 \leq_2 a_2$,
Case (5)	$z_2 \leq_2 a_2$,	$x_1^{\#} \leq_1 w_1$,	$w_1 \leq_1 a_1$,
Case (6)	$z_2 \leq_2 a_2$,	$x_1^{\#} \leq_1 w_1$,	$w_2 \leq_2 a_2$,
Case (7)	$z_2 \leq_2 a_2$,	$x_2^{+} \leq_2 w_2$,	$w_1 \leq_1 a_1$,
Case (8)	$z_2 \leq_2 a_2$,	$x_2^{+} \leq_2 w_2$,	$w_2 \leq_2 a_2$.

<u>Ad (1)</u>. $x_1 \leq_1 a_1$ and $x_1^{\#} \leq_1 a_1$ imply $[a] = [1]$.

<u>Ad (2)</u>. $[w] = \{w_1, w_2\}$ and $[a] = \{a_1, a_2\}$. Hence $x_1 \leq_1 z_1 \leq_1 a_1$ and $x_1^{\#} \leq_1 w_1 \leq_1 a_1$. Therefore $a_1 = 1$ and $[a] = [1]$.

<u>Ad (3)</u>. $[x] = \{x_1, x_2\}$ and $[w] = \{w_1, w_2\}$. Hence $x_1 \leq_1 z_1 \leq_1 a_1$ and $x_1^{\#} \leq_1 w_1 \leq_1 a_1$. Therefore $a_1 = 1$ and $[a] = [1]$.

<u>Ad (4)</u>. $[x] = \{x_1, x_2\}$ and $[a] = \{a_1, a_2\}$. Hence $x_1 \leq_1 z_1 \leq_1 a_1$ which implies $x_2 \leq_2 a_2$. But $x_2^{+} \leq_2 a_2$ and hence $a_2 = 1$. Consequently $[a] = [1]$.

<u>Ad (5)</u>. $[a] = \{a_1, a_2\}$ and $[z] = \{z_1, z_2\}$. Hence $x_1 \leq_1 z_1 \leq_1 a_1$ and $x_1^{\#} \leq_1 w_1 \leq_1 a_1$. Therefore $a_1 = 1$ and $[a] = [1]$.

<u>Ad (6)</u>. $[z] = \{z_1, z_2\}$ and $[w] = \{w_1, w_2\}$. Hence $x_1 \leq_1 z_1$ which implies $z_1^{\#} \leq_1 x_1^{\#} \leq_1 w_1$ and therefore

$z_2^+ \leq_2 w_2 \leq_2 a_2$. But $z_2 \leq_2 a_2$ and hence $a_2 = 1$. Consequently $[a] = [1]$.

<u>Ad (7)</u>. $[x] = \{x_1, x_2\}$, $[w] = \{w_1, w_2\}$, $[z] = \{z_1, z_2\}$, and $[a] = \{a_1, a_2\}$. Hence $x_1 \leq_1 z_1 \leq_1 a_1$ and $x_1^\# \leq_1 w_1 \leq_1 a_1$. Therefore $a_1 = 1$ and $[a] = [1]$.

<u>Ad (8)</u>. $[x] = \{x_1, x_2\}$ and $[z] = \{z_1, z_2\}$. Hence $x_2 \leq_2 z_2 \leq_2 a_2$, $x_2^+ \leq_2 w_2 \leq_2 a_2$. Therefore $a_2 = 1$ and $[a] = [1]$.

Hence $[x] \vee [x]'$ exists and equals $[1]$. Dually, $[x] \wedge [x]'$ exists and equals $[0]$.

Clearly $':L \longrightarrow L$ is an involution since $^\#:L_1 \longrightarrow L_1$ and $^+:L_2 \longrightarrow L_2$ are involutions. The verification that $':L \longrightarrow L$ is anti-automorphic is immediate. For if $[z]:[x] \leq [y]$, then we may assume $x_1 \leq_1 z_1$ and we need only check the cases $z_1 \leq_1 y_1$ and $z_2 \leq_2 y_2$. In the former case $x_1 \leq_1 y_1$ implies $y_1^\# \leq_1 x_1^\#$; hence $[y]' \leq [x]'$. In the latter case $y_2^+ \leq_2 z_2^+$ and $z_1^\# \leq_1 x_1^\#$ imply that $[y]' \leq [x]'$. Hence $':L \longrightarrow L$ is an orthocomplementation for the poset L.

We now assume that the S_i are proper suborthomodular lattices of L_i. Then, by Lemma 1.3.2, the order ortho-isomorphism θ becomes a join-meet ortho-isomorphism. This allows us to make statements such as the following: if $[x] = \{x_1, x_2\}$ and $[y] = \{y_1, y_2\}$, then

$(x_1 \vee_1 y_1)\theta = x_2 \vee_2 y_2$, and $(x_1 \wedge_1 y_1)\theta = x_2 \wedge_2 y_2$, i.e., $[x_1 \vee_1 y_1] = [x_2 \vee_2 y_2]$, and $[x_1 \wedge_1 y_1] = [x_2 \wedge_2 y_2]$.

2.1.15 <u>Lemma</u>. Let $\{0,1\}_i \leq S_i \prec L_i$ $(i = 1, 2)$. Let $[x] \in S$ and $[y] \in S$ so that $[x] = \{x_1, x_2\}$ and $[y] = \{y_1, y_2\}$. Then the following obtains:

(1) $[x] \vee [y]$ exists in L, and

$[x] \vee [y] = [x_1 \vee_1 y_1] = [x_2 \vee_2 y_2]$;

(2) $[x] \wedge [y]$ exists in L, and

$[x] \wedge [y] = [x_1 \wedge_1 y_1] = [x_2 \wedge_2 y_2]$.

<u>Proof</u>. Let $[x] = \{x_1, x_2\}$ and $[y] = \{y_1, y_2\}$. Then, since S_i is a sublattice of L_i $(i = 1, 2)$, $x_1, y_1 \in S_1$ implies $x_1 \vee_1 y_1 \in S_1$; $x_2, y_2 \in S_2$ implies $x_2 \vee_2 y_2 \in S_2$. Moreover $(x_1 \vee_1 y_1)\theta = x_1\theta \vee_2 y_1\theta = x_2 \vee_2 y_2$, hence $x_1, y_1 \leq_1 x_1 \vee_1 y_1$ implies $[x], [y] \leq [x_1 \vee_1 y_1] = [x_2 \vee_2 y_2]$.

Now suppose that $[x], [y] \leq [z]$. The following statements are valid regardless of the composition of the elements of L which implement $[x] \leq [z]$ and $[y] \leq [z]$.

If $[z] = \{z_1\}$ or if $[z] = \{z_1, z_2\}$, then $x_1 \leq_1 z_1$ and $y_1 \leq_1 z_1$. In this case it follows that $x_1 \vee_1 y_1 \leq_1 z_1$, and hence $[x_1 \vee_1 y_1] \leq [z]$.

If $[z] = \{z_2\}$, then $x_2 \leq_2 z_2$ and $y_2 \leq_2 z_2$. In this case it follows that $x_2 \vee_2 y_2 \leq_2 z_2$, and hence $[x_2 \vee_2 y_2] \leq [z]$.

We have shown that, for $[x],[y] \in S$, $[x],[y] \leq [x_1 \vee_1 y_1] = [x_2 \vee_2 y_2]$, and if $[x],[y] \leq [z]$ then $[x_1 \vee_1 y_1] \leq [z]$. Hence we conclude that $[x] \vee [y]$ exists and equals $[x_1 \vee_1 y_1]$.

2.1.16 <u>Lemma</u>. Let $\{0,1\}_i \leq S_i < L_i$ $(i = 1, 2)$. Let $[x] \in S$ and $[y] \in L-S$, then $[x] = \{x_1, x_2\}$ and either $[y] = \{y_1\}$ or $[y] = \{y_2\}$.

(1) If $[y] = \{y_1\}$, then $[x] \vee [y]$ exists in L and $[x] \vee [y] = [x_1 \vee_1 y_1]$;

(2) if $[y] = \{y_2\}$, then $[x] \vee [y]$ exists in L and $[x] \vee [y] = [x_2 \vee_2 y_2]$;

(3) if $[y] = \{y_1\}$, then $[x] \wedge [y]$ exists in L and $[x] \wedge [y] = [x_1 \wedge_1 y_1]$;

(4) if $[y] = \{y_2\}$, then $[x] \wedge [y]$ exists in L and $[x] \wedge [y] = [x_2 \wedge_2 y_2]$.

<u>Proof</u>. <u>Ad (1)</u>. $[x] = \{x_1, x_2\}$ and $x_1, y_1 \leq_1 x_1 \vee_1 y_1$ imply that $[x],[y] \leq [x_1 \vee_1 y_1]$. Now assume $[x],[y] \leq [z]$. Let these inequalities be implemented by $[u]$ and $[v]$, respectively. If $[z] = \{z_1\}$ or if $[z] = \{z_1, z_2\}$, then $x_1 \leq_1 z_1$ and $y_1 \leq_1 z_1$; hence $x_1 \vee_1 y_1 \leq_1 z_1$ which implies that $[x_1 \vee_1 y_1] \leq [z]$. If $[z] = \{z_2\}$, then $[v] = \{v_1, v_2\}$, $y_1 \leq_1 v_1$, $v_2 \leq_2 z_2$, and $x_2 \leq_2 z_2$; $x_1 \vee_1 y_1 \leq_1 x_1 \vee_1 v_1$ and $(x_1 \vee_1 v_1)\theta = x_2 \vee_2 v_2 \leq_2 z_2$; hence $[x_1 \vee_1 y_1] \leq [z]$.

Hence $[x] \vee [y]$ exists and equals $[x_1 \vee_1 y_1]$.

A similar argument proves (2). (3) and (4) are dual.

2.1.17 <u>Lemma</u>. Let $\{0,1\}_i \leq S_i < L_i$ $(i = 1, 2)$.

(1) If $[x] = \{x_1\}$ and $[y] = \{y_1\}$, then $[x] \vee [y]$ exists in L and $[x] \vee [y] = [x_1 \vee_1 y_1]$.

(2) If $[x] = \{x_2\}$ and $[y] = \{y_2\}$, then $[x] \vee [y]$ exists in L and $[x] \vee [y] = [x_2 \vee_2 y_2]$.

(3) If $[x] = \{x_1\}$ and $[y] = \{y_1\}$, then $[x] \wedge [y]$ exists in L and $[x] \wedge [y] = [x_1 \wedge_1 y_1]$.

(4) If $[x] = \{x_2\}$ and $[y] = \{y_2\}$, then $[x] \wedge [y]$ exists in L and $[x] \wedge [y] = [x_2 \wedge_2 y_2]$.

<u>Proof</u>. We need only prove (1) since (2) follows by symmetry; (3) and (4) are dual.

Clearly $[x],[y] \leq [x_1 \vee_1 y_1]$. Now assume that $[x],[y] \leq [z]$ where $[u]$ and $[v]$ are the elements of L which, respectively, implement the inequalities. Then both $[u]$ and $[v]$ have representatives in u_1 and v_1, respectively, in L_1. $x_1 \leq_1 u_1$ and $y_1 \leq_1 v_1$ imply $x_1 \vee_1 y_1 \leq_1 u_1 \vee_1 v_1$. If $[z] = \{z_1\}$ or if $[z] = \{z_1,z_2\}$, then clearly $[u_1 \vee_1 v_1]:[x_1 \vee_1 y_1] \leq [z]$. If $[z] = \{z_2\}$, then $[u] = \{u_1,u_2\}$, $[v] = \{v_1,v_2\}$, and $u_2 \vee_2 v_2 \leq_2 z_2$; hence $[u_1 \vee_1 v_1]:[x_1 \vee_1 y_1] \leq [z]$. Consequently, in this case, $[x] \vee [y]$ exists and equals $[x_1 \vee_1 y_1]$.

2.1.18 <u>Definition</u>. Let $S_1 < L_1$, and $S_2 < L_2$. Then we say that S_1 and S_2 are <u>corresponding sections</u> of L_1 and L_2 if and only if there exist $M_i \subset S_i$ $(i = 1, 2)$ such that

(1) $M_1\theta = M_2$,

(2) $S_1 = \bigcup_{m\in M_1} S_{m^\#}$ and $S_2 = \bigcup_{m\in M_2} S_{m^+}$, and

(3) $\theta|_{S_{m^\#}}:S_{m^\#}\cong S_{(m\theta)^+}$ is an ortho-isomorphism for each $m\in M_1$.

<u>Warning</u>. Although a section need not, in general, be a sublattice, in order for S_1 and S_2 to be corresponding sections of L_1 and L_2, S_1 and S_2 must be sub-orthomodular lattices of L_1 and L_2, respectively.

2.1.19 <u>Lemma</u>. Assume that S_1 and S_2 are corresponding sections of L_1 and L_2. Under this assumption if $[x],[y]\in L$ are such that $[x] = \{x_1\}$ and $[y] = \{y_2\}$, then $[x] \not\leq [y]$.

<u>Proof</u>. Suppose the statement is false. Then there exists $[z]$ such that $[z]:[x] \leq [y]$, $[x] = \{x_1\}$, and $[y] = \{y_2\}$. It follows that $[z] = \{z_1,z_2\}$, $x_1 \leq_1 z_1$, $z_2 \leq_2 y_2$, $z_1\in S_1$, and $z_2\in S_2$. Hence $z_1\in S_{m^\#}$ and $z_2\in S_{(m\theta)^+}$ for some $m\in M_1$, so that either $m^\# \leq_1 z_1$ or $z_1 \leq_1 m$. If $m^\# \leq_1 z_1$, then $(m\theta)^+ \leq_2 z_2 \leq_2 y_2$. Therefore $y_2\in S_{(m\theta)^+} \subset S_2$ and hence $[y]\in S$, which is a contradiction. If $z_1 \leq_1 m$,

then $x_1 \leq_1 z_1 \leq_1 m$; and consequently $x_1 \in S_{m\#} \subset S_1$, i.e., $[x] \in S$. This is a contradiction. Therefore no such $[z]$ exists and $[x] \not\leq [y]$.

2.1.20 <u>Corollary</u>. Assume that S_1 and S_2 are corresponding sections of L_1 and L_2. Under this assumption if $[x],[y] \in L$ are such that $[x] = \{x_1\}$ and $[y] = \{y_2\}$, then $U([x],[y]) \subset S$.

<u>Proof</u>. Suppose there exists $[u] \in U([x],[y])$ such that $[u] \notin S$. Then either $[u] = \{u_1\}$ or $[u] = \{u_2\}$. If $[u] = \{u_1\}$, then $[y] \leq [u]$, contradicting Lemma 2.1.19. If $[u] = \{u_2\}$, then $[x] \leq [u]$, again contradicting Lemma 2.1.19. Hence no such $[u]$ exists and $U([x],[y]) \subset S$.

2.1.21 <u>Proposition</u>. Let S_1 and S_2 be corresponding sections of L_1 and L_2. Then L is an orthomodular poset.

<u>Proof</u>. By Proposition 2.1.14 L is an orthocomplemented poset. Hence we need only show that

(1) if $[e],[f] \in L$ and $[e] \perp [f]$, then $[e] \vee [f]$ exists, and

(2) if $[e],[f] \in L$ and $[e] \leq [f]$, then $[f] = [e] \vee ([f]' \vee [e])'$.

<u>Ad (1)</u>. If $[e] \perp [f]$, then $[e] \leq [f]'$. If there exists $e_1 \in [e]$, then by Lemma 2.1.19 there exists $f_1 \in [f]$, and by one of Lemmata 2.1.15, 2.1.16, or 2.1.17,

$[e]\vee[f]$ exists (and equals $[e_1 \vee_1 f_1]$). If $[e] = \{e_2\}$, then by Lemma 2.1.19, there exists $f_2 \in [f]$ and hence by one of Lemmata 2.1.15, 2.1.16, or 2.1.17, $[e]\vee[f]$ exists (and equals $[e_2 \vee_2 f_2]$).

Ad (2). Assume that $[x] \leq [y]$. We will prove that $[y] = [x]\vee([x]\vee[y]')'$. We need only consider two cases:

Case (i) $[x]$ and $[y]$ both have a representative in L_1.

Case (ii) $[x] = \{x_1\}$ and $[y] = \{y_2\}$.

Ad (i). $x_1 \leq_1 y_1$ and hence, by Lemmata 2.1.15, 2.1.16, and 2.1.17, since L_1 satisfies the OMI,

$$[x]\vee([x]\vee[y]')' = [x_1]\vee([x_1]\vee[y_1^{\#}])' = [x_1]\vee[x_1 \vee_1 y_1^{\#}]'$$
$$= [x_1]\vee[x_1^{\#} \wedge_1 y_1] = [y_1] = [y].$$

Ad (ii). By Lemma 2.1.19 this case cannot obtain.

Since the OMI is satisfied in all consistent cases, L is an orthomodular poset.

2.1.22 Theorem. Assume that S_1 and S_2 are corresponding sections of L_1 and L_2. Moreover, assume L_i is complete and that S_i is a subcomplete sub-orthomodular lattice of L_i $(i = 1, 2)$. Then L is an orthomodular lattice.

Proof. Since L is an orthomodular poset, to show that L is an orthomodular lattice we need only show that $[x]\vee[y]$ exists for all $[x],[y] \in L$. Let $[x],[y] \in L$. Exactly one of the following must occur:

(1) $[x],[y] \in S$,

(2) $[x] \in S$, $[y] \in L-S$ (re-label $[x]$ and $[y]$ if necessary),

(3) $[x],[y] \in L-S$.

<u>Ad (1)</u>. By Lemma 2.1.15 $[x] \vee [y]$ exists.

<u>Ad (2)</u>. By Lemma 2.1.16 $[x] \vee [y]$ exists.

<u>Ad (3)</u>. There are four cases:

Case (i) $[x] = \{x_1\}$ and $[y] = \{y_1\}$,

Case (ii) $[x] = \{x_1\}$ and $[y] = \{y_2\}$,

Case (iii) $[x] = \{x_2\}$ and $[y] = \{y_1\}$,

Case (iv) $[x] = \{x_2\}$ and $[y] = \{y_2\}$.

By symmetry we need only consider Cases (i) and (ii).

<u>Ad (i)</u>. By Lemma 2.1.17 $[x] \vee [y]$ exists.

<u>Ad (ii)</u>. First note that $U([x],[y])$ is non-empty since it contains $[1]$, and that, by Corollary 2.1.20, $U([x],[y]) \subset S$. Let $M_1 = \{z_1$:there exists $[z] \in U([x],[y])$ such that $[z] = \{z_1,z_2\}\}$. Now $\inf_1 M_1$ (as computed in L_1) exists since L_1 is complete. But since S_1 is subcomplete, $\inf M_1$, as computed in S_1, exists and equals $\inf_1 M_1$. Let $z^{(1)} = \inf_1 M_1$. Let $z^{(2)} = z^{(1)}\theta$, let $[z]_0 = [z^{(1)}] = [z^{(2)}]$, and let $M_2 = \{z_2$:there exists $[z] \in U([x],[y])$ such that $[z] = \{z_1,z_2\}\}$. Then by Lemma 1.3.2 $z^{(2)} = \inf_2 \{z_1\theta : z_1 \in M_1\} = \inf_2 M_2 = \inf M_2$, as computed in S_2 since S_2 is subcomplete.

We claim that $U([z]_0) = U([x],[y])$. For, $x_1 \leq_1 \inf_1 M_1 = z^{(1)}$ and $y_2 \leq_2 \inf_2 M_2 = z^{(2)}$ imply that $[x],[y] \leq [z]_0$. Hence $U([z]_0) \subset U([x],[y])$. Moreover, if $[z] \in U([x],[y])$, then by Corollary 2.1.20 $[z] = \{z_1, z_2\}$, $z^{(1)} \leq_1 z_1$, and $z^{(2)} \leq_2 z_2$; hence $[z]_0 \leq [z]$, i.e., $[z] \in U([z]_0)$. Therefore $U([z]_0) = U([x],[y])$. Consequently L is an orthomodular lattice.

2.1.23 <u>Example</u>. If S_i is not a section of L_i, then L need not be a lattice. For example, for i = 1, 2, let L_i be the Boolean lattice given in Figure 2.

$L_i =$

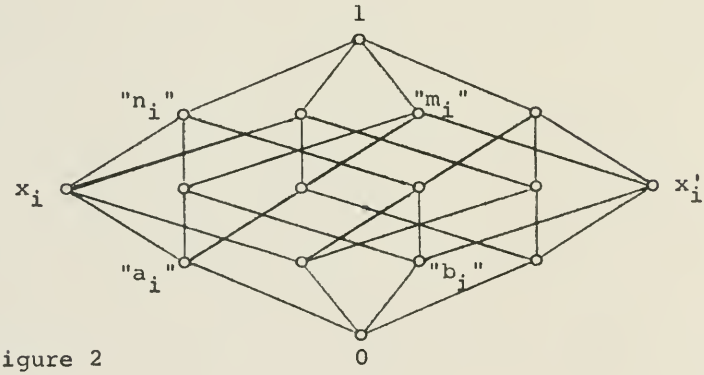


Figure 2

Let $S_i = \{0,1,x_i,x_i'\}$. Then S_i is not a section of L_i. Form L as in Definition 2.1.4. (The quotes about a_i, b_i, m_i, and n_i indicate that $a_1\theta \neq a_2$, $b_1\theta \neq b_2$, $m_1\theta \neq m_2$, and $n_1\theta \neq n_2$ in violation of Notation 2.1.7.) Then $[\text{"}a_1\text{"}] \vee [\text{"}b_2\text{"}]$ does not exist since $\{[\text{"}m_1\text{"}],[\text{"}n_2\text{"}],[1]\} = U([\text{"}a_1\text{"}],[\text{"}b_2\text{"}])$ has no smallest element.

2.1.24 <u>Example</u>. The following example illustrates the fact that if the sub-orthomodular lattice S_i is not a subcomplete sub-orthomodular lattice of the complete orthomodular lattice L_i $(i = 1, 2)$, then L need not be a lattice.

Let L_1 be the power set of an infinite set M, and let S_1 be the sub-orthomodular lattice of L_1 consisting of all finite or co-finite subsets of L_1. (Recall that a co-finite subset of M is a subset of M whose complement in M is finite.) Let L_2 be a disjoint "copy" of L_1. Then there exists a natural ortho-isomorphism $\varphi:L_1 \cong L_2$. Let $\theta = \varphi|_{S_1}$ and let $S_2 = S_1\theta$. Then $\theta:S_1 \cong S_2$ is an ortho-isomorphism mapping S_1 onto S_2. Clearly S_1 and S_2 are corresponding sections of L_1 and L_2. Moreover, L_i is complete, but S_i is not a subcomplete sub-orthomodular lattice of L_i $(i = 1, 2)$. Form L as in Definition 2.1.4.

To show that L is not a lattice, consider two subsets X and Y of M which are neither finite nor co-finite and such that $X \cup Y$ is not co-finite. Let x be the element of L_1 which corresponds to X and let y be the element of L_2 which corresponds to Y. Then $x \in L_1 - S_1$ and $y \in L_2 - S_2$. Suppose there is a $[z] \in L$ such that $[z] = [x] \vee [y]$. Then by Corollary 2.1.20, $[z] \in S$, and hence $[z] = \{z_1, z_2\}$. Let Z be the subset of M corresponding to z_1 (and z_2). Since $X \cup Y \subset Z$, Z is a co-finite subset of M. But, since $X \cup Y$ is not co-finite, there exists $w \in Z - (X \cup Y)$ such that $Z - \{w\} \supset X \cup Y$.

Let $V = Z-\{w\}$ and let v be the element of L_1 which corresponds to V. Then $[v] \in U([x],[y])$ but $[z] \not\leq [v]$. This contradicts the fact that $[z] = [x] \vee [y]$. Hence $[x] \vee [y]$ does not exist and L is not a lattice.

The poset L given in the above example is of interest for another reason. By Proposition 2.1.21 L is an orthomodular poset. Hence L is an example of an orthomodular poset which is not a lattice. The only other known example of such a poset is a finite poset given by M. F. Janowitz in [5].

2.1.25 <u>Example</u>. Let L_1 and L_2 be given by the following Hasse diagrams:

$L_1 =$

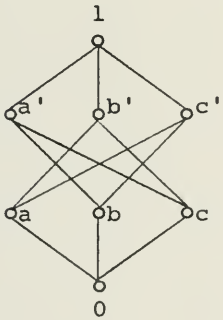
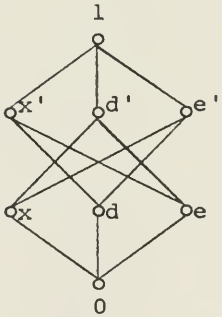


Figure 3



$= L_2$

Figure 4

Let $S_1 = S_{c'}$, let $S_2 = S_{x'}$, and let $c\theta = x$ (thereby determining θ). Apply Theorem 2.1.22 to L_1, L_2, S_1, S_2, and θ (noting that all of the hypotheses are satisfied) to obtain

the following orthomodular lattice (for simplicity of notation, we write z for [z]).

L =

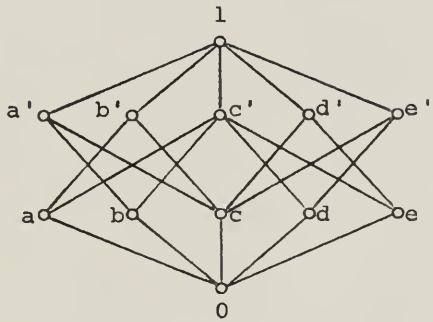


Figure 5

This lattice will be used in the following proposition.

2.1.26 <u>Proposition</u>. Let the hypotheses of Theorem 2.1.22 be satisfied, then the lattice L of the conclusion of Theorem 2.1.22 is complete. Moreover, if L_1 and L_2 (of that theorem) are atomic, then L is atomic and the atoms of L are of the form [m] where m is an atom of either L_1 or L_2; however, not every element of L of the form [x], where x is an atom of L_1 or L_2, need be an atom of L.

<u>Proof</u>. To show that L is complete, let M be any subset of L. Let $N = \{[x] \in M : \text{there exists } x_1 \in [x]\}$, let $N_1 = \{x : x \in L_1 \text{ and } [x] \in N\}$, let $n = \sup_1 N_1$ (the supremum exists since L_1 is complete), let $P = M - N$, let

$P_2 = \{x : x \in L_2,\ [x] \in M, \text{ and } [x] = \{x_2\}\}$, and let $p = \sup_2 P_2$ (the supremum exists since L_2 is complete).

We claim that sup N exists and equals $[n]$. It is clear that $[n] \geq [x]$ for all $[x] \in N$. Assume that $[z] \geq [x]$ for all $[x] \in N$. If there exists $z_1 \in [z]$, then $z_1 \geq_1 x_1$ for all $x \in N_1$; hence, in this case, $z_1 \geq_1 n$ so that $[z] \geq [n]$. If $[z] = \{z_2\}$, then for each $x \in N_1$ there exists $[w]^x = \{w_1^x, w_2^x\}$ such that $z_2 \geq_2 w_2^x$ and $w_1^x \geq_1 x$. Hence $w_1^x \geq_1 n$ so that $[w] \geq [n]$. Consequently sup N exists and equals $[n]$.

We now claim that sup P exists and equals $[p]$. It is clear that $[p] \geq [x]$ for all $[x] \in P$. Assume that $[z] \geq [x]$ for all $[x] \in P$. If there exists $z_2 \in [z]$, then $z_2 \geq_2 x$ for all $x \in P_2$; hence, in this case, $z_2 \geq_2 p$ so that $[z] \geq [p]$. If $[z] = \{z_1\}$, then, for each $x \in P_2$, there exists $[w]^x = \{w_1^x, w_2^x\}$ such that $z_1 \geq_1 w_1^x$ and $w_2^x \geq_2 x$. Hence $w_2^x \geq_2 p$ so that $[z] \geq [p]$. Consequently sup P exists and equals $[p]$.

Since L is a lattice $[n] \vee [p]$ exists in L. We claim that $[n] \vee [p] = \sup M$. Clearly $[n] \vee [p] \geq [x]$ for all $x \in M$. If $[z] \geq [x]$ for all $[x] \in M$, then $[z] \geq [n]$, $[z] \geq [p]$, and therefore $[z] \geq [n] \vee [p]$. Consequently sup M exists and equals $[n] \vee [p]$. Since L is orthocomplemented, inf M also exists. Therefore L is complete.

To show that L is atomic provided L_1 and L_2 are

atomic, let $[x]$ be any non-zero element of L. Then we may assume there exists a representative, x_1, of $[x]$ from L_1. Since L_1 is atomic, there exists an atom $a_1 \in L_1$ such that $a_1 \leq_1 x_1$. Either $[a_1]$ is an atom of L or there exists $[y] \in L$ such that $[0] < [y] < [a_1]$. Assume the latter, then $[y] = \{y_2\}$; and since L_2 is atomic, it follows that y_2 dominates some atom b_2 of L_2. Then $[0] < [b_2] \leq [y] < [a_1] \leq [x]$. Now $[b_2]$ is an atom of L, for, if it were not then either a_1 or b_2 would fail to be an atom in L_1 or L_2, respectively. Hence every non-zero element of L dominates some atom $[m]$ of L where m is either an atom of L_1 or an atom of L_2.

To show that not every element of L of the form $[x]$, where x is an atom of L_1 or L_2, need be an atom of L consider the following lattices:

L =

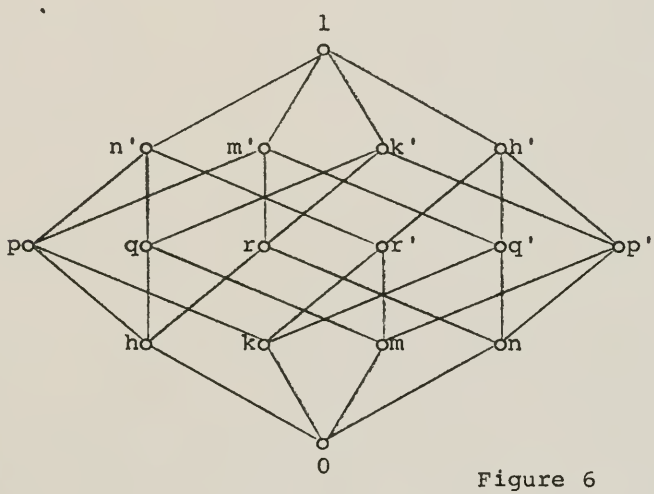


Figure 6

Let L_1 be the (Boolean) lattice given in Figure 6, let $S_1 = \{0,1,p,p',k,k',h,h'\}$, let L_2 be the (orthomodular) lattice given in Figure 5, let $S_2 = \{0,1,a,a',b,b',c,c'\}$, and let

$\theta = \{(0,0),(1,1),(a,p'),(b',k'),(c',h'),(a',p),(b,k),(c,h)\}$.

Apply Theorem 2.1.22 to obtain the lattice given in Figure 7. (Change the notation by dropping the brackets and representing all two-element equivalence classes by the representative in L_1.)

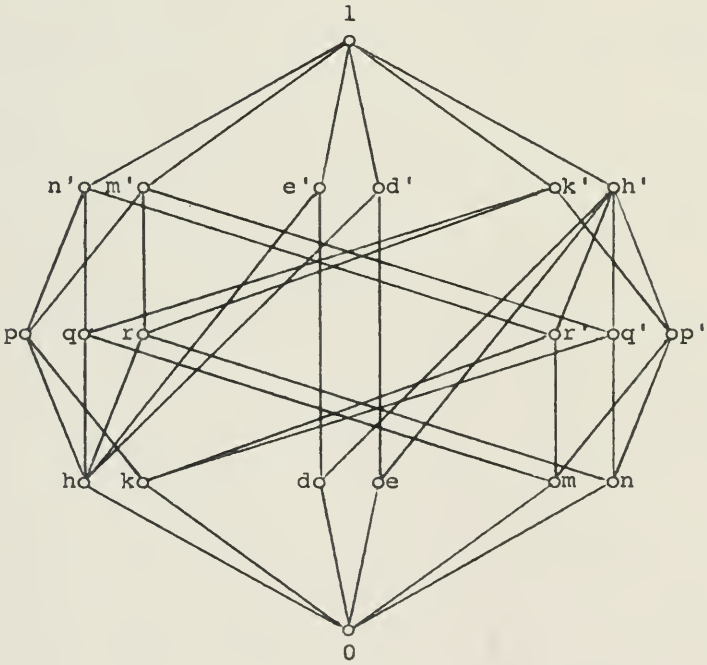


Figure 7

Now $[p'] = [a]$ is not an atom of L but a is an atom of L_2. Hence not every element of L of the form $[x]$, where x is an atom of L_1 or L_2, need be an atom of L.

2. A Partial Converse

2.2.1 Remark. We have shown that if an equivalence relation is defined on the union of two disjoint orthomodular lattices in such a way that elements of isomorphic sections (of a certain type) are equivalent to one another, then, upon "dividing out" this equivalence relation, a new orthomodular lattice is obtained. (We pictorially think of the elements of one section as being "pasted" to the corresponding elements of the other section.)

We now prove a partial converse of Theorem 2.1.22. We maintain Convention 2.1.1 and make the further assumption that L (cf. Definition 2.1.4) is an orthomodular lattice. After obtaining some preliminary more general information, we make the additional assumption that L_1 and L_2 are complete Boolean lattices.

The preliminary information necessary (cf. Lemma 2.2.2, Corollary 2.2.3, and Lemma 2.2.4) is concerned with comparabilities which may not obtain and restrictions on those that may. These observations form the crux of our main arguments.

2.2.2 <u>Lemma</u>. Assume that L is an orthomodular lattice and that $S_i < L_i$. Let $[z],[x],[y],[u] \in L$ be such that $[z] \leq [x],[y]$ and $[x],[y] \leq [u]$.

(1) If $[x] = \{x_1\}$ and $[y] = \{y_2\}$, then $[u] \in S$.

(2) If $[x] = \{x_2\}$ and $[y] = \{y_1\}$, then $[u] \in S$.

(3) If $[x] = \{x_1\}$ and $[y] = \{y_2\}$, then $[z] \in S$.

(4) If $[x] = \{x_2\}$ and $[y] = \{y_1\}$, then $[z] \in S$.

<u>Proof</u>. (2) follows from (1) by symmetry; (3) and (4) are dual to (1) and (2), respectively. Hence we need only prove (1).

Suppose that (1) is false. Then either $[u] = \{u_1\}$, or $[u] = \{u_2\}$ obtains. We may assume that $[u] = \{u_1\}$. Now, $u_1 \geq_1 x_1$, and there exists $[w] \in S$ such that $u_1 \geq_1 w_1$ and $w_2 \geq_2 y_2$. Let $[a] = [u] \wedge [y]'$, then $[a]' = [y] \vee [u]'$. Since L is an orthomodular lattice and

(I) $$[y] \leq [w] \leq [u],$$

by the OMI and DOMI we have

(II) $$[u] = [y] \vee ([u] \wedge [y]') = [y] \vee [a]$$

and

(III) $$[y] = [u] \wedge ([y] \vee [u]') = [u] \wedge [a]'.$$

Moreover, because of (I), by taking the infimum of both sides of (III) with $[w]$, we obtain

$$\begin{aligned} [y] &= [w] \wedge [y] = [w] \wedge ([u] \wedge [a]') \\ &= ([w] \wedge [u]) \wedge [a]' = [w] \wedge [a]', \end{aligned}$$

i.e.,

(IV) $$[y] = [w] \wedge [a]'.$$

If there exists $a_1 \in [a]$, then, since $S_i < L_i$ (cf. Lemmata 2.1.15 and 2.1.16), by (IV) we have $[y_2] = [w_1] \wedge [a_1] = [w_1 \wedge_1 a_1]$. Hence $[y] \in S$, contradicting the fact that $[y] = \{y_2\}$. If there exists $a_2 \in [a]$, then, since $S_i < L_i$ (cf. Lemmata 2.1.15 and 2.1.16), by (II) we have $[u_1] = [y_2] \vee [a_2] = [y_2 \vee_2 a_2]$. Hence $[u] \in S$ contradicting the fact that $[u] = \{u_1\}$.

2.2.3 <u>Corollary</u>. Assume that L is an orthomodular lattice and that $S_i < L_i$. If $x_1 \in L_1 - S_1$ and $y_2 \in L_2 - S_2$, then $[x_1] \vee [y_2] \in S$. If $x_2 \in L_2 - S_2$ and $y_1 \in L_1 - S_1$, then $[x_2] \vee [y_1] \in S$.

<u>Proof</u>. To prove the first statement let $[x_1] \vee [y_2] = [u]$ in Lemma 2.2.2. The second statement follows similarly.

2.2.4 <u>Lemma</u>. Assume that L is an orthomodular lattice and that $S_i < L_i$. Then there do not exist $[a], [b] \in L$ such that both $[a] \leq [b]$ and either

(1) $[a] = \{a_1\}$, and $[b] = \{b_2\}$, or

(2) $[a] = \{a_2\}$, and $[b] = \{b_1\}$.

<u>Proof</u>. By symmetry we need only prove (1). Suppose the statement is false, then there exists $[x] \in L$ such that $[x]:[a] \leq [b]$, $[a] = \{a_1\}$, and $[b] = \{b_2\}$. Then $[x] \in S$, and, since $[a] \leq [x] \leq [b]$, by the DOMI we have $[a] = [b] \wedge ([a] \vee [b]')$. Hence

$$[a] = [x]\wedge[a] = [x]\wedge([b]\wedge([a]\vee[b]')) = ([x]\wedge[b])\wedge([a]\vee[b]')$$
$$= [x]\wedge([a]\vee[b]'),$$

and therefore

$$[a] = [x]\wedge([a]\vee[b]').$$

But by Corollary 2.2.3 $[a]\vee[b]'\in S$ and $[x]\in S$. Hence $[a]\in S$, which is a contradiction. Therefore no such elements exist.

Thus ends our preliminary results. We are now prepared to deal with Boolean lattices. Henceforth, in addition to Convention 2.1.1 and the assumption that L is an orthomodular lattice, we assume that L_1 and L_2 are complete Boolean lattices. In each of the following results all assumptions augmenting Convention 2.1.1 will be made explicit. With the exception of direct reference to the lattices L_1 and L_2 of Convention 2.1.1, we will denote a Boolean lattice by the symbol B.

2.2.5 <u>Lemma</u>. Let B be a Boolean lattice, and let T be a proper sub-orthomodular lattice of B. If $z\in B$ is such that $B(0,z) \subset T$, then there exists $m\in B(z,1)$ such that $m\notin T$.

<u>Proof</u>. Suppose the statement is false. Then there exists $z\in B$ such that $B(0,z) \subset T$ and $B(z,1) \subset T$. Then $B(0,z') \subset T$ since T is a sub-orthomodular lattice of B. Let $x\in B$. Then $x = (x\wedge z)\vee(x\wedge z')$ and consequently $x\in T$ since

$x \wedge z \in T$ and $x \wedge z' \in T$. Therefore $B \subset T$, i.e., $B = T$, which contradicts the fact that T is a proper sublattice of B.

2.2.6 <u>Lemma</u>. Let B be a complete Boolean lattice. Let T be a subcomplete sub-orthomodular lattice of B. Let $M = \{x \in B : B(0,x) \subset T\}$. Let $c = \sup M$. Then $c \in M$ and, for all $x \in B$, $B(0,x) \subset T$ if and only if $x \leq c$.

<u>Proof</u>. To prove that $c \in M$, it suffices to show that $B(0,c) \subset T$. Let $a \leq c$. Then $a = a \wedge c = a \wedge (\sup M) = \sup \{a \wedge m : m \in M\}$. But, for all $m \in M$, $(a \wedge m) \in T$ since $a \wedge m \leq m$ and $B(0,m) \subset T$. Moreover, since T is a subcomplete sub-orthomodular lattice of B, $a \in T$. Consequently $B(0,c) \subset T$.

Now if $x \in B$ is such that $B(0,x) \subset T$, then $x \in M$. Hence $x \leq c$. Moreover, if $x \in B$ is such that $x \leq c$, then $B(0,x) \subset B(0,c) \subset T$.

2.2.7 <u>Lemma</u>. Let each L_i of Convention 2.1.1 be a complete Boolean lattice, let each S_i be a subcomplete sub-orthomodular lattice of L_i, let $M_i = \{x \in L_i : L_i(0,x) \subset S_i\}$, and let $c_i = \sup M_i$ $(i = 1, 2)$. Then

(1) $L_i(0,c_i) \subset S_i$, i.e., $c_i \in M_i$,

(2) $L_i(0,x_i) \subset S_i$ if and only if $x_i \leq_i c_i$,

(3) $x \in M_1$ if and only if $x\theta \in M_2$, and

(4) $c_1\theta = c_2$.

<u>Proof</u>. (1) and (2) are immediate consequences of Lemma 2.2.6.

To prove that if $x \in M_1$ then $x\theta \in M_2$, assume that the statement is false. Then there exist $x \in M_1$ and $m \in L_2(0,x\theta)-S_2$. Also, by Lemma 2.2.5 there exists $n \in L_1-S_1$ such that $x < n < 1$. Then $[m] < [x] < [n]$ which contradicts Lemma 2.2.4 since $m \in L_2-S_2$, $x \in S_1$, and $n \in L_1-S_1$. Hence if $x \in M_1$, then $x\theta \in M_2$. A similar argument utilizing θ^{-1} in place of θ proves the converse. Hence (3) is proved.

Now by (3) and the fact that θ is a complete join homomorphism (cf. Lemma 1.3.2) $c_1\theta = \sup\{m\theta : m \in M_1\} = \sup\{m : m \in M_2\} = c_2$. Hence (4) is proved.

2.2.8 <u>Lemma</u>. Let each L_i of Convention 2.1.1 be a complete Boolean lattice, let each S_i be a subcomplete sub-orthomodular lattice of L_i, let $M_i = \{x \in L_i : L_i(0,x) \subset S_i\}$, and let $c_i = \sup M_i$ $(i = 1, 2)$. Then the following statements are valid.

(1) If $x \in S_1 - S_{c_1^{\#}}$, then there exists $a \in L_1 - S_1$ such that $a <_1 x$.

(2) If $x \in S_1 - S_{c_1^{\#}}$, then there exists $b \in L_1 - S_1$ such that $x <_1 b$.

(3) If $x \in S_2 - S_{c_2^{+}}$, then there exists $a \in L_2 - S_2$ such that $a <_2 x$.

(4) If $x \in S_2 - S_{c_2^+}$, then there exists $b \in L_2 - S_2$ such that $x <_2 b$.

<u>Proof</u>. <u>Ad (1)</u>. Let $x \in S_1 - S_{c_1^\#}$, then $x \not\leq c$. Hence by Lemma 2.2.7 part (2) $L(0,x) \not\subset S_1$, i.e., there exists $a \in L_1 - S_1$ such that $a <_1 x$.

<u>Ad (2)</u>. Let $x \in S_1 - S_{c_1^\#}$ and suppose that there does not exist $b \in L_1 - S_1$ such that $x <_1 b$. Then $L(x,1) \subset S_1$ so that $L(0,x^\#) \subset S_1$. Hence by Lemma 2.2.7 part (2) $x^\# \leq_1 c_1$ and therefore $x \in S_{c_1^\#}$, which is a contradiction. The result follows.

<u>Ad (3) and (4)</u>. These are proved by noting the symmetry of the hypotheses.

2.2.9 <u>Proposition</u>. Let each L_i of Convention 2.1.1 be a complete Boolean lattice, let each S_i be a subcomplete sub-orthomodular lattice of L_i, let $M_i = \{x \in L_i : L_i(0,x) \subset S_i\}$, and let $c_i = \sup M_i$ $(i = 1, 2)$. Then $S_1 = S_{c_1^\#}$ and $S_2 = S_{c_2^+}$.

<u>Proof</u>. By the symmetry of the hypotheses we need only prove that $S_1 = S_{c_1^\#}$. By definition of c_1, $S_{c_1^\#} \subset S_1$. To show that $S_1 \subset S_{c_1^\#}$, assume that the statement is false. Then there exists $x \in S_1 - S_{c_1^\#}$. By Lemma 2.2.8 part (1) there exists $a \in L_1 - S_1$ such that $a <_1 x$. Moreover, $x\theta \in S_2 - S_{c_2^+}$, and by Lemma 2.2.8 part (4) there exists $b \in L_2 - S_2$ such that $x\theta <_2 b$. Then $[a] < [x] < [b]$, $a \in L_1 - S_1$, $x \in S_1$, and $b \in L_2 - S_2$ which contradicts Lemma 2.2.4.

The following theorem is now immediate.

2.2.10 <u>Theorem</u>. Let $(L_1, \leq_1, {}^{\#})$ and $(L_2, \leq_2, {}^{+})$ be two disjoint complete Boolean lattices each of cardinal number strictly larger than two. Let $S_i < L_i$ $(i = 1, 2)$ be subcomplete sub-orthomodular lattices of L such that there exists an ortho-isomorphism $\theta: S_1 \cong S_2$. Let $L_0 = L_1 \cup L_2$, let $P_1 = \{(x,y) \in L_0 \times L_0 : y = x\theta\}$, let $P = \Delta \cup P_1 \cup P_1^{-1}$. (By Proposition 2.1.3 P is an equivalence relation on L_0.) Let $L = L_0/P$. Then L is an orthomodular lattice if and only if there exists $c_1 \in L_1$ and $c_2 \in L_2$ such that $S_1 = S_{c_1^{\#}}$ and $S_2 = S_{c_2^{+}}$.

<u>Proof</u>. Assume that L is an orthomodular lattice. Let $M_i = \{x \in L_i : L_i(0,x) \subset S_i\}$, and let $c_i = \sup M_i$ $(i = 1, 2)$. Then by Proposition 2.2.9 $S_1 = S_{c_1^{\#}}$ and $S_2 = S_{c_2^{+}}$. Conversely, if $S_1 = S_{c_1^{\#}}$ and $S_2 = S_{c_2^{+}}$, then by Theorem 2.1.22 L is an orthomodular lattice.

3. Variation on a Theme

In case S_1 and S_2 consist of only two elements $\{0,1\}$, L is said to be the <u>horizontal sum</u> (<u>disjoint sum</u>) of L_1 and L_2, and L is written $HS(L_1, L_2)$. The simplicity of the operation HS allows us to make a more general definition.

2.3.1 <u>Definition</u>. Let $(L_\alpha, \leq_\alpha)$, $\alpha \in I$, be a family of

orthomodular lattices of cardinality larger than 2, indexed by the set I, where $^{\#}I > 1$, such that if α and β are distinct elements of I, then $L_\alpha \cap L_\beta = \emptyset$. Let $L_0 = \bigcup_{\alpha \in I} L_\alpha$, let 1_α and 0_α denote the unit and zero, respectively, of L_α. Define a relation P_1 on L_0 by

$$P_1 = \{(x,y) \in L_0 \times L_0 : \text{there exist } \alpha,\beta \in I \text{ such that } x = 1_\alpha \text{ and } y = 1_\beta, \text{ or } x = 0_\alpha \text{ and } y = 0_\beta\} \cup \Delta$$

where $\Delta = \{(x,x) : x \in L_0\}$. P_1 is clearly an equivalence relation. Let $L = L_0/P$. Then L is called the <u>horizontal sum</u> (<u>disjoint sum</u>) of the lattices L_α, written $L = HS(L_\alpha : \alpha \in I)$.

2.3.2 <u>Remark</u>. If $[x],[y] \in L$, then we write $[x] \leq [y]$ in case there exist representatives $x \in [x]$, $y \in [y]$, and $\alpha \in I$ such that $x,y \in L_\alpha$ and $x \leq_\alpha y$, where $\leq_\alpha$ is the partial ordering in L_α.

$\leq$ is easily seen to be a partial order for L; moreover, for any $\alpha \in I$, $[0_\alpha]$ and $[1_\alpha]$ are the zero and unit, respectively, for L. Since $[0_\alpha]$ and $[1_\alpha]$ are independent of α, we write $[0]$ for $[0_\alpha]$, $[1]$ for $[1_\alpha]$.

The orthocomplementations in L_α induce an orthocomplementation in L defined as follows:

$$\text{for } [x_\alpha] \in L,\ [x_\alpha]' = [x_\alpha']$$

where x_α' is the orthocomplement of x_α in L_α. Since, for $0_\alpha, 1_\alpha \in L$, $0_\alpha' = 1_\alpha$ and $1_\alpha' = 0_\alpha$, and since all other

equivalence classes are singletons, $':L \longrightarrow L$ is a well-defined orthocomplementation for L.

2.3.3 <u>Lemma</u>. Let $L = HS(L_\alpha : \alpha \in I)$. Then L is an orthomodular lattice.

<u>Proof</u>. Let $[x],[y] \in L$. Then an immediate consequence of the definition of $\leq$ is the fact that $[x] \vee [y]$ exists and

$$[x] \vee [y] = \begin{cases} [x \vee_\alpha y] \text{ if there exists } \alpha \in I \text{ such that } x,y \in L_\alpha. \\ [1] \text{ if there exists } \alpha,\beta \in I \text{ such that } \alpha \neq \beta, \\ \qquad x \in L_\alpha, \text{ and } y \in L_\beta. \end{cases}$$

Since L is an orthocomplemented poset such that the supremum of every pair of elements exists in L, it follows that L is an orthocomplemented lattice. To prove that L is an orthomodular lattice, let $[x] \leq [y]$. If $[x] = [0]$, $[x] = [1]$, $[y] = [0]$, or $[y] = [1]$, then the computation of the OMI is trivial. If none of these hold, then $[x]$ and $[y]$ are singletons whose only representatives are x and y, respectively, and there exists $\alpha \in I$ such that $x,y \in L_\alpha$. Hence

$$[x] \vee ([y] \wedge [x]') = [x] \vee ([y] \wedge [x']) = [x] \vee [y \wedge_\alpha x']$$
$$= [x \vee_\alpha (y \wedge_\alpha x')] = [y].$$

Therefore L is an orthomodular lattice.

2.3.4 <u>Definition</u>. Let P be an orthocomplemented poset. Then a section S of P is said to be <u>simple</u> if and only if $S - \{0,1\}$ consists only of atoms and co-atoms of P.

The following theorem is a variation of Theorem 2.1.22. The function θ is no longer required to be an order homomorphism, and additional assumptions are made on L_1, L_2, S_1, and S_2.

2.3.5 <u>Theorem</u>. Let S_i be a simple section of the orthomodular lattice L_i $(i = 1, 2)$, and let $\theta:S_1 \cong S_2$ be an ortho-isomorphism. Assume that $L_1 = HS(L_\alpha:\alpha \in I)$, for some index set I, and that, for any $\alpha \in I$, $S_1 \cap L_\alpha$ has cardinal number 2 or 4. Moreover, assume that $0\theta = 0$ and that $x \in S_1$ and x an atom of L_1 imply $x\theta$ is an atom of L_2. Define P, R, L, and ' as in Definitions 2.1.2, 2.1.4, and 2.1.11. Then L is an orthomodular lattice.

<u>Proof</u>. The proof that P is an equivalence relation is exactly the same as in Proposition 2.1.3. However, since θ is not an order isomorphism, the proof that R is a partial order is not the same as Proposition 2.1.9, but it follows the latter proof closely enough that we shall, in general, only point out those parts which are different. (In this proof the symbols (I), (II), (III), and (IV) refer to the comparabilities bearing these labels and appearing in the proof of Proposition 2.1.9; moreover, all unidentified elements are defined in the corresponding parts of the proof of Proposition 2.1.9.)

(1) R is reflexive since $[x]:[x]R[x]$.

(2) To prove that R is anti-symmetric, assume

that $[z]:[x]R[y]$ and $[w]:[y]R[x]$. We must show that $[x] = [y]$. There are nine cases as listed in Proposition 2.1.9. We may assume $\{[x],[y]\}\cap\{[0],[1]\} = \emptyset$.

<u>Case (1)</u>. $[x] = \{x_1\}$ and $[y] = \{y_1\}$. Since S_1 is a simple section (I) obtains.

<u>Case (2)</u>. $[x] = \{x_1\}$ and $[y] = \{y_2\}$. $[z]$ must be an atom or a co-atom of L. Each possibility is contradictory.

<u>Case (3)</u>. $[x] = \{x_1\}$ and $[y] = \{y_1,y_2\}$. $[x]R[y]$ implies $[y]$ is a co-atom and $[y]R[x]$ implies $[y]$ is an atom. Hence $[x] = [y]$ (which in this case is a contradiction).

<u>Case (4)</u>. $[x] = \{x_2\}$ and $[y] = \{y_1\}$. As in Case (2) this is contradictory.

<u>Case (5)</u>. $[x] = \{x_2\}$ and $[y] = \{y_2\}$. Since S_2 is a simple section (II) obtains.

Cases (6), (7), and (8) are resolved as in Case (3).

<u>Case (9)</u>. $[x] = \{x_1,x_2\}$ and $[y] = \{y_1,y_2\}$. Suppose $[x] \neq [y]$. Then $[x]R[y]$ implies x_1 is an atom but not a co-atom of L_1 and y_1 is a co-atom but not an atom of L_1; $[y]R[x]$ implies y_1 is an atom but not a co-atom of L_1. This is a contradiction.

(3) To prove that R is transitive we must again consider the sixteen cases provided by (IV). Cases (1),

(3), (4), and (5) are proved exactly as before. Case (2) is resolved as follows: If $[x] = [0]$, then $[r]:[x]R_1^2[y]$. If $[x] = [y]$, then $[x]:[x]R_1^2[y]$. If $[y] = [1]$, then $[p]:[x]R_1^2[y]$. Hence we may assume $[0] \neq [x]$, $[x] \neq [y]$, and $[y] \neq [1]$. Then $[x] = [p]$ or $[p] = [y]$. In the former case $[p] = [q]$ implies $[q]:[x]R_1^2[y]$, and $[p] \neq [q]$ implies $[q] = [r] = [y]$ and $[p]:[x]R_1R_2[y]$. In the latter case $[p] = [q] = [r] = [y]$ and $[p]:[x]R_1R_2[y]$.

In Case (6), if $[x] = [0]$, $[x] = [y]$, or $[y] = [1]$, then $[r]:[x]R_1R_2[y]$. Hence we may assume $[x] \neq [0]$, $[x] \neq [y]$, and $[y] \neq [1]$. Since $[p] = \{p_1,p_2\}$, $[q] = \{q_1,q_2\}$, $[r] = \{r_1,r_2\}$, $x_1 \leq_1 p_1$, $p_2 \leq_2 q_2$, $q_1 \leq_1 r_1$, and $r_2 \leq_2 y_2$, it follows that $[p]$, $[q]$, and $[r]$ are atoms or co-atoms. Hence either $[p] = [q]$ or $[q] = [r]$. If $[p] = [q]$ and $[p]$ is an atom, then $[p] = [x]$ and $[r]:[x]R_1R_2[y]$; if $[p] = [q]$ and $[p]$ is a co-atom, then $[p] = [y]$ and $[p]:[x]R_1R_2[y]$. Hence we may assume $[p] \neq [q]$. It follows that $[x] = [p]$ and $[q] = [r] = [y]$ and therefore $[q]:[x]R_1R_2[y]$. Consequently, in all cases, $([x],[y]) \in R_1R_2$.

In Case (7) we may assume $[x] \neq [0]$, $[x] \neq [y]$, and $[y] \neq [1]$. If $[p] = [r]$, then $[p]:[x]R_2^2[y]$. If $[p] \neq [r]$, then $[x] = [p]$, $[r] = [y]$, and $[q]:[x]R_1^2[y]$. Hence $([x],[y]) \in R_1^2 \cup R_2^2 \subset R$.

In Case (8) we may assume $[x] \neq [0]$, $[x] \neq [y]$,

and $[y] \neq [1]$. If $[p] = [r]$, then $[p]:[x]R_2^2[y]$. If $[p] \neq [r]$, then $[x] = [p]$, $[r] = [y]$, and $[q]:[x]R_2R_1[y]$.

Since we have not made use of the hypotheses that $L_1 = HS(L_\alpha:\alpha\in I)$ and that, for any $\alpha\in I$, $^{\#}(S_1\cap L_\alpha) = 2$ or 4, we may obtain Cases (9) - (16) by symmetry of hypotheses. Consequently L is a poset.

We must now prove that $':L \longrightarrow L$ is a well-defined orthocomplementation. That it is well-defined is proved exactly as in Proposition 2.1.14.

Assume that $[x],[x]' \leq [a]$. We must prove that $[a] = [1]$. Suppose that $[a] \neq [1]$. Then $[x] \neq [0],[1]$. If $[a] = \{a_1\}$, then, since the only two-element equivalence classes are $[0]$, $[1]$, and $[m]$ where $[m]$ is either an atom or a co-atom of L, either $[x] = \{x_1\}$ or $[x] = \{x_1,x_2\}$. If $[x] = \{x_1\}$, then we have $x_1,x_1^{\#} \leq_1 a_1$ and hence $[a_1] = [a] = [1]$. If $[x] = \{x_1,x_2\}$, then x_1 is either zero or an atom of L_1; it cannot be zero since $[x]' \leq [a]$, hence x_1 is an atom of L_1 and therefore $x_1^{\#}$ is a co-atom of L_1 which contradicts the fact that $[x]' \leq [a] = \{a_1\}$. If $[a] = \{a_2\}$, a similar argument yields a contradiction. If $[a] = \{a_1,a_2\}$, then a_1 is either an atom or a co-atom of L_1. It cannot be an atom since $[x],[x]' \leq [a]$. Hence a_1 is a co-atom of L_1. If $[x] = \{x_1\}$, then $x_1,x_1^{\#} \leq_1 a_1$ and hence $[a] = [1]$. If $[x] = \{x_2\}$, then there exist $[w],[z]$ such that $x_2 \leq_2 w_2$, $w_1 \leq_1 a_1$, $x_2^{+} \leq_2 z_2$, and $z_1 \leq_1 a_1$. Now

$[w]$ and $[z]$ must be co-atoms of L, hence $w_1 = z_1 = a_1$ and $w_2 = z_2$. Therefore $x_2, x_2^+ \leq_2 z_2$ and hence $[z] = [a] = [1]$. If $[x] = \{x_1, x_2\}$, then $[x]$ is an atom of L, $[a]$ is a co-atom of L, and $[x]' \leq [a]$ yields a contradiction. Hence $[a] = [1]$, and consequently $[x] \vee [x]'$ exists and equals $[1]$. Dually $[x] \wedge [x]'$ exists and equals $[0]$. Therefore $': L \longrightarrow L$ is an orthocomplementation.

To show that L is a lattice we need only show that $[x] \vee [y]$ exists for every $[x], [y] \in L$. Hence assume $[x], [y] \in L$. We need only consider the following six cases:

Case (1)	$[x] = \{x_1\}$	and	$[y] = \{y_1\}$,
Case (2)	$[x] = \{x_1\}$	and	$[y] = \{y_2\}$,
Case (3)	$[x] = \{x_1\}$	and	$[y] = \{y_1, y_2\}$,
Case (4)	$[x] = \{x_2\}$	and	$[y] = \{y_2\}$,
Case (5)	$[x] = \{x_2\}$	and	$[y] = \{y_1, y_2\}$,
Case (6)	$[x] = \{x_1, x_2\}$	and	$[y] = \{y_1, y_2\}$.

<u>Ad (1)</u>. $[x_1 \vee_1 y_1] \geq [x], [y]$. If $[z] \geq [x], [y]$, then there exists $z_1 \in [z]$ such that $z_1 \geq_1 x_1 \vee_1 y_1$ and hence $[z] \geq [x_1 \vee_1 y_1]$. For, if $[z] = \{z_2\}$, then there exists $[w] = \{w_1, w_2\}$ such that $z_2 \geq_2 w_2$ and $w_1 \geq_1 x_1$; in this case $[w]$ is a co-atom of L or $[1]$ which yields a contradiction since $[z] = \{z_2\}$.

<u>Ad (2)</u>. Let $[z] \in U([x], [y])$. It follows that $[z] = [1]$ or $[z] = \{z_1, z_2\}$ is a co-atom of L. We may assume $[z]$ is a co-atom of L. Since, by hypothesis,

$L_1 = HS(L_\alpha : \alpha \in I)$, $x_1 \in L_\beta$ for some $\beta \in I$; it follows that $\#(S_1 \cap L_\beta) = 4$, $z_1 \in S_1 \cap L_\beta$ and $[z]$ is unique with these properties. Consequently $[z] = [x] \vee [y]$.

<u>Ad (3)</u>. Note that $[x_1 \vee_1 y_1] \geq [x],[y]$. Let $[z] \geq [x],[y]$. If $[z] = \{z_1\}$, then $[y]$ is an atom of L, $z_1 \geq_1 y_1$, and $z_1 \geq_1 x_1$; hence $z_1 \geq_1 x_1 \vee_1 y_1$ and $[z] \geq [x_1 \vee_1 y_1]$. The case $[z] = \{z_2\}$ is contradictory since $[z] \geq [x]$ and S_i is a simple section of L_i $(i = 1, 2)$. If $[z] = \{z_1, z_2\}$, then $[z]$ is a co-atom of L, $z_1 \geq_1 x_1$, and $z_1 \geq_1 y_1$; hence $[z] \geq [x_1 \vee_1 y_1]$.

<u>Ad (4)</u>. The proof, in this case, is similar to that of (1).

<u>Ad (5)</u>. The proof, in this case, is similar to that of (3).

<u>Ad (6)</u>. We may assume that $\{[x],[y]\} \cap \{[0],[1]\} = \emptyset$ and that $[x] \neq [y]$. We claim that $[x] \vee [y] = [x_2 \vee_2 y_2]$. Clearly $[x_2 \vee_2 y_2] \geq [x],[y]$. Let $[z] \geq [x],[y]$. We may assume that neither $[x]$ nor $[y]$ is a co-atom in L, and hence that both $[x]$ and $[y]$ are atoms in L. $[z] \neq \{z_1\}$, since $[z] = \{z_1\}$ contradicts the fact that $\#(L_\alpha \cap S_1) \leq 4$, for every $\alpha \in I$, since x_1, y_1 are (in this case) atoms of the same L_α. If $[z] = \{z_2\}$, then $z_2 \geq_2 x_2, y_2$ so that $z_2 \geq_2 x_2 \vee_2 y_2$ and hence $[z] \geq [x_2 \vee_2 y_2]$. If $[z] = \{z_1, z_2\}$, then $[z] = 1$ or $[z]$ is a co-atom of L. We may assume that $[z] \neq 1$; hence $[z]$ is a co-atom of L. The

time has come to note that $x_1 \vee_1 y_1 = 1$ since $\#(S_1 \cap L_\alpha) \leq 4$ for each $\alpha \in I$. Hence $z_2 \geq_2 x_2$ and $z_2 \geq_2 y_2$; therefore $z_2 \geq_2 x_2 \vee_2 y_2$ and $[z] \geq [x_2 \vee_2 y_2]$. Hence in this case $[x] \vee [y] = [x_2 \vee_2 y_2]$. Consequently L is a lattice.

To show that L satisfies the OMI, assume that $[x] \leq [y]$. We must show that $[y] = [x] \vee ([y] \wedge [x]')$. We may assume that $[0] < [x] < [y] < [1]$. We consider nine cases.

<u>Case (1)</u>. $[x] = \{x_1\}$, and $[y] = \{y_1\}$. Since S_1 is simple, all computations occur in L_1 which is orthomodular. Hence the OMI is satisfied.

<u>Case (2)</u>. $[x] = \{x_1\}$, and $[y] = \{y_2\}$. Then there exists $[z] = \{z_1, z_2\}$ such that $x_1 \leq_1 z_1$ and $z_2 \leq_2 y_2$. But this contradicts the fact that S_i is a simple section $(i = 1, 2)$.

<u>Case (3)</u>. $[x] = \{x_1\}$, and $[y] = \{y_1, y_2\}$. Then $[y]$ is a co-atom of L. It follows that $x_1 \leq_1 y_1$ and hence that the OMI is satisfied since all computations are made in L_1.

<u>Case (4)</u>. $[x] = \{x_2\}$, and $[y] = \{y_1\}$. The proof is similar to that of Case (2).

<u>Case (5)</u>. $[x] = \{x_2\}$, and $[y] = \{y_2\}$. The proof is similar to that of Case (1).

<u>Case (6)</u>. $[x] = \{x_2\}$, and $[y] = \{y_1, y_2\}$. The proof is similar to that of Case (3).

<u>Case (7)</u>. $[x] = \{x_1, x_2\}$, and $[y] = \{y_1\}$. The

proof is similar to that of Case (3).

Case (8). $[x] = \{x_1, x_2\}$, and $[y] = \{y_2\}$. The proof is similar to that of Case (3).

Case (9). $[x] = \{x_1, x_2\}$, and $[y] = \{y_1, y_2\}$. Then $[x]$ is an atom of L, $[y]$ is a co-atom of L, and $[y] \neq [x]'$. It follows that $x_2 \dot{\leq}_2 y_2$ since $L_1 = HS(L_\alpha : \alpha \in I)$ and $\#(S_1 \cap L_\alpha) \leq 4$ for each $\alpha \in I$. Therefore all computations are made in L_2 and hence the OMI is satisfied.

Consequently L is an orthomodular lattice.

2.3.6 Remark. Let $L = HS(L_\alpha : \alpha \in I)$. If at least one L_α admits a chain of length 3, then L is non-modular.

Proof. Assume that L admits a chain of length 3. Then there exist $a, b \in L_\alpha - \{0,1\}$ such that $a < b$. Let $c \in L_\beta - \{0,1\}$ for any $\beta \neq \alpha$. Then $M(c,b)$ does not hold. Hence L is not modular.

The only (previously) known finite orthomodular non-modular lattices are those horizontal sums admitting at least one chain of length 3 and the following lattice (see Figure 10) given by R. P. Dilworth in [1]. Rather than exhibit this lattice "out of the blue" we construct it by applying Theorem 2.3.5.

2.3.7 Example. Let L_1 and L_2 be the following orthomodular lattices:

$L_1 =$

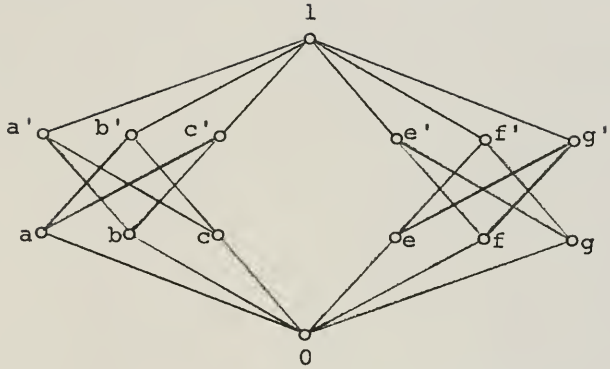


Figure 8

$L_2 =$

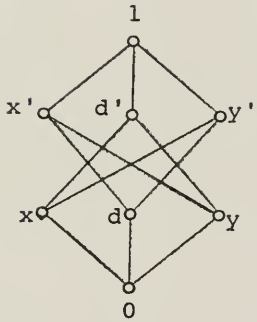


Figure 9

Let $S_1 = \{0,1,c,c',e,e'\}$ and $S_2 = \{0,1,x,x',y,y'\}$. Define $\theta: S_1 \cong S_2$ by $\theta = \{(0,0),(1,1),(c,x),(c',x'),(e,y),(e',y')\}$. Note that the hypotheses of Theorem 2.3.5 are satisfied. Hence the lattice given in Figure 10, L, is an orthomodular lattice. Re-labeling the equivalence classes $\{a\}$, $\{b\}$, $\{c,x\}$, $\{d\}$, $\{e,y\}$, $\{f\}$, and $\{g\}$ by a, b, c, d, e, f, and g, respectively, we obtain the following diagram for L:

L =

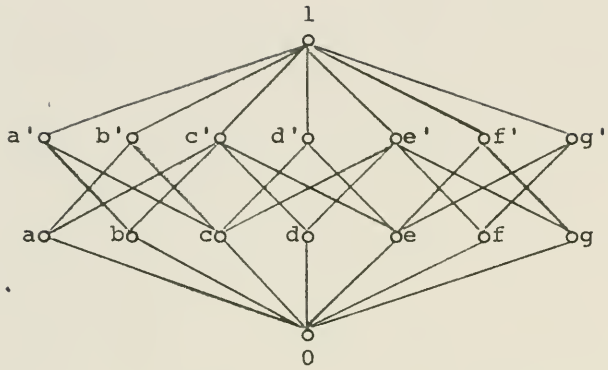


Figure 10

L is frequently denoted by the symbol D_{16}. Note that L is non-modular since M(g,a') fails to hold.

Note that D_{16} may be constructed from 2^3 by two applications of Theorem 2.1.22. The first application of Theorem 2.1.22 yields the lattice given in Figure 5. Form

L_2 =

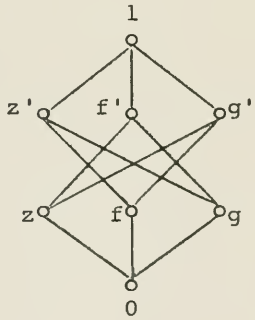


Figure 11

Let L_1 be the lattice given in Figure 5, let

$S_1 = \{0,1,e,e'\}$, let L_2 be the lattice given in Figure 11, let $S_2 = \{0,1,z,z'\}$, and let $\theta = \{(0,0),(1,1),(e,z),(e',z')\}$. Apply Theorem 2.1.22 and obtain D_{16} (given in Figure 10) by dropping the brackets and representing two-element equivalence classes by the representative in L_1.

In [3], the process of applying Theorem 2.1.22 with $L_2 = 2^3$ and with simple four-element sections was baptized <u>adjunction of a crown</u> to the lattice L_1. We now prove a theorem which generalizes the method of adjoining crowns and which allows us to exhibit a countable family of non-isomorphic finite orthomodular non-modular lattices. The construction is a generalization of the second method, exhibited above, of constructing D_{16}.

2.3.8 <u>Theorem</u>. For each $i = 1, 2, 3$, let L_i be a complete orthomodular lattice admitting a chain of length 3, let $x_i \in L_i$ be such that the principal section S_{x_i} is a proper sub-orthomodular lattice of L_i, and let $y_1 \in L_1$ be such that $S_{y_1} \cap S_{x_1} = \{0,1\}$. Let there exist $a,b \in L_i - S_{x_i}$, $i = 2$ or 3, such that $a < b$. Let S_{x_1} be ortho-isomorphic to S_{x_2} and let S_{y_1} be ortho-isomorphic to S_{x_3}. Let $\bar{L}$ be the orthomodular lattice obtained by applying Theorem 2.1.22 to L_1, L_2, S_{x_1}, and S_{x_2}; then the principal section

$\overline{S}_{y_1} = \{[z]\in\overline{L}: z\in S_{y_1}\}$ is a proper sub-orthomodular lattice (which is ortho-isomorphic to S_{y_1}) of $\overline{L}$. Let L be the orthomodular lattice obtained by applying Theorem 2.1.22 to $\overline{L}$, L_3, $\overline{S}_{y_1}$, and S_{x_3}. Then L is non-modular.

<u>Proof</u>. S_{y_1} is ortho-isomorphic to $\overline{S}_{y_1}$. For, the function $x \longrightarrow [x]$ is clearly an ortho-isomorphism. To show that it is an epimorphism, assume $[z]\in\overline{S}_{y_1}$; then we may assume that $[y_1] \leq [z]$. Since $S_{y_1}\cap S_{x_1} = \{0,1\}$, $[y_1] = \{y_1\}$. We claim that there exists $z_1\in[z]$. For, if $[z] = \{z_2\}$, then there exists $[w] = \{w_1,w_2\}$ such that $y_1 \leq_1 w_1$ and $w_2 \leq_2 z_2$. Then $w_1\in S_{y_1}\cap S_{x_1}$. Hence $w_1 = 1$ and therefore $z_2 = 1$ which contradicts the fact that $[z] = \{z_2\}$. Consequently there exists $z_1\in[z]$ and $y_1 \leq_1 z_1$; therefore $z_1\in S_{y_1}$ and $z_1 \longrightarrow [z]$. Hence $x \longrightarrow [x]$ is an epimorphism.

There exist $a,b\in L_2-S_{x_2}$ such that $a <_2 b$, and, in $\overline{L}$, $[a] = \{a\}$ and $[b] = \{b\}$; moreover, in L, $[[a]] = \{\{a\}\}$ and $[[b]] = \{\{b\}\}$. Now there exists $c\in L_3-S_{x_3}$; $[[c]] = \{\{c\}\}$. Let $[[m]] = [[b]]\wedge[[c]]$. By the dual of Corollary 2.1.20, $[[m]]\in S$. Hence $[[m]]$ has a representative $[m]\in\overline{S}_{y_1}$. It follows that $[m] \leq [y_1]'$. Hence if $[m] \neq 0$, then since $S_{y_1}\cap S_{x_1} = \{0,1\}$, $[m]$ is a

singleton. But $[c]$ is a singleton. Hence by Lemma 2.1.19, $[m] \not< [c]$. This is a contradiction. Hence $[m] = [0]$. Similarly, $[[a]] \vee [[c]] = 1$. Now $[[a]] < [[b]]$ and

$$[[a]] \vee ([[c]] \wedge [[b]]) = [[a]] \vee 0 < 1 \wedge [[b]]$$
$$= ([[a]] \vee [[c]]) \wedge [[b]].$$

Hence $M([[c]],[[b]])$ does not obtain and consequently L is non-modular.

CHAPTER 3

THE REPRESENTATION OF ORTHOCOMPLEMENTED POSETS BY SETS

1. Γ-sequences

In [7] Zierler and Schlessinger prove that every orthocomplemented poset P may be (in some sense) "embedded" in a Boolean lattice of sets. The set utilized is the set of all order ortho-homomorphisms of P into the two-element Boolean lattice. The proof is made by noting that the kernel of such homomorphisms is a maximal ideal (of a special type) and by developing some of the properties of these ideals.

In the sequel we give a proof for this theorem which is valid in orthocomplemented posets which are, in some sense, "not too wide." In particular, our proof holds for all finite orthocomplemented posets. The advantage of our method is that it prescribes an algorithm for generating the order ortho-homomorphisms. Because of this we are able to associate a certain matrix, called a deterministic Travis matrix, of zeroes and ones with such posets. The columns of this matrix, partially ordered vector-wise, form an orthocomplemented poset which is ortho-isomorphic to the given poset.

3.1.1 <u>Convention</u>. Let P be an orthocomplemented poset.

3.1.2 <u>Notation</u>. (1) Let $\tilde{N}$ denote the set of natural numbers, let $\tilde{R}$ denote the set of real numbers, and let $[0,1]$ denote the closed unit interval in $\tilde{R}$.

(2) Let 2^P denote the power set of P.

(3) If X and Y are two sets and if $(x,y) \in X \times Y$, then by $\pi_1((x,y))$ we mean x, and by $\pi_2((x,y))$ we mean y.

(4) For $M \subset P$, let χ_M denote the characteristic function of the set M, i.e., $\chi_M : P \longrightarrow \tilde{R}$ is defined as follows: for $x \in P$,

$$\chi_M(x) = \begin{cases} 1 \text{ if } x \in M. \\ 0 \text{ if } x \notin M. \end{cases}$$

3.1.3 <u>Definition</u>. Let $x \in P$. Define $f_x : S_x \longrightarrow \{0,1\} \subset \tilde{R}$ as follows: for $z \in S_x$, let $f_x(z) = \chi_{L(x,1)}(z)$.

3.1.4 <u>Remark</u>. Let $x \in L$.

(1) If $x = 0$, then $f_x(z) = 1$ for all $z \in S_x$.

(2) If $x \neq 0$, then, for $z \in S_x$, $f_x(z) = \begin{cases} 1 \text{ if } x \leq z. \\ 0 \text{ if } x \leq z'. \end{cases}$

(3) The range of f_x is a subset of $[0,1]$.

<u>Proof</u>. These are immediate consequences of the fact that $S_x = L(x,1) \cup L(0,x')$, the definition of f_x, and

the fact that P is an orthocomplemented poset.

3.1.5 <u>Definition</u>. A <u>Γ-sequence</u> on P is a function $\gamma:\tilde{N} \longrightarrow P\times 2^P$ such that the following conditions hold:

(1) If $\gamma(i) = (b,B)$, then $b = 0$ if and only if $B = \emptyset$.

(2) If $\gamma(i) = (b,B)$, then $B \neq \emptyset$ implies $b\in B$.

(3) $\pi_2(\gamma(1)) = P$.

(4) If $\gamma(i) = (0,\emptyset)$, then $\gamma(i+1) = (0,\emptyset)$.

(5) If $\gamma(i) = (b,B)$, $b \neq 0$, then $\gamma(i+1) = (c,C)$ where $C = B-S_b$.

If γ is a Γ-sequence, then define

$$B_\gamma = \{\pi_1(\gamma(i)):i\in\tilde{N}\}-\{0\}$$

and

$$D_\gamma = \bigcup_{x\in B_\gamma} S_x.$$

3.1.6 <u>Notation</u>. In the sequel γ always denotes a Γ-sequence. If only one γ is under discussion, then we sometimes write b_i for $\pi_1(\gamma(i))$ and B_i for $\pi_2(\gamma(i))$; then $\gamma(i) = (b_i,B_i)$.

3.1.7 <u>Lemma</u>. Let γ be a Γ-sequence. Then the following statements are valid.

(1) $B_{n+1} = P-(\bigcup_{j=1}^{n} S_{b_j})$.

(2) If $i < j$ and $B_i \neq \emptyset$, then $B_i \supsetneq B_j$.

(3) If $^{\#}P < \infty$, then there exists $m \in \tilde{N}$ such that $n \geq m$ implies $B_n = \emptyset$.

(4) If $^{\#}P < \infty$, then $D_\gamma = P$.

(5) If $a, b \in B_\gamma$, $a \neq b$, then $a \notin S_b$ or $b \notin S_a$.

(6) If $a, b \in B_\gamma$, $x \in S_a \cap S_b$, then $f_a(x) = f_b(x)$.

<u>Proof</u>. <u>Ad (1)</u>. If $n = 1$, then

$B_2 = B_1 - S_{b_1} = L - (\bigcup_{j=1}^{1} S_{b_j})$. Assume that the formula is valid for $n = k-1$. If $b_k \neq 0$, then

$B_{k+1} = B_k - S_{b_k} = (P - (\bigcup_{j=1}^{k-1} S_{b_j})) - S_{b_k} = P - (\bigcup_{j=1}^{k} S_{b_j})$. If $b_k = 0$, then $B_k = \emptyset$ and therefore $B_{k+1} = \emptyset$. But

$P - (\bigcup_{j=1}^{k} S_{b_j}) \subset P - S_{b_k} = P - P = \emptyset$. Hence $B_{k+1} = \emptyset = P - (\bigcup_{j=1}^{k} S_{b_j})$.

<u>Ad (2)</u>. Since $b_i \in B_i \cap S_{b_i}$,

$B_i \supsetneq B_i - S_{b_i} = B_{i+1} \supset B_j$.

<u>Ad (3)</u>. Suppose the statement is false. Then there exists a Γ-sequence γ such that $B_i \neq \emptyset$ for all $i \in \tilde{N}$ and by part (2) $P = B_1 \supsetneq B_2 \supsetneq B_3 \supsetneq \ldots$. Hence $^{\#}P > {}^{\#}B_1 > {}^{\#}B_2 > {}^{\#}B_3 > \ldots$, and $^{\#}B_i > 0$ for all $i \in \tilde{N}$. But this is a contradiction since there does not exist an infinite strictly decreasing sequence of natural numbers less than the natural number $^{\#}(P)+1$.

Ad (4). Suppose the statement is false. Then there exists $x \in P - D_\gamma$. Let $m = \inf\{i : b_i = 0\}$. Now $m > 1$ since $B_1 = P \neq \emptyset$ (cf. Definition 3.1.5), $B_m = \emptyset$, and $b_{m-1} \neq 0$. But, by part (1), $x \in L - D_\gamma = L - (\bigcup_{i=1}^{m-1} S_{b_i}) = B_m = \emptyset$ which is a contradiction.

Ad (5). Let $a = \pi_1(\gamma(i))$ and $b = \pi_2(\gamma(j))$; then $i \neq j$. If $i < j$, then $b \in B_j \subset B_{i+1} = B_i - S_a$; hence $b \notin S_a$. If $j < i$, then $a \in B_i \subset B_{j+1} = B_j - S_b$; hence $a \notin S_b$.

Ad (6). Suppose the statement is false. Then we may assume that $f_a(x) = 1$ and $f_b(x) = 0$. Hence $a \leq x$ and $b \leq x'$; therefore $a \leq x \leq b'$. Consequently $b \in S_a$ and $a \in S_b$ which contradicts part (3).

3.1.8 Definition. (1) For each Γ-sequence γ, define $g_\gamma : D_\gamma \longrightarrow \{0,1\}$ by $g_\gamma(x) = f_b(x)$ whenever $b \in B_\gamma$ and $x \in S_b$.

(2) If γ, δ are Γ-sequences, we define $\gamma \equiv \delta$ in case $D_\gamma = D_\delta$ and $g_\gamma(x) = g_\delta(x)$ for all $x \in D_\gamma$. Clearly $\equiv$ is an equivalence relation. Denote the equivalence class containing γ by $[\gamma]$.

(3) Let $\Sigma = \{[\gamma] : D_\gamma = L\}$. Each $[\gamma] \in \Sigma$ is said to be a Γ-state; Σ is called the Γ-state space for P.

3.1.9 Lemma. Let γ be a Γ-sequence. Then

(1) $g_\gamma : D_\gamma \longrightarrow \{0,1\}$ is a well-defined function from D_γ into $[0,1]$.

(2) If $x \in D_\gamma$, then $x' \in D_\gamma$, $g_\gamma(x') = 0$ if $g_\gamma(x) = 1$, and $g_\gamma(x') = 1$ if $g_\gamma(x) = 0$.

(3) $g_\gamma(0) = 0$ and $g_\gamma(1) = 1$.

(4) If $x \in D_\gamma$, $g_\gamma(x) = 1$, $y \in P$, and $x \leq y$, then $y \in D_\gamma$ and $g_\gamma(y) = 1$.

<u>Proof</u>. <u>Ad (1)</u>. By Lemma 3.1.7 part (6), g_γ is well-defined. The domain is D_γ by definition; the range is a subset of $[0,1]$ by Remark 3.1.4.

<u>Ad (2)</u>. If $x \in D_\gamma$, then $x \in S_b$ for some $b \in B_\gamma$. Hence by Lemma 1.3.6 $x' \in S_b \subset D_\gamma$. Moreover, since $b \leq x$ if and only if $x' \leq b'$, and $x \leq b'$ if and only if $b \leq x'$, it follows that $f_b(x') = 1$ if and only if $f_b(x) = 0$ and $f_b(x') = 0$ if and only if $f_b(x) = 1$. The result follows.

<u>Ad (3)</u>. Since $1 \in S_b$ for all $b \in P$, it follows that $f_b(1) = 1$ for all $b \in P$ and hence that $g_\gamma(1) = 1$. Therefore by part (2) $g_\gamma(0) = 0$.

<u>Ad (4)</u>. Since $g_\gamma(x) = 1$, there exists $b \in B_\gamma$ such that $0 < b \leq x$. Hence $b \leq y$, and $y \in S_b$; therefore $y \in D_\gamma$ and $g_\gamma(y) = 1$.

<u>Condition I</u>. For every Γ-sequence γ, $D_\gamma = P$.

<u>Condition II</u>. (1) If $a \in P - \{0\}$, then there exists

a Γ-sequence γ such that $\pi_1(\gamma(1)) = a$ and $D_\gamma = P$.

(2) If $a,b \in P-\{0\}$, $b \notin S_a$, then there exists a Γ-sequence γ such that $\pi_1(\gamma(1)) = a$, $\pi_1(\gamma(2)) = b$, and $D_\gamma = P$.

3.1.10 <u>Lemma</u>. Condition I implies Condition II.

<u>Proof</u>. Assume that Condition I holds.

(1) Let $a \in P$. Define $\gamma(1) = (a,P)$. For $n > 1$, let $F_n = \pi_2(\gamma(n-1)) - S_{\pi_1(\gamma(n-1))}$ and define

$$\gamma(n) = \begin{cases} (0,\emptyset) & \text{if } F_n = \emptyset. \\ (f,F_n) \text{ for any } f \in F_n & \text{if } F_n \neq \emptyset. \end{cases}$$

Then γ is a Γ-sequence, $\pi_1(\gamma(1)) = a$, and, by Condition I, $D_\gamma = P$.

(2) Let $a,b \in P$ be such that $b \notin S_a$. Define $\gamma(1) = (a,P)$ and $\gamma(2) = (b,P-S_a)$. For $n > 2$, let $F_n = \pi_2(\gamma(n-1)) - S_{\pi_1(\gamma(n-1))}$ and define

$$\gamma(n) = \begin{cases} (0,\emptyset) & \text{if } F_n = \emptyset. \\ (f,F_n) \text{ for any } f \in F_n & \text{if } F_n \neq \emptyset. \end{cases}$$

Then γ is a Γ-sequence, $\pi_1(\gamma(1)) = a$, $\pi_1(\gamma(2)) = b$, and, by Condition I, $D_\gamma = P$.

The following examples of horizontal sums show that the converse of Lemma 3.1.10 is not valid and that there are posets which satisfy neither.

3.1.11 <u>Example</u>. Let $L = HS(2^3_i : i \in \tilde{N})$. A Γ-sequence on L satisfying (1) (or (2)) of Condition II may be easily constructed by noting the following fact: for every set of atoms A of L consisting of exactly one atom of each 2^3_k, there exists a Γ-sequence γ such that $B_\gamma = A$ (and hence $D_\gamma = L$). But the Γ-sequence defined by

$\gamma(n) = (a_{2n}, L - \bigcup_{i=1}^{n-1} S_{a_{2i}})$ where a_{2n} is an atom of 2^3_{2n} has the property that

$$D_\gamma = \bigcup_{n=1}^{\infty} 2^3_{2n} \subsetneq L.$$

Hence L satisfies Condition II but not Condition I.

3.1.12 <u>Example</u>. Let $L = HS(2^3_\alpha : \alpha \in I)$ where I is any uncountable index set. Then L satisfies neither Condition I nor Condition II since, for any Γ-sequence γ, D_γ is a countable union of finite sets and hence countable, whereas L is uncountable. It follows that for every Γ-sequence γ, $D_\gamma \subsetneq L$.

3.1.13 <u>Remark</u>. (1) Condition I implies that $[\gamma] \in \Sigma$ for every Γ-sequence γ. Condition II implies that, for every $a \in P - \{0\}$, there exists $[\gamma] \in \Sigma$ such that $g_\gamma(a) = 1$; and, for every $a, b \in P - \{0\}$ such that $b \notin S_a$, there exists $[\gamma] \in \Sigma$ such that $g_\gamma(a) = g_\gamma(b) = 1$.

(2) By Lemma 3.1.7 part (4), if $\#P < \infty$, then

Condition I holds. Hence, by Lemma 3.1.10, if $\#P < \infty$, then Condition II holds.

3.1.14 <u>Lemma</u>. Let P satisfy Condition II, and let $e,f \in P$. Then $e \leq f$ if and only if $g_\gamma(e) \leq g_\gamma(f)$ for all $[\gamma] \in \Sigma$.

<u>Proof</u>. Assume that $e \leq f$. We must prove that $g_\gamma(e) \leq g_\gamma(f)$ for all $[\gamma] \in \Sigma$. Let $[\gamma]$ be any fixed element of Σ, and let $\gamma \in [\gamma]$. We may assume that $\{e,f\} \cap \{0,1\} = \emptyset$ and that $g_\gamma(e) = 1$. We must show that $g_\gamma(f) = 1$. But $1 = g_\gamma(e)$ implies that there exists $b \in B_\gamma$ such that $f_b \subset g_\gamma$ and $f_b(e) = 1$. Hence $b \leq e$ so that $b \leq f$; therefore $f_b(f) = 1$ and consequently $g_\gamma(f) = 1$.

Conversely, assume that $g_\gamma(e) \leq g_\gamma(f)$ for all $[\gamma] \in \Sigma$. We must prove that $e \leq f$. Suppose the statement is false. Then $e \not\leq f$, $e \neq 0$, and $f \neq 1$. If $f = 0$, then by Lemma 3.1.9, $g_\gamma(f) = 0$ for all $[\gamma] \in \Sigma$; hence by hypothesis $g_\gamma(e) = 0$ for all $[\gamma] \in \Sigma$ and consequently, by Condition II, $e = 0$ which contradicts the fact that $e \neq 0$. Therefore $f \neq 0$. If $f \leq e'$, then $e \leq f'$ and, by Condition II, there exists $[\gamma] \in \Sigma$ such that $g_\gamma(e) = 1$; hence by Lemma 3.1.9 $g_\gamma(f') = 1$ and $g_\gamma(f) = 0$; consequently $1 = g_\gamma(e) \not\leq g_\gamma(f) = 0$ which is a contradiction. Therefore $f \not\leq e'$ and consequently $f \notin S_e$. Hence $f' \notin S_e$ and by Condition II there exists $[\gamma] \in \Sigma$ such that $g_\gamma(e) = g_\gamma(f') = 1$; therefore by

Lemma 3.1.9 $1 = g_\gamma(e) \leq g_\gamma(f) = 0$ which is a contradiction. Consequently $e \leq f$.

3.1.15 Theorem. Let P be an orthocomplemented poset satisfying Condition II, and let $e,f \in P$. Then there exists a set X and a function $\Phi : P \longrightarrow 2^X$ such that

(1) $0\Phi = \emptyset$ and $1\Phi = X$,

(2) $e'\Phi = X-(e\Phi)$,

(3) $e \leq f$ if and only if $e\Phi \subset f\Phi$.

<u>Proof</u>. Let $X = \Sigma$. Define Φ as follows:

for $e \in P$, $e\Phi = \{[\gamma] \in \Sigma : g_\gamma(e) = 1\}$.

(1) follows from Lemma 3.1.9. (2) follows from the fact that $[\gamma] \in e'\Phi$ if and only if $g_\gamma(e') = 1$, and $g_\gamma(e) = 0$ if and only if $[\gamma] \notin e\Phi$. To prove (3), note that if $e \leq f$, then $g_\gamma(e) = 1$ implies $g_\gamma(f) = 1$ for all $[\gamma] \in \Sigma$. Therefore $e\Phi \subset f\Phi$. Conversely, assume that $e\Phi \subset f\Phi$ and that $[\gamma] \in \Sigma$. Then $g_\gamma(e) = 1$ implies $[\gamma] \in e\Phi$; hence $[\gamma] \in f\Phi$, so that $g_\gamma(f) = 1$. Therefore $g_\gamma(e) = 1$ implies $g_\gamma(f) = 1$. Consequently, $g_\gamma(e) \leq g_\gamma(f)$ for all $[\gamma] \in \Sigma$, and hence by Lemma 3.1.14, $e \leq f$.

2. Order Ortho-homomorphisms

3.2.1 <u>Definition</u>. Let P be an orthocomplemented poset. Let $\{0,1\} \subset \tilde{R}$, and define $0' = 1$ and $1' = 0$; then $\{0,1\}$ may be regarded as an orthocomplemented poset.

(1) Let $H = \{\alpha : P \longrightarrow \{0,1\} : (i)\ e'\alpha = (e\alpha)'$, (ii) $e \leq f$ implies $e\alpha \leq f\alpha$, (iii) if $\{x_i\}$ is a (not necessarily countable) chain, if $\inf \{x_i\}$ exists, and if $x = \inf \{x_i\}$, then $x_j\alpha = 1$ for every element x_j of the chain $\{x_i\}$ implies $x\alpha = 1\}$.

(2) For each $\alpha \in H$, let

$E_\alpha = \{x \in P : x\alpha = 1$ and $y < x$ imply $y\alpha = 0\}$.

(3) For each $\alpha \in H$, let

$F_\alpha = \{x \in P$: there exists a chain $\{x_i\}$, maximal with respect to $x_i\alpha = 1$, such that x is the smallest element of $\{x_i\}\}$.

3.2.2 <u>Lemma</u>. If $\alpha \in H$, then $0\alpha = 0$ and $1\alpha = 1$.

<u>Proof</u>. It is sufficient to prove $0\alpha = 0$. Suppose the statement is false. Then $0\alpha = 1$. Since $0 \leq 1$ implies $0\alpha \leq 1\alpha$, it follows that $1\alpha = 1$ and hence that $1 = 1\alpha = 0'\alpha = (0\alpha)' = 1' = 0$ which is a contradiction.

3.2.3 <u>Lemma</u>. Let P be an orthocomplemented poset such that if $\{x_i\}$ is a chain in P, then $\inf \{x_i\}$ exists in P. Let $\alpha \in H$. Then the following statements obtain.

(1) $E_\alpha \neq \emptyset$ and $E_\alpha = F_\alpha$.

(2) For $y \in P$, $y\alpha = 1$ if and only if there exists $x \in E_\alpha$ such that $x \leq y$.

<u>Proof</u>. <u>Ad (1)</u>. Let $\alpha \in H$. Then $1\alpha = 1$. $\{1\}$ is a chain in P; extend it to a chain $\{x_i\}$, maximal with respect to $x_i\alpha = 1$. Let $x = \inf \{x_i\}$. Since $x_i\alpha = 1$, it follows that $x\alpha = 1$. By maximality, $x \in \{x_i\}$, therefore x is the smallest element of $\{x_i\}$. Since $x\alpha = 1$, if $y < x$, it follows that $y\alpha = 0$; hence $x \in E_\alpha$ and therefore $E_\alpha \neq \emptyset$. Clearly $E_\alpha \supset F_\alpha$.

Suppose that $E_\alpha \not\subset F_\alpha$. Then there exists $x \in E_\alpha$ such that for every chain $\{x_i\}$, maximal with respect to $x_i\alpha = 1$, x is not the smallest element of $\{x_i\}$. Now $x \in E_\alpha$ implies $x\alpha = 1$. Since x is not the smallest element of any such maximal chain, there exists $y < x$ such that $y\alpha = 1$. But $x \in E_\alpha$ and $y < x$ imply $y\alpha = 0$ which is a contradiction. Hence $E_\alpha = F_\alpha$.

<u>Ad (2)</u>. The necessity of the condition is clear. To prove the sufficiency, note that, since $y\alpha = 1$, $\{y\}$ may be extended to a chain $\{y_i\}$, maximal with respect to $y_i\alpha = 1$. As above, define $x = \inf \{y_i\}$. Then $x \in \{y_i\}$ and $x \in E_\alpha$.

3.2.4 <u>Corollary</u>. Let P be an orthocomplemented poset such that if $\{x_i\}$ is a chain in P, then $\inf \{x_i\}$ exists in P. Then for any $\alpha \in H$, $P = \bigcup_{x \in E_\alpha} S_x$.

<u>Proof</u>. Suppose the statement is false. Let $y \in P - (\bigcup_{x \in E_\alpha} S_x)$, then $y' \in P - (\bigcup_{x \in E_\alpha} S_x)$. We may assume $y\alpha = 1$.

By Lemma 3.2.3 there exists $x \in E_\alpha$ such that $x \leq y$. Then $y \in S_x$ which is a contradiction.

For each Γ-state $[\gamma]$, the corresponding function $g_\gamma : P \longrightarrow \{0,1\}$ is an order ortho-homomorphism (provided we regard g_γ as mapping onto $\{0,1\}$, and define $0' = 1$ and $1' = 0$). In what follows we prove that every order ortho-homomorphism from a finite orthocomplemented poset into $\{0,1\}$ arises in this way.

3.2.5 <u>Proposition</u>. Let P be a finite orthocomplemented poset, let α be any order ortho-homomorphism mapping P onto $\{0,1\}$ (regarded as a subset of $[0,1]$ with $0' = 1$ and $1' = 0$). Then there exists a unique Γ-state $[\gamma]$ such that $g_\gamma = \alpha$.

<u>Proof</u>. We first note that the third condition on the elements of H (cf. Definition 3.2.1 part (1)) is redundant since P is finite; hence H is, in fact, the set of all order ortho-homomorphisms mapping P onto $\{0,1\}$.

Well-order E_α; since P is finite, E_α may be written as $\{b_1, b_2, \ldots, b_n\}$ for some $n \in \tilde{N}$, where $b_i \neq b_j$ if $i \neq j$. Define $\gamma : \tilde{N} \longrightarrow P \times 2^P$ by $\gamma(i) = (b_i, B_i)$ where $b_i \in E_\alpha$ if $1 \leq i \leq n$, $b_i = 0$ if $i > n$, $B_1 = P$, and $B_{i+1} = B_i - S_{b_i}$ for all $i \in \tilde{N}$. By the definition of E_α and the fact that $0\alpha = 0$ for all $\alpha \in H$, it follows that $b_i \neq 0$ for

$i = 1, \ldots, n$; hence γ is a Γ-sequence. By Corollary 3.2.4, $[\gamma]$ is a Γ-state. Finally, by Definition 3.1.8 parts (1) and (3) and Definition 3.2.1 part (2), it follows that $g_\gamma = \alpha$ and that $[\gamma]$ is unique.

3. Travis Matrices

The following definition is motivated by certain tables which appear in the 1962 Wayne State University Master's Thesis of R. D. Travis [6].

3.3.1 <u>Definition</u>. Let Σ^* and P^* be two non-empty sets. Let $':P^* \longrightarrow P^*$ be a mapping of period 2. By a <u>Travis matrix</u> over $(\Sigma^*, P^*, ')$ we mean a function $T:\Sigma^* \times P^* \longrightarrow [0,1]$ which satisfies the following postulates. (For simplicity we write $\alpha(e)$ for $T(\alpha,e)$.)

(1) $0 \leq \alpha(e) \leq 1$ for all $\alpha \in \Sigma^*$, $e \in P^*$.

(2) $\alpha(e') = 1 - \alpha(e)$ for all $\alpha \in \Sigma^*$, $e \in P^*$.

(3) If $e, f \in P^*$ are such that $\alpha(e) = \alpha(f)$ for all $\alpha \in \Sigma^*$, then $e = f$.

(4) If $\alpha, \beta \in \Sigma^*$ are such that $\alpha(e) = \beta(e)$ for all $e \in P^*$, then $\alpha = \beta$.

(5) There exists $0 \in P$ such that $\alpha(0) = 0$ for all $\alpha \in \Sigma^*$.

(6) $k = \inf\limits_{\substack{e \in P^* \\ e \neq 0}} \left(\sup\limits_{\alpha \in \Sigma^*} (\alpha(e))\right) > \frac{1}{2}$.

We define the <u>"weak" partial order induced on P^* by Σ^*</u> as follows: Let $e, f \in P^*$, then $e \leq^* f$ if and only if

$\alpha(e) \leq \alpha(f)$ for all $\alpha \in \Sigma^*$.

If the range of T is $\{0,1\}$, then T is said to be a <u>deterministic</u> Travis matrix over $(\Sigma^*, P^*, ')$.

3.3.2 <u>Lemma</u>. $(P^*, \leq^*, ')$ is an orthocomplemented poset.

<u>Proof</u>. Let $e,f,g \in P^*$. Since $\alpha(e) = \alpha(e)$ for all $\alpha \in \Sigma^*$, $e \leq^* e$. If $e \leq^* f$ and $f \leq^* e$, then $\alpha(e) \leq \alpha(f)$ and $\alpha(f) \leq \alpha(e)$ for all $\alpha \in \Sigma^*$; hence $\alpha(e) = \alpha(f)$ for all $\alpha \in \Sigma^*$ and therefore $e = f$ by Postulate (3) of Definition 3.3.1. Moreover, if $e \leq^* f$ and $f \leq^* g$, then $\alpha(e) \leq \alpha(f) \leq \alpha(g)$ for all $\alpha \in \Sigma^*$; hence $e \leq^* g$. Consequently $\leq^*$ partially orders P^*. Since $\alpha((e')') = 1 - [1 - \alpha(e)] = \alpha(e)$ for all $\alpha \in \Sigma^*$, $e'' = e$. If $e \leq^* f$, then $\alpha(e) \leq \alpha(f)$ for all $\alpha \in \Sigma^*$ and $1 - \alpha(f) \leq 1 - \alpha(e)$ for all $\alpha \in \Sigma^*$, i.e., $\alpha(f') \leq \alpha(e')$ for all $\alpha \in \Sigma^*$; hence $f' \leq^* e'$.

By Postulate (5) there exists $0 \in P$. Define 1 to be $0'$; $0 \leq^* e'$ for all $e \in P$ implies that $e \leq^* 1$ for all $e \in P$. We claim that $e \vee e'$ exists and equals 1. Suppose $e, e' \leq f$; we must prove that $f = 1$. Suppose that $f \neq 1$. Then $f' \neq 0$ and $\frac{1}{2} < k \leq \sup\{\alpha(f') : \alpha \in \Sigma^*\}$. Therefore there exists $\alpha \in \Sigma^*$ such that $\frac{1}{2} < \alpha(f') = 1 - \alpha(f)$; hence $\alpha(f) < \frac{1}{2}$. Since

(I) $$\alpha(e) \leq \alpha(f) < \tfrac{1}{2},$$

it follows that

(II) $$1 - \alpha(e) \leq \alpha(f) < \tfrac{1}{2}.$$

By adding the extremes of the inequalities (I) and (II) we

obtain $1 < 1$ which is a contradiction. Therefore $f = 1$.

Similarly, $e \wedge e'$ exists and $e \wedge e' = 0$. Consequently $(P^*, \leq^*, ')$ is an orthocomplemented poset.

3.3.3 <u>Definition</u>. Let T denote the matrix $T: \Sigma \times P \longrightarrow \{0,1\}$ defined by $T([\gamma], e) = g_\gamma(e)$.

3.3.4 <u>Lemma</u>. T is well-defined.

<u>Proof</u>. If $[\gamma], [\delta] \in \Sigma$ and $[\gamma] = [\delta]$, then $D_\gamma = D_\delta$ and $g_\gamma(e) = g_\delta(e)$ for all $e \in P$.

3.3.5 <u>Theorem</u>. If P satisfies Condition II, then T is Travis matrix over $(\Sigma, P, ')$.

<u>Proof</u>. We must show that the following statements are valid.

(1) $0 \leq g_\gamma(e) \leq 1$ for all $[\gamma] \in \Sigma$, $e \in P$.

(2) $g_\gamma(e') = 1 - g_\gamma(e)$ for all $[\gamma] \in \Sigma$, $e \in P$.

(3) If $e, f \in P$ are such that $g_\gamma(e) = g_\gamma(f)$ for all $[\gamma] \in \Sigma$, then $e = f$.

(4) If $[\gamma], [\delta] \in \Sigma$ are such that $g_\gamma(e) = g_\delta(e)$ for all $e \in P$, then $[\gamma] = [\delta]$.

(5) There exists $0 \in P$ such that $g_\gamma(0) = 0$ for all $[\gamma] \in \Sigma$.

(6) $k = \inf\limits_{\substack{e \in P \\ e \neq 0}} (\sup\limits_{[\gamma] \in \Sigma} g_\gamma(e)) > \frac{1}{2}$.

Ad (1). Since $[\gamma]\in\Sigma$, the domain of g_γ is P. The inequalities follow immediately from the fact that, for any $b\in B_\gamma$, $e\in P$, $f_b(e)$ equals either 0 or 1.

Ad (2). Let $e\in P$ and $[\gamma]\in\Sigma$. Then the result follows immediately from Lemma 3.1.9.

Ad (3). By Lemma 3.1.14 $g_\gamma(e) \leq g_\gamma(f)$ for all $[\gamma]\in\Sigma$ implies that $e \leq f$, and $g_\gamma(f) \leq g_\gamma(e)$ for all $[\gamma]\in\Sigma$ implies that $f \leq e$. Hence $e = f$.

Ad (4). If $[\gamma],[\delta]\in\Sigma$, then $D_\gamma = D_\delta = P$. Therefore, since $g_\gamma(e) = g_\delta(e)$ for all $e\in D_\gamma$, $[\gamma] = [\delta]$ by Definition 3.1.8.

Ad (5). Since P is an orthocomplemented poset, there exists $0\in P$ such that $0 \leq b$ for all $b\in P$. Hence, for any Γ-sequence γ, $0 \leq b'$ for all $b\in B_\gamma$. Therefore $f_b(0) = g_\gamma(0) = 0$ for all $[\gamma]\in\Sigma$ and for all $b\in B_\gamma$.

Ad (6). For any $e\in P$ such that $e \neq 0$, there exists a Γ-sequence γ such that $g_\gamma(e) = 1$ by Condition II. Hence $\sup\limits_{[\gamma]\in\Sigma^\gamma} g_\gamma(e) = 1$. Since this holds for all non-zero $e\in P$, it follows that $\inf\limits_{\substack{e\in P\\ e\neq 0}} (\sup\limits_{[\gamma]\in\Sigma} (g_\gamma(e))) = 1 > \frac{1}{2}$.

3.3.6 Corollary. Let P satisfy Condition II, let $\leq^*$ be the "weak" partial order induced on P by Σ (cf. Definition 3.3.1), and let $e,f\in P$. Then $e \leq f$ if and only if $e \leq^* f$.

Proof. By Lemma 3.1.14 $e \leq f$ if and only if

$g_\gamma(e) \leq g_\gamma(f)$ for all $[\gamma] \in \Sigma$. But by Definition 3.3.1, since the T of Definition 3.3.3 is a Travis matrix for $(\Sigma, P, ')$, $g_\gamma(e) \leq g_\gamma(f)$ for all $[\gamma] \in \Sigma$ if and only if $e \leq^* f$. Therefore $e \leq f$ if and only if $e \leq^* f$.

4. Examples

We now give deterministic Travis matrices which correspond to some of the posets previously mentioned. Figures 12, 13, 14, 15, and 16 are deterministic Travis matrices for the posets given in Figures 1, 3, 5, 8, and 10, respectively. Figure 17 is the Hasse diagram of the orthomodular poset which is not a lattice, given by Janowitz [5] and mentioned in Example 2.1.24; Figure 18 is a deterministic Travis matrix for it.

	0	e	f	1	e'	f'
S_1	0	1	1	1	0	0
S_2	0	0	1	1	1	0
S_3	0	0	0	1	1	1

Figure 12

	0	a	b	c	1	a'	b'	c'
s_1	0	1	0	0	1	0	1	1
s_2	0	0	1	0	1	1	0	1
s_3	0	0	0	1	1	1	1	0
s_4	0	0	0	0	1	1	1	1

Figure 13

	0	a	b	c	d	e	1	a'	b'	c'	d'	e'
s_1	0	1	0	0	1	0	1	0	1	1	0	1
s_2	0	1	0	0	0	1	1	0	1	1	1	0
s_3	0	1	0	0	0	0	1	0	1	1	1	1
s_4	0	0	1	0	1	0	1	1	0	1	0	1
s_5	0	0	1	0	0	1	1	1	0	1	1	0
s_6	0	0	1	0	0	0	1	1	0	1	1	1
s_7	0	0	0	1	0	0	1	1	1	0	1	1
s_8	0	0	0	0	1	0	1	1	1	1	0	1
s_9	0	0	0	0	0	1	1	1	1	1	1	0
s_{10}	0	0	0	0	0	0	1	1	1	1	1	1

Figure 14

	0	a	b	c	e	f	g	1	a'	b'	c'	e'	f'	g'
s_1	0	1	0	0	1	0	0	1	0	1	1	0	1	1
s_2	0	1	0	0	0	1	0	1	0	1	1	1	0	1
s_3	0	1	0	0	0	0	1	1	0	1	1	1	1	0
s_4	0	1	0	0	0	0	0	1	0	1	1	1	1	1
s_5	0	0	1	0	1	0	0	1	1	0	1	0	1	1
s_6	0	0	1	0	0	1	0	1	1	0	1	1	0	1
s_7	0	0	1	0	0	0	1	1	1	0	1	1	1	0
s_8	0	0	1	0	0	0	0	1	1	0	1	1	1	1
s_9	0	0	0	1	1	0	0	1	1	1	0	0	1	1
s_{10}	0	0	0	1	0	1	0	1	1	1	0	1	0	1
s_{11}	0	0	0	1	0	0	1	1	1	1	0	1	1	0
s_{12}	0	0	0	1	0	0	0	1	1	1	0	1	1	1
s_{13}	0	0	0	0	1	0	0	1	1	1	1	0	1	1
s_{14}	0	0	0	0	0	1	0	1	1	1	1	1	0	1
s_{15}	0	0	0	0	0	0	1	1	1	1	1	1	1	0
s_{16}	0	0	0	0	0	0	0	1	1	1	1	1	1	1

Figure 15

Figure 16

	0	a	b	c	d	e	f	g	1	a'	b'	c'	d'	e'	f'	g'
s_1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1
s_2	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1	0
s_3	0	1	0	0	1	0	0	0	1	0	1	1	0	1	1	1
s_4	0	1	0	0	0	1	0	0	1	0	1	1	1	0	1	1
s_5	0	1	0	0	0	0	1	0	1	0	1	1	1	1	0	1
s_6	0	1	0	0	0	0	0	1	1	0	1	1	1	1	1	0
s_7	0	1	0	0	0	0	0	0	1	0	1	1	1	1	1	1
s_8	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1
s_9	0	0	1	0	1	0	0	1	1	1	0	1	0	1	1	0
s_{10}	0	0	1	0	1	0	0	0	1	1	0	1	0	1	1	1
s_{11}	0	0	1	0	0	1	0	0	1	1	0	1	1	0	1	1
s_{12}	0	0	1	0	0	0	1	0	1	1	0	1	1	1	0	1
s_{13}	0	0	1	0	0	0	0	1	1	1	0	1	1	1	1	0
s_{14}	0	0	1	0	0	0	0	0	1	1	0	1	1	1	1	1
s_{15}	0	0	0	1	0	0	1	0	1	1	1	0	1	1	0	1
s_{16}	0	0	0	1	0	0	0	1	1	1	1	0	1	1	1	0
s_{17}	0	0	0	1	0	0	0	0	1	1	1	0	1	1	1	1
s_{18}	0	0	0	0	1	0	1	0	1	1	1	1	0	1	0	1
s_{19}	0	0	0	0	1	0	0	1	1	1	1	1	0	1	1	0
s_{20}	0	0	0	0	1	0	0	0	1	1	1	1	0	1	1	1
s_{21}	0	0	0	0	0	1	0	0	1	1	1	1	1	0	1	1
s_{22}	0	0	0	0	0	0	1	0	1	1	1	1	1	1	0	1
s_{23}	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0
s_{24}	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1

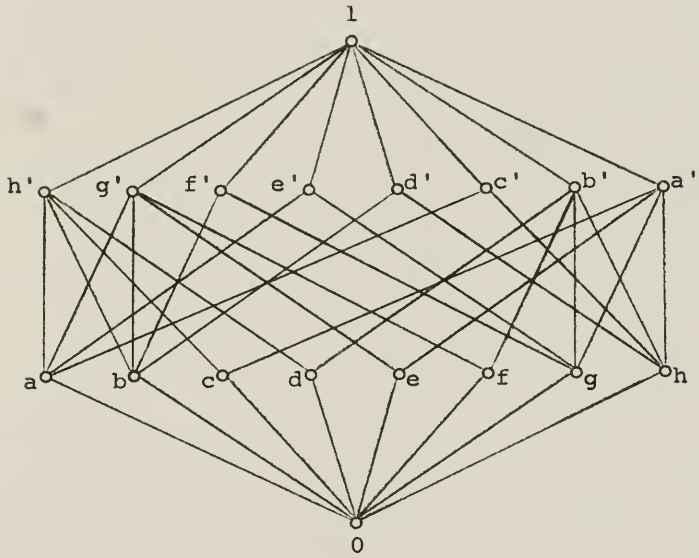


Figure 17

Figure 18

	0	a	b	c	d	e	f	g	h	1	a'	b'	c'	d'	e'	f'	g'	h'
s_1	0	1	1	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1
s_2	0	1	0	0	1	0	1	0	0	1	0	1	1	0	1	0	1	1
s_3	0	1	0	0	1	0	0	0	0	1	0	1	1	0	1	1	1	1
s_4	0	1	0	0	0	0	1	0	0	1	0	1	1	1	1	0	1	1
s_5	0	1	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1
s_6	0	0	1	1	0	1	0	0	0	1	1	0	0	1	0	1	1	1
s_7	0	0	1	1	0	0	0	0	0	1	1	0	0	1	1	1	1	1
s_8	0	0	1	0	0	1	0	0	0	1	1	0	1	1	0	1	1	1
s_9	0	0	1	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1
s_{10}	0	0	0	1	1	1	1	0	0	1	1	1	0	0	0	0	1	1
s_{11}	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1
s_{12}	0	0	0	1	1	0	0	0	0	1	1	1	0	0	1	1	1	1
s_{13}	0	0	0	1	1	0	1	0	0	1	1	1	0	0	1	0	1	1
s_{14}	0	0	0	1	1	0	0	1	0	1	1	1	0	0	1	1	0	1
s_{15}	0	0	0	1	0	1	0	0	0	1	1	1	0	1	0	1	1	1
s_{16}	0	0	0	1	0	1	1	0	0	1	1	1	0	1	0	0	1	1
s_{17}	0	0	0	1	0	0	1	0	0	1	1	1	0	1	1	0	1	1
s_{18}	0	0	0	1	0	0	0	1	0	1	1	1	0	1	1	1	0	1
s_{19}	0	0	0	1	0	0	0	0	0	1	1	1	0	1	1	1	1	1
s_{20}	0	0	0	0	1	1	1	0	0	1	1	1	1	0	0	0	1	1
s_{21}	0	0	0	0	1	1	0	0	0	1	1	1	1	0	0	1	1	1
s_{22}	0	0	0	0	1	0	1	0	0	1	1	1	1	0	1	0	1	1
s_{23}	0	0	0	0	1	0	0	1	0	1	1	1	1	0	1	1	0	1
s_{24}	0	0	0	0	1	0	0	0	0	1	1	1	1	0	1	1	1	1
s_{25}	0	0	0	0	0	1	1	0	1	1	1	1	1	1	0	0	1	0
s_{26}	0	0	0	0	0	1	1	0	0	1	1	1	1	1	0	0	1	1
s_{27}	0	0	0	0	0	1	0	0	1	1	1	1	1	1	0	1	1	0
s_{28}	0	0	0	0	0	1	0	0	0	1	1	1	1	1	0	1	1	1
s_{29}	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1	0	1	0
s_{30}	0	0	0	0	0	0	1	0	0	1	1	1	1	1	1	0	1	1
s_{31}	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0	0
s_{32}	0	0	0	0	0	0	0	1	0	1	1	1	1	1	1	1	0	1
s_{33}	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	0
s_{34}	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1

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BIOGRAPHICAL SKETCH

Richard Joseph Greechie was born the son of Joseph H. and Anna M. Greechie on April 12, 1941 in Boston, Massachusetts.

In June of 1958 he was graduated from Boston College High School. In June of 1962 he received the degree of Bachelor of Arts from Boston College. In September of 1962 he enrolled in the Graduate School of Wayne State University where he met Professor David Foulis. Having become interested in the work of Professor Foulis he followed him in September of 1963 to the University of Florida. At that time he enrolled in the Graduate School of the University of Florida and worked toward the degree of Doctor of Philosophy.

He is a member of the Mathematical Association of America and the American Mathematical Society.

This dissertation was prepared under the direction of the Chairman of the Candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

April 23, 1966

Dean, College of Arts and Sciences

Dean, Graduate School

Supervisory Committee:

Chairman